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### Prudential Policy with Distorted Beliefs

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PRUDENTIAL POLICY WITH DISTORTED BELIEFS

By

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# Prudential Policy with Distorted Beliefs\*

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June 2022

## Abstract

This paper studies leverage regulation when equity investors and/or creditors have distorted beliefs relative to a planner. We characterize how the optimal regulation responds to arbitrary changes in investors'/creditors' beliefs, relating our results to practical scenarios. We show that the optimal regulation depends on the type and magnitude of such changes. Optimism by investors calls for looser leverage regulation, while optimism by creditors, or jointly by both investors/creditors, calls for tighter leverage regulation. Our results apply to environments with i) planners with imperfect knowledge of investors'/creditors' beliefs, ii) monetary policy, iii) bailouts and pecuniary externalities, and iv) endogenous beliefs.

**JEL Codes:** G28, G21, E61, E52

**Keywords:** prudential policy, distorted beliefs, leverage regulation, robust optimal policy, monetary policy, bailouts, pecuniary externalities.

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# 1 Introduction

Financial markets have experienced recurrent booms and busts throughout their history. A growing literature identifies alternative rationales for welfare-improving prudential policy in these scenarios. For instance, investors may borrow or invest too much if they do not internalize the full social cost of future fire sales or aggregate demand shortfalls (e.g., Lorenzoni, 2008; Dávila and Korinek, 2018; Korinek and Simsek, 2016; Farhi and Werning, 2016), or if they expect government support in a downturn (e.g., Farhi and Tirole, 2012; Bianchi, 2016). In practice, policy responses involve regulating leverage decisions and managing monetary policy.

In this paper, we characterize optimal policy responses when investors and creditors have distorted beliefs, relative to those of a planner, about the returns to investment. The role played by individual beliefs in determining financial and real decisions has drawn increased attention since the global financial crisis of 2008, connecting with earlier work by Kindleberger (1972) and Minsky (1986). A widespread view is that exuberant beliefs about house prices helped fuel the boom in subprime lending that preceded the crisis, and that it would have been valuable to combat such exuberance by decreasing leverage, perhaps by imposing a leverage cap on financial institutions or households.<sup>1</sup> However, there is little formal analysis on the form of the optimal policy in exuberant times.

Moreover, different forms of belief exuberance may call for different policy responses. For example, the car rental company Hertz filed for bankruptcy in May 2020, which made its stock effectively worthless. Nevertheless, retail investors seemed willing to buy Hertz stock at rising prices. The SEC counteracted this apparently distorted valuation by banning Hertz from selling additional shares. In other words, the regulator prevented Hertz from *decreasing* its leverage, which is the opposite of setting a leverage cap.<sup>2</sup>

We present a tractable model in which equilibrium leverage and investment are endogenously determined as a function of the beliefs of equity investors and creditors over future states of nature. Investors fund investment in risky capital with a mixture of their own equity and debt from creditors. Three forces determine the trade-off between debt and equity finance. First, creditors are more patient than investors, which encourages investors to borrow. Second, the deadweight losses associated with investors defaulting on their debt make borrowing costly. Third, the differences in beliefs between investors and creditors determine how each group values cash flows, which affects leverage choices non-trivially. Initially, we characterize the optimal policy for a social planner who can

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<sup>1</sup>For empirical evidence that analyzes the role played by beliefs, see, for example, Cheng, Raina and Xiong (2014); Greenwood and Hanson (2013); Lopez-Salido, Stein and Zakrajsek (2017); Baron and Xiong (2017).

<sup>2</sup>See Page 27 for further details on this case.

only impose a leverage cap. In an extension, which we describe below, we also explore the role of monetary policy.

Two key objects fully characterize the equilibrium of our model. First, investment is determined by a levered version of Tobin’s  $q$ , which measures the joint market value of equity and debt per unit of investment. Second, the (private) marginal value of leverage plays a dual and critical role in our analysis. When the leverage cap does not bind, the marginal value of leverage optimally trades off the three forces described above to determine equilibrium leverage. When the leverage cap binds, the marginal value of leverage determines the sensitivity of investment to changes in the leverage limit, a *policy elasticity* in the sense of [Hendren \(2016\)](#), which proves to be a critical input for our normative results.

Using tools from variational calculus, we characterize the response of both objects to arbitrary changes in investors’ and/or creditors’ beliefs. This method is a useful contribution in its own right, since we obtain interpretable equations that can describe the consequences of flexible changes in beliefs, which may include many heuristics and biases that have been considered in behavioral economics.<sup>3</sup> Our results reveal nuanced effects, whereby both the type and extent of belief changes affect equilibrium behavior. For instance, changes in creditors’ beliefs near the default boundary are particularly important when distress costs are large, while changes in investors’ beliefs about downside (default) states are not relevant for market valuations.<sup>4</sup>

Our characterization of the equilibrium reveals a fundamental asymmetry whereby optimism (in a hazard-rate sense) among equity investors decreases the marginal value of leverage, while optimism among creditors increases it. That is, when leverage regulation does not bind, optimism among investors reduces leverage in equilibrium, while optimism among creditors increases leverage in equilibrium. This result also implies that investment becomes less sensitive to binding leverage limits when equity investors are exuberant, but more sensitive when creditors investors are exuberant.<sup>5</sup> Perhaps surprisingly, when considering an identical change in both investors’ and creditors’ beliefs, the changes in leverage and investment are qualitatively the same as in the case in which only creditors’

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<sup>3</sup>See [Xiong \(2013\)](#) and [Simsek \(2021\)](#) for recent surveys on beliefs and speculation. Variational calculus is used widely in economics, primarily to solve optimal control problems, including in the analysis of optimal taxation (e.g., [Golosov, Tsyvinski and Werquin, 2014](#)). To our knowledge, we are the first to employ these methods to explore the impact of arbitrary changes in beliefs on equilibrium outcomes and welfare.

<sup>4</sup>The upside/downside distinction is related to the analysis of [Simsek \(2013a\)](#), but not identical. We explicitly relate our positive results to his in Section D.5 of the Online Appendix.

<sup>5</sup>The positive implications of our model go some way towards reconciling the mixed empirical evidence on the relationship between risk-taking and leverage in the banking sector: bank capital is effective in curbing risk-taking incentives on average (e.g., [Jiménez et al., 2014](#)), but not to smooth out the largest booms and busts (e.g., [Jorda et al., 2021](#)).

beliefs change. This result is driven by the fact that creditors, by virtue of being more patient, attach a higher value to future payoffs, making their beliefs more important at the margin.

We then present our normative results, which are the central contribution of this paper. Formally, we study the second-best problem of a utilitarian social planner who can impose a leverage cap but cannot control the level of investment. The planner computes investors' and creditors' welfare using beliefs for each group that are potentially different from the beliefs that these agents use to make decisions. In addition to belief distortions (internalities), the planner also takes into account any externalities that leveraged investments impose on the broader economy.<sup>6</sup> One interpretation of our results is that investors and creditors have distorted beliefs and that the planner's beliefs are correct. Following this logic, our results can be interpreted as characterizing an optimal paternalistic policy, although we also consider alternative interpretations.

The marginal welfare effect of increasing the leverage cap is the sum of two components. The first is the *inframarginal* effect of more leverage on existing units of investment. This component captures how varying the leverage cap modifies the planner's valuation of pre-existing investment at the margin. The second is the *incentive* effect, which arises because varying the leverage cap impacts investment in equilibrium. For example, a tighter leverage cap reduces equilibrium investment, which is perceived to improve welfare for the planner when the value of investment perceived by investors and creditors is higher than the value perceived by the planner. The incentive effect hinges on the policy elasticity of investment with respect to the leverage cap.

Our central normative result determines the desirability of tightening or relaxing leverage caps in response to changes in beliefs. We show that the same objects that determine leverage and investment in equilibrium — the value of investment and the marginal value of leverage — also determine the normative implications of changes in beliefs. First, the *inframarginal* welfare effect of more leverage is proportional to the change in the value of investment. This effect implies tighter optimal leverage limits in response to a change in beliefs if (i) equilibrium investment increases, and (ii) the private marginal benefit of leverage is higher than the marginal benefit perceived by the planner. Second, the *incentive* effect is proportional to the sensitivity of investment to leverage regulation, which is linked to the marginal value of leverage. This effect implies tighter optimal leverage limits in response to a change in beliefs if (i) the sensitivity of investment to leverage regulation increases, and (ii) the private value of investment is higher than the

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<sup>6</sup>Our baseline model considers reduced-form externalities, capturing various potential spillovers from leveraged investment to the wider economy. In Section 4.3, we also consider micro-founded models of externalities generated by government bailouts and fire sales, which yield similar economic insights.

value perceived by the planner. The overall economics are subtle. For example, the same belief distortions can motivate tighter leverage regulation via the inframarginal effect, but more relaxed regulation via the incentive effect. Indeed, a core insight from our analysis is to show that, in a second-best world, the motivation for “leaning against the wind” when policymakers suspect over-optimism is not clear-cut. Nevertheless, our model provides sharp policy prescriptions in three relevant scenarios.

First, we consider a “debt exuberance” scenario. In this scenario, in which only creditors become optimistic relative to investors and the planner, optimal leverage limits are always binding and decreasing in the extent of creditors’ optimism. This is because optimism among creditors leads to an overvaluation of debt, and increases the sensitivity of investment to leverage policy. Second, in an “equity exuberance” scenario, in which only equity investors become optimistic relative to creditors and the planner (in a hazard-rate sense), it is not optimal to impose a binding leverage limit unless there are strong marginal externalities associated with leverage.<sup>7</sup> This case is the result of two forces. On one hand, if externalities are weak, then the planner wishes to push investors towards issuing more debt and less equity against inframarginal units of investment, because optimistic equity investors (wrongly) consider debt to be undervalued. On the other hand, equity optimism means that leverage limits become a blunt tool for the purpose of disciplining excessive investment. Finally, as discussed above, changes in creditors’ beliefs dominate marginal valuations in a “joint exuberance” scenario, again leading to tighter optimal regulation.

Our baseline model can further be used to give a positive — as opposed to normative — interpretation to optimal policy, by considering potential distortions in the planner’s beliefs. Formally, we characterize how the optimal policy responds to changes in the beliefs used by the planner to compute welfare. The results are ambiguous when the planner is subject to equity exuberance, leading to excessively tight leverage limits only when the inframarginal effect dominates incentive considerations at the margin. By contrast, debt or joint exuberance on behalf of the planner always implies excessively lenient leverage regulation.

We consider four extensions to our baseline framework. First, in order to explore the challenges associated with measuring agents’ belief distortions, we assume that the planner has imperfect knowledge of investors’ and creditors’ beliefs. Initially, we allow the planner to condition her policy on endogenous investment. In this case, we demonstrate that optimal policy must be conducted under residual uncertainty about policy elasticities. Indeed, an investment boom can be driven either by equity exuberance, with a low

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<sup>7</sup>Indeed, this scenario generates a case for leverage floors or, conversely, limits on equity issuance. Our results can be used to rationalize recent policy interventions that limit equity issuance, as in the case of Hertz mentioned above.

associated elasticity, or debt exuberance, with a high elasticity. We further analyze the case in which the planner cannot attach probabilities to the realizations of different belief shocks. In this case, we characterize *robust optimal* policies in the sense of [Woodford \(2010\)](#) and [Adam and Woodford \(2012\)](#), which maximize welfare while expecting a worst-case realization of investors' beliefs.

Second, we assume that the government has the ability to affect the risk-free interest rate via monetary policy. Investment remains sensitive to a monetary tightening (an increase in interest rates) because this policy raises the cost of leverage and reduces the private value of investment. We show that when investors and/or creditors become optimistic (pessimistic) it is optimal to increase (lower) interest rates, which improves welfare by reducing (increasing) investment. This insight is especially relevant in an equity exuberance scenario, in which leverage regulation cannot be used to improve welfare. These results connect our paper to the literature on monetary policy as a prudential tool (e.g., [Stein, 2013](#); [Caballero and Simsek, 2019](#)).

Our third set of extensions studies alternative micro-foundations for the externalities that leveraged investments impose on the broader economy. First, we consider the possibility of ex-post government bailouts without commitment. In addition to introducing an externality, bailouts modify investors' and creditors' decisions and change the role played by their beliefs. In this case, we show that belief distortions in good states of the world become especially important for policy. Second, we consider a model along the lines of [Lorenzoni \(2008\)](#), which features liquidity-constrained investors and potential asset fire sales. We show that our main formulae extend to this setting, which features a distributive pecuniary externality in the sense of [Dávila and Korinek \(2018\)](#). In addition, we show that the planner considers the policy elasticity of fire sale prices when choosing leverage caps. We show that his new effect favors even tighter caps with debt/joint exuberance, but not with equity exuberance.

In our final extension, we consider the possibility that agents' beliefs respond endogenously to increases in investment. In this case, all responses to belief distortions in our model are amplified by a multiplier term, which we characterize in terms of the responsiveness of market valuations to aggregate investment. We show that, if market values respond linearly to aggregates, this effect amplifies *both* the inframarginal and incentive effects of leverage regulation, thus leaving the trade-off between them unchanged relative to the baseline model. Additional nuanced effects, which we characterize in the Online Appendix, arise in the nonlinear case, in which the multiplier itself depends on belief distortions.



**Related Literature** Our paper is related to several literatures. Our approach to computing welfare is related to a growing literature that explores the normative implications of belief heterogeneity. [Bianchi, Boz and Mendoza \(2012\)](#) study paternalistic and non-paternalistic macroprudential policies in an environment with pecuniary externalities caused by collateral constraints. [Brunnermeier, Simsek and Xiong \(2014\)](#) develop a criterion to detect speculation under heterogeneous beliefs, which is also used in [Simsek \(2013b\)](#), [Heimer and Simsek \(2019\)](#), and [Caballero and Simsek \(2020\)](#) to provide normative assessments of financial innovation, leverage restrictions on trading, and stabilization policy, respectively. [Gilboa, Samuelson and Schmeidler \(2014\)](#) propose an alternative criterion to detect speculation. [Dávila \(2014\)](#) characterizes the optimal financial transaction tax for a given planner’s belief in an environment with heterogeneous beliefs. [Campbell \(2016\)](#), [Farhi and Gabaix \(2020\)](#), and [Exler et al. \(2019\)](#) also explore paternalistic policies in a household context, while [Haddad, Ho and Loualiche \(2020\)](#) do so in the context of technological innovations.

Another relevant strand of work studies the relationship between beliefs and leverage, including the contributions of [Geanakoplos \(1997, 2003, 2009\)](#), [Fostel and Geanakoplos \(2008, 2012, 2015, 2016\)](#), [Simsek \(2013a\)](#), and [Bailey et al. \(2019\)](#). Methodologically, we provide, to our knowledge, the first use of variational (Gateaux) derivatives to explore the impact of arbitrary changes in beliefs on equilibrium outcomes and welfare. Several of our findings are connected to the well-developed literature on government bailouts, which includes the contributions of [Farhi and Tirole \(2012\)](#), [Bianchi \(2016\)](#), [Chari and Kehoe \(2016\)](#), [Keister \(2016\)](#), [Gourinchas and Martin \(2017\)](#), [Cordella, Dell’Ariccia and Marquez \(2018\)](#), [Dávila and Walther \(2020\)](#), and [Dovis and Kirpalani \(2020\)](#), among others. We provide a novel analysis of how bailouts and belief distortions interact, and how they jointly shape the optimal regulatory policy. The recent work of [Krishnamurthy and Li \(2020\)](#) and [Maxted \(2020\)](#) quantitatively explores the role of beliefs on shaping business cycles in environments with financial frictions. In contrast to these two papers, our main contribution is normative and our model emphasizes the differences between investors’ and creditors’ beliefs.

Finally, our results also contribute to the literature that explores the interaction between monetary and regulatory policy. The recent work of [Caballero and Simsek \(2019\)](#) is the closest to this part of our analysis. While they study the design of macroprudential and monetary policy in a model with nominal rigidities and aggregate demand effects, we instead consider optimal policies in a model of risky credit with a rich specification of beliefs. [Farhi and Werning \(2020\)](#) also explore the role of monetary policy in an environment with nominal rigidities and belief distortions.

**Outline** The structure of the paper is as follows. Section 2 introduces our baseline model, characterizes its equilibrium, and describes some key positive properties of the model. Section 3 presents the central welfare effects that determine the optimal leverage regulation when a planner can observe agents’ beliefs. Section 4 outlines various further extensions and Section 5 concludes. All proofs and derivations are in the Appendix.

## 2 Baseline Model

### 2.1 Environment

**Agents, preferences and endowments.** There are two dates  $t \in \{0, 1\}$  and a single consumption good (dollar), which serves as numeraire. There are two types of agents: a unit measure of investors, indexed by  $I$ , and a unit measure of creditors, indexed by  $C$ . There is also a social planner/regulator/government, who sets leverage regulation. We denote the possible states of nature at date 1 by  $s \in [\underline{s}, \bar{s}]$ , where  $\underline{s} \geq 0$ . As described below,  $s$  corresponds to the realization of the returns to investors’ technology.

Both investors and creditors are risk-neutral. The lifetime utility of investors is

$$c_0^I + \beta^I \mathbb{E}^I [c_1^I(s)], \quad (1)$$

where  $c_0^I$  and  $c_1^I(s)$  denote the consumption of investors and  $\mathbb{E}^I[\cdot]$  denotes the expectation under the investors’ beliefs, whose determination is described below. The lifetime utility of creditors is

$$c_0^C + \beta^C \mathbb{E}^C [c_1^C(s)] - \Delta(b)k, \quad (2)$$

where  $c_0^C$  and  $c_1^C(s)$  denote the consumption of creditors and  $\mathbb{E}^C[\cdot]$  denotes the expectation under the creditors’ beliefs. The term  $\Delta(b)k$ , which we discuss in detail below, captures any externality that investors’ financing and investment choices impose on creditors. We assume that  $0 < \beta^I < \beta^C \leq 1$ , so that investors are less patient than creditors.

The endowments of the consumption good of investors and creditors at dates 0 and 1 are respectively given by  $\{n_0^I, n_1^I(s)\}$  and  $\{n_0^C, n_1^C(s)\}$ . Creditors’ and investors’ endowments are such that their consumption is never negative.<sup>8</sup>

**Investment technology.** Investors can invest at date 0 to create  $k \geq 0$  units of productive capital. This investment in capital yields  $sk$  dollars in state  $s$  at date 1, so that  $s$  also denotes the gross return on capital investment. As in canonical “Tobin’s  $q$ ”

<sup>8</sup>In Section D.5 of the Online Appendix, we study an alternative scenario in which the non-negativity constraint of investors’ consumption at date 0 binds. In that case, the equity contribution of investors is effectively capped and our positive results can be mapped to those in Simsek (2013a).

models of investment, creating  $k$  units of capital at date 0 requires  $k + \Upsilon(k)$  dollars, where  $\Upsilon(k)$  is a convex adjustment cost that satisfies  $\Upsilon(0) = 0$ ,  $\lim_{k \rightarrow 0} \Upsilon'(k) = 0$ ,  $\Upsilon'(k) \geq 0$ , and  $\Upsilon''(k) > 0$ . The combination of an investment technology that scales linearly with capital and a convex adjustment cost allows investors to separate their financing decisions from their investment decisions, as shown in Lemma 1 below.

**Financial contracts.** Investors finance their investment by issuing bonds with face value  $b$  and price  $Q(b)$  per unit of investment. Therefore, the total face value of debt issued is  $bk$ , the total amount raised via borrowing at date 0 is  $Q(b)k$ , and an investor’s leverage ratio is simply  $b$ . Any remaining financing is obtained with an equity contribution from the investor’s endowment.<sup>9</sup> Note that we assume a form of market segmentation/limited participation, in the sense that creditors cannot fund investors using equity.<sup>10</sup>

At date 1, after the state  $s$  is realized, investors decide whether to default. If investors default, creditors seize all of the investors’ productive capital and receive  $\phi s$  per unit of investment, where  $0 \leq \phi \leq 1$ . The remainder  $(1 - \phi)s$  measures the deadweight loss or cost of distress associated with default.

The difference in discount factors  $\beta^C - \beta^I > 0$  between creditors and investors translates into a benefit from issuing debt or, equivalently, into a cost of equity issuance.<sup>11</sup> This difference guarantees that investors always borrow in equilibrium. As explained in Section D.1 of the Online Appendix, with additional regularity conditions, our results extend to environments in which belief differences are the single rationale for investors to borrow.

The externality  $\Delta(b)k$  that investors impose on creditors depends on both leverage and the scale of investment.<sup>12</sup> Our reduced-form definition captures various spillovers from leveraged investment. For example, if investors are interpreted as banks in our model, then the externality can represent the common concerns that financial distress in a leveraged bank can cause distress in other financial institutions or in the broader economy. Accordingly, we assume throughout the paper that the externality associated with leveraged investment is negative, i.e.,  $\Delta(b) \geq 0$ , and increasing in the leverage ratio,

<sup>9</sup>In Section D.6.1 of the Online Appendix, we explain how to introduce inside equity in our model.

<sup>10</sup>An objective of part of the literature that studies belief disagreements is to endogenously determine from assumptions on beliefs which agents are borrowers and lenders in equilibrium. By segmenting equity investors from creditors at the onset, we implicitly allow for other rationales that motivate some agents to borrow or lend in equilibrium, and then consider the impact of changes in beliefs.

<sup>11</sup>There are readily available theories that make issuing debt beneficial/issuing equity costly. For example, a demand for “money-like” claims (Gorton and Pennacchi, 1990; Stein, 2012; DeAngelo and Stulz, 2015) or safe claims (Caballero and Farhi, 2018; Caballero, Farhi and Gourinchas, 2017), or the market discipline brought by debt (Diamond and Rajan, 2001).

<sup>12</sup>The assumption that this distortion only impacts creditors and is linear in capital investments  $k$  simplifies the exposition, but does not affect the qualitative insights of our analysis.

i.e.,  $\frac{d\Delta(b)}{db} \geq 0$ .

**Budget constraints.** In this environment, the budget constraint of investors at date 0 is given by

$$c_0^I + k + \Upsilon(k) = n_0^I + Q(b)k, \quad (3)$$

where  $Q(b)$  denotes the price of debt per unit of investment, which is determined by creditors. Similarly, the budget constraint of investors at date 1 in state  $s$  is given by

$$c_1^I(s) = n_1^I(s) + \max\{s - b, 0\}k. \quad (4)$$

This equation already reflects the fact that investors exercise their option to default whenever  $s < b$ . In that case, creditors seize all capital and its returns, and investors consume only their endowments  $n_1^I(s)$ .

**Beliefs.** We adopt a flexible approach to model the perceptions of investors and creditors over future states of nature. Formally, we assume that investors perceive the cumulative distribution over future states to be  $F^I(s)$ , while creditors perceive it to be  $F^C(s)$ . The distributions  $F^I(s)$  and  $F^C(s)$  can differ from each other and from the true distribution, which we do not have to specify to derive the majority of our results.

The advantage of this flexible approach is that it allows us to analyze the consequences of different belief configurations without taking a stance on the exact process of belief formation. We assume that these distributions are continuously differentiable, with densities  $f^I(s) > 0$  and  $f^C(s) > 0$  defined on  $s \in [\underline{s}, \bar{s}]$ .<sup>13</sup>

**Leverage regulation.** The planner is able to impose a leverage cap on investors at date 0. This cap is the central object of study in this paper. Formally, the planner requires that investors set  $b \leq \bar{b}$ , where  $1 - \bar{b}$  is the minimal permitted ratio of equity contribution to risky investment. This constraint imposes a leverage cap, or equivalently, a minimal equity contribution per unit of investment. In Section 3, we discuss in detail the role of the beliefs over future states  $s$  that the planner uses to evaluate welfare.

