Fall 10-1-2021

The Wordline Effective Field Theory of Spinning Gravitational Sources

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Abstract

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2021

The advent of gravitational wave physics has raised great interest in efficient calculations of gravitational dynamics. In particular, the worldline effective field theory (EFT) has proven to be powerful for describing the dynamics of compact binary inspirals. In this thesis, we report on progress in this method on including rotating gravitational sources. It has been shown that a connection exists between the radiative amplitudes from spinless classical sources in Yang-Mills theory and dilaton-gravity theory, inspired by the double copy construction in the scattering amplitude community. We generalize this result to spinning sources and find that an additional axion channel is necessary for the connection to be established. The spectrum coincides with that of the low energy limit of string gravity, and we deduce that the worldline EFTs correspond to the low energy limit of classical string theories. Furthermore, we show that tidal effects also admit a double copy structure. On the other hand, there has been new progress on incorporating dissipative effects into the worldline EFT. We generalize this construction to describe rotating objects and apply it to describe the absorptive effects of Kerr black holes by matching with graviton absorption probabilities calculated by Teukolsky equations. Using the resulting EFT, we reproduce the correct mass and spin absorption rates under general backgrounds. We demonstrate the utility of this EFT by computing new results for the dissipative equation of motion and power transfer in non-relativistic black hole binaries, starting at 5 and 2.5 post-Newtonian orders respectively.

The Wordline Effective Field Theory of Spinning Gravitational Sources

A Dissertation
Presented to the Faculty of the Graduate School
of
Yale University
in Candidacy for the Degree of
Doctor of Philosophy

by
Jingping Li

Dissertation Director: Walter Goldberger

December 2021

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Acknowledgements

First and foremost, I am indebted to my advisor, Prof. Walter Goldberger, for offering all the research opportunities and for his generous guidance on physics and many other aspects of acaemic career. I am grateful to Prof. Witold Skiba for offering side projects, helpful advice, and taking the time to be one of my referees and committee members. I would also like to thank Prof. David Poland for writing the letters and joining the committee, Prof. David Moore for being part of the committee, Prof. Riccardo Penco for a very careful reading and numerous suggestions, and Prof. Ira Rothstein for offering the job opportunity and collaboration on a project. I am also thankful to my other collaborators, Siddarth Prabhu and Alex Sandomirsky.

I would like to express my appreciation to all past and present members of the particle theory group at Yale for many insightful discussions and maintaining a lively working environment and to all the great friends from the department, from Yale, and friends I have been acquainted with from before, for keeping me companied and helping out when I was in need. I have to express my gratitude to Ying Xing for her companion and support, especially during the COVID-19 pandemic. Last but not least, I have to thank my parents, Li Li and Liqing Weng, and all the other members of my family. Without their support, I would never have advanced to this stage of my academic endeavor.

Introduction

The first direct detection of gravitational waves by Advanced LIGO [1] on the centennial of general relativity marks the beginning of a new era of multi-messenger astronomy and possibly explorations into fundamental questions of gravity. At the moment, the detectable signals primarily come from compact binaries of black holes and neutron stars. While these are extremely energetic events, their astronomical distance from us significantly weakens the signals. Therefore, accurate waveform templates are required in the searches. Unfortunately, as the problem is two-body in nature and highly dynamical, the full Einstein equation is extremely difficult if not impossible to solve due to the lack of symmetries.

While recent developments in numerical relativity have enabled the full-detailed treatment of the later stages (known as the merger and ringdown phases, Fig. 1) of the binary evolution where the gravitational field is strong [2–4], these numerical methods are not designed for the long inspiral phase (Fig. 1) and running them for the long timescales in this phase can be expensive [5]. Fortunately, the full solutions to the Einstein field equation are not always relevant. In this regime, as the bodies are still moving non-relativistically, simplification can be made by neglecting contributions at high orders of the ratio between the characteristic velocity and the speed of light $v/c \ll 1$. This is known as the post-Newtonian (PN) approximation and the expansion at $(v/c)^{2n}$ power is usually called nPN order (for a comprehensive review on various approaches to PN waveform templates, see Blanchet [6]). This idea traces back to the pioneers in the early days of general relativity [7], but the progress has been relatively slow. Therefore, an efficient framework would be very helpful.

A major breakthrough was made by the seminal work of Goldberger and Rothstein [8, 9] where they realized that the problem in this regime can be solved systematically

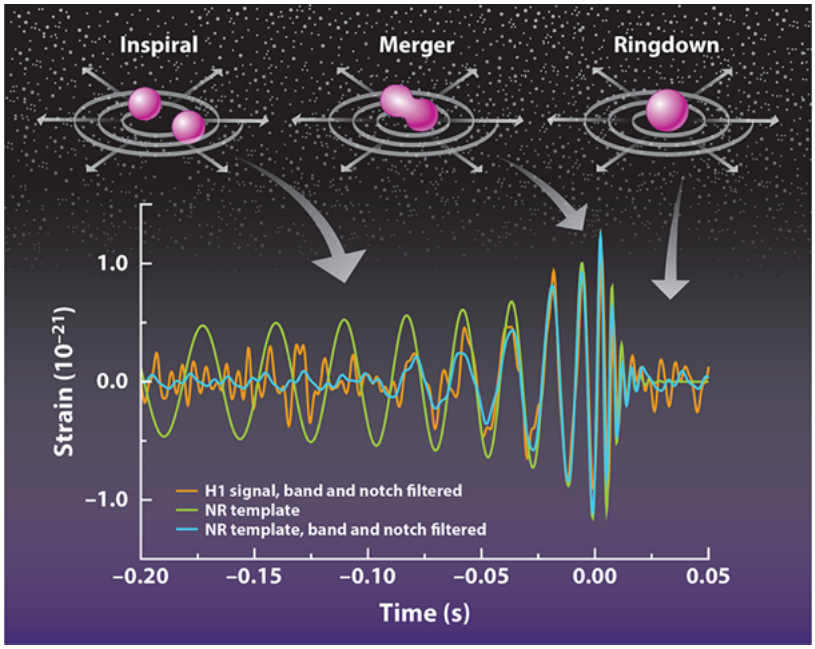

Figure 1: The stages of binary merger (source: aps.org)

by using effective field theory (EFT) methodologies. EFT is a powerful framework for precision calculations of long-distance observables in problems with separation of scales, where the short-distance dynamics is decoupled [10, 11]. The framework emerged in the context of chiral perturbation theory in particle physics [12], where the short-distance gluon and quark degrees of freedom of quantum chromodynamics (QCD) are decoupled and one can focus on the simpler long-distance hadronic degrees of freedom. Throughout the years, this framework has seen many successful applications in particle phenomenology and many comprehensive reviews [13, 14] and seminars [15] exist for this broad topic.

Their crucial realization for the binary inspiral is that the PN expansion parameter $v/c \ll 1$ provides the separation of scales in this problem. Specifically, the characteristic sizes of the compact bodies a (bounded by their Schwarzschild radii $a \geq 2GM/c^2$), the orbital radius r, and the wavelength of gravitational waves λ are related by

$$\frac{v^2}{c^2} \sim \frac{GM}{r} \leq \frac{a}{2r}, \quad \frac{v}{c} \sim \frac{r}{\lambda},$$

where the first comes from the virial theorem and the second from the multipole expanding a non-relativistic system. They used the first hierarchy $a \ll r$ to replace the full theory at

scale a by a point-particle model at scale r and constructed an EFT on the particle worldline. Using the second $r \ll \lambda$, they separated the gravitational field into short-distance potential modes and long-distance radiation modes and obtained the effective Lagrangian for long-distance dynamics by integrating out the former using Feynman diagrams. Their results were in agreement with direct PN calculations in full general relativity. Although these calculations were strictly dealing with point particles, they demonstrated that the worldline EFT framework also allows the inclusion of finite-size corrections corresponding to tidal deformations of the underlying extended objects. This method reduced the complicated general relativity of many-body systems into systematic calculations in perturbative field theory and quickly led to extensive generalizations and higher PN order calculations (for comprehensive reviews, see Porto [16] and Levi [17]). In particular, ref. [18] established the worldline EFT with spinning degrees of freedom. This is important as angular momentum plays crucial roles in many theoretically and phenomenologically interesting systems, such as black holes [19].

In this thesis, we describe recent work on the spinning worldline EFT along two parallel directions. The one, outlined in Section 0.1, revolves around developing the so-called double copy theory, originally discovered in the study of scattering amplitude, for the worldline EFT calculations. The results presented here are based on Refs. [20–22]. In Section 0.2, we summarize the second direction, which involves incorporating dissipative effects in the worldline EFTs. The extension of this framework to spinning black holes that is discussed in this thesis is based on Ref. [23].

0.1 Developments in the spinning classical double copy

While the EFT computations have been proven to be efficient, higher-order calculations are still expected to get troublesome due to the proliferation of graviton Feynman rules, so a goal is to simplify them. To this end, it was noticed that the modern scattering amplitude community has been developing a large toolbox for this purpose over the last 20 years (for an introductory review, see Elvang and Huang [24]). In particular, Bern, Carrasco, and Johansson (BCJ) conjectured the so-called BCJ relations, which imply the

color-kinematic duality of Yang-Mills scattering amplitudes [25]. It has also been shown that the duality enables perturbative gravity amplitudes to be constructed by replacing the color factors in Yang-Mills amplitudes with any kinematic numerators that satisfy the duality [26]. Therefore, perturbative gravity calculations are simplified in this way due to the much nicer Feynman rules in Yang-Mills theory (for a broad review, see [28]).

Naturally, one might ask whether these conclusions in scattering amplitudes have analogs in classical observables that can simplify computations therein. It was first discovered that, by similar replacements, stationary solutions to the Einstein field equation in Kerr-Schild form can be generated from stationary solutions to Yang-Mills equation [27]. Since then, various advances were made on expanding the boundaries of the Kerr-Schild double copy (see the later part of the review [28]). More recently, there has also been important progress on integrating various techniques from scattering amplitudes to obtain the conservative part of the potential of a two-body system [29] based on the ideas from [30]. Following this method, Bern et al. [31, 32] obtained results higher-order in the gravitational coupling for the first time. This was also generalized in various other ways, such as to systems with tidal effects [33, 34].

The discussions above could not consider radiation, since radiating systems are neither stationary nor conservative. For this purpose, Goldberger and Ridgway [35] studied radiative solutions in Yang-Mills theory resulting from the scattering of classical charges described by worldline EFTs. They found that a set of color-kinematic mappings perturbatively relate them to the radiative solutions from classical scattering in a dilaton-gravity theory. The dilaton is a scalar field that is necessary to balance the degrees of freedom on both sides of the double copy mapping.

We generalize the classical double copy to spinning classical sources using the construction introduced in [18] and find that the same set of rules apply only when an additional Kalb-Ramond radiation channel is included [20] with the Lagrangians fixed in specific forms [21]. The Kalb-Ramond field is a 2-form gauge field with anti-symmetric indices, which we call an axion for short. The axion is also needed to balance the degrees of freedom but was decoupled previously as index anti-symmetry was disallowed in the absence of spin. The axion-dilaton-graviton theory with fixed couplings matches the low energy spectrum

of string theory. Furthermore, the worldline coupling of spin and Yang-Mills field strength allows the interpretation of the worldline theory as an EFT of a rotating classical string with charges attached at the endpoints. Thus, it is natural to ask if the worldline theory fixed on the gravitational side has a similar interpretation. However, it is well known that closed strings have two spinning sectors and they couple to the axion (graviton) with opposite (same) sign. In contrast, there is no such distinction in the worldline EFT.

Fortunately, by examining the underlying structure of the bi-adjoint scalar radiation, it was discovered that the classical radiation amplitude also exhibits color-kinematic duality [36]. Therefore, we are allowed to use the kinematic factors with different spinning sectors as long as color-kinematic duality is obeyed [22]. In this method, the worldline theory we fixed in the old method is interpreted as setting one of the spinning sectors to zero. Furthermore, when we set the two sectors equal to each other, the axion once again decouples as the opposite signs cancel each other out. In the stringy interpretation, we notice that this theory is equivalent to an unoriented string theory where the axion channel is known to decouple. This provides a potential way to eliminate the unwanted axion for the purpose of phenomenological applications. In addition, we find that using different kinematic numerators in the mappings also allows us to establish the double copy when tidal effects are included, although the resulting gravitational theory requires the corrections from the three channels to be fixed to specific ratios.

0.2 Dissipative worldline theory of compact spinning objects

The worldline EFTs above consisting of long-distance degrees of freedom only describe elastic processes. Intuitively, there is no degree of freedom left to transfer the mechanical energy and angular momentum to. However, dissipative effects are an important part of the full theory. They include processes such as the emission and absorption by black hole horizons and neutron stars. In particular, for the most phenomenologically relevant case of black holes, these effects are associated with distortions of their horizons and can lead to significant effect on the dynamics. Specifically, absorption effects have been found to appear at 4PN order (relative to the well-known quadrupole radiation [39]) for non-

spinning black holes [37, 38] and 2.5PN for near extremally spinning ones [38]. The results are obtained by solving the Teukolsky-Press [40] or Regge-Wheeler-Zerilli [41] equation in black hole perturbation theory. These methods are devised for single black hole systems and generalizations to interacting many-body cases are not straightforward. Therefore, an EFT framework is desirable for fulfilling this role.

To achieve this goal, worldline operators depending on internal dynamics were introduced [42] (in the spinning case, by [43, 44] and with tidal corrections, by [45]). Including short-distance degrees of freedom in the EFT would seem to defeat the purpose. It was shown that only some easily extractable long-distance properties of the internal dynamics are necessary to calculate other long-distance properties. Recently, ref. [46] further extended this method, using the Schwinger-Keldysh or "in-in" formalism [47], to describe Hawking radiation [48]. The same method was used to compute the scattering from virtual Hawking radiation [49] and calculate the reaction force from the horizon absorption [50].

These latest efforts have been focusing on non-spinning setups. We extend the spinning construction to the reaction force computations [23]. By introducing a convenient basis for the spin-dependent (in-in) correlation function, we are able to extract the correlator and hence the retarded Green's function for the worldline graviton quadrupole operator in a low-frequency expansion but to all orders in the spin parameter. Using this information, we derive the spin and mass dissipation equation under background fields at the lowest order in curvature and found agreements with results derived using black hole perturbation theory. The conclusions are also consistent with the recent proofs of the vanishing static tidal response of Kerr black holes [52–54]. Finally, as an application, we derive the rate of power and spin absorption rate in a PN system by considering the case where the background is generated dynamically by an orbiting partner. We find that for binaries near extremal black holes, the leading PN expansion for the power transfer is of 2.5PN order, which is enhanced from the non-spinning 4PN results from non-spinning worldline EFT [50], both agreeing with the black hole perturbation theory calculations.

Outline of the thesis

The rest of the thesis is organized in the following manner. In Chapter 0.2, we briefly review the basics of perturbative general relativity as well as the application of worldline EFT methods which lays the groundwork for the remaining chapters. For the latter purpose, we start with the point-particle action coupled to gravity and then we survey the generalization to spinning degrees of freedom which is essential to the description of rotating bodies, finite-size corrections relevant to tidal effects of extended objects, and dissipation in the worldline theory which characterizes the internal dynamics of the gravitating sources.

The remaining chapters are devoted to reporting on the recent progress. Chapter 0.4.4 describes the classical double copy in systems with spinning sources. We motivate the discussion by a brief introduction to the color-kinematic duality and the double copy construction of scattering amplitudes as well as the latter's counterpart in the context of classical radiation from non-rotating sources. We then give a more detailed description of the calculations and results in the spinning case. In Chapter 0.7.4, we discuss the interpretation of the worldline EFT with spin in terms of classical strings and generalize the framework to deformable extended objects. In Chapter 0.10, we shift our focus to progress on constructing worldline EFTs for the dissipative effects of Kerr black holes and demonstrate their applications in various scenarios. Finally, Chapter 0.13 concludes the thesis and discusses the outlook of these projects.

The appendices supplement the main text with some lengthier results that are not crucial but might be useful as references to check the validity of the main calculations.

Notations and conventions

In this thesis, we use the mostly minus metric convention. The Minkowski metric is denoted by η, while the curved coordinate counterpart is denoted by g. We choose the Levi-Civita tensor convention as $\epsilon_{0123} = 1$. The covariant derivative of general coordinate transformations is denoted by ∇_μ. Overhead dots represent total (covariant) derivative, e.g. $\dot{x} \equiv \frac{dx(\lambda)}{d\lambda}$ (or $\frac{Dx(\lambda)}{D\lambda}$ for the covariant one). Indices in the middle of the Greek alphabet $(\mu, \nu, \dots)$ label components in spacetime coordinate basis and Latin ones $(a, b, \dots)$ label the local

frame Lorentz indices. Round brackets $(\dots)$ over the indices denote symmetrizations while square ones $[\dots]$ are anti-symmetrizations, and we assume the Einstein summation convention whenever indices are repeated. On a separate note, in Chapter 0.4.4 where we also encounter gauge theory, it should be clear from the context that the Latin labels $(a, b, \dots)$ on gauge theory variables denote the adjoint indices of the gauge group. Other than in Chapter 0.4.4, we assume that the spacetime dimension is 4, and the Planck mass is $m_{Pl}^2 = 1/32\pi G_N$ Finally, we use natural units where $\hbar = c = 1$.

The Worldline Effective Field Theory of Gravity

In this chapter, we briefly describe the general framework of the worldline EFT applied to calculations in perturbative gravity. In Section 0.3, we start with an introduction to the key ideas in general relativity, including gravitational waves and perturbative gravity as a massless spin-2 field theory. Section 0.4 reviews the construction of worldline EFT to describe generic gravitational sources including spin, finite-size corrections, and dissipative effects.

0.3 Gravitational waves and perturbative gravity

0.3.1 General relativity and gravitational waves

The central equation in general relativity is the Einstein field equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{1}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the gravitating matter, $R = R_{\mu\nu}g^{\mu\nu}$ and $R_{\mu\nu} = R_{\mu\rho\nu\sigma}g^{\rho\sigma}$ are the Ricci curvature scalar and tensor respectively, defined through the Riemann curvature tensor

$$R^{\mu}{}_{\rho\nu\sigma} = \partial_\nu \Gamma^{\mu}{}_{\rho\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\rho\nu} + \Gamma^{\mu}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\rho\sigma} - \Gamma^{\mu}{}_{\sigma\lambda}\Gamma^{\lambda}{}_{\rho\nu}, \tag{2}$$

where $\Gamma^{\mu}_{\ \rho\sigma}$ is the metric compactible Levi-Civita connection which is expressed in terms of the metric as

$$\Gamma^{\mu}_{\ \rho\sigma} = \frac{1}{2} g^{\mu\lambda}(\partial_\rho g_{\lambda\sigma} + \partial_\sigma g_{\lambda\rho} - \partial_\lambda g_{\rho\sigma}). \tag{3}$$

Therefore, this is a second-order non-linear partial differential equation of the $g_{\mu\nu}(x)$ which is sourced by $T_{\mu\nu}(x)$. In principle, if we are given the matter distribution described by $T_{\mu\nu}$, we could solve for a curved spacetime $g_{\mu\nu}$. For later convenience, another useful curvature is the Weyl tensor which is the traceless version of the Riemann tensor

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}) + \frac{1}{3}R g_{\mu[\rho}g_{\sigma]\nu}. \tag{4}$$

In the region where the field is weak, the metric can be decomposed as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x), \tag{5}$$

so that $|h_{\mu\nu}| \ll 1$. The inverse $g^{\mu\nu}$ is an infinite series in $h^{\mu\nu} = h_{\rho\sigma}\eta^{\mu\rho}\eta^{\nu\sigma}$, leading to an infinite expansion of the field equation (1)

$$\frac{1}{2}\partial^2 \bar{h}_{\mu\nu} - \frac{1}{2}\partial^\rho \partial_{(\mu}\bar{h}_{\nu)\rho} + \frac{1}{2}\eta_{\mu\nu}\partial^\rho\partial^\sigma \bar{h}_{\rho\sigma} = 8\pi G_N T_{\mu\nu} + \mathcal{O}(\bar{h}^3), \tag{6}$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h^{\lambda}_{\ \lambda}\eta_{\mu\nu}$ and $\mathcal{O}(\bar{h}^3)$ denotes all the higher-order terms. The central symmetry principle of general relativity is general covariance or diffeomorphism which is the invariance of physics under the change of coordinate systems $x \to x'(x)$. To see that this is a gauge symmetry, since the metric transforms as

$$g_{\mu\nu}(x) \to \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x), \tag{7}$$

under infinitesimal transformation $x \to x + \xi(x)$, the perturbed field transforms as

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \tag{8}$$

This is essentially the gauge transformation for this field which we can use to impose gauge

conditions. The relevant gauge condition in this thesis is the deDonder gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$, under which Eq. (6) becomes

$$\frac{1}{2}\partial^2 \bar{h}_{\mu\nu} = 8\pi G_N T_{\mu\nu} + \mathcal{O}(\bar{h}^3). \tag{9}$$

On earth, in a galaxy far, far away from the source, where gravity is weak and the matter is not dense, we have a wave equation

$$\partial^2 \bar{h}_{\mu\nu} \approx 0. \tag{10}$$

These are the gravitational waves that we are observing.

By solving the leading-order equation for h field, it is possible to show that the time-averaged leading-order radiative power is given by the celebrated quadrupole formula [39]

$$\langle P \rangle = \frac{G_N}{5} \langle \dddot{Q}_{ij} \dddot{Q}_{ij} \rangle, \tag{11}$$

where the mass quadrupole is $Q_{ij}(t) = \int d^3\vec{x} T_{00}(t, \vec{x}) x^i x^j$ or $Q_{ij}(t)$. The textbook derivation can be found in any standard reference, e.g. [56].

0.3.2 Perturbative gravity as a massless spin-2 field theory

In field theory applications, it is more convenient to use Lagrangian formalism. For general relativity, we have the Einstein-Hilbert action

$$S_{EH} = -\frac{1}{16\pi G_N} \int d^4x \sqrt{g} R. \tag{12}$$

The expanded Lagrangian is essentially a self-interacting classical field theory. The terms quadratic in h are the kinetic terms leading to the propagator which in momentum space is given by

$$D_{\mu\nu,\rho\sigma}(k) = \frac{16\pi i G_N}{k^2} (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}), \tag{13}$$

and the higher-order-in-h terms give an infinite tower of vertices whose vertex rules can be derived by taking an arbitrary number of variational derivatives

$$\frac{\delta^n S_{EH}}{\delta h(x_1)\dots\delta h(x_n)}. \tag{14}$$

Given the matter action S_M, the energy-momentum tensor is calculated using

$$T^{\mu\nu}(x) = -\frac{2}{\sqrt{g}}\frac{\delta S_M}{\delta h_{\mu\nu}(x)}. \tag{15}$$

The gauge invariance under Eq. (8) signals redundancies arising from the masslessness of the field. Furthermore, it can be shown that the deDonder gauge does not exhaust the gauge freedom and the remaining gauge redundancies can set the trace to zero. Therefore, we have a symmetric-traceless tensor field which in the Language of irreducible representations of Lorentz group, is a spin-2 representation. This massless spin-2 field described by the Einstein-Hilbert action is the graviton. In fact, the classic results [55, 57] tell us that the classical Lagrangian for any massless spin-2 field theory is uniquely given by the Einstein-Hilbert action.

0.4 The worldline effective field theory

In this thesis, we focus on the first hierarchy $a \ll r$, i.e. the worldline EFT description of an isolated body at a longer-distance scale r (e.g. the background curvature radius, the radiation wavelength, the orbital scale, or the impact parameter) compared to its characteristic size a. The central idea is that in the full theory of general relativity, the energy-momentum tensor describing a compact object is a localized distribution over spacetime and highly curved gravitational background in the vicinity whose dynamical perturbations are difficult to solve. However, these details are happening at a short-distance (or high-energy) scale, relative to the (long-distance or low-energy) scale at which the object interacts with the environment. By the old wisdom, they are well approximated by point particles.

The modern incarnation of this ancient idea is the method of EFT. By the cornerstones of EFT, high-energy physics decouples from the low-energy dynamics and observables. Its

effects are systematically encoded by the non-dynamical Wilson coefficients c_i of an infinite tower of operators $\mathcal{O}_i(x, g(x))$ consisting of the low-energy dynamical variables $x = x(\lambda)$ and the field $g(x)$ in the effective action

$$S_{eff} \supset \sum_i c_i \int d\lambda \mathcal{O}_i(x, g(x)), \tag{16}$$

where λ parametrizes the worldline of the particle. The choices of these operators are only restricted by the symmetries of the low energy description and redundancies due to operators which vanish under the free theory equation of motion[1].

By dimensional analysis, these operators usually have positive mass dimensions. To compensate, the Wilson coefficients have positive powers of the characteristic size a, which has a negative mass dimension in natural units. Consequently, the contributions are organized by powers of a/r where r is the characteristic long-distance scale in this problem. Therefore, we can calculate results to arbitrary accuracy by truncating the infinite tower at appropriate finite orders. The advantage of this method is that we can obtain the EFTs of the isolated bodies relative easily sweeping the short-distance physics under the rug completely, and systematically apply only the EFTs to many-body interactions without having to worry about the difficult dynamics of the strong field regions.

Finally, we obtain the Wilson coefficients by matching observables, such as correlation functions, calculated using these operators in the effective interaction with the ones obtained from the full theory, under the same background field in both cases [8]. The process is greatly simplified by choosing a convenient setup, for instance, where the worldline is staying at rest at the coordinate origin and arbitrary background field configuration. The reason is that these coefficients are independent of the setups being considered.

1. In an interacting theory, the free equation of motion equals higher-order interactions. We can use field redefinitions and the full equations of motion to trade the operators with higher-order ones which are readily included in the infinite tower. The result is only a shift in the higher-order Wilson coefficients which are unknown at this stage anyway. Therefore, we can assume that this was done beforehand and directly set to zero the vanishing operators under the free equation [65, 66].

0.4.1 Point particles in the first-order formalism

To construct the operators, we need to identify the low-energy degrees of freedom first. A systematic way of doing this is to consider them as spontaneously breaking the full symmetry and identify the degrees of freedom with the corresponding Goldstones using the coset construction [59]. However, we take a more pedestrian route here by starting from well-known results. The relevant symmetries are Lorentz and parity invariance, worldline reparametrization invariance, plus possible internal symmetries. The simplest worldline Lagrangian is the traditional integral over the proper time

$$S_{pp} = -m \int d\tau = -m \int d\lambda \sqrt{\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}, \tag{17}$$

for some differentiable parameter λ along the particle worldline. The square root makes it harder to handle and we note that by introducing an auxiliary field $e(\lambda)$ called the einbein, we have an equivalent quadratic formulation

$$S_{pp} = -\int d\lambda [e^{-1} \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + e m^2], \tag{18}$$

which generates Eq. (17) by back substituting the solution to its own equation of motion $\frac{\delta S_{pp}}{\delta e} = 0$. The einbein varies covariantly with reparametrization, thereby serving to guarantee worldline reparametrization. In constructing the Lagrangian, worldline reparametrization is always guaranteed as long as we balance the powers of $d\lambda$ and e.

The action Eq. (18) is in the second-order formalism. Expecting to introduce spin angular momentum in the following, which is a first-order dynamical variable, we compute the canonical momentum and Legendre transform into the first-order Lagrangian

$$S_{pp} = -\int d\lambda \{\dot{x} \cdot p - \frac{1}{2} e (p^2 - m^2)\}. \tag{19}$$

We notice that now the einbein also serves as a Lagrange multiplier that imposes the mass-shell constraint as the Hamiltonian in the Legendre transformation. In fact, this is classified by Dirac [58, 60] as a first-class constraint whose Poisson bracket with all other constraints vanish, and it has been argued that first-class constraints usually generate local

transformations [61, 62] which is worldline reparametrization in the present case.

0.4.2 Spinning degrees of freedom of the free Lagrangian

Spin is the canonical conjugate to rotation which is a subgroup of Lorentz transformation. Therefore, to incorporate rotation, we introduce worldline frame fields $e^a_\mu(\lambda)$ which are similar to their usual spacetime cousins in that they satisfy the property

$$\eta_{ab} e^a_\mu(\lambda) e^b_\nu(\lambda) = \eta_{\mu\nu}, \tag{20}$$

and $e^\mu_a(\lambda)$ are their inverses

$$e^a_\mu e^\nu_a = \delta^\nu_\mu, \quad e^a_\mu e^\mu_b = \delta^a_b. \tag{21}$$

Its first derivative is defined as

$$\Omega^{ab} = \eta^{\mu\nu} e^a_\mu \frac{de^b_\nu}{d\lambda}. \tag{22}$$

We note that by $\frac{d}{d\lambda}(\eta^{\mu\nu} e^a_\mu e^b_\nu) = 0$, Ω^{ab} is antisymmetric and the associated canonical momentum is the spin S_{ab}, which is also anti-symmetric. Thus, the first two terms in the Lagrangian are

$$\int d\lambda \{-\dot{x}^\mu e^a_\mu p_a + \frac{1}{2} S^{ab} \Omega_{ab}\}. \tag{23}$$

By counting the degrees of freedom, we see an overcount. A generic rank-2 anti-symmetric tensor in 4 dimensions has 6 degrees of freedom while we are expecting 3 which are conjugate to 3 spatial rotations parametrized by Euler angles. The resolution to this problem is to impose constraints. While various choices are possible, we choose the covariant constraints

$$S^{ab} p_a = 0, \tag{24}$$

where we see that in the rest frame where the spatial components of the momentum vanish, they are proportional to $S^{a0} = 0$, eliminating the 3 redundant components with a being spatial indices. Again, they are imposed as a Hamiltonian term

$$\int d\lambda e \lambda_a S^{ab} p_b, \tag{25}$$

where λ_a is the Lagrange multiplier. They have non-vanishing Poisson brackets among themselves and are second-class. Thus, they are not associated with any local symmetry but only serve to eliminate redundant generalized coordinates.

With the kinematics settled, we would like to write down the EFT for an isolated rotating body. Following the principles of EFT and traditional Hamiltonian formalism, we construct all terms using the canonical momenta, namely p_a and S^{ab}, and due to the constraint, the only non-vanishing Lorentz invariant combinations are their quadratics. It was shown by Hanson and Regge [64] that the constraint always reduces to a mass-shell constraint $p^2 = M^2(S^2)$ and the general spinning free Lagrangian is therefore given by

$$S_{pp} = \int d\lambda \{-\dot{x}^\mu e^a_\mu p_a + \frac{1}{2} S^{ab}\Omega_{ab} + e\lambda_a S^{ab} p_b + \frac{1}{2} e(p^2 - M^2(S^2))\}. \tag{26}$$

$M^2(S^2)$ is the so-called Regge function that encodes the deformation of the underlying object under its own spin. The detailed form does not affect the low-energy dynamics, so we leave it as it is. We note that while the mass shell constraint is not the unique choice, by non-linear redefinitions of the variables, it is possible to cast the other possibilities into this form, which results in a simple momentum equation of motion.

0.4.3 Interactions, finite-size corrections

To couple to gravity, in addition to Lorentz invariance, we also require general covariance. This is achieved simply by replacing the Minkowski metric with generic ones $\eta_{\mu\nu} \to g_{\mu\nu}(x)$ (the local metric η_{ab} remains unchanged since by the equivalence principle, it is always locally Minkowskian), derivatives by covariant ones $\partial_\mu \to \nabla_\mu$. The worldline derivative also gets covariantized as

$$\frac{d}{d\lambda} e^a_\mu \to \frac{D}{D\lambda} e^a_\mu = \frac{d}{d\lambda} e^a_\mu - \dot{x}^\nu \Gamma^\sigma_{\mu\nu} e^a_\sigma. \tag{27}$$

The covariantized free Lagrangian naturally generates the mass monopole and spin dipole couplings.

Now the most general operators that we should consider should be constructed out of p_a, S^{ab}, the covariant gradients ∇_a, the Levi-Civita tensors ϵ_{abcd} (in d dimensions this has

to be modified to having d indices) and the gravito-electric and magnetic fields defined from the Weyl tensor

$$E^{ab} = \frac{1}{p^2} C^{cadb} p_c p_d, \tag{28}$$

$$B^{ab} = \frac{1}{2p^2} \epsilon^{acde} C_{cd}{}^{bf} p_e p_f. \tag{29}$$

The symmetry principle requires that all these indices are in the corotating frame [59]. The Ricci curvatures are redundant as discussed previously due to the vacuum Einstein equation

$$R_{\mu\nu} = 0. \tag{30}$$

At linear order in curvature and time derivatives, the parity invariant possibilities are an infinite set of operators of the form

$$\sum_{n=1}^{\infty} \int d\lambda e \{ c_n^E S^{a_1} \dots S^{a_{2n}} \nabla_{a_3} \dots \nabla_{a_{2n}} E_{a_1 a_2}$$
$$+ c_{2n}^B S^{a_1} \dots S^{a_{2n+1}} \nabla_{a_3} \dots \nabla_{a_{2n+1}} B_{a_1 a_2} \}, \tag{31}$$

where $S^a = \epsilon^{abcd} S_{bc} p_d / \sqrt{p^2}$. (They are the higher-spin multipoles whose effects were investigated by Levi and Steinhoff [67].) They are the only possibilities as using the vacuum equations of motion, any other contraction would either vanish or lead to higher-order time derivatives.

