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Metastable states, quasi-stationary and soft measures, mixing time asymptotics via variational principles

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METASTABLE STATES,
QUASI-STATIONARY DISTRIBUTIONS AND SOFT MEASURES

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ABSTRACT. We establish metastability in the sense of Lebowitz and Penrose under practical and simple hypotheses for Markov chains on a finite configuration space in some asymptotic regime. By comparing restricted ensemble and quasi-stationary measures, and introducing soft measures as interpolation between the two, we prove asymptotic exponential exit law and, on a generally different time scale, asymptotic exponential transition law. By using potential-theoretic tools, and introducing "(κ, λ)-capacities", we give sharp estimates on relaxation time, as well as mean exit time and transition time. We also establish local thermalization on shorter time scales.

1. METASTABILITY AFTER LEBOWITZ AND PENROSE

1.1. Phenomenology and modelization. Lebowitz and Penrose characterized *metastable thermodynamic states* by the following properties [4]:

    (a) only one thermodynamic phase is present,
    (b) a system that starts in this state is likely to take a long time to get out,
    (c) once the system has gotten out, it is unlikely to return.

We can think, for example, about freezing fog made of small droplets in which only one phase is present (liquid phase) that remains for a long time in such a state (until collision with ground or trees, forming then hard rime) and that once frozen will typically not return to liquid state before pressure or temperature have changed.

To model such a state they considered in [4] a deterministic dynamics with equilibrium measure μ. First, they associated with the metastable phase a subset $\mathcal{R}$ of the configuration space, and described this metastable state by the *restricted ensemble* $\mu_\mathcal{R} = \mu(\cdot|\mathcal{R})$. Second, they proved that the escape rate from $\mathcal{R}$ of the system started in $\mu_\mathcal{R}$ is maximal at time $t = 0$, and that this initial escape rate is very small. Last, they used standard methods of equilibrium statistical mechanics to deal with (c). As an estimate of the returning probability to the metastable state they used the fraction of members of the *equilibrium* ensemble that have configurations in $\mathcal{R}$ and they noted ([4], Section 8):

> This amounts to assuming that a system whose dynamical state has just left $\mathcal{R}$ is no more likely to return to it than one whose dynamical state was never anywhere near $\mathcal{R}$. The validity of this assumption, at least in the short run, is dubious, but at least it provides us with some indication of what to expect.

In this paper we want to give a different model for the same phenomenology that overcomes the last difficulty. We will work with stochastic processes rather than deterministic dynamics, but the Lebowitz-Penrose modelization will be our guideline. We will try to recover this phenomenology under simple and practical hypotheses only. Since the study of metastability has been considerably enriched after the Lebowitz and Penrose work, we want also to incorporate in our modeling as much as possible of what was previously achieved. We will then make a brief and partial review of these achievements. Our goals and starting ideas will depend on this review but not our proofs, since we want to make this paper as self-contained as possible. Our model and results are presented in Section 2. Examples of applications are given in Section 7.

1.2. **A partial review.** Since the Lebowitz and Penrose paper, an enormous amount of work has been done to describe the metastability phenomenon. In particular Cassandro, Galves, Olivieri and Vares introduced the path-wise approach, which focused, in the context of stochastic processes, on time averages associated with an asymptotic exponential law [6]. This was further developed by the pioneering works of Neves and Schonmann who studied the typical paths for stochastic Ising model in a given volume in the low temperature regime ([8], [10]). This work was then extended to higher dimensions, infinite-volume and fixed-temperature regimes, locally conservative dynamics and probabilistic cellular automata ([15], [13], [19], [27], [33], [45]).

As developed in [28], a crucial role was played by large deviation tools inherited from Wentzell and Freidlin in their reduction procedure from continuous stochastic processes to finite-configuration-space Markov chains with exponentially small transition rates [7]. This is especially true for very-low-temperature regimes, but the same kind of reduction procedure made it possible to deal in various cases with large-volume rather than low-temperature limits (see [18] for the Curie-Weiss model under random magnetic field, see [28] for further examples).

Then, using potential-theoretic rather than large deviations tools, Bovier, Eckhoff, Gayrard and Klein developed a set of general techniques to compute sharp asymptotics of the expected value of asymptotic exponential laws associated with the metastability phenomenon, and revisited (after [5], [12]) the relation between spectrum of the generator of the stochastic dynamics and metastability ([20], [23], [26]). This allowed, for example, to go beyond logarithmic asymptotics for stochastic Ising models in the low-temperature regime ([21], [29]) and to prove the first rigorous results in the fully conservative case ([38]), to deal with metastability for the random hopping-time dynamics associated with the Random Energy Model ([22]), to make a detailed analysis of Sinai's random walk spectrum ([32]), and to extend the study of the disordered Curie-Weiss model to the case of continuous magnetic field distribution ([34], [44]). We refer to [48] for a comprehensive account of this approach.

We then reached an essentially complete comprehension of the metastability phenomenon in at least two classes of models: very low temperature dynamics in finite fixed volumes and large volume or continuous-configuration space dynamics that can be reduced via a Wentzell-Freidlin procedure to the previous case. Of course, specific and often nontrivial computations have to be made for each specific model, but there exists a general approach to the problem that is developed in [28] and, as far as the potential-theoretic part is concerned, [20], [23], and [26] together with [41], [46] that bridges between potential theory and typical path description by reinforcing and generalizing the results of [11] (and it is worth noting that [41], after [37, 43], contemplates also the case of polynomially small rather than only exponentially small transition probabilities). For both classes of models, like one-dimensional metastable systems as considered in [32] or [31], a recurrence property for a very localized subset of the configuration space (single configurations identified to metastable states in the first case, small neighborhoods of the dynamics attractors in the second case) plays an important role.

Beyond these two classes of models there are many limit cases, special cases, and partial results. For example, in [22] we are far from a finite-fixed-volume situation but single configurations can still be identified with metastable states and have still enough mass at equilibrium for potential-theoretic or renewal techniques to work. This is not the case in [38], where potential-theoretic tools give only expected values of some hitting times when the system is started from some specific harmonic measures that are very different from what one would expect to be a "metastable state" (here, like in the sequel and following Lebowitz and Penrose, we mean a whole measure when referring to a metastable "state" and not to a single configuration of the configuration space). Any kind of exponential law is presently also lacking in this case. The same difficulty is faced in [34], but it is overcome in [44] by mean of a specific coupling argument that gives point-wise estimates and opens the way to the exponential law. We also note that [46] develops some general martingale ideas to deal with the same issues within the framework built from [37, 41]. In fact, though working with a different setup, we share some of the leading ideas developed in [46], which uses some of the key objects that we introduced through this work. Such ideas also inspired [47] where the non-reversible situation is contemplated. Finally, the beautiful paper by Schonmann and Shlosman [19] achieves the *tour de force* of using essentially equilibrium statistical mechanics computations to deal with the dynamical problem of metastability. In this case also the exponential law is lacking as well

as sharp estimates on the relaxation time, and even the simple formulation of such properties is not completely obvious in this fixed-temperature and vanishing-magnetic-field regime.

1.3. **Starting ideas.** In the present paper we want to elaborate some tools to describe the metastability phenomenon beyond the case of a dynamics with a recurrence property for a very localized subset of the configuration space. We will focus on exponential laws and sharp asymptotics of their expected values. We note that the exponential law itself suggests some kind of recurrence property. If it is not a recurrence property for a very localized subset, it has to be in some sense a recurrence property to a whole "spread measure". And this measure should coincide with our metastable state. Now, following Lebowitz and Penrose, if we associate the metastable state with some subset $\mathcal{R}$ of the configuration space $\mathcal{X}$, then, considering property (b), we have at least two candidates to describe our metastable state: one is the restricted ensemble $\mu_\mathcal{R} = \mu(\cdot|\mathcal{R})$, the other is the quasi-stationary distribution

$$\mu_\mathcal{R}^* = \lim_{t \to +\infty} P_{\mu_\mathcal{R}}(X(t) \in \cdot | \tau_{\mathcal{X}\backslash\mathcal{R}} > t) \tag{1.1}$$

where $X(t)$ is the configuration of the system at time t and $\tau_{\mathcal{X}\backslash\mathcal{R}}$ is the exit time of $\mathcal{R}$ (we will be more precise in the next section). Notice that Eq. (1.1) provides the stationarity of $\mu_\mathcal{R}^*$ for the process conditioned to not having exit $\mathcal{R}$,

$$P_{\mu_\mathcal{R}^*}(X(t) \in \cdot | \tau_{\mathcal{X}\backslash\mathcal{R}} > t) = \mu_\mathcal{R}^*, \tag{1.2}$$

and thus explains the name of quasi-stationary distributions.

The main advantage of $\mu_\mathcal{R}$ is that $\mu_\mathcal{R}$ is often an explicit measure one can compute with, while $\mu_\mathcal{R}^*$ is only implicitly defined. The main advantage of μ_R^* is that the exit law of $\mathcal{R}$ for the system started in μ_R^* *is* an exponential law. Our first results will then start, as in [9], with a comparison between μ_R and μ_R^*. We will give simple and practical hypotheses to ensure that they are close in some sense, then we will be able to prove some kind of recurrence property for μ_R^*. In doing so we will also answer some problems left open in [9] (see our comment after formula (2.31)). All this will be done in the simplest possible setup: considering a Markov process on a finite configuration space in some asymptotic regime (including the possibility of sending to infinity the cardinality of the configuration space).

In the present work we will essentially build on the ideas of four different papers: [4] for the formulation of the problem, [6] for the focus on exponential laws, [20] for the introduction of potential-theoretic techniques in the metastability field to get sharp estimates on some mean hitting times, and Miclo's work [40] where some concepts of local equilibrium, and "hitting times" of such equilibriums, are introduced. As far as this last paper is concerned, it will only work as a source of inspiration: we will not require a full spectrum knowledge, and we will not introduce any notion of dependence of a local equilibrium on the initial condition. Finally, we note that the idea of considering quasi-stationary measures as metastable states was already contemplated in [24]. Even though some of our results echo some of [24], we were not able to make any clear comparison, essentially because of the much more analytical point of view of [24] and the many hypotheses introduced in the results of [24]. We note that [24] deals with a much more general setup than ours since the authors consider non-reversible Markov processes on a continuous configuration space, while we look at reversible Markov processes on finite configuration space. However, the reason why we assume reversibility is to be able to use potential-theoretic results to get sharp estimates on mean times via variational principles, a question that is not considered in [24].

1.4. **Two new objects.** In this section we provide a brief explanation on the two main new objects of this paper, that we will progressively describe in the sequel : (κ, λ)-*capacities* and *soft-measures*. To understand their meaning, beyond their definitions, we can start from the main formula introduced in the context of metastable dynamics by [20]. Given a reversible and irreducible Markov process $X : t \mapsto X(t) \in \mathcal{X}$, and for any two disjoint and non-empty subsets A and B of $\mathcal{X}$, it holds

$$\mathbb{E}_{\nu_A}[\tau_B] = \frac{\mu(V_{A,B})}{C(A,B)}, \tag{1.3}$$

where ν_A is the so-called harmonic measure on A (which actually depends also on B), τ_B is the hitting time of B, $\mu(V_{A,B})$ is the mean value, w.r.t. the equilibrium measure μ, of the "equilibrium potential" between A and B, and $C(A,B)$ is the capacity between A and B. This formula had in particular two crucial advantages. First, it allowed to describe the metastability phenomenon

essentially only through the computation of mean hitting times. Second, the most relevant part in the right-hand side is the capacity appearing in the denominator, which has the key property of satisfying two variational principles which, in turn, can be used to get sharp estimates just by using test functions to obtain upper bounds and test flows to obtain lower bounds. Using this formula one has however to cope with three interlinked difficulties, which, depending on the considered model, can or cannot be easily overcame:

  i) the choice of the family of sets A and B can be delicate;
  ii) there is in general no variational principle to help in estimating the mean potential at the numerator of the right-hand side;
  iii) the harmonic measure ν_A is in general very different from the natural measures associated with metastability, say $\mu_{\mathcal{R}}$ or $\mu_{\mathcal{R}}^*$.

Let us rapidly explain these three points. Formula (1.3) is in its very nature associated with the Markov process *stopped at time* τ_B. Our previous discussion explains why it will not be sufficient just to choose $B = \mathcal{X} \setminus \mathcal{R}$. Thus, in general, one has to consider a family of sets B that are "deep inside $\mathcal{X} \setminus \mathcal{R}$", and for a symmetric reason, the family of sets A should be chosen "deep inside $\mathcal{R}$" too. But then, the deeper are these chosen sets, the harder turns the estimation of the mean potential. Moreover, while $\mu_{\mathcal{R}}$ and $\mu_{\mathcal{R}}^*$ are usually concentrated deep inside $\mathcal{R}$ (and A), the harmonic measure ν_A of formula (1.3) has support on the *border* of A. In general, this makes uneasy the comparison between the Markov process started from ν_A and the system started from a "metastable equilibrium". We point out that this last difficulty is actually the exact counterpart in A of the fact that (1.3) deals with a Markov process stopped in B (this possibly not obvious fact can be well understood by looking at the proof of (1.3)).

The two objects that we introduce in this work, are partially intended to deal with these difficulties. The (κ, λ)-capacities are capacities computed in an extended network that is associated with a Markov process *stopped at rate λ in B* and for which κ plays a symmetrical role in A (just like, when discussing (1.3), we noted that the fact that ν_A was concentrated on the border of A was the counterpart of the fact that (1.3) was dealing with a process stopped in B). We will then be able to build on (1.3) with a Markov process that *can* penetrate B. This will allow to simply choose $B = \mathcal{X} \setminus \mathcal{R}$, rather than a family of subsets of $\mathcal{X} \setminus \mathcal{R}$, and to compute the "mean transition times" from metastable to stable states by estimating the (κ, λ)-capacities. Symmetrically, the parameter κ will be used to deal with measures that are concentrated deep inside $A = \mathcal{R}$. In addition, the parameter λ will be used to interpolate between the restricted ensemble $\mu_{\mathcal{R}}$ (at $\lambda = 0$) and the quasi-stationary measure $\mu_{\mathcal{R}}^*$ (at $\lambda = +\infty$). These interpolating measures will be our soft-measures; they are the quasi-stationary measures of the trace on $\mathcal{R}$ of the process killed outside $\mathcal{R}$ at rate λ. In some sense, they are intended to keep the idea of characterizing metastability through the computation of mean hitting times, for which we can benefit of the classical potential theory and of its variational principles. Though formula (1.3) will be used in our proofs, we will derive new (asymptotic) equations expressing these mean hitting times in terms of quantities that satisfy two-sided variational principles only, and do not involve mean potentials.

Finally, we stress that the difficulties arising when using (1.3) to describe a metastable dynamics will not magically disappear by using soft-measures or (κ, λ)-capacities instead of standard capacities. They are actually deferred into the estimation of the local relaxation times, called $\gamma_{\mathcal{R}}^{-1}$ and $\gamma_{\mathcal{X} \setminus \mathcal{R}}^{-1}$ in the sequel. However, in doing so, we can benefit from the huge mathematical literature dealing with the computation of rates of convergence to equilibrium. In this paper we will also prove a new Poincaré inequality, adding one more tool in this respect. And at this point, we should stress that the hypotheses that these *local* relaxation times should satisfy in order to apply our results, do not require sharp estimates. Rough estimates will be enough to find large windows in which choosing our parameters κ and λ to obtain, through the use of variational principles, sharp estimates on the *global* relaxation time (γ^{-1} in the sequel).

2. MODEL AND RESULTS

2.1. Quasi-stationary measure and restricted ensemble.
We consider a continuous-time Markov process X on a finite set $\mathcal{X}$ with generator defined by

$$\mathcal{L}f(x) = \sum_{y \in \mathcal{X}} p(x, y)(f(y) - f(x)) \tag{2.1}$$

for x in $\mathcal{X}$ and $f : \mathcal{X} \to \mathbb{R}$, and where p is such that

$$\sum_y p(x,y) = 1. \tag{2.2}$$

Since $\mathcal{X}$ is finite, any generator can be written like in (2.1) up to time rescaling. We assume that X is irreducible and reversible with respect to some probability measure μ, we denote by $\langle \cdot , \cdot \rangle$ the scalar product in $\ell^2(\mu)$, by $\| \cdot \|$ the associated 2-norm, by $\mathcal{D}$ the Dirichlet form defined by

$$\mathcal{D}(f) = \langle f, -\mathcal{L}f \rangle = \frac{1}{2} \sum_{x,y \in \mathcal{X}} c(x,y) \left[f(x) - f(y) \right]^2 \tag{2.3}$$

where each conductance $c(x,y)$ is equal to

$$c(x,y) = \mu(x)p(x,y), \tag{2.4}$$

and by γ the spectral gap

$$\gamma = \min_{\mathrm{Var}_\mu(f) \neq 0} \frac{\mathcal{D}(f)}{\mathrm{Var}_\mu(f)}. \tag{2.5}$$

For $\mathcal{R} \subset \mathcal{X}$ we define in each $x \in \mathcal{R}$ the escape probability (or rate)

$$e_\mathcal{R}(x) = \sum_{y \notin \mathcal{R}} p(x,y) \tag{2.6}$$

and we denote by $X_\mathcal{R}$ the *reflected process* (or *restricted process*) with generator given by

$$\mathcal{L}_\mathcal{R} f(x) = \sum_{y \in \mathcal{R}} p_\mathcal{R}(x,y)(f(y) - f(x)) \tag{2.7}$$

for x in $\mathcal{R}$ and $f : \mathcal{R} \to \mathbb{R}$, and where, for all x, y in $\mathcal{R}$,

$$p_\mathcal{R}(x,y) = \begin{cases} p(x,y) & \text{if } x \neq y, \\ p(x,x) + e_\mathcal{R}(x) & \text{if } x = y. \end{cases} \tag{2.8}$$

We will only consider subsets $\mathcal{R}$ such that both $X_\mathcal{R}$ and $X_{\mathcal{X} \setminus \mathcal{R}}$ are irreducible and we note that $X_\mathcal{R}$ inherits from X the reversibility property with respect to the restricted ensemble

$$\mu_\mathcal{R} = \mu(\cdot | \mathcal{R}). \tag{2.9}$$

We identify $\ell^2(\mu_\mathcal{R})$ with the subset of $\ell^2(\mu)$ of functions $f : \mathcal{X} \to \mathbb{R}$ such that $f|_{\mathcal{X} \setminus \mathcal{R}} \equiv 0$ and we denote by $\langle \cdot , \cdot \rangle_\mathcal{R}$, $\| \cdot \|_\mathcal{R}$, $\mathcal{D}_\mathcal{R}$, $c_\mathcal{R}(x,y)$ and $\gamma_\mathcal{R}$ the associated scalar product, 2-norm, Dirichlet form, conductances for x, y in $\mathcal{R}$ and spectral gap.

