

A critical analysis of how the potential of Dynamic
Geometry Software as a visualisation tool may
enhance the teaching of Mathematics.

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ABSTRACT

Visualisation in the mathematics classroom has its own pedagogical value and plays a significant role in developing mathematical intuition, thought and ideas. Dynamic visualisation possibilities of current digital technologies afford new ways of teaching and learning mathematics. The freely available *GeoGebra* software package is highly interactive and makes use of powerful features to create objects that are dynamic, and which can be moved around on the computer screen for mathematical exploration. This research study was conceptualised within the GeoGebra Literacy Initiative Project (GLIP) – an ICT teacher development project in Mthatha in the Eastern Cape, South Africa. The focus of this study was on how *GeoGebra* could be used as a teaching tool by harnessing its powerful visualisation capacity. In the study, selected GLIP teachers collaboratively developed *GeoGebra* applets, then implemented and evaluated them. The research methodology took the form of action research cycles in which the design, implementation and evaluation of successive applets determined the data gathering and analysis process. My data consisted mainly of recorded observations and reflective interviews. The underlying theoretical foundation of this study lies in constructivism, which aligned well with the conceptual and analytical framework of Kilpatrick et al.'s (2001) description of teaching proficiency.

An in-depth analysis of my classroom observations resulted in multiple narratives that illuminated how teachers harnessed the visualisation capabilities inherent in the software. My findings showed that dynamic visualisation and interactivity afforded by the use of technology are key enabling factors for teachers to enhance the visualisation of mathematical concepts. My analysis across participants also showed that technical difficulties often compromised the use of technology in the teaching of mathematics. The significance of this research is its contribution to the ongoing deliberations of visualisation and utilisation of technological resources, particularly through the empowerment of a community of teachers. The findings recognised that the integration of technology required appropriate training, proper planning and continuous support and resources for the teaching of mathematics. This action research provided insightful information on integrating Dynamic Geometry Software (DGS) tools in mathematics classrooms that could be useful to teachers and curriculum planners.

Keywords: Dynamic Geometry Software, Visualisation, Mathematical Teaching Proficiency, Constructivism.

*Asato Maa Sad-Gamaya
Tamaso Maa Jyotir-Gamaya*
(Sanskrit verse from an ancient sacred text)

Meaning:

*O Lord! Keep me not in the World of Unreality but lead me towards the Reality of Self,
Keep me not in the Ignorant State of Darkness but lead me towards the Light of Knowledge.*

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DECLARATION OF ORIGINALITY

I, Beena Deepak Mavani (Student number 615D0004), declare that this doctoral thesis entitled: “A critical analysis of how the potential of Dynamic Geometry Software as a visualisation tool may enhance the teaching of Mathematics”, is my own work written in my own words. It has not been submitted in any form for another qualification or any assessment to another university or institution. Where I have drawn on the words or ideas of others, these have been fully acknowledged and referenced in the manner required by the Rhodes University Department of Education Guide to referencing.

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List of Abbreviations

CAPS	Curriculum and Assessment Policy Statement
CU	Conceptual Understanding of Core Knowledge
DBE	Department of Basic Education, South Africa
DGS	Dynamic Geometric Software
GLIP	GeoGebra Literacy Initiative Programme
ICT	Information and Communications Technology
IF	Instructional Fluency
MP	Mathematical Proficiency
NCTM	National Council of Teachers of Mathematics, USA
VL	Visual Literacy

CHAPTER ONE

INTRODUCTION

1.1 INTRODUCTION TO THE RESEARCH

The ever-growing use of technology in all realms of life has also affected teaching styles around the globe. Research shows that ICT¹ can have a positive influence on the teaching of mathematics, and has the potential for conceptually enabling many children to see and access a variety of mathematical ideas (Jones, 2000; Keong, Horani & Daniel, 2005).

Alsina and Nelsen (2006, p. 121) argue that the interactive possibilities of technology may foster rich constructions of mathematical concepts and develop visual thinking. Dynamic Geometric Software (DGS) is a technological tool that provides an opportunity for discovery and exploration, in particular opening up new possibilities for visual experiences in mathematics teaching (and learning). Alsina and Nelsen (2006) also suggest that visualisation in the classroom has its own pedagogical values. In the heuristic of mathematical discovery, internal visualisation (such as imagination and drawing pictures in the mind) and external visualisation (such as making use of sketches and drawings) plays a major role in developing intuition and problem-solving skills.

The purpose of this research study is to:

- analyse how teachers make use of technology-aided visualisation for effective teaching;
- analyse the advantages and weaknesses of using DGS in the teaching and learning of mathematics;
- interpret, through elements of an action research approach, the pedagogical practices and instructional fluencies of selected teachers when they employ visual features of technological tools in a constructivist classroom;

¹. I use the acronym ICT (Information and Communication Technology) referring to both hardware and software of computer technology.

I use the word “technology” and computer technology synonymously with the same meaning.

- contribute to the growth of a community of proactive teachers in Mthatha collaborating in building ICT resources (applets), aligned with the South African curriculum content, within the context of a teacher intervention programme named GLIP.

This research study analyses how DGS can enhance teaching by exploiting the power of visualisation in a constructivist classroom. The research study also examines critical aspects of this software that limits the usage of visualisation in the mathematics classroom.

1.2 THE CONTEXT OF THE RESEARCH STUDY

The National Council of Teachers of Mathematics (NCTM) (2000) recognises that “[t]echnology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students’ learning” (p. 24). Digital technologies afford new ways of teaching and learning mathematics. Besides the computational ability of ICT tools enabling teachers and learners to easily execute routine procedures, it also allows teachers to foster a learning environment for rich conceptual understanding in mathematics.

The South African Department of Basic Education (DBE) Curriculum and Assessment Policy Statement [CAPS] (2011) identifies mathematics as a priority subject, as “it helps to develop mental processes that enhance logical and critical thinking” (p. 8). Mathematical problem solving also “enables us to understand the world around us”. The DBE (2011) has implemented ICT based teaching and learning as part of the National Strategy of MST (Mathematics, Science and Technology). The Ministerial Committee (2013, p. 11) however reports that the Eastern Cape Province faces a serious shortage of qualified and proficient teachers who should be implementing this strategy, particularly in the mathematics classroom. Further, the Ministerial Committee (2013) suggests that ICT resources in schools are not adequate, and where available, they are not used to their full potential. It is thus of particular significance that this study speaks to this policy. This research study, with its teacher intervention programme, will hopefully provide significant input into how teachers can harness interesting ICT resources for proficient teaching in mathematics.

1.2.1 GLIP – GeoGebra Literacy Initiative Project

GLIP is a teacher development project explicitly designed for interested teachers to use *GeoGebra*, a dynamic geometry software, as a teaching and learning tool for mathematics. *GeoGebra* is an open-source program that is freely available to anyone for the teaching and learning of

mathematics. GLIP trains teachers to not only use *GeoGebra*, but also to harness its interactive capacity to develop tailor-made applets for their teaching. Applets are small programs written in Java that can be easily embedded in web pages. Thus, with internet connectivity, teachers and learners can access these applets at any time and in any place.

My fellow researcher and I drive the GLIP with the support of the NRF SARChI Chair in Mathematics Education at Rhodes University. We were also involved in training the GLIP teachers in *GeoGebra* skills. The project was launched in November 2015 with the participation of twelve mathematics teachers from one school in Mthatha, Eastern Cape.

There were two phases in the GLIP initiative. Phase 1 involved basic *GeoGebra* training of teachers and their learners while Phase 2 focussed particularly on the use of the *GeoGebra* in classrooms. The first phase of GLIP consisted of introductory training in *GeoGebra*. This training helped teachers to familiarise themselves with the software. During this phase, we encouraged the teachers to explore and play with different options available in *GeoGebra* to realise and appreciate its potential in mathematics classrooms. In Phase 1, teachers were introduced to existing applets and were encouraged to use them. Teachers also reflected on how they could use these applets in their classrooms. After this, in Phase 2, we trained teachers how to develop applets. The applets were then piloted and implemented in the classroom for learner engagement and exploration. We also trained learners in this phase before the teachers started using them in classrooms.

The training materials for the first phase of this workshop were adapted from the website www.geogebra.org, downloaded on the 26th August 2015. We obtained the consent of Markus Hohenwarter (the founder of *GeoGebra*) to adapt and use these for our training purposes.

The second phase consisted of the design and implementation of applets in classrooms, involving several cycles. In this phase, the participating teachers started developing and using *GeoGebra* applets in their classrooms. The development of these applets determined the sequence of the GLIP cycles. A GLIP cycle commenced with the selection of a topic or sub-topic, determined by the annual teaching plan. The next step of the cycle was the planning of pedagogical strategies to be considered while designing the applet. The teachers brainstormed on how to teach using the *GeoGebra* technology. One or two teachers developed the applet and drove a discussion about the applet. The applet would then be modified whenever required. Once the applet was developed, it was used by all the participant teachers in their classrooms. After teaching with applets, the

teachers would again come together to reflect on and evaluate their lessons. This reflection informed the development of the next cycle of applets. The flow of development and deployment of applets is discussed further under data collection, Section 3.4.1 (p. 60).

This research process for this thesis ran parallel to the second phase of GLIP. GLIP was thus my empirical field. After the training in Phase 1, I assumed the role of a participant in the GLIP.

My specific research focus was on the teachers implementing these applets, while my fellow researcher focussed on the learners as they engaged with these applets. Both studies foregrounded the harnessing of the visualisation opportunities of these applets, firstly, in the context of learning (my fellow researcher's study) and, secondly, in the context of teaching (the focus of my study).

The entire GLIP project plan is available on the Rhodes University website <https://www.ru.ac.za/mathsedchair/communitydevelopment/>.

1.3 CONCEPTUAL LANDSCAPE – VISUALISATION

This study focusses on the role of visualisation in teaching and learning mathematics through ICT, in particular on the use of *GeoGebra*. Visualisation is a critical aspect of any mathematical activity (Arcavi, 2003; Duval, 1999; Guzmán, 2002). Many mathematical ideas, such as number operations and pattern generalisation, are born from concrete and visualisable situations. Presmeg (1986a) recommends that “when a topic is first taught, a visual representation often aids understanding”(p. 302) . The popular “input-output diagram” that many teachers use in teaching functions is an example of how visualisation can contribute to the illumination and understanding of a mathematical concept, in this case, a function. Arcavi (2003) emphasises that the visual representation of information in a graph, for example, enables us to comprehend a scenario or relationship between two processes without reading sequentially and logically printed words.

Nevertheless, Presmeg (2014, p. 153) warns that “visualization is not self-explanatory and is very important for teachers.” Guzmán (2002) concurs and cautions that mathematical images need to be interpreted and understood. The teacher plays a vital role in mediating mathematical meaning to its visual representation. Teachers should guide their learners to the underlying mathematical meanings contained in a given representation. The proper use of visualisation needs training in order to illustrate and mediate whole mathematical structures.

Some learners may attach irrelevant details to the visual images and do not necessarily see what teachers want them to see (Arcavi, 2003; Presmeg, 1986b). They face the difficulty of identifying the abstract relationships where vague images obscure mathematical meanings and they remain tied to the irrelevant details in the concrete imagery. My study thus emphasises the importance of the pedagogical role of instructions in the learning of mathematical concepts in a visual environment. I argue that DGS can play an important role in overcoming such difficulties. Arcavi (2003) asserts that technology may assist in developing visualisation skills so that learners can better ‘see’ mathematical concepts and ideas, helping to overcome the limitation of visual perception.

I discuss at length a working definition of visualisation in the literature review chapter of this thesis. I argue in the next section that ICT can be used effectively to enhance the appreciation of mathematical ideas and concepts through its inherent visualisation capacity.

1.4 TECHNOLOGY – DYNAMIC GEOMETRIC SOFTWARE

Incorporating educational technology such as ‘dynamic geometry software’ (DGS) in mathematics education is gaining traction among researchers and curriculum planners. DGS was developed with educational purposes in mind (Ruthven, Hennessy & Deaney, 2008), to support the teaching and learning of mathematics (Hoyles & Noss, 2003). DGS is particularly suited to make use of other facilities such as dragging and sliding points and lines on the computer screen. These features in DGS allow a continuous reconstruction and transformation of figures on the computer screen by direct manipulation with the mouse (or by touch on a touchpad). One of the most powerful and widely recognized educational components of DGS is visualisation.

There are many DGS packages available for teaching and learning. This particular research study focusses on *GeoGebra*, which is an open-source dynamic mathematics software that allows teachers and learners to use in any way they wish. This software requires only a Java plug-in, and therefore virtually runs on any operating system such as Windows, Mac OS or Android. Furthermore, the DBE (2013) also recommends the use of *GeoGebra* as a technological tool for teaching and learning mathematics.

Multiple perspectives in *GeoGebra* allow the integration of dynamic representations, thus creating a conceptually rich learning environment. Therefore, in teaching mathematics, the software provides valuable opportunities for accessing and understanding different representations of

concepts. For instance, the dynamic visualisation possibilities of the software are made visible when a function graph, for example, is dragged around the screen, and the algebraic notation and table values change simultaneously and dynamically. Thus, *GeoGebra* provides a resourceful tool for mathematics teachers to integrate mathematical content and pedagogical strategies for fostering understanding (Bu et al., 2012, p. 91). Stols and Kriek (2011) also acknowledge that one of the potentials of a dynamic learning environment is the ability to focus on interlinked parts of a mathematical idea, thus enriching visualisation.

1.5 TEACHING PROFICIENCY – AN ANALYTICAL FRAMEWORK

The research study makes use of an adapted analytical model of mathematical teaching proficiency proposed by Kilpatrick, Swafford and Findell (2001). Proficiency in teaching is related to effectiveness, consistently helping learners to learn worthwhile mathematical content. Kilpatrick et al. (2001) proposed a framework consisting of five interwoven and interdependent strands of teaching proficiency. Due to the limited scope of this research, I adapted Kilpatrick et al.'s two strands of teaching for mathematical proficiency to analyse teacher practice using DGS tools. They are:

1) *Conceptual understanding of core knowledge* required in the practice of teaching.

Kilpatrick et al. (2001) note the following:

The kinds of knowledge that make a difference in teaching practice and in students' learning are an elaborated, integrated knowledge of mathematics, a knowledge of how students' mathematical understanding develops, and a repertoire of pedagogical practices that take into account the mathematics being taught and how students learn it (p. 381).

An integrated and functional grasp of mathematical ideas and concepts is crucial in the teaching of mathematics. A proficient teacher makes use of appropriate resources, including ICT tools, which can facilitate this integrated knowledge to develop learners' mathematical proficiency.

Kilpatrick et al. (2001) propose that three kinds of knowledge are crucial for teaching school mathematics: knowledge of mathematics, knowledge of learners and knowledge of instructional practice. Teachers' mathematical knowledge and their capacity to use it in teaching is crucial in developing learner's mathematical proficiency. Teachers need to know how learners think and be aware of their conceptions and misconceptions.

2) *Instructional fluency* in carrying out basic instructional routines.

The second strand in teaching proficiency is the development of instructional routines. These routines, in the context of this study, are grounded in mathematical activity. Teachers, having acquired a repertoire of teaching approaches, can readily draw upon them as they interact with learners in teaching mathematics. Proficient teachers have a clear vision of the goals of instructions, and Kilpatrick et al. (2001) assert:

They need to be able to use their knowledge flexibly in practice to appraise and adapt instructional materials, to represent the content in honest and accessible ways, to plan and conduct instruction, and to assess what students are learning. (p. 369)

Many mathematical ideas are abstract and often appear to be rather obscure. Thus, proficiency in teaching entails a unique capacity to deconstruct highly abstract mathematical ideas and make them visible. Ball and Bass (2000) support the argument that teaching mathematics requires the ability to “deconstruct one’s own mathematical knowledge into less polished and final form, where elemental components are accessible and visible” (p. 98).

1.6 RESEARCH PROBLEM AND SETTING

Technology does however not replace a mathematics teacher. Instead, the teacher plays an important role in providing learning opportunities in a technology-rich learning environment. The NCTM (2000) emphasises that “[t]he effective use of technology in the mathematics classroom depends on the teacher” (p. 25). The effective use of DGS dramatically increases the possibilities of teaching mathematics that engages learners.

Nevertheless, in the literature I find the use of ICT in developed countries or cities either for teacher training or for professional development courses (Stols, et al. (2008), Ndlovu et al. (2013)). In South Africa, Stols, Mji and Wessels (2008) report on intervention programmes on DGS for in-service teachers from previously disadvantaged schools in the Pretoria region. They examined changes in teachers’ knowledge of geometry when using DGS and found that the perceived usefulness of the technology influenced teachers’ beliefs concerning the use of the software. In exploring the experiences of mathematics pre-service teachers using DGS, Ndlovu, Wessels and de Villiers (2013) reported challenges faced by the participants in their micro-teaching sessions. Even though *GeoGebra* is open-source software and available free of cost, there is little empirical

research in South Africa — especially in Eastern Cape — of its efficiency in teaching secondary school mathematics. Up to now, little is known about how teachers have used (or are using) DGS in actual classrooms in previously disadvantaged communities.

Ndlovu et al. (2013) also recommend that “[t]eachers working in professional learning communities can develop common identities and overcome their lack of confidence if they work collaboratively together and constantly engage in reflective practice” (p. 241). In the South African context, little research has been conducted on teachers collaboratively developing technological resources and then implementing them in classrooms.

Research by Gueudet and Trouche (2011) with French teachers evidenced that the collective work of the teachers contributed to the development of their teaching practices. I concur with them who observed that the use of DGS is not so widespread in schools. Gueudet and Trouche (2011) say that “this use appears quite complex to a number of teachers, and the ‘ordinary’ teacher training programme seems helpless to change this situation” (p. 399). They addressed these issues by putting forward the notion of ‘teachers empowering teachers’, aiming to foster the use of technology, particularly DGS, in mathematics classrooms. They conceptualised collaborative work on developing ICT resources as a way of teacher training in technology., This has inspired my work within the GLIP community in empowering teachers to make use of DGS in their mathematics classrooms and conduct this research project. GLIP envisages the collaborative engagement among teachers as a way forward in closing the gap between having access to technology and adapting it for effective and practical use in mathematics classrooms. Thus, it is important to critically look at ways of integrating the use of DGS within classroom practice, and then analyse how teaching takes place.

The overall goal of this research is to investigate whether and how dynamic software-aided visualisation can enhance the teaching of Grade 11 mathematics.

In pursuance of this goal, the following specific research questions guided the study:

- How can Dynamic Geometry Software such as *GeoGebra* be used as a visualisation tool to teach Grade 11 mathematics?
- What enabling and constraining factors do Grade 11 teachers encounter when using *GeoGebra* as a visualisation tool to teach Grade 11 mathematics?

1.6.1 Significance of the research

This action research study involves “teachers working together to improve educational practices” (Mertler, 2017). The significance of the study is to empower teachers with the dynamic mathematics software *GeoGebra* and enable them to incorporate its powerful visualisation capacity in their teaching of mathematics. In the GLIP, members generate new applets or modify available applets for teaching specific mathematical content. These applets are created in collaboration with all the teachers in GLIP. The teachers in GLIP brainstorm on innovative teaching techniques using these applets. During the implementation stage in the GLIP, my fellow researcher focussed his research on learners’ interactions with these *GeoGebra* applets, while I explored how teachers used *GeoGebra* as a visualisation tool in enhancing teaching proficiency. In the process of this study, a community of proactive teachers was formed, for creating and sharing applets relevant to the South African context. It is envisaged that this community will substantially contribute *GeoGebra* applets to the existing pool of online resources. I argue in this thesis that the teaching strategies using applets generated through my action research cycles, including implementation and reflection of instructional methods using ICT, will enrich the community of teaching practice. This research is a contribution to teacher training and teacher development programmes through effective integration of ICT into the mathematics classroom.

Research (Cuban et al., 2001; Hohenwarter & Lavicza, 2007) indicates that despite the numerous benefits of using ICT in mathematics education, the process of embedding ICT in classrooms is slow and complicated. Ensuring access only to technology is insufficient for the successful integration of technology into teaching and learning mathematics. Access needs to be accompanied by meaningful teacher support and mentoring. This is because teachers often feel that their skills and knowledge are limited or inadequate, and thus often refrain from using technology in their teaching (Hennessy et al., 2005; Stols et al., 2015). It is therefore critical to be aware of the vital role of appropriate professional development programmes, like GLIP, to integrate ICT into teaching practices.

Furthermore, it is hoped that this study speaks to the curriculum designers and policymakers who need to be informed of the importance and relevance of incorporating ICT as a pedagogical tool, in particular *GeoGebra*, into teacher in-service training courses.

Through this project, the participating teachers are empowered to use technology in teaching and learning mathematics. The research study thus contributes to an improved understanding of collaborative engagement among teachers in the context of using ICT as a teaching tool.

1.7 CONSTRUCTIVISM - THEORETICAL FRAMEWORK

A constructivist theoretical perspective informs this research study. There is a significant amount of research on the use of computer tools within a constructivist perspective (Bu et al., 2012; Hoyles & Noss, 2003). Technological environments can bring about pedagogical changes that involve practical application of new materials and methods. From a constructivist perspective, teachers design instructional activities to engage learners in doing things and as well as in thinking about what they are doing. Little emphasis is placed on transmitting information and more on developing learners' understanding of concepts and their skills. Thus, the constructivist approach aligns well with teaching with ICT technologies as it requires hands-on teaching and learning strategies, where the teacher and learner are actively engaged in constructing knowledge. This approach underpins both the intervention in this study and assumptions about teaching proficiency. Furthermore, DGS is a powerful teaching and learning medium and appears to create opportunities for creative thinking (Stols & Kriek, 2011).

In Chapter Two, I unpack and discuss more on each of these four perspectives, namely Visualisation, Technology, Teaching Proficiency and Constructivism.

1.8 METHODOLOGY

This research study employs a qualitative approach to data analysis and is oriented within the interpretive research paradigm. The interpretive paradigm enabled a rich understanding of situations (classroom and planning) where teachers utilised their teaching and technological skills and made decisions accordingly to understand their practices in a technology-based classroom. This study aimed ultimately to gain insights into the visualisation process in building mathematical ideas when teachers used co-developed applets in classrooms.

This research also has elements of an action research process. The data collection and analysis occurred in three GLIP cycles. A variety of data collection techniques were utilised during the process – audio-video recordings, stimulated reflective interviews and field notes. The data were analysed with specific reference to a framework of teaching proficiency grounded in the Kilpatrick

et al. (2001) analytical framework. I found this model useful for analysing my data as it builds on several dimensions of general pedagogical models of teaching practice. This analysis process is presented in a sequence of vignettes that serve to characterise multiple perspectives of the research. A more detailed description of the research methodology is provided in Chapter Three.

1.9 STRUCTURE OF THE THESIS

There are seven chapters in this thesis, including this chapter.

Chapter Two: Literature Review

This chapter provides the conceptual and theoretical frameworks that underpin my research study. Firstly, several pertinent issues related to the conceptual background of visualisation in teaching and learning of mathematics are examined. This leads to the discussion on the role of technology and its visualisation capabilities in mathematics classrooms. Thirdly, I discuss the fundamental concepts of Kilpatrick et al.'s (2001) framework of mathematical teaching proficiency that form the analytical framework of my study. I conclude the chapter with a discussion on the theory of constructivism.

Chapter Three: Methodology

Chapter Three provides a detailed discussion of the action research process that was used in this study. Methods of data collection such as classroom observation and stimulated recall interviews are described. I then examine the methods of data analysis and discuss strategies to enhance the validity and reliability of the research study. I address ethical considerations that pertain to this research. The chapter concludes by examining some of the challenges that arose during the data collection process.

Chapter Four, Five and Six: Analysis of teaching using DGS

These three chapters of similar structure constitute my analysis of data for each mathematical topic taught. I briefly discuss the planning and designing of the applets and the lesson. I then analyse participants' engagements with the co-developed applets and with the learners, by employing my analytical framework which I adapted from the literature. Thereafter, I present the findings concerning the research questions for every GLIP cycle. Each chapter concludes with an explanation and discussion of the results.

Chapter Seven: Conclusion and Implications

In this concluding chapter, I summarise the findings, consolidating the discussions from the previous three chapters together. I examine issues and limitations and propose recommendations from the research. I also discuss the contributions to the field and suggest directions for future research.

CHAPTER TWO

LITERATURE REVIEW

2.1 INTRODUCTION

In this chapter, I discuss the conceptual and theoretical frameworks that underpin my research study. Firstly, the conceptual background of visualisation and its relevance in teaching and learning are examined. Secondly, the role of technology and its effective use in mathematics classrooms is discussed. Thirdly, the fundamental concepts of Kilpatrick et al.'s (2001) framework of mathematical proficiency that forms the analytical framework of my study are deliberated upon. Finally, the theory of constructivism is characterised within the context of visualisation in a technological classroom. Refer to Figure 2.1 below.

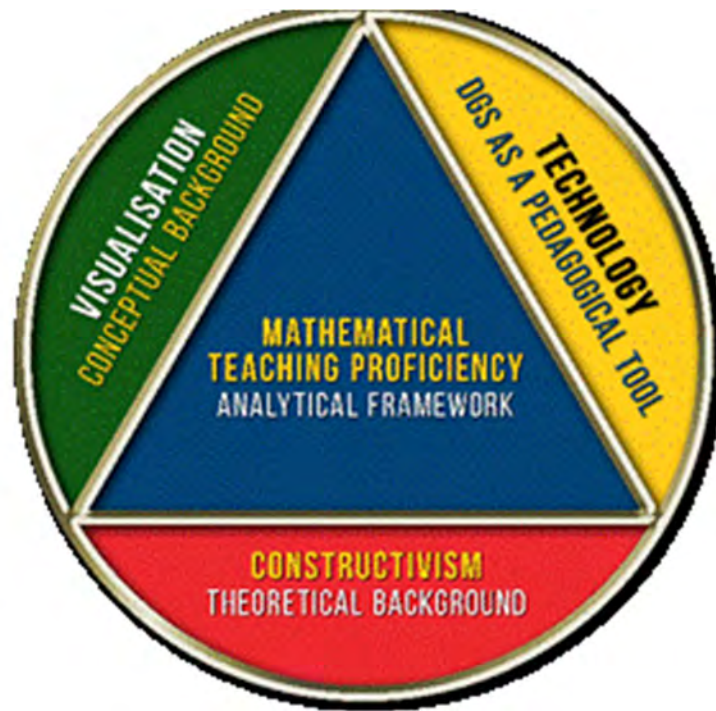


Figure 2.1: Conceptual and theoretical frameworks

2.2 CONCEPTUAL LANDSCAPE – VISUALISATION

This study focusses on the role of visualisation in teaching and learning mathematics through ICT, in particular, the use of *GeoGebra*. The importance of visualisation is gathering momentum among teachers and researchers.

2.2.1 What is Visualisation?

A Chinese proverb says, “a picture is worth ten thousand words”. Arcavi (2003) emphasises that the visual representation of information in a graph, for example, enables us to comprehend a scenario or relationship between two processes without reading sequentially and logically printed words. He also sees mathematics as a subject which represents real-world scenarios, many of which appear to be visual. Mathematics makes use of a variety of visual imagery.

2.2.1.1 Visual Imagery

Presmeg (1986a) defined visual imagery as a mental schema in the presence or absence of an object. For Presmeg, visualisation includes processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics. The ability to transform visual images mentally can be enhanced by external visual support (Dreyfus, 1993). Visualisation entails the coherent handling of an internal visual image and an external representation corresponding to the mathematical structure. However, Guzmán (2002) warns that “the degree of correspondence between the mathematical situation and the concrete way of representation, can be more or less closed, natural, symbolic, even more or less personal and perhaps incommunicable” (p. 5). Dreyfus (1993) surmises that when the images are associated with correct interpretations, “it helps to structure and solve problems specifically and knowledge in general” (p. 10). Kosslyn and Koeing (1992, p. 148) contend that “[m]any of the uses of imagery in reasoning involve anticipating the consequences of an action or event, which rely on our ability to transform patterns in our images”. Transforming representations implies a new perspective by rotation, translation, compression or even folding the image. Teaching about visual and spatial thinking is significant as it supports mathematical reasoning. The role of imagery in problem solving is also highlighted in the works of Krutetskii (1976), Presmeg (1986a) and Presmeg and Balderas-Cañas (2001).

In his study, Kurtetskii (1976) found that in problem-solving, pupils who lack the ability to use visual images, instead “use a harder and more complicated logical-analytical method of solution when relying on an image would give a much simpler solution” (p. 317). A visual-pictorial approach to solving problems is a powerful tool when engaging with mathematical ideas and reasoning. In her study with 54 learners, Presmeg (1986b) found that the majority of her learners used mnemonic functions of imagery. One form of memory imagery which was found to be particularly useful was the imagery of formulae – in the notebook, textbook or chalkboard. Vivid imagery, for instance a ‘fan diagram’ of trigonometric ratios of special angles 0° , 30° , 45° , 60° and 90° , has mnemonic advantages in learning mathematics. In their research, Presmeg and Balderas-Cañas (2001) reveal that the use of ‘pattern imagery’ and ‘dynamic imagery’ were effective in the learning of mathematics. Presmeg (2006) suggests imagery that serves an abstract function stripped of concrete details, significantly facilitates mathematical generalisation. Though visual imagery has potential strengths, Presmeg warns that ‘visualisation is not self-explanatory and is very important for teachers.’

Lean and Clements (1981), agreeing with Hebb (1972), define visual imagery as “the occurrence of mental activity corresponding to the perception of an object, but when the object is not present to the sense organ” (p. 267). They emphasise the significance of visual imagery in teaching and learning mathematics, by stating that “many highly original and significant creations of the human mind have been largely the result of nonverbal mental representations (mainly visual imagery)”. According to Lean and Clements (1981), mathematical ability generally comprises of general intelligence, visualisation and spatial ability to generate and formulate mental images. They distinguished the ability to form ‘memory’ images and the ability to form ‘abstract’ images, and if both these abilities exist, then the flexibility to switch between them (visual vs abstract), could be an essential factor in solving mathematical problems.

2.2.1.2 Defining Visualisation

Arcavi (2003) suggests that visualisation offers a method of seeing the unseen. For example, a graphical representation of tabular data helps us to see and understand some broad features of the data. A graph assists us in visualising a particular mathematical relationship. It is crucial, however, to understand that visualisation as a process does not exclude the process of verbalisation. To the contrary, the two processes may well complement each other, although the focus of this study is on visualisation.

For the purpose of this study, the following definition of visualisation by Arcavi (2003) is adopted:

Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings. (p. 217)

Duval (1999) observes that visual perception is complex, and at times, perplexing. Although we live in a three-dimensional world, many diagrammatic representations and images appear to be two dimensional, i.e. we can only see one side of an object. For a complete understanding of the object, the observer should ‘move around’ the object and visualise the other sides of the object. In my experience, many of my learners find this type of visualisation difficult. Visual perception needs exploration through physical movements, whereas visualisation can fully comprehend the object at once. According to Duval “understanding involves grasping the whole structure and there is no understanding without visualisation” (p. 13).

2.2.2 Visual Literacy

Visual literacy (VL) is more than seeing images: it is the process of attributing meaning to the perceived images. Vermeersch and Vandembroucke (2015) consider visual literacy as “a thinking process, one that implies specific cognitive actions like interpretation and reflection, understanding and comprehension, awareness,...” (p. 112). This concerns the use of perceived images with the intention of making something new out of that image information. It is a matter of visualising new images or reconstructing existing images. From a theoretical perspective, visual literacy “incorporates the philosophical, psychological, and physiological aspects of learning” (Avgerinou & Pettersson, 2011, p. 3). Acknowledging VL as a skill or an ability, Avgerinou and Pettersson indicate that critical viewing and thinking, imaging, visualising, inferring, constructing meaning and communicating must be learned for true comprehension. Thus, we can infer from the above arguments that VL, in a broader sense, resonates with Arcavi’s (2003) definition of visualisation in mathematics.

A recent research study published by the National Research Council (US) (2006) believes that spatial thinking is a fundamental and necessary mode of thought applicable throughout a life span in everyday life, in work situations, and science. Furthermore, this study recommends that we must foster a generation that is visually and spatially literate. While visual and spatial thinking is central

to mathematics and science, it is a “fundamental process skill that transcends the bounds of particular disciplines” (p. 134).

Presmeg and Duval, who are researchers of visualisation, surmise that visualisation can be developed and taught in classrooms. Avgerinou and Pettersson (2011) concur: “ ... VL skills are (a) learnable, (b) teachable, and (c) capable of development and improvement” (p. 4). Duval (1999, p. 14) argued that “visualisation (in mathematics) involves grasping directly the whole configuration of relations and in discriminating what is relevant in it.” However, Vermeersch and Vandembroucke (2015) suggest that “[v]isual perception, like seeing the different graphic elements of an image on a canvas, needs training” (p. 112).

2.2.3 Visualisation in Mathematics

Mathematical concepts are rich in visual representations. Diagrams frequently accompany mathematical thinking (Krutetskii, 1976). In order to solve mathematical problems, one should perceive and use clear mental pictures. The ability to visualise abstract mathematical ideas helps to solve complicated problems quickly and accurately. Krutetskii (1976), however, found in his study of Soviet school children, that “mathematically able students have no need for visualising objects or patterns even when the mathematical relation ‘suggests’ visual concept” (p. 317).

On the other hand, Guzmán (2002) considers visualisation to be a powerful communication and learning tool. He argues that mathematical activities like analysing a problem, problem-solving, demonstration of a problem and handling theorems involve some form of visual activity. Guzmán further argues that “mathematical concepts, ideas, methods, have a great richness of visual relationships that are intuitively representable in a variety of ways” (p. 3). He emphasises that using different visual images to analyse, differentiate and manipulate concepts, enables one to solve problems in a versatile manner. Visualisation is therefore useful in teaching and learning mathematics. Underlying mathematical ideas are born from concrete and visualisable situations, because visualisation is a critical aspect in mathematical activity through which one can explore different structures of concrete reality. The initial perceptions of similarities in the real objects or situations can guide learners to abstraction and symbolic representation. Thus, for Guzmán (2002), visualisation in mathematics is the ability to relate and handle both concrete and corresponding abstract objects. Fischbein (1993) echoes that training in visualisation is aimed not only to organise data at hand into meaningful structures, but it is also an important factor guiding the analytical

development of a solution. Bishop (1980) also concurred on the effects of diagrammatic training with learners and showed that it encouraged pupils to visualise the mathematical problems and their solutions.

Duval (2013) states that visualisation is the ability to discriminate what is irrelevant and what is relevant in a given context. For example, in Figure 2.2, AE, AF and BC are tangents to a circle. A question can be posed on whether the perimeter of the triangle ABC is equal, greater or smaller than the sum of the line segments AE and AF (adapted from (Duval, 1999, p. 19)).

In the first instance, a learner may find it difficult to answer this, since the lengths are not given. However, the moment the sub-configuration of BC is considered as two segments, $BC = BD + DC$; it is easy to infer that BD and BE are congruent, by virtue of the behaviour of two tangents from a common point. Visualisation lies in discarding the irrelevant segments AB and AC, and focussing on the line segments BE and BD as shown in Figure 2.3. Thus $BC = BE + CF$, and so the perimeter of triangle ABC will be equal to the sum of AE and AF. Teachers thus need to organise appropriate learning sequences to help learners “to embrace the whole range of variations of the conditions of a problem and to bring out the various factors that make them clear” (Duval, 1999, p. 19).

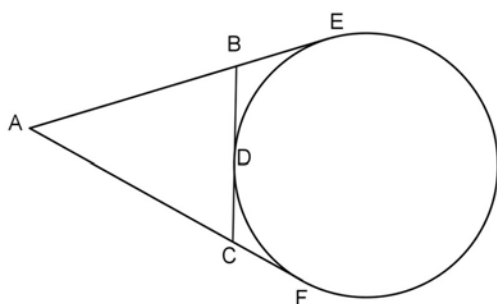


Figure 2.2: Tangents to a circle

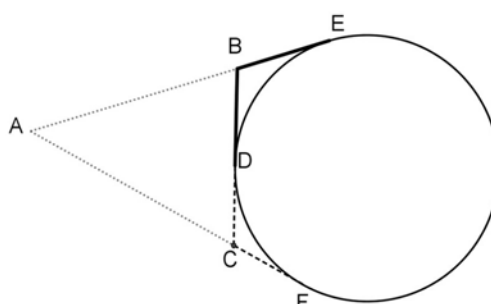


Figure 2.3: Tangents to a circle focusing on relevant segments

Some topics in higher secondary mathematics need greater visualisation abilities than other topics. Consider the following function $f(x) = x^2 - 4x + 3$, having the turning points at (2;-1). If asked to find the turning point of $f(x + 3) - 2$, learners typically find a new equation and recalculate the turning points algebraically. However, having a clear mental image of the parabola can assist in translating the turning point by three units left and two units down, resulting in the point (-1;-

3). Although the question had no mention of graphs, strategic use of visualisation can simplify the problem and yield appropriate solutions. Research and my own experience show that the following Grade 11 topics are found to be difficult for learners with poor visualisation practices:

- Sketching of graphs (Lean & Clements, 1981);
- Interpreting graphs (e.g. find the equation of a given graph, for which value of x is $f(x) > 0$, for which value of x is $f(x) \cdot g(x) < 0$);
- Translation and reflection of graphs;
- Interpreting two-dimensional impacts of three-dimensional trigonometry/geometry situations (angle between two planes) (Lean & Clements, 1981);
- Interpreting dimensions of three-dimensional geometric solids (to distinguish between height (h) and slant height (s)).

2.2.4 The Use of Visualisation in Teaching

Krutetskii's (1976) rich and diverse mathematical tasks that include problems on visualisation are a valuable resource for teachers. When a teacher does not include visual-pictorial aspects of mathematical concepts, it can lead learners to memorise analytic definitions and what Krutetskii fears: "apparently results in a certain one-sidedness in their mathematical development" (p. 321). Hence, Presmeg (1986a) recommends that "when a topic is first taught, a visual representation often aids understanding" (p. 302). The tree diagrams that we use in probability exercises contribute to the understanding of concepts. Such diagrammatic visualisation is essential, for as Guzmán (2002) argues "[I]n this kind of visualisation our mental objects and their mutual relationships concerning the aspects which are of interest for us are merely represented by diagrams that constitute a useful help in our thinking processes" (p. 8)

Presmeg (1986a) defined mathematical visualisation as "the extent to which a person prefers to use visual methods when attempting mathematical problems which may be solved by both visual and non-visual methods" (p. 298). She classified individuals concerning mathematical 'visuality' into three groups – namely non-visualisers, visualisers and a middle group. It is thus vital for teachers to choose their teaching strategies carefully to suit these individual differences. Invariably, teachers face different types (with respect to visuality) of learners in their classes. Presmeg (2014) reveals that visualisers in a class of a nonvisual teacher are like fish out of water, and their learning

is compromised. Teachers therefore need to bring different teaching styles into their practice to accommodate all learners. I concur with Presmeg (1986a, 2014) that when teachers prefer non-visual modes of teaching, visual learners encounter difficulties in abstracting and generalising mathematical ideas. Hence, teaching and learning are not always effective. Perceiving that learners need visual methods in learning, Presmeg and Balderas-Cañas (2001) recommend that teachers embed visual methods into their pedagogy. Presmeg (2014) also argues that visual thinking is fruitful in promoting creativity. Thus, there is a need for teachers to take advantage of visualisation for effective teaching and learning of mathematics. Lean and Clements (1981) also raise the following concern: “[w]hich is the best form of instruction for a person who prefers a visual mode of response (or, similarly, a verbal-logical mode)?” (p. 270).

At the beginning of this chapter, I noted that “a picture is worth ten thousand words”, but Guzmán (2002) cautions that mathematical images need to be interpreted and understood. The teacher plays a vital role in lending mathematical meaning to its visual representation. Students may attach irrelevant details to the visual images and do not necessarily see what teachers want them to see (Arcavi, 2003; Presmeg, 1986b). As mentioned earlier, ‘visualisation’ is not self-explanatory; teachers should guide their students to the underlying mathematical meanings entailed in a given representation. Arcavi (2003) suggests that “the competence to disentangle contexts” (p. 234) enables us to ‘see the unseen’. Dörfler (1991) argues that one of the main goals of teaching mathematics is to add meanings and concepts to the ‘image schemata’.

Duval (2013) argues that words can only indicate visual representations of ideal objects. The challenge in teaching, especially geometry, is the predominance of perceptual recognition of shapes and objects that blocks out the mathematical requirement of seeing things through words or symbolic notations. Duval (2013) contends that “geometrical activity involves at least two registers of semiotic representation... the two registers in question are natural language and non- iconic figures” (p. 32). Thus it is essential to develop coordination between words (language) and configuration (visualisation), although this may require extended training. There are two kinds of tasks involved in geometric problem-solving. One is recognising the shape or figure, and the other is recognising the properties of the shape or figure. Duval surmises that “...visual recognition at a glance is the first cognitive condition for solving problems in elementary geometry” (p. 27). It is,

therefore, necessary to teach students to focus primarily on the relevant properties for problem-solving.

The essence of teaching that promotes the effective use of visualisation in mathematics classrooms is in making connections between visual and symbolic representations of the same mathematical notion (Presmeg, 2006). Presmeg emphasises that connection making includes bringing in relationships between mathematical ideas, making use of multiple representations and prior knowledge; and above all, making connections to the real world. One of the key elements of proficient teaching is the concept of seeing connections across mathematical topics and across grades (Ball et al., 2008; Schoenfeld & Kilpatrick, 2008). When we make connections between mathematics topics and the real world in a classroom, we employ visualisation for effective learning.

The use of multiple representations in classrooms is another strategy to make effective use of visualisation. I believe that a teacher should have a thorough understanding of the use of multiple representations in mathematics. Teppo and Heuvel-Panhuizen (2014) propose that there are a variety of number line representations that can be employed in the school curriculum; nevertheless, each model serves different purposes. For example, a directed number line model supports the demonstration of operations with integers, a rational number line supports reasons of the density property of numbers. As teachers embed these different models in their classroom activities, Teppo and Heuvel-Panhuizen (2014) accentuate that they become “aware of the representational nuances and types of meaning that are both explicit and implicit within the contexts of specific number line tasks” (p. 56).

2.2.5 Difficulties in Using Visualisation in Teaching

According to Presmeg (1986b), when a topic is first taught, a visual presentation often aids understanding, but the practice of the procedure or formula may lead to habituation when an image is no longer necessary. Nonetheless, students may face difficulties in the generalisation of the mathematical ideas when they are unable to combine the concrete details and the abstract objects. They remain tied to the irrelevant details in the concrete imagery (Dreyfus, 1993). Aspinwall, Shaw and Presmeg (1997) note that “[a]n uncontrollable image, then, is one which is beyond the volition of the cognizing individual” (p. 303). Images are ‘uncontrollable’ in the sense that they

appear in an individual's thought irrelevant to the context and can also persist in the face of contrary evidence. Aspinwall et al. (1997) demonstrate a case where 'uncontrollable' or involuntary images obscured meanings in calculus. The student's difficulty was a result of his 'uncontrollable' image of the graphic representation for parabolas becoming a 'straight line up' [$y = x^2$ and $y' = 2x$]. My experience in the geometry classroom also concurs with this difficulty, where learners are unable to identify the angle at the circumference subtended by an arc, thus failing to apply *the angle at the centre* theorem – another case of an 'uncontrollable' image of a standard diagram. It emphasises the importance of the pedagogical role of graphical instructions in the learning of mathematical concepts. I argue that dynamic geometry can play an important role in overcoming such difficulties and this will be discussed in detail in the sections below.

Arcavi (2003) argues that when visualising 'conceptually rich images' (p. 235), the cognitive demand is high. The students may have to put effort into switching flexibly between visual representations and analytical properties, which is at the core of understanding mathematics (Arcavi, 2003; Guzmán, 2002). Besides, students do not rely on visual methods on the grounds "that they are not always procedurally 'safe'" (Arcavi, 2003, p. 235), and hence prefer formal analytical methods. Here I discuss the reasons or factors that affect these difficulties related to visualisation. Firstly, the low status accorded to visual images in mathematics classrooms (Arcavi, 2003; Dreyfus, 1991; Presmeg, 1986b, 2014). Teachers prefer non-visual methods of teaching and learning, as Presmeg (1986a) notes "practice of the procedure or formula may lead to habituation when an image is no longer necessary" (p 302). Arcavi (2003) resonates that "many teachers may feel that analytic representations, which are sequential in nature, seem to be more pedagogically appropriate and efficient" (p 236). The school curriculum assessment tasks favour the analytical mode where solutions in visual mode are discouraged. When I solved the inequality $x^2 + 3x - 10 \leq 0$ as a diagram of the final solution, the learners asked "Do you think we will get full marks for only this diagram?". Refer to the adjacent Figure 2.4

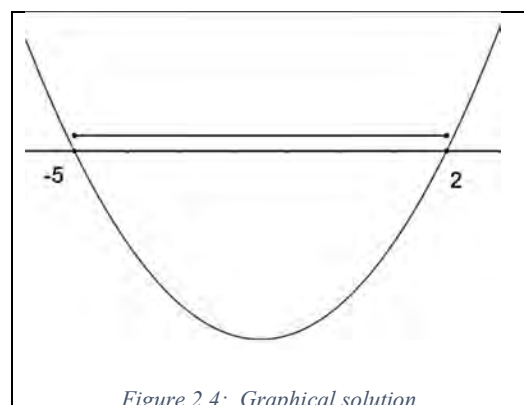


Figure 2.4: Graphical solution

There is thus evidence that many teachers do not make use of visualisations in mathematics classrooms. Secondly, mathematics textbooks subscribe to a verbal-logical argument. Even if there are images, these images do not necessarily support the dynamic nature of visualisation (Guzmán, 2002). I concur with Guzmán, as the textbooks that I use do not support the dynamic nature of visualisation. Let me illustrate with an appropriate example: Find the values of ' k ' for $f(x) = k$ have unequal roots where $f(x)$ can be any cubic graph. One of the solution strategies is to sketch the straight line (k - parallel to the x -axis), look for the intersection points with $f(x)$ which is a cubic graph, and dynamically move the line to visualise the solution. Very infrequently a hint of two graphs may be provided along with the answers in the textbooks and study guides.

Arcavi (2003) asserts that technology may assist in developing visualisation skills so that students can better 'see' mathematical concepts and ideas, helping to overcome their limitation of visual perception. Thus, visualisation can sharpen our understanding. By hearing the description of a concept, our imagination creates an image which we attach to the description. I argue in this study that seeing these concepts with the help of ICT technology, for example, could sharpen our imaginations and subsequent enjoyment, and develop a better understanding of the concept. I will argue in the next section that ICT technologies can be used effectively to enhance the appreciation of mathematical ideas and concepts through their inherent visualisation capacity.

2.3 TECHNOLOGY - A TEACHING TOOL

Incorporating technology in mathematics education is gaining traction among researchers and curriculum planners. "Technology potentially opens up new observation possibilities for teachers, allowing them to focus on students' investigations and thinking strategies while solving mathematical problems" (NCTM, 2000, p. 25). Technology brings in a repository of information materials that includes textbooks, audio-video files, online learning websites, simulation software, dynamic software and many more, into the classrooms. These rich resources of information open doors for teachers to enhance and reflect on their mathematical teaching practises. The National Curriculum Statement (Curriculum and Assessment Policy Statement [CAPS], 2011) recognises the potential that technology offers for curriculum delivery, and this is reflected in the teaching guidelines as follows:

Generate as many graphs as necessary, initially by means of point-by-point plotting, supported by available technology, to make and test conjectures and hence generalise the

effects of the parameter which results in a horizontal shift and that which results in a horizontal stretch and/or reflection about the y -axis (p 12).

Technology can enhance learning skills for teachers, as Bu et al. (2012) assert – a technological environment provides an opportunity for teachers to learn, reflect and relearn mathematics and pedagogical practices. Technology can transform teaching practices by shifting the focus from a teacher-oriented, to a learner-centred classroom environment.

There is a wealth of research reports that provide empirical evidence to indicate the different potentialities of technology (Bhagat & Chang, 2015; Cuban et al., 2001; Mariotti, 2000; Naidoo & Govender, 2014). These research papers are, in general, optimistic about the learning benefits of technology in mathematics. The identified potentials are: visualisation and stimulation (Arcavi, 2003; Naidoo & Govender, 2014); self-confidence and motivation (Cuban et al., 2001; Ruthven et al., 2008); and the development of deep understanding (Hollebrands, 2007).

Kendal and Stacey (2001) maintain that technology can be used ‘functionally’ and ‘pedagogically’ in mathematics classrooms. The functional use of technology is to carry out complex and routine procedures for solving problems. The pedagogical use is to develop a conceptual understanding of mathematical ideas. Sherman (2012) underscores the importance of a teacher to sustain students’ mathematical engagement with technological tools. Healy and Hoyles (2001) are concerned about the use of technological tools without teacher intervention /instruction for it may “lead students down into mathematical cul-de-sacs from which they can neither progress nor backtrack” (p. 237). Technology offers students an open world in which they can explore, nevertheless Balacheff and Kaput (1996) argue that technology “does not guarantee that specific learning will occur” (p. 483). Sherman (2012) concurs that the students’ engagement with the computer may result in the formulation of ‘shaky conjectures’. Inevitably, the student may focus on constructed objects or representations irrelevant to mathematics (Hoyles & Noss, 2003) and overlook the teacher’s intention (Hölzl, 2001). Thus, the role of a teacher in designing a mathematical task is decisive as it may have a direct impact on supporting students’ mathematical thinking. Hölzl suggests that in a guided discovery setting a teacher helps students to recognise the invariant properties, thus learning mathematical concepts by exploring. Teachers can leverage technology to support their instructions and illustrate to highlight some mathematical principles. Laborde (2001, p. 289)

ascertains that it is important for teachers to see the “role of technology as a way to facilitate the making of conjectures rather than a part of solution of tasks.”

The corpus of research studies on technology is focussed on student interaction with technology whereas the teacher dimension is rare (Hennessy, Ruthven & Brindley, 2005; Lagrange et al., 2003). The integration of technology into classrooms is a complex process combined with an already intricate teaching and learning process (Hennessy et al., 2005; Laborde, 2001).

The technological innovation aimed to improve teaching techniques entails classroom modifications. For these modifications to be implemented, Cuban et al. (2001, p. 816) recognises that “few of these reforms noted the workplaces within which teachers laboured, involved teachers in the design itself, allocated sufficient resources to develop teachers’ capacity to implement the desired changes.” I concur with Hoyles and Noss (2003) and Laborde (2001) that for integrating technology into the classroom, tools need to be designed with the teachers and not for the teachers. Gueudet and Trouche (2011) recommend a teacher training programme for integrating technology where teachers collaboratively design lessons and reflect on feedback from the lesson. Thus, development programmes like GLIP become essential for teachers to integrate technology into their teaching methods and broaden their instructional repertoires.

2.3.1 Dynamic Geometry Software

The type of educational technology used in this study is called ‘dynamic geometry software’ (DGS). Arcavi (2003) asserts that there is a need for cognitive technologies that help transcend the limitations of the mind. Such ‘technologies’ are those that use dynamic visual tools to better ‘visualise’ mathematical concepts and ideas. DGS was developed with educational purposes in mind, providing a setting in which students can construct and experiment with mathematical objects and relationships (Ruthven et al., 2008). These systems have become increasingly common classroom tools to support the teaching and learning of geometry (Hoyles & Noss, 2003). Key to effective DGS is an interface that affords direct manipulation of artefacts that can be dragged around the computer screen using a mouse, but keeping its constructed properties and underlying mathematical relationships preserved (Moreno-armella et al., 2008). DGS is particularly suited to making use of other facilities such as dragging and sliding points and lines on the computer screen. Dragging is a key feature of DGS. It allows a continuous reconstruction of figures on the computer screen by direct manipulation with the mouse (or by touch on a touchpad). The interface of DGS

provides an opportunity for students to conjecture and generalise by clicking and dragging objects, which dynamically re-draws and updates information on the screen (Hölzl, 2001). The system also provides feedback that encourages exploration and engagement with the mathematical concept (Moreno-armella et al., 2008; Sarama & Clements, 2009).

Ruthven et al. (2008) argue that DGS helps to produce accurate figures and images, including their measurements such as angle and length. Accordingly, it offloads the tedious skill required for accurate drawing and measuring and enables students to focus attention on the mathematical properties of the constructed objects (Tall, 1993). In their study, Ruthven et al. (2008) find that teachers value dynamic software for its contribution to guide students to discover mathematical properties. The following example alludes to the benefits of dynamic software in comparison with traditional discourses using static diagrams: in elementary calculus, for example, we teach our learners using symbolic notations that at stationary points, the first derivative is zero. The derivative of a function is the rate of change of $y = f(x)$ (dependent variable) with respect to x (independent variable), and is the gradient of the tangent to the function f at x . Being allowed to explore this aspect using DGS may arouse the students' curiosity, inviting mathematical reasoning. We can draw a cubic graph and plot two points on the graph. These two points are joined by a straight line, and DGS reads its gradient directly. As we drag these points closer to coincide and become a tangent to the graph at that point, the gradient of the straight line also changes dynamically. Having now established that the first derivative is the gradient of the tangent, we can now use another tool in DGS, '*tangent to the function at a point*'. The teacher can now motivate the learners to move the point around and focus on the gradient of the tangents at various points, including the turning points. This may lead them to formulate that the gradient of the tangent at turning points is zero, hence the symbolic notation $f'(x) = 0$, to find stationary points. Here, as Hölzl (2001) asserts, "drag mode develops in mutual dependence with the ability to grasp a mathematical situation – a learning process that is characterised by different layers of conceptions" (p. 83). A teacher may find difficulty in demonstrating such multiple, dynamic visual representations using static diagrams. By providing appropriate guidance, teachers not only lead students to convincing heuristic evidence in computer systems but also help them to understand why the proposition is true (de Villiers, 1998). This guided discovery conveys an insight that makes sense to students and they feel empowered.

2.3.2 *GeoGebra* as a DGS Tool

There are many DGS packages available for teaching and learning such as *Geometer's Sketchpad*, *Cabri* and *GeoGebra* (Bu et al., 2010; Ruthven et al., 2008). This study focusses on *GeoGebra* as a dynamic geometric software for two reasons. Firstly, *GeoGebra* is an open-source dynamic mathematics software which means it is available free of charge without any licensing issues. It allows teachers and learners to use it in any way they wish. Secondly, it is a platform-independent software which means that it runs on virtually any operating system as it requires only a Java plug-in. As observed earlier, the South African Department of Education also recommends the use of *GeoGebra* as a technological tool for teaching and learning mathematics.

Although this software was developed in the early 21st century to incorporate geometry, algebra and calculus, it has now been further developed incorporating other mathematical perspectives such as statistics, probability and three-dimensional graphics. *GeoGebra* has rapidly gained popularity among teachers and researchers around the world, as it is easy to use and combines many aspects of different mathematical ideas (Hohenwarter & Lavicza, 2007). Multiple perspectives in *GeoGebra* allow the integration of dynamic representations for a conceptually rich learning environment that supports the exploration, construction, and evaluation of mathematical models and simulations (Jones et al., 2009). It provides a resourceful, dynamic learning environment for mathematics teachers to integrate mathematical content and pedagogical strategies for the purpose of teaching mathematics for understanding (Bu et al., 2012, p. 91). It also offers a compelling opportunity for teachers to create interactive learning environments. It provides a learning environment where mathematics content and pedagogy are deeply intertwined in the practice of mathematics teaching and learning.

GeoGebra, a non-commercial software package, having the potential to impact teaching and learning, has led to the formation of many self-supporting user communities (Hohenwarter & Lavicza, 2007). The users of these communities provide and get extensive support and guidance, especially to overcome technical and teaching difficulties. The *GeoGebra* community has a large number of international users and developers – from almost 200 countries – developing and sharing teaching materials. Bu et al. (2012) observe that participants of the *GeoGebra* community not only actively invent and experiment with new ways of teaching mathematics but are themselves learning or relearning mathematics through their applets and manipulatives (which will

be deliberated upon in the following section). The technical advantage is that these applets can be easily embedded in a web-based application (Duffin, 2010), because of their small size and ability to run on any JavaScript-enabled browser. *GeoGebra* applets are available online which can be accessed across the world by teachers, learners and researchers alike (*GeoGebra*, n.d.).

Tasks in *GeoGebra* are useful when learners, for instance, are encouraged to uncover the hidden relationships (Hölzl & Schäfer, 2013). The interactive nature of dynamic applets have the potential to promote students' understanding, which would have been difficult in a static environment. With the help of *GeoGebra*, teachers can ask open-ended questions like 'what-if' and 'what-if-not', thereby supporting and guiding students to discover properties themselves. In *GeoGebra*, there are tools available that enable users to construct objects in many different ways.

For example, the parallel line tool is used to construct parallel lines; the polygon tool is used to construct polygons, etcetera. *GeoGebra* allows the flexibility to enable or disable certain tools in a dynamic worksheet. At one end we can create an applet with no tools available, while at the other, students can be given reasonably open-ended problems with tools to construct and solve them. Teachers need to consider the pedagogical implications and learning opportunities available when setting tasks using *GeoGebra*. Sherman (2010) suggests that students' prior mathematical knowledge and their skill with *GeoGebra* must also be looked into when such tasks are given to students.

With *GeoGebra*, abstract ideas may be reified not only by making connections between the real and abstract world, but also between different mathematical concepts. When engaging with *GeoGebra*, students are able to move between representations fluently with understanding. Researchers (Naidoo & Govender, 2014; Ruthven et al., 2008; Sherman, 2010; Stols & Kriek, 2011) concur that *GeoGebra* is a powerful teaching and learning tool that can help conceptual development and enhance mathematics teaching.

2.3.3 *GeoGebra* Applets as Manipulatives and Visualisation Tools

One of the important features of *GeoGebra* is that it provides a visual interface to create interactive applets. Applets offer a powerful platform for mathematics teachers to develop content-rich pedagogical strategies for mathematical instruction. These applets can be integrated into the mathematics classroom in the form of virtual manipulatives. Moyer et al. (2002) define virtual manipulatives as a web-based, visual, dynamic interactive environment which helps to construct

mathematical knowledge. Virtual manipulatives take the form of applets', 'mathlets', (Durmuş & Karakirik, 2006) or 'dynamic worksheets'. The mere presence of dynamic software does, however, not guarantee the acquisition of knowledge. Moyer (2001) argues that teachers play a crucial role in creating interactive mathematics environments that provide students with representations that enhance thinking. Teachers need to be skilled in working with virtual manipulatives with interactive capabilities to facilitate opportunities for constructing knowledge in their classrooms.

Rivera (2011) confirms that the conceptual content of the visual manipulatives matters as individuals need to establish a valid and consistent mapping between the visual representation and the corresponding concept. Sarama and Clements (2009, p. 147) argue on affordances of computer manipulatives as “embodying the processes children are to develop and internalise as mental actions.” The flexibility of virtual manipulatives allows learners to mirror mental ‘actions on objects’ thereby develop ‘integrated-concrete knowledge’. Sedig (2009) refers to this flexibility of computer representations as ‘malleable’ and ‘interactive’. We have already seen that visualisations in mathematics are hidden in the mathematical structures. Sedig suggests that one of the ways of decoding the latent meaning and structures is “adjusting the visualisation to the needs of the users is by designing it to be malleable – i.e., interactive” (p. 344) by performing epistemic actions like “rearrange and reorganise it, add to it, look at it from different perspectives, focus on and look for specific elements within it” (p. 345). In concurrence, Villarreal (2000) suggests that “the computer has come to restore the value of the process of visualization in mathematics education” (p.3). Thus, dynamic interactive applets make hidden mathematical properties visible (Arcavi, 2003) , providing opportunities for exploring relationships (Duffin, 2010; Sedig, 2009). We have thus seen that *GeoGebra* applets can facilitate experimentation and hence create an interactive environment of ‘learning by doing’. Nevertheless, Sarama and Clements (2009) argue that these manipulatives are only useful when used in “comprehensive, well-planned, instructional settings. Their physicality is not important—their *manipulability* and *meaningfulness* make them educationally effective” (p. 148).

In mathematics, these manipulatives can be used in a wide variety of topics. Figure 2.5 below, a typical applet, is an example of a dynamic *GeoGebra* visualisation that allows teachers to guide students to discover the relationship between the angle at the centre of a circle and the angle at the circumference of the circle subtended by the same arc BC . The learner can drag the slider α to

change the size of angle $B\hat{O}C$ and observe what happens to angle $B\hat{D}C$, hence verifying its relationship with angle $B\hat{O}C$. By engaging with this activity, learners may discover that in spite of changing the magnitude of the angles, the relationship between $B\hat{O}C$ and $B\hat{D}C$ is preserved i.e. that the angle at the centre of a circle is always twice the angle at the circumference of the circle. The above applet can be accessed online at <http://ggbm.at/t2Qf2dDV>. Teachers can thus use dynamic visualisation features of geometric software to help students formulate mathematical ideas through discovery and experimentation (Freiman et al., 2010). The interactive and dynamic visualisation capabilities of technological devices may foster rich constructions and develop visual thinking (Alsina & Nelsen, 2006). The dynamic visual aspects of *GeoGebra* applets can also complement a discourse with a visual demonstration which would have been difficult to convey in a traditional manner of text and the static medium of teaching and learning (Gage, 2010).

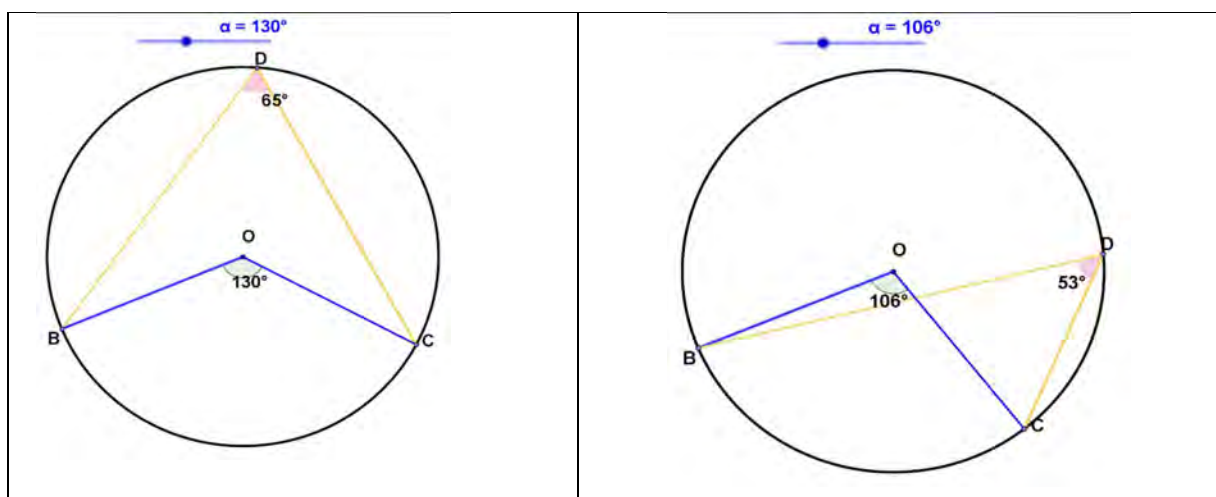


Figure 2.5: Angle at the centre applet

Durmuş and Karakirik (2006) argue that the use of virtual manipulatives not only increase students' conceptual understanding and problem-solving skills but also promotes their positive attitudes towards mathematics. Manipulatives can provide 'concrete experiences' that focus attention and increase motivation. Virtual manipulatives can provide an interactive environment for solving problems by making connections between mathematics concepts, operations and the real world. Teacher demonstration of manipulatives alone is, however, insufficient for conceptual understanding. Learners need to be allowed to work with these manipulatives and discover the mathematical concepts and relations themselves. Virtual manipulatives are particularly well suited for an interactive environment where students can pause, reflect, experiment and solve problems

to make connections between mathematical concepts because the devices on which the manipulatives are operated can be stopped at any time. It cannot be overemphasised that every learner should get an opportunity to engage with manipulatives. By doing so, they enrich their understanding of concepts and relations. Giving credit to Norman (1993), Durmuş and Karakirik (2006, p.) argue that computers “help us not only to make sense out of what we have experienced and what we know but also to compose new knowledge by adding new representations, modifying old ones, and comparing the two” (p 3).

According to Duffin (2010), one the strengths of using applets is that visual representations can be linked to help draw attention to the relationships between two representations and thus deepen understanding. In a *GeoGebra* applet, we can link all the forms of an equation of a straight line ($ax + by + c = 0$; $y = mx + c$) and its graph on the same screen. Thus, it supports and enhances the role of visual representations in learning with understanding. Stols and Kriek (2011) also recognise that one of the potentials of the dynamic environment is the ability to focus on interlinked parts of a geometric shape – thus enriching visualisation.

2.3.4 Limitations of Technology

2.3.4.1 Limited implementation

For the majority of teachers, merely providing technology is insufficient for the successful integration of technology into their teaching (Cuban et al., 2001). Teachers generally have the optimistic attitude that technology-supported teaching is powerful, and that it will promote visualisation and student understanding of mathematics. Teachers are generally cautious that, “[t]echnology makes mathematics teaching easier” (Gueudet & Trouche, 2011, p. 400). Often they feel that their skills and knowledge about technology are limited or inadequate, and may refrain from using technology in their teaching (Hennessy et al., 2005; Stols et al., 2015). From my observations, many of today’s teachers and students have unlimited access to computers, laptops and tablets. But despite technology becoming increasingly available to teachers, they often do not make optimal use of this technology for educational purposes (Stols et al., 2015). In particular, Stols et al. find that the use of *GeoGebra* in South African classrooms is not investigated at all. Technological devices are mostly used as for consumer purposes but not often for effective teaching and learning purposes. Teachers use them for administrative purposes and monitoring of students’ assessment, but not for teaching and learning. As mentioned earlier, there is no dearth

of teaching and learning resources available on the information highway, but the effectiveness of these materials is a serious concern. Teachers do not always have the time and expertise to search and screen the millions of internet sources for suitable quality materials (Stols et al., 2015).

2.3.4.2 Software Limitations

DGS may not be suitable for all tasks, and its use depends on the teacher. Most DGS, including *GeoGebra*, have shortcomings. Berger (2011) warns that “opportunities for learning mathematics in a technological environment may be constrained if the design of the task is not appropriate” (p. 111). Ruthven (2003) finds ‘approximation’ as a shortcoming in DGS, as he contends: “while it is possible to specify the degree of rounding with which measures are displayed, this rarely coincides exactly with the screen representation of objects” (p. 11). When calculating the area of a triangle, the lengths of the altitude and sides are displayed on the screen rounded off by two decimals, but the original values are preserved in the system. Consequently, when calculating the area of the triangle, the user value often varies from that of the system.

Another instance where *GeoGebra* is limited is the plotting of graphs of discontinuous functions at a point(s), so $\frac{(x^3-1)}{(x-1)}$ is sketched as a parabola without the ‘hole’ at $x = 1$. However, it sketches appropriately $\frac{(x^3-1)}{(x+1)}$ appropriately. Teachers may be aware of such situations in DGS where representations are inconsistent with our knowledge of mathematics. Hence, Berger (2011, p. 112) surmises “[i]t is thus desirable that teachers, researchers and task designers critically evaluate the affordances and constraints of computer-based tasks before they are implemented in the classroom.”