At the quadratic order in curvature, the possible curvature terms are [8]

$$\int d\lambda e \{ c_E E_{ab} E^{ab} + c_B B_{ab} B^{ab} \}, \tag{32}$$

where by parity invariance, terms like $\int d\lambda e E_{ab} B^{ab}$ are not considered. They lead to deviations from the point-particle geodesic in general relativity implying tidal forces in action. Thus, the Wilson coefficients $c_{E,B}$ encode the leading order tidal deformations which are finite-size corrections.

0.4.4 Internal dynamics and dissipative effects

Using these Lagrangians, it can be verified by examining the resulting equations of motion that S^2 and hence $M^2(S^2)$ do not change over time. This means that there cannot be any dissipative effects such as absorption and emission which should affect the mass shell.

To cope with this shortcoming, it is assumed that there exist some internal dynamics which are gapless modes described by the worldline variables $X(\lambda)$ and $\dot{X}(\lambda)$ [42,46]. Their most generic coupling to the external background, similar to Eq. (31), are composite multipole operators classified by the symmetry group

$$\sum_{n=0}^{\infty}\int d\lambda e\{Q^{aba_1\ldots a_n}_{E,(n)}(X,e^{-1}\dot{X};p,S)\nabla_{a_1}\ldots\nabla_{a_n}E_{ab} \\ +Q^{aba_1\ldots a_n}_{B,(n)}(X,e^{-1}\dot{X};p,S)\nabla_{a_1}\ldots\nabla_{a_n}B_{ab}\}, \tag{33}$$

where $Q^{aba_1\ldots a_n}_{E/B,(n)}$ are symmetric-traceless representations that are parity even/odd [42–44] and they depend on the momenta (p,S) through the choice of reference frame and the spinning axis. The reason that this form takes care of all possibility is that no specific form of these objects is being assumed, so all operators in the same representations can be defined into the same operator.

The detailed dynamics of $Q_{E/B}$ is determined by the full theory of X, and it is expected to be dependent on the actual physics such as the strong nuclear coupling in neutron stars. Fortunately, it turns out that the only information we need is the correlation functions. Therefore, we leave its Lagrangian unknown

$$\int d\lambda e\{L_X(X,e^{-1}\dot{X})+L_{int}(X,e^{-1}\dot{X},S^2)\}. \tag{34}$$

The effect of this term is to make the mass shell constraint X-dependent as well

$$\frac{1}{2}\int d\lambda e\{p^2-M^2(S^2)+L_X(X,e^{-1}\dot{X})+L_{int}(X,e^{-1}\dot{X},S^2)\} \\ =\frac{1}{2}\int d\lambda e\{p^2-M'^2(S^2,X,e^{-1}\dot{X})\}, \tag{35}$$

which indeed grants the possibility of dissipation.

The Classical Double Copy and the Spinning Generalization

In this chapter, we report on the classical double copy construction of radiation amplitudes from spinning worldline EFTs. To motivate the discussion, Section 0.5 starts with a brief review on the original double copy in the context of scattering amplitudes. In Section 0.6, we describe the setups of classical double copy in the original work which would become useful when we consider the spinning generalizations. Finally, in Section 0.7, we outline the important steps in the spinning calculations. This chapter is based on [20, 21].

0.5 The double copy in scattering amplitudes

We describe the central idea of the BCJ color-kinematic duality and the double copy construction in their original context of scattering amplitudes. We normalize the Yang-Mills action as

$$S_{YM} = -\frac{1}{4}\int d^d x F^a_{\mu\nu} F_a^{\mu\nu}, \tag{36}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$ is the Yang-Mills field strength with A^a_μ the components of the gauge connection $A_\mu = A^a_\mu T^a$. T^a and f^{abc} are the generators and the structure constants associated with the Lie algebra of the gauge group. g is the coupling constant. The field is usually also called a gluon due to its association with QCD, and the theory is gauge-invariant under the non-abelian gauge transformations

$$A_\mu \to G A_\mu G^{-1} + \frac{i}{g}(\partial_\mu G) G^{-1}, \tag{37}$$

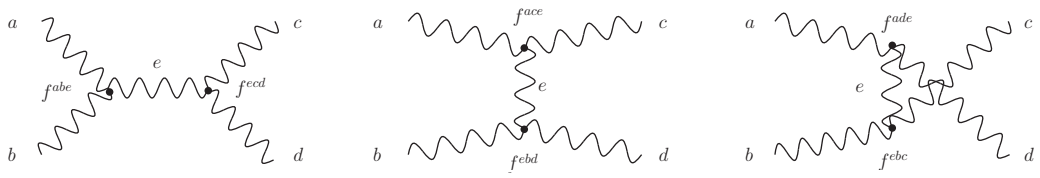

Figure 2: The s, t, u channel graphs of a 4-point scattering

which represents the fact that gluons are massless, at least in the weak coupling limit. Expanding out the field strength, we have terms cubic and quartic in A^a_μ which generate 3- and 4-point interaction Feynman vertex rules and the vertices depend on one and two factors of the structure constants f^{abc}.

For any tree-level scattering amplitude in Yang-Mills theory, the scattering amplitude can be expressed in terms of [25]

$$\mathcal{A}^{(tree)} = \sum_{g \in trivalent} \frac{c(g)n(g)}{d(g)}, \tag{38}$$

where $c(g)$'s are, due to QCD, called color numerators, consisting of group theoretic objects such as the structure constants f^{abc}, $n(g)$'s are kinematic numerators depending on polarizations $\epsilon^\mu(k_i)$ and momenta k_i of the particles being scattered and $1/d(g)$'s are the propagator pole structures which guarantees the theory is described by a local Lagrangian. The argument g labels all the trivalent diagrams corresponding to this scattering process. The reason tetravalent graphs corresponding to the quartic vertices are not considered is that they contain the same set of color numerators as the trivalent graphs and they can be absorbed into trivalent ones by modifying the definitions of $n(g)$.

The $c(g)$'s satisfy some algebraic relations since the structure constants obey anti-symmetry under the exchange of color indices

$$c(g) = -c(\bar{g}), \tag{39}$$

where $\bar{g}$ represents graphs with any two color states exchanged and Jacobi identities

$$f^{abe}f^{ecd} + f^{ace}f^{ebd} + f^{ade}f^{ebc} = 0. \tag{40}$$

For example, in 4-point processes, the trivalent graphs are the traditional s, t, u channel graphs Fig. 2, and this translates to the relations

$$c(g_s) + c(g_t) + c(g_u) = 0. \tag{41}$$

the trivalent graphs are for the It has been shown that the so-called BCJ relations of these amplitudes imply that the $n(g)$s satisfy an identical set of algebraic relations [25] which in the 4-point example are

$$n(g) = -n(\bar{g}), \tag{42}$$

$$n(g_s) + n(g_t) + n(g_u) = 0. \tag{43}$$

This property is known as the color-kinematic duality of gauge theory scattering amplitudes. The surprising duality is well-established at tree level since the BCJ relations have been proven in various ways.

To understand the consequence of this duality, we first note that gauge invariance Eq. (37) is manifest in the Yang-Mills amplitude $\mathcal{A}^{(tree)}$ as the invariance under the shift $\epsilon^\mu(k_i) \to \epsilon^\mu(k_i) + k_i^\mu$ in $n(g)$s. The establishment of this property crucially depends on the algebraic relations of $c(g)$s. Color-kinematic duality tells us that by the replacement

$$c(g) \mapsto \tilde{n}(g), \tag{44}$$

where $\tilde{n}(g)$s are duality-satisfying kinematic numerators with polarizations $\tilde{\epsilon}$, we obtain a new local tree-level amplitude

$$\mathcal{M}^{(tree)} = \sum_{g \in trivalent} \frac{n(g)\tilde{n}(g)}{d(g)}. \tag{45}$$

There is always a symmetric traceless part of the direct product $\epsilon^{\mu\nu}(k_i) \subset \epsilon^\mu(k_i)\tilde{\epsilon}^\nu(k_i)$. The color-kinematic duality then guarantees that $\epsilon^{\mu\nu}(k_i) \to a^\mu k_i^\nu + k_i^\mu b^\nu$, for arbitrary vectors a^μ and b^ν, which is essentially the gauge invariance in Eq. (8) in momentum space. As discussed in Section 0.3.2, this local spin-2 massless field theory could only be the graviton.

0.6 The classical double copy

In this section, we quickly review the setups of classical double copy construction introduced in [35]. We neglect some of the details since many will be covered in the next section on spinning generalization.

0.6.1 The actions and equations of motion

The Yang-Mills theory

When the Yang-Mills is coupled to some charges described by the action S_{wl}, the sourced Yang-Mills equation is given by

$$D_\nu F_a^{\nu\mu}(x) = g J_a^\mu(x), \tag{46}$$

where $D_\mu = \partial_\mu + igA_\mu^a T^a$ is the gauge covariant derivative and the current on the right-hand side is given by

$$J_a^\mu(x) = \frac{\delta S_{wl}}{\delta A_\mu^a(x)}. \tag{47}$$

The simplest Lagrangian is constructed by introducing a worldline color charge $c_\alpha^a(\lambda)$, as given by

$$S_{wl} \supset -\sum_\alpha \int d\lambda \dot{x}_\alpha^\mu c_\alpha^a A_\mu^a(x_\alpha), \tag{48}$$

where from here on, the subindices $(\alpha, \beta, \dots)$ label different particles. Strictly speaking, there should also be kinetic terms for the charge $c^a(\lambda)$ that controls its dynamics[2]. However, it is possible to obtain the same information model-independently from the covariant conservation of charge $\dot{x}_\alpha^\mu D_\mu c_\alpha^a = 0$ which leads to

$$\dot{c}_\alpha^a = g f^{abc} \dot{x}_\alpha^\mu A_\mu^b(x_\alpha) c_\alpha^c. \tag{49}$$

2. For instance,we can have a gauge-invariant action $\int \lambda i \dot{x}^\mu \psi^\dagger D_\mu \psi$, and the color charge is identified as $c^a = \psi^\dagger T^a \psi$ [68].

Therefore, the kinetic term is left implicit. From this action, the current is given by

$$J_a^\mu(x) = \sum_\alpha \int d\lambda c_\alpha^a \dot{x}_\alpha^\mu \delta^d(x - x_\alpha). \tag{50}$$

The other two equations of motion are

$$\dot{p}_\alpha^\mu = g c_\alpha^a F_{a\nu}^\mu(x_\alpha)\dot{x}_\alpha^\nu, \tag{51}$$

$$\dot{x}_\alpha^\mu = e p_\alpha^\mu. \tag{52}$$

Lastly, in Lorenz gauge $\partial^\mu A_\mu^a = 0$, Eq. (46) reduces to

$$\partial^2 A_a^\mu = g\tilde{J}_a^\mu, \tag{53}$$

where the non-linear terms in A_μ^a are also absorbed into the definition of $\tilde{J}$ and the effective gluon propagator is given in momentum space by

$$D_{\mu\nu}(k) = -\frac{i\eta_{\mu\nu}}{k^2}. \tag{54}$$

The dilaton-gravity theory

In general dimensions, we introduce a scalar field called dilaton in addition to the Einstein-Hilbert action

$$S = -2m_{Pl}^{d-2} \int d^d x\sqrt{g}\{R - (d-2)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\}, \tag{55}$$

where the Planck mass is defined as $m_{Pl} = (32\pi G)^{-\frac{1}{d-2}}$. The graviton propagator now become

$$D_{\mu\nu,\rho\sigma}(k) = \frac{i}{2m^{d-2}k^2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \frac{2}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma}), \tag{56}$$

and the scalar propagator is

$$D(k) = \frac{i}{4m_{Pl}^{d-2}(d-2)k^2}. \tag{57}$$

Of course, the coupling in Eq. (55) is arbitrary. This specific value is the one that establishes the double copy. In fact, the motivation of working in general dimensions and adding the dilaton is to subtract away the d dependence introduced by graviton propagators. The corresponding field equations are

$$\nabla^{\mu}\nabla_{\mu}\phi = -\frac{1}{4m_{Pl}^{d-2}(d-2)}J, \tag{58}$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{4m_{Pl}^{d-2}}(T_{\mu\nu} + T_{\phi\mu\nu}). \tag{59}$$

where the dilaton energy-momentum tensor isgiven by

$$T_{\phi}^{\mu\nu} = 4(d-2)m_{Pl}^{d-2}\{\partial_{\mu}\phi\partial_{\nu}\phi + \frac{1}{2}g_{\mu\nu}(\partial\phi)^2\}. \tag{60}$$

These equations can be recasted as

$$\partial^2\phi = -\frac{1}{4m_{Pl}^{d-2}(d-2)}\tilde{J}, \tag{61}$$

$$\frac{1}{2}\partial^2\bar{h}_{\mu\nu} - \frac{1}{2}\partial^{\rho}\partial_{(\mu}\bar{h}_{\nu)\rho} + \frac{1}{2}\eta_{\mu\nu}\partial^{\rho}\partial^{\sigma}\bar{h}_{\rho\sigma} = \frac{1}{4m_{Pl}^{d-2}}\tilde{T}_{\mu\nu} \tag{62}$$

where all the terms containing higher orders in the fields are absorbed into the modified sources $\tilde{J}$ and $\tilde{T}_{\mu\nu}$. Similarly, the dilaton source is given by

$$J(x) = \frac{\delta S_{wl}}{\delta\phi(x)}, \tag{63}$$

where the worldline action is

$$S_{wl} = \int ds e^{\phi}\{-\dot{x}^{\mu}e_{\mu}^{a}p_a + \frac{1}{2}e(p^a p_a - m^2)\}. \tag{64}$$

Again, the coupling to dilaton could have been arbitrary, but this choice of e^{ϕ} is the one that works in the end. The equations of motion are

$$\dot{p}^{\nu}_{\alpha} = -\left[\Gamma^{\nu}_{\rho\sigma}\dot{x}^{\rho} + (\dot{x}^{\nu}\partial_{\sigma} - \dot{x}_{\sigma}\partial^{\nu})\phi\right] p^{\sigma}, \tag{65}$$

$$\dot{x}^{\mu}_{\alpha} = e p^{\mu}_{\alpha}. \tag{66}$$

0.6.2 Radiation amplitude from scattering

The radiation is the solution to the field equation at asymptotic null infinity. From the equations of motions Eqs. (46), (61), (62), we notice that any one of the fields abstracted by $\Psi(x)$ is schematically solved by

$$\Psi(x) \propto \int d^d k e^{-ik\cdot x} \mathcal{D}(k) \tilde{\mathcal{J}}(k), \tag{67}$$

where $\mathcal{D}(k)$ is the propagator and $\tilde{\mathcal{J}}(k)$ is the redefined currents. Since the propagators are all $\mathcal{D}(k) \propto 1/k^2$, at asymptotic spatial infinity, we have (in 4 dimensions)

$$\lim_{|\vec{x}|\to\infty} \Psi_P(x) \propto \frac{1}{|\vec{x}|} \int d\omega e^{-i\omega t} \mathcal{A}_P(k), \tag{68}$$

where $\mathcal{A}_P(k) = g\epsilon_P(k)\tilde{\mathcal{J}}(k)|_{k^2=0}$ is the radiation amplitude in the polarization channel of ϵ_P, defined as being proportional to the on-shell current [35]. In addition, it also gives the radiated power spectrum in phase space

$$\Delta P^{\mu} = \int_k \theta(k^0)\delta(k^2)|\mathcal{A}_P(k)|^2, \tag{69}$$

where $\int_k \equiv \int \frac{d^d k}{(2\pi)^d}$ is the phase space integral. Therefore, the radiation amplitude captures all the information about the radiative fields. It is also an amplitude in the sense that interpreting $\Psi(x) = \langle\Psi(x)\rangle$ as a 1-point function, it can be obtained in a standard LSZ reduction way

$$\mathcal{A}_P(k) = \lim_{k^2\to 0} \epsilon_P(k)\mathcal{D}^{-1}(k) \int_k e^{ik\cdot x}\langle\Psi(x)\rangle. \tag{70}$$

At this point, there is a caveat that the boundary conditions on the propagators are not specified. In this classical system, they are expected to be retarded due to classical causality. However, in this section, the boundary condition does not play any role, so we leave it ambiguous. Specifically, we define the Yang-Mills radiation amplitude as

$$\epsilon_{P,\mu}(k)\mathcal{A}^{\mu}_{a}(k) = g\epsilon_{P,\mu}(k)\tilde{J}^{\mu}_{a}(k), \tag{71}$$

where $\tilde{J}^{\mu}_{a}(k) = \int d^{d}x e^{-ik\cdot x}\tilde{J}^{\mu}_{a}(x)$ and similarly, the dilaton and graviton radiation amplitudes as

$$\mathcal{A}_{d}(k) = -\frac{1}{2m_{Pl}^{(d-2)/2}(d-2)^{1/2}}J(k), \tag{72}$$

respectively.

While the standard Feynman rules take good care of the non-linear interactions, a problem remains that in the original currents such as Eq. (50) also depend on dynamical variables which have to satisfy the equations of motion. For generic orbits, it is not obvious how to solve them, so the simpler case with particles coming from infinity scattering and flying off to infinity. For this setup, the dynamical variables are written as

$$x^{\mu}_{\alpha}(\lambda) = b^{\mu}_{\alpha} + ep^{\mu}_{\alpha}\lambda + z^{\mu}_{\alpha}(\lambda), \tag{73}$$

$$p^{\mu}_{\alpha}(\lambda) = p^{\mu}_{\alpha} + \bar{p}^{\mu}_{\alpha}(\lambda), \tag{74}$$

$$c^{a}_{\alpha}(\lambda) = c^{a}_{\alpha} + \bar{c}^{a}_{\alpha}(\lambda). \tag{75}$$

where the time independent factors are the initial data at $\lambda \to \infty$. In particular, b_{α} is the impact parameter of the scattering particle. We can use the lower-order deflections to calculate the fields $\langle\Psi(x)\rangle$ and substitute them into the equations of motion to solve for higher-order deflections and reiterate. In this way, the dynamical variables and hence the source currents become perturbative as well. For the Yang-Mills equations of motion, the expansion parameter is roughly

$$\frac{g^2c^2}{Eb^{d-3}} \ll 1, \tag{76}$$

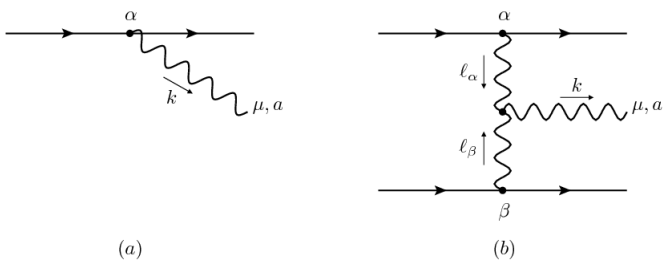

Figure 3: The leading order gluon radiation diagrams, excerpted from [35]. The straight lines with arrow always represent the classical particle worldlines, while the curved lines are gluons.

while the expansion parameter for the field equation is

$$g^2 c b^{4-d} \ll 1, \tag{77}$$

where c is the typical size of the color charges, b the impact parameters, and E the scattering energy scale. Therefore, we can identify the two when $c \sim Eb$, in which case the problem is identical to an expansion in g. In the gravitational case, the expansion parameter is

$$\frac{E}{4m_{Pl}^{d-2} b^{d-3}} \ll 1, \tag{78}$$

so it is equivalent to an expansion in

$$\kappa = \frac{1}{2m_{Pl}^{\frac{d-2}{2}}}. \tag{79}$$

Yang-Mills radiation

The formal solution to the field equation Eq. (46) is

$$\langle A_a^\mu(x) \rangle = -g \int_k \frac{i e^{-ik\cdot x}}{k^2} \tilde{J}_a^\mu(k), \tag{80}$$

from this, we normalize the radiation amplitude as $\mathcal{A}_a^\mu(k) = g\tilde{J}_a^\mu(k)$, such that the

$$\langle A_a^\mu(x) \rangle = -\int_k \frac{i e^{-ik\cdot x}}{k^2} \mathcal{A}_a^\mu(k). \tag{81}$$

For Yang-Mills radiation, [35] found the two diagrams in Fig. 3 at the leading non-trivial order. The straight lines represent the particle worldlines and the vertices on them are therefore the source currents. Due to the perturbative expansion described above, the currents are an infinite series organized by powers of g^2. In Fig. 3(a), the contribution comes from leading-order deflections while in Fig. 3(b) the leading order results arise from interactions between the fields sourced by undeflected charges. The result is given by

$$\begin{aligned}\mathcal{A}_a^\mu(k)\Big|_{\mathcal{O}(g^3)} &= -ig^3\sum_{\alpha,\beta}\int_{\ell_\alpha,\ell_\beta}\mu_{\alpha,\beta}(k)\bigg\{(c_\alpha\cdot c_\beta)\frac{\ell_\alpha^2}{k\cdot p_\alpha}c_\alpha^a\bigg(-(p_\alpha\cdot p_\beta)\left\{\ell_\beta^\mu-\frac{k\cdot\ell_\beta}{k\cdot p_\alpha}p_\alpha^\mu\right\}\\ &\quad + k\cdot p_\alpha p_\beta^\mu - k\cdot p_\beta p_\alpha^\mu\bigg) + i(p_\alpha\cdot p_\beta)f^{abc}c_\alpha^b c_\beta^c\frac{\ell_\alpha^2}{k\cdot p_\alpha}p_\alpha^\mu\\ &\quad + [c_\alpha,c_\beta]^a\left[2(k\cdot p_\beta)p_\alpha^\mu-(p_\alpha\cdot p_\beta)\ell_\alpha^\mu\right],\end{aligned} \tag{82}$$

where the first two lines come from Fig. 3(a) and the last from Fig. 3(b). The commutator means the combination $[c_\alpha,c_\beta]^a\equiv if^{abc}c_\alpha^b c_\beta^c$. In this expression, we have introduced the shorthand notations

$$\mu_{\alpha,\beta}(k)=\left[(2\pi)\delta(\ell_\alpha\cdot p_\alpha)\frac{e^{i\ell_\alpha\cdot b_\alpha}}{\ell_\alpha^2}\right]\left[(2\pi)\delta(\ell_\alpha\cdot p_\alpha)\frac{e^{i\ell_\beta\cdot b_\beta}}{\ell_\beta^2}\right](2\pi)^d\delta^d(\ell_\alpha+\ell_\beta-k). \tag{83}$$

Dilaton-gravity radiation

The formal solution to Eq. (62) is given by

$$\langle h_{\mu\nu}(x)\rangle = 2\kappa^2\int_k\frac{e^{-ik\cdot x}}{k^2}\left[\tilde{T}_{\mu\nu}(k)-\frac{1}{d-2}\eta_{\mu\nu}\tilde{T}^\sigma{}_\sigma(k)\right], \tag{84}$$

so we define the graviton radiation amplitude by $\mathcal{A}_g(k)=-\kappa\epsilon_{\mu\nu}\tilde{T}^{\mu\nu}(k)$, such that

$$\epsilon^{\mu\nu}\langle h_{\mu\nu}(x)\rangle = -2\kappa\int_k\frac{e^{-ik\cdot x}}{k^2}\mathcal{A}_g(k).$$

The diagrams contributing to the leading order are summarized in Fig. 4. Fig. 4(a) results from radiations caused by deflections of the worldline variables. Fig. 4(b), (c) are due to higher (than linear) field dependence in the worldline EFT. Finally, Fig. 4(d), (e) are the consequence of field interactions in spacetime.

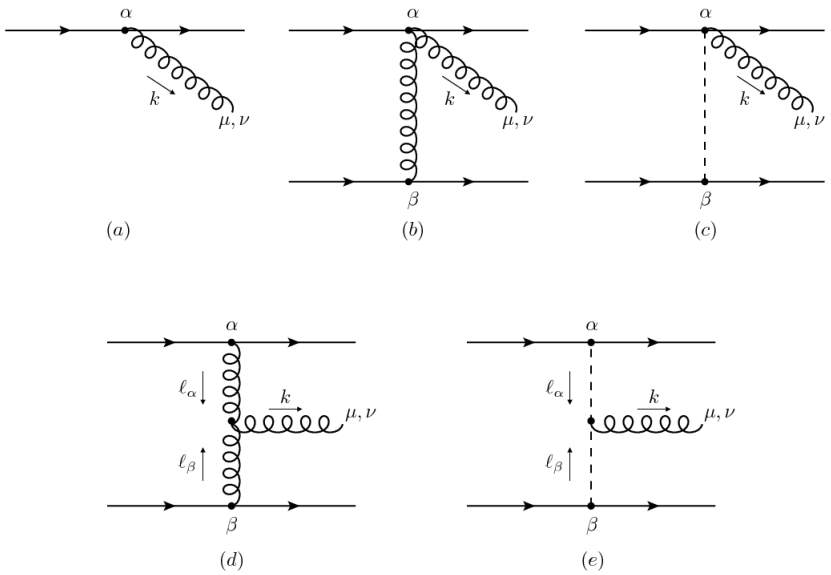

Figure 4: The leading order graviton radiation diagrams, excerpted from [35]. The dashed lines are dilatons and the coiled lines represent gravitons.

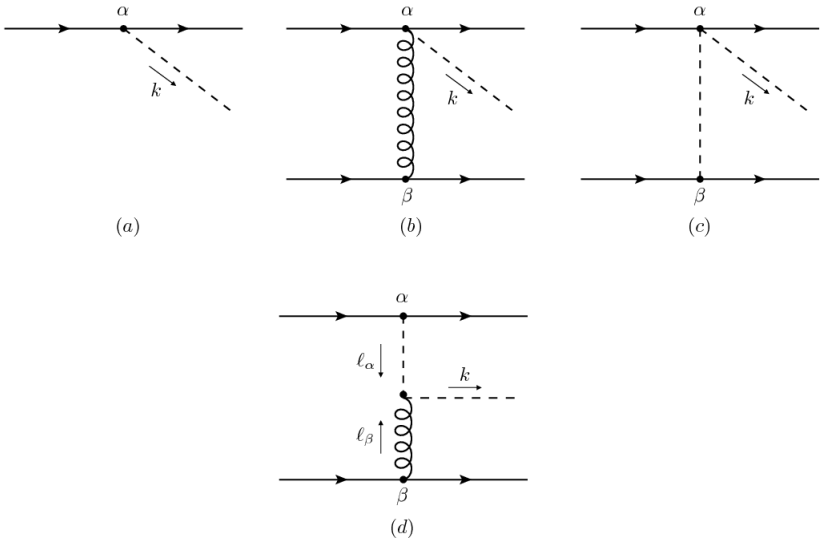

Figure 5: The leading order dilaton radiation diagrams, excerpted from [35].

Similarly, the formal solution to Eq. (61) is given by

$$\langle\phi(x)\rangle = \frac{\kappa^2}{d-2}\int_k \frac{e^{-ik\cdot x}}{k^2}\tilde{J}(k), \tag{85}$$

and we define the dilaton radiation amplitude as $\mathcal{A}_d(k) = -\frac{\kappa}{(d-2)^{1/2}}\tilde{J}(k)$. The relevant diagrams are summarized in Fig. 5 which are also very similar to the graviton diagrams in nature. The results are not particularly relevant to the discussions at hand and are omitted. They appear as Eqs. (50), (55) in Section III of [35].

0.6.3 The classical double copy

The conjectured mapping rules are

$$c_\alpha^a \;\mapsto\; p_\alpha^\mu, \tag{86}$$

$$f^{abc} \;\mapsto\; \Gamma^{\mu\nu\rho}(-k,\ell_\alpha,\ell_\beta), \tag{87}$$

$$g \;\mapsto\; \kappa, \tag{88}$$

where $\Gamma^{\mu\nu\rho}(-k,\ell_\alpha,\ell_\beta)$ is the kinematic factor of the 3-point vertex rule in Yang-Mills theory, given by

$$\Gamma^{\mu\nu\rho}(-k,\ell_\alpha,\ell_\beta) \equiv \frac{1}{2}\left[(\ell_\beta-\ell_\alpha)^\mu\eta^{\nu\rho} + (\ell_\alpha+k)^\rho\eta^{\mu\nu} - (\ell_\beta+k)^\nu\eta^{\mu\rho}\right].$$

The mappings convert the radiation amplitude Eq. (82) into a new one $\mathcal{A}^{\mu\nu}(k)$ which has to satisfy Ward identities on both indices $k_\mu\mathcal{A}^{\mu\nu}(k) = k_\mu\mathcal{A}^{\nu\mu}(k) = 0$ to guarantee the gauge invariance of the resulting theory. Unfortunately, it does not work out naively. However, as radiation amplitude is only defined on-shell, we may add to Eq. (82) any gauge terms that vanish on-shell. Therefore, there is a freedom of adding any term proportional to k^μ since it always vanishes when contracted to k_μ due to the $k^2 = 0$ on-shell condition. By adding the term

$$\hat{\mathcal{A}}_a^\mu(k) = -ig^3\sum_{\alpha,\beta}\int_{\ell_\alpha,\ell_\beta}\mu_{\alpha,\beta}(k)\frac{1}{2}(c_\alpha\cdot c_\beta)c_\alpha^a\frac{\ell_\alpha^2}{k\cdot p_\alpha}(p_\alpha\cdot p_\beta)k^\mu, \tag{89}$$

one can check that the resulting amplitude

$$\mathcal{A}_a^\mu(k) + \hat{\mathcal{A}}_a^\mu(k) \mapsto \mathcal{A}^{\mu\nu}(k) \tag{90}$$

indeed satisfies the Ward identities.

To see how this relates to the gravitational radiation, it shall be decomposed into polarization channels. The direct product of the two copies of vector polarizations

$$\epsilon_\mu \tilde{\epsilon}_\nu = \epsilon_{\mu\nu} + a_{\mu\nu} + \frac{\epsilon \cdot \tilde{\epsilon}}{d-2} \pi_{\mu\nu}, \tag{91}$$

is decomposed into the symmetric traceless channel $\epsilon_{\mu\nu} \equiv \frac{1}{2}(\epsilon_\mu \tilde{\epsilon}_\nu + \epsilon_\nu \tilde{\epsilon}_\mu) - \frac{\epsilon \cdot \tilde{\epsilon}}{d-2} \pi_{\mu\nu}$, the anti-symmetric channel $a_{\mu\nu} \equiv \frac{1}{2}(\epsilon_\mu \tilde{\epsilon}_\nu - \epsilon_\nu \tilde{\epsilon}_\mu)$, and the trace $\pi_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{(k_\mu q_\nu + k_\nu q_\mu)}{k.q}$ where q_μ is an arbitrary reference vector that is not parallel to k_μ. It was verified that $\mathcal{A}^{\mu\nu}(k)$ is in fact symmetric, and the non-vanishing symmetric traceless and trace channels match with the graviton and dilaton amplitudes on-shell [35]

$$\epsilon_{\mu\nu} \mathcal{A}^{\mu\nu}(k) = \mathcal{A}_g(k), \tag{92}$$

$$\frac{\epsilon \cdot \tilde{\epsilon}}{d-2} \pi_{\mu\nu} \mathcal{A}^{\mu\nu}(k) = \mathcal{A}_d(k). \tag{93}$$

0.7 The spinning generalization

In this section, we report on generalizing this construction to spinning sources [20, 21].

0.7.1 The spinning setup and equations of motion

Yang-Mills theory

As discussed in the previous section, there are an infinite number of operators that can be constructed with spins which are not higher-order in g or κ coupling. To avoid this issue, we focus on the regime where the object is rotating slowly so that we could expand in powers of spin and truncate the results to linear-order in spin. At this order in spin, there is only

one possible operator in the Yang-Mills worldline EFT

$$S = -\frac{g}{2}\int d\lambda e c_\alpha^a S_\alpha^{\mu\nu} F_{\mu\nu}^a(x_\alpha). \tag{94}$$

While the coefficient is again arbitrary in principle, this is the value that allows a consistent double copy. Incidentally, in $d = 4$, this can be written as

$$\frac{g}{m}\int dt c^a \vec{S}\cdot\vec{B}^a, \tag{95}$$

which is the non-abelian analog to the Dirac gyromagnetic ratio $g_D = 2$.

