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Studying a Tumor Growth Partial Differential Equation via the Black–Scholes Equation

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Received: 20 May 2020; Accepted: 27 May 2020; Published: 16 June 2020



Abstract: Two equations are considered in this paper—the Black–Scholes equation and an equation that models the spatial dynamics of a brain tumor under some treatment regime. We shall call the latter equation the tumor equation. The Black–Scholes and tumor equations are partial differential equations that arise in very different contexts. The tumor equation is used to model propagation of brain tumor, while the Black–Scholes equation arises in financial mathematics as a model for the fair price of a European option and other related derivatives. We use Lie symmetry analysis to establish a mapping between them and hence deduce solutions of the tumor equation from solutions of the Black–Scholes equation.

Keywords: Lie symmetry analysis; equivalence transformation; invariant solutions; Black–Scholes equation; glioblastoma

1. Introduction

The study of the most common and malignant brain tumor, glioblastoma, also known as glioblastoma multiforme (GBM), and that of option pricing, may be done in tandem. In both cases, partial differential equations (PDEs) are the central vehicle for mathematically studying the dynamics of the phenomena. The models that are used to study tumor dynamics and responses to treatment are often expressed in terms of PDEs [1–6]. Additionally, in financial mathematics, the evolution of the option value can be modeled via a PDE as a function of time and price of the underlying asset (see, for example, [7–9]). The connection through PDEs between mathematical models of glioblastomas and those of option prices can be exploited, courtesy of Lie symmetry analysis [10–19], to study one model through another arising from the “unrelated” field. There are algorithms in Lie symmetry analysis that allow one to identify “similar” differential equations and to exploit this phenomenon to study the equations side by side. However, while the theory on the use of admitted symmetries to relate differential equations has been available [10], we are not aware of any serious application of the theory. In this paper we make a contribution in this regard.

In general, given a differential equation to analyze, one may want to start by identifying a class of differential equations to which the given equation belongs, with a view to benefiting from the analysis already done on some of the equations that belong to that class. This might require finding a suitable mapping between the differential equation of interest and an equation from the class. Such mappings are typically realizable as equivalence transformations between equations that have similar symmetry structures. An equivalence transformation is essentially a change of variables that maps a differential equation in a given class to another equation in the same class, and relates properties of the two equations. Lie point (or contact) symmetries admitted by differential equations provide a means by which to identify equivalent differential equations and to construct equivalent transformations between them when such transformations exist [10,12,20–25]. This is an important

application of Lie symmetry analysis, whereby a given differential equation is mapped to another equation, typically of a simpler structure. For example, in [20,21,23,24] Lie symmetry analysis is used to determine whether or not there exists an invertible mapping of a given nonlinear PDE to some linear PDE, or a linear PDE with variable coefficients to a PDE with constant coefficients. Other studies that deal with the characterization of related differential equations via their symmetries include studies by Mahomed [25], Andriopoulos and Leach [26], and Dimas et al. [27].

Two equations are considered in this paper—the very well-known and studied Black–Scholes equation [7] and an equation that models the spatial dynamics of GBM under some treatment regime [2–4]. The two equations considered are (superficially) different and arise in very different contexts. The Black–Scholes equation is central to the mathematical modeling of options and other derivatives. It is perhaps the best known equation in financial mathematics and has been studied extensively. In its simplest form, the Black–Scholes equation is a (1 + 1) linear parabolic equation,

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx \frac{\partial u}{\partial x} - ru = 0, \tag{1}$$

where $u = u(x, t)$ is the fair option price depending on the current value of the underlying asset x and time t . As for the other parameters in the equation, σ is the market volatility of the underlying asset price, and is assumed to be constant for a given interest rate r . Our other equation of interest is a particular case of a mathematical model, first proposed by Wein and Koplow [2–4], which tracks the spatial dynamics of an infused cytotoxic treatment and its effect on a brain tumor and the surrounding normal tissue; namely,

$$\frac{\partial w}{\partial \tau} = \frac{L}{z^2} \frac{\partial}{\partial z} \left(z^2 \frac{\partial w}{\partial z} \right) + Mw. \tag{2}$$

Other authors have worked on mathematical models of various aspects regarding the spread of brain tumors [1,5,6]. In [6] the Adomian decomposition method (ADM) is applied to solve a variant of Equation (2). In Equation (2), $w = w(z, \tau)$ is the concentration of tumor cells at location z at time τ , and L is the diffusion coefficient for GBM. L is a proxy of the invasiveness of the GBM cells. The tumor spread is assumed to be spherically symmetric in this model, and z measures the distance from the center (i.e., the origin of the GBM). M is a parameter that represents the resultant effect of the proliferation rate of the tumor and the (therapy-dependent) killing rate of the tumor cells.