A recent research project, published at MIT by Carter, Greenberg, and Walker (2016) found in their randomised experiment that computer devices can have a substantial negative effect on academic performance. This adverse effect may be due to the ineffective use of this technology by the teacher. It is thus critical to be aware of the vital role of technology in classrooms and appropriate professional development programmes like GLIP. Easily accessible and modifiable teaching materials are essential to integrate technology into teaching practices.

2.4 TEACHING PROFICIENCY

One of the general aims of the national curriculum of South Africa (Curriculum and Assessment Policy Statement [CAPS], 2011) is “[a]ctive and critical learning: encouraging an active and critical approach to learning, rather than rote and uncritical learning of given truths”, and

specifically the teaching and learning of mathematics aims to develop, “an appreciation of beauty and elegance of mathematics”, and “deep conceptual understandings in order to make sense of mathematics” (p. 12). Spillane (2000) also agrees that students need to appreciate mathematical activity as more than computation. For students to achieve the above-mentioned standards of performance, Simon (2001) suggests that “the teacher's role is to guide and afford students' direct apprehension of mathematics” (p. 163). In the following section, I will discuss some perspectives of proficient teaching approaches offered by different scholars and researchers, that could be used to guide teachers' instructions enabling their students to make sense of mathematics.

2.4.1 Perspectives of Proficient Teaching

Teaching mathematics is not doing mathematics in front of the students on the chalkboard, it involves additional mathematics knowledge, competencies and skills (Hill, Sleep, Lewis, and Ball, 2007). Shulman (1986, 1987) propounded this as ‘pedagogical content knowledge’; Ball et al., (2008) refer to as ‘specialised content knowledge’; Ma (2010) offers ‘profound knowledge of mathematics’, and Kilpatrick et al. (2001) and Schoenfeld and Kilpatrick (2008) refer to as ‘mathematical teaching proficiency’.

For Shulman (1987) the knowledge base of a teacher represents “the blending of content and pedagogy into an understanding of how particular topics, problems or issues are organised, represented and adapted to the diverse interest and abilities of learners, and presented for instruction” (p. 8). Effective teaching takes place when a teacher can transform his or her content knowledge into pedagogically powerful teaching strategies. Shulman (1986) distinguishes three categories of content knowledge required for teaching: a) subject matter content knowledge; b) pedagogical content knowledge; and c) curricular knowledge. He proposes pedagogical content knowledge (PCK) as “the particular form of content knowledge that embodies the aspects of content most germane to its teachability” (p. 9). It is about representing ideas and formulating subject matter to make it comprehensible to high school students. Here the teacher draws upon a variety of instructional approaches like examples, non-examples, manipulatives, simulations, or inquiry methods that can lead to the educational outcomes identified by the teacher. An important role of a teacher is to probe and provoke students' creativity. Teaching includes a series of activities during which it provides opportunities for learning. Shulman (1987) argues that a knowledge base lies in the capacity of a teacher to understand student representations and to

respond to their ideas. Thus, Kilpatrick et al. (2001) argue that what makes teachers proficient is the skilled use of knowledge in teaching and learning.

Ball et al. (2008, p. 398) argue that “Teachers must know rationales for procedures, meanings for terms, and explanations for concepts.” Proficient teaching requires a special kind of skill or expertise for explaining to the learners what they do and why they do it. Teaching becomes effective when representing the meaning of the content that shows what the algorithm means and why the algorithm works. Ball et al. (2008) refers to this as ‘mathematical knowledge for teaching’, which has been identified, refined and rooted in the works of Shulman on the broader concept of content knowledge for teaching. It is the specialised knowledge of mathematics required to carry out the teaching of mathematics. Ma (2010, p. 18) agrees that “[s]pecial features of a teacher’s subject matter knowledge are derived from the task of promoting student learning.” Ball et al.’s (2008) analysis of teachers’ practice indicates that:

The mathematical knowledge needed for teaching is not less than that needed by other adults. In fact, knowledge for teaching must be detailed in ways unnecessary for everyday functioning. In short, a teacher needs to know more, and different, mathematics – not less. (p. 396)

For example, arithmetic patterns and geometric patterns are used for teaching simple interest and compound interest. An accountant uses simple interest and compound interest formulae, quite oblivious of their links with these underlying patterns. This illustrates the unique mathematical processes and tasks that teachers must do in order to teach mathematics, “yet foreign to most well-educated adults” (Ball et al. 2008, p. 398). To facilitate learning, teachers make explicit the connections between and among mathematical topics that remain hidden for non-teachers (Ma, 2010). Thus, general mathematical knowledge or the ability of an educated adult may be insufficient for teaching mathematics. Hill et al. (2007) concur that mathematically literate people can solve real-life problems but formulating real-life situations to illustrate mathematical reasoning and insight is limited to teachers. The domain of specialised content knowledge entails the unpacking of mathematical knowledge that is taught to students as they develop understanding. Thus, a teacher must understand the structures of the domain or idea being taught and the principle ideas and skills required in the domain, so that new ideas can be added (Shulman, 1987).

In most countries of the world, teachers teach across numerous grades. Schoenfeld and Kilpatrick (2008) argue that this practice provides teachers with a sense of curricular continuity and

mathematical depth which shapes their lessons . It is essential that a teacher knows what aspects of the domain have been taught and where the content leads to. For example, drawing a graph to represent simple interest and compound interest growth over a period of time in earlier Grades 8 or 9, will then provide a window to understanding functions and graphs that students will learn in Grade 10. Schoenfeld and Kilpatrick (2008) observe that the more advanced teacher's mathematical knowledge is, the better the teacher can make connections between mathematical topics. This form of knowledge Ball et al. (2008, p. 403) refer to as horizon knowledge – “an awareness of how mathematical topics are related over the span of mathematics included in the curriculum.” Proficient teacher knowledge of school mathematics should be broad and deep (Schoenfeld & Kilpatrick, 2008). It is broad in the sense that proficient teachers can represent concepts in multiple ways and make appropriate and rich connections to other topics. The example of graphic representation of simple interest as a straight line graph and compound interest as an exponential graph to would enable students to distinguish easily the difference between these two types of interest, and also to provide an insight into the fact that book-value in compound decay (reducing balance depreciation) can be never be zero, an asymptote of the function. They suggest that the characteristics of proficient teachers are:

- having a broad and connected mathematical content knowledge;
- being aware of learner's prior knowledge;
- being aware of the entire curriculum sequence;
- being able to introduce a new mathematical idea effectively;
- being able to develop understanding in multiple ways;
- help to work with misunderstanding to develop correct understanding.

By adopting the above practice, a deeper and broader understanding of core mathematical ideas evolves. This awareness allows teachers to plan for effective ways to introduce a new mathematical concept so that it fits with existing knowledge. Shulman (1987, p. 9) observes “this view of sources of content knowledge necessarily implies that the teacher must have not only *depth* of understanding with respect to the particular subjects taught, but also a *broad* liberal education that serves as a framework for old learning and as a facilitator for new understanding” (italics added for emphasis). Hennessey, Higley and Chesnut (2012) emphasise that links to prior

knowledge and multiple representations in teaching mathematics not only strengthens learning experiences but also increases retention and application.

Proficiency in teaching is related to effectiveness, consistently helping students to learn worthwhile mathematical content. Kilpatrick et al. (2001) propose five interwoven strands of teaching for mathematical proficiency, arguing that effective teaching practice can be developed and nurtured (p. 369). Teachers' mathematical teaching proficiency is crucial in developing students' mathematical proficiency. Teachers need to know mathematics in ways that enable them to help students learn. This specialised knowledge of mathematics includes conceptual underpinnings of ideas that help students learn mathematics with understanding.

2.4.2 Strands of Teaching Proficiency

In the context of proficient teaching, the five strands proposed by Kilpatrick et al. (2001) are:

- conceptual understanding of the core knowledge required in the practice of teaching;
- *fluency* in carrying out basic instructional routines;
- *strategic competence* in planning effective instruction and solving problems that arise during instruction;
- *adaptive reasoning* in justifying and explaining one's instructional practices and in reflecting on those practices so as to improve them; and
- *productive disposition* toward mathematics, teaching, learning, and the improvement of practice (p. 380).

Due to the limited scope of this study, teaching proficiency with respect to using DGS will only be analysed against the first two strands articulated above.

2.4.2.1 Conceptual Understanding of Core Knowledge (CU)

Kilpatrick et al. propose that three kinds of knowledge are crucial for teaching school mathematics: knowledge of mathematics, knowledge of students and knowledge of instructional practice. Knowledge of mathematics includes knowledge of mathematical facts, concepts and procedures, and the relationships among them. "Teachers certainly need to be able to understand concepts correctly and perform procedures accurately, but they also must be able to understand the conceptual foundations of that knowledge" (p. 371). A teacher must be aware that mathematical

ideas can be represented in multiple ways. Teachers' mathematical knowledge and their capacity to use it in teaching are crucial in developing students' mathematical proficiency. Teachers need to know how students think and be aware of their conceptions and misconceptions. One of the common mistakes that students make in algebra, for example, is that $(a + b)^2 = a^2 + b^2$. The learners overlook the binomial product of expressions, $(a + b)(a + b)$, and confuse it with the laws of exponents, $(a \cdot b)^2 = a^2 \cdot b^2$. The teacher needs to identify such common errors and diagnose reasons as to why this mistake is so prevalent. The teacher may provide counterexamples to clarify and confirm that $(2 + 5)^2 \neq 2^2 + 5^2$. The teacher also needs to develop a set of responses for helping students in removing such misconceptions. An example of an appropriate response would be to use visualisation processes to represent $(a + b)(a + b)$ as the area of a rectangle, as in Figure 2.6.

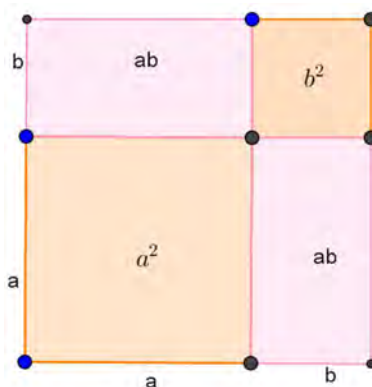


Figure 2.6: Visualising the square of a binomial

Generating and testing examples and counterexamples in teaching involves mathematical reasoning (Hill et al., 2007; Shulman, 1987). Knowledge must be connected so that it can be used intelligently. This integrated knowledge of mathematics has implications for teaching. Proficient teaching entails thinking and identifying multiple ways of representing concepts. A proficient teacher needs to make use of all the resources, such as DGS, that can facilitate this integrated knowledge to develop students' mathematical proficiency. A strong foundation of conceptual understanding of mathematics makes it possible for teachers to understand and use students' mathematical solutions, explanations, and questions constructively.

2.4.2.2 Instructional Fluency (IF)

Instructional fluency refers to the knowledge and skills acquired by a teacher and also knowing when and how to use procedures accurately, efficiently and appropriately in a given situation. A teacher who has acquired a repertoire of teaching approaches can readily draw upon them as they interact with learners in teaching mathematics. Proficient teachers have a clear vision of the goals of instructions, Kilpatrick et al (2001) argue:

They need to be able to use their knowledge flexibly in practice to appraise and adapt instructional materials, to represent the content in honest and accessible ways, to plan and conduct instruction, and to assess what students are learning. (p. 369)

These routines help in developing mathematical behaviours, like how to respond to a learner with serious misconceptions or to deal with learners who lack basic skills. Teachers should be aware of common misconceptions and how to deal with them. Teaching also entails an explanation of procedures, illuminating students' misconceptions and addressing these misconceptions. Often, students develop procedures of computation which may not be mathematically valid (Hill et al., 2007). Teachers must be able to examine the procedure and judge whether the students' methods are mathematically appropriate; however, spotting an incorrect procedure or answer alone is insufficient: interpreting student errors and evaluating alternative algorithms is not all that teachers do (Ball et al., 2008). A proficient teacher must be able to figure out what steps the student might have taken to produce this error, provide reasons and then determine an appropriate response. Hill et al. (2007, p. 123) surmise “[t]eaching requires making the content accessible, interpreting students' questions and productions, and being able to explain or represent ideas and procedures in multiple ways.”

Given that teachers have access to several approaches to teaching, if one does not work, they should be able to switch to another approach. According to Ball and Bass (2000), fluency in teaching mathematics is the ability of a teacher to “deconstruct mathematical knowledge where elemental components are accessible and visible” (p. 98). Although abstraction is central in mathematics, it may obscure the roots of the knowledge. Ball and Bass (2000) observe that a proficient teacher must be able to work backwards from abstract ideas and unpack their constituent elements. Thus, proficiency in teaching involves a unique capacity to understand and appreciate a student's insight and deconstruct highly abstract knowledge, making hidden elements visible.

Teaching requires mathematical knowledge and the design and sequence of mathematical tasks for instruction. Teachers should know which examples to start with and which examples to use to take students more deeply into the content, understanding, as Kilpatrick et al. (2001, p. 381) argue “what students understand about those ideas, or how they learn them.” The combined knowledge of teaching and mathematics enables a teacher to identify what different methods and procedures are appropriate to teach a specific idea (Ball et al., 2008). Teachers evaluate the instructional

advantages (and possibly are aware of the disadvantages) of representations to link the underlying mathematical structures. Thus, in describing the practise of a mathematics teacher, Ball et al. (2008, p. 401) suggest that “[d]uring a classroom discussion, a teacher must decide when to pause for more clarification when to use a student’s remark to make a mathematical point, and when to ask a new question or pose a new task to further students’ learning. Each of these decisions requires coordination between the mathematics at stake and the instructional options and purposes at play.”

Ma (2010) accentuates that a teacher with a profound understanding of mathematics is not only aware of the conceptual structure and underlying attitudes of mathematics, but is also able to teach these to students. Teachers with weak a conceptual knowledge of mathematics tend to develop only procedural fluency of learners. A teacher whose knowledge is rooted in only procedures cannot “describe a conceptually directed teaching strategy” (p. 34). However, a teacher with a profound understanding of a topic knows not only alternative ways of calculation but also is aware of why these procedures make sense for calculation. Teachers’ comprehensive knowledge of a topic may contribute to students’ opportunities to learn it.

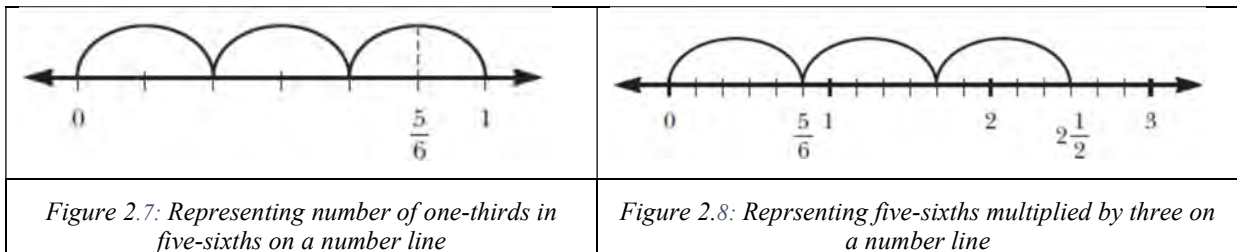
2.4.3 Teaching Proficiency in Relation to Visualisation and the Use of ICT.

Teaching is a complex activity with interrelated components (Kilpatrick et al., 2001), and visualisation is crucial in developing mathematical ideas. Proficient teachers are able to unpack mathematical content and make visible to students the ideas behind the procedures. Ball et al. (2008, p. 400) argue that “[t]eachers, however, must hold unpacked mathematical knowledge because teaching involves making features of particular content visible to and learnable by students.” Teaching entails decompressing mathematical knowledge for the students as they develop procedures with understanding. In the case of teaching the division of fractions, for example, $\frac{5}{6} \div \frac{1}{3}$, (Hill et al., 2007) maintain that teaching the algorithm of ‘invert-and-multiply’ is the compressed knowledge while decompressing it involves an explanation of the mathematical reasons tacit in this method. One of the possible rationales would be the application of students’ understanding that when we multiply both the dividend and the divisor with the same number, the quotient remains unchanged. Thus:

$$\frac{5}{6} \div \frac{1}{3} = \frac{5}{6} = \frac{5}{6} \times \frac{3}{1} = \frac{5 \times 3}{6 \times 1} = \frac{5}{2} \times \frac{3}{1} = 2 \frac{1}{2}$$

The symbolic rationale of this procedure is simple and does not necessarily illuminate students (Arcavi, 2003). He formulates that visualisation can accompany symbolic development. Hence, we may prompt questions like ‘How many $\frac{1}{3}$ ’s are there in $\frac{5}{6}$?’ Or pose a story problem – ‘ $\frac{1}{3}$ of an oil bottle weighs $\frac{5}{6}$ kg, then how much is the total weight of oil in the bottle?’ Figure 2.7 is the visual representation of ‘how many $\frac{1}{3}$ ’s in $\frac{5}{6}$ ’. Here in the number line, the area between 0 and 1 is divided into six equal parts, and two parts are $\frac{1}{3}$. We see from the figure that two full jumps and half a jump of $\frac{1}{3}$ make $\frac{5}{6}$. Thus, there are two-and-a-half $\frac{1}{3}$ ’s in $\frac{5}{6}$. Figure 2.8 is the representation of $\frac{5}{6}$ kg as the one-third of the bottle. Three one-thirds of five-sixths put together will give the full weight of oil in the bottle, which is $2\frac{1}{2}$ kg. Putting it differently, we have done this by adding five-sixths thrice or $\frac{5}{6}$ multiplied by 3, ie. $\frac{5}{6} \times \frac{3}{1}$. It is equivalent to the algorithm ‘invert-and-multiply’.

<https://www.geogebra.org/m/sxvvrnRw>.



The concrete image of the problem and its solution can be “an essential factor for creating the feeling of self-evidence”, thus visualisation becomes necessary “to see the unseen in symbols and words” Arcavi (2003, p. 220). The deliberate use of ‘unpacked knowledge’ – as Presmeg (2006) argues – the inclusion of non-essential pictorial diagrams prompts students’ use of intuition to support their visual thinking. Relating a mathematical problem to a real-world situation, as seen in the above example, promotes visualisation according to Presmeg (2006). He observes that “the essence of teaching that promotes visualisation in classrooms is captured in the word connections” (p. 214) Again, Arcavi (2003), Presmeg (2006, 2014) and Schäfer (2016) have stressed the need for multiple representations for effective use of visualisation in mathematics classrooms. In their

teaching of mathematics, teachers must have knowledge of visual methods in mathematics in ways that allow them to unpack ideas. Shulman (1987, p. 7) surmises that ways of teaching by representing ideas are “so that unknowing come to known, those without understanding can comprehend and discern, and the unskilled can become adept.”

Visualisation and simulation are the primary advantages of using dynamic software that supports the teaching and learning of abstract mathematics concepts (Naidoo & Govender, 2014). Teachers can play an essential and active role in analysing, designing and developing applets to be used by the learners effectively. Naidoo & Govender (2014) assert that teachers’ involvement in observing learners’ engagement with applets is essential, whether or not they are enthusiastic and critical in their use of tools. Teachers should be able to integrate dynamic mathematics software into their practice, thereby enabling learners to construct and visualise mathematical ideas. Bu et al. (2010) assert that engaging students in cognitive tasks with *GeoGebra* applets, for example, reinforces the importance of visualisation of mathematical concepts.

Examining different teaching approaches using technology, Kendal and Stacey (2001) observe that teachers typically make their own pedagogical choices about how to incorporate technology into their classrooms. Some teachers embrace technology as it enables them to increase their repertoire of different ways of teaching rules and procedures for student academic achievement, while other teachers find the capabilities of technology superior for developing conceptual understanding of mathematical ideas. Bu et al. (2012) propose that the role of *GeoGebra* is a pedagogical tool to integrate content and pedagogy in mathematics teaching and learning. It provides an opportunity to critically reflect on teaching practices and, “we are relearning or even diagnosing our own understanding of mathematics” (p. 92).

Proponents of DGS also suggest that it has the potential to promote a discovery method of teaching and learning. Hoyles and Noss (2003) affirm that teachers should use DGS as a pedagogic tool for the exploration of mathematical concepts and ideas for any mathematical domain. However, I have argued earlier that for effective implementation, technological tools should be designed with teachers. Ruthven et al. (2008, p. 299) agree that “[t]eachers who have worked closely with the designers and advocates of these forms of geometrical software have developed approaches to teaching and learning which reflect this association of the technology with an exploratory pedagogical orientation.” Hennessy et al. (2005, p. 187) recommend, based on their research with

English teachers, “a coherent and supportive community of practice associated with using ICT effectively, regularly, and consistently.” The findings of Gueudet and Trouche (2011) corroborate that within the frame of the community of practice where teachers collectively work to prepare lessons using DGS tools, indicates an ‘evolution’ in teachers’ pedagogy. For Gueudet and Trouche, the perspective of ‘teacher empowering teacher’ as a way of teacher education about technology appeared powerful in bridging the gap between having access to resources and appropriating them for usage in the classroom.

Thus GLIP is an important platform and forum that offers an opportunity for teachers to plan their technological lessons focussing on how technological-based tools can be implemented in classrooms that positively promotes the teaching and learning process of mathematics.

2.5 THEORETICAL FRAMEWORK – CONSTRUCTIVISM

A constructivist theoretical perspective informs this study. Boaler (2009) argues that a traditional approach to teaching is characterised by chalk and talk methods where the teacher typically explains and demonstrates the mathematical content of a lesson and the students watch, consume and practice the problems, mostly in silence. However, Jaworski (1994) argues that whereas this method of direct instruction develops only ‘lower-level mathematical skills’ like computation, deep conceptual understanding requires ‘higher-order skills’. The key idea of constructivism is that children construct their knowledge. Thus, Shulman (1987) argues that teaching involves an exchange of ideas and that students do not passively grasp ideas. Teachers need to present ideas in order to provoke the constructive process of the student. Jaworski (1994) also asserts that learners do not only absorb ideas presented by their teacher but rather that they create their knowledge. When provided with opportunities, students can make their own discoveries and representations. Shulman suggests that the role of teacher is to then respond actively and creatively to those student representations. Habib (2012) argues that substituting traditional teaching methods with active approaches, which is inherent in a constructivist approach, exploits the intuitive and applied activities related to life. This active approach aligns well with teaching with ICT technologies as it requires hands-on teaching and learning strategies, where the teacher and learner are actively engaged in constructing knowledge.

Goldin (1990) states that in a constructivist classroom, learning involves a constructive processes. A constructive process is one in which an individual organises and restructures his/her own

experience through their construction of knowledge. The constructivist approach views that learners already have a certain amount of knowledge – they do not come to class as empty vessels to be filled. The transfer of knowledge is therefore not one of filling empty vessels, but one of individual knowledge construction. Understanding is acquired through the creation of knowledge and learners take ownership of their learning. My emphasis is on teachers' understanding children's development of mathematical knowledge, based on constructivist findings. As Wright (2000, p. 144) indicates "Teachers are viewed as active learners and are encouraged to examine their own teaching practices and to focus on the mathematics children can do and how they do it." When teachers teach mathematics making more use of hands-on activities and promoting student autonomy and responsibility (Confrey, 1990), it may impact significantly on student learning.

2.5.1 Key Ideas of Constructivism

The literature on the theory of constructivism differentiates between radical constructivism and social constructivism (see for e.g. Hennessey et al., 2012; Jaworski, 1994).

Based on Piaget's cognitive adaptation, von Glasersfeld (1990) proposed two principles of radical constructivism:

- 1) Knowledge is not passively received either through the senses or by way of communication. Knowledge is actively built up by the cognising subject.
- 2) The function of cognition is adaptive, tending towards fit or viability.

In simpler terms, the first principle says that we all construct our own knowledge (Jaworski, 1994). The learner's new knowledge draws on prior knowledge and experiences – learners are not blank slates. Constructivism is a theory of knowing that attempts to show that knowledge can only be generated by experience. It implies that learners construct their knowledge by actively participating in a classroom situation. Wright (2000) agrees that knowledge is the result of learners' activities rather than of the passive reception of instruction.

The second principle says that an individual learns by adapting. From a constructivist point of view, mathematical concepts are constructed that 'fit' or are viable with our real-world experiences. Giving credit to Piaget, von Glasersfeld (1981, 2000) argues that an individual constructs knowledge as an adaption to their subjective experience. Knowledge construction and adaptation are results of cognitive structuring, acknowledged by Piaget's genetic epistemology. Knowledge

results from individual construction by modification of experiences or ideas. This principle emphasises that it is only possible to know the world through experiences.

However, Goldin (1990) disagrees with the above view of constructivism. He argues that “[t]he hypothesis that the construction of knowledge by human beings takes place possibly involving several developmental stages - when offered as part of a scientific theory of competence acquisition is logically independent of radical constructivist epistemology” (p 42). The above two principles do not, however, take into account the contextual influences on learning or recognise the individual differences in student understanding. Hardy and Taylor (1997) also argue that tenets of radical constructivism do not offer an adequate explanation of the influence of socio-cultural aspects of learning and the role of language in a classroom environment.

Taylor and Williams (1993) evolve a constructivist related theory of learning based on three principles, where the third principle derives from social perspectives and linguistic influences on learning. They consider that “[t]his principle acknowledges the sociocultural and socioemotional contexts of learning, highlights the central role of language in learning and identifies the learner as an interactive co-constructor of knowledge” (p. 135). The negotiation of concepts influences the students’ mathematical development, and knowledge is constructed intersubjectively. Elaborating on intersubjectivity, Jaworski (1994, p. 211) considers that “[i]ntersubjective or 'taken-as-shared' knowledge can be seen as a product of such interaction where participants seem to agree on certain interpretations represented through discourse and non-verbal communication.” Through negotiation and social interaction, individual knowledge may be challenged, and new knowledge constructed.

Identifying social interaction as an essential component in the process of teaching and learning, von Glasersfeld (1990, p. 26) observes that the constructive approach of teaching “takes the form of encouraging and orienting students’ constructive effort rather than curtailing their autonomy by presenting ready-made results as the only permitted path.” Then the teaching “aims at the students conceptual fit with the consensual domain of the particular field, a fit which, from the teacher's perspective, constitutes understanding.”

Vygotsky (1978) puts forward the notion of the *zone of proximal development* (ZPD) to explain human learning:

The distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem-solving under adult guidance, or in collaboration with more capable peers. (p. 86)

Children are capable of doing much more in a collective activity or under the guidance of an adult but limited to their developmental level. The essential aim of school learning is to create these intellectual levels

...learning awakens a variety of internal developmental processes that are able to operate only when the child is interacting with people in his environment and in cooperation with his peers. Once these processes are internalized, they become part of child's independent developmental achievement. (Vygotsky, 1978, p. 90)

Vygotsky implies that the role of a teacher, through appropriate instructions and classroom settings, is to provide learning opportunities for a child so that he or she can achieve higher conceptual levels than which would have been impossible to achieve otherwise. Shulman (1987) also maintains, as we have already observed in earlier paragraphs, that the teachers' interest in students' constructions can throw light on their conceptual structures. Jaworski (1994) aligns with this theory, using the metaphor of scaffolding that could enhance students' ability to handle problems without relying on the teacher. She interprets scaffolding "in terms of a teacher's offering of strategies for thinking and learning, rather than for grasping a particular skill or concept" (p. 31).

I concur with Hardy and Taylor (1997, p. 140) that "it is worth realising that neither social nor the individual components of learning necessarily supersedes the other." It is rather, as Jaworski (1994, p. 212) perceives, "individual construction, within a community rich in socio-cultural influences." The constructivist perspective on teaching and learning of mathematics envisions "both teacher and students in an ongoing collaborative endeavour" (Taylor & Williams, 1993, p. 136) that "contribute significantly to individual students' construction of meaning" (Jaworski, 1994, p. 211). As teachers interact both with the learners and with the computer tools, they are involved in a mathematical experience and may make an important contribution to knowledge construction.

2.5.2 Constructivism and Technology

Technology provides an opportunity to do and teach mathematics. Sutherland (2007) states that many educational technologies such as *GeoGebra* have been developed for teaching and learning mathematics as they enables students to develop 'symbolic tools' for solving mathematical

problems. Such tools become ‘internalised’ to the extent that they can be drawn upon later in other problem-solving situations. The process of internalisation is central to constructivism (Marti, 1996, p. 60). Internalisation is the development of knowledge involving a simultaneous re-organisation of internal mental spaces and external forms of knowledge.

Hoyles (2005), in the context of technological teaching and learning, suggests that mathematical concepts need to be inextricably interwoven with applets. When planning a task for the learner that facilitates the construction of mathematical ideas, a teacher should pay careful attention to include the two points mentioned below. First, the tools embedded must be ‘just enough’ to illuminate structures and relationships while not solving the task completely. Second, the tasks should be developed so that they foster students’ engagement with mathematical ideas. Drawing on constructivist ideas, Hoyles (2005, p. 141) observes that the student-computer interchange engages learners “in a dialectical relationship of action on the objects and thought.” Thus, dynamic software tools can provide insight into mathematical structures and relationships. When learners are active participants in learning using computer tools, they learn better and retain information longer. *GeoGebra* applets offer students an opportunity to do more than listen to the teacher during class. Hoyles (2005) also observes that virtual manipulatives are mediators and form a crucial part of internalising knowledge. Mathematical meanings are inextricably interwoven with the computer tools which enable learners to identify mathematical invariants and make connections.

Following on Vygotsky’s ideas on the use of tools, Noss and Hoyles (1996) synthesise that meaning in mathematics can be constructed and shared through interaction with computer tools. They argue, “[t]he computer is not only a set of tools which defines allowable actions and their expression; it plays a role in communicating the actions, sharing and re-negotiating mathematical expression and facilitating the (co-)construction of mathematical meanings” (p. 228). Furthermore, from Vygotsky’s perspective of the use of tools, Bu et al. (2012) argue that *GeoGebra* has gradually evolved from a technical tool that supports teaching and learning into a pedagogical tool that facilitates a mathematics teacher’s classroom practise.

Noss and Hoyles (1996, p. 154) demonstrate in their research with teachers using computers (Microworld), “how meanings for the computer and mathematics were created by teachers, and how these shaped the relationship between the computational media and the mathematical knowledge they were responsible for teaching.” In their research, the teachers interacted with the

computer tools and reconstructed the classroom settings before taking on board, with the students, those aspects of the mathematical meaning-making process. Noss and Holyes also note that teachers construct new ways of learnable and connected mathematics that the computer tools offer. Through active construction at the computer, conceptual learning of mathematical ideas occurs. Carpenter and Lehrer (1999) ascertain that when learners are given some control over the tasks they engage in and are allowed to reflect on their activity, learners develop a sense of ownership and have control over their learning. Thus, from a constructivist perspective, activities designed in applets involve students in doing things as well as making them think about what they are doing.

In a technology-based constructivist classroom, there is less emphasis on transmitting information, but more on developing students' understanding of concepts and their skills. Recent research work of Jaworski (2015) confirms that DGS can engage students in a deeper understanding of mathematical meanings. Instead of dispensing knowledge and focussing on the acquisition of established facts, technology enables teachers to design support programmes that can help students towards a greater depth of understanding of a concept. When applets and software programs like DGS are used in a classroom setting, teachers go beyond chalk and talk. However, Jaworski's students provided mixed responses to *GeoGebra*. One group of students, for example, asserted that in-depth mathematical meaning is unnecessary – “just because I understand maths better doesn't mean I'll do better in the exam” (p. 268). Another group of students reflected that using *GeoGebra* provided a dynamic visual representation, helping them to ‘spot patterns and trends’ that would otherwise have been missed. Promoting active learning promotes mastery of content with reasoning and understanding.

Dynamic mathematics software is a powerful teaching and learning medium and appears to create opportunities for creative thinking (Stols & Kriek, 2011). Stols (2007) further argues that new technologies can help learners to visualise difficult-to-understand concepts and also help teachers to create an active learning environment. In such environments, students do more than listen: they are encouraged to explore. When such technology is used in the classroom, the focus is on developing higher-order thinking skills where students are actively engaged in analysing and synthesising their knowledge.

Teaching mathematics with technology opens up new questions about the teacher's theory of learning and its impact on student learning (Kendal & Stacey, 2001). Their analysis of teachers

indicates that the pedagogical choices about working with technological tools are consistent with their teaching approaches. Different fundamental conceptions of mathematics associated with teaching approaches influence their particular choices while using technology – about what to emphasise, and how to incorporate the graphical, numerical and symbolic algebra capabilities of computer tools like DGS into their lessons. When a teacher focusses only on mathematical rules and procedures, he/she tends to ‘automatise’ computational procedures, demonstrating the use of technology by listing step-by-step routines and preferring symbolic representations. But when a teacher focusses on the conceptual understanding of mathematical ideas and student constructions of meaning, he/she encourages students to discover and develop their intuitive ideas. The teacher uses computer media to link different representations of mathematical ideas. The teacher arranges computer classrooms in which each student explores mathematical content and develops logical constructions of the concept as Kendal and Stacey (2001, p. 155) describe: “[e]ach student appeared to conjecture, analyse, negotiate meaning with other students in the class, make decisions, draw conclusions, and prove ideas”.

Students’ engagement in learning involves cognitive tasks and the manipulation of conceptual ideas. When technological tools are used effectively, students can authenticate their understandings (Jonassen & Strobel, 2006). The emerging technological tools, like DGS, with its inherent exploratory possibilities, may provide opportunities for a heuristic approach to constructing mathematical meaning. Thereby it may allow students to engage in active learning processes in which they build their own meanings and develop understanding. However, Khine (2003) cautions that students should not be left alone to explore; instead, teachers should provide guidance and facilitate students for learning to take place. Thus, in a constructivist classroom, the role of a teacher is to provide opportunities and facilitate learning.

2.5.3 Constructivism and Visualisation

Arcavi (2003) sees visualisation as a process of constructing knowledge that promotes understanding. Therefore, visualisation ‘as a process’ provokes intellectual activity and construction of knowledge. Presmeg (2006) stresses the need for visual representations in mathematics education. She also considers multiple representations of mathematical concepts as an integral part of a visualisation strategy of teaching and learning mathematics. In their research study on allowing learners to generate their own representations, Watson and Mason (2005)

emphasise that learners' active engagement in constructing their own examples enables them to probe and generalise mathematical ideas more deeply than if they were merely provided with facts. In a constructivist classroom, abstract ideas that involve examples and counterexamples are built upon students' representations. The construction of multiple examples by learners themselves, as Watson and Mason argue, not only facilitates conceptual learning but also enables learners to make links among concepts.

However, Duval (2013) asserts that as understanding involves grasping the whole structure, there is no understanding without visualisation. While a single activity focusses on one or some units and properties of a mathematical concept, visualisation leads to "grasping directly the whole configuration of relation" (Duval, 1999, p. 14). For example, when an activity is confined to computation only, the learner may not look at the whole mathematical structure, but only one aspect thereof. It is, therefore, essential to provide learning opportunities for learners to make connections and experience the whole mathematical structure of a particular problem. With an example, Duval argues that the construction to draw the graphs $y = 2x + 2$ and $y = x + 2$ by computing (perhaps by using a table) may not lead learners to understand the representation as a whole and the fact that the two graphs only differ from each other by virtue of their gradients. The use of visualisation enables learners to discriminate between the equations conceptually.

The scholarly works of Arcavi (2003), Presmeg (1986b, 2014), Duval (1999, 2013) and other researchers have provided compelling empirical evidence of the vital role of visualisation in developing an understanding of mathematical concepts and knowledge. Their research found strong evidence that learners do employ visual strategies (though different) to construct meaningful conceptual ideas. In particular, Presmeg (1986a), in commending the use of visual imagery, claims that the embodiment of abstract ideas in a concrete image can be effective in learning mathematics, i.e. the use of visualisation can facilitate the learning of mathematical contents with understanding. The essence of Vygotsky's ZPD shows that initial mastery of, for example, the four arithmetic operations, provides the basis for the subsequent development of a variety of complex processes of abstract algebra. Thus, concreteness is a necessary and inevitable step towards abstract thinking (Vygotsky, 1978).

The applets designed in GLIP engage learners to explore and discover mathematical ideas and subtleties. Thus, technological tools like *GeoGebra* can provide visual learning experiences and

allow learners to take control of their own learning (Bransford et al., 2000). This aligns well with the principles of constructivism.

2.6 SUMMARY OF THE CHAPTER

In this chapter, I have deliberated on the conceptual and theoretical framework that underpins the study. Firstly, several relevant issues related to visualisation within the context of teaching and learning mathematics were discussed. Secondly, the pros and cons of using a computer tool, DGS in particular, as a visualisation and teaching tool were discussed. This led to the discussion of teaching proficiency, focussing on Kilpatrick et al.'s model that informs the study. Finally, I discussed common features and ideas of the constructivist theory of teaching and learning and also highlighted its potential connections with the other elements that inform the study. Key issues established in this chapter will be employed to gather and analyse the data, which will be discussed in the following chapter on methodology.

CHAPTER THREE

RESEARCH METHODOLOGY

3.1 INTRODUCTION

This research study is conceptualised within the GeoGebra Literacy Initiative Project (GLIP) – an ICT teacher development project in Mthatha, Eastern Cape. The focus of this research was on how teachers use DGS as a teaching tool in harnessing its powerful visualisation capacity. As visualisation plays a vital role in developing mathematical ideas, the focus in the design and implementation of selected applets was on their visual appeal and capacity to enhance teaching proficiency. My research aim was specifically to explore how teaching took place, concerning selected teachers' interactions with the *GeoGebra* and with their learners. As the participating teachers implemented their applets in a teaching environment, their teaching practice was analysed in terms of the applets' visualisation power, grounded on strands of teaching proficiency (Kilpatrick et al., 2001) such as teaching for conceptual understanding and instructional fluency.

The overall goal of this research was to investigate whether and how dynamic software as a visualisation tool aided and enhanced the teaching of Grade 11 mathematics. In pursuance of this goal, the following specific questions guided the research:

- ❖ How can Dynamic Geometry Software such as *GeoGebra* be used as a visualisation tool to teach Grade 11 Mathematics?
- ❖ What enabling and constraining factors do Grade 11 teachers encounter when using *GeoGebra* as a visualisation tool to teach Grade 11 Mathematics?

This chapter presents the orientation of the research and the methods employed in conducting the research. I describe the analytical framework based on Kilpatrick et al.'s (2001) teaching proficiency in Table 3.3 (p.69). The sampling techniques and the participant selection procedures are elaborated upon. An account of procedures for the collection and analysis of the data is also provided. I discuss too, appropriate strategies to enhance the validity and reliability of the research study. Ethical considerations that pertain to this study are discussed. I also articulate some of the challenges and tensions that arose during the data collection process.

3.2 RESEARCH ORIENTATION

The study is situated within an interpretive research paradigm and oriented within the qualitative research framework. The interpretative research paradigm that underpinned this study coheres with the nature of the research enquiry. Cohen, Manion and Morrison (2007) assert that “to understand the subjective world of human experience” (p. 21) is fundamental in the context of an interpretative paradigm. From this perspective, I observed from the inside and through the direct experiences of the teachers involved. The research examined situations through the eyes of the participant teachers rather than the researcher, and this is particularly important within an interpretative research paradigm (Cohen et al., 2007). This interpretative research study involved describing and analysing teachers’ interactions with *GeoGebra* and with the learners in selected mathematics lessons. For this research, the elements of interpretive research that resonated were that events and situations are unique. Thus, the essential characteristic underpinning the data collection and analysis of this research is the consideration of all responses and actions (by participants) with a genuine interest in understanding their perspectives of classroom interactions. My analyses are interpretative renderings of reality.

Merriam (2009) suggests that in qualitative research, the emphasis is on “understanding how people interpret their experiences, how they construct their worlds, and what meaning they attribute to their experiences” (p. 14). The primary purpose of qualitative research is to know more about a phenomenon that could eventually inform teaching practice. The research focussed on participant teachers’ mathematical experiences using technological tools. Their classroom interactions with the *GeoGebra* applets were therefore observed and analysed to understand how participant teachers made use of technology-aided visualisation for effective teaching. Merriam (2009, p. 16) asserts that “the product of qualitative inquiry is *richly descriptive*” (italics in original). There are accordingly descriptions of the context, the participants involved, and the activities of interest.

Furthermore, Denzin and Lincoln (2005) assert that qualitative research involves interpretive approaches in natural conditions, attempting to gain a better understanding of the participants and their actions. The preferred paradigm allowed me to interpret and understand participants’ experiences in the context of integrating *GeoGebra* in teaching in mathematics classrooms. Through observations and interactions with the selected teachers, I aimed to comprehend how they

integrated technological tools in teaching mathematical ideas and the development of their teaching proficiency. For a given lesson, I probed and researched the meanings they made of their teaching with *GeoGebra*. At this level, qualitative research involves an interpretative approach as Denzin and Lincoln (2005) underscore: “qualitative researchers study things in their natural settings, attempting to make sense of or interpret, phenomena in terms of the meanings people bring to them” (p. 3). In concurrence, Stake (2010, p. 63) observes that qualitative research is about how things happen and “happenings are experiences, and the researcher needs to probe the assertions until the experience is credible.”

3.3 RESEARCH METHODOLOGY

3.3.1 Action Research Process

An important objective of GLIP is to empower teachers and then inform teaching practice by incorporating technological tools in mathematics teaching. GLIP is comprised of teachers from different schools, my fellow researcher and myself. This study involved an intensive and a long period of interactions with three GLIP teachers from two different schools. I adopted an action research approach for this research. Such an approach enabled me to involve the participants in the design and implementation of the *GeoGebra* applets. Merriam (2009) says that individuals engage in an action research to understand subtleties in educational processes. In concurrence, Winter (1998) suggests that action research is relevant in education. Winter observes that “in order to develop our practice we must seek to empower our students and thereby empower ourselves on a justifiable basis” (p. 55). Educational action research has the potential of engaging and empowering teachers in the process of teaching (Mertler, 2017). Thus, the participant teachers in this research followed the process of continuous learning. I collected rich, holistic information from the perspective of participants in a natural classroom environment during a sequence of teaching cycles.

Educational researchers (Cohen et al., 2007; O’Leary, 2004; Scott & Usher, 2003) have stressed the situational and participatory nature of action research. O’Leary (2004) argues that the central feature of action research is a collaboration among the research participants, where there is minimal distinction between the researcher and researched, while seeking solutions to improve professional practice. Mertler (2017) argues that action research allows teachers to study and reflect on their own instructional methods in order to improve their quality of effective teaching.

Accordingly, action research is mainly about critically examining, in a collaborative manner, what teachers are doing, why they are doing it, and the effects of their actions.

Mertler (2017) proposes a four-stage process of conducting action research: 1) the planning stage; 2) the acting stage; 3) the developing stage; 4) the reflecting stage. The process of action research is cyclical and spiralling in nature and is portrayed in Figure 3.1. Whereas action research has a clear starting point, it does not have a clear finishing point. Generally, the central theme of the research would be implemented in termly or even annual cycles. A given research project may continue to go through a series of cycles of implementation, evaluation, and revision, spiralling from one semester or year to the next (Mertler, 2017). In the GLIP project, a group of teachers came together intending to change and empower themselves by using *GeoGebra* in teaching mathematics. The empirical field of this study was thus the GLIP, which occurred in cycles defined by the applets that the teachers developed. The action research cycles were therefore determined by when the participating teachers implemented the applets in their teaching.

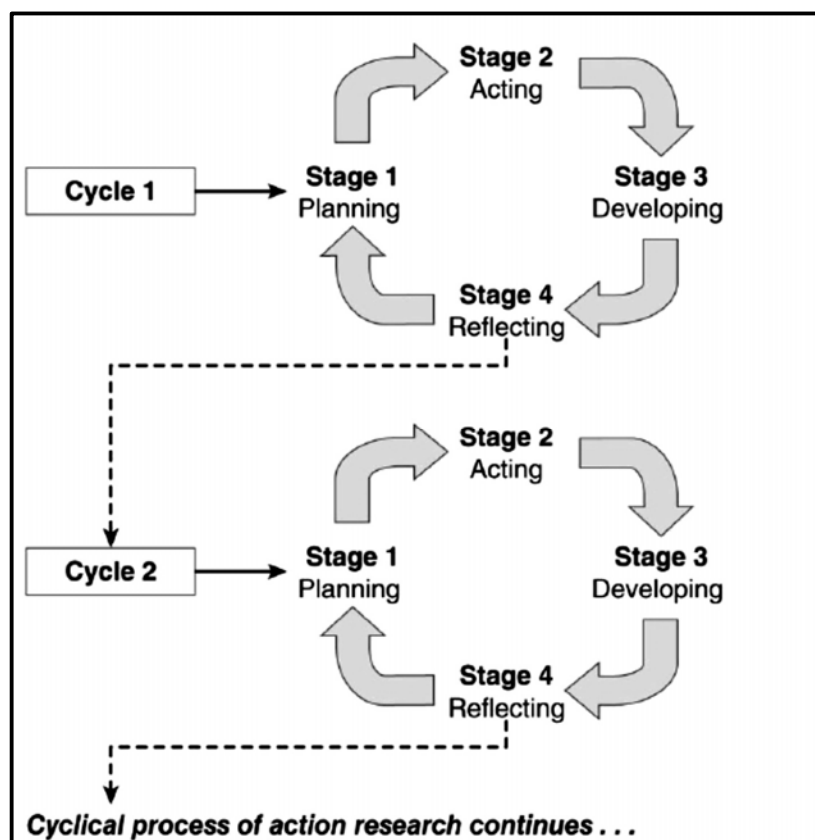


Figure 3.1: The cyclic and spiralling process of Action Research

Source: Mertler, 2017 (p. 88).

Figure 3.2 below is a diagrammatic representation of the elements of action research adapted for this study as it overlaps with some of the GLIP cycles. In this research, I refer to GLIP **cycles** and Action Research **steps**. The first step of the Action Research process is the planning step PLAN. Here the participating teachers discussed the pedagogical strategies to be used for implementing the designed applets in the classroom. The second step – ACT – is the implementation of the developed applets in the classrooms for teaching and learning. In this step, I gathered relevant data discussed in the section below. The third step – REFLECT – is the reflective step where the teachers shared their experiences of using applets in the classrooms. The fourth step – INTERVIEW – is an individual interview with the teachers using stimulated recall techniques discussed in the section below.

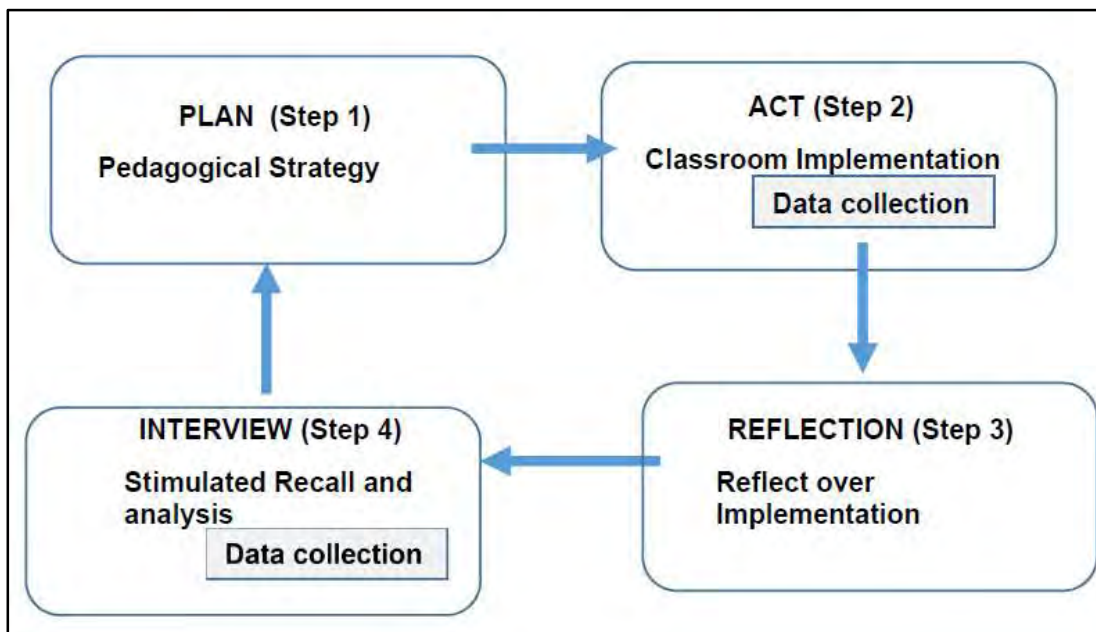


Figure 3.2: Sketch of Action Research Steps

3.3.2 Research Location

The empirical field of the research was the GLIP initiative. The participants who were involved in the GLIP project (as described in the Section 1.2.1 on p. 2) were from two schools, namely School A and School B, in the Mthatha region. Therefore, my research was located in these two schools. Firstly, I obtained approval for the research study from the Higher Degrees Committee of Rhodes University. My fellow researcher and I approached the principals of four schools in Mthatha where

functional computer laboratories were available and briefed them about the research project. All the schools' principals commended us for the idea of using their technological resources for teaching and learning purposes. They agreed in principle to the conduction of research on their premises. Of these schools, two were independent schools and two were public schools. Secondly, I sought permission from the Eastern Cape Department of Basic Education (ECDBE) to conduct research in the two selected secondary schools in the Mthatha region (see Annexure I). The GLIP programme was already operational by the time my application to research in the two schools was under consideration with ECDBE. In parallel, ethical approval was sought by the Ethics Committee of my University.

Meanwhile, interestingly, the governing body of an independent school did not provide us with permission to research in their school, stating that they were not willing to '*share our secrets of success stories.*' A teacher from this school, nonetheless, continues to be an active member of the GLIP programme. Thirdly and finally, the principals of the other two schools permitted us to conduct training and workshops with the teachers and the learners. At the time of issuing the formal letter granting us permission, the principal of School B commented "*We are glad that these computers are now used for learning purposes.*"

Thus, we ended up with two schools, namely School A and School B. Both schools offer the NCS – the South African national curriculum.

School A is an independent mixed gender school, comprising of about 1350 learners in 2017. The school practices a non-elitist admission policy and admits learners on 'first-come-first-serve' basis. The school offers classes from Grade 8 to 12 and is relatively well-resourced and well maintained. There are three functional computer laboratories with interactive whiteboard facilities and projectors. The regular maintenance of these computers (about 250) is outsourced. However, there are regular glitches in the efficient use of the interactive boards. The school offers two computer-related courses, namely CAT (Computer Applications and Technology) and IT (Information Technology), with the computer labs being used mainly for these courses. Apart from the designated subjects, the school also provides computer literacy lessons for an hour a week for all learners from Grade 8 to Grade 10. There were twelve mathematics teachers in the school; six of whom were teaching Grade 11 in 2017. The training of teachers in the context of this project in

the school on using *GeoGebra*, was started in November 2015 and extended to March 2016. We provided training to Grade 10 and Grade 11 learners during the second half of the year 2016.

School B is a public senior secondary school, offering classes from Grade 8 to 12. During 2017, the learner population was about 1200 learners of mixed gender. Adequately resourced, the school has one computer laboratory with 80 computers and 20 laptops, equipped with a data projector and a makeshift white screen and a chalkboard. At the time of this research, the school offered only one computer course, namely CAT. In August 2016, we trained all the mathematics teachers in this school on *GeoGebra*. We trained Grade 10 learners of school B during the September and October 2016 period. Subsequently, in early Feb 2017, we further trained the learners, then in Grade 11, of the participating teachers.

3.3.3 Participant selection and participants

The selection of schools and participant teachers was interlinked. The research required teachers to be proficient at using computers, and the schools of the respective teachers also needed to have adequate technological resources to implement the GLIP programme. Our interactions with the teachers in Mthatha allowed us to identify the teachers proficient in computers. The research required purposeful sampling of the participants owing to the nature of participants, their proficiency in computers and the availability of resources at their schools. According to Patton (2002), “[t]he logic and power of purposeful sampling lie in selecting information-rich cases for study in depth” (p. 232). The adoption of a purposeful sampling technique ensures that the appropriate teachers generate information-rich data that illuminates the research questions. Merriam (2009) emphasises that “[p]urposeful sampling is based on the assumption that the investigator wants to discover, understand, and gain insight and therefore must select a sample from which the most can be learned” (p. 77). Thus, the sample selection in this qualitative research is small and purposeful.

In GLIP, there were five teachers from three different schools. The training of the teachers at their schools enabled me to identify the potential candidates for this research project. At the initial stage of GLIP in 2016 and 2017, we invited only a few teachers who were proficient in computers, and they were willing to be participants in the research. Thus, two teachers, Antony and George from school A, and two teachers, Paul and John from school B, participated in the research project.

Abraham, though a GLIP member, could not participate in the research as his school declined our request to research in their premises, as discussed in the above section.

3.3.4 Brief background of participant teachers

In this section I introduce the research participants who volunteered their time and knowledge for the research project.

Antony from School A is vibrant and passionate about mathematics. He claimed that he had completed his engineering degree, but due to a lack of adequate facilities in the university, the course was not approved by the Department of Higher Education. Thus, Antony ended up without a bachelor's degree. However, in 2017, he enrolled for a BEd majoring in mathematics with a reputable university in South Africa. He started teaching in 2012 and has taught in two other schools around his hometown and Mthatha. Although Antony had undergone training in *GeoGebra* before the launch of GLIP and had used the software in preparing question papers, he had never thought of using it in his classroom. Prior to this project, Antony had used technological resources only once, when he had received a PowerPoint presentation from a teacher workshop. At school A, Antony is responsible for teaching mathematics from Grade 9 to 12.

George, from School A, is a committed and passionate school mathematics teacher. He has a BEd in mathematics education. George was in the fourth year of teaching at a school. He believes that his role as a teacher is to emphasise learners' divergent and creative thinking, and therefore he often coaches his learners to showcase their projects in maths and science fairs. George's core responsibilities at the school are teaching mathematics from Grade 8 to 11. George considered his participation in the research as an opportunity to learn teaching strategies that would have a positive impact on learners.

John expressed his desire to use the software for teaching and learning mathematics effectively after GLIP training. However, he was an irregular attender of GLIP meetings. John claimed that he was doing the final module of bachelor's degree in mathematics education. Owing to his absence during GLIP meetings, I did not include him in the analysis.

Paul, from school B, is an energetic mathematics teacher with a teaching experience of twelve years. Paul would refer to himself as an expert in the mathematical content and good at teaching it. At the time of this research, Paul held a diploma in education, and was pursuing a bachelor's

degree in mathematics. His peers consider him as a techno-savvy teacher. He often uses ‘Graph’ software for assessment purposes and claims that he is versatile in it. Occasionally, Paul took his learners to the computer laboratory where there is a repository of past question papers. Paul had also previously attended training in *GeoGebra* but had never used it in classrooms. At school B, he is responsible for teaching mathematics from Grade 8 to 12. His additional responsibilities include providing teachers with strategies that shape effective teaching.

A versatile and committed teacher, Abraham has a teaching experience of eight years. He holds a Higher Certificate in Education. Abraham believes that teaching mathematics requires some fun and laughter in classrooms. He offers extra mathematics lessons outside the school where he uses a projector and laptop to project past question papers or static images from different sources. GLIP meetings were held at his place where he held private lessons. However, Abraham’s school management refused to give us permission to conduct research in their school. Therefore, I could not observe and film his lessons. He continued to be a GLIP member and was enthusiastic in teaching and learning using technological tools. Abraham’s contribution to the GLIP is significant hence I included his profile in this section.

3.4 RESEARCH DESIGN

This research involved an in-depth study of practices of a group of teachers as they implemented technological resources in the form of *GeoGebra* applets in their classrooms. Therefore, I needed to establish a status of a ‘trusted person’ among the selected group of teachers. Mertler (2017) prescribes “[T]he act of becoming a trusted person in the setting actually makes the researcher a participant in that setting or group, as well” (p. 176). The GLIP meetings provided an opportunity to interact with each other, so within this context, I took an active role as a member of the team, involved in the discussions of pedagogical aspects and design of the applets. My participation as a team-member of GLIP provided me with two outcomes: one – I had the opportunity to learn first-hand what went on in that setting; and two – to establish an excellent rapport and trusted relationship with the participant teachers.

Nonetheless, in the process of classroom data collection, I changed my role from a participant to act completely as an observer, i.e. as a typical researcher-observer, I had little interaction with the participant teacher.

The research took place when the GLIP teachers implemented their applets on different topics in their classrooms. The schools in the Mthatha district had a standard annual teaching plan which guided us in determining the topics of discussion in the GLIP sessions, and hence the data collection schedule. In this thesis, I report on three topics in Grade 11 that the teachers taught with *GeoGebra*, namely i) circle geometry, 2) equations and inequalities and 3) straight lines.

The empirical field of the research is the GLIP programme, and in the following section, I discuss the GLIP model of implementation of technological resources in classrooms.

3.4.1 The GLIP Cycles and Data Collection

In Chapter One, I discussed the GLIP (GeoGebra Literacy Initiative Programme), which is a teacher development project explicitly designed for teachers to use *GeoGebra*. There were two phases in the GLIP – a training phase and an implementation phase. Data collection for this research lay in the implementation phase of GLIP, where teachers used *GeoGebra* applets as a pedagogical tool in classrooms. The GLIP cycles were iterative and comprised of six components: 1) identifying the mathematical domain; 2) discussing the pedagogical imperative; 3) designing and planning; 4) implementation; 5) reflection and 6) refinement. The empirical field of this research was thus the GLIP program, which occurred in cycles defined by the applets that the teachers developed. In this thesis, I report on three GLIP cycles with the participating teachers.

I primarily used two techniques for my data collection – classroom observation and ‘stimulated recall’ interview. Classroom observation, where field notes were taken, captured the data of the teachers’ implementation of applets in classrooms in the form of audio-video recordings. Stimulated recall interviews provided the setting to facilitate the reflective process (O’Brien, 1993). A stimulated recall interview technique was a reflective session where the teachers and I watched the videos of their teaching and interpreted the lessons. This method of stimulated recall prompted teachers to recollect and explain the strategies they used during the lesson.

What follows is a description of the process of the GLIP cycles and data collection techniques for gathering data to answer the research questions. Newby (2014) argues that “data are the material out of which we construct our research argument” (p. 146).

Domain: The topics were selected based on the annual teaching plan that aligned with the CAPS document issued by the Department of Basic Education (2011). In the GLIP meetings, we

discussed how we could harness aspects of *GeoGebra* into the design of our lessons in order to make the concepts visible and develop conceptual understanding and procedural fluency. However, appropriate topics had to be carefully selected as “not all problems are best investigated with *GeoGebra*” (Bu et al., 2012).

Pedagogy and Design: In the GLIP meetings, teachers discussed and designed the applets incorporating the relevant and appropriate pedagogy in the applets. These discussions were important to the preparation of the lessons. I audio recorded these discussions for my specific research purposes.

Implementation and Classroom Observation: Each teacher taught his lessons with the co-developed applets. For each mathematical domain, I observed two of the three lessons, and I thus observed eight lessons in total from four cycles. Due to the volume of the data generated, I present only six lessons from three cycles. During the classroom observations, I took the role of a non-participant observer. I collected the data in the form of audio-video recordings. I used an exclusive audio recording device placed inside each teacher’s pocket, to record the voices of the teachers. I used two cameras to capture the video of the lessons – firstly, a fixed camera on a tripod facing the projector screen and secondly a hand-held camera which followed the teacher as he taught. My fellow researcher volunteered to operate the moving camera and walked around the classroom. Throughout the observed and video-recorded lessons, we neither interfered nor interacted with the teacher or learners during their classroom activities. As Scott and Usher (2003) argue, “this detached stance allows observers to gain a more comprehensive view of what is being observed ” (p. 101). This dimension of a purely observational role during implementation allowed me to gain a more objective view of the classroom interactions. We had informed the classes beforehand, and they were accustomed to our presence in their space.

Reflection and Stimulated Recall interview: After the lesson using the applets, the participant teachers reflected on the applets and the presentation of their lesson. This reflection took place in our GLIP meetings. These meetings offered an opportunity for the teachers to reflect in detail on the experiences of their lessons. They also provided critical insights into and data about the efficacy of the applets as a visualisation tool.

Merriam (2009) argues that “[I]nterviewing is often the primary data collection strategy in qualitative studies” (p. 17). Within a span of three to four days after the classroom observation, I

arranged a meeting with the teacher concerned for a stimulated recall interview. Meanwhile, I watched and analysed the audio-video recording of the lesson before the interview. I interviewed the participating teachers individually to gather information and clarification about their experiences of the teaching using the applet, concerning how the use of the applets facilitated conceptual and instructional fluency. This stimulated recall interview allowed me to explore and analyse the teachers' interactions and engagement with the applets against the first two strands of Kilpatrick et al.'s proficient teaching viz. conceptual understanding and instructional fluency. I asked open-ended questions that followed up with probing for further details.

Refinement: After each cycle of implementation and its respective reflection, I consolidated the data obtained in that cycle. The data informed me as to how far the teachers were able to create a learning environment using *GeoGebra* applets and whether the teachers were able to implement the intended pedagogical practices. This step also provided me with an opportunity to review and consolidate what was achieved in that particular cycle. This step of the GLIP cycle helped us (my fellow researcher and I) to plan and refine the next cycle. For instance, after the first cycle, teachers realised that their learners required more training in computers and *GeoGebra* in particular. Furthermore, the teachers' reflections showed that learners' very limited experience with the computer keyboards impeded the mathematics intended in the lesson. Therefore, we made sure that the applets that were developed for later cycles required minimal keyboard inputs from the learners.

The following Table 3.1 indicates the components of the GLIP cycles where the action research took place.

<i>Table 3.1: Table to show where data collection occurs</i>			
Components within each GLIP cycle		Action Research	Data Collection
1.	Selection of Domain	✗	✗
2.	Pedagogical Imperative (GLIP meeting)	✓	✓
3.	Design of Applets (GLIP meeting)	✗	✓
4.	Implementation (Classroom Observation)	✓	✓
5.	Reflection (GLIP meeting)	✓	✗
6.	Interview (Individual)	✗	✓
7.	Refinement (GLIP meeting)	✓	✗

The diagrammatic representation of data collection within the GLIP cycle is as given in Figure 3.3.

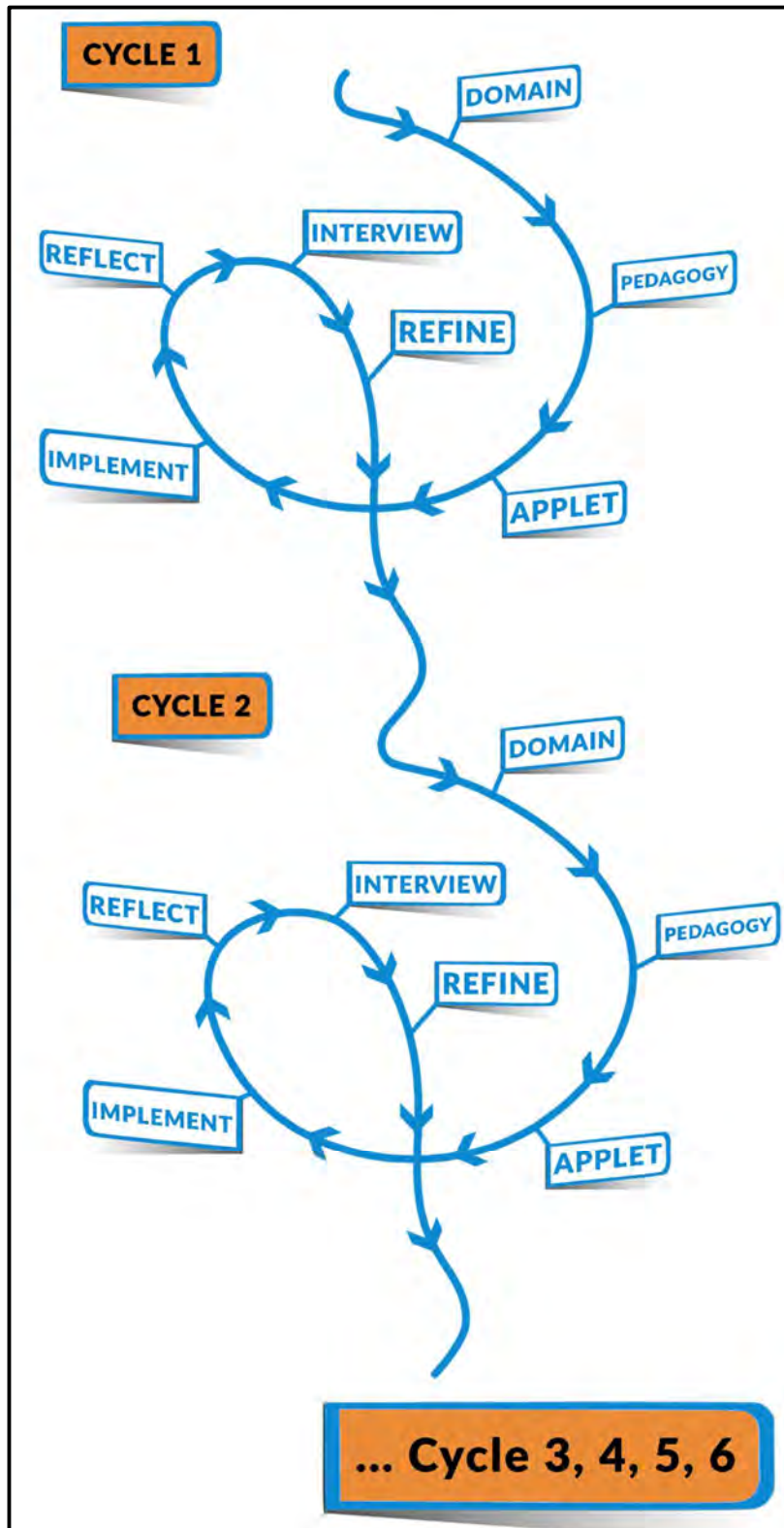


Figure 3.3: GLIP Cycles – the empirical field

3.4.2 The Pilot Study

After the first phase of the GLIP workshop training the teachers, the feedback which GLIP received was very encouraging and positive. The teachers were enthusiastic about *GeoGebra* as a teaching and learning tool in their mathematics classes. Two teachers, including one of the participant teachers – George from School A – used the pre-designed applets in his Grade 9 classes as a pilot to teach the area of a triangle. At the time, my analytical framework was not yet fully developed but the pilot of using *GeoGebra* assisted me in identifying and coding the teacher interactions with the applets and with the learners. These interactions are further discussed in Section 3.5.1.2 (p. 67) when I dealt with the full development of the analytical framework.

Furthermore, after the approval of the framework by Rhodes University, I also piloted a lesson by George in Grade 10 where George and I designed the applets for teaching linear functions. After the lesson, we watched the recording of the lesson together and conducted a mock stimulated recall interview process. The second pilot further helped me to refine the data gathering techniques and the analytical framework, thus sharpening my data gathering and analysis protocols.

3.5 DATA ANALYSIS

The data analysis protocol involves organising and explaining the data in terms of the participants' perspectives, accounting for categories, noting patterns, regularities and themes (Cohen et al., 2007). Newby (2014) considers data analysis as a process through which we make sense of data and communicate the essence of what it reveals, and “get our data to release the information we need to answer our research question” (p. 395).