Note that the planner cannot directly control the scale  $k$  of investment. We therefore focus on a second-best policy problem, in which  $k$  remains a free choice variable for investors.<sup>14</sup> It is possible to justify this assumption, for example, because the private sector

<sup>13</sup>We assume continuous and strictly positive densities to simplify the exposition of our analysis of belief perturbations (see Section 2.3). Our results go through, with some additional technical conditions, if we impose weaker assumptions, such as absolute continuity of the relevant distributions.

<sup>14</sup>In Section D.3 of the Online Appendix, we describe the form of the first-best policy, in which the planner can also control investment directly.

has superior information about investment opportunities (e.g., [Walther, 2015](#)). Perhaps for this reason, all relevant regulatory constraints in practice (e.g., capital requirements, leverage limits, liquidity coverage ratios, and net stable funding requirements in Basel III) focus on ratios of financial institutions’ assets to liabilities. Similarly, household finance regulations are based on loan-to-value and debt-to-income ratios. All of these regulatory tools leave the scale of investment unconstrained, as in our model.

**Equilibrium definition.** Given a leverage limit  $\bar{b}$ , an *equilibrium* in this economy is defined by an investment decision,  $k \geq 0$ , a leverage decision,  $b \leq \bar{b}$ , and a default decision rule such that i) investors maximize expected utility subject to their budget constraints while taking into account that any debt issued is valued by creditors, and ii) creditors value investors’ debt breaking even in expectation.

Our notion of equilibrium, in which borrowers internalize that their borrowing decisions affect their cost of financing in equilibrium, is standard in models of default (e.g., [Dubey, Geanakoplos and Shubik, 2005](#); [Aguiar and Amador, 2013](#); [Livshits, 2015](#)). In the body of the paper, we proceed as if the environment considered here is well-behaved in the sense that optimal leverage choices are finite. We discuss the necessary regularity conditions in [Section D.1](#) of the Online Appendix.

## 2.2 Equilibrium characterization

In the Appendix — see [Equations \(33\) through \(38\)](#) — we include a detailed formulation of the investors’ problem. Here, we introduce [Lemma 1](#), which presents a reformulation of the investors’ problem whose solution directly characterizes equilibrium leverage and investment.

**Lemma 1.** *[Investors’ problem] Equilibrium leverage and investment are given by the solution to the following reformulation of the problem faced by investors:*

$$\max_{b,k} [M(b) - 1] k - \Upsilon(k) \tag{5}$$

$$s.t. \quad b \leq \bar{b}, \tag{6}$$

where  $M(b)$  is given by

$$M(b) = \underbrace{\beta^I \int_b^{\bar{s}} (s - b) dF^I(s)}_{equity} + \underbrace{\beta^C \left( \int_b^{\bar{s}} b dF^C(s) + \phi \int_s^b s dF^C(s) \right)}_{debt}. \tag{7}$$

Intuitively, it is possible to fully characterize the equilibrium of the model by

incorporating the default decision of investors at date 1 and the pricing of debt by creditors into the investors' date 0 problem. First, notice that investors optimally default at date 1 whenever  $s < b$ , and repay when  $s \geq b$ .<sup>15</sup> Therefore,  $M(b)$  can be interpreted as the sum of the market value of equity and debt per unit of investment net of adjustment costs, expressed as a function of leverage  $b$ .

The first component of  $M(b)$  in Equation (7) corresponds to the present value of the equity payoffs, as perceived by investors. Since equity investors are only paid in the non-default states, this integral is over states in which  $s \geq b$ . The second component of  $M(b)$  in Equation (7) corresponds to the present value of the debt payoffs, as perceived by creditors. Since creditors are paid in both non-default and default states, this second component of  $M(b)$  accounts for both scenarios. When investors do not default ( $s \geq b$ ), creditors receive the promised  $b$  per unit of investment  $k$ . When investors default ( $s < b$ ), creditors receive  $\phi s$  per unit of investment  $k$ , which accounts for the deadweight losses of default. Importantly, while debt payoffs are valued using the creditors' discount factor  $\beta^C$  and beliefs  $F^C(s)$ , equity payoffs are valued using the investors' discount factor  $\beta^I$  and beliefs  $F^I(s)$ .

Lemma 1 clearly highlights that the leverage decision of investors is independent of their investment decision. That is, first, investors choose the level of  $b$  that maximizes  $M(b)$ . Next, given the optimal choice of  $b$ , investors choose  $k$  to maximize Equation (5). In Proposition 1, we formalize how the solution to the investors' problem characterizes equilibrium leverage and investment.

**Proposition 1.** *[Equilibrium leverage and investment] Equilibrium leverage  $b^*$  and equilibrium investment  $k^*$  are respectively given by the solution to*

$$\frac{dM(b^*)}{db} = \mu \tag{8}$$

$$M(b^*) = 1 + \Upsilon'(k^*), \tag{9}$$

where  $\mu$  is the Lagrange multiplier on the leverage constraint, which we have expressed as  $bk \leq \bar{b}k$ . When the investors' leverage constraint doesn't bind,  $b^* < \bar{b}$  and  $\mu = 0$ . When the investors' leverage constraint binds,  $b^* = \bar{b}$  and  $\mu \geq 0$ .

Equation (8) equates the marginal value of leverage per unit of investment and the Lagrange multiplier  $\mu$  associated with the leverage constraint.<sup>16</sup> Two forces determine the

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<sup>15</sup>Our analysis extends to settings with more nuanced optimal default decisions, for example, when investors receive government assistance in states in which they would otherwise default (see Section 4.3.1), or when creditors have recourse to the market value of collateral (see Section D.6.2 of the Online Appendix).

<sup>16</sup>Note that  $M(b)$  can be expressed in terms of i) a Modigliani-Miller valuation term, which is

marginal value of leverage per unit of investment,  $\frac{dM(b)}{db}$ , characterized in Equation (10):

$$\frac{dM(b)}{db} = \underbrace{\beta^C \int_b^{\bar{s}} dF^C(s) - \beta^I \int_b^{\bar{s}} dF^I(s)}_{\substack{\text{marginal benefit of leverage} \\ \text{(valuation difference)}}} - \underbrace{(1 - \phi) \beta^C b f^C(b)}_{\substack{\text{marginal cost of leverage} \\ \text{(distress)}}}. \quad (10)$$

The first force arises due to the differences in valuation between investors and creditors. By increasing the leverage ratio  $b$ , an investor is able to raise in present value terms  $\beta^C \int_b^{\bar{s}} dF^C(s)$  dollars per unit invested, whose repayment cost in present value terms corresponds to  $\beta^I \int_b^{\bar{s}} dF^I(s)$ . When investors have common beliefs, this first force is proportional to the difference in discount factors  $\beta^C - \beta^I > 0$ . When investors have common discount factors, this first force is proportional to the difference in the probability of repayment between creditors and investors  $\int_b^{\bar{s}} dF^C(s) - \int_b^{\bar{s}} dF^I(s)$ . In this case, investors find it attractive to increase leverage when they perceive non-default states to be relatively less likely than creditors, since the amount of funds they can raise from creditors,  $\int_b^{\bar{s}} dF^C(s)$ , is higher than the perceived repayment,  $\int_b^{\bar{s}} dF^I(s)$ .

The second force corresponds to the marginal increase in deadweight losses associated with defaulting more frequently after increasing leverage. These two forces guarantee that equilibrium leverage is strictly positive and finite, with  $0 < b^* < \infty$ , even in a laissez-faire scenario in which the leverage constraint is not binding.<sup>17</sup>

Equation (9), which is a levered version of Tobin's marginal  $q$ , characterizes the optimal investment choice  $k^*$ . Its left-hand side,  $M(b^*)$ , captures the marginal benefit to investors associated with a marginal increase in investment given an optimal leverage choice. Its right-hand side,  $1 + \Upsilon'(k^*)$ , simply corresponds to the marginal cost of such an increase, which captures the direct cost of investing and the adjustment cost.

Proposition 1 highlights that  $\frac{dM(b)}{db}$  and  $M(b)$  are the key objects that determine the equilibrium of our model. Interestingly, the marginal value of leverage per unit of investment,  $\frac{dM(b)}{db}$ , plays a dual role, depending on whether the leverage constraint binds.

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independent of the leverage choice, ii) a term that captures the marginal benefit of leverage, due to differences in discount factors or beliefs, and iii) a term that captures the marginal cost of leverage, due to the cost of distress associated with default:

$$M(b) = \underbrace{\beta^C \int_{\underline{s}}^{\bar{s}} s dF^C(s)}_{\substack{\text{Modigliani-Miller} \\ \text{valuation}}} - \underbrace{\int_b^{\bar{s}} (s - b) (\beta^C dF^C(s) - \beta^I dF^I(s))}_{\substack{\text{benefit of leverage} \\ \text{(valuation difference)}}} - \underbrace{(1 - \phi) \beta^C \int_{\underline{s}}^b s dF^C(s)}_{\substack{\text{cost of distress}}}.$$

<sup>17</sup>To see this, note that  $\left. \frac{dM(b)}{db} \right|_{b=0} = \beta^C - \beta^I > 0$ , so that  $b > 0$  is always optimal. Moreover, in Section D.1 of the Online Appendix, we show that finite leverage is guaranteed whenever the costs of distress are sufficiently large relative to the returns on investment.

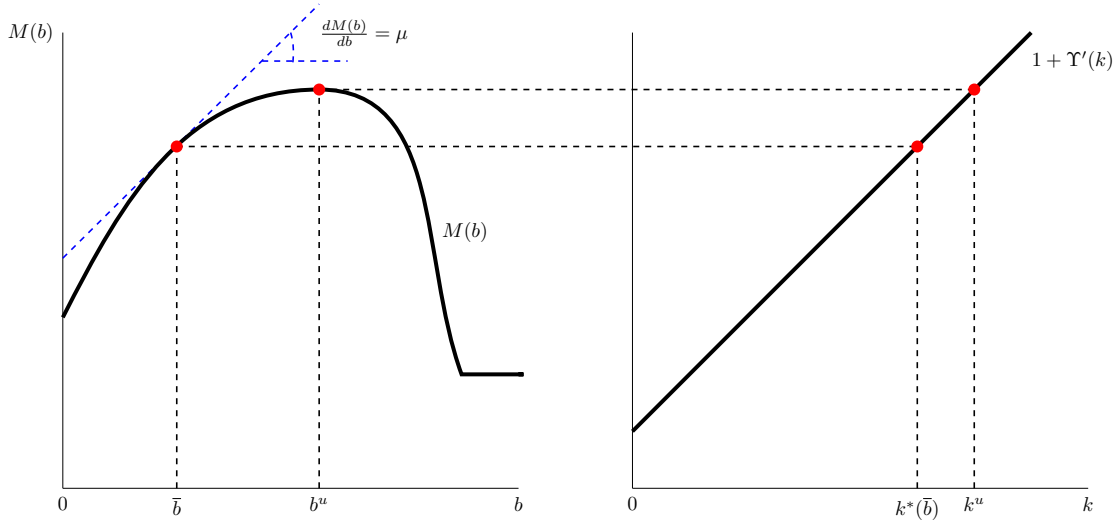


Figure 1: Sensitivity of investment to leverage limit

**Note:** Figure 1 illustrates the joint determination of equilibrium leverage (left panel) and investment (right panel), formalized in Proposition 1. Figure 1, which is designed to illustrate Lemma 2, should be read from left to right. Changes in the leverage limit  $\bar{b}$  around the laissez-faire optimum, denoted here by  $b^u$  — with the superscript  $u$  standing for unregulated — are associated with no changes in the level of investment, since  $\left. \frac{dM(b)}{db} \right|_{b=b^u} = 0$  in that case. As shown in Lemma 2, changes in the leverage limit  $\bar{b}$  away from the laissez-faire optimum induce changes in the level of investment that are increasing in the slope of  $\left. \frac{dM(b)}{db} \right|_{b=\bar{b}}$  and decreasing in the value of  $\Upsilon''(k)$ , which determines the slope of  $1 + \Upsilon'(k)$  in the right panel.

If the leverage constraint does not bind, then the solution to  $\frac{dM(b^*)}{db} = 0$  determines equilibrium leverage. If the leverage constraint binds, then the marginal value of leverage per unit of investment,  $\frac{dM(\bar{b})}{db}$ , determines the *sensitivity* of equilibrium investment to changes in the leverage limit,  $\frac{dk^*}{d\bar{b}}$ , as formalized in Lemma 2.

**Lemma 2.** [*Sensitivity of investment to leverage limit*] *If the leverage constraint binds, then the sensitivity of investors' investment to the leverage limit  $\bar{b}$  is given by*

$$\frac{dk^*}{d\bar{b}} = \frac{1}{\Upsilon''(k^*)} \frac{dM(\bar{b})}{db} \geq 0. \quad (11)$$

Hence, loosening (tightening) the leverage constraint increases (decreases) investment in proportion to the marginal value of leverage per unit of investment. Figure 1 illustrates these effects. Lemma 2 is helpful because  $\frac{dk^*}{d\bar{b}}$  will be an important input for our normative results.

In summary, taken together, Proposition 1 and Lemma 2 imply that understanding the behavior of  $M(b)$  and  $\frac{dM(b)}{db}$  is sufficient to determine i) equilibrium leverage and



investment, and ii) the sensitivity of investment to changes in a binding leverage limit. Next, we focus on characterizing how  $M(b)$  and  $\frac{dM(b)}{db}$  vary in response to changes in beliefs. These comparative statics provide the main ingredients for our normative analysis below.

### 2.3 Comparative statics and positive implications

Our ultimate goal is to understand how equilibrium outcomes (e.g., leverage and investment) and welfare vary in response to changes in beliefs, which are infinite-dimensional objects. Since we have specified flexible distributions of investors' and creditors' beliefs, we will characterize the responses of leverage and investment — and later of welfare — to changes in beliefs using variational (Gateaux) derivatives. Formally, we consider perturbations of beliefs of the form

$$F^j(s) + \varepsilon G^j(s),$$

where  $F^j(s)$  denotes the original cumulative distribution function of  $s$  for agents in group  $j \in \{I, C\}$ , the variation  $G^j(s)$  represents the direction of the perturbation of beliefs, and  $\varepsilon \geq 0$  is a scalar. When  $G^j(s) < 0$ , it is natural to say that the perturbed beliefs are locally more optimistic for state  $s$ , since the probability assigned to states equal or lower than  $s$  is now lower. Figure 2 illustrates an arbitrary perturbation of  $F^j(s)$ . We consider variations  $G^j(s)$  that are continuously differentiable and satisfy  $G^j(\underline{s}) = G^j(\bar{s}) = 0$ . These conditions ensure that perturbed beliefs are still valid cumulative distribution functions for small enough values of  $\varepsilon$ , as we formally show in Section D.2 of the Online Appendix.

A variational (or Gateaux) derivative is defined as follows (e.g., [Luenberger, 1997](#)). For concreteness, consider the market value  $M(b; F^j)$  per unit of investment in Equation (7), where we have made explicit its dependence on the beliefs of group  $j \in \{I, C\}$  of agents. Its variational derivative in the direction of a perturbation  $G^j(s)$  is denoted  $\frac{\delta M}{\delta F^j} \cdot G^j$  and defined as

$$\frac{\delta M}{\delta F^j} \cdot G^j \equiv \lim_{\varepsilon \rightarrow 0} \left[ \frac{M(b; F^j + \varepsilon G^j) - M(b; F^j)}{\varepsilon} \right].$$

Intuitively, the variational derivative of  $M(b)$  measures the change in the market value per unit of investment when we perturb the beliefs among  $j$ -agents by a small amount in the direction of  $G^j$ . The same definition applies to the change in the marginal value of leverage  $\frac{dM(b)}{db}$ , which we denote by  $\frac{\delta(\frac{dM}{db})}{\delta F^j} \cdot G^j$ . Applying a variational implicit function theorem to the conditions that characterize the equilibrium in Proposition 1, we similarly obtain the variational derivatives of leverage and investment, as shown in Lemma 3.

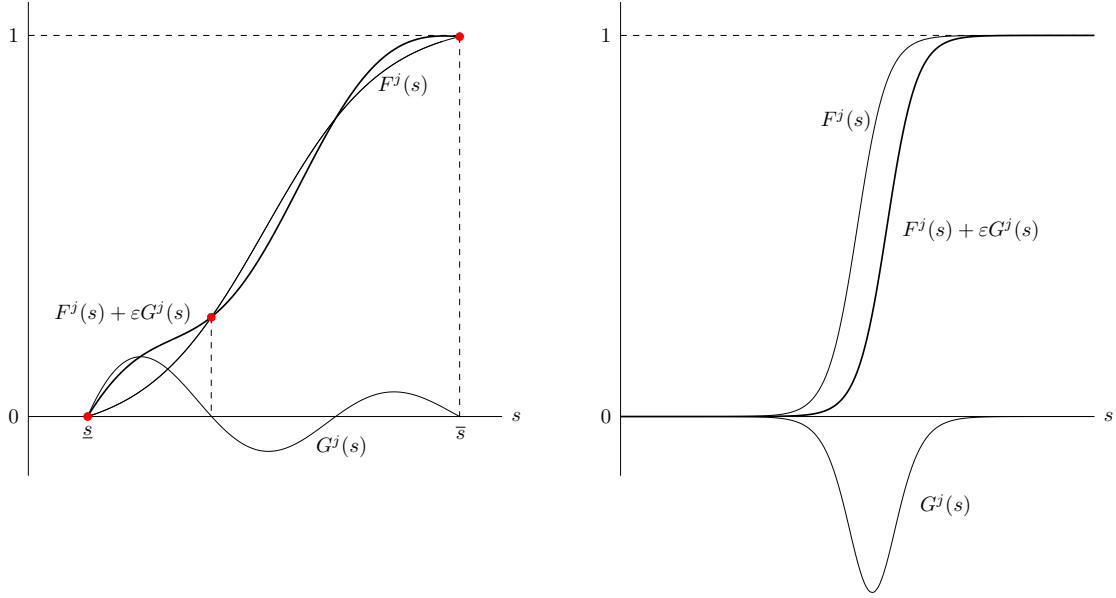


Figure 2: Beliefs' perturbations/variations (arbitrary and hazard-rate dominant)

**Note:** The left panel of Figure 2 illustrates an arbitrary perturbation/variation of beliefs, starting from the distribution of beliefs with cdf  $F^j(s)$  in the direction of  $G^j(s)$ . Note that  $G^j(s)$  is continuously differentiable and satisfies  $G^j(\underline{s}) = G^j(\bar{s}) = 0$ . The right panel of Figure 2 illustrates a hazard-rate dominant perturbation, such as those considered, for instance, in Propositions 3 and 6 below. Hazard-rate dominance is formally defined on Page 18. Note that an increase in the mean of a normal distribution, for a fixed variance, generates a hazard-rate dominant perturbation. Note also that hazard-rate dominant perturbations also satisfy first-order stochastic dominance, but the converse is not true. As explained in the text, note that  $G^j(s) < 0$  can be understood as local optimism at state  $s$ .

**Lemma 3.** *[Sensitivity of leverage and investment to beliefs] The responses of equilibrium investment to investors' and creditors' beliefs are characterized by the variational derivatives*

$$\frac{\delta k^*}{\delta F^I} \cdot G^I = \frac{\frac{\delta M}{\delta F^I} \cdot G^I}{\Upsilon''(k^*)} \quad \text{and} \quad \frac{\delta k^*}{\delta F^C} \cdot G^C = \frac{\frac{\delta M}{\delta F^C} \cdot G^C}{\Upsilon''(k^*)}.$$

Moreover, if the leverage constraint does not bind, then the responses of equilibrium leverage satisfy

$$\frac{\delta b^*}{\delta F^I} \cdot G^I = \frac{\frac{\delta(\frac{dM}{db})}{\delta F^I} \cdot G^I}{-\frac{d^2 M}{db^2}} \quad \text{and} \quad \frac{\delta b^*}{\delta F^C} \cdot G^C = \frac{\frac{\delta(\frac{dM}{db})}{\delta F^C} \cdot G^C}{-\frac{d^2 M}{db^2}}.$$

Lemma 3 shows that the same perturbation of beliefs impacts leverage and investment through different channels. Investment changes in proportion to the variational derivative

of the total valuation  $M(b)$ .<sup>18</sup> Leverage, if it is not determined by a binding constraint, changes in proportion to the variational derivative of the marginal value of leverage  $\frac{dM}{db}$ . This subtle distinction will be a key driver of our normative results. Next, in Proposition 2, we characterize the key variational derivatives.

**Proposition 2.** a) [Variational derivatives: Market value] The market value per unit of investment changes in response to variations in investors' and creditors' beliefs according to

$$\frac{\delta M}{\delta F^I} \cdot G^I = -\beta^I \int_b^{\bar{s}} G^I(s) ds \quad (12)$$

$$\frac{\delta M}{\delta F^C} \cdot G^C = -\beta^C \left[ (1 - \phi) b G^C(b) + \phi \int_{\underline{s}}^b G^C(s) ds \right]. \quad (13)$$

b) [Variational derivatives: Marginal value of leverage] The marginal value of leverage changes in response to variations in investors' and creditors' beliefs according to

$$\frac{\delta \left( \frac{dM}{db} \right)}{\delta F^I} \cdot G^I = \beta^I G^I(b) \quad (14)$$

$$\frac{\delta \left( \frac{dM}{db} \right)}{\delta F^C} \cdot G^C = -\beta^C G^C(b) \left( 1 + (1 - \phi) b \frac{g^C(b)}{G^C(b)} \right), \quad (15)$$

where  $g^C(b) = G^{C'}(b)$ .

The general characterization in Proposition 2 shows that both the *type* and the *magnitude* of changes in beliefs are critical to understanding the behavior of leverage and investment. Part a) of Proposition 2 shows that market values  $M(b)$  respond to the variation of investors' beliefs  $G^I(s)$  in solvent states  $s \geq b$  — in Equation (12) — or to the variation of creditors' beliefs  $G^C(s)$  in default states  $s < b$  — in Equation (13). Moreover, there is a special role for creditors' optimism  $-G^C(b)$  at the default boundary, which measures the degree to which creditors understate the probability of default.<sup>19</sup> Finally, Equations (12) and (13) show that the response of  $M(b)$  to both investors' and creditors' beliefs inherits the sign of  $-G^j(s)$ , that is, market values rise with the local optimism of any agent.

By contrast, part b) of Proposition 2 points to a fundamental asymmetry between the

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<sup>18</sup>A first look at Equation (9) may imply that  $\frac{\delta k^*}{\delta F^I} \cdot G^I$  also depends on  $\frac{\delta b^*}{\delta F^I} \cdot G^I$ , since  $\frac{\delta k^*}{\delta F^I} \cdot G^I = \frac{\frac{dM}{db} \frac{\delta b^*}{\delta F^I} \cdot G^I + \frac{\delta M}{\delta F^I} \cdot G^I}{\Upsilon''(k^*)}$ . However, the term  $\frac{dM}{db} \frac{\delta b^*}{\delta F^I} \cdot G^I$  is always 0, either because leverage is optimally chosen (and  $\frac{dM}{db} = 0$ ) or because the constraint binds (and  $\frac{\delta b^*}{\delta F^I} \cdot G^I = 0$ ).

<sup>19</sup>This property of Equation (13) arises because of the costs of distress in our model, which imply that creditors' payoffs are discontinuous in  $s$  at the default boundary whenever  $\phi < 1$ .

responses of leverage to creditors' and investors' beliefs. Equations (14) and (15) imply that the marginal value of leverage  $\frac{dM(b)}{db}$  decreases when investors are optimistic about the probability of default — since it inherits the sign of  $G^I(b)$  — but increases when creditors are optimistic — since it inherits the sign of  $-G^C(b)$ . Notice also that the response of the marginal value of leverage depends only on local belief changes at the default boundary.

