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Dimension and Ramsey results in partially ordered sets.

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DIMENSION AND RAMSEY RESULTS IN PARTIALLY ORDERED SETS

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DEDICATION

This thesis or dissertation is dedicated to my parents

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ABSTRACT

DIMENSION AND RAMSEY RESULTS IN PARTIALLY ORDERED SETS

Sida Wan

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In this dissertation, we have two major parts. One is the dimension results on different classes of partially ordered sets. We developed new tools and theorems to solve the bounds on interval orders using different number of lengths. We also discussed the dimension of interval orders that have a representation with interval lengths in a certain range. We further discussed the interval dimension and semi dimension for posets.

In the second part, we discussed several related results on the Ramsey theory of grids, the results involve the application of Product Ramsey Theorem and Partition Ramsey Theorem.

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CHAPTER 1

INTRODUCTION

In the first chapter, we introduce the basic concepts and notations of graphs and partially ordered sets. Further, related theorems that are used in the thesis will be provided. Some examples and proofs will be given for a more detailed introduction.

1.1.1 Introduction

In mathematics, graph theory is the study of graphs, which are mathematical structures used to model relations between objects.

Graph theory has many applications in different fields. In mathematics, graph theory can be used in geometry topology. Algebraic graph theory has close relation with graph theory that can be applied to many areas such as dynamic systems and complexity. In data science, graph structures can be used to store data, allowing data in the storage to be linked together directly and, in many cases, making the operation much more efficient. In statistics, graph theory is commonly used too. For example, in exponential random graph modeling, the different structural features in the network graph are studied to build the models for analysis and prediction.

Graph theory is also widely used in sociology for social network analysis because of its advantage in representing the social network structure.

Other commonly used fields for graph theory would include physics and chemistry, where it's an excellent model to study the three-dimensional structures of complicated simulated atomic systems. Statistics on graph-theoretic properties related to the topology of the atoms can be gathered and analyzed quantitatively. It's also commonly used in Linguistics and biology. [1]

1.1.2 Basic Notations

A graph is a pair $G = (V, E)$, where V is the ground set of the vertices of G and E is the set of edges. We can consider the elements of E to be 2-element subsets of V, therefore $E \subseteq V \times V$. We can denote $(u, v) \in E$ if an edge exists between the two vertices u and v in the ground set V. We also call u and v adjacent in such a case. A graph G is usually pictured by drawing a dot for each vertex in V and a line connecting two vertices if there is an edge between them.

Here is an example. (see Figure 1.1). The graph in the figure has a ground set $V = 1, 2, 3, 4$ and edge set $E = \{(1, 3), (2, 3), (2, 4)\}$.

1.1.3 Subgraphs and Induced Subgraphs

Let $G = (V, E)$ and $H = (V', E')$ be two graphs. H is a subgraph of G if $V' \subset V$ and $E' \subset E$. If $G \neq H$, then H is a proper subgraph of G. Further, H is an induced subgraph of G if for any two vertices x, y in V', x and y are adjacent in H if and only if they are adjacent in G.

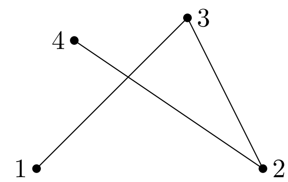


Figure 1.1: Graph example

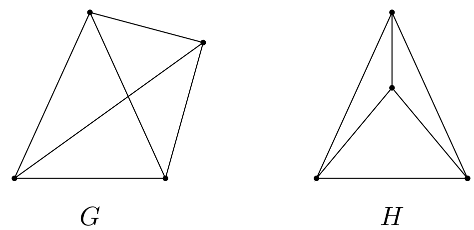


Figure 1.2: Planar Graphs

1.1.4 Graph Isomorphism

Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be two graphs. If there exits a bijection $f : V_1 \to V_2$ such that for any two vertices x and y in G, x and y are adjacent in G if and only if $f(x)$ and $f(y)$ are adjacent in H.

1.1.5 Plane and Planar Graphs

A graph $G = (V, E)$ is a plane graph if:

1: V is a set of points in the plane $\mathbf{R}^2$.

2. Every edge is a curve between two vertices, where the two points are the endpoints. Different edges have different sets of endpoints.

3. The interior of an edge contains no vertex and no point of any other edge.

Further, a graph is called planar if it is isomorphic to a plane graph. Embedding a graph G in a planar graph H is called planar embedding or drawing.

In Figure 1.2, Graph G is not a plane graph, but it is planar since it can be drawn as a plane graph H as it is shown below.

1.1.6 Contractions and Minors

A subdivision of a graph G is a graph obtained from G by replacing some edges of G with new paths between their ends, where none of these paths has an inner vertex in $V(G)$ or on another new path.

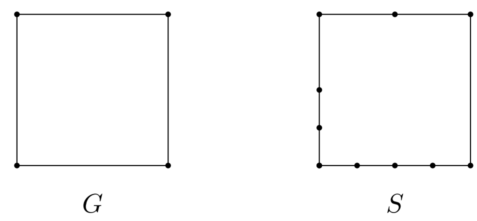


Figure 1.3: Subdivisions

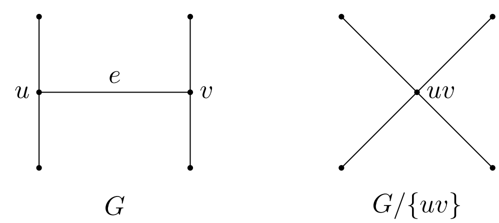


Figure 1.4: Contractions and minors

In the example in Fig 1.3, S is a subdivision of G.

Let S be a subdivision of graph G; if a graph H contains a S as a subgraph, then G is a topological minor of H.

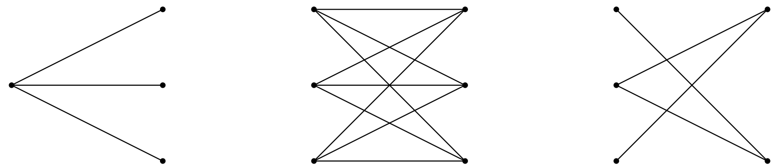
An edge contraction is an operation that removes an edge from a graph and, in the meantime, merges the two vertices that were previously connected by that edge. Here is an example where the edge e between the two vertices u and v in the graph G is contracted, resulting in a new graph $G/\{uv\}$ (see Figure 1.4).

A graph H is called a minor of graph G if H can be obtained from G by deleting edges and vertices and contracting edges.

1.1.7 Kuratowski's Throrem

Let $G = (V, E)$ be a graph; if each pair of the vertices in V is connected by an edge, then G is a complete graph. We use K^n to denote the complete graph with n vertices. For example, a triangle would be a complete graph with three vertices; and can be denoted by K^3. The graph shown in Figure 1.2 is K^4.

A bipartite graph, also called a bigraph, is a graph whose vertices can be decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent. Meanwhile, if every vertex of the first set is connected to every vertex of the second set, we call such a graph a complete bipartite graph. We use $k_{n,m}$ to denote a complete bipartite graph where one of the two sets contains



$K_{1,3}$ $K_{3,3}$ a non-complete bipartite graph

Figure 1.5: Bipartite posets

n vertices and the other set contains m vertices. Here are some examples of bipartite graphs. (see Figure 1.5)

Kuratowski and Waquer proved the following theorem that characterized the planar graph. The following statements are equivalent for a graph G:

(1) G is a planar;

(2) G contains neither K^5 nor $K_{3,3}$ as a minor;

(3) G contains neither K^5 nor $K_{3,3}$ as a topological minor;

1.2.1 Basic Notations

A partially ordered set $\mathbf{P}$ is a pair (X, P) where X is a set, and P is a reflexive, asymmetric and transitive binary relation on X. Here X is called the ground set, and P is the partial order on X. For convenience, we usually call partially ordered sets posets. In this thesis, we will just consider finite posets, where the number of elements in the ground set of the given poset is finite. To be more specific on the order relations, for any $x, y \in X$, $(x, x) \in P$ because P is reflexive. We write $x \leq y$ in P when $(x, y) \in P$. The notation $x < y$ is used when $x \leq y$ in P and $x \neq y$.

1.2.2 Example

Here is an example. Let $X = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b, c\}\}$, and $P = \{(x, y) \in X \times X : x \subseteq y\}$. $\mathbf{P} = (X, P)$ is a partially ordered set where the ground set is X and is ordered by set inclusion.

1.2.3 Comparable and Incomparable

Let $\mathbf{P} = (X, P)$ be a poset and $x, y \in X$ where $x \neq y$. We say x and y are comparable if either $x < y$ or $y < x$. Otherwise we say x and y are incomparable, denoted by $x \| y$. We say x is covered by y in $\mathbf{P}$ if $x < y$ in P and there is no vertex $z \in X$ such that $x < z$ in P and $z < y$ in P.

1.2.4 Subposet

If Y is a nonempty subset of X, we denote the restriction of P to Y by $P(Y)$, which is a partial order on Y and poset $(Y, P(Y))$ is called a subposet of $\mathbf{P}$.

1.2.5 Upset and Downset

Let $\mathbf{P} = (X, P)$ be a poset, and $x \in X$, the upset of x is $U(x) = \{y \in X : x < yinP\}$, similarly, the downset of x is $D(x) = \{y \in X : y < x \ in \ P\}$

1.2.6 Chains and Antichains

Let $\mathbf{P} = (X, P)$ be a poset. We call P a chain if every distinct pair of points in X is comparable. We also call P to be a linear order or a total order on X if $\mathbf{P}$ is a chain. Similarly, we call P an antichain if every distinct pair of points in X is incomparable. Let $\mathbf{P}' = (Y, P(Y))$ be a subposet of

P. The subset Y is a chain if the subposet $\mathbf{P'}$ is a chain, Y is an antichain if the subposet P' is an antichain.

1.2.7 Height, Length, and Width of a Poset

The height of a poset is the number of vertices in a maximum chain, while the length of the poset is one less than the height. The width of a poset is the number of vertices in a maximum antichain.

1.2.8 Minimal and Maximal, Minimum and Maximum

A point $x \in X$ is a maximal point in **P** if there is no point $y \in X$, such that $x < y$ in P. Similarly, a point $x \in$ is a minimal point if there is no point $y \in X$, such that $y < x$ in P. An element $x \in X$ is a maximum (or greatest) element of **P** if for every $y \in X$, we have $y \leq x$. Similarly, an element $x \in X$ is a minimum (or least) element of **P** if $x \leq y$ for every $y \in X$.

1.2.9 Dual of a Poset

We denote P^d to be the dual of the partial order P. $P^d = (y, x) : (x, y) \in P$. $\mathbf{P}^d = (X, P^d)$ is the dual of the poset **P**.

1.2.10 Poset Isomorphism

Let $\mathbf{P} = (X, P)$ and $\mathbf{Q} = (Y, Q)$. If there exists a bijection $f : X \to Y$, such that, $x_1 < x_2$ in P if and only if $f(x_1) < f(x_2)$ in Q. Then poset **P** is isomorphic to poset **Q**, we will write it as $(X, P) \cong (Y, Q)$.

1.2.11 Dilworth's Chain Covering Theorem

Dilworth Characterized the width of any finite poset in terms of partition of such poset into a minimum number of chains. It's a fundamental tool to study the dimension of posets and many other fields. Here is Dilworth's chain covering theorem and its dual version for antichains.

Theorem 1.2.1. *If* **P** *is a poset with width w, then there exists a partition $X = C_1 \cup C_2 \cup \cdots \cup C_w$, where C_i is a chain for $i = 1, 2, \ldots w$.*

Proof. We proof the theorem by using induction on $|x|$.

If $|X| = 1$, the result is trivial. Assume validity for $|X| < k$ and let $\mathbf{P} = (X, P)$ be a poset with $k + 1$ vertices. The case where $width(X, P) = 1$ is trivial, hence let $width(X, P) > 1$ without loss of generality. Let C be a chain in $\mathbf{P}$, $width(X - C, P(X - C)) = w < n$, then by the assumption, we can partition $X - C$ into w chains. Then we can partition X into $w + 1$ chains since we simply just need to add the chain C into that partition. Since $w + 1 \leq n$, so we have a partition of X into at most n chains. Meanwhile X can be partitioned into at least n chains since $\mathbf{P}$ has width n, this complete the proof for the case when $width(X - C, P(X - C)) < n$.

When $width(X - C, P(X - C)) = n$. If there is a loose point x in $\mathbf{P}$. Then there is a trivial chain $C = \{x\}$ and the $width(X - C, P(X - C)) < n$. So we might just assume that there is no loose point in this case for when it happened; it belongs to the case above, which is already proven.

Choose a maximal point y and a minimal point x where $x < y$ in P. Let $C = \{x, y\}$, hence by our assumption, $width(X - C, P(x - C)) = n$. Hence the poset $\mathbf{Q} = (X - C, P(X - C))$ contains an antichain with n elements, name is $A = \{x_1, x_2, \ldots, x_n\}$. By the induction hypothesis, we can partition each of $U[A]$ and $D[A]$ into n chains, and notice that $U[A] \cap D[A] = [A]$. Let the two partition to be $U[A] = c_1 \cup c_2 \cup \cdots \cup c_n$ and $D[A] = l_1 \cup l_2 \cup \cdots \cup l_n$ where $x_i \in c_i \cap l_i$ for $i = 1, 2, \ldots, n$. This gives us the partition $X = (c_1 \cup l_1) \cup (c_2 \cup l_2) \cap \cdots \cup (c_n \cap l_n)$ which is the desired chain and completes the prove. $\square$

Here is the dual version for partitioning antichains.

Theorem 1.2.2. *If $\mathbf{P}$ is a poset with height h, then there exists a partition $X = A_1 \cup A_2 \cup \cdots \cup A_h$, where A_i is an antichain for $i = 1, 2, \ldots h$.*

1.2.12 Hasse Diagram

Let x, y be two vertices in a poset $\mathbf{P} = (X, P)$, we say y covers x is $x < y$ in P and there is no such $z \in X$, such that $x < z < y$ in P.

A Hasse diagram is a form of a drawing of a poset $\mathbf{P}$. We usually choose an Euclidean plane for each vertex x and y, where $x < y$ in P, we put x below y in the plane, and we draw a line or curve that goes upward from x to y whenever Y covers x.

Here is an example of Hasse diagram (see Figure 1.6) of a $\mathbf{P} = (X, P)$ with $X = \{a, b, c, d, e, f, g\}$.

Observe that $a < c$ in P; $b \| d$ in P. There is no minimum or maximum elements in poset $\mathbf{P}$, but we do have minimal elements a, b and d and maximal elements f and g.

1.2.13 Extensions and Realizers

Let $\mathbf{P} = (X, P)$ be a poset, Q is an extension of partial order P if $P \subseteq Q$, that is $x < y$ in P implies $x < y$ in Q for all $x, y \in X$. Q is a linear extension of P if Q is a total order.

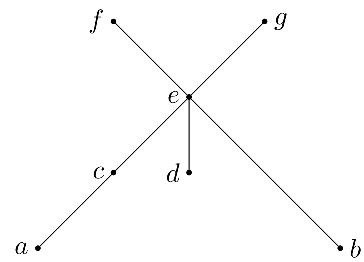


Figure 1.6: Poset Example

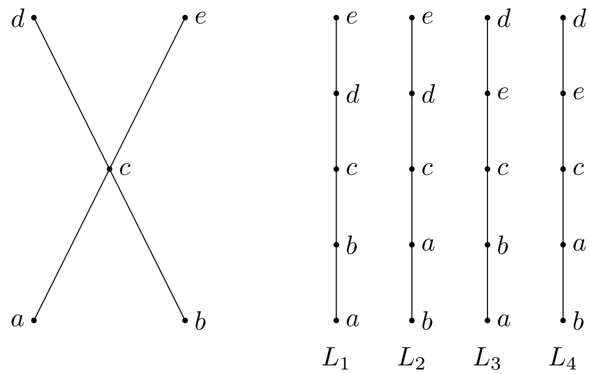


Figure 1.7: Linear Extensions

It's relatively easy to see that for every partial order P on X, there exists a linear extension of P. Further, let $x, y \in X$ and $x \| y$ in P, there exists a linear extension L_1 of P such that $x < y$ in L_1 and there exists a linear extension L_2 of P such that $y < x$ in L_2.

1.2.14 Poset Dimension

For a poset $\mathbf{P} = (X, P)$, Dushnik and Miller [12] introduced the concept of dimension for poset. The dimension of poset $\mathbf{P}$ is denoted by $\dim(\mathbf{P})$, which is the least positive integer t for which there exists a family of linear extensions $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of P such that the intersection of the t linear orders is p, i.e., $P = \cap\mathcal{R} = \bigcap_{i=1}^{t} L_i$.

1.2.15 Example

Here is a simple poset $\mathbf{P} = (X, P)$, where $X = \{a, b, c, d, e\}$. The Hasse diagram is drawn in the figure (see Figure 1.7). There are 4 linear extensions of P: L_1, L_2, L_3, L_4. The dimension of $\mathbf{P}$ is larger than 1 since it's not a chain, and we can observe that L_1, L_4 forms a realizer of poset $\mathbf{P}$ since all the incomparable pairs are reversed in the two linear extension. Hence poset $\mathbf{P}$ has dimension 2.

1.2.16 Standard Example

Dushnik and Miller used S_n to denote the standard example with $2n$ vertices where $S_n = (X, P)$ is a height two poset. $X = \{a_1, a_2, ..., a_n\} \cup \{b_1, b_2, ..., b_n\}$, and $a_i < b_j$ in P if and only if $i \neq j$, for $i, j = 1, 2, ..., n$. Here is a picture of S_4 below(see Figure 1.8).

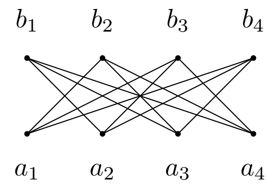


Figure 1.8: S_4

It is known that the dimension of a typical example S_n is n. Also, notice that the standard example S_n is n-irreducible, which means removing any point of S_n will reduce the dimension of the poset.

1.2.17 Alternating cycles

Let $P = (X, P)$ be a poset, and let $inc(X, P) = \{(x, y) \in X \times X : x \| y \ in \ P\}$. An alternating cycle with length k in (X, P) is a sequence of ordered pairs $(x_i, y_i) : 1 \leq i \leq k$ from $inc(X, P)$ with $y_i \leq x_{i+1}$ in P (cyclically, i.e $y_i \leq x_{i+1}$ in P for $i = 1, 2, ..., k-1$ and $y_k \leq x_1$ in P). If $y_i \leq x_j$ if and only if $j = i + 1$ (cyclically), then we call it a strict alternating cycle.

Here is an example shown in Figure 1.9.

Observe that $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is an alternating cycle but not a strict alternating cycle since $y_1 < x_2$ but also $y_1 < x_3$.

Meanwhile we have a strict alternating cycle $(x_1, y_1), (x_2, y_3)$.

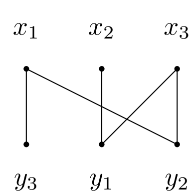


Figure 1.9: Alternating cycle

1.2.18 Transitive Closure

Let R be a binary relation on a set X. We use $tr(R)$ to denote the transitive closure of R, where $tr(R) = \{(x, y) \in X \times X :$ there exist a sequence $a_0, a_1, \ldots, a_n$ such that $(a_i, a_{i+1}) \in R$ for $i = 1, 2, \ldots, n-1$ with $a_0 = x$ and $a_n = y$.

Here is an important lemma proved by Trotter and Moore in [50].

Lemma 1.2.3. *Let $\mathbf{P} = (X, P)$ be a poset and let $\mathbf{S} \subseteq inc(X, P)$. The following three statements are equivalent:*

(1) $tr(P \cup \mathbf{S})$ is not a partial order on X.

(2)S contains an alternating cycle.

(3)S contains a strict alternating cycle.

Proof. To proof (2) $\to$ (1) first. Suppose $\mathbf{S}$ contains an alternating cycle $\{(x_i, y_i) : 1 \leq i \leq k\}$ (cyclically). Let $Q = tr(P \cup S)$. Suppose Q is a partial order on X, then Q is transitive. Since (y_i, x_{i+1}) is in P, we have $(x_i, x_{i+1}) \in Q$, which implies $x_1 = x_2 = \ldots = x_n$. Meanwhile since $(x_i, y_i) \in Q$ and $(y_i, x_{i+1}) \in Q$ implies $x_i = y_i = x_{i+1}$ for $i = 1, 2, \ldots, k$ which is clearly a contradiction. Hence (2) $\to$ (1)

Then, we shall prove (1) $\to$ (3). Let $Q = tr(P \cup S)$, since Q is not a partial order on X, and Q is reflexive and transitive, Q must fail to be antisymmetric. Then there exists s sequence $a_1, a_2, \ldots, a_n$ of points in X, not all of them are the same point, such that $(a_i, a_{i+1}) \in \mathbf{S}$ for $i = 1, 2, \ldots, n$(cyclically). Take a sequence $b_1, b_2, \ldots, b_s$ which has minimal length. We will construct a strict alternating cycle from the sequence.

Without loss of generality, assume $(b_1, b_2) \in \mathbf{S}$. let $x_1 = b_1$ and $x_2 = b_2$. Suppose we have defined (x_i, j_i) with $y_i = u_j$ $(2 \leq j \leq s)$. If $(b_j, b_{j+1}) \in \mathbf{S}$, let $x_{i+1} = y_i = b_j$. If $(b_j, bj+1) \in \mathbf{P}$, then $(b_{j+1}, bj+2) \in \mathbf{S}$, since otherwise it would violet the minimality. Let $x_{i+1} = b_{j_1}$ and $y_{i+1} = b_{j+2}$, this construction gives us a strict alternating cycle. Hence (1) $\to$ (3).

Since (3) $\to$ (2) is the trivial case, the proof of the lemma is complete. $\square$

1.2.19 Irreducible posets

A poset $\mathbf{P}$ is said ot be $t-irreducible$ for some integer $t \geq 2$, if $\dim(\mathbf{P}) = t$, and for every nonempty proper subposet $\mathbf{Q}$ of $\mathbf{P}$, $\dim(\mathbf{Q}) < t$.

A poset is irreducible if it is $t-irreducible$ for some integer $t > 2$.

The only 2-irreducible poset is the 2-element antichain. Trotter and Moore gave a collection of all $3-irreducible$ posets using Gallai's list of forbidden subgraphs for comparability graphs [48]. In contrast, Kalle and Rival used different approaches to determine all the 3-irreducible posets [27]. However, it is hopeless to determine the lists of all 4-irreducible posets for how complex a 4-irreducible poset can get.