Merriam (2009) argues that “the much preferred way to analyse data in a qualitative study is to do it simultaneously with data collection” (p. 171). Data that have been analysed while being collected are likely to immediately illuminate the research questions. Following these recommended methods, I viewed and reviewed each classroom observation, made notes and reflected on the data, and identified things to pursue and ask the participants during the stimulated recall interview. Furthermore, I also recognised and became more conscious of observing or looking out for specific things in the subsequent cycle of data collection. Prior to the actual data collection with the participant teachers, I piloted each cycle of applets with my fellow researcher. He took a few of his learners and taught them using the co-developed applets, and I observed those lessons and

made notes of hunches, ideas and things to look out for in the classroom observation of the relevant cycle. With the recursive process of data collection and analysis, I was able to bring greater focus to data collection, and thus, the analysis became more intense as the study progressed.

3.5.1 The Step-by-Step Process of Data Analysis

The process of data analysis involved four steps, namely 1) managing the data; 2) categorising the data and organising the data based on the categories identified; 3) reporting on the data in the form of narratives and horizontal analysis; and finally the last step 4) interpreting the data. The following sections expand on this protocol of step-by-step data analysis.

3.5.1.1 Managing Data

The collection of qualitative data was done through multimedia recording. The multimedia recording of classroom observation comprised of two video- and one voice recorder, while stimulated recall interviews involved only a voice recorder. Researchers (for instance, Merriam, 2009; Newby, 2014) have recommended that transcribing one's own data is another means of engaging with the data and generating insights. I personally did all the transcriptions of the multimedia recordings. Firstly, I transcribed the recordings of the classroom observations into one lesson transcriptions and labelled them, followed by the transcriptions of the reflective interviews with the participant teachers. In the labelling of transcriptions, I observed the following protocol: each transcription label had four characters; the first two characters were CL or RI, where CL referred to classroom observation, and RI referred to stimulated recall interview. The third character in the label was a number that referred to the number of the action research cycle, and the last character was again an alphabet symbol that referred to the first letter of the name of the participant teacher. For instance, the transcription CL1A referred to the transcription of the classroom observation of Antony's lesson in cycle one, while RI1A referred to the transcription of the reflective interview with Antony in cycle one.

After converting the multimedia recordings to text, I read through the transcriptions along with the media source to ensure the accuracy of the transcriptions. This also enabled me to add the teachers' non-verbal gestures that may be relevant to data analysis.

For this research thesis, I report on three GLIP cycles. In each cycle, I observed and recorded two participant teachers. The quantum of data generated and analysed in three research cycles is summarised in Table 3.2. The analytical instrument, CUIF, is explained in the following section.

Table 3.2: Quantum of data per research cycle

Data Sources	Step of Empirical Action Research cycle	Analytical Instruments	Quantum of data
Voice Memo (audio recording)	1) Pedagogical Imperative		1 × 3
	2) Design of applets		1 × 3
Multimedia Recording and Transcription	1) Classroom Observation	CUIF	2 × 3
	2) Reflective interview	CUIF	2 × 3

3.5.1.2 Organising the Data, Using an Analytical Framework CUIF

As mentioned in the above section, I scrupulously reviewed and transcribed verbatim data from all sources. Thereafter, I organised the data by assigning codes using my analytical framework. Merriam (2009) defines coding in qualitative analysis as “[c]oding is nothing more than assigning some sort of shorthand designation to various aspects of your data so that you can easily retrieve specific pieces of the data” (p. 173). The objective of coding is to identify segments of data which is a potential answer or connected to the issue under scrutiny. According to Newby (2014), when we put codes together, we obtain insights into the research questions that may not be available from the raw data.

I developed an analytical tool, entitled the CUIF – see Table 3.3. It contains a framework of deductive codes and observable indicators, amalgamating Kilpatrick et al.’s (2001) model of teaching proficiency with concepts of visualisation derived from previous research by Arcavi (2003), Presmeg (1986b, 2006, 2014) and Duval (2013, 2014). I used a colour coding system to assign each of the identified data segments to the observable indicators in order to look for similarities and differences as well as odd occurrences. The codes then enabled me to explore how participant teachers shaped their instructional practices while integrating technological tools in their mathematical classrooms, and the challenges they faced during the implementation of co-developed applets. The analytical framework, CUIF, was refined as observable indicators emerged (or overlapped) from the analysis of pilot data.

I coded the participant teachers’ interactions with the applets and responses to the learning situations according to the coding system given in the CUIF, as shown in Table 3.3. The teachers’

mathematical teaching proficiency was analysed in relation to their interactions with the applets and with the learners, and the kinds of visualisation capabilities inherent in the applets that they demonstrated. In order to help me to recognise the two strands of mathematical teaching proficiency (Core Understanding and Instructional Fluency), I expanded the definitions and descriptions of what Kilpatrick et al. (2001) meant by these strands. I excluded the other three strands of Kilpatrick et al.'s teaching proficiency model from my analysis, owing to the limited scope of this research study. I therefore only analysed teaching proficiency with respect to using *GeoGebra* against the first two strands articulated above. It is recognised that these two strands are interwoven, and the combination thus represents a synthesis of mathematical teaching proficiency with visualisation.

Table 3.3: The CUIF Analytical Instrument

The TWO strands of Kilpatrick et al.'s (2001) teaching proficiency	Indicators of the TWO strands of teaching proficiency
<p>Conceptual Understanding (CU) - refers to knowledge of mathematics, knowledge of students and knowledge of instructional practices.</p>	<p>CK: (Knowledge) demonstrates accurate explanations of mathematical concepts, operations, relations, terms and notations that are useful to learners.</p> <p>CMR: (Multiple representation) represents mathematical ideas in multiple ways.</p> <p>CC: (Connections) emphasizes relationships among concepts and makes connections, links to prior learning and references to real life situations.</p> <p>CD: (Discovery) encourages and engages learners in tasks to explore and discover concepts.</p> <p>CM: (Mis-conceptions) identifies common student misconceptions and is prepared to remove such misconceptions.</p> <p>Non-example: Explaining Mnemonic techniques is not a core knowledge of a teacher.</p> <p>CR: (Real-life) makes references to real life situations.</p> <p>CS: (Switch) Switches between abstract and concrete reification.</p>
<p>Instructional Fluency (IF) is the ability to use knowledge flexibly in practice to appraise and adapt instructional materials (Kilpatrick et al. , 2001, p. 369)</p>	<p>FA: (Acquisition) a range of instructional materials which can be drawn from fluently as teachers interact with students.</p> <p>FM: (Mis-conception) responds to a learner with serious misconceptions.</p> <p>FL: (Literacy) able to deal with students who lack basic skills.</p> <p>FR: (Responsive) flexible and responsive to students' queries and provides alternate methods or procedures to the same problem.</p> <p>FP: (Procedure) explains and encourages learners to use and select appropriate algorithms, formulae, procedures or conventions accurately, efficiently and flexibly.</p> <p>FF: Feedback to assure that the learners understand or are able to perform tasks, prompting answers.</p>

The CU and IF strand of proficiency for teaching mathematics entails a set of interrelated above-mentioned 'codes' in Table 3.3. These codes are discussed briefly hereunder.

CK (Knowledge): A teacher provides clear and efficient explanations of mathematical concepts, relations, notations and procedures that are useful to learners.

CMR (Multiple representation): A teacher represents mathematical situations in different ways or uses multiple representations, models or real objects. The use of multiple representations in classrooms is another strategy to make effective use of visualisation.

CC (Connections): Making connections across mathematical topics and across the curriculum is one of the key elements of proficient teaching. When teachers make connections between mathematics topics and the real world in a classroom, they employ visualisation for effective teaching and learning.

CD (Discovery): A teacher engages learners and encourages them to explore and discover concepts. Here, the teacher acknowledges constructivism as a theory that knowledge is the result of learners' activities rather than the passive receipt of instruction.

CM (Misconceptions): From their experiences, teachers have identified likely misconceptions around a mathematical idea and are able to unpack them in the classroom.

CR (Real-life): This code relates to teaching activities where teachers devise a real-world context. Visualisation in teaching results in making connections with everyday or real-life situations.

CS (Switch): Switching between the abstract and concrete can be thought of as ways in which teachers direct mathematical generalisation combining the concrete details and the abstract ideas. Visualisation in mathematics is the ability to relate and handle both concrete and corresponding abstract objects.

FA (Acquisition): Fluency in acquisition stands for those instances when teachers fluently draw from a range of acquired instructions as they interact with the applets and the learners.

FM (Misconceptions): Fluency in handling misconceptions signifies a teacher's responses to a learner who gives an answer that demonstrates serious misconceptions. (The misconceptions addressed here refer to individual learner's misconceptions not identified by the teacher during the planning of the lesson).

FL (Literacy): Fluency in basic computer skills corresponds to those interactions where teachers deal with learners who lack basic skills in using computers for learning purposes. The indicator **FL** was not initially planned; however, after piloting I realised that the teachers spent a lot of time

and energy helping learners to use computers correctly. I do not endeavour to identify specific skills or knowledge required by mathematics teachers.

FR (Responsive): Fluency in response denotes teachers' responses to situations or questions from learners that arise during their lessons. Responses could include providing alternate methods or procedures for the same problem.

FP (Procedure): Fluency in procedures signifies how teachers not only carry out the routine procedure but also explain and encourage learners to use and select an appropriate algorithm that is efficient and effective.

FF (Feedback): Fluency in soliciting feedback from learners helps a teacher to ensure that the learners understand or are able to perform a task. This is important when a teacher is working in a constructivist precept that encourages learners to observe and talk about mathematical view points.

The classroom observation provided an idea of how the participant teachers engaged with the co-developed *GeoGebra* applets. These were video recorded, and field notes were taken. I reviewed the videos and the video transcriptions to describe and document the unique or common patterns of the teaching practices. I identified the basic units of data, and then I coded. I categorised the data and tagged it by linking the teachers' interactions with the applets and with the learners, using the indicators mentioned in the CUIF analytical framework, shown in Table 3.3. This level of analysis helped me to gain an in depth understanding of their teaching practices.

3.5.1.3 Reporting the Data – Vertical and Horizontal Analysis

There are three analysis chapters – Chapters Four, Five and Six – one for each cycle. For each chapter, I first briefly present the development and design of the applets for that specific cycle. The GLIP meetings helped me to understand how teachers planned their lessons to develop conceptual understanding and procedural fluency in mathematical concepts. More importantly, the analysis of the audio recorded GLIP meetings (second and third steps of the empirical cycle) enabled me to uncover what happened during the planning sessions.

I used a vertical (descriptive) analysis for each teacher to narrate their implementation of applets in classrooms. According to Newby (2014), narrative analysis enables a researcher to process the data in ways that extract the most relevant information in order to address the research question.

The narrative accounts enabled events and actions to be conveyed as they unfolded. The narrative approach allowed me an in depth understanding of my research questions.

Nevertheless, individual observation and analysis of one participant cannot be considered in isolation (Cohen et al., 2007). I consolidated the data in my horizontal analysis as I put together all the main issues that arose amongst the participants. A horizontal analysis is a cross-referencing of what transpired from a vertical analysis (Merriam, 2009). Consequently, in the horizontal analysis I looked for commonalities, different methods of teaching and recurring patterns across my participants.

In this way, my vertical and horizontal analysis complemented each other. The vertical analysis enabled me to present a detailed and comprehensive picture of each participant teacher. In my horizontal analysis, cross-referencing of what transpired from the vertical analysis enabled me to understand teachers' interactions and responses as they used applets for teaching and learning mathematics. I consider this blend of analysis an effective approach to grow my own insights into the participants' worlds.

3.5.1.4 Interpreting the Data (Answering the Research Questions)

The above analysis protocol allowed me to interpret the data in order to answer the research questions. Using this technique, I could identify enabling and disabling aspects of using technological tools in mathematics classrooms. According to Newby (2014), in social science research it is only by examining the lives, actions and statements of people can we understand the world from their perspective. Thus, the interpretation of data underpins the qualitative approach to research.

At the end of each cycle representing a particular mathematical domain, I then present a section to discuss the answers to my research questions.

3.6 VALIDITY AND RELIABILITY

Newby (2014) argues that validity in educational research means to “convince others that the processes we used to collect our data and the analysis we used to convert them into evidence are reasonable and, then, that the argument we construct to reach a conclusion is logical” (p. 97). From this perspective, the validity of research becomes intricate and multifaceted and thus requires careful consideration. Cohen et al. (2007) echo that validation and reliability of data is an essential

key to effective research that has a substantial effect on the interpretation of data and the authenticity of research. Reliability, as Merriam (2009) describes, is an essential consideration of any research and indicates the trustworthiness or quality of the research. I adopted different strategies to present my research study's validity and reliability, and the following paragraphs are devoted to discussing these issues:

Rigour in action research, Mertler (2017) describes as “[t]he extent to which it reaches a standard of quality is directly related to the usefulness of the research findings for its intended audience” (p. 68). In general, rigour refers to the validity, quality and credibility of the research. Maxwell (2009) suggests that rigour in action research is typically based on procedures of checking to ensure that the results are not biased. Mertler (2017) proposes that one of the strategies to deal with ‘**bias**’ validity threat is by reporting a situation revealing the views of the participants. I included multiple methods to ensure validity and reliability of the research findings. Reflective interviews and GLIP meetings gave me an opportunity to clarify meanings held by the participant teachers. Whenever any discord existed, such as coding of a teacher interaction as CU or IF, I took the conundrum to the GLIP meetings, and we discussed it and reached a consensus. I also asked the participant teachers to review the analyses and results from the action research process. The idea here was to solicit feedback on my analysis from the participant teachers, ensuring validity and credibility through ‘**member checking**’. So the strategy of ‘member checking’ was an important way of ruling out the possibility of misinterpreting the meaning of the participant teachers’ interactions and responses (Maxwell, 2009). It also provided an opportunity for the participant teachers to further explain, extend or curtail the information that had been reported on in the analysis.

Furthermore, my fellow researcher and I discussed my preliminary analysis and I consulted with him about the validity of my findings. His involvement was invaluable to me as it ensured that I remained consistent and rigorous in my analysis. Cohen et al. (2007) explain that **triangulation** in social sciences is an “attempt to map out, or explain more fully, the richness and complexity of human behaviour by studying it from more than one standpoint” (p. 141). The involvement of my fellow researcher was a form of ‘triangulation’, which contributed to the validity and reliability of the findings. The triangulation of methods can assist the researcher in generating reliable evidence. In general, data collection techniques that have proved their worth to be valid and reliable include observations and interviews (Cohen et al., 2007; Newby, 2014). The use of multiple methods of

data collection is a powerful way of demonstrating the validity of research, especially qualitative research (Cohen et al., 2007). For this reason, the data collection techniques for this research include multimedia video recordings of classroom observations (in fact, I used two video cameras and an exclusive audio recording device), field notes and stimulated recall reflective interviews.

Mertler (2017) suggests that “[p]rolonged engagement and persistent observation” helps participants fully understand the outcomes of an action research process. The sustained presence of the researcher with the participants therefore provides extended opportunities to share their experiences. I interacted with the participating teachers for quite a long time before the actual data collection, and therefore I was “close to reality” (Maxwell, 2009). The long-term interaction yielded considerable data which was rich, detailed, and varied enough to render an analysis process which was valid and reliable.

I have discussed in Section 3.3.1 (p. 53) that an action research process is, by its nature, cyclical. In order to develop adequate credibility, it is critical to proceed through several cycles, where the earlier cycles inform the later cycles (Mertler, 2017). With three cycles reported on in this thesis, the GLIP team learned continuously, from one cycle to another, thus enhancing the validity and reliability of the research findings.

In this section, I have examined the character and principles of research design that define the qualities of valid research. I am convinced that my data collection techniques are appropriate and the analysis protocol has a rigour that substantiates the claims that I have made.

3.7 ETHICAL CONSIDERATIONS

Maxwell (2009) believes that “ethical concerns should be involved in every aspect of design” (p. 216). In agreement with him, Merriam (2009) argues that ensuring validity and reliability in qualitative research involves investigating in an ethical manner. I followed all of Rhodes University’s ethical protocols for conducting research. I obtained relevant **permission** from the Department of Basic Education, Eastern Cape, to conduct research in public schools. The relevant documents are in Annexure I.

The principle of **informed consent** implies a participant’s right to freedom and withdrawal from the study. Correspondingly, Newby (2014) clarifies that the participants should have a full understanding of their commitment. When my fellow researcher and I approached the schools and

the identified teachers, we explained our research agenda and the research process to them. We also informed them in writing that they were under no obligation to continue their involvement until the end of the project and guaranteed their confidentiality. In order to ensure that the participants' consent was voluntary and informed, I obtained a signed consent from the school and from the participant teachers, in Annexure II.

Moreover, I also obtained informed parental consent of the learners concerned, in the form of signed letters, as learners were recorded during my classroom observations. In Paul's class, a learner did not return the signed consent letter and therefore did not attend the lessons during the classroom observations even though this learner had participated in the introductory training in *GeoGebra* and the training interludes between the cycles.

Disseminating findings can raise certain ethical issues. I ensured the **anonymity** of the schools and the participants to protect their identity. The audiovisual recordings, the stimulated recall interview recordings, the transcripts and other data are stored safely and secured on my password protected computer. I intend to store the data for five years, after which I will destroy the data sets.

Cohen et al. (2007) assert that ethical researchers should behave towards their participants "...in such a way as to preserve their dignity as humans"(p. 58). The bottom line of ethical research is my own integrity and the participants' welfare and dignity.

3.8 CHALLENGES IN DATA COLLECTION

The first challenge in the data collection process that I noticed was the erratic computer terminals and lack of maintenance of computer laboratories in schools. Teachers often took on the role of technical assistant in helping learners to overcome technical glitches rather than expounding on mathematical content. Even so, as the hardware problem persisted, learners' incompetency in using computer input devices such as the keyboard and mouse posed a threat to the smooth interchange of mathematical ideas in the class.

Nonetheless, we overcame the situation as we provided further training in *GeoGebra* before the implementation of each cycle. For this additional training, the participant teachers had to have flexible schedules, and it was not always feasible to arrange these training sessions. To a large extent, we addressed the challenge of learner incompetency in using computers by ensuring that the collaboratively designed applets required minimum input (by way of typing using keyboards)

from the learners. Interestingly, Antony, after a lesson using *GeoGebra* applets, informed his learners that using keyboards was equivalent to learning mathematics.

Yet another challenge or drawback of the research is that sample selection did not include female teachers. Coincidentally all the participants were males. There were no female teachers in the GLIP project at the time of data collection, nonetheless, gender discrimination or gender-based analysis was not on my research agenda. As a woman, I wished to empower a few women through my research process. I approached a few female teachers in Mthatha to participate in our project; however, they were not willing to spend time and energy on the GLIP meetings owing to other commitments. Gender was not a factor in this research and the dominance of male teachers in the sampling was neither purposeful nor deliberate.

3.9 CONCLUSION

The empirical component of this study centres on an intervention programme that capitalises on the community of practice inherent in the GLIP. The study is situated within the qualitative research framework and oriented within an interpretive research paradigm. The research methodology took the form of action research cycles in which the design, implementation and evaluation of successive applets determined the data gathering and analysis process. I developed an analytical tool synthesising Kilpatrick et al.'s (2001) model of teaching proficiency and concepts of visualisation. I employed this analytical tool to understand teachers' classroom practices and emerging patterns of interactions that surfaced during the implementation of applets. This methodology enabled me to interpret how participant teachers shaped their instructional practices while integrating technological tools in mathematical classrooms, and the challenges they faced during the implementation of co-developed applets.

In the following Chapters Four, Five and Six, I present the data analysis of three GLIP cycles.

CHAPTER FOUR

TEACHING MATHEMATICS USING DGS – DATA PRESENTATION AND ANALYSIS – CYCLE 1

4.1 INTRODUCTION

The overall goal of this study was to investigate whether, and how, DGS-aided visualisation can enhance the teaching of mathematics. The GLIP model of action research cycles, as discussed in Chapter Three, enabled me to gather data for three topics from the three participating teachers. The GLIP meeting started in December 2016, and the data collection of the first cycle commenced from January 2017, with the third cycle ending in August 2017.

Table 4.1 below is a summary of the data capturing process that is discussed and analysed in this chapter and the following one - Chapter Five and Chapter Six (i.e. a chapter for a cycle). In these chapters, I structure and discuss the data gathered during each step of the action research cycle shown in Chapter Three in Figure 3.3 (p. 64) (i.e. planning, implementation and reflective interview). The GLIP meeting started in December 2016, and the data collection of the first cycle commenced from January 2017, with the third cycle ending in August 2017.

Table 4.1: Summary of the data collection and analysis

No.	Topic	Teacher		
		Antony	Paul	George
1	Angle at the centre – Chapter Four	Data available and analysed	Data available and analysed	Data available and not analysed
2	Quadratic inequalities and nature of roots - Chapter Five	Data available and analysed	Data available and analysed	Data not collected
3	Equations of a straight line – Chapter Six	Data not collected	Data available and analysed	Data available and analysed
	Total of analysed lessons	2	3	1

In Chapters Four, Five and Six, I examine the data drawn from the video lessons and reflective interviews for two teachers (per cycle) and then provide an overall picture highlighting visualisation and teaching proficiency across the teachers for each cycle.

This research study generated several applets for each cycle. ‘Applet i,j’ refers to an applet where ‘i’ and ‘j’ denote the ‘i’th applet of the ‘j’th GLIP cycle, for example, Applet 3.1 thus refers to the first applet in the third cycle.

This chapter, Chapter Five and Chapter Six are similarly structured as follows:

Section 4.2 – This section briefly describes the GLIP meeting where the planning and designing of the applets took place, and teachers implementing these applets in the observed lessons. For each selected topic, there was a discussion among GLIP members which is described in this subsection. The design sub-section elaborates upon some technical aspects of the software and pedagogical aspects that emerged during the construction of the applets.

Section 4.3 – This section presents data and associated analysis in the form of narratives for each of the two participants.

Section 4.4 – This section analyses data across participants, pulling together the indicators of visualisation and mathematical teaching proficiency, thus complimenting the narratives.

Section 4.5 – This section answers the research questions.

Section 4.6 – Concluding remarks are made in this final section.

A synopsis of the methodology and analytical framework would be helpful for the reader at this stage of the thesis.

4.1.1 Synopsis of the methodology

The GLIP cycles which coincided with my action research, framed the research design. The narrative analysis that follows, unfolded in chronological order, i.e. as the events happened in real time. The narratives of each participating teacher describe how the selected teacher implemented the co-developed applets in his classroom, and his interaction with the applets and with the learners. The narratives for each teacher are in the form of a prose account, interspersed with figures for the relevant applets and excerpts of transcripts and analyses.

This research study focussed on two strands of Kilpatrick et al.’s (2001) mathematical teaching proficiency model, namely Core Understanding (CU) and Instructional Fluency (IF). The analytical instrument CUIF in Table 3.3 on p. 69 was used to analyse the data. For instance, the indicator ‘Connections’ (CC) under Core Understanding indicates a teacher teaching new

mathematical concepts by bringing in connections between prior knowledge and real-life situations. On the other hand, the indicator ‘Responsive’ (FR) under Instructional Fluency indicates that while interacting with a learner, the teacher responds flexibly to a learner’s queries and may provide an alternative explanation or procedure. In Chapter Two, Section 2.4.3 (p. 39), I discussed teaching proficiency in relation to visualisation. Thus, the use and allocation of the indicators described in the CUIF enabled me to interrogate the proficiency of the teachers’ practices in terms of how they integrated the visualisation capabilities of *GeoGebra* into their classrooms.

4.2 CYCLE 1: - Angle at the Centre theorem

Theorem: The angle subtended by an arc at the centre of a circle is twice the angle subtended by the same arc at the circumference.

4.2.1 Discussion – GLIP Meeting

The participant teachers met for the first time after the training on *GeoGebra* to discuss how to use the software in their classrooms. There were only three teachers, namely Antony, Paul and George, and two researchers, present in this meeting. George had managed to hire a venue for our meetings away from our schools but still accessible to all of us. The lighting of the classroom was not suitable for display on the projector, and there was significant noise from children in the adjacent classroom. Later, after our meeting, we learned from the venue manager that during non-school days, including Saturdays, parents leave younger children there to be supervised. Since the venue was not available during regular working days, we decided to look for another venue for the next round of discussions.

4.2.1.1 Identification of concepts

We decided to focus on only one theorem out of the seven circle geometry theorems in the Grade 11 curriculum, which was the *angle at the centre theorem*. All the participants agreed that *GeoGebra* might be useful for their learners to first investigate the theorem before providing the formal proof of the theorem. We debated at length about whether the learners should construct geometric figures such as circles and line segments, and measure angles on their own during the lesson and then explore the theorem, or whether they should use a preset applet that containing circles and measured angles. The learners could then investigate the relationship between the angle at the centre and the angle on the circumference of the circle. Other options also emerged – like

using the ‘Navigation bar’ akin to the animation feature of the presentation software. This option would be best suited for a classroom demonstration where we could do away with the ‘burden’ of repeated drawings. Without settling the discord, we moved on to designing the applets. All the participants were working on their computers, and one of the computers was connected to a projector that would display their constructions.

4.2.1.2 Designing and Planning

Antony suggested that we should encourage learners to remember the basic geometric concepts of circles like radius, chord, arc and circumference. He would draw a circle with a centre, the angle at the centre, the angle at the circumference, measure the angles and then would encourage learners to identify the relationships between them. George added that the learners must be able to understand the properties of a radius – that they are equal everywhere in a given circle. He suggested that as teachers, we have to guide some of the learners to identify the line from the centre to the circumference as a radius. We constructed a circle, an arc, line segments and angles using the *GeoGebra* tools. Alongside, in Figure 4.1, is the first applet that we designed to explore the *angle-at-the centre theorem*.

Paul also suggested that with the applet, the learners could move the points around the circle and look for the ratio between the angles. Moving those points around was an essential part of the DGS and would help the learners to see that the angle at the circumference can be anywhere on the same side of the arc.

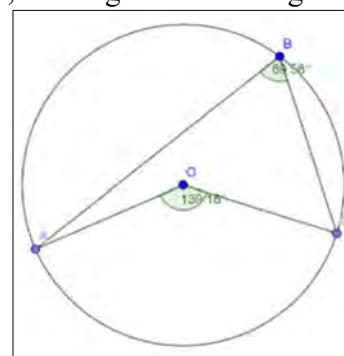


Figure 4.1: Applet 1.1

Paul came forward with a common misunderstanding evident in many learners. He noticed that even when the angle is not at the centre of the circle, or the angle is not on the circumference of the same circle, many learners had the tendency to still apply the theorem. Antony then suggested that we could subtend angles inside and outside the circle from the same arc but not necessarily on the circumference or at the centre. The learners could then move the new point around the screen and verify whether these angles were related to the angle at the centre or the angle on the circumference. We could thus emphasise that the theorem holds only for the angle at the centre of a circle and the angle on the circumference of the

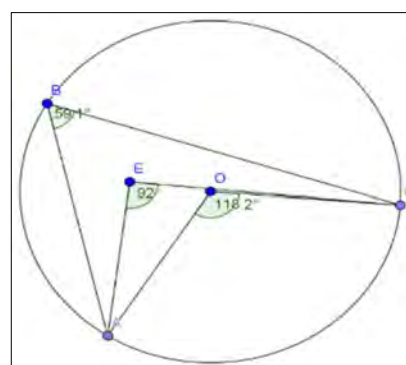


Figure 4.2: Applet 1.2

same circle, thereby addressing these misconceptions. The Applet 1.2, in Figure 4.2 addressed these misconceptions. In this applet, ‘E’ is an independent point, implying that we can drag the point around the screen without any restrictions. Thus, the teachers established that the theorem holds only for the angle at the centre and the angle on the circumference of the same circle.

Paul showed us on the projector that by constructing another angle at the circumference, we could relate this to the next theorem that the angles subtended by the same chord or arc are equal. Antony commented that it would be straightforward if we could do that so that the learners might be able to link these two theorems by themselves. Thus, we collaboratively developed the third applet of the day, Applet 1.3 (shown in Figure 4.3).

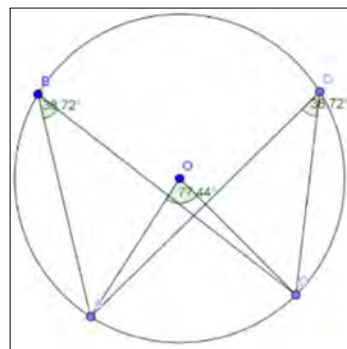


Figure 4.3: Applet 1.3

Antony said that he would move the angle at the circumference to the other side of the arc to make learners understand which angles are referred to in the theorem. Paul pointed out that when we move the angle to the other side, the software measures the reflex angle, and therefore we would have to remeasure the angle at the circumference. George commented that too many angles might confuse the learners and suggested therefore that the reflex angle could be hidden, or else draw another point on the other side of the arc. However, we found a way out of the impasse by colouring the angles, resulting in Applet 1.4 shown in Figure 4.4.

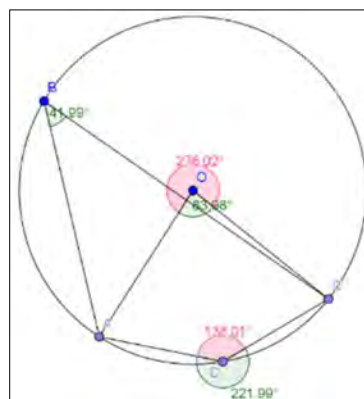


Figure 4.4: Applet 1.4

George proposed that we could deploy the same applet to introduce the concept of the cyclic quadrilateral and its opposite angle property. This Applet 1.4 proves Figure 4.4 the *cyclic quadrilateral theorem* – *opposite angles of a cyclic quadrilateral are supplementary*).

In this collaborative fashion, we designed four applets for the understanding of: 1) *angle at the centre theorem*; 2) *angles subtended by the same chord*; 3) *misapprehensions*; and 4) *cyclic quadrilateral property*. Nevertheless, Antony interrupted and said that we could add one more to

our list when we drag the arc to form a semi-circle. We could then guide learners to investigate a particular case of the *angle at the centre theorem* (i.e. *an angle subtended by a diameter is 90°*).

George was unhappy when he dragged the chord to form a diameter. He said that due to rounding off, the angles were not showing as exactly 90° and it was challenging to drag points to a desired angle at the centre. He brought to our attention that the angle at the centre might not be double the angle at the circumference due to a rounding off mistake in the software. All of us then realised the same and attempted to round off different decimal places which were not satisfactory. Paul suggested that we could draw a straight line passing through the centre of the circle to get a diameter and then draw the angle at the circumference. Paul's idea functioned only for the semi-circle issue; however, the rounding off errors existing in the previous applets was still a concern.

Antony proposed that we could change the angle at the centre to integer values. Then we could overcome the rounding off anomaly of the software. The 'slider' tool allowed us to move the angle at the centre in integer values from 0° to 360° , and we began to construct a new applet. Nonetheless, as Antony was constructing, he commented that the point on the circumference could be dragged around the circle including the other side of the arc, and hence became distorted. The software calculated the reflex angle and could therefore confuse the learner. Paul suggested restricting the point within the desired arc. Thus, after a process of trial and error using different tools, we were able to design Applet 1.5 shown in Figure 4.5.

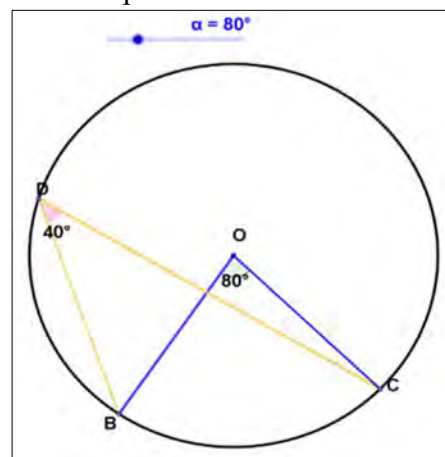


Figure 4.5: Angle at the centre theorem in Applet 1.5

The critical mathematical expression was the angle at the circumference on the same side of the arc. Then we simply needed to create an arc in the figure. All the participants were able to construct it and agreed that it was a useful learning tool. However, on the following day, George came back seeking clarification on using the slider as he was not successful in constructing the applet when he tried to do it again. He did not remove the default 45°

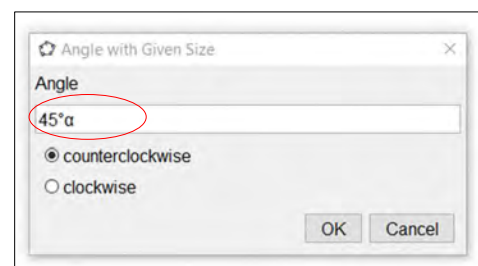


Figure 4.6: Error creating the slider angle

while making an angle at the centre and instead added α along with it, thus measuring an angle different from the slider angle, as in Figure 4.6. Later, when new teachers joined the GLIP community, another two teachers made the same error.

The discussion moved on to providing a proof of the theorem. There was a consensus among the participants that a proof was required, and it would be done at the end of the lesson. Antony quipped that the learners should first understand the relationship between those two angles and then he would provide the proof explaining to them why the relationship existed. George observed that a step by step procedure for a proof was helpful for the learners to grasp the gist of mathematics. We all embarked on creating an applet explaining the proof for the *angle at the centre theorem*.

Paul volunteered to demonstrate his constructions using the projector. All of us agreed that we might not use the actual values of the angles but generalise them as 'x' and 'y'. But the applet looked cumbersome with diagrams and text, so we used the 'Graphic 2' option in *GeoGebra* to display the text, and in this way, we could clearly demarcate between figures and text. The 'play' tool of the software enhanced the applet as the teachers could present the applet as if they were drawing and writing one step after the other. [The play mode can be activated as follows: Select 'Options' from the 'Menu' Bar, then select 'Advanced'. A screen pops up. Select the 'Graphics' tab under 'Navigation bar for Construction steps' Click (check) 'Show' and the 'Play' button]. The navigation bar would enable us to display the diagrams and the text in the order of our (pre)constructions. Antony commented that the learners would then be able to appreciate the reasons behind the *angle at the centre theorem*. Applet 1.8 is thus the proof for the *angle at the centre theorem* shown in Figure 4.19 (p. 107).

4.2.1.3 Practice Applets

Moving ahead with the discussion, we decided on two particular questions for learners to practice. We all agreed that the problems shown by the learners involved the application of properties that they should be aware of such as the *sum of interior angles of a triangle*. The practice question applets are discussed in detail below under implementation. Applet 1.6 and Applet 1.7 are the first and second practice applets, shown in Figure 4.12 (p. 92), Figure 4.13 (p. 93). The preparation of these practice applets involved some knowledge of programming using Java script; consequently they were designed by the researchers themselves. However, Antony and Paul supplied us with the questions.

All three teachers then implemented the applets we had developed, in their classrooms. Due to the limited scope of this thesis and the volume of the data, I analysed only two lessons per cycle. In this cycle I analysed Antony's and Paul's lessons. The first vignette is that of Antony's lesson, followed by a vignette of Paul's lesson.

4.2.2 Analysis of lesson by Antony

4.2.2.1 Act 1 Scene 1 ~ Teaching through Investigation

There were 34 learners in the class, and the lesson schedule was for two hours. Being a novice in a technological classroom (computer lab), Antony faced several challenges during the class. Some desktops and keyboards were not working. Almost 25 minutes were lost in settling down the learners and overcoming initial technical glitches with the computer systems. Antony's lesson focused on investigating the *angle at the centre theorem* using *GeoGebra* with the learners who had had a prior introduction to circle geometry theorems. The teacher helped the learners to construct geometrical figures in *GeoGebra* and attended to other technical issues. Figure 4.8 and Figure 4.7 are snapshots of Antony's classroom and his interactions with the learners.



Figure 4.7: Antony interacting with learners



Figure 4.8: Antony's classroom

4.2.2.2 Act 1 Scene 1 ~ Applet 1

The teacher started the lesson using *GeoGebra* and indicated that they would be investigating the *angle at the centre theorem*. The extract of the transcript below indicates how the teacher introduced the lesson. He asked his class to state the theorem, and a learner stated the theorem

(CC), relating it to his prior learning. The teacher confirmed the theorem by repeating the definition of the theorem (CK) using appropriate mathematical terminology, in lines 12–13 in .

Table 4.2.

Table 4.2: Transcript (CL1A) – stating the theorem

5- 7	Antony: Ladies and Gentlemen let us all open our GeoGebra files. Here we want to investigate centre theorem. Can anyone please state theorem 3 for us?
10	L: The angle at the centre is double the angle at the circumference provided they are subtended by the same chord or arc. (Some learners are referring to the textbook as a learner states the theorem)
12-13	Antony: The angle at the centre is double the angle at the circumference provided they are subtended by the same chord or arc CK.

By adopting an exploratory approach, the teacher engaged the learners in the construction of circles, arcs and angles (CD). He made explicit links between these concepts and the concrete construction on the computer screen. The teacher also responded to learners (FL) who struggled to construct geometric shapes in *GeoGebra*. The teacher frequently ensured (FF) that the learners could follow his directions by providing step by step instructions and simultaneously displaying his constructions on the projector. The learners engrossed themselves in the construction and manipulation of the applets they constructed, while following his directions.

Antony presented geometric concepts, combining them with technological tools. When he asked the learners if they could remember what an arc was (CC), as evident in lines 39–44 in Table 4.3, he was reminding the learners of their prior knowledge of mathematical concepts. Guiding the learners to plot two points on the circumference of the circle, he encouraged the learners to see and visualise the arc (CS), thus concretising the concept of an arc. Antony employed a similar strategy to construct and visualise the angle at the centre and the angle on the circumference (CS) of a circle.

Table 4.3: Transcript (CL1A) – Encouraging learners in the construction of circles and angles

15-17	Antony: Then, that is what we want to investigate using GeoGebra CD. Now we go to the circles, you see there those icons, on the top, there is a circle with the centre through a point. You click there on the circle, then draw your circle with a centre. Are we all there FF?
24-25	Learner raising a concern and teacher attends to it. [inaudible] there is a technical problem here (with the desktop) FL.
39-44	Antony: Then what we can do we can put 2 points along the circumference so that we create an arc CK. Do you know an arc ne? L: Yes sir Antony: Then you put a point on the circumference, and that becomes point A and Point C and see that one is going to be our arc CS.
48-55	Antony: You go to the line segment, and then you join A and the centre. Are we able to join those two? FF Join again point C and O as you can see now we are having an angle at the centre CS. If

	<i>you struggling, then simply indicate by lifting up your hand right. Then now we want to have an angle which is going to be the angle at the circumference. And that is being subtended by the same chord or arc</i> CK.
58-61	<i>(A learner raises a hand). Antony is now helping the learner on the desktop</i> FR <i>No... this is not at the circumference, we want on the circumference</i> FM. <i>(Some learners can construct angle at the centre diagram, drag the point on the circumference while a few learners struggle.)</i>

Antony asked the learners to raise their hands if they struggled with the construction (FF), and he moved around the classroom. He guided learners while constructing line segments and measuring angles: “Click the arrow”, “This point is on the circle.” In *GeoGebra*, it is crucial to click the ‘Move’ tool after a particular construction. If the ‘Move’ tool is not selected, then the diagrams will be bizarre and lose control over the figure. This excerpt highlighted that Antony had become adept at technology (CK).

Antony then instructed his students to measure both angles, i.e. the angle at the centre and the angle at the circumference of the circle they had constructed on their screens using the measuring tool in *GeoGebra*. Most of the learners initially found this challenging because the tool to measure angles has to select points in an anti-clockwise direction. Despite this, they soon developed the appropriate techniques to construct and measure the required angles. The teacher encouraged the learners to take ownership of their constructions (CD). Meanwhile, he corrected (FR) the learner who plotted the point near the circumference by emphasising that the ‘point should be **on** the circumference’, which helped the learner to overcome his misconception (FM). Refer to lines 101–103 in Table 4.4.

The teacher proceeded by motivating the learners to compare the angle at the centre and the angle subtended at circumference (CD) and identified the relationship between these angles. In this way he also referred to their prior knowledge of comparisons of numbers such as less than and greater than, and ratio and proportion (CC). He encouraged learners to discover the theorem by asking questions like “What is it that we see?”, “Compare those angles”, “What can you say about the relationship between those two?”, exhibiting multiple questioning strategies (CMR & FA). Although one of the learners accurately stated that the angle at the centre is twice the angle at the circumference, the teacher verified this with values from a randomly selected learner (FP) and elaborated on the procedure, thereby reifying the theorem. In lines 119–120 in Table 4.4, he then concluded by adding that the “Angle at the circumference is half of the angle that is at the centre”

(CK, CMR & FA). This reverse statement of the theorem indicated that the teacher could represent the theorem in multiple ways. The teachers' proficiency at presenting the theorem in different ways would enable the learners to select appropriate algorithms in problem-solving, i.e. in order to find the unknown angle between the angle at the centre and the angle at the circumference, they would need to double or halve the known angle.

Table 4.4: Transcript (CL1A) – Measuring angles

67- 68	Antony: Go to line segment then join those points A and B and join C and B to construct angle at circumference. FR. (Teacher is moving around helping the learners). Go to line click there, click the arrow. The arrow. Then segment, then join from here to there FL.
70-72	Antony: Do we all have a diagram of that nature. Anyway it won't be like that...it is not a must it is like that CS it can be any because you are able to move that point B along the circumference CD.
80- 91	Antony: Let us measure those angles at the centre that is angle AOC. There is a symbol of that angle then you move in a clockwise direction. Ok, did you manage to measure that angle FF? L: No, No. Antony: You are supposed to click 3 points to show the angle. (Teacher attends to students having difficulty with this).
101-103	L: Angle at the circumference is not half that of the middle one. Antony: Ok, [pause] How can that happen? now can you measure that angle at the circumference FP Your point was not on the circumference. FR. we want on the circumference" FM.
109-120	Antony: Ok now after measuring those angles angle at the centre and the angle at the circumference, what is it that we see CD? Compare those angles FP and tell me what is the ratio that you can say about the relationship CC between those two angles CD. L: The angle at the centre is double the angle at the circumference. Antony: What is the size of the angle at the centre one person? FF. L: 71,44 Antony: Then what is the size of the angle at the circumference? FF. L: 35,72 Antony: Then, what is the relationship between the two? CD L: The angle at the centre is double the angle at the circumference. Antony: The angle at the centre is double the angle at the circumference CK. Or the angle at the circumference is the half of the angle that is at the centre. CMR & FA Angle at the centre is bigger than angle at the circumference, always. When you solve the angles you must know it FP Can you all see that? FF

4.2.2.3 Act 1 Scene 1 ~ Applet 2

There is a common misunderstanding among some learners: applying the *angle at the centre theorem* even when the angle is not at the centre. In order to remove misconceptions (CM) such as this, Antony asked the learners to plot a point D inside the circle and measure the angle \widehat{ADC} , as in Figure 4.9. He emphasised that the angle was subtended by the arc AC (CK), demonstrating accurate mathematical relations and notations. The teacher ensured that the learners could follow his instructions and accordingly, construct the figure(FF), while he prompted them to ask questions

shown in line 131 in Table 4.5. Antony motivated his learners to identify any relationships (CD) between angle \widehat{ADC} and the angle at the centre or angle at the circumference. One of the learners dragged the point around the screen within the circle, on the circumference, into the centre and outside the circle, and concluded that “the angle in the circle is not half the angle at the centre of the circle because the angle is not in the circumference”. Thanks to the dynamic nature of the virtual manipulative, the learner was able to make the discovery.

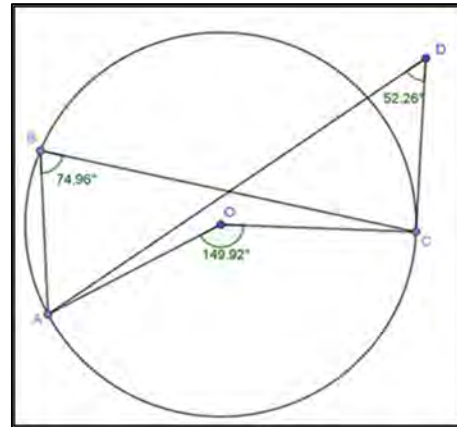


Figure 4.9: Addressing Misconceptions

Table 4.5: Transcript (CL1A) – Addressing the misconception

128-131	Antony: (Giving instructions to plot a point D inside a circle and join then to arc AC) Then you measure that angle, measure the angle that is formed by that one. Measure that angle remember when you measure that angle we talk of an arc then the angle is always subtended the angle will be between A and C CK between that arc. So that angle is ADC. When measuring that angle ADC then measure that angle. Are we done? FF
135-158	Antony: Ok, now is there any relationship now between the new angle and the angle at the centre? CD? Can you relate now the new angle and the angle at the centre? Is there any relationship between those two? Is there any relationship between those two angles the new angle FA & CD? The angle at the centre or the angle at the circumference? yes L: (not audible) Antony: Can you say that to everyone who is here? FF L: I am saying the angle the angle in the circle is not half the angle at the centre of the circle because the angle is not in the circumference.

The elaboration below of the theorem provided evidence of interplay between CK, CMR, FA and CM. The teacher removed the misconception (CM) and thereby threw more light onto the theorem, explicitly stating “One angle must be at the centre, and the other angle must be on the circumference” (CK). By paraphrasing the theorem (CMR), as evident in lines 147–148 in Table 4.6, he reinforced the concept and provided evidence of his range of instructional repertoire.

Table 4.6: Transcript (CL1A) – Paraphrasing the theorem

144-148	Antony: She says there is no relationship between those two angles because now the new angle is not in the circumference CM, but it is inside the circle. So there will be no relationship between those two angles. Because Theorem 3 specifically states that the angle must be at the centre and the other angle must be in the circumference CK & CM. So it proves that immediately the angle we are talking about is not on the circumference then you can't relate those two angles CMR & FA.
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Antony proceeded with the lesson, motivating learners to play with the points on the circumference A , B and C (CD), allowing the learners to discover the variant and invariant properties. This exploration by the learners, supported by the teacher's instructions, concretised the concept (CS) that the angle at the circumference would be invariant if subtended by the same chord or arc (CK). Refer to line 172 in Table 4.7. Further, the learners also established through exploration that the angles changed if the chord or arc was changed, but still maintained the ratio between the angle at the centre and the angle on the circumference (CD).

Table 4.7: Transcript (CLIA) – Emphasising the invariant property

152-172	<p>Antony: Then you go to that point which is on the circumference, and you move it along the circumference, and you see if there are any changes? CD & CM The angles...are there any changes? Don't mind angle D...look at the angle at the centre and the circumference. Are there any changes between those two? CD & FF</p> <p>L: No no</p> <p>Antony: OK. There are no changes. It means even if you move that angle at the circumference as long as it is still subtended by the same chord or arc, then it means then those two angles remain the same. CS</p> <p>What happens now if we move the point of the arc. Let us move the arc and see what is going to happen between those two angles CD, the angle at the centre and the angle at the circumference. can you see? FF</p> <p>L: Yes,</p> <p>Antony: What is happening?</p> <p>L: The angles are changing.</p> <p>Antony: Which one is changing?</p> <p>L: All angles are changing.</p> <p>Antony: Both of them they are changing because they are being subtended by the same chord or arc. Both of them they are changing if you are moving that arc or you are changing that arc you are actually changing the size of those two angles because those two angles they depend on that arc. If you move that arc both angles, they are changing in size. Their ratio does not change CC</p> <p>But if you move the angle at the circumference, nothing is going to happen because it is still subtended by the same arc CK. Their ratio does not change</p>
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4.2.2.4 Act 1 Scene 1 ~ Applet 3 ~ Same side of the Arc

The ensuing investigation revealed Antony's knowledge of instructional practices (CU). Asking learners to plot a point 'F' on the arc CA and to measure the angle $C\hat{F}A$, as in Figure 4.10, he used accurate mathematical terminology (CK). Using the precise mathematical term 'same segment' (CK), he warned the learner that the *angle at the centre* theorem did not apply in that scenario (CM). Besides this, he used in multiple representations of the theorem when he stated "Both angles cannot be inside" (CM), as in lines 210–211 of Table 4.8. He also emphasised that the angle cannot be outside the circle. This notion of 'inside, outside' may not be mathematically appropriate terms, but provided a clear procedure useful in problem solving (FP). He also reified the misconception by measuring and comparing the angles (CS). He challenged the learners to measure the correct angle that could be related to angle $C\hat{F}A$.

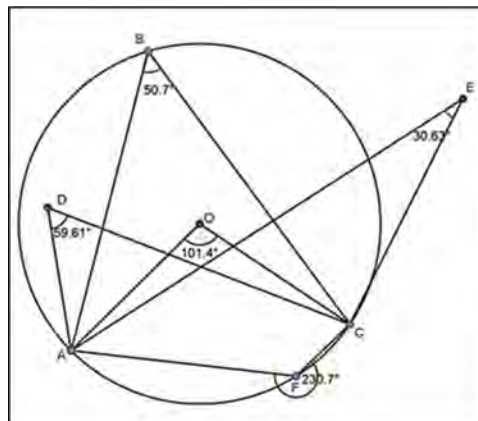


Figure 4.10: Emphasising the same side of the arc

Table 4.8: Transcript (CL1A) – Emphasising the same side of the arc

200-206	<p>Antony: Now let us put a point F on the circumference, on arc CA. can we see that point?</p> <p>L: Yes sir</p> <p>Antony: Let us join AF and FC. Listen those angles we always say that they should be subtended by the same chord or arc. I am talking about the arc AC. Alright, then let us measure now that angle. There is an angle that is formed there let us measure that inside angle CK.</p>
208-218	<p>Antony: While measuring that angle, you are supposed to start from CFA, want to measure that angle CFA. If you can see now, this angle is subtended by the same chord or arc. But this angle is on the same segment CK therefore if you relate now this angle and this angle you can't say angle at the centre now is twice angle at the circumference CM & CS. They must not both be inside CM. You can see now if you consider polygon AFCO now both these angles they are now inside CS. If we talk of Theorem 3, the other angle must be on the outside, and the other angle must be on the inside. Whenever you talk of the angle at the centre being twice as the angle at the circumference, you must check if the other angle is inside and the other angle is outside. Because immediately if both angles are inside, then that one is no longer theorem 3 CM. Because you say that theorem three the other angle must be inside the polygon and the other angle must be outside FP. Do we agree?</p> <p>L: Yes</p>
223-233	<p>Antony: Ok, now can you please measure for me the outside angle, the correct angle we can relate to angle F, the correct angle CD measure that angle FP.</p> <p>I am saying you must measure the angle that you are going to relate to angle F. That is the angle we are talking about is the angle at the centre FR & FM. Measure the correct angle at the centre.</p> <p>Antony: You managed ne? Ok, then now what is the relationship between the new angle that you measure and that angle F FF? So what is the relationship?</p> <p>L: Angle at the centre is double the angle at the circumference.</p>

	<p>Antony: Yes...that angle is double the angle at the circumference. So here what we are actually learning is that the angle that is at the centre and the angle that is at the circumference; the other one must be on the inside and the other one must be outside FP. You see, it is inside out. That is the easiest way to understand [this] theorem.</p>
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4.2.2.5 Act 1 Scene 1 ~ Applet 3 ~ Cyclic Quadrilateral

Antony continued with the same manipulative and sparked the learners' curiosity by asking them **(CD)** to relate the two angles to cyclic quadrilaterals. A learner came up with an accurate definition of a cyclic quadrilateral. Nevertheless, he referred the learner to the properties of angles in a cyclic quadrilateral **(FA)**. In other words, the teacher responded **(FR)** by guiding the learner to focus on the angle properties of a cyclic quadrilateral **(FP)**. When the learner stated that the opposite angles are supplementary, Antony confirmed the concept using another mathematical term **(CK & CMR)**. Thus, without using a formal proof, he linked the *angle at the centre theorem* to the properties of a cyclic quadrilateral **(CC)**.

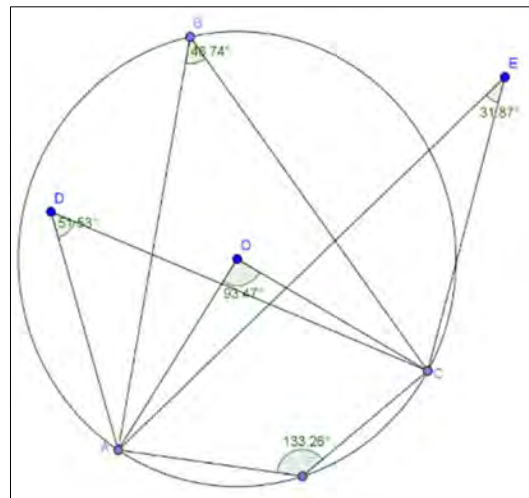


Figure 4.11: Identifying cyclic quadrilaterals

The teacher asked the learners to identify a cyclic quadrilateral among the quadrilaterals in the diagram in Figure 4.11. Once again, a learner came up with a correct answer **(ABCF)**, but the teacher responded **(FR)** by guiding the learner to focus on the angle properties of a cyclic quadrilateral **(FP)**. This time, the teacher calculated the sum of the interior opposite angles and then explicitly stated the converse theorem of the cyclic quadrilateral in lines 264–266 of Table 4.9. Here the teacher proved not only the *interior angle of cyclic quadrilateral theorem* **(FP)**, but also its converse **(CMR & CK)**.

Table 4.9: Transcript (CL1A) – Identifying cyclic quadrilaterals

243-253	<p>Antony: I want us to relate now these two angles CD that is angle F and angle B. What did we say about cyclic quadrilaterals? FF</p> <p>L: All four points are on the circumference</p> <p>Antony: yeah, all for points are on the circumference. And if a quadrilateral is having all four vertices on the circumference, then we say that quadrilateral is cyclic. Then what about the angle CD & CMR? What did we say about the angles? We said...</p> <p>L: Opposite angles add up to 180 degree</p>
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	<p>Antony: Opposite angles, they add up to 180° CK. Now can we add these two angles, angle F and angle B and see what is going to happen there, they give you 180 degrees.</p>
254-266	<p>Antony: Between these quadrilaterals or else those polygons that are there, which one is a cyclic? FF Here I am having ADCF as a quadrilateral and am having AOCF that is another quadrilateral. And having ABCF which is another quadrilateral. Then I am having three quadrilaterals there. Then which one is a cyclic quadrilateral? CD Between those 3? L: ABCF Antony: ABCF Why? L: Because all the points are on the circumference. Antony: Let's talk of angles FR L: Two opposite interior angles are supplementary. L: Opposite interior angles are supplementary Antony: opposite interior angles are supplementary. CK This one in my case is $124,86^\circ$, and the other one is $55,14^\circ$. Which if you add them you get a what? 180° CMR. Thus that proves that one is a cyclic quadrilateral. That one is a cyclic quadrilateral.</p>

4.2.2.6 Act 1 Scene 1 ~ Practice Applets

Antony provided the learners with two applets to practise on the questions related to the *angle at the centre theorem*. Figure 4.12, showing Applet 1.6, was the first practice question that the

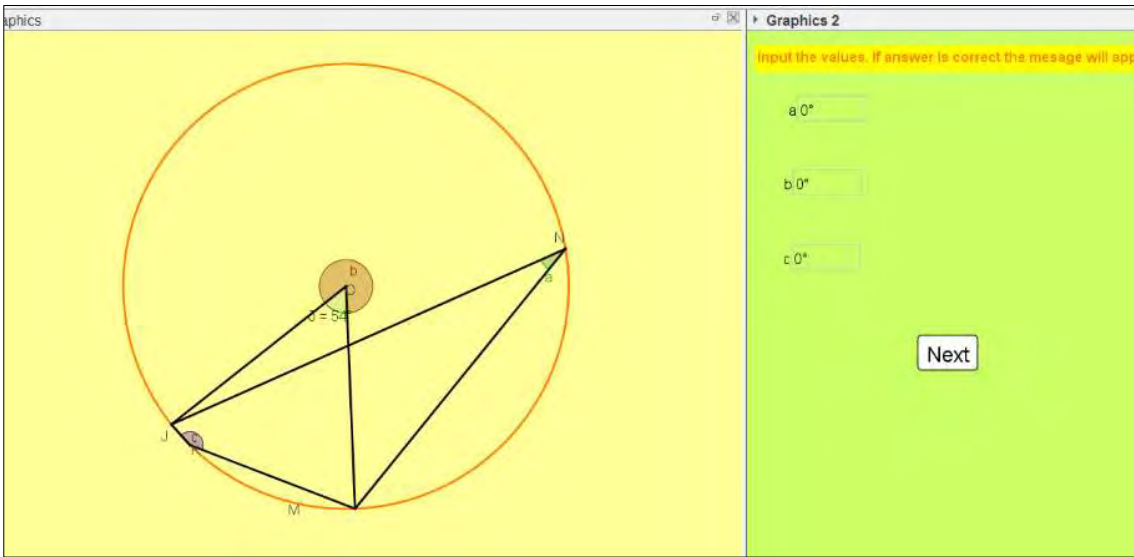


Figure 4.12: First practice applet – Applet 1.6

learners worked on with enthusiasm. Antony explained and clarified to the learners as he was moving around from desk to desk. He attended to the queries of the learners, and thus his words and actions indicated **FR** and **FP**. However, more often than not, he avoided providing direct answers and instead, guided the learners through strategic questioning. Before he asked learners to attempt the second practice applet – Applet 1.7 – he drew the attention of learners. He clarified the *angle in a semicircle theorem* (**FR** & **FP**), demonstrating accurate explanations of mathematical

concepts (CK) and justifying its links to the *angle at the centre theorem* (CC). When a learner sought assistance to solve angle $B\hat{A}E$, Antony did not provide a direct solution but directed her (FP & FR) to the *semicircle theorem* using the different term ‘diameter’ (CMR) shown in Figure 4.13. Furthermore, he supplied an appropriate procedure to find the unknown angle (FP), but only provided an implied link to the *sum of the angles property* of a triangle (CC). He also used accurate mathematical term ‘corollary’, (CK) connecting the *angle at the centre theorem* with the angle subtended by a diameter. Nevertheless, he was flexible enough to use the layman’s description of the *semicircle theorem* by using the phrase ‘form a certain angle’ (CMR), as in line 318 in Table 4.10.

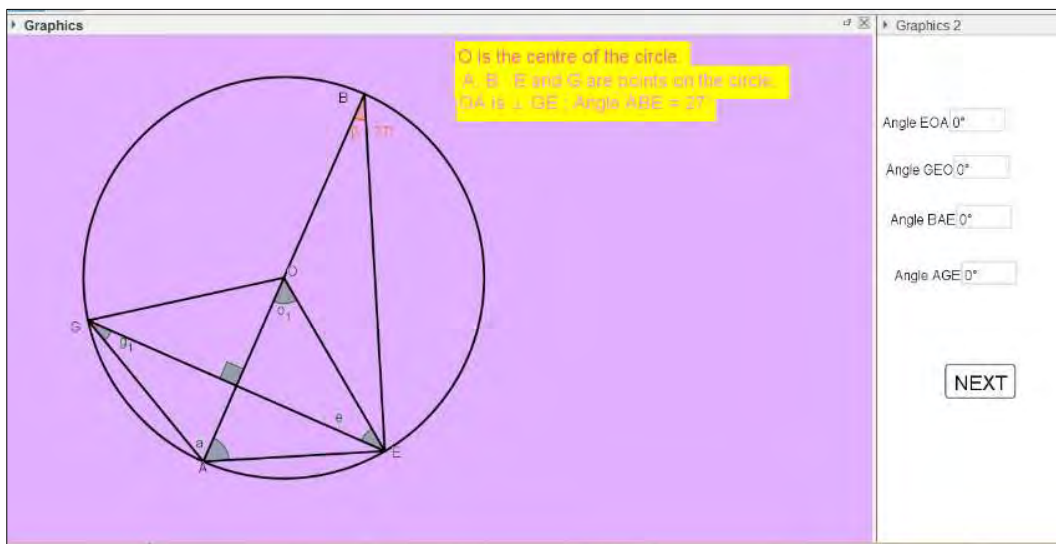


Figure 4.13: Second practice applet – Applet 1.7

Table 4.10: Transcript (CLIA) – Establishing a corollary of the theorem

303-323	<p>Antony: (moving around and helps the learners in solving the questions) The angle subtended by a diameter is 90 degrees. CK Because we say that subtended at the centre is twice the angle at the circumference and diameter is a straight line and angle subtended is therefore 90 degrees. CC & FA (pause)</p> <p>Antony: We said that the angles subtended by the same chord or arc are equal CK, i.e. theorem, Theorem 4, i.e. Theorem 4. So whenever you deal with chords, you are supposed to check if the angles are not subtended by the same arc or chord. FP & CK If they are subtended by the same chord, then it means that they are supposed to be equal.</p> <p>L: What is angle BAE (practice 2 question)?</p> <p>Antony: What is the value of the angle that is subtended by the diameter? FF (A pause). An angle subtended by a diameter? FR & FA & CMR</p> <p>Learners: 90 degrees.</p> <p>Antony: It is a right angle CMR. Therefore, it means this angle is 90 (BEA), and you already know that one (B) and then you find this one (BAE). The angle subtended by a diameter is 90 degrees FP & FF. You need to know that information. That is the converse, sorry, corollary, of theorem 3 CC. The angle subtended by a diameter is 90 degrees CK. If a diameter forms a certain</p>
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	<p><i>angle on the circumference, that angle is supposed to be 90 degrees</i> CMR <i>Do you have GeoGebra in our tablets, for those who have tablets?</i></p> <p><i>Learners: No, no.</i></p> <p><i>Antony: You don't have GeoGebra. You are supposed to install GeoGebra so that we can be able to give you these applets, and then you play around on them for practice purposes. Then it means to install GeoGebra in our tablets so that it can be given these applets and then you play around.</i></p>
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4.2.3 Analysis of the lesson by Paul

4.2.3.1 Act 1 Scene 2 ~ Teaching through Exploration

There were 48 learners in the class, and the lesson was scheduled for one and a half hours. Paul faced several challenges during the class. Some desktops and monitors were not working; therefore, some learners had to share the computers. Some learners were working on their laptops, but these were not connected, so Paul had to share the files one by one. He did not anticipate this situation, and it took some time for all learners to access the applets. Those learners who were using desktops were using a shared folder to access the files. Paul's desktop connected to the projector did not function, therefore, he set up his laptop and connected it to the projector. Almost 20 minutes were lost before he started to focus on the *angle at the centre theorem* using *GeoGebra*. Paul wanted to provide an overview of the *angle at the centre theorem* and the theorems that are linked to it. Throughout the lesson, he employed predesigned applets. While displaying the applets with a data projector, Paul instructed the learners to explore the different aspects of the theorem. He initially asked learners to do an investigative pen and paper task on the topic before this lesson.

4.2.3.2 Act 1 Scene 2 ~ Applet 1.5

The teacher started the lesson using the *GeoGebra* applet and indicated that the lesson aimed to gain an overview of the *angle at the centre theorem*. The extract of the transcript below indicates how Paul introduced the lesson. He encouraged learners to remember the pen and paper investigation on circle geometry. Straight away, he started on the theorem and asked the learners to identify the angle at the centre and the angle at the circumference from the predesigned applet as in Figure 4.5 (p. 82), thereby linking the concepts with the figure (CC and CS). He encouraged the learners to identify the mathematical term for 'BC' (CD & CS) and used different terminology for a portion of the circle as an 'arc' and 'segment' as shown in lines 49–50 in Table 4.11. When asking the whole class to divide the angle at the centre by two and verify whether it measured the same as the angle at the circumference, one learner did not follow the instructions. Paul attended

specifically to that learner and showed him the angle at the centre, the angle at the circumference, and their relationship (FL).

He then motivated the learners to drag the point 'D' on the circumference (CD) and to verify whether it maintained the ratio of 2:1 between the angles. Some learners moved the entire circle instead of moving the point on the circle, but the teacher noticed this and help them to restore the applet, showing his skill in using the software (FR). He showed them that irrespective of the position of the angle at the circumference on the same side of the arc, the *angle at the centre theorem* holds (CS). He employed layman's language such as 'other side of the angle' to visualise the positions of the angle at the centre and the angle at the circumference. After that, Paul accurately stated the theorem (CK), line 67 in Table 4.11.

Table 4.11: Transcript (CLIP) – Exploring and stating the theorem

29- 50	<p>Paul: Right! We are logged in and opened applet 1.1 L: Yes sir, Yes Paul: We have opened that applet 1.1? L: Yes Paul: Let us go to Applet 1.1. Still open? L: Yes Paul: Right? What we want to do here, you still remember the investigation that you did (referring to a previous class or task on an investigation using pen and paper) looking at Euclidean geometry. L: Yes Paul: Do you remember? L: Yes Paul: We want to take an overview of all the theorems here. Today we can, and as you can see there, there are two angles. What is the angle O called? (CC, CS) L: That one is an angle at the centre Paul: And angle D is called? (CC, CS) L: Circumference Paul: Angle on the circumference. Yes, D is on the circumference, and angle CDB is called angle on the circumference. What is BC called? (CC, CS, FF) L: Segment. L: Arc Paul: Here, BC is arc, or segment (CMR). If you can quickly check with your calculator or without calculator divide that the angle at the centre by two and can you see what you get?</p>
54-67	<p>Paul: Select D and drag it along the circle. Just drag it. (CD) L:(inaudible). Paul: It must move. Is it moving? Let us see if the angle O at the centre divided by 2 is angle D at the circumference L: Yes. yes L: Mine is not. [inaudible]. Paul: Don't make it bigger. Right. (FR) (Inaudible sounds). Paul: No matter what position you are at as long as you are on the circumference (CMR) and that one is standing from the centre (referring to the projector) and as long as its on the other side (CS), you see that that one D is on the other side. The angle subtended by an arc at the centre of a circle is twice the angle subtended by the same arc at the circumference (CK). L: Yes sir, Yes. Paul: As long as you move your D you will find (CD) that Paul and L: The angle at the centre is twice the angle at the circumference. (CK)</p>

Next, the teacher wanted to change the angle at the centre and asked learners to ‘move alpha’ (see Table 4.12). Once the learners moved the angle at the centre, they discovered (CD), that the angles maintained their relationship according to the theorem (see Table 4.12). He also exhibited his mathematical knowledge as well as his ability to deal with learners’ lack of basic understanding (CK & FL), by correcting a learner who stated that it was the angle on the circumference that changed when the slider was dragged and not the circumference. Paul drew the learners’ attention to the fact that the arc ‘BC’ also changed when he moved the slider. He emphasised and concretised the concept (CS & CD), by encouraging them to ‘move D’, the angle on the circumference, to different positions for a given angle at the centre.