We will show that Equations (1) and (2) are equivalent and transformable into each other. In fact “every” property of one equation can be transformed into a corresponding property of the other. In particular, every solution of one equation can be transformed into a solution of the other, which is what this paper is about. The equivalence between the two equations is suggested by isomorphism of symmetry Lie algebras admitted by the equations. We use Lie symmetry analysis to construct a point transformation that maps the Black–Scholes equation to the brain tumor equation and consequently every solution of the Black–Scholes equation to a corresponding solution of the brain tumor equation.

The paper is organised as follows. In Section 2, we introduce Lie symmetry analysis of differential equations. Analysis of our two equations of interest is done in Section 3 wherein we determine Lie point symmetries admitted by the equations and construct an equivalence transformation between them. In Section 4, we perform the calculus of recovering invariant solutions of the tumor equation from invariant solutions of the Black–Scholes equation. Finally, we give concluding remarks in Section 5.

2. Preliminaries of Lie Symmetry Analysis

The Lie symmetry analysis of differential equations, on which the work reported here is based, has its origins in studies by the Norwegian mathematician Sophus Lie who began to investigate continuous groups of transformations that leave differential equations invariant [10,18,19]. At the center of Lie symmetry methods is the notion of invariance of differential equations under continuous

groups. Consider, for example, a general second-order PDE with one dependent variable u and two independent variables (x, t) ,

$$\Delta(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0, \quad u = u(x, t), \tag{3}$$

which is a prototype for Equations (1) and (2). A group of continuous point transformations

$$\tilde{x} = f(x, t, u, \varepsilon), \quad \tilde{t} = g(x, t, u, \varepsilon), \quad \tilde{u} = h(x, t, u, \varepsilon), \tag{4}$$

where ε is the group parameter, is a symmetry of Equation (3) if the equation has the same form in the new variables \tilde{x}, \tilde{t} , and \tilde{u} , or equivalently, if the group transformations (4) transform any solution of Equation (3) into another solution of Equation (3). Thanks to a theorem due to Lie (the first fundamental theorem of Lie [10]), we can always assume that the parametrization of the group (4) is in such a way that it is uniquely defined by the first-order differential operator

$$X = \xi(x, t, u)\partial_x + \zeta(x, t, u)\partial_t + \eta(x, t, u)\partial_u, \tag{5}$$

where

$$\xi = \left. \frac{\partial f}{\partial \varepsilon}(x, t, u; \varepsilon) \right|_{\varepsilon=0}, \quad \zeta = \left. \frac{\partial g}{\partial \varepsilon}(x, t, u; \varepsilon) \right|_{\varepsilon=0}, \quad \eta = \left. \frac{\partial h}{\partial \varepsilon}(x, t, u; \varepsilon) \right|_{\varepsilon=0}.$$

The operator (5) is called the infinitesimal generator of the Lie group (4), and

$$\tilde{x} = x + \varepsilon\xi(x, t, u), \quad \tilde{t} = t + \varepsilon\zeta(x, t, u), \quad \tilde{u} = u + \varepsilon\eta(x, t, u) \tag{6}$$

is the infinitesimal transformation of the Lie group. We often call the operator (5) a symmetry of Equation (3). The first fundamental theorem of Lie essentially says that a Lie group of point transformations is “equivalent” to its infinitesimal transformation, and also to its infinitesimal generator.

A straightforward algorithm is available for finding symmetries admitted by a differential equation. It is based on the fact that if the operator (5) is an infinitesimal generator of the group (4), then the group is admitted by Equation (3) if and only if

$$X^{(2)}\Delta = 0 \quad \text{when} \quad \Delta = 0, \tag{7}$$

where $X^{(2)}$ is the second extension of X . Newcomers to Lie group methods are encouraged to read more on the subject from the many available books on the subject, including [10–19].

Knowledge of Lie groups admitted by a given differential equation can be used for many things, including the application reported in this paper; namely, identifying and relating equivalent differential equations.