1.2.20 A/B

Let $\mathbf{P} = (X, P)$ be a poset and let A and B be a subsets of X. Let L be a linear extension of P, we say A is over B in L if $a > b$ in L whenever $a \in A$, $b \in B$ and $a \| b$ in P, it's usually denoted by A/B.

Here is an important lemma proved by Hiraguchi [25].

Lemma 1.2.4. *Let $\mathbf{P} = (X, P)$ be a poset and let $C \subset X$ be a chain. Then there exists linear extensions L_1, L_2 of P, such that:*

(i) $y < x$ in L_1 for every $x, y \in X$ with $x \in C$ and $x \| y$ in P.

(ii) $y > x$ in L_1 for every $x, y \in X$ with $x \in C$ and $x \| y$ in P.

Proof. Let $S_1 = \{(y, x) \in X \times X : x \in C, x \| y \text{ in } P\}$, it's easy to see that there is no alternating cycle in S_1, hence by lemma 1.2.3, the transitive closure Q of $(P \cup S_1)$ is a partial order on X. Take any linear extension of Q would give us a L_1 that satisfy the conditions in (i). Similar way to find an L_2. □

Lemma 1.2.5. *Let $\mathbf{P} = (X, P)$ be a poset. If x is a maximum (or minimum) element of the partially ordered set (X, P), then* $\dim((X - x, P(X - x)) = \dim(\mathbf{P})$

Lemma 1.2.6. *Let $\mathbf{P} = (X, P)$ be a poset and there are subsets X_1 and X_2 of subsets of X such that $P = P(X_1) \cup P(X_2)$, and elements in X_1 are incomparable to elements in X_2, then* $\dim(\mathbf{P}) = \max(\dim(P_1), \dim(P_2))$.

Lemma 1.2.7. *(Hiraguchi [25])Let $\mathbf{P} = (X, P)$ be a poset and C_1, C_2 be disjoint chains in the poset. If $C_1 \| C_2$ in $\mathbf{P}$, i.e., for all $x \in C_1$ and all $y \in C_2$, we have $x \| y$ in P. Then there exist linear extensions L_1 abd L_2 of P such that:*

1. *X/C_1 and C_2/X in L_1, and*
2. *X/C_2 and C_1/X in L_2.*

The proof is relatively easy, for the first condition, observe that there is no such element s such that $x < s < y$ for some $x \in C_1$ and $y \in C_2$ since $x \| y$ no matter which x, y we pick. No alternating cycle will be formed; hence the linear extension L_1 can be constructed. The same argument for L_2.

There is an immediate consequence of the lemma.

Theorem 1.2.8. *Let $\mathbf{P} = (X, P)$ be a poset and C_1, C_2 be disjoint chains and $C_1 \| C_2$ in P. Then*

$\dim(\mathbf{P}) \leq \dim(X - (C_1 \cup C_2), P(X - (C_1 \cup C_2)))$ *+2*

Theorem 1.2.9. *Interpolation Property Let $\mathbf{P} = (X, P)$ be a poset, let Y be a subset of X, and Q be an extension of $P(Y)$. Then there exists an extension R of P such that $R(Y) = Q$.*

Proof. Let $S = inc(X, P) \cap Q$, since Q is an extension of $P(Y)$, then S contains no alternating cycle. Let $R = tr(P \cup S)$, then R is a partial order on X by Lemma 1.2.3. Hence R is an extension of P such that $R(Y) = Q$. □

When we have the linear extension of a subposet, it's easy to apply the interpolation property to extend the linear extension of the subposet to a linear extension of the parent poset.

Here is a theorem proved by Hiraguchi using interpolation property. It's known as the continuity property of regular dimension on posets.

Theorem 1.2.10. *Let $\mathbf{P} = (X, P)$ be a poset with at least 2 vertices, let $x \in X$. Then*

$$\dim(X, P) \leq 1 + \dim(X - \{x\}, P(X - \{x\}))$$

Proof. Let $Y = X - \{x\}$, and $\mathbf{Q} = (Y, P(Y))$. Assum $\dim(Q) = t$. Let $\mathbf{R} = \{L_1, L_2, \ldots, L_t\}$ be a realizer of Q. Apply interpolation property to obtain linear extensions $S_1, S_2, \ldots, S_{t-1}$ where $S_i(Y) = L_i$ for $i = 1, 2, \ldots, t-1$. Let $S_t = L_t(D(x)) < \{x\} < M_t(Y - D(x))$ and let $S_{t+1} = M_t(Y - U(x)) < \{X\} < M_t(U(x))$ (see Figure 1.10. $R' = \{S_1, S_2, \ldots, S_{t+1}\}$ is a realizer of P. To prove this, take any pair (a, b) in $\mathbf{P}$. If $a < b$ in P, then $a < b$ in each of $L_i's$, since we constructed $S_i's$ by interpolation lemma, it's clear that $a < b$ in each of $S_i's$. Now assume $(a, b) \in inc(X, P)$, if $x \in a, b$, without loss of generality, say $x = a$, we can easily see that $x < b$ in S_t, and $x > b$ in S_{t+1}. Assume that $x \notin a, b$, if a, b got reversed in $\{L_1, L_2, \ldots, L_{t-1}\}$, then it's reversed in $\{S_1, S_2, \ldots, s_{t-1}\}$ by interpolation property. Without loss of generality, the only case we need to consider about is that when $a < b$ in $\{L_1, L_2, \ldots, L_{t-1}\}$ and $b < a$ in L_t. Hence we have $a < b$ in $\{S_1, S_2, \ldots, S_{t-1}\}$. If $a < b$ in S_t and S_{t+1}, then R' would fail to be a realizer of $\mathbf{P}$. But this requires $a \in D(x)$ and $b \in U(x)$, which would implie that $a < x < b$, contradicts to the assumption that $a \| b$ in $\mathbf{P}$. Hence every incomparable pair is reversed in R', which completes the proof.

□

A similar technique, along with the interpolation property, can prove the continuity of interval order dimension by W.T Trotter, as the reader will see in the future chapters.

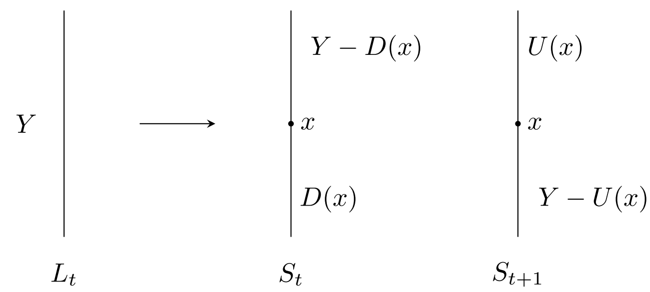


Figure 1.10: Continuity

Here is the removable conjecture for posets, one of the most well-known conjectures in the poset field.

Conjecture 1.2.11. *If* $\mathbf{P} = (X, P)$ *is a poset with* $|X| \geq 3$*. Then there exist distinct points* $x, y \in X$ *such that* $\dim(\mathbf{P}) \leq \dim(X - \{x, y\}, P(x - \{x, y\}))$.

Trotter proved the removable pair conjecture for interval order dimension, the proof will be provided in chapter 3, Csaba Biro proved the removable pair conjecture for fractional dimension [4], but the original conjecture remains open.

A couple of theorems were also proved that the removal of a pair for specific posets would decrease the dimension of the poset by at most 1.

Lemma 1.2.12. *Let* $\mathbf{P} = (X, P)$ *be a poset, and let* x_1, x_2 *be distinct maximal elements in the poset. If* $D(x_1) \subseteq D(x_2)$*, then* $\dim(X, P) \leq 1 + \dim(X - \{x_1, x_2\}, P(X - \{x_1, x_2\}))$

Proof. Let $t = \dim(X - \{x_1, x_2\}, P(X - \{x_1, x_2\})$.

Take a realizer $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of $(X - \{x_1, x_2\}, P(X - \{x_1, x_2\}))$. Let $S_i = L_i < \{x_2\} < \{x_1\}$ for each $i = 1, 2, \ldots t$, clearly $S_i's$ are linear extensions of P. Then let $S_{t+1} = D(x_1) < \{x_2\} < D(x_2) - D(x_1) < \{x_2\} < X - (D(x_1) \cup D(x_2))$, it's easy to check that S_{t+1} is a linear extension of P and $\{S_1, S - 2, \ldots, S_{t+1}\}$ is a realizer of $\mathbf{P}$. □

1.2.21 Hiraguchi's Inequality

Dilworth [11] proved the following fundamental theorem on the dimension of posets.

Theorem 1.2.13. *Let* $\mathbf{P} = (X, P)$ *be a poset with width* w*. Then* $\dim((P)) \leq w$.

Proof. By Dilworth's antichain partitioning theorem, we can partition X into w antichains. Denoted by $X = C_1 \cup C_2 \cup \cdots \cup C_w$. We will construc the realizer of $\mathbf{P}$ with w linear extensions of P. By lemma 1.2.4, we can take a linear extension L_i of P such that C_i/X in L_i for each $i = 1, 2, \ldots, w$. Observe that $\mathcal{R} = \{L_1, L_2, \ldots, L_w\}$ is a realizer of $\mathbf{P}$, since for each $(x, y) \in inc(X, P)$, there exists an $j \in [w]$ such that $x \in C_j$ and then $x < y$ in L_j by the construction of linear extensions. Hence completed the proof. □

Here is a fundamental theorem on the bound of a poset with relatively few vertices.

Theorem 1.2.14. *Let $\mathbf{P} = (X, P)$ be a poset. If the cardinal of the ground set $|x| \leq 5$, then* $\dim(\mathbf{P}) \leq 2$.

Proof. First, if $|X| = 3$, suppose the poset has dimension three, since $\dim(\mathbf{P} < \text{width}(P)$, the poset has to be an antichain of three elements which would have dimension 2.

If $|X| = 4$. Suppose the poset has dimension three. There is a 3-element chain, and lemma1.2.6 implies that the poset has the fourth vertex as a minimum or maximum element, for otherwise, there are disconnected points, and the maximal dimension would be the maximal dimension of a poset with three elements, which would be two as it is proved above. However, lemma 1.2.5 told us that the poset has dimension two if the poset has a maximum (or minimum) element.

If $|X| = 5$, by lemma 1.2.5, there is no such a maximum or minimum element. Otherwise, the dimension of the poset would be two. If there is an antichain of four elements, the dimension of such poset would be two followed by the similar argument as above. Hence, we might assume that there is an antichain of three elements. Without loss of generality, we might assume that the poset has three minimal elements and two maximal elements with a connected graph by lemma1.2.6. There are only four posets up to isomorphism in this case. All of them are of dimension two. □

1.2.22 Example

If $\mathbf{P} = (X, P)$ is a poset with $|X| \leq 7$, then $dim(\mathbf{P}) \leq 4$

The following are a couple of essential theorems proved by Hiraguchi.

Theorem 1.2.15. *Let $\mathbf{P} = (X, P)$ be a poset and let $C \subseteq X$ be a chain with $X - C \neq \emptyset$. Then:*

$$\dim(X, P) \leq \dim(X - C, P(X - C)) + 2$$

Proof. The proof also applies interpolation lemma. Suppose $\dim(X - C, P(X - C)) = t$, we shall also build a realizer $\mathcal{R} = S_1, S_2, \ldots, S_t$ for $(X - C, P(X - C))$ first. Then by interpolation property, take a linear extension L_i of S_i for P for each $i = 1, 2, \ldots, t$, where $L_i(X - C) = M_i$. Then by

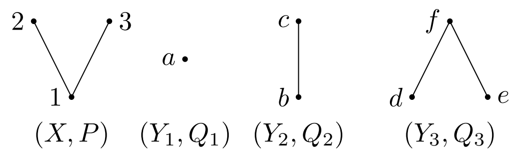


Figure 1.11: (X, P) and $\mathcal{F}$

lemma 1.2.4, take linear extensions L_{t+1} and L_{t+2} where C/X in L_{t+1} and X/C in L_{t+2}. it's easy to check that $\{L_1, L_2, \ldots, L_{t+1}, L_{t+2}\}$ is a realizer of $\mathbf{P}$. □

Before introducing the following dimension theorem on posets concerning the cardinality of the vertices, we shall introduce the lexicographic sum and decomposable.

1.2.23 Lexicographic Sum

Let $\mathbf{P} = (X, P)$ be a poset and let $\mathcal{F} = \{(Y_x, Q_x : x \in X\}$ be a family of posets indexed by the points in (X, P). Define the lexicographic sum of $\mathcal{F}$ over (X, P), denoted by $\sum_{x\in(X,P)}(Y_x, Q_x)$ as the poset (Z, P_z) where $Z = \{(x, y) : x \in X, y \in Y_x\}$ and $(x_1, y_1) < (x_2, y_2)$ in P_z if and only one of the following two holds:

1. $x_1 < x_2$ in P.
2. $x_1 = x_2$ and $y_1 \leq y_2$ in Q_x

1.2.24 Example of Lexicographic Sum

This may look complicated in first glance, but it is actually straightforward like its name. Here is a simple example (see Figure 1.11). We have a poset $\mathbf{P}$ whose ground set is $\{1, 2, 3\}$, and a family of posets $\mathcal{F} = \{(Y_x, Q_x : x \in X\}$ indexed by the points in (X, P), i.e., $\{1, 2, 3\}$. Where $(Y_1, Q_1), (Y_2, Q_2)$ and (Y_3, Q_3) is shown in the picture too and have gounnd sets $\{a\}$, $\{b, c\}$ and $\{d, e, f\}$ respectively.

The lexicographic sum of $\mathcal{F}$ over (X, P) is the poset $\mathbf{P_3} = (Z, P_z)$ in the picture (see Figure 1.12).

Notice that $(1, a) < (3, d)$, $(1, a) < (3, f)$ and $(1, a) < (3, e)$ in the poset of the lexicographic sum $\mathbf{P_3}$ because $1 < 3$ in (X, P). $(3, d) < (3, f)$ since $3 = 3$ in P and $d < f$ in Q_3.

The following lemma was proved by Hiraguchi [25].

Lemma 1.2.16. *Let $P = (X, P)$ be a poset and let $\mathcal{F} = \{(Y_x, Q_x : x \in X\}$ be a family of posets indexed by the points in (X, P). Then*

$\dim(\sum_{x\in(X,P)}(Y_x, Q_x)) = \max\{\dim(X, P), \max\{\dim(Y_x, Q_x) : x \in X\}\}$

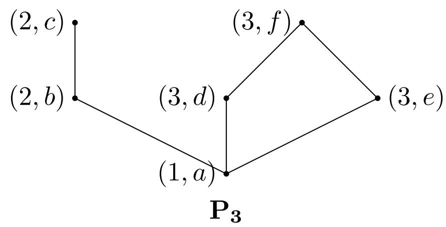


Figure 1.12: Lexicographic Sum of $\mathcal{F}$ over (X, P)

Proof. Let $t = \max\{\dim(X, P), \max\{\dim(Y_x, Q_x) : x \in X\}\}$. Let the poset $(Z, P_z) = \sum_{x \in (X,P)} (Y_x, Q_x)$. Take a realizer $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of P. For each $x \in X$, let $\{M_{x1}, M_{x2}, \ldots, M_{xt}\}$ be a realizer of Q_x. For each i in $[t]$. We have the linear extension L_i for (X, P), replace reach x in the linear extension L_i by the linear extension M_{xi} to gain a new linear extension S_i on the set $Z = \{(x, y) : x \in X, y \in Y_x\}$, i.e., the ground set of the poset that denote the lexicographic sum. Repeat this for all the pair $(L_i, M_i)'s$ to build $S_i's$, where $i \in [t]$. It's obvious that $\{S_1, S_2, \ldots, S_t\}$ is a realizer for the lexicographic sum. $\square$

We call a lexicographic sum $\sum_{x \in (X,P)} (Y_x, Q_x)$ trivial is $|X| = 1$ or $|Y_x| = 1$ for all $x \in X$.

1.2.25 Decomposable

We say that a poset (Z, P) is decomposable with respect to lexicographic sums if it is isomorphic to a nontrivial lexicographic sum; otherwise, it is indecomposable with respect to lexicographic sums.

The following proposition is straightforward and fundamental.

Proposition 1.2.17. *Let* $\mathbf{P}$ *be a* $t-$ *irreducible poset for some* $t \geq 2$. *Then* $\mathbf{P}$ *is indecomposable with respect to lexicographic sums.*

Here are some important observations on lexicographic sums. It's easy to see that the only indecomposable disconnected poset is the 2-element antichain trivial case.

The only indecomposable poset with a greatest (or least) element is a 1-element poset. Take the poset with a greatest element case, use the notations mentioned above for lexicographic sum. Suppose the poset has more than one element. In that case, it is isomorphic to a lexicographic sum where $|X| = 1$ and Y_x is the whole poset. Similar proof for the posets with a least element.

1.2.26 Duplicated holdings

Let $\mathbf{P} = (X, P)$ be a poset, $x, y \in X$. We say that x and y are duplicated holdings if $U(x) = U(y)$ and $D(x) = D(y)$.

This group of vertices that have the same upset and downset is an equivalence relation on X. In a poset with no duplicated holdings, all equivalence classes are singletons. It's also easy to see that the only irreducible poset with duplicated holdings is a 2-element antichain.

Theorem 1.2.18. *Let* $\mathbf{P} = (X, P)$ *be a poset with* $|X| \geq 4$. *Then* $\dim(\mathbf{P}) \leq |X|/2$ *[25].*

Proof. The theorem was proved by contradiction. Suppose the theorem is not true, choose a poset $\mathbf{P} = (X, P)$ with $|X|$ as small as possible where $t = \dim(\mathbf{P}) > |X|/2$. Hence the poset $\mathbf{P}$ is $t - irreducible$ by definition. From theorem 1.2.14 and exercise 1.2.22 , we know that $t \geq 4$, since $t > 2$, if $t = 3$, the smallest $|x|$ has to be 6, but in this case $\dim(\mathbf{P}) \leq |X|/2$, which is a contradiction. So let's assume that $t \geq 4$. Thus, (X, P) is a connected indecomposable poset. Hence there is no maximum element if $\mathbf{P}$.

Assume x_1, x_2 be two distinct maximal elements of $\mathbf{P}$. Suppose that $D(x_1) \subseteq D(x_2)$. By lemma1.2.12 and the assumption, we have:

$$|X|/2 < t = \dim(X, P) \leq \dim(X - \{x_1, x_2\}, P(X - \{x_1, x_2\})) + 1 \leq |X - \{x_1, x_2\}| + 1 = |X|/2$$

Which is clearly a contradiction. Which implies that $D(x_1)$ and $D(x_2)$ are not a subset of each other by symmetry.

Hence there is an element $a \in D(x_1) - D(x_2)$ and an element $b \in D(x_2) - D(x_1)$. Then the two chains $C_1 = \{a, x_1\}$ and $C_2 = \{b, x_2\}$ that are disjoint chains and $C_1 \| C_2$. Hence by lemma 1.2.7 we conclude that:

$$|X|/2 < t = \dim(X, P) \leq \dim(X - (C_1 \cup C_2), P(X - (C_1 \cup C_2))) + 2 \leq |X|/2$$

Which is a contradiction that completes the proof. □

Theorem 1.2.19. *(Kimble [30], Trotter [47]) Let* $\mathbf{P} = (X, P)$ *be a poset and let* $A \subseteq X$ *be an antichain. Then* $\dim(\mathbf{P}) \leq \max\{2, |X - A|\}$.

Theorem 1.2.20. *(Trotter [47]) Let* $\mathbf{P} = (X, P)$ *be a poset and let* $A = \max(X, P)$. *If* $X - A \neq \emptyset$. *Then* $\dim(\mathbf{P}) \leq 1 + \text{width}(X - A, P(X - A))$.

Theorem 1.2.21. *(Trotter [47]) Let* $\mathbf{P} = (X, P)$ *be a poset and let* $A \subseteq X$ *be an antichain and* $X - A \neq \emptyset$. *Then* $\dim(\mathbf{P}) \leq 1 + 2\,\text{width}(X - A, P(X - A))$.

1.2.27 Critical pairs

Let $\mathbf{P} = (X, P)$ be a poset, let $(x, y) \in inc(X, P)$. We call the ordered pair (x, y) a critical pair in poset $\mathbf{P}$ if:

1. $z < x$ in P implies $z < y$ in P, and
2. $y < w$ in P implies $x < w$ in P.

for all $z, w \in X - x, y$.

Here is a proposition proved by Rabinovitch and Rival [39].

Proposition 1.2.22. *Let $P = (X, P)$ be a poset and let $\mathcal{R}$ be a family of linear extensions of P. Then the following statements are equivalent:*

(1). $\mathcal{R}$ is a realizer of P.

(2). For every critical pair (x, y) in the poset $\mathbf{P}$, there is a $L \in \mathcal{R}$ for which $y < x$ in L.

Proof. (1) $\rightarrow$ (2) is trivial since every incomparable pair is reversed in some linear extension of a realizer. Now suppose (2) hold, we want to show that $\mathcal{R}$ is a realizer of poset $\mathbf{P}$. Take an arbitrary incomparable pair (u, v). It is sufficient to show that $v < u$ in some $L \in \mathcal{R}$.

We do induction on $n = f(u, v)$, where $n = |[(D(u) \cap I(V)] \cup [U(v) \cap I(u)]|$. If $n = 0$, then (u, v) is a critical pair, we are good. Assume $v > u$ in some $L \in \mathcal{R}$ when $n \leq k$. Consider a pair (u, v) with $f(u, v) = k + 1$. There is either a v_1 where $v_1 > v$ and $v_1 \| u$ or there exists an u_1 such that $u_1 < u$ and $u_1 \| v$. In the first case $f(u_1, v) \leq k$, then there is a linear extension L in $\mathcal{R}$ such that $v < u_1 < u$. Similarly, in the latter case, we have $f(u, v_1) \leq k$, then there is a linear extension L in $\mathcal{R}$ such that $v < v_1 < u$. $\square$

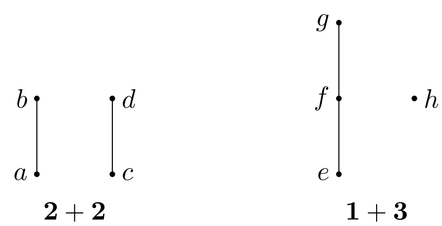


Figure 1.13: $\mathbf{1+3}$

1.3. INTERVAL ORDERS AND SEMIORDERS

1.3.1 Interval orders

A poset $\mathbf{P} = (X, P)$ is an interval order if there is a function f which assigns each $x \in X$ a closed interval $[l_x, r_x]$ of the real line $\mathcal{R}$ such that $x < y$ in P if and only if $r_x < l_y$ in $\mathcal{R}$.