The general characterizations in Proposition 2 — and their welfare counterparts in Section 3 — are valuable because they can be used to explore the impact of different changes in beliefs, for instance, changes in the perception of volatility or rare events. In order to provide sharper insights, in Proposition 3 we analyze perturbations  $G^j(s)$  that induce optimism in the sense of hazard-rate dominance. Formally, an absolutely continuous distribution  $F^j(s)$  becomes more optimistic in the sense of hazard-rate dominance if the hazard rate  $h^j(s) \equiv \frac{f^j(s)}{1-F^j(s)}$  decreases for all  $s$  (e.g., Shaked and Shanthikumar, 2007). This is a stronger requirement than first-order stochastic dominance, but a weaker requirement than the monotone likelihood ratio property.<sup>20</sup>

**Proposition 3.** *[Differential impact of optimism by investors and creditors] Optimism in this proposition is defined in the sense of hazard-rate dominance.*

- a) Debt exuberance: *When creditors become more optimistic, both the market value of investment  $M(b)$  and the marginal value of leverage  $\frac{dM(b)}{db}$  increase.*
- b) Equity exuberance: *When investors become more optimistic, the market value of investment  $M(b)$  increases but the marginal value of leverage  $\frac{dM(b)}{db}$  decreases.*
- c) Joint exuberance: *When investors and creditors have common beliefs and both become equally more optimistic, both the market value of investment  $M(b)$  and the marginal value of leverage  $\frac{dM(b)}{db}$  increase.*

Proposition 3 confirms that optimism/exuberance by any agent, in a hazard-rate sense, increases market values  $M(b)$ .<sup>21</sup> Meanwhile, investor optimism decreases the marginal value of leverage  $\frac{dM(b)}{db}$ , while creditor optimism increases it. Intuitively, optimism on the credit-supply side (i.e., by creditors) makes borrowing cheaper and encourages leverage through a substitution effect. However, optimism on the credit-demand side leads investors to believe that the likelihood of defaulting is lower and that the borrowing conditions offered by creditors are unfavorable, encouraging them to increase their equity contribution.

<sup>20</sup>Therefore, in terms of variational derivatives, a perturbation  $G^j(s)$  induces optimism in a hazard-rate sense if  $\frac{\delta h^j(s)}{\delta F} \cdot G^j \leq 0$  for all  $s$ . We present a detailed characterization of this property in Section D.4 of the Online Appendix.

<sup>21</sup>The results regarding the market value  $M(b)$  also hold when exuberance is defined in a first-order stochastic dominance (FOSD) sense. Hazard-rate dominance is the appropriate stochastic order in our setting due to the costs of distress, which scale with  $(1 - \phi)$  and are critical for the marginal value of leverage. Indeed, when  $\phi = 1$ , all our results apply in an FOSD sense.

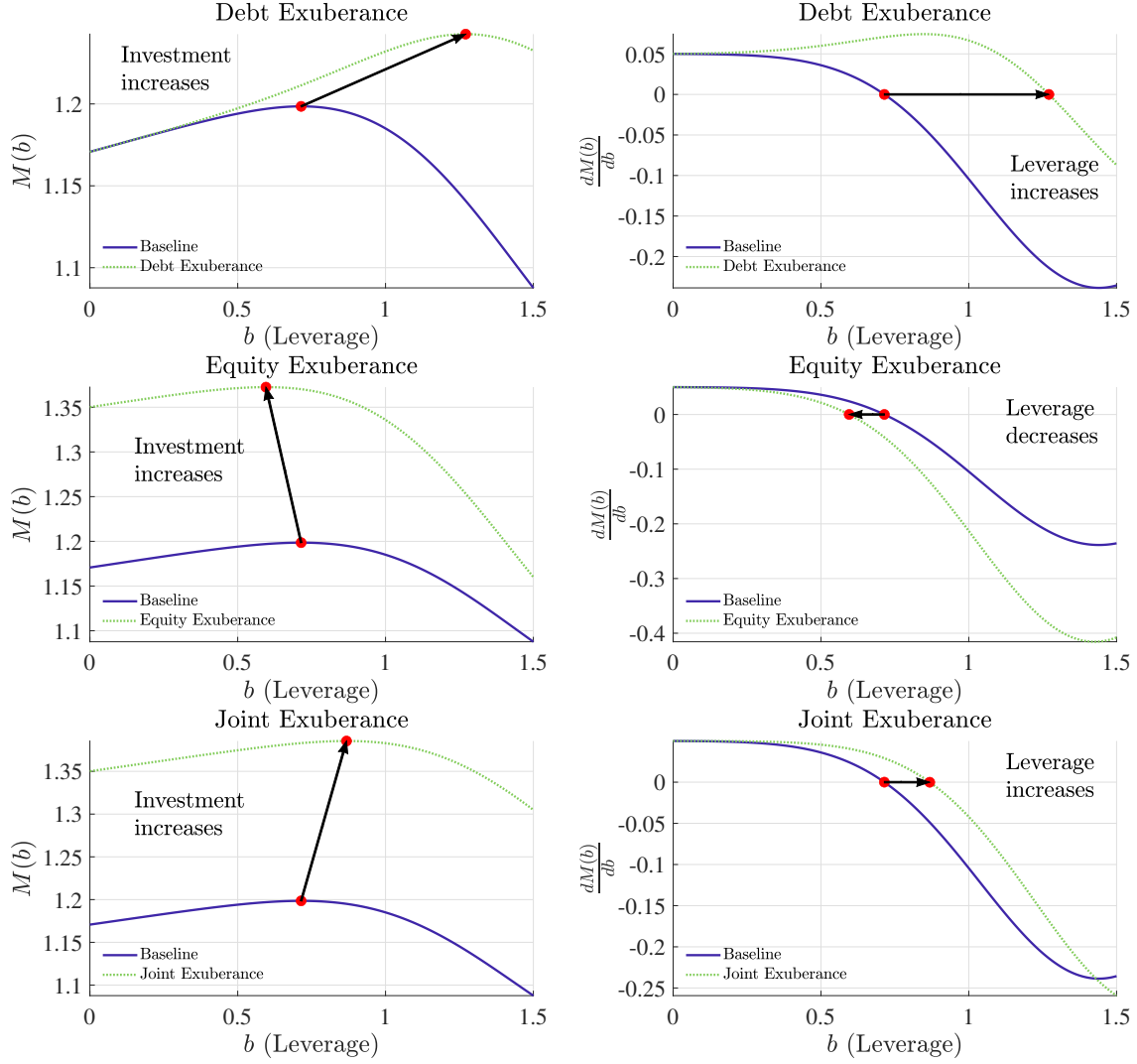


Figure 3: Differential impact of investors' and creditors' optimism

**Note:** Figure 3 illustrates the results of Proposition 3. The left plots in Figure 3 show  $M(b)$ , the market value of debt and equity per unit of investment, as a function of leverage  $b$ . The right plots in Figure 3 show  $\frac{dM(b)}{db}$ , the marginal value of leverage, as a function of leverage  $b$ . We assume that beliefs about  $s$  are normally distributed, with means indexed by  $\mu$  and standard deviations indexed by  $\sigma$ , and that investment costs are given by  $\frac{k^2}{2\varphi}$ . The parameters used in all plots are:  $\beta^I = 0.9$ ,  $\beta^C = 0.95$ ,  $\phi = 0.8$ ,  $\varphi = 1$ , and  $\sigma^I = \sigma^C = 0.4$ . The baseline beliefs are  $\mu^I = \mu^C = 1.3$ . The debt exuberance scenario corresponds to  $\mu^I = 1.3$  and  $\mu^C = 1.5$ . The equity exuberance scenario corresponds to  $\mu^I = 1.5$  and  $\mu^C = 1.3$ . The joint exuberance scenario corresponds to  $\mu^I = 1.5$  and  $\mu^C = 1.5$ .

The proposition establishes a surprising additional result in a joint exuberance scenario, in which both investors and creditors become more optimistic starting from a common belief assessment. The sentiments of creditors dominate in this scenario, and the comparative statics are qualitatively the same as for debt exuberance. This result is driven by the relative patience of creditors, who attach a higher value to future payoffs, and whose beliefs are therefore more important for overall valuations at the margin.<sup>22</sup> We expect the logic behind this result to apply more broadly. In general, any additional force that drives creditors to become lenders in equilibrium (e.g., intertemporal substitution) will also make them value future payoffs more. Figure 3 illustrates these results.

These subtle distinctions will be key to our analysis of optimal policy below. The response of the marginal value of leverage to belief variations is especially important because it also determines the sensitivity of investment to leverage regulation — see Lemma 2. Moreover, the results so far have interesting positive implications in their own right. We discuss these briefly before we present the main results of the paper.

Combining Lemma 3 with Proposition 3, we immediately obtain the following result:

**Corollary 1.** *[Positive implications of optimism by investors and creditors] Assume that the leverage constraint does not bind. In the equity exuberance scenario in Proposition 3, equilibrium investment increases but leverage decreases. In the debt and joint exuberance scenarios, both equilibrium investment and leverage increase.*

Corollary 1 implies that in expansions fueled by equity exuberance, investment and leverage decouple, and become *negatively* correlated. It is useful to examine the relationship between this implication of our model and related results in the existing literature. In our model, exuberant investors decide to invest more, but consider debt to be excessively expensive. Hence, they find it optimal to reduce leverage, and to fund the marginal increase in investment by reducing consumption at date 0. Relatedly, Bailey et al. (2019) consider a model with fixed investment, and show that investor optimism is associated with lower leverage, an implication which they confirm empirically for households' leverage choices. By contrast, in the model of Simsek (2013a), investors have a fixed amount of equity capital or, equivalently, face a binding non-negativity constraint on their consumption. In his analysis, investment can be increased only by additional borrowing, so that the correlation between investment and leverage is always positive, as we discuss further in Section D.5 of the Online Appendix.

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<sup>22</sup>For instance, consider the special case where there are no costs of distress ( $\phi = 1$ ). In this case, adding up Equations (14) and (15) with a common variation  $G^C(b) = G^I(b)$  shows that the total marginal change in  $\frac{dM(b)}{db}$  is  $(\beta^C - \beta^I) G(b) < 0$ .

### 3 Optimal Leverage Regulation

In this section, which contains the core contributions of this paper, we study the problem of a social planner who can set the leverage limit  $\bar{b}$ . Formally, social welfare for the planner is given by the sum of utilities of investors and creditors, which are computed using the planner's probability assessments for each of the agents. That is, the planner computes investors' welfare assessing the likelihood of events using a distribution  $F^{I,P}(s)$ . Similarly, the planner computes creditors' welfare assessing the likelihood of events using a potentially different distribution  $F^{C,P}(s)$ .

This approach allows us to explore a wide range of normative objectives. For instance, a planner that respects agents' beliefs will set

$$F^{I,P}(s) = F^I(s) \quad \text{and} \quad F^{C,P}(s) = F^C(s).$$

Alternatively, a planner who uses the true/objective distribution of investment returns, which we denote here by  $F(s)$ , to compute social welfare will set

$$F^{C,P}(s) = F^{I,P}(s) = F(s).$$

Given this approach, we can now characterize the optimal policy, which takes into account the fact that investors and creditors make decisions under their own beliefs,  $F^I(s)$  and  $F^C(s)$ , but evaluates the consequences of these decisions using the planner's beliefs,  $F^{I,P}(s)$  and  $F^{C,P}(s)$ , which are taken as primitives.

It will become evident that our conclusions do not depend directly on the true distribution of  $s$ . However, one natural interpretation of our results is that investors and creditors have distorted beliefs and that the planner's beliefs are correct. Following this logic, our results can be interpreted as characterizing an optimal paternalistic policy. While this interpretation is useful to illustrate some of the underlying economics cleanly, our formal results have multiple interpretations. In particular, at the end of this section, we study the impact of changes in the planner's beliefs on the optimal regulation, which can be interpreted as describing how the optimal policy changes when the planner's beliefs depart from the correct beliefs. Moreover, in Section 4.1, we consider the problem of a planner who has imperfect knowledge of investors' and creditors' beliefs. Both of these exercises illustrate how our results can also be used to explore the limitations of paternalism.

### 3.1 Planner's problem

The first step is to formulate the planner's problem. The planner chooses the leverage limit  $\bar{b}$ , taking into account that leverage and investment decisions react to this policy. Indeed, the optimality conditions (8) and (9) jointly define equilibrium leverage and investment as implicit functions  $b^*(\bar{b})$  and  $k^*(\bar{b})$  of the policy. Lemma 4 formally characterizes social welfare from the perspective of the utilitarian planner as a function of the leverage cap  $\bar{b}$ .

**Lemma 4.** *[Planner's problem] The planner's problem can be expressed as*

$$\max_{\bar{b}} W(b^*(\bar{b}), k^*(\bar{b})),$$

where social welfare  $W(b, k)$  is given by

$$W(b, k) = [M^P(b) - \Delta(b) - 1]k - \Upsilon(k),$$

and where  $M^P(b)$  denotes the present value of payoffs under the planner's beliefs

$$M^P(b) = \beta^I \int_b^{\bar{s}} (s - b) dF^{I,P}(s) + \beta^C \left( \int_b^{\bar{s}} b dF^{C,P} + \phi \int_{\underline{s}}^b s dF^{C,P}(s) \right).$$

Lemma 4 shows that the planner's objective mimics the objective of the reformulated investors' problem introduced in Lemma 1 after incorporating the planner's beliefs and the externality  $\Delta(b)k$ . This result is intuitive, but not obvious. The welfare of investors and creditors as perceived by the planner depends on their actual beliefs through the equilibrium choices of investment and leverage, but the social valuation of investment in  $M^P(b)$  depends only on the planner's beliefs.

There are three observations worth highlighting. First, note that whenever leverage regulation is binding, social welfare depends on investors' and creditors' beliefs only through the investment choice  $k^*$  since, in that case,  $b^*(\bar{b}) = \bar{b}$  is directly controlled by the planner. Second, note that if there are no externalities ( $\Delta(b) \equiv 0$ ), then the economy is constrained efficient for a planner that respects agents' beliefs, since  $M^P(b) = M(b)$  in that case. Finally, note that Lemma 4 relies on assigning equal welfare weights to all agents. While the linearity of preferences makes this criterion natural, we explain how our results generalize in the Appendix.

### 3.2 Marginal welfare effects

Proposition 4 presents the marginal welfare effect of varying the leverage cap,  $\frac{dW}{db}$ , which will be central to our analysis of optimal policy.



**Proposition 4.** *[Marginal welfare effect of varying the leverage cap] The marginal welfare impact of increasing the leverage cap, whenever the leverage cap is binding, is*

$$\frac{dW}{d\bar{b}} = \underbrace{\left[ \frac{dM^P(\bar{b})}{db} - \frac{d\Delta(\bar{b})}{db} \right] k^*(\bar{b})}_{\text{Inframarginal Effect}} + \underbrace{\left[ M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b}) \right] \frac{dk^*(\bar{b})}{d\bar{b}}}_{\text{Incentive Effect}}. \quad (16)$$

Proposition 4 shows that knowledge of  $M^P(b)$ ,  $M(b)$ , and  $k^*(\bar{b})$  and their derivatives is sufficient to determine whether it is desirable to increase or decrease the leverage cap. We refer to the first term in Equation (16) as the inframarginal effect. This term captures how varying the leverage cap modifies the planner's valuation of pre-existing investment at the margin. We refer to the second term in Equation (16) as the incentive effect. This term captures the investment response associated with a change in  $\bar{b}$ . Lemma 2 implies that the sign of the incentive effect term is determined by the sign of the difference  $M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b})$ . For example, this term is negative when investors and/or creditors are optimistic relative to the planner, or when there are strong negative externalities associated with leverage.

Furthermore, Proposition 4 implies a simple test for whether the marginal welfare effect of raising the leverage cap is positive:

$$\frac{dW}{d\bar{b}} > 0 \iff \frac{d \ln [M^P(\bar{b}) - \Delta(\bar{b})]}{db} + \left[ \frac{M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b})}{M^P(\bar{b})} \right] \frac{d \ln k(\bar{b})}{d\bar{b}} > 0. \quad (17)$$

Equation (17) reveals three sufficient statistics that determine whether leverage regulation should become tighter or looser. The first corresponds to the marginal social benefit  $\frac{d \ln [M^P(\bar{b}) - \Delta(\bar{b})]}{db}$  of leverage. The second is the wedge  $\frac{M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b})}{M^P(\bar{b})}$ , which measures the proportional difference between the planner's and the agents' perception of the present value of investment. The third is the semi-elasticity  $\frac{d \ln k(\bar{b})}{d\bar{b}}$  of investment to leverage requirements. While the first sufficient statistic is exclusively a function of the planner's beliefs, the second one requires a direct assessment of belief differences between the agents' and the planner's beliefs, while the third one could be directly recovered from a regression of log investment on exogenous leverage limit changes. If one could measure or infer the elements of Equation (17), it would be possible to provide explicit quantitative guidance on the optimal leverage regulation.

Finally, note that if one starts from the laissez-faire allocation, the incentive effect vanishes and  $\frac{d \ln k(\bar{b})}{d\bar{b}} = 0$ , as illustrated in Figure 1 above. In that case, the only sufficient

statistic that characterizes the marginal effect is  $\frac{d \ln[M^P(b) - \Delta(b)]}{db}$  evaluated at the laissez-faire optimum. This is an interesting observation, since it is sufficient for the planner to form an assessment over the social marginal value of leverage, which does not require the planner to know the beliefs of investors or creditors.

Proposition 4 illustrates again that, in the absence of externalities, the single rationale for regulation is the difference in beliefs between the planner and the agents in the economy. Indeed, if  $\Delta(b) = 0$  and the planner and the agents have the same set of beliefs, the incentive effect vanishes — since  $M^P(\bar{b}) = M(\bar{b})$  — and the inframarginal effect is always positive whenever the constraint binds, so it is optimal never to set a leverage cap.

### 3.3 The impact of beliefs on optimal regulation

This subsection introduces the main results of the paper. It characterizes how a change in beliefs by investors or creditors modifies the form of the optimal leverage regulation. We begin by analyzing how the marginal effect of leverage regulation on welfare  $\frac{dW}{db}$  responds to changes in beliefs. Since beliefs are infinite-dimensional objects in our analysis, we focus on the variational derivative of  $\frac{dW}{db}$  with respect to beliefs in Proposition 5. Under appropriate regularity conditions, one can translate claims about the marginal response of  $\frac{dW}{db}$  to changes in beliefs into implications for the optimal leverage regulation; see, for example, Proposition 6 and our discussion in the Appendix.

**Proposition 5.** *[Impact of beliefs on leverage regulation: General characterization] The change in the marginal welfare effect of varying the leverage cap, whenever the leverage cap is binding, in response to a change in beliefs by either investors or creditors,  $j = \{I, C\}$ , is given by*

$$\begin{aligned} \frac{\delta \frac{dW}{db}}{\delta F^j} \cdot G^j &= \left[ \frac{dM^P(\bar{b})}{db} - \frac{d\Delta(\bar{b})}{db} - \frac{dM(\bar{b})}{db} \right] \left[ \frac{\delta k^*(\bar{b})}{\delta F^j} \cdot G^j \right] \\ &+ \left[ M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b}) \right] \left[ \frac{\delta \frac{dk^*}{db}}{\delta F^j} \cdot G^j \right] \end{aligned} \quad (18)$$

The characterization of marginal welfare effects in Proposition 5 permits a clearer assessment of how, and why, the rationale for leverage regulation changes when investors' and/or creditors' beliefs change. Changes in investors' or creditors' beliefs affect the marginal welfare effect of leverage regulation through two channels, namely, through the change in optimal investment  $k^*(\bar{b})$  and the change in the sensitivity of investment to policy  $\frac{dk^*}{db}$ . These effects correspond to the inframarginal and incentive effects identified

in Proposition 4. Intuitively, when deciding how to adjust the leverage policy in response to a change in beliefs, the planner must assess i) the extent to which the change in beliefs affects investment behavior and the desirability of regulation, and ii) the extent to which the change in beliefs affects the sensitivity or effectiveness of regulation.

The comparative statics in Lemma 3 imply that the variational derivatives of  $k^*(b)$  and  $\frac{dk^*(\bar{b})}{db}$  are directly related to the variational derivatives of the market value of investment  $M(b)$  and the marginal value of leverage  $\frac{dM}{db}$ . Therefore, the effects of investors' beliefs on leverage regulation inherit the nuanced patterns associated with  $\frac{\delta M}{\delta F^j} \cdot G^j$  and  $\frac{\delta \frac{dM}{db}}{\delta F^j} \cdot G^j$ , characterized in Section 2. As implied by our detailed discussion of Propositions 2 and 3, the type of belief variation considered is important. In particular, the magnitude of the variation at the default boundary,  $G^j(b)$ , plays a special role, and there is a fundamental asymmetry between investors and creditors.

It is particularly instructive to consider the case of quadratic adjustment costs. Therefore, for the remainder of this section, we assume that  $\Upsilon(k) = \frac{k^2}{2\varphi}$ . In that case, Equation (18) simplifies to

$$\begin{aligned} \frac{\delta \frac{dW}{db}}{\delta F^j} \cdot G^j &= \varphi \cdot \left[ \frac{dM^P(\bar{b})}{db} - \frac{d\Delta(\bar{b})}{db} - \frac{dM(\bar{b})}{db} \right] \left[ \frac{\delta M(\bar{b})}{\delta F^j} \cdot G^j \right] \\ &+ \varphi \cdot \left[ M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b}) \right] \left[ \frac{\delta \frac{dM}{db}}{\delta F^j} \cdot G^j \right] \end{aligned} \quad (19)$$

We can now directly rely on the results in Proposition 3 to characterize the effect of belief changes on the optimal regulation. Following Proposition 3, we consider three scenarios in Proposition 6.

**Proposition 6.** *[Impact of beliefs on optimal regulation: Specific scenarios]*

a) *Debt exuberance:* Assume that investors and the planner share common beliefs  $F^I(s) = F^{I,P}(s) = F^{C,P}(s)$ , and creditors' beliefs are more optimistic than the planner's beliefs in a hazard-rate sense. Then, increased optimism by creditors implies  $\frac{\delta \frac{dW}{db}}{\delta F^C} \cdot G^C < 0$ . Hence, the optimal leverage cap is binding and decreasing in optimism.

b) *Equity exuberance:* Assume that creditors and the planner share common beliefs  $F^C(s) = F^{C,P}(s) = F^{I,P}(s)$ , and investors' beliefs are more optimistic than the planner's beliefs in a hazard-rate sense. Then, increased optimism by investors implies  $\frac{\delta \frac{dW}{db}}{\delta F^I} \cdot G^I > 0$

if and only if:

$$\frac{dM^P(\bar{b})}{d\bar{b}} - \frac{dM(\bar{b})}{d\bar{b}} - \frac{d\Delta(\bar{b})}{d\bar{b}} > \underbrace{\left[ M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b}) \right]}_{<0} \underbrace{\frac{\frac{\delta \frac{dM}{d\bar{b}}}{\delta F^j} \cdot G^j}{-\frac{\frac{\delta \frac{dM}{d\bar{b}}}{\delta F^j} \cdot G^j}}}_{>0}. \quad (20)$$

It is never optimal to impose a binding leverage cap if (20) is satisfied for all  $\bar{b}$ , which is always true when externalities  $\Delta(\bar{b})$  are sufficiently small.

c) *Joint exuberance:* Assume that creditors and investors share common beliefs  $F^C(s) = F^I(s) = F^0(s)$  that is more optimistic than the planner's beliefs  $F^{C,P}(s) = F^{I,P}(s) = F^P(s)$  in a hazard-rate sense. Then, as in the debt exuberance scenario, increased optimism by investors and creditors implies  $\frac{\delta \frac{dW}{d\bar{b}}}{\delta F^0} \cdot G^0 < 0$ . Hence, the optimal leverage cap is binding and decreasing in optimism.