3. Derivation of an Equivalence Transformation Relating Equations (1) and (2)

Lie point symmetries of the Black–Scholes Equation (1) and the tumor Equation (2) are easily obtained. Using Program Lie [28], for example, we determine that symmetries of the Black–Scholes equation are

$$\begin{aligned} \Sigma_1 &= \partial_t, & \Sigma_2 &= x\partial_x \\ \Sigma_3 &= 2t\partial_t + (\ln x + Dt)x\partial_x + 2rtu\partial_u, \\ \Sigma_4 &= \sigma^2tx\partial_x + (\ln x - Dt)u\partial_u, \\ \Sigma_5 &= 2\sigma^2t^2\partial_t + 2\sigma^2tx \ln x\partial_x + [(\ln x - Dt)^2 + 2\sigma^2rt^2 - \sigma^2t]u\partial_u, \\ \Sigma_6 &= u\partial_u, & \Sigma_\phi &= \phi(x, t)\partial_u, \end{aligned} \tag{8}$$

where $D = r - \sigma^2/2$ and ϕ is any solution of the Black–Scholes equation, and those of the tumor Equation (2) are

$$\begin{aligned}
 \Gamma_1 &= \partial_t, & \Gamma_2 &= w\partial_w, & \Gamma_3 &= \partial_z - (w/z)\partial_w, \\
 \Gamma_4 &= 2\tau\partial_\tau + z\partial_z + 2M\tau w\partial_w, \\
 \Gamma_5 &= 4L\tau^2\partial_\tau + 4L\tau z\partial_z + (4LM\tau^2 - 6L\tau - z^2)w\partial_w \\
 \Gamma_6 &= 2L\tau\partial_z - (2L\tau/z + z)w\partial_w, & \Gamma_\psi &= \psi(z, \tau)\partial_w,
 \end{aligned}
 \tag{9}$$

where ψ is a solution of the tumor equation.

The algebra of the symmetries of the Black–Scholes and tumor equations is $\{sl(2, \mathbb{R}) \oplus_s W\} \oplus_s \infty A_1$, where $W = \langle X_1, X_2, X_3 \rangle$ is the Heisenberg–Weyl algebra with [29]

$$[X_1, X_2] = 0, \quad [X_1, X_3] = 0, \quad [X_2, X_3] = X_1.$$

For representation of the Heisenberg–Weyl algebra in the Mubarakzhanov classification scheme, see [30].

The commutator table for the finite part of the Lie algebra, $sl(2, \mathbb{R}) \oplus_s W$, wherein the homogeneity symmetry is excluded, for the Black–Scholes symmetries, is presented in Table 1. The corresponding table for symmetries of the tumor equation is Table 2.

Table 1. Commutator table for the basis operators Σ_i of the Black–Scholes equation.

	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6
Σ_1	0	0	$2\Sigma_1 + 2r\Sigma_6 + D\Sigma_2$	$\sigma^2\Sigma_2 - D\Sigma_6$	$2\sigma^2\Sigma_3 - 2D\Sigma_4 - \sigma^2\Sigma_6$	0
Σ_2	0	0	Σ_2	Σ_6	$2\Sigma_4$	0
Σ_3	$-2\Sigma_1 - 2r\Sigma_6 - D\Sigma_2$	$-\Sigma_2$	0	Σ_4	$2\Sigma_5$	0
Σ_4	$-\sigma^2\Sigma_2 + D\Sigma_6$	$-\Sigma_6$	$-\Sigma_4$	0	0	0
Σ_5	$-2\sigma^2\Sigma_3 + 2D\Sigma_4 + \sigma^2\Sigma_6$	$-2\Sigma_4$	$-2\Sigma_5$	0	0	0
Σ_6	0	0	0	0	0	0

Table 2. Commutator table for the basis operators Γ_i of the tumor equation.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
Γ_1	0	0	0	$2\Gamma_1 + 2M\Gamma_2$	$4L\Gamma_4 - 6L\Gamma_2$	$2L\Gamma_3$
Γ_2	0	0	0	0	0	0
Γ_3	0	0	0	Γ_3	$2\Gamma_6$	$-\Gamma_2$
Γ_4	$-(2\Gamma_1 + 2M\Gamma_2)$	0	$-\Gamma_3$	0	$2\Gamma_5$	Γ_6
Γ_5	$6L\Gamma_2 - 4L\Gamma_4$	0	$-2\Gamma_6$	$-2\Gamma_5$	0	0
Γ_6	$-2L\Gamma_3$	0	Γ_2	$-\Gamma_6$	0	0