1.3.2 $\mathbf{2+2}$ and $\mathbf{1+3}$

A $\mathbf{2+2}$ poset is a poset with two chains of length two that are incomparable. Similarly, a $\mathbf{1+3}$ is a poset with a chain of length three and a vertex that is incomparable to all the vertices in the chain. Below is the Hasse diagram of the two special posets (see Figure 1.13).

Clearly, if $\mathbf{P}$ is an interval order, then it does not contain a $\mathbf{2+2}$ as a subposet. Here is a short proof. Suppose the $\mathbf{2+2}$ has four vertices a, b, c, d as the ground set like the picture in Figure 1.13. then $r_a < l_b$. If we determine the interval for vertex c first, then $l_c < r_a$ and $r_c > l_b$ for c is incomparable to both a and b. Since $c < d$ in P, $l_d > r_c$, then $l_d > r_c > l_b > r_a$, hence we have $d > a$ in P by the definition of interval orders, which is a contradiction. Similarly, if we determine the interval for vertex d first, we could not find an interval for c. Hence the proof is completed. Further, Fishburn proved that a poset $\mathbf{P}$ is an interval order if and only if it does not contain a $\mathbf{2+2}$ as a subposet.

If there are duplicated holdings in an interval order, then we can assign the same intervals to the duplicated holdings. Also, it is easy to prove that adding duplicated holdings to an interval order with a dimension of at least two does not change the dimension of the interval order. Hence, we are more interested in interval orders with no duplicated holdings in most cases.

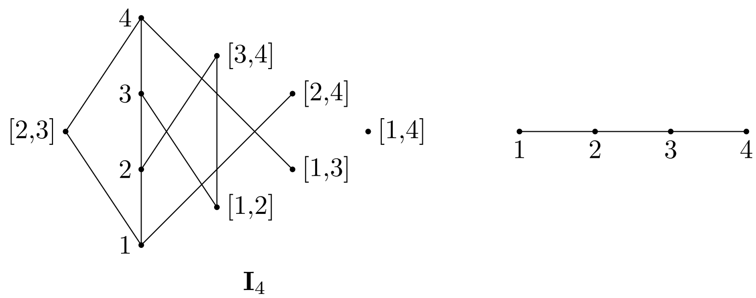


Figure 1.14: $\mathbf{I}_4$

1.3.3 Proposition

Let $\mathbf{P}$ be a poset. Recall that $D(x) = \{y \in X : y < x \text{ } in \text{ } P\}$ and $U(x) = \{y \in X : y > x \text{ } in \text{ } P\}$ The following statements are equivalent.

(1). $\mathbf{P}$ is an interval order.

(2). $\mathcal{D}(\mathbf{P}) = \{D(x) : x \in X\}$ is a chain in $\mathbf{2}^X$.

(3). $\mathcal{U}(\mathbf{P}) = \{U(x) : x \in X\}$ is a chain in $\mathbf{2}^X$.

1.3.4 Canonical interval order

We use $\mathbf{I}_n$ to denote the canonical interval order determined by the set of all closed intervals with distinct integer end points from $[n]$. Where $[n] = 1, 2, \cdots, n$. Here is an example of $\mathbf{I}_4$ (see Figure 1.14):

Theorem 1.3.1. *The dimension of canonical interval orders is unbounded.*

Proof. We can apply Ramsey's theorem for the proof.

Recall the following form of Ramsey's theorem:

For integer s, l, m, there exists a number $n_0 = R(s, l, m)$ such that if $n > n_0$, then no matter how we partition the n elements into l(or fewer) parts, there is an s elements subset of the n elements, where every m element subsets of the i elements are in the same part.

Assume that there is an upper bound k for canonical interval orders. For an interval order $\mathbf{I}_n$, let $L_1, \cdots, L_k$ be a realizer of $\mathbf{I}_n$. For each triple a, b, c with $a < b < c$, where $a, b, c \in [n]$. We assign the set a, b, c to class i where L_i is the first linear extension in which $[b, c]$ is less that $[a, b]$. There will be k(or fewer) classes.

Take $m = 3$, $s = 4$, $l = k$. By Ramsey's theorem, there exists an n_o, for all $n > n_0$, there is a set with 4 elements say a, b, c, d with $(a < b < c < d)$, such that every 3-elements subsets of it is in the

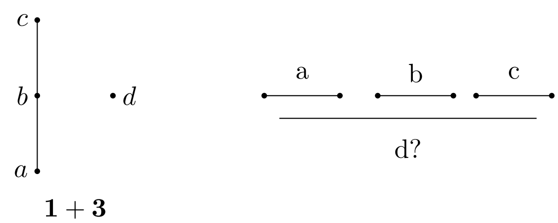


Figure 1.15: $\mathbf{1}+\mathbf{3}$

same class say t. Hence for the subset a, b, c, $[b, c] < [a, b]$ in L_t; for the subset b, c, d, $[c, d] < [b, c]$ in L_t. They would imply that $[c, d] < [a, b]$ in L_t, which is a contradiction. Hence the dimension of canonical interval orders is unbounded. $\square$

Furedi, Hajnal and Trotter gave an estimate of the canonical interval dimension in 1991, which also provides an estimate of the growth rate of the dimension of canonical interval orders. The theorem is shown below:

Theorem 1.3.2. $\dim\{\mathbf{I}_n\} = lglgn + (\frac{1}{2} + o(1))lglglgn$

The following result is immediate:

Proposition 1.3.3. *Let $\mathbf{P} = (X, P)$ be an interval order with $|X| = n$. Then $\mathbf{P}$ is isomorphic to a subposet of the canonical interval order $\mathbf{I}_{2n}$.*

Kierstead and Trotter proved the following theorem: if an interval order has a sufficiently large dimension, then it contains any given interval order with a certain number of vertices.[29]

Theorem 1.3.4. *Let $\mathbf{P} = (X, P)$ be an interval order with $|X| = n$. If $\mathbf{Q}$ is an interval order with $\dim(\mathbf{Q}) \geq 30n - 6$, then $\mathbf{Q}$ contains a subposet isomorphic to $\mathbf{P}$.*

1.3.5 Semiorders

An interval order $\mathbf{P} = (X, P)$ is called a semiorder if $\mathbf{P}$ has an interval representation where each interval has the same length.

Usually, we denote the length by 1, which can be changed by scale. An immediate observation is that any poset that contains a $\mathbf{1}+\mathbf{3}$ as a subposet is not a semiorder.

See the graph in Figure 1.15, in any representation of the $\mathbf{1}+\mathbf{3}$, the interval for d must properly include the interval for b, which is impossible.

Further, Scott and Suppes [45] proved that an interval order $\mathbf{P}$ is a semiorder if and only if $\mathbf{P}$ does not contain $\mathbf{1}+\mathbf{3}$ as a subposet.

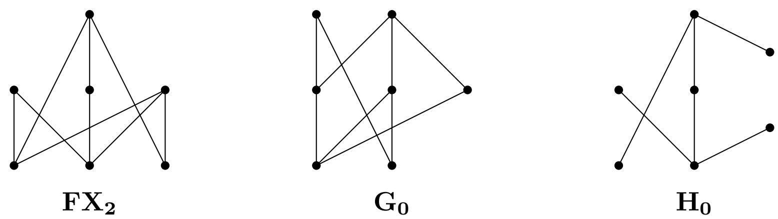


Figure 1.16: Semiorder with dimension 3

1.3.6 dimension theory for semiorders and interval orders

Rabinovitch developed several tools for the dimension of interval orders, and he proved a couple of important bounds on interval orders and semiorders. Let us first look at some of the fundamental properties of interval orders.

Theorem 1.3.5. *(Rabinovitch [38]) Let* $\mathbf{P} = (X, P)$ *be an interval order and let* A *and* B *be 2 disjoint subsets of the ground set* X*. Then there exists a linear extension* L *of* P *with* A/B *in* P*.*

Proof. Notice that if A and B are subsets of X and L is a linear extension of$P$, we say A is over B in L and denote it byA/B if $a > b$ in L whenever $a \in A$, $b \in B$ and $a \| b$ in L.

To proof the theorem, let $S = \{(a, b) \in A \times B : a \| b \ in \ P\}$. We claim that S does not contain a strict alternating cycle. To show this, suppose to the contrary that there is a strict alternating $\{(a_i, b_i) : 1 \leq i \leq k\}$ in S. Then $b_i < a_j$ in P if and only if $j = i + 1$ cyclically. Then the subposet of $\mathbf{P}$ with ground set $\{b_1, a_2, b_k, a_1\}$ is a $\mathbf{2} + \mathbf{2}$, which is a contradiction that proves the claim.

The by Lemma 1.2.3, take L as any linear extension of $tr(P \cup S)$. □

Theorem 1.3.6. *(Rabinovitch [38]) Let* $\mathbf{P}$ *be a semiorder. Then* $\dim(\mathbf{P}) \leq 3$

The theorem is a straightforward consequence of the Theorem 2.6.1 shown in the later section.

Further, Rabinovitch provided three 3-irreducible semiorders.

Theorem 1.3.7. *(Rabinovitch [38]) Let* $\mathbf{P}$ *be a semiorder, then* $\dim(\mathbf{P}) \leq 2$ *unless* $\mathbf{P}$ *contains one or more of the following subposets (see Figure 1.16).*

Although poset can have large dimensions with small height, for example, the standard example can have an arbitrary large dimension with just a height of 2. The following theorem by Robinovitch shows that it's not the same case for interval orders.

Theorem 1.3.8. *(Rabinovitch [38]) If* $\mathbf{P} = (X, P)$ *is an interval order of height* h*, then* $\dim(\mathbf{P}) \leq h + 1$.

Proof. We will construct a realizer of size $h+1$ for $\mathbf{P}$. By the dual version of Dilworth's theorem, we can partition the poset into h antichains by successively removing the minimal elements, denoted by $X = A_1 \cup A_2 \cup \cdots \cup A_h$, to be more specific, A_1 is the set of minimal elements of X, A_2 is the set of minimal elements of $X - A_1$,... so on and so forth. For each $i = 1, 2, \ldots, h$, apply Theorem 1.3.5, let L_i be a linear extension of P with $A_i/(X - A_i)$ in L_i. Then let $L_{h+1} = L_1^d(A_1) < L_1^d(A_2) < \cdots < L_1^d(A_h)$, recall that $L_1^d(A_1)$ is the dual of L_1^d that restricts on A_1. Clearly, $\{L_1, L_2, \ldots, L_{h+1}\}$ is a realizer of $\mathbf{P}$. $\square$

CHAPTER 2

A BOUND ON THE DIMENSION OF INTERVAL ORDERS

2.1. INTRODUCTION

Interval order and interval graphs are two widely studies classes of partially ordered sets and undirected graphs. Researchers in diverse fields such as mathematics, computer science, engineering, and the social sciences have investigated in structural, algorithmic, combinatorial problems associated with them. Semiorders are special interval orders we introduced in the first chapter. Semiorders were introduced and applied in mathematical psychology by Duncan Luce.

2.2. MOTIVATION

In a recent paper by Keller, Trenk, and Young [26], the authors proved that the dimension of interval orders that have a representation with interval lengths 0 and 1 have dimension at most 3. At the end of their paper, they proposed two problems. (1) Find a good bound on the dimension of interval orders whose representation uses intervals of length r and s, where $r, s > 0$. (2) Find a good bound on the dimension of interval orders that have a representation using at most r different lengths.

The results of this chapter were by Csaba Biro and Sida Wan. [6]

Let $f(r)$ denote the best bound in problem (2). In [26], the authors gave a simple upper bound $f(r) \leq 3r + \binom{r}{2}$. We provide a better bound; we also noticed that the bound is related to not just the number of lengths but also the relation between the lengths, hence it is natural to discuss the dimension of interval orders that have a representation with interval lengths in a certain range.

In this chapter, we use the function "lg" to denote the logarithm of base 2.

We start with several fundamental theorems for interval orders.

Lemma 2.3.1. *Let* $mathbf P = (X, P)$ *be a poset, the following conditions are equivalent. [24]*

(1) $\mathbf{P}$ *has no subposet that is isomorphic to* $\mathbf{2} + \mathbf{2}$.

(2) For any elements a, b *in* $\mathbf{P}$, *either* $D(a) \subseteq D(b)$ *or* $D(b) \subseteq D(a)$.

(3) For any elements a, b *in* $\mathbf{P}$, *either* $U(a) \subseteq U(b)$ *or* $U(b) \subseteq U(a)$.

(4) $\mathbf{P}$ *is an interval order.*

Proof. Statement (3) and (4) are equivalent because of duality. Statement (1) and statement (2) are immediate equivalents. We shall prove that statement (2) and statement (4) are equal.

First, suppose that (4) is true. Let $x, y \in X$. Let $\mathbf{I}$ be an interval order representation of $\mathbf{P}$. For every $w \in X$, let a_w, b_w be the left and right endpoints respectively. If $a_x < a_y$, then $D(x) \subseteq D(y)$, otherwise, $D(y) \subseteq D(x)$.

Not suppose statement (2) holds for poset $\mathbf{P}$. We want to show that $\mathbf{P}$ is an interval order. Let $Q = \{D(x) : x \in X\}$, let $|Q| = m$. Define a linear order L on Q, such that $D(a) < D(b)$ in L if $D(a) \subsetneq D(b)$. Label the sets in Q so that $D_1 < D_2 < \cdots < D_m$ in L. For each $x \in X$, let $[i, j]$ be the interval that represent x, where $D(x) = D_i$ and $j = m$ if x is a maximal element. Otherwise, let $D_{j+1} = \cap\{D(y) : x < y \text{ in } P\}$. □

Theorem 2.3.2. *Let* $\mathbf{P}$ *be a poset of height* h. *Then there exists a partition*

$X = A_1 \cup A_2 \cup \cdots \cup A_h$,

with A_j *an antichain for every* $j \in [h]$

2.4. TWIN-FREE AND DISTINGUISHING REPRESENTATIONS

Let $\mathbf{P}$ be an interval order, and fix a representation for $\mathbf{P}$. Let $x, y \in X$ be such that the same interval is assigned to both. We call x and y a twin (of points). If a representation does not have any twins, we call it twin-free. A representation of an interval order is distinguishing, if every real number occurs at most once as an endpoint of an interval of the representation, i.e. no two intervals share an endpoint. A distinguishing representation is, of course, twin-free.

Let $\mathbf{P} = (X, P)$ be a poset, and $x, y \in X$. We say x, y have duplicated holdings if $\{z \in X : z > x\} = \{z \in X : z > y\}$ and $\{z \in X : z < x\} = \{z \in X : z < y\}$; in other words, the upsets and the downsets of x and y are the same. If $\mathbf{P}$ is an interval order with a representation in which x and y are twins, then they have duplicated holdings. So if an interval order has no duplicated holdings, then every representation is twin-free.

One important property of two elements with duplicated holdings is that we may discard one of them without reducing the dimension (the dimension is at least 2). We will use this property later by assuming that some poset, for which we are proving an upper bound for its dimension, has no duplicated holdings.

Since this paper studies interval orders for which the lengths of the intervals are not arbitrary, we introduce the following notations. Let $S \subseteq \mathbb{R}^+ \cup \{0\}$, $S \neq \emptyset$. An S-representation of a poset $\mathbf{P}$ is an interval representation, in which every interval length is in S.

It is easy to see that every interval order has a distinguishing $\mathbb{R}^+$-representation. Things get less obvious with restrictions introduced. A simple example would be a $\{0\}$-representation of an antichain of size at least 2, which cannot be made distinguishing, or even twin-free. We will prove that—essentially—this is the only problem case.

Following Fishburn and Graham [16], we will use the notation $C(S)$ to denote the family of posets that have an S-representation. As a special case, $C([\alpha, \beta])$ denotes the family of posets for which there is a representation with intervals of lengths between α and β (inclusive). We will use the short hands $C[\alpha, \beta] = C([\alpha, \beta])$, and $C(\alpha) = C([1, \alpha])$.

The following observation is obvious due to the scalability of intervals in a representation.

Observation 2.4.1. *$C[\alpha, \beta] = C[m\alpha, m\beta]$, for all $m \in \mathbb{R}$.*

With these notations, $C(\mathbb{R}^+)$ is the family of interval orders, and for $s \neq 0$, $C(\{s\}) = C(\{1\}) = C(1)$ is the family of semiorders.

Now we are ready to prove the theorem that shows that—in most cases—we can assume that a poset has a distinguishing representation.

Theorem 2.4.2. *Let $S \subseteq \mathbb{R}^+ \cup \{0\}$, $S \neq \emptyset$.*

1. *Every poset $\mathbf{P} \in C(S)$ that has a twin-free free S-representation also has a distinguishing S-representation.*

2. *If $0 \notin S$, then every poset $\mathbf{P} \in C(S)$ has a distinguishing S-representation.*

Proof. Let $S \subseteq \mathbb{R}^+ \cup \{0\}$, $S \neq \emptyset$, and let $\mathbf{P} \in C(S)$. Consider an S-representation of $\mathbf{P}$; with a slight abuse of notation, the multiset of intervals in this representation will also be referred to as $\mathbf{P}$. We will define two symmetric operations that we will perform repeatedly. These will be used to decrease the number of common endpoints of the intervals. After this, we enter a second phase, in which we remove twins, if possible.

Left and right compression

Let $c \in \mathbb{R}$, and $\epsilon > 0$. Let $L = \{x \in P : l_x < c\}$, and let $R = P - L$. Define $L' = \{[l_x + \epsilon, r_x + \epsilon] : x \in P\}$. Let $P' = L' \cup R$, a multiset of intervals. The operation that creates P' from P is what we call "left compression" with parameters c and ϵ.

We can similarly define right compressions. Let $R = \{x \in P : r_x > c\}$, and let $L = P - R$. Define $R' = \{[l_x - \epsilon, r_x - \epsilon] : x \in P\}$. Let $P' = L \cup R'$ to define the operation of right compression.

Lemma 2.4.3. *Let $\mathbf{P}$ be a poset (representation), $c \in \mathbb{R}$, and let $\epsilon = \frac{1}{2}\min\{|a-b| : a$ and b are distinct endpoints$\}$. Let $\mathbf{P}'$ be the left (right) compression of $\mathbf{P}$ with parameters c and ϵ. Then $\mathbf{P}$ and $\mathbf{P}'$ represent isomorphic posets.*

Proof of lemma. We will do the proof for left compressions. The argument for right compressions is symmetric.

Notice that if a and b are two endpoints of intervals of $\mathbf{P}$, then their relation won't change, unless $a = b$. More precisely, if $a < b$ in $\mathbf{P}$ then the corresponding points in $\mathbf{P}'$ will maintain this relation, similarly for $a > b$.

So if x and y are two intervals in $\mathbf{P}$ with no common endpoints, then their (poset) relation is maintained in $\mathbf{P}'$.

Now suppose that x and y are intervals with some common endpoints. There are a few cases to consider.

If $l_x = l_y$ then either $x, y \in L$ or $x, y \in R$, so either both are shifted, or neither. Therefore $x \| y$ both in $\mathbf{P}$ and in $\mathbf{P}'$.

Now suppose $l_x \neq l_y$; without loss of generality $l_x < l_y$. Also assume $r_x = r_y$. Then $l_x + \epsilon < l_y$, so $x \| y$ both in $\mathbf{P}$ and in $\mathbf{P}'$.

The remaining case is, without loss of generality, $r_x = l_y$. Then $r_x + \epsilon < r_y$ (unless $l_y = r_y = r_x$, which was covered in the second case), so, again $x \| y$ both in $\mathbf{P}$ and in $\mathbf{P}'$. $\square$

Now we return to the proof of the theorem. We will perform left and right compressions until no common endpoints remain except for twins. Let x, y be two intervals with a common endpoint, but $x \neq y$. Let $\epsilon = \frac{1}{2} \min\{|a - b| : a \text{ and } b \text{ are distinct endpoints}\}$, as above.

- If $l_x = l_y$ and $r_x \neq r_y$, perform a right compression with $c = \min\{r_x, r_y\}$ and ϵ.
- If $r_x = r_y$ and $l_x \neq l_y$, perform a left compression with $c = \max\{l_x, l_y\}$ and ϵ.
- If $l_x < r_x = l_y < r_y$ (or vice versa) either a left or a right shift will work with $c = r_x = l_y$.

Note that even though the definition of ϵ looks the same in every step, the actual value will change as the representation changes. Indeed, it is easy to see that ϵ is getting halved in every step.

If $\mathbf{P}$ started with a twin-free representation, then we have arrived to a distinguishing representation, so part 1 is proven.

If $\mathbf{P}$ had twins, those are still present at the representation. Let x and y be identical intervals of the representation, and let $\epsilon = \frac{1}{2} \min\{|a - b| : a \text{ and } b \text{ are distinct endpoints}\}$ again. If $0 \notin S$, then the length of x (and hence the length of y) is positive. Note that this length is at least ϵ. Move x by ϵ to the right, that is, replace x with the interval $[l_x + \epsilon, r_x + \epsilon]$. The new representation will not have the x,y twin and represents the same poset. Repeat this until all twins disappear. $\square$

Let $\mathbf{I}$ be a representation of an interval order (X, P). Kierstead and Trotter [28] defined choice functions: a choice function f on $\mathbf{I}$ is an injection $f : X \to \mathbb{R}$ such that $l_x \leq f(x) \leq r_x$ in $\mathbb{R}$. For a given choice function f, define the linear order $L(f)$ by setting $x < y$ in $L(f)$ if and only if $f(x) < f(y)$ in $\mathbb{R}$. It is easy to see that for each choice function f on $\mathbf{I}$, $L(f)$ is a linear extension of P. Indeed, for every $x, y \in X$, $x < y$ in P, I_x always lies to the left of I_y, hence for any choice function f, we have $f(x) < f(y)$.

In [28], the following lemma is proven, which is specific to interval orders. We provide a different proof here, which hopefully provides some more insight.