Table 4.12: Transcript (CLIP) – Exploring the theorem further

68-100	<p>Paul: Move that alpha and check it (CD).</p> <p>L: (Inaudible)</p> <p>L: (moving the text narration instead of moving the slider). Move it. Do you see what is happening? What is happening when you move?</p> <p>(Inaudible. There is some talk among some learners).</p> <p>Paul: Don't move alpha. The dot there next to alpha. Click on it and drag. FR</p> <p>L: Ooh yeah. (Laugh)</p> <p>Paul: Now, do you see that?</p> <p>L: Yes sir.</p> <p>Paul: Move that alpha and check. Move it. Move It. And as you move it check the relationship (CD) between those two angles. Are they maintaining the relationships that you are saying.?</p> <p>L: (moving the slider and exclaiming) Yes!</p> <p>Paul: Does that show the angle at the centre is twice the angle at the circumference (CS)?</p> <p>L: Yes yes I see that (excited)</p> <p>(Inaudible).</p> <p>Paul: You can play around by moving the slider alpha and then move D. See what is happening? I want you to check for different values of alpha which is the angle at the centre (CD). Move the slider from one end to the other. Check what is happening?</p> <p>L: The angle at the centre is changing.</p> <p>Paul: Ok. What else is changing (CD)?</p> <p>L: Circumference.</p> <p>Paul: Angle on the circumference (CK). What else?</p> <p>L: (Talking inaudible).</p> <p>L: (After moving the slider from one end to another, he brings to $\alpha = 180^\circ$ (more than twice he comes to this angle 180°))</p> <p>Paul: Who is calling me?</p> <p>(Inaudible).</p> <p>Paul: Arc is also changing (CC). You can see that. This is the theorem that you were investigating in your investigation (task). That you were investigating the theorem that state that angle at the centre is twice the angle at the circumference (CK). Are we together?</p> <p>L: No response.</p> <p>Paul: You can move that or just, move your alpha see what is happening? I want you to check for different values of alpha (CD). Say 60°, 90°, 150°, 180°. Move your D. It may turn the same (CS). No matter what you do or move. Even if you move C, is C moving? No. It's only that D is moving.</p>
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4.2.3.3 Act 1 Scene 2 ~ Applet 1.5 ~ Angle on a Semi-Circle

Paul continued to encourage learners to move the slider to different values (CD) of the angle at the centre. He asked learners to conjecture (CD) about the angle on the circumference when the angle at the centre was 180° . He then motivated them to move the angle at the centre to 180° (CS) and verify for themselves their conjectures. This is recorded in lines 116–123 in Table 4.13. Paul sparked the learners' interest by asking them to identify the line 'BC' (FF & CC) shown in Figure 4.14. The learners correctly recognised 'BC' as a hypotenuse, but the teacher directed their attention to the line 'BC' as part of the circle. He specified that the *semi-circle theorem* was a special case of the *angle at the centre theorem* (CC). He stated the *semi-circle theorem* accurately (CK), in two different ways and using mathematical terminology. (CMR). Although he encouraged learners to generalise what they observed (CD), ended the lesson by linking (CC & CMR) the hypotenuse of a right-angled triangle to the diameter of a circle.

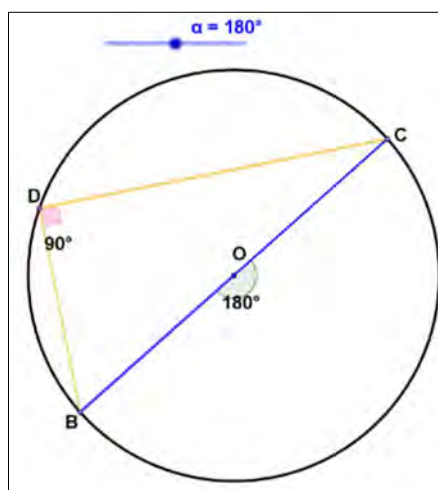


Figure 4.14: Angle on a semicircle

Table 4.13 Transcript (CLIP) – Linking hypotenuse and diameter

116-135	<p>Paul: What will be angle D if our alpha is at 180° (CD)? I want you to calculate. CC</p> <p>L: It's 90°</p> <p>Paul: Yeah Now move alpha to 180 (CS)</p> <p>L: [inaudible]</p> <p>Paul: What is the angle at the centre (FF)?</p> <p>L: 180°</p> <p>Paul: And what is the angle at the circumference?</p> <p>L: 90°</p> <p>Paul: So there would be so many questions that will be linked to angle 90°. You can see that angle at the centre is now 180 (CS). Therefore, line segment BC is called, what? (FF CC)</p> <p>L: Hypotenuse</p> <p>Paul: Yeah, its hypotenuse when you see the triangle. What is BC called when you see the circle? (FF, CC)</p> <p>L: Diameter.</p> <p>Paul: Yes, BC is also a diameter (CC, CMR). Are we together? CS</p> <p>L: Yes sir</p> <p>Paul: So there would be so many questions that will be linked to this (FP). You can be asked to prove that it is a diameter (FA). Is that ok? BC can be diameter of a circle if BC can form 90° (CMR). This a special theorem of angle at the centre (CC). You can also say it as an angle subtended by diameter on the circumference or angle formed by a semicircle is 90° (CK, CMR)</p> <p>L: Semicircle is 90°</p>
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140-147	<p>Paul: all of them come from the angle at the centre is twice the angle at the circumference. CC</p> <p>Paul: Are we together?</p> <p>Paul: Can we conclude anything else? CD</p> <p>L: The angle at the centre is twice</p> <p>Paul: Yes we already investigated that</p> <p>L: Angle made by the semicircle is 90 degrees.</p> <p>Paul: We also have seen that angle subtended by a diameter is 90° CMR. Anything else?</p> <p>L: [Talking inaudibly]</p> <p>Paul: Hypotenuse of a right-angled triangle will be the diameter of a circle. CC, CMR</p>
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4.2.3.4 Act 1 Scene 2 ~ Applet 1.2

Paul dealt with two common misconceptions using the second applet. Firstly, he explained to the learners that the *angle at the centre theorem* did not hold if the angles were not at the centre or on the circumference respectively, as shown in Figure 4.15. Referring to angles $A\hat{C}B$ and $A\hat{F}B$ on the applet, he reiterated that if the angles were on the same side of the circumference, the angle at the centre would be twice their value CK.

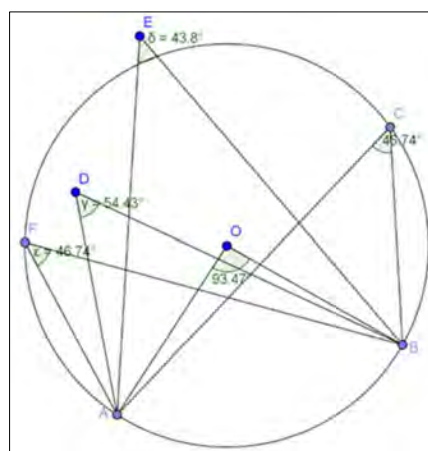


Figure 4.15: Addressing misconceptions A

Paul asked the learners to check whether two times the value of angle $A\hat{D}B$ (a point inside the circle) would yield the value of the angle at the centre CS. He also used the procedure of *half the angle at the centre* to verify whether it would give the value of angle $A\hat{E}B$ (a point outside the circle). He showed learners to use ‘half’ and ‘double’ appropriately CS, CMR & FA. He emphasised that the theorem holds only for the angle at the centre and other angles on the circumference CK & CM and not anywhere else (lines 173-174 in Table 4.14). Paul encouraged the learners to drag the points to change the angle at the centre. He convinced them that only angles on the circumference $A\hat{C}B$ and $A\hat{F}B$ were half the angle at the centre, but not $A\hat{D}B$ and $A\hat{E}B$ CM & CS, even though they are formed by the same arc BA.

In dealing with the second misconception related to the same side of the arc of the *angle at the centre theorem*, Paul asked the learners to drag one of the points on the circumference around the circle (CD) as in Figure 4.16. He linked the figure to their prior knowledge on reflex angles (CC & CD) and encouraged them to calculate the remaining angle (CC) using the appropriate procedure (FA). In this way, he proved to them that the revolutionary angle of \hat{C} , which in this case was $118,5^\circ$ was certainly was not half of 123° (the angle at the centre) (CS & CM).

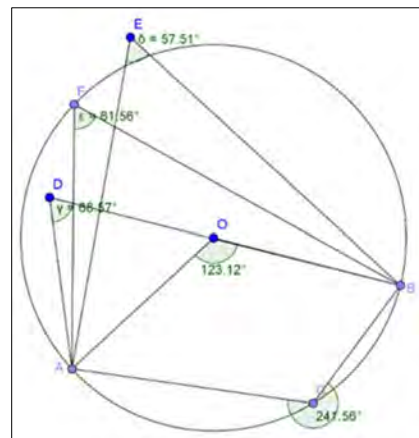


Figure 4.16: Addressing misconceptions B

He drew the attention of the learners to the fact that \hat{F} , at $61,5^\circ$ was still half of 123° (CK). Paul asked the learners to identify the applicable angle at the centre (FR & CD) with respect to \hat{C} on the circumference. Purposefully avoiding rounding-off errors, he then asked them to find the revolutionary angle of \hat{D} , and verified that half of its value (237°) would be \hat{C} ($118,5^\circ$) (FA & CK).. He used two concrete examples to match up the appropriate angle at the centre with the angle on the circumference in order to apply the *angle at the centre theorem* correctly – thus addressing the second misconception. This is recorded in lines 213–214 of Table 4.14. He concluded by restating the *angle at the centre theorem* with emphasis on the same side of the arc (CK & CM).

Table 4.14: Transcript (CLIP) – Addressing misconceptions

154-189	<p>Paul: Just close Applet 1.1. Then you go Applet 1.2</p> <p>L: (talking)</p> <p>Paul: Applet 1.2 you must see that diagram once you opened.</p> <p>L: Yes</p> <p>Paul: Right. So you can see here. Angle C and angle F. No matter how many angles we have as long as it's on the circumference, it will twice the angle at the centre (CK, CMR).</p> <p>L: (confused)</p> <p>Paul: Now, can you see angle D. Can you see angle D?</p> <p>L: Yes</p> <p>Paul: Is [it] on the circle or inside or outside the circle? (FF)</p> <p>L: Inside</p> <p>Paul: If you multiply that angle, do you get angle O? (CS, CM)</p> <p>L: (after a pause) No.</p> <p>Paul: Look at angle E. Look at angle E.</p> <p>L: Yes</p> <p>Paul: Is it inside, outside or on the circumference (FF)?</p> <p>L: Outside</p> <p>Paul: Is it half the angle at O? (CS, CM, CMR)</p> <p>L: No.</p>
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	<p>Paul: It is just for you understand that it is only the angle at the circumference not anywhere else that holds the theorem. (FA,CK,CM). Are we together?</p> <p>L: Yes</p> <p>Paul: Is that clear?</p> <p>L: Yes</p> <p>Paul: So if you can check and you can play around with B (CD)and if you check, B.</p> <p>L: [inaudible murmur]</p> <p>Paul: Can you say what is happening (FF)?</p> <p>L: (no response)</p> <p>Paul: Yes, of course, angle D and E are changing, and both come from the same arc BA, but the change is not the half that one of at the centre (CM, CS). Are we together?</p> <p>L: Yes</p> <p>Paul: So please its only angle at the circumference that will be half the angle at the centre. And normally we say that in the same segment (CK). Because all the angles there and we are looking at all the angles that are either above you notice (CMR)</p>
191-214	<p>Paul: Right. Move C to go below just as I have done there. Right. What is happening? (CD)</p> <p>L: The angle is reflex now.</p> <p>Paul: Right but is it related to the angle at the centre?(FA, CC, CD)</p> <p>L: [inaudible]</p> <p>Paul: Let us calculate the revolution angle of C (CC). (pause) Can you now see that angle is no longer related to that angle at the centre (CC CK) which is 123 on the projector? (learners are busy with their calculations using values on their computer). And my angle C is 241,5. Please look at the projector. So what is the value of the remaining angle of C(CC)?</p> <p>L: 118,5.</p> <p>Paul: Right. Is it related to 123°?</p> <p>L: No. No.</p> <p>Paul: So it is no longer related to angle 123. It will be related to which angle(FR, CD)?</p> <p>L: [inaudible].</p> <p>Paul: We see that 118,5 is not half of angle 123 (CM). Look at the projector there. Can you see this angle C is no longer half this one at the centre (CM)? It will be half of the other one, Reflex O(CK). Can you see reflex angle O? Calculate the reflex of O?</p> <p>L: 236</p> <p>Paul: So It will be 360-123, i.e. 237 (FA)</p> <p>L: Yes</p> <p>Paul: Let us take another example. Move B. Ok That will be 140 at the centre. So this angle will be 250 not related to 140. 140 is related to just F only, which is 70. Are we together? So that is we say it will be on the same side of your chord or arc (CK, CM), (pause) angle of the centre is twice the angle subtended by the arc on the same side of the arc on the circumference.</p>

4.2.3.5 Act 1 Scene 2 ~ Applet 1.3

In the third applet of the lesson (Figure 4.17), Paul showed the learners how the theorem ‘angles on the circumference subtended by the same segment or arc are equal’ is linked to the ‘angle at the centre theorem’. Paul asked them to notice that the angles \widehat{ADB} and \widehat{ACB} coming from the same arc AB were equal (CK & CS), and then he stated the theorem, (lines 235–236 in Table 4.15). He repeatedly reminded them that the angles on the circumference must originate from the same arc in order to be equal (CM & FA). He also linked this to the angle at the centre theorem and applied it to prove that $\widehat{D} = \widehat{C}$ (CC & CK). Further, Paul encouraged learners to move the points around to confirm that the angles from the same segment were equal (CD).

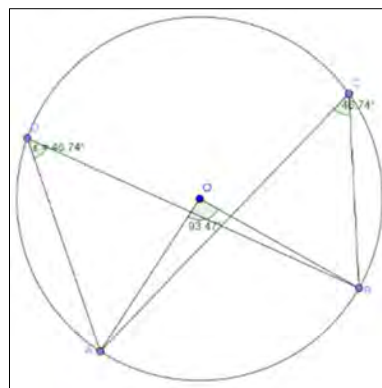


Figure 4.17: Angles subtended by the same arc

Table 4.15: Transcript (CLIP) – Establishing a corollary of the theorem

222-240	<p>Paul: This is just the same thing that we are talking about. That the angle ...</p> <p>Paul & L: ... at the centre is twice the angle at the circumference.</p> <p>Paul: And as you can see D and C because they are all coming from AB, they become equal (CK, CS). And that theorem that we say that angles in the same ...</p> <p>Paul & L: ... segment are equal.</p> <p>Paul: The angles subtended by the same segment on the same side of the circumference of the chord are equal (CK). Can you see that?</p> <p>L: Yes</p> <p>Paul: As long as they start from the same what? (does not seem to wait for the response) Arc or chord (CM & FA). If we check if I use mine AB, it goes to O and O is related to D. Its twice the D? But at the same time, O is twice C (CC). Therefore, C and D are equal (CK)</p> <p>L: Equal</p> <p>Paul: That is proof that we are going to use that the angles in the same segment are ...</p> <p>Paul & L: ... equal</p> <p>Paul: Are we together?</p> <p>L: Yes.</p> <p>Paul: Right. You can now play around and whatever you can and move around. (Pause) You can move any point whatever you want to move (CD). It won't change. Those angles C and D would just be the same. Can you see that?</p>
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4.2.3.6 Act 1 Scene 2 ~ Applet 1.4

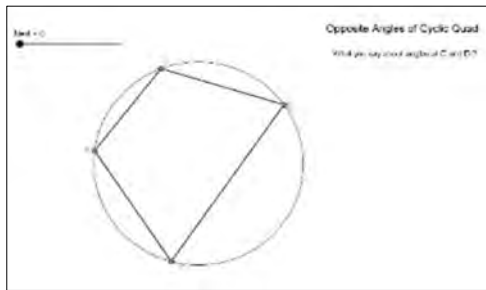
Paul moved on to the next applet, shown in Figure 4.18. This applet was refined a day before the lesson by Paul, consequently, this fourth applet presented by Paul was different from the previous one in Figure 4.4. He requested that there should be text in the applet to prove the *opposite angles of a cyclic quadrilateral theorem*. It is therefore a five-step applet, allowing learners to make

conjectures and then verify them. However, Paul went straight to the last step of the applet to display the proof of the theorem, as shown in Step 5 of Figure 4.18. Later, he admitted that a time factor that did not allow him to involve learners in that applet.

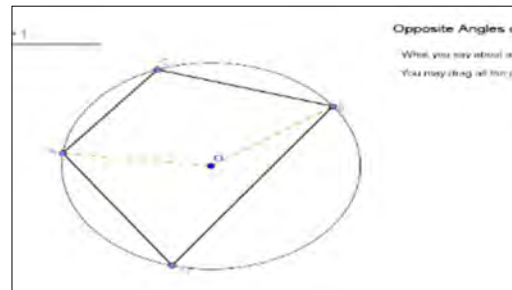
Paul asked the learners to focus on the relevant arc and the angles it subtended at the centre and on the circumference. He relied on colour to connect to the *angle at the centre theorem* (CC & CS). The learners were encouraged to move the points around on the circumference and verify whether the opposite angles added up to 180° (CD & CMR). The discussion is recorded in lines 252–253, Table 4.16. After that, he stated the theorem using apt mathematical terms (CK). He used both the terms 180° and ‘supplementary’, thereby coding CMR.

Table 4.16: Transcript (CLIP) – Opposite angles of a cyclic quadrilateral

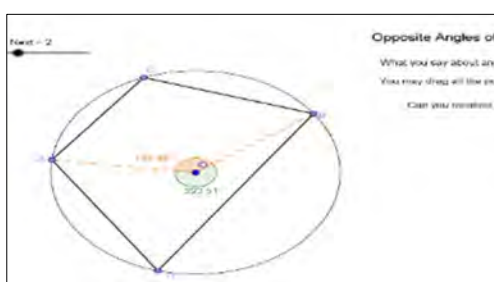
245-259	<p>Paul: Open that one 1.4. Here we are now looking at both sides. If you notice C, angle C. Can you see angle C?</p> <p>L: Yes</p> <p>Paul: It is half of that angle θ, the big angle here, different colours. Can you see that angle? Look at the arc AB. It subtends an angle O at the centre and C and D on the circumference. They are highlighted as the same colour. This angle O, green colour, is twice that of angle C. and this one reflex O, orange colour, is twice that of angle D(CC).</p> <p>L: Yes</p> <p>Paul: If you can move, you can move, that is why it is already there for yourself to move. It is changing what is happening? (CD) [Inaudible]. Right. Opposite angles add up to 180°. (CMR) Is it clear now?</p> <p>L: Yes</p> <p>Paul: This is the proof of the theorem that states the opposite angles of a cyclic quadrilateral are supplementary (CK). It comes from the angle at the centre theorem (CC).</p>
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Step 1



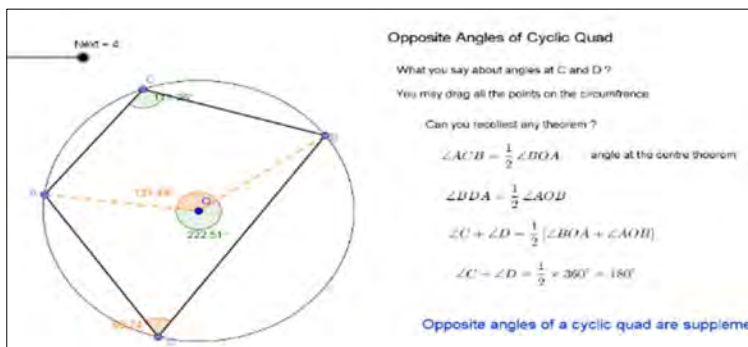
Step 2



Step 3



Step 4



Step 1

Figure 4.18: Opposite angles of a cyclic quadrilateral

4.2.3.7 Act 1 Scene 2 ~ Unknown applet

Extending the theorems on cyclic quadrilaterals, Paul asked his learners to open another applet which he had found on the internet. The GLIP teachers did not generate this applet. This applet on the *exterior angle of a cyclic quadrilateral* displayed all the four exterior angles and their equivalent interior opposite angles in a cyclic quadrilateral. Paul claimed that he downloaded it from a website and thought it would be useful. He reminded his learners how an exterior angle was formed (CC) as the learners engaged with the applet. He asked the learners to click the checkboxes on the applet one after the other and encouraged them to relate the exterior angle so

produced with its corresponding interior angle in the cyclic quadrilateral (CD & CS). Refer to lines 274–282 in Table 4.17. He then stated the theorem (CK) – ‘The exterior angle of a cyclic quadrilateral is equal to its interior opposite angle’. After showing all the four exterior angles in the applet, Paul focussed on one pair of exterior and interior opposite angles. At the projector, he explained the reason why they are equal by placing a ruler on the line DC (Set 3 of the figure), so producing an exterior angle. Paul reminded the learners about the adjacent angles of a straight line property (CC), in this case, $ext.\hat{C} + int.\hat{C} = 180^\circ$. He then linked this to the interior angles of the cyclic quadrilateral property (CC), i.e. $int.\hat{E} + int.\hat{C} = 180^\circ$. In this way, he proved that $ext.\hat{C} = int.\hat{E}$ (CS). He concluded the lesson by stating that theorem once again (CK).

Table 4.17: Transcript (CLIP) –Exterior angle property of cyclic quadrilaterals

266-294	<p>Paul: Go to GeoGebra. Applet 2. Then you look for the exterior angle of a cyclic quad. Are we there?</p> <p>L: Yes sir.</p> <p>Paul: If you click that Set one. Can you see what's happening? Please take note of exterior angle that will be formed by producing existing sides of a cyclic quad (CC)</p> <p>L: Cyclic quad</p> <p>Paul: Are we together? Click the set one. Now click on set two. And see what is happening there(CD). Can you see it?</p> <p>L: Yes</p> <p>Paul: And can you see how they are related(CD)? The exterior angle is equal to (pause) opposite interior angle (CK).</p> <p>L: Interior angle</p> <p>Paul: Are we together?</p> <p>L: Yes</p> <p>Paul: Look at set 3. Just click at set 3. Can you see what is happening? (Pause, helps a learner on the computer). Look at set 4. [Pause] And once we are there. If you check your angle C interior angle C and interior angle E. they add up to 180° (CK).</p> <p>Paul & L: 180</p> <p>Paul: Let us just try to make it. But it is not coming to 90° now. But you can quickly check that they are supplementary (CMR). Is that clear? (a pause) Let me do this. (go to the projector and showing with a ruler) This angle (pointing to C) and this angle (pointing to E) are supplementary add up to 180° (CC). Reason Opposite angles of a cyclic quad. Then this angle (exterior angle C) and this angle (pointing to interior angle C) because they are on a straight line add up to 180° (CC)</p> <p>L: 180</p> <p>Paul & L : 180</p> <p>Paul: So if these two add to 180°. But this one int. C and this one (int E) are 180. Therefore, this one (ext C) must be equal to this one (int E) (CS). This theorem we are looking at Exterior angle of a cyclic quadrilateral is equal to interior opposite angle (CK).</p> <p>Paul & L: Interior opposite angle</p>
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4.2.3.8 Act 1 Scene 2 ~ Practice Applets

Paul provided learners with two applets to practice on the circle geometry theorems starting with the first practice applet, Applet 1.6. He encouraged the learners to look at the angle at the centre and he moved the angles on the circumference. He then answered a question (FA) and informed

learners that the applet would not respond when inputting a wrong answer but would indicate a correct answer. The learners soon became engrossed with the first one (Figure 4.12). Paul encouraged them to move the points on the circumference if they were not sure and learn by themselves whenever they made mistakes (CD). He moved around the classroom, helping learners who had difficulty in solving the questions. He did not solve the problems for them; instead, he asked questions, such as “Where is the angle at the centre?” and “What do you know about angles around a point?” (FF & FL). This is shown in the transcript in Table 4.18. When a learner answered that ‘c’ was 50°, the teacher pointed out her mistake (FM & FR). He also corrected another learner’s mistake (FL & FP) and told him how opposite angles of a cyclic quadrilateral (FP & FA) were related to the *angle at the centre theorem* (FL & CC). In another instance, Paul addressed a learner who wrongly connected the *angle at the centre* to the *angle on the circumference* (FM),, by instructing the learner on how to link and apply the *angle at the centre theorem* (FR, FP & FA), and also reminded the learner about the angles around a point (FL & CC).

Table 4.18: Transcript (CLIP) – Attending learners in solving practise questions

361-371	<p>Paul: Why are you saying $c=50$ (FM)? Opposite angles of a cyclic quad are not equal, but they are supplementary. You are supposed to subtract from 180 (FR, FP, FL). You remember, ne? Most of these theorem starts from the angle at the centre (FL, CC). Ne? That is why we say opposite angles of cyclic quadrilaterals are supplementary (FP, FA). Please do as many questions as possible. You also link it to your investigation. You link it to the investigation that you are doing. Guys if you are done with that there is also practice 2.</p> <p>L: Oh !</p> <p>Paul: If you can do questions on one, then you can do more questions from two. You just minimise it. Why are you saying $b=110^\circ$ (FM)? Is angle b at the centre? Learn to relate angles with chords and arcs (FR, FA). Look at the arc or rotate it and then apply the theorem. The angle at the centre must be on the other side of the angle on the circumference. (FA)</p> <p>L: So this is 220, am I correct?</p>
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Meanwhile, some learners embarked upon the next practice question, Applet 1.7 shown in Figure 4.13. Paul’s strategy was to ask leading questions, such as “What is the sum of the angles in a triangle?” and “What can you say about an angle in a semicircle?”. The transcript in Table 4.19 shows this (FL & FR). He responded to learners’ queries and led them to basic ideas. Besides this, he provided methods with explanations on where to focus and identify the related angles (FL & FM), and then apply the theorem (FP) as in the transcript in Table 4.19.

Table 4.19: Transcript (CLIP) – Responding to learners’ methods

397-400	<p>Paul: Can you see AE let us look at all the angles that are formed by AE (FM). First one is what GI we have OI at the centre (FL). We have B also subtended (CMR) by AE, which is given there ($B=36^\circ$) (FL). Are we together? Remember angle at the centre is twice angle at the circumference. Can you see that? So this angle OI would be twice this B (FP).</p>
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Paul also asked for reasons when a learner answered correctly. When a learner replied with the value of \hat{GEO} , he sought the reason behind the procedure (FF). The learner provided a rationale for the method he had chosen, i.e. the *sum of the angles of a triangle property*; however, Paul provided an alternative method – using the exterior angle of a triangle (CMR & CC) to solve the required angle as in the transcript in Table 4.20. He thus exhibited a blend of conceptual and procedural knowledge.

Table 4.20: Transcript (CLIP) – Providing alternate methods

405-413	<p>Paul: You got the answer to \hat{GEO}. What is the reason? (FF)</p> <p>L: Can we see a right triangle there (referring to triangle OE intersection of AB and GE)? We already know this angle (referring to $o1$ the other is 90 degree). Is that ok? And say the sum of angles of a triangle is 180°, ne?</p> <p>Paul: 180. Or can we see this one?</p> <p>L: Yes</p> <p>Paul: So this angle $O1 + \hat{GEO}$ is equal to what?</p> <p>L: 90</p> <p>Paul: What is the reason? The exterior angle of a triangle (CMR, CC).</p>
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Even when he was seeking help to solve \hat{BAE} , Paul prompted the learner to identify the relevant chord, angle and triangle. He first asked the learner to identify the chord AB (FF & FP) as in the transcript in Table 4.21, then, after recognising AB as diameter, Paul reminded him of the angle subtended by a diameter (FP). Another question followed to locate the 90° angle on the circumference. He encouraged the learner to focus on the triangle and apply the properties of a triangle (FP). The learner was unable to recognise \hat{BEA} as 90° until the teacher guided him to see it. Thus, in this episode, Paul demonstrated his range of instructional repository (FA).

Table 4.21: Transcript (CLIP) – Guiding learners to solve

415-424	<p>Paul: What sort of a line is AB (FM, FF)?</p> <p>L: Diameter</p> <p>Paul: Diameter and that diameter is giving you an angle at the circumference. Which angle is made by AB? (FP)</p> <p>L: It's 90°.</p> <p>Paul: Where is that 90°? (FF) Where will you get that 90°? AB is what? The diameter, of course. Diameter subtends 90° at the circumference (FM). Where are you going to get that 90°?</p> <p>L: E</p> <p>Paul: At E. Can you see? The whole of E this this this is all together 90°. Can you see that? And can you see that ABE is a triangle? Can you use those concepts to answer this question to find \hat{BAE}? (FP)</p>
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4.2.3.9 Act 1 Scene 2 ~ Applet 6 ~ Proof of the theorem

In the last lap of the lesson, Paul elaborated on the formal proof of the theorem (CK). Figure 4.19 below is the pre-designed applet displaying the step by step procedure for the formal proof of the *angle at the centre theorem*. This applet has a ‘play’ button that allows a user to proceed manually one step after the other, providing time for instructions and explanations wherever required.

Paul asked the learners (CC), to identify the properties of triangle ΔAOC and ΔAOB (CD), which the learners correctly identified as isosceles triangles. He continued to display the proof as seen in Figure 4.19 (can be accessed online at <https://ggbm.at/Umu9XrU>). To provide further insight into the theorem, he asked the learners to reopen the earlier applet in Figure 4.3 and reiterated the fact, established earlier, that the angle subtended inside or outside the circle by the arc is not half the

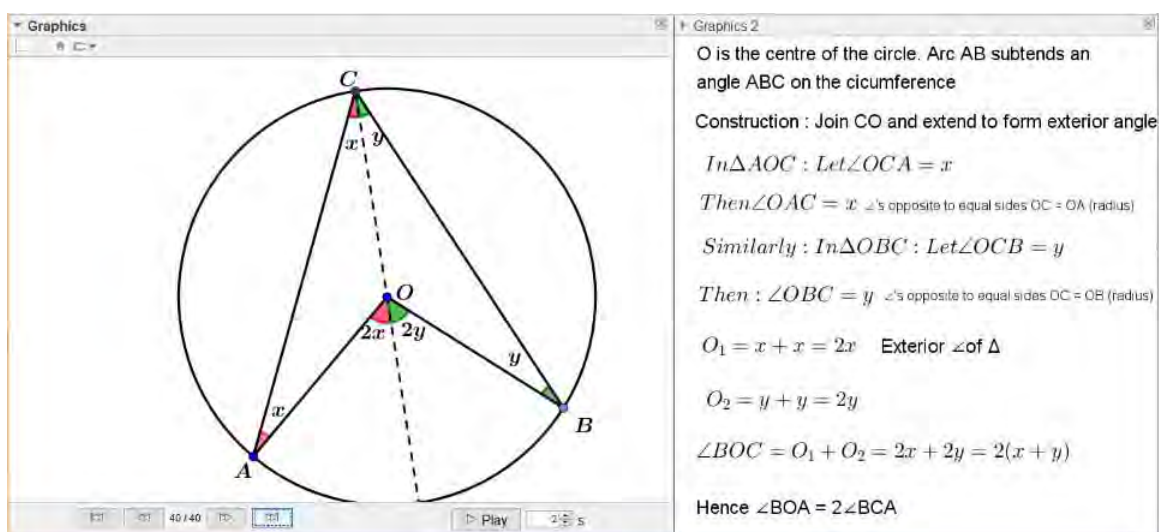


Figure 4.19: Justifying angle at the centre theorem in Applet 1.8

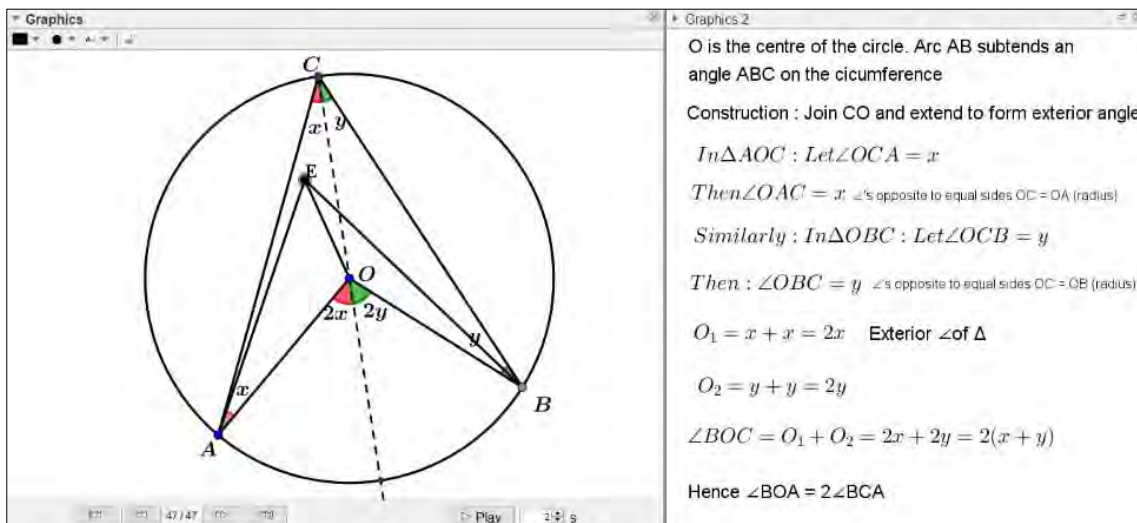


Figure 4.20: Highlighting the role of the radius

angle at the centre (CK). Though he asked them (CD) for a reason, there was hardly any response amongst the learners, except for some whispering. He toggled back to the proof applet (Figure 4.19), and then he plotted an additional point E (Figure 4.20) inside the circle and joined it with the centre, OE , and to the arc, AE and BE . He brought the attention of the learners to the additional point E and that triangles $\triangle AOE$ and $\triangle BOE$ did not form isosceles triangles (CK, CC). He emphasised this by dragging point E around the screen (CS).

4.3 HORIZONTAL ANALYSIS ACROSS THE LESSONS

The above narrative analysis shows how the two teachers implemented the applets that they had collaboratively designed, in their classrooms. In this section, the discussion is on the insights gleaned from the above narratives across the participants. The horizontal analysis helps to look for commonalities and differences across the lessons. Furthermore, it enhances the research analysis to understand how the participant teachers implemented *GeoGebra* applets as a visualisation tool for teaching mathematics.

As the analysis unfolded, different characteristics of teaching proficiency in relation to visualisation were evident. The analysis of data in Cycle 1 suggested that there were similarities and differences across the participants in their indicators of CU and IF.

Here, the analysis is interspersed with a few instances from each lesson which validates the evidence of visualisation and teaching proficiency.

4.3.1 The unfolding of core understanding (CU)

Kilpatrick et al. (2001) argue that possessing proficiency in core understanding when teaching mathematics enables teachers to integrate their knowledge of content and knowledge of processes to increase their learners' mathematical understanding.

4.3.1.1 Emphasis on teaching for core knowledge (CK)

In both classrooms, the teachers engaged learners with applets that featured important circle geometry theorems, such as the *angle at the centre theorem*, corollaries of this theorem, the *opposite angles of a cyclic quadrilateral theorem*, and others. Both Antony and Paul provided opportunities for learners to engage with the applets, so that they could better understand and visualise circle geometry theorems. I put forward two similar instances from the observed lessons, where teachers encouraged learners to drag the points and grasp several aspects of the *angle at the centre theorem*. Firstly, the subtended angle on the circumference was dragged and showed visually that the size of the angles remained invariant. Secondly, dragging the arc altered the sizes of the angle at the centre and the angle at the circumference dynamically, but the ratio between them remained unchanged. They complemented the applets by articulating definitions of the theorems using appropriate mathematical terminology.

Kilpatrick et al. (2001) argue that “In the course of their work as teachers, they must understand mathematics in ways that allow them to explain and unpack ideas in ways not needed in ordinary adult life” (p. 371). Thus, knowledge of mathematics for teaching includes the knowledge of ways that mathematical ideas can be represented. The analysis of the lessons showed that the teachers gave careful attention to identifying the concepts that they needed to teach effectively and provided instructions that provided opportunities for learning mathematics. Knowing the mathematics for teaching, including considering the objectives of the lesson, the teachers were able to facilitate the conceptual foundations of theorems. Guided by their proficiency in teaching mathematical knowledge, they facilitated their learners' development of mathematical ideas.

4.3.1.2 Emphasis on representing multiple ideas (CMR)

Research indicates that representing mathematical ideas and solving a problem in multiple ways is an attitude of proficient teachers (Kilpatrick et al., 2001). Antony and Paul generated multiple examples and diagrams through dragging the points in the applet. Evidently, the dynamic, multiple figures allowed different entry points for learners to engage with geometrical ideas. The teachers,

thus, simplified the properties of circle geometry before arguing and proving the theorems. Together with rich, dynamic aspects in the applets, they provided learning opportunities that promoted mathematical proficiency.

I put forward three facets of the lessons where the teachers employed different perspectives to develop a conceptual and procedural understanding of circle geometry concepts. Antony paraphrased the *angle at the centre theorem* (in lines 119–120 in Table 4.4 on p. 87), and thus provided a different perspective of the theorem. Secondly, both teachers used the perspective of ratio and proportion to relate the *angle at the centre* to the *angle at the circumference*. This sort of cross-curricular understanding between geometry and arithmetic is central to the role of teaching, as it provides teachers with mathematical knowledge useful for teaching. Thirdly, both teachers used different terms for supplementary angles and right angles during their interactions with the learners. Teachers encouraging a geometrical conceptual understanding from various perspectives, expand learners' mathematical ideas and thus develop their mathematical proficiency and connect relate several concepts in mathematics.

While engaging with practice applets, (Applet 1.6 in Figure 4.12 on p. 92; Applet 1.7 in Figure 4.13 on p. 93), both the teachers encouraged learners to solve the problems in multiple ways. With practice applets, they motivated learners to apply different methods to determine the angles. Paul, especially, (for instance in lines 405–413 in Table 4.20 on p. 106) provided an alternative method to one of the learners – calculating the angle at the centre without applying the *angle at the centre theorem* (he used the *exterior angle of a triangle property*). His explanation of the solution from a different viewpoint led his learners towards a more flexible understanding of geometric concepts. I concur with Kilpatrick et al. (2001) who argue that proficient mathematics teachers understand that there is not a fixed method for learning mathematics, but rather, multiple ways to approach a mathematical concept.

4.3.1.3 Emphasis on connecting concepts (CC)

In lessons on circle geometry, I observed that the teachers expounded different aspects of geometry while dealing with the *angle at the centre theorem*. My participant teachers invoked prior learning of aspects of geometry, such as the *sum of angles in a triangle* and *angles around a point*, as they interacted with the applets and the learners. Antony set the stage for additional considerations of fundamental aspects of geometry such as the arc of a circle and a vertex of an angle when he

motivated his learners to construct circles, chords and angles. While discussing the special case of the angle at the centre, Paul linked it to a right-angled triangle (transcript in Table 4.13 on p. 97). Furthermore, Paul emphasised that when applying the *exterior angle of an isosceles triangle property*, it formed the *angle at the centre theorem* (in lines 89–97 of Table 4.23, p. 120). Both teachers appreciated a teaching strategy that revisited and reinforced basic geometric ideas. Such connections are crucial aspects of meaningful mathematical teaching and learning. Nevertheless, in the context of teaching, Presmeg (2014, p. 153) warns that “visualization is not self-explanatory and is very important for teachers”. Therefore, it is incumbent for teachers to unpack mathematical ideas and make abstract ideas visualisable to learners.

Furthermore, the teachers’ instructions suggested how they linked the *opposite-angles of cyclic quadrilateral theorem* with the *angle at the centre theorem*. Ma (2010) notes that a teacher with profound understanding “has a general intention to make connections among mathematical concepts and procedures, from simple and superficial connections between individual pieces of knowledge to complicated and underlying connections among different mathematical operations and subdomains.” (p. 104). Thus, these participant teachers composed interconnected knowledge packages which formed a solid network of different properties and theorems in geometry.

4.3.1.4 Emphasis on teaching for exploring and discovering concepts (CD)

From the perspective of constructivism, learners do not merely absorb what the teacher instructs them to do; instead, they interpret and generate meanings in their own way (Boaler, 2009; Jaworski, 1994). Contextually, the participant teachers offered opportunities for their learners to manipulate and explore with the applets. Antony allowed geometric constructions in software; thus, encouraging the learners to take ownership of their learning. He motivated learners to identify any relationships between an angle inside the circle and an angle at the centre or at the circumference. A learner was able to make the discovery, through dragging the points, that “[t]he angle in the circle is not half the angle at the centre of the circle because the angle is not on the circumference”. Antony’s lesson was intellectually demanding and challenging for learners (and exciting too). He allowed learners to generate and discover various viewpoints, and the interactions that ensued showed that he presented a coherent picture of geometrical properties. His lesson went beyond the articulation and manifestation of mathematical facts. The analysis of Antony’s lesson supports a technology-based constructivist classroom, where there is more emphasis on developing learners’ mathematical proficiency. I concur with Mariotti (2000) who concludes that the

experience of geometrical constructions in a computer environment “provides a context in which the development of the meaning of Geometry theorem may be achieved” (p. 48).

Interestingly, learners in both classes established through exploration, that the angles changed if the chord or arc was changed, but that the relationship between the angle at the centre and the angle at the circumference did not change. Mariotti (2000) also emphasises the importance of ‘dynamic exploration’ in learning geometrical concepts and mathematical proofs.

The interactions showed that the properties of circle geometry were made explicit by the teacher, using the tools of the software. The role of the teacher is crucial in order to direct the objectives of the lesson and to guide learners towards the geometrical meaning of the figures.

4.3.1.5 Emphasis on tackling common misconceptions (CM)

Both Antony and Paul addressed the misconceptions as discussed and identified at the GLIP meetings (Section 4.2.1.2 on p. 80). Employing the relevant applets, they addressed the misconceptions i) applying the *angle at the centre theorem* for angles not at the centre or the circumference of a circle; and ii) thinking that the angle at the centre is twice the angle at the circumference even if the angles are not subtended by the same segment. The participants were fluent in mathematics, but in this context, they also exhibited their knowledge of learners and how they learned. Kilpatrick et al. (2001) maintain that core understanding of a proficient teacher relates to knowledge of learners and “the approaches that are typical for students of a given age and background, their common conceptions and misconceptions” (p. 378). The participant teachers could engage their learners in meaningful mathematical learning. Antony and Paul were aware of their learners’ mathematical thinking and common misconceptions to which they were sensitive and able to address. A striking feature was that both the teachers addressed the misconceptions, albeit using different approaches.

Antony allowed learners time and opportunity to explore and discover the misconceptions as mentioned above, related to the *angle at the centre theorem*. He made the learners aware of these misconceptions through their own constructions and discoveries (lines 21–32 of Table 4.22 on p. 119). Antony prompted his learners to discover and encouraged them to share their findings (see the transcript in Table 4.5 on p. 88 and the discussion in Section 4.3.1.4 on p. 111). From a constructivist perspective, these instances also illustrate the assertion by Jaworski (1994) that learners do not only absorb ideas presented by their teacher but rather that they create their own

knowledge. When provided with opportunities, the learners in Antony's class made their discoveries and representations. Thus, the role of a teacher is to respond actively and creatively to these student representations (Shulman, 1987).

Paul also held the view that learners should explore and construct their own mathematical knowledge; however, he did not provide enough time and opportunity for them to explore and come up with their own ideas. He managed to address the two misconceptions related to the theorem by showing the learners these misconceptions using concrete examples and counterexamples (transcript in Table 4.14 on p. 99).

4.3.1.6 Emphasis on teaching which relates to real-life situations (CR)

When involved in giving instructions and emphasising the theorems, the teachers did not make real-life references or connect the theorems today-to-day realities. I will discuss, along with the other cycles, the absence of this code as I consolidate my findings in my last chapter.

4.3.1.7 Emphasis on switching between abstract and concrete (CS)

Any representation of geometrical concepts remains abstract in the sketch, and it is lost when looking at the final product of the construction (Kokol-Voljc, 2007). Axioms of geometry such as 'two points determine a line', 'three points determine an angle' and other facts were practised in Antony's class using *GeoGebra*. Using the appropriate *GeoGebra* tools, Antony guided learners to learn and visualise the characteristics of circles, arcs and angles, and they learned the importance of direction in measuring the value of an angle (for instance, in lines 80–91 in Table 4.4 on p. 87). He made explicit links between these concepts and the concrete constructions in *GeoGebra*. Through activities performed with the applets, this teacher supported the development of geometrical concepts and thus made the main characteristic features of *angle at the centre theorem* visualisable. The use of *GeoGebra* the process of construction of the geometric figures concretised the respective concepts.

There are significant advantages of using 'pre-designed' applets, as seen in the analysis of Paul's lesson. Paul considered that allowing his learners to do the constructions using software tools might not pay off. He believed that many learners were from families with scarce computer access. He felt that the learners might get bogged down by certain intricacies of the software and he feared that this might cloud the learning objective. Nonetheless, for the learners, an essential part of the construction of geometric figures were be lost. Antony argues in his reflective interview that the

concrete constructing activities are crucial for the process of developing abstract concepts, (see lines 45–53 in Table 4.22 on p. 119).

4.3.2 The unfolding of instructional fluency (IF)

The second strand in Kilpatrick et al. (2001)'s framework of mathematical teaching proficiency is instructional fluency. In this section, I consolidate different pertinent indicators of IF (instructional fluency) that were made evident in this cycle.

4.3.2.1 Emphasis on acquiring an instructional repertoire (FA)

Mavani, Mavani and Schäfer (2018) report on “how pedagogical styles and personal theories influenced the use of technology in mathematical classrooms” (p. 9). The applets and the lessons were designed collaboratively with the teachers, however, Paul and Antony had their own reasons for integrating technology into the milieu of classroom activity. Both believed that applets could enhance the grasp of the circle geometry theorems, but they adopted different approaches when implementing these applets. Antony favoured learners constructing geometric figures in order to help them build a more profound and basic understanding of geometric ideas. He enacted the geometric essentials through well-structured use of *GeoGebra*. Antony's instruction provoked learner engagement and allowed exploration of these artefacts to attain conceptual understanding (see the discussion in Section 4.3.1.4 on p. 111).

Pre-designed applets were desirable for Paul to avoid the computer intricacies that might mask the relevant objectives of the lesson. He led the class straight away into the investigation, amplifying and visualising the circle geometry theorems. This was similar to the theme “focusing on overarching issues and accentuating important features” identified by Ruthven et al. (2009), highlighting the contribution of digital tools in teaching and learning. Paul facilitated mathematical understanding among learners by motivating exploration with the use of pre-designed applets as he felt that learners might have come with irrelevant ideas (see transcript in Table 4.23 on p. 120). Accordingly, this practice supports the idea that learners may focus on constructed objects or representations irrelevant to mathematics (Hoyles & Noss, 2003) and overlook the teacher's intention (Hölzl, 2001). Furthermore, Hölzl (2001) asserts that in a guided discovery setting a teacher helps learners to recognise the mathematical properties, thus teaching mathematical concepts by allowing exploration. As Hölzl (2001) highlights, in this episode, Paul employed

applets in the empirical confirmation of theorems. It is thus evident that *GeoGebra* rendered a pedagogical space for teachers that allows learners to engage in geometrical explorations.

In a DGS environment, the pedagogical challenge is to design activities that involve learners, either to construct geometric objects or to manipulate pre-designed geometric objects (Leung, 2011). Pedagogical complexity arises from the need to develop new ways of teaching effectively. Further scrutiny of the lessons across the participants, thus highlighted how pedagogical styles and acquired instructional routines influenced the use of technology in these mathematical classrooms. The pedagogical practices of the teachers determine the implementation of technological tools. Nonetheless, Mason (2013) stipulates that appropriate instructional routines are required when a teacher directs an activity using an applet, to maximise the learning potential for learners.

4.3.2.2 Emphasis on tackling a learner's misconception (FM)

The practice applets reinforced the concepts discussed about the *angle at the centre theorem*. These applets also provided opportunities for the teachers to ponder over the learning of their learners. Managing learners, when they make mistakes and have misconceptions in real contexts of practice, demanded a sort of in-depth knowledge and skill from the participant teachers. Both Antony and Paul attended to learners' mistakes individually and corrected them. When a learner wrongly construed that the opposite angles of a cyclic quadrilateral are equal, Paul corrected him: "*Opposite angles of a cyclic quad are not equal, but they are supplementary... most of these theorems start from the angle at the centre [theorem].*" (lines 361–362 of the transcript CL1P). The teacher not only corrected the learner, but also explained why the *opposite angles of a cyclic theorem* worked. Antony also moved around the classroom as the learners engaged with the practice applets. When some learners could not identify the diameter and then apply the *semi-circle theorem* in practice Applet 2 (see Figure 4.13 on p. 93), Antony set out to expound on the *angle subtended by a diameter theorem* and convincingly connected it to its root – the *angle at the centre theorem*. The teachers, nevertheless, reflected that the learners were occupied with the practice applets and enjoyed doing them (see the transcript in Table 4.22 on p. 119 and Table 4.23 on p. 120). There was learning in the process.

These theorems are not enough by themselves. Teachers are required to instruct learners on how to use them to apply them in different situations. Merely identifying an inaccurate procedure or answer is not what the participant teachers did, as discussed in the above paragraph; they not only

corrected the learners, but also explained why the theorems worked. Ball et al. (2008) surmise that a proficient teacher figures out what procedure the learner might have taken to produce this particular error, provides reasons and then determines an appropriate response. According to Ball and Bass (2000), instructional fluency in mathematics is the ability of a teacher to “deconstruct mathematical knowledge where elemental components are accessible and visible” (p. 88). Even though the theorems are known by different names and definitions, their roots are common. The participant teachers unpacked knowledge, making mathematical content accessible to learners. Thus, the findings from the first cycle show that proficiency in teaching involves a unique capacity to understand and appreciate learners’ insights and deconstruct abstract knowledge, making obscure elements visible.

4.3.2.3 Emphasis on teaching basic skills (FL)

I observed that for many a learner, computers were a challenge. In Antony’s lesson, while he was busy providing instructions to learners on the computer (transcript in Table 4.4, p. 87), the circles, chords and the angles looked bizarre on some of the learner’s screens as they could not manipulate the mouse competently. Lines 34–41 in Table 4.22 on p. 119, show Antony’s concern about the learners’ lack of computer competence. In one instance, a learner could not plot a point exactly on the circumference of the circle and therefore, did not agree with the teacher that the angles subtended by the same chord were equal. At first, the teacher was confused, but later realised that the mistake made by the learner (lines 101–103 in Table 4.4 on p. 87). This unexpected challenge from the learners highlights the research showing that a teacher has to become adept at technology. Despite Paul not allowing geometric constructions in the software, his learners still faced challenges in controlling the movement of the mouse and the keyboard, throwing the figure out of proportion. Many a time, the teacher had to intervene and fix the applet. In lines 68–73 in Table 4.12 (p. 96), learners moved the text ‘alpha’ instead of moving the angle ‘alpha’ – which created confusion for a while. However, Paul noticed the mistake and immediately corrected this by telling them to move the slider by clicking on the dot, showing his technical expertise in responding to the learners’ unexpected actions. Interestingly, Paul also highlighted the lack of basic computer skills of learners in lines 123–128 of Table 4.23 (p. 120).

Interestingly, at the closure of the lesson, Antony recommended that the learners install *GeoGebra* on their tablets or phones so that they could learn on how to use the software (lines 320–323 in

Table 4.10 on p. 93). Learners have to be competent in handling computers for the purpose of learning mathematics.

4.3.2.4 Emphasis on responding to learners' thinking (FR)

Kilpatrick et al.'s (2001) framework considers teacher's actions to trigger learners' thinking and calls for the proficient teacher to respond to learners' actions on mathematical concepts or tasks. While learners were interacting with the practice applets, the teachers moved around the laboratory, attending to the learners' queries. Antony scaffolded his learners through deliberate questioning, while Paul provided procedures with explanations on how to identify and relate the angles.

Antony's responses to a learner's observation on cyclic quadrilaterals showed several interesting features where there was a subtle interweaving of procedures and concepts. He asked the learners to distinguish a cyclic quadrilateral among the quadrilaterals in the diagram (Figure 4.11 and lines 243–266 in Table 4.9 on p. 91). In response to a learner's answer, he started with the procedure of calculating the sum of the opposite angles. After a discussion on various perspectives related to the concept of a cyclic quadrilateral, he concluded by explicitly stating the converse theorem of a cyclic quadrilateral. His thoughtful responses showed his ability to intertwine knowledge of a specific geometric concept related to the procedure.

Paul's response, in lines 361–371 in Table 4.18 (p. 105), also revealed interesting features. A learner appeared to not recognise the *angle at the centre* related to the *angle at the circumference* in the practice applet. Paul's response indicated the importance of the rotation of the arc to see the angle. He provided the learner with a relevant experience to visualise "*The angle at the centre must be on the other side of the angle on the circumference*". I found that the teacher's response was rooted in building the learners' conceptual and procedural knowledge.

Thus, the teachers' attention and responses to the learners showed justifications and explanations about applications of properties or strategies to be considered in solving problems. I concur with Kilpatrick et al. (2001), who argue that through interactions with learners, proficient teachers take responsibility for the mathematical actions that occur in response to learners' mathematical ideas.

4.3.2.5 Emphasis on understanding procedures (FP)

Kilpatrick et al. (2001) argue that instructional activity should be able to provide a satisfactory explanation and justify the reason behind the procedure, so that learners make sense of the problem

situation. I found that both teachers provided explanations and justifications for the procedures while interacting with the practice applets.

However, interestingly, Antony favoured blending procedures along with the discussion of the theorems, i.e. even before learners started solving the practice applets. Discussing theorems and concepts on circle geometry using the applets, Antony highlighted certain procedures that are useful for solving angles in the applets to follow. Throughout the discussion about applets, he consistently explained how to solve problems. For instance, in lines 119–120 and 231–233 in Table 4.8 (p. 90), he directed the learners to a strategy of relating angles while applying the theorem to determine the angles. This notion of ‘inside, outside’, ‘angle at the centre bigger’ gave a clear procedure useful in problem-solving. Lines 311–314 in Table 4.10 (p. 93) record Antony solving an angle in the practice Applet 1.7, where he gave an accurate explanation of a procedure and justified its linkage to the *angle at the centre theorem*. Rittle-Johnson and Alibali (1999) find that “[w]hen children are empowered with fundamental concepts, they are able to solve novel problems on their own” (p. 189).

Paul, on the other hand, provided explanations and justifications for the procedures while solving problems in the practice applets. For instance, see lines 405–413 and 415–424 in Table 4.21 (p. 107), where he provided his learners with relevant concepts to solve an angle in the practice applet. Paul believed that practice leads to understanding, as he reflected “*put more questions to practise. More they practise more they understand*” (line 117, RIIP, my original transcript). This finding is in concurrence with Kilpatrick et al. (2001, p. 168) who provide an example to ascertain how learners build on practice in extending their mathematical proficiency to include new concepts.

The analysis of the lesson shows that the teachers possessed a mathematical proficiency for teaching that helped them identify the essential components of geometrical concepts that were useful for learners in using and connecting these essential ideas.

4.3.2.6 Emphasis on inviting feedback (FF)

Antony used effective questioning that invoked the learners’ responses and allowed them to develop their mathematical thinking. He deliberately elicited responses from a range of learners so that individual learner’s ideas contributed to further discussions on geometrical concepts. Paul, in contrast, did virtually all the talking, offering only a few opportunities for responses. He did ask questions but did not wait for his learners to respond. It would have been interesting to see how

the learners would have answered his questions, had he waited. Ball and Bass (2000) affirm that invoking learner's responses plays a pivotal role in the teaching and learning of mathematics. The emphasis is on instructional activities that are relevant and invite responses, to lead learners to observe and make conjectures.

Table 4.22: Transcript (RIIA) – Antony's reflection on the first lesson

21-32	<p>R: What are a few things that you liked most in the lesson?</p> <p>Antony: I also liked that when the angles subtended by the same chord inside or outside the circle. Even if it is subtended by the same chord or arc if it is not on the circumference then, therefore we cannot relate the angle at the centre with the angles, not on the circumference. They were exploring with the angles inside the circle, what is happening with the angle from the same arc outside the circle. They were comparing those angles, and they were able to find out that there is no relationship. The relationship is only when the angle is subtended by the arc or chord on the circumference and the angle at the centre subtended by the same chord or arc. You cannot relate the angles subtended by the same chord or arc that are not on the circumference. If we permit them to explore, they can do wonders. When the point on the circumference was dragged to other segments, the measures of the angle changed. Surprisingly, many learners were able to measure the angle at the centre and were helping others who could not do it.</p>
34-41	<p>Antony: Some of the learners are not computer literate. If we can maybe if we can make them use computers in general and then the software. Maybe some of them even lack the basic skills, cannot open the GeoGebra file, and use of a mouse is worse. A learner could not plot points properly and was saying that the angles in the same segment are not equal. When I noticed his screen, the point on the circumference was not exactly on the circle and later, I saw his drawings had become disfigured. The same learner could not drag properly as if he was using the mouse for the first time. Some did not know what is a right-click or a left-click. It means that they did not take those computer classes seriously. They must get used to it.</p>
45-53	<p>R: So you think you would not allow them to construct the circles and line segments in your next lesson on this topic?</p> <p>Antony: It is time-consuming if we allow them (referring to learners) to construct circles, segments and measure angles in the software. They do not understand that an angle comprises of three points or two lines and is measured in the anti-clockwise direction. Hence many struggled to measure them in the computer. But most of them were catching up, there was learning in the process. They struggled to measure the angles. But they must know that its middle point where the angles are measured. In the software, it depends on the order of points you select, clockwise or anticlockwise. So it is important for them to learn how to measure those angles.</p>
55-57	<p>Antony: I like that part of software most where it was possible to move the angles around the circumference and see that the measure of the angle did not change. It was not possible on a chalkboard classroom, and we talk, talk and talk and they do not understand. But they can now see what we are talking about.</p>
58-67	<p>R: What differences did you find between the chalkboard and GeoGebra in your lesson delivery?</p> <p>Antony: Numerical values of the angles change dynamically with the change in an arc, I could see that most of them in class could actually relate and see by themselves that the angle at the centre is twice that angle at the circumference.</p> <p>... So when they are using GeoGebra, they can better visualise which angles to relate, and they can see the inside angle at the centre will not be double that of on the circumference, they can measure and see now what we are talking about.</p>
71-78	<p>R: How do you think those practice questions were useful or successful?</p> <p>Antony: It was successful in the sense that kids were keen to use it, and they were talking about it for a long time. Even during the practice questions, some learners were still trying to move the points around the circumference to relate the angles. But I wonder what if the question was in hard copy. However, they were enjoying those questions and excited to share their correct answers. While in</p>

	<i>notebooks, they would not even show me, perhaps they were not confident. But now, here they know it is a correct answer because the computer tells them that it is correct. I think it will boost their confidence and they now understand better those theorems.</i>
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Table 4.23: Transcript (RIIP) – Paul’s reflection on Cycle 1

14-17	<i>R: you choose to use the pre-designed applets in the class, what made you decide to do this? Paul: pre-designed applets the advantage is that though the learners, you know, some of them may appear to be technologically ahead, but it will take time for them to draw and measure. So pre-designed applets will save time. I should say it is a time factor.</i>
43-48	<i>R: How do you think that collaboration among the teachers helped you with the lesson? What was the influence? Paul: It was helpful. Now we know what the important thing is to do. And already the pre-designed applets will not now take time. If we allow learners to draw circles, segments etcetera, themselves they may be drawing angles in the opposite segment, maybe they are drawing on the same segment. There is always a danger that they will speak of things that might not be relevant.</i>
56-64	<i>R: How effectively was the concept conveyed? Especially when you compare it with the investigation that you gave. Paul: Yeah, with investigation some of them they could not understand the concept. Measuring the angles. One concept was to measure the angle at the centre and suppose to fold it and then compare the angles, I mean angles on the circumference and angle at the centre, they were not getting it. But with this applet, it was correctly visualised. They could see what we are talking about. And I can see when they were doing those exercise applets, they could pick it up. Only a few struggled, but a push made them do it. Those exercises were useful as it reinforced the concepts that were discussed in the class.</i>
73-82	<i>R: How did GeoGebra help or did not help your teaching? Paul: Oh no, it helps a lot. R: Was it a barrier? Paul: No, no, it was not a barrier. Actually, the learners can see what we are talking about. We can manipulate so many things, I mean we ask them what would happen if it is like this, ok then let us check it, like that. You see some of the learners are used to or recognise only if it is at the top. So with this, you can shift those angles, it remains the same. So no matter where you are as long as you are on the same side of the segment, it still holds the theorem. Yeah, the dynamic feature of the software. It is like this whenever you ask to draw a right-angled triangle they recognise only the standard form but if we twist it, its something else.</i>
89-97	<i>R: What are the common mistakes that the learners were making or what misconceptions did they have while you were interacting with them? Paul: Some of the learners were failing to see the relationship of angles, you know, once they start circle geometry, they divorce lines, triangles likewise saying, ‘this is not my cup of tea.’ So if you put a triangle in a circle, oh oh. R: They do not see the triangle. Paul: They still think of circle geometry. Some of them could not see the exterior angle of a triangle. These are some of the mistakes. They are not linking with previous learning. That is the thing. That linkage is broken. They don’t link that as I see.</i>
123-128	<i>Paul: The learners require more exposure to computers. They come from places where there is no electricity. These learners require more training in using computers. You know, some of them couldn’t handle a mouse. That child couldn’t control the mouse as she moved the point, the figure was becoming large, and then it became very small. Very often, I have to tell her to close the file and reopen without saving. I mean they don’t know how to minimise or maximise a window. Many of them lack basic computer skills.</i>

4.4 DISCUSSION ON THE RESEARCH QUESTIONS

In this section, I set out to answer the two research questions. The first one concerns the teaching of mathematics integrating DGS as a visualisation tool. Specifically, the first research question is:

4.4.1 How can Dynamic Geometry Software such as *GeoGebra* be used as a visualisation tool to teach Grade 11 Mathematics?

The analysis of the lessons in Cycle 1 of the two participant teachers, shows that the participant teachers adopted different teaching approaches as they implemented co-developed *GeoGebra* applets in mathematics classrooms.

In his first lesson, Antony directed his learners to geometric principles through the process of construction, using the software before establishing the specified outcomes. The discussion in Section 4.3.1.7 (p. 113), describes his instructions guiding the learners to learn and visualise geometric properties through concrete constructions. Indeed, Antony was overly optimistic about the ease with which learners were able to use *GeoGebra*. Instead, learners encountered difficulties in constructing and measuring angles in *GeoGebra*. Despite these difficulties, Antony engaged with them. During the reflective interview, he emphasised that the learners visualised the geometric concepts as they constructed them. Echoing Duval (2014), evidence shows that visualisation based on the construction of configurations aid in identifying and understanding specific geometric properties. An alternative approach in the teaching of geometry, that promotes learners to construct and draw figures using conventional tools or computer tools, becomes essential. The teaching instructions during the initial stages of introducing geometry at school must focus on the development of ‘geometric visualisation’ (Duval, 2013) before aiming at the acquisition of theorems.

Paul considered the software a valuable tool for learners to visualise several features of the theorem. I discussed in Section 4.3.1.3 (p. 110) how Paul brought in connections between ‘diameter’ and ‘hypotenuse’ while discussing the *semi-circle theorem* using the Applet 1.4. Such connections between a new idea and an existing idea are crucial when learning geometric concepts. When we make connections between mathematical topics, we employ visualisation (Presmeg, 2006). Significantly, by utilising the dynamic visualisation power of technology, Paul provided

opportunities for learners to make connections and experience the whole mathematical structure of a particular theorem.

Neither of the teachers conceived of *GeoGebra* as an efficient tool to produce static figures but used it instead to generate figures; and using its measurement facility, to highlight invariant properties by dragging points. The two teachers successfully designed their lessons, capitalising on the dynamic visualisation capabilities of DGS, guiding learners to generate multiple examples which were enabled through the dragging of points. The teachers inspired their learners to observe and take note of the measurements of the angles provided by the software as they dragged the points, thus enabling them to see the variant and invariant features of the *angle at the centre theorem* in different situations (see the discussion in Section 4.3.1.1 on p. 109). With static diagrams, it is more difficult to convey this message, but by dragging their constructions, learners learned to distinguish, as Gawlick (2002, p. 85) expands: “diagrams drawn in an empirical way from diagrams resulting from the use of geometric primitives.” Laborde (2001, p. 293), one of the key developers of *Cabri* (the first generation DGS), highlighted the role of visualisation through dragging as a “visual amplifier where it was easier to observe the variant and invariant properties during the deformation of the diagram by the drag mode than in a static paper-and-pencil diagram.”

Participant teachers tapped into the dynamic visualisation capabilities of *GeoGebra*, aiming at deepening their learners’ conceptual understanding. Thus, the findings are consistent with Stols and Kriek (2011), who recognised that one of the potentials of a dynamic environment is its ability to focus on interlinked parts of a geometric shape, thus enriching visualisation.

The following discussion is on the second research question.

4.4.2 What enabling and constraining factors do Grade 11 teachers encounter when using GeoGebra as a visualisation tool to teach Grade 11 Mathematics?

4.4.2.1 Constraints

There were two sets of constraints that the teachers encountered during the integration of *GeoGebra* in mathematics classrooms, namely: i) software constraints and ii) technical challenges.

Software Constraints

There were two software features in *GeoGebra* that posed challenges to the teachers as they implemented applets in classrooms. First, as we discussed in the GLIP meetings the rounding-off feature in *GeoGebra* can lead to inadequate precision in the display of the measured values of angles, and thus lead to unconvincing results e.g. $\frac{1}{2} \times 123,23 = 61,62$ and not the expected value of 61,615. It was therefore a challenge for the teachers when the rounded-off values obscured the actual relationships between the angles. Paul overcame the constraints with his ready-made applets and provided learners with these applets to explore further on their computers. With the pre-designed circles and angles measured, Paul made the properties more discernible and convincing to learners. The slider tool the tool allowed him to move the angle at the centre in integer values and in this way he tackled this limitation of *GeoGebra*.

The second software constraint is that *GeoGebra* measures angles in a particular direction, clockwise or anti-clockwise. Hence, when the angle at the circumference was dragged to the other side of the arc, the software measured its reflex angle. Ruthven et al. (2008) also report on this issue (for any DGS) in their research. Their participant teachers were thus restricted to purposefully dragging to the same side of the arc. Antony prompted his learners to identify and measure the angle at the centre that would be twice the angle at the circumference on a minor arc, and thus overcame the problem. He reflected that most of his learners were able to identify and measure the angle at the centre related to the angle at the circumference on the minor arc (see lines 29–32 in Table 4.22 on p. 119).

Technical Challenges

The teachers also encountered challenges owing to the lack of their learners' computer proficiency. In the reflective interviews, the participant teachers raised their concerns that learners struggled to control the figures on their screen (see the discussion in Section 4.3.2.3, p. 116). Both Antony and Paul believed that learners might get bogged down by certain intricacies of the computer, and feared that this could mask the teaching and learning objectives of the lessons. However, despite software tools sometimes being challenging for new users, the learners struggling to do geometric constructions or control geometric figures in *GeoGebra*, they still learned in the process.

There were other technical issues related to the use of technological resources in classrooms. In both lessons, teachers experienced a delay in kicking off the lesson straight away, as the learners

took time to settle down. Technological glitches were inevitable such as faulty input and output devices and non-operating terminals. A teacher thus needs to be flexible in dealing with faulty computers in classrooms. Teaching and learning time was definitely affected by technical issues raised by the learners. Allowing learners to manipulate the applets creates a meaningful learning experience; nevertheless, it may be a constraint when a teacher is not proficient in the software.