We denote by $p_\mathcal{R}^*$ the sub-Markovian kernel on $\mathcal{R}$ such that, for all x, y in $\mathcal{R}$,

$$p_\mathcal{R}^*(x,y) = p(x,y). \tag{2.10}$$

We know from [3] and the Perron-Frobenius theorem that there exists $\phi_\mathcal{R}^* > 0$ such that $1 - \phi_\mathcal{R}^*$ is the spectral radius of $p_\mathcal{R}^*$ and that there is a unique *quasi-stationary measure* $\mu_\mathcal{R}^*$ such that $\mu_\mathcal{R}^* p_\mathcal{R}^* = (1 - \phi_\mathcal{R}^*)\mu_\mathcal{R}^*$. In addition we have, for all x, y in $\mathcal{R}$ and $t \geq 0$, with $\tau_{\mathcal{X} \setminus \mathcal{R}}$ the exit time from $\mathcal{R}$, i.e., the hitting time of $\mathcal{X} \setminus \mathcal{R}$,

$$\lim_{t \to +\infty} P_x(X(t) = y | \tau_{\mathcal{X} \setminus \mathcal{R}} > t) = \mu_\mathcal{R}^*(y), \tag{2.11}$$

$$P_{\mu_\mathcal{R}^*}(\tau_{\mathcal{X} \setminus \mathcal{R}} > t) = e^{-\phi_\mathcal{R}^* t}, \tag{2.12}$$

$$\mu_\mathcal{R}^*(e_\mathcal{R}) = \phi_\mathcal{R}^*. \tag{2.13}$$

The limit in (2.11) is called a *Yaglom limit* after Yaglom showed the existence of such limits in the case of branching processes [2]. In our context of finite state spaces, the existence of such a limit, that does not depend on the starting point x, simply follows from the Perron-Frobenius theorem. In Sections 2.3 and 6 these properties will be rederived in a slightly more general context.

Our first result states that if $1/\phi_\mathcal{R}^*$, the mean exit time for the system started in $\mu_\mathcal{R}^*$, is large with respect to $1/\gamma_\mathcal{R}$, the relaxation time of the reflected process, then the quasi-stationary measure $\mu_\mathcal{R}^*$

is close to the restricted ensemble $\mu_{\mathcal{R}}$. This is similar to Lemma 10 (b) in [9]. More precisely, for all x in $\mathcal{R}$, let us define

$$\varepsilon_{\mathcal{R}}^* = \frac{\phi_{\mathcal{R}}^*}{\gamma_{\mathcal{R}}} \tag{2.14}$$

$$h_{\mathcal{R}}^*(x) = \frac{\mu_{\mathcal{R}}^*(x)}{\mu_{\mathcal{R}}(x)} \tag{2.15}$$

and notice that $h_{\mathcal{R}}^*$ is a right eigenvector of $p_{\mathcal{R}}^*$ with eigenvalue $1 - \phi_{\mathcal{R}}^*$. We prove the following.

Proposition 2.1. *If $\varepsilon_{\mathcal{R}}^* < 1$, then*

$$\mathrm{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*) = \|h_{\mathcal{R}}^* - \mathbb{1}_{\mathcal{R}}\|_{\mathcal{R}}^2 \leq \frac{\varepsilon_{\mathcal{R}}^*}{1 - \varepsilon_{\mathcal{R}}^*} \tag{2.16}$$

Proof. See Section 3.1. □

Remark. When proving that $\varepsilon_{\mathcal{R}}^*$ goes to 0 in some asymptotic regime (for example when the cardinality of the configuration space goes to infinity like in [6], when some parameter of the dynamics goes to 0 like the temperature in [8] or when both happen like in [36]) one has to give upper bounds on $\phi_{\mathcal{R}}^*$ and lower bounds on $\gamma_{\mathcal{R}}$. $\phi_{\mathcal{R}}^*$ satisfies a variational principle through which one can get such upper bounds using suitable test functions. In particular, since one can often easily compute with $\mu_{\mathcal{R}}$, and $e_{\mathcal{R}}$ is often explicit, one can usually estimate

$$\phi_{\mathcal{R}} = \mu_{\mathcal{R}}(e_{\mathcal{R}}) \tag{2.17}$$

and then bound $\phi_{\mathcal{R}}^*$ with $\phi_{\mathcal{R}}$. In some cases, for example in the low-temperature regime, this estimate will already be good enough. More generally and precisely, we have the following lemma, that we prove in Section 3.2.

Lemma 2.2. $\phi_{\mathcal{R}}^* = \min\limits_{\substack{f \neq 0 \\ f_{|\mathcal{X} \backslash \mathcal{R}} = 0}} \dfrac{\mathcal{D}(f)}{\|f\|^2} \leq \dfrac{1}{\mathbb{E}_{\mu_{\mathcal{R}}}\left[\tau_{\mathcal{X} \backslash \mathcal{R}}\right]} \leq \phi_{\mathcal{R}}.$

Lower bounds on $\gamma_{\mathcal{R}}$ can be more difficult to obtain. However we note, first, that rough lower bounds will often be sufficient to our ends, second, that the new Poincaré inequality we will prove in this paper (Theorem 2.10) can be used to this purpose (see Section 7.2).

As a consequence of this first result we can control the convergence rate of the Yaglom limit in (2.11). We note that, by the reversibility of X with respect to μ, $p_{\mathcal{R}}^*$ is a self-adjoint operator on $\ell^2(\mu_{\mathcal{R}})$ and has real eigenvalues. By the Perron-Frobenius theorem, this implies the existence of a spectral gap $\gamma_{\mathcal{R}}^* > 0$ equal to the difference between the first and the second largest eigenvalue of $p_{\mathcal{R}}^*$.

Proposition 2.3. *If $\varepsilon_{\mathcal{R}}^* < \frac{1}{3}$, then*

$$\frac{1}{\gamma_{\mathcal{R}}^*} \leq \frac{1}{\gamma_{\mathcal{R}}} \left\{ \frac{1 - \varepsilon_{\mathcal{R}}^*}{1 - 3\varepsilon_{\mathcal{R}}^*} \right\}. \tag{2.18}$$

Proof. See Section 3.3. □

Remark. Since, after the static study made in [42], we intend to apply our results to the dynamical study of the cavity algorithm introduced in [30], for which finite-volume effects are of first importance, we need to give asymptotics with quantitative control of corrective terms. This produces quite long formulas and to simplify the reading we put between curly brackets any terms that go to 1 in a suitable asymptotic regime.

Then we set

$$\zeta_{\mathcal{R}}^* = \min_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) h_{\mathcal{R}}^{*\,2}(x) = \min_{x \in \mathcal{R}} \mu_{\mathcal{R}}^*(x) h_{\mathcal{R}}^*(x), \tag{2.19}$$

which is the mass of the smallest atom of the measure $\mu_{\mathcal{R}}$ biased by $h_{\mathcal{R}}^{*\,2}$, and define, if $\varepsilon_{\mathcal{R}}^* < 1/3$ and for any $\delta \in]0, 1[$,

$$T_{\delta,\mathcal{R}}^* = \frac{1}{\gamma_{\mathcal{R}}} \left(\ln \frac{2}{\delta(1-\delta)\zeta_{\mathcal{R}}^*} \right) \left\{ \left(1 + \sqrt{\frac{\varepsilon_{\mathcal{R}}^*}{1 - \varepsilon_{\mathcal{R}}^*}} \right) \left(\frac{1 - \varepsilon_{\mathcal{R}}^*}{1 - 3\varepsilon_{\mathcal{R}}^*} \right) \right\} \tag{2.20}$$

to get point-wise mixing estimates for Yaglom limits.

Theorem 2.4 (Mixing towards quasi-stationary measure). *If $\varepsilon_{\mathcal{R}}^* < 1/3$, then for all $x, y \in \mathcal{R}$ and $\delta \in]0, 1[$,*

$$\left| \frac{\mathbb{P}_x(X(t) = y \,|\, \tau_{\mathcal{X} \setminus \mathcal{R}} > t)}{\mu_{\mathcal{R}}^*(y)} - 1 \right| < \delta \quad \text{as soon as} \quad t > T_{\delta, \mathcal{R}}^* \tag{2.21}$$

Proof. See Section 3.4. □

Remark. In words, this says that either the system leaves $\mathcal{R}$ before time $T_{\delta, \mathcal{R}}^*$, or it is described after that time by $\mu_{\mathcal{R}}^*$ in the strongest possible sense. This theorem is useful only if one can provide upper bounds on $T_{\delta, \mathcal{R}}^*$. Bounding $T_{\delta, \mathcal{R}}^*$ depends on the control we have on $\varepsilon_{\mathcal{R}}^*$ and on this new parameter $\zeta_{\mathcal{R}}^*$. As far as the latter is concerned, we note that it only appears in the formula through its logarithm. Crude or very crude estimates of $\zeta_{\mathcal{R}}^*$ will then often be sufficient. One has for example the following lemma.

Lemma 2.5.

i) *With $\zeta_{\mathcal{R}} = \min_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x)$ and $\alpha_{\mathcal{R}} = \max_{x \in \mathcal{R}} e_{\mathcal{R}}(x)$, it holds*

$$\ln \frac{1}{\zeta_{\mathcal{R}}^*} \le \ln \frac{4}{\zeta_{\mathcal{R}}} + \frac{\alpha_{\mathcal{R}}}{\gamma_{\mathcal{R}}} \left[\ln \frac{4 \varepsilon_{\mathcal{R}}^*}{(1 - \varepsilon_{\mathcal{R}}^*) \zeta_{\mathcal{R}}} \right]_+, \tag{2.22}$$

where the brackets $[\cdot]_+$ stand for the positive part.

ii) *If $p(x, x) > 0$ for all $x \in \mathcal{R}$, then*

$$\ln \frac{1}{\zeta_{\mathcal{R}}^*} \le \ln \frac{1}{\min_{x \in \mathcal{R}} \mu_{\mathcal{R}}^{*\,2}(x)} \le 2 \Delta_{\mathcal{R}} D_{\mathcal{R}}, \tag{2.23}$$

where

$$\begin{aligned} \Delta_{\mathcal{R}} &= \max\{- \ln p_{\mathcal{R}}(x, y) \,:\, p_{\mathcal{R}}(x, y) > 0, \, \forall x, y \in \mathcal{R}\} \\ D_{\mathcal{R}} &= \min\{k \ge 0 \,:\, p_{\mathcal{R}}^k(x, y) > 0, \, \forall x, y \in \mathcal{R}\} \end{aligned}.$$

Proof. See Appendix A. □

Also, since $h_{\mathcal{R}}^*$ is superharmonic on $\mathcal{R}$ with respect to $\mathcal{L}$ (see Appendix A), it reaches its minimum on the internal border of $\mathcal{R}$,

$$\partial_- \mathcal{R} = \{x \in \mathcal{R} \,:\, \exists y \notin \mathcal{R}, p(x, y) > 0\}. \tag{2.24}$$

Then we always have

$$\zeta_{\mathcal{R}}^* \ge \left(\min_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) \right) \left(\min_{x \in \partial_- \mathcal{R}} h_{\mathcal{R}}^*(x) \right)^2, \tag{2.25}$$

and in the special case when $\partial_- \mathcal{R}$ reduces to a singleton, this gives, by (2.13) and (2.17),

$$\zeta_{\mathcal{R}}^* \ge \min_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) \left(\frac{\phi_{\mathcal{R}}^*}{\phi_{\mathcal{R}}} \right)^2. \tag{2.26}$$

Bounding $\zeta_{\mathcal{R}}^*$ from below essentially reduces in this case to giving a lower bound on ϕ_R^*, which is one of the main goals of this paper (see Theorem 2.9).

We will make a special choice for the parameter δ in (2.20): we define

$$T_{\mathcal{R}}^* = T_{\varepsilon_{\mathcal{R}}^*, \mathcal{R}}^*. \tag{2.27}$$

We then have

$$\phi_{\mathcal{R}}^* T_{\mathcal{R}}^* \le \varepsilon_{\mathcal{R}}^* \left(\ln \frac{3}{\varepsilon_{\mathcal{R}}^* \zeta_{\mathcal{R}}^*} \right) \left\{ \left(1 + \sqrt{\frac{\varepsilon_{\mathcal{R}}^*}{1 - \varepsilon_{\mathcal{R}}^*}} \right) \left(\frac{1 - \varepsilon_{\mathcal{R}}^*}{1 - 3\varepsilon_{\mathcal{R}}^*} \right) \right\} \tag{2.28}$$

as soon as $\varepsilon_{\mathcal{R}}^* < 1/3$. We will sometimes refer in the sequel to the regime $\phi_{\mathcal{R}}^* T_{\mathcal{R}}^* \ll 1$. Equation (2.28) provides a sufficient and practical condition for being in such a regime. We close this section with a first asymptotic exponential law in this particular regime.

Theorem 2.6 (Asymptotic exit law). *For any probability measure ν on $\mathcal{R}$, define $\pi_{\mathcal{R}}(\nu) = \mathbb{P}_\nu(\tau_{\mathcal{X} \setminus \mathcal{R}} < T_{\mathcal{R}}^*)$. If $\varepsilon_{\mathcal{R}}^* < 1/3$, then, for all $t \ge \phi_{\mathcal{R}}^* T_{\mathcal{R}}^*$,*

$$\begin{cases} \mathbb{P}_\nu(\tau_{\mathcal{X} \setminus \mathcal{R}} > \frac{t}{\phi_{\mathcal{R}}^*}) \le (1 - \pi_{\mathcal{R}}(\nu)) e^{-t} \left\{ e^{\phi_{\mathcal{R}}^* T_{\mathcal{R}}^*} (1 + \varepsilon_{\mathcal{R}}^*) \right\} \\ \mathbb{P}_\nu(\tau_{\mathcal{X} \setminus \mathcal{R}} > \frac{t}{\phi_{\mathcal{R}}^*}) \ge (1 - \pi_{\mathcal{R}}(\nu)) e^{-t} \left\{ e^{\phi_{\mathcal{R}}^* T_{\mathcal{R}}^*} (1 - \varepsilon_{\mathcal{R}}^*) \right\} \end{cases}.$$

Proof. See Section 4.1 □

Remark. The theorem gives more than an asymptotic exponential exit law. It says that, provided $\pi_{\mathcal{R}}(\nu)$ converges to some limit and in the regime $\phi_{\mathcal{R}}^* T_R^* \ll 1$, the normalized mean exit time $\phi_{\mathcal{R}}^* \tau_{\mathcal{X} \backslash \mathcal{R}}$ converges in law to a convex combination between a Dirac mass in 0 and an exponential law with mean 1.

As an example of an application we can consider the case of the restricted ensemble.

Lemma 2.7. *It holds*

$$\pi_{\mathcal{R}}(\mu_{\mathcal{R}}) = \mathbb{P}_{\mu_{\mathcal{R}}}(\tau_{\mathcal{X} \backslash \mathcal{R}} \leq T_{\mathcal{R}}^*) \leq \frac{1}{2}\sqrt{\frac{\varepsilon_{\mathcal{R}}^*}{1 - \varepsilon_{\mathcal{R}}^*}} + \phi_{\mathcal{R}}^* T_{\mathcal{R}}^* . \tag{2.29}$$

Proof. See Section 4.2. □

This shows an asymptotic exponential exit law in the regime $\phi_{\mathcal{R}}^* T_{\mathcal{R}}^* \ll 1$ for the system started in the restricted ensemble.

Another consequence of Theorem 2.6 is that, in the regime $\phi_{\mathcal{R}}^* T_{\mathcal{R}}^* \ll 1$, $\mu_{\mathcal{R}}^*$ asymptotically maximizes the mean exit time on the set of all possible starting measures. This can be seen in a different way by following [9]. Consider, for any $t > 0$ the natural coupling up to time $t \wedge \tau_{\mathcal{X} \backslash \mathcal{R}}$ between X started from a measure ν and a process that starts from $X(0)$, follows the law of the reflected process up to time t, and then the same law as the original process. This last process cannot escape from $\mathcal{R}$ before X and we get,

$$\mathbb{E}_{\nu}\left[\tau_{\mathcal{X} \backslash \mathcal{R}}\right] \leq t + \left(1 + \frac{e^{-\gamma_{\mathcal{R}} t}}{\zeta_R}\right) \mathbb{E}_{\mu_{\mathcal{R}}}\left[\tau_{\mathcal{X} \backslash \mathcal{R}}\right] , \tag{2.30}$$

with, as previously, $\zeta_{\mathcal{R}} = \min_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x)$. Using Lemma 2.2 and optimizing in t one gets

$$\mathbb{E}_{\nu}\left[\tau_{\mathcal{X} \backslash \mathcal{R}}\right] \leq \frac{1}{\phi_{\mathcal{R}}^*}\left\{1 + \varepsilon_{\mathcal{R}}^* + \varepsilon_{\mathcal{R}}^* \ln \frac{1}{\varepsilon_{\mathcal{R}}^* \zeta_R}\right\} . \tag{2.31}$$

We already mentioned that $\phi_{\mathcal{R}}^*$ can be estimated from above by using test functions in a variational principle (see Lemma 2.2). One of the questions raised in [9] is that of upper bounds on mean exit times, i.e., that of lower bounds for $\phi_{\mathcal{R}}^*$. This is the question we will now adress.

2.2. (κ, λ)**-capacities, mean exit times and a new Poincaré inequality.** In this section we introduce a new object which extends the notion of capacity between sets. For any $\kappa, \lambda > 0$ and $A, B \subset \mathcal{X}$, we first extend the electrical network $(\mathcal{X}, c)$, with $c(x, y) = \mu(x)p(x, y) = \mu(y)p(y, x)$ for all distinct $x, y \in \mathcal{X}$, into a larger electrical network $(\tilde{\mathcal{X}}, \tilde{c})$ by attaching a dangling edge $(a, \bar{a})$ with conductance $\kappa\mu(a)$ to each $a \in A$ and a dangling edge $(b, \bar{b})$ with conductance $\lambda\mu(b)$ to each $b \in B$ (this extension is related with some Markov chain modification considered in [14]. More precisely, we add $|A| + |B|$ nodes and edges to the network by setting

$$\tilde{\mathcal{X}} = \mathcal{X} \cup \{\bar{a} : a \in A\} \cup \{\bar{b} : b \in B\}$$

and, for all distinct $\tilde{x}, \tilde{y} \in \tilde{\mathcal{X}}$ we define

$$\tilde{c}(\tilde{x}, \tilde{y}) = \begin{cases} c(x, y) & \text{if } (\tilde{x}, \tilde{y}) = (x, y) \in \mathcal{X} \times \mathcal{X} \\ \kappa\mu(a) & \text{if } (\tilde{x}, \tilde{y}) = (a, \bar{a}) \text{ for some } a \in A \\ \lambda\mu(b) & \text{if } (\tilde{x}, \tilde{y}) = (b, \bar{b}) \text{ for some } b \in B \\ 0 & \text{otherwise} \end{cases} . \tag{2.32}$$

This extended network is naturally associated with *a family* of "two level Markov processes" that evolve like X in $\mathcal{X}$, "go down" from A and B in $\mathcal{X}$ to $\bar{A}$ and $\bar{B}$ at rate κ and λ respectively, and "go up" from $\bar{A}$ and $\bar{B}$ to A and B in $\mathcal{X}$ at some rates tuning the equilibrium measure of such processes in $\tilde{\mathcal{X}}$. (We will use in the proof of our results this liberty in choosing the rates of this family of processes associated with this unique extended electrical network.)