We now wish to find an invertible mapping

$$z = \alpha(x, t, u), \quad \tau = \beta(x, t, u), \quad w = \varphi(x, t, u) \tag{10}$$

which transforms Equation (1) into Equation (2). To exploit symmetries in the search for this mapping, we need to find suitable bases $\{X_i\}$ and $\{Y_i\}$ for the Lie algebras of the Black–Scholes and the tumor equations, respectively, so that their commutator tables are identical in the sense that if

$$[X_\alpha, X_\beta] = C_{\alpha\beta}^\gamma X_\gamma \quad \text{then} \quad [Y_\alpha, Y_\beta] = C_{\alpha\beta}^\gamma Y_\gamma,$$

with the same structure constants $C_{\alpha\beta}^\gamma$, $\alpha, \beta, \gamma = 1, \dots, 6$ [10,12]. We achieve this via the following renaming and scaling of the symmetries in (8) and (9):

$$\begin{aligned}
 X_1 = \Sigma_3 &\longleftrightarrow a_1 \Gamma_4 = Y_1 \\
 X_2 = \Sigma_6 &\longleftrightarrow a_2 \Gamma_2 = Y_2 \\
 X_3 = \Sigma_4 &\longleftrightarrow a_3 \Gamma_6 = Y_3 \\
 X_4 = \Sigma_2 &\longleftrightarrow a_4 \Gamma_3 = Y_4 \\
 X_5 = \Sigma_5 &\longleftrightarrow a_5 \Gamma_5 = Y_5 \\
 X_6 = \Sigma_1 + D\Sigma_2 + r\Sigma_6 &\longleftrightarrow a_6(\Gamma_1 + M\Gamma_2) = Y_6
 \end{aligned}
 \tag{11}$$

where

$$a_1 = 1, \quad a_2 = 3, \quad a_3 = i\sigma\sqrt{\frac{3}{2\lambda L}}, \quad a_4 = \frac{a_3}{3}, \quad a_5 = \frac{\sigma^2}{2\lambda L}, \quad a_6 = \lambda, \quad (\lambda \text{ arbitrary}).$$

The renaming and scaling of symmetries in (11) was done by inspection, thanks to Mathematica [31]. The nonzero commutators for the new basis of the finite part of the Lie algebra of the Black–Scholes equation (respectively tumor equation, with X_i replaced by Y_i) are given in Table 3.

Table 3. Commutator table for the new basis $\{X_i\}$ of the Lie algebra of the Black–Scholes equation (and tumor equation with X_i replaced by Y_i).

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	X_3	$-X_4$	$2X_5$	$-2X_6$
X_2	0	0	0	0	0	0
X_3	0	0	0	$-X_2$	0	$-\sigma^2 X_4$
X_4	0	0	0	0	$2X_3$	0
X_5	0	0	0	0	0	$\sigma^2 X_2 - 2\sigma^2 X_1$
X_6	0	0	0	0	0	0

The implication of the alignment in (11) is that the symmetry Lie algebras generated by the two basis symmetries in (8) and (9) are isomorphic. This phenomenon provides a set of necessary conditions out of which the mapping (10) can be constructed (if it exists). In the case of our two equations of interest, Equations (1) and (2), we note that they belong to the class of linear parabolic equations (by virtue of them admitting 6 to ∞ symmetries) that are reducible to the standard heat equation (see [32,33] and the references therein),

$$W_T = \Omega W_{XX}, \tag{12}$$

an equation that necessarily admits a two-dimensional Abelian subalgebra generated by translations of the independent variables,

$$\Psi_1 = \partial_T \quad \text{and} \quad \Psi_2 = \partial_X. \tag{13}$$

Therefore, each of Equations (1) and (2) must admit two symmetries

$$X_i = \zeta_{i1}\partial_x + \zeta_{i2}\partial_t + \eta_i\partial_u \quad \text{and} \quad X_j = \zeta_{j1}\partial_x + \zeta_{j2}\partial_t + \eta_j\partial_u$$

that are equivalent (in the sense defined in [10,12]) to Ψ_1 and Ψ_2 in (13); i.e., X_i and X_j generate an Abelian subalgebra. Furthermore, the coefficients ζ_{i1} , ζ_{i2} , ζ_{j1} , and ζ_{j2} must be such that

$$\det \begin{pmatrix} \zeta_{i1} & \zeta_{i2} \\ \zeta_{j1} & \zeta_{j2} \end{pmatrix} \neq 0 \tag{14}$$

if the mapping (10) is (to be) invertible. The symmetries X_4 and X_6 of Equation (1) generate an Abelian subalgebra and satisfy (14), as do the symmetries Y_4 and Y_6 of Equation (2). (Another pair of

two-dimensional Abelian subalgebras of which the basis symmetries satisfy (14) and may be used to search for a map (10) is $\langle X_3, X_5 \rangle$ and $\langle Y_3, Y_5 \rangle$ for the Black–Scholes and tumor equations, respectively).