Lemma 2.5.1. *Let (X, P) be an interval order, $X = X_1 \cup X_2 \cup \cdots \cup X_s$ be a partition. Let L_i be a linear extension of $P(X_i)$ where $i = 1, 2, \ldots, s$. Then there exists a linear extension L of P such that $L(X_i) = L_i$.*

Proof. We will prove the lemma for $s = 2$; the case of $s > 2$ then follows by induction.

Let X_1, X_2, L_1, L_2 be defined as in the lemma. Define the relation $E = L_1 \cup L_2 \cup P$, and the directed graph $G = (X, E)$. It is sufficient to show that G has no directed closed walk; indeed, if that is the case, the transitive closure T of G is an extension of the poset $\mathbf{P}$, and any linear extension L of T will satisfy the requirements of the conclusion of the lemma.

Suppose for a contradiction that G contains a directed closed walk. Since neither $G[X_1]$ nor $G[X_2]$ contains a directed closed walk, every directed closed walk in G must have both an X_1X_2 and an X_2X_1 edge. We will call these edges *cross-edges*. Let C be a directed closed walk in G with the minimum number of cross-edges.

As we noted, C contains at least one X_1X_2 edge; let (a, b) be such an edge. Let (c, d) be the first X_2X_1 edge that follows (a, b) in C. Observe that $c < d$, $a < b$ in P, and $b \leq c$ in L_2. If $d = a$, then $c < d = a < b$ in P, which would contradict $b \leq c$ in L_2. If $d > a$ in L_1, then we could eliminate the path $ab \ldots cd$ in C, replacing it with the single-edge path ad, and thereby decreasing the number of cross-edges in C, contradicting the minimality of C. (See Figure 2.1.)

So we concluded that $d < a$ in L_1, and recall that $b \leq c$ in L_2. If $b \leq c$ in P, then $a < b \leq c < d$ would contradict $d < a$ in L_1. (In particular, $b \neq c$.) Obviously, $b \not> c$ in P, so $b \| c$ in P. Similar argument shows that $d \| a$ in P. Hence the set $\{a, b, c, d\}$ induces a $\mathbf{2} + \mathbf{2}$ in $\mathbf{P}$, a contradiction. $\square$

Let (X, P) be a poset, and $X = Y \cup Z$ be a partition of X. We say that Y is over Z in an linear extension L of P if $y > z$ in L whenever $y \in Y$, $z \in Z$ and $y \| z$ in P. Using choice functions,

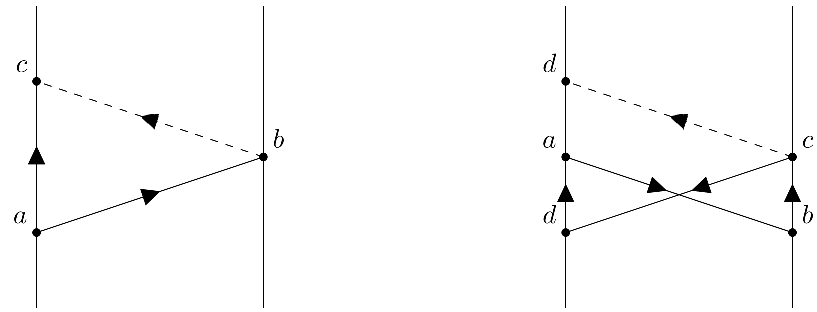


Figure 2.1: Minimal oriented cycles

Kierstead and Trotter [28] provided a shorter proof of a lemma below due to Rabinovitch [36]. We will include the proof for completeness.

Lemma 2.5.2. *Let (X, P) be an interval order, and $X = X_1 \cup X_2$ be a partition of X, where $\mathbf{P}_1 = (X_1, P(X_1))$, $\mathbf{P}_2 = (X_2, P(X_2))$. Then*

$$\dim(\mathbf{P}) \leq \max\{\dim(\mathbf{P}_1), \dim(\mathbf{P}_2)\} + 2.$$

Proof. Consider a distinguishing representation of (X, P), and let $t = \max\{\dim(\mathbf{P}_1), \dim(\mathbf{P}_2)\}$. By Lemma 2.5.1, there exists a family $\mathcal{R}$ of t linear extensions of P, such that the restriction of the linear extensions in $\mathcal{R}$ to each X_i form a realizer of P_i, for $i = 1, 2$. Then define two choice functions f_1 and f_2, where $f_1(x) = l_x$, $f_2(x) = r_x$ for every $x \in X_1$; $f_1(y) = r_y$, $f_2(y) = l_y$ for every $y \in X_2$. Let $L_1 = L(f_1)$, $L_2 = L(f_2)$. Clearly, $\mathcal{R} \cup \{L_1, L_2\}$ is a realizer of P. $\square$

The following theorem is given by Kiestead and Trotter [28], the proof was done by induction on the number of vertices in the paper. here we provide an alternative proof of the existence of a choice function directly from the representation of the interval order.

Theorem 2.5.3. *Let $\mathbf{P} = (X, P)$ be an interval order with no duplicated holdings, let $\mathbf{I}$ be a distinguishing representation of $\mathbf{P}$. If L is an arbitrary linear extension of P, then there exists a choice function f on $\mathbf{I}$, such that $L(f) = L$.*

Proof. Without loss of generality, label the ground set X by the linear extension $L = x_1 x_2 \ldots x_n$. Let I be the function that maps each $x \in X$ to a closed interval I_x in $\mathbb{R}$. If I_x is an interval with a positive length, then let l_x be the left endpoint of $I(x)$ and r_x be the right endpoint of I_x. Otherwise, we say $I_x \in D$ if it is a zero length interval and let m_x denote the real value of I_x in $\mathbb{R}$. Meanwhile, let ϵ be the smallest difference between any two endpoints in $\mathbf{I}$. Since $\mathbf{I}$ is a distinguishing representation, we have $\epsilon > 0$. Without further due, find a choice function for $\mathbf{I}$ that gives us L. For convenience, let $f_i = f(x_i)$ for $i = 1, 2 \cdots n$. First, define:

$$f_i = \begin{cases} m_{x_1}, & I_x \in D \\ l_{x_1} + \epsilon/2, & I_x \notin D \end{cases}$$

If x_1 is a zero length interval in $\mathbf{I}$, then $f_1 = m_{x_1} \in [l_x, r_x]$, for $l_{x_1} = m_{x_1} = r_{x_1}$ in this case. Otherwise, if $x_1 \notin D$, $l_{x_1} + \epsilon/2 < l_{x_1} + \epsilon < r_{x_1}$ by our definition. Then, for $i = 2, 3, \ldots n$, define:

$$f_i = \begin{cases} m_{x_i}, & I_{x_i} \in D \\ \max\{f(i-1) + \epsilon/2^i, l_{x_i} + \epsilon/2^i\}, & I_{x_i} \notin D \end{cases}$$

We shall check if $f_i \in [l_{x_i}, r_{x_i}]$ for every $i = 1, 2, \cdots n$. We call f_i to be good if $f_1 \in [l_{x_i}, r_{x_i}]$. We have already shown that f_1 is good. We will proceed by induction. We will first show that $f(2)$ is good. If one or both of I_{x_1} and I_{x_2} are zero length intervals, it is clear that f_1 and f_2 are good. Assume that they are both intervals with positive length. If $x_2 > x_1$ in P, then $f_2 = l_{x_2} + \epsilon/4 < l_{x_2} + \epsilon \leq r_{x_2}$, f_2 is good. Otherwise, if $x_1 \| x_2$ in P, there are 2 cases, either $l_{x_1} < l_{x_2} < r_{x_1}$ or $l_{x_2} < l_{x_1} < r_{x_2}$. In the first case, $l_{x_2} > l_{x_1} + \epsilon/2 + \epsilon/4$, hence $f_2 = l_{x_2} + \epsilon/4$, f_2 is good. In the second case $f_2 = f_1 + \epsilon/4 = l_{x_1} + \epsilon/2 + \epsilon/4 < l_{x_1} + \epsilon < r_{x_2}$, hence f_2 is also good. Now, assume $f_i's$ are good for $i = 1, 2, \ldots, k-1$, $(0 < k \leq n)$, need to show that f_k is also good. If $I_{x_{k-1}}$ is a zero length interval, either $x_{k-1} < x_k$ or $x_{k-1} \| x_k$ in P, it's clear that f_k is good. Let's assume $I_{x_{k-1}}$ has positive length. If after we take the maximum we obtain $f_k = l_{x_n} + \epsilon/2^n$, then f_k is good. The case we need to check is the one that $f_k = f_{k-1} + \epsilon/2^k > l_{x_k} + \epsilon/2^k$ and meanwhile $f_{k-1} + \epsilon/2^k$ is not in $[l_{x_k}, r_{x_k}]$, i.e. $f_{k-1} + \epsilon/2^k > r_{x_k}$. But we will show that this is impossible. Since for f_{k-1} there exists a interval I_{x_s}, $0 < s < k-1$ (see Figure 2.2) , such that $f_{k-1} < l_{x_s} + \epsilon/2 + \epsilon/2^2 + \cdots < l_{x_s} + \epsilon$. And we have $x_s < x_{k-1} < x_k$ in L, hence $x_k \| x_s$. Then $f(x_{k-1}) - \epsilon < l(x_s) < r(x_k) < f_{x_{k-1}}$, notice that $r(x_k) = m(x_k)$ if x_k is a zero length interval, but both case give us $r(x_k) - l(x_s) < \epsilon$ which is a contradiction. Hence f_k is good, we have $f_i \in [l_{x_i}, r_{x_i}]$ for each $i = 1, 2, \ldots, n$. For the rest of the proof, it's easy to see that $f_i \geq f_{i-1} + \epsilon/2^{i-1} > f_{i-1}$, hence $L(f) = L$.

$\square$

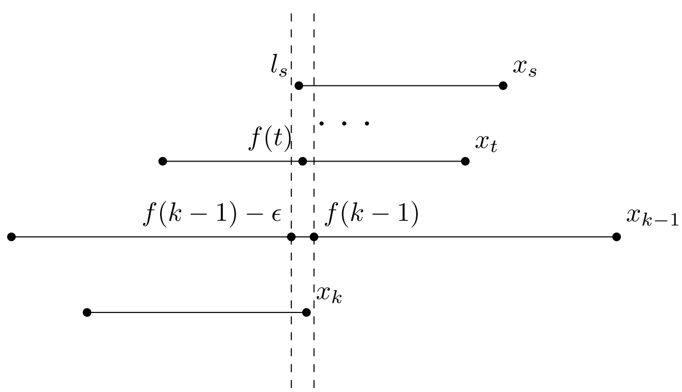


Figure 2.2: Existence of Linear Extensions

2.6. DIMENSION OF INTERVAL ORDERS USING TWO LENGTHS

In [26], the following theorem is proven.

Theorem 2.6.1. *If* $\mathbf{P}$ *is an interval order that has representation such that every interval is of length 0 or 1, then* $\dim(\mathbf{P}) \leq 3$.

In [26], the authors defined two disjoint sets of incomparable pairs neither of which contains an alternating cycle, hence there exist linear extensions that reverse all the incomparable pairs in each of the sets. The remaining incomparable pairs can be reversed in one extra linear extension. Here we provide a shorter proof using a choice function, which gives the three linear extensions that realize the interval order directly.

Proof. Let $\mathbf{P}$ be a twin-free interval order, and let $\mathbf{I}$ be a distinguishing representation of $\mathbf{P}$ which only consist of length 0 and 1 intervals. Let poset $\mathbf{U} = (U, \mathbf{P_U})$ be the subposet of $\mathbf{P}$ consisting of all the points represented by intervals of length 1 in $\mathbf{I}$, and $\mathbf{D}$ be the subposet of $\mathbf{P}$ consisting all the points represented by intervals of length 0 in I. Let D be the ground set of $\mathbf{D}$. For each element $x \in D$, use R_x to denote the unique real number in the interval representing x. Partition U into antichains $A_1, A_2, \ldots, A_t$ by taking the minimal elements successively. It is easy to see that $x < y$ in P for every $x \in A_i$, $y \in A_{i+2}$. Let $A_{\text{odd}} = \{x \in U : x \in A_i \text{ for some } i \in [t] \text{ with } i \text{ odd}\}$, and $A_{\text{even}} = U - A_{\text{odd}}$.

Let f_1, f_2 be choice functions on $\mathbf{I}$, defined as follows.

$$f_1(x) = \begin{cases} l_x, & x \in A_{\text{odd}} \\ r_x, & x \in A_{\text{even}} \\ R_x, & x \in D \end{cases}$$

$$f_2(x) = \begin{cases} r_x, & x \in A_{\text{odd}} \\ l_x, & x \in A_{\text{even}} \\ R_x, & x \in D \end{cases}$$

Then, let $L_1 = L(f_1)$, $L_2 = L(f_2)$, hence L_1 and L_2 are both linear extensions of P. It is clear that each incomparable pair $\{x, y\}$, where $x \in A_{\text{odd}}$, $y \in A_{\text{even}}$, is reversed in the two linear extensions, as well as the incomparable pairs $\{x, y\}$, for which $x \in U$, $y \in D$. the only incomparable pairs need to be reversed are the ones that both of the points are in the same A_i, we can reverse all of them in one linear extension, and finally we use interpolation lemma to interpolate all the points in the set D to the last linear extension:

$$L_3 = L_1^d(A_1) < L_1^d(A_2) < \cdots < L_1^d(A_t) \cup D.$$

Hence, $\{L_1, L_2, L_3\}$ is a realizer of $\mathbf{P}$. $\square$

2.7. DIMENSION OF INTERVAL ORDERS WITH REPRESENTATION USING MULTIPLE POSITIVE LENGTHS

Let $r, s > 0$. Recall that $C(\{r, s\})$ denotes the class of interval orders that have a representation, in which every interval is of length r or s. Rabinovitch [37] proved that the dimension of a semiorder is at most 3. Here, we prove the following bound on the dimension of posets in $C(\{r, s\})$.

Proposition 2.7.1. *Let* $\mathbf{P} \in C(\{r, s\})$. *Then* $\dim(\mathbf{P}) \leq 5$.

Proof. Let $\mathbf{P} \in C(\{r, s\})$, and consider a representation of $\mathbf{P}$. We can partition $\mathbf{P}$ into the union of 2 semiorders, $\mathbf{S_r}$ and $\mathbf{S_s}$, which consist of intervals of length only r and s, respectively. Since the dimension of a semiorder is at most 3, apply Lemma 2.5.2 to conclude

$$\dim(P) \leq \max\{\dim(S_r), \dim(S_s)\} \leq 5.$$

□

Let $f(r)$ be the maximum dimension of interval orders having a representation consisting of intervals of at most r different positive lengths. By partitioning the interval orders into the union of r different semiorders, then using similar techniques, we have the following bound for $f(r)$.

Proposition 2.7.2. $f(r) \leq \lceil \lg r \rceil + 3$.

If these bounds are tight is not known.

Recall that $C(\alpha)$ is the family of posets that have a representation with intervals of lengths between 1 and α.

Theorem 2.8.1. *Let* $\mathbf{P} = (X, P)$ *be a interval order with a representation such that each interval is of length* 1 *except for one interval, which is of length between 0 and 2 (inclusive). Then* $\dim(\mathbf{P}) \leq 3$.

Proof. We may assume that $\mathbf{P}$ has no duplicated holdings. By Theorem 2.4.2, it has a distinguishing representation; fix one of these. Let x_0 be the interval whose length is not 1. Let m_0 be the midpoint of x_0, and let A_0 be the set of intervals that contain m_0. (See Figure 3.)

Let $U_0 = \{x \in X : l_x > m_0\}$, and let $D_0 = \{x \in X : r_x < m_0\}$. Let A_1 be the set of minimal elements of U_0, and let $U_i = U_{i-1} - A_i$, where A_i is the set of minimal elements of U_{i-1} for $i = 1, 2, \ldots, k$. Similarly, let B_1 be the set of maximal elements of D_0, and let $D_i = D_{i-1} - B_i$, where B_i is the set of maximal elements of D_{i-1} for $i = 1, 2, \ldots, s$. Hence we have a partition P_1 of $\mathbf{P}$: $B_s \cup \cdots \cup B_1 \cup A_0 \cup A_1 \cup \cdots \cup A_k$.

For any elements x and y, where $x \in A_0$, $y \in A_2$, we have $x < y$ in P. Indeed, if y is in A_2, there must be an element w in A_1, such that, $m_0 < l_w < r_w < l_y$. Since w has length 1, we have $l_y > m_0 + 1$. And given that x_0 has length between 0 and 2 inclusive with midpoint m_0, we have $r_x \leq m_0 + 1 < l_y$. By symmetry, it can be proved that $x < y$ for every $x \in B_2$, $y \in A_0$. In addition, from the property of semiorders, $x < y$ for every $x \in A_i$, $y \in A_{i+2}$, and for every $x \in B_{j+2}$, $y \in B_j$, where $i = 1, 2, \ldots, k-2$, $j = 1, 2, \ldots, s-2$. Finally, for every $x \in B_1$, $y \in A_1$, clearly $x < y$ since $r_x < m_0 < l_y$.

Hence if we relabel the partition P_1 from left to right to be $S_1 \cup \cdots \cup S_n$, we have $x < y$ for every $x \in A_i$, $y \in A_{i+2}$, where $i = 1, 2, \ldots, n-2$. Meanwhile each S_i is an antichain. Then, apply a similar method as the one in the proof of Theorem 2.6.1. Let f_1, f_2 be choice functions on $\mathbf{I}$, which define as follows:

$$f_1(x) = \begin{cases} l_x, & x \in S_{\text{odd}} \\ r_x, & x \in S_{\text{even}} \end{cases}$$

$$f_2(x) = \begin{cases} r_x, & x \in S_{\text{odd}} \\ l_x, & x \in S_{\text{even}} \end{cases}$$

Let $L_1 = L(f_1)$, $L_2 = L(f_2)$, and let $L_3 = L_1^d(A_1) < L_1^d(A_2) < \cdots < L_1^d(A_t)$. Clearly $\{L_1, L_2, L_3\}$

is a realizer of $\mathbf{P}$. $\square$

Theorem 2.8.2. *Let* $\mathbf{P} \in C(2)$. *Then* $\dim(\mathbf{P}) \leq 4$.

Proof. Let $\mathbf{P} \in C(2)$, where $\mathbf{P} = (X, P)$. Fix a distinguishing representation of $\mathbf{P}$. We will think of the elements of $\mathbf{P}$ as intervals, and we will use the notation l_x, r_x to denote the left and right endpoints of x in X, respectively.

We again apply the technique of partitioning the poset by successively removing minimal elements. To be precise, we let A_1 be the set of minimal elements of X, and let $P_1 = P(X - A_1)$. We define A_i recursively as follows: assuming that A_{i-1} and P_{i-1} are defined, we let A_i be the set of minimal elements of X_{i-1}, and we let $X_i = X_{i-1} - A_i$.

We will show that for all i, $A_i < A_{i+3}$; that is, whenever $x \in A_i$ and $y \in A_{i+3}$, we have $x < y$.

Let i be a positive integer, and let $x \in A_i$, $y \in A_{i+3}$. Since $x \not> y$, we prove that x and y cannot be incomparable. There exists $z_2 \in A_{i+2}$ such that $z_2 < y$; and so on, $z_1 \in A_{i+1}$ with $z_1 < z_2$, and $z_0 \in A_i$ with $z_0 < z_1$.

Note that

$$l_y > r_{z_2} \geq l_{z_2} + 1 > r_{z_1} + 1 \geq l_{z_1} + 2 > r_{z_0} + 2.$$

Since $z_0, x \in A_i$, we have $z_0 \| x$, so $l_x \leq r_{z_0}$. From these we conclude that $l_y > l_x + 2$. If $x \| y$, then $r_x \geq l_y$, which would make the length of x more than 2. So, we conclude $x < y$, as desired.

We define three linear extensions with choice functions that reverses most critical pairs. Let the choice functions f_0, f_1, f_2 be defined by

$$f_i(x) = \begin{cases} r_x, & x \in A_j \text{ with } j \equiv i \mod 3 \\ l_x, & \text{otherwise.} \end{cases}$$

Let L_1, L_2, L_3 be the linear extensions defined by these choice functions.

If $x \| y$, and $x \in A_i$, $y \in A_j$ with $i \neq j$, then $l_x < l_y \leq r_x$, which means that they will appear in both order in one of L_1, L_2, L_3. So we only need to reverse critical pairs that appear in a single A_i. This can be done with one extra linear extension:

$$L_4 = L_1^d(A_1) < L_1^d(A_2) < \cdots < L_1^d(A_t).$$

$\square$

It is open whether there is a poset in $C(2)$ that is actually four-dimensional. It feels unlikely that

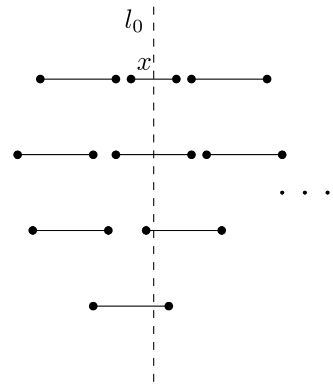


Figure 2.3: Proof illustration

the addition of all numbers between 1 and 2 as possible lengths would not increase the dimension from semiorders, but finding a four dimensional poset in $C(2)$ has resisted our efforts.

Recall $C[\alpha,\beta]$ denote the class of interval orders that can be represented with intervals of lengths in the range $[\alpha,\beta]$. Note that $C(\alpha) = C[1,\alpha]$. Use $f(C[\alpha,\beta])$ to denote the least upper bound of the dimension of posets in the class $C[\alpha,\beta]$. We just proved that $f(C[1,2]) \leq 4$.

Theorem 2.8.3. *For $t \geq 2$, $f(C(t)) = f(C[1,t]) \leq 2\lceil \lg\lg t\rceil + 4$.*

Proof. Let $n = 2^{2^{\lceil \lg\lg t\rceil}}$. Since $n \geq t$, it is clear that $f(C[1,t]) \leq f(C[1,n])$. We will show by induction that $f(C[1,n]) \leq 2\lceil \lg\lg n\rceil + 4 = 2\lg\lg n + 4$.