4.4.2.2 Enabling factors

The critical enabling factor in *GeoGebra* that adds pedagogical value, is that the software makes it possible to create a dynamic construction and simultaneously retain its defined characteristics under manipulation by dragging with the computer mouse. Dragging is a key feature of *GeoGebra*. It allows a continuous reconstruction of figures on the computer screen by direct manipulation with the mouse (or by touch on a touchpad). Hölzl (2001, p. 83) asserts that the “drag mode develops in mutual dependence with the ability to grasp a mathematical situation – a learning process that is characterised by different layers of conceptions.” The pedagogical utility of the drag facility, as evident in this research study (see the discussion in Section 4.3.2.1, p. 114), vindicates the above assertion made by Hölzl (2001).

Furthermore, I concur with Mariotti (2000) that the intrinsic logic of a DGS, “expressed by its reaction to the dragging test, induces learners to shift the focus onto the procedure, and in doing so, it opens up to a theoretical perspective”. (p. 32).

Of further significance is the interactive facility of *Geogebra*. The software responds to instructions given by the user. The measurement of an angle, the length of line segments, etc. are considered as the responses provided by the software. The interactive quality of working with a computer allowed the teachers to secure learners in actively engaging with the geometric ideas. The teachers capitalised on the interactivity capability of the software to drive away any misconceptions related to the theorem. They engaged their learners in measuring the angles and seeing what happened when they dragged points on the circumference to the other side of the arc. Research by Deaney, Ruthven, and Hennessy (2006) has also shown that “interactivity and the immediate visual feedback afforded by the use of technology” (p. 475) and was seen by the teachers as playing a significant role in contributing towards enhancing learners’ conceptual understanding. Paul admitted that dynamic features of the software enable a learner to see and apply the *angle at the centre theorem* in the non-standard diagram (lines 70–79 of the transcript in

Table 4.23, p. 120). Presmeg (1986b), in her study on ‘visualisers’, observes a sort of difficulty experienced by learners: “An image of a standard figure may induce inflexible thinking which prevents the recognition of a concept in a non-standard diagram” (p. 44). In this case, the teacher (Paul) acknowledges such inflexible thinking by learners where the learners could not recognise the *angle at the centre theorem* when the diagram did not conform to a standard diagram (the angle at the circumference above the angle at the centre). Nonetheless, the dynamic manipulation enabled in the software may aid to preclude ‘inflexible thinking’.

4.5 CONCLUSION

The main objective of this chapter was to present the data through in-depth narratives of the participating teachers’ use of technological tools in the teaching of circle geometry theorems. A detailed analysis of the teachers’ lessons, vertically (individually) and then horizontally (in relation to each other), indicated that there were similarities and differences across the participants in their teaching approaches as they implemented *GeoGebra* applets in their classrooms. As the analysis unfolded, different characteristics of teaching proficiency in relation to visualisation were evident. Furthermore, the analysis of their responses during the reflective interviews identified interesting pedagogical strategies for the visualisation of mathematical concepts. However, these components of teaching proficiency and visualisation revealed pedagogical tensions between the teachers’ knowledge of mathematics and the knowledge of their learners.

The next chapter presents an in-depth analysis of Cycle 2, which is on the topic of Quadratic Equations and Inequalities.

CHAPTER FIVE

TEACHING MATHEMATICS USING DGS – DATA PRESENTATION AND ANALYSIS – CYCLE 2

5.1 INTRODUCTION

In the previous chapter, I recorded in a narrative style and analysed the participant teachers' implementation of applets on circle geometry in their classrooms. This chapter aims to analyse and answer the research questions on the topic 'Quadratic Inequalities and the Nature of Roots'. I recommend that the reader to browse through the recap methodology in Section 4.1.1 (p. 78) of the previous chapter, in order to be reminded of the methodological processes that guided my analysis. This chapter follows a similar structure to Chapter 4, (see p. 77).

5.2 Cycle 2: - Quadratic Inequalities and the Nature of Roots

The next topic in the sequence of the teaching plan was quadratic equations and inequalities. In the discussion that follows, the quadratic equation is represented by $ax^2 + bx + c = 0$ and its roots are given by the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The expression under the surd $b^2 - 4ac$ (generally represented as Δ) is called the discriminant.

5.2.1 GLIP Meeting

By the time we reviewed Cycle 1 of the GLIP, two more teachers had joined our group – John and Abraham. There were thus seven members that participated in the meeting focussing on the second cycle. John was already a participating teacher in my research study', however, Abraham was not a formal participant teacher in my research, but a member of GLIP. Abraham has a facility in town where he conducts extra lessons, and our meeting on Cycle 2 was held there. All the teachers used their laptops during this meeting, while Abraham connected his laptop to a projector and displayed his work onto the screen.

There are many windows to *GeoGebra*, such as algebra, geometry and others. We found the Computer Algebra System (CAS) window of *GeoGebra* suitable for teaching and learning quadratic equations and inequalities. Primarily, the CAS window of *GeoGebra* is used for symbolic computations, such as factorisation and solutions to equations in one variable. An

important feature of *GeoGebra* is that the same equations and expressions can be used and shared between the different windows. For example, an equation $f(x) = 2x^2 + 3x - 5$ is defined and solved for x in one window, then the same equation is considered as a function, and its graph is displayed in the graphics window. Furthermore, the solutions are indicated in coordinate form and plotted on the graph, as shown in Figure 5.1.

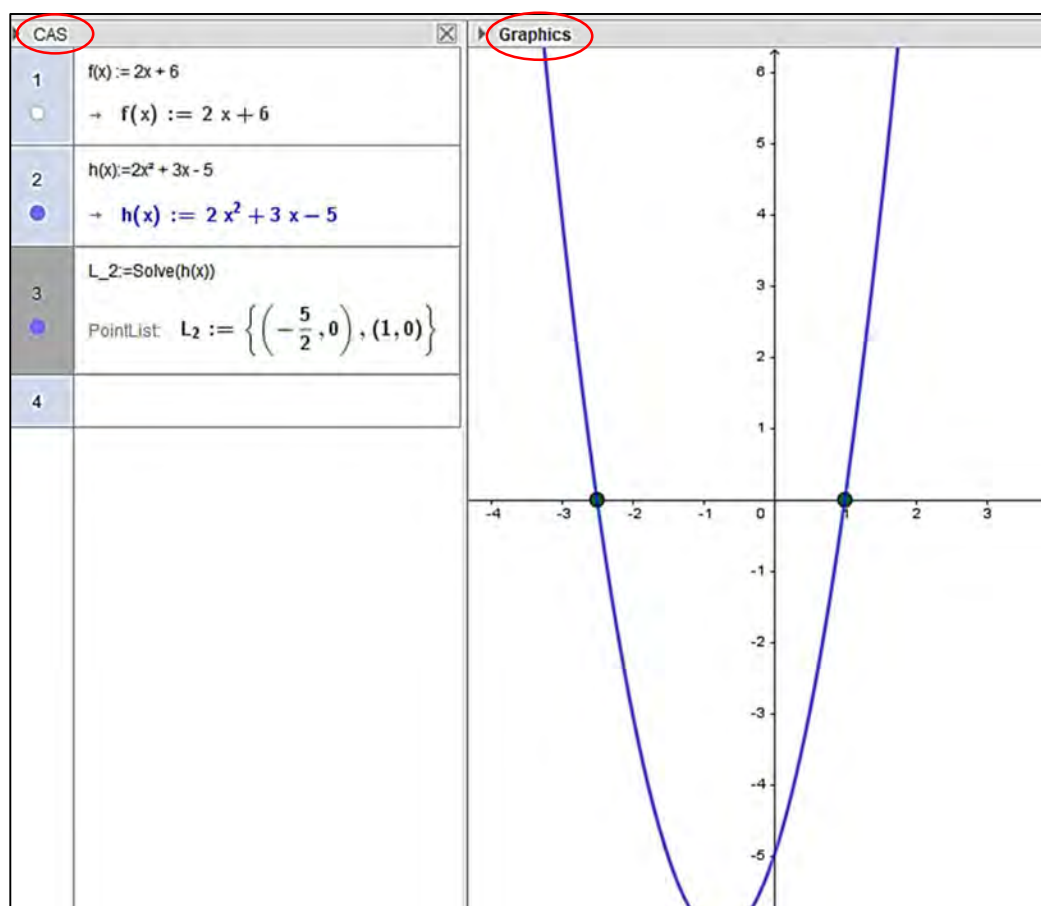


Figure 5.1: The CAS and Graphics window in GeoGebra

One of the recommendations of the GLIP teachers after the first cycle was to train the learners on using *GeoGebra* specific to the topic that was being taught. My fellow researcher and I designed an hour-long training programme on using CAS within *GeoGebra*. The handout we produced for training learners can be accessed in the Annexure III under the heading ‘Training Interlude - CAS’. However, the planned training could not be accommodated in both schools because of the full timetable of the learners and teachers, and the non-availability of the computer lab. Our

observation calendar was planned at the beginning of the year and could not be altered. The teachers therefore issued the training handout during their lessons.

5.2.1.1 Identification of concepts

We discussed the sub-topics we could use with *GeoGebra* and decided on the following concepts:

- Solutions to quadratic equations
- Nature of roots
- Simultaneous equations
- Quadratic Inequalities

Since all these concepts were seen as important by the teachers, we decided that they would be formally introduced in class and *GeoGebra* would then serve to reinforce the concepts through visualising the underlying mathematical ideas.

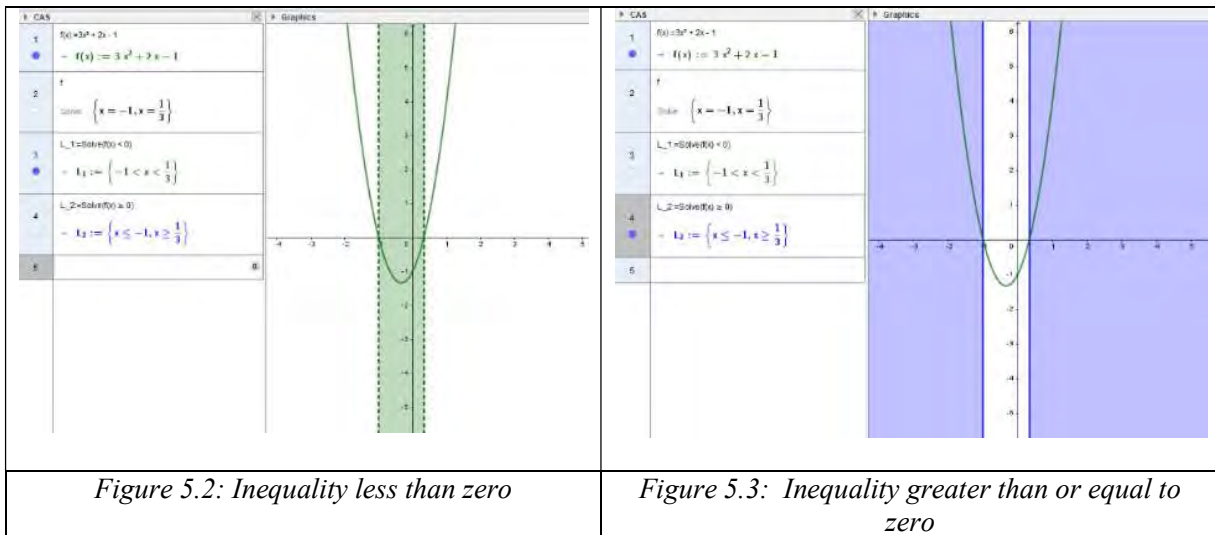
5.2.1.2 Planning and Designing

It is important to note that the software does not provide a direct step-by-step procedure to solve the equations, but instead only displays the solutions once the equations have been entered. Antony felt that it was key for learners' conceptual understanding to observe that the x –intercepts of a particular graph are also the solutions to the given equation. The other teachers were more interested in the procedures of how to solve a quadratic equation. They advocated for designing an applet that would take the learners through a step-by-step procedure in solving a quadratic equation, using the quadratic formula. After arguments and counterarguments on the design of the applet, we agreed that providing a procedure alone to solve a quadratic equation would not harness the visualisation capabilities of the software.

George then put forward the idea of a worksheet that would reinforce the different types of equations. The roots of these equations would include real or non-real roots, equal or unequal roots, and rational or irrational roots. The teachers would initially provide the necessary instructions in class to solve quadratic equations using pen and paper. The following day, the learners would explore the same quadratic equations in the worksheet, using *GeoGebra* in the computer lab. We entrusted George to prepare such a worksheet on quadratic equations.

Antony suggested that the same equations that appeared on the quadratic equations worksheet be used to solve quadratic inequalities at a later stage. He demonstrated on the screen how he used the software to solve the quadratic inequality $3x^2 + 2x - 1 < 0$. All the teachers agreed that

solving inequalities was easier to visualise and understand when graphs were used to determine the solutions. Therefore, we expected learners to visualise inequalities, as in Figure 5.2 and Figure 5.3. Furthermore, solutions in algebraic notation and graphical representation were shown in *GeoGebra*. The teachers believed that this could lead to the idea that the critical values of any quadratic inequality coincide with the roots of its equation or the x –intercepts of its graph.



Abraham then suggested that such a worksheet could be extended to discuss the nature of roots of quadratic equations. He proposed a layout of the worksheet, and after a discussion, we agreed on the final design of the worksheet. The final version of the worksheet can be accessed at WS-1 Annexure-IV. Teachers would instruct learners to type the quadratic equations into *GeoGebra*, solve the equations and then obtain their corresponding graphs. The learners were expected to fill in the columns 4, 5 and 6 on the worksheet, based on their observations in *GeoGebra*. Columns 3 and 4 would have been filled in during the previous lesson in the classroom.

GeoGebra can be used effectively to visualise the concept of classification of roots of quadratic equations. Thus, Applet 2.1 was designed for the purposes of visualising the nature of quadratic roots and for engaging the learners (see Figure 5.4). In this applet, we used sliders for the parameters ‘ a ’, ‘ b ’ and ‘ c ’, for the standard coefficients of a quadratic equation. We expected that learners would be able to relate the values of the discriminant with the numerical values of the x –intercepts of the graph. This is shown in Figure 5.4, with the answers in columns 5 and 6 on worksheet WS-1).

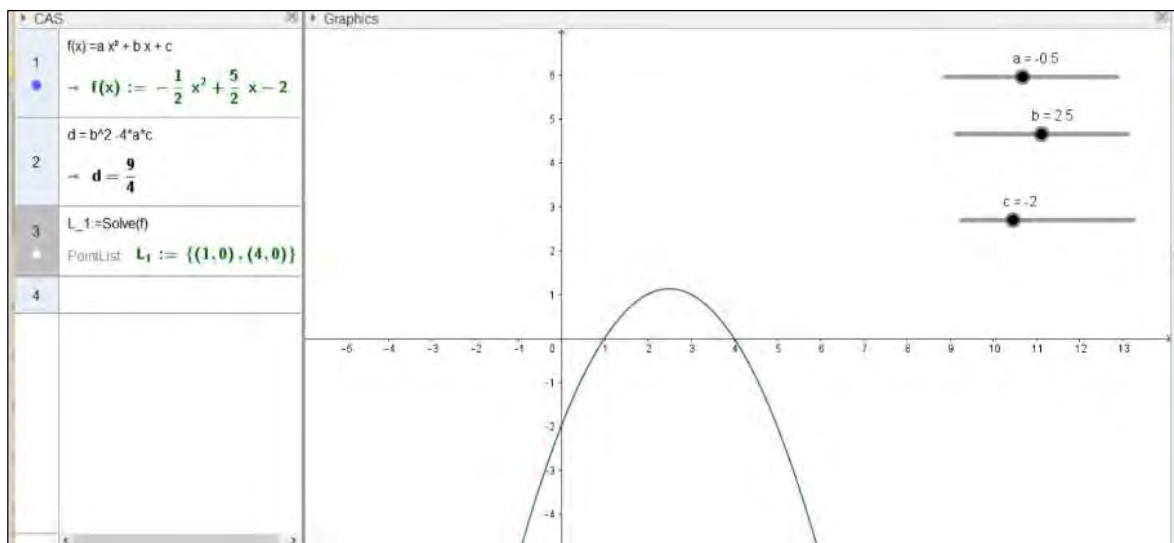
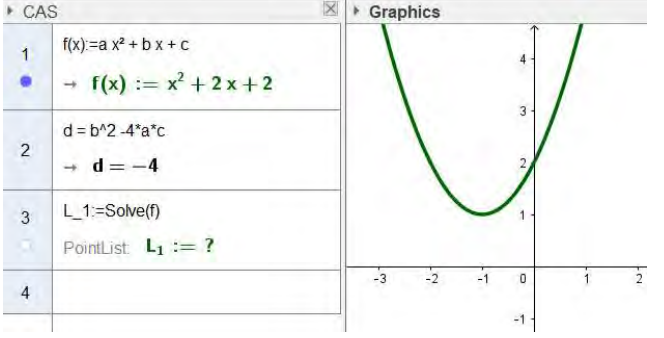
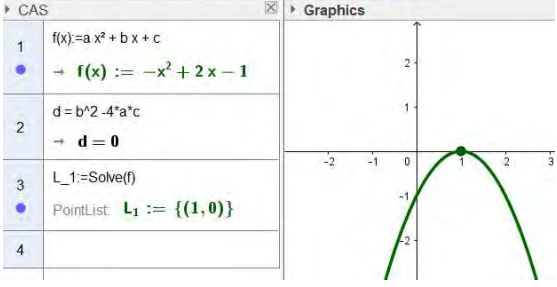
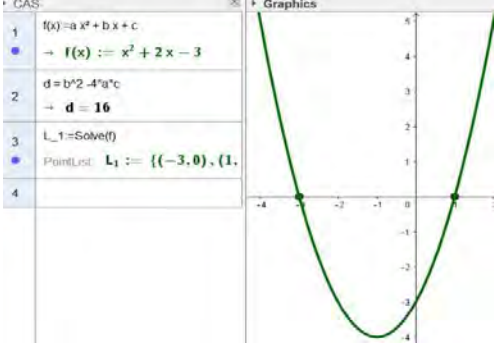
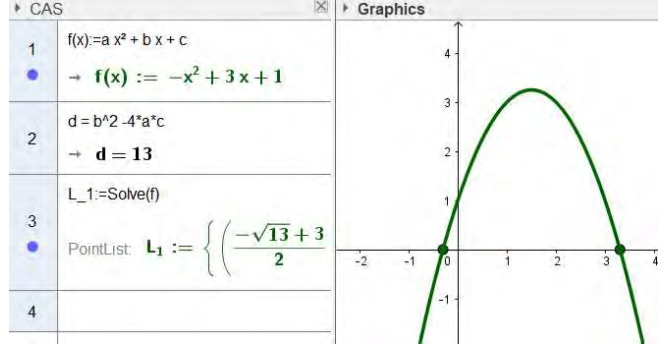


Figure 5.4: Exploring the nature of quadratic roots Applet 2.1

The Applet 2.1 provided the value of the discriminant, the solutions of the equation (in the CAS window) and the graphical representation of the equation (in the graphics window). The emphasis of this applet was to: i) classify the roots of a quadratic equation into real and non-real (imaginary) roots; and ii) distinguish between rational and irrational roots. Table 5.1 summarises the classification of roots of a quadratic equation. When the value of the discriminant is less than zero, roots are non-real (or imaginary), and there will be no x -intercepts on the graph, as in Figure 5.5. When the value of the discriminant is equal to zero, roots are real, rational and equal, and there will be only x -intercepts in the graph, as in Figure 5.6. When the value of the discriminant is greater than zero and a perfect square, roots are real, rational and unequal, and there will be two x -intercepts in the graph, as in Figure 5.7. When the value of the discriminant is greater than zero, roots are real, irrational and unequal, and there will be two x -intercepts in the graph, as in Figure 5.8.

Table 5.1: Summary of the nature of roots of a quadratic equation

Discriminant $\Delta = b^2 - 4ac$	Nature of roots	Graphical representation in Applet 2.1
$\Delta < 0$	Non-real	 <p>Figure 5.5: Non-real roots</p>
$\Delta = 0$	Real, rational and equal	 <p>Figure 5.6: Real rational and equal roots</p>
$\Delta > 0$, a perfect square	Real, rational and unequal	 <p>Figure 5.7: Real rational and unequal roots</p>
$\Delta > 0$, not a perfect square	Real, irrational and unequal	 <p>Figure 5.8: Real and irrational roots</p>

5.2.1.3 Practice Applets

The second applet – Applet 2.2 – we developed, was for practising the nature of roots of a quadratic equation based on its graph, shown in Figure 5.9. The premise behind this applet was that the absolute value of the leading term of the quadratic expression is a prime number. The quadratic expression is also displayed on the graphics window.

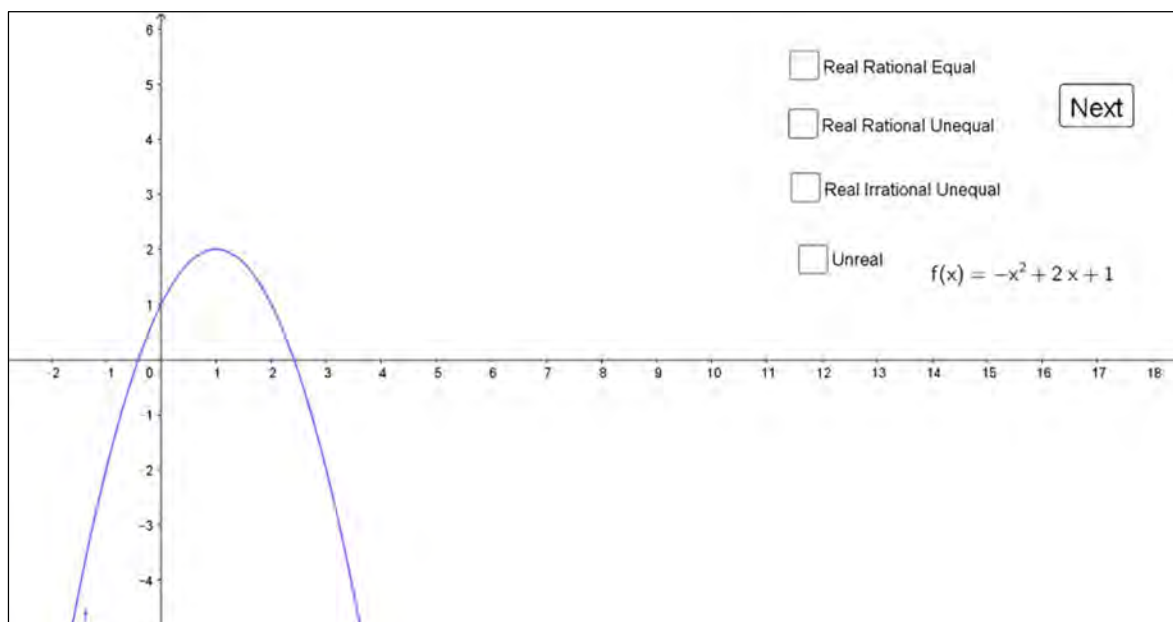


Figure 5.9: Practice question on the nature of roots in Applet 2.2

All the teachers implemented the applets in their classrooms. Due to the limited scope of this thesis, I recorded and analysed only two classes – Antony’s and Paul’s lessons. A vignette of Antony’s class is presented first, followed by a vignette of Paul’s lesson.

The learners were aware of the procedures to solve quadratic equations and inequalities prior to the observed lessons. In previous grades they had learned about the parabola of the form $f(x) = ax^2 + b$.

5.2.2 Analysis of the lesson by Antony

There were 34 learners in the class, and the lesson was scheduled for one hour; however, it lasted for 70 minutes. The learners appeared to be very excited at the beginning of the session, as they entered the computer lab. The desktops that were not working had been marked, thanks to the lab technicians and the learners could easily identify those that were working and those that were not.

There were, as a consequence, fewer computer–hardware related glitches in this lesson compared to Antony’s earlier lesson.

Antony had issued worksheet WS-1 to the learners one day ahead of this lesson. The quadratic equations were solved in the class, and the columns 2 and 3 were filled in before this recorded lesson. He had also briefed the learners on what he expected from them with this worksheet. He mentioned what was meant by the value of $b^2 - 4ac$, a surd in the quadratic formula, as the determining factor for roots of a quadratic equation. He distributed the handout (Annexure III) as they entered the lab.

5.2.2.1 Act 2 Scene 1 ~ Linking equations and graphs

Antony started the lesson by introducing the CAS window in *GeoGebra*. He provided clear instructions to his learners on how to choose the CAS window and went through ‘the commands’ in the software as laid out in the handout. He often repeated his instructions and reminded the learners to follow them accurately. However, some learners struggled to use the keyboards. A combination of keys were required to type symbols such as * (for multiplication), ^ (for exponent), and < (for less than). For typing the linear equation, $2x + 5$, the learners were required to use 2, *, x , + and then 5. When learners were not able to type the equation successfully, Antony repeated his instructions, shown in lines 22–24 in Table 5.2. It was very evident to me that Antony had to teach basic computer literacy skills to the learners (see the transcript). To his credit, Antony often checked whether the learners were on track during the lesson (FF). Antony used a linear function to link its equation to its graph (CC).

Once learners were able to construct the straight line, Antony proceeded to solve the equation, as shown in lines 32–33 of Table 5.2. Line 39 records Antony using different mathematical expressions such as ‘ x –values’ and ‘solve for x ’, to demonstrate different terminologies for the solutions to the equation (CMR). He then demonstrated to his learners the relevant tool in the software to find the solution. Antony made references to the graph on the screen and showed them where the solution to the equation appeared on the graph as the x –intercept, thus making the solution visual. (CS & CC). It was thus evident that the visual aspect of an equation in the form of a graph provided a rich and powerful resource to generate understanding. Furthermore, Antony encouraged the learners to write the x –intercepts in coordinate form, thus linking algebra to coordinate geometry, as in lines 47–50 in Table 5.2 (CC and CMR).

Table 5.2: Transcript (CL2A) – Linking equations and graphs

22 - 24	<p>Antony: I said 2, shift 8 Press shift same time as eight don't press shift then 8 (FR)</p> <p>L: Oh! (Laughing)</p> <p>Antony: I said shift 8, I didn't say shift then 8, then the symbol * will appear</p>
26-27	<p>Antony: Can you see the equation $2x+5$? Now there is a white dot. Can you see that white dot? Click there. Then a straight line will appear. This is a straight line. A function (CS, CC). $2x+5$ is a linear function (CMR).</p>
29-31	<p>Antony: Are you still struggling to show that line? FF</p> <p>L: Yes. (Inaudible Helping learners)</p> <p>Antony: Click the white dot the straight line will come indicating that it is a straight line.</p>
32-33	<p>Antony: Right, I hope you are all there (FF). Then after that, you should turn the page. We are interested in finding the values of x as we are using this to solve for the values for x.</p>
37- 43	<p>Antony: Here is tool $x=$ Click on the tool $x=$. There keep your cursor here in front of $x=$ and solve for x. That $x = -5/2$. First, type the equation and click on the white button to show the graph of the equation. After then, we want to find the value of x, that is we need to solve (CMR). So, we put the cursor in front of the last typed equation then press $x=$ which stands for solve. Now you want to know where exactly the point is (CS). This will give you the value of the $x -$ intercept (CC). The x-intercept is $x = -5/2$ (CC). if you are not sure you again click there that white button and it shows the coordinate and where exactly it is.</p>
44 - 46	<p>L: I don't see it.</p> <p>Antony: Where is $x=-5/2$ is? Does it show you the point? Are you now able to show the point? (FR)</p> <p>L: Yes</p>
47 - 50	<p>Antony: if you can show the point, that is how you solve for x? ne. That is how we solve for x even if (isiXhosa inaudible) $x = -5/2$; $2x+5=0$; take 5 to another side of the equation $2x=-5$ we divide by 2 to get exactly the same solution $x = -5/2$ (FP). Then that one we write as a coordinate $(-5/2;0)$ (CC, CMR).</p>

In the remaining part of the lesson, Antony shifted his focus from linear equations to quadratic equations and inequalities. As is evident in lines 54–57 in Table 5.3, Antony provided clear instructions on how to type in a quadratic equation. He immediately linked it to its graph (CK & CC). Meanwhile, Antony also had to attend the learners' inadvertent use of the 'caps' and 'num lock' keys (FR). His ability to respond to his learners' errors was evident here: he had to repeat the instructions to the class to ensure that the 'caps' key was off in lines 61–62 and 78–80 in Table 5.3. Antony continued with his explanation of the connection between solutions and $x -$ intercepts (CC & CMR), as evident in lines 63–66 (in Table 5.3). However, his explanation of an interesting piece of mathematics was hindered and bogged down by the learners' lack of the required computer illiteracy skills. They struggled to type x^2 , which required a combination of keys 'shift' and '6' to write the superscript (lines 67–71 of the transcript below). This further confirmed Antony's ability to deal with learners who lacked the necessary computer skills (FL). The interaction in lines 72–76 (in Table 5.3) showed the development of learners' conceptual understanding of the interconnectedness of roots of an equation and the $x -$ intercepts (CK).

Table 5.3: Transcript (CL2A) – Quadratic equations

54-59	<p>Antony:Can we now open a new file? Go to file, new file, do not save. File new is to open a new file. I want us to input a quadratic equation, this quadratic equation of ours let it be $2x^2$ you press 2, press shift 8 for multiplication, x squared is going to be shift 6 and 2. That one is $2x^2$. Then -x and -1. (pause. Learners murmur inaudibly). $2x^2 - x - 1$ that is the equation now if you want to see the graph there is that white dot there. You click then it will show the graph. It will show the quadratic function, a parabola graph. (CK CC).</p>
60 - 62	<p>L: [Mr Antony] I do not see the graph here. Antony: (Helping learners) Some of us used capital letter X you will have to use small letter x (FR). Capital letter X will not give the correct result. You are forced to use small letter x.</p>
63 - 66	<p>Antony: I hope all of us are having that quadratic function, the graph is there (CS). Now if you want to know the x-intercepts of that graph, again take your cursor there, and we go to solve which is x= this will give you the x co-ordinate or the x-intercepts (CC, CMR). The solutions of the equations are? Now, are you all having those values? What are the values of x? (FE).</p>
67 - 71	<p>L: Sir, I am not getting it Antony: (Helping learners) You must use shift 6 and 2 for x squared. I repeat, x squared is going to be shift 6 and 2. L: No. Antony: Press x then shift 6 together and then 2.</p>
72-76	<p>Antony: Immediately you put the cursor on tool x= it shows that solve. So we are getting those x co-ordinates (CMR). L: $x = -1/2$ and $x = 1$ L: y-intercept is -1 (IsiXhosa) L: Yeah! a parabola graph</p>
78-80	<p>Antony: you have capital letter X. (isiXhosa listen carefully. I have been repeating. See here). The caps lock key is blinking so, it will always be capital letters (FR). Caps lock must be off. Yes, now type small letter x. then shift 6 together. Yeah, you got it.</p>

Antony focused on the solutions to a quadratic equation that demanded learners to think critically and discover the salient features in the quadratic formula. He strengthened learning experiences through asking probing questions and encouraging the learners to explore mathematical concepts. Antony explained different methods to solve a quadratic equation lines 81–83 of Table 5.4 (CC & FP). He then asked them to think of a situation when they would not be able to solve a quadratic equation (CD). Antony did not explain his answer to the question until he received some responses from the learners.

Meanwhile, he also engaged with different terminologies for solutions to provoke the learners' interest (CMR). Antony tactfully asked, "When will you have non-real values?", another expression for no solutions – thereby displaying his ability to switch between different mathematical ideas that would help learners to deepen their mathematical proficiency. In this way Antony was introduced the nature of roots of a quadratic equation. When a learner used the word 'negative', he highlighted the expression under the surd in the quadratic formula that determined

if the roots were real or non-real (CK). As the learners engaged with different equations in the worksheet, Antony moved around the classroom, helping learners wherever required.

Conscious of using *GeoGebra* for teaching and learning, Antony also incorporated his own knowledge and his teaching expertise into the mathematics classroom. He provided opportunities for learners to develop conjectures and encouraged them to make connections to the acquired knowledge on real and non-real values of solutions of a quadratic equation. He provided the necessary instructions about writing down the observations from the software into the worksheet in lines 94–98 of Table 5.4. Antony elaborated on the solutions and salient features of the graph, using appropriate mathematical language. In lines 104–105 in Table 5.4, he emphasised conceptual links between the real roots of a quadratic equation and the x –intercepts of its parabola (CC & CMR). Some of the learners appeared to be confused about the equation $y = 2x^2 + 3x + 2$, as it did not have real roots. Therefore, he repeated his explanations of the relationship between roots and the x –intercepts (FP & CK), emphasising key mathematical concepts and procedures (lines 110–113 in Table 5.4). Furthermore, he encouraged the learners to switch between abstract algebra and the concrete graph (CS).

Table 5.4: Transcript (CL2A) – Roots of the quadratic equation

81-93	<p>Antony: Ok, now let us see. That one is $f(x)$ this is how we solve for x immediately you start solving for x we will be able to factorise, and in some cases, we won't be able to factorise and then you are going to use the quadratic formula or solve for x by completing the square (CC, FP). The quadratic formula is $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Also, in some extreme cases you will not be solved, isn't it?</p> <p>Antony: I am waiting for the response. When will I not be able to solve? (FA, CD)</p> <p>L: Don't know.</p> <p>Antony: When will you have non-real values? (CD, CMR)</p> <p>L: It is negative.</p> <p>Antony: What is negative?</p> <p>L: That thing.</p> <p>Antony: When we use the quadratic formula to solve, if $b^2 - 4ac$ is negative; we will not find the real values of x (CK). Are we together? In other words, the roots are said to be non-real and will discuss further later. Now let us refer to the worksheet that we did in the class yesterday. (pause).</p>
94-98	<p>Antony: Ok. Now let use the equations from the worksheet, and I want you to write down the x-intercepts column 3. Verify solutions in the computer and your answer in column 1.</p> <p>L: Which equation, sir?</p> <p>Antony: Look, the worksheet. The first equation is $2x^2 - 5x + 3$, use that and then observe in CAS what's the solutions and the x –intercept. Put that in column 3. Verify your solutions with column 1.</p>
101- 105	<p>L: Sir, here I don't see any x-intercepts.</p> <p>Antony: What is the equation? You started from the bottom (of the worksheet).</p> <p>L: Yeah.</p> <p>Antony: doesn't matter. You see now; there are no solutions and hence no x-intercepts (CC). So now when there are no real values of x, the parabola does not intersect the x-axis (CC, CMR).</p>
109 - 113	<p>L: why the graph does touch the x-axis for this equation?</p>

<p><i>Antony: Listen here (IsiXhosa). Look at the last equation, $(2x^2 + 3x + 2)$ you can see that there are no solutions, and there are no x-intercepts (FP CS). We can, therefore, say that for no real values of x of a quadratic equation, its graphs also do not have the x-intercepts (CMR). (IsiXhosa Remember) the solutions are the x-intercepts (CK). I can now see so many hands.</i></p>
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5.2.2.2 Act 2 scene 1 ~ Quadratic Inequalities

Antony moved to the next agenda of the lesson, i.e. quadratic inequalities. He continued with the quadratic equation $y = 2x^2 - 5x + 3$ with its graph and its solutions already on the projector screen. He then provided step by step instructions on how to type the inequality into the CAS window. He often ensured that learners followed his instructions (FF). Antony gestured the ‘greater than’ and ‘less than’ symbols with his fingers, emphasising mathematical terms and notations (CK). He then typed $f(x) \geq 0$ in row 3 of the CAS window. Learners found it difficult to type the inequality symbol ‘ \geq ’; and some of them typed ‘.’, while others typed ‘+’ instead of ‘=’. Antony explained to the learners how to use the appropriate keys on the keyboard, thus responding to their lack of the necessary computer literacy skills (FL). This is shown in lines 135–136 and 148–149 of Table 5.5.

As Antony expounded on inequalities, a few learners were murmuring. Apparently, they did not follow what the teacher was explaining. Nevertheless, when Antony solved the inequality, he asked his learners to articulate the solution, thereby encouraging learners to take ownership of their knowledge (CD). In lines 150–151 (in Table 5.5), he corrected a learner in the use of precise mathematical terms and notations (CK & FA).

Antony was always willing to help learners to clarify their doubts, exhibiting his skill in responding to the learners’ queries (FR). In lines 154–158 (in Table 5.5), he used alternative methods to explain the solutions of the inequality from the graph. He claimed, however, that he wanted to address a certain misconception related to inequalities. Antony used *GeoGebra* as a visualisation tool and showed the learners how the graph could be highlighted for $f(x) \geq 0$, thus concretising the concept (CS). Referring to the graph, he also used terms like ‘left of $-\frac{1}{2}$ ’ for the solution $x \leq -\frac{1}{2}$ (CMR). This indicates that Antony employed different strategies to elaborate on the solutions to the inequality.

Table 5.5: Transcript (CL2A) – Quadratic inequalities greater than zero

119- 125	<p>Antony: Then, after that, as it says equations and inequality, we need to solve for the inequality. Let us solve the inequality. We at (row) 3 we simply say $f(x)$ f, shift 9, x, shift 0. Are we all having $f(x)$? my equation, i.e. $f(x)$ here it is $2x^2 - x - 1$</p> <p>L: Yes, then, sir.</p> <p>T: Press f, shift 9, x, shift 0 then it will be $f(x)$. (Pause). Now if you do have $f(x)$ let us say shift, we all know the sign of greater than or equal to? It is greater than and this less than (making gestures with fingers). CK CM</p>
127-136	<p>Antony: If you check your keyboard, there is greater > do you see greater there?</p> <p>L: Yes</p> <p>Antony: Greater?</p> <p>L: No.</p> <p>Antony: [Here] You press shift and full stop. There is a full stop there or a dot in your keyboard, whatever that you call it. Then it is press shift and full stop (FL). It will give you > and then press equal to and 0. Now that one will be $f(x) \geq 0$. Are we all there?</p> <p>L: Yes</p> <p>Antony: (Helping learners) Press equal to, then 0, then you press enter. It will show you the inequality symbol FL. Does it show the inequality equation?</p>
142 - 149	<p>Antony: Then now if you say it must solve. Let us start by solving. I guess now we are all able to solve. Are you able to solve? There is inequality $2x^2 - x - 1 \geq 0$ are you all having that</p> <p>L: Yes</p> <p>L: in IsiXhosa - What is this about?</p> <p>L: (murmuring inaudibly).</p> <p>Antony: Do you have a solution? Are we all having the solution to that inequality? (Helping learners). What have you pressed? It's a plus symbol; we want equal to symbol. No. you must not use shift (FL). Alright. (pause) Can someone tell me the solution? (CD)</p>
150 - 151	<p>L: That x is greater than one and less than the negative half.</p> <p>Antony: Yes, here x is less than or equal to negative half and x is greater than or equal to 1 (CK FA)</p>
154 - 158	<p>L: Sir, I do not see it.</p> <p>Antony: That is $x \leq -\frac{1}{2}$ and $x \geq 1$. Left of $x = -\frac{1}{2}$ and right of $x = 1$. (CMR). Immediately to do that one, you will press in that white button. It will highlight your solution. If you click that button, it will highlight your solutions (CS). Then the highlighted part is going to be the one that gives the values of x when $f(x) \geq 0$.</p>

Antony further explained the inequality $f(x) < 0$ using the software. This explanation demonstrated evidence of CK as he used accurate mathematical terms and notations (lines 170–171 in Table 5.6). Switching between graphs and solutions (CS), he encouraged learners to link the solutions to the highlighted part of the graph. Another instance of tapping into visual and algebraic representations in *GeoGebra* (CS) that struck me, was when he emphasised the differences between the solutions of the ‘less than’ and ‘less than or equal to’ inequality. He asked probing questions to identify the differences between ‘less than’ ($<$) and ‘less than or equal to’ (\leq) and encouraged the learners to make inferences from the graph. As is evident in lines 179–193 in Table 5.6, the learners engaged with the task of identifying the differences. They immediately identified the differences in *GeoGebra*, as the software sketched vertical lines at the boundary values. *GeoGebra* draws a broken line to indicate exclusion and a solid line to indicate

the inclusion, of the boundary values of an interval, as shown in Figure 5.2 and Figure 5.3 on p.130. Indeed, to enhance conceptual understanding, Antony furnished the implication, in mathematical terms, of the learner's response (CS & CK). Accurate terms were used to explain the concepts of inclusion and exclusion of the boundary values of the inequality.

Table 5.6: Transcript (CL2A) – Quadratic inequalities less than zero

164 – 171	<p>Antony: Now if we can do the same thing now say $f(x) < 0$ $f(x)$ less than 0 as question 4 [row 4]. Then after that, you will solve for that. Press enter then solve, and after that, you show now where exactly is $f(x)$ less than zero (CS). And solve for that one. If you want to solve for that one, it will show a different colour. If it is not clear, you can hide that one of $f(x) \geq 0$. You go back to that of $f(x) \geq 0$ and you hide that one by pressing that dot, then it will hide the one for $f(x) \geq 0$ and show only the one $f(x) < 0$. Only the highlighted part of the graph is the solution of $f(x) < 0$ that is $-\frac{1}{2} < x < 1$, x lies between $-\frac{1}{2}$ and 1 (CK). Only the highlighted part.</p>
174 – 176	<p>Antony: But then, if you are vigilant enough, then you will see that there is a difference between \leq and $<$. Is there anyone who can tell me what is it that they see between the solution of $f(x) \leq$ and $f(x) <$? There is a solution for $f(x) \leq 0$ and there is a solution for $f(x) < 0$ (CD).</p>
179 – 193	<p>L: The inequality and the x-intercepts (isiXhosa inaudible) Antony: Anyone? I am waiting for a response. Anyone? There are vertical lines that are drawn there at first we were looking for $f(x) \leq 0$ and there were vertical lines. Now we are looking for $f(x) < 0$ there are still vertical lines (CD, CMR). What is the difference between those two lines? L: There is no 'equal to' sign. L: $f(x) \geq 0$ is that first solution, there is a separation (IsiXhosa) L: (inaudible) Antony: That one is, Tell them. Tell the whole class Antony: The one is? The one is a L: Broken line Antony: Broken line and the other is a L: Solid line Antony: The other is a solid line. That solid line indicates that there is also an equal sign (CS). The solid line indicates that the boundary values are included in the solution, Immediately, there is no equal sign then the line will only be a broken line to indicate that the (boundary values) are excluded (CS, CK).</p>

As the lesson progressed, Antony also underscored the distinguishing features between $f(x) \geq 0$ and $f(x) \leq 0$, (lines 194–207 in Table 5.7). He expounded on the differences in the solutions using precise mathematical terms and notations that were useful to learners. He often made references to the graph and moved between $f(x) \geq 0$ and $f(x) \leq 0$. His explanation focussed on the boundary values and demonstrated how these values were critical in solving the inequalities. Nevertheless, he also employed different terms (CMR) such as 'left of (line 201 in Table 5.7) and from 'negative infinity' (line 206 in Table 5.7) to convey the concept of inequality. Furthermore, he emphasised the significance of the visual representation, i.e. the parabola graph of the quadratic

function. I argue that technologies such as DGS that use visual tools may enable learners to ‘visualise’ the mathematical concepts and ideas in a more meaningful way.

Lines 208–213 (in Table 5.7) capture the evidence of Antony developing mathematical ideas and practising the interweaving of CU and IF so that learners could grasp the underlying concepts and the procedures connected to these concepts. He explained that the boundary values are the x –intercepts of the graph when solving the quadratic inequality for ‘less than’ or ‘greater than’ zero. Kilpatrick et al. (2001) argue that understanding the rules that underlie ideas “makes it possible for learners to use mathematics reliably” (p. 162).

Meanwhile, Antony proceeded to the next agenda planned for the day, i.e. the nature of roots. However, a pair of learners were still discussing inequalities. I deviate for a moment to relate something interesting from this pair of learners. When Antony noticed this pair of learners, he reiterated his explanation on solving the inequality (lines 243–250 in Table 5.7). This exchange suggested that the learners required more time to engage with the inequalities as they internalised the concepts. The screen recording of these learners (the data of my fellow researcher) showed that these two learners discussed at great length, using the equations provided in the worksheet, even as the teacher moved on to discuss the nature of roots. Seemingly, the interactive environment provided by Antony was an opportunity for learners to explore and engage with mathematical concepts, which is **CD**.

Table 5.7: Transcript (CL2A) – Distinguishing ‘greater than’ and ‘less than’ zero

194 - 202	<p>Antony: Now, let us look at the difference between $f(x)$ greater than zero and $f(x)$ less than zero. There are two points to indicate now the solution. You can even name those points; there is no problem you can name them. That is why now we find out that we are saying for less than, the solution will be restricted between two points (CK). You can see that it is between the first and second point. And for greater than it will be separated (CK). There will be a separation. Now if you observe here, these highlighted parts are bounded between two lines for less (CS). But for greater, there is a separation between those highlighted parts. There is a part that is highlighted on the left, and there is a part that is highlighted on the right.</p> <p>L: (murmuring inaudibly)</p>
203 - 207	<p>L: What is this separation?</p> <p>Antony: Let me show you that one. That one is for greater. Can you see that between these highlighted parts there is a separation for greater? (CD) The solution is x less than or x greater than, i.e. look at the x-axis it is negative infinity to this boundary value which is $-1/2$ here, which is less than $-1/2$. The second part is from 1 to positive infinity which is greater than 1 (CK CMR).</p>
208 - 213	<p>L: (murmuring inaudibly) It's from infinity to negative half. How about less than zero?</p> <p>Antony: But when we talk of less than zero, those points the highlight is bounded by two points (CK FR FP). (pause). Remember, when you solve an inequality such as less than zero you give the boundaries within which the solutions lie (CK FA) and boundary values are given by the x –intercepts of the graph (FP). When you solve an inequality for greater than zero, you give the</p>

	<i>separation from the boundary values (CK, FA) and again boundary values are provided by the x –intercepts of the graph (FP).</i>
243 - 250	<i>L: Greater than is from here to here.</i> <i>L: Less than 0 is from here to here</i> <i>Antony: Negative half to positive one, greater than is from there to negative half and from there to positive one.</i> <i>L: You mean from infinity to negative half.</i> <i>Antony: For greater than zero, it is from infinity to negative half.</i> <i>L: It's from negative infinity to negative half and from one to infinity.</i> <i>Antony: Look its x element of between negative infinity and negative half and one to positive infinity.</i>

5.2.2.3 Act 2 Scene 1 ~ Applet Nature of Roots

Having a lesson on a Friday afternoon, Antony made some humorous comments that attracted some laughter among learners, in lines 218–223 of Table 5.8. He wittily used the opportunity to bring some smiles to the faces of the learners as well as alert them to their lack of computer literacy skills.

Antony balanced procedures and concepts quite well, when he asked his learners to recall the formula for the discriminant and then explained how the formula worked. He provided step by step instructions in constructing sliders for the parameters of a quadratic equation. Thereafter, in the CAS window, he typed the quadratic equation in standard form in the first row and the formula for the discriminant in the second row. Despite this, the learners misinterpreted his instructions, and many of them could not synchronise the equation or the discriminant with the appropriate parameters.

Antony wanted to encourage his learners to make conjectures about the graphical interpretation of the relation between the discriminant and the roots of a quadratic equation. His mathematical instructions did, however, not receive much attention as the learners were weighed down by the intricacies of the software (in lines 266–271 and 289–294 of Table 5.8) . It appeared that he gave up correcting the mistakes that the learners were making on the computers, as he asked learners to share computers with other more skilled learners and who were able to synchronise sliders of the parameters and the quadratic functions successfully.

Shown in lines 104–105 and 344 and 347 (in Table 5.8), Antony emphasised conceptual links between the nature of roots of a quadratic equation and the x –intercepts of its parabola (CC & CMR). In lines 352–353 (in Table 5.8), he exemplified the strand of conceptual understanding, when he made visual the situation where the roots were equal and connected to the parabola,

having only one x –intercept (CS). In fact, he inspired his learners to drag the sliders representing the parameters of a quadratic equation, to have different values of the discriminant, and then asked them to compare this with the x –intercepts of the graph (CD). In this way he established mathematical relationships between the algebraic expression and the graph. His interactions with his learners using *GeoGebra* led to mathematical abstractions through emphasising relationships among and between concepts (CK and CMR).

In his interactions with the learners, Antony further demonstrated solid links between the value of the discriminant and the properties of numbers. Specifically, he drew on learners’ prior knowledge on square numbers (CC) to enhance the knowledge on the nature of roots of a quadratic equation (CK). Antony used probing questions in lines 329–345 in Table 5.8, that required learners to think critically to establish conceptual links among mathematical ideas (CD). He also invited feedback from the learners. He intervened effectively with ideas that structured the learning process, when he stated, ‘*listen to her, she is linking to the properties of natural numbers*’. Antony also displayed a sound skill in procedures and making links to other mathematical ideas, refer to lines 354–371 in Table 5.8, which were essential for developing learners’ mathematical proficiency (FP). Thus, my analysis complements Ruthven et al.’s (2009) study which emphasised that “the teachers still played an important role in inducting and supporting students in the use of the software for mathematical purposes” (p.291).

Table 5.8: Transcript (CL2A) – Discussing the nature of roots

104-105	Antony You see now; there are no solutions and hence no x -intercepts (CC). So now when there are no real values of x , the parabola does not intersect the x -axis (CC, CMR).
218-223	Antony: No question because you are rushing to go home? Or no question because you understand what we are doing? L: Aha! Understand. Both, both. Antony: I can see that some of you are afraid of these computers. L: (laughing). Antony: Maybe you think that this is a real mouse
263-265	L: It is not like that on the screen. Antony: I think you must undo. Yes, undo, and now click on the ‘Move’ tool. You see that arrow symbol, the first one. Yeah.
266-271	Antony: Then now let us press $d = b^2 - 4ac$, the discriminant because now we are determining the nature of roots. Remember, we discussed earlier that in the quadratic formula, inside the square root is $b^2 - 4ac$ which we call it as the discriminant. (CK) Antony: What is your problem? Look, your equation is not changing (Helping learners). Antony: We said that, so many hands. Ok. What do you mean by roots? So many problems with the computer. The roots are the x values (CC, CK).
276-277	Antony: I can see that many of you are still struggling to have the sliders and the equation. Please share with your neighbour who has done it.

289-294	<p>Antony: Ok, I move now 'a'. We all managed to get $b^2 - 4ac$. Now let us see what is going to happen if we change 'a' as it is can you see $b^2 - 4ac$ the value of $b^2 - 4ac$. What is the value of $b^2 - 4ac$?</p> <p>L: -3</p> <p>L: Why the discriminant is 99/5 whereas for others its -3.</p> <p>Antony: Type again. It's the same result. We can find that later, now share the computer with [].</p>
332-336	<p>Antony: Yes, therefore, when the discriminant is equal to zero, then the x –intercepts coincide (CK, CC). Helping learners (inaudible) as you move c you can see that the graph is moving up or down. And when our $b^2 - 4ac$ is equal to zero; the two x –intercepts of the quadratic equation or function are also equal (CK, FP). For those who did not understand how to drag your 'a' to 1, 'b' to 2 and 'c' to 1. What is the value of 'd' (CD, FA)?</p>
344	<p>The roots are non -real that is the roots do not exist, or the graph does not cut the x-axis (CC, CMR).</p>
347-353	<p>Antony: The roots are real, and that is the roots exist, and the graph intersects the x-axis (CC, CMR). What else have we seen? (CD)</p> <p>L: When $d = 0$ the roots are equal</p> <p>Antony: Yes and then the graph? (CD) How many roots or x-intercepts we have when $b^2 - 4ac$ is zero (FA)?</p> <p>L: One</p> <p>Antony: Yes. So, we can say that when $b^2 - 4ac$ is zero, the roots are equal (FP). And the graph has only one x –value on the x-axis (CC, CMR). Ok, Are there any questions? (a pause).</p>
354 - 371	<p>Antony: Ok. For real roots, we further classify equal unequal and rational and irrational roots. We have seen $b^2 - 4ac = 0$ we say roots are equal. When we say $b^2 - 4ac > 0$ the roots are unequal (FP). Let me show with an example. I take the equation from the worksheet. Drag the points. Drag the points to $a = -3$, $b = 5$ and $c = 1$. Look at the second one on the worksheet $-3x^2 + 5x + 1 = 0$. The roots are? (FF)</p> <p>L: Real unequal</p> <p>Antony: $b^2 - 4ac > 0$ The roots are real and that if the roots exist and the graph intersects the x –axis (CC). Here the roots are real unequal and irrational. Why irrational? (FF)</p> <p>L: d is not a square number.</p> <p>Antony: Yes.</p> <p>L: (isiXhosa inaudibly)</p> <p>Antony: Yes, listen to her. She is linking to the properties of natural numbers (CC).</p> <p>L: I am saying that the roots are irrational because $d = 13$ is not a square number.</p> <p>Antony: In conclusion, we say that when d is greater than zero roots are unequal (FP). When d is not a square number (CC), roots are irrational (FP). When d is a square number, the roots are (a pause) (FF)</p> <p>L: Rational.</p> <p>Antony: Roots are rational. Any questions?</p>
392 - 399	<p>Antony: This is like mathematics as it is mathematics. Now we are doing mathematics using computers. But what is happening is some of us are unable to use computers. So we have to first teach how to use the computers before you go to the actual use of mathematics. But if you can keep on doing this. How to do such programs using GeoGebra. Then we will become perfect and then we will enjoy more of this. Otherwise for today unless there are some more questions we can call it is a</p> <p>L: day</p> <p>Antony and L : Thank you</p>

By the time Antony asked his learners to use the practise applet, the lesson already exceeded the allotted time. He motivated his learners to apply their understanding to answer questions in the

applet and ended the lesson by emphasising the importance of using computers in learning mathematics.

5.2.3 Analysis of lesson by Paul

Paul's lesson was held on the following day. Antony had cautioned him to confine the lesson to only one topic owing to time constraints. Therefore, Paul chose the topic of inequalities during the planned, recorded session. The nature of roots was scheduled for the next day but was not recorded.

There were 48 learners in this class. However, a few had to share the computers due to the malfunctioning of some computer terminals. Some laptops were also made available. This time the desktops and the laptops were switched on before the lesson, thus saving some time and there were fewer computer-hardware related glitches compared to the earlier lesson. The lesson was scheduled outside regular school hours for 40 minutes, and it lasted for 35 minutes.

We incorporated a change in the handout to include a word of caution: to 'use the small letter 'x' and not capital letter 'X' while keying in the equation'. However, the new handout could not be printed in time due to technical problems at Paul's school. Once the learners settled down, Paul distributed the worksheet WS-1. Prior to this lesson, Paul had discussed solving quadratic equations and inequalities with his learners.

5.2.3.1 Act 2 Scene 2 ~ Quadratic equations and graphs

Paul started the lesson with the quadratic equation $y = x^2 - x - 6$, giving clear instructions on how to enter the equation in *GeoGebra* in lines 4–5 and 7–8 in Table 5.9. His computer screen was simultaneously projected onto a screen as he typed in the quadratic equation. As with Antony's class, the learners struggled to key in the quadratic equation in *GeoGebra*. In lines 10–13 (in Table 5.9), Paul demonstrated his proficiency in computers, as he quickly responded to a learner's misapprehension in using the keys. The learners' computer illiteracy was prevalent throughout the lesson and affected the smooth flow of teaching and learning. When Paul wanted to unpack the relationship between a quadratic equation and its graph, he was interrupted on several occasions by the learners who became bogged down by the technical intricacies of using the computer keyboards, shown in lines 24–25, 30–31 and 58–63 in Table 5.9. On the video, I observed that he was moving around the learners quite briskly, helping them with the keys and the software tools, and at the same time explaining relevant mathematical concepts to the learners.

The interaction with the learners using *GeoGebra*, that emerged in lines 50–52 in Table 5.9 revealed the learners were developing conceptual understanding. I observed that Paul built on the learners’ prior knowledge (CC) to facilitate the connections between algebra and graphs (CC). As he solved the equation in *GeoGebra* using the ‘solve’ tool, Paul directed them to generalise the number of solutions to a quadratic equation (CS), (maximum of two solutions) as evident in lines 58 and 62 in Table 5.9. He carefully chose words such as ‘ x –values’ and ‘solutions’ to represent ideas in multiple ways (CMR), as evident in lines 62–63 of Table 5.9.

Besides this, in line 89 in Table 5.9, Paul emphasised the connections between the algebraic solutions and the x –intercepts of the graph (CC). Here, a learner suggested that the solutions were the intercepts of the graph; however, Paul responded, amending it to ‘ x –intercepts’, to be more specific (FR).

In this episode, Paul was able to draw on the capabilities of *GeoGebra* to engage learners with conceptual links between algebra and graphs. The activities and explanations of the concepts were mathematically appropriate and useful for learners to understand the ideas involved.

Table 5.9: Transcript (CL2P) – Quadratic equations

1 - 8	<p>Paul: Today, we are focussing on quadratic equations and inequalities. We will discuss the nature of roots tomorrow. Now, open <i>GeoGebra</i> those who have not opened. I see some of us have already opened it. Go to the extreme right; there is an arrow there; I want you to choose the option written CAS. Can you see it? It is the CAS ne. Now what you want to do? We want to look at solving. There is one equation that I want us to punch quickly. I want us to punch $x^2 - x - 6$.</p> <p>L: Yes sir, select CAS then?</p> <p>Paul: Right. Let us use the equation $x^2 - x - 6$ How do you punch x^2? You press your x ne then you press shift and six then you press two can you see that?</p>
10 - 13	<p>Paul: Press shift and six aha! This six on top here look, or this pattern (showing cap ‘^’) it is on top of six, is that ok? Have you seen it?</p> <p>L: Sir, I do not get it. How do we type squared?</p> <p>Paul: If you press shift and six the one on top, not six on the number lock. FL</p>
24 - 31	<p>Paul: There is a circle which is open. Click that one and see what happens? CD</p> <p>L: Sir, I don’t get it.</p> <p>Paul: Your laptop is not responding. Let us restart and meanwhile share with other pupils.</p> <p>L: What’s happening here?</p> <p>Paul: Go to the right and select CAS then type $x^2 - x - 6$. The x is a shift and six ne then two and then $-x - 6$ then press enter. Can you click on that open circle? Just highlight it and see what will happen?</p> <p>L: Sir. (Inaudible isiXhosa) There is no...</p> <p>Paul: There is no what? Can’t you see it under 1? Yeah, you did not press enter. FL</p>
47 - 52	<p>Paul: Once you press your x shift and six, then two to get your x^2, then you press $-x - 6$ then you say (inaudible) once you do that you press enter ne? It must give you that equation. Then if you click the open circle, you get the what? FF</p> <p>L: Graph</p> <p>Paul: The graph ne? What is the shape called? CC</p> <p>L: Parabola, sir.</p>

	Paul: Parabola ne. Which means that that is the graph of that equation ne? CC
56 - 63	Paul: It's[Computer] slow. Ok. Once you highlight that you select $x=$. can you see $x=$. It must give you answers. L: yes Paul: How many solutions does a quadratic equation have? CD How many solutions do you notice here? FF L: Yes. L: No Paul: You click $x=$. Highlight that here, and you get your answers. Yeah! Highlight there and click $x=$. Two values of x ne. So once you highlight you must get your solutions ne?? Two solutions ne, (CMR) can you see them?
68 - 76	L: How do you get two numbers, sir? Paul: You highlight here then you press this, so you get your answer. So did you get your two answers? L: Yes Paul: Where can we see those answers can be indicated on your graph (CD). Just click that open circle again; you will see what will happen. Have you seen that? CD CS L: No. L: Yes Paul: How do you solve in GeoGebra? Highlight it; then you click, did you see where it is written $x=$ on your toolbar on top there.
84 - 89	Paul: So, what did you notice? CD And want to solve. So, you can see that the solutions are $x=-3$ and $x=2$. So now look again there is a circle. Can you click on that circle? Just highlight it and see what will happen? CD L: Yes, sir. Paul: Click that one. Just click on that circle. And see what will happen? CD You see that x -intercepts ne? FR So solutions are the x -intercepts of the parabola graph. CC CS

Paul then took the learners to the next level of mathematical understanding as he asked his learners to engage further with the equations in the worksheet. He encouraged and motivated his learners to do hands-on activities in *GeoGebra*, as evident in lines 91 and 101–104 in Table 5.10. Paul guided the learners (**CD**) to comprehend the graphical implication of a quadratic equation with one real root, as shown in lines 110–120 in Table 5.10, thus enriching their understanding through concepts and relations (**CC**). This shows that Paul enabled the learners to reach a higher level of conceptual understanding (**CK**). The analysis of the lesson, thus, affirms that the interactive nature of dynamic applets has the potential to ‘interweave content and pedagogy’ (Ball & Bass, 2000), which would have been difficult in a static environment.

Table 5.10: Transcript (CL2P) – Quadratic roots and intercepts

91- 93	Paul: You can now type new equations from the worksheet and check the intercepts. CD L: Sir, which equation? Paul: This one on the worksheet (inaudible).
99- 104	Paul: Write down the observations in your worksheet ne? L: Where, sir? Paul: I want you to write on the third column – observe the graph and write the x -intercepts. Is that ok? I tell you it is going very interesting to know these equations and it's the shape of the graph.

	<p>L: Yes, sir.</p> <p>Paul: Don't do in another row you can retype in row 1. The graph and intercepts will change automatically. CD</p>
110-120	<p>L: Here, there is one solution.</p> <p>Paul: What is the equation?</p> <p>L: this equation $x^2 + 2x + 1$</p> <p>Paul: For the equation $x^2 + 2x + 1$, there is only one solution.</p> <p>L: yeah! I can see that.</p> <p>Paul: Ok you don't worry you see when you type this equation ne? Now listen. You get only one solution, and what do you see in the graph? CD</p> <p>L: (-1, 0).</p> <p>Paul: Yes, its solution is $x = -1$, and there is only one intercept, as you said. Look at that graph it's intercepted on the x-axis and is the turning point ne? CK CC</p> <p>L: Yeah! Wow! I can see that now.</p>

5.2.4 Act 2 Scene 2 ~ Quadratic inequalities and graphs

Paul moved on to the next session of the lesson, i.e. quadratic inequalities. He continued with the equation on the screen, $y = x^2 - x - 6$, retaining the solutions and the graph. He used the following row in the CAS to get the inequality as $f(x) > 0$ (where $f(x) = x^2 - x - 6$). He provided clear instructions on how to enter the inequality, shown in lines 125–126 and 134–135 in Table 5.11. He repeated the instructions regarding the keyboards when he realised that the learners were fumbling with the keys. Paul guided (**CD**) his learners to recognise the inequality as the shaded part of the existing graph. However, his directions to link the graphs did not receive much attention as the learners were still weighed down by the intricacies of the computer, as evident in lines 129–131 and 176–179 in Table 5.11. The learners' idiosyncratic ways of using the keyboards continued to hinder the teacher from mediating the mathematical content smoothly. Once the learners settled on the correct inequality, Paul guided them to visualise the algebraic notation of the inequality and the equivalent highlighted region of the graph (**CC**). Lines 137–140 (in Table 5.11) bear evidence to the fact that Paul used accurate mathematical terms and notations that were useful to the learners (**CK**).

When he dealt with the 'less than' inequality, he made use of different terminologies such as 'middle region', 'below x - axis', in lines 162 and 165 of Table 5.11. His proficiency was demonstrated by his use of multiple ways (**CMR**) to make the ideas comprehensible to learners. Furthermore, he showed them the region of the graph that was below the x - axis for $f(x) \geq 0$, recorded in line 165 in Table 5.11; thus, he effectively switched between the abstract and concrete, which is **CS**. In my view, toggling between 'greater than' and 'less than' inequality allowed

learners to develop useful procedural fluency to solve inequalities; hence I coded line 167 (in Table 5.11) as **FA**.

In lines 185–192 (in Table 5.11), Paul expounded on the differences between less than ($<$) and less than or equal to (\leq) and made visual references (**CS**) to the dotted vertical lines. He thus explained the exclusion of the critical values of the inequality, using appropriate mathematical terms (**CK**). However, it is evident in lines 201–210 in Table 5.11, that he could not explain as he wanted to since his laptop was not responding. It took a while before his computer provided the expected graph.

Meanwhile, a learner raised a concern that the graph did not highlight the solution to the inequality $2x^2 + 3x + 2 \leq 0$, the last equation in the worksheet WS-1. Lines 193–199 (in Table 5.11) are an interesting excerpt that testify to the fact that learners can work on their own in a technological environment, and a teacher might need to scaffold them on the mathematical concepts that enhance their knowledge. Here, Paul displayed his flexibility in responding to a learner’s concern, (**FR**), and he confirmed the conclusions made by the learner with an added accurate explanation (**CK**). Furthermore, as he explained to the learner, he made visual reference to the mathematical concept of non-real roots, i.e. that there were no x –intercepts of the graph for that instance.

Table 5.11: Transcript (CL2P) – Quadratic inequalities

123-126	Paul: Next thing that you want to do is to make that one an inequality. Ok if you are using CAS 3 go to 4. Click here, and your cursor should flash like that ne. So we want to make that one an inequality. Can you see it is already written $f(x)=?$ Press f, open bracket, we use shift and nine then x. This key, zero with shift, to close the bracket. And greater than 0. For greater than symbol use shift full stop and then press 0.
127- 135	Paul: So, you click $f(x)$ or type $f(x)$ then you select >0 , and press enter. It should be like that. Every time you punch ne there will be nothing then you click the circle you click this one it will give you the answers. And it is shaded now that is greater than 0. L: [Mr Paul], look here. Paul: (Helping learners). What! You typed capital letter F FM . Small letter there. When we say $f(x)$, we will be using this one. Ahha! Just type $f(x)$ there. Open bracket, what are you doing? Start f, open bracket again, shift 9 then x, then close bracket shift 0, and finally greater than shift full stop and then 0.
137- 140	Paul: Now, enter. So the shaded region is the set of your solution. CC Can you see that? That will be x less than what? $x < -2$ and $x >$ CK L:3 Paul: $x > 3$. Is that ok? FF
162- 167	Paul: Now click your circle there. It will shade the middle one. CMR Can you see now that what we were talking about that if it is inequality it says \leq ne. L: Yes Paul: We want the part which is below the x-axis. CS And you can see that is the one which is shaded. L: Yes, sir. Paul: So if you unclick and go back to the previous one you see where it will shade CD FA

176- 179	<p>Paul: Something that you don't see that. $f(x) \leq 0$, press enter. Why is not capturing? Oh! There we are. There we are. Oh! you pressed $<+$, you need to press \leq FR</p> <p>L: Oh! Oh! Ok Ok. Thanks (giggling).</p> <p>Paul: Press \leq, equal to sign. You don't have to press shift. Simply press equal to sign.</p>
185- 192	<p>Paul: Can you see something? Unclick the other one. What can you see about the lines? Those vertical lines. What can you say about vertical lines?</p> <p>L: (inaudible)</p> <p>Paul: They are dotted meaning that we don't need those solutions CK CS</p> <p>L: oh! Yes.</p> <p>Paul: We don't need the values of -2 and 3. Then if you go to number 5 and click and also check there the part of the graph that is shaded is the one above the x-axis. (CMR) Are we together? The solution is $x < -2$ or $x > 3$.</p>
193- 199	<p>L: Teacher, why it doesn't highlight?</p> <p>Paul: What you notice here, ne? What's the inequality?</p> <p>L: It's ≤ 0 This equation, last one from this worksheet.</p> <p>Paul: So you see the graph is always above the x-axis, so for all values of x. And here it's different it's greater than zero, So you see the graph is always positive, (CMR) so for all values of x.</p> <p>L: So, no solutions am I ok here.</p> <p>Paul: Yes, you can also say here that there no real values of x because there are no x-intercepts. FR CK</p>
201- 210	<p>T: We are now previously greater than zero go and unclick there you might see this one. And click this one ≤ 0. You must not have a situation like this ne (showing two equations showing together) FR</p> <p>L: Yes</p> <p>Paul: Unclick the first one, and you see that we have dotted lines meaning that we don't need -2 and 3 in our solution. CK FA. And your answers will be you can even click less than or equal to and say solve. Is it going to solve? Let me check. Where is my answer now? It is not giving me the answers. It will just show only your graph ne? Right, what I am emphasising is that once you have less than zero, it will shade below the x-axis. (CMR). Some problem, so it is not solving less than or equal to the graph. But still, you can click on your computer it will show you. (Focussing on his system rather than learners)</p>
224- 226	<p>Paul: Now you see that graph is shaded below the x-axis, ne. (CS)</p> <p>L: And the lines are not dotted</p> <p>Paul: Yes the lines are not dotted as we are looking for \leq</p>

5.3 HORIZONTAL ANALYSIS ACROSS LESSONS

In the previous section, the descriptions of the classroom integration of the applets in Cycle 2 revealed how the participant teachers made use of the potential of *GeoGebra*. This section presents a horizontal analysis that interprets the findings from the preceding section. The horizontal analysis of the data, strand-wise, showed that the participants demonstrated traits of core understanding (CU) and evidenced their instructional fluency (IF). This discussion takes the form of presenting instances that best illustrate the pertinent strand of mathematical teaching proficiency. This form of analysis enabled me to gain deeper insights to obtain answers to my research questions.