Proposition 6 shows a clear distinction between the effects of debt and equity exuberance. In the case of debt exuberance, there are two effects. First, investors consider creditors' beliefs to be excessively optimistic, which leads them to take too much leverage from a social perspective. Thus, the inframarginal effect decreases the social benefit of encouraging leverage. Second, the incentive effect becomes stronger with exuberance due to the increased sensitivity of investment to leverage. Both effects imply that debt exuberance decreases the marginal benefit of permitting leverage. Starting from a case with common beliefs, in which there is no rationale for a binding leverage cap, we therefore find that debt exuberance makes a binding cap more desirable.

In the case of equity exuberance, the incentive effect is reversed due to the decreased sensitivity of investment to leverage. Moreover, the inframarginal effect is also potentially reversed, because investors find the borrowing conditions offered by creditors unattractive, which can lead them to take too little leverage from the planner's perspective. Indeed, this occurs precisely when the marginal leverage externality  $\frac{d\Delta(\bar{b})}{d\bar{b}}$  is not too large, in the sense that the inequality in Equation (20) is satisfied. Notice, in particular, that  $\frac{dM^P(\bar{b})}{d\bar{b}} - \frac{dM(\bar{b})}{d\bar{b}} > 0$  in the case of equity exuberance, so that Equation (20) is guaranteed to hold when externalities are small enough. In this case, equity exuberance serves to make a binding cap *less* desirable. We conclude that it is never optimal to impose a binding cap if the primary motive for intervention are belief distortions/internalities, as opposed to externalities. Instead, it would be optimal to impose a binding leverage floor, although this is not a policy we have considered in our analysis so far.

Finally, we show that the case of joint exuberance leads to the same qualitative result as the case of debt exuberance. As discussed in the context of Proposition 3, this occurs

because creditors are more patient and their valuation of marginal default states is higher than the valuation of investors. As a result, joint exuberance also supports a tightening of a binding leverage cap.

We note that all of our results extend to the special case in which there is no investment margin, so that  $k^*(\bar{b}) \equiv \bar{k}$  is a constant. In this case, regulatory considerations are driven only by the inframarginal effect. As a result, a binding leverage cap remains optimal in the case of debt/joint exuberance, and the optimal policy in the case of equity exuberance remains ambiguous, depending on the magnitude of externalities.

Figure 4 illustrate how the optimal regulation responds to changes in investors' and creditors' beliefs. It compares equilibrium leverage and investment without regulation and under the optimal regulation when varying the beliefs of investors, creditors, or both, while holding fixed the planner's beliefs. To highlight the role of beliefs in shaping the optimal regulation, we assume that other externalities are zero, i.e,  $\Delta(b) = 0$ . The top row of Figure 4 shows the levels of leverage and investment without regulation. The bottom row shows leverage and investment under the optimal leverage regulation. More precisely, as explained in the note that describes Figure 4, assuming that the beliefs about  $s$  are normally distributed, the horizontal axis represents the perceived expected return on investment by investors in the equity exuberance scenario, by creditors in the debt exuberance scenarios, and by both investors and creditors in the joint exuberance scenario. Perhaps surprisingly, the optimal leverage cap in a joint exuberance scenario is tighter than the cap in a debt exuberance scenario. This result is driven by the scale of investment: while investment is lower whenever leverage is regulated, it is still that case that investment in the joint exuberance scenario is significantly larger than in the debt exuberance scenario. Therefore, because investment is larger in the joint exuberance case, the inframarginal effect defined in Equation (16) becomes more important, so reducing leverage becomes highly desirable for the planner. Intuitively, when investment is large, the same reduction in leverage generates a larger overall welfare gain since it applies to more units of capital.

**Equity, debt, and joint exuberance in practice** One can interpret the recent experience of Hertz and the associated intervention by the SEC as a manifestation of an equity exuberance scenario.<sup>23</sup> In June 2020, there seemed to be retail investors willing

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<sup>23</sup>For an account of the equity issuance of Hertz and the response of the SEC, see, for instance, <https://www.wsj.com/articles/hertz-sold-29-million-in-stock-before-sec-stepped-in-11597100128>. Similar concerns about exuberant equity valuations have emerged recently, when retail investors generated rising demand for stocks such as Gamestop, which have gained popularity via social media. See, for instance, <https://www.reuters.com/article/us-gamestop-stocks/gamestop-to-capitalize-on-stonks-rally-with-1-billion-stock-sale-plan-idUSKBN2BS0SN>.

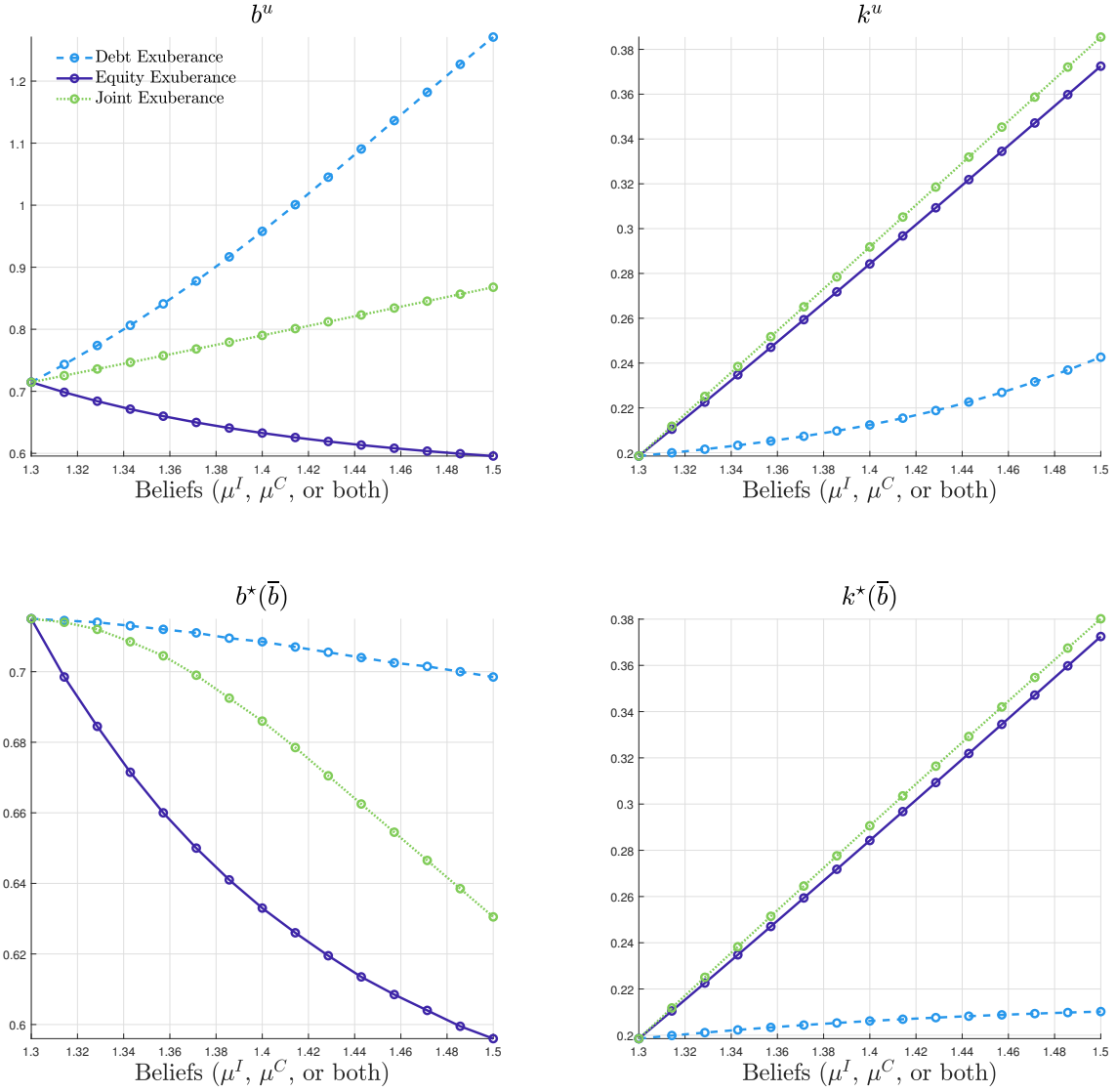


Figure 4: Impact of beliefs on optimal regulation: comparative statics

**Note:** Figure 4 compares equilibrium leverage and investment without regulation and under the optimal regulation when varying the beliefs of investors, creditors, or both, while holding fixed the planner's beliefs. The top left and top right plots respectively show equilibrium leverage and investment without regulation,  $b^u$  and  $k^u$ , where the superscript  $u$  stands for unregulated. The bottom left and bottom right plots respectively show equilibrium leverage and investment under the optimal regulation,  $b^*(\bar{b})$  and  $k^*(\bar{b})$ . As shown in Proposition 6, note that  $b^*(\bar{b}) = b^u$  in the equity exuberance scenario and  $b^*(\bar{b}) = \bar{b}$  in both the debt exuberance and the joint exuberance scenario. We assume that beliefs about  $s$  are normally distributed, with means indexed by  $\mu$  and standard deviations indexed by  $\sigma$ , and that investment costs are given by  $\frac{k^2}{2\varphi}$ . The parameters used in all plots are:  $\beta^I = 0.9$ ,  $\beta^C = 0.95$ ,  $\phi = 0.8$ ,  $\varphi = 1$ , and  $\sigma^I = \sigma^C = 0.4$ . The planner's beliefs are fixed in all plots at  $\mu^{I,P} = \mu^{C,P} = 1.3$  and  $\sigma^{I,P} = \sigma^{C,P} = 0.4$ . The baseline beliefs are  $\mu^I = \mu^C = 1.3$ . The debt exuberance outcome in each plot varies the expected return on investment perceived by creditors,  $\mu^C$ , between 1.3 and 1.5, while keeping  $\mu^I = 1.3$ . The equity exuberance outcome in each plot varies the expected return on investment perceived by investors,  $\mu^I$ , between 1.3 and 1.5, while keeping  $\mu^C = 1.3$ . The joint exuberance outcome in each plot varies at the same the expected return investment perceived by both investors and creditors,  $\mu^I = \mu^C$ , between 1.3 and 1.5. To highlight the role of beliefs in shaping the optimal regulation, we assume that other externalities are zero, i.e.,  $\Delta(b) = 0$ .

to purchase Hertz’s stock even though the company had declared bankruptcy and it was unlikely that equityholders would receive any funds at all. Hertz’s management, seeking to maximize the firm’s value, promptly decided to sell shares in the open market. The regulator (in this case, the SEC) intervened by vetoing the equity issuance, which can be interpreted as a cap on equity (a leverage floor), as predicted by Proposition 6. One could argue that the SEC was only considering the welfare of equity investors, while the planner in our model considers the welfare of both equity investors and creditors. As we describe in the Appendix, the conclusion that a cap on equity is optimal is also valid under that criterion.<sup>24</sup> In the case of equity-driven booms, our analysis additionally implies that the optimality of leverage caps is determined by whether the policy-maker views the market failure primarily as an externality, such as the spillovers from systemically important banks to the macroeconomy, or as an internality/distorted beliefs, such as households’ possibly misguided expectations about house price growth.

Alternatively, one can interpret the boom in subprime lending that preceded the global financial crisis of 2008 as a debt exuberance or joint exuberance scenario (e.g., [Cheng, Raina and Xiong, 2014](#)). Our results would have provided a clear rationale for limiting leverage in the years prior to 2007/2008 above and beyond what other externalities may call for. Note that, in our model, periods with compressed credit spreads — which have been shown to forecast negative excess returns ([Greenwood and Hanson, 2013](#); [Lopez-Salido, Stein and Zakrajsek, 2017](#)) — may be a symptom of debt exuberance. Interestingly, our illustration in Figure 4 suggests that periods of joint exuberance, in which leverage and investment would increase simultaneously without regulation, are those in which the optimal leverage regulation ought to be tighter.<sup>25</sup>

### 3.4 Distortion in the planner’s beliefs

So far, our analysis has focused on characterizing optimal policy as a function of investors’ and creditors’ beliefs, holding constant the beliefs under which the planner evaluates optimal policy. We close this section by characterizing the impact of changes in the beliefs  $F^{I,P}(s)$  and  $F^{C,P}(s)$  that the planner uses to evaluate the welfare of investors and creditors when setting the optimal policy. One can use the comparative statics below to explore misguided paternalism, namely how the optimal policy changes when the planner’s

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<sup>24</sup>There is a large literature on behavioral corporate finance that explores the roles of beliefs for equity issuance, see e.g., [Baker and Wurgler \(2002, 2013\)](#). Many findings in this literature can be mapped to the equity exuberance scenario.

<sup>25</sup>We should caveat that our illustration of the results in Figure 4 assumes a parametric distribution to specify agents’ beliefs — unlike most of the results in the paper. While we do not include a full quantitative exploration in the paper, we conjecture that debt exuberance by itself may have stronger quantitative effects in models in which beliefs feature a thick left tail (e.g., rare disasters).

beliefs become distorted.

For brevity, we focus here on comparative statics in three scenarios, which are the analogue to our treatment of investors' and creditors' beliefs in Proposition 6. Under the same conditions on  $\Upsilon(\cdot)$  used to derive Proposition 6, we provide the following characterization:

**Proposition 7.** *[Impact of the planner's beliefs on optimal regulation: Specific scenarios]*

a) Planner's debt exuberance: Consider changes in the belief  $F^{C,P}(s)$  that the planner uses to evaluate creditors' utility, and hold fixed the belief  $F^{I,P}(s)$  that the planner uses to evaluate investors' utility. Then, increased optimism by the planner in a hazard-rate sense implies  $\frac{\delta \frac{dW}{db}}{\delta F^{C,P}} \cdot G^{C,P} > 0$ . Hence, the planner's debt exuberance always leads to more lenient perceived optimal regulation.

b) Planner's equity exuberance: Consider changes in the belief  $F^{I,P}(s)$  that the planner uses to evaluate investors' utility, and hold fixed the belief  $F^{C,P}(s)$  that the planner uses to evaluate creditors' utility. Then, increased optimism in a hazard-rate sense by the planner can imply either  $\frac{\delta \frac{dW}{db}}{\delta F^{I,P}} \cdot G^{I,P} > 0$  or  $\frac{\delta \frac{dW}{db}}{\delta F^{I,P}} \cdot G^{I,P} < 0$ . Hence, it is ambiguous whether planner's equity exuberance leads to tighter or looser perceived optimal regulation.

c) Planner's joint exuberance: Assume that the planner uses a single belief  $F^P(s) = F^{I,P}(s) = F^{C,P}(s)$  to evaluate agents' utility. Then, increased optimism by the planner in a hazard-rate sense implies  $\frac{\delta \frac{dW}{db}}{\delta F^P} \cdot G^P > 0$ . Hence, the planner's joint exuberance always leads to more lenient perceived optimal regulation.

The economic intuition for this result follows from our characterization of the marginal welfare effect  $\frac{dW}{db}$  in Proposition 4. The marginal effect is increasing in the marginal value of leverage  $\frac{dM^P(\bar{b})}{db}$  under the planner's beliefs, and also increasing in the planner's total valuation of investment  $M^P(\bar{b})$ .<sup>26</sup> Both equity and debt exuberance on behalf of the planner increase the total valuation  $M^P(\bar{b})$ . However, using a parallel argument to Proposition 3, we can show that the marginal value  $\frac{dM^P(\bar{b})}{db}$  is decreasing with debt exuberance, but increasing with equity exuberance in the planner's beliefs. Hence, the overall effects of the planner's equity exuberance are ambiguous, while debt exuberance always leads to an increase in  $\frac{dW}{db}$ , meaning a more lenient perceived optimal policy. As in our previous analysis, the case of joint exuberance inherits the properties of debt exuberance, because the beliefs of patient creditors dominate the relevant valuations.

Note that the results in Propositions 6 and 7 jointly provide a general characterization of the impact of the planner's beliefs and beliefs of investors and creditors. Hence, we

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<sup>26</sup>This follows because  $k^*(\bar{b}) > 0$  and, by investors' first-order condition in Equation (8),  $\frac{dk^*(\bar{b})}{db} \geq 0$ .



can extract insights about several scenarios that may arise if the planner is not perfectly rational. First, consider a scenario where investors and creditors are rational but the planner’s beliefs are distorted. In this case, a laissez-faire policy is clearly optimal, but the planner might wrongly “over-regulate” by imposing a binding constraint. Proposition 7 shows when over-regulation is possible. For example, pessimism in the planner’s beliefs about creditors’ payoffs (in the sense of the hazard rates of  $F^{C,P}(s)$ ) always brings about over-regulation by decreasing the perceived benefit  $\frac{dW}{db}$  of permitting more leverage.

Second, one can imagine a model where private agents agree with the planner’s beliefs, but those beliefs themselves are irrational. In this case, it is optimal to impose a binding leverage constraint if there is debt exuberance or joint exuberance among private agents (see Proposition 6). However, since the planner is also exuberant, she continues to regard a laissez-faire policy as optimal, and will therefore “under-regulate”, failing to impose a constraint. Finally, in a mixed scenario where agents and the planner are irrational but disagree with one another, either over-regulation or under-regulation becomes possible, and Proposition 7 delineates both cases.

## 4 Extensions

In this section, we consider four extensions to the baseline model introduced in Section 2. First, we consider a planner who does not perfectly observe investors’ and creditors’ beliefs. Second, we allow the planner to conduct monetary policy to change the cost of leverage. Third, we introduce alternative micro-foundations for externalities, namely implicit government subsidies/bailouts and fire sales externalities. Finally, we consider the possibility that beliefs are endogenous to aggregate activity.

### 4.1 Imperfect knowledge of beliefs

In this section, the physical setup of the economy is identical to our baseline model, but the planner cannot observe the pair of distribution functions  $\theta \equiv \{F^I, F^C\} \in \Theta$ , where  $\Theta$  denotes the set of possible belief constellations. We assume that the planner controls leverage by imposing an equality constraint  $b = \bar{b}$  on investors. In this environment, the planner wishes to choose  $\bar{b}$  so as to maximize  $W(\bar{b}, k^*(\bar{b}; \theta))$ , where we write  $k^*(\bar{b}; \theta)$  for equilibrium capital investment when beliefs are  $\theta$ . The mapping from the policy  $\bar{b}$  to the equilibrium outcome  $k^*$  is now uncertain from the planner’s perspective.<sup>27</sup>

First, we characterize optimal policy when the planner can make probabilistic

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<sup>27</sup>Investment is determined by the first-order condition  $M(\bar{b}; \theta) - 1 = \Upsilon'(k^*(\bar{b}; \theta))$ , where  $M(\bar{b}; \theta)$  is defined as in the baseline model for any pair  $\theta = \{F^I, F^C\}$ .

assessments about  $\theta$ , based on her prior beliefs and on the observation of endogenous equilibrium quantities. Second, we characterize robust optimal policy, which is derived by considering the “worst case” beliefs  $\theta$  using a similar approach to [Woodford \(2010\)](#).

#### 4.1.1 Optimal policy conditional on endogenous investment

Suppose that the planner attaches a prior probability measure  $\mu$  to different realizations of beliefs  $\theta$ . We consider a policy game in which i) beliefs  $\theta$  are realized, ii) the planner chooses the leverage requirement  $\bar{b}$ , iii) investors and creditors determine equilibrium investment  $k$ , and iv) the planner observes the realization of  $k$ . Since the planner is able to extract information about  $\theta$  by observing equilibrium capital investment, we are interested in *consistent* policies, for which the planner does not want to deviate from her chosen  $\bar{b}$  after observing  $k$ .<sup>28</sup> Despite the static nature of our model, this notion captures the practical idea that regulators adjust their policy based on observed aggregate quantities.

Consistent policies must satisfy the following first-order condition, which we derive formally in Online Appendix [C.1.1](#):

$$\left[ \frac{dM^P(\bar{b})}{db} - \frac{d\Delta(\bar{b})}{db} \right] k^*(\bar{b}^*; \theta) + [M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b}^*; \theta)] \mathbb{E}_\mu \left[ \frac{dk^*(\bar{b}; \theta)}{d\bar{b}} \middle| I(\bar{b}^*, \theta) \right] = 0, \quad (21)$$

where  $\mathbb{E}_\mu [\cdot | I(\bar{b}^*, \theta)]$  denotes the planner’s expectation over the set of beliefs according to her information set  $I(\bar{b}^*, \theta)$ , which is defined formally in the Appendix. Equation (21) contains the same inframarginal and incentive effects as in our baseline model. The level of capital investment  $k^*(\bar{b}^*; \theta)$  in the inframarginal effect is known to the planner who observes equilibrium outcomes. Similarly, the market valuation of capital  $M(\bar{b}^*; \theta)$ , which enters the incentive effect, is revealed to the planner by its one-to-one relationship with investment. However, the key policy elasticity  $\frac{dk^*(\bar{b}; \theta)}{d\bar{b}}$ , which determines the sensitivity of capital to the leverage requirement, is unknown to the planner and evaluated in expectation.

Therefore, a planner who can condition policy on equilibrium investment is not uncertain about the extent of over/underinvestment induced by belief distortions, but has residual uncertainty about the type of distortion. For example, a pair of beliefs  $\theta = \{F^I, F^C\}$  that induces the same level of equilibrium investment could be driven either by equity exuberance or debt exuberance. If the planner considers equity exuberance more likely a priori, then her expectation of  $\frac{dk^*(\bar{b}; \theta)}{d\bar{b}}$  in equilibrium is lower, which implies

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<sup>28</sup>An alternative notion of optimal policy in this environment assumes that the planner pre-commits to a mapping  $\bar{b}(k)$ , as in [Hauk, Lanteri and Marcet \(2021\)](#). This definition leads to a modified formula for marginal welfare effects, which we analyze in Section [C.1.1](#) of the Online Appendix.

a weaker incentive effect.<sup>29</sup>

#### 4.1.2 Robust optimal policy

Alternatively, suppose that the planner chooses  $\bar{b}$  to maximize welfare under the *worst case* beliefs that she considers plausible:

$$\max_{\bar{b}} \min_{\theta \in \Theta} W(\bar{b}, k^*(\bar{b}; \theta)) \quad (22)$$

Following [Woodford \(2010\)](#), we specify the plausible set  $\Theta$  by a maximum distance between the planners' and the agents' beliefs in the sense of relative entropy (see [Online Appendix C.1.2](#)).

Applying the envelope theorem to Equation (22), and noting that the constraint set  $\Theta$  does not depend on  $\bar{b}$ , we can write the marginal welfare effect of varying  $\bar{b}$  as

$$\begin{aligned} \frac{d}{d\bar{b}} \min_{\theta \in \Theta} W(\bar{b}, k^*(\bar{b}; \theta)) &= \left[ \frac{dM^P(\bar{b})}{d\bar{b}} - \frac{d\Delta(\bar{b})}{d\bar{b}} \right] k^*(\bar{b}; \hat{\theta}(\bar{b})) \\ &\quad + \left[ M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b}) \right] \frac{\partial k^*(\bar{b}; \hat{\theta}(\bar{b}))}{\partial \bar{b}}, \end{aligned}$$

where  $\hat{\theta}(\bar{b})$  denotes the solution to the minimization problem in Equation (22). Thus, the planner considers the same inframarginal and incentive effects as in the baseline model, but now evaluates them at the worst-case beliefs.