Definition 2.8. The (κ, λ)-capacity, $C_\kappa^\lambda(A, B)$, is defined as the capacity between the sets $\bar{A}$ and $\bar{B}$ in the electrical network $(\tilde{\mathcal{X}}, \tilde{c})$, and then is given, according to Dirichlet principle, by

$$
\begin{aligned}
C_\kappa^\lambda(A, B) &= \min_{\tilde{f}:\tilde{\mathcal{X}}\mapsto\mathbb{R}} \left\{ \frac{1}{2} \sum_{\tilde{x},\tilde{y}\in\tilde{\mathcal{X}}} \tilde{c}(\tilde{x},\tilde{y})[\tilde{f}(\tilde{x}) - \tilde{f}(\tilde{y})]^2 ; \tilde{f}_{|_{\bar{A}}} = 1 , \tilde{f}_{|_{\bar{B}}} = 0 \right\} \\
&= \min_{f:\mathcal{X}\mapsto\mathbb{R}} \mathcal{D}(f) + \kappa \sum_{a\in A} \mu(a)[f(a) - 1]^2 + \lambda \sum_{b\in B} \mu(b)[f(b) - 0]^2 \\
&= \min_{f:\mathcal{X}\mapsto\mathbb{R}} \mathcal{D}(f) + \kappa\mu(A)\mathbb{E}_{\mu_A}\left[(f_{|_A} - 1)^2\right] + \lambda\mu(B)\mathbb{E}_{\mu_B}\left[(f_{|_B} - 0)^2\right] .
\end{aligned}
\tag{2.33}
$$

Remarks.

   i) Since all the points of $\bar{A}$ and $\bar{B}$ are at potential 1 and 0 respectively in formula (2.33), they are electrically equivalent and we could have defined the (κ, λ)-capacity between A and B by adding just two nodes to the electrical network $(\mathcal{X}, c)$. However, our definition with dangling edges will be more useful in the sequel.

   ii) A (κ, λ)-capacity is in some sense easy to estimate since it satisfies a two-sided variational principle. On one hand, by definition, it is the infimum of some functional, and any test function will provide an upper bound. On the other hand it is the supremum of another functional on flows from $\bar{A}$ to $\bar{B}$, which are antisymmetric functions of oriented edges with null divergence in $\mathcal{X}$, i.e., on functions $\tilde{\psi} : \tilde{\mathcal{X}} \times \tilde{\mathcal{X}} \mapsto \mathbb{R}$ such that for all $x \in \tilde{\mathcal{X}} \setminus (\bar{A} \cup \bar{B})$, $\mathrm{div}_x\tilde{\psi} = \sum_{\tilde{x}\in\tilde{\mathcal{X}}} \tilde{\psi}(x, \tilde{x}) = 0$. This is Thomson's principle that goes back to [1], Chapter 1, Appendix A (see also lecture notes [35] for a more modern presentation or textbook [16] for the proof of an almost equivalent result). Letting

$$
\tilde{\mathcal{D}}(\tilde{\psi}) = \tfrac{1}{2} \sum_{\tilde{x},\tilde{y}\in\tilde{\mathcal{X}}} \frac{\tilde{\psi}(\tilde{x},\tilde{y})^2}{\tilde{c}(\tilde{x},\tilde{y})} ,
$$

   be the energy dissipated by the flow $\tilde{\psi}$ in the network $(\tilde{\mathcal{X}}, \tilde{c})$, and $\tilde{\Psi}_1(\bar{A}, \bar{B})$ the set of unitary flows from $\bar{A}$ to $\bar{B}$, that is, the set of flows $\tilde{\psi}$ from $\bar{A}$ to $\bar{B}$ such that

$$
\sum_{\bar{a}\in\bar{A}} \mathrm{div}_{\bar{a}}\tilde{\psi} = \sum_{\bar{a}\in\bar{A}} \sum_{\tilde{x}\in\tilde{\mathcal{X}}} \tilde{\psi}(\bar{a}, \tilde{x}) = 1 = -\sum_{\bar{b}\in\bar{B}} \mathrm{div}_{\bar{b}}\tilde{\psi} = -\sum_{\bar{b}\in\bar{B}} \sum_{\tilde{x}\in\tilde{\mathcal{X}}} \tilde{\psi}(\bar{b}, \tilde{x}) ,
\tag{2.34}
$$

   we have

$$
C_\kappa^\lambda(A, B) = \max_{\tilde{\psi}\in\tilde{\Psi}_1(\bar{A},\bar{B})} \tilde{\mathcal{D}}(\tilde{\psi})^{-1} .
\tag{2.35}
$$

   If $A \cap B = \emptyset$ this gives

$$
\begin{aligned}
C_\kappa^\lambda(A, B) &= \max_{\psi\in\Psi_1(A,B)} \left(\mathcal{D}(\psi) + \sum_{a\in A} \frac{(\mathrm{div}_a\psi)^2}{\kappa\mu(a)} + \sum_{b\in B} \frac{(\mathrm{div}_b\psi)^2}{\lambda\mu(b)} \right)^{-1} \\
&= \max_{\psi\in\Psi_1(A,B)} \left(\mathcal{D}(\psi) + \tfrac{1}{\kappa\mu(A)}\mathbb{E}_{\mu_A}\left[\left(\tfrac{\mathrm{div}\psi}{\mu_A}\right)^2\right] + \tfrac{1}{\lambda\mu(B)}\mathbb{E}_{\mu_B}\left[\left(\tfrac{\mathrm{div}\psi}{\mu_B}\right)^2\right] \right)^{-1} ,
\end{aligned}
\tag{2.36}
$$

   where $\Psi_1(A, B)$ is the set of unitary flows ψ from A to B and

$$
\mathcal{D}(\psi) = \tfrac{1}{2} \sum_{x,y\in\mathcal{X}} \frac{\psi(x,y)^2}{c(x,y)} .
$$

   Then, any test flow provides a lower bound on $C_\kappa^\lambda(A, B)$.

   iii) We know ([16], [35]) that the infimum and supremum in (2.33) and (2.36), are realized, respectively, by the equilibrium potential $V_\kappa^\lambda = \mathbb{P}_{(\cdot)}(\ell_A^{-1}(\sigma_\kappa) < \ell_B^{-1}(\sigma_\lambda))$, where ℓ_A^{-1} and ℓ_B^{-1} are the right continuous inverses of the local times in A and B, while σ_κ and σ_λ are independent exponential times with rates κ and λ, and by its associated normalized current

$$
-\tfrac{c\nabla V_\kappa^\lambda}{C_\kappa^\lambda(A,B)} : (x, y) \in \mathcal{X} \times \mathcal{X} \longmapsto \tfrac{c(x,y)}{C_\kappa^\lambda(A,B)}(V_\kappa^\lambda(x) - V_\kappa^\lambda(y)) .
\tag{2.37}
$$

   We will say more on such quantities in the next section.

iv) The previous definitions and observations extend to the case when κ and λ are equal to $+\infty$. In that case we identify $\bar{A}$ with A in the extended network if $\kappa = +\infty$, or $\bar{B}$ with B if $\lambda = +\infty$, and we drop the infinite upper or lower index in the notation, so that, for example, $C_\kappa(A, B) = C_\kappa^\infty(A, B)$. However, when κ and λ are both equal to $+\infty$, to avoid any ambiguity we need to require that $A \cap B = \emptyset$. In that case the notation becomes $C(A, B) = C_\infty^\infty(A, B)$ and we recover indeed the usual notion of capacity.

We then get sharp asymptotics on mean exit times for the system started in the quasi-stationary measure.

Theorem 2.9 (Mean exit time estimates). *For all $\kappa > 0$, it holds*

$$\frac{C_\kappa(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}{\mu(\mathcal{R})} \left\{ 1 - \varepsilon_\mathcal{R}^* - \frac{\kappa}{\gamma_\mathcal{R}} \right\} \leq \phi_\mathcal{R}^* \leq \frac{C_\kappa(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}{\mu(\mathcal{R})} \left\{ 1 - \frac{C_\kappa(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}{\kappa\mu(\mathcal{R})} \right\}^{-2} \tag{2.38}$$

Proof. See Section 5. □

Remarks.

i) In the regime $\varepsilon_\mathcal{R}^* \ll 1$, one can choose κ such that $\phi_\mathcal{R}^* \ll \kappa \ll \gamma_\mathcal{R}$ and infer, by the lower bound in (2.38), that $\kappa \gg \frac{C_\kappa(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}{\mu(\mathcal{R})}$. In turns, this yields an asymptotical matching upper bound.

ii) Both bounds are in some sense easy to estimate since capacities satisfy a two-sided variational principle. Moreover, compared with the formula for mean exit time provided by potential-theoretic techniques (see, e.g., [20]), the above inequalities require no residual average potential estimates. (Such estimates, as well as some harmonic measures will only play a role in the *proof* of the theorem.)

Our (κ, λ)-capacities provide also spectral gap estimates and a new general Poincaré inequality. For $\kappa, \lambda > 0$ and $A, B \subset \mathcal{X}$ we set

$$\phi_\kappa^\lambda(A, B) = \frac{C_\kappa^\lambda(A, B)}{\mu(A)\mu(B)} = \phi_\lambda^\kappa(B, A). \tag{2.39}$$

Theorem 2.10 (Relaxation time estimates). *For all $\kappa, \lambda > 0$ and any $\mathcal{R} \subset \mathcal{X}$ such that $X_\mathcal{R}$ and $X_{\mathcal{X}\backslash\mathcal{R}}$ are both irreducible Markov processes,*

$$\begin{cases} \frac{1}{\gamma} \geq \frac{1}{\phi_\kappa^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})} \left\{ 1 - \frac{C_\kappa(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}{\kappa\mu(\mathcal{R})} - \frac{C^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}{\lambda\mu(\mathcal{X}\backslash\mathcal{R})} \right\}^2 \\ \frac{1}{\gamma} \leq \frac{1}{\phi_\kappa^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})} \left\{ 1 + \max\left(\frac{\kappa + \phi_\kappa^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}{\gamma_\mathcal{R}}, \frac{\lambda + \phi_\kappa^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}{\gamma_{\mathcal{X}\backslash\mathcal{R}}} \right) \right\} \end{cases}. \tag{2.40}$$

Proof. See Section 5. □

Remarks.

i) Without loss of generality, we can assume $\mu(\mathcal{R}) \leq \mu(\mathcal{X}\backslash\mathcal{R})$ so that, by (2.39), $\phi_\kappa^\lambda \leq 2C_\kappa^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})/\mu(\mathcal{R})$. Then, as a consequence of the previous theorem and of the monotonicity in κ and λ of (κ, λ)-capacities, we get matching bounds on $1/\gamma$ in the regime $\varepsilon_\mathcal{R}^* + \varepsilon_{\mathcal{X}\backslash\mathcal{R}}^* + \phi_\mathcal{R}^*/\gamma_{\mathcal{X}\backslash\mathcal{R}} \ll 1$. One can indeed choose κ such that $\phi_\mathcal{R}^* \ll \kappa \ll \gamma_\mathcal{R}$, just as for Theorem 2.9 (Remark i)), and λ such that $\phi_\mathcal{R}^*, \phi_{\mathcal{X}\backslash\mathcal{R}}^* \ll \lambda \ll \gamma_{\mathcal{X}\backslash\mathcal{R}}$. In addition and like previously, all the relevant quantities can be estimated by two-sided variational principles.

ii) The lower bound is a generalization of the classical isoperimetrical estimate that is recovered for $\kappa = \lambda = +\infty$.

iii) The upper bound is a new Poincaré inequality. This inequality, or an easy-to-derive version when one divides the configuration space into more than two subsets, echoes Poincaré inequalities given in [25]. We are not able to compare in full generality our result with that of [25] but we note that because of the presence of some global parameter called γ in [25] one gets generally in our metastable situation an extra *factor* $1/\min(\gamma_\mathcal{R}, \gamma_{\mathcal{X}\backslash\mathcal{R}})$ by applying the results of [25].

iv) The proof of this upper bound, when considering more than two subsets, extends verbatim to obtain the following result.

Lemma 2.11. *If $\mathcal{R}_1$, $\mathcal{R}_2$, ..., $\mathcal{R}_m$ form a partition of $\mathcal{X}$ for which each of the restricted processes $X_{\mathcal{R}_i}$ is irreducible, if κ_1, κ_2, ..., κ_m are positive real numbers and if we write γ_i for $\gamma_{\mathcal{R}_i}$ and $\phi(i,j) = C_{\kappa_i}^{\kappa_j}(\mathcal{R}_i, \mathcal{R}_j)/(\mu(\mathcal{R}_i)\mu(\mathcal{R}_j))$, then*

$$\frac{1}{\gamma} \leq \left(\frac{1}{2} \sum_{i \neq j} \frac{1}{\phi(i,j)} \right) \left\{ 1 + \frac{\max_i \frac{1}{\gamma_i} \left\{ 1 + \sum_{j \neq i} \frac{\kappa_i}{\phi(i,j)} \right\}}{\frac{1}{2} \sum_{i \neq j} \frac{1}{\phi(i,j)}} \right\}. \tag{2.41}$$

2.3. **Soft measures, local thermalization, transition and mixing times.** We address now the difficulty raised by Lebowitz and Penrose. Whatever the measure we choose to describe our metastable state, restricted ensemble or quasi-stationary measure, it is associated with some subset $\mathcal{R}$ of the configuration space. Then there is an ambiguity when one looks at property (b): what is "getting out" of the metastable state? One is tempted to say that it corresponds in our model to the exit from $\mathcal{R}$. But doing so we are very unlikely to modelize in any satisfactory way property (c): we can expect that "on the edge", when the system just exited $\mathcal{R}$, it has probabilities of the same order to "proceed forward" and thermalize in $\mathcal{X} \setminus \mathcal{R}$ and to go "backward" and thermalize in $\mathcal{R}$. Thus we would like to define what would be a "true escape" from $\mathcal{R}$. Theorem 2.4 suggests an answer in the regime $\phi^*_{\mathcal{X} \setminus \mathcal{R}} T^*_{\mathcal{X} \setminus \mathcal{R}} \ll 1$. We could define the true escape as the first excursion of length $T^*_{\mathcal{X} \setminus \mathcal{R}}$ inside $\mathcal{X} \setminus \mathcal{R}$. Since time randomization is almost always a good idea, we are led to consider a random timer, which is independent of the dynamics and has exponential distribution in order to keep the Markovianity of the process. The timer starts when the dynamics exits $\mathcal{R}$, but if it does not ring before returning to $\mathcal{R}$, the excursion to $\mathcal{X} \setminus \mathcal{R}$ is ignored in the sense that it is not considered a "true escape" from $\mathcal{R}$. A "true escape" happens only when the timer rings during one of the excursion outside $\mathcal{R}$. This will lead to an extension of the concept of quasi-stationary distribution that interpolate between $\mu^*_{\mathcal{R}}$ and $\mu_{\mathcal{R}}$ and we will see (Theorem 2.19 below) that the system will actually be close to equilibrium the first time when the timer will ring during an excursion outside $\mathcal{R}$: it will have truly escaped from metastability.

For any $A \subset \mathcal{X}$ we call

$$\ell_A(t) = \int_0^t \mathbb{1}_A(X(s)) ds \tag{2.42}$$

the local time spent in A up to time t and we denote by ℓ_A^{-1} the right-continuous inverse of ℓ_A:

$$\ell_A^{-1}(t) = \inf\{s \geq 0 : \ell_A(s) > t\}. \tag{2.43}$$

Recall that the process X can be seen as the process updated, according to its discrete version with transition probability matrix p, at each ring of a Poissonian clock with intensity 1. Let us then call τ the first ringing time. For σ_λ an exponential time with mean $1/\lambda$ that is independent from X, we define for all x and y in $\mathcal{R}$

$$p^*_{\mathcal{R},\lambda}(x,y) = \mathbb{P}_x \left(X(\tau^+_{\mathcal{R}}) = y, \ell_{\mathcal{X} \setminus \mathcal{R}}(\tau^+_{\mathcal{R}}) \leq \sigma_\lambda \right) \tag{2.44}$$

with $\tau^+_{\mathcal{R}}$ the return time in $\mathcal{R}$ after the first clock ring, i.e., $\tau^+_{\mathcal{R}} = \tau + \tau_{\mathcal{R}} \circ \theta_\tau$ with θ the usual shift operator. We also define, for all x in $\mathcal{R}$,

$$e_{\mathcal{R},\lambda}(x) = \mathbb{P}_x(\ell_{\mathcal{X} \setminus \mathcal{R}}(\tau^+_{\mathcal{R}}) > \sigma_\lambda) = 1 - \sum_{y \in \mathcal{R}} p^*_{\mathcal{R},\lambda}(x,y) \tag{2.45}$$

and for all x and y in $\mathcal{R}$

$$p_{\mathcal{R},\lambda}(x,y) = \begin{cases} p^*_{\mathcal{R},\lambda}(x,y) & \text{if } x \neq y, \\ p^*_{\mathcal{R},\lambda}(x,x) + e_{\mathcal{R},\lambda}(x) & \text{if } x = y. \end{cases} \tag{2.46}$$

The Markov process $X_{\mathcal{R},\lambda}$ on $\mathcal{R}$ with generator defined by

$$\mathcal{L}_{\mathcal{R},\lambda} f(x) = \sum_{y \in \mathcal{R}} p_{\mathcal{R},\lambda}(x,y)(f(y) - f(x)) \tag{2.47}$$

is reversible with respect to $\mu_{\mathcal{R}}$ and has spectral gap

$$\gamma_{\mathcal{R},\lambda} = \min_{\text{Var}_{\mu_{\mathcal{R}}}(f) \neq 0} \frac{\mathcal{D}_{\mathcal{R},\lambda}(f)}{\text{Var}_{\mu_{\mathcal{R}}}(f)} \tag{2.48}$$

where

$$\mathcal{D}_{\mathcal{R},\lambda}(f) = \frac{1}{2} \sum_{x,y} c_{\mathcal{R},\lambda}(x,y)(f(x) - f(y))^2 \tag{2.49}$$

with

$$c_{\mathcal{R},\lambda}(x,y) = \mu_{\mathcal{R}}(x) p_{\mathcal{R},\lambda}(x,y) = p_{\mathcal{R},\lambda}(y,x) \mu_{\mathcal{R}}(y). \tag{2.50}$$

In addition we define

$$\mathcal{T} := \ell_{\mathcal{X}\setminus\mathcal{R}}^{-1}(\sigma_\lambda) \tag{2.51}$$

$$\tau_{\mathcal{X}\setminus\mathcal{R},\lambda} = \ell_{\mathcal{R}}(\ell_{\mathcal{X}\setminus\mathcal{R}}^{-1}(\sigma_\lambda)). \tag{2.52}$$

We will refer to $\tau_{\mathcal{X}\setminus\mathcal{R},\lambda}$ as the *transition time*, since, for suitable choices of λ, this is the time spent by the process in $\mathcal{R}$ before "truly escaping" from $\mathcal{R}$, as seen by formula (2.65) in Theorem 2.19 below.

Remark. While $\mathcal{T}$ is the global time such that the time spent in $\mathcal{X} \setminus \mathcal{R}$, during possibly many excursions, is equal to σ_λ, the time $\tau_{\mathcal{X}\setminus\mathcal{R},\lambda}$ is the local time on $\mathcal{R}$ associated to $\mathcal{T}$. On one hand, it may thus look natural to address the study toward the characterization of the global time $\mathcal{T}$. On the other hand, the transition time $\tau_{\mathcal{X}\setminus\mathcal{R},\lambda}$ is a natural generalization of the exit time $\tau_{\mathcal{X}\setminus\mathcal{R}}$, given in such a way that when the time $\tau_{\mathcal{X}\setminus\mathcal{R},\lambda}$ is reached, not only the dynamics has exited $\mathcal{R}$ but it has also spent in $\mathcal{X} \setminus \mathcal{R}$ a time equal to σ_λ. With a little effort, we will then derive for $\tau_{\mathcal{X}\setminus\mathcal{R},\lambda}$ similar results to those we have obtained for $\tau_{\mathcal{X}\setminus\mathcal{R}}$, and in particular its asymptotic exponential law (see Theorem 2.17). At this point one may then think to derive information on $\mathcal{T}$ by the identity

$$\mathcal{T} = \sigma_\lambda + \tau_{\mathcal{X}\setminus\mathcal{R},\lambda},$$

but since the random variables σ_λ and $\tau_{\mathcal{X}\setminus\mathcal{R},\lambda}$ are not independent, this representation of $\mathcal{T}$ is not immediately useful. However, we will show that for a suitable range of λ the global time $\mathcal{T}$ and the local time $\tau_{\mathcal{X}\setminus\mathcal{R},\lambda}$ are asymptotically of the same order, which is also the order of the relaxation time (see Theorem 2.19 and remark below).