The CUIF framework in Table 3.3 in Chapter Three (p. 69), serves as a reminder of the observable indicators used to analyse the data.

Paul did not include the ‘nature of roots’ section of the topic in his observed lesson. He arranged that particular lesson on the following day – this was however not recorded. Despite this, he shared his experiences on the lesson on the ‘nature of roots’ during the reflective interview, and I thus included this in the current horizontal analysis.

5.3.1 The unfolding of core understanding (CU)

In this section, I consolidate different pertinent indicators of CU (core understanding) evidenced in this cycle.

5.3.1.1 Emphasis on teaching for core knowledge (CK)

In order to solve the given inequalities, both my participating teachers adopted two conceptual ideas: 1) every inequality comes from an equality (Alsina & Nelsen, 2009); and 2) expressing the conclusion using inequality notations. They both made use of the tools available in *GeoGebra* to explain the solutions to the inequality, using appropriate mathematical terms (lines 194–207 in Table 5.7 on p. 140 and lines 137–140 in Table 5.11 on p. 148). Thus, the intended mathematical knowledge of solving an inequality, which has been referred to as ‘mathematical knowledge for teaching’ by (Ball et al., 2008, p. 394), was evident among the participant teachers.

By treating equations as algebraic functions first, and then providing a graphical representation of these functions using *GeoGebra*, my participant teachers provided learners with a rich and comprehensive learning environment and opportunity. Their instructions illustrate that the lessons did more than teaching mathematical terms and notations. Their interactions with the applets and with the learners developed a classroom environment that supported “teaching for understanding” (Schoenfeld & Kilpatrick, 2008, p. 19).

5.3.1.2 Emphasis on representing multiple ideas (CMR)

Kilpatrick et al. (2001) argue that “mathematical ideas are enhanced through multiple representations” (p. 95). The classical algebra-graph approach to functions is an illustration of multiple representations. Teachers designed *GeoGebra* applets that dynamically linked algebraic expressions and graphical representations (in Figure 5.2 on p. 129 and Figure 5.6 on p. 131). As the learners engaged with the different equations in the worksheet, the teachers encouraged them to connect algebraic equation and graphs, for instance, see Table 5.4 (p. 136).

Schäfer (2016) considers that one of the key elements of proficient teaching is making use of multiple representations that include bringing in causal relationships between mathematical ideas. I noted that both teachers capitalised on the dynamic graphing capabilities of algebraic equations in *GeoGebra*. Specifically, they distinguished between graphical situations of i) equal and unequal roots and ii) real and non-real roots. Using the Applet 2.1, Antony allowed learners to think critically and relate different representations to each other, such as the roots of the equation, the discriminant and the x –intercepts of the parabola, shown in lines 347–353 of Table 5.8 (p. 142) and in lines 88–99 in Table 5.12 (p. 157).

I thus concur with Hennessey et al. (2012) that multiple representations in teaching mathematics not only strengthens learning experiences but also increases retention and application.

5.3.1.3 Emphasis on connecting concepts (CC)

Owing to the participant teachers “... knowing school mathematics in-depth and breadth” (Schoenfeld & Kilpatrick, 2008), they were able to make effective connections to other topics. The teachers structured their lesson instructions and equations so that it would allow them to connect certain concepts such as equivalence of equal roots and the turning point. They were aware that the mathematics they were engaging in would lead learners into certain concepts required for interpreting graphs in Grade 12 calculus.

Furthermore, Antony made an instructional decision to start his lesson with a linear equation in *GeoGebra*. He stated “*I always prefer to move from known to unknown. This would make them to understand easier.*” (line 125, RI2A, my original transcripts). Paul started with a quadratic equation; however, in his reflection he decided “*Before using a quadratic, I will begin with a linear inequality. In our next meeting, we must discuss on an applet linear inequality*” (Table 5.13, lines 35–36, p. 158). One important consideration regarding their interactions was that the teachers considered what their learners were familiar with and how their learners made progress in learning content. This connection concurs with a study by Schoenfeld and Kilpatrick (2008) who record “[t]hat practice provides the teachers with a sense of curricular and mathematical depth, which shapes the ways in which they orient their current instruction” (p. 6).

5.3.1.4 Emphasis on teaching for exploring and discovering concepts (CD)

The role of a teacher is to probe and provoke a learner's creativity (Shulman, 1987). In a technology-based classroom, there is less emphasis on transmitting information, and more on developing learners' conceptual understanding and skills through exploration and engagement.

In Antony's reflection (lines 35–47 in Table 5.12 on p. 157), where he highlighted the excitement of learners when they understood how the x –intercepts of the graph related to the roots of the algebraic equation. It is evident here, that the learners came up with their own ideas when they were provided with an opportunity to engage with the tasks. This implies that the learners constructed their knowledge by actively participating in a classroom situation. Wright (2000) agrees that knowledge is the result of learners' activities rather than of the passive reception of instruction.

With Applet 2.1, Antony encouraged his learners to explore different equations and observe their graphs. He prompted them to make conjectures on the nature of roots and waited until he received the appropriate response from the learners. Here Antony leveraged *GeoGebra* to support his instructions and stimulated the learners' creativity to highlight mathematical principles. I therefore find that technology enables teachers to design tasks that can support learners to explore and make sense of mathematical ideas. In agreement, Laborde (2001) ascertains that it is important for teachers to see the “role of technology as a way to facilitate the making of conjectures rather than a part of solution of tasks” (p. 289).

Paul also motivated his learners to explore different equations in the worksheet and observe their respective graphs Table 5.9 on p. 145, in lines 84–89 show him motivating learners to draw comparisons between the CAS window and the graphics window. Although he prompted the learners to respond, he did not receive an immediate response; he then provided the mathematical concept, in this case that solutions to the equation are the x –intercepts of its graph.

Furthermore, *GeoGebra* is considered as a pedagogical tool, as Khine (2003) observes – that in a constructivist classroom, the role of a teacher is to provide opportunities and facilitate learning.

5.3.1.5 Emphasis on tackling common misconceptions (CM)

Ball et al. (2008) identify “knowledge of content and students” (p. 401), which is “knowledge of common student conceptions and misconceptions about particular mathematical content”. In our GLIP meetings, when reflecting on the lesson implementation, we identified three common

misconceptions related to inequalities. One is the notational error when learners are often confused with the ‘greater than’ or ‘less than’ symbol. The second misconception lies in the conclusion when representing the solution to quadratic inequalities, especially the inequality notation. The third misconception is the procedure of solving inequalities: learners apply the rule of equivalent equations, multiplying an equation by any (positive or negative) number, which leads to incorrect results when applied to inequalities. This third misconception was, however, not addressed by the teachers during the lesson observation, and neither of the applets addressed this misconception.

During Antony’s lesson, he attempted to address the first misconception by making gestures with his fingers, recorded in lines 123–125 in Table 5.5 (p. 138). He addressed the second misconception in the inequality notation of the solution, in lines 154–158 (in Table 5.5), by referring to the algebraic solution and the highlighted part of the graph. Referring to the graph, he also used terms like ‘left of $-\frac{1}{2}$ ’ for the solution $x \leq -\frac{1}{2}$. In the stimulated recall interview, (lines 100–104 in Table 5.12 on p. 157), Antony observed that learners found it difficult to represent the solution to inequalities and he showed his learners how to interpret the solution from the graph. It is evident that Antony identified his learners’ difficulties and discussed them in an effort to make learners aware of them. I concur with Kilpatrick et al. (2001) and Ball et al. (2008) that teachers need to be sensitive to the misconceptions that learners that bring to mathematics. Proficient teaching demands knowledge of the junction between content and the learners.

5.3.1.6 Emphasis on teaching which relates to real-life situations (CR)

This code relates to teaching activities that refer to real-life situations. I found little evidence of teachers making references to day-to-day realities or illustrating the application of inequalities. I will discuss this further when I summarise my findings in my last chapter.

5.3.1.7 Emphasis on switching between abstract and concrete (CS)

Appropriate to the abstract nature of a mathematical idea, each concept has several concrete images (Kilpatrick et al., 2001). In lines 185–192 (of Table 5.11, p. 148), Paul expounded on the differences between ‘less than’ and ‘less than or equal to’ by making visual references to the ‘dotted’ vertical lines on the graph. Antony asked his learners to identify the differences between ‘less than’ and ‘less than or equal to’ and make inferences from the graph, as is evident in lines 179–193 of Table 5.6 (p. 139). Furthermore, Antony used multiple terms for the x –values of the inequality, such as critical values, boundary values and x –intercepts, making references to the

graphs, thus affording his learners a deeper understanding. The flexibility to switch between visual and abstract images could be a significant factor in understanding concepts and useful for solving mathematical problems. This argument is consistent with Sarama and Clements (2009) findings that virtual manipulatives facilitate learners to develop “integrated-concrete knowledge” (p. 146).

5.3.2 The unfolding of instructional fluency (IF)

In this section, I consolidate the different pertinent indicators of IF (instructional fluency) evident in this cycle.

5.3.2.1 Emphasis on acquiring an instructional repertoire (FA)

Kilpatrick et al. (2001) consider instructional routines as one of the basic components in teaching as “teachers who have acquired a repertoire of instructional routines can readily draw upon them as they interact with students in teaching mathematics” (p. 382). Teachers have access to a range of pedagogical strategies in developing conceptual and procedural understanding and know when they are appropriate in different situations. Here I found two different strategies adopted by the teachers:

Paul provided additional equations apart from the worksheet for the learners to practise in *GeoGebra*. He reflected: “*More equations were required [than] in the worksheet*” (Table 5.13, line 31, p. 158). This implies that more equations in the worksheet could reveal the underlying concepts useful for solving problems. This practice is consistent with the observation by Rittle-Johnson and Alibali (1999) that “[p]rocedural knowledge may only lead to greater conceptual knowledge under certain circumstances, such as after extensive experience using the procedure, or when the relationship between the procedure and the underlying concepts is relatively transparent” (p. 177).

By questioning critically, Antony led the learners towards identifying non-real roots of quadratic equations. He guided them in reasoning about the properties of numbers and the nature of roots and revealed the underlying concept by linking with the x – intercepts of the graph. Here, conceptual instruction occurred prior to working with the different equations. Researchers (Carpenter & Lehrer, 1999; Kilpatrick et al., 2001) have shown that instructions that emphasise understanding procedures before using them enhance both conceptual and procedural knowledge

Nevertheless, the overriding goal of these differing strategies was the development of mathematical proficiency.

5.3.2.2 Emphasis on tackling a learner's misconception (FM)

This indicator refers to a learner with serious misconceptions which were not anticipated by the teacher. In my observation, I could not locate any episode where the teachers intervened to address a particular misconception of a learner. However, in Section 5.3.1.5 (p. 152), I have discussed teachers addressing the common misconceptions.

5.3.2.3 Emphasis on teaching basic skills (FL)

This indicator within IF deals with a teacher's knowledge of how to deal with learners who lack critical prerequisite skills in using computers for learning purposes. Recent research (Hennessy et al., 2005; Stols et al., 2015) reports that teachers are confident in using technological tools but not so confident in using them with learners. Nonetheless, my participant teachers gained confidence after their first lesson, and GLIP meetings supported them in building up their confidence.

Antony and Paul were very vigilant with their learners' use of computers. They noticed learners' errors such as typing a capital letter X instead of small letter x ; '6' instead of '^'; '=' instead of '+'; (owing to an inability to use a combination of keys), to name a few. In a GLIP meeting, Antony reflected that "*For a minute I was lost. I could not understand what was happening when a learner claimed that he could not see any graphs, but his equation was correct. Then I saw it was capital letter X.*" It requires expertise to identify such nuances in typed equations. During the lesson, I saw Paul briskly moving around to replace mouse and keyboard, signifying the teacher's role as a 'technical assistant'.

However, when Antony realised that he was losing focus on the 'mathematics' while dealing with his learners' computer illiteracy skills, he no longer provided technical assistance and asked his learners to share with other learners who were doing it correctly.

Antony made a closing remark to the class: "*This is like mathematics as it is mathematics. Now we are doing mathematics using computers. But what is happening is some of us are unable to use computers. So we have to first teach how to use the computers before you go to the actual use of mathematics*" (line 392–395, L2A in Table 5.8, p. 142).

It suggested that using computers in a mathematics classroom itself is learning mathematics. Technical knowhow such as skills in using keyboards for mathematical equations becomes necessary for effective integration of technology.

5.3.2.4 Emphasis on responding to learners' thinking (FR)

Ball et al. (2008) recognise that good teaching entails responding to learners' questions. I put forward two instances in the data that revealed how teachers responded to learners' queries. This resulted in a productive classroom discussion among learners and teachers. The discussion, in lines 203–213 in Table 5.7 (p. 140), shows that Antony responded to a concern from a learner who could not locate the 'separation' in the graph. He manipulated *GeoGebra* to generate different graphs, instantly representing the different situations necessitated by the learner's concern. He provided alternative methods of interpreting the inequality from the graph that involved terms such as ' $-\infty$ ' and ' ∞ ', ' x – intercepts' and 'interval'. Antony made appropriate responses to the learner's questions that reflected the promotion of understanding of mathematical procedures in solving inequalities.

In Paul's lesson, in lines 110–120 of Table 5.10 (p. 146), a learner could not understand the situation where a quadratic equation had only a single solution (remember, the teacher had said to the class that there were two solutions for quadratic equations). The interaction between the teacher and the learner provided evidence that he convincingly guided the learner through using different representations in the applet. He exploited the opportunity to visualise the special mathematical situation and thus enhanced the knowledge of the learner.

Here, teachers acknowledged and appreciated learners' critical thinking. They were responsive to learners' questions and provided alternative methods or explanations, exploiting *GeoGebra* to visualise the mathematical idea. They tackled the situations that arose during the lesson and capitalised on the opportunities that would be beneficial to the learners in solving equations and inequalities. I concur with the observations of Schoenfeld and Kilpatrick (2008) that “[t]he more that teachers are attuned to their students' understanding, the more they can create learning environments that are responsive to them” (p. 15).

5.3.2.5 Emphasis on understanding procedures (FP)

Ball et al. (2008) inform us that “being able to carry out [a] procedure is necessary, but not sufficient, for teaching” (p. 396). Inequalities are often easier to understand if we use graphs to

determine the solution. There was evidence that the teachers focussed on the procedure of interpreting inequalities from the graph generated in the applet. They corresponded between algebraic inequalities and graphical representation, and thus effectively developed visual procedures in solving inequalities.

Interestingly, while explaining ‘greater than’ and ‘less than’, both teachers efficiently switched between the graphs and the algebraic solutions. They developed mathematical ideas and procedures intertwining CU and IF, allowing the learners an opportunity to grasp the underlying concepts. Thus, the analysis suggests that toggling between the graphs allowed learners to understand the underlying concepts and procedures. Teachers successfully capitalised on the interactive potentials of *GeoGebra* to represent the meaning of the procedure “every inequality comes from equality” (Alsina & Nelsen, 2009), discussed in Section 5.3.1.5 on p. 152). These findings on teaching for the understanding of procedures highlight what Kilpatrick et al. (2001) propose, of proficient teachers as “be[ing] able to unpack mathematical content and make visible to students the ideas behind the concepts and procedures” (p. 428).

5.3.2.6 Emphasis on inviting feedback (FF)

Kilpatrick et al. (2001) consider that a proficient teacher sets a routine that encourages learners to observe and talk about the mathematical situation. The routine practice of soliciting feedback from learners helps a teacher to assure that the learners understand or are able to perform a task. Antony asked questions such as “*What is the value of x ?*”, “*Are you still struggling to show that line?*”, thereby monitoring the progress of the class. Similarly, Paul asked questions such as “*Anyone with a question so far?*” “*How many solutions do you notice here?*”, thereby prompting responses from learners.

These sets of routines aided teachers to check on the progress of the learners. It served as an opportunity for the learners to open up and respond or ask questions.

Table 5.12: Transcript (RI2A) – Antony reflecting on the second cycle

37-42	<p><i>R: Let us see this part of video recording. They are speaking in isiXhosa. What is happening? I can see that they are excited.</i></p> <p><i>Antony: As I said earlier, they explore themselves. They are excited because that one saw the x-intercept and is sharing with her friend. Otherwise, what I hear and see on the recording, I see in the background that they are getting something. They are learning (pause) they can pick up something. I can see that [they] are excited yoo this is straight-line see this is x-intercept. I think that one is helping them.</i></p>
88-99	<p><i>R: Ok. What was the best one that worked?</i></p>

	<p><i>Antony: hmm hmm, the algebraic expression and its graph, solution and x-intercepts are that I think it worked. It was supposed to be effective.</i></p> <p><i>R: You mean graph and the solution?</i></p> <p><i>Antony: They use like GeoGebra they can punch in the equations, and you see everything there, like that, is what we mean when we say the point of intersection. This is what is nature of roots when roots are undefined or non-real that what it looks like non-intersection with the x-axis. $b^2 - 4ac < 0$ is non-real now they don't know what are talking about. Now when we show them in computer practically that they can see, "yeah ok ok this why roots are not real because the graph is not intersecting x-axis." So when they change the values of a, b and c they can see how the graph is moving above (or below) the x-axis and that the discriminant becomes negative. In that manner, it is helping them a lot.</i></p>
100-104	<p><i>R: So. How do you think this program GeoGebra helped you in the lesson, like delivering the content? Like preparation and presentation of the lesson?</i></p> <p><i>Antony: learners have difficulties in writing the correct notations when they write the final answer. Now they can see that x is less than 1 mean that graph, is highlighted on the left of x=1. If they read from the graph, they will be to answer the inequality correctly.</i></p>

Table 5.13: Transcript (RI2P) – Paul's reflection on Cycle 2

25-27	<p><i>Paul: Yeah! I think that learner picked it up. Now, they can see what will happen when roots are equal. Even my Grade 12 find it difficult to understand that part. I think I must use them in their lesson. Well, it will be good if these learners remember what they see and learn here next year also.</i></p>
28 - 31	<p><i>R: What did not work during the lesson?</i></p> <p><i>Paul: I spent a lot of time in assisting learners using the greater than less than symbols. So I was trying to explain that they were supposed to do and what not. Otherwise, the lesson went fine. I think challenges that learners face is the use of computers. More equations were required in the worksheet.</i></p>
47-55	<p><i>Paul: I used the applet that we have designed. They were playing around with those a, b and c. What do you call them?</i></p> <p><i>R: You mean sliders</i></p> <p><i>Paul Yeah! Sliders. It was useful for them; they could even see that when $a = -1$ it's a frown and $a = 1$ it's a smile concave up. It saves us a lot of time explaining. They see what we are talking about. It must make them understand those concepts.</i></p> <p><i>R: So, what worked well with that lesson?</i></p> <p><i>Paul: I see that learners were excited as they see the graphs. They can now see what we mean by non-real roots when the discriminant is less than zero.</i></p>

5.4 DISCUSSION ON THE RESEARCH QUESTIONS

In this section, I set out to answer the two research questions. The first one concerns the teaching of mathematics integrating DGS as a visualisation tool.

5.4.1 How can Dynamic Geometry Software such as *GeoGebra* be used as a visualisation tool to teach Grade 11 Mathematics?

Guzmán (2002) considers visualisation to be a powerful teaching and learning tool of mathematics. My analysis of my participants teaching in Cycle 2 undoubtedly confirms Guzmán's view.

The use of multiple representations is a strategy to capitalise on the power of visualisation. The data analysis shows how the mathematical discourse was developing to incorporate multiple representations of equations and inequalities. In DGS, mathematical representations are inherently dynamic, and the participating teachers designed applets that dynamically linked algebraic expressions and graphical representations, as mentioned in the discussion in Section 5.3.1.2 (p. 150). The teachers thus effectively integrated learning tools that allowed learners to visually engage mathematically relevant features of quadratic equations and inequalities. I concur with Drijvers (2012) who recognise the importance of “dynamic phenomena that invite mathematical reasoning, in many cases concerning the relationships between multiple representations of the same mathematical object” (p. 489).

The essence of the teaching that promotes the effective use of visualisation in mathematics classrooms is ‘making connections’ (Presmeg, 2006). The participant teachers triggered (or intended to trigger) previously learned concepts and also connected to other future topics, thus employing visualisation for enhanced teaching and learning (discussion in Section 5.3.1.3 on p. 151). The ease with which the equations could simply be typed (or by dragging the sliders in Applet 2.1) to generate graphs in *GeoGebra*, contributed to this connection-making. The more equations and graphs that were used, the richer the connections and more meaningful opportunities the teacher provided for learners to make sense of mathematical concepts.

Guzmán (2002) underscores that underlying mathematical ideas are born from concrete and visualisable situations. Visualisation is a critical aspect in mathematical activity, through which one can explore different structures of concrete reality. My analysis implies that the visualisation of algebraic equations and inequalities affords teachers to make abstract concepts visual (see Section 5.3.1.7, p. 153). In accord, Presmeg (1986a) claims that the embodiment of abstract ideas in a concrete image can facilitate teaching and learning mathematical content with understanding.

Rivera (2011) characterised the graphical solution process as a synthesis of symbolic and visual approaches in solving inequalities. The goal of instruction in solving inequalities showed that the teachers focussed on the procedure of interpreting inequalities from the graph generated in the applet. Thus, the essence of teaching visual thinking is evident in these observed lessons using *GeoGebra* applets. The facilitation of making these kinds of visual connections in the teaching and

learning of mathematics, thus made “optimal use of the strengths of visual processing” (Presmeg, 1986b, p. 46), by using *GeoGebra*.

5.4.2 What enabling and constraining factors do Grade 11 teachers encounter when using *GeoGebra* as a visualisation tool to teach Grade 11 Mathematics?

5.4.2.1 Constraining factors

The graphs in Applet 2.1 provided a clear understanding of real and non-real roots as intersections and non-intersections of the graphs with the x –axis. Likewise, for equal roots, the turning point of the parabola was on the x –axis, and for unequal roots, the turning point was elsewhere in the plane. These values were visually discernible in the applet on the cartesian plane. The classification of roots of a quadratic equation however includes rational and irrational values but it was not possible to discriminate irrational numbers from decimal numbers on the x –axis. *GeoGebra* rounds off any number to the specified degree of precision. The point, for instance ‘ $1+\sqrt{2}, 1 - \sqrt{2}$ ’ is displayed as ‘2.41, 0.41’ rounded off to two decimal places. Thus, visual evidence of the x –intercept of the graph does not show as a rational or irrational number. It was not possible to make out an irrational number on the x -axis with Applet 2.1. Antony, however, overcame this situation. He guided learners to make use of the worksheet and link the value of the discriminant with the roots of the equation. Thereafter, he prompted learners through probing questions to link the value of the discriminant with the square numbers.

In solving inequalities as in Figure 5.10, by default *GeoGebra* highlights the region on the cartesian plane where the inequality holds true. The participant teachers used the same procedure in their classrooms. While we were reflecting after the lessons (an essential step of GLIP cycles), we realised that the software was not highlighting the solutions on the x –axis. We discovered later, thanks to the *GeoGebra* user community, the use of ‘object properties’ in *GeoGebra*. With this realisation, we were able to highlight the solution of inequalities on the x –axis as equivalent to the visual representation of inequalities on a number line (Figure 5.11). Researchers such as Berger (2011) and Bu et al. (2012), have noted that *GeoGebra* is a user-friendly software package. However, there are subtle technicalities that could pose challenges to the teachers.

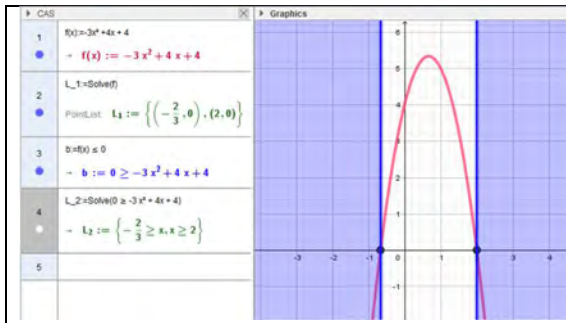


Figure 5.10: Graph of an inequality highlighted vertically

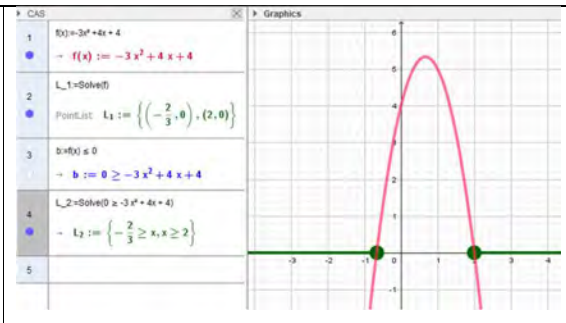


Figure 5.11: Graph of an inequality highlighting the solution on x -axis

Interestingly, both teachers exploited the default inequality representation in the graph. Referring to the graph, they used the terms ‘broken lines’ or ‘dotted lines’ to reinforce the inclusion or exclusion of critical values of the inequality.

As in other cycles, the lack of good computer literacy constrained the flow of the lessons substantially. This is an important factor to consider when introducing new technologies into one’s teaching repertoire.

5.4.2.2 Enabling factors

By using tools in *GeoGebra* such as ‘Graphics’ and ‘Solve’, enabled both Antony and Paul were enabled to visually demonstrate the key concepts and procedures in solving quadratic inequality.

GeoGebra allowed a meaningful environment to obtain the graph of any equation and its solution. It is tedious to generate multiple graphs and solve the equations by hand on the chalk board or on paper. Thus, this elimination of the cognitive workload allowed an opportunity for the teachers to engage immediately with their learners in developing mathematical concepts. In Applet 2.1, by moving the slider of a parameter (a , b and c , the coefficients of a quadratic equation), the function, the graph, the solutions, the intercepts and the discriminant are all updated dynamically. Thus, *GeoGebra* potentially allows for developing insights into variant and invariant properties. This is not the case with the images in mathematics textbooks – they are static and by their very nature do not support the dynamic nature of visualisation (Guzmán, 2002), enacted by the teachers using the Applet 2.1.

Researchers such as Drijvers (2012) and Jones et al. (2009) argue for the potential of dynamic geometry software to move between multiple representations. With easy-to-use tools, *GeoGebra*

allows the user to see the algebraic solution of an equation or an inequality and its graph representations on a screen simultaneously, thus allowing the teachers to switch flexibly between algebraic notations and graphs. In Applet 2.1, various representations were automatically linked, which were dynamically manipulated with the help of sliders, to inform and scaffold learners to build a coherent network of mathematical connections. Thus, *GeoGebra* serves to illuminate mathematical ideas. In alignment, Bu et al. (2012) recognise that “GeoGebra enhances the learning environment with its multiple representations and computational abilities” (p. 98). A teacher may find it difficult to demonstrate such multiple, dynamic visual representations using static diagrams.

A further enabling feature of *GeoGebra* that was evident in this study was how it facilitated the construction of knowledge through active engagement. Learners had to take responsibility for their own learning by engaging with the software as instructed.

Further, the use of *GeoGebra* in this study encouraged learners to interact with each other. This was evident when small groups interacted with each other and when the more computer literate learners assisted those who were struggling with their keyboards.

5.5 CONCLUSION

With the help of *GeoGebra* and the co-developed worksheet, participant teachers engaged learners with different quadratic equations and inequalities, supporting and guiding them to understand their properties. The analysis of the lessons showed evidence of how the teachers used the software to make visible various mathematical properties associated with quadratic equations and inequalities. Using *GeoGebra*, teachers facilitated visual connections in the teaching and learning of mathematics. They each considered the pedagogical implications and learning opportunities available while engaging learners using *GeoGebra* and implemented these in their own unique ways. Showing the solutions of inequalities, *GeoGebra* by default highlighted the region on the cartesian plane where the inequality held true. It would have been better however, if it had highlighted the solution on the x – axis. Nevertheless, the benefit of multiple dynamic representations afforded in *GeoGebra*, enhanced teaching and learning experiences in algebraic expressions and equations.

In the following chapter, I present an in-depth narrative analysis of the next topic – equations of parallel and perpendicular lines.

CHAPTER SIX

TEACHING MATHEMATICS USING DGS – DATA PRESENTATION AND ANALYSIS – CYCLE 3

6.1 INTRODUCTION

In the previous chapter, I narrated and analysed the participant teachers' implementation of applets on quadratic inequalities and the nature of roots in their classrooms. This chapter aims to analyse and answer the research questions on the topic equations of parallel and perpendicular lines.

The recap methodology (Section 4.1.1 on p. 78) of Chapter Four. This chapter follows a similar structure to Chapter 4, as is described on p. 77.

6.2 CYCLE 3 – EQUATIONS OF PARALLEL AND PERPENDICULAR LINES

The next topic that the participating teachers selected was analytical geometry. It included how to determine the equations of parallel and perpendicular lines and the angle of inclination of a straight line. This chapter covers the last cycle that I analysed for this research; however, the GLIP teachers continued to meet after this cycle, improving the existing applets and designing new ones.

6.2.1 GLIP Discussion

Again, as in the previous discussion, the teachers strongly recommended training for the learners, not necessarily only on *GeoGebra* but on general aspects of computer usage and literacy. We thus designed an hour-long training programme on using coordinates and straight lines in *GeoGebra*. The handout for training the learners can be accessed in the Annexure V under the heading 'Training Interlude – Coordinates and Straight Lines'. Prior to the observed lessons, the participating teachers, my and fellow researcher and I engaged learners with *GeoGebra* for an hour, using the abovementioned handout.

The teachers also felt they needed a whiteboard to write on during the lessons. Paul usually uses a chalkboard in his lesson, but he claimed that his hands would become dusty and hence not appropriate for using a laptop or projector. In Antony's school computer laboratory, there were no writing boards (chalkboard or whiteboard), but only one smartboard which required a smart pen

to write on. Unfortunately, the smart pen was faulty, and Antony had to recalibrate it now and then. He was regrettably unsuccessful at calibrating the smart pen and the board and gave up using the smartboard. Abraham, though not a research participant, also raised a concern that his learners complained that they could not see what was projected on the screen clearly due to the bright reflection. He arranged to put dark curtains up to cover the windows and changed the lighting.

The GLIP discussion on this topic and design of applets was spread over two days, and there was subsequent communication through emails among the participants.

6.2.1.1 Identification of concepts

During their earlier phases of schooling, learners had learned how to sketch straight lines of the form $y = mx + c$, the gradient-intercept form. The teachers acknowledged that their learners preferred to use the gradient formula $m = \frac{y_2 - y_1}{x_2 - x_1}$, to determine the gradient between two points (x_1, y_1) and (x_2, y_2) . Nonetheless, they unanimously raised the concern that most of their learners found it challenging to understand the concept of slope of a line as a ratio of vertical change to horizontal change. Again, all of us agreed that for effective instruction to happen, it was better to use pre-designed applets. We decided on the following topics to be incorporated in the applets: i) concept of a gradient of a line; ii) interpretation of gradient from a given line; iii) relation of gradients of parallel lines; iv) relation of gradients of perpendicular lines; v) equations of parallel and perpendicular lines; and vi) relation between gradient and angle of inclination. The angle of inclination is defined as the angle that a line makes with the positive axis, where angle is measured in an anti-clockwise direction.

6.2.1.2 Designing and Planning

Abraham put forward the suggestion of reinforcing the concept of gradient as a ‘rise over run’ (a colloquial terminology for the ratio of vertical change to horizontal change). Abraham, meanwhile, searched on the internet and found an applet by Tim Brsenenki (<https://www.geogebra.org/m/gV9GRCRN>) which we all accepted for implementation in our classrooms. Shown in Figure 6.1 and Figure 6.2, the Applet 3.1 is a cartesian grid devoid of axes, where a gradient or slope can be randomly generated. A slope and a gradient in this thesis are synonymous and are used interchangeably. On successfully dragging the line using the points, to the given gradient, the applet indicates the slope accordingly. The teachers believed that this applet would thus further engage the learner with the concept of ‘rise over run’.

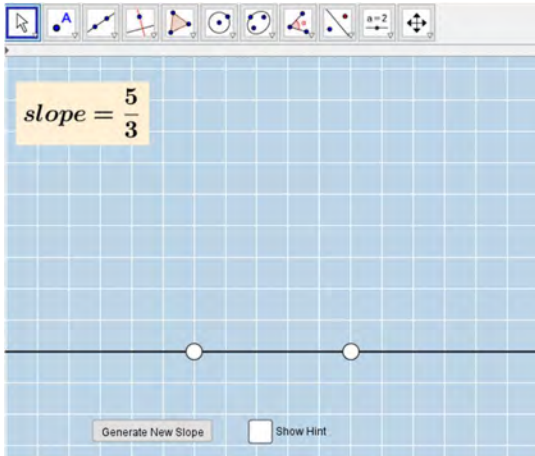


Figure 6.1: Applet 3.1

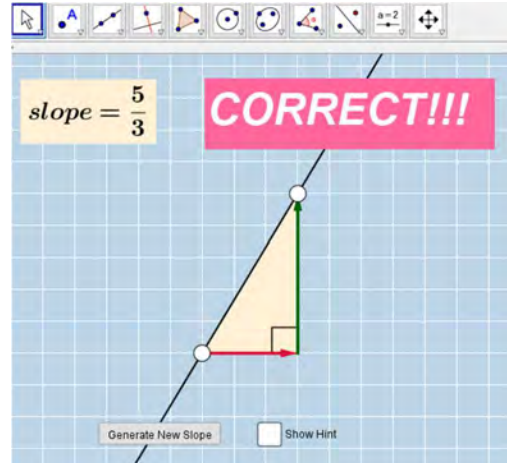


Figure 6.2: Dragging the line for a given gradient in Applet 3.1

Paul put forward the idea of replicating the slider model (as in Applet 2.1) for straight lines. He argued that using *GeoGebra* would be enlightening to learners to show how the change in parameters affected a straight line. The movement of straight line graphs would help the learners to discover the relationship between the parameters of the equation and graphs. However, George also believed that learners in Grade 11 required more engagement where they played around with gradients and intercepts. Paul thus suggested another applet where the equation of a straight line was provided. The learners would then identify the gradient from this given equation and then sketch the line accordingly. Paul and Antony worked together in generating the applet with input from the researchers as some basic programming skills in using logical operators were required. Thus, the second applet, Applet 3.2, in Figure 6.3 and Figure 6.4, was generated. Antony quipped that this applet would help the learners to get a better grasp of parameters in the straight lines.

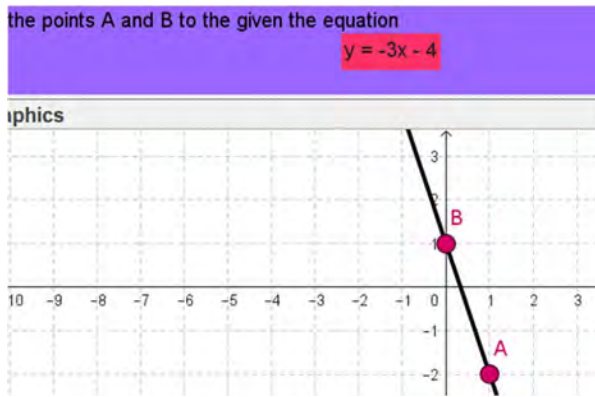


Figure 6.3: Applet 3.2

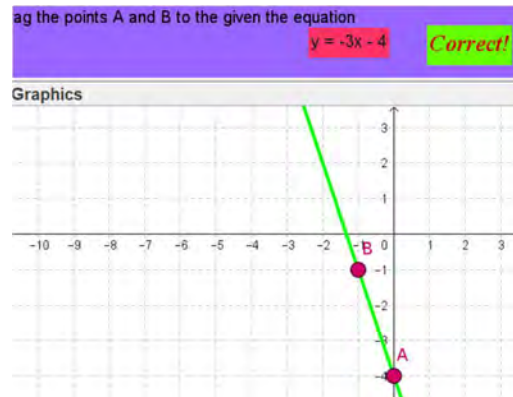


Figure 6.4: Sketching the line for a given gradient in Applet 3.2

The third applet, Applet 3.3 was designed to identify the gradient from a straight line (Figure 6.5 and Figure 6.6). Teachers held the view that this reverse process was necessary in order to comprehend the concepts involved in straight lines. The equation was not provided, nor any points on the line which was drawn on a cartesian grid. In the first version of the applet, axes were un-graduated (numbers on the axes were hidden). Later, however, Paul felt that the learners might struggle to identify the gradient without any points; hence, in the final output, we graduated axes. The learners were expected to identify the gradient of randomly generated lines and put the ratio into the input box provided. Abraham felt that a learner must be able to recognise the slope as a positive or negative without using the formula. He explained (as if in a classroom) that a positive slope was looking up (mimicking as he lifted his head) and a negative slope as looking down (again mimicking as he put his head down).



Figure 6.5: Applet 3.3



Figure 6.6: Identifying the gradient in Applet 3.3

Antony believed that only three structured applets should be used in a lesson. Paul suggested that as he may not get the opportunity to use computer the lab on consecutive days, he wanted to cover parallel and perpendicular lines and the angle of inclination on the same day. Abraham suggested using a worksheet model in investigating parallel and perpendicular lines, so teachers quickly developed the worksheet WS-2, which can be accessed in Annexure VI. The Applet 3.4 was also designed, incorporating both parallel and perpendicular lines on the same applet. Applet 3.4 (Figure 6.7 and Figure 6.8) was meant for investigating the relationships between the ‘gradient and intercept’ of the given line and with that of parallel and perpendicular lines. The learners would work in conjunction with the paper-based worksheet WS-2, in which they were expected to make conclusions based on their observations in the applet.

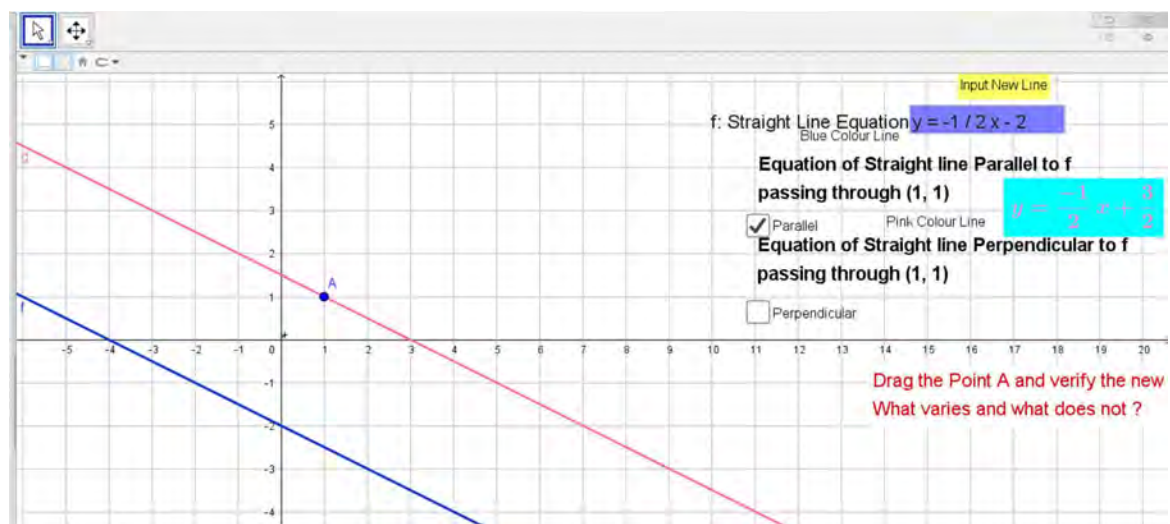


Figure 6.7: Investigating gradients of parallel lines in Applet 3.4

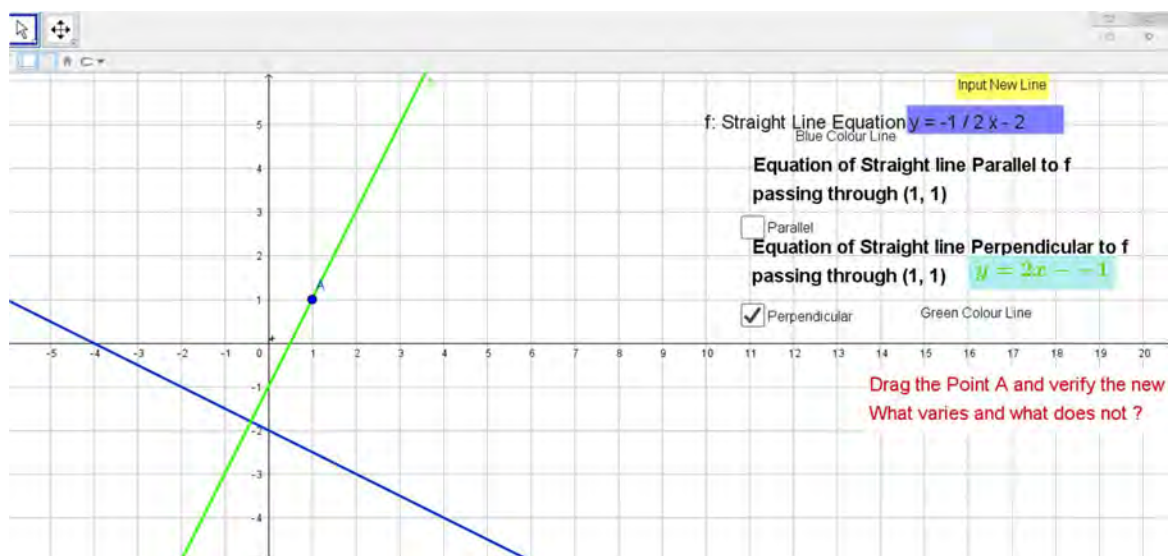


Figure 6.8: Investigating gradients of perpendicular lines in Applet 3.4

Paul also believed that concepts built through the applets might not be enough to develop the procedures for determining the equations of parallel and perpendicular lines. The teachers agreed on an applet that would aid learners with step-by-step procedures in finding the equation for straight lines. It was entrusted upon the researchers to design the applet as it required some advanced programming skills. Thus, the Applet 3.5, as shown in Figure 6.9, provided procedures for finding the equations for parallel or perpendicular lines and then displaying the required parallel or perpendicular lines. Paul joined us in designing the applet as he was keen on learning the advanced scripting used in *GeoGebra*. Later on, we also developed another applet, Applet 3.6 (Figure 6.10), that allowed learners to sketch parallel and perpendicular lines. The learners were expected to determine the slope of the given line and then sketch parallel or perpendicular lines accordingly.

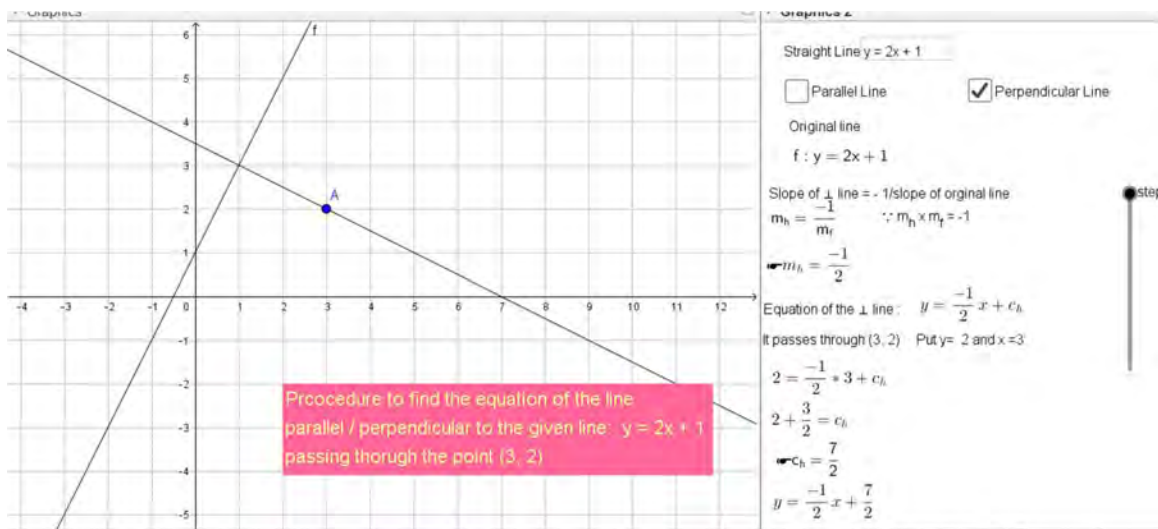


Figure 6.9: Step by step procedure in finding the equations of parallel and perpendicular lines and the required line sketched in Applet 3.5

However, the Applet 3.6, intended for learners to put into practice their understanding of slopes and equations for parallel and perpendicular lines, was prone to errors, and it did not function as intended on the desktops of both the schools. This was first identified in Paul's lesson, but we could only rectify the bug after both lessons were recorded. The issue was compatibility of the teachers' version of *GeoGebra* with that of the computers at the schools. Apparently, Paul's computer had been updated to a newer version of *GeoGebra* while the internet at school worked with an older version.

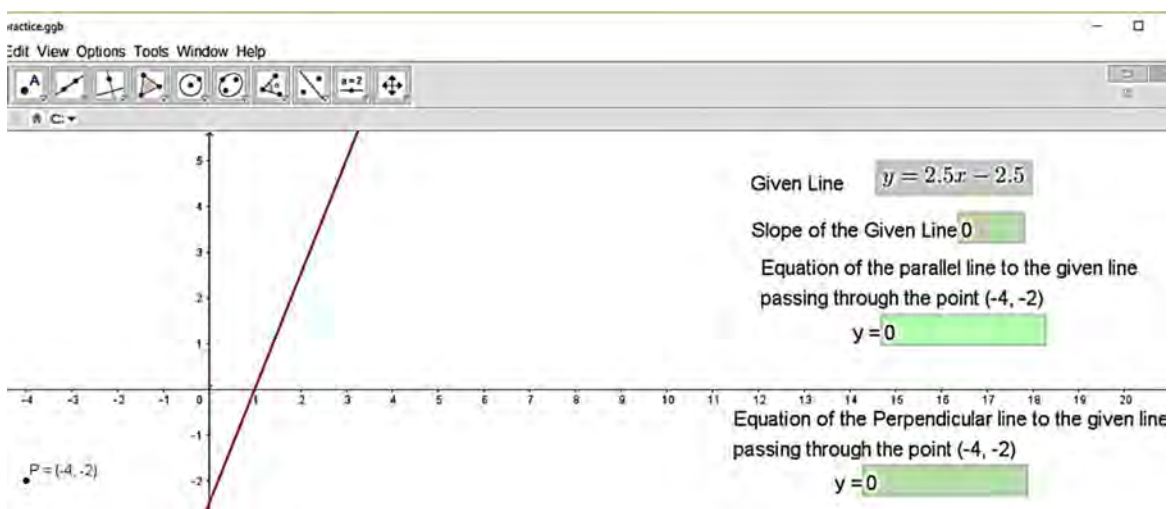


Figure 6.10: Sketching the parallel and perpendicular line through a given point in Applet 3.6

The last concept in this topic was the angle of inclination of a straight line. The inclination of a line is the anti-clockwise angle which it makes with the positive direction of the x –axis. Abraham proposed to design an applet that would establish connections with trigonometric ratios, thus, Applet, 3.7 was designed to highlight the relationship between the angle of a straight line and its gradient. The applet would allow a learner to explore and visualise that both *tangent* and *gradient* are the same ratio, as displayed in Figure 6.11 and Figure 6.12. As the ensuing discussion on using the applet in the classroom, many interesting potential pedagogical practices and opportunities became evident. Abraham maintained that this applet might also emphasise that an acute angle would have a positive gradient while an obtuse angle would have a negative gradient. George held that it could also explain gradients for horizontal and vertical straight lines. Antony suggested that we could also prove that angles of inclination of parallel lines are equal (i.e. corresponding angles equal) and thus their gradients would be equal too.

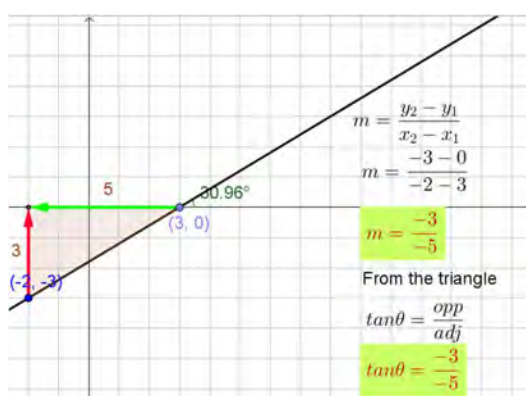


Figure 6.11: Applet 3.7

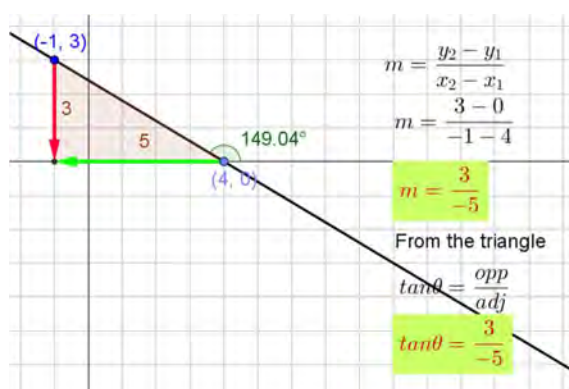


Figure 6.12: Exploring the relationship between slope and tangent of the angle of inclination in Applet 3.7

In this following section, I analyse the lessons of two teachers, Paul and George. Antony could not attend school for two weeks during the scheduled lessons due to personal reasons, and hence I could not record and analyse his classes. Prior to the observed lessons, both the teachers conducted training for the learners for an hour on coordinates of the graph and straight lines using *GeoGebra*. The training resource can be accessed in Appendix V.

6.2.2 Analysis of Paul's lesson

Paul arranged two lessons using these applets. Applets 3.1 to 3.3 were used during the first lesson, and applets 3.4 to 3.7 were used in the second lesson; the second lesson being held three days after the first lesson. On both days, 46 learners attended the lessons (two learners were absent). There were no delays, and once the learners found their seats and desktops, the lesson started promptly. Most of the computer systems were up and running prior to the learners entering the lab. They simply needed to log in to the computers and access the required files. By then, the learners knew which computers were working, but some of them still had to share desktops. On the first day, Paul was obligated to attend a meeting, so the lesson only lasted for 30 minutes instead of the scheduled 40 minutes. Paul moved around the computer lab from learner to learner to clarify any questions they had. Figure 6.13 is a snapshot of Paul's classroom, showing the projector and a chalkboard adjacent to each other.

At the beginning of the second lesson, Paul handed out the worksheet WS-2, for investigating parallel and perpendicular lines. Unfortunately, Paul could not use Applet 3.6 as the desktops were not compatible with the new version of *GeoGebra* in which this applet was designed. Learners were more inclined to play around with the points on the applet rather than writing down the observations.



Figure 6.13: Paul's classroom

6.2.2.1 Act 3 Scene 1 ~ Applet 3.1

Paul began the lesson by mentioning the objective of the activity for this particular lesson. He wanted to foster a precise and richer understanding of gradients and equations of lines. From Lines 8–12 in Table 6.1, we see that he began with Applet 3.1 and reminded his learners about the definition of a gradient. I classified this as **CK**. As he encouraged his learners to recollect and connect to prior knowledge (**CC**) on properties of rational numbers, he also demonstrated how to do the activity in the applet (**FP**). For the next few minutes, he paused and watched how learners they were working on the applet. He helped those learners who could not sketch the line.

He demonstrated only one strategy in Applet 3.1 – of dragging the horizontal movement towards the right and the vertical movement up for positive gradient, and down for the negative gradient. I classified this as **FP**. Paul illustrated the concept of gradient using multiple representations and terminologies (**CMR**) such as ‘vertical change’ and ‘horizontal change’. Thus, the sequence of interactions suggested that Paul employed a range of instructions, (**FA**), to address the mathematical proficiency of learners. Furthermore, he showed flexibility in his interactions with his learners (**FL**), who lacked the basic skills of moving the grid, as shown in lines 35–42 of Table 6.1.

As the learners progressed in successfully sketching lines in the applet, he directed their attention to the sign of the gradient and its slope. In lines 57–58 (in Table 6.1), he encouraged the learners to explore and visualise the lines of negative and positive gradients. He encouraged learners to explore (**CD**) and switch between concrete lines and abstract ideas (**CS**). Once the learners were familiar with the sketched lines, he explained how the vertical change affected the sign of the gradient (lines 61–62 in Table 6.1). He thus led the learners to estimate the gradient for a given line. Lines 61–62 also illustrate that Paul offered learners an opportunity to make sense of mathematical ideas and develop a richer understanding of the concept of a gradient of a line, that would be useful to them in solving future problems.

Table 6.1: Transcript (CL3P1) – Interaction with Applet 3.1

1-7	<p><i>Paul: Good morning, I was delayed at the meeting. Let us settle down quickly. (Learners are moving around). T: Ok, what is wrong with you? Are you scared of computers ne? L: Ahha Paul: Basically, what we to do today, boys and girls, is to summarise that we were doing in class. The exercise that is designed for you. Right so once (pause) once have you opened that? Once you do that, start with the exercise that is there. We will relearn gradients and equations of straight lines so that we have precise knowledge. Right!</i></p>
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8-21	<p>Paul: Right open the first one, you see 3-slope-run-rise. Right with this one now it is giving us slope. Remember slope is your gradient ne? I will come (a learner asking for help)</p> <p>L: Yes</p> <p>Paul: Right. What we are going to do here you see the gradient that you are given. The gradient what you are given is, say -6. Remember when you say -6 our gradient comes from. CK So if you got a whole number like -6 it means that your change in y is -6 and change in x is 1. Are we together? It means that your change in x is what?</p> <p>L: 1</p> <p>Paul: That answer that we can use the one on the board is CC (he writes - n the chalkboard but refrains further and walk back to projector and then to his laptop). So, what you can do you see these two points here this one and this one</p> <p>L: Yes</p> <p>Paul: this one moves along the x-axis I said along the x-axis is 1. Then because it is negative, I will go down. And from there I have to move six units. FP</p>
35-42	<p>Paul: yes your numerator is your change in y FL</p> <p>L: Oh oh</p> <p>Paul: change in y means you either go up or down. So if it is negative, give it to the numerator FP. Change in x means you move right.</p> <p>L: When I called you, you did not respond</p> <p>Paul: Ok, baba. Your gradient is 7/6 move six right say 1 2 3 4 5 6 . then up 1 2 3 4 5 6 7 see FL</p> <p>L: Yeah ! Wow.</p>
43-45	<p>Paul: The numerator is change in y and denominator change in x. So guys look here at the screen once again. Here the slope is -5/3 ,ne? So I move three right make it straight line horizontal ne CMR. Negative -5 ne I move 5 down vertical CMR ne. Look its correct. Right. Are we together?</p>
54-62	<p>Paul: So please the denominator is your change in x ne. So you count. Here, for example, you count five units 1 2 3 4 5 then you go down six units is that ok. Please ensure that you sign the register. (Pause)</p> <p>L: I did not get it .</p> <p>Paul: Please also check the slope if your gradient is negative. That will tell you how to move your points. Is that ok? You check ne; you know slope if it is negative you must know how it is slanted CMR FM. Once again, if it was positive move right then up. Remember the denominator move right then you see the way the lines are sloping with your gradient.?</p> <p>L: Yes.</p> <p>Paul: Ok. Once you see that your gradient is negative, you should have a picture of how your line should be slanting FM. As x moves to the right, if y moves down, then gradient is negative. CMR CK Are we together?</p>

6.2.2.2 Act 3 Scene 1 ~ Applet 3.2

Paul scrupulously sequenced the applets to support the learners' construction of mathematical ideas around the equation of straight lines. Next in the sequence he used Applet 3.2 to give the learners an opportunity to put into practise the concept of straight lines. Lines 65–74 in Table 6.2 exemplify the evidence of CU. Paul solicited previously learned concepts of straight lines, (**CD** and **CC**), and thus made links with learners' prior knowledge. He used mathematically appropriate language when he discussed horizontal and vertical lines. Furthermore, Paul made use of his experience in dealing with these situations, which is classified as **FA**. Applet 3.2 randomly generated lines and the first equation appeared on the teacher's screen. Paul made a general statement of the mathematical fact that lines of the type where is $y = k$ is a horizontal line, and

lines of the type $x = k$ is a vertical line, where k is a constant. I argue that these kinds of abstract statements while referring to the lines on the computer screen helps learners to enhance their understanding of these types of lines.

It was apparent that not all learners were able to sketch horizontal and vertical lines successfully. Lines 104–108 in Table 6.2 show where Paul had to repeat his explanations on horizontal and vertical lines, and this time, he provided related concepts in detail: “*all y-values on the line will be -3*” (CMR) as a reason for having horizontal lines.. Besides this, Paul drew fluently on his range of instructions to unpack the ideas that made the basic underlying concepts visible, which is classified as FA. It is interesting to note that a learner came up with a valid explanation for the line to be vertical, as evident in lines 124–126 of Table 6.2. This aligns well with Kilpatrick et al.’s (2001) assertion that all learners could comprehend mathematical concepts if instructions are presented with the relevant underlying principles.

His interaction with the learners revealed that Paul emphasised the intercept-point method of sketching a line of the form in Applet 3.2. He provided accurate explanations such as ‘intercept’ and a ‘point on the line’, which was useful for learners when they engaged with the applet. In lines 97–99 (Table 6.2), Paul made connections between expressions and their graphical representation, which is classified as CC. In lines 92–94 (Table 6.2), Paul demonstrated knowledge of mathematical procedures (FP), as he explained the sketching of the line using the substitution method in determining a point on the line.

Paul interacted with the learners as they engaged with the applets. I could see in the video that he was moving around and clarifying their questions. His voice was inaudible, but I could hear words such as ‘vertical’, ‘horizontal’, ‘intercept’, ‘substitution’, etcetera. As he was explaining to a learner on using the substitution method to find another point on the line, a learner quipped, “*how can be $x = 1$?*” (lines 122–124 in Table 6.2). I saw elements of CU and IF at play in this instance. Paul made use of the situation and displayed his range of instructions (FA), as he addressed the whole class to emphasise a mathematical concept, which is classified as CK. He unpacked a fundamental idea on the classification of variables as independent and dependent variables. This classification of variables could be very crucial in understanding ‘functions’, another topic in mathematics, hence also coded as CC. The extract below clearly shows how Paul addressed the

development of mathematical proficiency among learners, which is at the core of teaching proficiency.

Table 6.2: Transcript (CL3P1) – Interactions with learners and Applet 3.2

65-74	<p>Paul: You go to the next one which 4-line-sketch. If you open what is it saying? Drag point A and B to the given equation. For example, you are given $y=1$. Remember from what you did $y=1$ is a what? CC Is it a vertical line or a horizontal line? CD It must cut at $y=$ if it is given as $y=1$ for example. The one on the screen is written $y=1$ he?</p> <p>L: Yes sir</p> <p>Paul: so, you drag you take your A you also take your B is that ok? Once you have moved there, it should say correct. Please just use the two points that you are given. Please remember $x=$ a constant is a vertical line; FA $y=$ a constant is a what? A horizontal line. CK</p>
76-82	<p>Paul: yes, a horizontal line. Drag a point A to $y=1$ see it should be on the y-axis. And then B drag it. But why it is not correct on the screen.</p> <p>L: he points on $y=0$.</p> <p>Paul: is it? That means I must see a doctor.</p> <p>L: Yes, you are getting old.</p> <p>Paul: Alright, so you see ne it a horizontal line. CS Please follow the equation. (Helping learners). Let me explain once again. You will be moving the points ne. (Background murmuring inaudibly.)</p> <p>Paul: What I was explaining ne. $x=$ constant is a vertical line and $y=$ constant a horizontal line CK</p> <p>CM (Pause helping learners inaudible).</p>
90-94	<p>Please follow the equation. The easiest way to do that once you are given with an equation that says, for example, $y=4x-1$ take your A or B to -1 because it will pass there ne -1 on the y-axis. The y-intercept there CK. Then try to substitute if you put one there (for x) you will get 3. Take B to 3 sorry (1;3). Go to (1,3) and at $y=3$. Move again, move again to point (1,3). My B is at -1 and then A I am moving to (1,3). FP (Pause). See the line is correct ne.</p>
97-100	<p>Paul: Why can't you drag. Control your mouse. (Pause). So I am saying if you are given an equation $y=x+3$, for example, you start by taking your A to $y=3$. Remember, your constant is your y-intercept. CC Is that clear? Then you take the other point you check now by substituting. Is that ok? Are we together? And then click the new line.</p>
104-108	<p>L: No, no. How you do this?</p> <p>Paul: your line $y=-3$. Drag the point to -3. And also drag B to anywhere -3. No, like that. Listen here, Listen. When the equation is $y=-3$, remember it's a horizontal line. All the values on the lines will have y-values equal to 3 CMR. Therefore, you first drag to y-intercept at $y=-3$ and then drag the other point where $y=-3$, and it becomes a horizontal line. FA</p>
114-124	<p>Paul: ...the best way is if you are given $y=-x-2$ ne FL take your A to -2 on the y-axis, -2 on the y-axis the intercept. Take another point one value which is not 0 for x. for example, -1 if you put $x=-1$ there once you move there you will get your $y=-1$, FP. This one I am sure people are getting some problems he? I see some people must familiarise with this concept.</p> <p>L: Can I put $x=1$?</p> <p>Paul: yes any value other $x=0$</p> <p>L: How can be $x=1$?</p> <p>Paul: x can any value? CK for $x=1$ $y=-3$ FP. (speaking loudly) Listen carefully, x can take any value ne. Remember $y=mx+c$, y is dependent on the value of x. and x is called an independent and y is dependent. Are we together? Right. CC (lowering his voice) Now here So drag your B to (1,-3), and A is already at $y=-2$. FR</p>
125-127	<p>L: [name] do all the values on the line $x=-1$ have x-values equal to -1.</p> <p>Paul: Yes. Yes. You already have A at and now drag B anywhere here to become a vertical line. Yeah, you see. There you are? Continue to the next line.</p>

6.2.2.3 Act 3 Scene 1 ~ Applet 3.3

Paul proceeded to the final applet of the day – Applet 3.3. He reiterated the definition of a gradient of a straight line (line 142 in Table 6.3). However, he adopted a procedural approach to encourage the learners to use accurate techniques in finding the gradient, as evident in lines 142–143 in Table 6.3. I coded the lines as FP. Paul did not provide a mathematical explanation for the signs of gradients. Instead, he offered a visual procedure in identifying the sign of a gradient of a line – looking at the direction of the line. He repeated the visual ideas that he had discussed regarding the gradient of lines, while interacting with Applet 3.3. In lines 145–146 in Table 6.3, Paul encouraged the learners to see the right-angled triangle (CS) made by the line when it intersected the axes.

Throughout the lesson, I found that learners did not struggle with the computers, except for a few instances. The notable instance was when the learners could not input the fraction symbol ‘forward slash’, and the teacher had to intervene and help them, as in lines 147–149 in Table 6.3.

Table 6.3: Transcript (CL3P1) – Interactions with learners and Applet 3.3

138-144	Paul: Close that one and take the next one from here. Yeah, that one 5-slope-linear-equation. Right. L: (inaudible). Paul: This one will give you a line. You find the gradient. You type and press enter. Once it is correct, it will give the correct message. Then click Next for the new line. Please sign the register ne. (Pause) Please you punch in the gradient that is shown and press enter. Remember, the gradient is. CK Please the best way is just to count. From the origin, you count along the x-axis, and you count along the y-axis. FP (Moving around helping learners).
145-146	Paul: Make a right-angle triangle every time. FA You check where it is cutting the y-axis and then check where it is cutting the x-axis. CS
147-149	L: How do I input the fraction? Paul: you see the question mark symbol. You press that one. No. don't use the shift key. Simply press that one. Listen ne, all of you. For a fraction two over three, press two then slash then three ne. FL
151-153	Paul: Please remember the slope of the line will give you the sign of the gradient. Always treat the x-axis as your positive value. Is that ok? It is y that change from -ve to +ve. Is that ok? FP Your x always move to the right. Make sure that you see where it is cutting ne?
156-159	Paul: Please, today we are ending here. You check on that question on the slope and make sure that you master that. We will find another lesson so that we proceed. There are other things that we need to do there. Parallel lines, perpendicular lines as well as quadrilaterals. So I will tell you when. We are going to give you an investigation of parallel lines and perpendicular lines ne? I am just going to print it and give you now But for now we are done.

6.2.2.4 Act 3 Scene 2 ~ Applet 3.4

At the opening of this lesson, Paul reminded the learners about their previous lesson about the slope of a line using *GeoGebra*. He engaged the learners with Applet 3.4 that encouraged learners

to explore and discover concepts related to ‘gradients and intercepts’ of parallel and perpendicular lines. He suggested that learners drag point *A* (in Applet 3.4) at different points and then understand the variant and invariant properties of parallel and perpendicular lines. Lines 21–22 and 35–37 in Table 6.4 suggest that he specifically directed the attention of the learners to the gradient properties of parallel and perpendicular lines, which is coded **CD**, thereby promoting the development of conceptual understanding among the learners. Paul encouraged the learners to explore (lines 17–19 in Table 6.4), prompting them to make connections between the applets and the intended mathematical concepts.

The worksheet was designed for the learners to write down their observations and then draw appropriate mathematical conclusions. Paul reminded them to write the equations of parallel and perpendicular lines in the worksheet (lines 35–37). Interestingly, Paul provided an idea on how the worksheet was going to help them in making conclusions by saying “*you can see the whole picture*”.