To gain further insights, we characterize the worst-case beliefs in two steps. First, we note that by [Lemma 4](#), welfare for any given  $\bar{b}$  depends on beliefs only through their impact on  $k^*$ . In the case with quadratic adjustment costs, welfare is a quadratic function of  $k^*(\bar{b}; \theta)$ . Thus, we can equivalently define the worst-case belief as the value of  $\theta$  that induces the largest deviation from the planner's preferred value of  $k^*$ . Moreover, noting that  $k^*$  is a linear function of the market valuation  $M(b; \theta)$ , we can write the worst case beliefs in terms of the largest mis-valuation of capital investments:

$$\hat{\theta}(\bar{b}) = \arg \max_{\theta \in \Theta} |M(\bar{b}; \theta) - M^P(\bar{b})|, \quad (23)$$

Therefore, robust optimal policy in our model depends on a simple sufficient statistic, namely, the largest possible distortion to market valuations. Second, to characterize the

<sup>29</sup>In the [Online Appendix](#), we further contrast this result with a case in which the planner cannot adjust her policy to equilibrium outcomes. In that case, the planner is relatively less informed, and her uncertainty about the joint realization of  $M(\bar{b}^*; \theta)$  and  $\frac{dk^*(\bar{b}; \theta)}{d\bar{b}}$  generate additional nuances for optimal policy.

deeper properties of this distortion, we analyze the first-order conditions that characterize worst-case beliefs. These can be written as

$$1 + \log m^I(s) = v^I + v^0 \beta^I \max \{s - \bar{b}, 0\} \quad (24)$$

$$1 + \log m^C(s) = \begin{cases} v^C + v^0 \beta^C \bar{b}, & s \geq \bar{b} \\ v^C + v^0 \beta^C \phi s, & s < \bar{b}, \end{cases} \quad (25)$$

where  $m^j(s)$  denotes the proportional distortion (i.e., a Radon-Nikodym derivative) of agent  $j$ 's beliefs relative to the planner's, and where  $v^0$ ,  $v^I$ , and  $v^C$  are Lagrange multipliers defined in the Online Appendix.

Equation (24) characterizes the worst-case distortion  $m^I(s)$  to investors' beliefs. For a given  $\bar{b}$ , investors' beliefs about default states with  $s < \bar{b}$  do not affect market valuations or capital investment. Hence, it is optimal in those states to implement a fixed distortion  $m^I(s) = e^{v^I - 1}$ .<sup>30</sup> The optimal distortion in solvent states, by contrast, scales with the value of investors' equity claim  $s - \bar{b}$ . Equation (25) characterizes the worst case for creditors' beliefs. Once again, in order to maximize capital distortions, these conditions imply that belief distortions are scaled with the value of creditors' claims in each (default or solvent) state of the world. In conclusion, we find that a planner gears robust optimal policy towards belief distortions that most heavily affect market valuations. The worst-case belief distortions correlate with the market valuations of debt and equity claims.

## 4.2 Monetary policy

In our baseline model, the natural interest rate in debt markets in our model is fixed at  $r^* = \frac{1 - \beta^C}{\beta^C}$ . For a reduced-form treatment of monetary policy, similar to [Farhi and Tirole \(2012\)](#), we now assume that the planner can distort the interest rate to  $r \neq r^*$  in addition to setting a leverage cap  $\bar{b}$ . The deadweight loss generated by interest rate distortions is a convex function  $\mathcal{L}(r) \geq 0$ , with  $\mathcal{L}(r^*) = 0$ .

We focus on the welfare effect of interest rate policy when beliefs are distorted. Writing  $W(\bar{b}, r)$  for welfare, we obtain the following marginal welfare benefit of varying interest

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<sup>30</sup>If the worst-case capital distortion in problem (23) features overinvestment with  $M(\bar{b}; \theta) > M^P(\bar{b})$ , then this fixed distortion satisfies  $m^I(s) < 1$ , and vice versa.

rates:<sup>31</sup>

$$\frac{\partial W(\bar{b}, r)}{\partial r} = [M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b}, r)] \frac{dk^*}{dr} - \mathcal{L}'(r), \quad (26)$$

where  $M(b, r)$  is the adjusted market value of investment, which we characterize in Online Appendix C.2. Equation (26) shows that the welfare effect of monetary policy depends on the valuation wedge  $M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b}, r)$  and the policy elasticity  $\frac{dk^*(\bar{b}, r)}{dr}$ . Unlike the policy elasticity with respect to leverage caps, on which we have focused above, this elasticity is always negative because an increase in  $r$  increases investors' cost of debt finance.

Importantly, optimism in either a debt or equity exuberance scenario implies that  $\frac{\partial W(\bar{b}, r)}{\partial r}$  becomes larger. This is the result of two forces. First, the valuation wedge becomes larger in absolute value. Second, increased optimism implies that the effect of the interest rate  $r$  on bond values is stronger when bond values are elevated, so that investment becomes more sensitive to monetary policy. Both forces increase the planner's incentive to raise (lower) interest rates in response to optimistic (pessimistic) beliefs.

This result stands in contrast to leverage caps, to which investment becomes *less* sensitive in an equity exuberance scenario. Therefore, monetary policy is a natural substitute when optimism blunts the effectiveness of leverage regulation. These results connect our paper to the literature on monetary policy as a prudential tool when leverage policy is constrained (e.g., Stein, 2013; Caballero and Simsek, 2019). We show that monetary policy can be useful by affecting investment, and is particularly effective in cases where the use of leverage caps is *endogenously* constrained by distorted beliefs.

### 4.3 Alternative micro-foundations for externalities

Our baseline model features a reduced-form externality  $\Delta(b)$ , which is meant to capture various spillovers from leveraged investment to the broader economy. Here, we consider two alternative micro-foundations, namely, government bailouts and fire sales externalities. In each case, we show that similar equations as in our baseline model describe the relevant welfare effects, and highlight the novel economic insights. We present formal derivations in Section C.3 of the Online Appendix.

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<sup>31</sup>Unlike in Proposition 5, there is no inframarginal effect associated with raising the interest rate in our model because changes in the value of debt are welfare-neutral transfers. Accordingly, Equation (26) exclusively contains an incentive effect.

### 4.3.1 Government bailouts

Assume that at date 1, after the state  $s$  is realized, the government makes a transfer  $t(b, s)$  to investors. The funds for this transfer are raised using a tax  $(1 + \kappa)t(b, s)$  on creditors, where  $\kappa > 0$  measures the marginal deadweight loss associated with taxation.<sup>32</sup> In this formulation, the externality  $\Delta(b)$  that investors impose on creditors per unit of capital is defined as follows:

$$\Delta(b) = (1 + \kappa) \beta^C \int_{\underline{s}}^{\bar{s}} t(b, s) dF^{C,P}(s). \quad (27)$$

In addition, one needs to adjust the market value  $M(b)$  of investment and the planner's equivalent valuation  $M^P(b)$ , taking into account that investors and creditors benefit from bailouts. Modulo this adjustment, which we derive in Online Appendix C.3.1, our characterizations of marginal welfare effects go through unchanged. Equation (18) continues to characterize how belief distortions affect the marginal social value of a leverage cap.

The novel economic effect of bailouts arises because they change investors' marginal incentives in response to belief distortions. In the appendix, we show how one can generalize the key variational derivatives  $\frac{\delta M(b)}{\delta F^j} \cdot G^j$  and  $\frac{\delta dM}{\delta F^j} \cdot G^j$  to account for this change. Intuitively, bailouts attenuate the importance of *downside* belief distortions, in which investors rely on implicit government support. Consequently, bailouts also increase the importance of belief distortions among investors relative to those among creditors.

We now focus on the instructive special case in which bailouts are perfectly targeted towards avoiding bankruptcy, so that  $t(b, s) = \max\{b - s, 0\}$ . This scenario corresponds to interpreting investors as “too big to fail” (TBTF). In this case, the valuation of (safe) debt is independent of creditors' beliefs  $F^C(s)$ . We obtain the following condition in terms of investors' beliefs:

$$\frac{\delta dW}{\delta F^I} \cdot G^I \leq 0 \iff \frac{G^I(b)}{\int_{s^*(b)}^{\bar{s}} G^I(s) ds} \leq \frac{\beta^C(1 + \kappa) - \beta^I F^{I,P}(b) + \beta^I F^I(b)}{\gamma(b) + \beta^I \int_{s^*(b)}^{\bar{s}} (F^{I,P}(s) - F^I(s)) ds}. \quad (28)$$

If the inequality in (28) holds, then a leverage cap is optimal when investors' beliefs are distorted in a TBTF scenario. Equation (28), which is the analogue to Equation (20) in the baseline model, shows that the incentive to cap leverage is stronger when the “downside” belief change in marginal bailout states  $G^I(b)$  is small relative to the overall “upside” belief change in solvent states  $\int_{s^*(b)}^{\bar{s}} G^I(s) ds$ . Intuitively, the inframarginal term in the marginal welfare effect of a leverage cap scales with the level of investment  $k^*(b)$ , which in

<sup>32</sup>If investors default, creditors seize all of the investors' resources — including any government transfer — receiving  $\phi s + t(b, s)$  per unit of investment. Similar results obtain if the tax is paid by investors.

this scenario is determined purely by investors' beliefs about upside states. Therefore, large upside optimism generates a strong incentive to constrain leverage. By contrast, changes in beliefs about downside/default states mostly result in a decreased sensitivity  $\frac{dk^*(b)}{db}$  of investment to leverage regulation. Thus, large downside optimism makes regulation less attractive at the margin.

### 4.3.2 Fire sales/Pecuniary externalities

Here we extend our model along the lines of [Lorenzoni \(2008\)](#) to introduce the possibility that investors must sell capital at depressed prices before it yields its final return. In this framework, which we describe in detail in Section C.3.2 of the Online Appendix, investors and creditors interact over three time periods  $t = 0, 1, 2$ . At the initial date  $t = 0$ , as in our baseline model, investors create  $k_0$  units of capital and sell bonds to creditors. At  $t = 1$ , investors face a deterministic reinvestment requirement  $\xi \geq 0$  per unit of capital, and have no further access to external finance. Hence, they must sell  $k_0 - k_1$  of their investments to outside buyers, whose technology and preferences we describe in detail in the appendix. At  $t = 2$ , any capital that has not been sold yields  $s$  per unit to investors. The beliefs about  $s$  held by investors, creditors, and the social planner are defined as in the baseline model.

At date 1, investors' remaining capital holdings  $k_1$  are given by

$$k_1 = \underbrace{\left(1 - \frac{\xi}{q}\right)}_{\equiv \lambda} k_0. \quad (29)$$

Equation (29) states that investors are able to retain a fraction  $\lambda$  of their capital, which is in turn an increasing function of the market price  $q$  of capital at date 1. If market prices of capital decrease at date 1, then a larger fraction of investors' capital must be sold, which generates a pecuniary externality. In equilibrium, for a given leverage cap  $\bar{b}$ , leverage  $b^*(\bar{b})$ , capital investments  $k_0^*(\bar{b})$  and the market price  $q^*$  (or equivalently the retained fraction  $\lambda^*(\bar{b})$ ) are determined simultaneously by investors' optimality condition in problem (31) and by market clearing at date 1.

In the Online Appendix, we characterize the equilibrium and derive that the marginal welfare effect of varying  $\bar{b}$  is given by

$$\frac{1}{\lambda} \frac{dW}{d\bar{b}} = \underbrace{\left[ \frac{dM^P(\bar{b})}{d\bar{b}} + M^P(\bar{b}) \frac{d \log \lambda^*}{d\bar{b}} \right]}_{\text{Inframarginal effect}} k_0^*(\bar{b}) + \underbrace{\left[ M^P(\bar{b}) - M(\bar{b}) \right]}_{\text{Incentive effect}} \frac{dk_0^*(\bar{b})}{d\bar{b}}, \quad (30)$$

where the valuations  $M(b)$  and  $M^P(b)$  are defined as in the baseline model.<sup>33</sup> Equation (30) has the same structure as in the baseline model, containing an inframarginal and an incentive effect. In the inframarginal effect, the reduced-form marginal externality  $-\frac{d\Delta(b)}{db}$  is measured by  $M^P(\bar{b})\frac{d\log\lambda^*}{d\bar{b}}$ , and is therefore determined by the general equilibrium response of fire sale prices to the planner's policy  $\bar{b}$ . This feature introduces  $\frac{d\log\lambda^*}{d\bar{b}}$  as an additional policy elasticity into the planner's problem. By contrast, the fire sales externality does not affect the incentive effect.<sup>34</sup>

A novel effect in this extension is that the policy elasticity  $\frac{d\log\lambda^*}{d\bar{b}}$  can change in response to belief distortions. In the appendix, we show that there is a one-to-one relationship between  $\lambda^*(\bar{b})$  and the market valuation  $M^*(\bar{b})$  of capital. This relationship is typically *decreasing*, since larger capital investments lead to more severe asset sales at date 1, meaning that a lower share  $\lambda$  can be retained by investors. Accordingly, the policy-elasticity  $\frac{d\log\lambda^*}{d\bar{b}}$  decreases whenever the marginal value  $\frac{dM^*(\bar{b})}{d\bar{b}}$  increases. Therefore, in an equity exuberance scenario,  $\frac{d\log\lambda^*}{d\bar{b}}$  increases as market valuations become less sensitive to leverage. *Ceteris paribus*, the inframarginal welfare effect becomes larger and reduces the welfare benefit of a binding leverage cap. By the converse argument, a binding leverage cap becomes (even) more attractive in a debt/joint exuberance scenario.

#### 4.4 Endogenous beliefs

In this section, we extend our baseline model by assuming that investors' and creditors' beliefs are represented by the distribution  $F^j(s; K)$ , which potentially depends on aggregate capital investment  $K$ . In equilibrium, we require consistency of these beliefs in the sense that  $K = k^*$ , where  $k^*$  is the investment chosen by an individual investor. However, individual investors take  $K$  as given when making decisions at date 0.

The characterization of the investors' problem in Lemma 1 is unchanged. However, equilibrium capital investments are now defined implicitly by

$$\Upsilon'(k^*) = M(b^*; k^*) - 1,$$

where  $M(b, K)$  denotes the usual market valuation of investments when beliefs depend

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<sup>33</sup>We define the total face value of investors' debt as  $b \cdot (\lambda k)$ , so that  $b$  is borrowing per unit of *retained* capital. With this modification, we show in the Online Appendix that investors solve the problem

$$\max_{b, k_0} [\lambda M(b) - 1] k_0 - \Upsilon(k_0) \text{ subject to } b \leq \bar{b}. \quad (31)$$

We can then show, in analogy to Lemma 4, that the planner solves the same problem with  $M(b)$  replaced by  $M^P(b)$ .

<sup>34</sup>This contrast is natural because agents correctly anticipate the *level* of fire sale prices in equilibrium but, being infinitesimal, neglect their collective marginal effect on prices.



on  $K$ . The key difference to the baseline model is that any shock that increases capital investments generates an amplified response through its effect on beliefs. For instance, whenever the leverage cap is binding, we have

$$\frac{dk^*}{d\bar{b}} = \underbrace{\left( \frac{1}{\Upsilon''(\cdot) - \frac{\partial M}{\partial K}} \right)}_{\equiv A} \frac{dM}{d\bar{b}},$$

where the multiplier  $A$  captures the amplification generated by endogenous belief responses, and is increasing in the sensitivity  $\frac{\partial M}{\partial K}$  of market valuations to aggregate investment.

In Section C.4 of the Online Appendix, we re-derive our main results in this setting. We again focus on how the marginal welfare effect  $\frac{dW}{db}$  of varying the leverage cap responds to a distortion in beliefs. An instructive special case is where market valuations are linear in  $K$  and adjustment costs are quadratic, which implies that the multiplier  $A$  is a constant.<sup>35</sup> In this case, we show that

$$\begin{aligned} \frac{\delta \frac{dW}{db}}{\delta F^j} \cdot G^j = A \left\{ \left[ \frac{dM^P(\bar{b})}{db} - \frac{d\Delta(\bar{b})}{db} - \frac{\partial M(\bar{b}; k^*)}{\partial b} \right] \left[ \frac{\delta M(\bar{b}; k^*)}{\delta F^j} \cdot G^j \right] \right. \\ \left. + \left[ M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b}) \right] \left[ \frac{\delta \frac{\partial M}{\partial b}}{\delta F^j} \cdot G^j \right] \right\}. \end{aligned} \quad (32)$$

Notice that Equation (32), which describes how  $\frac{dW}{db}$  responds to an arbitrary distortion in beliefs  $F^j$ , effect takes the same shape as in Equation (19). However, the whole expression is now multiplied by  $A = \frac{1}{\Upsilon''(\cdot) - \frac{\partial M}{\partial K}}$ , compared to  $\varphi = \frac{1}{\Upsilon''(\cdot)}$  in the baseline model. If beliefs respond optimistically to aggregate investment with  $\frac{\partial M}{\partial K} > 0$ , then the magnitude of welfare effects is amplified. Conversely, pessimistic responses dampen the welfare effects.

However, it is useful to note that the sign of the welfare effect, which determines whether leverage caps become attractive after belief distortions, continues to be determined by the sum of inframarginal and incentive effects, and is the same as in the baseline model. Therefore, the endogeneity of beliefs in Equation (32) affects only the magnitude of any response to belief distortions.

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<sup>35</sup>In the case where  $M(b; K)$  is not linear in  $K$ , which we characterize in the Appendix, we additionally account for the response of  $A$  to changes in beliefs. We show that this response works in favor of a leverage cap whenever  $\frac{\delta \frac{\partial M}{\partial K}}{\delta F^j} \cdot G^j > 0$ . For tractability, we assume that the distortion  $G^j$  applied to  $F^j(s; K)$  is invariant to the level of  $K \geq 0$ .

## 5 Conclusion

This paper characterizes leverage regulation and monetary policy in environments in which equity investors' and creditors' beliefs differ from the beliefs of a planner. We show that the optimal policy response to changes in beliefs depends on the type and the magnitude of the beliefs held by investors and creditors. We show that optimism by creditors, or jointly by both investors and creditors, is associated with a tighter optimal leverage cap, while optimism by investors is associated with loosening the optimal leverage cap, unless externalities associated with leveraged investments are sufficiently strong.

We also explore the role of changes in the planner's beliefs for the optimal policy response, and explore the properties of (robust) optimal policy when the planner has imperfect knowledge of agents' beliefs. Among various extensions, we further show that monetary tightening can act as a useful substitute for financial regulation since increased optimism by either equity investors or creditors is associated with higher incentives to raise interest rates. We also consider scenarios in which beliefs are endogenous.

There are many fruitful avenues for further research on the design of prudential policies with distorted beliefs. For instance, exploring many of the effects that we have identified in this paper in a dynamic quantitative framework that includes multiple rationales for regulation is one of the most promising ones. Lastly, we hope that the variational approach that we have introduced here to characterize the impact of general changes in beliefs can be used in other scenarios.

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## A Proofs and Derivations: Section 2

### Proof of Lemma 1 [Investors' problem]

The problem that investors face at date 0, after anticipating their optimal default decision, can be expressed as follows:

$$V(\bar{b}) = \max_{b, k, c_0^I, c_1^I(s)} c_0^I + \beta^I \int c_1^I(s) dF^I(s), \quad (33)$$

subject to budget constraints at date 0 and in each state  $s$  at date 1, the creditors' debt-pricing equation, the non-negativity constraint of consumption at date 0, and the leverage constraint set by the planner:

$$c_0^I + k + \Upsilon(k) = n_0^I + Q(b)k \quad (\lambda_0) \quad (34)$$

$$c_1^I(s) = n_1^I(s) + \max\{s - b, 0\}k, \forall s \quad (35)$$

$$Q(b) = \beta^C \left( \int_b^{\bar{s}} b dF^C(s) + \int_{\underline{s}}^b \phi s dF^C(s) \right) \quad (36)$$

$$c_0^I \geq 0 \quad (\eta_0) \quad (37)$$

$$bk \leq \bar{b}k \quad (\mu). \quad (38)$$

Note that the debt-pricing equation  $Q(b)$  is derived from the behavior of creditors. In Lagrangian form, this problem can be expressed as follows:

$$\begin{aligned} \mathcal{L}^I = & c_0^I + \beta^I \int_b^{\bar{s}} (s - b) dF^I(s) k - \lambda_0 (c_0^I - n_0^I - Q(b)k + k + \Upsilon(k)) \\ & + \eta_0 c_0^I + \mu k (\bar{b} - b). \end{aligned} \quad (39)$$

Equations (5) and (6), as well as the results in Lemma 1, follow directly from Equation (39) when  $\eta_0 = 0$ .

### Proof of Proposition 1 [Equilibrium leverage and investment]

Equation (8) follows from maximizing Equation (5) subject to Equation (6) (multiplied by  $k$ ). Equations (9) and (10) follow from differentiating Equation (5) with respect to  $k$  and (7) with respect to  $b$ . These conditions are necessary for optimality and sufficient under the regularity conditions described in the Online Appendix. It follows immediately from Equation (7) that a necessary regularity condition for investment to be positive is that  $M(b^*) > 1$ .

**Proof of Lemma 2 [Sensitivity of investment to leverage limit]**

Differentiating Equation (9) with respect to  $\bar{b}$  implies that  $\frac{dM}{db}(b^*) \frac{db^*}{db} = \Upsilon''(k^*) \frac{dk^*}{db}$ . Equation (11) follows immediately by rearranging this expression and noticing that  $\frac{db^*}{db} = 1$  when  $\mu > 0$  and  $\frac{db^*}{db} = 0$  when  $\mu = 0$ .

**Proof of Lemma 3 [Sensitivity of leverage and investment to beliefs]**

Note the first-order condition for leverage can be written as  $\frac{dM}{db}(b^*; F^I, F^C) = 0$ . An application of the implicit function theorem implies that

$$\frac{\delta b^*}{\delta F^I} \cdot G^I = \frac{\frac{\delta(\frac{dM}{db})}{\delta F^I} \cdot G^I}{-\frac{d^2M}{db^2}}.$$

The same approach applies when (variationally) differentiating with respect to  $F^C$ .

Similarly, the first-order condition for leverage can be written as  $\frac{dM}{db}(b^*; F^I, F^C) = 0$ . An application of the implicit function theorem implies that

$$\frac{\delta k^*}{\delta F^I} \cdot G^I = \frac{\frac{dM}{db} \left( \frac{\delta b^*}{\delta F^I} \cdot G^I \right) + \frac{\delta M}{\delta F^I} \cdot G^I}{\Upsilon''(k^*)} = \frac{\delta M}{\delta F^I} \cdot G^I.$$

Notice that this derivation exploits the fact that  $\frac{dM}{db} = 0$ . The same approach applies when (variationally) differentiating with respect to  $F^C$ .

**Proof of Proposition 2 [Variational derivatives]**

For completeness, we include here the counterparts of Equations (7) and (10), making explicit their dependence on  $F^I(s)$  and  $F^C(s)$ :

$$M(b; F^I, F^C) = \beta^I \int_b^{\bar{s}} (s - b) dF^I(s) + \beta^C \left( \int_b^{\bar{s}} b dF^C + \int_{\underline{s}}^b \phi s dF^C(s) \right)$$

$$\frac{dM(b; F^I, F^C)}{db} = \beta^C \int_b^{\bar{s}} dF^C(s) - (1 - \phi) \beta^C b f^C(b) - \beta^I \int_b^{\bar{s}} dF^I(s).$$

We compute  $\frac{\delta M}{\delta F^I} \cdot G^I$  as follows:

$$\begin{aligned} \frac{\delta M}{\delta F^I} \cdot G^I &\equiv \lim_{\varepsilon \rightarrow 0} \frac{M(b; F^I + \varepsilon G^I, F^C) - M(b; F^I, F^C)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\beta^I \int_b^{\bar{s}} (s-b) d(F^I(s) + \varepsilon G^I(s)) - \beta^I \int_b^{\bar{s}} (s-b) dF^I(s)}{\varepsilon} \\ &= \beta^I \int_b^{\bar{s}} (s-b) dG^I(s) = -\beta^I \int_b^{\bar{s}} G^I(s) ds. \end{aligned}$$

where the last equality follows after integrating by parts.