We therefore proceed to determine functions α , β and φ in the mapping (10) by requiring that the mapping takes X_4 to Y_4 and X_6 to Y_6 ; i.e.,

$$\begin{aligned} X_4 \alpha(x, t, u) &= Y_4 z, & X_4 \beta(x, t, u) &= Y_4 \tau, & X_4 \varphi(x, t, u) &= Y_4 w, \\ X_6 \alpha(x, t, u) &= Y_6 z, & X_6 \beta(x, t, u) &= Y_6 \tau, & X_6 \varphi(x, t, u) &= Y_6 w. \end{aligned} \tag{15}$$

The equations in (15) are determining equations for the functions α , β , and φ in (10). Written explicitly, they are six first-order PDEs; namely,

$$\kappa - x \alpha_x = 0 \tag{16}$$

$$\beta_x = 0 \tag{17}$$

$$\kappa \varphi + x \alpha \varphi_x = 0 \tag{18}$$

$$\alpha_t + r u \alpha_u + D x \alpha_x = 0 \tag{19}$$

$$\beta_t + r u \beta_u + D x \beta_x - \lambda = 0 \tag{20}$$

$$\varphi_t + r u \varphi_u + D x \varphi_x - \lambda M \varphi = 0, \tag{21}$$

where

$$\kappa = i \sqrt{6 \lambda L} / \sigma.$$

The solution of the system in Equations (16)–(21) is easy, albeit tedious. We obtain

$$\alpha = \kappa (\ln x - D t), \quad \beta = \lambda t + \delta_1, \quad \varphi = \frac{\delta_2 u^{\lambda M/r}}{\ln x - D t}, \tag{22}$$

where δ_1 and δ_2 are arbitrary constants. We set

$$\lambda = \frac{r}{M} \quad \text{and} \quad \delta_2 = \frac{1}{\kappa}, \tag{23}$$

the first assignment being done to preserve linearity of the map while the second is innocuous but convenient. From Equations (22) and (23) the map in (10) becomes

$$z = \kappa (\ln x - D t), \quad \tau = \frac{r}{M} t + \delta_1, \quad w = \frac{u}{\kappa (\ln x - D t)}. \tag{24}$$

We now invert the mapping (24) and use the chain rule to compute formulae for the relevant partial derivatives of u with respect to x and t in terms of partial derivatives of w with respect to z and τ . We obtain

$$x = \bar{\alpha}(z, \tau, w) = \exp \left[\frac{z}{\kappa} + \frac{D M}{r} (\tau - \delta_1) \right] \tag{25}$$

$$t = \bar{\beta}(z, \tau, w) = \frac{M}{r} (\tau - \delta_1) \tag{26}$$

$$u = \bar{\varphi}(z, \tau, w) = w z \tag{27}$$

and

$$u_x = \kappa (u + z u_z) / \bar{\alpha}(z, \tau, w) \tag{28}$$

$$u_t = r z u_\tau / M - \kappa D (u + z u_z) \tag{29}$$

$$u_{xx} = \left[\kappa^2 (2 u_z + z u_{zz}) - \kappa (u + u_z) \right] / \bar{\alpha}(z, \tau, w)^2. \tag{30}$$