We know by the Perron-Frobenius theorem that the spectral radius of $p_{\mathcal{R},\lambda}^*$ is a simple positive eigenvalue that is smaller than or equal to 1 and has left and right eigenvectors with positive coordinates. We call it $1 - \phi_{\mathcal{R},\lambda}^*$ and denote by $\mu_{\mathcal{R},\lambda}^*$ the unique associated left eigenvector that is also a probability measure on $\mathcal{R}$. We then have the following lemma.

Lemma 2.12. *It holds*

i) $\phi_{\mathcal{R},\lambda}^* = \mu_{\mathcal{R},\lambda}^*(e_{\mathcal{R},\lambda})$;

ii) $\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\tau_{\mathcal{X}\setminus\mathcal{R},\lambda} > t) = e^{-t\phi_{\mathcal{R},\lambda}^*}$, $\forall t \geq 0$;

iii) $\lim_{t\to\infty} \mathbb{P}_x(X \circ \ell_{\mathcal{R}}^{-1}(t) = y \,|\, \tau_{\mathcal{X}\setminus\mathcal{R},\lambda} > t) = \mu_{\mathcal{R},\lambda}^*(y)$, $\forall x, y \in \mathcal{R}$.

Proof. See Section 6.1. $\qquad\qquad\square$

We say that $\mu_{\mathcal{R},\lambda}^*$ is a quasi-stationary measure associated with a soft barrier, or a soft quasi-stationary measure, or, more simply, a *soft measure*. Indeed, $\mu_{\mathcal{R},\lambda}^*$ is the limiting distribution of the process conditioned to survival when it is killed at rate λ outside $\mathcal{R}$. So, the hardest quasi-stationary measure associated with $\mathcal{R}$, corresponding to $\lambda = +\infty$, is the quasi-stationary measure $\mu_{\mathcal{R}}^*$, while the softest measure, corresponding to $\lambda = 0$, is the restricted ensemble $\mu_{\mathcal{R}}$ ($\phi_{\mathcal{R},0}^* = 0$ and $\mu_{\mathcal{R},0}^*$ is the equilibrium measure associated with $p_{\mathcal{R},0}^* = p_{\mathcal{R},0}$, which is reversible with respect to $\mu_{\mathcal{R}}$). More precisely we have the following.

Lemma 2.13. *The function* $\lambda \in [0, +\infty] \mapsto \mu_{\mathcal{R},\lambda}^* \in \ell^2(\mu_{\mathcal{R}}^*)$ *is a continuous interpolation between the restricted ensemble* $\mu_{\mathcal{R}}$ *and the quasi-stationary distribution* $\mu_{\mathcal{R}}^*$. *In particular, for any* $\lambda_0 \in [0, +\infty]$ *and* $y \in \mathcal{R}$, *we have*

$$\lim_{\lambda\to\lambda_0} \mu_{\mathcal{R},\lambda}^*(y) = \mu_{\mathcal{R},\lambda_0}^*(y) \tag{2.53}$$

and for all $x \in \mathcal{R}$ *it holds the limit commutation property*

$$\lim_{\lambda\to\lambda_0} \lim_{t\to\infty} \mathbb{P}_x(X \circ \ell_{\mathcal{R}}^{-1}(t) = y \,|\, \tau_{\mathcal{X}\setminus\mathcal{R},\lambda} > t) = \lim_{t\to\infty} \lim_{\lambda\to\lambda_0} \mathbb{P}_x(X \circ \ell_{\mathcal{R}}^{-1}(t) = y \,|\, \tau_{\mathcal{X}\setminus\mathcal{R},\lambda} > t). \tag{2.54}$$

Proof. See Section 6.2. $\qquad\qquad\square$

Analogously to what was done in the case $\lambda = +\infty$ we set $\varepsilon_{\mathcal{R},\lambda}^* = \phi_{\mathcal{R},\lambda}^*/\gamma_{\mathcal{R},\lambda}$, $h_{\mathcal{R},\lambda}^* = \mu_{\mathcal{R},\lambda}^*/\mu_{\mathcal{R}}$ and we call $\gamma_{\mathcal{R},\lambda}^*$ the gap between the largest and the second eigenvalue of $p_{\mathcal{R},\lambda}^*$ (since $p_{\mathcal{R},\lambda}^*$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ it has only real eigenvalues). We also define $\phi_{\mathcal{R},\lambda} = \mu_{\mathcal{R}}(e_{\mathcal{R},\lambda})$.

Proposition 2.14. *The parameters $\gamma_{\mathcal{R},\lambda}$, $\phi_{\mathcal{R},\lambda}^*$, $\varepsilon_{\mathcal{R},\lambda}^*$ and $\phi_{\mathcal{R},\lambda}$ depend continuously on λ. In addition, when λ decreases to 0, so do $\phi_{\mathcal{R},\lambda}^*$, $\varepsilon_{\mathcal{R},\lambda}^*$ and $\phi_{\mathcal{R},\lambda}$, while $\gamma_{\mathcal{R},\lambda}$ increases.*

Proof. See Section 6.3. □

The proofs of Sections 3 carry over this more general setup, and we get, by defining the analogous $T_{\delta,\mathcal{R},\lambda}^*$ and $\alpha_{\mathcal{R},\lambda}$ (while ζ_R, which is associated with $\mu_{\mathcal{R}}$ rather than $\mu_{\mathcal{R}}^*$, has no "λ-extension"), the following theorem.

Theorem 2.15 (Mixing towards soft measures). *For all $\lambda \geq 0$, $\phi_{\mathcal{R},\lambda}^* \leq \phi_{\mathcal{R}}^*$, $\gamma_{\mathcal{R},\lambda} \geq \gamma_{\mathcal{R}}$ and $\varepsilon_{\mathcal{R},\lambda}^* \leq \varepsilon_{\mathcal{R}}^*$, Proposition 2.1, Proposition 2.3, Theorem 2.4 and Lemma 2.5 hold with an extra index λ and writing $X \circ \ell_{\mathcal{R}}^{-1}$ instead of X.*

Remark. By continuity and monotonicity, the hypotheses $\varepsilon_{\mathcal{R},\lambda}^* < 1$ and $\varepsilon_{\mathcal{R},\lambda}^* < 1/3$ are always satisfied for λ small enough.

We are now ready to deal with local thermalization: we will identify a "short" time scale on which, for any given starting point, the system will relax towards a mixture of "local equilibria" that are our quasi-stationary measures with soften barriers.

For a given $\kappa \geq 0$, let σ_κ be an exponential time with mean $1/\kappa$ which is independent from X and from σ_λ. We think to σ_k as to the random time which enters in the construction of soft measures over $\mathcal{X} \setminus \mathcal{R}$, in the same way the random time σ_λ entered in the construction of soft measure over $\mathcal{R}$. We define inductively, for $\kappa, \lambda \geq 0$, the stopping times τ_i for $i \geq 0$:

$$\tau_0 = 0, \tag{2.55}$$
$$\tau_1 = \ell_{\mathcal{R}}^{-1}(\sigma_\kappa) \wedge \ell_{\mathcal{X} \setminus \mathcal{R}}^{-1}(\sigma_\lambda), \tag{2.56}$$
$$\tau_{i+1} = \tau_i + \tau_1 \circ \theta_{\tau_i} \tag{2.57}$$

Then for $\delta \in (0,1)$ we call i_0 the smallest $i \geq 1$ such that one of the two following conditions holds,

$$i) \qquad X(\tau_i) \in \mathcal{R} \text{ and } \ell_{\mathcal{R}}(\tau_i) - \ell_{\mathcal{R}}(\tau_{i-1}) > T_{\delta,\mathcal{R},\lambda}^*, \tag{2.58}$$
$$ii) \qquad X(\tau_i) \notin \mathcal{R} \text{ and } \ell_{\mathcal{X} \setminus \mathcal{R}}(\tau_i) - \ell_{\mathcal{X} \setminus \mathcal{R}}(\tau_{i-1}) > T_{\delta,\mathcal{X} \setminus \mathcal{R},\kappa}^*, \tag{2.59}$$

and we set $\tau_\delta = \tau_{i_0}$.

Theorem 2.16 (Local thermalization). *For any $\delta \in (0,1)$ and any probability measure ν on $\mathcal{X}$, if $\varepsilon_{\mathcal{R},\lambda}^* < 1/3$ and $\varepsilon_{\mathcal{X} \setminus \mathcal{R},\kappa}^* < 1/3$, then*

$$\max \left(\max_{x \in \mathcal{R}} \left| \frac{\mathbb{P}_\nu(X(\tau_\delta) = x \mid X(\tau_\delta) \in \mathcal{R})}{\mu_{\mathcal{R},\lambda}^*(x)} - 1 \right|, \max_{x \in \mathcal{X} \setminus \mathcal{R}} \left| \frac{\mathbb{P}_\nu(X(\tau_\delta) = x \mid X(\tau_\delta) \notin \mathcal{R})}{\mu_{\mathcal{X} \setminus \mathcal{R},\kappa}^*(x)} - 1 \right| \right) < \delta. \tag{2.60}$$

Moreover if $\xi = \max \left(e^{\kappa T_{\delta,\mathcal{R},\lambda}^*} - 1, e^{\lambda T_{\delta,\mathcal{X} \setminus \mathcal{R},\kappa}^*} - 1 \right) < 1$, it holds

$$\mathbb{P}_\nu \left(\tau_\delta > t \left(\tfrac{1}{\kappa} + \tfrac{1}{\lambda} \right) \right) \leq e^{-t} \left\{ \frac{1}{1 - \xi} \right\}. \tag{2.61}$$

Proof. See Section 6.4. □

Remark. For κ and λ small enough, we have $\varepsilon_{\mathcal{R},\lambda}^* < 1/3$ and $\varepsilon_{\mathcal{X} \setminus \mathcal{R},\lambda}^* < 1/3$. Then, when κ and λ decrease to 0, we have non-increasing upper bounds on $T_{\delta,\mathcal{R},\lambda}^*$ and $T_{\delta,\mathcal{X} \setminus \mathcal{R},\kappa}^*$. As a consequence, the condition $\xi < 1$ will always be satisfied for κ and λ small enough and the theorem says that starting from any configuration the system is close to a random mixture of two states ($\mu_{\mathcal{R},\lambda}^*$ and $\mu_{\mathcal{X} \setminus \mathcal{R},\kappa}^*$, close to $\mu_{\mathcal{R}}$ and $\mu_{\mathcal{X} \setminus \mathcal{R}}$ respectively) after a time of order $T_{\delta,\mathcal{R},\lambda}^* + T_{\delta,\mathcal{X} \setminus \mathcal{R},\kappa}^*$.

As previously we make special choices for the parameter δ and we set

$$T_{\mathcal{R},\lambda}^* = T_{\varepsilon_{\mathcal{R},\lambda}^*,\mathcal{R},\lambda}^* \quad \text{and} \quad T_{\mathcal{X} \setminus \mathcal{R},\kappa}^* = T_{\varepsilon_{\mathcal{X} \setminus \mathcal{R},\kappa}^*,\mathcal{X} \setminus \mathcal{R},\kappa}^* \tag{2.62}$$

We then have, as soon as $\varepsilon^*_{\mathcal{R},\lambda} < 1/3$,

$$\phi^*_{\mathcal{R},\lambda} T^*_{\mathcal{R},\lambda} \leq \varepsilon^*_{\mathcal{R},\lambda} \left(\ln \frac{3}{\varepsilon^*_{\mathcal{R},\lambda} \zeta^*_{\mathcal{R},\lambda}} \right) \left\{ \left(1 + \sqrt{\frac{\varepsilon^*_{\mathcal{R},\lambda}}{1 - \varepsilon^*_{\mathcal{R},\lambda}}} \right) \left(\frac{1 - \varepsilon^*_{\mathcal{R},\lambda}}{1 - 3\varepsilon^*_{\mathcal{R},\lambda}} \right) \right\} . \tag{2.63}$$

Now the proofs of Section 4 carry over this more general setup and we get asymptotic exponential laws for the transition time $\tau_{\mathcal{X}\backslash\mathcal{R},\lambda}$.

Theorem 2.17 (Asymptotic transition law). *For all $\lambda \geq 0$, Theorem 2.6, Lemma 2.7 and inequality (2.31) hold with an extra index λ.*

We can also give sharp estimates on the mean transition time and asymptotics of the mixing time.

Theorem 2.18 (Mean transition time estimates). *For all $\kappa, \lambda > 0$, setting $\phi^\lambda_\kappa = \phi^\lambda_\kappa(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})$ (recall (2.39)), it holds*

$$\begin{cases} \phi^*_{\mathcal{R},\lambda} \geq \frac{C^\lambda_\kappa(\mathcal{R},\mathcal{X}\backslash\mathcal{R})}{\mu(\mathcal{R})} \left\{ \frac{1 - \mu(\mathcal{R}) - 2\phi^*_{\mathcal{R},\lambda}/\lambda}{1 - \mu(\mathcal{R})} \right\} \left\{ 1 - \max\left(\frac{\kappa + \phi^\lambda_\kappa}{\gamma_\mathcal{R}}, \frac{\lambda + \phi^\lambda_\kappa}{\gamma_{\mathcal{X}\backslash\mathcal{R}}} \right) \right\} , \\ \phi^*_{\mathcal{R},\lambda} \leq \frac{C^\lambda_\kappa(\mathcal{R},\mathcal{X}\backslash\mathcal{R})}{\mu(\mathcal{R})} \left\{ 1 + \varepsilon^*_{\mathcal{R},\lambda} + \varepsilon^*_{\mathcal{R},\lambda} \ln \frac{1}{\varepsilon^*_{\mathcal{R},\lambda} \zeta_\mathcal{R}} + \frac{\phi^*_{\mathcal{R},\lambda}}{\kappa} \right\} . \end{cases} \tag{2.64}$$

Proof. See Section 6.5. □

Remarks

 i) In the regime $\varepsilon^*_\mathcal{R} + \varepsilon^*_{\mathcal{X}\backslash\mathcal{R}} + \phi^*_\mathcal{R}/\gamma_{\mathcal{X}\backslash\mathcal{R}} \ll 1$ and assuming $\mu(\mathcal{R}) \leq \mu(\mathcal{X}\backslash\mathcal{R})$ one can choose κ and λ in such a way that $\phi^*_\mathcal{R} \ll \kappa \ll \gamma_\mathcal{R}$ and $\phi^*_\mathcal{R}, \phi^*_{\mathcal{X}\backslash\mathcal{R}} \ll \lambda \ll \gamma_{\mathcal{X}\backslash\mathcal{R}}$, and then we get matching bounds provided $\varepsilon^*_{\mathcal{R},\lambda} \ll \ln(1/\zeta_\mathcal{R})$. Once again, all the relevant quantities can be estimated via a two-sided variational principle.
 ii) This logarithmic term in the upper bound looks spurious. An upper bound without such a term should hold but we were not able to derive it.

Theorem 2.19 (Mixing time asymptotics). *For $\kappa, \lambda > 0$ and any $x \in \mathcal{X}$, we define $\mathcal{T} = \ell^{-1}_{\mathcal{X}\backslash\mathcal{R}}(\sigma_\lambda)$ and $\nu_x = \mathbb{P}_x(X(\mathcal{T}) = \cdot)$. Then, if $\varepsilon^*_{\mathcal{X}\backslash\mathcal{R},\kappa} < 1/3$*

$$\|\nu_x - \mu_{\mathcal{X}\backslash\mathcal{R}}\|_{TV} \leq \frac{1}{2}\varepsilon^*_{\mathcal{X}\backslash\mathcal{R},\kappa} + \lambda T^*_{\mathcal{X}\backslash\mathcal{R},\kappa}, \tag{2.65}$$

$$\|\nu_x - \mu\|_{TV} \leq \mu(\mathcal{R}) + \sqrt{\frac{\varepsilon^*_{\mathcal{X}\backslash\mathcal{R},\kappa}}{1 - \varepsilon^*_{\mathcal{X}\backslash\mathcal{R},\kappa}}} + \lambda T^*_{\mathcal{X}\backslash\mathcal{R},\kappa} . \tag{2.66}$$

In addition, if

$$\eta = \mu(\mathcal{R}) + 2 \left(\sqrt{\frac{\varepsilon^*_{\mathcal{X}\backslash\mathcal{R},\kappa}}{1 - \varepsilon^*_{\mathcal{X}\backslash\mathcal{R},\kappa}}} + \lambda T^*_{\mathcal{X}\backslash\mathcal{R},\kappa} \right) < \frac{1}{2}, \tag{2.67}$$

then, with

$$t_{\text{mix}} = \inf_{t \geq 0} \left\{ \max_{x \in \mathcal{X}} \|\mathbb{P}_x(X(t) = \cdot) - \mu\|_{TV} \leq \frac{1}{2} \left(\eta + \frac{1}{2} \right) \right\}, \tag{2.68}$$

we have

$$t_{\text{mix}} \leq \frac{2}{\phi^*_{\mathcal{R},\lambda} \left(\frac{1}{2} - \mu(\mathcal{R}) \right)} \left\{ 1 + \varepsilon^*_{\mathcal{R},\lambda} + \varepsilon^*_{\mathcal{R},\lambda} \ln \frac{1}{\varepsilon^*_{\mathcal{R},\lambda} \zeta_R} + \frac{\phi^*_{\mathcal{R},\lambda}}{\lambda} \right\} . \tag{2.69}$$

Proof. See Section 6.6. □

Remark. The theorem makes sense in the regime $\varepsilon^*_\mathcal{R} + \varepsilon^*_{\mathcal{X}\backslash\mathcal{R}} + \phi^*_\mathcal{R}/\gamma_{\mathcal{X}\backslash\mathcal{R}} \ll 1$. One can then choose λ such that $\phi^*_{\mathcal{R},\lambda} \ll \lambda \ll \gamma_{\mathcal{X}\backslash\mathcal{R},\kappa}$. If $\lambda T^*_{\mathcal{X}\backslash\mathcal{R},\kappa} \ll 1$ then $(1 + 2\eta)/4$ can be made as close as $(1 + 2\mu(\mathcal{R}))/4 < 1/2$ as we want. If $\varepsilon^*_{\mathcal{R},\lambda} \ln(1/\zeta_\mathcal{R}) \ll 1$, then the theorem provides the correct order for the mixing time, since the spectral gap goes like $\phi^*_{\mathcal{R},\lambda}/\mu(\mathcal{X}\backslash\mathcal{R})$ and $\mu(\mathcal{X}\backslash\mathcal{R}) \geq 1/2$.