The field equation remains the same, but the gauge current becomes

$$J_a^\mu(x) \equiv -\frac{1}{g}\frac{\delta}{\delta A_\mu^a(x)} S_{pp} \tag{96}$$

$$= \sum_\alpha \int dx_\alpha^\mu c_\alpha^a \delta^d(x - x_\alpha) + \int d\lambda e S_\alpha^{\mu\nu} D_\nu \left[c_\alpha^a \delta^d(x - x_\alpha)\right]. \tag{97}$$

Using the covariant continuity equation $D_\mu J_a^\mu = 0$, we find the new conservation equation for the charges

$$(\dot{x}_\alpha \cdot D)\, c_\alpha^a = \frac{g}{2} e f^{abc} S_\alpha^{\mu\nu} F_{\mu\nu}^b c_\alpha^c. \tag{98}$$

The energy-momentum tensor is given by

$$T^{\mu\nu}(x) \equiv -2\frac{\delta}{\delta g_{\mu\nu}(x)} S_{wl}\Big|_{g=\eta}$$

$$= \sum_\alpha \int dx^{(\mu} p^{\nu)} \delta^d(x - x_\alpha) + \int dx^{(\mu} S^{\nu)\sigma} \partial_\sigma \delta^d(x - x_\alpha) \tag{99}$$

$$+ g\int d\lambda \delta^d(x - x_\alpha) c_\alpha^a F^a{}_\sigma{}^{(\mu} S^{\nu)\sigma}. \tag{100}$$

Given the Yang-Mills energy-momentum tensor $T_{YM}^{\mu\nu}$, a corollary of Noether's theorem is that under arbitrary support X , $\int d^d x X_\nu \partial_\mu \left(T^{\mu\nu} + T_{YM}^{\mu\nu}\right) = 0$ is true when equations of motion are satisfied. From this fact, we can extract the equations of motion

$$\dot{p}_\alpha^\mu = g_s c_\alpha^a F_a^{\mu\nu} \dot{x}_{\alpha\nu} + \frac{g}{2} e S_\alpha^{\rho\sigma} c_\alpha^a D^\mu F_{\rho\sigma}^a, \tag{101}$$

$$\dot{S}^{\mu\nu}_\alpha = \dot{x}^\nu_\alpha p^\mu_\alpha - \dot{x}^\mu_\alpha p^\nu_\alpha + gec^a_\alpha F^a_\sigma{}^\mu S^{\nu\sigma}_\alpha - gec^a_\alpha F^a_\sigma{}^\nu S^{\mu\sigma}_\alpha. \tag{102}$$

The remaining equation for the position variable is most conveniently obtained by requiring the constraint to hold along the worldline

$$\frac{d}{ds}\left(S^{\mu\nu}_\alpha p_{\alpha\nu}\right) = 0, \tag{103}$$

and substitute in the equations above to solve for $\dot{x}^\mu_\alpha$. In solving these problems, we assume that the spin also separates into the initial data and the deflections

$$S^{\mu\nu}_\alpha(\lambda) = S^{\mu\nu}_\alpha + \bar{S}^{\mu\nu}_\alpha(\lambda). \tag{104}$$

String-gravity theory

In this case, we include the previously ignored anti-symmetric gauge field channel denoted by $B_{\mu\nu} = -B_{\nu\mu}$ which has an anti-symmetric gauge transformation

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \xi_\nu - \partial_\nu \xi_\mu. \tag{105}$$

The complete action that works for the double copy is given by

$$S_g = -2m_{Pl}^{d-2}\int d^dx\sqrt{g}\left[R - (d-2)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{e^{-4\phi}}{12}H_{\mu\nu\sigma}H^{\mu\nu\sigma}\right], \tag{106}$$

where $H_{\mu\nu\sigma} = 6\partial_{[\mu}B_{\nu\sigma]}$ is the gauge-invariant field strength tensor. We call this the string-gravity action since it is precisely the low-energy effective action of the massless common sector of oriented string theories. In that context, the $B_{\mu\nu}$ field is called the Kalb-Ramond 2-form field, which we will call an axion since in 4 dimensions, the field strength is equivalent to an odd-parity scalar field $H_{\mu\nu\sigma} = \epsilon_{\mu\nu\sigma\rho}\partial^\rho\phi$ known by the same name [69]. We note that this is somewhat expected due to its roles in the double copy of gauge theory amplitudes [70] as well as in Kerr-Schild double copy [71]. For the worldline theory, we also have one additional possibility

$$S_{wl} \supset \frac{1}{4}\sum_\alpha \int d\lambda S^{\mu\nu}_\alpha \dot{x}^\sigma_\alpha H_{\mu\nu\sigma} e^{-2\phi}. \tag{107}$$

Incidentally, this coupling together with the graviton coupling term has a simpler geometric interpretation in terms of the "string frame" spin $\tilde{S}_{\mu\nu} = e^{2\phi} S_{\mu\nu}$, which is given by

$$S_{wl} \supset \frac{1}{2} \int dx^{\rho} C^{+}{}_{\mu\nu\rho} \tilde{S}^{\mu\nu}. \tag{108}$$

The symmetric part $C^{+\rho}{}_{(\mu\nu)} = \tilde{\Gamma}^{\rho}{}_{\mu\nu}$ corresponds to the Levi-Civita connection compatible with string frame metric $\tilde{g}_{\mu\nu} = e^{2\phi} g_{\mu\nu}$ and the torsion $T^{\rho}{}_{\mu\nu} = C^{+\rho}{}_{[\mu\nu]} = e^{-2\phi} H^{\rho}{}_{\mu\nu} \equiv \tilde{H}^{\rho}{}_{\mu\nu}$ corresponds to the axion field strength.

The field equations arising from these actions are given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{4 m_{Pl}^{d-2}} (T_{\mu\nu} + T_{\phi}^{\mu\nu} + T_{B}^{\mu\nu}), \tag{109}$$

$$\nabla^{\mu} \nabla_{\mu} \phi = -\frac{1}{4 m_{Pl}^{d-2} (d-2)} J + \frac{e^{-4\phi}}{6(d-2)} H^2, \tag{110}$$

$$\nabla_{\lambda} (e^{-4\phi} H^{\mu\nu\lambda}) = \frac{1}{m_{Pl}^{d-2}} J^{\mu\nu}, \tag{111}$$

where the currents latter two currents are given by

$$J \equiv \sum_{\alpha} \int ds \left(\dot{x}_{\alpha}^{\mu} p_{\alpha\mu} e^{\phi} + S_{\alpha}^{\mu\nu} \dot{x}_{\alpha}^{\sigma} H_{\mu\nu\sigma} e^{-2\phi} \right) \frac{\delta^d (x - x_{\alpha})}{\sqrt{g}}, \tag{112}$$

$$J^{\mu\nu} \equiv \sum_{\alpha} \frac{1}{4} \int ds (S_{\alpha}^{\lambda\mu} \dot{x}_{\alpha}^{\nu} + S_{\alpha}^{\nu\lambda} \dot{x}_{\alpha}^{\mu} + S_{\alpha}^{\mu\nu} \dot{x}_{\alpha}^{\lambda}) \nabla_{\lambda} \left[e^{-2\phi} \frac{\delta^d (x - x_{\alpha})}{\sqrt{g}} \right]. \tag{113}$$

The energy-momentum tensors are given by

$$\begin{aligned} T^{\mu\nu} =& \sum_{\alpha} \int dx_{\alpha}^{(\mu} p_{\alpha}^{\nu)} \frac{\delta^d (x - x_{\alpha})}{\sqrt{g}} e^{\phi} + \int dx_{\alpha}^{(\mu} S_{\alpha}^{\nu)\alpha} \nabla_{\alpha} \left[\frac{\delta^d (x - x_{\alpha})}{\sqrt{g}} \right] \\ & - \frac{1}{2} \int dx_{\alpha}^{\sigma} H_{\rho\lambda\sigma} g^{\lambda(\mu} S_{\alpha}^{\nu)\rho} e^{-2\phi} \frac{\delta^d (x - x_{\alpha})}{\sqrt{g}}, \end{aligned} \tag{114}$$

$$T_{B}^{\mu\nu} = -m_{Pl}^{d-2} e^{-4\phi} \{ H^{\mu\rho\sigma} H^{\nu}{}_{\rho\sigma} - \frac{1}{6} H^2 g^{\mu\nu} \}, \tag{115}$$

and $T_{\phi}^{\mu\nu}$ remains unchanged. Using the covariant conservation $\int d^d x \sqrt{g} X_{\nu} \nabla_{\mu} \Big(T_{pp}^{\mu\nu} + T_{B}^{\mu\nu} +$

$T_\phi^{\mu\nu}\Big) = 0$, we obtain the full equations of motion for the momentum and spin

$$\begin{aligned}\dot{p}_\alpha^\mu =& p_{\alpha\nu}\dot{x}_\alpha^\nu\partial^\mu\phi - p_\alpha^\mu\dot{x}_\alpha^\sigma\partial_\sigma\phi - \Gamma^\mu_{\sigma\rho}\dot{x}_\alpha^\sigma p_\alpha^\rho + \frac{1}{2}R^\mu{}_{\nu\lambda\sigma}\dot{x}_\alpha^\nu S_\alpha^{\sigma\lambda}e^{-\phi} \\ &+\frac{1}{4}\dot{x}_\alpha\cdot\nabla(g^{\mu\lambda}H_{\lambda\sigma\rho}S_\alpha^{\sigma\rho}e^{-2\phi})e^{-\phi} - \frac{1}{2}S_\alpha^{\lambda\rho}\dot{x}_\alpha^\sigma H_{\lambda\rho\sigma}e^{-3\phi}\partial^\mu\phi,\end{aligned} \tag{116}$$

$$\begin{aligned}\dot{S}_\alpha^{\mu\nu} =& p_\alpha^\mu\dot{x}_\alpha^\nu e^\phi - p_\alpha^\nu\dot{x}_\alpha^\mu e^\phi - \Gamma^\mu_{\sigma\rho}S_\alpha^{\rho\nu}\dot{x}_\alpha^\sigma - \Gamma^\nu_{\sigma\rho}S_\alpha^{\mu\rho}\dot{x}_\alpha^\sigma \\ &-\frac{1}{2}H_{\lambda\sigma\rho}g^{\lambda\mu}S_\alpha^{\nu\rho}\dot{x}_\alpha^\sigma e^{-2\phi} + \frac{1}{2}H_{\lambda\sigma\rho}g^{\lambda\nu}S_\alpha^{\mu\rho}\dot{x}_\alpha^\sigma e^{-2\phi}.\end{aligned} \tag{117}$$

The position equation is again obtained by requiring the spin constraint Eq. (103).

For the axion field equation (111), we choose the gauge $\partial_\mu B^{\mu\nu} = 0$ and the equation becomes

$$\partial^2 B^{\mu\nu}(x) = 4\kappa^2\tilde{J}^{\mu\nu}(x), \tag{118}$$

and correspondingly, we define the axion radiation amplitude by

$$\mathcal{A}_a = 2\kappa a_{\mu\nu}\tilde{J}^{\mu\nu}(k). \tag{119}$$

0.7.2 Yang-Mills radiation from spinning charges

To shorten the writings, we introduce the notations

$$S_\alpha^{\mu\nu}p_\nu \equiv (S_\alpha\wedge p)^\mu, \tag{120}$$

$$k_\mu S_\alpha^{\mu\nu}p_\nu \equiv (k\wedge p)_\alpha. \tag{121}$$

The leading order current corresponds to the Feynman diagrams in Figs. 6(a), (b) which is given by

$$\tilde{J}_a^\mu(k)\big|_{\text{Fig. 6(a)+(b)}} = \sum_\alpha\int d\lambda\, e^{ik\cdot x_\alpha}c_\alpha^a\left[\dot{x}_\alpha^\mu - i(S_\alpha\wedge k)^\mu\right]. \tag{122}$$

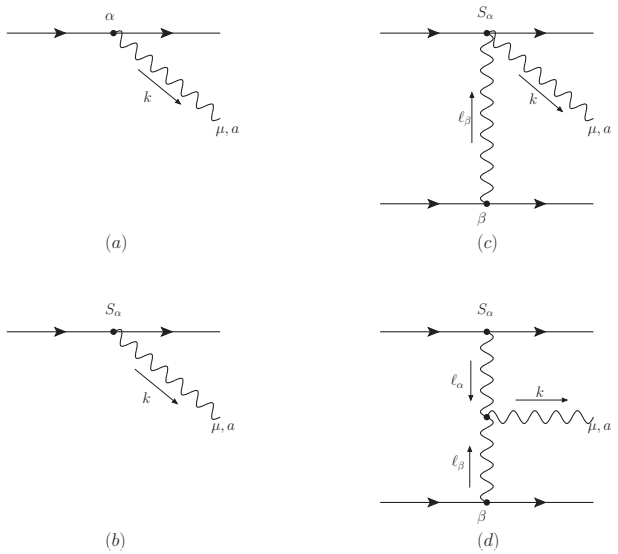

Figure 6: The $\mathcal{O}(\kappa^3, S^1)$ gluon radiation diagrams

It sources the field at the intermediate order

$$\begin{aligned} A_a^\mu(x)\big|_{\mathcal{O}(g^1)} &= g\int \frac{d^d\ell}{(2\pi)^d}\frac{e^{-i\ell\cdot x}}{\ell^2}\tilde{J}_a^\mu(\ell)\big|_{\mathcal{O}(g_s^1)} \\ &= g\sum_\alpha \int d\lambda e\frac{d^d\ell}{(2\pi)^d}\frac{e^{-i\ell\cdot(x-x_\alpha)}}{\ell^2}c_\alpha^a\left[p_\alpha^\mu + i\kappa_\alpha(S_\alpha\wedge\ell)^\mu\right], \end{aligned} \tag{123}$$

which by the equations of motion, leads to deflections: for colors

$$\dot{\tilde{c}}_\alpha^\mu\big|_{\mathcal{O}(g^2,S^0)} = -g^2\sum_{\beta\neq\alpha}\int_l 2\pi\delta(\ell\cdot p_\beta)\frac{e^{-i\ell\cdot(b_{\alpha\beta}+e\lambda p_\alpha)}}{\ell^2}f^{abc}c_\alpha^b c_\beta^c(p_\alpha\cdot p_\beta), \tag{124}$$

$$\dot{\tilde{c}}_\alpha^\mu\big|_{\mathcal{O}(g^2,S^1)} = -ig^2\sum_{\beta\neq\alpha}\int_l 2\pi\delta(\ell\cdot p_\beta)\frac{e^{-i\ell\cdot(b_{\alpha\beta}+e\lambda p_\alpha)}}{\ell^2}f^{abc}c_\alpha^b c_\beta^c(p_{\alpha\mu}\ell_\nu S_\beta^{\mu\nu} + p_{\beta\nu}\ell_\mu S_\alpha^{\mu\nu}), \tag{125}$$

for momenta

$$\dot{\tilde{p}}_\alpha^\mu\big|_{\mathcal{O}(g^2,S^0)} = ig^2\sum_\beta(c_\alpha\cdot c_\beta)p_{\alpha\nu}\int_\ell 2\pi\delta(\ell\cdot p_\beta)\frac{e^{-i\ell\cdot(b_{\alpha\beta}+e\lambda p_\alpha)}}{\ell^2}(\ell^\mu p_\beta^\nu - \ell^\nu p_\beta^\mu), \tag{126}$$

$$\dot{\tilde{p}}_\alpha^\mu\big|_{\mathcal{O}(g^2,S^1)} = g^2\sum_\beta(c_\alpha\cdot c_\beta)\int_l 2\pi\delta(\ell\cdot p_\beta)\frac{e^{-i\ell\cdot(b_{\alpha\beta}+e\lambda p_\alpha)}}{\ell^2}[p_{\alpha\nu}(\ell^\mu\ell_\sigma S_\beta^{\nu\sigma} - \ell^\nu\ell_\sigma S_\beta^{\mu\sigma}) + S_\alpha^{\rho\sigma}p_{\beta\sigma}\ell_\rho\ell^\mu] \tag{127}$$

for positions

$$\dot{z}^\mu_\alpha - e\bar{p}^\mu_\alpha\big|_{\mathcal{O}(\eta^2,S^1)} = 0, \tag{128}$$

(we note that this would be non-zero if $g_D \neq 2$) and for spins

$$\dot{\bar{S}}^{\mu\nu}_\alpha\big|_{\mathcal{O}(\eta^2,S^1)} = \dot{z}^\nu_\alpha p^\mu_\alpha - \dot{z}^\mu_\alpha p^\nu_\alpha + ig^2 \sum_\beta \int_l 2\pi\delta(\ell\cdot p_\beta)\frac{e^{-i\ell\cdot(b_{\alpha\beta}+e\lambda p_\alpha)}}{\ell^2}$$
$$\times (c_\alpha\cdot c_\beta)([\ell_\sigma p^\mu_\beta - \ell^\mu p_{\beta\sigma}]S^{\nu\sigma}_\alpha - [\ell_\sigma p^\nu_\beta - \ell^\nu p_{\beta\sigma}]S^{\mu\sigma}_\alpha). \tag{129}$$

Substituting these deflections into the lower order amplitude, we find

$$\mathcal{A}^\mu_a(k)\big|_{\text{Fig. 6(a)+(b)};\mathcal{O}(g^3,S^1)} = \sum_\alpha \int d\lambda e^{ik\cdot x_\alpha}\frac{i}{k\cdot\dot{x}_\alpha}\left[\dot{c}^a_\alpha \dot{x}^\mu_\alpha + c^a_\alpha\left\{\ddot{x}^\mu_\alpha - \frac{k\cdot\ddot{x}_\alpha}{k\cdot v_\alpha}v^\mu_\alpha\right\}\right]\Bigg|_{\mathcal{O}(g^3,S^1)} \tag{130}$$

$$+\frac{1}{k\cdot\dot{x}_\alpha}\left[\dot{c}^a_\alpha(S_\alpha\wedge k) + c^a_\alpha\left\{(\dot{S}_\alpha\wedge k) - \frac{k\cdot\ddot{x}_\alpha}{k\cdot\dot{x}_\alpha}(S_\alpha\wedge k)^\mu\right\}\right]\Bigg|_{\mathcal{O}(g^3,S^1)}, \tag{131}$$

where the explicit results are given by Eqs. (316), (317) in Appendix .1. At this order $O(g^3)$ in perturbation, there are two contributions from diagrams without deflections in the particle trajectories. The first of these is from Fig. 6(c),

$$\mathcal{A}^\mu_a(k)\big|_{\text{Fig. 6(c)};\mathcal{O}(g^3,S^1)} = ig_s^2 \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} \int \mu_{\alpha\beta}(k)[c_\alpha,c_\beta]^a \ell^2_\alpha (S_\alpha\wedge p_\beta)^\mu. \tag{132}$$

The second of the zero deflection contributions is from the diagram with the triple vertex in Fig. 6(d),

$$\mathcal{A}^\mu_a(k)\big|_{\text{Fig. 6(d)};\mathcal{O}(g^3,S^1)} = -ig^3 \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} \int \mu_{\alpha\beta}(k)[c_\alpha,c_\beta]^a\{2(k\cdot p_\beta)(S_\alpha\wedge\ell_\alpha)^\mu$$
$$+(\ell_\alpha\wedge p_\beta)_\alpha(\ell_\alpha-\ell_\beta)^\mu + 2(\ell_\alpha\wedge\ell_\beta)_\alpha p^\mu_\beta\}. \tag{133}$$

Combining them together, we find the results in terms of a sum of two different color

structures

$$\mathcal{A}_a^\mu(k)\big|_{\mathcal{O}(g^3,S^1)} = ig^3 \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \int d\mu_{\alpha\beta}(k) \left[(c_\alpha \cdot c_\beta) c_\alpha^a \mathcal{A}_s^\mu + [c_\alpha, c_\beta]^a \mathcal{A}_{adj}^\mu \right], \tag{134}$$

with

$$\begin{aligned}
\mathcal{A}_{adj}^\mu \;\equiv\; & -\left[(\ell_\alpha \wedge p_\beta)_\alpha (\ell_\beta - \ell_\alpha)^\mu - \frac{\ell_\alpha^2}{k\cdot p_\alpha}(\ell_\beta \wedge p_\beta)_\alpha p_\alpha^\mu - \frac{\ell_\beta^2}{k\cdot p_\beta}(\ell_\alpha \wedge p_\beta)_\alpha p_\beta^\mu + \ell_\alpha^2 (S_\alpha \wedge p_\beta)^\mu \right] \\
& + 2(k\cdot p_\beta)\left[(S_\alpha \wedge \ell_\alpha)^\mu - \frac{(k\wedge \ell_\alpha)_\alpha}{k\cdot p_\beta} p_\beta^\mu \right] + \frac{\ell_\alpha^2}{k\cdot p_\alpha}(p_\alpha \cdot p_\beta)(S_\alpha \wedge k)^\mu,
\end{aligned} \tag{135}$$

and

$$\begin{aligned}
\mathcal{A}_s^\mu \;\equiv\; & +\ell_\alpha^2 \left[(S_\beta \wedge \ell_\beta)^\mu - \frac{(k\wedge \ell_\beta)_\beta}{k\cdot p_\alpha} p_\alpha^\mu \right] \\
& + \frac{\ell_\alpha^2}{k\cdot p_\alpha}\left[(k\cdot p_\beta)\left\{ (S_\alpha \wedge \ell_\beta)^\mu - \frac{(k\wedge \ell_\beta)_\alpha}{k\cdot p_\beta} p_\beta^\mu \right\} - (k\cdot \ell_\beta)\left\{ (S_\alpha \wedge p_\beta)^\mu - \frac{(k\wedge p_\beta)_\alpha}{k\cdot p_\beta}\ell_\beta^\mu \right\} \right] \\
& - \frac{\ell_\alpha^2}{k\cdot p_\alpha}\left[(\ell_\beta \wedge p_\beta)_\alpha \left\{ \ell_\beta^\mu - \frac{k\cdot \ell_\beta}{k\cdot p_\alpha} p_\alpha^\mu \right\} + (k\cdot p_\beta)(S_\alpha \wedge k)^\mu \right] + \kappa_\beta \frac{\ell_\alpha^2}{k\cdot p_\alpha}(\ell_\beta \wedge p_\alpha)_\beta \left[\ell_\beta^\mu - \frac{k\cdot \ell_\beta}{k\cdot p_\alpha} p_\alpha^\mu \right] \\
& + \frac{\ell_\alpha^2}{(k\cdot p_\alpha)^2}(p_\alpha \cdot p_\beta)(k\cdot \ell_\beta)(S_\alpha \wedge k)^\mu.
\end{aligned} \tag{136}$$

0.7.3 String-gravity radiation from spinning particles

As we are focusing on linear-order in spin, there is no need to consider the deflection from the axion field. The reason is that it is always sourced at one order in spin by Eq. (113) and couples to the equations of motion Eqs. (116), (117) at another which is already second order in spin.

The graviton and dilaton fields at the lowest order are given by

$$\langle h^{\mu\nu}(x)\rangle\big|_{\mathcal{O}(\kappa^2)} = 2\kappa^2 \sum_\alpha \int \frac{d^d\ell}{(2\pi)^d} \frac{e^{-i\ell\cdot(x-x_\alpha)}}{\ell^2} \left[p_\alpha^\mu p_\alpha^\nu - \frac{p_\alpha^2}{d-2}\eta^{\mu\nu} - \frac{i}{2}\left\{ p_\alpha^\mu (S_\alpha \wedge \ell)^\nu + p_\alpha^\nu (S_\alpha \wedge \ell)^\mu \right\} \right], \tag{137}$$

$$\langle \phi(x)\rangle\big|_{\mathcal{O}(\kappa^2)} = \frac{\kappa^2}{(d-2)} \sum_\alpha p_\alpha^2 \int \frac{d^d\ell}{(2\pi)^d} \frac{e^{-i\ell\cdot(x-x_\alpha)}}{\ell^2}. \tag{138}$$

which in the equations of motion upto linear order in spin

$$\dot{p}^\mu_\alpha\big|_{\mathcal{O}(S^0)+\mathcal{O}(S^1)} = p_{\alpha\nu}\dot{x}^\nu_\alpha\partial^\mu\phi - p^\mu_\alpha\dot{x}^\sigma_\alpha\partial_\sigma\phi - \Gamma^\mu_{\sigma\rho}\dot{x}^\sigma_\alpha p^\rho_\alpha + \frac{1}{2}R^\mu{}_{\nu\lambda\sigma}\dot{x}^\nu_\alpha S^{\sigma\lambda}_\alpha e^{-\phi}, \tag{139}$$

$$\dot{S}^{\mu\nu}_\alpha\big|_{\mathcal{O}(S^0)+\mathcal{O}(S^1)} = p^\mu_\alpha\dot{x}^\nu_\alpha e^\phi - p^\nu_\alpha\dot{x}^\mu_\alpha e^\phi - \Gamma^\mu_{\sigma\rho}S^{\rho\nu}_\alpha\dot{x}^\sigma_\alpha - \Gamma^\nu_{\sigma\rho}S^{\mu\rho}_\alpha v^\sigma_\alpha, \tag{140}$$

leads to deflections in momentum

$$\dot{\bar{p}}^\mu_\alpha\big|_{\mathcal{O}(\kappa^2,S^0)} = i\kappa^2\sum_{\beta\neq\alpha}\int_\ell 2\pi\delta(\ell\cdot p_\beta)\frac{e^{-i\ell\cdot(b_{\alpha\beta}+e\lambda p_\alpha)}}{\ell^2}\bigg[-\frac{p^2_\beta}{d-2}(\ell\cdot p_\alpha)p^\nu_\alpha + 2(p_\alpha\cdot p_\beta)(\ell\cdot p_\alpha)p^\nu_\beta - (p_\alpha\cdot p_\beta)^2\ell^\nu\bigg], \tag{141}$$

$$\begin{aligned}\dot{\bar{p}}^\mu_\alpha\big|_{\mathcal{O}(\kappa^2,S^1)} = \kappa^2\sum_{\beta\neq\alpha}\int_\ell 2\pi\delta(\ell\cdot p_\beta)\frac{e^{-i\ell\cdot(b_{\alpha\beta}+e\lambda p_\alpha)}}{\ell^2}\bigg[&\frac{p^2_\beta}{d-2}(\ell\cdot p_\alpha)(S_\alpha\wedge\ell)^\mu \\ &+[(\ell\wedge p_\beta)_\alpha - (\ell\wedge p_\alpha)_\beta][(\ell\cdot p_\alpha)p^\mu_\beta - (p_\alpha\cdot p_\beta)\ell^\mu] + (\ell\cdot p_\alpha)(p_\alpha\cdot p_\beta)(S_\beta\wedge\ell)^\mu\bigg],\end{aligned} \tag{142}$$

in position

$$\dot{z}^\mu_\alpha - e\bar{p}^\mu_\alpha\big|_{\mathcal{O}(\eta^2,S^1)} \equiv -\frac{i\kappa^2}{(d-2)}\sum_{\beta\neq\alpha}m^2_\beta\int_\ell 2\pi\delta(\ell\cdot p_\beta)\frac{e^{-i\ell\cdot(b_{\alpha\beta}+e\lambda p_\alpha)}}{\ell^2}(S_\alpha\wedge\ell)^\mu,$$

and in spin

$$\begin{aligned}\dot{\bar{S}}^{\mu\nu}_\alpha\big|_{\mathcal{O}(\eta^2,S^1)} = i\kappa^2\sum_{\beta\neq\alpha}\int_\ell 2\pi\delta(\ell\cdot p_\beta)\frac{e^{-i\ell\cdot(b_{\alpha\beta}+e\lambda p_\alpha)}}{\ell^2}\bigg[&(\ell\cdot p_\alpha)\Big\{p^\nu_\beta(S_\alpha\wedge p_\beta)^\mu - p^\mu_\beta(S_\alpha\wedge p_\beta)^\nu\Big\} \\ &+(p_\alpha\cdot p_\beta)\Big\{p^\nu_\beta(S_\alpha\wedge\ell)^\mu - p^\mu_\beta(S_\alpha\wedge\ell)^\nu + \ell^\mu(S_\alpha\wedge p_\beta)^\nu - \ell^\nu(S_\alpha\wedge p_\beta)^\mu\Big\} \\ &+\frac{p^2_\beta}{d-2}\Big\{p^\mu_\alpha(S_\alpha\wedge\ell)^\nu - p^\nu_\alpha(S_\alpha\wedge\ell)^\mu - \ell^\mu(S_\alpha\wedge p_\alpha)^\nu + \ell^\nu(S_\alpha\wedge p_\alpha)^\mu - 2(\ell\cdot p_\alpha)S^{\mu\nu}_\alpha\Big\}\bigg].\end{aligned} \tag{143}$$

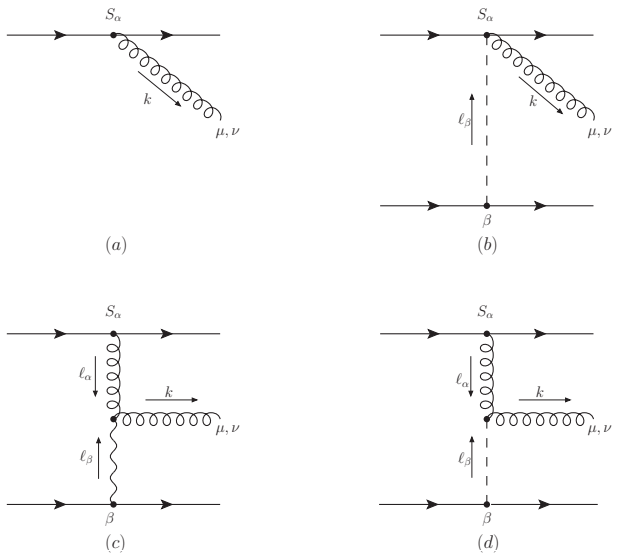

Figure 7: The $\mathcal{O}(\kappa^3, S^1)$ axion radiation diagrams. The coiled lines here are now axions and the thin curved lines represent the gravitons.

The axion channel

The leading order radiation has contributions from Figs. 7(a)-(d). The contribution from Fig. 7(a) is due to deflections induced by the leading order fields. It is given by

$$\mathcal{A}_a(k)\big|_{\text{Fig. 7(a)};\mathcal{O}(\kappa^3,S^1)} = \frac{i}{2}\kappa^3 a_{\mu\nu}\sum_\alpha \int d\tau_\alpha e^{ik\cdot x_\alpha}\frac{k_\lambda}{k\cdot\dot{x}_\alpha}\Bigg[-\frac{k\cdot\ddot{x}_\alpha}{k\cdot\dot{x}_\alpha}S_\alpha^{\lambda\mu}\dot{x}_\alpha^\nu + S_\alpha^{\lambda\mu}\ddot{x}_\alpha^\nu \tag{144}$$

$$+\dot{S}_\alpha^{\lambda\mu}\dot{x}_\alpha^\nu + \text{cyclic permutations } (\mu,\nu,\lambda)\Bigg]\Bigg|_{\mathcal{O}(\eta^2,S^1)}. \tag{145}$$

We can explicitly evaluate these expressions by substituting deflections derived previously. The resulting expressions are given by Eqs. (318), (319) in Appendix .2.