Upon substituting Equations (25)–(30) into Equation (1), Equation (1) is transformed into the PDE

$$\frac{\partial w}{\partial \tau} = \frac{3L}{z^2} \frac{\partial}{\partial z} \left(z^2 \frac{\partial w}{\partial z} \right) + Mw, \tag{31}$$

which is not quite the tumor equation, Equation (2). It is, however, easy to determine the appropriate scaling of z , τ , and w in Equation (31) to obtain the tumor equation. This is easily achieved by replacing z and w by $\sqrt{3}z$ and $3w$, respectively. We also conveniently (but w.l.o.g.) set $\delta_1 = 0$ so that the map from Equations (25)–(27) becomes

$$x = e^{\frac{\sqrt{3}z}{\kappa} + \frac{DM\tau}{r}}, \quad t = \frac{M\tau}{r}, \quad u = 3\sqrt{3}wz, \tag{32}$$

which is the desired mapping that transforms the Black–Scholes equation into the tumor equation. Consequently, any solution $u = \Theta(x, t)$ of the Black–Scholes Equation (1) is transformed via the mapping (32) into a solution

$$\begin{aligned} w(z, \tau) &= \frac{1}{3\sqrt{3}z} u(x(z, \tau), t(z, \tau)) \\ &= \frac{1}{3\sqrt{3}z} u\left(e^{\frac{\sqrt{3}z}{\kappa} + \frac{DM\tau}{r}}, \frac{M\tau}{r}\right) \end{aligned} \tag{33}$$

of the tumor Equation (2).

4. Recovering Invariant Solutions of Equation (2) from Those of Equation (1)

In this section we transform four solutions of the Black–Scholes equation into solutions of the tumor equation. The Black–Scholes solutions used were constructed from symmetries of the Black–Scholes equation as invariant solutions via the well-known algorithm [10,11,16–18]. The symmetries used are X_1 , X_3 , X_5 , and X_6 from (11).

4.1. Solution from $X_3 = \sigma^2 t x \partial_x + (\ln x - D t) u \partial_u$

The Black–Scholes Equation (1) admits the solution

$$u(x, t) = \frac{C_1}{x^{D/\sigma^2} \sqrt{t}} \exp \left\{ \frac{(D + \sigma^2)^2 t}{2\sigma^2} + \frac{\ln^2 x}{2\sigma^2 t} \right\}, \tag{34}$$

where C_1 is an arbitrary constant. This solution arises from X_3 as an invariant solution. According to (33) the solution in (34) is transformed into a solution of the tumor equation as follows:

$$\begin{aligned} w(z, \tau) &= \frac{1}{3\sqrt{3}z} u\left(e^{\frac{\sqrt{3}z}{\kappa} + \frac{DM\tau}{r}}, \frac{M\tau}{r}\right) \\ &= \frac{C_1 \sqrt{r/M}}{3\sqrt{3}} \exp \left[\frac{M(D + \sigma^2)^2 \tau}{2r\sigma^2} - 2D \left(\frac{\sqrt{3}z}{\kappa} + \frac{DM\tau}{r} \right) \right. \\ &\quad \left. + \frac{r}{M\tau} \ln\left(e^{\frac{\sqrt{3}z}{\kappa} + \frac{DM\tau}{r}}\right)^2 \right] (z\sqrt{\tau})^{-1} \end{aligned} \tag{35}$$

After simplification, the solution (35) of the Black–Scholes equation is reduced to the following solution of the tumor equation:

$$w(z, \tau) = \frac{K_1}{\sqrt{\tau} z} \exp \left\{ M\tau - \frac{z^2}{4L\tau} \right\}, \tag{36}$$

where K_1 is arbitrary constant.

4.2. Solution from $X_1 = (D t + \ln x) x \partial_x + 2t \partial_t + 2r t u \partial_u$

This symmetry of the Black–Scholes Equation (1) leads to the following invariant solution of the equation:

$$u(x, t) = e^{rt} \left[C_1 + C_2 \operatorname{erfi} \left(\frac{D t - \ln x}{\sqrt{2} \sqrt{t} \sigma} \right) \right], \tag{37}$$

where C_1 and C_2 are arbitrary constants, and erfi is the imaginary error function. Proceeding as we did in Section 4.1 above, we found that solution (37) was transformed according to (33) into a solution of the tumor equation:

$$w(z, \tau) = \frac{e^{M\tau}}{z} \left[K_1 + K_2 \operatorname{erfi} \left(\frac{z}{2 \sqrt{L\tau}} \right) \right], \tag{38}$$

where K_1 and K_2 arbitrary constants.

4.3. Solution from $X_6 = D x \partial_x + \partial_t + r u \partial_u$

In this case the corresponding invariant solution of the Black–Scholes equation is

$$u(x, t) = e^{rt} [C_2 + C_1 (\ln x - D t)], \tag{39}$$

where C_1 and C_2 are arbitrary constants, and is transformed according to (33) into the following solution of the tumor equation:

$$w(z, \tau) = e^{M\tau} \left(K_1 + \frac{K_2}{z} \right), \tag{40}$$

where K_1 and K_2 arbitrary constants.