Let us finally summarize our results. To have a mathematical model of the metastability phenomenon described by properties (a)-(c), we first consider a reversible Markov process on a finite state space $\mathcal{X}$, and a subset $\mathcal{R}$ of $\mathcal{X}$ such that $\mu(\mathcal{R}) < \mu(\mathcal{X}\backslash\mathcal{R})$, with μ the equilibrium measure of the process. We the describe metastable states by soft measures associated with $\mathcal{R}$ in the regime $\varepsilon^*_\mathcal{R} + \varepsilon^*_{\mathcal{X}\backslash\mathcal{R}} + \phi^*_\mathcal{R}/\gamma_{\mathcal{X}\backslash\mathcal{R}} \ll 1$. In this regime all soft measures are close to the restricted ensemble

(Theorem 2.15). If we choose κ and λ such that $\phi_{\mathcal{R}}^* \ll \kappa \ll \gamma_{\mathcal{R}}$ and $\phi_{\mathcal{R}}^*, \phi_{\mathcal{X}\backslash\mathcal{R}}^* \ll \lambda \ll \gamma_{\mathcal{X}\backslash\mathcal{R}}$ then we can show

    i) local thermalization towards the soft measure $\mu_{\mathcal{R},\lambda}$ or $\mu_{\mathcal{X}\backslash\mathcal{R},\kappa}$ starting from any configuration in $\mathcal{X}$ and on a *short time scale* $\frac{1}{\kappa} + \frac{1}{\lambda}$ (Theorem 2.16),

    ii) exponential asymptotic transition time on a *long time scale* $\frac{1}{\phi_{\mathcal{R},\lambda}^*} \sim \frac{\mu(\mathcal{R})}{C_\kappa^\lambda(\mathcal{R},\mathcal{X}\backslash\mathcal{R})}$ (Theorems 2.17 and 2.18),

    iii) return time to metastable state on a *still longer time scale* $\frac{1}{\phi_{\mathcal{X}\backslash\mathcal{R},\kappa}^*} \sim \frac{\mu(\mathcal{X}\backslash\mathcal{R})}{C_\kappa^\lambda(\mathcal{R},\mathcal{X}\backslash\mathcal{R})}$ (Theorem 2.18 applied to $\mathcal{X}\backslash\mathcal{R}$ in place of $\mathcal{R}$).

In addition relaxation and mixing times are of the same order as the mean transition time (Theorems 2.10 and 2.19) - in particular the relaxation time has the same exact asymptotic up to a factor $\mu(\mathcal{X}\backslash\mathcal{R})$ - while exit times are on long, but generally shorter, time scale (Theorem 2.9). And we note once again, that all relevant quantities can be estimated via two-sided variational principles.

3. ANALYSIS IN $\ell^2(\mu_{\mathcal{R}})$

3.1. **Proof of Proposition 2.1.** We recall that the reflected process $X_{\mathcal{R}}$ is reversible w.r.t. $\mu_{\mathcal{R}}$ with spectral gap $\gamma_{\mathcal{R}}$. In particular, for any function $f \in \ell^2(\mu_{\mathcal{R}})$, we have the Poincaré inequality $\mathrm{Var}_{\mu_{\mathcal{R}}}(f) \leq \frac{1}{\gamma_{\mathcal{R}}}\mathcal{D}_{\mathcal{R}}(f)$, where $\mathcal{D}_{\mathcal{R}}(f)$ is the Dirichlet form of f given by

$$\mathcal{D}_{\mathcal{R}}(f) = \langle f, -\mathcal{L}_{\mathcal{R}}f\rangle_\mu = \sum_{x,y \in \mathcal{R}} \mu_{\mathcal{R}}(x)f(x)(\delta_x(y) - p_{\mathcal{R}}(x,y))f(y)\,. \tag{3.1}$$

Applying the Poincaré inequality to $h_{\mathcal{R}}^*$, and exploiting the definition of $p_{\mathcal{R}}$ and $p_{\mathcal{R}}^*$, we get

$$
\begin{aligned}
\mathrm{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*) &\leq \frac{1}{\gamma_{\mathcal{R}}}\mathcal{D}_{\mathcal{R}}(h_{\mathcal{R}}^*) = \frac{1}{\gamma_{\mathcal{R}}}\sum_{x,y\in\mathcal{R}}\mu_{\mathcal{R}}(x)h_{\mathcal{R}}^*(x)\left(\delta_x(y) - p_{\mathcal{R}}(x,y)\right)h_{\mathcal{R}}^*(y)\\
&= \frac{1}{\gamma_{\mathcal{R}}}\left(\mu_{\mathcal{R}}(h_{\mathcal{R}}^{*\,2}) - \sum_{x,y\in\mathcal{R}}\mu_{\mathcal{R}}^*(x)p_{\mathcal{R}}(x,y)h_{\mathcal{R}}^*(y)\right)\\
&= \frac{1}{\gamma_{\mathcal{R}}}\left(\mu_{\mathcal{R}}(h_{\mathcal{R}}^{*\,2}) - \sum_{x,y\in\mathcal{R}}\mu_{\mathcal{R}}^*(x)(p(x,y) + \delta_x(y)e_{\mathcal{R}}(x))h_{\mathcal{R}}^*(y)\right)\\
&\leq \frac{1}{\gamma_{\mathcal{R}}}\left(\mu_{\mathcal{R}}(h_{\mathcal{R}}^{*\,2}) - \sum_{x,y\in\mathcal{R}}\mu_{\mathcal{R}}^*(x)p_{\mathcal{R}}^*(x,y)h_{\mathcal{R}}^*(y)\right)\,.
\end{aligned}
\tag{3.2}
$$

From the last line, using that $\mu_{\mathcal{R}}^*$ is a left eigenvector of $p_{\mathcal{R}}^*$ with eigenvalue $(1 - \phi_{\mathcal{R}}^*)$ and that $\mu_{\mathcal{R}}^*(h_{\mathcal{R}}^*) = \mu_{\mathcal{R}}(h_{\mathcal{R}}^{*\,2})$, we get

$$\mathrm{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*) \leq \frac{\phi_{\mathcal{R}}^*}{\gamma_{\mathcal{R}}}\mu_{\mathcal{R}}(h_{\mathcal{R}}^{*\,2}) = \frac{\phi_{\mathcal{R}}^*}{\gamma_{\mathcal{R}}}\left(\mathrm{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*) + 1\right)\,. \tag{3.3}$$

Finally, rearranging the terms in the above inequality and from the hypothesis $\varepsilon_{\mathcal{R}}^* = \frac{\phi_{\mathcal{R}}^*}{\gamma_{\mathcal{R}}} < 1$, we obtain the required upper bound.

3.2. **Proof of Lemma 2.2.** Let us denote by $\mathcal{L}_{\mathcal{R}}^*$ the sub-Markovian generator associated to the kernel $p_{\mathcal{R}}^*$. For any function $f \in \ell^2(\mu_{\mathcal{R}})$, this is defined as

$$(\mathcal{L}_{\mathcal{R}}^*f)(x) = -f(x) + \sum_{y\in\mathcal{R}}p_{\mathcal{R}}^*(x,y)f(y)\,, \tag{3.4}$$

and we have the following useful lemma:

Lemma 3.1. *For all $f \in \ell^2(\mu_{\mathcal{R}})$, it holds*

$$\mathcal{D}_{\mathcal{R}}(f) \leq \frac{\mathcal{D}(f)}{\mu(\mathcal{R})} = \langle f, -\mathcal{L}_{\mathcal{R}}^*f\rangle_{\mathcal{R}}\,. \tag{3.5}$$

Proof of Lemma 3.1 . For all $x, y \in \mathcal{R}$ with $x \neq y$, $p_{\mathcal{R}}(x,y) = p(x,y)$. Then we have

$$
\begin{aligned}
\mathcal{D}_{\mathcal{R}}(f) &= \frac{1}{2} \sum_{x,y \in \mathcal{R}} \mu_{\mathcal{R}}(x) p_{\mathcal{R}}(x,y) \left[f(x) - f(y) \right]^2 \\
&= \frac{1}{2} \sum_{x,y \in \mathcal{R}} \mu_{\mathcal{R}}(x) p(x,y) \left[f(x) - f(y) \right]^2 ,
\end{aligned}
\tag{3.6}
$$

since only the terms in $x \neq y$ matter in this sum. Thus, extending the sum to all $x, y \in \mathcal{X}$,

$$
\mathcal{D}_{\mathcal{R}}(f) \leq \frac{1}{2} \sum_{x,y \in \mathcal{X}} \mu_{\mathcal{R}}(x) p(x,y) \left[f(x) - f(y) \right]^2 \leq \frac{\mathcal{D}(f)}{\mu(\mathcal{R})} ,
\tag{3.7}
$$

and this provides the stated upper bound.

To prove the equality, we recall that the space $\ell^2(\mu_{\mathcal{R}})$ is identified with the subset of functions $f \in \ell^2(\mu)$ with $f_{|\mathcal{X} \setminus \mathcal{R}} \equiv 0$. Since, for all $x, y \in \mathcal{R}$, it holds that $\mu_{\mathcal{R}}(x) = \mu(x)/\mu(\mathcal{R})$ and $p_{\mathcal{R}}^*(x,y) = p(x,y)$, we have

$$
\begin{aligned}
\frac{\mathcal{D}(f)}{\mu(\mathcal{R})} &= \frac{1}{\mu(\mathcal{R})} \sum_{x,y \in \mathcal{X}} \mu(x) f(x) \left(\delta_x(y) - p(x,y) \right) f(y) \\
&= \sum_{x,y \in \mathcal{R}} \mu_{\mathcal{R}}(x) f(x) \left(\delta_x(y) - p_{\mathcal{R}}^*(x,y) \right) f(y) \\
&= \langle f, -\mathcal{L}_{\mathcal{R}}^* f \rangle_{\mathcal{R}} ,
\end{aligned}
\tag{3.8}
$$

which concludes the proof. $\qquad\square$

We can now proceed with the proof of Lemma 2.2. Since $1 - \phi_{\mathcal{R}}^*$ is the largest eigenvalue of $p_{\mathcal{R}}^*$, we have

$$
\phi_{\mathcal{R}}^* = \min_{\substack{f:\mathcal{R}\to\mathbb{R} \\ f \neq 0}} \frac{\langle f, -\mathcal{L}_{\mathcal{R}}^* f \rangle_{\mathcal{R}}}{\langle f, f \rangle_{\mathcal{R}}} ,
\tag{3.9}
$$

then the equality in Lemma 2.2 is a consequence of Lemma 3.1. Taking $f = \mathbb{1}_{\mathcal{R}}$ as test function in (3.9), we get

$$
\begin{aligned}
\phi_{\mathcal{R}}^* &\leq \sum_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) \left(1 - \sum_{y \in \mathcal{R}} p_{\mathcal{R}}^*(x,y) \right) \\
&= \sum_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) \left(1 - \sum_{y \in \mathcal{R}} p(x,y) \right) \\
&= \sum_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) e_{\mathcal{R}}(x) = \phi_{\mathcal{R}} ,
\end{aligned}
\tag{3.10}
$$

and it remains to prove that $\mathbb{E}_{\mu_R} \left[\tau_{\mathcal{X} \setminus \mathcal{R}} \right]$ lies between $1/\phi_{\mathcal{R}}$ and $1/\phi_{\mathcal{R}}^*$.

Since, for any $k \in \mathbb{N}$, $(1 - \phi_{\mathcal{R}}^*)^k$ is the largest eigenvalue of $(p_{\mathcal{R}}^*)^k$, the same argument gives

$$
1 - (1 - \phi_{\mathcal{R}}^*)^k \leq \sum_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) \left(1 - \sum_{y \in \mathcal{R}} (p_{\mathcal{R}}^*)^k (x,y) \right)
\tag{3.11}
$$

namely,

$$
(1 - \phi_{\mathcal{R}}^*)^k \geq \sum_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) \sum_{y \in \mathcal{R}} (p_{\mathcal{R}}^*)^k (x,y) .
\tag{3.12}
$$

By summing on $k \geq 0$ and with $\hat{X}$ the discrete time version of X, such that X follows $\hat{X}$ at each ring of a Poissonian clock of intensity 1, we have, with obvious notation,

$$
\frac{1}{\phi_{\mathcal{R}}^*} = \sum_{k \geq 0} (1 - \phi_{\mathcal{R}}^*)^k \geq \sum_{k \geq 1} \mathbb{P}_{\mu_{\mathcal{R}}} (\hat{\tau}_{\mathcal{X} \setminus \mathcal{R}} \geq k) = \mathbb{E}_{\mu_{\mathcal{R}}} [\hat{\tau}_{\mathcal{X} \setminus \mathcal{R}}] \geq \mathbb{P}_{\mu_{\mathcal{R}}} (\hat{\tau}_{\mathcal{X} \setminus \mathcal{R}} = 1) = \frac{1}{\phi_{\mathcal{R}}} .
\tag{3.13}
$$

3.3. **Proof of Proposition 2.3.** The second smallest eigenvalue of the sub-Markovian generator $\mathcal{L}_{\mathcal{R}}^{*}$, $\phi_{\mathcal{R}}^{*} + \gamma_{\mathcal{R}}^{*}$, satisfies the variational formula

$$\phi_{\mathcal{R}}^{*} + \gamma_{\mathcal{R}}^{*} = \min\left\{ \frac{\langle f, -\mathcal{L}_{\mathcal{R}}^{*} f\rangle_{\mathcal{R}}}{\langle f, f\rangle_{\mathcal{R}}} : f \neq 0, \langle f, h_{\mathcal{R}}^{*}\rangle_{\mathcal{R}} = 0 \right\} \tag{3.14}$$

$$= \min\left\{ \langle f, -\mathcal{L}_{\mathcal{R}}^{*} f\rangle_{\mathcal{R}} : \langle f, h_{\mathcal{R}}^{*}\rangle_{\mathcal{R}} = 0, \langle f, f\rangle_{\mathcal{R}} = 1 \right\}$$

Let f be a function on $\mathcal{R}$ that realizes the minimum in the above definition, with $\langle f, f\rangle_{\mathcal{R}} = 1$. Since $\langle f, h_{\mathcal{R}}^{*}\rangle_{\mathcal{R}} = 0$, we have

$$\langle f, h_{\mathcal{R}}^{*} - \mathbb{1}_{\mathcal{R}}\rangle_{\mathcal{R}} = -\langle f, \mathbb{1}_{\mathcal{R}}\rangle_{\mathcal{R}} = -\mu_{\mathcal{R}}(f)$$

and then, by the Cauchy-Schwartz inequality together with Proposition 2.1,

$$\mu_{\mathcal{R}}^{2}(f) \leq \|f\|_{\mathcal{R}}^{2} \cdot \|h_{\mathcal{R}}^{*} - \mathbb{1}_{\mathcal{R}}\|_{\mathcal{R}}^{2} \leq \frac{\varepsilon_{\mathcal{R}}^{*}}{1 - \varepsilon_{\mathcal{R}}^{*}}. \tag{3.15}$$

Now, writing the orthogonal decomposition $f = \mu_{\mathcal{R}}(f) + g$, with $\mu_{\mathcal{R}}(g) = 0$, we have

$$1 = \|f\|_{\mathcal{R}}^{2} = \mu_{\mathcal{R}}^{2}(f) + \|g\|_{\mathcal{R}}^{2}$$

and thus, from (3.15),

$$\|g\|_{\mathcal{R}}^{2} = 1 - \mu_{\mathcal{R}}^{2}(f) \geq 1 - \frac{\varepsilon_{\mathcal{R}}^{*}}{1 - \varepsilon_{\mathcal{R}}^{*}} = \frac{1 - 2\varepsilon_{\mathcal{R}}^{*}}{1 - \varepsilon_{\mathcal{R}}^{*}}.$$

Using g as a test function in

$$\gamma_{\mathcal{R}} = \min\left\{ \frac{\mathcal{D}_{\mathcal{R}}(h)}{\|h\|_{\mathcal{R}}^{2}} : h \neq 0, \mu_{\mathcal{R}}(h) = 0 \right\}, \tag{3.16}$$

we get

$$\gamma_{\mathcal{R}} \leq \frac{1 - \varepsilon_{\mathcal{R}}^{*}}{1 - 2\varepsilon_{\mathcal{R}}^{*}} \mathcal{D}_{\mathcal{R}}(g) = \frac{1 - \varepsilon_{\mathcal{R}}^{*}}{1 - 2\varepsilon_{\mathcal{R}}^{*}} \mathcal{D}_{\mathcal{R}}(f). \tag{3.17}$$

From Lemma 3.1, and using that f was chosen in order to have $\langle f, -\mathcal{L}_{\mathcal{R}}^{*} f\rangle_{\mathcal{R}} = \phi_{\mathcal{R}}^{*} + \gamma_{\mathcal{R}}^{*}$, we get

$$\gamma_{\mathcal{R}} \leq \frac{1 - \varepsilon_{\mathcal{R}}^{*}}{1 - 2\varepsilon_{\mathcal{R}}^{*}} \langle f, -\mathcal{L}_{\mathcal{R}}^{*} f\rangle_{\mathcal{R}} = \frac{1 - \varepsilon_{\mathcal{R}}^{*}}{1 - 2\varepsilon_{\mathcal{R}}^{*}} (\phi_{\mathcal{R}}^{*} + \gamma_{\mathcal{R}}^{*}). \tag{3.18}$$

Setting $\phi_{\mathcal{R}}^{*} = \varepsilon_{\mathcal{R}}^{*} \gamma_{\mathcal{R}}$ and rearranging the terms in the last inequality, we get

$$\left(\frac{1 - 3\varepsilon_{\mathcal{R}}^{*} + \varepsilon_{\mathcal{R}}^{*}{}^{2}}{1 - 2\varepsilon_{\mathcal{R}}^{*}} \right) \gamma_{\mathcal{R}} \leq \left(\frac{1 - \varepsilon_{\mathcal{R}}^{*}}{1 - 2\varepsilon_{\mathcal{R}}^{*}} \right) \gamma_{\mathcal{R}}^{*},$$

which, under the hypothesis $\varepsilon_{\mathcal{R}}^{*} < 1/3$, implies

$$\frac{1}{\gamma_{\mathcal{R}}^{*}} \leq \frac{1}{\gamma_{\mathcal{R}}} \left\{ \frac{1 - \varepsilon_{\mathcal{R}}^{*}}{1 - 3\varepsilon_{\mathcal{R}}^{*}} \right\}.$$

3.4. **Proof of Theorem 2.4.** The proof is based on a classical trick to control mixing times with relaxation times. For any probability measure ν on $\mathcal{R}$, any $f : \mathcal{R} \to \mathbb{R}$ such that $\mu_{\mathcal{R}}^{*}(f) \neq 0$ and any $s, t \geq 0$, one can check that

$$\mathbb{E}_{\nu}[f(X(s+t))\mathbb{1}_{\{\tau_{\mathcal{X}\backslash\mathcal{R}} > s+t\}}] - \mu_{\mathcal{R}}^{*}(f)\mathbb{P}_{\nu}(\tau_{\mathcal{X}\backslash\mathcal{R}} > s+t)$$

$$= \sum_{y \in \mathcal{R}} \left(\mathbb{P}_{\nu}(X(s) = y, \tau_{\mathcal{X}\backslash\mathcal{R}} > s) - \mathbb{P}_{\nu}(\tau_{\mathcal{X}\backslash\mathcal{R}} > s)\mu_{\mathcal{R}}^{*}(y) \right) \tag{3.19}$$

$$\times \left(\mathbb{E}_{y}[f(X(t))\mathbb{1}_{\{\tau_{\mathcal{X}\backslash\mathcal{R}} > t\}}] - \mathbb{P}_{y}(\tau_{\mathcal{X}\backslash\mathcal{R}} > t)\mu_{\mathcal{R}}^{*}(f) \right).$$