The other contributions to the axion amplitude, at this order in perturbation, come from diagrams with no deflections in the trajectories of the particles. Fig. 7(b), with an intermediate dilaton, corresponds to

$$\mathcal{A}_a(k)\big|_{\text{Fig. 7(b)};\mathcal{O}(\eta^2,S^1)} = -\frac{i\kappa^3 a_{\mu\nu}}{2(d-2)}\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} m_\beta^2\int d\mu_{\alpha\beta}(k)\ell_\alpha^2\Big[(k\cdot p_\alpha)S_\alpha^{\mu\nu} - 2(S_\alpha\wedge k)^\mu p_\alpha^\nu - (\mu\leftrightarrow\nu)\Big]. \tag{146}$$

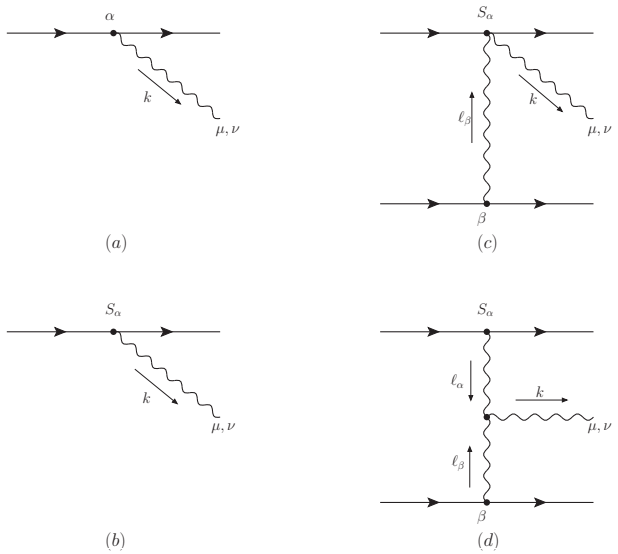

Figure 8: The $\mathcal{O}(\kappa^3, S^1)$ graviton radiation diagrams

The two 3-point vertex diagrams in Figs. 7(c), (d) contribute

$$\mathcal{A}_a(k)\big|_{\text{Fig. 7(c)};\mathcal{O}(\eta^2,S^1)} = -\frac{2i\kappa^3 a_{\mu\nu}}{d-2}\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} p_\beta^2 \int \mu_{\alpha\beta}(k)\Big[\Big\{(k\cdot\ell_\alpha)(S_\alpha\wedge\ell_\alpha)^\mu p_\alpha^\nu - (k\cdot p_\alpha)(S_\alpha\wedge\ell_\alpha)^\mu \ell_\alpha^\nu + (k\wedge\ell_\alpha)_\alpha p_\alpha^\mu \ell_\alpha^\nu\Big\} - (\mu\leftrightarrow\nu)\Big], \tag{147}$$

$$\begin{aligned}\mathcal{A}_a(k)\big|_{\text{Fig. 7(d)};\mathcal{O}(\eta^2,S^1)} = -i\kappa^3 a_{\mu\nu}\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\int \mu_{\alpha\beta}(k)\Big[&(p_\alpha\cdot p_\beta)\Big\{(k\cdot\ell_\alpha)(S_\alpha\wedge\ell_\alpha)^\mu p_\beta^\nu - (k\cdot p_\beta)(S_\alpha\wedge\ell_\alpha)^\mu\ell_\alpha^\nu\\ &- (k\wedge\ell_\alpha)_\alpha l_\alpha^\mu p_\beta^\nu\Big\} + (k\cdot p_\beta)^2(S_\alpha\wedge\ell_\alpha)^\mu p_\alpha^\nu - (k\cdot p_\alpha)(k\cdot p_\beta)(S_\alpha\wedge\ell_\alpha)^\mu p_\beta^\nu\\ &+ (k\cdot p_\beta)(k\wedge\ell_\alpha)_\alpha p_\alpha^\mu p_\beta^\nu + (k\cdot p_\beta)(\ell_\alpha\wedge p_\beta)_\alpha \ell_\alpha^\mu p_\alpha^\nu - (k\cdot p_\alpha)(\ell_\alpha\wedge p_\beta)_\alpha \ell_\alpha^\mu p_\beta^\nu\\ &+ (k\cdot\ell_\alpha)(\ell_\alpha\wedge p_\beta)_\alpha p_\alpha^\mu p_\beta^\nu - \frac{2p_\beta^2}{d-2}\Big\{(k\cdot\ell_\alpha)(S_\alpha\wedge\ell_\alpha)^\mu p_\alpha^\nu - (k\cdot p_\alpha)(S_\alpha\wedge\ell_\alpha)^\mu\ell_\alpha^\nu\\ &+ (k\wedge\ell_\alpha)_\alpha p_\alpha^\mu\ell_\alpha^\nu\Big\} - (\mu\leftrightarrow\nu)\Big].\end{aligned} \tag{148}$$

The graviton channel

Similarly, the gravitational radiation receives contributions from Fig. 8(a)-(d). The contribution from Figs. 8(a), (b) come from deflections to the particle spin and trajectory due to

the leading order fields. This comes out to be

$$\mathcal{A}_g(k)\big|_{\text{Fig. 8(a)+(b)};\mathcal{O}(\kappa^3,S^1)} = -\kappa\epsilon_{\mu\nu}\sum_\alpha \int d\lambda e^{ik\cdot x_\alpha}\Bigg[\frac{-i}{k\cdot\dot{x}_\alpha}\Bigg\{\frac{k\cdot\ddot{x}_\alpha}{k\cdot\dot{x}_\alpha}\dot{x}^\mu_\alpha p^\nu_\alpha - \dot{v}^\mu_\alpha p^\nu_\alpha - \dot{x}^\mu_\alpha \dot{p}^\nu_\alpha\Bigg\} \tag{149}$$

$$- \frac{k_\rho}{k\cdot\dot{x}_\alpha}\Bigg\{\frac{k\cdot\ddot{x}_\alpha}{k\cdot\dot{x}_\alpha}\dot{x}^\mu_\alpha S^{\nu\rho}_\alpha - \dot{x}^\mu_\alpha S^{\nu\rho}_\alpha\Bigg\} + (\mu\leftrightarrow\nu)\Bigg]\Bigg|_{\mathcal{O}(\kappa^3,S^1)}. \tag{150}$$

These contributions are explicitly given by Eqs. (320), (321) in Appendix .2.

There are two contributions from emission off bulk vertices with the particles not suffering any deflections. The first of these is from Fig. 8(c) with an intermediate graviton. This contributes

$$\mathcal{A}_g(k)\big|_{\text{Fig. 8(c)};\mathcal{O}(\kappa^3,S^1)} = -2\kappa\epsilon_{\mu\nu}\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\int \mu_{\alpha\beta}(k)\frac{\ell^2_\alpha}{2}\Big[(p_\alpha\cdot p_\beta)\Big\{(S_\alpha\wedge p_\beta)^\mu \ell^\nu_\beta - (S_\alpha\wedge\ell_\beta)^\mu p^\nu_\beta\Big\}$$
$$- (k\cdot p_\alpha)(S_\alpha\wedge p_\beta)^\mu p^\nu_\beta + \frac{m^2_\beta}{d-2}(S_\alpha\wedge\ell_\beta)^\mu p^\nu_\alpha + (\mu\leftrightarrow\nu)\Big]. \tag{151}$$

The final contribution is from the graviton triple vertex diagram in Fig. 8(d). As in [**?**], in computing this contribution, we use the background field gauge 3-point vertex. This gives

$$\mathcal{A}_g(k)\big|_{\text{Fig. 8(d)};\mathcal{O}(\eta^2,S^1)} = -2\kappa\epsilon_{\mu\nu}\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\int \mu_{\alpha\beta}(k)\Bigg[(p_\alpha\cdot p_\beta)\Bigg\{(\ell_\alpha\cdot\ell_\beta)(S_\alpha\wedge\ell_\alpha)^\mu p^\nu_\beta + (k\cdot p_\beta)(S_\alpha\wedge\ell_\alpha)^\mu\ell^\nu_\alpha$$
$$+ (\ell_\alpha\wedge\ell_\beta)_\alpha p^\mu_\beta l^\nu_\alpha + (\ell_\alpha\wedge p_\beta)_\alpha \ell^\mu_\alpha\ell^\nu_\alpha - \frac{1}{2}\ell^2_\alpha(\ell_\alpha\wedge p_\beta)_\alpha\eta^{\mu\nu}\Bigg\} - (k\cdot p_\beta)^2(S_\alpha\wedge\ell_\alpha)^\mu p^\nu_\alpha$$
$$+ (k\cdot p_\alpha)(k\cdot p_\beta)(S_\alpha\wedge\ell_\alpha)^\mu p^\nu_\beta + (k\cdot p_\alpha)(\ell_\alpha\wedge p_\beta)_\alpha p^\mu_\beta\ell^\nu_\alpha - (k\cdot p_\beta)(\ell_\alpha\wedge p_\beta)_\alpha p^\mu_\alpha\ell^\nu_\alpha$$
$$- (k\cdot p_\beta)(\ell_\alpha\wedge\ell_\beta)_\alpha p^\mu_\alpha p^\nu_\beta + (k\cdot p_\alpha)(\ell_\alpha\wedge\ell_\beta)_\alpha p^\mu_\beta p^\nu_\beta - (\ell_\alpha\cdot\ell_\beta)(\ell_\alpha\wedge p_\beta)_\alpha p^\mu_\alpha p^\nu_\beta$$
$$+ \frac{p^2_\beta}{d-2}(S_\alpha\wedge\ell_\alpha)^\mu p^\nu_\alpha + (\mu\leftrightarrow\nu)\Bigg]. \tag{152}$$

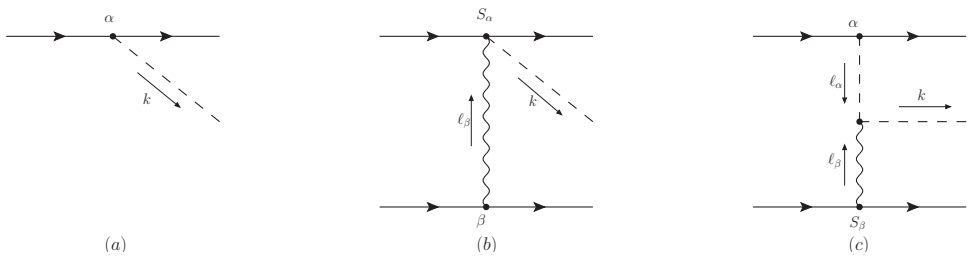

Figure 9: The $\mathcal{O}(\kappa^3, S^1)$ dilaton radiation diagrams

The dilaton channel

Lastly, the leading order dilaton radiation amplitude is summarized in Fig. 9. The first gives contributions to the dilaton amplitude from trajectory deflections,

$$\mathcal{A}_d(k)\big|_{\text{Fig. 9(a)};\mathcal{O}(\kappa^3,S^1)} = \sum_\alpha \int d\lambda e e^{ik\cdot x_\alpha} \frac{i}{k\cdot\dot{x}_\alpha}\left[\frac{k\cdot\ddot{x}_\alpha}{k\cdot\dot{x}_\alpha}\dot{x}_\alpha\cdot p_\alpha - \ddot{x}_\alpha\cdot p_\alpha - \dot{x}_\alpha\cdot\dot{p}_\alpha\right]\Bigg|_{\mathcal{O}(\eta^2,S^1)}. \tag{153}$$

We substitute in the momentum deflections and find the result in Appendix .2, Eq. (322).

The other two contributions are calculated at zero deflections in the particle trajectories. Fig. 9(b) involving an intermediate graviton contributes

$$\mathcal{A}_d(k)\big|_{\text{Fig. 9(b)};\mathcal{O}(\kappa^3,S^1)} = \frac{2i\kappa^3}{(d-2)^{1/2}} \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}} \int \mu_{\alpha\beta}(k)\ell_\alpha^2 (p_\alpha\cdot p_\beta)(\ell_\beta\wedge p_\alpha)_\beta. \tag{154}$$

Finally, we have Fig. 9(c) involving the 3-point graviton-dilaton vertex in the bulk, contributing

$$\mathcal{A}_d(k)\big|_{\text{Fig. 9(c)};\mathcal{O}(\kappa^3,S^1)} = -\frac{2i\kappa^3}{(d-2)^{1/2}} \sum_{\substack{\alpha\beta\\ \alpha\neq\beta}} p_\alpha^2 \int \mu_{\alpha,\beta}(k)(k\cdot p_\beta)(\ell_\alpha\wedge\ell_\beta)_\beta. \tag{155}$$

0.7.4 The spinning classical double copy

Applying the same set of mapping rules in Eqs. (86), (87), (88) and leaving the spin unchanged, we find that once again, it is necessary to include an additional gauge term

$$\hat{\mathcal{A}}_a^\mu\Big|_{\mathcal{O}(S^1)} = \frac{1}{2}g^3\sum_{\alpha,\beta}\int \mu_{\alpha,\beta}(k)\epsilon_\mu\left[(c_\alpha\cdot c_\beta)c_\alpha^a\frac{\ell_\alpha^2}{k\cdot p_\alpha}\{(\ell_\alpha\wedge p_\beta)_\alpha+(\ell_\beta\wedge p_\alpha)_\beta\}k^\mu\right], \tag{156}$$

so that the mapped amplitude

$$\mathcal{A}_a^\mu+\hat{\mathcal{A}}_a^\mu\Big|_{\mathcal{O}(S^1)}\mapsto \mathcal{A}^{\mu\nu}\Big|_{\mathcal{O}(S^1)}, \tag{157}$$

is gauge-invariant. Decomposing by the polarization tensors, we have therefore successfully confirmed that the predictions match with each channel directly calculated from string-gravity

$$a_{\mu\nu}\mathcal{A}^{\mu\nu}(k)\Big|_{\mathcal{O}(S^1)} = \mathcal{A}_a(k)\Big|_{\mathcal{O}(S^1)}, \tag{158}$$

$$\epsilon_{\mu\nu}\mathcal{A}^{\mu\nu}(k)\Big|_{\mathcal{O}(S^1)} = \mathcal{A}_g(k)\Big|_{\mathcal{O}(S^1)}, \tag{159}$$

$$\frac{\epsilon\cdot\tilde{\epsilon}}{d-2}\pi_{\mu\nu}\mathcal{A}^{\mu\nu}(k)\Big|_{\mathcal{O}(S^1)} = \mathcal{A}_d(k)\Big|_{\mathcal{O}(S^1)}, \tag{160}$$

at linear order in spin expansion [21].

Classical strings and extended objects

While the leading order calculations are surely successful, the color-kinematic mapping rules proposed above are not applied to the radiation amplitudes. The problem is that, unlike their scattering amplitude cousins in Section 0.5, these rules cannot guarantee gauge invariance by construction and there is a possibility of failure at higher orders.

The issue was resolved by Shen [36]. By examining the radiation amplitudes in the so-called cubic bi-adjoint scalar theory [72], it was discovered that color-kinematic duality also exists for these classical radiation amplitudes. Exploiting the duality, an updated version of classical double copy was constructed, and was verified to hold at the next-to-leading order.

In this chapter, we re-analyze the results derived in the previous chapter, using this revised version of classical double copy as described in Section 0.8. In Section 0.9, we discuss how the generalization to the spinning case has a possible interpretation in terms of classical strings. Furthermore, in Section 0.10, we apply the new classical double copy to worldline EFT with finite-size corrections and show that it works under specific conditions. This chapter is a modified version of [22].

0.8 The color-kinematic duality of radiation amplitudes

To facilitate the duality for radiation amplitudes, we begin with reviewing the classical bi-adjoint scalar radiation and its mapping to the Yang-Mills radiation amplitude. From this result, we describe how the color-kinematic duality can be established for radiation amplitudes.

0.8.1 Bi-adjoint scalar radiation

Inspired by the works on scattering amplitudes [73] and Kerr-Schild double copy [71], Ref. [72] considered the massless bi-adjoint scalar field theory with a cubic interaction

$$S \supset -\frac{y}{3}\int d^dx f^{abc}\tilde{f}^{\tilde{a}\tilde{b}\tilde{c}}\phi^{a\tilde{a}}\phi^{b\tilde{b}}\phi^{c\tilde{c}}, \tag{161}$$

that is invariant under two independent $G\times\tilde{G}$ global symmetry groups acting on $\phi^{a\tilde{a}}$ in their respective adjoint representation. In this theory, there are also two copies of color charges c^a_α, $\tilde{c}^{\tilde{a}}_\alpha$, respectively transforming in the adjoint representations of G and $\tilde{G}$. Given these variables, the simplest worldline term that is consistent with the symmetries is given by (suppressing particle labels)

$$S_{wl} \supset y\int d\lambda e(c\cdot\phi\cdot\tilde{c}), \tag{162}$$

where we have introduced the shorthand notation $c\cdot\phi\cdot\tilde{c} = (c\cdot\phi)^{\tilde{a}}\tilde{c}^{\tilde{a}} = c^a(\phi\cdot\tilde{c})^a = c^a\phi^{a\tilde{a}}\tilde{c}^{\tilde{a}}$.

The field equation is

$$\partial^2\phi^{a\tilde{a}} - yf^{abc}\tilde{f}^{\tilde{a}\tilde{b}\tilde{c}}\phi^{b\tilde{b}}\phi^{c\tilde{c}} = -y\mathcal{J}^{a\tilde{a}}, \tag{163}$$

where the source from the worldline action is given by

$$\mathcal{J}^{a\tilde{a}}(x) = \sum_\alpha\int d\lambda e c^a_\alpha\tilde{c}^{\tilde{a}}_\alpha\delta^d(x-x_\alpha). \tag{164}$$

The equations of motion are given by

$$\dot{p}^\mu_\alpha = -ec^a_\alpha\tilde{c}^{\tilde{a}}_\alpha\partial^\mu\phi^{a\tilde{a}}, \tag{165}$$

$$\dot{x}^\mu_\alpha = ep^\mu_\alpha, \tag{166}$$

$$\dot{c}^a_\alpha = -yf^{abc}c^c_\alpha\tilde{c}^{\tilde{b}}_\alpha\phi^{b\tilde{b}}, \tag{167}$$

$$\dot{\tilde{c}}^{\tilde{a}}_\alpha = -y\tilde{f}^{\tilde{a}\tilde{b}\tilde{c}}\tilde{c}^{\tilde{c}}_\alpha c^b_\alpha\phi^{b\tilde{b}}. \tag{168}$$

Similarly to the Yang-Mills calculations, in the regime $c\sim\tilde{c}\sim Eb$, the expansion is orga-

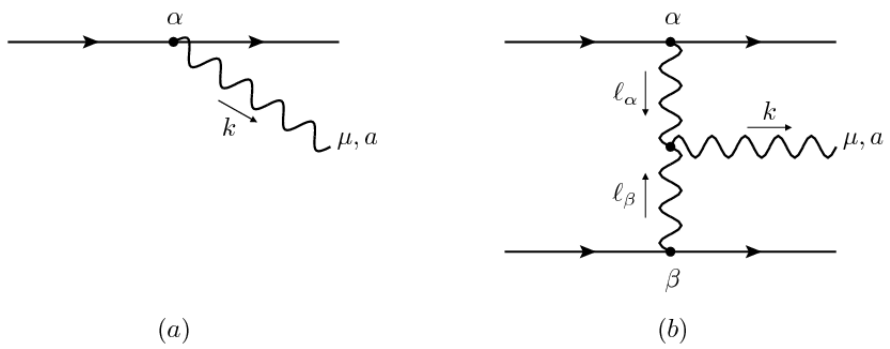

Figure 10: The leading order bi-adjoint scalar radiation diagrams

nized by a single parameter proportional to the coupling constant y.

The calculations are entirely analogous to the other cases, and the two leading order diagrams in Fig. 10 lead to the total amplitude in the form

$$\mathcal{A}^{a\tilde{a}}(k) = -iy^3 \sum_{\alpha,\beta} \int_{\ell_{\alpha,\beta}} \mu_{\alpha,\beta}(k) \Bigg[((c_\alpha \cdot c_\beta)c_\alpha)^a \ell_\alpha^2 \frac{k \cdot \ell_\beta}{(k \cdot p_\alpha)^2} ((\tilde{c}_\alpha \cdot \tilde{c}_\beta)\tilde{c}_\alpha)^{\tilde{a}} + [c_\alpha, c_\beta]^a (-1)[\tilde{c}_\alpha, \tilde{c}_\beta]^{\tilde{a}}$$
$$+ [c_\alpha, c_\beta]^a \frac{\ell_\alpha^2}{k \cdot p_\alpha} ((\tilde{c}_\alpha \cdot \tilde{c}_\beta)\tilde{c}_\alpha)^{\tilde{a}} + ((c_\alpha \cdot c_\beta)c_\alpha)^a \frac{\ell_\alpha^2}{k \cdot p_\alpha} [\tilde{c}_\alpha, \tilde{c}_\beta]^{\tilde{a}} \Bigg]. \tag{169}$$

By using a similar set of mappings

$$\tilde{c}_\alpha^a \;\mapsto\; p_\alpha^\nu, \tag{170}$$

$$\tilde{f}^{\tilde{a}\tilde{b}\tilde{c}} \;\mapsto\; \Gamma^{\mu\nu\rho}(-k, \ell_\alpha, \ell_\beta), \tag{171}$$

$$y \;\mapsto\; g, \tag{172}$$

the Yang-Mills amplitude is retrieved

$$\mathcal{A}^{a\tilde{a}}(k) \mapsto \mathcal{A}^{a\mu}(k). \tag{173}$$

0.8.2 The classical color-kinematic duality

The key observation is that the bi-adjoint radiation amplitude Eq. (169) is factorizable in a matrix form

$$\mathcal{A}^{a\tilde{a}}(k) = y^3 \sum_{\alpha,\beta} \int (C_a)^T P \tilde{C}_{\tilde{a}}, \tag{174}$$

where the two copies of the "color numerators" written as column vectors are

$$C^a = \begin{pmatrix} (c_\alpha \cdot c_\beta) c_\alpha^a \\ [c_\alpha, c_\beta]^a \end{pmatrix}, \; \tilde{C}^{\tilde{a}} = \begin{pmatrix} (\tilde{c}_\alpha \cdot \tilde{c}_\beta) \tilde{c}_\alpha^{\tilde{a}} \\ [\tilde{c}_\alpha, \tilde{c}_\beta]^{\tilde{a}} \end{pmatrix}, \tag{175}$$

and the "propagator matrix"

$$P = -i\mu_{\alpha,\beta}(k) \begin{pmatrix} \ell_\alpha^2 \frac{k\cdot\ell_\beta}{(k\cdot p_\alpha)^2} & \frac{\ell_\alpha^2}{k\cdot p_\alpha} \\ \frac{\ell_\alpha^2}{k\cdot p_\alpha} & -1 \end{pmatrix}, \tag{176}$$

consisting of a mixture of poles from both the worldline and spacetime propagators. Using these results, the Yang-Mills radiation amplitude Eq. (82) is also found to be factorizable as

$$\mathcal{A}^{a\mu}(k) = g^3 \sum_{\alpha,\beta} \int (C_a)^T P N^\mu, \tag{177}$$

where the kinematic numerators are now written as

$$N^\mu = \begin{pmatrix} (p_\alpha \cdot p_\beta) p_\alpha^\mu \\ (k\cdot p_\alpha) p_\beta^\mu - (k\cdot p_\beta) p_\alpha^\mu + \frac{1}{2}(p_\alpha \cdot p_\beta)\ell_\alpha^\mu - \frac{1}{2}(p_\alpha \cdot p_\beta)\ell_\beta^\mu \end{pmatrix}. \tag{178}$$

Now we can see that the second components both enjoy anti-symmetry under the exchange $\alpha \leftrightarrow \beta$, which is the manifestation of color-kinematic duality in the classical case. The mappings from bi-adjoint to Yang-Mills and from Yang-Mills to dilaton-graviton straightforwardly become

$$\tilde{C}^{\tilde{a}} \mapsto N^\mu, \quad C^a \mapsto N^\nu. \tag{179}$$

Indeed, the resulting prediction

$$\mathcal{A}^{\mu\nu}(k) = g^3 \sum_{\alpha,\beta} \int (N^\nu)^T P N^\mu, \tag{180}$$

corresponds to the dilaton gravity amplitude. Now we understand that the role of the extra gauge term Eq. (89) is to equip the numerators with algebraic properties dual to the color factors.

The proposal further generalizes to the n-th order perturbative amplitude. We organize the bi-adjoint theory amplitude into

$$\mathcal{A}_n^{a\tilde{a}}(k) = y^{2n+1} \sum_{ij} C_i^a P^{ij} \tilde{C}_j^{\tilde{a}}. \tag{181}$$

from which we could extract $P^{ij}(k)$. For the gauge theory, we construct all possible $N_i^\mu(k)$ that obey similar particle interchange symmetries and kinematic Jacobi identities as the corresponding color objects C_i^a do, and we make the first replacement $\tilde{C}^{\tilde{a}} \mapsto N^\mu$ to obtain a Yang-Mills amplitude

$$\mathcal{A}_n^{\mu a}(k) = g^{2n+1} \sum_{ij} C_i^a P^{ij} N_j^\mu, \tag{182}$$

such that Ward identity $k_\mu \mathcal{A}_n^{\mu a}(k) = 0$ is satisfied. This then automatically guarantees that a second replacement $C^a \mapsto N^\nu$ generates a consistent radiation amplitude

$$\mathcal{A}_n^{\mu\nu}(k) = \kappa^{2n+1} \sum_{ij} N_i^\mu(k) P^{ij}(k) N_j^\nu(k), \tag{183}$$

whose spectrum contains a graviton by the color-kinematic duality. By explicit calculations, this conjecture was verified to be valid at the next-to-leading order [36].

0.9 The spinning worldline theory and classical strings

0.9.1 The color-kinematic dual radiation amplitude with spin

We combine the two Yang-Mills radiation amplitude Eq. (82), (134) together with the extra gauge terms Eqs. (89), (156) result can be put into the form

$$\mathcal{A}^{\mu,a}(k) = g^3 \sum_{\alpha,\beta} \int (C^a)^T P N_R{}^\mu \tag{184}$$

where the effects of spin are encoded in the new numerator structure $N_R = N + N_S$, with N^μ as in Eq. (178) and

$$N_S{}^\mu = i \left(\begin{array}{l} (\ell_\beta \wedge p_\alpha)_\beta p_\alpha^\mu - (\ell_\beta \wedge p_\beta)_\alpha p_\alpha^\mu - (p_\alpha \cdot p_\beta)(S_\alpha \wedge k)^\mu + (k \cdot p_\alpha)(S_\alpha \wedge p_\beta)^\mu \\ (k \cdot p_\beta)(S_\alpha \wedge \ell_\alpha)^\mu + (\ell_\alpha \wedge \ell_\beta)_\beta p_\alpha^\mu - \frac{1}{2}(\ell_\alpha \wedge p_\beta)_\alpha (\ell_\beta - \ell_\alpha)^\mu - (\alpha \leftrightarrow \beta). \end{array} \right). \tag{185}$$

Remarkably, the propagator matrix is identical to Eq. (176) found in the absence of spin and the second kinematic numerator satisfies the correct anti-symmetry condition. Now we understand that the specific choice of gyromagnetic ratio of $g_D = 2$ and the extra gauge terms conspire to provide a color-kinematic dual factorizable amplitude.

Then substitution $C^a \mapsto N^\mu$ in Eq. (184) gives an amplitude

$$\mathcal{A}^{\mu\nu}(k) = \kappa^3 \sum_{\alpha,\beta} \int (N^\nu)^T P N_R{}^\mu, \tag{186}$$

which is consistent with the results for string-gravity radiation in the spacetime theory defined by Eq. (106).

We further notice that this construction allows a generalization to a wider class of spinning extended objects. We recall that as described in Section 0.5, the kinematic numerator structure in the original double copy construction replacements need not be the one in the original one, but any one that possesses color-kinematic duality. In the present case, we can use the mapping $C^a \mapsto N_L^\mu$, where in general the consistent numerators $N_L{}^\mu$ do not coincide with $N_R{}^\mu$, but are instead taken from a different gauge theory radiation amplitude. The gravitational radiation field in this more general situation is given by

$$\mathcal{A}^{\mu\nu}(k) = \kappa^3 \sum_{\alpha,\beta} \int (N_L{}^\nu)^T P N_R{}^\mu. \tag{187}$$

The duality once again guarantees gauge invariance $k_\mu \mathcal{A}^{\mu\nu} = k_\nu \mathcal{A}^{\mu\nu} = 0$.

In particular, we can take N_R^μ and N_L^μ to possess two independent sets of spin degrees of freedom $S_R^{\mu\nu}$ and $S_L^{\mu\nu}$ but identical otherwise (same color charges and momenta). The result in Eq. (186) is recovered by taking $S_L^{\mu\nu} = 0$ and $S_R^{\mu\nu} = S^{\mu\nu}$, while taking $S_L^{\mu\nu} = S^{\mu\nu}$ and $S_R^{\mu\nu} = 0$ flips the order of ϵ and $\tilde{\epsilon}$ leading to a sign change in the axion channel. This

corresponds to coupling the spin to a different connection

$$S_{wl} \supset \frac{1}{2}\int dx^{\rho} C^{-}{}_{\mu\nu\rho}\tilde{S}^{\mu\nu}, \tag{188}$$

where $C^{-\rho}{}_{(\mu\nu)} = \tilde{\Gamma}^{\rho}{}_{\mu\nu}$ and torsion $T^{\rho}{}_{\mu\nu} = C^{-\rho}{}_{[\mu\nu]} = -\tilde{H}^{\rho}{}_{\mu\nu}$. On the other hand, the choice

$$S_L^{\mu\nu} = S_R^{\mu\nu} = \frac{1}{2}S^{\mu\nu}, \tag{189}$$

yields an amplitude $\mathcal{A}^{\mu\nu}(k) = \mathcal{A}^{\nu\mu}(k)$ corresponding to spinning particles with vanishing axion couplings.

0.9.2 Classical strings

The form of the string-gravity action radiation in Eq. (106) is clearly alluding to certain connections to string theory. As it turns out, the more general situation with independent spins $S_L^{\mu\nu}$ and $S_R^{\mu\nu}$ coupled to gravitational fields via a term

$$S_{wl} \supset -\frac{1}{4}\int dx^{\lambda}(S_L^{\mu\nu} - S_R^{\mu\nu})H_{\mu\nu\lambda}e^{-2\phi}. \tag{190}$$

also has a similar interpretation. Namely, the form of this coupling and the $g_D = 2$ chromomagnetic coupling in Eq. (94) suggests that they are in fact a classical closed string interacting with the string-gravity background and a classical open string coupled to a gauge field.

Open strings

To provide evidence for this claim, we consider first an open string which is in a "semi-classical" configuration (i.e. a state where a large portion of the stringy oscillator spectrum are highly occupied and the string invariant mass M_s is large in string units). If this object is placed in an external field A_{μ}^{a} whose typical time and distance scales are large compared to the string length l, we can describe it systematically as a point source carrying gauge-invariant interactions with the gauge field. We focus on the interactions linear in A_{μ} and consider for simplicity the case of abelian gauge symmetry.

In the full theory of the extended string, we couple the gauge field by attaching Chan-Paton charges q_σ localized at the string endpoints $\sigma = 0, \pi$ (g_o is the open string coupling constant)

$$S_{int} = g_o \sum_{\sigma=0,\pi} q_\sigma \int dX^\mu(\tau,\sigma) A_\mu(X(\tau,\sigma)). \tag{191}$$

In the Polyakov gauge in which the free string equations of motion are simply $(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0$, subject to Virasoro constraints $\partial_\tau X \cdot \partial_\sigma X = 0$, and $(\partial_\tau X)^2 = -(\partial_\sigma X)^2$, we may expand the solution in terms of oscillator modes

$$X^\mu(\tau,\sigma) = x^\mu + l^2 p^\mu \tau + Z^\mu(\tau,\sigma), \tag{192}$$

with oscillator contribution $Z^\mu = il \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau} \cos n\sigma$, and expanding about the center-of-mass coordinate $x^\mu(\tau) = x^\mu + l^2 p^\mu \tau$. In particular, the current induced by the moving string is the "vertex operator"

$$J^\mu(k) = \int d^d x e^{ik\cdot x} J^\mu(x) = g_o \sum_{\sigma=0,\pi} q_\sigma \int d\tau \partial_\tau X^\mu(\sigma,\tau) e^{ik\cdot X(\sigma,\tau)}, \tag{193}$$

and the multipole limit corresponds to the formal expansion in powers of $k^\mu \to 0$ and Z^μ. Choosing coordinates with $x^\mu = 0$, the leading order term in this expansion is

$$J^\mu(k \to 0) \simeq g_o \sum_\sigma q_\sigma \int d\tau e^{il^2 k\cdot p\tau} l^2 p^\mu + \mathcal{O}(Z^1, k\cdot Z^1) = g_o Q (2\pi)\delta(k\cdot p) p^\mu, \tag{194}$$

which is of course the current of a static point particle with total charge $Q = \sum_\sigma q_\sigma$.

At the next order in the expansion, the string motion generates an electric dipole moment of the form

$$J^\mu(k \to 0)|_E \simeq -ig_o \sum_\sigma q_\sigma \int d\tau e^{il^2 k\cdot p\tau} \left(k\cdot p Z^\mu - (k\cdot Z)p^\mu\right) + \mathcal{O}(Z^2, k\cdot Z^2) \tag{195}$$

after integration by parts. Inserting the mode expansion for Z^μ, this is a sum of delta functions $\delta(l^2 k\cdot p - n)$ with $n \neq 0$ and therefore vanishes at frequencies $\omega \equiv k\cdot v \ll M_s^{-1} l^{-2}$. So there is no permanent (time-independent) electric dipole moment, as expected from time-

reversal symmetry. At the same order in the multipole expansion but next order in powers of Z^μ, we also have a term

$$J^\mu(k\to 0)|_B \simeq g_o\sum_\sigma q_\sigma \int d\tau e^{il^2k\cdot p\tau} ik\cdot Z\partial_\tau Z^\mu + \mathcal{O}(k\cdot Z^2) \tag{196}$$

$$\simeq -\frac{i}{2}g_o k_\nu \sum_\sigma q_\sigma \int d\tau e^{il^2k\cdot p\tau}\left(Z^\mu\partial_\tau Z^\nu - Z^\nu\partial_\tau Z^\mu\right) + \mathcal{O}(k\cdot Z^2) \tag{197}$$

Here we have discarded a term which, upon integration by parts, acquires a factor of $(k\cdot p)$ and is suppressed in the static limit $\omega \ll M_s^{-1}l^{-2}$. Given that

$$Z^\mu \dot{Z}^\nu\Big|_{\sigma=0,\pi} = -il^2\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu\alpha_{-n}^\nu + \text{time dependent terms}, \tag{198}$$

the part of the moment that remains after rapid oscillations of order the string scale are averaged out is proportional to the angular momentum of the string about the center of mass $x^\mu = 0$,

$$J^\mu(k\to 0)|_B = ig_oQ(2\pi)\delta(k\cdot p)k_\nu S^{\mu\nu} \tag{199}$$

with intrinsic spin $S^{\mu\nu} \gg \hbar$ given in terms of the oscillator modes by [75]

$$S^{\mu\nu} = \int_0^\pi d\sigma\left(Z^\mu\partial_\tau Z^\nu - Z^\nu\partial_\tau Z^\mu\right) = -i\sum_{n=1}^\infty\frac{1}{n}\left(\alpha_{-n}^\mu\alpha_n^\nu - \alpha_{-n}^\nu\alpha_n^\mu\right). \tag{200}$$

To summarize, we have found the current induced by the open string in the long-wavelength limit

$$J^\mu(k) = g_oQ(2\pi)\delta(k\cdot p)\left[1 + ik_\nu S^{\mu\nu}\right] + \cdots. \tag{201}$$

This is precisely the form one finds for a pointlike startic particle with current $J^\mu(x) = \delta S_{pp}/\delta A_\mu(x)$ and

$$S_{pp} = g_oQ\int dx^\mu A_\mu(x) + \frac{1}{2}g_oQ\int d\tau S^{\mu\nu}F_{\mu\nu}(x). \tag{202}$$

In particular, in $d = 4$ dimensions, if we make the identification $g_o = g$, the spin-dependent terms corresponds to a magnetic moment interaction with gyromagnetic ratio equal to Dirac's value $g_D = 2$, consistent with earlier classical [76] and quantum mechanical [77]

string theory results. The non-abelian generalization of this calculation should proceed along the same lines, with a Wilson line inserted between the string endpoints in order to ensure gauge invariance, and the effective description, in that case, is then the chromomagnetic interaction in Eq. (94).

