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Riemann Versus Lebesgue Integrals

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Riemann Integrals

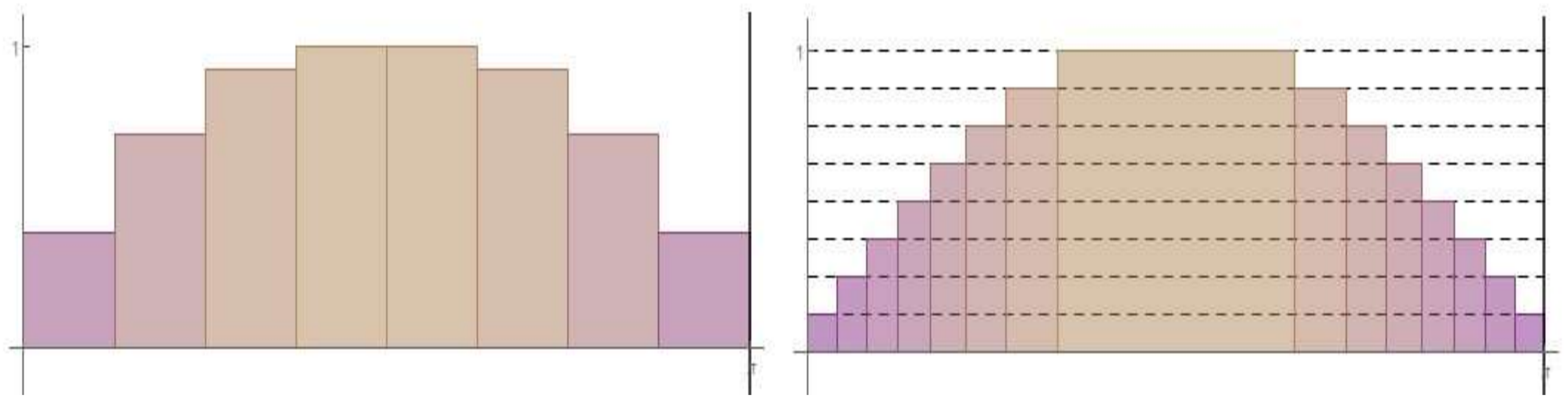
- Riemann integration partitions the domain of the function into a series of smaller intervals.
- Then, the height of the function is taken at a specific point within each interval.
  - Leftmost, rightmost, midpoint, minimum, and maximum are the most common.
- For each interval, multiply that function height by the width of the interval to generate the area of a rectangle. The sum of each of these rectangles is the approximated area for the integral.

Lebesgue Integrals

- Lebesgue integration uses a set of points in the range of the function for its procedure.
- To take the Lebesgue integral, the "measure" of each range value within the function must be determined.
  - The measure of a range value is the total length of the function where that value is the closest value in the set.
  - "Closest" in this context means it is either the largest range value under or the smallest range value above the function at any given domain value, depending on which Lebesgue approximation is used.
- By taking the product of the range value and its measure, the total area of all rectangles with a height of the range value is obtained. The sum of each of these areas gives the Lebesgue approximation for the area of the integral.

Comparison

- As the number of intervals within the given partition increase, the approximation approaches the true value of the integral. This is true for both Riemann and Lebesgue integrals.



A comparison of the Riemann and Lebesgue approximations for "y = sin(x)" from [0,π], using eight divisions.

- Every function which is integrable using the Riemann method is integrable with the Lebesgue method. However, there are some functions that are not Riemann integrable, but are Lebesgue integrable. An example is the Dirichlet function:

$$g(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

The Dirichlet function is discontinuous at every single point on the real number line, so it is impossible to determine the sum of the heights of Riemann approximated rectangles as we increase the number of partitions to infinity. However, we can look at the function using Lebesgue integration.

- The range of the function is simply the values 0 and 1.
- Since the measure of the values at a height of 0 do not affect the area, we can ignore these.
- The measure of the value 1 on the Dirichlet function is 0, because the rational numbers are a series of singular isolated points.
- Thus, the Lebesgue integral of the Dirichlet function is 0.

Conclusions

- There are two main types of integration: Riemann and Lebesgue. Both have their advantages and drawbacks.
  - Riemann integration is simpler to learn and understand but does not quite have the power to determine the integral of some functions.
  - Lebesgue integration is more powerful than Riemann integration, but requires a deeper understanding of mathematics and analysis to understand.
- Although students are taught one way to integrate, there are more tools at their disposal later in their studies which could help in solving more complex and analytically rigorous problems. It is important to recognize these tools so they will not go unused.

Future Work

A continuation of this project would consist of a deeper exploration into measure theory, such as what it means for a function which has values that do not have a measure, and what those functions look like.

References

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- Adams, Malcolm, and Victor Guillemin. *Measure Theory and Probability*. Birkhäuser, 1996.
- https://demonstrations.wolfram.com/RiemannVersusLebesgue/