Indeed, one can rewrite the right-hand side of the above equality as the sum of four terms, two of which coincide with the two terms in the left-hand side by the Markov property, while the other two terms cancel using the quasi-stationarity property, i.e.

$$\mathbb{E}_{\mu_{\mathcal{R}}^{*}}\left[f(X(t)) \,|\, \tau_{\mathcal{X}\backslash\mathcal{R}} > t \right] = \mu_{\mathcal{R}}^{*}(f). \tag{3.20}$$

As a consequence, by the Cauchy-Schwartz inequality one gets

$$
\begin{aligned}
\Big| \mathbb{E}_\nu [f(X(s+t)) \mathbb{1}_{\{\tau_{\mathcal{X}\setminus\mathcal{R}}>t\}}] - \mu_\mathcal{R}^*(f) &\mathbb{P}_\nu(\tau_{\mathcal{X}\setminus\mathcal{R}} > s+t) \Big| \\
&\leq \left\| \frac{\mathbb{P}_\nu(X(s)=\cdot\,,\,\tau_{\mathcal{X}\setminus\mathcal{R}}>s)}{\mu_\mathcal{R}(\cdot)} - \mathbb{P}_\nu(\tau_{\mathcal{X}\setminus\mathcal{R}} > s) h_\mathcal{R}^*(\cdot) \right\|_\mathcal{R} \\
&\times \left\| \mathbb{E}_{(\cdot)}[f(X(t)) \mathbb{1}_{\{\tau_{\mathcal{X}\setminus\mathcal{R}}>t\}}] - \mathbb{P}_{(\cdot)}(\tau_{\mathcal{X}\setminus\mathcal{R}} > t) \mu_\mathcal{R}^*(f) \right\|_\mathcal{R} .
\end{aligned}
\tag{3.21}
$$

We now estimate these two factors. Noting that

$$
\mathbb{P}_\nu(X(s) = \cdot\,,\, \tau_{\mathcal{X}\setminus\mathcal{R}} > s) = \nu e^{s\mathcal{L}_\mathcal{R}^*}(\cdot) \quad \text{and} \quad \mathbb{E}_{(\cdot)}[f(X(t))\mathbb{1}_{\{\tau_{\mathcal{X}\setminus\mathcal{R}}>t\}}] = e^{t\mathcal{L}_\mathcal{R}^*} f(\cdot) ,
$$

and diagonalizing the self-adjoint operator $\mathcal{L}_\mathcal{R}^*$ in an orthonormal basis, one gets

$$
\left\| \frac{\mathbb{P}_\nu(X(s)=\cdot\,,\,\tau_{\mathcal{X}\setminus\mathcal{R}}>s)}{\mu_\mathcal{R}(\cdot)} - \left\|\frac{\nu}{\mu_\mathcal{R}}\right\|_\mathcal{R} \frac{h_\mathcal{R}^*}{\|h_\mathcal{R}^*\|_\mathcal{R}} \cos\theta_\nu \, e^{-\phi_\mathcal{R}^* s} \right\|_\mathcal{R}^2 \leq \left\|\frac{\nu}{\mu_\mathcal{R}}\right\|_\mathcal{R}^2 \sin^2\theta_\nu e^{-2s(\phi_\mathcal{R}^*+\gamma_\mathcal{R}^*)} ,
\tag{3.22}
$$

with $\theta_\nu \in [0,\pi/2[$ such that $\left\|\frac{\nu}{\mu_\mathcal{R}}\right\|_\mathcal{R} \|h_\mathcal{R}^*\|_\mathcal{R} \cos\theta_\nu = \langle \frac{\nu}{\mu_\mathcal{R}}, h_\mathcal{R}^* \rangle = \nu(h_\mathcal{R}^*)$, and

$$
\left\| \mathbb{E}_{(\cdot)}[f(X(t))\mathbb{1}_{\{\tau_{\mathcal{X}\setminus\mathcal{R}}>t\}}] - \|f\|_\mathcal{R} \frac{h_\mathcal{R}^*}{\|h_\mathcal{R}^*\|_\mathcal{R}} \cos\theta_f \, e^{-\phi_\mathcal{R}^* t} \right\|_\mathcal{R}^2 \leq \|f\|_\mathcal{R}^2 \sin^2\theta_f e^{-2t(\phi_\mathcal{R}^*+\gamma_\mathcal{R}^*)}
\tag{3.23}
$$

with $\theta_f \in [0,\pi] \setminus \{\pi/2\}$ such that $\|f\|_\mathcal{R} \|h_\mathcal{R}^*\|_\mathcal{R} \cos\theta_f = \langle f, h_\mathcal{R}^* \rangle = \mu_\mathcal{R}^*(f)$.
Moreover, since

$$
\mathbb{P}_\nu(\tau_{\mathcal{X}\setminus\mathcal{R}} > s) = \mu_\mathcal{R}\left(\frac{\mathbb{P}_\nu(X(s)=\cdot\,,\,\tau_{\mathcal{X}\setminus\mathcal{R}}>s)}{\mu_\mathcal{R}(\cdot)} \right) ,
$$

by the Cauchy-Schwartz inequality and using (3.22) we get

$$
\begin{aligned}
\left| \mathbb{P}_\nu(\tau_{\mathcal{X}\setminus\mathcal{R}} > s) - \left\|\frac{\nu}{\mu_\mathcal{R}}\right\|_\mathcal{R} \frac{\cos\theta_\nu}{\|h_\mathcal{R}^*\|_\mathcal{R}} e^{-\phi_\mathcal{R}^* s} \right| &= \left| \mu_\mathcal{R}\left(\frac{\mathbb{P}_\nu(X(s)=\cdot\,,\,\tau_{\mathcal{X}\setminus\mathcal{R}}>s)}{\mu_\mathcal{R}(\cdot)} - \left\|\frac{\nu}{\mu_\mathcal{R}}\right\|_\mathcal{R} \frac{h_\mathcal{R}^*}{\|h_\mathcal{R}^*\|_\mathcal{R}} \cos\theta_\nu \, e^{-\phi_\mathcal{R}^* s} \right) \right| \\
&\leq \left\langle \mathbb{1}_\mathcal{R}, \left| \frac{\mathbb{P}_\nu(X(s)=\cdot\,,\,\tau_{\mathcal{X}\setminus\mathcal{R}}>s)}{\mu_\mathcal{R}(\cdot)} - \left\|\frac{\nu}{\mu_\mathcal{R}}\right\|_\mathcal{R} \frac{h_\mathcal{R}^*}{\|h_\mathcal{R}^*\|_\mathcal{R}} \cos\theta_\nu \, e^{-\phi_\mathcal{R}^* s} \right| \right\rangle_\mathcal{R} \\
&\leq \left\|\frac{\nu}{\mu_\mathcal{R}}\right\|_\mathcal{R} \sin\theta_\nu e^{-s(\phi_\mathcal{R}^*+\gamma_\mathcal{R}^*)} .
\end{aligned}
\tag{3.24}
$$

Using inequalities (3.22) and (3.24), we finally get

$$
\begin{aligned}
\left\| \frac{\mathbb{P}_\nu(X(s)=\cdot\,,\,\tau_{\mathcal{X}\setminus\mathcal{R}}>s)}{\mu_\mathcal{R}(\cdot)} - \mathbb{P}_\nu(\tau_{\mathcal{X}\setminus\mathcal{R}} > s) h_\mathcal{R}^*(\cdot) \right\|_\mathcal{R} & \\
&\hspace{-9em}\leq \left\| \frac{\mathbb{P}_\nu(X(s)=\cdot\,,\,\tau_{\mathcal{X}\setminus\mathcal{R}}>s)}{\mu_\mathcal{R}(\cdot)} - \left\|\frac{\nu}{\mu_\mathcal{R}}\right\|_\mathcal{R} \frac{h_\mathcal{R}^*}{\|h_\mathcal{R}^*\|_\mathcal{R}} \cos\theta_\nu \, e^{-\phi_\mathcal{R}^* s} \right\|_\mathcal{R} \\
&\hspace{-9em}\quad + \left\| \left(\mathbb{P}_\nu(\tau_{\mathcal{X}\setminus\mathcal{R}} > s) - \left\|\frac{\nu}{\mu_\mathcal{R}}\right\|_\mathcal{R} \frac{\cos\theta_\nu}{\|h_\mathcal{R}^*\|_\mathcal{R}} e^{-\phi_\mathcal{R}^* s} \right) h_\mathcal{R}^* \right\|_\mathcal{R} \\
&\hspace{-9em}\leq \left\|\frac{\nu}{\mu_\mathcal{R}}\right\|_\mathcal{R} (1 + \|h_\mathcal{R}^*\|_\mathcal{R}) \sin\theta_\nu \, e^{-s(\phi_\mathcal{R}^*+\gamma_\mathcal{R}^*)} .
\end{aligned}
\tag{3.25}
$$

which provides an estimate of the first factor in (3.21).

To what concerns the second factor, noting that

$$
\mathbb{P}_{(\cdot)}(\tau_{\mathcal{X}\setminus\mathcal{R}} > t) = \mathbb{E}_{(\cdot)}[\mathbb{1}_\mathcal{R}(X(t)) \mathbb{1}_{\{\tau_{\mathcal{X}\setminus\mathcal{R}}>t\}}]
$$

and that, from the definition of $\cos\theta_f$ applied to $f = \mathbb{1}_\mathcal{R}$,

$$
\cos\theta_{\mathbb{1}_\mathcal{R}} = \frac{1}{\|h_\mathcal{R}^*\|_\mathcal{R}} \quad \text{and} \quad \sin^2\theta_{\mathbb{1}_\mathcal{R}} = \frac{\|h_\mathcal{R}^*\|_\mathcal{R}^2 - 1}{\|h_\mathcal{R}^*\|_\mathcal{R}^2} = \frac{\mathrm{Var}_{\mu_\mathcal{R}}(h_\mathcal{R}^*)}{\|h_\mathcal{R}^*\|_\mathcal{R}^2} ,
$$

from inequality (3.23) we get

$$
\left\| \mathbb{P}_{(\cdot)}(\tau_{\mathcal{X}\setminus\mathcal{R}} > t) - \frac{h_\mathcal{R}^*}{\|h_\mathcal{R}^*\|_\mathcal{R}^2} e^{-\phi_\mathcal{R}^* t} \right\|^2 \leq \frac{\mathrm{Var}_{\mu_\mathcal{R}}(h_\mathcal{R}^*)}{\|h_\mathcal{R}^*\|_\mathcal{R}^2} e^{-2t(\phi_\mathcal{R}^*+\gamma_\mathcal{R}^*)} .
\tag{3.26}
$$

Then, from inequalities (3.23) and (3.26),

$$\left\| \mathbb{E}_{(\cdot)}[f(X(t)) \, \mathbb{1}_{\{\tau_{\mathcal{X}\backslash\mathcal{R}}>t\}}] - \mathbb{P}_{(\cdot)}(\tau_{\mathcal{X}\backslash\mathcal{R}} > t)\mu_{\mathcal{R}}^*(f) \right\|_{\mathcal{R}}$$

$$\leq \left\| \mathbb{E}_{(\cdot)}[f(X(t))\mathbb{1}_{\{\tau_{\mathcal{X}\backslash\mathcal{R}}>t\}}] - \|f\|_{\mathcal{R}} \frac{h_{\mathcal{R}}^*}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} \cos\theta_f \, e^{-\phi_{\mathcal{R}}^* t} \right\|_{\mathcal{R}}$$

$$+ \left\| \mathbb{P}_{(\cdot)}(\tau_{\mathcal{X}\backslash\mathcal{R}} > t)\|f\|_{\mathcal{R}} \|h_{\mathcal{R}}^*\|_{\mathcal{R}} \cos\theta_f - \|f\|_{\mathcal{R}} \frac{h_{\mathcal{R}}^*}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} \cos\theta_f \, e^{-\phi_{\mathcal{R}}^* t} \right\|_{\mathcal{R}} \quad (3.27)$$

$$\leq \|f\|_{\mathcal{R}} \sin\theta_f e^{-t(\phi_{\mathcal{R}}^*+\gamma_{\mathcal{R}}^*)} + \|f\|_{\mathcal{R}} \|h_{\mathcal{R}}^*\|_{\mathcal{R}} \frac{\sqrt{\mathrm{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*)}}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} \cos\theta_f \, e^{-t(\phi_{\mathcal{R}}^*+\gamma_{\mathcal{R}}^*)}$$

$$= \left(\|f\|_{\mathcal{R}} \sin\theta_f + \frac{\mu_{\mathcal{R}}^*(f)\sqrt{\mathrm{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*)}}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} \right) e^{-t(\phi_{\mathcal{R}}^*+\gamma_{\mathcal{R}}^*)} .$$

which provides an estimate of the second factor in (3.21).

Inserting (3.25) and (3.27) in (3.21), we then obtain

$$\left| \mathbb{E}_\nu[f(X(s+t))\mathbb{1}_{\{\tau_{\mathcal{X}\backslash\mathcal{R}}>t\}}] - \mu_{\mathcal{R}}^*(f)\mathbb{P}_\nu(\tau_{\mathcal{X}\backslash\mathcal{R}} > s+t) \right|$$

$$\leq \|\tfrac{\nu}{\mu_{\mathcal{R}}}\|_{\mathcal{R}}(1 + \|h_{\mathcal{R}}^*\|_{\mathcal{R}}) \sin\theta_\nu \left(\|f\|_{\mathcal{R}} \sin\theta_f + \frac{\mu_{\mathcal{R}}^*(f)\sqrt{\mathrm{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*)}}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} \right) e^{-(s+t)(\phi_{\mathcal{R}}^*+\gamma_{\mathcal{R}}^*)} . \quad (3.28)$$

To conclude our proof we will make two more steps. First notice that from (3.24) one also gets that, for any $t \geq 0$,

$$\mathbb{P}_\nu(\tau_{\mathcal{X}\backslash\mathcal{R}} > t) \geq \|\tfrac{\nu}{\mu_{\mathcal{R}}}\|_{\mathcal{R}} \left(\frac{\cos\theta_\nu}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} e^{-t\phi_{\mathcal{R}}^*} - \sin\theta_\nu \, e^{-t(\phi_{\mathcal{R}}^*+\gamma_{\mathcal{R}}^*)} \right) .$$

In particular, as soon as the following condition is verified

$$\sin\theta_\nu \, e^{-t(\phi_{\mathcal{R}}^*+\gamma_{\mathcal{R}}^*)} \leq \delta \frac{\cos\theta_\nu}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} e^{-\phi_{\mathcal{R}}^* t} , \quad (3.29)$$

that is

$$\|h_{\mathcal{R}}^*\|_{\mathcal{R}} \tan\theta_\nu \, e^{-\phi_{\mathcal{R}}^* t} \leq \delta , \quad (3.30)$$

it holds

$$\mathbb{P}_\nu(\tau_{\mathcal{X}\backslash\mathcal{R}} > t) \geq (1-\delta)\|\tfrac{\nu}{\mu_{\mathcal{R}}}\|_{\mathcal{R}} \frac{\cos\theta_\nu}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} e^{-\phi_{\mathcal{R}}^* t} . \quad (3.31)$$

Now, dividing both terms of (3.28) by $\mu_{\mathcal{R}}^*(f)\mathbb{P}_\nu(\tau_{\mathcal{X}\backslash\mathcal{R}} > s+t)$, we reach an inequality that controls the Yaglom limit and that, provided condition (3.30) holds and then using the last inequality, reads as

$$\left| \frac{\mathbb{E}_\nu[f(X(t)) \,|\, \tau_{\mathcal{X}\backslash\mathcal{R}}>t]}{\mu_{\mathcal{R}}^*(f)} - 1 \right| \leq \frac{1+\|h_{\mathcal{R}}^*\|_{\mathcal{R}}}{1-\delta} \tan\theta_\nu \left(\tan\theta_f + \sqrt{\mathrm{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*)} \right) e^{-\gamma_{\mathcal{R}}^* t} . \quad (3.32)$$

As a final step we apply this inequality to $\nu = \delta_x$ and $f = \delta_y$. For this choice of ν and f, and by definition of θ_ν and θ_f, one has

$$\tan\theta_\nu \leq \frac{1}{\cos\theta_\nu} = \frac{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}}{\sqrt{\mu_{\mathcal{R}}(x)h_{\mathcal{R}}^*(x)}} \quad \text{and} \quad \tan\theta_f \leq \frac{1}{\cos\theta_f} = \frac{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}}{\sqrt{\mu_{\mathcal{R}}(y)h_{\mathcal{R}}^*(y)}} ,$$

Thus, from (3.32), we obtain that under condition (3.30)

$$\left| \frac{\mathbb{P}_x\left((X(t)=y) \,|\, \tau_{\mathcal{X}\backslash\mathcal{R}}>t\right)}{\mu_{\mathcal{R}}^*(y)} - 1 \right| \leq e^{-\gamma_{\mathcal{R}}^* t} \frac{(1+\|h_{\mathcal{R}}^*\|_{\mathcal{R}})\|h_{\mathcal{R}}^*\|_{\mathcal{R}}^2}{(1-\delta)\sqrt{\mu_{\mathcal{R}}(x)h_{\mathcal{R}}^*(x)}} \left(\frac{1}{\sqrt{\mu_{\mathcal{R}}(y)h_{\mathcal{R}}^*(y)}} + \frac{\sqrt{\mathrm{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*)}}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} \right)$$

$$\leq e^{-\gamma_{\mathcal{R}}^* t} \frac{1}{1-\delta} \left(1 + \sqrt{1 + \frac{\varepsilon_{\mathcal{R}}^*}{1-\varepsilon_{\mathcal{R}}^*}} \right) \left(1 + \frac{\varepsilon_{\mathcal{R}}^*}{1-\varepsilon_{\mathcal{R}}^*} \right) \frac{1}{\sqrt{\zeta_{\mathcal{R}}^*}} \left(\frac{1}{\sqrt{\zeta_{\mathcal{R}}^*}} + \sqrt{\frac{\varepsilon_{\mathcal{R}}^*}{1-\varepsilon_{\mathcal{R}}^*}} \right) , \quad (3.33)$$

where in the second line we used that $\|h_{\mathcal{R}}^*\|_{\mathcal{R}} \geq 1$, the estimate given in Proposition 2.1, and we introduced the quantity $\zeta_{\mathcal{R}}^*$ defined in (2.19).

The right-hand side of the last inequality is smaller than δ as soon as

$$t \geq \frac{1}{\gamma_{\mathcal{R}}^*} \left[\ln\frac{2}{\delta(1-\delta)\zeta_{\mathcal{R}}^*} + \ln\left(\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{\varepsilon_{\mathcal{R}}^*}{1-\varepsilon_{\mathcal{R}}^*}} \right) \left(1 + \frac{\varepsilon_{\mathcal{R}}^*}{1-\varepsilon_{\mathcal{R}}^*} \right) \left(1 + \sqrt{\frac{\zeta_{\mathcal{R}}^*\varepsilon_{\mathcal{R}}^*}{1-\varepsilon_{\mathcal{R}}^*}} \right) \right) \right] , \quad (3.34)$$

which also implies (3.30).