Closed strings

Since the bulk action on the gravity side is related to closed strings, we would naturally expect that the worldline action is also somehow related. Following the notations in [75], In the background field action, the closed string coupling to axion is given by

$$S_B = -\frac{1}{2\pi l^2}\int d\tau \int_0^\pi d\sigma \epsilon^{\alpha\beta} B_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu, \tag{203}$$

where l is the string length. From Eq. (338) in Appendix .3, we see that the low energy effective vertex generated by this action can be written as

$$J^{\mu\nu}(k) \simeq \frac{\pi}{2}\delta(k\cdot p)k_\lambda[p^\mu(S_L^{\nu\lambda} - S_R^{\nu\lambda}) + p^\nu(S_L^{\lambda\mu} - S_R^{\lambda\mu})]. \tag{204}$$

where we have denoted the two sets of closed string angular momentum generators by

$$S_L^{\mu\nu} = -i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu), \tag{205}$$

$$S_R^{\mu\nu} = -i\sum_{n=1}^{\infty}\frac{1}{n}(\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu - \tilde{\alpha}_{-n}^\nu \tilde{\alpha}_n^\mu), \tag{206}$$

this vertex corresponds to the worldline action

$$S_a = -\frac{1}{4}\int dx^\lambda (S_L^{\mu\nu} - S_R^{\mu\nu})H_{\mu\nu\lambda}. \tag{207}$$

This agrees with Eq. (190) apart from the dilaton factor. But the discrepancy arises only because this expression is in the string frame and it is related to the previous result by a Weyl rescaling. This is expected since the spacetime action Eq. (106) is also in this so-called Einstein frame metric. Similarly, as shown in Appendix .4, the low energy contribution to

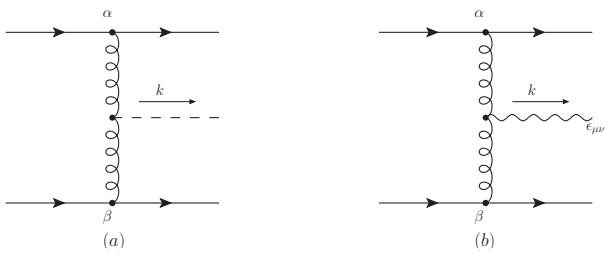

Figure 11: The diagrams containing internal axion lines

the energy-momentum tensor correspond to the correct coupling to gravity.

We note that although S_L cannot be set to exactly zero when S_R is not due to the Virasoro constraints arising from the string worldsheet conformal symmetry, states where $S_R \ll S_L$ do exist. The construction is also very reminiscent of Kawai-Lewellen-Tye (KLT) relations [79], in that the spin-2 radiation amplitude can be obtained from the product of two independent spin-1 radiation amplitudes. In fact, the KLT relations are the source of inspiration for BCJ. Thus, this provides another parallel between the classical and quantum field theory results.

0.9.3 Unoriented strings and the decoupling of axion

In addition to the generic description above, there is also the interesting special case

$$S_L = S_R = \frac{1}{2}S. \tag{208}$$

In this setting, the two kinematic numerators are identical and hence axion radiation is absent.

We speculate that by imposing the constraint Eq. (208), we might be able to consistently remove axion radiation. The axion is sourced by spins and only has spacetime interaction vertices with two axion insertions. To linear order in spin, the decoupling of axion is then trivial, since these Feynman rules do not allow internal axion lines to interfere with graviton and dilaton radiation. At the next order $\mathcal{O}(S^2)$, internal axion lines start to enter through diagrams such as Fig. 11 start to enter as well as through equations of motion, but this is

somewhat expected, since

$$\begin{aligned}\tilde{\epsilon}_\nu \epsilon_\mu \mathcal{A}^{\nu\mu} &= \sum_{i,j} \big(N_i^\nu(S^0) + N_i^\nu(S^1) + N_i^\nu(S^2)\big) P_{ij} \big(N_j^\mu(S^0) + N_j^\mu(S^1) + N_j^\mu(S^2)\big) + \mathcal{O}(S^3) \\ &= \ldots + \sum_{i,j} \big[N_i^\nu(S^1) P_{ij} N_j^\mu(S^1) + N_i^\nu(S^2) P_{ij} N_j^\mu(S^0) + N_i^\nu(S^0) P_{ij} N_j^\mu(S^2)\big] + \mathcal{O}(S^3).\end{aligned} \tag{209}$$

It is easy to see that $N_i(S^1) P_{ij} N_j(S^1)$ gives a new contribution at $\mathcal{O}(S^2)$ comparing to the results if we were to construct the double copy using a single sector of spin. It is possible that there are cancellations among these additional contributions at the next order.

In fact, one might consider the parity transformation operator on these classical quantities, defined as

$$\Omega \ : \ S_R \mapsto S_L, \tag{210}$$

and construct the projection operator

$$P = \frac{1}{2}(1 + \Omega). \tag{211}$$

Acting with this on the two sets of Lorentz generators yields Eq. (208). This is analogous to the projection which yields unoriented strings. If this correspondence to unoriented strings holds at all orders, axion radiation would indeed be absent, as is the case for unoriented string spectrum.

0.10 The classical double copy of extended objects

The classical double copy based on color-kinematic duality is also applicable to worldline EFTs containing finite-size corrections [22]. We explore its feasibility in the following.

At leading (quadratic in fields) order, the finite-size operators determine the linear response, namely the multipole moments induced on the finite-size object by an external field configuration. In the bi-adjoint theory, the linear response operators consistent with

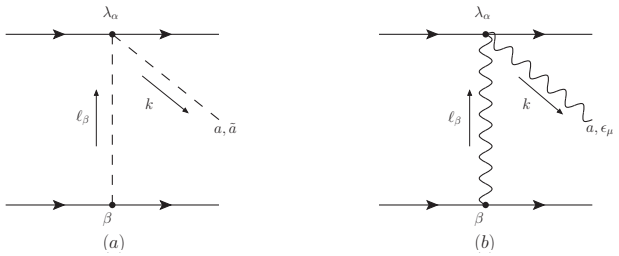

Figure 12: The Leading order diagrams for bi-adjoint and Yang-Mills radiation amplitudes induced by finite-size worldline operators.

$G \times \tilde{G}$ symmetry arise at zeroth order in spacetime derivatives, and consist of the four terms

$$S = \frac{1}{2} y^2 \sum_i \lambda_i \int d\lambda e \mathcal{O}^{BS}_{(i)}, \tag{212}$$

where

$$\mathcal{O}^{BS}_{(1)} = \phi^{a\tilde{a}}\phi^{a\tilde{a}}, \quad \mathcal{O}^{BS}_{(2)} = (c\cdot\phi)^{\tilde{a}}(c\cdot\phi)^{\tilde{a}}, \quad \mathcal{O}^{BS}_{(3)} = (c\cdot\phi)^{\tilde{a}}(c\cdot\phi)^{\tilde{a}}, \quad \mathcal{O}^{BS}_{(4)} = (c\cdot\phi\cdot\tilde{c})^2, \tag{213}$$

and λ_i^α are set of dimensionless coupling constants. For clarity, we have suppressed the particle label α on the coupling constants and action. We have also introduced the shorthand notation $(c\cdot\phi)^{\tilde{a}} = c_a\phi^{a\tilde{a}}$, $(\phi\cdot\tilde{c})^a = \phi^{a\tilde{a}}\tilde{c}_{\tilde{a}}$, and $c\cdot\phi\cdot\tilde{c} = c_a\tilde{c}_{\tilde{a}}\phi^{a\tilde{a}}$. We normalize these operators with prefactor y^2 and impose the coupling constant mapping rules $y \mapsto g \mapsto \kappa$ under the double copy.

It is straightforward to compute the contribution to long-distance radiation from the terms in Eq. (212). We work to linear order in the parameters λ_i and consider a scattering event where the field generated by a second point source deforms the extended object. The time-dependence of the induced moments then sources scalar radiation. Diagrammatically, the situation is depicted in Fig. 12(a). To linear order in the finite-size couplings, the amplitude is given by

$$\mathcal{A}^{a\tilde{a}} = y^3 \sum_{\alpha,\beta} \int \tilde{C}^T_a \Lambda C_a, \tag{214}$$

where we have defined the color numerators as

$$C^a = \begin{pmatrix} c^a_\beta \\ (c_\alpha \cdot c_\beta) c^a_\alpha \end{pmatrix}, \; \tilde{C}^{\tilde{a}} = \begin{pmatrix} \tilde{c}^{\tilde{a}}_\beta \\ (\tilde{c}_\alpha \cdot \tilde{c}_\beta) \tilde{c}^{\tilde{a}}_\alpha \end{pmatrix}, \tag{215}$$

as well as the 2×2 matrix Λ_α of Wilson coefficients and a propagator prefactor for particle α

$$\Lambda = \mu_{\alpha,\beta}(k) \ell^2_\alpha \begin{pmatrix} \lambda^\alpha_1 & \lambda^\alpha_3 \\ \lambda^\alpha_2 & \lambda^\alpha_4 \end{pmatrix}. \tag{216}$$

Likewise, in gauge theory, the linear response of a (spinless) extended color sources in the long-wavelength limit can be described by including the operators

$$S = \frac{1}{4} g^2 \sum_i \lambda_i \int d\lambda e \mathcal{O}^{YM}_{(i)}, \tag{217}$$

where now

$$\begin{aligned} \mathcal{O}^{YM}_{(1)} &= F^a_{\mu\nu} F^{\mu\nu}_a, \; \mathcal{O}^{YM}_{(2)} = (c \cdot F)_{\mu\nu} (c \cdot F)^{\mu\nu}, \\ \mathcal{O}^{YM}_{(3)} &= p^\mu p^\nu F^a_{\mu\sigma} F^{a\sigma}_\nu, \; \mathcal{O}^{YM}_{(4)} = p^\mu p^\nu (c \cdot F)_{\mu\sigma} (c \cdot F)_\nu{}^\sigma. \end{aligned} \tag{218}$$

These operators induce contributions to radiation in scattering as shown in Fig. 12(b). By explicit calculations, we find that the result exhibits a factorization property which parallels that found in the bi-adjoint case,

$$\mathcal{A}^{a\mu} = g^3 \sum_{\alpha,\beta} \int \tilde{C}^T_a \Lambda_\alpha N^\mu. \tag{219}$$

where C^a_i, $i = 1, 2$ is the set of color factors given in Eq. (215) while the kinematic numerators are defined to be

$$N^\mu = \begin{pmatrix} (k \cdot p_\beta) \ell^\mu_\beta - (k \cdot \ell_\beta) p^\mu_\beta \\ \frac{1}{2}(p_\alpha \cdot p_\beta)[(k \cdot p_\alpha) \ell^\mu_\beta - (k \cdot \ell_\beta) p^\mu_\alpha] + \frac{1}{2}(k \cdot p_\alpha)[(k \cdot p_\beta) p^\mu_\alpha - (k \cdot p_\alpha) p^\mu_\beta] \end{pmatrix}. \tag{220}$$

In order to account for the full finite-size effects in gravity (including all the axionic

operators), it is also necessary to consider the radiation induced by the interactions of the spin of the probe particle β with the extended object α. Working to linear order in spin [3], the amplitude is represented by diagrams of the same topology as in Fig. 12(b) where the off-shell gluon is emitted from the chromomagnetic coupling of particle β. The result is then of the same form as Eq. (219), but the kinematic factor is shifted by a term linear in $S_\beta^{\mu\nu}$, $N^\mu \to N^\mu + N_S{}^\mu$, where

$$N_S{}^\mu = i \begin{pmatrix} (k\cdot\ell_\beta)(S_\beta\wedge\ell_\beta)^\mu - (\ell_\alpha\wedge\ell_\beta)_\beta \ell_\beta^\mu \\ \frac{1}{2}(\ell_\beta\wedge p_\alpha)_\beta[(k\cdot p_\alpha)\ell_\beta^\mu - (k\cdot\ell_\beta)p_\alpha^\mu] + \frac{1}{2}(k\cdot p_\alpha)[(k\cdot p_\alpha)(S_\beta\wedge\ell_\beta)^\mu - (\ell_\alpha\wedge\ell_\beta)_\beta p_\alpha^\mu] \end{pmatrix}. \tag{221}$$

The respective results in Eqs. (214, 219) suggest a set of mapping rules between finite-size objects in bi-adjoint scalar and Yang-Mills theory. Namely, making the replacement $\tilde{C}^a \mapsto N^\mu$ maps the finite-size amplitude $\mathcal{A}^{a\tilde{a}}$ of bi-adjoint theory to the radiation field $\mathcal{A}^{a\mu}$ in gauge theory[4]. We note that the Wilson coefficients in Eqs. (214, 219) are in principle different. However, this mapping relation gives a direct correspondence between the two sets $\lambda_i \mapsto \lambda_i^{YM} \equiv \lambda_i$. Given this, it is then natural to take a further step $\mathcal{A}^{a\mu} \mapsto \mathcal{A}^{\mu\nu}$ while demanding λ_i unchanged, where

$$\mathcal{A}^{\mu\nu} = \kappa^3 \sum_{\alpha,\beta} \int (N_L{}^\nu)^T \Lambda N_R{}^\mu, \tag{222}$$

with $N_{L,R}{}^\mu = N_{S_{L,R}}{}^\mu$. Because $k_\mu \mathcal{A}^{\mu\nu} = k_\nu \mathcal{A}^{\mu\nu} = 0$ this defines a consistent radiation field in a theory of finite-size sources coupled to massless fields $(\phi, g_{\mu\nu}, B_{\mu\nu})$.

Given the structure of the mapping $\mathcal{A}^{a\tilde{a}} \mapsto \mathcal{A}^{a\mu}$, which takes finite-size operators in bi-adjoint theory with no derivatives to two-derivative operators in gauge theory, we expect that the double copy amplitude $\mathcal{A}^{\mu\nu}$ encodes finite-size effects corresponding to a total of

3. We ignore finite-size operators built out of spin at this order in gauge or finite-size couplings. It is easy to see that there is no kinematic numerator with a dual representation at linear order in spin. However, we might still need include such terms at higher orders in perturbation theory.

4. Notice that this mapping takes operators with no derivatives on $\phi^{a\tilde{a}}$ to operators involving gradients of A_a^μ in Yang-Mills. Including an operator of the form, e.g., $y^2 \int d\tau (\partial_\mu \phi^{a\tilde{a}})^2$ in scalar theory yields a radiation amplitude $\mathcal{A}^{a\tilde{a}} = y^3 \int \mu_{\alpha,\beta} \ell_\alpha^2 (k\cdot\ell_\beta) \tilde{C}_1^{\tilde{a}} C_1^a$ whose propagator structure $\ell_\alpha^2(k\cdot\ell_\beta)$ does not match with any of the terms in Eq. (219) Rather, it corresponds to a four-derivative operator $\int d\tau D_\sigma F^a_{\mu\nu} D^\sigma F_a^{\mu\nu}$ which yields an amplitude of the form $g^3 \int \mu_{\alpha,\beta} \ell_\alpha^2 (k\cdot\ell_\beta) C_1^a N_1^\mu$ consistent with the color-kinematics substitution $\tilde{C}^{\tilde{a}} \mapsto N^\mu$.

four derivatives acting on the fields $(\phi, g_{\mu\nu}, B_{\mu\nu})$. To determine the precise form of the finite-size response encoded in $\mathcal{A}^{\mu\nu}$, we therefore start with the most general set of four-derivative diffeomorphism invariant worldline operators that are quadratic in these fields. Since we are not considering spin-dependent finite-size operators in the gauge theory, we also limit ourselves to spin-independent gravitational higher-dimensional operators. At the four-derivative level, the complete set of terms allowed by diffeomorphism invariance is

$$S = \sum_i \tilde{\lambda}_i \int d\lambda e \tilde{\mathcal{O}}_{(i)}, \tag{223}$$

where now we have twelve Wilson coefficients $\tilde{\lambda}_i$ corresponding to ten positive definite operators

$$\tilde{\mathcal{O}}_{(G_1)} = \frac{1}{4}(R_{\mu\nu\rho\sigma})^2, \tilde{\mathcal{O}}_{(G_2)} = \frac{1}{4}(R_{\mu\nu\rho\sigma}p^\sigma)^2, \tilde{\mathcal{O}}_{(G_3)} = \frac{1}{4}(R_{\mu\nu\rho\sigma}p^\nu p^\sigma)^2, \tag{224}$$

$$\tilde{\mathcal{O}}_{(D_1)} = \frac{1}{2}(\nabla_\mu\nabla_\nu\phi)^2, \tilde{\mathcal{O}}_{(D_2)} = \frac{1}{2}(p^\mu\nabla_\mu\nabla_\nu\phi)^2, \tilde{\mathcal{O}}_{(D_3)} = \frac{1}{2}(p^\mu p^\nu\nabla_\mu\nabla_\nu\phi)^2, \tag{225}$$

$$\tilde{\mathcal{O}}_{(A_1)} = \frac{1}{6}(\nabla_\sigma H_{\mu\nu\rho})^2, \tilde{\mathcal{O}}_{(A_2)} = \frac{1}{4}(\nabla_\sigma H_{\mu\nu\rho}p^\rho)^2, \tilde{\mathcal{O}}_{(A_3)} = \frac{1}{6}(\nabla_\sigma H_{\mu\nu\rho}p^\sigma)^2, \tilde{\mathcal{O}}_{(A_4)} = \frac{1}{4}(\nabla_\sigma H_{\mu\nu\rho}p^\sigma p^\rho)^2, \tag{226}$$

and to two terms that mix the graviton with the dilaton or axion

$$\tilde{\mathcal{O}}_{(GD)} = (p^\rho p^\sigma R_{\mu\rho\nu\sigma})\nabla^\mu\nabla^\nu\phi, \tag{227}$$

$$\tilde{\mathcal{O}}_{(GA)} = (p^\sigma R_{\mu\nu\rho\sigma})(p_\lambda\nabla^\mu H^{\nu\rho\lambda}). \tag{228}$$

We note that as we are dealing with a general dimensional scenario, we are not classifying the operators by parities as we did in Eqs. (28, 29) for the case of $d = 4$.

The amplitudes corresponding to radiation from the induced multipoles at zeroth and first orders in spin are calculated from the diagrams in Figs. 13, 14 respectively. It turns out that the individual amplitudes corresponding to each of the operators in Eq. (223) do not factorize in the way that would be expected from color-kinematics. However, by taking linear combinations of operators, it is possible to construct amplitudes where the only kinematic numerators that arise coincide with those that appear in the gauge theory

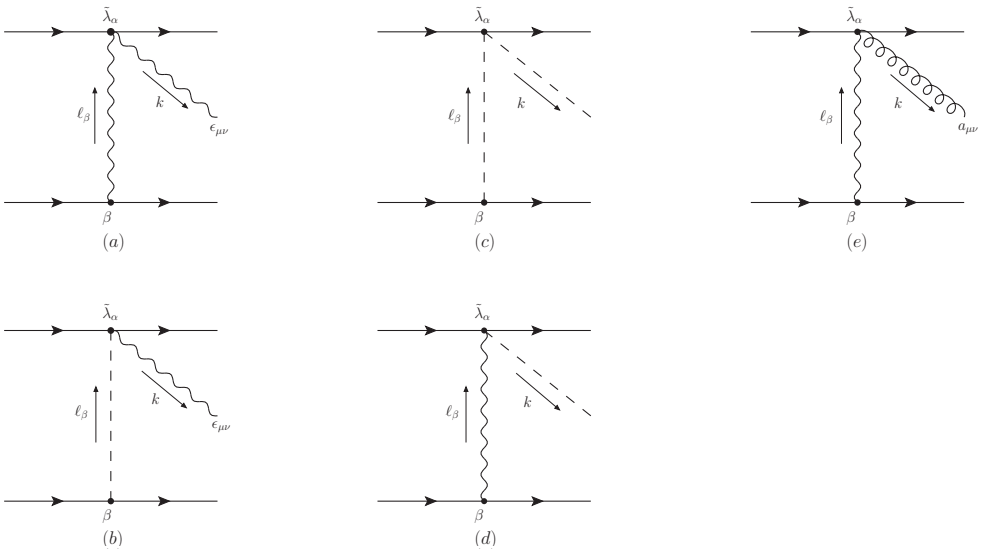

Figure 13: The leading order radiation diagrams for non-spinning sources induced by finite-size effects.

(in Eqs. (220), (221). For this choice of operators coefficients, the amplitude in the gravity theory agrees with the prediction of the double copy given by Eq. (219).

As an explicit example, we consider the case with positive axion-spin coupling, $S_L^{\mu\nu} = 0$ and $S_R^{\mu\nu} = S^{\mu\nu}$. The explicit calculations are reported in Appendix .6. The result is that the gravitational Wilson coefficients are related to the finite-size coupling on the gauge theory side by the relations

$$\lambda_1^\alpha = 2\tilde{\lambda}_{G_1}^\alpha = \frac{\tilde{\lambda}_{D_1}^\alpha}{d-2} = 4\tilde{\lambda}_{A_1}^\alpha, \tag{229}$$

$$\lambda_2^\alpha = 2\tilde{\lambda}_{G_2}^\alpha = -2\tilde{\lambda}_{GD}^\alpha = 2\tilde{\lambda}_{D_1}^\alpha = \frac{2\tilde{\lambda}_{D_2}^\alpha}{d-4} = 4\tilde{\lambda}_{GA}^\alpha = 4\tilde{\lambda}_{A_2}^\alpha = 8\tilde{\lambda}_{A_3}^\alpha, \tag{230}$$

$$\lambda_3^\alpha = 2\tilde{\lambda}_{G_2}^\alpha = -2\tilde{\lambda}_{GD}^\alpha = 2\tilde{\lambda}_{D_1}^\alpha = \frac{2\tilde{\lambda}_{D_2}^\alpha}{d-4} = -4\tilde{\lambda}_{GA}^\alpha = 4\tilde{\lambda}_{A_2}^\alpha = 8\tilde{\lambda}_{A_3}^\alpha, \tag{231}$$

$$\lambda_4^\alpha = 2\tilde{\lambda}_{G_3}^\alpha = -\,4\tilde{\lambda}_{GD}^\alpha = 4\tilde{\lambda}_{D_1}^\alpha = -2\tilde{\lambda}_{D_2}^\alpha = \frac{4\tilde{\lambda}_{D_4}^\alpha}{d-2} = 4\tilde{\lambda}_{A_4}^\alpha. \tag{232}$$

We note that although the full set of operators in the string gravity background are not independent, there are still more free coefficients than the number of purely gravitational operators. Thus it should be possible to characterize the full gravitational tidal response at the linear level, provided one could project out the fields $\phi, B_{\mu\nu}$ in a systematic way that does not introduce new constraints among the gravitational tidal operators.

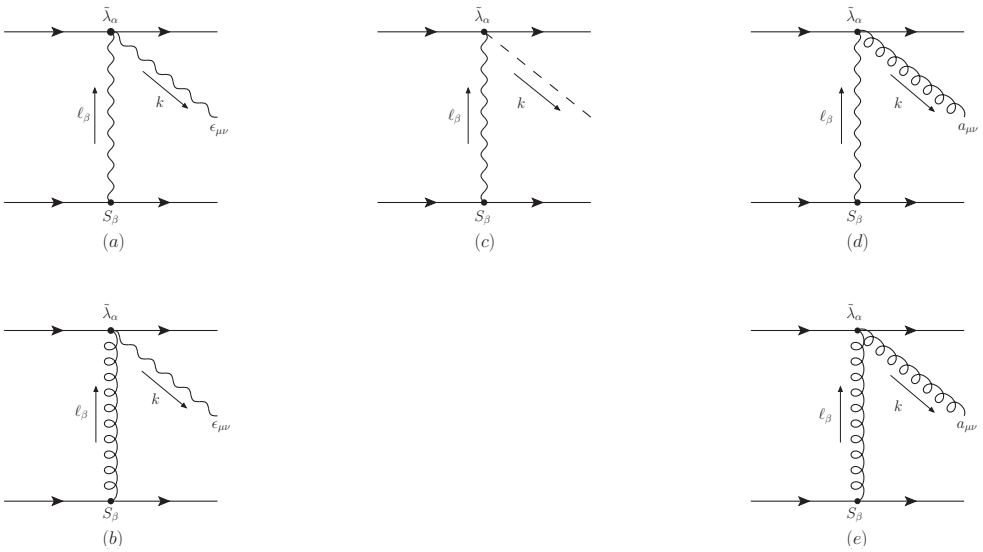

Figure 14: The leading order radiation diagrams for linear-order spinning sources induced by finite-size effects.

It is also interesting to note that, at least for some of these relations, there is a geometrical pattern. In terms of the non-minimal connection $C^+_{\mu\nu\rho}$ connection defined in Eq. (108), the above relations imply that the independent operators in the gravitational double can be expressed as

$$S = \frac{1}{8}\sum_i \lambda_i \int d\lambda e \mathcal{O}^{SG}_{(i)}, \tag{233}$$

where

$$\mathcal{O}^{SG}_{(1)} = (\tilde{R}^+_{\mu\nu\rho\sigma})^2, \tag{234}$$

$$\mathcal{O}^{SG}_{(2)} = (\tilde{R}^+_{\mu\nu\rho\sigma}\dot{x}^\sigma)^2 + \tilde{\mathcal{O}}_{(GA)} + \frac{1}{4}\tilde{O}_{(A_2)} + \frac{1}{12}\tilde{O}_{(A_3)}, \tag{235}$$

$$\mathcal{O}^{SG}_{(3)} = (\tilde{R}^+_{\mu\nu\rho\sigma}\dot{x}^\sigma)^2 - 3\tilde{\mathcal{O}}_{(GA)} + \frac{1}{4}\tilde{O}_{(A_2)} + \frac{1}{12}\tilde{O}_{(A_3)}, \tag{236}$$

$$\mathcal{O}^{SG}_{(4)} = (\tilde{R}^+_{\mu\nu\rho\sigma}\dot{x}^\rho\dot{x}^\sigma)^2. \tag{237}$$

However, while the dilaton dependence has been completely absorbed into the curvature associated with the non-minimal connection $C^+_{\mu\nu\rho}$, it does not seem possible to modify the torsion in order to simplify the axion-dependent terms.

An Effective Theory of Kerr Horizons

In this chapter, we turn to the the description of dissipative effects in the worldline EFT with spins. While the corresponding worldline EFT has been constructed for non-spinning [42] and slowly rotating objects [43], it has not been done for generic spinning objects. As we have mentioned, the near extremally rotating black holes have 1.5PN order enhanced absorption effects relative to the non-spinning case [38], so it is more phenomenologically relevant to consider the case with arbitrarily large spin (only bounded by extremality). Therefore, the goal is to generalize the method used in [50] to the case with generic value of spin and calculate the dissipative effect in that framework.

In the following, we start with describing the treatment of worldline EFT using the in-in formalism in Section 0.11. Then we implement the construction for rotating (Kerr) black holes in Section 0.12. As a validity check, we verify that at the leading order, the resulting dissipative equations of motion from the worldline EFTs are consistent with the ones derived using black hole perturbation theory. Furthermore, in Section 0.13, we derive the leading-PN-order dissipative dynamics for the momentum and angular momentum, and calculate the associated energy and spin transfer. This chapter is adapted from [23].

0.11 Dissipative effects for spinning compact objects and the in-in formalism

For reader's convenience, we repeat the relevant action

$$S = \int d\lambda \{ -\dot{x}^\mu e^a_\mu p_a + \frac{1}{2} S^{ab} \Omega_{ab} + e\lambda_a S^{ab} p_b + \frac{1}{2} e(p^2 + L_X(X, e^{-1}\dot{X}))\} + S_{int}, \tag{238}$$

where the interaction term contains

$$S_{int} = -\int d\lambda e \{ Q^{ab}_E(X) E_{ab} + Q^{ab}_B(X) B_{ab} \} + \dots . \tag{239}$$

In the following, we will denote $ds = ed\lambda$.

A caveat in the dissipative problem is that the safest assumption about the worldline momentum and spin is that they are also composite operators $p(X)$ and $S(X)$, as naturally, the overall momentum and spin of a composite object is not independent of its internal dynamical modes. For instance, the spin of a classical string is dependent on their oscillator modes as discussed in Section 0.9.2. However, it is possible to choose a generalized coordinate system such that the external variables become independent of the internal ones. In the string theory example, the center of mass momentum is separated from the other degrees of freedom by imposing appropriate boundary conditions for the oscillator modes. In any case, as it turns out, if we interpret these variables as the in-in expectation values $p^\mu = e^\mu_a \langle p^a(X) \rangle$ and $S^{\mu\nu} = e^\mu_a e^\nu_b \langle S^{ab}(X) \rangle$, there would not be any difference in the end.

The appropriate framework to understand how the internal modes affect the dynamics with a given initial condition is the in-in formalism which is designed to calculate expectation values at a fixed time (as opposed to transition amplitudes). The corresponding path integral formulation is the Schwinger-Keldysh [47] closed time path which essentially path integrate over both the forward and the backward evolution, as opposed to the conventional Feynman path integral over only the forward path when taking the in-out expectation value. We encode the full set of classical variables by $\Delta = (x^\mu, e^a_\mu)$, and the Schwinger-Keldysh

effective action Γ is given by

$$\exp\left[i\Gamma[\Delta,e,h;\tilde{\Delta},\tilde{e},\tilde{h}]\right]=\int_{ini}\mathcal{D}X\mathcal{D}\tilde{X}\exp\left[iS[\Delta,e,h,X]-iS[\tilde{\Delta},\tilde{e},\tilde{h},X]\right], \tag{240}$$

where we are integrating over an additional copy of variables $\tilde{X}$ which corresponds to the back evolution and the boundary conditions are fied at the initial time. In principle, there are also corrections from integrating out dynamical gravitons, but those effects are higher orders and not considered in the present discussions.

The resulting functional $\Gamma[\Delta,e,h;\tilde{\Delta},\tilde{e},\tilde{h}]$ determines the classical equations of motion by

$$\frac{\delta}{\delta\Delta}\Gamma[\Delta,e,h;\tilde{\Delta},\tilde{e},\tilde{h}]\Big|_{\substack{\Delta=\tilde{\Delta}\\ e=\tilde{e},h=\tilde{h}}}=0. \tag{241}$$

The path integral makes the moments $Q_{E/B}(X)$ appear in the equations of motion in terms of $\langle Q_{E/B}(X)\rangle$ which is an in-in correlation function defined as

$$\langle Q_{E/B}(X)\rangle=\int\mathcal{D}X\mathcal{D}\tilde{X}\exp\left[iS[\Delta,e,h,X]-iS[\tilde{\Delta},\tilde{e},\tilde{h},\tilde{X}]\right]Q_{E/B}(X). \tag{242}$$

Just as in the usual quantum field theory correlation functions calculated by Feynman propagators, in the perturbative regime, this can be calculated using the Schwinger-Keldysh propagators

$$\langle\mathcal{O}_a(\lambda)\mathcal{O}_b(\lambda')\rangle=\begin{pmatrix}\langle T\mathcal{O}(\lambda)\mathcal{O}(\lambda')\rangle & \langle\mathcal{O}(\lambda')\mathcal{O}(\lambda)\rangle\\ \langle\mathcal{O}(\lambda)\mathcal{O}(\lambda')\rangle & \langle\tilde{T}\mathcal{O}(\lambda)\mathcal{O}(\lambda')\rangle\end{pmatrix}, \tag{243}$$

where T and $\tilde{T}$ represents time and anti-time orderings, and the subindices label the first and the second copy. Explicitly, perturbative expansion gives

$$\langle Q_E^{ab}(s)\rangle=i\int ds'\{\langle TQ_E^{ab}(\lambda)Q_E^{cd}(\lambda')\rangle-\langle Q_E^{ab}(\lambda')Q_E^{cd}(\lambda)\rangle\}E_{cd}(x(s'))+\mathcal{O}(E^2). \tag{244}$$

We find that

$$\langle Q_E^{ab}(s)\rangle=\int ds' G_{R,E}^{ab,cd}(s-s')E_{cd}(x(s')), \tag{245}$$

with the retarded Green's function given by

$$G^{ab,cd}_{R,E}(s-s') = -i\theta(s-s')\langle[Q^{ab}_E(s), Q^{cd}_E(s')]\rangle$$
$$= i(\langle TQ^{ab}_E(s)Q^{cd}_E(s')\rangle - \langle Q^{ab}_E(s')Q^{cd}_E(s)\rangle) \tag{246}$$

In the low-energy regime, where the rate of variation of external fields is way slower than the internal dynamics, the low-energy effective description should regard the internal dynamics as instantaneous and Green's function should be derivatives over δ-functions. Integrating by parts, we find a derivative expansion

$$\langle Q^{ab}_E(s)\rangle = \Lambda^{ab,cd}_0 E_{cd} + \Lambda^{ab,cd}_1 \frac{d}{ds}E_{cd} + \dots, \tag{247}$$

and these $\Lambda^{ab,cd}_{0,1}$ are the structures that we would like to obtain to determine the dynamics. In particular, the former encodes the static tidal response, which is essentially described by the tidal operators introduced in the previous sections, while the latter includes the dissipative effects. We note that in calculating $\frac{d}{ds}E_{cd}$, the derivative acts on both the field and the frames

$$\frac{D}{Ds}E_{ab} \;=\; e_a{}^\mu e_b{}^\nu \frac{D}{Ds}E_{\mu\nu} - e^{-1}\Omega_a{}^c E_{cb} - e^{-1}\Omega_b{}^c E_{ac}. \tag{248}$$

Variation with respect to the einbein gives the condition

$$\frac{\delta}{\delta e}\Gamma[\Delta, e, h; \tilde{\Delta}, \tilde{e}, \tilde{h}]\Big|_{\substack{\Delta=\tilde{\Delta}\\ e=\tilde{e}, h=\tilde{h}}} = \langle p^2 - H_X - H_{int}\rangle = 0, \tag{249}$$

where the Hamiltonians are given by

$$H_X = -\frac{\delta}{\delta e}\int d\lambda e L_X(X, e^{-1}\dot{X}) = \dot{X}\frac{\partial L_X}{\partial \dot{X}} - L_X, \tag{250}$$

and at the quadrupole order

$$H_{int} = -\frac{\delta}{\delta e}\int d\lambda e\{Q^{ab}_E E_{ab} + Q^{ab}_B B_{ab}\}. \tag{251}$$

We observe that the mass shell condition is given by $M^2 \equiv \langle p^2 \rangle = \langle H_X \rangle$ in the absence of interactions. As the mass is now determined by the internal dynamics, interactions that change it could affect the mass by $\langle H_{int} \rangle$, an apparent manifestation of dissipation predicted in this framework.