For $n = 2$, the statement reduces to Theorem 2.8.2. Let $n > 2$ be an integer. Note that in this case $\lceil \lg\lg t\rceil \geq 1$, so $n \geq 4$ and a square. Let $m = \sqrt{n} = 2^{2^{\lceil \lg\lg t\rceil - 1}}$. If $\mathbf{P} \in C[1,n]$, then we can partition intervals of a representation of $\mathbf{P}$ into "short" intervals of length at most m, and "long" intervals of length at least m. (Intervals of length m, if any, can be placed arbitrarily.) By Lemma 2.5.2, Observation 2.4.1, and the hypothesis,

$$f(C[1,n]) \leq \max\{f(C[1,m]), f(C[m,n])\} + 2 = f(C[1,m]) + 2 \leq$$
$$2(\lceil \lg\lg t\rceil - 1) + 4 + 2 = 2\lceil \lg\lg t\rceil + 4.$$

□

2.9. CLASSIFICATION OF INTERVAL ORDERS

Another interesting topic is the classification of interval orders of class $C(\alpha)$ ($\alpha \leq 1$), where $C(\alpha)$ denote the finite interval orders using interval length in $[1, \alpha]$.

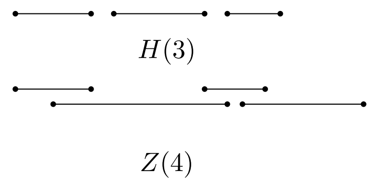


Figure 2.4: Zig chian

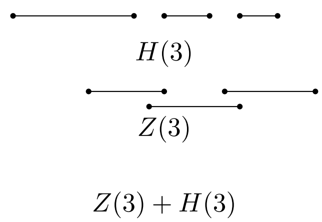


Figure 2.5: $Z(3) + H(3)$

2.10. NOTATION

We shall define some notations that will be convenient for the next theorem.

2.10.1 Chains and Zig Chains

In this section let $H(s)$ be a chain that has a vertices. Let $Z(m)$ be a poset with b vertices, denoted by $\{x_1, x_2, \ldots, x_m\}$ such that $x_i \| x_{i+1}$ and $a_x < a_{x+1}$ for each $i \in [1, m-1]$ and $x_j < x_{j+2}$ for each $j \in [1, m-2]$. We call it zig chain.

2.10.2 Example

With slight abuse of notation, we also use $H(s)$ to denote the representation of interval orders that is a chain with s intervals. Here are some examples of chains and zig chains (see Figure 2.4).

We say a zig chain captures a chain if the leftmost interval of the zig chain is incomparable to the left end interval of the chain and the rightmost end interval of the zig chain is incomparable to the rightmost end interval of the chain. To be precise, let $H(s)$ be a chain with s vertices $\{x_1, x_2, \ldots, x_s\}$ such that $x_1 < x_2 < \cdots < x_s$. Let I_1 be the interval representation of $H(s)$. Let $Z(m)$ be a zig chain with m vertices $\{y_1, y_2, \ldots, y_m\}$ such that $y_i \| y_{i+1}$ and $a_y < a_{y+1}$ for each $i \in [1, m-1]$ and $y_j < y_{j+2}$ for each $j \in [1, m-2]$. We say $Z(m)$ captures $H(s)$ if $a(y_1) < b(x_1)$ and $b(y_m) > a(x_s)$.

Here is an example of a $Z(3)$ captures $H(3)$ (see Figure 2.5). We call such poset $Z(3) + H(3)$.

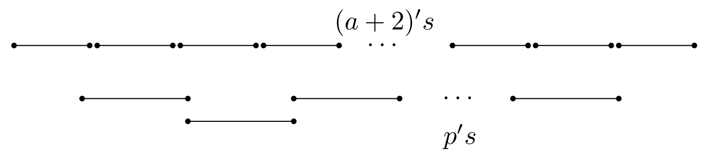


Figure 2.6: Zig chain example

Theorem 2.10.1. *Let $1 < \alpha_1 < \alpha_2$, where α_1, α_2 are rational numbers. Then there exists an interval order that is in $C(\alpha_2)$ but not in $C(\alpha_1)$.*

Proof. Let α_1, α_2 be rational numbers that are larger than 1 and $\alpha_2 > \alpha_1$. Here is how we found the poset that is in $C(\alpha_2)$ but not in $C(\alpha_1)$.

Write α_1 and α_2 in fraction form. Take a common multiplier of the denominator of α_1 and α_2. Without loss of generality, assume the commons multiplier is p, and $\alpha_1 = \frac{a}{p}$, $\alpha_2 = \frac{b}{p}$. a and b are integers and $a < b$.

Take an interval order like the one in Figure 2.6.

First, We construct a zig chain with p intervals, denote it by $Z(a)$ for convenience, and each interval of $Z(a)$ has length $\frac{b}{p}$. The total length of the zig chain is larger than $b - u$ for some $u > 0$. We can properly arrange the zig chain to make u as small as possible, let $0 < u < b - a$. Then we construct a chain with $a + 2$ unit intervals, denote it by $H(a + 2)$ for convenience. We can make the total length of a chain of a unit interval length as short as $a + \epsilon$, where $\epsilon < b - u - a$. Then we can make the zig chain $Z(a)$ captures $H(p+2)$. However, with an antichain consisting of p interval of length $\frac{a}{p}$, the total length of such antichain is less than a, there is no way we can make such antichain capture $H(p+2)$. Hence the poset $Z(a) + H(p+2)$ is the poset that is in $C(\alpha_2)$ but not in $C(\alpha_1)$. $\square$

In the paper [16], Fishburn and Graham classified the interval graphs under expanding length restrictions. They use $C(\alpha)$ to denote the finite interval graphs representable as intersection graphs of closed real intervals with lengths in $[1, \alpha]$. The points of increase for the class C are the rational $\alpha \geq 1$. They also found the irreducible graphs for each rational α, with $\alpha = p/q$ where p and q are relatively prime. Similar techniques can be used to classify the interval orders in $C(\alpha)$. And even the classification of split semiorders that we are going to introduce in the next chapter.

CHAPTER 3

INTERVAL DIMENSION AND SEMI DIMENSION

3.1. INTRODUCTION

3.1.1 Interval Dimension

Let $\mathbf{P} = (X, P)$ be a poset, the interval dimension of $\mathbf{P}$, denoted by $\mathrm{Idim}(\mathbf{P}) = t$, is the least t such that there exists t interval orders $\{I_1, I_2, \ldots, I_t\}$, where $\mathbf{P} = I_1 \cap I_2 \cap \cdots \cap I_t$.

Notice that each interval order has interval dimension 1. And as we have proved in the previous section by Ramsey theorem, interval order can have arbitrarily large regular dimensions. Hence the gap between the interval dimension and regular dimension can be arbitrarily large.

3.1.2 Semi Dimension

Similarly, let $\mathbf{P} = (X, P)$ be a poset, the semi-dimension of $\mathbf{P}$, denoted by $\mathrm{Sdim}(\mathbf{P}) = k$, is the least k such that there exists s interval orders $\{S_1, S_2, \ldots, S_k\}$, where $\mathbf{P} = S_1 \cap S_2 \cap \cdots \cap S_k$.

3.1.3 Observation

A couple of observations can be made immediately since a linear extensions can be considered as an semiorder or an interval order. Meanwhile, semiorders are interval orders, hence for any poset $\mathbf{P}$, $\mathrm{Idim}(P) \leq \mathrm{Sdim}(P) \leq \dim(P)$.

3.1.4 Properties

Here are several properties of the Interval dimension proved by Bogart and Trotter [49]. Let $\mathbf{P} = (X, P)$ be a poset, $x \in X$. Let A be an antichain and C be a chain of the poset. Let (u, v) be a critical pair and $Y \subseteq X$, $Q = P(Y)$, then

1. $\mathrm{Idim}(\mathbf{Q}) \leq \mathrm{Idim}(\mathbf{P})$.
2. $\mathrm{Idim}(\mathbf{P}) \leq 1 + \mathrm{Idim}(X - x, P(X - x)$.

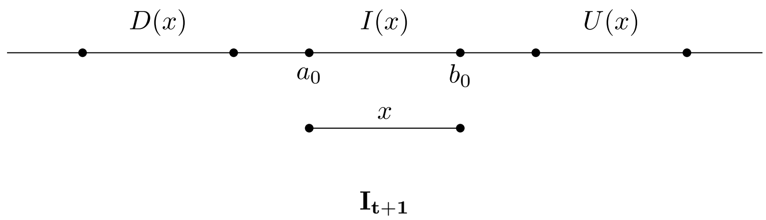


Figure 3.1: Alternative proof

3. $\text{Idim}(\mathbf{P}) \leq 1 + \text{Idim}(X - \{u, v\}, P(X - \{u, v\}))$.
4. $\text{Idim}(\mathbf{P}) \leq 2 + \text{Idim}(X - C, P(X - C))$.
5. $\text{Idim}(\mathbf{P}) \leq \max\{2, |X - A|\}$.
6. $\text{Idim}(\mathbf{P}) \leq 2\,\text{width}(X - A) - 1$ when $X - A \neq \emptyset$.
7. $\text{Idim}(\mathbf{P}) \leq \text{width}(X - A)$ if $A = \max(\mathbf{P})$ or $A = \min(\mathbf{P})$ and $X - A \neq \emptyset$.
8. $\text{Idim}(\mathbf{P}) = \text{Idim}(\mathbf{P}^d)$.

For the second property, we provide one easier proof using the properties of interval orders. Here is how:

Proof. Let $\text{Idim}(X - x, P(X - x)) = t$, take a interval order realizer $\mathcal{I}' = \{I'_1, I'_2, \ldots, I'_t\}$ of $\mathbf{P}$. By interpolation property, take an interval order I for X where $I_i(X - x) = I'_i$ for each $i = 1, 2, \ldots, t$. Since for the interval order I'_t, there is an representation of it say $\mathbf{I'_t}$, where each vertex in $y \in X - x$ is represented by an interval $[a_y, b_y]$, a_y, b_y are real numbers. Let a_0 to be the least left end point in the interval representation, and b_0 the be the right most end point, and $l_0 = b_0 - a_0 + 1$. Let $I(x) = \{y \in X - x : y \| x \ in \ P\}$. Shift all interval representations for $D(x)$ in $\mathbf{I'_t}$ l_0 units to the left, we can do this because for each $u \in D(x)$ and $v \in I(x)$, either $u \| v$ or $u < v$ in P, otherwise $v < u < x$, then v would be in the set $D(x)$. Similarly, we can shift all interval representations for $U(x)$ in $\mathbf{I'_t}$ l_0 units to the right. Then Take such new representation and add the interval for x to be $[a_0, b_0]$ (see Figure 3.1). Call the new interval order representation $\mathbf{I_{t+1}}$, where by definition, there is an interval order for such representation, name it I_{t+1}. Clearly, $\{I_1, I_2, \ldots, I_t, I_{t+1}\}$ is an interval order representation or (X, P).

□

The third property, which is the removable conjecture for interval order dimension, is proved by Trotter [49]. Here is the proof:

Proof. Take a critical pair u, v from the poset $\mathbf{P} = (X, P)$, such that for every vertex $w \in X$, $w > u$ implies $w > v$; and $w < v$ implies that $w < u$. Let $\mathbf{Q} = (X - \{u, v\}, P(X - \{u, v\})$, and let

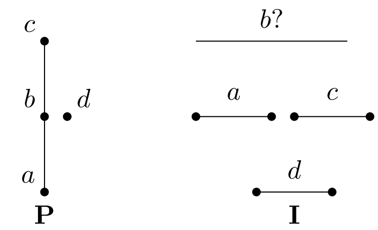


Figure 3.2: Interpolation lemma problem

Idim $\mathbf{Q} = t$. Choose an interval order realizer $\mathcal{I}' = \{I'_1, I'_2, \dots, I'_t\}$ for poset $\mathbf{Q}$, where each I'_i is an interval order representation for the extension of $\mathbf{Q}$.

By interpolation property, choose interval order I_i for X where $I_i(X - \{u, v\}) = I'_i$ for each $i = 1, 2, \dots, t$. Define the function that maps the vertices from X to closed real intervals $\mathbf{C}$. Partition X into five subposets as the following:

$X_1 = \{x \in X : x > u\}$, $X_2 = \{x \in X : x \| v, x > v\}$, $X_3 = \{x \in X : x \| u, x \| v\}$

$X_4 = \{x \in X : x < u, x \| v\}$, $X_5 = \{x \in X : x < b\}$.

Let $|X_i| = n_i$ for $i = 1, 2, 3, 4, 5$. Choose open intervals:

$C_1 = (2, 3)$, $C_2 = (1, 2)$, $C_3 = (-1, 1)$, $C_4 = (-2, -1)$, $C_5 = (-3, -2)$.

For n_i vertices in X_i, take a linear extension for the subposet, and choose n_i intervals from C_i where $I_{n_1} < I_{n_2} < \cdots < I_{n_i}$ to represent such linear order. Finally, take $I_u = [-1, 2]$, $I_v = [-2, 1]$. Combine these intervals to be the representation I_{t+1} of poset (X, P). It's easy to check that $\mathbf{I} = \{I_1, I_2, \dots, I_{t+1}\}$ is a realizer of poset $\mathbf{P} = (X, P)$. □

However, such a method cannot be used to prove the removable conjecture of the semi dimension. In fact, Trotter give the continuity theorem for semi dimension, where he stated that $\mathrm{Sdim}(X) \leq 1 + \mathrm{Sdim}((X - x), P(X - x))$ for a poset $P = (X, P)$ and any vertex $x \in X$. The proof used interpolation property, which is not true for semiorders because of the unique feature of semiorders that they all must have the same length. Here is an example.

3.1.5 Example

Take a simple $\mathbf{3} + \mathbf{1}$ poset $P = (X, P)$ as below, where $X = (a, b, c, d)$ as they are labeled in picture 3.2

For the subposet $(Y, P(Y))$, where $Y = a, c, d$. We find a representation $\mathbf{I}$ where the distance between the right endpoint of a and left endpoint of c is less than 1. There is no way that we can "fit" an interval to represent b that have the same unit length in $\mathbf{I}$. Hence the interpolation lemma does not apply to semiorders.

Hence we cannot prove the continuity of the semi dimension. This problem is still open, and a lot of dimension theorems concerning semi dimension have to be re-examined.

In this part, we give some of the results on semi dimensions.

Theorem 3.2.1. *Let* $\mathbf{P} = (X, P)$ *be a poset with height 2. Then* $\dim(\mathbf{P})+1 \leq \mathrm{Idim}(\mathbf{P}) = \mathrm{Sdim}(\mathbf{P})$.

Notice that for a height two poset, the regular dimension is not bounded, so as the interval and semi dimension, one example would be the standard example. And the canonical interval orders also have unbounded dimensions, which its interval order dimension is just 1. Hence the gap between the interval order dimension and regular dimension is unbounded. Here is the proof of the theorem.

Proof. We will proof $\mathrm{Idim}(\mathbf{P}) = \mathrm{Sdim}(\mathbf{P})$ first.

We already know that $\mathrm{Idim}(\mathbf{P}) \leq \mathrm{Sdim}(\mathbf{P})$. Here we just need to show that $\mathrm{Sdim}(\mathbf{P}) \leq \mathrm{Idim}(\mathbf{P})$. Assume that $\mathrm{Idim}(\mathbf{P}) = t$. Let A be the set of all minimal elements in X, and $B = X/A$. Take an interval order realizer $\mathbf{I}$ for $\mathbf{P}$. Where $\mathbf{I} = \{I_1, I_2, \ldots, I_t\}$, where $I_i's$ are distinguishing representations. Without loss of generality, take an arbitrary I_i, let a be the real value of the least left end point in I_i, and let b be the right most end point in I_i. For the intervals that represents vertex x. If $x \in A$, then extend the left end point of x to a; if x is in B, extend the right end point of x to b. Observe that the new interval representation is still an extension of P and it preserves all the incomparabilities. And in this new representation there is no $\mathbf{3}+\mathbf{1}$ in it, hence we can use a semiorder S_i to represent it. Further, for any incomparable pair (u, v), if they are "overlapped" in I_j, they are still "overlapped" in S_j. If $u < v$ in I_s and $v < u$ in I_k, then they will be "overlapped" in either S_s or S_k. Hence after we apply the transformation to all interval orders in $\mathbf{I}$, $\{S_1, \ldots, S_t\}$ is clearly a semiorder realizer of poset $\mathbf{P}$.

Then, it is sufficient to show that $\dim(\mathbf{P}+1) \leq \mathrm{Idim}(\mathbf{P})$. Take a arbitrary interval order realizer $\mathbf{I}$ for $\mathbf{P}$ as it was in the previous part, $\mathbf{I} = \{I_1, I_2, \ldots, I_t\}$, where $I_i's$ are distinguishing representations. A and B defined the same. For each I_i, where $i = 1, 2, \ldots, t$, take a choice function f_i: $C_i \to \mathbf{R}$, where C_i is the set of closed intervals in I_i that represent the vertices in $\mathbf{P}$. Let $f_i(x) = r(x)$ for all $x \in A$; $f_i(x) = l(x)$ for all $x \in B$. Recall that $r(x)$ is the right endpoint of the interval that represents x, and $l(x)$ is the left endpoint of the interval that represents x. Define $L_i = L(f_i)$ as the notation we used before, which is the linear extension which orders $x \in X$ as the real value of each $f_i(x)$. Take $L_{t+1} = {L_1}^d(A) < {L_1}^d(B)$. Clearly, $\{L_1, L_2, \ldots, L_{t+1}\}$ is a regular realizer for $\mathbf{P}$.

□

In the last chapter, we proved that the bound of the regular dimension for the class of interval order $C[1, 2]$ is 4. Here we will give the semi dimension bound for the same class.

$\boxed{B_1A_2}$ $\boxed{C_1A_3}$ $\boxed{B_4A_5}$ $\boxed{C_4A_6}$ $\boxed{B_7A_8}$ $\boxed{C_7A_9}$ $\cdots$

Figure 3.3: Block 1

$\boxed{A_1}$ $\boxed{B_2A_3}$ $\boxed{C_2A_4}$ $\boxed{B_5A_6}$ $\boxed{C_5A_7}$ $\boxed{B_8A_9}$ $\cdots$

Figure 3.4: Block 2

Theorem 3.2.2. *Let* $\mathbf{P} = (X, P)$ *be a poset that is in class* $C[1,2]$*, then* $\mathrm{Sdim}(\mathbf{P}) \leq 3$

Proof. Partition $\mathbf{P}$ into n parts by recursively removing the minimal elements, i.e., $X = A_1 \cap P_1 \cap \cdots \cap A_n$, where $A_1 = \min(X)$, $A_2 = \min(X/A_1)$, $A_3 = \min(X/A_1/A_2), \ldots$.

Observe that for any $x \in A_i$, $y \in A_{i+3}$, $x < y$ in P. For if $x|\ y$, there must be $a \in A_{i+1}$ and $b \in A_{i+2}$, such that $l_b > r_a$, and $r_b < l_y$. Also see that $l_x < l_a$, since otherwise a would be in A_i. Meanwhile $r_x > l_y$, then $r_x - l_x > (r_b - l_b) + (r_a - l_a)$ which is larger that 2, a contradiction.

Another observation would be that for every $x \in A_i$, $y \in A_{i+2}$ and $z \in A_{i+1}$, if $x \| y$, then $x \| z$. For if $x < z$, $l_z > r_x > l_y$ for some $y \in A_{i+2}$, then z would be in A_{i+2} which is a contradiction again.

We can now build the semiorder realizer for $\mathbf{P}$.

For the first semiorder, partition $A_i = B_i \cup C_i$ for $i = 1(\mod 3)$, such that for all $x \in C_i$, $x \| y$ for some $y \in A_2$, and $B_i = A_i / C_i$. Put the vertices into "blocks" like below (see Figure 3.3).

Since each block is an induced subposet of $\mathbf{P}$ with vertices in the corresponding partitions, and it is also an interval order of height 2, then it can be represented by a semiorder, say S_1.

Similarly, For the second semiorder, partition $A_i = B_i \cup C_i$ for $i = 2(mod\ 3)$, such that for all $x \in C_i$, $x \| y$ for some $y \in A_2$, and $B_i = A_i / C_i$. Put the vertices into "blocks".

Build corresponding S_2 likewise.

For $i = 0(mod\ 3)$, similarly, $A_i = B_i \cup C_i$, for all $x \in C_i$, $x \| y$ for some $y \in A_2$, and $B_i = A_i / C_i$. Put the vertices into "blocks" as in Figure 3.5.

Build corresponding S_3 from the block likewise. It's clear that $\{S_1, S_2, S_3\}$ is a realizer of poset $\mathbf{P}$. $\square$

$\boxed{A_1}$ $\boxed{A_2}$ $\boxed{B_3A_4}$ $\boxed{C_3A_5}$ $\boxed{B_6A_7}$ $\boxed{C_6A_8}$ $\cdots$

Figure 3.5: Block 3

We already know that there is a poset in the class $C[1,2]$ that has a semi dimension larger or equal than 2. For example, the $\mathbf{1}+\mathbf{3}$ poset. But whether the bound is 2 or 3 reminds open.

The same technique can be used to prove the following theorem.

Theorem 3.2.3. *Let* $\mathbf{P} = (X, P)$ *be a poset that is in class* $C[1,k]$*, where* k *is a positive integer that is larger than 1. Then* $\mathrm{Sdim}(\mathbf{P}) \leq k+1$

We will leave the proof to the reader. The semi dimension bound for the poset class $C[1,2]$ is not tight. If there is a poset in $C[1,2]$ that has semi dimension 3 reminds open. In the theorem above, when $k \geq 4$, the bound discovered by this method is less tight than the bound in theorem 2.8.3. Hence we shall lower the bound of semiorder for posets in $C[1,k]$ to such bound when k is larger than 4.

The continuity of the semi dimension of a poset is still not clear because we cannot apply interpolation lemma. Here is a theorem for a poset with semi dimension 1.

Recall the next lemma for the two-point removal theorem for regular dimension. The proof is straightforward using interpolation property.

Lemma 3.2.4. *Let* x_1, x_2 *be distinct elements in a poset* $\mathbf{P} = (X, P)$*. If* $D(x_1) \subseteq D(x_2)$*, then* $\dim(X, P) \leq \dim(X - \{x_1, x_2\}, P(X - \{x_1, x_2\})) + 1$*.*

Proof. Let $t = \dim(X - \{x_1, x_2\}, P(X - \{x_1, x_2\}))$, and let $\{L_1, L_2, \ldots, L_t\}$ be a realizer of poset $(X - x, P(X - x))$. Let $M_i = L_i < \{x_2\} < \{x_1\}$, for $i = 1, 2, \ldots, t$. Let $M_{t+1} = D(x_1) < \{x_1\} < D(x_2) - D(x_1) < \{x_2\} < X - (D(x_1) \cup D(x_2))$. It's easy to check that $\{M_1, M_2, \ldots, M_{t+1}\}$ is a realizer of $\mathbf{P}$. □

The following lemma for semi dimension is an immediate consequence.