We compute  $\frac{\delta M}{\delta F^C} \cdot G^C$  as follows:

$$\begin{aligned} \frac{\delta M}{\delta F^C} \cdot G^C &\equiv \lim_{\varepsilon \rightarrow 0} \frac{M(b; F^I, F^C + \varepsilon G^C) - M(b; F^I, F^C)}{\varepsilon} \\ &= \beta^C \left( \int_b^{\bar{s}} b dG^C(s) + \int_{\underline{s}}^b \phi s dG^C(s) \right) \\ &= -\beta^C \left( (1-\phi) b G^C(b) + \phi \int_{\underline{s}}^b G^C(s) ds \right). \end{aligned}$$

where the last equality follows after integrating by parts.

We compute  $\frac{\delta \frac{dM}{db}}{\delta F^I} \cdot G^I$  as follows:

$$\frac{\delta \frac{dM}{db}}{\delta F^I} \cdot G^I = \lim_{\varepsilon \rightarrow 0} \frac{(-\beta^I \int_b^{\bar{s}} d(F^I + \varepsilon G^I)) - (-\beta^I \int_b^{\bar{s}} dF^I)}{\varepsilon} = \beta^I \left( -\int_b^{\bar{s}} dG^I(s) \right) = \beta^I G^I(b).$$

We compute  $\frac{\delta \frac{dM}{db}}{\delta F^C} \cdot G^C$  as follows:

$$\frac{\delta \frac{dM}{db}}{\delta F^C} \cdot G^C = \beta^C \left( \int_b^{\bar{s}} dG^C(s) - (1-\phi) b g^C(b) \right) = -\beta^C \left( G^C(b) \left( 1 + (1-\phi) b \frac{g^C(b)}{G^C(b)} \right) \right),$$

where the last equality follows after integrating by parts.

### Proof of Proposition 3 [Differential impact of optimism by investors and creditors]

a) From Lemma 3, it follows that  $\frac{\delta b^*}{\delta F^C} \cdot G^C$  and  $\frac{\delta k^*}{\delta F^C} \cdot G^C$  will have the same sign as  $\frac{\delta \left( \frac{dM}{db} \right)}{\delta F^C} \cdot G^C$  and  $\frac{\delta M}{\delta F^C} \cdot G^C$ , respectively. From Equations (13) and (15), if creditors are



optimistic in a hazard-rate sense,  $G^C(\cdot) \leq 0$ , so it is sufficient to show that

$$1 + (1 - \phi) b \frac{g^C(b)}{G^C(b)} \geq 0.$$

At an interior optimum with common beliefs, Equation (10) implies that

$$\frac{dM}{db} = \beta^C - \beta^I - \beta^C (1 - \phi) b \frac{f^C(b)}{1 - F^C(b)} = \mu \geq 0$$

or, equivalently,

$$\beta^C - \beta^I \geq \beta^C (1 - \phi) b \frac{f^C(b)}{1 - F^C(b)}.$$

As shown in Section D.4 of the Online Appendix, hazard-rate dominance implies that  $\frac{f^C(s)}{1 - F^C(s)} \geq -\frac{g^C(s)}{G^C(s)}$ , so the following relation holds:

$$\beta^C - \beta^I \geq -\beta^C (1 - \phi) b \frac{g^C(s)}{G^C(s)},$$

which implies that

$$1 + (1 - \phi) b \frac{g^C(s)}{G^C(s)} \geq \frac{\beta^I}{\beta^C} \geq 0.$$

It is then immediate that  $\frac{\delta(\frac{dM}{db})}{\delta F^C} \cdot G^C > 0$  and  $\frac{\delta M}{\delta F^C} \cdot G^C > 0$ , and therefore  $\frac{\delta b^*}{\delta F^C} \cdot G^C > 0$  and  $\frac{\delta k^*}{\delta F^I} \cdot G^I > 0$ .

b) From Lemma 3, it follows that  $\frac{\delta b^*}{\delta F^I} \cdot G^I$  and  $\frac{\delta k^*}{\delta F^I} \cdot G^I$  will have the same sign as  $\frac{\delta(\frac{dM}{db})}{\delta F^I} \cdot G^I$  and  $\frac{\delta M}{\delta F^I} \cdot G^I$ , respectively. From Equations (12) and (14), if investors are optimistic in a hazard-rate sense,  $G^I(\cdot) \leq 0$ , and it is immediate that  $\frac{\delta(\frac{dM}{db})}{\delta F^I} \cdot G^I < 0$  and  $\frac{\delta M}{\delta F^I} \cdot G^I > 0$ , and therefore  $\frac{\delta b^*}{\delta F^I} \cdot G^I < 0$  and  $\frac{\delta k^*}{\delta F^I} \cdot G^I > 0$ .

c) Suppose that  $F^C = F^I = F^0$ . Then the effect of joint exuberance on  $\frac{dM}{db}$  is

$$\begin{aligned} \frac{\delta\left(\frac{dM}{db}\right)}{\delta F^0} \cdot G &= \frac{\delta\left(\frac{dM}{db}\right)}{\delta F^I} \cdot G + \frac{\delta\left(\frac{dM}{db}\right)}{\delta F^C} \cdot G \\ &= -G(b) \left( \beta^C - \beta^I + \beta^C (1 - \phi) b \frac{g(b)}{G(b)} \right). \end{aligned}$$

Since optimism in a hazard-rate sense implies that  $G(b) \leq 0$ , we need to show that

$$\beta^C - \beta^I + \beta^C (1 - \phi) b \frac{g(b)}{G(b)} \geq 0.$$

At an interior optimum with common beliefs, Equation (10) implies that

$$\frac{dM}{db} = \beta^C - \beta^I - \beta^C (1 - \phi) b \frac{f^0(b)}{1 - F^0(b)} = \mu \geq 0,$$

or, equivalently,

$$\beta^C - \beta^I \geq \beta^C (1 - \phi) b \frac{f^0(b)}{1 - F^0(b)}.$$

As shown in Section D.4, hazard-rate dominance implies that  $\frac{f^0(s)}{1 - F^0(s)} \geq -\frac{g^0(s)}{G^0(s)}$ , so the following relation holds:

$$\beta^C - \beta^I \geq -\beta^C (1 - \phi) b \frac{g^0(s)}{G^0(s)},$$

which implies, as required, that

$$\beta^C - \beta^I + \beta^C (1 - \phi) b \frac{g^0(s)}{G^0(s)} \geq 0.$$

## B Proofs and Derivations: Section 3

### Proof of Lemma 4 [Planner's problem]

The planner's objective is given by the sum of investors' and creditors' expected utility. Formally, ignoring constant terms that depend only on endowments, we have  $W = u^{I,P} + u^{C,P}$ , where  $u^{I,P}$  and  $u^{C,P}$  are given by

$$\begin{aligned} u^{I,P} &= \left[ Q(b) - 1 + \beta^I \int_b^{\bar{s}} (s - b) dF^{I,P}(s) \right] k - \Upsilon(k) \\ u^{C,P} &= \left[ -Q(b) + \beta^C \left( \int_b^{\bar{s}} b dF^{C,P} + \int_{\underline{s}}^b \phi s dF^{C,P}(s) \right) \right] k - \Delta(b) k, \end{aligned}$$

which imply that

$$W = \left[ \underbrace{\beta^I \int_b^{\bar{s}} (s - b) dF^{I,P}(s) + \beta^C \left( \int_b^{\bar{s}} b dF^{C,P} + \int_{\underline{s}}^b \phi s dF^{C,P}(s) \right)}_{\equiv M^P(b)} - 1 - \Delta(b) \right] k - \Upsilon(k).$$

The results in Lemmas 4 follow immediately.

Note that a planner that exclusively values the welfare of investors simply maximizes  $u^{I,P}$ , taking as given  $Q(b)$  as defined in Equation (36). This is as if the planner decided to set  $F^{C,P}(s) = F^C(s)$ . This observation is useful when relating our results to the Hertz

scenario on page 27. A planner that assigns different welfare weights to investors and creditors simply maximizes a linear combination of  $u^{I,P}$  and  $u^{C,P}$ .

**Proof of Proposition 4 [Marginal welfare effect of varying the leverage cap]**

The result follows directly by totally differentiating the characterization of the planner's objective in Lemma 4, applying the envelope theorem, and noting that  $\frac{db^*}{db} = 1$  whenever the leverage cap is binding.

**Proof of Proposition 5 [Impact of beliefs on optimal regulation: General characterization]**

The variational derivative of Equation (16) with respect to beliefs  $F^j$  for  $j \in \{I, C\}$  is

$$\begin{aligned} \frac{\delta \frac{dW}{db}}{\delta F^j} \cdot G^j &= \left[ \frac{dM^P(\bar{b})}{db} - \frac{d\Delta(b)}{db} \right] \left[ \frac{\delta k^*(\bar{b})}{\delta F^j} \cdot G^j \right] \\ &+ \left[ M^P(\bar{b}) - \Delta(b) - M(\bar{b}) \right] \left[ \frac{\delta \frac{dk^*}{db}}{\delta F^j} \cdot G^j \right] - \left[ \frac{\delta M(\bar{b})}{\delta F^j} \cdot G^j \right] \frac{dk^*(\bar{b})}{d\bar{b}}. \end{aligned}$$

Notice that we can express optimal investment as  $k^*(\bar{b}) = \Psi(M(\bar{b}) - 1)$ , where  $\Psi(\cdot)$  is the inverse function of  $\Upsilon'(\cdot)$ . This implies that

$$\frac{\delta k(\bar{b})}{\delta F^j} \cdot G^j = \Psi'(\cdot) \left[ \frac{\delta M(\bar{b})}{\delta F^j} \cdot G^j \right] \quad \text{and} \quad \frac{dk(\bar{b})}{d\bar{b}} = \Psi'(\cdot) \frac{dM(\bar{b})}{db}.$$

Dividing these two expressions, we get

$$\frac{\frac{\delta k(\bar{b})}{\delta F^j} \cdot G^j}{\frac{dk(\bar{b})}{d\bar{b}}} = \frac{\frac{\delta M(\bar{b})}{\delta F^j} \cdot G^j}{\frac{dM(\bar{b})}{db}},$$

or equivalently,

$$\left[ \frac{\delta M(\bar{b})}{\delta F^j} \cdot G^j \right] \frac{dk^*(\bar{b})}{d\bar{b}} = \frac{dM(\bar{b})}{db} \left[ \frac{\delta k^*(\bar{b})}{\delta F^j} \cdot G^j \right].$$

Combining our results, we obtain the required expression in Equation (18).

Whenever the planner's objective is well-behaved in  $\bar{b}$ , establishing that  $\frac{\delta \frac{dW}{db}}{\delta F^j} \cdot G^j > 0$  guarantees that optimal leverage regulation involves a looser leverage cap. Formally, this

is the case whenever (i) the planner's objective is quasi-concave in  $\bar{b}$  and  $\frac{\delta dW}{\delta F^j} \cdot G^j > 0$  evaluated at the optimal (second-best) policy or (ii) welfare takes any shape and  $\frac{\delta dW}{\delta F^j} \cdot G^j > 0$  for all  $\bar{b}$ . As is standard in normative exercises, the planner's objective need not be quasi-concave without imposing additional restrictions on primitives — even though we find the problem to be well-behaved when simulating the model for standard functional forms and belief distributions.

**Proof of Proposition 6 [Impact of beliefs on optimal regulation: Specific scenarios]**

First, consider the debt and joint exuberance scenarios. By Proposition 3, debt or joint exuberance (in a hazard rate sense) increases  $M(b)$  and also the marginal value of leverage  $\frac{dM(b)}{db}$  for all  $b$ .

Moreover, we have

$$\frac{dM^P(\bar{b})}{db} < \frac{dM(\bar{b})}{db} \Rightarrow \left[ \frac{dM^P(\bar{b})}{db} - \frac{d\Delta(b)}{db} - \frac{dM(\bar{b})}{db} \right] < 0$$

and also

$$M^P(\bar{b}) < M(\bar{b}) \Rightarrow M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b}) < 0.$$

Moreover, for any marginal increase in debt/joint exuberance represented by the distortion  $G^j$ , we have

$$\frac{\delta M(\bar{b})}{\delta F^j} \cdot G^j > 0, \quad \frac{\delta \frac{dM}{db}}{\delta F^j} \cdot G^j > 0.$$

Substituting into Equation (19),<sup>36</sup> we obtain that

$$\begin{aligned} \frac{\delta dW}{\delta F^j} \cdot G^j &= \varphi \cdot \left[ \underbrace{\frac{dM^P(\bar{b})}{d\bar{b}} - \frac{d\Delta(b)}{d\bar{b}} - \frac{dM(\bar{b})}{d\bar{b}}}_{<0} \right] \left[ \underbrace{\frac{\delta M(\bar{b})}{\delta F^j} \cdot G^j}_{>0} \right] \\ &+ \varphi \cdot \left[ \underbrace{M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b})}_{<0} \right] \left[ \underbrace{\frac{\delta \frac{dM}{db}}{\delta F^j} \cdot G^j}_{>0} \right] < 0, \end{aligned}$$

as required.

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<sup>36</sup>In general, note that  $\frac{\delta \frac{dW}{db}}{\delta F^j} \cdot G^j = \Psi'(\cdot) \frac{\delta \frac{dM}{db}}{\delta F^j} \cdot G^j + \Psi''(\cdot) \frac{\delta M(\bar{b})}{\delta F^j} \cdot G^j$ . When adjustment costs are quadratic,  $\Psi'(\cdot) = \varphi$  — a scalar defined in the text — and  $\Psi''(\cdot) = 0$ .

Second, consider the equity exuberance scenario. Repeating our steps above, we find that

$$\begin{aligned} \frac{\delta \frac{dW}{db}}{\delta F^j} \cdot G^j &= \varphi \cdot \left[ \underbrace{\frac{dM^P(\bar{b})}{d\bar{b}} - \frac{d\Delta(\bar{b})}{d\bar{b}} - \frac{dM(\bar{b})}{d\bar{b}}}_{\leq 0, \equiv A} \right] \left[ \underbrace{\frac{\delta M(\bar{b})}{\delta F^j} \cdot G^j}_{> 0, \equiv B} \right] \\ &+ \varphi \cdot \left[ \underbrace{M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b})}_{< 0, \equiv C} \right] \left[ \underbrace{\frac{\delta \frac{dM}{db}}{\delta F^j} \cdot G^j}_{< 0, \equiv D} \right]. \end{aligned}$$

and we can write

$$\frac{\delta \frac{dW}{db}}{\delta F^j} \cdot G^j > 0 \Leftrightarrow AB + CD > 0 \Leftrightarrow A > -C \frac{D}{B},$$

which is equivalent to

$$\frac{dM^P(\bar{b})}{d\bar{b}} - \frac{d\Delta(\bar{b})}{d\bar{b}} - \frac{dM(\bar{b})}{d\bar{b}} > - \left[ M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b}) \right] \frac{\frac{\delta \frac{dM}{db}}{\delta F^j} \cdot G^j}{\frac{\delta \frac{dM}{db}}{\delta F^j} \cdot G^j}.$$

The results in this proposition follow directly by combining the comparative statics in Propositions 3 with the general characterization in Proposition 5 and Equation (19). Our conclusions about optimal policy follow directly, because those results provide signs for  $\frac{\delta \frac{dW}{db}}{\delta F^j} \cdot G^j > 0$  for all  $\bar{b}$ .

**Proof of Proposition 7 [Impact of the planner's beliefs on optimal regulation: Specific scenarios]**

Each case in the proposition holds constant the beliefs of creditors and investors. Hence, the terms  $M(\bar{b}) > 0$ ,  $k^*(\bar{b}) > 0$ , and  $\frac{dk^*(\bar{b})}{d\bar{b}} \geq 0$  in the marginal welfare effect  $\frac{dW}{db}$  (see Proposition 4) are also held fixed. It is then clear that  $\frac{dW}{db}$  is increasing in the planner's marginal value  $\frac{dM^P(\bar{b})}{d\bar{b}}$  of leverage and weakly increasing in the planner's valuation  $M^P(\bar{b})$  of investment.

By a parallel argument to Proposition 3, it follows that (i)  $M^P(\bar{b})$  increases with the planner's equity exuberance, debt exuberance, and joint exuberance, and that (ii)  $\frac{dM^P(\bar{b})}{d\bar{b}}$  increases with the planner's debt exuberance or joint exuberance but decreases with the planner's equity exuberance. This establishes the claims in the proposition.

# ONLINE APPENDIX

## FOR ONLINE PUBLICATION ONLY

### C Extensions

#### C.1 Imperfect knowledge of beliefs

##### C.1.1 Optimal policy conditional on endogenous investment

For technical consistency, we assume here that  $\Theta$  is a closed set of pairs of continuous cumulative distribution functions. We can equip this set with the sup norm to make it a metric space, meaning that we define probability measures over the Borel sets in  $\Theta$ , including the planner's probability measure  $\mu$ .

Let  $I(\bar{b}, \theta) = \{\tilde{\theta} \in \Theta : k^*(\bar{b}, \tilde{\theta}) = k^*(\bar{b}, \theta)\}$  be the set of beliefs under which the equilibrium response to policy  $\bar{b}$  is the same as under  $\theta$ . For any given  $\theta$ , a consistent policy  $\bar{b}^*$  has to solve the following fixed point problem:

$$\bar{b}^* \in \arg \max_{\bar{b}} \mathbb{E}_{\mu} \left[ W(\bar{b}, k^*(\bar{b}, \tilde{\theta})) \middle| I(\bar{b}^*, \theta) \right]. \quad (40)$$

For any interior solution to the maximization problem in (40), we have the first-order condition

$$\mathbb{E}_{\mu} \left[ \left[ \frac{dM^P(\bar{b})}{db} - \frac{d\Delta(\bar{b})}{db} \right] k^*(\bar{b}; \tilde{\theta}) + [M^P(\bar{b}) - \Delta(b) - M(\bar{b}; \tilde{\theta})] \frac{dk^*(\bar{b}; \tilde{\theta})}{d\bar{b}} \middle| I(\bar{b}^*, \theta) \right] = 0. \quad (41)$$

A consistent policy  $\bar{b}^*$  must also satisfy this condition, and substituting, we obtain

$$\mathbb{E}_{\mu} \left[ \left[ \frac{dM^P(\bar{b})}{db} - \frac{d\Delta(\bar{b})}{db} \right] k^*(\bar{b}^*; \tilde{\theta}) + [M^P(\bar{b}) - \Delta(b) - M(\bar{b}^*; \tilde{\theta})] \frac{dk^*(\bar{b}^*; \tilde{\theta})}{d\bar{b}} \middle| I(\bar{b}^*, \theta) \right] = 0.$$

Equation (21) follows by noting that  $k^*(\bar{b}^*; \tilde{\theta})$  is a constant conditional on the information set  $I(\bar{b}^*, \theta)$ , and that the mapping between capital and  $M(\bar{b}^*; \tilde{\theta})$  is invertible, so that we can remove all terms except  $\frac{dk^*(\bar{b}; \tilde{\theta})}{d\bar{b}}$  from the expectation operator — note that  $\frac{dM^P(\bar{b})}{db}$  and  $M^P(\bar{b})$  depend on the planner's beliefs, not on investors' or creditors' beliefs.

**Alternative policy problem 1/No conditioning on endogenous investment:**  
The marginal welfare effect of varying  $\bar{b}$  when the planner does not observe realized

investment is given by the equivalent of (41) without conditioning on any information:

$$\begin{aligned}\mathbb{E}_\mu \left[ \frac{dW}{d\bar{b}} \right] &= \mathbb{E}_\mu \left[ \left[ \frac{dM^P(\bar{b})}{db} - \frac{d\Delta(\bar{b})}{db} \right] k^*(\bar{b}; \tilde{\theta}) + [M^P(\bar{b}) - \Delta(b) - M(\bar{b}; \tilde{\theta})] \frac{dk^*(\bar{b}; \tilde{\theta})}{d\bar{b}} \right] \\ &= \left[ \frac{dM^P(\bar{b})}{db} - \frac{d\Delta(\bar{b})}{db} \right] \mathbb{E}_\mu [k^*(\bar{b}; \tilde{\theta})] + [M^P(\bar{b}) - \Delta(b) - \mathbb{E}_\mu [M(\bar{b}; \tilde{\theta})]] \mathbb{E}_\mu \left[ \frac{dk^*(\bar{b}; \tilde{\theta})}{d\bar{b}} \right] \\ &\quad - \text{Cov}_\mu \left[ M(\bar{b}; \tilde{\theta}), \frac{dk^*(\bar{b}; \tilde{\theta})}{d\bar{b}} \right].\end{aligned}$$

We note that the marginal benefit of varying  $\bar{b}$  does not have a certainty equivalent property, because the planner must consider the covariance between agents' valuation of investment  $M(\bar{b}; \tilde{\theta})$  and the policy elasticity  $\frac{dk^*(\bar{b}; \tilde{\theta})}{d\bar{b}}$ . Intuitively, it is more valuable to impose a binding leverage cap if beliefs that induce overinvestment also induce a high policy elasticity. For instance, if  $\theta$  is a scalar parameter so that a higher  $\theta$  shifts creditors' beliefs in the sense that of hazard rate dominance, then Proposition 3 implies that this covariance is positive, meaning that uncertain debt exuberance further strengthens the case for imposing a binding leverage cap. By contrast, if  $\theta$  is a parameter that shifts (equity) investors' beliefs in a hazard rate sense, then this covariance is negative, which makes a binding leverage cap (even) less attractive.

**Alternative policy problem 2/Commitment:** [Hauk, Lanteri and Marcet \(2021\)](#) study a planner who maximizes expected welfare  $\mathbb{E}[W(\tau, s, A)]$ , where  $\tau$  is a scalar policy variable,  $s = h(\tau, A)$  is a quantity/signal determined in equilibrium, and  $A$  is the state of the economy. The planner observes  $s$  but not  $A$ , and commits to a mapping  $\tau = \mathcal{R}(s)$  based on the observable signal. In our setting, we can map  $\tau \rightarrow \bar{b}$  to the (binding) leverage requirement,  $s \rightarrow k^*$  to equilibrium capital investment, and  $A \rightarrow \theta = \{F^I, F^C\}$  to any uncertainty about agents' beliefs. Under appropriate regularity conditions, a variational argument in Proposition 2 in [Hauk, Lanteri and Marcet \(2021\)](#) implies that an optimal commitment satisfies the following first-order condition:

$$\mathbb{E} \left[ \frac{\frac{\partial W}{\partial b} k^* + \frac{\partial W}{\partial k} \frac{dk^*}{db}}{1 - \frac{dk^*}{db} \mathcal{R}'(k^*)} \bigg| k^* \right] = 0.$$

Evaluating the derivatives, we obtain

$$\mathbb{E} \left[ \frac{\left( \frac{dM^P(\bar{b})}{db} - \frac{d\Delta(\bar{b})}{db} \right) k^* + \left( M^P(\bar{b}) - \Delta(b) - M(\bar{b}; \tilde{\theta}) \right) \frac{dk^*}{d\bar{b}}}{1 - \frac{dk^*}{db} \mathcal{R}'(k^*)} \bigg| k^* \right] = 0,$$

where all terms inside the expectation are evaluated at  $\bar{b} = \mathcal{R}(k^*)$ . Since the first term in the numerator is known conditional on  $k^*$ , and defining the random variable  $\Omega = \left(1 - \frac{dk^*}{db} \mathcal{R}'(k^*)\right)^{-1}$ , we can re-write this condition as

$$\begin{aligned} & \mathbb{E}[\Omega | k^*] \left( \frac{dM^P(\bar{b})}{db} - \frac{d\Delta(\bar{b})}{db} \right) k^* \\ & + \mathbb{E}[\Omega | k^*] \mathbb{E} \left[ \left( M^P(\bar{b}) - \Delta(b) - M(\bar{b}; \hat{\theta}) \right) \frac{dk^*}{d\bar{b}} \middle| k^* \right] \\ & + \text{Cov} \left[ \Omega, \left( M^P(\bar{b}) - \Delta(b) - M(\bar{b}; \hat{\theta}) \right) \frac{dk^*}{d\bar{b}} \right] = 0. \end{aligned}$$

Noting that  $M(\bar{b}; \hat{\theta})$  is also known conditional on  $k^*$ , we can further simplify to obtain

$$\begin{aligned} & \left( \frac{dM^P(\bar{b})}{db} - \frac{d\Delta(\bar{b})}{db} \right) k^* \\ & + \left( M^P(\bar{b}) - \Delta(b) - M(\bar{b}; \hat{\theta}) \right) \left\{ \mathbb{E} \left[ \frac{dk^*}{d\bar{b}} \middle| k^* \right] + \text{Cov} \left[ \hat{\Omega}, \frac{dk^*}{d\bar{b}} \right] \right\} = 0, \end{aligned}$$

where  $\hat{\Omega} = \frac{\Omega}{E[\Omega | k^*]}$ . This condition is the same as Equation (21), except for the final term, which depends on  $\text{Cov} \left[ \Omega, \frac{dk^*}{d\bar{b}} \right]$ . As discussed in more detail in [Hauk, Lanteri and Marcet \(2021\)](#),  $\Omega$  corrects the first-order condition for the change in the probability distribution of  $k^*$  when the planner alters her ex-ante commitment. If this change of measure is correlated with policy elasticities, it introduces an additional marginal welfare effect. By contrast, a planner in a “consistent” equilibrium, in the sense that we discuss in the text, does not take this change into account since she is able to adjust her policy *ex-post*, after  $k^*$  is realized.