Explicitly, for the momentum equation of motion, we vary with respect to x^μ and find

$$\frac{D}{Ds}p^\mu = -\frac{1}{2}R^\mu{}_{\lambda\rho\sigma}\frac{dx^\lambda}{ds}S^{\rho\sigma} + e^a{}_\rho e^b{}_\sigma \left[\langle Q^E_{ab}\rangle \nabla^\mu E^{\rho\sigma} + \langle Q^B_{ab}\rangle \nabla^\mu B^{\rho\sigma}\right]. \tag{252}$$

The first term on the right is the usual Mathisson-Papapetrou-Dixon force [63, 80]. By contracting both sides with p_μ, we could verify that indeed only the interaction part contribute to the dissipation of mass

$$\frac{d}{ds}M^2 = 2e^a{}_\rho e^b_\sigma \left[\langle Q^E_{ab}\rangle (p\cdot\nabla)E^{\rho\sigma} + \langle Q^B_{ab}\rangle (p\cdot\nabla)B^{\rho\sigma}\right] \neq 0. \tag{253}$$

To obtain the spin equation of motion consistently, we take the variation with respect to the parameters $\theta^{ab} = -\theta^{ba}$ from $\delta e^a{}_\mu = \theta^a{}_b e^b{}_\mu$ and find

$$\frac{D}{Ds}S^{\mu\nu} = \frac{dx^\nu}{ds}p^\mu - \frac{dx^\mu}{ds}p^\nu + 2e^\mu{}_a e^\nu{}_b \left[\langle Q^E_{cd}\rangle \frac{\delta}{\delta\theta_{ab}}E_{cd} + \langle Q^B_{cd}\rangle \frac{\delta}{\delta\theta_{ab}}B_{cd}\right], \tag{254}$$

with the field variations given by

$$\frac{1}{2}\langle Q^E_{cd}\rangle \frac{\delta}{\delta\theta^{ab}}E^{cd} = \langle Q^E_{c[a}\rangle E^c{}_{b]} - \langle Q^E_{cd}\rangle \frac{p_{[a}\epsilon_{b]}{}^{ce}B^d{}_e}{\sqrt{p^2}}, \tag{255}$$

$$\frac{1}{2}\langle Q^B_{cd}\rangle \frac{\delta}{\delta\theta^{ab}}B^{cd} = \langle Q^B_{c[a}\rangle B^c{}_{b]} - \langle Q^B_{cd}\rangle \frac{p_{[a}\epsilon_{b]}{}^{ce}E^d{}_e}{\sqrt{p^2}}. \tag{256}$$

We find similarly that the only interaction part leads to the dissipation of spin

$$\frac{d}{ds}S^2 = 4\langle Q^E_{ab}\rangle E^{bc}S^a{}_c + 4\langle Q^B_{ab}\rangle B^{bc}S^a{}_c \neq 0. \tag{257}$$

Furthermore, we also need to find the other set of Hamiltonian equations that determines the velocities $\frac{dx^\mu}{ds}$ and Ω_{ab} which are obtained by variations with respect to their conjugate

momenta (the in-in expectation values). The position equation of motion is easy to find

$$\begin{aligned}
&\frac{dx^\mu}{ds} + \frac{1}{2p^2} R_{\nu\lambda\rho\sigma} \frac{dx^\lambda}{ds} S^{\mu\nu} S^{\rho\sigma} - \frac{p^\mu}{p^2} p \cdot \frac{dx}{ds} \\
&= \frac{1}{p^2} \langle Q^E_{ab} \rangle \left(2\sqrt{p^2} e^\mu{}_d \epsilon^{dac} B^b{}_c + e^a{}_\rho e^b{}_\sigma S^{\mu\nu} \nabla_\nu E^{\rho\sigma} \right) \\
&+ \frac{1}{p^2} \langle Q^B_{ab} \rangle \left(2\sqrt{p^2} e^\mu{}_d \epsilon^{dac} E^b{}_c + e^a{}_\rho e^b{}_\sigma S^{\mu\nu} \nabla_\nu B^{\rho\sigma} \right),
\end{aligned} \tag{258}$$

but the momentum equation would have dependence on the choice of $\langle H_X + H_{int} \rangle$. We can circumvent this model dependence on the internal dynamics by directly obtaining the spin equation via matching to the full theory in the following applications to black holes.

0.12 Application to Kerr black holes

In this section, we apply the general framework described in the previous section to the case of a Kerr black hole with mass M and spin S. For the low-frequency observation characterized by ω, the EFT power counting consists of a double expansion in the small parameters $\kappa \equiv \hbar\omega/m_{Pl} \ll 1$ and $G_N M\omega \ll 1$. The former controls quantum gravity corrections, which are negligibly small for the classical applications here, while the latter gives an expansion parameter for classical finite-size effects. Thus, we focus on the leading order in $G_N M\omega$ power counting but zeroth order in κ. In terms of the dimensionless rotation parameter $\chi = S/G_N M^2$, our EFT works upto extremality (maximal rotation) $\chi^2 \leq 1$. Equivalently, our EFT is for arbitrary ω/Ω_H (with Ω_H the angular velocity of the horizon), and therefore extends previous work [43] to the $\Omega_H \gg \omega$ regime where these effects are PN enhanced relative to the non-spinning case.

0.12.1 Wightman function from absorption probability

Here we extract the two-point Wightman correlators of the composite operators $Q^{E,B}_{ab}$, by matching to the graviton absorption probability given in [81, 82]. The incoming graviton is assumed to be in a state with fixed angular momentum quantum numbers $(\ell, m, h = \pm 2)$, sharply localized about a frequency ω. Classically, the probability is simply the coefficient for absorption of a localized wavepacket in the given partial wave.

The calculation is most simply done in the rest frame. The probability is given by

$$p(1 \to 0) = \sum_X |\mathcal{A}(1 + M \to 0 + X)|^2, \tag{259}$$

where the S-matrix amplitude takes the form

$$\begin{aligned} i\mathcal{A}(1 + M \to 0 + X) &= \langle X; 0 | T \exp\left[iS_{int} \right] | M; 1 \rangle \\ &\approx - i \int ds \langle X | Q_E^{ab}(s) | M \rangle e^i{}_a(s) e^j{}_b(s) \langle 0 | E_{ij}(x^0(s), 0) | \lambda \rangle \\ &\quad + \text{magnetic}, \end{aligned} \tag{260}$$

at the lowest order in perturbation theory. Summing over the final states using the resolution of identity $\sum_X |X\rangle\langle X| = \mathbb{I}$, we get

$$\begin{aligned} p(1 \to 0) &= \sum_X \int ds ds' \langle M; \lambda | H_{int}(s) | X; 0 \rangle \langle X; 0 | H_{int}(s') | M; \lambda \rangle \\ &= \int ds ds' \langle Q^E_{a_1 b_1}(s) Q^E_{a_2 b_2}(s') \rangle e^{a_1}{}_{c_1}(s') e^{b_1}{}_{d_1}(s') \langle \lambda | E^{c_1 d_1}(x'^0, 0) | 0 \rangle \\ &\times e^{a_2}{}_{c_2}(s) e^{b_2}{}_{d_2}(s) \langle 0 | E^{c_2 d_2}(x^0, 0) | \lambda \rangle + \text{magnetic}. \end{aligned} \tag{261}$$

where working in the rest frame, we have the freedom to choose the coordinate time as the parametrization of the worldline

$$x^0(s) = M_\star s, \tag{262}$$

where $M_\star$ is some unimportant constant to take care of the dimensions. We explain the symbols in the following.

The graviton state $|\lambda\rangle$ with quantum numbers (ℓ, m, h) is defined as a wavepacket

$$|\lambda\rangle \equiv \int_0^\infty \frac{d\omega}{2\pi} \psi_\lambda(\omega) |\omega, \ell, m, h\rangle$$

where the spherical helicity eigenstates [83] and the wavepackets are normalized as

$$\langle \omega, \ell, m, h | \omega', \ell', m', h' \rangle = 2\pi \delta(\omega - \omega') \delta_{\ell\ell'} \delta_{mm'} \delta_{hh'}, \tag{263}$$

and

$$\int_0^\infty \frac{d\omega}{2\pi}|\psi_\lambda(\omega)|^2 = 1, \tag{264}$$

respectively such that $\langle\lambda|\lambda\rangle = 1$. The one-graviton-state matrix elements of the field strengths on the worldline are given by

$$\langle 0|B_{ab}(x^0,0)|\lambda\rangle = \pm i\langle 0|E_{ab}(x^0,0)|\lambda\rangle = \pm i\epsilon_{h,ij}(k)e^{-i|\vec{k}|x^0},$$

By using the Wigner D-matrices, defined as the matrix elements of the rotation operator $\hat{U}[R(\theta,\phi,\psi)]$ with Euler angles (θ,ϕ,ψ)

$$D^\ell_{m,h}(\theta,\phi,\psi) \equiv \langle \ell, m|\hat{U}[R(\theta,\phi,\psi)]|\ell,h\rangle, \tag{265}$$

we further decompose the polarization tensors oriented along $\hat{k} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ into

$$\epsilon_{h,ab}(k) = \langle a,b|2,h\rangle = \sum_{m=-2}^{2}\langle a,b|2,m\rangle D^{\ell=2}_{m,h}(\theta,\phi,0). \tag{266}$$

where $\langle i,j|$ is the abstract representation of the space of symmetric traceless tensors. In 4 dimensions and in the rest frame, this is normalized as

$$\langle a,b|c,d\rangle = \frac{1}{2}(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - \frac{2}{3}\delta_{ab}\delta_{cd}). \tag{267}$$

(A concrete representation is given in [44].) The third angle is set to zero as this on-axis rotation gives an unimportant phase factor. Furthermore, given that the one-graviton state and the spherical helicity eigenstates are related by

$$\langle \omega,\ell,m,h|k,h\rangle = (2\pi)^2\sqrt{\frac{2\ell+1}{2\pi\omega}}\delta(\omega - |\vec{k}|), \tag{268}$$

the wavepacket can be written as

$$\begin{aligned}\langle 0|E_{ab}(x^0,0)|\lambda\rangle &= \sum_{h'}\int \frac{d^3\vec{k}}{(2\pi)^3 2|\vec{k}|}\langle 0|E_{ab}(x^0,0)|k,h'\rangle\langle k,h'|\lambda\rangle \\ &= \frac{\sqrt{2\ell+1}}{4m_{Pl}}\psi_\lambda(x^0)\sum_{m=-2}^{2}\langle a,b|2,m'\rangle\int d\Omega D^{\ell *}_{m',h}(\theta,\phi,0)D^{\ell=2}_{m',h}(\theta,\phi,0),\end{aligned}$$

where we have written down the time domain representation

$$\psi_\lambda(x^0) = \int_0^\infty \frac{d|\vec{k}||\vec{k}|^{\frac{1}{2}}}{(2\pi)^{5/2}}e^{-i|\vec{k}|x^0}\psi_\lambda(|\vec{k}|). \tag{269}$$

We use the orthogonality relation

$$\int d\Omega D^{\ell *}_{m,p}(\theta,\phi,0)D^{\ell'}_{m',p}(\theta,\phi,0) = \frac{4\pi}{2\ell+1}\delta_{\ell\ell'}\delta_{mm'}, \tag{270}$$

to integrate over the solid angle and obtain

$$\langle 0|E_{ij}(x^0,0)|\lambda\rangle = \frac{\pi}{\sqrt{5}m_{Pl}}\delta_{\ell,2}\langle i,j|2,m\rangle\psi_\lambda(x^0). \tag{271}$$

In the rest frame, we may also align the spin along x^3-axis, we have the spatial components

$$e^a{}_c(s) = \begin{pmatrix} \cos\Omega x^0 & -\sin\Omega x^0 & 0 \\ \sin\Omega x^0 & \cos\Omega x^0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{272}$$

where Ω is the magnitude of angular velocity. In the abstract representation, we can trade the tensor representation for the unitary representation

$$e^a{}_c e^b{}_d\langle c,d|2,m\rangle = \langle a,b|\hat{U}[R_z^{-1}(\Omega x^0)]|2,m\rangle = e^{im\Omega x^0}\langle a,b|2,m\rangle. \tag{273}$$

Now the remaining unknown is the $\langle Q^E(s)Q^E(s')\rangle$ correlation function which we would like to express in terms of a set of form factors as arbitrary functions of χ. Notice that χ scales with inverse powers of $1/G_N$ and thus must be matched non-perturbatively. The form factors are enumerated by the possible tensor structures which can now depend upon

the direction of the spin, the magnitude of which is absorbed into the form factors. It is useful to expand this correlator into a basis of tensors that are invariant under rotations about the spin axis. Viewing the correlator as a linear map on the 5D space of traceless symmetric rank-$\ell = 2$ tensors (transverse to the black hole momentum p^a), a basis of tensors consists of the various powers of the generator J_3 of rotations. Because we are in the $\ell = 2$ representation of $SO(3)$, only the powers J_3^k for $k = 0, \ldots, 4$ are independent. For instance, $J_3^5 = 5J_3^3 - 4J_3$ and so on. Thus, our tensor basis consists of the identity tensor on the $\ell = 2$ space.

$$\langle a, b|c, d\rangle = \frac{1}{2}\left[\langle a|c\rangle\langle b|d\rangle + \langle a|d\rangle\langle b|c\rangle - \frac{2}{3}\langle a|b\rangle\langle c|d\rangle\right], \tag{274}$$

with $\langle a|b\rangle = \delta^a{}_b - p^a p_b/p^2$ the $\ell = 1$ identity matrix, together with the independent powers of the angular momentum J_3 in the $\ell = 2$ representation. In particular, the rotation generator in the Cartesian basis is

$$\langle a, b|J_3|c, d\rangle = \frac{1}{2}\left[\langle a|c\rangle\langle b|J_3|d\rangle + \langle a|d\rangle\langle b|J_3|c\rangle + \langle b|c\rangle\langle a|J_3|d\rangle + \langle b|d\rangle\langle a|J_3|c\rangle\right] \tag{275}$$

where in turn the angular momentum generator in the $\ell = 1$ space is $\langle a|J_3|b\rangle = is^c\epsilon_c{}^a{}_b = \frac{i}{\sqrt{p^2}}p^c s^d \epsilon_{cd}{}^a{}_b$ (we denote the spin direction by the unit spacelike vector $s^a = \delta^a{}_3$). The tensor $\langle a|J_3|b\rangle$ has eigenvalues $m = \pm 1, 0$ corresponding to the eigenvectors $v_\pm^a = \mp\frac{1}{\sqrt{2}}\left(\delta^a{}_1 \pm i\delta^a{}_2\right)$ and $v_0^a = s^a$ so it is normalized according to the usual conventions used in quantum mechanics. Higher powers, of the form $\langle a, b|J_3^k|c, d\rangle$. can be obtained from Eq. (275) by successive tensor contraction, e.g.

$$\langle a, b|J_3^2|c, d\rangle = \sum_{e,f}\langle a, b|J_3|e, f\rangle\langle e, f|J_3|c, d\rangle, \tag{276}$$

etc. We have defined these invariant tensors such that $\langle a, b|J_3|c, d\rangle$ is pure imaginary and Hermitian, and therefore our tensor basis satisfies the relation

$$\langle a, b|J_3^j|c, d\rangle = (-1)^j\langle c, d|J_3^j|a, b\rangle. \tag{277}$$

In this basis, the correlator then takes the form

$$\langle Q_E^{ab}(s)Q_{E,cd}(s')\rangle = M_\star^2 \sum_{j=0}^{4} A_{E,j}^+(s-s')\langle a,b|J_3^j|c,d\rangle, \tag{278}$$

where the functions $A_{E,j}^+(s-s')$ can depend on the magnitude of the particle spin as well as its mass. We will adopt an identical decomposition for the magnetic correlator $\langle Q_B^{ab}(s)Q_B^{cd}(s')\rangle$.

In the point particle limit where our EFT is valid, the form factors $A_k^+(s-s')$ are analytic in ω, i.e. can be represented as series of derivatives acting on the delta function $\delta(s-s')$ given the lack of long-time tails. Note that Hermiticity of the operators $Q_{E/B}^{ab}(s)$ implies that the frequency space Wightman function

$$W_{E/B}^{ab,cd}(\omega) = M_* \int ds e^{i\omega M_* s}\langle Q_{E/B}^{ab}(s)Q_{E/B}^{cd}(0)\rangle \tag{279}$$

obeys the reality condition

$$[W_{E/B}^{ab,cd}(\omega)]^* = W_{E/B}^{cd,ab}(\omega) \tag{280}$$

on the real ω-axis. Given the properties of our tensor basis, this implies that the frequency-dependent form factors $A_k^+(\omega) = M_* \int ds e^{i\omega M_* s} A_k^+(s)$ obey $[A_k^+(\omega)]^* = [A_k^+(\omega)]$ on the real axis.

Inserting the form Eq. (278) into $p(1\to 0)$, and using the fact that $J_3|\ell,m\rangle = m|\ell,m\rangle$, we obtain that

$$p(1\to 0) = \frac{4}{5}G_N\omega^5 \sum_{j=0}^{4} m^j \left(A_{E,j}^+(\omega-m\Omega) + A_{B,j}^+(\omega-m\Omega)\right). \tag{281}$$

The dependence on the shifted frequency $\omega - m\Omega$ reflects the transformation from the static frame to the rotating frame of the black hole where the correlators are defined. We can read off $A_k^+(\omega)$ by comparing powers of m in the result given in [81, 82]

$$p(1\to 0) \approx \frac{16}{225\pi} A_H (G_N M)^4 \omega^5 \left[1+(m^2-1)\chi^2\right]\left[1+\frac{1}{4}(m^2-4)\chi^2\right]\theta(\omega - m\Omega_H)\,(\omega - m\Omega_H)\,, \tag{282}$$

with $A_H = 4\pi(r_+^2+a^2) = 8\pi(G_N M)^2\left[1+\sqrt{1-\chi^2}\right]$ the area of the horizon, $\chi = a/G_N M = J/G_N M^2$ the dimensionless rotation parameter of the Kerr black hole, and $\Omega_H = 4\pi a/A_H$ the angular velocity of the horizon. This result is valid to all orders in the rotation parameter χ, but holds to leading order in $G_N M\omega \ll 1$. The factor of $\omega - m\Omega_H$, ensures that his result is valid in both the slow and rapidly rotating cases. We have inserted a step function into Eq. (282) to enforce the condition $\omega - m\Omega_H > 0$ so that the single-particle absorption probability is positive. Naively, this seems to imply that we can not trust our results in the super-radiant regime $\omega \ll \Omega_H$. However we can match in this regime for $m\Omega_H < 0$, which can then be continued for all m.

Comparison of $p(1 \to 0)$ with Eq. (281) suggests that we should identify the angular velocity in the EFT with the horizon angular velocity,

$$\Omega = \Omega_H = \frac{4\pi a}{A_H}, \tag{283}$$

which, together with Eq. (262) fixes the relation between the angular velocity Ω_{ab} and spin S^{ab} for a Kerr black hole,

$$e^{-1}\Omega^{ab} = g_{\mu\nu}e^a_\mu \frac{D}{Ds}e^\nu{}_b = \frac{4\pi}{A_H}\frac{M_*}{M}S^{ab}. \tag{284}$$

The non-vanishing frequency space response functions are then

$$A^+_{0,E}(\omega) = A^+_{0,B}(\omega) = \frac{2A_H}{45\pi G_N}(G_N M)^4(1-\chi^2)^2\theta(\omega)\omega, \tag{285}$$

$$A^+_{2,E}(\omega) = A^+_{2,B}(\omega) = \frac{A_H}{18\pi G_N}(G_N M)^4\chi^2(1-\chi^2)\theta(\omega)\omega, \tag{286}$$

$$A^+_{4,E}(\omega) = A^+_{4,B}(\omega) = \frac{A_H}{90\pi G_N}(G_N M)^4\chi^4\theta(\omega)\omega. \tag{287}$$

In obtaining this result, we have used the equality of the electric and magnetic responses that arises as a consequence of the Teukolsky equation [43]. We will check this below by comparing to known results obtained via different methods.

The step function $\theta(\omega)$ reflects that matching was performed under the assumption the graviton is quantized around the Boulware vacuum [84], corresponding to no (Hawking)

particle emission for $\omega - m\Omega_H > 0$. By contrast, matching in the Unruh state [85], where the black hole can emit Hawking radiation, would lead to Wightman response functions $A_+(\omega)$ that are non-vanishing even at $\omega < 0$. See [46] for a more detailed discussion of matching in the Unruh state. It is straightforward to check that in the Boulware state, the single-particle emission probability $p(0 \to 1)$ is given in the EFT by a formula like Eq. (281) that involves the Wightman correlators $A_+(m\Omega - \omega)$, leading to the prediction of a non-zero emission probability for superradiant modes with $\omega - m\Omega_H < 0$:

$$p(0 \to 1) \approx \frac{16}{225\pi} A_H (G_N M)^4 \omega^5 \left[1 + (m^2 - 1)\chi^2\right] \left[1 + \frac{1}{4}(m^2 - 4)\chi^2\right] \theta(m\Omega_H - \omega)\,(m\Omega_H - \omega)\,, \tag{288}$$

see [44] for more a detailed discussion of the worldline EFT in the regime of superradiant emission.

0.12.2 The causal response function

In the classical processes that we consider in this paper, the relevant correlator is the retarded Green's function

$$G_R^{ab,cd}(s - s') = -i\theta(s - s')\langle \Big[Q^{ab}(s), Q^{cd}(s')\Big] \rangle \tag{289}$$

rather than the Wightman functions obtained in the previous section. Because this is a real quantity, the frequency space causal response $G_R^{ab,cd}(\omega) = M_* \int ds e^{i\omega M_* s} G_R^{ab,cd}(s)$ satisfies the reality condition

$$\Big[G_R^{ab,cd}(-\omega)\Big]^* = G_R^{ab,cd}(\omega), \tag{290}$$

for real frequencies. Thus $\mathrm{Re}G_R^{ab,cd}(\omega)$ is an even function on the real ω-axis while $\mathrm{Im}G_R^{ab,cd}(\omega)$ is an odd function. The retarded Green's function is related to the two-point Wightman correlators by a dispersion relation of the form

$$G_R^{ab,cd}(\omega) = M_* \int ds e^{i\omega M_* s} G_R^{ab,cd}(s) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{W^{ab,cd}(\omega') - W^{cd,ab}(-\omega')}{\omega - \omega' + i\epsilon}, \tag{291}$$

	$\omega \leftrightarrow -\omega$	$ab \leftrightarrow cd$
$\mathrm{Re}\left[W^{ab,cd}(\omega) - W^{cd,ab}(-\omega)\right]$	odd	even
$\mathrm{Im}\left[W^{ab,cd}(\omega) - W^{cd,ab}(-\omega)\right]$	even	odd

Table 1: The properties of $W^{ab,cd}(\omega) - W^{cd,ab}(-\omega)$. Even/odd means that the function changes/does not change sign under the given transformation.

which, as a consequence, defines a function that is analytic for $\mathrm{Im}\omega \geq 0$ but singular on the lower-half complex-ω plane. Expanding out the dispersion relation in Eq. (291) into its real and imaginary parts, we find that in terms of the Wightman functions

$$\mathrm{Re}G_R^{ab,cd}(\omega) = \frac{1}{2}\mathrm{Im}\left[W^{ab,cd}(\omega) - W^{cd,ab}(-\omega)\right] + \mathrm{Pr}\int_0^\infty \frac{\omega' d\omega'}{\pi} \frac{\mathrm{Re}\left[W^{ab,cd}(\omega') - W^{cd,ab}(-\omega')\right]}{\omega^2 - \omega'^2} \quad (292)$$

and

$$\mathrm{Im}G_R^{ab,cd}(\omega) = -\frac{1}{2}\mathrm{Re}\left[W^{ab,cd}(\omega) - W^{cd,ab}(-\omega)\right] + \omega \cdot \mathrm{Pr}\int_0^\infty \frac{d\omega'}{\pi} \frac{\mathrm{Im}\left[W^{ab,cd}(\omega') - W^{cd,ab}(-\omega')\right]}{\omega^2 - \omega'^2} \quad (293)$$

This result follows from Eq. (280), which implies the exchange properties under the transformations $\omega \to -\omega$ or $ab \leftrightarrow cd$ listed in Table 1.

Note that in addition to the contribution of the worldline multipole operators, the physical response (as determined, for example, through measurements of the gravitational field at large distances) can also receive contributions from terms in the worldline action that are polynomial in E_{ab}, B_{ab} and/or their derivatives with respect to the parameter s. We will henceforth refer to these terms as "local", to make the distinction from terms in the action involving the internal degrees of freedom X.

Focusing on purely electric couplings, such local terms modify the low-frequency response by an analytic function $L^{ab,cd}(\omega)$ whose real part is *even* under $\omega \leftrightarrow -\omega$ or index interchange $ab \leftrightarrow cd$. Then using (277) we see that the local contribution to the real response can only involve the tensor structures $\langle a, b|J_3^j|c, d\rangle$ with $j = 0, 2, 4$. In particular, the static Love numbers of the black hole, which are identified with the local response at $\omega = 0$ (both from $L^{ab,cd}(\omega)$ and from Eq. (291)) cannot involve tensor structures that are linear or cubic in the spin. Alternatively, time reversal-invariance implies that terms odd

in spin vanish in the static limit.

The contribution from terms in the action also modifies the imaginary part by terms that are odd under either $\omega \to -\omega$ or $ab \leftrightarrow cd$ exchange. Despite possibly having a non-vanishing imaginary part, the local response $L^{ab,cd}(\omega)$ does not contribute to dissipation, as will be discussed below, and thus can not be matched using $p(1 \to 0)$ but instead must be fixed by matching to other observables in the full theory, for instance, elastic scattering of low-frequency gravitons off the black hole.

Because our matching procedure only fixes the Wightman function at low frequency, it does not completely determine the form of the retarded response function. In particular, matching to low-frequency absorption cannot yield information about the terms in Eq. (292) and Eq. (293) that involve principal part integrals over high arbitrarily high-frequency scales, where the EFT description necessarily breaks down. However, from Eq. (292) and Table 1, we see that the principal part integral contribution to $\mathrm{Re}G_R^{ab,cd}(\omega)$ is analytic at $\omega = 0$ (assuming the integral in Eq. (292) converges), and even under either $\omega \leftrightarrow -\omega$ or $ab \leftrightarrow cd$ exchange. Similarly $\mathrm{Im}G_R^{ab,cd}(\omega)$ is odd if we replace $\omega \leftrightarrow -\omega$ or $ab \leftrightarrow cd$. Consequently, the principal part contribution to $G_R^{ab,cd}(\omega)$ is physically indistinguishable (i.e. of the same form), from the local response $L^{ab,cd}(\omega)$ arising from adding local counterterms to the point particle action.

On the other hand, the calculable part of Eqs. (292), (293) gives rise to a genuinely non-local contribution to the retarded Green's function, of the form

$$G^{ab,cd}_{R,non-local}(\omega) = -\frac{i}{2}\left[W^{ab,cd}(\omega) - W^{cd,ab}(-\omega)\right]. \tag{294}$$

This object does not have the correct $ab \leftrightarrow cd$ index exchange properties to arise from curvature couplings in the point particle action, and cannot be absorbed into a local counterterm. It is in particular this function $G^{ab,cd}_{R,non-local}(\omega)$ that gives rise to dissipative effects in the EFT description of the black hole.

Ignoring the local contribution to the causal response, we obtain from Eqs. (285)-(287)

the result

$$G^{ab}_{R,Ecd}(\omega) = \frac{M^2 A_H}{45\pi G_N}(G_N M)^4\,(-i\omega)\cdot\langle a,b|(1-\chi^2)^2+\frac{5}{4}\chi^2(1-\chi^2)J_3^2+\frac{1}{4}\chi^4 J_3^4|c,d\rangle, \quad (295)$$

with an identical expression for the magnetic Green's function $G^{ab}_{R,Bcd}(\omega)$. This result is equivalent to the statement that, up to local terms, the quadrupole moment induced[5] by an external electric field is

$$\begin{aligned}\langle Q^{ab}_E(s)\rangle &= \int ds' G^{ab,cd}_{R,E}(s-s')E_{cd}(s') \\ &= \frac{A_H}{45\pi G_N}(G_N M)^4\langle a,b|(1-\chi^2)^2+\frac{5}{4}\chi^2(1-\chi^2)J_3^2+\frac{1}{4}\chi^4 J_3^4|c,d\rangle\frac{d}{ds}E_{cd}(x(s)),\end{aligned} \quad (296)$$

where the derivative here is in the co-rotating frame, see Eq. (248).

An identical formula relates the induced magnetic moment $\langle Q^{ab}_E(s)\rangle$ to the co-rotating components of magnetic curvature $B_{ab}(x(s))$ along the point particle worldline.

Taking the limit where the rotation of the black hole is larger than the intrinsic time dependence of the curvature, we may approximate

$$\frac{d}{ds}E_{ab} \approx -\Omega_a{}^c E_{cb} - \Omega_b{}^c E_{ac} = iM_*\Omega_H\langle a,b|J_3|c,d\rangle E_{cd}, \quad (297)$$

in which case the induced moment is of the form

$$M_*^{-1}\langle Q^{ab}_E(s)\rangle \approx \frac{4i(G_N M)^5}{45G_N}\chi\langle a,b|(1-2\chi^2)J_3+\frac{5}{4}\chi^2 J_3^3|c,d\rangle E_{cd}. \quad (298)$$

Despite appearances, Eq. (298) does not imply the existence of a non-vanishing static Love tensor for the Kerr black hole, since this relation cannot arise from local terms in the point particle action. In particular, a term such as $\int \chi ds E_{ab}\langle a,b|J_3^j|c,d\rangle E_{cd}$ for $j=1,3$ vanishes identically due to the antisymmetry under $ab\leftrightarrow cd$ of the tensor structures. We have verified, however, that Eq. (298) is consistent, for $\chi \ll 1$, with the results of [52] which

5. Because of spin, the Kerr black hole has an infinite series of permanent multipole moments [93], which in the point particle limit are equivalent to local spin-dependent worldline interactions that linearly in the curvature tensor. Here, by induced moment, we mean the shift in the value of the permanent moments that are generated when a background field $R_{\mu\nu\rho\sigma}\neq 0$ is turned on.

obtained the quadrupolar response of a slowly spinning Kerr black hole, at linear order in χ.

On the other hand, it is in principle possible that Kerr black holes have non-zero static Love numbers, but by symmetry those, would have to correspond to local worldline counterterms which in our basis take the form

$$S_{pp} \supset G_N^4 M^6 \int ds f_j(\chi^2) \chi^j E_{ab} \langle a, b | J_3^j | c, d \rangle E_{cd}, \tag{299}$$

with $j = 0, 2, 4$, as well as their magnetic counterparts. Here, we have defined $\chi = \frac{\sqrt{-S^\mu S_\mu}}{G_N p^2} \leq 1$, and $f_{0,2,4}(\chi^2)$ are functions analytic at $\chi^2 = 0$. The overall scaling $G_N^4 M^6$ is the characteristic magnitude of the static tidal response of a compact object. It is well known that the spin-independent term in Eq. (299) has vanishing Wilson coefficient, i.e $f_0(\chi^2 = 0) = 0$, [86–88]. Recently, ref. [53] has extended this calculation to arbitrary orders in spin (previous partial results can be found in [89]) and found, remarkably, that the all-local contributions to the static response function of the Kerr black hole are in fact vanishing as well.