Lemma 3.2.5. *Let* x_1, x_2 *be distinct elements in a poset* $\mathbf{P} = (X, P)$*. If* $D(x_1) \subseteq D(x_2)$*, then* $\mathrm{Sdim}(X, P) \leq \mathrm{Sdim}(X - \{x_1, x_2\}, P(X - \{x_1, x_2\})) + 1$*.*

The proof is almost the same except here we construct a semi realizer, here we do not actually use interpolation property, we just simply add the two maximal elements to the right end of each semiorder representation. The additional semiorder is constructed in the same way.

Here is another removable conjecture on special a condition derived from the regular dimension.

Lemma 3.2.6. *Let* x *be a minimal elements in a poset* $\mathbf{P} = (X, P)$*, and* y *is a maximal element of* $\mathbf{P}$*. If* $x \| y$*, then* $\mathrm{Sdim}(X, P) \leq \mathrm{Sdim}(X - \{x, y\}, P(X - \{x, y\}))$*.*

Proof. Let $t = \mathrm{Sdim}(X - \{x, y\}, P(X - \{x, y\}))$, and let $\{S_1, S_2, \ldots, S_t\}$ be a semi realizer of poset $(X - \{x, y\}, P(X - \{x, y\}))$. For each S_i where $i = 1, 2, \ldots, t$, take a semiorder I_i as a representation of S_i. Since $\mathbf{P}$ is a finite poset. We can find a unit interval representation for x and y, such that $x < S_i < y$, for each $i = 1, 2, \ldots, t$. It's clearly a semiorder since each interval is of length 1. Label such new semiorder M_i. Repeat this for all $M_i s$. Then, take the last semiorder M_t in the realizer, let $M_{t+1} = D(y) < y < S_t - D(y) - U(x) < x < U(x)$. In the representation of S_t, we push all the unit intervals in $D(y)$ to the far left and the ones in $U(x)$ to the far right, we then insert two unit intervals for x and y. Hence M_{t+1} is a semiorder by definition. Finally, it's clear that $\{M_1, M_2, \ldots, M_{t+1}\}$ is a semi realizer of $\mathbf{P}$. □

Theorem 3.2.7. *Let* $\mathbf{P} = (X, P)$ *be a poset. let* $x \in X$*. If* $\mathrm{Sdim}(X - x, P(X - x)) = 1$*, the* $\mathrm{Sdim}(\mathbf{P}) \leq 2$.

Proof. Let S_0 be one of the semiorder that represent $(X - x, P(X - x))$. Let $[a_u, b_u]$ be the unit interval that represent u in the semiorder for all $u \in X$. Notice that a_u, b_u are real numbers. Let the most left endpoint to be a_0, and the right most endpoint to be b_0. Let $l_0 = b_0 - a_0$. Let $U(x)$ and $D(x)$ be upset and downset of x, let $inc(x)$ be the set of vertices in X that are incompareble to x.

We will build the two semi realizers from S_0. In S_1, for each $u \in D(x)$, assign u an new semiorder $[a_u - l_0 - 3, b_u - l_0 - 3]$ and assign $[a_0 - 2, a_0 - 1]$ to x. We basically push all the downset the the far left, this adds the the relation to the poset (u, i) where for all u in $D(x)$ and for some i in $inc(x)$. Notice that this will not create any alternating cycles. Similarly, in S_2, assign v an new semiorder $[a_v + l_0 + 3, b_v + l_0 + 3]$ and assign $[b_0 + 1, b_0 + 2]$ to x. It's easy to check that $\{S_1, S_2\}$ is a realizer of $\mathbf{P}$. □

The technique is similar to the proof of continuity of regular dimension, where we break the last linear extension into two. Here we break the semiorder representation into two semiorders. We used "brute force" for the proof. However, we can build a more subtle realizer for the poset. Since we have the representation for the poset with semi dimension 1. Without loss of generality, assume it's a distinguished representation. Take the rightmost unit interval $[a_w, b_w]$ in the downset of x, all the unit intervals have the right endpoint that is less than a_w is in the downset. Hence we can just shift the whole downset by $1 + \epsilon$, and then assign the interval $[a_w, b_w]$ to x. This will be equivalent to the S_1 we build in the proof. Similarly proof can be done for S_2 by symmetry.

The following is a straightforward consequence of the theorem above.

Lemma 3.2.8. *Let* $\mathbf{P} = (X, P)$ *be a poset. If* $\mathrm{Sdim}(X - x, P(X - x)) = t$*, then* $\mathrm{Sdim}(X, P) \leq 2t$*.*

This is a very weak bound when t is large. If we can "fit" in x in every semiorder in the realizer for the poset $(X - x, P(X - x))$, then the dimension of poset $|mathbf{P}$ would be $t + 1$. But in the worst-case scenario, we can not interpolate x in any of the semiorders, so we would need to break all of them into two from the method above. This is not "semi-oeder" efficient at all, since we just need one additional semiorder for all the relations between x and the other vertices, the purpose of the rest semiorders are just to "fit" in x without changing the relations between the other vertices.

Here is a theorem that can be potentially useful to solve the continuity of semi dimension.

We begin with an easy lemma.

Lemma 3.2.9. *Let* $\mathbf{P} = (X, P)$ *be a semiorder. Let* $\mathbf{I}$ *be a semiorder representation of* $\mathbf{P}$*, such that the interval represents* $x \in X$ *is* $I(x) = [a_x, a_x + 1]$*. Let* $m > 0$ *and* $\mathbf{I}_m$ *be an interval representation of* X *such that* $I_m(x) = [ma_x, m(a_x + 1)]$*. Then* $\mathbf{I}_m$ *is a semiorder represetation of* $\mathbf{P}$*.*

Proof. Since $\mathbf{I}$ is a representation of $\mathbf{P}$, then $x < y$ in P if and only if $a_x + 1 < a_y$. Clearly, $m(a_x + 1) < ma_y$ if and only if $a_x + 1 < a_y$ for any $m > 0$. And in the representation $\mathbf{I}_m$, each interval has the same length m, hence $\mathbf{I}_m$ is a semiorder representation of poset $\mathbf{P}$. □

Theorem 3.2.10. *If* $\mathbf{P} = (X, P)$ *is a semiorder. Then* $\mathbf{P}$ *has a representation* $\mathbf{I}$ *such that, for all* $x \in$*X,* $|I(x) - k| < \epsilon$ *for somt integer* k *and any* $\epsilon > 0$*.*

This theorem is basically saying that for any semiorder, we can have a representation of such semiorder where all of the endpoints of the unit intervals are "$\epsilon - close$" to some integers. Here is the proof.

Proof. Take a distinguished representation of poset $\mathbf{P}$, fix an ϵ. Partition X into antichains $A_1, A_2, \ldots, A_m$ by taking the minimal elements successively. There is an interval $[a_1, b_1]$ where $a_1 < b_1$, such that $a_x < a_1$ and $b_x > b_1$ for each $x \in A_1$, where $[a_x, b_x]$ is the unit interval in the representation $\mathbf{I}$ for x (see Figure 3.6. Let $l = 1000n/\epsilon$. For all endpoint that are larger that b_1, add l to the real number that represent the endpoint. Hence each interval in the part A_1 becomes $[a_x, b_x + l]$, where $x \in A_1$. Meanwhile, for the interval that represents points in the other parts such as $[a_y, b_y]$ where $y \in A_2 \cup A_3 \cup, \ldots, \cup A_m$. It becomes $[a_y + l, b_y + l]$. Similarly, we find the interval $[a_2, b_2]$ in the part A_2, and add l the endpoints that have real value that is bigger than b_2. After we do this for all the parts, it's easy to check that we did not change the relation P between the pair of vertices. We will obtain a semiorder representation of $\mathbf{P}$ since each of the interval has length $1 + l$. Finally,

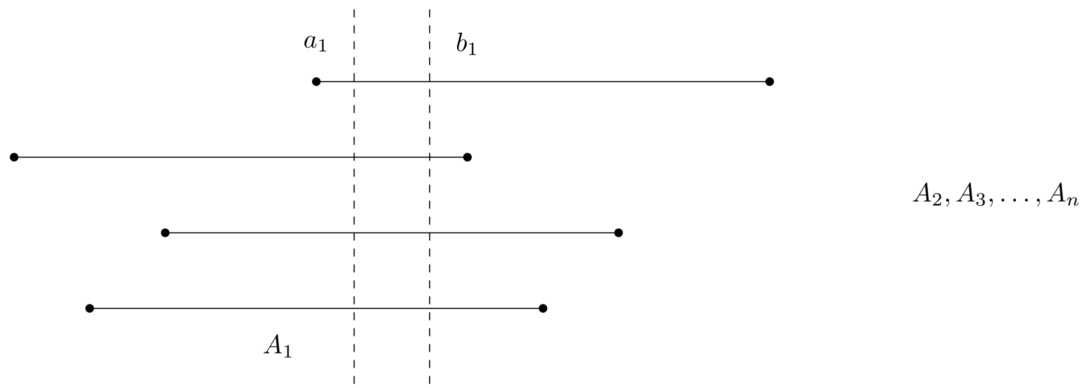


Figure 3.6: Stretch the intervals

divide the endpoints by $l + 1$. By Lemma 3.2.9, the new interval representation is a semiorder representation of $\mathbf{P}$ with unit length, let the most left endpoint to be 0, it's easy to check that for all $x \in$X, $|I(x) - k| < \epsilon$ for some integer k.

$\square$

The following is another interesting topic, split orders and split semiorders.

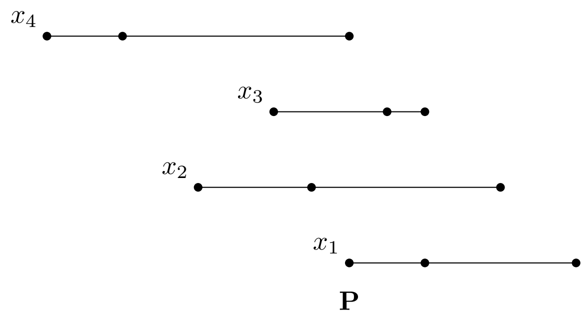


Figure 3.7: Split semiorder example

3.3. SPLIT ORDERS AND SPLIT SEMIORDERS

3.3.1 Introduction

We call a poset **P** a split interval order if there exists a function I that assigns a closed interval $I_x = [a_x, b_x]$ and a set $F = \{f_x : x \in X\}$ of real numbers to each $x \in X$, we will just call this set middle points, such that:

1. $x \in X$, $a_x < f_x < b_x$ for all $x \in X$, and
2. For all $x, y \in X$, $x < y$ in P if and only if $f_x < a_y$ and $b_x < f_y$ in **R**.

We call (I, F) a representation of **P**

3.3.2 Example

Here are some examples of split semiorders. (see Figure 3.7)

In the split semiorder representation of $\mathbf{P} = (X, P)$, there are 4 vertices $\{x_1, x_2, x_3, x_4\}$, the middle points of the interval represent the f_{x_i} for each x_i. By the definition of the relation of split semiorders, $x_4 < x_3$ since $a_{x_3} > f_{x_4}$ and $f_{x_3} > b_{x_4}$. $x_2 \not< x_1$ since $b_{x_2} \not< f_{x_1}$ even though $f_{x_2} < a_{x_1}$; and clearly $x_1 \not< x_2$, hence we have $x_1 \| x_2$ in P.

3.3.3 Split Semiorders

A poset $\mathbf{P} = (X, P)$ is a split semiorder when there exists a function U that assigns a closed real interval $U(x) = [a_x, a_x + 1]$ and a set $F = \{f_x : x \in X\}$ of real numbers to each $x \in X$, such that:

1. $x \in X$, $a_x < f_x < a_x + 1$ for all $x \in X$, and
2. For all $x, y \in X$, $x < y$ in P if and only if $f_x < a_y$ and $a_x + 1 < f_y$ in **R**.

It is a special case of interval orders where the length of the interval is unit length.

There are also bitolerance orders and tolerance orders that are also special poset classes. But here, we will try to focus on split interval orders and split semiorders.

Here are several useful theorems proved by Fishburn, Trotter in [18].

Theorem 3.3.1. *Every split semiorder and split interval order has a distinguishing representation.*

We say a poset $\mathbf{P} = (X, P)$ has a distinguishing representation (I, F) is $|\{a_x : x \in X\} \cup \{b_x : x \in X\} \cup \{f_x : x \in X\}| = 3|X|$. Which just means that all the end point and the middle point has different real values.

The other theorem in the same paper gives the forbidden structures for split interval orders and split semiorders.

Theorem 3.3.2. $\mathbf{m} + \mathbf{n}$ *is minimally forbidden for split semiorders if and only if it is* $\mathbf{2} + \mathbf{3}$ *or* $\mathbf{1} + \mathbf{4}$*;* $\mathbf{m} + \mathbf{n}$ *is minimally forbidden for split interval orders if and onlu if it is* $\mathbf{3} + \mathbf{3}$.

In the paper, the authors ask the open question for the characterization of split semiorders that have (U, F) representations in which all splitting points lie in a central range of their interval's midpoints such as $|f(x) - a(x) - \frac{1}{2}| \leq \lambda$ for fixed λ in $[0, 1/2)$.

The $\lambda = 0$ case is trivial. Since in that case, all interval representation would have unit interval length and the middle point would be exactly the midpoint. This gives us a semiorder.

In fact, for each rational number λ_1 and λ_2, where $\lambda_1 < \lambda_2$, let $\mathbf{Q_1}$ and $\mathbf{Q_2}$ be the class of poset where all splitting points lie inside $[m - \lambda_1, m + \lambda_1]$ and $[m - \lambda_2, m + \lambda_2]$ respectively. No matter how close λ_1 and λ_2 is, we can give a split semiorder that is in the class $\mathbf{Q_2}$ but not in the class $\mathbf{Q_1}$.

[todo]

In [17], Fishburn and Trotter proved the following theorem.

Theorem 3.3.3. *If* $\mathbf{P}$ *is a split semiorder, then* $\dim(\mathbf{P}) \leq 6$

Robinovitch gives examples of semiorders of linear dimension 3 [38], but we don't know whether there are split orders with dimension 6.

The authors also mentioned the following theorem in the paper.

Theorem 3.3.4. *If* $\mathbf{P} = (X, P)$ *is a split semiorder, then* $\mathrm{Idim}(\mathbf{P}) \leq 2$

Proof. The proof is straightforward. We take a distinguishing split semiorder representation $\mathbf{I}$. Where each $x \in X$ is represented by a close interval $[a_x, b_x]$ and a point f_x, where $a_x < f_x < b_x$. Let $\mathbf{I_1}$ be an interval order represent $\mathbf{P}$, wherer $I_{1x} = [a_x, f_x]$. Let $\mathbf{I_2}$ be an interval order represent $\mathbf{P}$, wherer $I_{2x} = [f_X, b_X]$. Clearly $\{\mathbf{I}_1, \mathbf{I}_2\}$ is a interval realizer of split semiorder $\mathbf{P}$. $\square$

From Theorem 3.3.3 we know that split semiorder has semi dimension less or equal than 6. But can we improve it? Or can we find a split semiorder that has semi dimension 6? In fact, the split semiorder with the largest semi dimension we could find is 3 so far, hence:

$3 \leq \max\{\mathrm{Sdim}(\mathbf{P} : \mathbf{P} \textit{ is a split semiorder}\} \leq 6$

CHAPTER 4

RAMSEY RESULTS ON POSETS

4.1. INTRODUCTION

Ramsey theory is a branch of mathematics that focuses on the appearance of order in a substructure given a structure of a known size. Problems in Ramsey theory usually ask: is there exists a large enough structure to guarantee a particular one? If there is, how large does the structure need to be?

Ramsey theory is very interesting. It also has real applications in the fields of communications, information retrieval in computer science, and decision making. [8]

Here is the Ramsey's original version concerning complete graphs.

4.1.1 Ramsey' original theorem

Theorem 4.1.1. *For every $r \in \mathbb{Z}$ there exists an $n \in \mathbb{Z}$ such that every graph with at least n vertices contains either K^r or $\bar{K^r}$ as an induced subgraph.*

4.1.2 An alternative statement and Ramsey number

Usually, we demonstrate the theorem for two colors, say blue and red, let r and s be two positive integers. Ramsey's theorem states that there exists a least positive integer, denoted by $R(r, s)$, such that for any complete graph K^n where $n > R(r, s)$, if we color all the edge of K^n by red or blue, there is a monochromatic read K^r or a blue K^s as an induced subgraph of K^n. Here $R(r, s)$ is the Ramsey number.

4.1.3 Example

Here are some examples.

Take a complete graph with six vertices (see Figure 4.1). No matter how you color the graph with red or blue, there will always be a monochromatic triangle (K^3). In fact, 6 is the least number

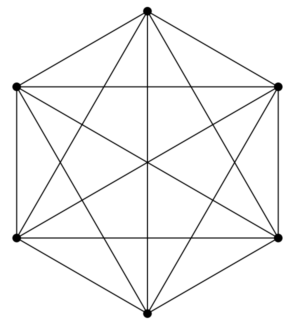


Figure 4.1: K^6

of vertices for a complete graph to guarantee that, hence the Ramsey number $R(3,3) = 6$.

A summary of known results of $R(r,s)$ can be found online, such as $R(3,4) = 9$, $R(4,4) = 18$. In fact, there are not too many of them are known, instead, when s and r get large, we might only be able to find a bounds for the Ramsey numbers so far. Such as $R(3,10)$ is in $[40,43]$, $R(5,5)$ is in $[43,49]$, $R(10,10)$ is in $[798,23556]$. As you can see, the bound of Ramsey number became hopelessly large even when $r = s = 10$.

In this article, we focus more on Ramsey problems concerning existence. However, the Ramsey number is convenient for proving the Ramsey's original theorem. Here is the proof of the theorem for the 2-color case.

Proof. We proceed by induction. From the definition of Ramsey number, it's clear that $R(n,2) = n$, since K^2 is just an edge. We then prove that $R(s,r)$ (there is a red Ks or a blue Kr as an induced subposet) exists by finding an explicit bound for it by the inductive assumption that $R(r-1,s)$ and $R(r,s-1)$ exits. It would be sufficient to prove that:

$R(r,s) \leq R(r-1,s) + R(r,s-1)$

Consider the complete graph with $R(r-1,s)+R(r,s-1)$ vertices and color all it's edges by red or blue randomly. Pick a vertex x from it, partition other vertices into 2 sets A and B. where (x,v) is blue for all $v \in A$ and (x,u) is red for all $u \in B$. Since $|A| + |B| + 1 = R(r-1,s) + R(r,s-1)$, then $|A| > R(r-1,s)$ or $|B| > R(r,s-1)$. In the first case, if there is a red K^s in A, then we are done; if there is a blue K^{r-1} in A, then $|A| \cup x$ has a blue K^r. Similarly, for the later case, if there is a blue K^r in B, then we are done. Otherwise, there is a red K^{r-1} in B, hence $B \cup x$ is a red K^r. This completes the proof. $\square$

4.2. RAMSEY THEOREM WITH MULTI-COLOR

A multi-color Ramsey number is a Ramsey number using three or more colors, The only two non-trivial Ramsey numbers for which the exact value is known, which are $R(3,3,3) = 17$ and $R(3,3,4) = 30$. The existence of $R(m,n,s)$ for any positive integer m,n,s is an easy consequence of the Lemma 4.7.1 in the Section 4.5.

Here is the original product Ramsey theorem by Graham, Rothschild and Spencer [22].

Theorem 4.3.1. *Let $k_1, k_2, \ldots, k_t$ be nonnegative integers. Let r and t be positive integers. Let $m_1, m_2, \ldots, m_t$ be integers with $m_i \geq K_i$ for $i = 1, 2, \ldots, t$. Then there exists and integer $R = R(r, t; k_1, k_2, \ldots, k_t; m_1, m_2, \ldots, m_t)$ so that if $X_1, X_2, \ldots, X_t$ are sets and $|X_i| \geq R$ for $i = 1, 2, \ldots, t$, then for every function $f : \binom{X_1}{k_1} \times \binom{X_2}{k_2} \times \cdots \times \binom{X_t}{k_t} \to [r]$, there exist an element $\alpha \in [r]$ and subsets $Y_1, Y_2, \ldots, Y_t$ of $X_1, X_2, \ldots, X_t$, respectively, so that $|Y_i| \geq m_i$ for $i = 1, 2, \ldots, t$ and f maps every elements of $\binom{Y_1}{k_1} \times \binom{Y_2}{k_2} \times \cdots \times \binom{Y_t}{k_t}$ to α.*

Here is an alternative statement [23] that is equal to the theorem above but in a grid perspective.

Theorem 4.3.2. *For all $k > 0, s_1, s_2, \ldots, s_k > 0, a_1, a_2, \ldots, a_l > 0$, and all $r > 0$, there exist $n_1, n_2, \ldots, n_k > 0$ so that, if $|B_i| \geq n_i$ and $1 \leq i \leq k$, and $[B_1]^{s_1} \times [B_2]^{s_2} \times \cdots \times [B_k]^{s_k}$ is $r-colored$, then there exist $A_i \subset B_i$ and $|A_i| = a_i$ so that $[A_1]^{s_1} \times [A_2]^{s_2} \times \cdots \times [A_k]^{s_k}$ is monochromatic.*

Notice here, $[B_i]$ is a set with B_i elements, and $[B_i]^{s_i}$ is the subset of $[B_i]$ that has s_i elements. $[B_1]^{s_1} \times [B_2]^{s_2} \times \cdots \times [B_k]^{s_k}$ is a k dimensional grid.

Proof. We proceed by induction on k. When $k = 1$, the theorem is equivalent to Ramsey's original theorem, which is proven. Fix $s_1, s_2, \ldots, s_k > 0, a_1, a_2, \ldots, a_l > 0, r$ and define correspondent $n_1, n_2, \ldots, n_{k-1}$ to meet the condition of the theorem. Then define n_k, M, where $M = r^T$, and $T = \binom{n_1}{s_1} \cdots \binom{n_{k-1}}{s_{k-1}}$. such that if $|B| \geq n_k$, for every M coloring of subset of $[n_k]$ with s_k elements, there is a subset A of B, so that every $[A]^{s_k}$ is of the same color.