### C.1.2 Robust optimal policy

We formally define the set of plausible beliefs  $\Theta$  in the planner’s problem (22) as follows:

$$\Theta = \left\{ \left( m^I, m^C \right) : \sum_{j \in \{I, C\}} \mathbb{E}^{j, P} \left[ m^j \log m^j \right] \leq \mathcal{D}, \mathbb{E}^{j, P} \left[ m^j \right] = 1 \right\}.$$

This set contains all beliefs such that the sum of relative entropies across investors and creditors, relative to the planner’s beliefs, is at most  $\mathcal{D}$ . The worst-case beliefs in problem



(23) must also solve the dual problem

$$\begin{aligned} \min_{\theta} \sum_{j \in \{I, C\}} \mathbb{E}^{j, P} \left[ m^j \log m^j \right] \text{ subject to} \\ k^* (\bar{b}; \theta) = \mathcal{K}, \\ \mathbb{E}^{j, P} \left[ m^j \right] = 1. \end{aligned}$$

where  $\mathcal{K}$  is the value of  $k^* (\bar{b}; \theta)$  that is achieved by the maximum in problem (23). Equivalently, noting that there is a one-to-one relationship between  $k^* (\bar{b}; \theta)$  and the market valuation  $M (\bar{b}; \theta)$ , and expanding the expected values, we can write this problem as

$$\begin{aligned} \min_{\theta} \sum_{j \in \{I, C\}} \int_{\underline{s}}^{\bar{s}} m^j (s) \log m^j (s) dF^{j, P} (s) \text{ subject to} \\ \beta^I \int_b^{\bar{s}} (s - b) m^I (s) dF^{I, P} (s) \\ + \beta^C \left( \int_b^{\bar{s}} b m^C (s) dF^{C, P} (s) + \phi \int_{\underline{s}}^b s m^C (s) dF^C (s) \right) = \mathcal{M}, \quad (v^0) \\ \int_{\underline{s}}^{\bar{s}} m^I (s) dF^{I, P} (s) = 1, \quad (v^I) \\ \int_{\underline{s}}^{\bar{s}} m^C (s) dF^{C, P} (s) = 1, \quad (v^C), \end{aligned}$$

where the variables in brackets denote the relevant Lagrange multipliers. Taking the first-order conditions yields Equations (24) and (25) in the text.

## C.2 Monetary policy

In the model with monetary policy described in Section 4.2, investors' problem is equivalent to the proof of Lemma 1, except that the price of debt is given by

$$Q (b, r) = \beta (r) \left( \int_b^{\bar{s}} b dF^C (s) + \phi \int_{\underline{s}}^b s dF^C (s) \right),$$

with  $\beta^C$  now given by  $\beta (r) = \frac{1}{1+r}$ . All our results in Section 2 therefore apply once we replace the market value of investment  $M (b)$  in investors' objective function with

$$M (b, r) = \beta^I \int_b^{\bar{s}} (s - b) dF^I (s) + \beta (r) \left( \int_b^{\bar{s}} b dF^C (s) + \phi \int_{\underline{s}}^b s dF^C (s) \right).$$

A parallel argument to Lemma 4, taking into account the deadweight loss  $\mathcal{L}(r)$  of interest rate distortions, establishes that the planner's problem is

$$\max_{\bar{b}, r} W\left(b^*(\bar{b}, r), k^*(\bar{b}, r), r\right),$$

where the welfare function is

$$W(b, k, r) = \left[ M^P(b, r) - \Delta(b) - 1 \right] k - \Upsilon(k) - \mathcal{L}(r).$$

Taking the total derivative with respect to  $r$  yields Equation (26) in the paper. Finally, totally differentiating investors' first-order conditions yields

$$\frac{dk^*(\bar{b}, r)}{dr} = \frac{1}{\Upsilon''(k^*)} \frac{dM(\bar{b}, r)}{dr} = \frac{1}{\Upsilon''(k^*)} \frac{\beta'(r)}{\beta(r)} Q(b, r),$$

which for a given value of  $r$  is increasing in the bond price  $Q(b, r)$ . We conclude that, as claimed in the text,  $\frac{dk^*(\bar{b}, r)}{dr}$  is not diminished by equity exuberance (which leaves  $Q(b, r)$  unchanged), and is increased by debt exuberance (which increases  $Q(b, r)$ ).

### C.3 Alternative micro-foundations for externalities

#### C.3.1 Ex-post government bailouts

With government bailouts, investors' problem is equivalent to the proof of Lemma 1, except for investors' date 1 budget constraints and the equilibrium price of debt, which are now given by

$$\begin{aligned} c_1^I(s) &= n_1^I(s) + \max\{s + t(b, s) - b, 0\} k, \forall s \\ Q(b) &= \beta^C \left( \int_{s^*(b)}^{\bar{s}} b dF^C(s) + \int_{\underline{s}}^{s^*(b)} (\phi s + t(b, s)) dF^C(s) \right), \end{aligned}$$

where  $s = s^*(b)$  solves the equation  $s + t(b, s) - b = 0$ , and is uniquely defined as long as  $t(b, s)$  is decreasing in  $s$  and increasing in  $b$ . All our results in Section 2 therefore apply once we replace the market value of investment  $M(b)$  in investors' objective function with

$$M(b) = \beta^I \int_{s^*(b)}^{\bar{s}} (s + t(b, s) - b) dF^I(s) + \beta^C \left( \int_{s^*(b)}^{\bar{s}} b dF^C + \int_{\underline{s}}^{s^*(b)} (\phi s + t(b, s)) dF^C(s) \right).$$

A parallel argument to Lemma 4 establishes that the planner's problem is

$$\max_{\bar{b}} W \left( b^* (\bar{b}), k^* (\bar{b}) \right),$$

where the welfare function is

$$W (b, k) = \left[ M^P (b) - \Delta (b) - 1 \right] k - \Upsilon (k),$$

with the externality  $\Delta (b)$  defined in Equation (27), and with the planner's valuation of investment defined as

$$\begin{aligned} M^P (b) &= \beta^I \int_{s^*(b)}^{\bar{s}} (s + t (b, s) - b) dF^{I,P} (s) \\ &+ \beta^C \left( \int_{s^*(b)}^{\bar{s}} b dF^{C,P} + \int_{\underline{s}}^{s^*(b)} (\phi s + t (b, s)) dF^{C,P} (s) \right). \end{aligned}$$

With these alternative definitions, we can repeat the steps leading to Propositions 4 and 5 to establish that the marginal welfare effects are exactly as described in Equations (16) and (18).

Finally, taking variational derivatives and integrating by parts, as in Lemma 3, we obtain the effect of belief distortions on market valuations as

$$\begin{aligned} \frac{\delta M}{\delta F^I} \cdot G^I &= \beta^I \int_{s^*(b)}^{\bar{s}} (s + t (b, s) - b) dG^I (s) \\ &= -\beta^I \int_{s^*(b)}^{\bar{s}} \left( 1 + \frac{\partial t (b, s)}{\partial s} \right) G^I (s) ds, \end{aligned}$$

and

$$\begin{aligned} \frac{\delta M}{\delta F^C} \cdot G^C &= \beta^C \left( \int_{s^*(b)}^{\bar{s}} b dG^C (s) + \int_{\underline{s}}^{s^*(b)} (\phi s + t (b, s)) dG^C (s) \right) \\ &= -\beta^C \left( (1 - \phi) s^* (b) G^C (s^* (b)) + \int_{\underline{s}}^{s^*(b)} \left( \phi + \frac{\partial t (b, s)}{\partial s} \right) G^C (s) ds \right). \end{aligned}$$

These expressions show that, if bailouts satisfy  $\frac{\partial t (b, s)}{\partial s} \leq 0$ , then the presence of bailouts attenuates the sensitivity of the market valuation  $M (b)$  to the changes in beliefs  $G^I (s)$  and  $G^C (s)$ . Moreover, if bailouts are convex in  $s$ , so that  $\frac{\partial t (b, s)}{\partial s}$  is larger in absolute value for low  $s$ , then the attenuation effect is skewed towards belief distortions in bad states. Intuitively, bailouts imply that agents' beliefs about downside risk become less important for market valuation.

We further obtain the effect of belief distortions on the marginal value of leverage as

$$\begin{aligned} \frac{\delta \frac{dM}{db}}{\delta F^I} \cdot G^I &= \beta^I \left( - \int_{s^*(b)}^{\bar{s}} dG^I(s) + \int_{s^*(b)}^{\bar{s}} \frac{\partial t(b, s)}{\partial b} dG^I(s) \right) \\ &= \beta^I \left( \left( 1 - \frac{\partial t(b, s^*(b))}{\partial b} \right) G^I(s^*(b)) - \int_{s^*(b)}^{\bar{s}} \frac{\partial^2 t(b, s)}{\partial b \partial s} G^I(s) ds \right), \end{aligned}$$

and

$$\begin{aligned} \frac{\delta \frac{dM}{db}}{\delta F^C} \cdot G^C &= -\beta^C G^C(s^*(b)) \frac{\partial s^*(b)}{\partial b} \left( 1 + \frac{\partial t(b, s^*(b))}{\partial s} + (1 - \phi) s^*(b) \frac{g^C(s^*(b))}{G^C(s^*(b))} \right) \\ &\quad - \beta^C \int_0^{s^*(b)} \frac{\partial^2 t(b, s)}{\partial b \partial s} G^C(s) ds. \end{aligned}$$

If bailouts satisfy  $\frac{\partial t(b, s)}{\partial b} \geq 0$ , then the effect of changes in beliefs over the marginal default state  $s^*(b)$  on the marginal valuation  $\frac{dM}{db}$  is attenuated towards zero by the presence of bailouts. In addition, both variational derivatives of  $\frac{dM}{db}$  contain a term with the sign of  $-\frac{\partial^2 t(b, s)}{\partial b \partial s} G^j(s)$  for  $j \in \{I, C\}$ . These terms arise because changes in beliefs affect investors' strategic incentive to take on leverage in order to increase bailouts. If the strategic incentive  $\frac{\partial t(b, s)}{\partial b}$  is decreasing in  $s$ , then optimism increases  $\frac{dM}{db}$ . Bailouts are often modeled as a convex function of the shortfall  $b - s$  of asset values from debt obligations. This directly implies  $\frac{\partial^2 t(b, s)}{\partial b \partial s} \leq 0$ .

### C.3.2 Fire sales/Pecuniary externalities

**Detailed description of the environment:** We consider an extension of our baseline model with three time periods  $t = 0, 1, 2$ . There are three type of agents: Investors, creditors and households. Investors' and creditors' preferences are given by

$$\begin{aligned} U^I &= c_0^I + \beta^I \mathbb{E}^I [c_1^I + c_2^I] \\ U^C &= c_0^C + \beta^C \mathbb{E}^C [c_1^C + c_2^C]. \end{aligned}$$

Investors' budget constraints at each date are

$$\begin{aligned} c_0^I &= n_0^I - k_0 - \Upsilon(k_0) + Q(b, \lambda) k_0 \\ c_1^I &= q(k_0 - k_1) - \xi k_0 \\ c_2^I &= \begin{cases} sk_1 - b\lambda k_0, & s \in \mathcal{N} \\ 0, & s \in \mathcal{D}. \end{cases} \end{aligned}$$

Investors are endowed with  $n_0^I$  at date 0. They raise  $Q(b, \lambda) k_0$  at date 0 from creditors, where  $\lambda = 1 - \frac{\xi}{q}$  represents the market price of capital at date 1, which investors take as given. This finances their expenditure on capital and consumption. At date 1, investors have no endowment and sell  $k_0 - k_1$  units of capital and pay the reinvestment requirement  $\xi$  per unit of  $k_0$ . At date 2, in non-default states (denoted  $s \in \mathcal{N}$ ), investors consume the difference between the value of their remaining capital and the face value of their bonds. As described in the text, we write  $b$  for the normalized face per unit of  $\lambda k$ .

Creditors' budget constraints are

$$\begin{aligned} c_0^C &= n_0^C - hQ(b, \lambda) k_0 \\ c_1^C &= 0 \\ c_2^C &= h \begin{cases} b\lambda k_0, & s \in \mathcal{N} \\ \phi s k_1, & s \in \mathcal{D}. \end{cases} \end{aligned}$$

Creditors buy a fraction  $h$  of investors' debt at date 0. They have no endowment or consumption at date 1. At date 2, they are repaid the face value of their debt if investors do not default, and extract a fraction  $\phi$  of the value of remaining capital otherwise.

Households are active at dates 1 and 2. Their preferences are given by

$$U^H = c_1^H + \mathbb{E}^H [c_2^H].$$

Their budget constraints are

$$\begin{aligned} c_1^H &= n_1^H - qk_1^H \\ c_2^H &= F(k_1^H). \end{aligned}$$

Households have an endowment at date 1, which they can spend on purchasing capital  $k_1^H$ . Capital purchases yield consumption  $F(k_1^H)$  at date 2. We assume that  $F(k_1^H)$  is concave and satisfies  $F'(0) < (1 - \phi)\mathbb{E}^I[s]$ . The latter inequality is a sufficient condition ensuring that, in equilibrium, investors will never sell more capital to households than is necessary to cover their reinvestment need.

**Equilibrium characterization:** At date 1, the non-negativity constraint for investors' consumption always binds. Setting  $c_1^I = 0$ , we obtain

$$k_1 = \left(1 - \frac{\xi}{q}\right) k_0 \equiv \lambda k_0.$$

This implies that investors' asset values at date 2 are

$$sk_1 = \lambda sk_0.$$

Optimally, the following condition therefore determines investors' default choice:

$$\begin{aligned} s \in \mathcal{D} &\Leftrightarrow sk_1 < b\lambda k_0 \\ &\Leftrightarrow s < b. \end{aligned}$$

By creditors' optimality conditions, the price of debt at date 0, per unit of  $k_0$ , satisfies

$$\begin{aligned} Q(b, \lambda) &= \beta^C \left[ \int_b^{\bar{s}} \lambda b dF^C(s) + \phi \int_{\underline{s}}^b \lambda s dF^C(s) \right] \\ &= \lambda \beta^C \left[ \int_b^{\bar{s}} b dF^C(s) + \phi \int_{\underline{s}}^b s dF^C(s) \right]. \end{aligned}$$

Following the steps leading to Lemma 1 in the baseline model, we now find that investors' problem can be re-written as

$$\max_{b, k_0} [\lambda M(b) - 1] k_0 - \Upsilon(k_0) \text{ subject to } b \leq \bar{b},$$

where  $M(b)$  is defined as in Equation (5). Investors' first-order conditions determining the optimal choice of  $b$  and  $k_0$  are therefore

$$\lambda \frac{dM(b)}{db} = \mu \tag{42}$$

$$\lambda M(b) - 1 = \Upsilon'(k_0). \tag{43}$$

In addition, households' first-order condition for capital purchases, which is given by

$$q = F'(k_1^H),$$

and substituting the market clearing requirement that  $k_1^H = k_0 - k_1$ , we obtain the equilibrium relationship

$$q = F'(k_0 - k_1) = F'\left(\frac{\xi}{q} k_0\right) = F'((1 - \lambda) k_0).$$

Using the definition  $1 - \frac{\xi}{q} = \lambda$ , we can write  $q = \frac{\xi}{1 - \lambda}$ , which implies that

$$\xi = (1 - \lambda) F'((1 - \lambda) k_0). \tag{44}$$

For any given  $\bar{b}$ , equilibrium is obtained by solving equations (42), (43) and (44) for the unknown variables  $k_0 = k_0^*(\bar{b})$ ,  $b = b^*(\bar{b})$  and  $\lambda = \lambda^*(\bar{b})$ . We can further write (43) as

$$k_0 = \Psi(\lambda M(b) - 1),$$

where  $\Psi(\cdot)$  is the inverse of  $\Upsilon'(\cdot)$ , and substitute into (44) to obtain

$$\xi = (1 - \lambda)F'((1 - \lambda)\Psi(\lambda M(b) - 1)),$$

This equation defines an implicit mapping from market valuations  $M(b)$  to equilibrium prices  $\lambda$ . Let  $\lambda = \Lambda(M(b))$  denote this mapping. Whenever the leverage cap is binding, we can now write equilibrium prices as

$$\lambda^*(\bar{b}) = \Lambda(M(\bar{b})), \quad (45)$$

which we will employ in our welfare analysis below.

**Welfare analysis/Derivation of Equation (30):** The planner maximizes the sum of agents' utilities in equilibrium. Ignoring exogenous endowments and repeating the steps leading to Lemma 4, we can write and simplify welfare as follows:

$$\begin{aligned} W &= U^I + U^C + U^H \propto [\lambda M^P(b) - 1] k_0 - \Upsilon(k_0) + F(k_1^H) - qk_1^H \\ &= [\lambda M^P(b) - 1] k_0 - \Upsilon(k_0) + F(k_1^H) - \frac{\xi}{1 - \lambda} k_1^H \\ &= [\lambda M^P(b) - 1] k_0 - \Upsilon(k_0) + F((1 - \lambda)k_0) - \xi k_0 \\ &\equiv W(b, k_0, \lambda). \end{aligned}$$

where the third line follows by substituting the market clearing condition  $k_1^H = k_0 - k_1$ . Notice that we can write

$$\begin{aligned} \frac{\partial W}{\partial \lambda} &= M^P(b) k_0 + \underbrace{\left\{ F'(k_1^H) - q \right\}}_{=0} \frac{dk_1^H}{d\lambda} \\ &= M^P(b) k_0, \end{aligned}$$

where the second line follows by substituting households' first-order condition for capital purchases. The planner sets  $\bar{b}$  to maximize

$$W(b^*(\bar{b}), k_0^*(\bar{b}), \lambda^*(\bar{b})).$$

Whenever the leverage cap is binding, totally differentiating this expression yields

$$\begin{aligned} \frac{dW}{d\bar{b}} &= \left[ \lambda \frac{dM^P(\bar{b})}{d\bar{b}} + M^P(\bar{b}) \frac{d\lambda}{d\bar{b}} \right] k_0^*(\bar{b}) + \left[ \lambda M^P(\bar{b}) - 1 - \Upsilon'(k_0^*(\bar{b})) \right] \frac{dk_0^*(\bar{b})}{d\bar{b}} \\ &= \left[ \lambda \frac{dM^P(\bar{b})}{d\bar{b}} + M^P(\bar{b}) \frac{d\lambda}{d\bar{b}} \right] k_0^*(\bar{b}) + \lambda \left[ M^P(\bar{b}) - M(\bar{b}) \right] \frac{dk_0^*(\bar{b})}{d\bar{b}}, \end{aligned}$$

where the second line substitutes investors' first-order condition. Dividing both sides by  $\lambda > 0$  yields Equation (30).

**Further welfare analysis** The response of equilibrium prices to  $\bar{b}$ , using the mapping derived in Equation (45), can be expressed as

$$\frac{d\lambda}{d\bar{b}} = \Lambda'(M(\bar{b})) \frac{dM(\bar{b})}{d\bar{b}}.$$

To obtain a tractable characterization of this effect to belief distortions, we use a log-linear approximation

$$\log \Lambda(M) \simeq \lambda_0 - \lambda_1 M.$$

We now obtain

$$\frac{d \log \lambda}{d\bar{b}} = \underbrace{\frac{\Lambda'(M(\bar{b}))}{\Lambda(M(\bar{b}))}}_{=-\lambda_1} \frac{dM(\bar{b})}{d\bar{b}}.$$

Hence, we find that the variational derivative of the novel term in Equation (30) is

$$\frac{\delta \left( \frac{d \log \lambda}{d\bar{b}} \right)}{\delta F^j} \cdot G^j = -\lambda_1 \left[ \frac{\delta \frac{dM(\bar{b})}{d\bar{b}}}{\delta F^j} \cdot G^j \right].$$

We note that, as discussed in the paper, this effect scales with the variational derivative of the marginal value of leverage, which we have characterized in Lemma 3.