In addition to the local contributions to the static response, there are also terms in the point particle action which modify $G_R^{ab,cd}(\omega)$ away from $\omega = 0$. The leading such terms at low frequency are of the form

$$S_{pp} \supset G_N^5 M^6 \int ds \chi^j E_{ab} \langle a, b | [i J_3]^j | c, d \rangle \dot{E}_{cd}, \tag{300}$$

with $j = 1, 3$. These local interactions not forbidden by symmetries (it is even under both parity and time reversal), and yield contributions to $\langle Q_E^{ab} \rangle$ of comparable magnitude to those in Eq. (296). However, unlike the terms in Eq. (296), the curvature couplings in Eq. (300) cannot give rise to dissipative effects, despite the fact that they contribute to $\mathrm{Im} G_{R,E}^{ab,cd}(\omega)$. The recent analysis of ref. [53] indicates that, for the Kerr black hole, terms such as those in Eq. (300) are also vanishing. Assuming the validity of the results in [53], it then follows that Eq. (295) completely characterizes the black hole response function at linear order in time derivatives but to all orders in spin.

0.12.3 Black hole dissipative dynamics in a tidal environment

Assuming that all the local contributions to black hole response are indeed zero [53], the complete equations of motion for a spinning black hole moving in a background gravitational field with curvature scale $\mathcal{R} \gg G_N M$ can be obtained straightforwardly by inserting the induced moments $\langle Q_{E,B} \rangle$ from Eq. (296) and its magnetic analog into Eqs. (252), (254). Because the resulting expressions are messy and not particularly illuminating, we will report instead on the implications of these equations for the rate of change of mass and spin that arise as a consequence of tidal interactions, given in Eqs. (253), (257). We consider separately the cases of a rapidly spinning black hole, $\mathcal{R}^{-1} \ll \Omega_H$ and $\chi \sim \mathcal{O}(1)$, as well as the opposite slow-spin limit $\mathcal{R}^{-1} \gg \Omega_H$, which necessarily requires that $|\chi| \ll 1$,

Using the relation in Eq. (262) between our parameter s and the proper time τ along the worldline of the rotating black hole, we find, in the large spin case

$$\begin{aligned} \frac{d}{d\tau} M &\approx \frac{8(G_N M)^5}{45 G_N} \chi \epsilon^{\mu\nu}{}_{\lambda} s^{\lambda} \left[(1+3\chi^2) E_{\mu\rho} \dot{E}^{\rho}{}_{\nu} + \frac{15}{4} \chi^2 E_{\mu\rho} s^{\rho} \dot{E}_{\nu\sigma} s^{\sigma} \right] + \text{magnetic} \\ &\quad + \mathcal{O}(G_N^7 M^7 / \mathcal{R}^7). \end{aligned} \tag{301}$$

This result, which is valid to all orders in spin, agrees with results found in Refs. [90, 91]. To linear order in χ it also agrees with results obtained in [43]. In the opposite, $\chi \to 0$ limit, we find instead

$$\frac{d}{d\tau} M \approx \frac{16}{45 G_N} (G_N M)^6 \left[\dot{E}_{\rho\sigma} \dot{E}^{\rho\sigma} + \dot{B}_{\rho\sigma} \dot{B}^{\rho\sigma} \right] + \mathcal{O}(\chi, G_N^6 M^6 / \mathcal{R}^6) \tag{302}$$

which receives corrections at linear order in $\chi \ll 1$ from radiative tail contributions to the EFT matching and to the Schwinger-Keldysh action. This is also in agreement with [51, 92].

For the torque induced on the black hole by the tidal background, we find from Eqs. (254), (296)

$$\begin{aligned} \frac{d}{d\tau} S &\approx -\frac{2}{45 G_N} (G_N M)^5 \chi \left[8(1+3\chi^2) E_{\rho\sigma} E^{\rho\sigma} + 3(4+17\chi^2) E_{\lambda\rho} E^{\lambda}{}_{\sigma} s^{\rho} s^{\sigma} + 15\chi^2 (E_{\rho\sigma} s^{\rho} s^{\sigma})^2 \right] \\ &\quad + \text{magnetic}. \end{aligned} \tag{303}$$

in the limit $\Omega_H \gg \mathcal{R}^{-1}$. In the opposite, $\chi \to 0$, Eq. (296) is dominated by the time

variation of $E_{\mu\nu}$ and the torque is instead

$$\frac{d}{d\tau}S \approx -\frac{8}{45G_N}(G_N M)^6 \epsilon^{\mu\nu}{}_{\lambda} s^{\lambda} \left[E_{\mu\rho}\dot{E}^{\rho}{}_{\nu} + B_{\mu\rho}\dot{B}^{\rho}{}_{\nu} \right] \tag{304}$$

Both Eqs. (303), (304) are in agreement with results obtained previously in [51, 92]. To go to next order in $G_N M/\mathcal{R} \ll 1$ would require the inclusion in both the EFT matching and Schwinger-Keldysh action of infrared divergent tail terms corresponding to graviton scattering off the black holes own gravitational field. Ref. [91] has reported a result for these next-to leading order corrections, although a discrepancy with their earlier results [90] obtained in a probe limit remains unsettled in the literature.

As another check of our results, note that from Eqs. (252), (254), we also find that in terms of the curvatures E_{ab}, B_{ab} in the rotating frame

$$\frac{d}{d\tau}M - \Omega_H \frac{d}{d\tau}S = \langle Q^E_{ab}\rangle \frac{D}{D\tau}E^{ab} + \langle Q^B_{ab}\rangle \frac{D}{D\tau}B^{ab}, \tag{305}$$

or by Eq. (296),

$$\begin{aligned} \frac{d}{d\tau}M - \Omega_H \frac{d}{d\tau}S &= \frac{A_H (G_N M)^4}{45\pi G_N} \frac{D}{D\tau} E_{ab} \langle a,b| (1-\chi^2)^2 + \frac{5}{4}\chi^2(1-\chi^2)J_3^2 + \frac{1}{4}\chi^4 J_3^4 |c,d\rangle \frac{D}{D\tau}E_{cd} \\ &\quad + \text{magnetic}. \end{aligned} \tag{306}$$

Because the even powers of the tensor $\langle a,b|J_3|c,d\rangle$ are positive definite, this quantity is manifestly positive in the physical region $\chi^2 \leq 1$. Therefore the change in the black hole area as a result of tidal interactions is also positive

$$\frac{d}{d\tau}A_H = \frac{2A_H}{M\sqrt{1-\chi^2}} \left[\dot{M} - \Omega_H \dot{S} \right] \geq 0, \tag{307}$$

as required on general grounds [94].

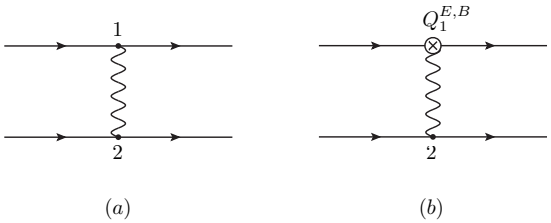

Figure 15: Potential exchange diagrams responsible for two-particle effective interaction. The graviton is mediated between the mass monopole of particle 2 and (a) mass monopole of 1 (b) the tidal quadrupole of 1 (a similar diagram with $1 \leftrightarrow 2$ has been omitted).

0.13 Post-Newtonian equations of motion for binary dynamics

The same worldline effective action formalism can also be applied to dissipation in dynamically generated spacetimes, i.e sourced by the particles themselves, rather than the fixed background field case discussed above. In order to do so, we have to include in the Schwinger-Keldysh functional an integral over the fluctuations of the gravitational field itself[6].

As an example, we will consider a binary system of black holes in the non-relativistic regime, with $v^2 \sim G_N M r/r \ll 1$. For illustration, we will focus on the regime of rapidly spinning black holes, with $\Omega_H \gg v/r$. The rotation parameters will be assumed to scale as $\chi \sim \mathcal{O}(1)$. Integrating out the potential graviton exchange between the black holes, Fig. 15(b), the two-particle interaction term reduces to [50]

$$\Gamma_{int} \approx -G_N m_1 m_2 \int dt \left[\frac{Q^{ab}_{E,1}(t)}{m_1^2} e_1{}^i{}_a e_1{}^j{}_b + (1 \leftrightarrow 2) \right] \partial_i \partial_j \frac{1}{|\vec{x}(t)|}, \tag{308}$$

with $\vec{x} = \vec{x}_1 - \vec{x}_2$, up to terms suppressed by more power of the velocities. Varying the in-in action, we obtain, in the linear response limit, an instantaneous non-conservative force on

6. The role of the in-in formalism to describe radiation reaction forces was first discussed in [95].

the black holes that is given by,

$$\begin{aligned}\vec{F}_1(t) &= \frac{\delta}{\delta\vec{x}_1(t)}\,\Gamma[\vec{x},\tilde{\vec{x}};e_{1,2},\tilde{e}_{1,2}]\Big|_{\vec{x}=\tilde{\vec{x}};e_{1,2}=\tilde{e}_{1,2}} \\ &\approx -G_N m_1 m_2\left[\frac{\langle Q^{ab}_{E,1}(t)\rangle}{m_1^2}e_1{}^a{}_j e_1{}^b{}_k + (1\leftrightarrow 2)\right]\nabla\partial_j\partial_k\frac{1}{|\vec{x}(t)|} = -\vec{F}_2(t), \quad (309)\end{aligned}$$

with $\vec{x} = \vec{x}_1 - \vec{x}_2$. Similarly, the torque on each black hole can be obtained from the Schwinger-Keldysh action by varying with respect to the frame $e^a{}_i$,

$$\frac{d}{dt}\vec{S}^i_1 = e^i_{1a}\epsilon^{abc}\frac{\delta}{\delta\theta_1{}^{bc}}\,\Gamma[\vec{x},\tilde{\vec{x}};e_{1,2},\tilde{e}_{1,2}]\Big|_{\vec{x}=\tilde{\vec{x}};e_{1,2}=\tilde{e}_{1,2}} \approx \frac{2G_N m_2}{m_1}e^i_{1a}e^j_{1b}e^k_{1c}\epsilon^{abd}\langle Q^{E,1c}{}_d\rangle\partial_j\partial_k|\vec{x}|^{-1} \quad (310)$$

On the right-hand side of this and the previous equation, the in-in expectation values in the PN limit can be obtained from Eq. (296), by inserting

$$E_{ab} = G_N m_2 e_1{}^i{}_a e_1{}^i{}_b\partial_i\partial_j|\vec{x}(t)|^{-1} \quad (311)$$

into $\langle Q^{ab}_{E,1}\rangle$, and similarly for the case of $\langle Q^{ab}_{E,2}\rangle$. This yields the result

$$\vec{F}_1(t) = -\vec{F}_2(t) = -\frac{8}{5}\frac{G_N^5 m_1^3 m_2^2}{|\vec{x}|^7}\left[1 + 3\chi_1^2 - \frac{15}{4}\chi_1^2\left(\vec{s}_1\cdot\frac{\vec{x}}{|\vec{x}|}\right)^2\right]\frac{\vec{x}}{|\vec{x}|}\times\vec{S}_1 + (1\leftrightarrow 2) \quad (312)$$

for the non-conservative force. The torque on each particle is

$$\frac{d}{dt}\vec{S}_1 = -\frac{8}{5}\frac{G_N^5 m_1^3 m_2^2}{|\vec{x}|^6}\left[1 + 3\chi_1^2 - \frac{15}{4}\chi_1^2\left(\vec{s}_1\cdot\frac{\vec{x}}{|\vec{x}|}\right)^2\right]\left[\vec{S}_1 - \frac{\vec{S}_1\cdot\vec{x}}{\vec{x}^2}\vec{x}\right]. \quad (313)$$

In Eq. (312), "$1\leftrightarrow 2$" has the meaning that we exchange the particle labels without changing the sign of $\vec{x}$. The PN equations of motion to linear order in the spin for an arbitrary composite object were first calculated in [45]. Our results at $\chi \ll 1$ agree with those of ref. [45] if one uses Eq. (285) with $\chi = 0$ to fix their dissipation parameter. The friction force $\vec{F}_{1,2}$ is a 5PN effect, while our result for the torque is 4PN relative to the leading order gravito-magnetic spin precession formula predicted by linearized GR.

As a simple consequence of Eq. (312), consider the mechanical power that is absorbed

or extracted by the black hole horizons

$$\frac{d}{dt}E = \sum_\alpha \vec{v}_\alpha \cdot \vec{F}_\alpha \approx \frac{8}{5}\frac{G_N^5 m_1^2 m_2}{|\vec{x}|^8}(m_1+m_2)\left[1+3\chi_1^2 - \frac{15}{4}\chi_1^2\left(\vec{s}_1\cdot\frac{\vec{x}}{|\vec{x}|}\right)^2\right]\vec{S}_1\cdot\vec{L} + (1\leftrightarrow 2), \tag{314}$$

where $\vec{L}$ is the orbital angular momentum about the center of mass. This results agrees to linear order with [43, 45]. Depending on the relative orientations between the spins and the orbit, the rate of change of energy can be positive or negative, reflecting the possibility of energy extraction from the black holes through the Penrose process. For example, if the spins are orthogonal to the orbital plane, dE/dt can be either positive or negative depending on whether the spins are aligned or anti-aligned with $\vec{L}$. Regardless, Eq. (314) enters at order v^5, or 2.5PN relative to leading order quadrupole radiation from the binary and, as is well known [38, 51], is enhanced relative to absorption in the case of non-rotating black holes by a factor of v^{-3}. A final check of these results is that the orbital angular momentum as predicted by Eq. (312) is given by

$$\frac{d}{dt}\vec{L} = \sum_\alpha \vec{x}_\alpha \times \vec{F}_\alpha \approx -\frac{d}{dt}(\vec{S}_1+\vec{S}_2), \tag{315}$$

with $d\vec{S}_{1,2}/dt$ given by Eq. (313). It therefore follows that the total angular momentum $\vec{J} = \vec{L} + \vec{S}_1 + \vec{S}_2$ is conserved, as should be expected given that the tidal dynamics we consider here does not involve any gravitational radiation out to infinity at leading PN order.

Conclusions and Outlook

In this thesis, we reported on recent progress in developing worldline EFT methodologies for describing gravitational dynamics and gravitational radiation involving many-body systems of compact spinning objects. In Chapter 0.2, we gave a brief introduction to the central ideas in perturbative gravity, the logic behind applying the worldline EFT framework in gravity, and how to describe spin, finite-size effect, and dissipation therein.

In Chapter 0.4.4, we described how the color-kinematic duality and double copy construction formulated in quantum field theory scattering amplitudes led to the proposal of classical double copy [35] in studying classical Yang-Mills and dilaton-gravity radiation amplitudes. We generalized the discussion in [35] to worldline EFT containing spin degrees of freedom and verified that the same procedure works at leading order, provided that we also include an axion channel and the couplings are fixed to specific spins. In particular, we found that the worldline spin-gauge field coupling corresponds to a gypromagnetic ratio of Dirac value $g_D = 2$ and, the spacetime couplings of the gravitational channels correspond to the low-energy EFT of string theory. The form of the string-gravity action is very suggestive of underlying connections to string theories, but the worldline couplings deviate from the ones expected in a worldline effective action of a closed string due to the fact that the latter has two spinning sectors while we only have one in the former.

This problem was resolved by the classical double copy based on color-kinematic duality [36]. In Chapter 0.7.4, we generalized this upgraded version to the spinning system and found that it is possible to incorporate both sectors. In this new language, an appropriate choice of Dirac gyromagnetic ratio is necessary for the color-kinematic duality to hold. We explicitly calculated the low-energy effective gauge vertex of a classical open string by

dimensionally reducing over the string length and found that it is in agreement with the Yang-Mills worldline EFT with $g_D = 2$. Similarly, the low-energy vertices from a closed string match with those from the string-gravity worldline EFT with two spin sectors. The particular case with spin sectors identified decouples from axion and can be understood as corresponding to unoriented strings. Furthermore, we showed that the modified classical double copy is applicable to worldline EFTs with finite-size corrections, provided that the Wilson coefficients are set to specific ratios.

Along a parallel direction, in Chapter 0.10, we extended the works from [42–44, 50] to construct the worldline EFT that describes dissipative dynamics of Kerr black holes. To do this, we matched with absorption probabilities calculated by black hole perturbation theory and extracted the in-in correlation functions for the worldline multipoles to all orders in spin using a convenient basis for these correlation functions. This information was substituted into the equation of motion to derive the dissipation of spin and mass and the results were found to be consistent with the existing literature at the lowest order. Finally, we demonstrated the utility of this framework by computing the dissipation in a PN binary and obtained a new result that is 2.5 PN enhanced from the non-spinning counterpart.

As the classical application of double copy is a burgeoning field at the moment, there are many potential topics for further investigations. For phenomenological applications, it would be useful to remove the axion and dilaton channels. The former might be achieved due to the underlying string theory interpretation, so it would be helpful to verify the stringy interpretation beyond the leading order results. For the latter, a proposed solution is to introduce a ghost field that cancels with the dilaton [96]. However, there is not a systematic construction of the ghost field and it is not obvious whether it holds at higher orders. Another possibility is to further the progress on applications to bound orbits as opposed to scattering sources [97] since the bounded binaries are much more relevant to observations compared to the scattering scenario. Moreover, as we have seen progress in the case of conservative dynamics, it might be useful if we could establish some direct connections between the double copy construction in scattering amplitude methods and radiative dynamics [98, 99].

There are also plenty of direct follow-ups to the Kerr horizon EFT program. As already

mentioned, we would like to perform higher-order calculations and resolve the discrepancies in existing literature [90, 91]. It would also be interesting to explore the phenomenological implications of our results, since the enhanced dissipative effect could be relevant to observations. Following the works of [46, 49], the spinning generalizations to the treatments on quantum gravitational effects in the worldline EFT framework is another exciting possibility. Finally, there have been efforts aimed at quantizing the worldline theory of spinless black holes to establish connections to S-matrix theory [100, 101] but the quantization has yet to be extended to spinning black holes with dissipative effects.

In summary, the worldline EFT has been proven to be a powerful tool for studying gravitational dynamics and calculating gravitational-wave observables and the generalization to spinning systems is essential to observations. In preparation for the increasing sensitivity of gravitational wave experiments in this new decade, it is surely a fruitful area of research to focus on.

Worldline graph calculations

In this appendix, we summarize the explicit results for the contributions to the total amplitudes that are due to radiation coming directly off the worldline described in Section 0.7.2 and 0.7.3. The two sections present the relevant contributions in the gauge and gravitational theories respectively.

.1 Gauge theory

The relevant contributions are from Figs. 6(a), (b) and are given by Eqs. (130), (131). These can be written down explicitly as

$$\begin{aligned}(130) = ig^2 \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \int d\mu_{\alpha\beta}(k) \Bigg[& [c_\alpha, c_\beta]^a \frac{\ell_\alpha^2}{k\cdot p_\alpha} \Big\{ \kappa_\alpha (\ell_\beta \wedge p_\beta)_\alpha p_\alpha^\mu - \kappa_\beta (\ell_\beta \wedge p_\alpha)_\beta p_\alpha^\mu \Big\} \\ & + (c_\alpha \cdot c_\beta) c_\alpha^a \Bigg\{ \frac{\ell_\alpha^2}{k\cdot p_\alpha} \Big[(\ell_\beta \wedge p_\beta)_\alpha \Big(\ell_\beta^\mu - \frac{k\cdot \ell_\beta}{k\cdot p_\alpha} p_\alpha^\mu \Big) \Big] \\ & - \frac{\ell_\alpha^2}{k\cdot p_\alpha} \Big[(k\cdot p_\alpha)(S_\beta \wedge \ell_\beta)^\mu + (\ell_\beta \wedge \ell_\alpha)_\beta p_\alpha^\mu + (\ell_\beta \wedge p_\alpha)_\beta \Big(\ell_\beta^\mu - \frac{k\cdot \ell_\beta}{k\cdot p_\alpha} p_\alpha^\mu \Big) \Big] + \Bigg\} \Bigg], \end{aligned} \tag{316}$$

$$\begin{aligned}(131) = ig^2 \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \int d\mu_{\alpha\beta}(k) \Bigg[& [c_\alpha, c_\beta]^a \frac{p_\alpha \cdot p_\beta}{k\cdot p_\alpha} \ell_\alpha^2 (S_\alpha \wedge k)^\mu \\ & + (c_\alpha \cdot c_\beta) c_\alpha^a \Bigg\{ \frac{\ell_\alpha^2}{k\cdot p_\alpha} \Big[\frac{p_\alpha \cdot p_\beta}{k\cdot p_\alpha} (k\cdot \ell_\beta) - (k\cdot p_\beta) \Big] (S_\alpha \wedge k)^\mu \\ & - \frac{\ell_\alpha^2}{k\cdot p_\alpha} \Big[(\ell_\alpha \wedge \ell_\beta)_\alpha p_\beta^\mu - (k \wedge p_\beta)_\alpha \ell_\beta^\mu - (k\cdot p_\beta)(S_\alpha \wedge \ell_\beta)^\mu + (k\cdot \ell_\beta)(S_\alpha \wedge p_\beta)^\mu \Big] \Bigg\} \Bigg]. \end{aligned} \tag{317}$$

.2 Gravitational theory

First, we look at the axion current. This receives contributions from the worldline diagram Fig. 7(a), given by Eqs. (144), (145). These compute to

$$(144) = \frac{i\kappa^3}{2}\epsilon^{\mu\nu}\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\int \mu_{\alpha\beta}(k)\ell_\alpha^2\Big[-\frac{(p_\alpha\cdot p_\beta)^2(k\cdot\ell_\beta)}{(k\cdot p_\alpha)^2}(S_\alpha\wedge k)^\mu p_\alpha^\nu$$
$$+\frac{p_\alpha\cdot p_\beta}{k\cdot p_\alpha}\Big\{(p_\alpha\cdot p_\beta)\ell_\beta^\nu-2(k\cdot p_\alpha)p_\beta^\nu+2(k\cdot p_\beta)p_\alpha^\nu\Big\}(S_\alpha\wedge k)^\mu-(\mu\leftrightarrow\nu)\Big], \tag{318}$$

$$(145) = \frac{i\kappa^3}{2}\epsilon^{\mu\nu}\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\int d\mu_{\alpha\beta}(k)\ell_\alpha^2\Bigg[\frac{p_\alpha\cdot p_\beta}{k\cdot p_\alpha}\Big\{(k\cdot\ell_\beta)(S_\alpha\wedge p_\beta)^\mu p_\beta^\nu-(k\cdot p_\beta)(S_\alpha\wedge\ell_\beta)^\mu p_\beta^\nu$$
$$-(k\wedge\ell_\beta)_\alpha p_\alpha^\mu p_\beta^\nu+(k\wedge p_\beta)_\alpha p_\alpha^\mu\ell_\beta^\nu\Big\}+(p_\alpha\cdot p_\beta)\Big\{(S_\alpha\wedge\ell_\beta)^\mu p_\beta^\nu-(S_\alpha\wedge p_\beta)^\mu\ell_\beta^\nu\Big\}$$
$$+(k\cdot p_\alpha)(S_\alpha\wedge p_\beta)^\mu p_\beta^\nu-(k\cdot p_\beta)(S_\alpha\wedge p_\beta)^\mu p_\alpha^\nu-(k\wedge p_\beta)_\alpha p_\alpha^\mu p_\beta^\nu$$
$$+\frac{p_\beta^2}{d-2}\Big\{2(S_\alpha\wedge k)^\mu-(k\cdot p_\alpha)S_\alpha^{\mu\nu}\Big\}-(\mu\leftrightarrow\nu)\Bigg]. \tag{319}$$

Next, we move to graviton radiation. Worldline contributions to the energy momentum pseudotensor are from Figs. 8(a), (b). The corresponding expressions Eqs. (149), (150) are respectively given by

$$(149) = -2i\kappa^3\epsilon^{\mu\nu}\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\int d\mu_{\alpha\beta}(k)\ell_\alpha^2\Bigg[\frac{(p_\alpha\cdot p_\beta)(k\cdot\ell_\beta)}{2(k\cdot p_\alpha)^2}\Big\{(\ell_\beta\wedge p_\beta)_\alpha-(\ell_\beta\wedge p_\alpha)_\beta\Big\}p_\alpha^\mu p_\alpha^\nu$$
$$-\frac{p_\alpha\cdot p_\beta}{k\cdot p_\alpha}\Big\{\Big((\ell_\beta\wedge p_\beta)_\alpha-(\ell_\beta\wedge p_\alpha)_\beta\Big)p_\alpha^\mu\ell_\beta^\nu+\frac{1}{2}(\ell_\alpha\wedge\ell_\beta)_\beta p_\alpha^\mu p_\alpha^\nu\Big\}$$
$$-\frac{k\cdot p_\beta}{2(k\cdot p_\alpha)}\Big\{(\ell_\beta\wedge p_\beta)_\alpha-(\ell_\beta\wedge p_\alpha)_\beta\Big\}p_\alpha^\mu p_\alpha^\nu$$
$$+(p_\alpha\cdot p_\beta)(S_\beta\wedge\ell_\beta)^\mu p_\alpha^\nu+\Big\{(\ell_\beta\wedge p_\beta)_\alpha-(\ell_\beta\wedge p_\alpha)_\beta\Big\}p_\alpha^\mu p_\beta^\nu$$
$$+\frac{m_\beta^2}{2(d-2)}(S_\alpha\wedge\ell_\beta)^\mu p_\alpha^\nu+(\mu\leftrightarrow\nu)\Bigg], \tag{320}$$

$$
\begin{aligned}
(150) = -2i\kappa^3\epsilon^{\mu\nu}\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\int d\mu_{\alpha\beta}(k)\ell_\alpha^2\Bigg[&\frac{(p_\alpha\cdot p_\beta)^2(k\cdot\ell_\beta)}{2(k\cdot p_\alpha)^2}(S_\alpha\wedge k)^\mu p_\alpha^\nu - \frac{(p_\alpha\cdot p_\beta)^2}{2(k\cdot p_\alpha)}(S_\alpha\wedge k)^\mu\ell_\beta^\nu\\
&- \frac{p_\alpha\cdot p_\beta}{2(k\cdot p_\alpha)}\Big\{(k\cdot p_\beta)\Big((S_\alpha\wedge k)^\mu + (S_\alpha\wedge\ell_\alpha)^\mu\Big)p_\alpha^\nu + (k\cdot\ell_\beta)(S_\alpha\wedge p_\beta)^\mu p_\alpha^\nu\\
&+ (\ell_\alpha\wedge\ell_\beta)_\alpha p_\alpha^\mu p_\beta^\nu - (k\wedge p_\beta)_\alpha p_\alpha^\mu\ell_\beta^\nu\Big\} + (p_\alpha\cdot p_\beta)(S_\alpha\wedge k)^\mu p_\beta^\nu + \frac{1}{2}(k\cdot p_\beta)(S_\alpha\wedge p_\beta)^\mu p_\alpha^\nu\\
&- \frac{1}{2}(k\wedge p_\beta)_\alpha p_\alpha^\mu p_\beta^\nu - \frac{m_\beta^2}{d-2}(S_\alpha\wedge k)^\mu p_\alpha^\nu + (\mu\leftrightarrow\nu)\Bigg].
\end{aligned}
\tag{321}
$$

Finally, the dilaton worldline graph Fig. 9(a) gives rise to the contribution in Eq. (153) and can be computed explicitly to be

$$
\begin{aligned}
(153) = -\frac{2i\kappa^3}{(d-2)^{1/2}}\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\int d\mu_{\alpha\beta}(k)\ell_\alpha^2\Bigg[&\frac{p_\alpha\cdot p_\beta}{(k\cdot p_\alpha)^2}p_\alpha^2(k\cdot\ell_\beta)\Big\{(\ell_\beta\wedge p_\alpha)_\beta - (\ell_\beta\wedge p_\beta)_\alpha\Big\}\\
&+ \frac{p_\alpha\cdot p_\beta}{k\cdot p_\alpha}p_\alpha^2(\ell_\alpha\wedge\ell_\beta)_\beta + \frac{k\cdot p_\beta}{k\cdot p_\alpha}p_\alpha^2\Big\{(\ell_\beta\wedge p_\beta)_\alpha - (\ell_\beta\wedge p_\alpha)_\beta\Big\} + 2(p_\alpha\cdot p_\beta)(\ell_\beta\wedge p_\alpha)_\beta\Bigg].
\end{aligned}
\tag{322}
$$

Total amplitudes in the gravitational theory

In this appendix, we display the total radiation amplitude in each channel as calculated in Section 0.7.2 and 0.7.3. First, the axion amplitude is given by

$$
\begin{aligned}
\mathcal{A}_a(k)\big|_{\mathcal{O}(\kappa^3,S^1)} = &-\frac{i\kappa^3 a_{\mu\nu}}{2}\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\int \mu_{\alpha\beta}(k)\Big[-\frac{(p_\alpha\cdot p_\beta)^2(k\cdot\ell_\beta)\ell_\alpha^2}{(k\cdot p_\alpha)^2}(S_\alpha\wedge k)^\mu p_\alpha^\nu\\
&+\frac{(p_\alpha\cdot p_\beta)\ell_\alpha^2}{k\cdot p_\alpha}\Big\{\Big((p_\alpha\cdot p_\beta)\ell_\beta^\nu-2(k\cdot p_\alpha)p_\beta^\nu+2(k\cdot p_\beta)p_\alpha^\nu\Big)(S_\alpha\wedge k)^\mu+(k\cdot\ell_\beta)(S_\alpha\wedge p_\beta)^\mu p_\beta^\nu\\
&-(k\cdot p_\beta)(S_\alpha\wedge\ell_\beta)^\mu p_\beta^\nu-(k\wedge\ell_\beta)_\alpha p_\alpha^\mu p_\beta^\nu+(k\wedge p_\beta)_\alpha p_\alpha^\mu\ell_\beta^\nu\Big\}\\
&+(p_\alpha\cdot p_\beta)\Big\{\ell_\alpha^2(S_\alpha\wedge\ell_\beta)^\mu p_\beta^\nu-\ell_\alpha^2(S_\alpha\wedge p_\beta)^\mu\ell_\beta^\nu+2(k\cdot\ell_\alpha)(S_\alpha\wedge\ell_\alpha)^\mu p_\beta^\nu\\
&-2(k\cdot p_\beta)(S_\alpha\wedge\ell_\alpha)^\mu\ell_\alpha^\nu-2(k\wedge\ell_\alpha)_\alpha l_\alpha^\mu p_\beta^\nu\Big\}+\ell_\alpha^2(k\cdot p_\alpha)(S_\alpha\wedge p_\beta)^\mu p_\beta^\nu\\
&-\ell_\alpha^2(k\cdot p_\beta)(S_\alpha\wedge p_\beta)^\mu p_\alpha^\nu-2(k\cdot p_\alpha)(k\cdot p_\beta)(S_\alpha\wedge\ell_\alpha)^\mu p_\beta^\nu+2(k\cdot p_\beta)^2(S_\alpha\wedge\ell_\alpha)^\mu p_\alpha^\nu\\
&-\ell_\alpha^2(k\wedge p_\beta)_\alpha p_\alpha^\mu p_\beta^\nu+2(k\cdot p_\beta)(k\wedge\ell_\alpha)_\alpha p_\alpha^\mu p_\beta^\nu+2(k\cdot p_\beta)(\ell_\alpha\wedge p_\beta)_\alpha\ell_\alpha^\mu p_\alpha^\nu\\
&-2(k\cdot p_\alpha)(\ell_\alpha\wedge p_\beta)_\alpha\ell_\alpha^\mu p_\beta^\nu+2(k\cdot\ell_\alpha)(\ell_\alpha\wedge p_\beta)_\alpha p_\alpha^\mu p_\beta^\nu-(\mu\leftrightarrow\nu)\Big].
\end{aligned}
\tag{323}
$$

The leading order graviton amplitude is given by

$$\mathcal{A}_g(k)\big|_{\mathcal{O}(\kappa^3,S^1)} = -i\kappa^3\epsilon_{\mu\nu}\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\int d\mu_{\alpha\beta}(k)\Bigg[\frac{(p_\alpha\cdot p_\beta)(k\cdot\ell_\beta)\ell_\alpha^2}{2(k\cdot p_\alpha)^2}\Big\{(p_\alpha\cdot p_\beta)(S_\alpha\wedge k)^\mu p_\alpha^\nu$$
$$+\Big((\ell_\beta\wedge p_\beta)_\alpha-(\ell_\beta\wedge p_\alpha)_\beta\Big)p_\alpha^\mu p_\alpha^\nu\Big\}-\frac{(p_\alpha\cdot p_\beta)\ell_\alpha^2}{k\cdot p_\alpha}\Big\{\frac{1}{2}(p_\alpha\cdot p_\beta)(S_\alpha\wedge k)^\mu\ell_\alpha^\nu$$
$$+\frac{1}{2}(k\cdot p_\beta)\Big((S_\alpha\wedge k)^\mu+(S_\alpha\wedge\ell_\alpha)^\mu\Big)p_\alpha^\nu+\frac{1}{2}(k\cdot\ell_\beta)(S_\alpha\wedge p_\beta)^\mu p_\alpha^\nu+\frac{1}{2}(\ell_\alpha\wedge\ell_\beta)_\beta p_\alpha^\mu p_\alpha^\nu$$
$$+\frac{1}{2}(\ell_\alpha\wedge\ell_\beta)_\alpha p_\alpha^\mu p_\beta^\nu+\Big((\ell_\beta\wedge p_\beta)_\alpha-(\ell_\beta\wedge p_\alpha)_\beta-\frac{1}{2}(k\wedge p_\beta)_\alpha\Big)p_\alpha^\mu\ell_\beta^\nu\Big\}$$
$$-\frac{(k\cdot p_\beta)\ell_\alpha^2}{2(k\cdot p_\alpha)}\Big\{(\ell_\beta\wedge p_\beta)_\alpha-(\ell_\beta\wedge p_\alpha)_\beta\Big\}p_\alpha^\mu p_\alpha^\nu+(p_\alpha\cdot p_\beta)\Big\{\frac{\ell_\alpha^2}{2}(S_\alpha\wedge\ell_\beta)^\mu p_\beta^\nu$$
$$-(\ell_\alpha\cdot\ell_\beta)(S_\alpha\wedge\ell_\alpha)^\mu p_\beta^\nu+(k\cdot p_\beta)(S_\alpha\wedge\ell_\alpha)^\mu\ell_\alpha^\nu+\frac{1}{2}\ell_\alpha^2(S_\alpha\wedge p_\beta)^\mu\ell_\beta^\nu$$
$$+(\ell_\alpha\wedge\ell_\beta)_\alpha p_\beta^\mu\ell_\alpha^\nu+(\ell_\alpha\wedge p_\beta)_\alpha\ell_\alpha^\mu\ell_\alpha^\nu-\frac{\ell_\alpha^2}{2}(\ell_\alpha\wedge p_\beta)_\alpha\eta^{\mu\nu}\Big\}$$
$$+\frac{1}{2}\Big\{\ell_\alpha^2(\ell_\beta\wedge p_\beta)_\alpha-\ell_\beta^2(\ell_\alpha\wedge p_\beta)_\alpha\Big\}p_\alpha^\mu p_\beta^\nu+\Big\{(k\cdot p_\beta)(S_\alpha\wedge\ell_\alpha)^\mu-\frac{\ell_\alpha^2}{2}(S_\alpha\wedge p_\beta)^\mu$$
$$+(\ell_\alpha\wedge p_\beta)_\alpha l_\alpha^\mu+(\ell_\alpha\wedge\ell_\beta)_\alpha p_\beta^\mu\Big\}\Big((k\cdot p_\alpha)p_\beta^\nu-(k\cdot p_\beta)p_\alpha^\nu\Big)+(\mu\leftrightarrow\nu)\Bigg], \tag{324}$$

and finally, we have the leading order dilaton amplitude

$$\mathcal{A}_d(k)\big|_{\mathcal{O}(\kappa^3,S^1)} = -\frac{i\kappa^3}{(d-2)^{1/2}}\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\int d\mu_{\alpha\beta}(k)p_\alpha^2\Bigg[\frac{p_\alpha\cdot p_\beta}{(k\cdot p_\alpha)^2}\ell_\alpha^2(k\cdot\ell_\beta)\Big\{(\ell_\beta\wedge p_\alpha)_\beta-(\ell_\beta\wedge p_\beta)_\alpha\Big\}$$
$$+\frac{k\cdot p_\beta}{k\cdot p_\alpha}\ell_\alpha^2\Big\{(\ell_\beta\wedge p_\beta)_\alpha-(\ell_\beta\wedge p_\alpha)_\beta\Big\}+\frac{p_\alpha\cdot p_\beta}{k\cdot p_\alpha}\ell_\alpha^2(\ell_\alpha\wedge\ell_\beta)_\beta+2(k\cdot p_\beta)(\ell_\alpha\wedge\ell_\beta)_\beta\Bigg]. \tag{325}$$

Classical closed strings

In this appendix, we give the explicit derivations of the low-energy limit of axion and graviton vertices for classical closed strings as discussed in Section 0.9.2.