Finally, from the hypothesis $\varepsilon_{\mathcal{R}}^* < 1/3$, from the concavity of the logarithm and of the square root function, and using that $\zeta_{\mathcal{R}}^* \leq 1$, then $\delta(1 - \delta) \leq 1/4$ and $\ln 8 \geq 1 + 5/(4\sqrt{2})$, after some computation one obtains that the condition (3.34) is implied by the stronger condition

$$t \geq \frac{1}{\gamma_{\mathcal{R}}^*} \left(\ln \frac{2}{\delta(1-\delta)\zeta_{\mathcal{R}}^*} \right) \left\{ 1 + \sqrt{\frac{\varepsilon_{\mathcal{R}}^*}{1-\varepsilon_{\mathcal{R}}^*}} \right\}, \tag{3.35}$$

which, in turns, follows from $t > T_{\delta,\mathcal{R}}^*$ by using Proposition 2.3.

4. AROUND THE EXPONENTIAL LAW

4.1. **Proof of Theorem 2.6.** We write

$$\mathbb{P}_\nu(\tau_{\mathcal{X}\backslash\mathcal{R}} > \tfrac{t}{\phi_{\mathcal{R}}^*}) = \pi_{\mathcal{R}}(\nu)\mathbb{P}_\nu(\tau_{\mathcal{X}\backslash\mathcal{R}} > \tfrac{t}{\phi_{\mathcal{R}}^*} \mid \tau_{\mathcal{X}\backslash\mathcal{R}} < T_{\mathcal{R}}^*)$$
$$+ (1 - \pi_{\mathcal{R}}(\nu))\mathbb{P}_\nu(\tau_{\mathcal{X}\backslash\mathcal{R}} > \tfrac{t}{\phi_{\mathcal{R}}^*} \mid \tau_{\mathcal{X}\backslash\mathcal{R}} > T_{\mathcal{R}}^*).$$

If $t \geq \phi_{\mathcal{R}}^* T_{\mathcal{R}}^*$, the first term in the r.h.s equals zero and we get

$$\mathbb{P}_\nu\left(\tau_{\mathcal{X}\backslash\mathcal{R}} > \tfrac{t}{\phi_{\mathcal{R}}^*}\right) = (1 - \pi_{\mathcal{R}}(\nu))\mathbb{P}_\nu\left(\tau_{\mathcal{X}\backslash\mathcal{R}} > \tfrac{t}{\phi_{\mathcal{R}}^*} \mid \tau_{\mathcal{X}\backslash\mathcal{R}} > T_{\mathcal{R}}^*\right).$$

By Theorem 2.4, we also have

$$\left| \mathbb{P}_\nu\left(\tau_{\mathcal{X}\backslash\mathcal{R}} > \tfrac{t}{\phi_{\mathcal{R}}^*} \mid \tau_{\mathcal{X}\backslash\mathcal{R}} > T_{\mathcal{R}}^*\right) - e^{-\phi_{\mathcal{R}}^*\left(\frac{t}{\phi_{\mathcal{R}}^*} - T_{\mathcal{R}}^*\right)} \right| \leq \varepsilon_{\mathcal{R}}^* e^{-\phi_{\mathcal{R}}^*\left(\frac{t}{\phi_{\mathcal{R}}^*} - T_{\mathcal{R}}^*\right)},$$

which together with the previous equality completes the proof.

4.2. **Proof of Lemma 2.7.** On the one hand we have

$$\mathbb{P}_{\mu_{\mathcal{R}}^*}(\tau_{\mathcal{X}\backslash\mathcal{R}} \leq T_{\mathcal{R}}^*) = 1 - e^{-\phi_{\mathcal{R}}^* T_{\mathcal{R}}^*} \leq \phi_{\mathcal{R}}^* T_{\mathcal{R}}^*. \tag{4.1}$$

On the other hand, denoting by $d_{TV}(\mu,\nu)$ the total variation distance between μ and ν, from the $\ell^2(\mu_{\mathcal{R}})$ estimate given in Proposition 2.1, together with the Cauchy-Schwartz inequality, we get

$$d_{TV}(\mu_{\mathcal{R}}, \mu_{\mathcal{R}}^*) = \frac{1}{2} \sum_{x \in \mathcal{R}} |\mu_{\mathcal{R}}(x) - \mu_{\mathcal{R}}^*(x)| = \frac{1}{2} \sum_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) |1 - h_{\mathcal{R}}^*(x)|$$
$$\leq \frac{1}{2}\mathrm{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}})^{1/2} \leq \frac{1}{2}\sqrt{\frac{\varepsilon_{\mathcal{R}}^*}{1 - \varepsilon_{\mathcal{R}}^*}}. \tag{4.2}$$

We then derive the desired result by using an optimal coupling.

5. WORKING WITH (κ, λ)-CAPACITIES

5.1. **Proof of the upper bound in Theorem 2.9.** Let $\tilde{X}$ denote the continuous-time Markov chain on $\tilde{\mathcal{X}}$ defined, for $\tilde{\kappa} > 0$, by the generator

$$(\tilde{\mathcal{L}}f)(\tilde{x}) = \begin{cases} \tilde{\kappa}(f(x) - f(\bar{x})) & \text{if } \tilde{x} = \bar{x} \in \bar{\mathcal{R}} \\ (\mathcal{L}f)(x) + \kappa(f(\bar{x}) - f(x)) & \text{if } \tilde{x} = x \in \mathcal{R} \\ (\mathcal{L}f)(x) & \text{if } \tilde{x} = x \in \mathcal{X}\backslash\mathcal{R} \end{cases}. \tag{5.1}$$

This is a reversible process with respect to a measure $\tilde{\mu}$ defined as

$$\tilde{\mu}(\tilde{x}) = \begin{cases} \mu(x) & \text{if } \tilde{x} = x \in \mathcal{X} \\ \frac{\kappa}{\tilde{\kappa}}\mu(x) & \text{if } \tilde{x} = \bar{x} \in \bar{\mathcal{R}} \end{cases}. \tag{5.2}$$

Note that $\tilde{\mu}$ is not a probability measure. Let us denote by $\tilde{\nu}_{\bar{\mathcal{R}}}$ the harmonic measure on $\bar{\mathcal{R}}$ associated with $\mathcal{X}\backslash\mathcal{R}$, i.e., the probability measure on $\bar{\mathcal{R}}$ defined by

$$\tilde{\nu}_{\bar{\mathcal{R}}}(\bar{x}) = \frac{-\tilde{\mu}(\bar{x})(\tilde{\mathcal{L}}\tilde{V}_\kappa)(\bar{x})}{C_\kappa(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})} \tag{5.3}$$

and with

$$\tilde{V}_\kappa(\tilde{x}) = \begin{cases} V_\kappa(x) = \mathbb{P}_x(\sigma_\kappa < \tau_{\mathcal{X}\backslash\mathcal{R}}) & \text{if } \tilde{x} = x \in \mathcal{R} \\ 1 & \text{if } \tilde{x} = \bar{x} \in \bar{\mathcal{R}} \\ 0 & \text{if } \tilde{x} = x \in \mathcal{X}\backslash\mathcal{R} \end{cases}.$$

With obvious notation, we then have

$$\mathbb{E}_{\tilde{\nu}_{\bar{\mathcal{R}}}}[\tilde{\tau}_{\mathcal{X}\setminus\mathcal{R}}] = \frac{\tilde{\mu}(\tilde{V}_\kappa)}{C_\kappa(\mathcal{R}, \mathcal{X}\setminus\mathcal{R})} \,. \tag{5.4}$$

Such kind of formula was introduced into the study of metastability in [20]. We refer to lecture notes [35] for a derivation.

Now setting $\nu(x) = \tilde{\nu}_{\bar{\mathcal{R}}}(\bar{x})$ for all $x \in \mathcal{R}$, we can write

$$\mathbb{E}_{\tilde{\nu}_{\bar{\mathcal{R}}}}[\tilde{\tau}_{\mathcal{X}\setminus\mathcal{R}}] = \tfrac{1}{\tilde{\kappa}} + \mathbb{E}_\nu[\tau_{\mathcal{X}\setminus\mathcal{R}}] + \mathbb{E}_\nu[\tau_{\mathcal{X}\setminus\mathcal{R}}] \cdot \kappa \cdot \tfrac{1}{\tilde{\kappa}} = \tfrac{1}{\tilde{\kappa}} + \mathbb{E}_\nu[\tau_{\mathcal{X}\setminus\mathcal{R}}](1 + \tfrac{\kappa}{\tilde{\kappa}}) \,,$$

where the first of the three summands stands for the mean time to go from $\bar{\mathcal{R}}$ to $\mathcal{R}$, the second one for the mean time spent when moving inside $\mathcal{R}$ before reaching $\mathcal{X}\setminus\mathcal{R}$, and the last one for the mean time spent moving back and forth between $\mathcal{R}$ and $\bar{\mathcal{R}}$. From (5.2) we also have

$$\frac{\tilde{\mu}(\tilde{V}_\kappa)}{C_\kappa(\mathcal{R}, \mathcal{X}\setminus\mathcal{R})} = \frac{\mu(V_\kappa) + \sum_{\bar{x}\in\bar{\mathcal{R}}} \tilde{\mu}(\bar{x})}{C_\kappa(\mathcal{R}, \mathcal{X}\setminus\mathcal{R})} = \frac{\mu(V_\kappa) + \tfrac{\kappa}{\tilde{\kappa}}\mu(\mathcal{R})}{C_\kappa(\mathcal{R}, \mathcal{X}\setminus\mathcal{R})} \,.$$

Inserting the above equalities in (5.4) and multiplying by $\tilde{\kappa}$, we then get

$$1 + \mathbb{E}_\nu[\tau_{\mathcal{X}\setminus\mathcal{R}}](\tilde{\kappa} + \kappa) = \frac{\tilde{\kappa}\mu(V_\kappa) + \kappa\mu(\mathcal{R})}{C_\kappa(\mathcal{R}, \mathcal{X}\setminus\mathcal{R})} \,. \tag{5.5}$$

Note that $\mu(\mathcal{R}), \mu(V_\kappa), C_\kappa(\mathcal{R}, \mathcal{X}\setminus\mathcal{R})$ and $\mathbb{E}_\nu[\tau_{\mathcal{X}\setminus\mathcal{R}}]$ do not depend on $\tilde{\kappa}$. Then, in the limit of a vanishing $\tilde{\kappa}$, it holds

$$1 + \kappa\mathbb{E}_\nu[\tau_{\mathcal{X}\setminus\mathcal{R}}] = \frac{\kappa\mu(\mathcal{R})}{C_\kappa(\mathcal{R}, \mathcal{X}\setminus\mathcal{R})} \,. \tag{5.6}$$

This already provides, by (2.31), an upper bound on $\phi_{\mathcal{R}}^*$.

To get the more practical upper bound stated in (2.38), let first note that dividing (5.5) by $\tilde{\kappa}$, and then sending $\tilde{\kappa}$ to $+\infty$, we get

$$\mathbb{E}_\nu[\tau_{\mathcal{X}\setminus\mathcal{R}}] = \frac{\mu(V_\kappa)}{C_\kappa(\mathcal{R}, \mathcal{X}\setminus\mathcal{R})} \,. \tag{5.7}$$

Together with (5.6), this implies $\frac{\mu(V_\kappa)}{C_\kappa(\mathcal{R}, \mathcal{X}\setminus\mathcal{R})} = \frac{\mu(\mathcal{R})}{C_\kappa(\mathcal{R}, \mathcal{X}\setminus\mathcal{R})} - \frac{1}{\kappa}$ or equivalently

Lemma 5.1.

$$\mu_{\mathcal{R}}(V_\kappa) = 1 - \frac{C_\kappa(\mathcal{R}, \mathcal{X}\setminus\mathcal{R})}{\kappa\mu(\mathcal{R})} \tag{5.8}$$

We now exploit the variational principle for $\phi_{\mathcal{R}}^*$ provided by Lemma 2.2 and take V_κ as test function. Noting that $\tilde{V}_\kappa$ is the equilibrium potential of the electrical network on $\tilde{\mathcal{X}}$ defined in (2.32), from (2.33) we get

$$\mathcal{D}(V_\kappa(x)) \le C_\kappa(\mathcal{R}, \mathcal{X}\setminus\mathcal{R}) \,. \tag{5.9}$$

By the Jensen inequality and (5.8),

$$\begin{aligned}
\|V_\kappa\|^2 &= \mu(\mathcal{R}) \sum_{x\in\mathcal{R}} \mu_{\mathcal{R}}(x)\mathbb{P}_x(\sigma_\kappa < \tau_{\mathcal{X}\setminus\mathcal{R}})^2 \\
&\ge \mu(\mathcal{R})\mu_{\mathcal{R}}(V_\kappa)^2 \\
&= \mu(\mathcal{R})\left(1 - \tfrac{C_\kappa(\mathcal{R}, \mathcal{X}\setminus\mathcal{R})}{\kappa\mu(\mathcal{R})}\right)^2 \,.
\end{aligned} \tag{5.10}$$

Finally inserting these inequalities in the variational principle for $\phi_{\mathcal{R}}^*$, we get the stated upper bound.

5.2. **Proof of the upper bound in Theorem 2.10.** For any $f \in \ell^2(\mu)$, we have

$$\begin{aligned}
\mathrm{Var}_\mu(f) &= \mu(\mathrm{Var}_\mu(f|\mathbb{1}_{\mathcal{R}})) + \mathrm{Var}_\mu(\mu(f|\mathbb{1}_{\mathcal{R}})) \\
&= \mu(\mathcal{R})\mathrm{Var}_{\mu_{\mathcal{R}}}(f_{|\mathcal{R}}) + \mu(\mathcal{X}\setminus\mathcal{R})\mathrm{Var}_{\mu_{\mathcal{X}\setminus\mathcal{R}}}(f_{|\mathcal{X}\setminus\mathcal{R}}) \\
&\quad + \mu(\mathcal{R})\mu(\mathcal{X}\setminus\mathcal{R})\left(\mu_{\mathcal{R}}(f_{|\mathcal{R}}) - \mu_{\mathcal{X}\setminus\mathcal{R}}(f_{|\mathcal{X}\setminus\mathcal{R}})\right)^2 \,.
\end{aligned} \tag{5.11}$$

Now, using the test function

$$\tilde{f} = \frac{f - \mu_{\mathcal{X}\setminus\mathcal{R}}(f_{|\mathcal{X}\setminus\mathcal{R}})}{\mu_{\mathcal{R}}(f_{|\mathcal{R}}) - \mu_{\mathcal{X}\setminus\mathcal{R}}(f_{|\mathcal{X}\setminus\mathcal{R}})}$$

in the definition (2.33) of (κ, λ)-capacity, we get

$$C_\kappa^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R}) \leq \mathcal{D}(\tilde{f}) + \kappa\mu(\mathcal{R})\mathrm{Var}_{\mu_\mathcal{R}}(\tilde{f}_{|\mathcal{R}}) + \lambda\mu(\mathcal{X}\backslash\mathcal{R})\mathrm{Var}_{\mu_{\mathcal{X}\backslash\mathcal{R}}}(\tilde{f}_{|\mathcal{X}\backslash\mathcal{R}})$$

$$= \left(\mu_\mathcal{R}(f_{|\mathcal{R}}) - \mu_{\mathcal{X}\backslash\mathcal{R}}(f_{|\mathcal{X}\backslash\mathcal{R}})\right)^{-2}\left(\mathcal{D}(f) + \kappa\mu(\mathcal{R})\mathrm{Var}_{\mu_\mathcal{R}}(f_{|\mathcal{R}}) + \lambda\mu(\mathcal{X}\backslash\mathcal{R})\mathrm{Var}_{\mu_{\mathcal{X}\backslash\mathcal{R}}}(f_{|\mathcal{X}\backslash\mathcal{R}})\right),$$

which provides an upper bound on $\left(\mu_\mathcal{R}(f_{|\mathcal{R}}) - \mu_{\mathcal{X}\backslash\mathcal{R}}(f_{|\mathcal{X}\backslash\mathcal{R}})\right)^2$. Applying that bound in Eq. (5.11), and from the definition of $\phi_\kappa^\lambda(A, B)$, we get

$$
\begin{aligned}
\mathrm{Var}_\mu(f) &\leq \mu(\mathcal{R})\mathrm{Var}_{\mu_\mathcal{R}}(f_{|\mathcal{R}}) + \mu(\mathcal{X}\backslash\mathcal{R})\mathrm{Var}_{\mu_{\mathcal{X}\backslash\mathcal{R}}}(f_{|\mathcal{X}\backslash\mathcal{R}}) \\
&+ \left(\mathcal{D}(f) + \kappa\mu(\mathcal{R})\mathrm{Var}_{\mu_\mathcal{R}}(f_{|\mathcal{R}}) + \lambda\mu(\mathcal{X}\backslash\mathcal{R})\mathrm{Var}_{\mu_{\mathcal{X}\backslash\mathcal{R}}}(f_{|\mathcal{X}\backslash\mathcal{R}})\right)\phi_\kappa^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})^{-1} \quad (5.12) \\
&\leq \frac{\mathcal{D}(f)}{\phi_\kappa^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})} + \frac{\mu(\mathcal{R})\mathcal{D}_\mathcal{R}(f_{|\mathcal{R}})}{\gamma_\mathcal{R}}\left(1 + \frac{\kappa}{\phi_\kappa^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}\right) + \frac{\mu(\mathcal{X}\backslash\mathcal{R})\mathcal{D}_{\mathcal{X}\backslash\mathcal{R}}(f_{|\mathcal{X}\backslash\mathcal{R}})}{\gamma_{\mathcal{X}\backslash\mathcal{R}}}\left(1 + \frac{\lambda}{\phi_\kappa^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}\right) \\
&\leq \frac{\mathcal{D}(f)}{\phi_\kappa^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}\left\{1 + \max\left(\frac{\phi_\kappa^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R}) + \kappa}{\gamma_\mathcal{R}}; \frac{\phi_\kappa^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R}) + \lambda}{\gamma_{\mathcal{X}\backslash\mathcal{R}}}\right)\right\},
\end{aligned}
$$

where in the last step we used that

$$\mathcal{D}(f) \leq \mu(\mathcal{R})\mathcal{D}_\mathcal{R}(f_{|\mathcal{R}}) + \mu(\mathcal{X}\backslash\mathcal{R})\mathcal{D}_{\mathcal{X}\backslash\mathcal{R}}(f_{|\mathcal{X}\backslash\mathcal{R}}).$$

The upper bound in (2.40) follows directly.