#### C.4 Endogenous belief distortions

When beliefs are endogenous to aggregate capital investments  $K$ , which individual investors take as given, we write the market value of capital (i.e., the equivalent to Equation



(7) as

$$M(b; K) = \beta^I \int_b^{\bar{s}} (s - b) dF^I(s; K) + \beta^C \left[ \int_b^{\bar{s}} b dF^C(s) + \phi \int_{\underline{s}}^b s dF^C(s; K) \right].$$

The first-order conditions of an individual investor are, as in the baseline model,

$$\begin{aligned} \frac{\partial M(b; K)}{\partial b} &= \mu \\ M(b; K) &= 1 + \Upsilon'(k). \end{aligned}$$

Substituting the consistency requirement that  $k = K$  and totally differentiating, we obtain

$$\begin{aligned} \frac{\partial M(\bar{b}; k)}{\partial b} + \frac{\partial M(\bar{b}; k)}{\partial K} \frac{dk}{d\bar{b}} &= \Upsilon''(k) \frac{dk}{d\bar{b}} \\ \Rightarrow \frac{dk^*}{d\bar{b}} &= \frac{\partial M(\bar{b}; k^*)}{\partial b} \left( \Upsilon''(k) - \frac{\partial M(\bar{b}; k)}{\partial K} \right)^{-1}. \end{aligned} \quad (46)$$

Similarly, the total variational derivative with respect to beliefs  $F^j$  satisfies

$$\begin{aligned} \frac{\delta M(\bar{b}; k)}{\delta F^j} \cdot G^j + \frac{\partial M(\bar{b}; k)}{\partial K} \left[ \frac{\delta k^*(\bar{b})}{\delta F^j} \cdot G^j \right] &= \Upsilon''(k) \left[ \frac{\delta k^*(\bar{b})}{\delta F^j} \cdot G^j \right] \\ \Rightarrow \frac{\delta k^*(\bar{b})}{\delta F^j} \cdot G^j &= \left[ \frac{\delta M(\bar{b}; k^*)}{\delta F^j} \cdot G^j \right] \left( \Upsilon''(k) - \frac{\partial M(\bar{b}; k)}{\partial K} \right)^{-1}. \end{aligned} \quad (47)$$

We can further compute, with quadratic adjustment costs,

$$\begin{aligned} \frac{\delta \frac{dk^*}{d\bar{b}}}{\delta F^j} \cdot G^j &= \frac{\delta \left( \frac{\partial M(\bar{b}; k)}{\partial b} \left( \Upsilon''(k) - \frac{\partial M(\bar{b}; k)}{\partial K} \right)^{-1} \right)}{\delta F^j} \cdot G^j \\ &= \left[ \frac{\delta \frac{\partial M}{\partial b}}{\delta F^j} \cdot G^j \right] \left( \Upsilon''(k) - \frac{\partial M(\bar{b}; k)}{\partial K} \right)^{-1} \\ &\quad + \frac{\partial M(\bar{b}; k)}{\partial b} \left( \Upsilon''(k) - \frac{\partial M(\bar{b}; k)}{\partial K} \right)^{-2} \frac{\delta \frac{\partial M}{\partial K}}{\delta F^j} \cdot G^j. \end{aligned} \quad (48)$$

Repeating the steps leading to Proposition 4, we find that the marginal welfare effect of varying  $\bar{b}$  is given by

$$\frac{dW}{d\bar{b}} = \left[ \frac{dM^P(\bar{b})}{db} - \frac{d\Delta(\bar{b})}{db} \right] k^*(\bar{b}) + [M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b})] \frac{dk^*}{d\bar{b}}.$$

Taking variational derivatives, we then obtain

$$\begin{aligned} \frac{\delta \frac{dW}{d\bar{b}}}{\delta F^j} \cdot G^j &= \left[ \frac{dM^P(\bar{b})}{db} - \frac{d\Delta(\bar{b})}{db} \right] \left[ \frac{\delta k^*(\bar{b})}{\delta F^j} \cdot G^j \right] \\ &+ [M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b})] \left[ \frac{\delta \frac{dk^*}{d\bar{b}}}{\delta F^j} \cdot G^j \right] \\ &- \frac{dk^*}{d\bar{b}} \left[ \frac{\delta M(\bar{b})}{\delta F^j} \cdot G^j \right]. \end{aligned} \quad (49)$$

Now we divide (46) by (47) to obtain

$$\frac{dk^*}{d\bar{b}} \left[ \frac{\delta M(\bar{b})}{\delta F^j} \cdot G^j \right] = \frac{\partial M(\bar{b}; k^*)}{\partial b} \left[ \frac{\delta k^*(\bar{b})}{\delta F^j} \cdot G^j \right].$$

Substituting into (49) establishes that

$$\begin{aligned} \frac{\delta \frac{dW}{d\bar{b}}}{\delta F^j} \cdot G^j &= \left[ \frac{dM^P(\bar{b})}{db} - \frac{d\Delta(\bar{b})}{db} - \frac{\partial M(\bar{b}; k^*)}{\partial b} \right] \left[ \frac{\delta k^*(\bar{b})}{\delta F^j} \cdot G^j \right] \\ &+ [M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b})] \left[ \frac{\delta \frac{dk^*}{d\bar{b}}}{\delta F^j} \cdot G^j \right]. \end{aligned} \quad (50)$$

Finally, substituting (47) and (48) into (50), we get

$$\begin{aligned} \frac{\delta \frac{dW}{d\bar{b}}}{\delta F^j} \cdot G^j &= \underbrace{\left( \Upsilon''(k) - \frac{\partial M(\bar{b}; k)}{\partial K} \right)^{-1}}_{\text{Multiplier } A} \left\{ \left[ \frac{dM^P(\bar{b})}{db} - \frac{d\Delta(\bar{b})}{db} - \frac{\partial M(\bar{b}; k^*)}{\partial b} \right] \left[ \frac{\delta M(\bar{b}; k^*)}{\delta F^j} \cdot G^j \right] \right. \\ &+ [M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b})] \left[ \frac{\delta \frac{\partial M}{\partial \bar{b}}}{\delta F^j} \cdot G^j \right] \left. \right\} \\ &+ [M^P(\bar{b}) - \Delta(\bar{b}) - M(\bar{b})] \frac{\partial M(\bar{b}; k^*)}{\partial b} \left( \Upsilon''(k) - \frac{\partial M(\bar{b}; k^*)}{\partial K} \right)^{-2} \frac{\delta \frac{\partial M}{\partial K}}{\delta F^j} \cdot G^j. \end{aligned} \quad (51)$$

The first two lines are the same as Equation (18), but is multiplied by the factor  $A$  defined in the text. The final term, which arises only if  $M(b; K)$  is not linear in  $K$ , is also amplified, and increases the magnitude of the incentive effect if a distortion increases the responsiveness of  $M$  to  $K$ .

## D Additional Proofs and Derivations

### D.1 Regularity conditions

Note that investors always find it optimal to choose non-negative leverage in equilibrium, since

$$\left. \frac{dM}{db} \right|_{b=0} = \beta^C - \beta^I > 0.$$

Therefore, for a given leverage constraint  $\bar{b}$ , our problem always features a solution for leverage in  $[0, \bar{b}]$  and a finite solution for investment, since  $\frac{d^2V}{dk^2} = -\Upsilon''(k) < 0$ . A sufficient condition that guarantees a finite solution without leverage regulation is that creditors perceive the net present value of investment to be negative if there is always default, that is,  $\beta^C \phi \mathbb{E}^C [s] < 1$ , since

$$\lim_{b \rightarrow \infty} M(b) = \beta^C \phi \mathbb{E}^C [s].$$

This sufficient condition extends directly to the environment with bailouts in Section 4.3.1 after imposing that bailouts are bounded above,  $t(b, s) \leq \bar{t}$ , and that investment has negative net present value if always in distress, even under the maximum bailout,  $\beta^C (\phi \mathbb{E}^C [s] + \bar{t}) < 1$ .

In order to explore the quasi-concavity of the investors' objective, it is useful to normalize  $\frac{dM}{db}$ , characterized in Equation (10), as follows:

$$J(b) = \frac{\frac{dM}{db}}{\beta^C (1 - F^C(b))} \equiv 1 - \frac{\beta^I}{\beta^C} \frac{1 - F^I(b)}{1 - F^C(b)} - (1 - \phi) b \frac{f^C(b)}{1 - F^C(b)},$$

where the normalization is valid for any non-zero level of  $b$ . Therefore, it follows that the quasi-concavity of the investors' objective can be established by characterizing the conditions under which  $J'(b)$  is negative. Note that

$$J'(b) = -\frac{\beta^I}{\beta^C} \frac{\partial}{\partial b} \left( \frac{1 - F^I(b)}{1 - F^C(b)} \right) - (1 - \phi) \left[ \frac{f^C(b)}{1 - F^C(b)} + b \frac{\partial}{\partial b} \left( \frac{f^C(b)}{1 - F^C(b)} \right) \right].$$

There are two sufficient conditions that, when jointly satisfied, guarantee that  $J'(b) < 0$ .

First, when the hazard rate of creditors' beliefs is monotone increasing, then

$$\frac{\partial}{\partial b} \left( \frac{f^C(b)}{1 - F^C(b)} \right) > 0.$$

Second, if investors are more optimistic than creditors in the hazard-rate sense, then

$$\frac{\partial}{\partial b} \left( \frac{1 - F^I(b)}{1 - F^C(b)} \right) > 0.$$

Therefore, when both conditions are satisfied, we have  $J'(b) < 0$ , which yields the result. We formally state this result as Lemma 5.

**Lemma 5. (*Single-peaked objective function without bailouts*)** *Suppose that there is no bailout, and:*

1. *Equity investors are weakly more optimistic than creditors in the hazard-rate order;*
2. *Creditors' hazard rate  $\frac{f^C(s)}{1 - F^C(s)}$  is increasing in  $s$ .*

*Then  $M(b)$  is single peaked.*

Notice that the solution for optimal leverage can be expressed in general as follows:

$$b = \frac{1}{(1 - \phi) \frac{f^C(b)}{1 - F^C(b)}} \left( 1 - \frac{\beta^I}{\beta^C} \frac{1 - F^I(b)}{1 - F^C(b)} \right).$$

Note also that whenever  $\beta^I = \beta^C$ ,  $\frac{dM}{db} \Big|_{b=0} = 0$ , but the rest of the results remain valid.

## D.2 Variations and cumulative distribution functions

For simplicity, we drop the superscript  $j$  and work with  $F(s)$  and  $G(s)$  in this appendix. Recall that a function  $F(s)$  is a cumulative distribution function if and only if it is non-decreasing, right-continuous, and satisfies  $F(\underline{s}) = 0$  and  $F(\bar{s}) = 1$ . We say a variation  $G(s)$  of beliefs is valid if, for small enough  $\varepsilon$ , the perturbed belief  $F(s) + \varepsilon G(s)$  remains a cumulative distribution function.

**Definition 1.** A right-continuous function  $G(s)$  is a *valid variation* of a cumulative distribution function  $F(s)$  if  $G(\bar{s}) = G(\underline{s}) = 0$ , and there exists an  $\bar{\varepsilon} > 0$  such that for all  $\varepsilon \in [0, \bar{\varepsilon}]$ , the following conditions are satisfied:

1.  $F(s) + \varepsilon G(s)$  is non-decreasing in  $s$ ;
2.  $0 \leq F(s) + \varepsilon G(s) \leq 1, \forall s$ .

The following lemma shows that our regularity conditions in the baseline model are sufficient to guarantee that all variations are valid.

**Lemma 6.** *(Regularity conditions on belief variations) If (i)  $F(s)$  and  $G(s)$  are continuously differentiable, (ii)  $f(s) = F'(s) > 0$ , and (iii)  $G(\bar{s}) = G(\underline{s}) = 0$ , then  $G(s)$  is a valid variation of  $F(s)$ .*

*Proof.* By assumption,  $f(s) = F'(s)$  and  $g(s) = G'(s)$  are continuous and therefore bounded on the interval  $[\underline{s}, \bar{s}]$ , so that we can define  $\underline{f} = \inf \{g(s) | s \in [\underline{s}, \bar{s}]\} > 0$  and  $\underline{g} = \inf \{f(s) | s \in [\underline{s}, \bar{s}]\}$ . For all  $s$ , we have

$$F'(s) + \varepsilon G'(s) = f(s) + \varepsilon g(s) \geq \underline{f} + \varepsilon \underline{g},$$

Hence,  $F(s) + \varepsilon G(s)$  is non-decreasing for all  $\varepsilon \leq \underline{f}/\underline{g} \equiv \bar{\varepsilon}$ .

Moreover, note that, for all  $\varepsilon \leq \bar{\varepsilon}$ , and for all  $s$ , we have

$$F(s) + \varepsilon G(s) = F(\underline{s}) + \underbrace{\varepsilon G(\underline{s})}_{=0} + \int_{\underline{s}}^s \underbrace{(f(s) + \varepsilon g(s))}_{\geq 0, \forall \varepsilon \leq \bar{\varepsilon}} ds \geq F(\underline{s}) = 0,$$

and similarly,

$$F(s) + \varepsilon G(s) = F(\bar{s}) + \underbrace{\varepsilon G(\bar{s})}_{=0} - \int_s^{\bar{s}} \underbrace{(f(s) + \varepsilon g(s))}_{\geq 0, \forall \varepsilon \leq \bar{\varepsilon}} ds \leq F(\bar{s}) = 1.$$

Hence,  $0 \leq F(s) + \varepsilon G(s) \leq 1$  for all  $\varepsilon \leq \bar{\varepsilon}$ , as required.  $\square$

### D.3 First-best corrective policy

The first-best problem when the planner can control both  $b$  and  $k$  is

$$\max_{b,k} W(b,k) = [M^P(b) - \Delta(b) - 1]k - \Upsilon(k),$$

with first-order conditions

$$\begin{aligned} \frac{dM^P(b^1)}{db} - \frac{d\Delta(b^1)}{db} &= 0 \\ M^P(b^1) - \Delta(b^1) - 1 &= \Upsilon'(k^1), \end{aligned}$$

where we denote by  $b^1$  and  $k^1$  the first-best leverage and investment. Formally, we consider an equilibrium with Pigouvian taxes  $\tau = (\tau_k, \tau_b)$ , where investors pay  $\tau_k k + \tau_b b$  at date 0 to the government, which is then rebated as a lump sum to either investors or creditors.

Investors solve

$$V(\tau) = \max_{b,k} [M(b) - 1]k - \Upsilon(k) - \tau_k k - \tau_b b,$$

with first-order conditions

$$\begin{aligned} \frac{dM(b)}{db} k &= \tau_b \\ M(b) - 1 &= \Upsilon'(k) + \tau_k. \end{aligned}$$

It follows that the corrective policy that achieves the first-best solution is

$$\begin{aligned} \tau_b &= \left[ \frac{dM(b^1)}{db} - \left( \frac{dM^P(b^1)}{db} - \frac{d\Delta(b^1)}{db} \right) \right] k^1 \\ \tau_k &= M(b^1) - (M^P(b^1) - \Delta(b^1)). \end{aligned}$$

#### D.4 Properties of hazard-rate dominant perturbations

We often rely on the following two properties of hazard-rate dominant variations/perturbations.

**Property 1** The hazard rate after an arbitrary perturbation of the form described in Section 2 of the paper is given by  $h(s) = \frac{f(s) + \varepsilon g(s)}{1 - (F(s) + \varepsilon G(s))}$ . Its derivative with respect to  $\varepsilon$  takes the form

$$\frac{dh(s)}{d\varepsilon} = \frac{g(s)}{1 - (F(s) + \varepsilon G(s))} + \frac{(f(s) + \varepsilon g(s)) G(s)}{(1 - (F(s) + \varepsilon G(s)))^2}.$$

In the limit in which  $\varepsilon \rightarrow 0$ , for hazard-rate dominance to hold, it must be the case that  $\lim_{\varepsilon \rightarrow 0} \frac{dh(s)}{d\varepsilon} < 0$ , therefore

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{dh(s)}{d\varepsilon} &= \frac{g(s)}{1 - F(s)} + \frac{f(s)}{1 - F(s)} \frac{G(s)}{1 - F(s)} < 0 \\ &\iff g(s) + \frac{f(s)}{1 - F(s)} G(s) < 0 \\ &\iff \frac{g(s)}{G(s)} + \frac{f(s)}{1 - F(s)} > 0 \\ &\iff \frac{f(s)}{1 - F(s)} > -\frac{g(s)}{G(s)}, \end{aligned}$$

where in the second-to-last line the sign of the inequality flips because  $G(s)$  is negative, since hazard-rate dominance implies first-order stochastic dominance.

**Property 2** Hazard-rate dominance implies that a perturbation increases  $\frac{1-F(s)}{1-F(b)}$ , where  $s > b$ . This implies that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\partial}{\partial \varepsilon} \left( \frac{1 - F(s) - \varepsilon G(s)}{1 - F(b) - \varepsilon G(b)} \right) = \frac{(-G(s))(1 - F(b)) - (1 - F(s))(-G(b))}{(1 - F(b))^2} \geq 0,$$

or equivalently

$$(-G(s))(1 - F(b)) \geq (1 - F(s))(-G(b)). \quad (52)$$

## D.5 Binding equity constraint

Whenever the investors' date 0 non-negativity constraint is binding, the total amount of equity is effectively fixed to  $n_0^I$ , and Lemma 1 ceases to hold. Equation (33) and Equations (34) through (38) remain valid in that case. For simplicity, we consider here the case without bailouts, no monetary policy, and  $\Upsilon(k) = 0$ . These assumptions imply that  $s^*(b) = b$ , and allow us to focus on equilibrium leverage.

Under those assumptions, when the date 0 non-negativity constraint binds, the problem that investors face can be expressed as

$$\max_{b,k} \beta^I \int_{s^*(b)}^{\bar{s}} (s - b) dF^I(s) k,$$

where  $k = \frac{n_0^I}{1-Q(b)}$  and  $Q(b) = \beta^C \left( \int_{s^*(b)}^{\bar{s}} b dF^C(s) + \phi \int_{\underline{s}}^{s^*(b)} s dF^C(s) \right)$ . Intuitively, investors maximize the leverage return on their initial wealth  $n_0^I$ . Under natural regularity conditions, the solution to this problem is given by the first-order condition on  $b$

$$\frac{1 - Q(b^*)}{Q_b(b^*)} = \frac{\int_{s^*(b^*)}^{\bar{s}} (s - b^*) dF^I(s)}{\int_{s^*(b^*)}^{\bar{s}} dF^I(s)}, \quad (53)$$

where  $Q_b(b) = \beta^C \left( \int_{s^*(b)}^{\bar{s}} dF^C(s) - (1 - \phi) s^*(b) f^C(s^*(b)) \right)$ . Equation (53) is the counterpart of Equation (11) in Simsek (2013a), after accounting for the cost of distress associated with bankruptcy. In this appendix, to highlight the differences with Simsek (2013a), we focus on the case of equity exuberance, although our approach can be used to study other scenarios. Formally, we consider the case in which  $F^C(s) = F^{C,P}(s) = F^{I,P}(s)$ .

In order to understand whether equilibrium leverage increases or decreases in response to a perturbation in investors' leverage, it follows from Equation (53) that it is sufficient to characterize the behavior of  $T(b) \equiv \frac{\int_{s^*(b)}^{\bar{s}} s dF^I(s)}{\int_{s^*(b)}^{\bar{s}} dF^I(s)} = \int_b^{\bar{s}} (s - b) \frac{f^I(s)}{1 - F^I(b)} ds$ . The change in

$T(b)$  induced by a change in investors' beliefs in the direction  $G^I$  is given by

$$\frac{\delta T}{\delta F^I} \cdot G^I = \frac{\int_b^{\bar{s}} (s-b) \left[ g^I(s) (1 - F^I(b)) - f^I(s) (-G^I(b)) \right] ds}{(1 - F^I(b))^2}.$$

If  $\frac{\delta T}{\delta F^I} \cdot G^I$  is positive (negative), leverage will increase (decrease). This characterization allows to consider any perturbation of beliefs. However, if we are interested in hazard-rate dominant perturbations, it can be shown that when investors become more optimistic in a hazard-rate sense and they are constrained on the amount of equity issued, leverage increases in equilibrium. Formally,  $\frac{\partial T}{\partial F^I} \cdot G^I \geq 0$  if

$$\left( \int_b^{\bar{s}} (s-b) g^I(s) ds \right) (1 - F^I(b)) - \left( \int_b^{\bar{s}} (s-b) f^I(s) ds \right) (-G^I(b)) \geq 0,$$

which is equivalent to

$$\left( \int_b^{\bar{s}} (-G^I(s)) ds \right) (1 - F^I(b)) - \left( \int_b^{\bar{s}} (1 - F^I(s)) ds \right) (-G^I(b)) \geq 0,$$

which follows by integrating (52) over  $s \in [b, \bar{s}]$ . This argument is an alternative way to formalize some of the main results in Simsek (2013a), in particular Theorems 4 and 5.

Finally, we can consider the normative implications of this case. In this scenario, the planner's objective can be written as  $\beta^I \int_{s^*(b)}^{\bar{s}} (s-b) dF^{I,P}(s) k$ . With a single degree of freedom, since  $b$  and  $k$  are connected via the date 0 budget constraint of investors, it is straightforward to show that an increase in optimism by investors in the hazard-rate sense calls for tightening leverage regulations.

## D.6 Alternative modeling assumptions

### D.6.1 Outside equity issuance

We consider an extension of our baseline model in which, in addition to investors and creditors, there are shareholders (denoted  $S$ ) who are able to invest in outside equity claims against investors' cash flows. The lifetime utility of a representative shareholder is  $c_0^S + \beta^S \mathbb{E}^S [c_1^S(s)]$ , where  $\mathbb{E}^S$  is the expectation under shareholders' beliefs  $F^S(s)$ . For simplicity, we continue to assume segmented markets: creditors do not invest in equity, and shareholders do not invest in bonds.

In addition to leverage  $b$ , investors choose a share  $\sigma \in [0, 1]$  of equity to retain, and sell a share  $1 - \sigma$  of equity claims to shareholders. The market value of outside equity in



equilibrium is then given by

$$P^S(b, \sigma) = (1 - \sigma) \beta^S \int_b^{\bar{s}} (s - b) dF^S(s).$$

By contrast, the market value of debt  $Q(b)$  remains unchanged from the baseline model, since the payoff to debtholders is unaffected by inside or outside ownership of equity shares. Repeating the steps leading to Lemma 1 in the text, we find that the following reformulation of the investors' problem characterizes the equilibrium:

**Lemma 7.** *[Investors' problem with outside equity issuance] Investors solve the following problem to decide their optimal investment, outside equity issuance, and leverage choices at date 0:*

$$\begin{aligned} V(\bar{b}) = \max_{b, k; \sigma \in [0, 1]} & [M(b, \sigma) - 1]k - \Upsilon(k) \\ \text{s.t. } & b \leq \bar{b} \quad (\mu), \end{aligned}$$

where  $\mu$  denotes the Lagrange multiplier on the leverage constraint imposed by the government (reformulated as  $bk \leq \bar{b}k$ ), and  $M(b, \sigma)$  is given by

$$\begin{aligned} M(b, \sigma) = & \underbrace{\sigma \beta^I \int_{s^*(b)}^{\bar{s}} (s - b) dF^I(s)}_{\text{inside equity}} + \underbrace{(1 - \sigma) \beta^S \int_{s^*(b)}^{\bar{s}} (s - b) dF^S(s)}_{\text{outside equity}} \quad (54) \\ & + \underbrace{\beta^C \left( \int_{s^*(b)}^{\bar{s}} b dF^C(s) + \phi \int_{\underline{s}}^{s^*(b)} s dF^C(s) \right)}_{\text{debt}}. \end{aligned}$$

Lemma 7 shows that investors continue to maximize the same objective as in the baseline model, but must first solve an auxiliary maximization problem in Equation (54), which determines the optimal value  $\sigma$  of the share of equity retained by insiders. The auxiliary problem is clearly linear in  $\sigma$ . Hence, for any given choice of  $b$ , it is either optimal to retain all shares ( $\sigma = 1$ ) or sell all shares to outsiders ( $\sigma = 0$ ), depending on the differences between insiders' and outsiders' discount factors and beliefs.

This result clarifies how our main results are affected by outside equity issuance. On the one hand, if inside and outside shareholders have the same preferences and beliefs, then investors are indifferent between all values of  $\sigma$ , and their problem reduces to the exact same problem as in the baseline model. In this case, all of our positive and normative results on the marginal effects of changes in beliefs carry over without modification.

On the other hand, if there are differences in preferences or belief disagreements

between insiders and outsiders, then investors' choices are affected only by marginal changes in the beliefs of (outside) shareholders if it is optimal to sell all shares with  $\sigma = 1$ , and only by marginal changes in their own beliefs if  $\sigma = 0$ . However, all of our results on the effects of equity exuberance continue to remain true after a modification to the definition of exuberance, namely, that *both* investors' beliefs  $F^I(s)$  and outsider shareholders' beliefs  $F^S(s)$  become more optimistic in the sense of hazard-rate dominance.

### D.6.2 Collateralized credit

In the body of the paper, we consider an environment in which creditors can seize from investors the full gross return on investment in case of default. If we assume that capital trades at a price  $q(s)$  at date 1, and that credit is collateralized exclusively by the market value of the investment at date 1, we can reformulate the two relevant equations in Equation (36) to accommodate collateralized borrowing as follows:

$$c_1^I(s) = n_1^I(s) + sk + \max\{q(s) - b, 0\}k, \quad \forall s$$

$$Q(b) = \beta^C \left( \int_{s^*(b)}^{\bar{s}} b dF^C(s) + \phi \int_{\underline{s}}^{s^*(b)} q(s) dF^C(s) \right),$$

where  $s^*(b)$  now solves  $q(s^*) = b$ . It is straightforward to extend our results to this case.