4.4. Solution from $X_5 = 2 \sigma^2 t x \ln x \partial_x + 2 \sigma^2 t^2 \partial_t + \left[(2 r t^2 \sigma^2 - \sigma^2 t + (\ln x - D t)^2) \right] u \partial_u$

The invariant solution of the Black–Scholes equation that arises from this symmetry is

$$u(x, t) = t^{-\frac{1}{2}} e^{\frac{(D^2+2r\sigma^2)t}{2\sigma^2}} x^{\frac{\ln x - 2Dt}{2t\sigma^2}} \left[C_1 + C_2 \ln(x^{1/t}) \right], \tag{41}$$

where C_1 and C_2 arbitrary constants. This solution is transformed according to (33) into the following solution of the tumor equation

$$w(z, \tau) = e^{M\tau - \frac{z^2}{4L\tau}} \left(\frac{K_1}{\sqrt{\tau} z} + \frac{K_2}{\tau^{3/2}} \right), \tag{42}$$

where K_1 and K_2 arbitrary constants.

We remark here that the solutions of the tumor equation derived from invariant solutions of the Black–Scholes equation are indeed (as we should expect) invariant solutions of the tumor equation associated with symmetries of the tumor equation obtainable from corresponding symmetries of the Black–Scholes equation via the constructed map.

When we invert the mapping (32), we obtain

$$z = \frac{\kappa (\ln x - D t)}{\sqrt{3}}, \quad \tau = \frac{r t}{M}, \quad w = \frac{u}{3 \kappa (\ln x - D t)}. \tag{43}$$

Therefore, any symmetry

$$X = \xi(x, t, u) \partial_x + \zeta(x, t, u) \partial_t + \eta(x, t, u) \partial_u \tag{44}$$

of the Black–Scholes equation is transformed under the mapping (43) into a symmetry of the tumor equation

$$\bar{X} = \bar{\zeta}(z, \tau, w) \partial_z + \bar{\zeta}(z, \tau, w) \partial_\tau + \bar{\eta}(z, \tau, w) \partial_w \quad (45)$$

where

$$\bar{\zeta}(z, \tau, w) = X(z) \Big|_{(43)}, \quad \bar{\zeta}(z, \tau, w) = X(\tau) \Big|_{(43)}, \quad \bar{\eta}(z, \tau, w) = X(w) \Big|_{(43)}.$$

The symmetries X_1 , X_3 , X_5 , and X_6 of the Black–Scholes equation used in Section 4 are transformed under the mapping (43) into corresponding symmetries of the tumor equation as follows:

$$X_1 \mapsto Y_1 - \frac{1}{3} Y_2, \quad X_3 \mapsto \frac{1}{\sqrt{3}} Y_3, \quad X_5 \mapsto Y_5, \quad X_6 \mapsto Y_6. \quad (46)$$

5. Concluding Remarks

When studying a given differential equation, it is prudent to explore equivalent differential equations that have been studied before, and then, if possible, lift solutions and/or other properties of interest of the known equation(s) via some mapping to the given equation. In this paper, we have done so with two equations that are supposedly different and arise in two very different settings. It so happens that the Black–Scholes and tumor equations, both of which admit $6 + \infty$ Lie point symmetries, can be reduced to the heat equation via an equivalence transformation. We have exploited an algorithm by means of which a variable coefficient PDE is transformed into a constant coefficient PDE to construct an equivalent transformation between the Black–Scholes and tumor equations. The transformation is then used to deduce four invariant solutions of the tumor equation from invariant solutions of the Black–Scholes equation.

The work reported in this paper contributes a nontrivial example to this important aspect Lie symmetry analysis of differential equations—the derivation via admitted Lie point symmetries and use of mappings between equivalent differential equations. In fact, many differential equations that have equivalent symmetry structures can be studied in a similar fashion.

Author Contributions: W.S. and T.F.N. worked together in the derivation of the mathematical results reported. Both authors have read and agreed to the published version of the manuscript.

Funding: The authors thank the Directorate of Research Development and Innovation of Walter Sisulu University for the financial support rendered towards publication of this article.

Acknowledgments: The authors would like to thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper.

Conflicts of Interest: The authors declare that there is no conflict of interest regarding the publication of this paper.

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