Then let $|B_i| = n_i$, if $|B_i| > n_i$, take a subset as the new B_i with n_i elements. Let χ be an r-coloring of $[B_1]^{s_1} \times [B_2]^{s_2} \times \cdots \times [B_k]^{s_k}$. Let χ' be a coloring on $[B_k]^{s_k}$ by $\chi'(E) = \chi'(E')$ if and only if

$\chi(S_1, S_2, \ldots, S_{k-1}, E) = \chi(S_1, S_2, \ldots, S_{k-1}, E')$ for all $S_i \in [B_i]^{s_k}$.

Observe that χ' is an $M-coloring$, then there is an $A_k \subset B_k$, where $|A_k| = a_k$, such that $[A_k]^{s_k}$ is monochromatic under χ'. Hence there is an χ'' for $S_i \in [B_i]^{s_i}$, such that

$\chi''(S_1, S_2, \ldots, S_{k-1}) = \chi(S_1, S_2, \ldots, S_{k-1}, E)$ for all $E \in [A_k]^{s_k}$.

Then by induction, there exist $A_1, A_2, \ldots, A_{k-1}$ so that $[A_1]^{s_1} \times [A_2]^{s_2} \times \cdots \times [A_{k-1}]^{s_{k-1}}$ is monochromatic under the color χ'', since χ is a $r-color$, hence $[A_1]^{s_1} \times [A_2]^{s_2} \times \cdots \times [A_k]^{s_k}$ is monochromatic under χ.

∎

4.3.1 Boolean Dimension

Boolean dimension is another interesting topic, it was first introduced by Gamnosi, Nešetřil and Talamo [19]. Here is an introduction.

From the previous chapters, we know that any poset can be expressed as an intersection of linear orders. Let $\mathbf{P} = (X, P)$ be a poset. For a positive integer d, let 2^d denote the set of all possible $0-1$ strings of length d. We also call such string bit strings. Let $\mathcal{B} = \{O_1, O_2, \ldots, O_d\}$ be a family of linear orders on the ground set of $\mathbf{P}$. One thing that needs to be noticed is that those linear orders need not to be linear extensions of P. We define a function $\phi : \mathbf{2}^d \to \{0, 1\}$. We form a bit string $b(x, y, \mathcal{B})$ of length d which has value 1 in the ith coordinate if and only if $x < y$ in O_i. The pair $(\mathcal{B}, \phi)$ is a Boolean realizer when for each distinct vertices x, y of poset $\mathbf{P}$, $x < y$ in P if and only of $\phi(b(x, y, \mathcal{B})) = 1$. The Boolean dimension of poset $\mathbf{P}$ is the least positive integer d for which $\mathbf{P}$ has a Boolean realizer $(\mathcal{B}, \phi)$ with $|\mathcal{B}| = d$. We denote the Boolean dimension of any poset $\mathbf{P}$ by $\operatorname{bdim} P$.

It is clear that for any poset $\mathbf{P}$, $\operatorname{bdim}(\mathbf{P}) \leq \dim(\mathbf{P})$. To prove this, simplit take a realizer $\mathcal{R} = \{L_1, L_2, \ldots, L_d\}$ of poset $\mathbf{P}$, which is also a family of linear orders by the definition. Take function ϕ where ϕ only maps the bit string with length d: $\{1, 1, \ldots, 1\}$ to 1, and other bit strings to 0.

It is trivially to see that $\operatorname{bdim}(\mathbf{P}) = 1$ if and only if $\mathbf{P}$ is a chain. It is an easy exercise to show that $\operatorname{bdim}(\mathbf{P}) = 2$ implies that $\dim(\mathbf{P}) = 2$. Gambosi, Nešetřil and Talamo [20] showed that $\dim(\mathbf{P}) = 3$ if and only if $\operatorname{bdim}(\mathbf{P}) = 3$.

We know that the dimension of standard example S_n is n. Here is an interesting theorem of the Boolean dimension of the standard example.

Theorem 4.3.3. *Let S_n be the standard example with $2n$ vertices. Then $\operatorname{bdim}(S_n) = n$ when $2 \leq n \leq 4$ and $\operatorname{bdim}(S_n) = 4$ when $n \geq 4$.*

4.4. MOTIVATION

The original motivation of this paper was two questions of Nešetřil and Pudlák from 1986. In their paper [33], they introduced the notion of the Boolean dimension of partially ordered sets (see the first paragraphs of Section 4.6). They proved an upper and a lower bound based on the number of points of the poset. At the end of their note they asked two questions. They did not voice their thoughts about which way the questions would go; nevertheless we rephrase the questions as "statements" for easier discussion.

Statement 4.4.1. *The Boolean dimension of planar posets is unbounded.*

Note that this is the opposite of what many researchers, possibly including Nešetřil and Pudlák, conjectured. For example in [7], the authors state that it is clear from the presentation of the question in [33] that they believed the answer should be "no".

We will say that a class $\mathcal{C}$ of posets has the Ramsey-property, if for all r and $P \in \mathcal{C}$, there is a poset $Q \in \mathcal{C}$, such that for every r-coloring of the comparabilities of Q, there is a subposet Q' of Q that is isomorphic to P such that every comparability of Q' is of the same color.

Nešetřil and Rödl [34] proved that the class of all posets has the Ramsey property. (In fact, this is just a consequence of their much stronger theorem.) Later we will show that this special consequence is also a rather direct consequence of the so-called Product Ramsey Theorem.

Nešetřil and Pudlák asked a second question in their paper, which we also phrase as a statement.

Statement 4.4.2. *The class of planar posets has the Ramsey property.*

Nešetřil and Pudlák pointed out that Statement 4.4.2 implies Statement 4.4.1, though they did not include the proof in their short article. We provide a proof of this implication in Section 4.6. We do this, after we set up the basic framework of the Ramsey-property for relational sets, and we prove a useful general lemma in Section 4.5.

We take a slight detour in Section 4.7 to discuss some related statements for planar graphs.

When we started to work on present research, we guessed both Statement 4.4.1 and 4.4.2 are false. Since we did not know how to approach the more general question, we decided we would attempt to disprove Statement 4.4.2. One natural way to do that would be to construct a planar poset P for which there is not an appropriate Q required by the Ramsey-property. A simple choice for a planar poset would be a 2-dimensional grid. However, we quickly realized that that is not a good example. In fact, we proved the following.

Proposition. *For each t positive integer, the class of t-dimensional grids have the Ramsey Property.*

As we will see in Section 4.8, this proposition is a relatively simple consequence of the Product Ramsey Theorem, and easily implies that the class of all posets has the Ramsey-property.

We considered developing a tool that would be more powerful than the Product Ramsey Theorem. The result of this attempt is the following conjecture.

Conjecture. *For all t, r, m, and l there exists an n such that for all r-colorings of the $\mathbf{m}^t$ subposets of $\mathbf{n}^t$, there is a monochromatic $\mathbf{l}^t$ subposet L. That is, every $\mathbf{m}^t$ subposet of L receives the same color.*

In Section 4.9, we prove the conjecture for the special case $t = 2$.

Theorem. *For all r, m, and l there exists an n such that for all r-colorings of the $\mathbf{m}^2$ subposets of $\mathbf{n}^2$, there is a monochromatic $\mathbf{l}^2$ subposet P. That is, every $\mathbf{m}^2$ subposet of P receives the same color.*

In Section 4.10 we, again, use the Product Ramsey Theorem to prove the following, somewhat counterintuitive result. This is a generalization of a classical result by Paoli, Trotter, and Walker [35].

Theorem. *Let X be a poset and let M be a linear extension of X. Furthermore, let k be a positive integer. Then there exists a grid $Y \cong \mathbf{n}^t$ such that for all $L_1, L_2, \ldots, L_k$ linear extensions of Y, there is a subposet X' of Y such that*

- *$X' \cong X$, evidenced by the embedding $f : X \to Y$;*
- *for all $i = 1, \ldots, k$, and for all $a, b \in X$, we have $a < b$ in M if and only if $f(a) < f(b)$ in L_i.*

We note that, although some of the problems studied may be interesting for infinite posets, in this paper every poset is finite. In fact, we will omit the word finite, and even when we say, for example "class of all posets", we mean "class of all finite posets".

4.5. RELATIONAL SETS

First, we define the Ramsey property of general classes of sets with relations to prove a general lemma that shows that the number of colors (as long as it is at least 2) does not matter.

Let X be a set, and r a positive integer. An *r-coloring* of X is a function $c : X \to [r]$, where $[r] = \{1, \ldots, r\}$. The elements of $[r]$ are called *colors*. Indeed, any function $g : X \to S$ where $|S| = r$ can be considered an r-coloring. A *relation* R on X is a subset $R \subseteq X \times X$. If $X' \subseteq X$, we use the usual notation $c|_{X'}$ and $R|_{X'}$ for the restriction of c and R (respectively) to the subset X'.

Let $\mathcal{C}$ be a class of ordered pairs (X, R), where X is a set, and R is a relation on X. We say that $\mathcal{C}$ has the *Ramsey Property*, if for all $(X, R) \in \mathcal{C}$ and for all r positive integer, there exists $(Y, S) \in \mathcal{C}$ such that for every r-coloring c of S, there exists a subset $Y' \subseteq Y$ such that if $S' = S|_{Y'}$, then $(Y', S') \cong (X, R)$, and for all $a, b \in S'$, we have $c|_{S'}(a) = c|_{S'}(b)$. Less formally $\mathcal{C}$ has the Ramsey property, if for all $\mathbf{X} \in \mathcal{C}$ there is a (larger) $\mathbf{Y} \in \mathcal{C}$, such that if we r-color the relations of $\mathbf{Y}$, we will find a monochromatic subrelation $\mathbf{X}'$ of $\mathbf{Y}$ that is isomorphic to $\mathbf{X}$. Monochromatic means that every relation of $\mathbf{X}'$ is assigned the same color. The set (with the relation) $\mathbf{Y}$ is called the *Ramsey set* of X.

With a slight abuse of notation, we will often write X for the pair $\mathbf{X} = (X, R)$.

After this definition one might think that the classical theorem of Ramsey could be rephrased by perhaps saying that the set of (complete) graphs have the Ramsey Property, using the usual definition of a graph as an irreflexive, symmetric relation. This is, however, not the case. Clearly, one can 2-color the relations of a graph by coloring the two directions of an edge opposite colors for each edge, and then no monochromatic edge will even be found.

On the other hand, it *is* possible to state Ramsey's Theorem with this terminology: it is the statement that the class of linear orders has the Ramsey Property. This will be a special case of our Proposition 4.8.2.

The following lemma is often useful when one is trying to prove that a class has the Ramsey Property.

Lemma 4.5.1. *Suppose $\mathcal{C}$ has the following property: for all $X \in \mathcal{C}$ there is a $Y \in \mathcal{C}$ and a positive integer $r_0 \geq 2$, such that if we r_0-color the relations of Y, we will find a monochromatic subrelation X' of Y that is isomorphic to X. Then $\mathcal{C}$ has the Ramsey Property.*

Proof. Let $\mathcal{C}$ be a class, $X \in \mathcal{C}$, and r a positive integer; we need to show that a Ramsey set Y can be found. We will do that by induction on r. If $r \leq 2$, then by the conditions there exists Y and

$r_0 \geq 2$. Since $r \leq r_0$, an r-coloring is a special r_0-coloring, so the statement follows.

Now let $r > 2$, and assume the statement is true for $r-1$. So there exists $Y \in \mathcal{C}$, such that if we $r-1$-color the relations of Y, there is a monochromatic subrelation X' of Y that is isomorphic to X.

We can use the hypothesis again for $Y \in \mathcal{C}$ and 2-colors. There exists a $Z \in \mathcal{C}$ such that any 2-coloring of the relations of Z yields a monochromatic copy of Y. We claim that Z is a correct choice for the original set X and r colors.

To see this, consider an r-coloring c of the relations of Z. Now recolor Z with only 2 colors based on the c: if $c(x) = 1$, use the color blue; otherwise use the color red. We know Z yields a monochromatic Y. If Y is blue, then we notice that X is a subrelation of Y, so we found a monochromatic X. If Y is red, then we revert to c to color the relations of Y with $r-1$ colors, and we find the monochromatic X this way. $\square$

We assume the readers are already familiar with basic notions of partially ordered sets and graph theory. We refer the reader to the monograph of Trotter [48], and the textbook of Diestel [10].

4.6. TWO QUESTIONS OF NEŠETŘIL AND PUDLÁK

Recall that the Iverson bracket is a notation that converts a logical proposition to 0 or 1: $[P] = 1$ if P is true, and $[P] = 0$, if P is false.

Let P be a poset, and let $(\mathcal{L}, S)$ be a pair where $\mathcal{L} = \{L_1, \ldots, L_d\}$, $(d \geq 1)$ is a set of linear orders of the elements of P, and S is a set of binary (0–1) strings of length d. For two distinct elements $x, y \in P$, let $P_i(x, y)$ be the proposition that $x < y$ in L_i. We call $(\mathcal{L}, S)$ a *Boolean realizer*, if for any two distinct elements x, y, we have $x < y$ in P if and only if $[P_1(x, y)][P_2(x, y)] \ldots [P_d(x, y)] \in S$. We call this binary string the *signature* of the pair (x, y). The number d is the cardinality or size of the Boolean realizer. The minimum cardinality of a Boolean realizer is the Boolean dimension of P, denoted by $\dim_B(P)$.

We note that there are minor variations in the the definition of Boolean realizers in the literature. (See next paragraph for citations.) With our definition, antichains are of Boolean dimension 1 (one can take $S = \emptyset$), chains are of Boolean dimension 1, and in general, $\dim_B(P) \leq \dim(P)$, because a Dushnik–Miller realizer P can be easily converted into a Boolean realizer of the same size by taking $S = \{11 \ldots 1\}$.

Boolean dimension and structural properties of posets have seen an increased interest in recent years, e.g. in [14], the authors showed that posets with cover graphs of bounded tree-width have bounded Boolean dimension. Further, in [3], the authors compared the Dushnik–Miller dimension, Boolean dimension, and local dimension in terms of tree-width of its cover graph, and in [31], the authors studied the behavior of Boolean dimension with respect to components and blocks.

As mentioned earlier, the following statement appeared without proof in [33]. We include a proof for completeness.

Proposition 4.6.1. *Statement 4.4.2 implies Statement 4.4.1.*

Proof. Assume that Statement 4.4.2 is true, but the Boolean dimension of planar posets is at most k. Let P be a planar poset whose Dushnik–Miller dimension is greater than k (such a poset is well-known to exist). By Statement 4.4.2, there is a planar poset Q such that any 2^k-coloring of the comparabilities of Q yields a monochromatic P.

Let $(\mathcal{L}, S)$ be a Boolean realizer of size k of Q, and let $\mathcal{L} = \{L_1, \ldots, L_k\}$. Color the comparabilities of Q with binary strings of length k as colors: if $x < y$ in Q, let the color of (x, y) be the signature of (x, y).

Now let P' be a subposet of Q such that $P' \cong P$ and every comparable pair of P' is of the same color, say $d_1 d_2 \ldots d_k$ (where d_i is the ith digit of the binary string). Let $M_i = L_i$ if $d_i = 1$, and let $M_i = L_i^d$ (the dual of L_i), if $d_i = 0$. It is routine to verify that $\{M_1, \ldots, M_k\}$ is a realizer of P, contradicting $\dim(P) > k$. $\square$

4.7. RAMSEY PROPERTY OF PLANAR GRAPHS

We noted earlier that our general notion of Ramsey Property is not very natural for studying graphs, because we can color the two directions of an edge with different colors. So it is natural to redefine the Ramsey Property specifically for classes of graphs.

We say that a class of graphs $\mathcal{C}$ has the *Ramsey Property*, if for all $G \in \mathcal{C}$ and for all r positive integers, there exists $H \in \mathcal{C}$ such that for every r-coloring of $E(H)$, there exists an induced subgraph G' of H such that $G' \cong G$, and every edge of G' is of the same color. We use the term "monochromatic" as before, and we call the graph H the Ramsey graph of G.

Ramsey's Theorem can be restated by saying the class of complete graphs has the Ramsey Property. The fact that the class of all graphs has the Ramsey Property is a more difficult statement, and it was proven around 1973 independently by Deuber [9], by Erdős, Hajnal and Pósa [13], and by Rödl [41].

We note that as for general relations, the analogous lemma is true and can be proven exactly the same way.

Lemma 4.7.1. *Suppose $\mathcal{C}$, a class of graphs, has the following property: for all $G \in \mathcal{C}$ there is a $H \in \mathcal{C}$ and a positive integer $r_0 \geq 2$, such that if we r_0-color the edges of H, we will find a monochromatic induced subgraph G' of H that is isomorphic to G. Then $\mathcal{C}$ has the Ramsey Property.*

Motivated by our problem on planar posets, we were curious whether the class of planar graphs has the Ramsey Property. This is not the case. In fact, as Axenovich et al. [2] pointed out, a result of Gonçalves [21] and the Four Color Theorem imply that if G has an appropriate H as in the definition, then G must be planar bipartite. But to just prove that the class of planar graphs does not have the Ramsey Property, we only need elementary tools.

Proposition 4.7.2. *The class of planar graphs do not have the Ramsey Property.*

Proof. Let G be a planar graph that is not bipartite. Now suppose that the class of planar graphs has the Ramsey Property. Then there exists a planar Ramsey graph H for G. Since $\chi(H) \leq 6$, one can decompose the edge set of H into $\binom{6}{2}$ bipartite graphs. (These numbers can obviously be improved.) Color the edges of H based on which of these bipartite graphs they are in. Let $G' \cong G$ be a monochromatic induced subgraph of H. Since the edges of G' use a single color, G' is bipartite, a contradiction. $\square$

4.8. RAMSEY PROPERTY OF GRIDS

We will use $\mathbf{k}$ to denote the k-element chain, and $\mathbf{k}^t$ for the poset that is the product of the k-element chain by itself t times. In more details, suppose the ground set of $\mathbf{k}$ is $X = \{x_1, x_2, \ldots, x_k\}$ with $x_1 < x_2 < \cdots < x_k$. Then the elements of $\mathbf{k}^t$ are t-tuples $(x_{i_1}, x_{i_2}, \ldots, x_{i_t})$ with $1 \leq i_l \leq k$ for all l, and $(x_{i_1}, \ldots, x_{i_t}) \leq (x_{j_1}, \ldots, x_{j_t})$ in $\mathbf{k}^t$ if and only if $i_l \leq j_l$ for all l.

This poset will be called the $\mathbf{k}^t$ grid. The number t is called the dimension of the grid. This coincides with the Dushnik–Miller dimension of the poset for $k \geq 2$, so it will not cause confusion.

Let the ground set of the poset $\mathbf{n}^t$ be the set of t-tuples of numbers from $[n]$. Let $S_1, S_2, \ldots, S_t$ be nonempty subsets of $[n]$. The subposet induced by the elements of $S_1 \times \cdots \times S_t$ is called a *subgrid* of $\mathbf{n}^t$. Of course, every subgrid is a subposet that is a grid, but the converse is not true.

We will use the Product Ramsey Theorem, which can be phrased with our terminology as follows.

Theorem 4.8.1 (Product Ramsey Theorem [23])**.** *For all t, r, m, and l there exists an n such that for all r-colorings of the $\mathbf{m}^t$ subgrids of $\mathbf{n}^t$, there is a monochromatic $\mathbf{l}^t$ subgrid L. That is, every $\mathbf{m}^t$ subgrid of L receives the same color.*

A relatively easy consequence is the following proposition.

Proposition 4.8.2. *For each t positive integer, the class of t-dimensional grids have the Ramsey Property.*

Before we present a proof, let us recall some classical results of poset theory. An $\mathbf{s}^t$ grid P has Dushnik–Miller dimension at most t. As such, it has a realizer $\{L_1, \ldots, L_t\}$, which can be used to embed P into $\mathbb{N}^t$: indeed, the ith coordinate of the element x can be chosen to be the position of x in L_i (that is, the size of the closed downset of x in L_i). In fact, this is an embedding into $\mathbf{l}^t$, where $l = s^t$. It also has the property that for each pair of integers i, j with $1 \leq i \leq t$ and $1 \leq j \leq l$, there is exactly one $x \in P$ such that the ith coordinate of x in the embedding is j.

Such embeddings will be called "casual". Here is the precise definition. Let t, s be positive integers, and let $l = s^t$. Let $P = \mathbf{s}^t$, and $Q = \mathbf{l}^t$. We define the usual projection functions: for $x \in Q$ and $1 \leq i \leq t$, the positive integer $\mathrm{Proj}_i(x)$ is the position (size of closed downset) of the ith coordinate of x in $\mathbf{l}$. An embedding $f : P \to Q$ is called *casual*, if for all i, j with $1 \leq i \leq t$ and $1 \leq j \leq l$, there is exactly one $x \in P$ such that $\mathrm{Proj}_i(f(x)) = j$.

So in a casual embedding there are no ties in coordinates. Note that we also require that there are no "unused" coordinates. So, a casual embedding of $\mathbf{s}^t$ into a grid $\mathbf{l}^t$ always has the property that $l = s^t$.

The existence of a casual embedding is typically proven non-constructively, though it is not difficult to construct one.

Now we are ready to prove Proposition 4.8.2.

Proof. Let t, r be positive integers. Let s be a positive integer, and P be an $\mathbf{s}^t$ grid. We will show that a Ramsey poset Q exists for P.

If $s = 1$, the theorem is trivial, so we assume $s \geq 2$.

We invoke the Product Ramsey Theorem for t and r as fixed above, for $m = 2$, and $l = s^t$, to get a number n. We claim that $Q = \mathbf{n}^t$ is a Ramsey poset for P.

To show this, we consider a coloring $c : C(Q) \to [r]$ of the comparabilities of Q; here $C(Q)$ denotes the set $\{(a, b) \in Q : a \text{ and } b \text{ are comparable}\}$. We will use this to define an r-coloring of the $\mathbf{2}^t$ subgrids of Q as follows. Let M be a $\mathbf{2}^t$ subgrid, with the least element a, and the greatest element b. Then we assign the color $c(a, b)$ to this subgrid.