He did not wait or ask for the learners to come up with their observations. Instead, he made mathematically appropriate conclusions using apt terminology, such as ‘the product of gradients is -1’ in lines 51–52 in Table 6.4 . Initially, I coded this statement as **CK**, which is a demonstration of a mathematical concept. After discussions with my fellow researcher and the GLIP teachers, however, I re-coded as a combination of **CK** and **FP**, where **FP** is the selection of an appropriate formula. Later, in lines 69–71, he repeated his conclusions. In the second instance of making mathematically appropriate statements, Paul explained to learners that the gradient of a perpendicular line would be the negative reciprocal of the given line. He changed the perspective to gradients of perpendicular lines, and I thus coded it as **CMR**.

Table 6.4: Transcript (CL3P2) – Discovering with Applet 3.4 using the worksheet

1-6	Paul: Good Morning. We have to finish where we left on Saturday. The password is there on the board. Please make sure you switch the user. Someone can use this system. (Helping learners to login). Anyone having a problem? Where are the other people? They are delaying us. Right, can we be settled, please? If you still remember we left, we were investigating slope of line. Then we said the next thing we investigate parallel lines and perpendicular lines. Can we go and open applet number 6? Can you see it? Not the one with Word programme. Check the one with GeoGebra icon
17-24	Paul: Right, what should you do? You follow drag point A and verify the new line. What varies, and what does not? CD Can you see that one? If you drag point A, you must see something. Go to point A. Why did you use this computer again, which does not show? You ended up using this one because this does not give you the option of other users here. You share this one. Right, what is happening? I want you to check as you check to move that point move it slowly. As you will be moving that point check what is happening to your gradient of the parallel line and the perpendicular line CD. Is that ok? L: Yes T: Check what is changing there. Can you see the changes?
30-32	Paul: You can insert a new equation there. You can punch $y = 2x+3$, the first equation on the worksheet. Can you see that? Just go there, you click and type the new equation. And you will see what will happen. Is it not working there? Can you see that?
35-37	Paul: So if you go to the blue one you can insert any equation that you want. And you investigate what happened to the parallel line and what happened to the perpendicular line CD. Please write down the observations in your worksheet so that you can see the whole picture. Is that ok?
43-52	Paul: There are three lines that you see there. Don't you see that? The gradient of the parallel line is not changing ne? L: What do we write? Paul: (IsiXhosa - Look) ne? When you drag A to (3,1). You see that changes in pink and green lines. Write down the point (3,1) and the equations of parallel lines and perpendicular lines. (pause- seemingly waiting for the learner to write down the equation.) L: Yes sir Paul: Right. Drag to A to different points, and you write down the parallel lines and perpendicular lines. Write your observation in the worksheet that I gave you. (Pause and then louder) And this is for you to understand that parallel lines have the same gradient CK and perpendicular line the product of the gradients is equal to -1. CK FP
66-71	Paul: Once you finished all the lines that are given in the worksheet, make appropriate conclusions on the other side. Is that ok? L: What do we write, sir? Paul: Just recheck the gradient of the parallel line and the perpendicular line. CD. Which means that as long as you got parallel lines, the gradients are always the same. Then as long as you have perpendicular lines, the gradient will always be the negative reciprocal CK CMR. Are we together?

As the interactions with Applet 3.4 unfolded, Paul encouraged his learners further, to go beyond the equations on the worksheet, and to explore and discover concepts for themselves concerning gradients of different types of straight lines. The interactions, as shown in the transcript in Table 6.5, testifies evidence of CU and IF. He made the most of the situation to take the opportunity to motivate learners to enhance their knowledge on the gradients of vertical and horizontal lines. He used the original line and then asked the learners to identify the gradients of its parallel and perpendicular lines. Learners were keen and eager to respond. He then used the equation of a vertical line and then asked the learners to identify the gradients of lines that were parallel and

perpendicular to it. After that, he asked them to generalise about the relationship of gradients of parallel and perpendicular lines, from the concrete ideas CS (lines 102–105 in Table 6.5). He thus allowed the learners to deepen their understanding of gradients of different kinds of straight lines through exploring the Applet 3.4. Apart from gradients, Paul also established the mathematical fact that vertical and horizontal lines are complementary lines, as shown in lines 82–85 of Table 6.5. The excerpt in Table 6.5 is also coded as FA as it exhibited his range of instructions to include vertical and horizontal lines in the explorations of parallel and perpendicular lines.

Table 6.5: Transcript (CL3P2) – Exploring gradients of vertical and horizontal lines with Applet 3.4

77-90	<p>Paul: Your line is cutting y-axis some people did not put that. Can you input there just say $y=4$? So you must see it will tell you the gradient of the perpendicular line. CD What is it? FA</p> <p>L: It is undefined CK</p> <p>Paul: It is undefined. Can you see that? Then put $x=0$ or $x=2$</p> <p>L: Where?</p> <p>Paul: Where did you input your equation. Remove that $y = 4$ and put $x=2$ what are you getting. CD (Helping a learner). So, if you put $x=2$ what are you getting? You see that you still get your three lines. Our point A (1,3) and this line at $x= 1$ is a vertical line that is parallel to it. Another one $y = 3$ is horizontal and perpendicular to it CK CS Are we together?</p> <p>L: Yes sir</p> <p>Paul: So now, can you check what is the gradient of the line that is parallel to that one?</p> <p>L: Undefined</p> <p>Paul: Undefined. CK What is the gradient of the perpendicular one?</p> <p>L: Zero CK</p>
99-105	<p>Paul: Which one will have a 0 gradient, and which one will have undefined? With your $x=0$, $x=8$ what is the gradient there is a question mark ne, which means that is undefined ne.</p> <p>L: Yes</p> <p>Paul: So all vertical lines, the gradient is undefined. CK All horizontal line the gradient is what CK</p> <p>L: Zero</p> <p>Paul: Zero is that ok? You can check with $y=4$ again. You see that your gradient will be zero. CS Then the gradient of its perpendicular one can you see is undefined, and that of a line parallel to it will be zero.</p>

6.2.2.5 Act 3 scene 2 ~ Applet 3.5

Paul then focussed the learners’ attention on determining the equations of parallel and perpendicular lines by using the Applet 3.5. It is evident from the following transcript in Table 6.6, that the interaction with the learners using the applet reinforced mathematical procedures in determining the equations of parallel and perpendicular lines. Initially, when interacting with the Applet 3.5, Paul provided specific technical instructions on how to use the applet such as “Click the parallel line, there is a box for the parallel line”, “click there that icon step.... Move it [slider] up slowly slowly slowly move it step by step”. He did not, however, provide mathematical explanations on how to find the required equations. Instead, he allowed the learners to explore the

procedures, as evident in lines 136 and 139 in Table 6.6. He evidently wanted them to understand the procedures by themselves, and hence he emphasised moving the slider ‘slowly’. This helped to make the required procedure clear.

However, it was evident from lines 141–153 in Table 6.6 that Paul was flexible and responsive to a learners’ concerns, particularly those who did not comprehend the procedure described in the applet. This is classified as **FR**. He provided an accurate explanation and encouraged the learner who was struggling, to select the most efficient method. As he explained the procedure, he also encouraged the learner by explaining the concepts that lay behind the procedure (learners were asked to write down the procedures in a notebook, but most of them did not have one). It was evident that Paul provided fluent instructions as he interacted with the learners. This I classified as **FA**.

Table 6.6: Transcript (CL3P2) – Interactions in Applet 3.5 exploring and explaining procedures

120-127	<p>Paul: Move just go there and say step and click there that icon step. There should be a tick on that box parallel line. Move it [slider] up slowly slowly slowly move it step by step you should see something. Yeah. I am very sorry for these people here you are first of all click. (helping learner). As you go up, it is telling you all the steps that you need to follow. Please take note of what is written there. That is the procedure that we follow when you are finding the equation. When you are done, you also go and check the perpendicular lines. And you check what will be happening. The procedure, look at the procedure step by step, then it will give you the equation, and it will draw the line. So, if you go to the perpendicular line, it will end up drawing a perpendicular line. Is that ok?</p>
133-139	<p>L: What is happening here? Paul: Yoo! What are these ones doing? Have you done the parallel line? L: Yes sir Paul: Right, please take note of the steps. Is that ok? CD L: Yes sir Paul: Right after this one you can even change. I am sure the equation of the straight line. You can change the equation there and put any and see what will happen. CD</p>
140-155	<p>L: I do not know what is happening? Paul: We need a gradient of the line and a point to find the line FP. We know parallel lines L: are equal. Paul: Yeah, their gradients are equal FP. Are we together? L: yes Paul: Ok. Right. Here, you see the gradient is 2. Therefore the equation is $y=2x+c$, and now put the point (2,-1) into this equation, and find the value of c which is -5 FP. Look at parallel drawn here. Ok. L: When do you use this equation? (Paul recalls that the equation learner wrote was). Paul: We can also use that method. FA Look, generally, given a fixed point (x_1, y_1), then any point (x, y) on the line, we know that gradient m equals the ratio of $m = \frac{y-y_1}{x-x_1}$. So we get this formula making $y - y_1 = m(x - x_1)$ Here it is shown a different method. Be open to this alternative $y = mx + c$. You chose anyone that you feel quicker; you get the same equation FR L: Ok, sir. Paul: Look at the line drawn here, sketch the situation and think. Do not blindly follow the formulae. Is that ok?</p>

6.2.2.6 Act 3 scene 2 ~ Applet 3.7

As mentioned, there were some glitches the Applet 3.6, as the computers in the school laboratory were not updated. Paul immediately recognised these problems in the applet and proceeded to the next applet, i.e. Applet 3.7. It is evident from his interactions with the learners throughout this episode (captured in Table 6.7) that he motivated and involved the learners in tasks to explore mathematical concepts. This is classified as **CD**. Paul engaged learners in a discussion about exploring the relationship between the slope and angle of inclination of a straight line by using the Applet 3.7. In lines 187–189 (in Table 6.7), he pointed out for the learners the salient features of a straight line when a point on the plane is dragged around the screen.

The excerpt of the interaction in Table 6.7 suggests that Paul focussed on developing mathematical concepts through exploration, using the Applet 3.7. For instance, in lines 237–238, he was encouraging his learners to drag the point and investigate different scenarios. These different scenarios would be useful to learners while solving problems. Through appropriate guidance, he drove home concepts relating to the gradient and angle of inclination of a straight line. For instance, in lines 232–235, he was guiding a learner to see the connection between the gradient and the angle of inclination of a vertical line. He then showed how the gradient is undefined and the angle is 90° for a vertical line. I have thus coded this as a combination of **CS** and **CK**. Interestingly in line 235: “*its gradient is also undefined, remember the earlier applet.*” as Paul connected the gradients of these lines to the concepts already explained in an earlier applet **CC**. Similarly, in the applet he also showed the learners how the gradient and angle of horizontal lines become zero (lines 198–203 and 248–250).

I have also coded the interactions in lines 187–192 and 194–208 in Table 6.7, as **FA**, since Paul encouraged the learners to see the relevant properties in a mathematical task and made connections with the concepts discussed in earlier applets.

Apart from teaching the learners about simple equations and linking slope and angle of inclination in graphs, Paul provided an opportunity for the learners to look beyond the formula, which I classified as **CK**. He connected the sign of the gradient with a range of value(s) for the angle of inclination in lines 228–229 in Table 6.7. Furthermore, he also drew the attention of the learners to the fact that the angle of inclination is restricted to between 0° and 180° (line 246 in Table 6.7).

My analysis allows me to concur with Kendal and Stacey (2001) that in mathematics classrooms, pedagogical use of technology can develop conceptual understanding of mathematical ideas.

Table 6.7: Transcript (CL3P2) – Interactions with Applet 3.7

181-185	<p>Paul: Right now, listen, can we close that applet the one that you have opened number 8. Let us go to number 9. Because number 8 there are some glitches somewhere. Click on nine slope tan exploration. With this one, we need to find our gradient, and we are linking it to our tangent. Is that ok?</p> <p>L: Yeah, there are some messages. What does it mean?</p>
187-192	<p>Paul: Can you go to that blue point where it is written (5:3) right. Move that one and see what is happening. Move it even change the sign of the slope. Move the point, and you will see the changes in the slope and the angle. CD Are we together? You must see which angle we are talking about CM</p> <p>Learners: Move the blue point slowly slowly slowly (and then giggling).</p> <p>Paul: Do what I am saying wanna. Take that blue point to the x-axis take it to the x-axis straight away. There is a blue point that you are moving to.</p>
194-208	<p>Paul: Move that one so that it [line] is a negative gradient. You see when your gradient is negative what is happening, play around with it and see CD. What happens when your point is along the x-axis? Just take that to the x-axis and see that point. Take it along the x-axis make it fly along the x-axis and see, can you see? What are your gradient and your angle? Yeah it depends where you have taken your line to</p> <p>L: Yes</p> <p>Paul: It must go to the positive x-axis. Take it to the other side, yes. What is your gradient? Is it shown there? Your gradient is zero. Is that ok?</p> <p>L: Yes yes</p> <p>Paul: What is your angle? Is it showing?</p> <p>L: No</p> <p>Paul: Must show somewhere. It will be at 0.</p> <p>L: Yes sir</p> <p>Paul: So the angle we are looking is with the positive axis. FA Are we together? In this (applet), you see the angle between the line and the positive x-axis.</p>
222-225	<p>Paul: Now, can you change I want us to look at what happens when you take that line along the y-axis. Go and move that original point of yours the (5;3) move it so that it goes along the y-axis. Along y-axis. What is your angle? CD Can you see that?</p> <p>L: Oh oh 90</p>
226-236	<p>Paul: Can you see that? If it is a perpendicular if your line is perpendicular to the x-axis, the angle will be 90. Is that ok? Can you see 90 there? You see that gradient is undefined and $\tan 90$ CD is also undefined. Right. (Pause). Play around with the point. Please can you link the sign of the gradient with your angle CD. If your gradient is negative, your angle is more than 90 CK. Is that ok?</p> <p>L: Yes sir</p> <p>L: What sir?</p> <p>Paul: What is your problem? I see. When your angle is 90, the gradient is undefined. Is that ok? Please take note of what is happening to your gradient to your tan ratio. Is that ok? Can you play around with that one and see what is happening to gradient and tan ratio? No matter where you are vertical lines is always 90 CK CS Right FR. And its gradient is also undefined, remember the earlier applet. CC Is that ok?</p> <p>L: yes sir</p>
237-241	<p>Paul: Move it around all the quadrants and see what is happening? Here just check and investigate, please. Investigate what happens when the gradient is negative. CD</p> <p>L: Now, what sir?</p> <p>Paul: So, if you take it to the other side to the negative side, it will show gradient -1. Look your angle is greater than 90. Obtuse angle. Ok. Can you see that angle is 135? $\tan 135$ is -1. Is that ok? CM</p>
243-250	<p>Paul: Move the point on the x-axis, and you will see the angle 180. And see the slope is zero. Are we together? A horizontal line is having a gradient of zero. The angle formed is 180. You can see that. Is that ok?</p> <p>L: Yes, I can see it now. $\tan 180$ is also zero.</p>

	<i>Paul: The size of your angle and also note that you will never exceed 180 CK. Your angle will be from 0 to 180, is that ok? No matter where you are horizontal straight line is always equal to 180 CK CS, and the gradient is zero. Is that ok?</i>
252-253	<i>Paul: I give you some more time to investigate a little more. Move and move what is going on? Move it. Move that point across the screen. CD</i>

6.2.3 Analysis of George's lessons

This was George's third lesson using *GeoGebra*. He had implemented the applets from the previous two cycles; however, his lessons were not recorded. George's class comprised of 35 learners and lasted for one hour and thirty minutes. The computer laboratory was more compact in size, and the learners complained about the blurred images on the projector screen. Either the teacher or a learner frequently switched the lights on and off to make clearer the images on the screen. George moved around the classroom, giving instructions and helping the learners who were struggling with the activity on the computers. The computers and their peripherals functioned well.

George drew conclusions after the activity with Applet 3.4, as the scheduled lesson ended. With a few minutes available, he asked his learners to engage with the Applet 3.5. However, some of the learners remained idle. George claimed that he organised another day for the learners to engage learners with Applet 3.7, but this was not recorded. Figure 6.14 shows the layout of George's classroom and Figure 6.15 shows the effect of lighting in the classroom as he switches off the light to make gestures during the lesson.



Figure 6.14: Layout of George's classroom

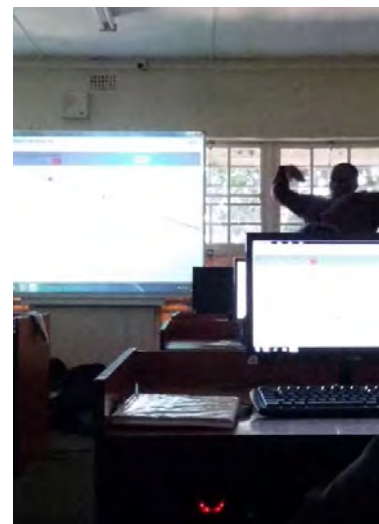


Figure 6.15: The effect of lighting in the classroom

6.2.3.1 Act 3 scene 3 ~Applet 3.1

The introductory conversation with the learners in lines 5–16 in Table 6.8 showed evidence of **CC** as George prompted the learners’ previously learned ideas on analytical geometry. He encouraged the learners to express the mathematical concepts that they already knew. He did not venture into Applet 3.1 straight away; but opened a new *GeoGebra* file and plotted a point (3;5), thus reminding learners about cartesian coordinates.

Thereafter, he opened the Applet 3.1 and asked his learners about the ratio of rise and run of a line. Initially, the learners did not follow him at all. Nonetheless, George engaged them in thinking and doing the activity in Applet 3.1, which is classified as **CD**. I argue that searching for a solution is essential in learning mathematics, and here the teacher challenged the learners to find the solution. The interaction showed evidence of **CK** and **CC** as George encouraged the learners to interact with the applets and realise the concept of the slope of a line. When a learner mentioned that slope was related to its steepness, he agreed with the answer, connecting the slope to the concept of ‘rise over run’.

When the learners sketched a straight line for a given slope using the applet, the concept became clear and concrete. It is evident from the interactions that the learners appreciated the mathematical ideas and there were ‘wow moments’ during the lesson, as shown in lines 41–65 of Table 6.8. George’s explanations and engagement with the applet enabled his learners to internalise the concept of slope of a line. Accurate definitions of a gradient of a line were fully explained by George. He engaged the learners with the applet rather than simply providing them with the formula required to find the gradient of a straight line. Furthermore, when the learners understood that the ratio of ‘rise over run’ was the same as the gradient of a line, they retorted with the required formula, as evident in lines 66–74 in Table 6.8. As the learners became familiar with the concept, George then introduced them to the practicality of using the formula to determine the gradient of a line (lines in 95–100 in Table 6.8).

Table 6.8: Transcript (CL3G) – Interaction with Applet 3.1

5-16	<p><i>George: Ok, the topic that we are going to look at today is analytical geometry. There are so many things that I like you to learn from GeoGebra. Before we start anything, I want you to go to the lower grade. You all know the cartesian plane isn't it. When you talk about the cartesian plane, we always talk about the point. If I plot a point on the cartesian plane, what comes to your mind when you hear the word point. There is a point on the cartesian plane.</i></p> <p><i>L: It is [has] got coordinates.</i></p> <p><i>George: It is [has] got the coordinates; which coordinates are they? CC</i></p> <p><i>L: x and y</i></p>
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	<p>George: x and y. Here on the screen the point, as you can see, it is $(3,5)$. How are they arranged first one is x coordinate and the second one is the y coordinate, CC alright? Those two combined will form what you call a coordinate. (Pause). What is x and what is y for the coordinate $(3,5)$? L: x is three and y is 5.</p>
22-24	<p>George: It is a straight line, isn't it. That is a straight line that I have drawn, isn't it? A linear graph. A linear function CMR. Now from the lower grade there is something that you are talking about? Run and rise. L: No.</p>
33-34	<p>George: What is that? Ok, how do you explain the slope of that line to me? FF L: It is the steepness.</p>
46-50	<p>George: But you are just telling me about the steepness [making gestures]. The steepness of a line. Look at this slope and that line formed there? Can you see that line and the triangle that is formed there? (clicks on the hint of the applet) Now I have given a slope of. The triangle is formed on the sides of the line, isn't it? Can you see it? I got this vertical side and that horizontal side. CS Can you see it? L: The line is the hypotenuse of the triangle.</p>
51-65	<p>L: No, I can't see the lines. The lights are too much. George: You mean, you cannot see the lines because of the lights. L: Yes George: (moves himself to the switch to put the lights off and talking) Now if you count the units from that horizontal one to the end how many are they? Only on those lines. L: 5 George: Five, do you agree? L: Yes, agreed George: What about those vertical ... L: Eight George: Still, nothing comes to your mind? CS The slope of the line remember I was talking about isn't it? CK L: Wow! oh!</p>
66-74	<p>George: The slope is vertical change over horizontal change CK CMR L: Yes yes George: (gestures) picture this as a cartesian plane you understand what I am saying? Picture this whole diagram as it is on a cartesian diagram. CS Now what comes to your mind when you talk about the slope of a line? The slope of a line is the gradient of that line. L: Oh. Oh George: It is the gradient of that line. Now how do you calculate the gradient of a line? How do you calculate the gradient of a straight line? CMR CK</p>
95-100	<p>George: If you are using the formula in grade 10, what will that formula be? The one that you were using in grade 10 to substitute gradient of a straight line, what is that? The formula for calculating the gradient. L: (inaudible) George: FP so in that case you see that the change is definitely is change in y that will be the rise moving up. Do you get it? Alright.</p>

George explained to his learners how a horizontal and a vertical change of points of a line formed a right-angled triangle (refer to Figure 6.2 p. 165). He linked the concept of the gradient of a line to a right-angled triangle in lines 46–50 in Table 6.8, which would prove useful for the learners to understand the relationship between the gradient and the tangent ratio. Interestingly, one of the learners in his class related the line to the hypotenuse of a triangle.

With Applet 3.1, George unpacked the basic concepts such as the gradient of a line and increasing and decreasing lines that were meaningful for the learners. As the lesson progressed, he further developed the concept of slope of a line. He challenged the learners to observe and identify the difference between lines having a positive or a negative slope. This I classified as **CD**. He used strategic questioning and encouraged the learners to explore, using the applet. Interestingly, the learners were intrigued by the challenge and came up with different responses. One of the responses can be accessed in the transcript in Table 6.9. That particular learner conjectured that a line with a positive slope increases to the right and a line with negative slope increases to the left. George was thus able to highlight an essential feature of a function graph by engaging learners in conjecturing and exploring the applet. He built on the ideas of the learners and provided an accurate explanation of increasing and decreasing straight line functions and their relation to its gradient. George's approach aligned well with Laborde (2001) who proposed that it was important for teachers to see the role of technology as a way to facilitate the making of conjectures.

The interactions with the learners and the applet as evident in the extract in Table 6.9, enabled the learners to enhance their conceptual understanding, which in turn developed their mathematical thinking and reasoning.

Table 6.9: Transcript (CL3G) – Discussion on increasing and decreasing lines with Applet 3.1

140-167	<p><i>George: ...from your observation when slope has a negative sign somewhere and when the slope is all positive what is the difference that you are noticing in those two lines? A line that is formed by a negative slope and a line that is formed by a positive slope what is the difference between those two? Try it out quickly and give me an answer CD</i></p> <p><i>L: (inaudible)</i></p> <p><i>George: It is what? I want you to refer to the lines.</i></p> <p><i>L: (inaudible)</i></p> <p><i>George: Can you please listen to what she is saying here?</i></p> <p><i>L: When there is a negative number, the slope the line changes to the left side.</i></p> <p><i>George: Can you stand up and show the direction that you are showing here?</i></p> <p><i>L: when there is a negative number there the slope changes to the left side and when there is a positive number.</i></p> <p><i>George: Which direction? Are you also checking? CD What is the difference? Can you come up again and say a bit louder? Others keep quiet and listen. There is no reason that you are using the computer and talking. Listen to her.</i></p> <p><i>L: when the slope is positive, it is increasing to the right.</i></p> <p><i>George: increasing to the right. (gestures) That line I am having, can you see it? What kind of slope is that?</i></p> <p><i>L: Negative slope.</i></p> <p><i>George: So if it is a positive slope, what is it supposed to be like?</i></p> <p><i>L It is increasing to the left.</i></p> <p><i>George: it is supposed to be in the other direction, isn't it. (gestures)</i></p> <p><i>L: Yes, yes.</i></p> <p><i>George: So when a slope is increasing the gradient that you always find is that is going to be a positive gradient (gestures) CC. The gradient of that line is positive if it is increasing. If it is decreasing, we</i></p>
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	<p>say the gradient is going to be a negative gradient; you understand that ne? CK We are talking increasing and decreasing functions based on x and y values. As x increases, if y values are also increasing, then we say its an increasing function. CC So when a slope is increasing the gradient that you always find is that is going to be a positive gradient. CK FR Alright.</p>
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6.2.3.2 Act 3 Scene 3 ~ Applet 3.2

The Applet 3.2 focussed on sketching a straight line, and George demonstrated on the screen how different lines were drawn using the applet. Initially, he drew a vertical line and a horizontal line and then chose the lines of the gradient-intercept form. He asked questions, proposed solutions and guided the learners to sketch the line, as in the conversation in lines 199–217 in Table 6.10. Thus, he provided a detailed explanation of the procedures of sketching the straight line, which I classified as FP. I was interested to see traces of conceptual understanding intertwined within his description of procedures on drawing a line. In line 204 (in Table 6.10), as he was explaining the y -intercept, he pointed out that the y –intercept was the point on the line that cut the y –axis. In fact, to emphasise the point on the axis, he explained using different terms and concepts, such as ‘ y cut’ and ‘crossing the y –axis’. This I coded as CMR.

After working out three equations on the projector with explanations, he then asked the learners to sketch the different lines on their own. George moved around the class, attending to and guiding learners individually. He did not do the work for them but encouraged the learners to suggest their own methods of sketching the line. Some learners appeared to rapidly in carry out the task efficiently. When he explained, using the projector for sketching the line of the form $y = mx + c$, he chose the gradient-intercept method; Nonetheless, the learners preferred to use the dual-intercept method to draw the lines. They were not able to sketch the line when the x –intercept could not be plotted in the applet (in the applet, points could be dragged only to integer values). The interactions in lines 227–240 in Table 6.10 show a learner who could not follow the gradient-intercept approach. George resorted to explaining the substitution method to determine the second point (the first point being the y –intercept). Lines 227–240, capture the evidence of George responding to the learners’ queries and his flexibility to provide alternative methods of sketching the line. This I classified as FR. The excerpt also throws light on the range of instructions that he employed fluently as he interacted with learners.

Table 6.10: Transcript (CL3G) – Interactions with the learners and engaging with Applet 3.2

<p>170-193</p>	<p>George: Now go to applet four quickly. We are using a gradient for this can you see that I have written and equation there? It is written $x=-2$ what do you do to draw a graph? A line which is given in that equation. How do you do that? Where do I find that line? It is a straight line, right? $x=-2$ where do I find it? Can someone help me? Or can someone describe the line? FF By showing a sign from where you are. Is it like this vertical line, or is it like this horizontal line? (showing vertical and horizontal lines) Or something else? FF Is it going in that direction or in that direction or something else? L: Something else George: that direction we have there? A graph with a positive gradient L: No no George: So it is this direction? FF L: yes George: this direction, this direction or something else? (gestures) What will be its gradient? Why? CD L: Something else George: Where is x on that cartesian plane is it one point? Now if you check there $x=-2$ where do I find it? Now when I go to this point there what is the value of x at this point. L: (inaudible) George: What if I go to that point there? All x-values are -2 on the line. It is a vertical line parallel to the y-axis. L: Parallel to the y-axis. George: What about gradient? A line has a gradient. But you can see now there is no change in x-value; therefore, anything divided by zero is undefined and has a gradient that is undefined. (Pause) George: A vertical line parallel to the y-axis. So, the graph is supposed to be that along that line? That is. Alright. (He skips two or three lines to a get desired one) go to the next one $y=1$ where is it? L: On the y-axis George: on the y-axis, how? Say it like this – passing through y-axis at $y=1$. FP</p>
<p>199-217</p>	<p>George: Ok, some questions are coming up. The equations of the form $y=mx+c$. I think you have met them in grade 10. What does m stand for? FF CC L: (inaudible) George: m is the gradient, isn't it? L: yes George: then c is what? What is c? c is the y-intercept. Where the graph crosses the y-axis. CS, CMR You need to use the graph and y-intercept to sketch a graph like that. Like this one, can you see it? L: $y= 3x+4$ George: $y= 3x+4$. So the y-intercept is going to be four the graph will cut y-axis at that point. Then now I move I now use the gradient to get the line. What is the gradient of that line? FF L: 3. George: what is that? 3? So using that takes how many units are you supposed to rise? FF L: 3 George: it is three and how many run and which direction? FF L: Right George: How many units, right? L: (silence) ...3GeorgeT: I move one-unit right. And see that it is correct. Let me get another one. FP.</p>
<p>227-240</p>	<p>L: I can't do this? George: Your y-intercept is correct at 3. Then look at your gradient it is -2, right? L: it is -2, but I not able to plot on the x-axis. George: Your gradient is -2 so move two down from 3 and remember -2 is $-2/1$. So move after two units down move one unit right. L: I do not understand. How can I plot x-intercept here? George: let us another method. You can substitute any point. Say $x= 1$ FR</p>

	<p>L: I put $x=$ one into the equation, and then George: What is your y? L: 1 George: so we have (1,1) a point and drag it to there and see. FA L: Yeah, I see. George: So you can determine any point of the line and then sketch the line. FP</p>
241-245	<p>L: Yeah, I see. George: [louder to the class] Any point on the line will satisfy the equation. Remember $y = mx + c$ represents a relation between y and x. Here on my board its $y = 3x + 4$ so any point on the line will have that relation when $x = -1$ then $(-1, 1)$ is a point on the line. [drags the point on the line] Alright FP [pause]. FP</p>

6.2.3.3 Act 3 scene 3 ~ Applet 3.4

Prior to interacting with the pre-designed applet on investigating the effect of the gradient and intercept of a straight line on parallel and perpendicular lines, George thought of encouraging learners to recollect their previously learned properties of these lines. In Table 6.11, I present the transcript that unfolded during the lesson, where the learners recalled concepts on parallel and perpendicular lines which they had learned previously. He drew three lines, of which two were parallel to each other, and the third perpendicular to the other two. It is interesting to observe the nature of interactions that George had with the learners. He asked learners to mention the known facts of parallel and perpendicular lines in lines 245–261 of Table 6.11. He added to the suggested ideas and provided an accurate mathematical concept of parallel lines. While doing so, he also brought in different ideas to emphasise the concept. For instance, George used phrases such as “same distance throughout”, and “always apart by the equal distance”. It endorsed George’s ability to represent ideas in different ways, and hence I coded this as **CMR**. In my opinion, such ideas helped the learners to not only deepen their understanding of these lines but also help them to comprehend other related concepts. There was a clear opportunity for the teacher to bring in real-life connections about these lines, but unfortunately this did not happen. Nonetheless, this excerpt of the interaction using *GeoGebra* demonstrated that he intended to induce learners to make sense of these lines.

Table 6.11: Transcript (CL3G) – Recollecting prior knowledge

245-261	<p>George: Now if I want to draw a parallel line to AB using <i>GeoGebra</i>, let me draw a parallel line first. I will draw a parallel line of AB through this point C. the way I supposed to do it is I go to that tool, and then I click the parallel line. I need to draw the line through C but is supposed to be parallel to this line there. There comes the parallel line. Now from what you can see there, what do you think parallel lines are? What do you mean by parallel lines? FF CC L: (inaudible) George: They are? What are the parallel lines? If I say, two lines are parallel, what does it mean? L: They are equal</p>
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	<p>George: Are they equal? L: They are apart George: they are apart from each other? Definition of parallel lines yes [name] L: (inaudible) George: Can you listen to [name] yes can you come again? L: They will never meet George: Ok. Parallel lines are lines that move in the same direction but never meet CK (gestures) They are equidistant from one another. Same distance throughout. They are always apart by the equal distance CK CMR. They will never meet, and they are moving in the same direction, alright?</p>
262 - 279	<p>George: Then, I also want to introduce another type of line. That line that I have drawn there. I am not sure if you can see from there? I measure the angle there. I draw two lines there. L: Yes George: Can someone read the angle I have measured there? Can you just come here? Readout that angle that is drawn there. Now can you see? Come and read it out? L: 90° George: That symbol you know if you see that symbol what does it mean? How many degrees are those? CK L: 90° George: So I am saying that those two angles are meeting at an angle of 90. What other word can I use instead of 90°? And English word a mathematical word FF L: At the right angle George: At right angle CC another word FF L: Perpendicular George: Those lines are perpendicular to one another, isn't it? CMR (gestures) L: yes George: So when lines are perpendicular they meet at an angle of L: 90. George: 90°, isn't it.?</p>

After a brief recap of parallel and perpendicular lines, George engaged the learners with the Applet 3.4 to explore and comprehend the properties of these lines. He distributed the worksheet WS-2 to every learner. George demonstrated on the projector how to input the equation, check the required line (parallel or perpendicular) and drag the points in the applet. The transcript of his step-by-step instructions consisted of 60 lines; however, he summarised the instructions which I have extracted in lines 349-355 in Table 6.12. He asked the learners to record their observations on the worksheet and encouraged them to make inferences from their interactions with the applet (lines 376–415 in Table 6.12). He challenged learners to articulate their mathematical ideas and inferences, which I classified as **CD**. The interaction among learners and George using the applet establish that he helped learners to formulate mathematical ideas through discovery and experimentation.

Interestingly, when George engaged with the learners, they could identify the variant and invariant properties of parallel lines (lines 422–425 in Table 6.13). When dealing with perpendicular lines, George was repeatedly seen urging learners to establish the relationship between the gradients of these lines. The learners could however only identify them as different and were not able to

articulate the negative reciprocal relationship of the gradients. George continued to persuade the learners to find conclusions from their observations, thus giving the learners an opportunity to explore their mathematical ideas, as evident in lines 459–468 in Table 6.13. Learners were hesitant to articulate their mathematical ideas; nonetheless, George offered strategies for thinking and learning that could enhance their ability to handle mathematical situations. The excerpt thus aligns well with the notion of using ‘scaffolding’ for teaching as articulated by Jaworski (1994), who interprets scaffolding “in terms of a teacher's offering of strategies for thinking and learning, rather than for grasping a particular skill or concept” (p. 31). The GLIP teachers considered “the product of gradients of perpendicular lines is -1”, as an algorithm useful for solving problems. Therefore, I have coded line 466 in Table 6.13 as **FP**. In lines 515–517, George alluded to this negative reciprocal relationship, and this endorsed his ability to represent mathematical ideas in different ways. I coded this **CMR**.

Furthermore, in line 468 (in Table 6.13), he drew attention to the perpendicular lines in the applet. I argue that he demonstrated elements of **CS**, as sketches are essential tools to think logically before applying the formulae directly.

He spent approximately 45 minutes on this applet alone. He evidently wanted to ensure that every learner grasped the underlying mathematical ideas of properties of parallel and perpendicular lines. By the time the learners made conclusions of their inferences, the lesson was over.

Table 6.12: Transcript (CL3G) –Interaction with learners engaging with Applet 3.4 to investigate

342-348	<p><i>L: why the gradient is 2? There is only one point given.</i> <i>George: Gradient is the coefficient of What is the equation of a straight line?</i> <i>T: Alright. It is the slope-intercept form of an equation. CK Alright. You compare it with the standard form which is $y=mx+c$ FP So what is the gradient? FF</i> <i>L: Is it not m.</i> <i>George: Here it is 2. Alright</i></p>
349-355	<p><i>George: ...now what you need to do is put in the equation. What is the equation/ where is the equation? This one you don't have to $y=2x+3$, so that is the line. The first one is this point (3;3). You move this point to that point. That is the parallel line. You need to write the equation of the parallel line. What is its slope write it? Then you click there again for the perpendicular line. This is the equation. Record the equation. What is the slope? Write it down. Same equation but different points now. Drag the point the equations will change. Record at least for three points for a given equation. Then you move to the next equation.</i></p>
360-368	<p><i>George: look your equation is not in standard form. The gradient is not equal to one.</i> <i>L: So we have to rearrange the equation.</i> <i>George: make y as the subject of the formula, the standard form of a straight line function.</i> <i>L: (calculates) $y = -x + 2$ whole over 3'.</i> <i>George: So now this can be rewritten as y equals negative one over three x plus two over three. What's the gradient?</i> <i>L: It's negative one over three.</i> <i>George: Type this equation here and see what is the slope? (pause) So in this case what is the slope?</i></p>

	L: Yeah, I agree FM
385-416	<p>George: You are looking at the paper now, $y=2x+3$ when you move the point through those 3 points. The first passing through (3 ;3) what is the first equation?</p> <p>L: $y=2x-3$</p> <p>George: $y=2x-3$ did you all get that?</p> <p>L: Yes</p> <p>George: For the second point, what is the equation of the parallel line?</p> <p>L: $y= 2x-7$</p> <p>George: $y = 2x-7$ then for the last point? What is the equation?</p> <p>L: $y=2x+5$</p> <p>George: $y=2x+5$. Now on this side, there is a no change in the slope. What is the slope of the original equation?</p> <p>L: 2</p> <p>George: 1st one?</p> <p>L:2</p> <p>George: 2nd one?</p> <p>L: 2</p> <p>George L:2</p> <p>George: it doesn't change, isn't it?</p> <p>L: Yes</p> <p>George: What is the slope of the first parallel line? $y=2x-3$</p> <p>L: 2</p> <p>George: equation of its perpendicular line?</p> <p>L: $y=$</p> <p>George: Then, what is its slope?</p> <p>L:(Silence)</p> <p>George: next equation we got is $y=2x-7$ what is the slope?</p> <p>L: 2</p> <p>George: Equation of its perpendicular line</p> <p>L: (no response)</p> <p>George: Then the slope of that?</p> <p>L:</p> <p>George: Then the final one $y=2x+5$. What is the slope?</p> <p>L: 2</p> <p>George: then the equation of its perpendicular</p> <p>L: $y=$</p> <p>George: will you write that 0?</p> <p>L: No</p> <p>George: you write $y=x$. What is the slope of that line?</p> <p>L: (inaudible)</p> <p>George: So I wanted to analyse those three and write down your suggestion on the paper. You write down now. CD</p>

Table 6.13: Transcript (CL3G) – Interaction with learners drawing inferences from their interaction with Applet 3.4

422-425	George: look at those different parallel lines. How does the slope relate to the slope of their original equation? [name] speak loudly. CD L: The parallel line the slope remains the same. The slope of parallel lines and the original line is also the same. The only thing that changes is the y-intercept.
442-449	George: what else do you observe? Now let's look at you got an idea. CD What did you observe? L: the slopes of perpendicular lines are the same. George: They are all the same? What is that slope? L: (silence) George: What about the original one? What can you say about the slope there? You said it is same for all parallel lines it is same for all perpendicular line. Can you tell me the relationship of the slope with the original one and the parallel one? CD Check the slope of the original one and check the slope of any parallel line and perpendicular line.
459-468	George: what about the original and the perpendicular? For instance, the slope of the original equation there is two the gradient of its perpendicular is. So how do you compare that with the first slope 2? How do you compare those 2? They are not the same. One is half, and the other is 2 L: Yes George: So how do you compare them? Those are not the same, isn't it? So how do you compare them? What is the product of the gradients? FF L: -1 George: For perpendicular lines, the product is -1. FP L: (inaudible) George: Look at those lines on the applet.
510-517	George: Then if you are comparing the two the perpendicular one and the original one, what did we say the first time? We say if we the product of the gradients, what do we get? L: -1 George: If you those two again, what are you getting? If you multiply x 3, what do you get? FF L: -1 George: Their product is -1. FP Or we can say the perpendicular line has a gradient of negative reciprocal. CK, CMR So what conclusion can you come up with? The conclusion is one related to parallel one and another related to perpendicular one. Separately yes.
528-532	George: did you hear what she said? L: No L: Slopes of parallel lines are always equal. Then the product slopes of two perpendicular lines equal to -1 L: -1 -1 George: so those are rules you need to remember always

6.3 HORIZONTAL ANALYSIS ACROSS LESSONS

In the previous section, the vertical analysis of the classroom implementation of *GeoGebra* applets in Cycle 3 revealed how the participant teachers made use of the co-developed applets to develop mathematical proficiency. This section presents a horizontal analysis that interprets the findings from the preceding section. The horizontal analysis of the data, strand-wise, shows that the participants demonstrated their core understanding (CU) and displayed traits of instructional fluency (IF). Horizontal analysis requires pulling codes together across the participants. However, the above analysis was dense, and therefore, my goal in this horizontal analysis is to compress

numerous instances into a few succinct expressions that best illustrate the pertinent strand of teaching proficiency. This form of analysis would enable me to gain deeper insights to obtain answers to my research questions.

The CUIF framework (Table 3.3 in Chapter Three on p. 69), serves as a reminder of the observable indicators used to analyse the data.

George arranged another lesson the following day for Applets 3.5, 3.6 and 3.7; however, the lesson not recorded. He shared his experiences of the lesson on the remaining applets during the reflective interview, and I thus included this in the current horizontal analysis.

6.3.1 The unfolding of core understanding (CU)

Kilpatrick et al. (2001) argue that possessing proficiency in core understanding in teaching mathematics enables teachers to integrate their knowledge of content and knowledge of processes to increase their learners' mathematical understanding.

6.3.1.1 Emphasis on teaching for core knowledge (CK)

Applet 3.1 served the purpose of engaging learners in understanding the gradient as a ratio of change. While sketching the line in the applet, Paul used only one approach, whilst George built upon individual learner's initial input. Their focus was not on applying the formula to determine the gradient of a line; instead, they led learners to link the formula. Both teachers motivated their learners to comprehend the concept of gradient as a ratio of change. Mathematical concepts were embedded in the applet that illuminated structures and relationships of straight lines and their gradients. With Applet 3.4, both teachers involved their learners with the applet and then established properties of parallel lines and perpendicular lines. They made mathematically appropriate conclusions while using apt terminology.

Spillane (2000) viewed that learners appreciate mathematical activity as more than computation. Owing to the visual and dynamic nature of the applets the participant teachers were able to engage their learners in order to focus on the teaching and learning goals of exploring and connecting the relationship between the gradient of a linear equation and its straight-line graph. The participant teachers of my research project know and do mathematics themselves, so their teaching proficiency in mathematics prepared them to facilitate their learners' development of mathematical ideas. Success in integrating technological tools requires, as Ball and Bass (2000) argue, "teachers

being able to use mathematical knowledge bridging the divide between content and pedagogy” (p. 99).

6.3.1.2 Emphasis on representing multiple ideas (CMR)

Kilpatrick et al. (2001) argue that proficient mathematics teachers understand that there is not a fixed method for learning mathematics but rather multiple ways to approach a mathematical concept. Teachers designed *GeoGebra* applets that dynamically linked linear equations and straight-line graphs (Applet 3.2, Applet 3.4). Notably, applet 3.7 represented geometrical properties of lines apart from algebra and graphs. The teachers guided their learners to appreciate the properties of straight lines as they correlated the equations, graphs and angles.

While engaging learners with Applet 3.1, the teachers used multiple ideas in explaining the gradient of a line, such as a formula, as a ratio of change in y over change in x , or as a ratio of vertical change over horizontal change. Specifically, George used terms such as ‘rise over run’ to emphasise the concept of the gradient of a straight line. In sketching the line in Applet 3.2, Paul and George both used two strategies; however, their strategies were different. Paul initially asked learners to use the dual-intercept method of sketching the line, while George initially explained the gradient-intercept method. The second strategy was the same – determining two points on the line and then sketching the line. Interestingly, both teachers explained two approaches in determining the gradient of a perpendicular line, while engaging with Applet 3.4. Specifically, George involved his learners with the Applet 3.4 in mathematical reasoning, in the context of conjecturing and generalising properties of lines, from multiple instances in the worksheet.

GeoGebra applets designed by the teachers provided dynamic graphing of linear equations, thereby benefitting teaching and learning mathematics, with the teachers using computer media to link different representations of mathematical ideas. I assert that mathematical software, supported by the instructions of the teacher, helps to provide better multiple and dynamic representations of abstract ideas and the links between symbols, variables and graphs.

6.3.1.3 Emphasis on connecting concepts (CC)

Mathematical knowledge of proficient teachers includes in-depth knowledge of where the content comes from and where it leads to (Schoenfeld & Kilpatrick, 2008). The participant teachers unpacked their mathematics knowledge for teaching that would allow learners to deepen their understanding of mathematical ideas. Discussing the gradient of a line, George and Paul motivated

learners to see the triangle formed in the applets owing to the change in values. The reference to a triangle while engaging with the concept of the slope would allow learners to make sense of the slope-tangent equation, and it would also lead them into the direction of mathematics of change. Teachers were aware that it might be useful for the learners to understand the notion of change in any curve when they encounter calculus in the following grade.

George and Paul developed the idea of ‘increasing and decreasing lines’ relying upon observations and exploration with the applets. George used the term ‘function’ to provide accurate explanations on ‘increasing and decreasing functions’ in relation to the gradient of the linear equation and the direction of the graph (line 23 in Table 6.8 on p. 184 and lines 166–167 in Table 6.9 on p. 186). Furthermore, Paul explained about dependent and independent variables, treating y as dependent on x , which is independent (lines 122–124 in Table 6.2, p. 175) – yet another fundamental aspect of ‘functions’. The teachers were able to extract the specific characteristics and structure of linear equations and guide the learners to similar mathematical structures in other areas of mathematics (functions and calculus, in this case). I concur with Kilpatrick et al. (2001) that proficient teachers use the curriculum to help learners connect mathematical ideas and progress to more in-depth and better-grounded mathematics.

While designing the Applet 3.7, we discussed how different mathematical concepts could be connected in Section 6.2.1.2 (p. 170 below the discussion on Applet 3.7). George and Paul leveraged the applet to support their instructions and thus highlighted several mathematical principles. An in-depth analysis of the lesson showed that these teachers did not connect the ‘angles of inclination’ concept with parallel and perpendicular lines, neither did they emphasise the relation of gradients of parallel lines by linking to the corresponding angles property of parallel lines, as discussed in the GLIP meeting. Possibly, more structured planning is required while integrating new tools in classrooms.

6.3.1.4 Emphasis on teaching for exploring and discovering concepts (CD)

Jonassen and Strobel (2006) argue that when technological tools are used effectively, learners can authenticate their understandings. The emerging technological tools, like DGS, with their inherent exploratory possibilities, provide opportunities for a heuristic approach to constructing mathematical meaning. Analysis of George’s interaction with the learners using the applets gave rise to some interesting insights. For instance, by using mathematical inferences like ‘positive and

negative slope' in Applet 3.1; 'variant and invariant properties of parallel and perpendicular lines' in Applet 3.4, showed that George was desirous of crafting an environment for learners to experience and share their mathematical ideas. When learners came forward with their ideas and observations, George did not affirm whether they were correct or incorrect. He built upon their responses as opportunities to raise mathematical concepts. In fact, unlike Paul, a significant mathematical outcome of his lesson went beyond the articulation and manifestation of mathematical facts. He allowed and challenged learners thinking to generate and discover different perspectives through exploration. This analysis confirms that the interaction between George and his learners was a way of developing reasoning, as well as placing a firm emphasis on exploring mathematical understandings.

The analysis of George's lesson aligns with a typical technology-based constructivist classroom, where there is less emphasis on transmitting information, but more on developing learners' understanding of concepts and their skills. I concur with Noss and Hoyles (1996), who note that teachers construct new ways of learnable and connected mathematics than the computer tools offer. Through active interaction with the computer teaching and learning, mathematical ideas occur.

6.3.1.5 Emphasis on tackling common misconceptions (CM)

Paul addressed two common misconceptions in considering the angle of inclination of a straight line, as shown in Table 6.15, lines 75–84 in (p. 204). For a negative gradient line, learners tend to identify the acute angle between the line and the negative x –axis instead of the obtuse angle between the line and the positive x –axis. In lines 237–241 in Table 6.7 (p. 182), Paul occupied his learners in exploring the angle for a negative gradient line with Applet 3.7. He directed them to generalise the fact that a negative gradient would have an obtuse angle of inclination.

Yet another common error that the teachers identified, included the gradients of vertical and horizontal lines. According to the participant teachers, learners misinterpret the gradient of a vertical line as 1 and that of the horizontal line as undefined, (see George's reflection in in Table 6.14, lines 96–105 (p. 204). Paul, in his lesson, in lines 233–236 and 249–250 in Table 6.7 (p. 182), emphasised the gradient in relation to its tangent ratio of vertical and horizontal lines. George claimed that he also engaged his learners with the applet to overcome such common errors. He specified such mistakes to the learners and instructed them to remember the applet when they were asked to interpret the gradient of vertical and horizontal lines.

Schoenfeld and Kilpatrick (2008) argue that a proficient teacher uncovers learners' mathematical ideas in a way that helps them see the mathematics from a learner's perspective. The analysis of the lessons suggests that proficiency requires understanding and recognising nuances of the learners' thinking as well as features of their thinking that lead them to misconceptions. Achieving such a balance requires, as Ball and Bass (2000) argue, "a teacher to be sensitive to learners' ideas, nature of mathematical reasoning and the need for the steps in argument to be developed" (p. 95). Thus, primarily, teachers would need to understand the ideas behind learners' thinking and be able to engage them in certain tasks that might help them to develop appropriate mathematical ideas.

6.3.1.6 Emphasis on teaching which relates to real-life situations (CR)

There is little evidence that teachers made real-life references or connected to day-to-day realities with the concept of gradient. I will discuss, along with the other cycles, this absence of code as I consolidate my findings in my last chapter.

6.3.1.7 Emphasis on switching between abstract and concrete (CS)

Ironically, Ball et al. (2008) argue that "[s]ome representations are especially powerful; others, although technically correct, do not open the ideas effectively to learners" (p. 392). In the following paragraphs, I discuss how the participant teachers could not achieve their teaching goal: the concrete idea of the relation between a pair of perpendicular lines.

In our GLIP discussion, we considered the negative reciprocal relationship between gradients of pair of perpendicular lines as an abstract concept. When the teachers prepared the worksheet, we anticipated that learners would write down the gradients of pair of perpendicular lines and then from multiple instances they would be able to generalise or relate this to the negative reciprocal relationship between the gradients. However, George lamented that "*they did not see the negative reciprocal relationship and product of gradients as -1*" (line 81 in Table 6.14, p. 204). The Applet 3.4 indicated the equations of a pair of perpendicular lines; nonetheless, learners did not appreciate the numerical association. The dynamic representation of perpendicular lines and the use of the worksheet proved to be insufficient in concretising the numerical association of gradients of a pair of perpendicular lines. It appeared that the teachers did not invoke the learners' previously learned concept of reciprocals of rational numbers before exploring the properties of perpendicular lines. The comprehension of the negative reciprocal relationship of gradients of perpendicular lines demanded higher-order thinking – understanding and linking the rational numbers.

Nonetheless, without much ado, Paul instructed his learners to verify the product of gradients of a pair of perpendicular lines. For Paul, learners simply need to plug in the values of gradients and thus verify the formula – ‘product of gradients is -1 ’ – or verify if negative reciprocal relationships hold for gradients of perpendicular lines.

Kilpatrick et al. (2001) argue that each abstract concept has several concrete images appropriate to the nature of a mathematical idea. In agreement, Presmeg (1986b) refers ‘concretising’ to the embodiment of an abstract idea as an image. The straight-line graph makes the abstract linear equation more concrete; however, the concept of gradient remains abstract in the line. For an analytical approach in sketching the line, it requires no more than determining two points on the line, obscuring the gradient of the line. The gradient-intercept approach in sketching the line directs our attention to the notion of gradient as a ratio of change. George expected that his learners would be able to sketch the line (in Applet 3.2) successfully, using this gradient-intercept method. However, he reflected that “*it was again a challenge because they could not use the slope to find another point on the graph*” (lines 73–74 in Table 6.14, p. 204). Neither the applet, nor the instructions from the teacher could lead the learners to concretise (or apply) the concept of gradient while sketching a line.

6.3.2 The unfolding of instructional fluency (IF)

In this section, I consolidate different pertinent indicators of IF (instructional fluency) that are evidenced in this cycle.

6.3.2.1 Emphasis on acquiring an instructional repertoire (FA)

Paul discussed vertical and horizontal lines using three applets. Firstly, he developed ideas on sketching the lines. Secondly, he explained parallel and perpendicular lines, such as the x –axis being parallel to horizontal lines and perpendicular to vertical lines. Thirdly, Paul engaged the learners in understanding the relationship between the gradient and the angle of inclination of these lines. He repeatedly yet scrupulously engaged with these concepts in three applets; meanwhile, George used only one applet to engage with the concepts relating to vertical and horizontal lines. Paul, however, also made abstract statements about these lines that would help learners to enhance their understanding of the concept. Effective teaching took place when he transformed his content knowledge into pedagogically, powerful teaching strategies. Here Paul drew upon a variety of instructional approaches using the applets, that led to the educational outcomes identified by him.

Thus, Kilpatrick et al. (2001) argue that what makes teachers proficient is the skilled use of knowledge in teaching and learning. Teaching includes a series of activities during which it provides opportunities for learning. Kendal and Stacey (2001) concur that in mathematics classrooms, the pedagogical use of technology is important to develop an understanding of mathematical ideas.

George introduced the gradient-intercept method of sketching the line of the form $y = mx + c$ in Applet 3.2. As learners became involved in sketching the line as per his instructions, he realised that they did not understand the procedure that he wanted them to adopt. He therefore changed his strategy: he instructed them to determine another point on the line and then join the line with the y –intercept (lines 227–240, Table 6.10, p. 188). George was disappointed that the learners did not follow the gradient-intercept method, as reflected during the stimulated recall interview (also discussed in Section 6.3.1.7 (p. 198). Nonetheless, this instance showed his flexibility in his repertoire of instructions that he could fluently employ as he interacted with learners. Kilpatrick et al. (2001) consider instructional routine as one of the essential components in teaching as “teachers who have acquired a repertoire of instructional routines can readily draw upon them as they interact with students in teaching mathematics” (p. 382).

6.3.2.2 Emphasis on tackling a learner’s misconception (FM)

Kilpatrick et al. (2001) argue that a teacher interacting with learners must be sensitive to “misconceptions that students have about the mathematical ideas, the difficulties they have in learning it” (p. 384). Both teachers moved from one learner to another and assisting them in engaging with the applets. They also provided explanations that directly addressed the learners’ misconceptions.

Lines 54–62 in Table 6.1 (p. 172), reflect Paul’s interaction with a learner while engaged with Applet 3.1; he used different terms and approaches in addressing the learner’s misunderstanding. Evidently the learner was confused by the movement of the points that matched the required gradient of the line. Paul provided different explanations such as “*move right then down*” for negative gradient; “*denominator move right*” “*picture of how your line is slanting*”, that prepared the learner to sketch the line. These instructions allowed the learner to engage with the applet and verify his mathematical learning.

In an interaction with a learner in Table 6.12 (p. 191), lines 360–368, George found that a learner misinterpreted the conventions of notations of equations: the learner had inadequately identified the gradient of a straight line in the worksheet. For the learner ‘the coefficient of x in the equation’, always represented the gradient of the line, irrespective of the form of the equation. George guided the learner to write the equation in $y = mx + c$ form, and thereafter the learner identified the gradient accurately. Furthermore, he encouraged the learner to cross-check the equation in the software. In this way, the learner’s misconception was corrected (in fact, the worksheet WS-2 directed learners to rearrange any equation to the $y = mx + c$ form).

A focus on the above instances acknowledges that particular interest in learner misconceptions is a key feature of a proficient teacher. Technical tools may support learning, but more is required from a teacher than merely watching learners working on their computers. A proficient teacher works with learners and engages them with the tools (applets in this case). My participant teachers proved to be sensitive to misconceptions held by individual learners. Concurring, Schoenfeld (2007) observes that “[r]ecognizing misconceptions is crucial if a teacher is to target instruction so that students can clarify their thinking and gain understanding” (p. 163).

6.3.2.3 Emphasis on teaching basic skills (FL)

Unlike in previous cycles, teachers did not face many computer illiteracy issues with the learners. Most of the applets required minimum input from the learners in the form of dragging and clicking. Applet 3.3 and Applet 3.4 required learners to input rational numbers for gradients and a few learners struggled in typing the forward slash on the computer. In Paul’s class, especially during the second lesson, some desktops were not functioning, and therefore more learners had to share systems. I discuss this issue of malfunctioning again when answering the research question in Section 6.4.2 (p. 208).

6.3.2.4 Emphasis on responding to learners’ thinking (FR)

Kilpatrick et al. (2001) observe that a proficient teacher “interprets students’ written work, analyses their reasoning, and responds to the different methods they might use in solving a problem” (p. 370). Applet 3.5 explained a procedure in determining the equation of a line by using the standard form $y = mx + c$; however, a learner was confused, and he could not follow that method. Instead, he had a different method in determining the equation of a straight line – the equation $y - y_1 = m(x - x_1)$ (lines 140–155 in Table 6.6 (p. 180)). The learner’s procedure was correct but

was emphasised neither by the teacher nor the applet; hence he raised his concern with the teacher. Paul, when he realised that the learner was using a different method, provided the learner with the reason behind the procedure. He made use of the applet to reassure the learner that his procedure was correct. In this way, Paul proficiently responded when he asked the learner to think of the situation before blindly following the procedure. He was therefore able to show flexibility in dealing with a method not included in the applet.

In lines 155–167 (Table 6.9, p. 186), George responded to a learner’s observation on increasing and decreasing lines. To the learners, all lines were increasing, either to the right or the left of y –axis. Interestingly, he did not disagree with the observations made by the learner, although the learner’s observation was logical and sensible. Nonetheless, George was aware that mathematical conventions speak of increasing and decreasing lines with respect to the x –axis and not y –axis; and he therefore tactfully responded to the situation. As discussed in Section 6.3.1.3 (p. 195), George connected the straight lines to functions and provided appropriate mathematical explanations and terms.

Here, teachers interpreted learners’ imprecise explanations, guided them to focus on essential mathematical points, and in this case, helped them to learn conventional terms. This reflects the teacher’s underlying belief that resonates with a “knowledge of students” as Kilpatrick et al. (2001) affirm. They argue that a proficient teacher is sensitive to the unique ways of learners’ thinking, learning and doing mathematics. ‘Knowledge of students’ overlaps substantially with ‘knowing students as thinkers’ a category of teaching proficiency identified by Schoenfeld and Kilpatrick (2008), but it encompasses many other ways of being proficient. Schoenfeld and Kilpatrick argue that classroom activities and interactions with individual learners are, by and large, a function of a teacher’s view of learners as thinkers. Thus, “[t]he more that teachers are attuned to their students’ understanding, the more they can create learning environments that are responsive to them” (p. 15).

6.3.2.5 Emphasis on understanding procedures (FP)

Ma (2010) describes what she calls “profound understanding of fundamental mathematics”, in terms of teaching for understanding procedures. According to Ma (2010), teaching for understanding procedures “lead[s] students to an understanding of the rationale of the procedure as well as the associated mathematical concepts” (p. 41). This emphasis on teaching allows

teachers to illustrate the rationale of the algorithm by their explanations and demonstration. The analysis of these lessons shows that the teachers used the applets to represent the mathematical concepts underlying the procedures. For instance, both teachers explained the procedure for sketching a line in Applet 3.2 and provided a reason for the procedure, as in lines 242–245 in Table 6.10 on p. 188 (George) and lines 114–124, Table 6.2 on p. 175 (Paul). Making use of the applet, George dragged a point on the line to establish the fact that coordinates of the point on the line satisfy the equation, thus revealing the underlying mathematical concept for the procedure of ‘substitution’ in sketching the line. Interestingly, Paul used phrases such as “*y is dependent on the value of x*”, while George used “*relation between y and x*”, implying that they believed in similar strategies of teaching for understanding the procedures.

“Mathematical knowledge *usable* for teaching” (italics in original p. 97), according to Ball and Bass (2000), refers to how a teacher is able to understand mathematics in order to weave content-related ideas into a coherent whole. Thus, teaching proficiency entails a special capacity to unpack one’s knowledge whilst interacting with applets and provide a comprehensible explanation to make sense for the learners.

6.3.2.6 Emphasis on inviting feedback (FF)

Inviting feedback is a powerful strategy for improving teaching instructions, especially in a technological environment. The routine practice of soliciting responses from learners helps a teacher to assure that the learners understand or are able to perform a task. The learners were interacting with the applets, therefore, teachers often sought responses from them. For instance, George asked questions such as “*Is it like this vertical line, or is it like this horizontal line?*”; “*What does m stand for?*”; “*What do you see here?*”, thereby ensuring their involvement in the learning of mathematical ideas. George initiated the interactions soliciting (tirelessly) information from the learners. Paul, too prompted learners for responses such as “*what is happening?*”; “*y= a constant is a what?*”: However, Paul did not wait for responses from his learners and often instead made relevant conclusions to the task by himself, while George waited patiently for the responses from the learners and then made appropriate mathematical conclusions.

Furthermore, these sets of routines served the purpose of constantly engaging the learners with the applets and thus aided teachers to check on the progress of the learners. It served as an opportunity for the learners to open up and respond or ask questions. I concur with Kilpatrick et al. (2001)

who consider that a proficient teacher sets a routine that encourages learners to observe and talk about the mathematical situation.

Table 6.14: RI3G George's reflection on Cycle 3

73-74	George: With the other applet, it was again a challenge because they could not use the slope to find another point on the graph.
80-83	R: what other challenge did you face? You already mentioned time was a factor for you. Anything else? George: They did not see the negative reciprocal relationship or the product of gradients as -1. I think I should have written down on the board as they did. I think we should have a writing board. Doing maths, we always want to write so that we can emphasis what we are talking about.
84-89	R: How did the technology help to support your teaching style? Or teaching Instruction? George: I keep on telling this thing of the gradient of the parallel line and perpendicular lines. That concept I don't think if I explain it to them, they would understand it. The way they got it using the applets, it helped me in telling about these concepts. Many of them visualised the changes and that it is increasing or decreasing. It would have been difficult to guess unless we draw a quite number of accurate straight lines on the chalkboard.
90-95	R: I hear that it helped in visualizing for the learners the relation between graph and gradient. What else worked in your favour or against your teaching plan? George: In this particular case, it helped. What happened now they talk about it in the class. They talked about the movement of lines and getting that or this correct. They talked about right-angled triangles that have seen in the computer and considered straight line as a hypotenuse. They could easily comprehend the concept of angle of inclination when I used the applet.
96-105	R: I could not record your lesson for the last applet. How did it go? George: I believe they could now comprehend how tan ratio is related to the slope of the line. I could see that after that lesson learners do interpret the gradients of vertical and horizontal lines correctly. Otherwise, they were so confused with those lines. R: Can you expand on those confusion that they have? How did the applet helped you? George: They wrongly interpret the gradient of line $x = 2$ as either 2 or 1. For the line $y = 2$, they consider as unidentified, as they there is no x . I asked them to see what usually mistakes they make. So I believe with this applet they could now see why the gradients are zero and undefined. They have to see the line. It will help them a lot as they have seen it and they can relate to tan ratio. I told them they must remember them whatever they have done in the class.

Table 6.15: RI3P Paul's reflection on cycle 3

29-36	R: Which applet do you think +worked best for you? You have already taken this topic with other classes. How do you feel about the lessons? Paul: There are certain concepts learners will quickly see on the computer. The first applet where we drag to form a line that would help them to see the difference between negative and positive gradient. With a computer, if something is captured, it is quite fast to convey your message. They can reopen the applet and practice once again. So, technology makes life easy if they want to learn. Chalkboard is limited we rub what we write, and they also rub it away from their minds (laughs). In computer, since they do it and see it will stay in mind.
75-84	R: I see that you were keen on vertical and horizontal lines in the Applet [3.7] tangent slope. What was your experience? Paul: That one was favourite. I tried to make one before, but I could not restrict between 0 and 180. Our meetings helped me to do it. It would tell them about the gradient and the angle, and they can move those lines and verify those values. It saves time for us. These computers make our life easy. Very often I have seen learners making mistakes in writing the angle of inclination for a negative gradient. So I emphasised here why we subtract from 180°. R: What other mistakes did you find among learners? Paul: Other common mistakes they make is gradient of line say $x = 3$, either they write gradient as 3 or 1. I believe those applets will help to make correct conclusions.

6.4 DISCUSSION ON RESEARCH QUESTIONS

6.4.1 How can Dynamic Geometry Software such as *GeoGebra* be used as a visualisation tool to teach Grade 11 Mathematics?

Teaching learners to visualise straight lines was a significant contribution in this cycle. Teachers, with the aid of their applets, successfully drew the attention of the learners to the formation of a triangle indicating the horizontal change and the vertical change. Teachers represented the geometrical properties of lines, allowing their learners to reason why the sign of the gradient affected the value of its angle of inclination. The teachers thus effectively integrated co-developed applets that allowed learners to comprehend the properties of straight lines as they visually related to the equations, graphs and angles shown in the discussion in Section 6.3.1.2 (p. 196). I concur with Drijvers (2012) who defines: “dynamic phenomena that invite mathematical reasoning, in many cases concerning the relationships between multiple representations of the same mathematical object” (p. 489).

Presmeg (2006) characterised the practices of visualisation in teaching: “[t]he visual teachers constantly made connections between the subject matter and other areas of thought, such as other sections of the syllabus” (p. 210). While introducing parallel and perpendicular lines, George prompted his learners to recollect mathematical facts about these lines, in the transcript in Table 6.11 (p. 189). He connected to the geometrical aspects of lines prior to embarking on analytical properties of lines. He brought in connections between different ideas, making use of prior knowledge. Presmeg considers that the word ‘connections’ captures the essence of teaching visualisation. Invoking of prior knowledge is a crucial aspect of meaningful mathematical teaching and learning, and in this way, visualisation in mathematics enables connections of a new concept with an existing concept.

Visualisation as a critical aspect of mathematical activity was demonstrated as the teachers emphasised relationships between mathematical ideas, making use of the co-developed applets. They linked the concept of slope of a line to a right-angled triangle, which would enhance the mathematical concepts related to the gradient of a line. If ‘seeing’ a triangle in the applets for the understanding of the concept of slope of a line was apparent, the concept of ‘increasing or

decreasing' remained latent. The mere presence of an applet did not guarantee that understanding of a concept would take place. George and Paul made use of the Applet 3.1 to engage with the concept of 'increasing or decreasing function', as discussed in Section 6.3.1.3 (p. 195). The teachers allowed their learners to visualise a positive or negative gradient of a straight line. I concur with Moyer (2001), who argues that teachers play a crucial role in creating interactive mathematics environments that provide learners with visual representations that enhance thinking.