.3 Axion coupling

The axion term in the background field action

$$S_B = g_c \int d\tau \int_0^\pi d\sigma \epsilon^{\alpha\beta} B_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \tag{326}$$

where we have denoted the closed string coupling as

$$g_c = -\frac{1}{2\pi l^2}. \tag{327}$$

The one point vertex corresponding to this part of the action is given by

$$J^{\mu\nu}(x) = i g_c \int d\tau \int_0^\pi d\sigma \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \delta^D(x - X). \tag{328}$$

The zeroth order is the solution to the free closed string mode expansion (sum of left and right movers)

$$X^\mu(\tau, \sigma) = x^\mu + l^2 p^\mu \tau + Z^\mu(\tau, \sigma), \tag{329}$$

where we have defined

$$Z^\mu(\tau,\sigma) = i\frac{l}{2}\sum_{n\neq 0}\frac{1}{n}e^{2in\sigma}(e^{-2in\tau}\alpha_n^\mu - e^{2in\tau}\tilde{\alpha}_{-n}^\mu), \tag{330}$$

$$Z'^\mu(\tau,\sigma) = -l\sum_{n\neq 0}e^{2in\sigma}(e^{-2in\tau}\alpha_n^\mu - e^{2in\tau}\tilde{\alpha}_{-n}^\mu), \tag{331}$$

$$\dot{Z}^\mu(\tau,\sigma) = l\sum_{n\neq 0}e^{2in\sigma}(e^{-2in\tau}\alpha_n^\mu + e^{2in\tau}\tilde{\alpha}_{-n}^\mu), \tag{332}$$

where prime and dot denotes the spatial and temporal derivatives with respect to the worldline coordinates. Using the same set of constraints for a rotating string at the origin, we have

$$\begin{aligned} J^{\mu\nu}(k) &= ig_c\int d^Dx e^{ik\cdot x}\int d\tau\int_0^\pi d\sigma(\dot{X}^\mu X'^\nu - X'^\mu\dot{X}^\nu)\delta^D(x-X) \\ &\simeq ig_c\int d\tau e^{ik\cdot pl^2\tau}\int_0^\pi d\sigma[(l^2p^\mu Z'^\nu - l^2Z'^\mu p^\nu) \\ &+ (ik\cdot Zl^2p^\mu + \dot{Z}^\mu)Z'^\nu - Z'^\mu(ik\cdot Zl^2p^\nu + \dot{Z}^\nu)]. \end{aligned} \tag{333}$$

Since the closed string has periodic boundary conditions, the linear terms $\oint d\sigma Z' = 0$ naturally. Hence, the lowest order consists of two powers of the modes

$$J^{\mu\nu}(k) = ig_c\int d\tau e^{ik\cdot pl^2\tau}\int_0^\pi d\sigma[(ik_\lambda Z^\lambda l^2p^\mu + \dot{Z}^\mu)Z'^\nu - (\mu\leftrightarrow\nu)]. \tag{334}$$

The first term is given by

$$\begin{aligned} &\int d\tau e^{ik\cdot pl^2\tau}ik_\lambda l^2p^\mu\int_0^\pi d\sigma Z^\lambda Z'^\nu \\ =&\int d\tau e^{ik\cdot pl^2\tau}\frac{l^4}{2}k_\lambda p^\mu\sum_{n,m\neq 0}\frac{1}{n}(e^{-2in\tau}\alpha_n^\lambda - e^{2in\tau}\tilde{\alpha}_{-n}^\lambda)(e^{-2im\tau}\alpha_m^\nu - e^{2im\tau}\tilde{\alpha}_{-m}^\nu)\int_0^\pi d\sigma e^{2i(n+m)\sigma} \\ =&\pi^2l^4k_\lambda p^\mu\sum_{n\neq 0}\frac{1}{n}[\delta(k\cdot pl^2)\alpha_n^\lambda\alpha_{-n}^\nu - \delta(k\cdot pl^2)\tilde{\alpha}_n^\lambda\tilde{\alpha}_{-n}^\nu + \delta(k\cdot pl^2-4n)\tilde{\alpha}_n^\lambda\alpha_n^\nu - \delta(k\cdot pl^2-4n)\alpha_n^\lambda\tilde{\alpha}_n^\nu] \\ \simeq&\pi^2l^4k_\lambda p^\mu\sum_{n\neq 0}\frac{1}{n}[\delta(k\cdot pl^2)\alpha_n^\lambda\alpha_{-n}^\nu - \delta(k\cdot pl^2)\tilde{\alpha}_n^\lambda\tilde{\alpha}_{-n}^\nu], \end{aligned} \tag{335}$$

since we are looking at $k \cdot p \ll 1/l^2$. For the second, we integrate by part where appropriate

$$\begin{aligned}
&\int d\tau e^{ik\cdot pl^2\tau} \int_0^\pi d\sigma \dot{Z}^\mu Z'^\nu \\
&= -\int d\tau \int_0^\pi d\sigma e^{ik\cdot pl^2\tau} Z^\mu \dot{Z}'^\nu - \int d\tau \int_0^\pi d\sigma e^{ik\cdot pl^2\tau} ik\cdot pl^2 Z^\mu Z'^\nu \\
&= \int d\tau \int_0^\pi d\sigma e^{ik\cdot pl^2\tau} Z'^\mu \dot{Z}^\nu \\
&\quad - \pi^2 k\cdot pl^4 \sum_{n\neq 0} \frac{1}{n}[\delta(k\cdot pl^2)\alpha_n^\mu \alpha_{-n}^\nu - \delta(k\cdot pl^2)\tilde{\alpha}_n^\mu \tilde{\alpha}_{-n}^\nu \\
&\quad + \delta(k\cdot pl^2 - 4n)\tilde{\alpha}_n^\mu \alpha_n^\nu - \delta(k\cdot pl^2 - 4n)\tilde{\alpha}_n^\mu \tilde{\alpha}_n^\nu]. \qquad (336)
\end{aligned}$$

The first two terms are constrained by the δ-function to vanish; thus, the remaining gives

$$\int d\tau \int_0^\pi d\sigma e^{ik\cdot pl^2\tau} \dot{Z}^\mu Z'^\nu - (\mu \leftrightarrow \nu) \simeq 0. \qquad (337)$$

Substituting the string coupling constant and the mode expansions, the one point vertex is then given by

$$J^{\mu\nu}(k) = ig_c \pi^2 l^4 \delta(k\cdot pl^2) k_\lambda \sum_{n\neq 0} \frac{1}{n}[p^\mu(\alpha_{-n}^\nu \alpha_n^\lambda - \tilde{\alpha}_{-n}^\nu \tilde{\alpha}_n^\lambda) + p^\nu(\alpha_{-n}^\lambda \alpha_n^\mu - \tilde{\alpha}_{-n}^\lambda \tilde{\alpha}_n^\mu)]. \qquad (338)$$

.4 Graviton coupling

In conformal gauge, the graviton coupling term looks like

$$S_g = -\frac{1}{2\pi l^2} \int d\tau \oint d\sigma h_{\mu\nu}(X) \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \qquad (339)$$

for a closed string. The one point vertex corresponding to this part of the action is given by

$$V^{\mu\nu}(x) = g_c \int d\tau \oint d\sigma \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \delta^D(x - X). \qquad (340)$$

The zeroth order is the solution to the free closed string mode expansion (sum of left and right movers)

$$X^\mu(\tau,\sigma) = x^\mu + l^2 p^\mu \tau + i\frac{l}{2}\sum_{n\neq 0}\frac{1}{n}e^{2in\sigma}(e^{-2in\tau}\alpha_n^\mu - e^{2in\tau}\tilde{\alpha}_{-n}^\mu). \tag{341}$$

Using the same set of constraints for a rotating string at the origin, we have

$$\begin{aligned} V^{\mu\nu}(k) &= g_c \int d^D x e^{ik\cdot x}\int d\tau \oint d\sigma \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \delta^D(x-X) \\ &\simeq g_c \int d\tau e^{ik\cdot p l^2\tau}\oint d\sigma [l^4 p^\mu p^\nu + i l^4 p^\mu p^\nu k\cdot Z + l^2 p^\mu \dot{Z}^\nu + l^2 \dot{Z}^\mu p^\nu \\ &- l^4 p^\mu p^\nu (k\cdot Z)^2 + i l^2 p^\mu k\cdot Z\dot{Z}^\nu + i l^2 p^\nu k\cdot Z\dot{Z}^\mu + \dot{Z}^\mu\dot{Z}^\nu - Z'^\mu Z'^\nu]. \end{aligned} \tag{342}$$

The lowest order gives the point particle contribution. The terms with one power of Z vanish upon closed loop integration of σ. The double time derivative gives the same contribtuion (up to high-frequency terms) as the double σ derivative and hence cancels each other. Explicitly, we have

$$\begin{aligned} \oint d\sigma \dot{Z}^\mu \dot{Z}^\nu &= l^2 \sum_{n\neq 0}(\alpha_n^\mu \alpha_{-n}^\nu + e^{2in\tau}\tilde{\alpha}_{-n}^\mu e^{2in\tau}\alpha_{-n}^\nu \\ &+ e^{-2in\tau}\alpha_n^\mu e^{-2in\tau}\tilde{\alpha}_n^\nu + e^{2in\tau}\tilde{\alpha}_{-n}^\mu e^{-2in\tau}\tilde{\alpha}_n^\nu). \end{aligned} \tag{343}$$

The term with two powers of k is suppressed and hence we have

$$\begin{aligned} V^{\mu\nu}(k) &\simeq g_c \int d\tau e^{ik\cdot p l^2\tau}\oint d\sigma [i l^2 p^\mu k\cdot Z\dot{Z}^\nu + i l^2 p^\nu k\cdot Z\dot{Z}^\mu] \\ &= -\frac{g_c l^4 \pi}{2}\int d\tau e^{ik\cdot p l^2\tau}[p^\mu k_\lambda \sum_{n,m\neq 0}\frac{1}{n}(\alpha_n^\lambda \alpha_{-n}^\nu + \tilde{\alpha}_n^\lambda \tilde{\alpha}_{-n}^\nu - e^{4in\tau}\tilde{\alpha}_{-n}^\lambda \alpha_{-n}^\nu + e^{-4in\tau}\alpha_n^\lambda \tilde{\alpha}_n^\nu) + (\mu\leftrightarrow\nu)] \\ &\simeq i\frac{g_c l^4}{2}2\pi^2\delta(k\cdot p l^2)[p^\mu k_\lambda (E^{\lambda\nu} + \tilde{E}^{\lambda\nu}) + (\mu\leftrightarrow\nu)]. \end{aligned} \tag{344}$$

The corresponding energy-momentum pseudo-tensor is negative two times this

$$\tilde{T}^{\mu\nu}(k) = 2\pi i\delta(k\cdot p)[k_\lambda (S_L + S_R)^{\lambda(\nu}p^{\mu)}]. \tag{345}$$

Radiation from higher-dimensional operators

In this appendix, we summarize the results for the gravitational radiation amplitudes due to finite-size operators as calculated in Section 0.10

.5 Spin-independent contributions

First we consider radiation induced by graviton or dilaton exchange only. Then, as depicted in Fig. 13(a), the quadratic graviton operators Eq. (224) lead to graviton radiation amplitudes

$$\mathcal{A}^{\mu\nu}_{G_1} = 2\kappa^3 \sum_{\alpha,\beta} \int_{\ell_{\alpha,\beta}} \mu_{\alpha,\beta}(k) \ell_\alpha^2 \Big[(k\cdot p_\beta)\ell_\beta^\mu - (k\cdot \ell_\beta)p_\beta^\mu\Big]\Big[(k\cdot p_\beta)\ell_\beta^\nu - (k\cdot \ell_\beta)p_\beta^\nu\Big], \tag{346}$$

$$\begin{aligned}\mathcal{A}^{\mu\nu}_{G_2} =& \kappa^3 \sum_{\alpha,\beta} \int_{\ell_{\alpha,\beta}} \mu_{\alpha,\beta}(k) \ell_\alpha^2 \Bigg\{ \Big[(p_\alpha\cdot p_\beta)[(k\cdot p_\alpha)\ell_\beta^\mu - (k\cdot \ell_\beta)p_\alpha^\mu] \\ &+ (k\cdot p_\alpha)[(k\cdot p_\beta)p_\alpha^\mu - (k\cdot p_\alpha)p_\beta^\mu]\Big]\Big[(k\cdot p_\beta)\ell_\beta^\nu - (k\cdot \ell_\beta)p_\beta^\nu\Big] \\ &- \frac{1}{d-2}\Big[(k\cdot p_\alpha)\ell_\beta^\mu - (k\cdot \ell_\beta)p_\alpha^\mu\Big]\Big[(k\cdot p_\alpha)\ell_\beta^\nu - (k\cdot \ell_\beta)p_\alpha^\nu\Big]\Bigg\},\end{aligned} \tag{347}$$

$$\begin{aligned}\mathcal{A}^{\mu\nu}_{G_3} =& \frac{\kappa^3}{2} \sum_{\alpha,\beta} \int_{\ell_{\alpha,\beta}} \mu_{\alpha,\beta}(k) \ell_\alpha^2 \Bigg\{ \Big[(p_\alpha\cdot p_\beta)[(k\cdot p_\alpha)\ell_\beta^\mu - (k\cdot \ell_\beta)p_\alpha^\mu] + (k\cdot p_\alpha)[(k\cdot p_\beta)p_\alpha^\mu \\ &- (k\cdot p_\alpha)p_\beta^\mu]\Big]\Big[(p_\alpha\cdot p_\beta)[(k\cdot p_\alpha)\ell_\beta^\nu - (k\cdot \ell_\beta)p_\alpha^\nu] + (k\cdot p_\alpha)[(k\cdot p_\beta)p_\alpha^\nu - (k\cdot p_\alpha)p_\beta^\nu]\Big] \\ &- \frac{1}{d-2}\Big[(k\cdot p_\alpha)\ell_\beta^\mu - (k\cdot \ell_\beta)p_\alpha^\mu\Big]\Big[(k\cdot p_\alpha)\ell_\beta^\nu - (k\cdot \ell_\beta)p_\alpha^\nu\Big]\Bigg\}.\end{aligned} \tag{348}$$

In addition, the graviton-dilaton operator Eq. (227) gives rise to the diagram Fig. 13(b) that corresponds to graviton radiation of the form

$$\mathcal{A}^{\mu\nu}_{G_D} = -\frac{\kappa^3}{d-2}\sum_{\alpha,\beta}\int_{\ell_{\alpha,\beta}} \mu_{\alpha,\beta}(k)\ell_\alpha^2\Big[(k\cdot p_\alpha)\ell_\beta^\mu - (k\cdot\ell_\beta)p_\alpha^\mu\Big]\Big[(k\cdot p_\alpha)\ell_\beta^\nu - (k\cdot\ell_\beta)p_\alpha^\nu\Big]. \quad (349)$$

We observe that the combinations $\mathcal{A}_{G_2} - \mathcal{A}_{G_D}$ and $\mathcal{A}_{G_3} - \frac{1}{2}\mathcal{A}_{G_D}$ yield results independent of d.

In the dilaton channel, the quadratic dilaton operators Eq. (225) allow diagrams of the form shown in Fig. 13(c), resulting in radiation amplitudes

$$\mathcal{A}_{D_1} = \frac{\kappa^3}{(d-2)^{3/2}}\sum_{\alpha,\beta}\int_{\ell_{\alpha,\beta}} \mu_{\alpha,\beta}(k)\ell_\alpha^2(k\cdot\ell_\beta)^2, \quad (350)$$

$$\mathcal{A}_{D_2} = \frac{\kappa^3}{(d-2)^{3/2}}\sum_{\alpha,\beta}\int_{\ell_{\alpha,\beta}} \mu_{\alpha,\beta}(k)\ell_\alpha^2(k\cdot\ell_\beta)(k\cdot p_\alpha)^2, \quad (351)$$

$$\mathcal{A}_{D_3} = \frac{\kappa^3}{(d-2)^{3/2}}\sum_{\alpha,\beta}\int_{\ell_{\alpha,\beta}} \mu_{\alpha,\beta}(k)\ell_\alpha^2(k\cdot p_\alpha)^4. \quad (352)$$

In addition, Fig. 13(d) with an insertion of the graviton-dilaton operator Eq. (227) also leads to radiation in the scalar channel,

$$\mathcal{A}_{D_G} = \frac{\kappa^3}{(d-2)^{1/2}}\sum_{\alpha,\beta}\int_{\ell_{\alpha,\beta}} \mu_{\alpha,\beta}(k)\ell_\alpha^2\Big[\big((p_\alpha\cdot p_\beta)(k\cdot\ell_\beta) - (k\cdot p_\alpha)(k\cdot p_\beta)\big)^2 - \frac{(k\cdot\ell_\beta)^2}{d-2} + \frac{2(k\cdot\ell_\beta)(k\cdot p_\alpha)}{d-2}\Big]. \quad (353)$$

Finally, diagram Fig. 13(e) with a single insertion of the graviton-axion operator Eq. (228) yields the axion radiation amplitude

$$\mathcal{A}^{\mu\nu}_{A_G} = 2\kappa^3\int_{\ell_{\alpha,\beta}} \mu_{\alpha,\beta}(k)\ell_\alpha^2\Big[(k\cdot p_\beta)\ell_\beta^\mu - (k\cdot\ell_\beta)p_\beta^\mu\Big] \times\Big[(p_\alpha\cdot p_\beta)[(k\cdot p_\alpha)\ell_\beta^\nu - (k\cdot\ell_\beta)p_\alpha^\nu] + (k\cdot p_\alpha)[(k\cdot p_\beta)p_\alpha^\nu - (k\cdot p_\alpha)p_\beta^\nu]\Big]. \quad (354)$$

We note that in matching the double copy to the dilaton channel, we have omitted contact terms with no propagator factors. Such terms yield integrals that are proportional

to to

$$\int \frac{d^d\ell_\alpha}{(2\pi)^d}\frac{d^d\ell_\beta}{(2\pi)^d}\Big[(2\pi)\delta(\ell_\alpha\cdot p_\alpha)e^{i\ell_\alpha\cdot b_\alpha}\Big]\Big[(2\pi)\delta(\ell_\alpha\cdot p_\alpha)e^{i\ell_\beta\cdot b_\beta}\Big](2\pi)^d\delta^d(\ell_\alpha+\ell_\beta-k)$$
$$=\int d\tau_\alpha d\tau_\beta e^{ik\cdot x_\beta^{(0)}}\int\frac{d^d\ell}{(2\pi)^d}e^{i\ell\cdot x_\alpha^{(0)}}e^{-i\ell\cdot x_\beta^{(0)}}=\int d\tau_\alpha d\tau_\beta e^{ik\cdot x_\beta^{(0)}}\delta^d(x_\alpha^{(0)}-x_\beta^{(0)}), \tag{355}$$

where the free particle paths are $x_\alpha^{(0)}=b_\alpha+v_\alpha\tau_\alpha$. Because we consider classical scattering at non-zero impact parameter $b_{\alpha\beta}=b_\alpha-b_\beta\neq 0$, such terms are identically zero.

.6 Spin-dependent contributionss

The spinning point sources now support internal graviton and axion exchange, corresponding to the diagrams in Fig. 14(a). These yield graviton emission amplitudes

$$\mathcal{A}^{\mu\nu}_{G_1}=2i\kappa^3\sum_{\alpha,\beta}\int_{\ell_{\alpha,\beta}}\mu_{\alpha,\beta}(k)\ell_\alpha^2\Big[(k\cdot p_\beta)\ell_\beta^\nu-(k\cdot\ell_\beta)p_\beta^\nu\Big]\Big[(k\cdot\ell_\beta)(S_\beta\wedge\ell_\beta)^\mu-(\ell_\alpha\wedge\ell_\beta)_\beta\ell_\beta^\mu\Big], \tag{356}$$

$$\begin{aligned}\mathcal{A}^{\mu\nu}_{G_2}=&\frac{i\kappa^3}{2}\sum_{\alpha,\beta}\int_{\ell_{\alpha,\beta}}\mu_{\alpha,\beta}(k)\ell_\alpha^2\Bigg\{\Big[(k\cdot p_\beta)\ell_\beta^\nu-(k\cdot\ell_\beta)p_\beta^\nu\Big]\\&\times\Big[(\ell_\beta\wedge p_\alpha)_\beta[(k\cdot p_\alpha)\ell_\beta^\mu-(k\cdot\ell_\beta)p_\alpha^\mu]+(k\cdot p_\alpha)[(k\cdot p_\alpha)(S_\beta\wedge\ell_\beta)^\mu-(\ell_\alpha\wedge\ell_\beta)_\beta p_\alpha^\mu]\Big]\\&+\Big[(p_\alpha\cdot p_\beta)[(k\cdot p_\alpha)\ell_\beta^\nu-(k\cdot\ell_\beta)p_\alpha^\nu]+(k\cdot p_\alpha)[(k\cdot p_\beta)p_\alpha^\nu-(k\cdot p_\alpha)p_\beta^\nu]\Big]\\&\times\Big[(k\cdot\ell_\beta)(S_\beta\wedge\ell_\beta)^\mu-(\ell_\alpha\wedge\ell_\beta)_\beta\ell_\beta^\mu\Big]\Bigg\},\end{aligned} \tag{357}$$

$$\begin{aligned}\mathcal{A}^{\mu\nu}_{G_3}=&\frac{i\kappa^3}{2}\sum_{\alpha,\beta}\int_{\ell_{\alpha,\beta}}\mu_{\alpha,\beta}(k)\ell_\alpha^2\Big[(p_\alpha\cdot p_\beta)[(k\cdot p_\alpha)\ell_\beta^\nu-(k\cdot\ell_\beta)p_\alpha^\nu]\\&+(k\cdot p_\alpha)[(k\cdot p_\beta)p_\alpha^\nu-(k\cdot p_\alpha)p_\beta^\nu]\Big]\Big[(\ell_\beta\wedge p_\alpha)_\beta[(k\cdot p_\alpha)\ell_\beta^\mu-(k\cdot\ell_\beta)p_\alpha^\mu]\\&+(k\cdot p_\alpha)[(k\cdot p_\alpha)(S_\beta\wedge\ell_\beta)^\mu-(\ell_\alpha\wedge\ell_\beta)_\beta p_\alpha^\mu]\Big].\end{aligned} \tag{358}$$

At linear order in spin, we can also have graviton radiation mediated by axion exhange, as in Fig. 14(b)

$$\begin{aligned}\mathcal{A}^{\mu\nu}_{G_A} =& i\kappa^3 \sum_{\alpha,\beta} \int_{\ell_{\alpha,\beta}} \mu_{\alpha,\beta}(k) \ell_\alpha^2 \Big\{ - \Big[(k\cdot p_\beta)\ell_\beta^\nu - (k\cdot \ell_\beta) p_\beta^\nu \Big] \\ & \times \Big[(\ell_\beta \wedge p_\alpha)_\beta [(k\cdot p_\alpha)\ell_\beta^\mu - (k\cdot \ell_\beta) p_\alpha^\mu] + (k\cdot p_\alpha)[(k\cdot p_\alpha)(S_\beta \wedge \ell_\beta)^\mu - (\ell_\alpha \wedge \ell_\beta)_\beta p_\alpha^\mu] \Big] \\ & + \Big[(p_\alpha \cdot p_\beta)[(k\cdot p_\alpha)\ell_\beta^\nu - (k\cdot \ell_\beta) p_\alpha^\nu] + (k\cdot p_\alpha)[(k\cdot p_\beta)p_\alpha^\nu - (k\cdot p_\alpha) p_\beta^\nu] \Big] \\ & \times \Big[(k\cdot \ell_\beta)(S_\beta \wedge \ell_\beta)^\mu - (\ell_\alpha \wedge \ell_\beta)_\beta \ell_\beta^\mu \Big] \Big\}. \end{aligned} \tag{359}$$

There is also dilaton radiation, from the diagram in Fig. 14(c) with one insertion of the graviton-dilaton mixing operator in Eq. (227)

$$\begin{aligned}\mathcal{A}_{D_G} =& \frac{i\kappa^3}{(d-2)^{1/2}} \sum_{\alpha,\beta} \int_{\ell_{\alpha,\beta}} \mu_{\alpha,\beta}(k) \ell_\alpha^2 \Big[(k\cdot p_\beta)(k\cdot p_\alpha) - (k\cdot \ell_\beta)(p_\alpha \cdot p_\beta) \Big] \\ & \times \Big[(\ell_\beta \wedge p_\alpha)_\beta (k\cdot \ell_\beta) + (\ell_\alpha \wedge \ell_\beta)_\beta (k\cdot p_\alpha) \Big]. \end{aligned} \tag{360}$$

Finally, there is spin-dependent axion radiation, involving insertions of the purely axionic operators in Eq. (226) in the diagram of Fig. 14(e). Two of these amplitudes readily factorize into products of the kinematic factors appearing in gauge theory

$$\mathcal{A}^{\mu\nu}_{A_1} = 4i\kappa^3 \sum_{\alpha,\beta} \int_{\ell_{\alpha,\beta}} \mu_{\alpha,\beta}(k) \ell_\alpha^2 \Big[(k\cdot p_\beta)\ell_\beta^\nu - (k\cdot \ell_\beta) p_\beta^\nu \Big] \Big[(k\cdot \ell_\beta)(S_\beta \wedge \ell_\beta)^\mu - (\ell_\alpha \wedge \ell_\beta)_\beta \ell_\beta^\mu \Big], \tag{361}$$

$$\begin{aligned}\mathcal{A}^{\mu\nu}_{A_4} =& 2i\kappa^3 \sum_{\alpha,\beta} \int_{\ell_{\alpha,\beta}} \mu_{\alpha,\beta}(k) \ell_\alpha^2 \Big[(p_\alpha \cdot p_\beta)[(k\cdot p_\alpha)\ell_\beta^\nu - (k\cdot \ell_\beta) p_\alpha^\nu] \\ & + (k\cdot p_\alpha)[(k\cdot p_\beta)p_\alpha^\nu - (k\cdot p_\alpha) p_\beta^\nu] \Big] \Big[(\ell_\beta \wedge p_\alpha)_\beta [(k\cdot p_\alpha)\ell_\beta^\mu - (k\cdot \ell_\beta) p_\alpha^\mu] \\ & + (k\cdot p_\alpha)[(k\cdot p_\alpha)(S_\beta \wedge \ell_\beta)^\mu - (\ell_\alpha \wedge \ell_\beta)_\beta p_\alpha^\mu] \Big]. \end{aligned} \tag{362}$$

The remaining two, $\mathcal{A}_{A_2}$ and $\mathcal{A}_{A_3}$, do not factorize into Yang-Mills kinematic factors. However, if we also include $\mathcal{A}_{A_G}$ (Fig. 14(d)) obtained from inserting the graviton-axion operator

Eq. (228), we find that the linear combinations

$$\mathcal{A}^{\mu\nu}_{A_G} + \mathcal{A}^{\mu\nu}_{A_2} + \frac{1}{2}\mathcal{A}^{\mu\nu}_{A_3} = 2i\kappa^3 \sum_{\alpha,\beta} \int_{\ell_{\alpha,\beta}} \mu_{\alpha,\beta}(k) \ell_\alpha^2 \Big[(k\cdot\ell_\beta)(S_\beta \wedge \ell_\beta)^\mu - (\ell_\alpha \wedge \ell_\beta)_\beta \ell_\beta^\mu \Big]$$
$$\times \Big[(p_\alpha \cdot p_\beta)[(k\cdot p_\alpha)\ell_\beta^\nu - (k\cdot\ell_\beta)p_\alpha^\nu] + (k\cdot p_\alpha)[(k\cdot p_\beta)p_\alpha^\nu - (k\cdot p_\alpha)p_\beta^\nu] \Big], \quad (363)$$

$$\mathcal{A}^{\mu\nu}_{A_G} - \mathcal{A}^{\mu\nu}_{A_2} - \frac{1}{2}\mathcal{A}^{\mu\nu}_{A_3} = -2i\kappa^3 \sum_{\alpha,\beta} \int_{\ell_{\alpha,\beta}} \mu_{\alpha,\beta}(k) \ell_\alpha^2 \Big[(k\cdot p_\beta)\ell_\beta^\nu - (k\cdot\ell_\beta)p_\beta^\nu \Big]$$
$$\times \Big[(\ell_\beta \wedge p_\alpha)_\beta [(k\cdot p_\alpha)\ell_\beta^\mu - (k\cdot\ell_\beta)p_\alpha^\mu] + (k\cdot p_\alpha)[(k\cdot p_\alpha)(S_\beta \wedge \ell_\beta)^\mu - (\ell_\alpha \wedge \ell_\beta)_\beta p_\alpha^\mu] \Big], \quad (364)$$

indeed factorize.

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