By the Product Ramsey Theorem, a monochromatic $\mathbf{l}^t$ subgrid exists; let this be called R. Let P' be a casually embedded copy of P into R; we claim P' is monochromatic. To see this, let $a < b$ in P'. Since a and b have distinct ith coordinates for each $i = 1, \ldots, t$ in R (and Q), they determine an $M(a, b)$ $\mathbf{2}^t$ subgrid of R (and Q). Due to the choice of R by the Product Ramsey Theorem, each $M(a, b)$ has the same color r_0, which, in turn, implies $c(a, b) = r_0$. □

We would like to note that Ramsey's classical theorem is a special case of Proposition 4.8.2 when $t = 1$.

The special case of the theorem of Nešetřil and Rödl now follows easily.

Corollary 4.8.3. *The class of all posets has the Ramsey Property.*

Proof. Let P be a poset. It is well-know that every poset is a subposet of a large enough Boolean lattice. The Boolean lattice of dimension d is the grid $\mathbf{2}^d$.

So first find a Boolean lattice B such that P is a subposet of B. Then use Proposition 4.8.2 to find a grid Q, a Ramsey Poset for B. A monochromatic subposet B clearly contains a monochromatic P, so the theorem follows. □

Furthermore, since every poset of Dushnik–Miller dimension d can be embedded into $\mathbf{k}^d$ for sufficiently large k, the following corollary is immediate.

Corollary 4.8.4. *The class of posets of dimension at most d has the Ramsey Property.*

(Of course, the corollary remains true if one replaces "at most" with "exactly".)

Unfortunately, none of the tools used here seem to be capable of grasping the complexities of planar posets, so the truth value of Statement 4.4.2 remains open.

4.9. RAMSEY THEORY OF GRID SUBPOSETS

During this research, we found a statement that would have powerful consequences. It is not a straight generalization of the Product Ramsey Theorem, but it seems to be more useful in many cases. Although the authors disagree on the truth value, for easier discussion we state it as a conjecture.

Conjecture 4.9.1. *For all t, r, m, and l there exists an n such that for all r-colorings of the $\mathbf{m}^t$ subposets of $\mathbf{n}^t$, there is a monochromatic $\mathbf{l}^t$ subposet L. That is, every $\mathbf{m}^t$ subposet of L receives the same color.*

Note that the difference between the Product Ramsey Theorem and this conjecture is that this conjecture replaces "subgrids" in the Product Ramsey Theorem with the more general "subposets".

We were able prove this conjecture for $t = 2$.

Theorem 4.9.2. *For all r, m, and l there exists an n such that for all r-colorings of the $\mathbf{m}^2$ subposets of $\mathbf{n}^2$, there is a monochromatic $\mathbf{l}^2$ subposet P. That is, every $\mathbf{m}^2$ subposet of P receives the same color.*

We break up the proof into smaller parts. The following lemma is interesting in its own right.

Lemma 4.9.3. *Let P be an $\mathbf{s}^2$ grid, whose ground set is represented by ordered pairs (i, j), with $0 \leq i, j \leq s-1$. Then P has only one realizer with two linear extensions. Namely, one linear extension of this realizer is the lexicographic order on the pairs of P, and the other is the colexicographic order (the coordinates are considered right-to-left).*

Proof. Define the following two sets of ordered pairs of incomparable elements in P.

$$I_1 = \big\{\big((1,0),(0,s-1)\big),\big((2,0),(1,s-1)\big),\ldots,\big((s-1,0),(s-2,s-1)\big)\big\}$$
$$I_2 = \big\{\big((0,1),(s-1,0)\big),\big((0,2),(s-1,1)\big),\ldots,\big((0,s-1),(s-1,s-2)\big)\big\}$$

Now let $(x_1, y_1) \in I_1$ and $(x_2, y_2) \in I_2$ be two arbitrary elements of these sets. With appropriate choices of $i, j \in \{0, \ldots, s-1\}$, we have

$$x_1 = (i+1, 0), \qquad y_1 = (i, s-1),$$
$$x_2 = (0, j+1), \qquad y_2 = (s-1, j).$$

Notice (x_1, y_1) and (x_2, y_2) cannot be reversed at the same time in a linear extension: indeed, they form an alternating cycle, because $x_1 \leq y_2$, and $x_2 \leq y_1$. So every pair in I_1 must be reversed in a single linear extension, and the same is true for I_2. There is only one linear extension that reverses every pair in I_1, and there is only one for I_2.

The linear extension that reverses all of I_1 is the lexicographic order of the pairs, and the one that reverses I_2 is the colexicographic order. $\square$

The key observation to the proof of Theorem 4.9.2 is the following.

Lemma 4.9.4. *Let P be an $\mathbf{s}^2$ grid, and let Q be an $\mathbf{l}^2$ grid with $l = s^2$. There is (up to automorphism) only one casual embedding of P into Q. That is, if $f, g : P \to Q$ are two casual embeddings, then $f(P) = g(P)$, and $f \circ g^{-1}$ (and $g \circ f^{-1}$) are automorphisms of P.*

Proof. We can think of Q as consisting of pairs (i, j) with $0 \leq i, j \leq l - 1$ with the natural order. We will use the usual projection functions $\mathrm{Proj}_1((i, j)) = i$, and $\mathrm{Proj}_2((i, j)) = j$.

Let f be a casual embedding of P into Q. Let $k \in \{1, 2\}$, and $h_k = \mathrm{Proj}_k \circ f$. The definition of a casual embedding exactly means that h_k is a bijection from P to $\{0, \ldots, l - 1\}$. Now consider the following linear order of the elements of P.

$$L_k = (h_k^{-1}(0), h_k^{-1}(1), \ldots, h_k^{-1}(l - 1))$$

First note that L_k is a linear extension of P, because f is an embedding. Due to the same reason, any pair of incomparable elements will be ordered opposite in L_1 and L_2, so $\{L_1, L_2\}$ is a realizer of P. It is also important to recall that the ordered pair of linear extensions (L_1, L_2) uniquely determines the casual embedding f. (See discussion after the statement of Proposition 4.8.2.)

Now let g be another casual embedding of P into Q, and let M_1, M_2 be the linear extensions determined by g, similarly as above. Since $\{M_1, M_2\}$ is also a realizer of P, Lemma 4.9.3 implies that either $L_1 = M_1$ and $L_2 = M_2$, or $L_1 = M_2$ and $L_2 = M_1$.

The former case is simple: since (L_1, L_2) uniquely determines f, and (M_1, M_2) uniquely determines g, the fact $(L_1, L_2) = (M_1, M_2)$ implies that $f = g$.

So now assume that $L_1 = M_2$, and $L_2 = M_1$. By Lemma 4.9.3, L_1 is either the lexicographic order, or the colexicographic order of P. By possibly swapping the roles of f and g, we may assume it is the former. With these assumptions, f and g are completely determined. It is easy to see that

$$f((i, j)) = (si + j, sj + i) \qquad \text{and} \qquad g((i, j)) = (sj + i, si + j).$$

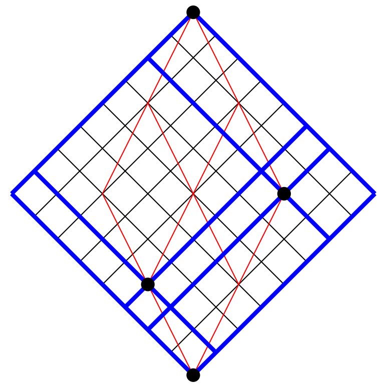


Figure 4.2: For the proof of Theorem 4.9.2. In this figure, $m = 2$, $l = 3$. The 9×9 grid is Q', the core P' is red. The points of the subposet D are marked by black dots. The blue grid is S.

If $(a, b) \in f(P)$, say $(a, b) = f((i, j))$, then $(a, b) = g((j, i))$, so $(a, b) \in g(P)$, and the converse follows the same way. This shows $f(P) = g(P)$. The last part of the statement follows from the fact that composition of isomorphisms is an isomorphism. $\square$

If the conditions of Lemma 4.9.4 are satisfied, then the subposet induced by the image $f(P)$ under a casual embedding of P into Q will be called the *core* of Q. Lemma 4.9.4 shows that the core of Q is uniquely determined by Q, so the usage of the definite article is justified.

Proof of Theorem 4.9.2. Let $M = m^2$, and $L = l^2$. By the Product Ramsey Theorem, there exists a positive integer n such that for all r-colorings of the $\mathbf{M}^2$ subgrids of $\mathbf{n}^2$, there is a monochromatic $\mathbf{L}^2$ subgrid. We claim that n satisfies the requirements of our theorem.

To see this, let c_1 be an r-coloring of the $\mathbf{m}^2$ subposets of $\mathbf{n}^2$. We will define an r-coloring c_2 on the $\mathbf{M}^2$ subgrids of $\mathbf{n}^2$. For each $Q \cong \mathbf{M}^2$ subgrid, let P be the core of Q. Let $c_2(Q) = c_1(P)$.

As noted earlier, the $\mathbf{n}^2$ grid has a monochromatic $\mathbf{L}^2$ subgrid under the coloring c_2; call this Q'. Here, monochromatic means that there exists a color r_0, such that for every $\mathbf{M}^2$ subgrid G of Q', we have $c_2(G) = r_0$. Let P' be the core of Q' (see Figure 4.2).

Clearly, $P' \cong \mathbf{l}^2$. It remains to be seen that every $\mathbf{m}^2$ subposet of P' received the same color under c_1.

Let D be an arbitrary $\mathbf{m}^2$ subposet in P'. Let

$$S_1 = \{\mathrm{Proj}_1(x) : x \in D\} \qquad S_2 = \{\mathrm{Proj}_2(x) : x \in D\}.$$

where $\mathrm{Proj}_i(x)$ is the ith coordinate of x in $\mathbf{n}^2$. Let $S = S_1 \times S_2$. Since P' is a casually embedded copy, $|S_i| = |D| = m^2 = M$, so $S \cong \mathbf{M}^2$, a subgrid of Q'. Therefore $c_2(S) = r_0$. On the other hand,

D is the core of S, so $c_1(D) = r_0$, which finishes the proof. $\square$

Clearly, the techniques used here heavily rely on the fact that $t = 2$. E.g. Lemma 4.9.3 and Lemma 4.9.4 are not true for $t > 3$. But the conjecture may still be correct.

Trotter [46] suggested that the conjecture is false for $t = 3$, offering the following idea for a counterexample. Let $t = 3$, $r = 2$, $m = 2$, and $l = 8$. Suppose the conjecture is true, and there is an appropriate n. Now color the $\mathbf{m}^t = \mathbf{2}^3$ subposets of $\mathbf{n}^3$ as follows. For a subposet P, consider the coordinates of the points in the $\mathbf{n}^3$. If there is no tie, then the coordinates define a realizer of P. There are multiple fundamentally different realizers of $\mathbf{2}^3$, so color P based on what kind of realizer its embedding defines. It does not matter how we group the realizer types into the two color groups, only that there exist two fundamentally different realizers that are colored different. If there is a tie in the coordinates, color P arbitrarily.

Now suppose we have a monochromatic $\mathbf{l}^3 = \mathbf{8}^3$. Within this $\mathbf{8}^3$, we can find both types of realizers, so it cannot be monochromatic.

However, it is not clear that this counterexample works. We conflate the embeddings into the $\mathbf{8}^3$ with the embedding into the $\mathbf{n}^3$. For any given $\mathbf{2}^3$, these two embeddings could be fundamentally different in terms of the realizers they generate. It may be possible to find a weird $\mathbf{8}^3$ such that every $\mathbf{2}^3$ subposet of that one has the same type of realizer generated, when we consider their embedding into the $\mathbf{n}^3$. Indeed, this is another interesting Ramsey-theoretical question.

4.10. MATCHING LINEAR EXTENSIONS

As evidenced in Section 4.9, it is interesting to consider linear extensions of posets, and how they behave in Ramsey-theoretical questions.

Theorem 4.10.1. *Let X be a poset and let M be a linear extension of X. Furthermore, let k be a positive integer. Then there exists a grid $Y \cong \mathbf{n}^t$ such that for all $L_1, L_2, \ldots, L_k$ linear extensions of Y, there is a subposet X' of Y such that*

- *$X' \cong X$, evidenced by the embedding $f : X \to Y$;*
- *for all $i = 1, \ldots, k$, and for all $a, b \in X$, we have $a < b$ in M if and only if $f(a) < f(b)$ in L_i.*

Loosely speaking, for any poset X and its linear extension M, one can find a large enough grid, so that no matter how we pick a fixed number of linear extensions of that grid, it has an X-subposet on which each linear extension conforms with M.

A special case of this theorem for $k = 1$ was proven by Paoli, Trotter and Walker [35]. The proof written in [35] contains an error: it attempts to use the infinite version of the Product Ramsey Theorem, which is false. However, the error is easily correctable by just using the finite version, and choosing appropriately large numbers. Our arguments follow their ideas with the necessary correction and generalizations. One can also prove this result using results of Rödl and Arman [42], but we believe that the proof provided here is more insightful.

We will need the following classical theorem by Rothschild [43] about partitions. To state this theorem, we will call a partition of a set into t parts, a *t-partition.*

Theorem 4.10.2. *Let $s \leq t$ be positive integers, and r a positive integer. Then there exists a positive integer k_0 such that for all $k \geq k_0$, no matter how one colors the s-partitions of $[k]$ with r colors, there exists a monochromatic t-partition in the following sense: any s-partition generated from that t-partition by unifying parts will have the same color.*

Now we are ready to prove Theorem 4.10.1.

Proof of Theorem 4.10.1. To start the proof, we pick X and M, although we will not use them at all in the first part of the proof. Let $s = \dim(X)$. We may assume that $s \geq 3$, for otherwise X can be embedded into a 3-dimensional poset, and apply the theorem to that.

We need to show that for large enough n and large enough t, the poset $\mathbf{n}^t$ has the prescribed property. We will determine the exact value of n and t later. For now, just let $Y = [n]^t$ for undetermined, but large n and t. Then let $L_1, \ldots, L_k$ be linear extensions of Y.

In the next steps, we will apply the Product Ramsey Theorem repeatedly to cut down Y. To do this, we will color the $\mathbf{2}^t$ subgrids (referred to as *hypercubes*) of Y with 2^k colors in each step.

Let H be a hypercube of Y. Then $H = C_1 \times \cdots \times C_t$, where $C_i = \{a_i, b_i\}$, and $a_i < b_i$. Every point of H is of the form $(c_1, \ldots, c_t)$, where $c_i = a_i$ or $c_i = b_i$. Once we fix an H hypercube in Y, we can identify the points with 0–1 strings (bit strings) of length t: we write 0 if $c_i = a_i$, and we write 1, if $c_i = b_i$.

We call two incomparable points *antipodal*, if they differ in every bit. So the pair $00\ldots0$, $11\ldots1$, is *not* antipodal, but every other pair with differing bits is. We can call the bit strings corresponding to these pairs of antipodal points, antipodal bit strings. There are $2^{t-1} - 1$ pairs of antipodal bit strings.

It will be important later that antipodal bit strings bijectively correspond to 2-partitions of $[t]$. Indeed, for $i \in [t]$, we can place i into the first or second part based on the ith bit.

Enumerate every pair of antipodal bit strings one by one. In each step, we will define a coloring of the hypercubes of Y with 2^k colors, then use the Product Ramsey Theorem to cut down Y to a smaller grid.

Let A be the current antipodal pair of bit strings. We define a coloring of the hypercubes of Y as follows. Let H be a hypercube. Recall that A identifies a pair of antipodal points A_H in H. For $i = 1, \ldots, k$, write 'G' (good), if L_i orders the points of A_H as it would be natural by the ith coordinate of the corresponding bit strings, write 'B' (bad) otherwise. We will have constructed a string of length k consisting of G's and B's. This is the "color" of H for the antipodal pair A.

As an example: suppose the current antipodal pair is 001101, 110010, and the color of the hypercube H is GBG. The antipodal pair 001101, 110010 determines a pair of points a, b of H, respectively. The color GBG means that $a < b$ in L_1, and $b < a$ in L_2, and L_3. In this example, $t = 6$ and $k = 3$.

The careful reader may get worried about the case when $k > t$. However, this is not a concern. We will see later that we can always choose a larger t, so we can ensure that $t \geq k$.

Now we have defined a coloring of the hypercubes of H for A. By the Product Ramsey Theorem, if n is large enough, there is a monochromatic grid, as large as we need. For now, let us just prescribe a very large grid, and we will determine that size later.

We will replace Y with this monochromatic grid, and we note the color of it. We will assign this color to the 2-partition that corresponds to the antipodal pair A. Then we move on to the next antipodal pair, do the coloring of the hypercubes of (the reduced) Y, apply the Product Ramsey Theorem, and produce a large monochromatic subgrid. Reduce Y again to this, and move on.

After going through every antipodal pair, we arrive at a final grid Y. This has the property that every hypercube in it is uniform with respect to the order of their antipodal points in the linear extensions L_i. We also colored every 2-partition of $[t]$ with 2^k colors.

Now we apply Theorem 4.10.2 to get t_0 such that if $t \geq t_0$, and if we 2^k-color the 2-partitions of $[t]$, then we can find a monochromatic $(s+k)$-partition. (Recall that $s = \dim(X)$.) The usage of t here is no accident: indeed, our original t was to be determined this way. Note that t only depends on k and s.

We do have a 2^k coloring defined on the 2-partitions of $[t]$, so now we determine the $(s+k)$-partition ψ, whose existence was guaranteed above. Recall that no matter how we unify parts in ψ to get a 2-partition, it will always have the same color r_0, which is a string of G's and B's of length k. Somewhat magically, it turns out that we can guarantee that r_0 is GG...G.

To see this, suppose that the ith digit of r_0 is B. Let $A \in \psi$ be the part for which $i \in A$, and let B and C be two other parts (recall $s \geq 3$, so $s+k \geq 3$). The partitions $\{B, [t] \setminus B\}$ and $\{C, [t] \setminus C\}$ are both colored r_0. Let the corresponding antipodal pairs of bit strings be b–b', and c–c': let b be the bit string that has 1's for indices in B, and 0's for the rest, and c be the bit string that has 1's for indices in C, and 0's for the rest.

Let H be a hypercube in the final, uniformized Y. Carry over the notation b, b', c, c' to denote the points of H corresponding to these bit strings. Then $b \| b'$, $c \| c'$, $b < c'$, and $c < b'$. In other words, $\{(b, b'), (c, c')\}$ is an alternating cycle. Yet, in L_i, the order of these pairs are "bad", and by the choice $i \in A$, the ith digit of the bit string b is 0, as well as the ith digit of the bit string c. So in L_i, we have $b > b'$, and $c > c'$, a contradiction. In other words, every digit of the color r_0 must be G.

We will use the monochromatic, and all-good partition ψ to embed X into Y. The parts of ψ are going to be groups of coordinates that are handled together. To do this, we will need Y to be large enough to accommodate the embedding. So, it is time to determine n. We must have chosen n to be large enough, so that after $2^{t-1} - 1$ repeated applications of the Product Ramsey Theorem, the remaining Y is isomorphic to $[n_0]^t$ with $n_0 \geq |X|$. Note that n only depends on k and t, and since t itself only depends on k and s, at the end n also only depends on k and s.

Let $M_1 = M_2 = \cdots = M_k = M$, and let $\{M_{k+1}, \ldots, M_{k+s}\}$ be a realizer of X. Clearly, each M_i is a linear extension of X, and $\cap M_i = X$. Also, label the parts of ψ with $A_1, \ldots, A_{k+s}$ in such a way that $[k] \subseteq A_1 \cup \cdots \cup A_k$. For each $x \in X$, find the position of x in M_i (from below). Let this number be h_i (for *height*). Then we map x to the element $(\chi_1, \ldots, \chi_t)$, where $\chi_j = h_l$ if $j \in A_l$. When we specifically want to emphasize the coordinates of the element x, we will write $\chi_j(x)$. Clearly,

$1 \leq \chi_j(x) \leq |X|$, so this is a mapping from X to Y. As we promised, coordinates with indices in the same part of ψ are grouped together so that they all get same value. It is also clear that this is an embedding.

It remains to be seen that for each i, L_i conforms M. Fix $i \in [k]$, and let $a \| b$ be elements of X so that $a < b$ in M. Consider the hypercube

$$H = \{\chi_1(a), \chi_1(b)\} \times \cdots \times \{\chi_t(a), \chi_t(b)\},$$

and the bit string

$$d = [\chi_1(a) > \chi_1(b)] \ldots [\chi_t(a) > \chi_t(b)].$$

(Here we used the Iverson bracket notation.)

We know that at least the first k digits of d are 0, in particular the ith digit is 0. Since the antipodal points corresponding to d and d' (the complement of d) in H are a and b, respectively, and since H was colored GG…G, it shows that $a < b$ in each of $L_1, \ldots, L_k$. This finishes the proof. $\square$

It may be tempting to attempt to generalize this theorem further. After all, it may seem that if we have k linear extensions of X, say $M_1, \ldots, M_k$, most of the proof still goes through. One may think that if we perform the embedding at the end carefully, we could make L_i conform with M_i for all $i = 1, \ldots, k$.

This, however, is not the case, and there is a very simple counterexample. Just choose any poset X that has at least two fundamentally different linear extensions M_1, M_2; we will make $k = 2$. Then, if an appropriate Y exists, we pick $L_1 = L_2$. Clearly, we cannot have both L_1 conforming with M_1, and L_2 conforming with M_2.

CHAPTER 5

CONCLUSION AND OPEN QUESTIONS

In this thesis, we provided many results on the dimension and Ramsey aspect of posets. We developed new tools and new theorems for the dimension of interval orders and semiorders and improved a couple of bounds on the dimension of classes of interval orders. We proved several results concerning semi dimensions for different classes of posets. In chapter 4, we proved Ramsey property of grids and Ramsey theory of grid subposets in dimension 2. And in the last part, we considered linear extensions of posets and how they behave in Ramsey questions.

We left several open questions that are interesting.

1. Is there interval orders with a representation that consists of intervals of two lengths that has dimension 4?

For updates. André Kézdy solved the problem with computer search. There exists a poset in $C(1, 8)$ that has dimension 4.

2. Is there an interval order in $C[1, 2]$ that has semi dimension 3?

3. Is there a split semiorder that has dimension 4?

4. Does semi dimension has continuity property?

5. Let $\mathbf{P} = (X, P)$ be a poset. Let $x \in X$. If $\text{Sdim}(X - x, P(X - x)) = 2$, is the semi dimension of $\mathbf{P} = 3$?

6. Is the Ramsey property true for grid subposets when $t = 3$?

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