Interestingly, both teachers commented on visualising the increasing or decreasing lines during the reflective interview, shown in lines 84–85 of Table 6.14 on p. 204 (George) and lines 31–33 in Table 6.15 on p. 204 (Paul). They considered the several dynamic lines sketched by the learners, (with the aid of the computers) as standing out as fundamental support in teaching and learning. A visual approach thus enhances the learning of straight lines. Researchers (Arcavi, 2003; Ruthven et al., 2008) accentuate that the visual trajectory in mathematics education is linked to the use of computers. Hence, Villarreal's (2000) verbalisation that "the computer has come to restore the value of the process of visualization in mathematics education" (p. 3) cannot be over-emphasised.

Rivera (2011) argues that in mathematics that the focus is on the transition from concrete situations to abstract ideas through a process of visualisation. Visual thinking takes one a step further from perception, to understanding and adding meaning to the images or concrete objects. From the perspective of the participant teachers, the dynamic movement of lines as the learners dragged the points, was enough to visualise the properties of parallel lines and perpendicular lines. Discussing gradients of perpendicular lines, both the teachers switched between the visual or functional relationship between gradients as the negative reciprocal and the numerical product of gradients as a negative one. However, as discussed in Section 6.3.1.7 (p. 198), the teachers' reflections on the lessons showed that their learners did not appreciate the relationship between a pair of perpendicular lines. Using an applet with a dynamic line and its rotated line at 90° about a point (say origin) and then exploring their gradients could be a feasible way to visualise the relationships of gradients of perpendicular lines.

Furthermore, the visually rich method of sketching a line did not receive much attention from the learners (in Section 6.3.1.7, CS, p. 199). Presmeg (1992) discusses several instances in which visualisation is a hindrance to mathematical understanding and solving problems. She describes how difficulties arise as a result of 'uncontrollable' images, or images that appear in students'

thoughts involuntarily that are inappropriate and prevent mathematical generalisation. The visual trajectory gradient of a line, introduced in the Applet 3.1 requires to be further developed and applied in different situations, as Applet 3.2 and Applet 3.3 did. Again, the visualisation related to the gradient of a line may be introduced to the learners at an earlier stage; it would provide opportunities to internalise the concept that could then be applied to the visual methods of sketching the lines.

The teachers preferred to construct straight line graphs that required only the computation of some coordinates, and then plotting the line. Duval (1999) notes that the construction of graphs is not visualisation in mathematics: “[c]onstructing a graph requires only to compute some coordinates and to plot a straight line” (p. 16). Visualisation in sketching lines entails the estimation of the direction of the line, based on salient characteristics such as gradient and intercepts. Both teachers succeeded in constructing the lines but did not allow learners to visualise the final representation of the linear equation.

Similarly, the visually salient property of parallel lines as a vertical displacement from one line to the other was not discussed by the teachers. Discussing visualisation as a process, Arcavi (2003) demonstrates how to “re-direct the attention from the notion of distance between parallel lines (as the length of a segment perpendicular to both) towards the notion of the vertical displacement from one line to the other” (p. 233). Visualisation refers to the development and use of the intervening conceptual structure – vertical displacement – that enables us to see the change of the y –intercept in parallel lines.

Summing up, from my point of view, applets were not a self-explanatory tool for identifying all potential mathematical understandings. Researchers (Presmeg, 2006; Duval, 1999) surmise that teachers who put a conscious effort into promoting visualisation, strive for a deeper understanding of mathematical ideas. George and Paul guided the students to the underlying mathematical meanings hidden in the given representations in the applets. The teachers initiated a productive discussion and interaction on the important mathematical characteristics of straight lines. The applets as visual aids supplement teaching and learning of mathematical concepts and procedures so the potential role of visualisation was recognised in these lessons using applets. The teachers played a vital role in lending mathematical meaning to visual representation. The analysis of the lessons demonstrates that the dynamic visual aspects of *GeoGebra* applets can also complement a

teaching and learning environment, that would have been difficult to convey in the traditional manner of text and the static media of teaching and learning. In agreeing, (Heid & Blume, 2008, p. 67), argue “a dynamical graphical model highlights aspects of the situation that were not as salient had we investigated it alone or even by modelling it symbolically.”

6.4.2 What enabling and constraining factors do Grade 11 teachers encounter when using *GeoGebra* as a visualisation tool to teach Grade 11 Mathematics?

6.4.2.1 Constraints

It was in this third cycle that the participant teachers encountered the most technical challenges.

Technical Challenges

The Applet 3.6 (on practising parallel and perpendicular lines) did not function properly during Paul’s lesson because the *GeoGebra* version on his laptop was not compatible with the computer desktops at school. If a computer lab is connected to the internet then the systems are automatically updated, however, in both schools, the authorities claimed that the internet was not working owing to various reasons. *GeoGebra* software is being continuously updated, and hence it is important for teachers to ensure the compatibility of the software on different devices.

Paul found the lack of computer literacy skills amongst his learners constraining. On several occasions, he said that the learners’ lack of computer skills hampered progress in the lesson. “*Sharing computer and fighting to type. If they do not get used to it, I don’t think we will make progress in making use of the lab*” (lines 42–43 RI3P, my original transcript).

Berger (2011) notes that in the context of using *GeoGebra*, “the syntactical and mathematical knowledge required to plot a graph in an appropriate window is less complex” (p. 116). However, the participant teachers’ experiences highlight that integrating technology in classrooms is demanding as the learners face ‘syntactical difficulties’ in inputting a simple equation.

George’s major constraint was ‘time’. He interacted with the learners and encouraged them to come up with ideas rather than him providing the ideas. He was guiding them to reach the mathematical outcomes of the lesson. He apparently wanted to cover the concepts related to the topic on a single day within two hours, but as he reflected “*Unlike the proper teaching time, these needs time. Too many applets and concepts to be done at the same time.*” (line 11, RI3G, my

original transcript). Nonetheless, he arranged extra lessons to complete the topic with the remaining applets. The major constraint that George faced was the difficulty in adequately covering the assigned content within the limited availability of class and laboratory time. However, this constraint can be successfully overcome through meticulous and thoughtful planning. Kilpatrick et al. (2001) also warn that one of the critical resources was time: “[i]f teachers are going to engage in inquiry, they need repeated opportunities to try out ideas and approaches with their students” (p. 399).

Nonetheless, Gordon (2009) has shown that more time spent on instructional sequences and activities promote the development of concepts and skills in mathematics.

6.4.2.2 Enabling Factors

As discussed earlier, Applet 3.1 was downloaded from the online resources, <https://www.geogebra.org/m/gV9GRCRN>, developed by Tim Bresenseki. The vibrant and active *GeoGebra* community provided an anchor for the teachers to seek support from the users of the software for teaching and learning purposes. The ability of the software to display a certain object – straight lines in this case – on satisfying a given condition, proved to be useful in Applet 3.1. Such capabilities of the software were exposed to the GLIP teachers, owing to the user community that supports the teachers. Hohenwarter and Lavicza (2007) suggest that “there is an extensive self-supporting user community that shares free interactive teaching materials on the GeoGebra Wiki website and supports fellow users through the user forum” (p. 51).

Researchers (Berger, 2011; Kilpatrick et al., 2001; Schäfer, 2016) have written about the value of technology to move between different representations of mathematical ideas, such as between algebra and graphs, that enhance conceptual understanding. Thus, teaching for developing mathematical proficiency is all about using **multiple representations**, i.e. looking at mathematical concepts from different perspectives. Furthermore, Berger (2011) underscores different pedagogical opportunities offered by multiple views of the same mathematical object in a DGS. Participant teachers tapped into this ability of *GeoGebra* and engaged learners in moving between different representations of a linear equation, that allowed the learners to see the same mathematical object in different ways, illuminating the properties of the straight line.

A static representation of a straight line is a snapshot of a certain instance of a gradient and intercept. In contrast, the **dynamic representation** of lines provides opportunities to see how the

parameters change (or do not change) by dragging. Paul explained this affordance of the software to address this perspective of teaching and learning: *“In chalkboard, I may draw a line or two, but on the computer, I can move the lines and the equations changed. They can now see what is happening and why the gradients have a rule for parallel lines and perpendicular lines”* (lines 68-70 RIP3). In his explanation, Paul identified an invaluable constituent of the software: the dynamic representation of objects such as straight lines, that permits one to observe the effects of the changing parameters of a straight line. George also agreed: *“I mean the computer makes it easier, drawing lines instantly. It would have been a tedious task in other classes where we have to calculate and draw different lines, and there would little time to reflect on important things”* (lines 38-40 RI3G, my original transcript).

Manually sketching different straight-line graphs may be cumbersome using algebraic computations, and hence, prone to mistakes. The use of *GeoGebra* eliminated the need for tedious hand sketching of graphs of different linear equations. The software was used in the lessons by the participant teachers to generate graphs of functions, allowing them to focus on the mathematical concepts. Agreeing Berger (2011) highlights, “this [dynamic representation] may give insight into invariant or variant properties of families of functions” (p. 114). The use of *GeoGebra* to represent linear equations graphically and dynamically potentially affords new ways of teaching and learning about straight-line properties. Related to this, Hoyles (2005) underscores how the constructivist ideas of using virtual manipulatives (applets) in classrooms can provide insight into mathematical relationships and thus form a crucial part of internalising knowledge.

I concur with Alsina & Nelsen (2006), who inform us that the interactive and dynamic visualisation capabilities of technological devices may foster rich constructions and develop visual thinking. Thus, *GeoGebra* stands out as a fundamental pedagogical tool that facilitates a mathematics teacher’s classroom practice.

6.5 CONCLUSION

This chapter dealt with the data presentation and gave an in-depth analysis of participant teachers’ interactions with the applets on equations of parallel and perpendicular lines, and insights gleaned from the reflective interviews with the teachers. In general, a detailed analysis of the video recordings indicated that there were similarities and differences between the participant teachers’ pedagogical practices and teaching proficiency. Furthermore, as the analysis unfolded, it showed

that *GeoGebra*, as a visualisation tool, enabled teaching and learning. Using the inherent visualisation capabilities of a DGS in the classroom is an effective strategy to enhance mathematical teaching for understanding concepts.

In the following concluding chapter, I mention certain implications and recommendations of using DGS in mathematics classrooms, as I summarise and reflect on the complete GLIP cycles.

CHAPTER SEVEN

CONCLUSION AND IMPLICATIONS

7.1 INTRODUCTION

This chapter aims to summarise the key findings gleaned from an action research process involving three cycles of teaching various Grade 11 topics in mathematics. I explore the implications of this research, the significance of the research, the new knowledge generated, and the limitations of this study. I also make recommendations that are relevant to the use of visualisation and technology in a mathematics education context and identify some areas for further research.

Three teachers from two schools in the Mthatha district took part in this action research project and implemented co-developed applets in their classrooms. In this context, my research aimed to answer the following two questions:

- 1) How can Dynamic Geometry Software such as *GeoGebra* be used as a visualisation tool to teach Grade 11 Mathematics?
- 2) What enabling and constraining factors do Grade 11 teachers encounter when using *GeoGebra* as a visualisation tool to teach Grade 11 Mathematics?

These questions demanded different conceptual and theoretical deliberations, considerations and understandings of mathematics education research. In the literature review (Chapter Two), I worked with following concepts fundamental to my research project, namely: i) visualisation as an overarching concept; ii) technology with DGS (*GeoGebra*) as a pedagogical and visualisation tool; iii) mathematical teaching proficiency as an analytical framework; and iv) constructivism, as a theoretical underpinning. Figure 7.1 portrays these conceptual elements that framed my research project.

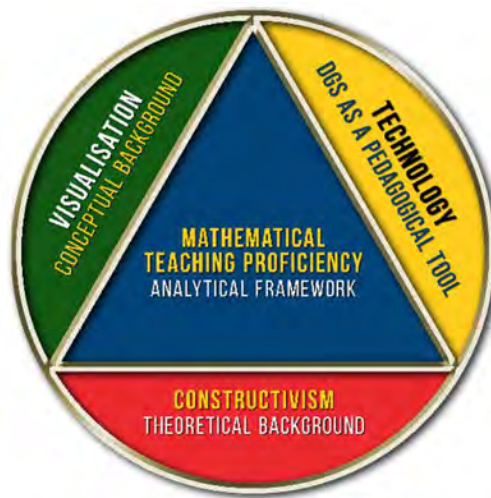


Figure 7.1: The conceptual framework of my research

7.2 REFLECTIONS ON THE KEY FINDINGS

Below is a synopsis of the findings relative to the research questions garnered from three cycles of technological integration in the selected mathematics classrooms.

7.2.1 Synopsis of Findings Related to Research Question 1

There were several features of visualisation that had a practical bearing on the teaching of mathematics. To summarise the discussion, I have grouped them into five different categories.

Conceptual Connections

An aspect of teaching that promotes the effective use of visualisation in mathematics classrooms is ‘making connections’ (Presmeg, 2006). We often employ visualisation when we make connections between mathematical topics. Connections between different representations play a crucial role in teaching and learning mathematics with understanding (Hiebert & Carpenter, 1992).

While discussing theorems in circle geometry, the participating teachers made use of their developed applets and built their lessons by making connections with different aspects of mathematics. In some instances, the participants connected the *semi-circle theorem* to the mathematical properties of a right-angled triangle. In other instance, the teacher combined different properties of triangles to justify the *angle at the centre theorem*. Furthermore, the teachers’ instructions suggested how they linked the *opposite angles of cyclic quadrilateral theorem* with

the *angle at the centre theorem*. Importantly, by utilising the dynamic visualisation power of technology, the teachers provided opportunities for learners to make connections and experience the whole mathematical structure of the *angle at the centre theorem*.

While introducing parallel and perpendicular lines on a cartesian plane, for example, George prompted learners to recollect geometrical facts about these lines. He brought in relationships between different ideas making use of his learners' prior knowledge. In some instances, as the teachers engaged with applets on straight lines, they also encouraged their learners to link the concept of rise over run with the sides of a right-angled triangle. Additionally, the right-angled triangle enabled the participant teachers to associate the concept of the gradient with the trigonometric ratio, another facet of mathematics. The teachers exploited their applets to extract the structure of linear equations and guided their learners to similar mathematical structures in other areas of mathematics, such as functions and calculus.

These episodes of making connections with different topics in mathematics characterise Presmeg's (2006) practices of visualisation. "The visual teachers constantly made connections between the subject matter and other areas of thought, such as other sections of the syllabus, and above all, the real world" (p. 210). Visualisation in mathematics enables the teacher to discover connections that a new concept has, with other existing concepts. The teachers all made a conscious effort in their classrooms to make connections across numerous mathematical topics.

Absence of real-life connections (indicator CR)

Presmeg (2006; 1992) believes that visualisation in teaching enables learners to start from a problem that they encounter in everyday life and solve it with the help of a mathematical situation. There was however a lack of evidence in the observed lessons that the participating teachers made connections with everyday or real-life situations. For instance, while dealing with the slope of a line, the use of stairs at school (or any other building) as a realistic example would have helped to understand slope, i.e. the rise of a step, in relation to the run. This real-life model would have reinforced the understanding of gradient, calculated by dividing the vertical height by the horizontal distance. Similarly, while dealing with circle geometry, a realistic situation in cinematography where the 'viewing angle' describes the angle of a scene, say a building being captured, could have been exploited. A cinematographer can capture the same building on film from different viewpoints, with the viewing angle unaltered. This is possible when the camera is

placed on the circumference of the circle and moved along the circular arc. The teachers could have connected this situation to the *angles subtended by the same arc on the circumference* and facilitated further discussion on its related theorems. Visualisation requires the coherent handling of internal image and external representation corresponding to the mathematical structure.

The analysis of lessons on integrating *GeoGebra* illuminated how teachers did harness the visualisation capabilities inherent in the software, but not for handling everyday life situations.

Geometric Constructions

Duval (2013) favours the teaching of geometry through the construction of figures using computer tools or conventional tools that promote visualisation in geometry. Visualisation based on the construction of shapes and figures helps to identify and understand specific geometric properties. The teachers highlighted how the software supported constructions of geometric shapes and figures. A sound understanding of geometric principles becomes necessary to use the tools in the software, and the use of tools reinforces the related geometric properties. If the learners are however not technically proficient, as was the case in my study, many learners would not have been bogged down by the intricacies of the software tools. Nonetheless, with the underlying belief that learners visualise the geometric concepts as they construct, one of my teachers – Antony – favoured the constructions of geometric shapes and figures despite experiencing these difficulties.

Visual Process

Research (Arcavi, 2003; Avgerinou & Pettersson, 2011) indicates that visualisation is a thinking process of creation, interpretation and reflection upon images.

Intriguingly, Duval (1999) discloses that the construction of graphs from a given equation is not enough to learn visualisation in mathematics. Visualisation in sketching straight lines enables the estimation of the direction of the line based on salient characteristics such as gradient. Exploring with the applets, the teachers leveraged technology to support their instructions to see how the change in the sign of the gradient affected the direction of the graph of a linear equation. In an interaction with another applet, George, for example, encouraged his learners to adopt the gradient-intercept approach in sketching the straight line for a given equation (here he relied on the visual elements of vertical and horizontal movement). From a pedagogical standpoint, the teachers' strategy illustrates that instructional routine is also about teaching ways of 'visual analysis' and

‘visual reasoning’ (Vermeersch & Vandenbroucke, 2015). From the perspective of constructivism, Moyer (2001) argues that teachers play a crucial role in creating interactive mathematics environments that provide learners with visual representations which enhance thinking.

In sum, there was strong evidence that the participating teachers promoted and facilitated visual thinking in their lessons using our co-developed applets. Research studies (Arcavi, 2003; Duval, 2013; Presmeg, 1986b) emphasise that teachers should employ visually oriented instructions in the classroom. The participant teachers designed the lessons and the applets that guided and allowed the learners to experience and develop their visual thoughts. The efforts put into this kind of visual teaching and learning reinforces the idea that mathematics is more than simply doing calculations, but also involves thinking and understanding.

Multiple Representations

The use of multiple representations in classrooms is another strategy which capitalises on the visualisation processes. Researchers (Presmeg, 2006); Schäfer, 2016)) inform us that multiple representations of mathematical concepts is an integral part of a visualisation strategy of teaching and learning mathematics. My participating teachers collaboratively developed applets that linked symbolic and visual representations and guided their learners to draw attention to the relationships between two and more representations. This aspect of visualisation corroborates with one of the powerful potentials of DGS, which Drijvers (2012) identifies as the “dynamic phenomena that invite mathematical reasoning, in many cases concerning the relationships between multiple representations of the same mathematical object” (p. 489). I discuss more on the multi-representational potential of *GeoGebra* in the following section as I summarise the key enabling factors, in section 7.2.2.2 (p. 218) below.

7.2.2 Synopsis of Findings Relative to Research Question 2

7.2.2.1 Key disabling factors

Software issues

In displaying non-integer numerical values, *GeoGebra* rounds off a number to the specified degree of precision. The rounding off in *GeoGebra* can lead to an inadequate precision of display of measured values which can lead to unconvincing results e.g. $\frac{1}{2} \times 123,23 = 61,62$ and not the expected value of 61,615. Thus it was a challenge for the teachers when rounding off values (in

the applet for exploring circle geometry theorems), which then obscured the actual relationships between the angle at the centre and the angle at the circumference.

When a point on the circumference, subtended from an arc, was dragged to the other side of the arc, the software then measured its reflex angle. In such situations, the applet did not reveal the actual relationship between the angles under investigation. However, the teachers overcame the above two constraints with their ready-made applets and provided learners with these applets to further explore on their computers.

Again, in the nature of roots applet, an irrational number was rounded off to a decimal number. Therefore, it was difficult to distinguish between a rational and an irrational number from the x –intercept of the graph. Thus, in the discussion on the nature of roots of a quadratic equation, the applet did not represent an irrational number visually, and the issue remained unresolved.

Lack of technical expertise

In solving inequalities, *GeoGebra*, by default, highlights the y –values of the graph. When discussing solutions to inequalities during the lesson, the software did not highlight the solutions on the x –axis. We noticed this confusion while we were reflecting in one of our GLIP meetings.

Similarly, one of the pre-designed applets on the topic of straight lines, did not function properly during the lesson. Two teachers designed the said applet on their personal laptops and shared them with the learners; however, the computers in the school laboratory were not updated with the latest *GeoGebra* version. *GeoGebra* is generally a user-friendly software package, but there are subtle technicalities that posed challenges to the teachers during their teaching.

Time Management

All the participating teachers in this action research project struggled with time management and found it difficult to cover the intended curriculum topics in their lessons. There are two reasons for this apparent mismanagement of time:

Firstly, the teachers encountered challenges owing to the low level of the learners' proficiency with computers. During the GLIP meetings and reflective interviews, the teachers raised their concerns that learners struggled to control the figures in the applets. The learners also faced difficulties in keying in mathematical expressions. Although the teachers had a plan to divide the activity for each applet over the stipulated time in a lesson, they ended up spending too much time

explaining basic computer literacy technicalities before they were able to teach. Berger (2011) raises similar concerns that technical difficulties may exacerbate the use of technology in the learning of mathematics.

Secondly, in both schools, the computer laboratory was shared with the computer department, and hence there was a conflict of resources. On the timetable, mathematics classes were not allocated computer laboratory time, and therefore teachers had to negotiate with the computer department for using their computer resources. Consequently, the teachers squeezed as many concepts as possible into one session when they were using the computer laboratory, They invariably ended up not completing the intended teaching. For instance, Antony could not deal with the geometric proofs that he had planned for discussion in circle geometry. In another instance, Paul cut short the lesson on inequalities and the nature of roots when he realised that the learners might not keep pace with the curriculum. In a third instance, on the topic of straight lines, Paul decided to split the concepts (he extended a lesson to a Saturday) while George extended the lesson beyond the stipulated time by almost 30 minutes and still did not complete the planned applets. It is thus important that when teachers are trained in the use of computer technology, that time management issues form an integral aspect of this training. It is also important that both teachers and learners have developed basic computer literacy skills when embarking on operating in a technology rich environment. I make concrete suggestions in this regard in the sections below.

7.2.2.2 Key enabling factors

Multiple Representations

Researchers (e.g. Drijvers, 2012; Jones et al., 2009) agree on the potential of DGS that allows users to create dynamically connected multiple representations of mathematical problems. With easy-to-use tools, the teachers designed applets that enabled their learners to see the equations and their graphs on adjacent windows of computer screens, thus allowing them to switch flexibly between algebraic notations and graphical representations. In the applet designed for discussing the nature of roots, various representations were automatically linked. With the slider tool, the teachers dynamically manipulated these representations. Thus, they built a coherent network of mathematical ideas – equations, graphs, roots, x – intercepts, and properties of numbers. Evidently, the benefit of dynamic multiple representations is in making algebraic objects such as expressions and equations more real and meaningful in teaching and learning.

Another instance of the developed applets that exploited multiple representations, was the applet designed to make sense of gradient-tangent relation of a straight line. The teachers saw opportunities to focus on developing mathematical ideas through manipulating linear graphs in multiple forms of representation – symbolic, graphical and numerical. The mathematical ideas that the teachers engaged in with their learners included acute angles and positive gradients, obtuse angles and negative gradients, and gradients and angles of vertical and horizontal lines. Participant teachers made use of this ability of the software by engaging learners in moving between different representations of a linear equation, that allowed the learners to see the same mathematical object in different ways, illuminating geometric and algebraic insights.

A key potential of *GeoGebra* is its capacity to assimilate geometric properties into an algebraic graph. The findings underscored different pedagogical opportunities offered by multiple views of the same mathematical object in the software. Ruthven (2003) values the use of technology to move between different representations, and emphasises “ [a] key potential of DGS is to construct geometric tools to support the analysis of function graphs” (p. 31). Very much in line with this, Heid and Blume (2008) conclude that technological tools that enable multiple representations “contribute to the improvement of visual reasoning skills” (p. 72).

Dynamic Representation

It is generally agreed that dragging is a key feature of any DGS (Hölzl, 2001; Hoyles & Noss, 2003). Dragging allows a continuous reconstruction of figures or graphs on the computer screen by direct manipulation with the mouse or with the ‘slider’ tool. Sliders allow users to manipulate the values of variables or parameters such as angles.

The manipulative ease of *GeoGebra* enabled mathematical exploration of circle geometry theorems in this study. The dynamic variation of dragging the points on the circumference made it easier to appreciate the mathematical relationship between the angles under investigation. The teachers designed their lessons, exploiting the dynamic representations inherent in *GeoGebra*, guiding learners to generate multiple examples. Even though *GeoGebra* is an efficient tool to produce static figures, it is a convenient tool to generate dynamic figures and their measurements that may help teachers to highlight variant and invariant properties under dragging. Laborde (2001, p. 293) agrees, underscoring the role of drag mode in DGS as an “amplifier where it was easier to

observe the variant and invariant properties during the deformation of the diagram by the drag mode than in a static paper-and-pencil diagram.”

Again, the applets on straight lines afforded dynamic representations of lines and their equations, providing opportunities to explore what happens when a certain parameter is changed by the dragging of lines. In one instance, the teachers exploited the potential of the software in exploring by scaffolding their learners to establish the gradient properties of parallel and perpendicular lines. Dragging of the line graphs changed the equations of the straight lines dynamically. Intriguingly, in another applet on the nature of roots, it was the opposite that happened. By changing a parameter of an equation with the help of sliders, teachers encouraged learners to observe the effects of the changes in the features of its graph. These applets illuminated the properties of an equation and its graph that are variant, and those that are not. I argue that DGS provides interesting tools that assist teaching and learning in a disciplined framework of mathematical reasoning.

By their very nature, textbooks make exclusive use of static diagrams and images, which do not support the dynamic nature of visualisation (Guzmán, 2002) that was enacted by my participating teachers using the co-developed applets.

Interactive facility

Any DGS interface has the potential for users to interact with the software. The use of *GeoGebra* allowed interactivity and provided immediate responses. The participating teachers all encouraged and directed learners to make explicit constructions or change certain parameters that could help them in making conjectures or explore certain mathematical ideas. My participants, in agreement with research by Deaney et al. (2006), also recognised that the “interactivity and the immediate visual feedback afforded by the use of technology” (p. 475) plays a significant role in contributing towards enhancing learners’ mathematical proficiency.

The practice applets allowed the learners to solve extra problems and the teachers reported that they were enthusiastic about attempting those questions. The applets validated the learners’ input, providing feedback messages accordingly and the learners were excited to share their methods and answers. This exchange provided opportunities for the learners to reflect on and learn from their mistakes, thus enhancing their mathematical proficiency. The visual and immediate response from the software allowed the learners to progress at their own pace and thus supported learners in their learning. In this way, teachers were able to address those learners who lacked basic skills.

Furthermore, this facility allowed the learners to test their knowledge with little guidance from a teacher.

On reflection, these applets however received only minimum input from the learners in the form of numbers, checkboxes or dragging. Often the learners simply engaged in trial and error methods and guessing the answer without making proper mathematical procedures or geometrical reasons. Teachers were aware of this pitfall.

User community

When working with software, it is natural to encounter difficulties unique to that software. An extensive and vibrant user community has developed around *GeoGebra*, which shares free interactive teaching and learning resource materials on the *GeoGebra* website, <http://www.geogebra.org>. Volunteers from this user community support fellow users through the user forum. Bu et al. (2012) observe that participants of their *GeoGebra* community, like my community of teachers, not only actively invent and experiment with new ways of teaching mathematics but are themselves learning or relearning mathematics through their applets.

When designing applets on straight lines, we adapted an applet from the user community. In another applet, we were unable to restrict the movement of points to have only integer values. A user suggested to use the tool ‘snap to grid’. While reflecting on the inequalities applet, we looked for applets in the user community forum. It was a learning curve for all of us when, on selecting ‘show on x –axis’ checkbox in ‘object properties’, the graphical solution of inequalities on the x –axis would be represented. We realised that we had missed a rather easy-to-use tool.

Dynamism

The term ‘dynamic’ seems to have permeated throughout the analysis of the teachers’ lessons. The teachers interacted with the applets to engage learners to make various mathematical properties, geometrical or algebraic, apprehensible. When a dynamic figure or graph was dragged, the teachers encouraged their learners to observe invariant or variant properties. The analysis shows that dynamic representation facilitates at least three things. Firstly, it enables the user to focus attention on the continuous dynamic variation of figures or graphs; secondly, it enables the user to generate discrete examples more efficiently than manual constructions; and thirdly, it prompts learner exploration and the formulation of conjectures. These three abilities of the software are

however, interdependent and positioned on a continuum. It must be stressed, however, that the ability of the software to prompt exploration is often dependent on the teacher's guidance and support.

7.3 SIGNIFICANCE AND CONTRIBUTION TO NEW KNOWLEDGE

The significance of this research contributes to the ongoing deliberations of visualisation and utilisation of technological resources, particularly through the empowerment of a community of teachers.

In reviewing pertinent literature relating to the integration of ICT tools in classrooms, there was little evidence of empirical research that had been carried out in previously disadvantaged communities such as in Mthatha, Eastern Cape. Thus, the knowledge gap that this research addresses, lies in its context of understanding the building of technological resources (applets) aligned with the South African curriculum content in areas where resources such as computers, are still novel and unreliable.

7.3.1 Utilising computer resources

Researchers (Hennessy et al., 2005; Stols et al., 2015) indicate that despite the numerous benefits of using ICT, teachers in general, do not make use of these resources as a teaching tool. Also, access alone to technology is insufficient for the integration of technology into classrooms. This action research project demonstrated that through a community of teachers (GLIP), it is however possible to make effective use of ICT in the available computer laboratories with carefully planned support mechanisms from teachers. It is thus crucial for teachers and school management to be aware of the resources and possibilities at their disposal for the successful integration of ICT resources into their teaching programme.

7.3.2 Pedagogical Beliefs

Examining different teaching approaches using technology, Kendal and Stacey (2001) observe that teachers typically make their own pedagogical choices about how to incorporate technology into their classrooms. The action research process of my study enabled my participating teachers to develop their own pedagogy in a computer environment. The findings showed that one teacher, for example, was keen on pre-producing applets on his own, whereas another teacher made use of the shared experiences with the software. Some of the teachers totally embraced technology as it

enabled them to increase their repertoire of different ways of teaching rules and procedures, while other teachers found the capabilities of technology superior for developing conceptual understanding of mathematical ideas. From my findings, it is thus significant for practising teachers to note, that the effectiveness of mathematics software in classrooms largely depends on the teachers' pedagogical motivation and beliefs, and their willingness to embrace this technology.

7.3.3 Significance in Mathematics Education

The insights gleaned from the analysis suggest that DGS has the potential to change the dynamics of a mathematics classroom. I argue that my research findings on teaching with DGS tools as revealed in the analysis, can inform all the stakeholders in mathematics education in South Africa about the enabling and constraining factors of implementing DGS in mathematics classrooms. The integration of technology requires proper planning, support, resources, and training for the teaching of mathematics.

My review of relevant research on visualisation indicates that the power of visualisation in interactive computer technology is pedagogically highly recommendable. The findings of my research confirm this and thus encourages teaching practice that incorporates the use of visualisation. I hope that teachers and researchers who read this thesis may gain insight into how *GeoGebra* can be used as a visualisation tool for the teaching and learning of mathematics. Visualisation is inherent in *GeoGebra*; nonetheless, it is incumbent on the teachers to unpack the mathematical ideas and make abstract ideas visible to learners. The dynamic aspects of technology, through the use of applets, in my case, stand out as key enabling factors for enhancing the visualisation of mathematical concepts. Hence, the integration of applets significantly reinforces the value of the process of visualisation in mathematics education.

7.3.4 Beneficiaries of the research

The participants and the GLIP members were direct beneficiaries of this action research project. Although they were aware of technological tools prior to the GLIP, they confined its use to administrative purposes only. This action research provided an opportunity for them to embrace technological tools and resources for their own teaching and learning purposes. Further beneficiaries were the two schools whose laboratories and computers we used. It is through our intervention that these resources were put to the use that they were originally intended for.

7.3.5 Enhancing teacher professional development through GLIP

My teachers were initially reluctant to use digital tools for one reason or another. This reluctance was also observed by Cuban et al. (2001) and Stols et al. (2015) in their work. GLIP meetings and discussions, however, motivated and inspired my participants to use *GeoGebra* in their classrooms. Collaboration among teachers empowered them to master the technical aspects of the software. GLIP offered them an opportunity to interact and share ideas, which in turn supported them to improve their teaching strategies in implementing applets into their mathematics lessons. Thus, this action research significantly contributed to the growth of a community of proactive teachers in Mthatha collaborating in designing, planning, and implementing ICT resources (as applets) for teaching and learning mathematics. The knowledge gleaned from the findings suggests that for the effective integration of technological tools in mathematics classrooms, collaboration among teachers can be very beneficial and in some instances is very necessary.

Mavani et al. (2018), reporting on the GLIP model of teacher development, acknowledge the collaborative teamwork among the teachers, that bridged the gap between having access to technology and adapting it for effective use in classrooms. The findings of my research concur with the perspective of ‘teacher empowering teacher’ as a way forward for teacher education on technology. The GLIP model testifies that a pragmatic approach, i.e. a sensible and realistic approach in harnessing technology for teaching and learning mathematics, is highly desirable. I hope that the research will be of interest to important stakeholders in mathematics education, teachers and curriculum planners on integrating DGS into mathematics classrooms.

7.3.6 Four perspectives of the research

Of further significance, is that this study provides insights into interwoven issues relating to four key educational perspectives: visualisation, ICT resources, mathematical teaching proficiency and the pedagogical significance of constructivism. Following on Vygotsky’s ideas on the metaphor of scaffolding, Jaworski (1994) writes about scaffolding “in terms of a teacher's offering of strategies for thinking and learning, rather than for grasping a particular skill or concept” (p. 31). This study found that the empowerment of teachers with appropriate ICT tools and visualisation strategies, created a pedagogical environment that allowed learners to construct their own knowledge. It is thus a strength of this study, that through using DGS, the selected teachers

developed their own pedagogies that enabled them to scaffold the learning process by offering strategies for thinking and learning in a dynamic ICT teaching and learning environment.

7.4 LIMITATIONS OF THE STUDY

Despite the initial intention and purpose not to generalise, it can be argued that the limitation of the study is its small sample size, and therefore findings cannot be generalised. The study occurred only in one town, two schools and involved only three teachers. The small sample size, however, enabled a unique and in-depth opportunity to understand and interpret the subtleties of teaching using technological tools.

The duration of data collection was also a limitation of the research. The small number of classrooms, relatively few lessons and the limited period of twelve months may have impacted the findings of the research. In the analysis, I considered only three cycles of GLIP due to the vast amount of data gathered from the four cycles of GLIP, that involved nine classroom observations.

Nevertheless, one of the strengths of the GLIP model of a community of practice approach, is that it can be replicated in a wide range of classroom settings.

7.5 RECOMMENDATIONS

This action research provided insightful information on integrating DGS tools in mathematics classrooms that could be useful to teachers and curriculum planners.

Against the background of my research findings, I believe that for teachers, strategic and proper planning is required for integrating DGS as an effective teaching and learning resource in the mathematics classroom.

7.5.1 Curriculum planning for schools

The research findings relative to the learners' technical struggles with the computers suggest that the use of ICT in mathematics education be introduced in earlier grades. In his research, my fellow researcher, whose specific research focus was on learners in a DGS environment, made similar recommendations. He recommended:

In general, learners struggled to control graphs and figures in the screen. It is important to note that computer use per se does not necessarily improve learners' knowledge. It needs to be supported by guidance of an expert or a knowledgeable person. As Klien, Nir-Gal and Darom (1998) conclude that young children's interaction with the computers alone does not

necessarily benefit children's development of higher thinking processes (Mavani, 2019, p. 253).

It is thus incumbent on curriculum planners to ensure that basic computers skills are mastered in the primary grades if learners are to make full use of complex and sophisticated educational software in secondary schools.

7.5.2 Curriculum Planning for Teacher Education

In order for teachers to integrate technological tools into their everyday teaching, they need to be effectively trained on the use of these tools and be able to critically evaluate the efficacy of these tools. I thus recommend that tertiary teacher training institutions make this training mandatory in their teacher education courses.

Mishra and Koehler (2006) note that “teachers need to know not just the subject matter they teach but also the manner in which the subject matter can be changed by the application of technology” (p. 1028). My participant teachers believed that they faced impediments owing to their own lack of knowledge of teaching practices using technological tools. Technological environments require that teachers adapt their instruction and teaching methods. Adequate training and collegial support may develop teachers' proficiency to integrate technology into their teaching practices. We need a clear curriculum for a teacher training programme, with innovative pedagogies fostering technological use in classrooms. I see a case for further research on the integration of technology into the curriculum of teacher training programmes. The findings from such empirical studies would further open up the complexities and subtleties of a technology-based pedagogy.

7.5.3 Professional development programmes

Time and again, research has called for more in-service computer training courses in teacher development programmes. Ongoing teacher development programmes are required to continue to develop teachers' skills in integrating technological tools in mathematics classrooms. Based on my empirical research results about the use of DGS tools, I advocate a collaborative model of teacher empowerment, very much in line with Gueudet and Trouche's (2011) ‘teacher empowering teacher’ model for the effective use of technology for mathematics teaching.

7.5.4 Timetable

As indicated in my findings, time management is often an underestimated element of an ICT pedagogy. Assude (2005) argues that “[t]he question of time management in integrating ICT is a key issue” (p. 184). As I have discussed in Section 7.2.2.1 (p. 217) on the issue of time management, I make recommendations on improving the teaching time of a mathematics class. Ideally, a dedicated computer classroom for the department of mathematics should be set up. This arrangement would aid mathematics teachers to engage their learners across grades in the computer classroom during their lessons without having a conflict with the other subjects, and spending too much time in setting up the guest laboratory for the specific purpose of the mathematics lesson. If this is not possible, then regular dedicated mathematics lessons need to be factored into the timetabling of the existing computer laboratory.

7.6 RECOMMENDATIONS FOR FURTHER RESEARCH

A rewarding feature of this research process has been its generation of avenues for further research which would include other mathematical topics and grades. My research model could well be replicated with other ICT clusters of teachers. This could apply to other school subjects as well. A further research opportunity that has arisen out of my research project is an exploration of how other collaborative in-service teacher models have contributed to effective pedagogies, also in other subjects.

Our existing GLIP teachers continue to interact with each other, sharing applets and pedagogical approaches that worked in their classes. We are exploring possibilities to provide online applets to learners. Remote or online teaching and learning approaches are becoming a feasible and necessary alternative to face-to-face classroom pedagogy, particularly in the context of the current COVID-19 pandemic and social distancing paradigm. This has opened up a huge research arena that could yield far-reaching findings as we are globally rethinking the way we teach and reach out to learners. The Java-based applets that we used could easily be migrated to any online platform. I recognise that online or remote teaching and learning is more than uploading educational resources on a cloud or a website in a digital format. Research into appropriate pedagogical strategies that are appropriate in this regard, would thus be desirable.

7.7 WAY FORWARD

Our GLIP program, with its associated research activities, has gained traction in the teaching community in the Mthatha region. The DBE district office in Mthatha involved us in empowering teachers with DGS tools in 2018-2019. In this teacher development programme, we trained 40 teachers for 20 hours from different schools in the district.

Of late, we are conducting an exclusive training for twelve mathematics teachers in a school for the implementation of DGS in the classroom. In this programme, we are endeavouring to use DGS tools across the mathematics curriculum in secondary schools, i.e. from Grade 8 to Grade 12.

An upcoming project is planned for training BEd students from Walter Sisulu University (WSU) Mthatha, on using technology tools in mathematics classrooms. The agenda is to design lessons incorporating DGS tools into their micro-teaching sessions.

7.8 CONCLUDING REMARKS

Most of the applets that we have generated are accessible on www.visual-maths.com and are freely downloadable materials for anybody to use in their classrooms

Antony and George are still in schools, using *GeoGebra* applets across the grades. Paul was promoted to a higher post in the department; however, he continues to encourage teachers to use *GeoGebra* applets in classrooms. Abraham negotiated with his school management to share the computer laboratory for his Grade 11 and Grade 12 mathematics lessons.

This research has afforded me the most rewarding experience in understanding pedagogical issues relating to mathematics teaching in general and integrating technology into teaching in particular. I have become acutely aware of the diverse visualisation processes made possible by the dynamic aspects of DGS. This dynamic visualisation has significantly contributed to my own pedagogical discourse and practice as an enthusiastic and committed mathematics teacher and researcher.



I tried to write in such way that a learner could always see the inner foundation of things he is learning, that he could find the source of the discovery and, consequently, understand everything as if he invented that by himself - G.W. Leibnitz

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Annexure I – Letter of permission to undertake research in secondary schools in Mthatha District



STRATEGIC PLANNING POLICY RESEARCH AND SECRETARIAT SERVICES

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Date: 26 January 2017

Mrs. Beena Deepak Mavani

P.O. Box 205

Mthatha

5099

Dear Mrs. Deepak Mavani

PERMISSION TO UNDERTAKE A DOCTORAL THESIS: A CRITICAL ANALYSIS OF HOW THE POTENTIAL OF DYNAMIC GEOMETRY SOFTWARE AS A VISUALISATION TOOL MAY ENHANCE THE TEACHING OF MATHEMATICS

1. Thank you for your application to conduct research.
2. Your application to conduct the abovementioned research in four Secondary Schools under the jurisdiction of the Mthatha District of the Eastern Cape Department of Education (ECDoE) is hereby approved based on the following conditions:
 - a. there will be no financial implications for the Department;
 - b. institutions and respondents must not be identifiable in any way from the results of the investigation;
 - c. you present a copy of the written approval letter of the Eastern Cape Department of Education (ECDoE) to the Cluster and District Directors before any research is undertaken at any institutions within that particular district;
 - d. you will make all the arrangements concerning your research;
 - e. the research may not be conducted during official contact time;
 - f. should you wish to extend the period of research after approval has been granted, an application to do this must be directed to Chief Director: Strategic Management Monitoring and Evaluation;




- g. your research will be limited to those institutions for which approval has been granted, should changes be effected written permission must be obtained from the Chief Director: Strategic Management Monitoring and Evaluation;
 - h. you present the Department with a copy of your final paper/report/dissertation/thesis free of charge in hard copy and electronic format. This must be accompanied by a separate synopsis (maximum 2 – 3 typed pages) of the most important findings and recommendations if it does not already contain a synopsis.
 - i. you present the findings to the Research Committee and/or Senior Management of the Department when and/or where necessary.
 - j. you are requested to provide the above to the Chief Director: Strategic Management Monitoring and Evaluation upon completion of your research.
 - k. you comply with all the requirements as completed in the Terms and Conditions to conduct Research in the ECDoE document duly completed by you.
 - l. you comply with your ethical undertaking (commitment form).
 - m. You submit on a six monthly basis, from the date of permission of the research, concise reports to the Chief Director: Strategic Management Monitoring and Evaluation.
3. The Department reserves a right to withdraw the permission should there not be compliance to the approval letter and contract signed in the Terms and Conditions to conduct Research in the ECDoE.
 4. The Department will publish the completed Research on its website.
 5. The Department wishes you well in your undertaking. You can contact the Director, Ms. NY Kanjana on the numbers indicated in the letterhead or email nykanjana@live.co.za should you need any assistance.



NY KANJANA
DIRECTOR: STRATEGIC PLANNING POLICY RESEARCH & SECRETARIAT SERVICES
FOR SUPERINTENDENT-GENERAL: EDUCATION



Annexure I – Continued Approval Letter from School A



DATE: 02 FEB 2017

To,
Beena Deepak Mavani
Rhodes University
Grahamstown

Ref: Yourself : Permission to conduct empirical research study


Dear Ms Beena,

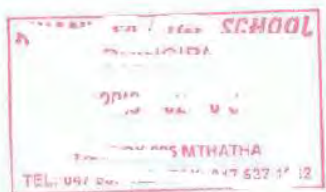
We refer to the above: -

The school management team has approved your request to conduct a research study in our school. Kindly ensure that your research activities do not affect the normal school activities.

You have to follow and adhere to a set of computer lab rules of the school.

Yours faithfully,


Principal



Principal _____ Email: _____

Annexure I – Continued Approval Letter from School B

MRS BEENA MAVANI & MR DEEPAK MAVANI
RHODES UNIVERSITY
GRAHAMSTOWN

DEAR SIR/ MADAM

Re: PERMISSION TO CONDUCT EMPIRICAL RESEARCH STUDY- YOURSELVES

The above matter refers:

The Management of the above named school has considered your request to conduct research at the school. Our understanding is that this will require a teacher and learners as participants. In principle the school has no objection to permit you for this purpose as it will also benefit teachers in the teaching of Mathematics as well as enhance learners' attainment in this subjects.

Wishing you well in your studies.

Yours faithfully

///

EASTERN CA

Annexure II – Letter to Teachers and Participant Consent Form

Beena Mavani
PhD. Candidate, Rhodes University

email: beenasivaram@yahoo.com
Cell : 0840502615

Subject: Information for participation in the PhD research project and request for participation consent

Dear colleague,

1. Introduction

I am a PhD. student in mathematics education in the Faculty of Education at Rhodes University. I humbly request your participation in an exciting research project titled 'integrating technology in mathematics classroom'. The research is a critical analysis of how the potential of technological tools may enhance teaching of mathematics.

2. Background

This study is part of the requirements for the completion of my PhD degree in mathematics education. One of the objectives of the research is to develop the appropriate ICT skills and use it effectively and strategically as both a teaching and learning tool of mathematics. For the purpose of research, we will be using dynamic geometry software, GeoGebra, an open source software.

The purpose of this research study is to:

- analyse how teachers make use of ICT technology-aided visualisation for proficient teaching,
- analyse the advantages and weaknesses of using Dynamic Geometry Software (DGS) in teaching and learning of mathematics,
- to interpret, the pedagogical practices and instructional fluencies when visual aspects of technological tools are employed in a classroom,
- contribute to the growth of a community of practice collaborating in building IT resources (applets) aligned with the South African curriculum content.

More specifically, I am looking at teachers' teaching for mathematical proficiency in relation to their classroom instructional practices using DGS. To accomplish the objectives of the research I need to observe you in practice. The focus will be on classroom teaching using GeoGebra applets in grade 10 and 11 classes. Further, I will video-record your lessons, since this will provide us with a more comprehensive recording of the lessons for a detailed analysis. This PhD study, with a focus on teaching, is paired with another PhD study, that focuses on aspects of learning with GeoGebra applets in mathematics education.

3. Description and Invitation

The study will take place during 2017. The data collection involves meetings among teachers deliberating on how to use DGS. At least three (3) of your lessons will be audio-video recorded as you go about teaching integrating technology. You will be also engaged in a post lesson reflection interview, during which we will reflect and evaluate on each of your lessons in order to gain insights into your lessons and discuss in-depth evidences of your effective instructional practices. The ultimate research objective is not to evaluate your teaching or your compliance to the curriculum as a mathematics teacher, but rather to analyse and discuss enabling and constraining factors that you encountered when using DGS as a teaching resource to teach mathematics. Your participation in this study will not affect your regular teaching in any way as observations are being planned in during normal class times, and interviews will be conducted at conveniently arranged times. The data will be strictly confidential and only you, the researchers, my supervisor and other participating teachers will have access to it.

4. Risks and Benefits

There are no foreseeable risks involved in participating in the study. Participating in this study will give you an opportunity to study and contribute to knowledge and understanding of effective teaching practices, and share your valuable experience with others. The study results will be used for informing and improving mathematics teachers' teaching practices integrating technology, GeoGebra in particular, within the South African education system.

5. Time involvement

There will be a short induction programme on introducing the key concepts of the research study for about thirty minutes. There will be meetings to discuss on how to use the GeoGebra applets. The lesson observations will be conducted during normal class times. I also anticipate that the reflective interviews might last for about an hour, and the timings will be negotiated with you. Due to complexity of the study, your involvement and commitment is very essential. You may also be contacted over phone or email to discuss any matter related to this study.

6. Participants' rights

Participating in this research is absolutely voluntary. That is, your participation is strictly optional and at your personal discretion. Hence, after acquainting yourself with this information, you may decide whether or not to take part in the study and give consent to that effect. You also have the right to withdraw from any part of the study at any time. Your identity will be confidential and pseudonyms or codes will be substituted for the names of the participants and the school.

7. Audio-video recordings

All aspects of the study will be strictly confidential. On completion of the project, all data collected will be archived and securely stored with only research team having access to it. The findings of my study may be communicated to you upon completion of my study. However, should you have any concerns about your participation or the conduct of this research project, please feel free to contact me (see my contact details above) or my research project supervisor Prof Marc Schäfer at M.Schafer@ru.ac.za.

8. Consent

Please complete, sign the attached consent form and return it to me at your earliest convenience. I will be happy to answer any questions or queries that you might have. Hoping for a favourable response!

Yours Sincerely,

Beena Mavani

Appendix: Participant consent form

I,, mathematics teacher of agree to participate in the PhD research project entitled: *Integrating Technology in mathematical classroom*.

In giving my consent I acknowledge that:

1. The procedures required for the project and the time involved have been spelled out to me, and any questions I had about this project have been answered to my satisfaction and expectation.
2. I have read the Information and Participant Consent Forms and have been given the opportunity to discuss the information and my involvement and level of participation in the research project with the researcher and other participating mathematics teachers who will analyse the recorded lesson videos with me.
3. I understand that I can withdraw from the study at any time at no cost without affecting my relationship with the researcher now and/or in the future.
4. I understand that the researcher will observe and record at least three of my lessons using GeoGebra applets, and involve me in post lesson reflective session, and that only the research team, their supervisor and other teachers involved in this study will have access to these data. I agree to:

- I agree to participate in meetings for discussing the pedagogical practices using GeoGebra applets.
- I give consent to observe and audio-video record them my lessons using DGS.
- I agree to be engaged in a reflective interview, after each lesson in order to have an in depth discussion and analysis on my teaching practice.
- I also give consent to audio-video record such interviews conducted, for the purpose of providing an accurate record of the interviews for later analysis and interpretations.

I understand that my involvement is strictly confidential and that no information about me or my school will be used in any way that reveals my identity or that of my school.

Name (participating teacher):

Signed (participating teacher): Date:

PARENT CONSENT – Research Study - Integrating Technology in Mathematics Classroom.

I, Mr/Mrs _____

parent of _____ in grade _____

- 1) do agree / do not agree to allow my child to participate in this project
- 2) do agree / do not agree to allow my child to be video-recorded.
- 3) do agree / do not agree to one-one discussion my child.

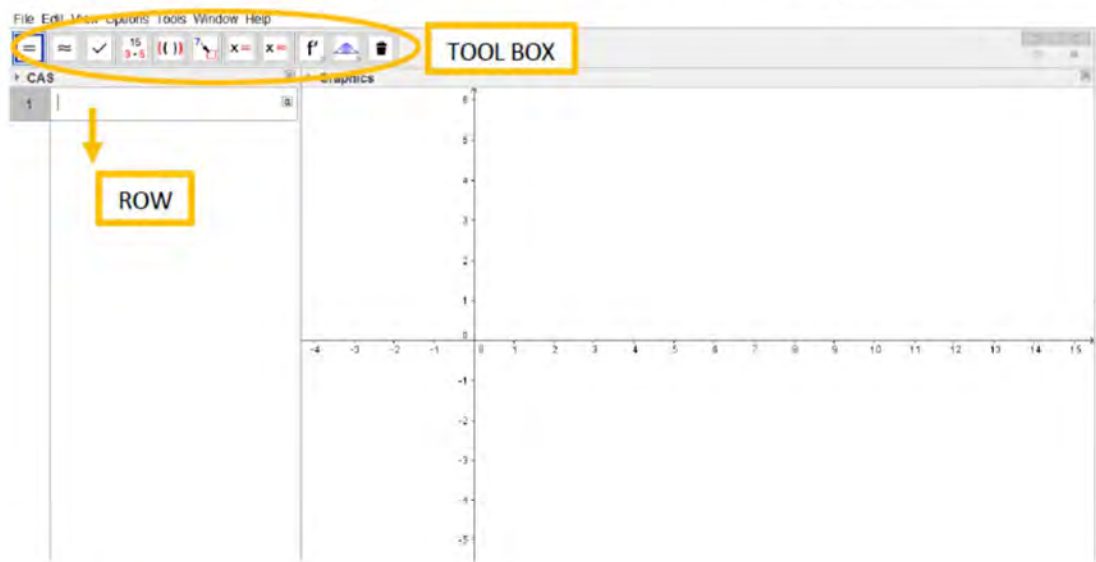
Parent's signature: _____ Date _____

If you have any questions about your rights as a research participant, please contact Rhodes University Ethical Standards Committee: email ethics-committe@ru.ac.za , phone 046-603 8055



Annexure III– Training Interlude – CAS

Introducing CAS

CAS is the acronym for Computer Algebra System. Choose CAS perspective by clicking on File and then click 'CAS', the following screen will appear. A CAS is a software that is used to manipulate mathematical expressions and symbols like manual computations.





TOOLS

	Symbolic Evaluation - It evaluates the expression in the CAS row.
	Solve for x - it solves one or more equations in CAS row.

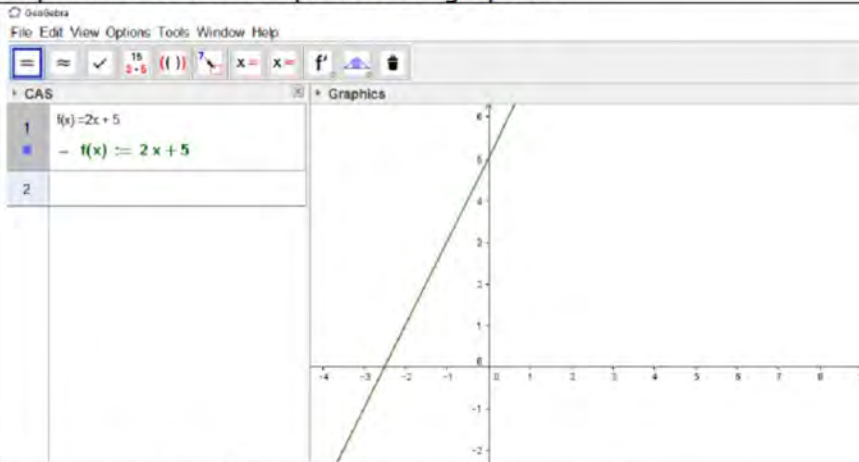
Instructions

1


Type an expression in Row: like $2x + 5$ then click 



- 2 Then click on the white button, called 'show button', below the row number - it implies to show the expression in 'graphics'.

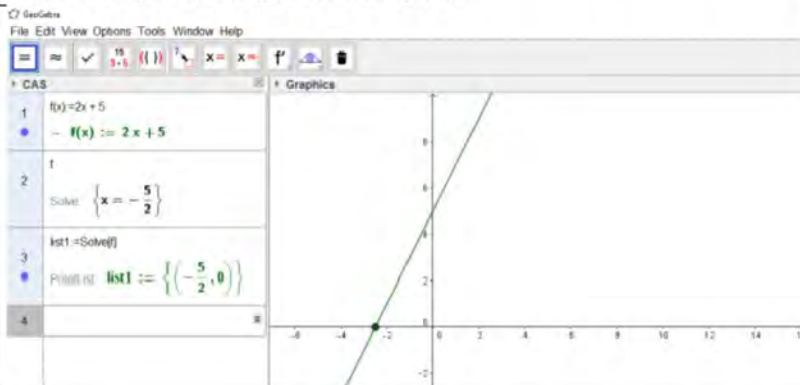


- 3 You may retype the whole expression once again or copy and paste the expression

or simply type 'f' (since the software allots it as f(x)) then select . It will solve the equation.

Solve: $\{x = -\frac{5}{2}\}$ will be displayed in row 2

- 4 In row 3 again type as typed in row 2 and then solve Then click on show button on row 3



You can now try using a different expression or equation say quadratic. When typing exponents, we use '^' which is obtained by using a combination of keys [Shift][6]. Thus to input $2x^2 - 4x + 3$, type as : $2x^2 - 4x + 3$

Annexure IV - WS-1 Worksheet Equations, Inequalities and Nature of Roots


In GeoGebra file, drag a , b , and c so as to match the following expression / equation

Equation	Solve for x	Calculate the value of $b^2 - 4ac$	Observe the graph – x-intercepts	Observe the value of d in CAS	Comment on the nature of roots of the equation
$x^2 + 2x - 3 = 0$					
$3x^2 + 5x + 1 = 0$					
$x^2 - 4x = 0$					
$2x^2 - 5x + 3 = 0$					
$-3x^2 + 4x + 4 = 0$					
$x^2 + 2x + 1 = 0$					
$-x^2 + 4x - 4 = 0$					
$2x^2 + 3x + 2 = 0$					

Annexure V - Training interlude – Coordinates and Straight Lines

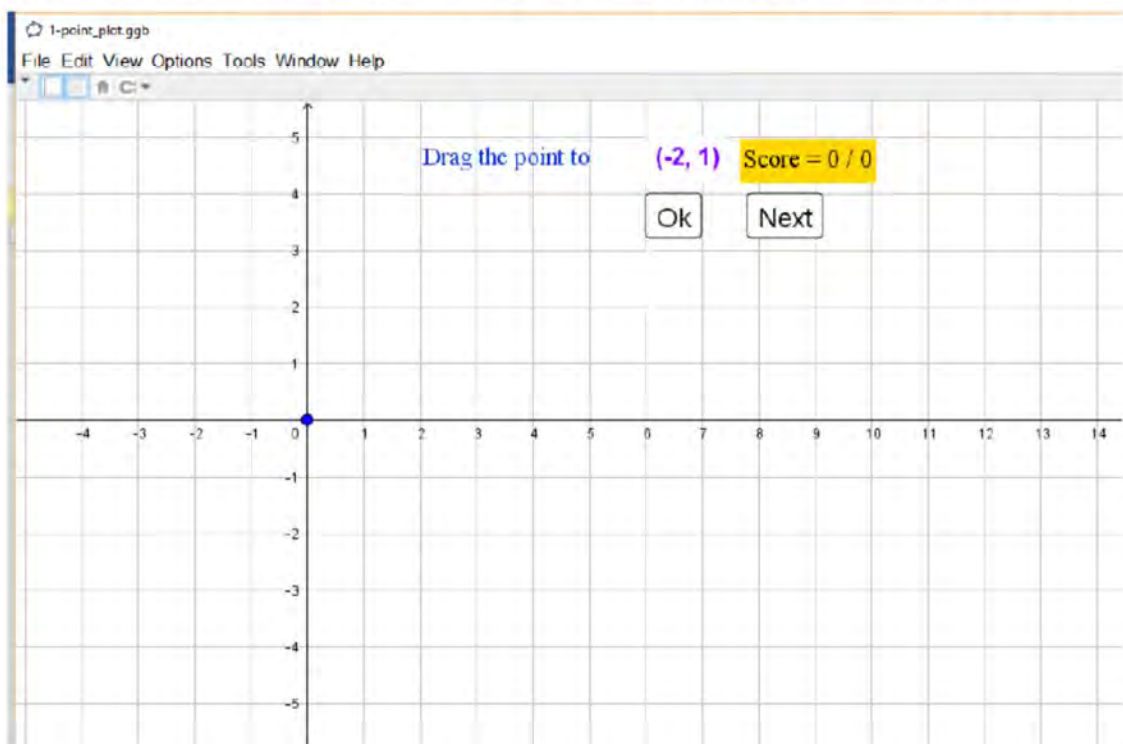
Familiarise with the coordinates

Choose Graphics perspective by clicking on File and then click Graphics.

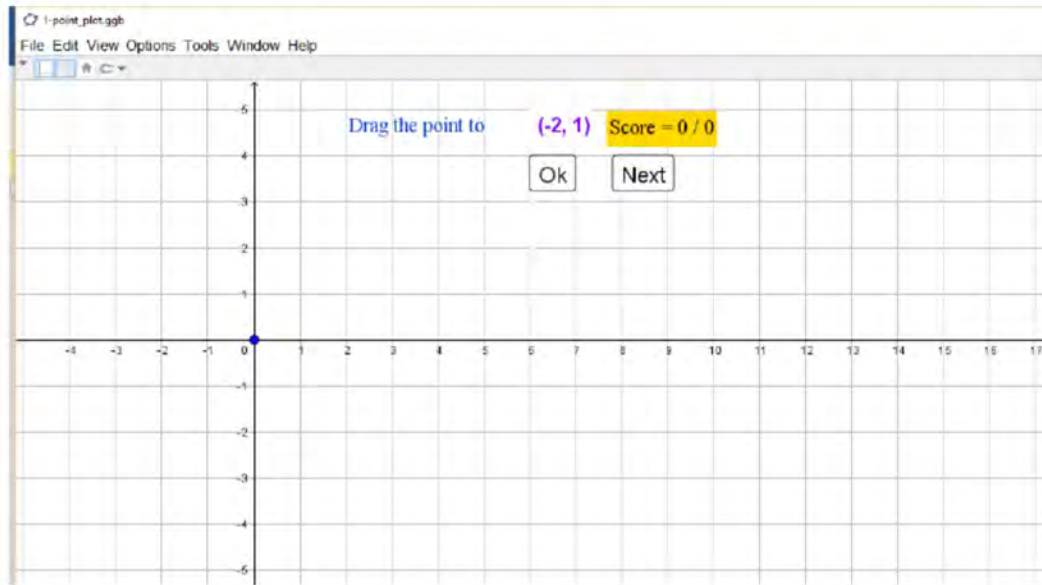
Select a point 'A'  on and click on the graph. Observe carefully the values of x and y . The coordinate form (x,y) is displayed. If not displayed, right click the point, scroll down to object properties, a pop-up window will appear, tick the second check box if not 'ticked' and from the drop-down options select 'value', then click ok button and close 'object properties' dialog box, the coordinates form will appear.

Drag this point around the screen and note down the coordinates of the point. Observe carefully where ' x ' values are negative and positive and similarly for ' y ' values. Also note in which quadrant where both values are positive, negative and one negative and other negative.

Open the applet 1 to drag the points as mentioned. Drag the blue colour point to the required point on the graph, eg, as in the figure drag the point to $(-2,1)$. After placing the point as required, click n the ;Ok' button and click on 'next' to move to next point to be plotted.

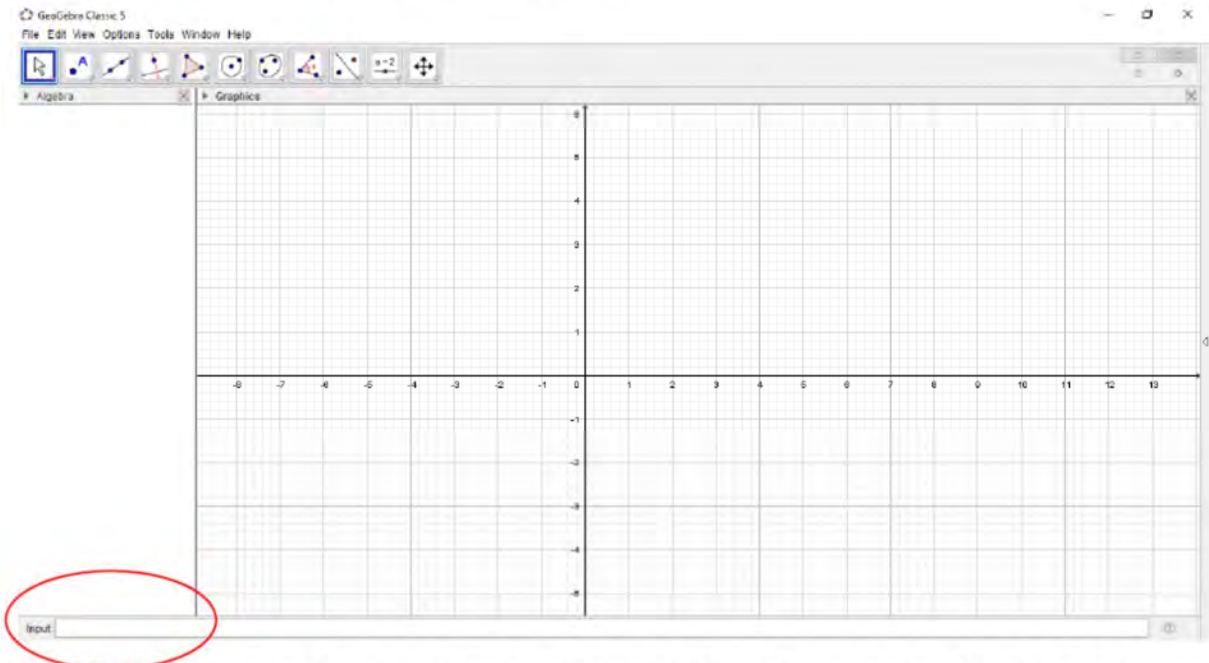


In the second applet, you have to interpret the point, and type into the space provided in coordinate form, eg. You are expected to type the coordinates of 'A'. After entering the ordered pair, click on 'Ok button, if the answer is put the software will provide feedback accordingly.



Parameters of a Linear Equation

Choose a new *GeoGebra* file. Then choose *Graphics* perspective by clicking on *File* and then click *Graphics*, the following screen will appear.



If 'Input' bar is not active or not visible, then go to *View Menu*, click on 'Input bar', to activate it.



Some important tips

Multiplication needs to be entered using an asterisk. Example: $a*x$

Asterisk symbol in keyboard is obtained by pressing shift and 8 together

GeoGebra is case sensitive! Uppercase and lowercase letters must not be mixed up.

The variable x or y within a function always need to be lower case. (e.g., $f(x) = 3*x + 2$ or $g: y = 2*x + 4$)

Confirm an expression you entered into the Input Bar by pressing the *Enter* key.

In input bar type: $y = x+1$. Then continuing the following equations :

$y = 2x + 1$; $y = 3x + 1$; $y = 5x + 1$; $y = -2x + 1$; $y = -4x + 1$; $y = -5x + 1$

Observe the lines carefully. Take note of the slant of the line, here the only change is on the coefficient of x and the constant remains unchanged.

Now let's delete all the lines and type new set of straight lines. This time the coefficient of x remain unchanged and the constant are changed.

$$y = 2x + 1 ; y = 2x + 3 ; y = 2x - 4 ; y = 2x - 2 ; y = 2x - 5 ; y = 2x$$

Observe the lines carefully. Take note of the y -intercepts of the line.

Delete these lines. You may now enter more lines on your own and observe the slope and intercepts of the lines.

Annexure VI - WS-2 Worksheet Parallel and Perpendicular Lines

Investigating Parallel and Perpendicular lines

Equation of a straight line : $y = mx + c$; m is the slope and c is the y-intercept. A line of the form

$ax + by = c$, its slope will be determined by making y as the subject of the formula ie $y = -\frac{a}{b}x + \frac{c}{b}$

Open the applet '6_parallel_perpendicular_lines_investigate'. Note: This applet will generate the parallel and the perpendicular lines for you. You may input the equation in the blue colour 'input box' and press enter. You are required to fill in the table below (next page) and your conclusions. You can drag the point 'A' anywhere around the screen, but fill up the table for at least three points for a line.

Conclusions from the observations

Parallel Lines (||) what changes you observed? What changed and what did not? How are the slopes related?

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Perpendicular Lines (⊥) what changes you observed? What changed and what did not? How are the slopes related?

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Sl No	Input Line	Slope	Point (A)	Equation of line	Slope	Equation of \perp line	Slope
1	$y = 2x + 3$						
2	$3y + x = 2$						
3	$y = -\frac{1}{2}x - 2$						
4	$y = 3x - 1$						
5	$5x + y = 3$						