5.3. Proof of the lower bound of Theorem 2.9.

From inequality (5.12) applied to $f = h_\mathcal{R}^*$ and $\lambda = +\infty$, and since $h_{\mathcal{R}|\mathcal{X}\backslash\mathcal{R}}^* = 0$, we get

$$\mathrm{Var}_\mu(h_\mathcal{R}^*) \leq \mu(\mathcal{R})\mathrm{Var}_{\mu_\mathcal{R}}(h_\mathcal{R}^*) + \frac{\mathcal{D}(h_\mathcal{R}^*)}{C_\kappa(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}\mu(\mathcal{R})(1 - \mu(\mathcal{R}))\left\{1 + \frac{\kappa}{\gamma_\mathcal{R}}\right\}. \quad (5.13)$$

On the other hand, by (5.11),

$$\mathrm{Var}_\mu(h_\mathcal{R}^*) = \mu(\mathcal{R})\mathrm{Var}_{\mu_\mathcal{R}}(h_\mathcal{R}^*) + \mu(\mathcal{R})(1 - \mu(\mathcal{R})).$$

Inserting this formula in (5.13), the term $\mu(\mathcal{R})\mathrm{Var}_{\mu_\mathcal{R}}(h_\mathcal{R}^*)$ becomes zero and then, dividing by $\mu(\mathcal{R})(1 - \mu(\mathcal{R}))$, we have

$$1 \leq \frac{\mathcal{D}(h_\mathcal{R}^*)}{C_\kappa(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}\left\{1 + \frac{\kappa}{\gamma_\mathcal{R}}\right\}, \quad (5.14)$$

or equivalently

$$C_\kappa(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})\left\{1 + \frac{\kappa}{\gamma_\mathcal{R}}\right\}^{-1} \leq \mathcal{D}(h_\mathcal{R}^*). \quad (5.15)$$

Now, dividing by $\mu(\mathcal{R})\|h_\mathcal{R}^*\|_\mathcal{R}^2$ and using that, by Prop. 2.1, $\|h_\mathcal{R}^*\|_\mathcal{R}^2 = \mathrm{Var}_{\mu_\mathcal{R}}(h_\mathcal{R}^*) + 1 \leq 1/(1 - \varepsilon_\mathcal{R}^*)$, we get

$$\frac{C_\kappa(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}{\mu(\mathcal{R})}\left\{\frac{1 - \varepsilon_\mathcal{R}^*}{1 + \frac{\kappa}{\gamma_\mathcal{R}}}\right\} \leq \frac{C_\kappa(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}{\mu(\mathcal{R})\|h_\mathcal{R}^*\|_\mathcal{R}^2}\left\{1 + \frac{\kappa}{\gamma_\mathcal{R}}\right\}^{-1} \leq \frac{\mathcal{D}(h_\mathcal{R}^*)}{\|h_\mathcal{R}^*\|_\mathcal{R}^2\mu(\mathcal{R})} = \phi_\mathcal{R}^*, \quad (5.16)$$

where the last equality comes from Lemma 3.1 and the fact that

$$\langle h_\mathcal{R}^*, -\mathcal{L}_\mathcal{R}^* h_\mathcal{R}^*\rangle_\mathcal{R} = \phi_\mathcal{R}^*.$$

Finally, using the convexity of the function $x \mapsto \frac{1}{1+x}$, we obtain

$$\frac{C_\kappa(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}{\mu(\mathcal{R})}\left\{1 - \varepsilon_\mathcal{R}^* - \frac{\kappa}{\gamma_\mathcal{R}}\right\} \leq \phi_\mathcal{R}^*, \quad (5.17)$$

which concludes the proof of the lower bound in (2.38).

5.4. **Proof of the lower bound in Theorem 2.10.** We use the test function V_κ^λ, for which we know that (see (2.33))

$$\mathcal{D}(V_\kappa^\lambda) + \kappa\mu(\mathcal{R})\mathbb{E}_{\mu_\mathcal{R}}\left[\left(V_{\kappa\,|_\mathcal{R}}^\lambda - 1\right)^2\right] + \lambda\mu(\mathcal{X}\backslash\mathcal{R})\mathbb{E}_{\mu_{\mathcal{X}\backslash\mathcal{R}}}\left[\left(V_{\kappa\,|_{\mathcal{X}\backslash\mathcal{R}}}^\lambda - 0\right)^2\right] = C_\kappa^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})$$

so that

$$\mathcal{D}(V_\kappa^\lambda) \leq C_\kappa^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})\,. \tag{5.18}$$

We then look for a lower bound on $\mathrm{Var}_\mu(V_\kappa^\lambda)$. From (5.11) we have

$$\mathrm{Var}_\mu(V_\kappa^\lambda) \geq \mu(\mathcal{R})\mu(\mathcal{X}\backslash\mathcal{R})\left(\mu_\mathcal{R}(V_{\kappa\,|_\mathcal{R}}^\lambda) - \mu_{\mathcal{X}\backslash\mathcal{R}}(V_{\kappa\,|_{\mathcal{X}\backslash\mathcal{R}}}^\lambda)\right)^2\,,$$

thus we need to estimate $\mu_\mathcal{R}(V_{\kappa\,|_\mathcal{R}}^\lambda)$ and $\mu_{\mathcal{X}\backslash\mathcal{R}}(V_{\kappa\,|_{\mathcal{X}\backslash\mathcal{R}}}^\lambda)$.

By the monotonicity in λ, for all $x \in \mathcal{R}$ we get

$$V_\kappa^\lambda(x) = \mathbb{P}_x(\ell_\mathcal{R}^{-1}(\sigma_\kappa)) < \ell_{\mathcal{X}\backslash\mathcal{R}}^{-1}(\sigma_\lambda)) \geq \mathbb{P}_x(\sigma_\kappa < \tau_{\mathcal{X}\backslash\mathcal{R}}) = V_\kappa(x)\,,$$

which implies, together with Lemma 5.1,

$$\mu_\mathcal{R}(V_{\kappa\,|_\mathcal{R}}^\lambda) \geq 1 - \frac{C_\kappa(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}{\kappa\mu(\mathcal{R})}\,.$$

In the same way we have

$$\mu_{\mathcal{X}\backslash\mathcal{R}}(V_{\kappa\,|_{\mathcal{X}\backslash\mathcal{R}}}^\lambda) \leq \frac{C^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}{\lambda\mu(\mathcal{X}\backslash\mathcal{R})}\,.$$

Altogether, we finally get

$$\gamma \leq \frac{\mathcal{D}(V_\kappa^\lambda)}{\mathrm{Var}_\mu(V_\kappa^\lambda)} \leq \frac{C_\kappa^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}{\mu(\mathcal{R})\mu(\mathcal{X}\backslash\mathcal{R})}\left(1 - \frac{C_\kappa(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}{\kappa\mu(\mathcal{R})} - \frac{C^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}{\lambda\mu(\mathcal{X}\backslash\mathcal{R})}\right)^{-2}\,. \tag{5.19}$$

6. Working with soft measures

6.1. **Proof of Lemma 2.12.** If $\lambda = 0$ the first statement holds trivially since, in that case, $\phi_{\mathcal{R},\lambda}^* = 0 = \mu_{\mathcal{R},\lambda}^*(e_{\mathcal{R},\lambda})$. If $\lambda > 0$, we can write

$$\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\tau_{\mathcal{X}\backslash\mathcal{R},\lambda} \leq t) = \sum_{k\geq 1}\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(N_\mathcal{R}(t) \geq k)(1 - \phi_{\mathcal{R},\lambda}^*)^{k-1}\mu_{\mathcal{R},\lambda}^*(e_{\mathcal{R},\lambda})\,,$$

where $N_\mathcal{R}(t)$ is the number of clock rings inside $\mathcal{R}$ for the Poissonian clock associated to X. Taking the limit as $t \to \infty$ in the above equation, we get that

$$1 = \mu_{\mathcal{R},\lambda}^*(e_{\mathcal{R},\lambda})/\phi_{\mathcal{R},\lambda}^*\,,$$

which provides identity i).

Let us now define the operator $\mathcal{L}_{\mathcal{R},\lambda}^*$ on $\ell^2(\mu_\mathcal{R})$ as

$$(\mathcal{L}_{\mathcal{R},\lambda}^* f)(x) = -f(x) + \sum_{y\in\mathcal{R}}p_{\mathcal{R},\lambda}^*(x,y)f(y) \qquad \forall x \in \mathcal{R}, f \in \ell^2(\mu_\mathcal{R}) \tag{6.1}$$

and notice that, for any probability measure ν on $\mathcal{X}$, it holds

$$\mathbb{E}_\nu\left[f(X \circ \ell_\mathcal{R}^{-1}(t))\mathbb{1}_{\{\tau_{\mathcal{X}\backslash\mathcal{R},\lambda}>t\}}\right] = \nu\left(e^{t\mathcal{L}_{\mathcal{R},\lambda}^*}f\right)\,. \tag{6.2}$$

The exponential law given in ii) follows from the above identity applied to $\nu = \mu_{\mathcal{R},\lambda}^*$ and $f = \mathbb{1}_\mathcal{R}$.

Finally, since $1 - \phi_{\mathcal{R},\lambda}^*$ is a simple eigenvalue equal to the spectral radius of $p_{\mathcal{R},\lambda}^*$, for any $x, y \in \mathcal{R}$ and in the large t regime, we have

$$\mathbb{P}_x(X \circ \ell_\mathcal{R}^{-1}(t) = y\,, \tau_{\mathcal{X}\backslash\mathcal{R},\lambda} > t) \sim c_x\mu_{\mathcal{R},\lambda}^*(y)e^{-t\phi_{\mathcal{R},\lambda}^*}\,, \tag{6.3}$$

where $c_x\mu_{\mathcal{R},\lambda}^*$ is the canonical projection of δ_x on the one-dimensional eigenspace associated with $\mu_{\mathcal{R},\lambda}^*$ (c_x is strictly positive as a consequence of the positivity of $\mu_{\mathcal{R},\lambda}^*$). From (6.3), and taking the limit when t goes to infinity, it follows

$$\lim_{t\to\infty}\mathbb{P}_x(X \circ \ell_\mathcal{R}^{-1}(t) = y\,|\,\tau_{\mathcal{X}\backslash\mathcal{R},\lambda} > t) = \mu_{\mathcal{R},\lambda}^*(y)\,.$$

6.2. **Proof of Lemma 2.13.** The result is once again a consequence of the Perron-Frobenius theorem. Let $\chi_\lambda(y)$ denote the characteristic polynomial of $\mathcal{L}^*_{\mathcal{R},\lambda}$, which can be written as $\chi_\lambda(y) = (y + \phi^*_{\mathcal{R},\lambda})a(y)$. If $a(y) = (y + \phi^*_{\mathcal{R},\lambda})q(y) + a(-\phi^*_{\mathcal{R},\lambda})$ is the Euclidian division of $a(y)$ by $(y - \phi^*_{\mathcal{R},\lambda})$, we have the Bézout identity

$$\frac{1}{a(-\phi^*_{\mathcal{R},\lambda})}a(y) - \frac{1}{a(-\phi^*_{\mathcal{R},\lambda})}q(y)(y + \phi^*_{\mathcal{R},\lambda}) = 1. \tag{6.4}$$

In particular, for any $x \in \mathcal{R}$, $\frac{1}{a(-\phi^*_{\mathcal{R},\lambda})}\delta_x a(\mathcal{L}^*_{\mathcal{R},\lambda}) = c_x \mu^*_{\mathcal{R},\lambda}$ is the canonical projection of δ_x on the eigenspace associated to $\mu^*_{\mathcal{R},\lambda}$, and since $c_x > 0$ as previously noticed, we have

$$\mu^*_{\mathcal{R},\lambda} = \frac{\delta_x a(\mathcal{L}^*_{\mathcal{R},\lambda})}{\sum_{y \in \mathcal{R}} \delta_x a(\mathcal{L}^*_{\mathcal{R},\lambda}) \mathbb{1}_{\{y\}}}. \tag{6.5}$$

Since $a(y) = \frac{\chi_\lambda(y)}{(y + \phi^*_{\mathcal{R},\lambda})}$, the above equation expresses the map $\lambda \mapsto \mu^*_{\mathcal{R},\lambda}$ as a composition of continuous functions of λ.

6.3. **Proof of Proposition 2.14.** As far as $\phi_{\mathcal{R},\lambda}$ is concerned, continuity and monotonicity follow from continuity and monotonicity of $e_{\mathcal{R},\lambda}(x)$ for any $x \in \mathcal{R}$. We then consider the other parameters. The continuity follows from the continuity of the eigenvalues as root of the characteristic polynomial. To prove the monotonicity, we notice that when λ decreases to zero, $p^*_{\mathcal{R},\lambda}(x,y)$ grows for all x and y in $\mathcal{R}$ as well as $c_{\mathcal{R},\lambda}(x,y)$ for any distinct $x, y \in \mathcal{R}$. From the variational characterization of $\phi^*_{\mathcal{R},\lambda}$, i.e.

$$\phi^*_{\mathcal{R},\lambda} = \min \left\{ \langle f, -\mathcal{L}^*_{\mathcal{R},\lambda} f \rangle_{\mathcal{R}} : \langle f, f \rangle_{\mathcal{R}} = 1, f > 0 \right\} \tag{6.6}$$

$$= \min_{\substack{\langle f,f \rangle_{\mathcal{R}}=1 \\ f>0}} \sum_{x,y \in \mathcal{R}} \mu_{\mathcal{R}}(x)f(x)\left(f(x) - \sum_{y \in \mathcal{R}} p^*_{\mathcal{R},\lambda}(x,y)f(y) \right) \tag{6.7}$$

where the restriction $f > 0$ comes from the fact that, by the Perron-Frobenius theorem, the right eigenvector has positive coordinates, we see that $\phi^*_{\mathcal{R},\lambda}$ is decreasing in λ. Similarly, using

$$\gamma_{\mathcal{R},\lambda} = \min \left\{ \tfrac{1}{2} \sum_{x,y \in \mathcal{R}} c_{\mathcal{R},\lambda}(x,y)(f(x) - f(y))^2 : \mathrm{Var}_{\mu_{\mathcal{R}}}(f) = 1 \right\}, \tag{6.8}$$

we see that $\gamma_{\mathcal{R},\lambda}$ is increasing in λ. As a consequence $\varepsilon^*_{\mathcal{R},\lambda}$ is decreasing in λ, and we have

$$\varepsilon^*_{\mathcal{R},0} = \frac{\phi^*_{\mathcal{R},0}}{\gamma_{\mathcal{R},0}} = \frac{\mu^*_{\mathcal{R},0}(e_{\mathcal{R},0})}{\gamma_{\mathcal{R},0}} = 0. \tag{6.9}$$

6.4. **Proof of Theorem 2.16.** Proof of (2.60): We first write

$$\mathbb{P}_\nu(X(\tau_\delta) = x \mid X(\tau_\delta) \in \mathcal{R}) = \frac{1}{\mathbb{P}_\nu(X(\tau_\delta) \in \mathcal{R})} \sum_{i \geq 0} \sum_{x_i \in \mathcal{X}} \mathbb{P}_\nu(i_0 > i, X(\tau_i) = x_i) \tag{6.10}$$

$$\times \quad \mathbb{P}_{x_i}(X \circ \ell_{\mathcal{R}}^{-1}(\sigma_\kappa) = x, \ell_{\mathcal{R}}^{-1}(\sigma_\kappa) < \ell_{\mathcal{X}\backslash\mathcal{R}}^{-1}(\sigma_\lambda), \sigma_\kappa > T^*_{\delta,\mathcal{R},\lambda})$$

Now, conditioning on σ_κ and setting $\mathbb{P}_{x_i}^{\sigma_\kappa} = \mathbb{P}_{x_i}(\cdot \mid \sigma_\kappa)$, we get

$$\mathbb{P}_\nu(X(\tau_\delta) = x \mid X(\tau_\delta) \in \mathcal{R}) = \frac{1}{\mathbb{P}_\nu(X(\tau_\delta) \in \mathcal{R})} \sum_{i \geq 0} \sum_{x_i \in \mathcal{X}} \mathbb{P}_\nu(i_0 > i, X(\tau_i) = x_i)$$

$$\times \mathbb{E}\left[\mathbb{P}_{x_i}^{\sigma_\kappa}(X \circ \ell_{\mathcal{R}}^{-1}(\sigma_\kappa) = x, \ell_{\mathcal{R}}^{-1}(\sigma_\kappa) < \ell_{\mathcal{X}\backslash\mathcal{R}}^{-1}(\sigma_\lambda), \sigma_\kappa > T^*_{\delta,\mathcal{R},\lambda}) \right]$$

$$= \frac{1}{\mathbb{P}_\nu(X(\tau_\delta) \in \mathcal{R})} \sum_{i \geq 0} \sum_{x_i \in \mathcal{X}} \mathbb{P}_\nu(i_0 > i, X(\tau_i) = x_i) \tag{6.11}$$

$$\times \mathbb{E}\left[\mathbb{1}_{\{\sigma_\kappa > T^*_{\delta,\mathcal{R},\lambda}\}} \mathbb{P}_{x_i}^{\sigma_\kappa}(X \circ \ell_{\mathcal{R}}^{-1}(\sigma_\kappa) = x \mid \sigma_\kappa < \tau_{\mathcal{X}\backslash\mathcal{R},\lambda}) \mathbb{P}_{x_i}^{\sigma_\kappa}(\sigma_\kappa < \tau_{\mathcal{X}\backslash\mathcal{R},\lambda}) \right],$$

where the second equality comes from the independence between X, σ_κ and σ_λ. Since

$$\mathbb{P}_\nu(X(\tau_\delta) \in \mathcal{R}) = \sum_{i \geq 0} \sum_{x_i \in \mathcal{X}} \mathbb{P}_\nu(i_0 > i, X(\tau_i) = x_i) \mathbb{E}\left[\mathbb{1}_{\{\sigma_\kappa > T^*_{\delta,\mathcal{R},\lambda}\}} \mathbb{P}_{x_i}^{\sigma_\kappa}(\sigma_\kappa < \tau_{\mathcal{X}\backslash\mathcal{R},\lambda}) \right], \tag{6.12}$$

from (6.11) we get

$$
\frac{\mathbb{P}_\nu(X(\tau_\delta) = x \mid X(\tau_\delta) \in \mathcal{R})}{\mu^*_{\mathcal{R},\lambda}(x)} - 1 = \frac{1}{\mathbb{P}_\nu(X(\tau_\delta) \in \mathcal{R})} \sum_{i \geq 0} \sum_{x_i \in \mathcal{X}} \mathbb{P}_\nu(i_0 > i, X(\tau_i) = x_i)
$$

$$
\times \mathbb{E}\left[\mathbb{1}_{\{\sigma_\kappa > T^*_{\delta,\mathcal{R},\lambda}\}} \mathbb{P}^{\sigma_\kappa}_{x_i}(\sigma_\kappa < \tau_{\mathcal{X}\backslash\mathcal{R},\lambda}) \left(\frac{\mathbb{P}^{\sigma_\kappa}_{x_i}(X \circ \ell^{-1}_{\mathcal{R}}(\sigma_\kappa) = x \mid \sigma_\kappa < \tau_{\mathcal{X}\backslash\mathcal{R},\lambda})}{\mu^*_{\mathcal{R},\lambda}(x)} - 1 \right) \right].
$$

(6.13)

An analogous expression can be found for $\frac{\mathbb{P}_\nu(X(\tau_\delta)=x \mid X(\tau_\delta)\in\mathcal{X}\backslash\mathcal{R})}{\mu^*_{\mathcal{X}\backslash\mathcal{R},\kappa}(x)} - 1$. The result then follows from Theorem 2.15, and in particular from the equivalent of Theorem 2.4.

To prove inequality (2.61), we first state the following lemma.

Lemma 6.1. *Let $T > 0$ and $\{\sigma_i : i \geq 1\}$ be a sequence of independent exponential random variables of rate κ such that $e^{\kappa T} - 1 < 1$. If $N = \min\{i \geq 1 : \sigma_i > T\}$, then*

$$
\mathbb{P}\left(\sum_{i=1}^N \sigma_i > \frac{t}{\kappa} \right) \leq \frac{e^{-t}}{1 - (e^{\kappa T} - 1)}
$$

(6.14)

Proof of Lemma 6.1. Using the property of the exponential distribution, we have

$$
\mathbb{P}\left(\sum_{i=1}^N \sigma_i > \frac{t}{\kappa} \right) = \sum_{n \geq 1} \mathbb{P}(N = n) \mathbb{P}\left(\sum_{j=1}^n \sigma_i > \frac{t}{\kappa} \,\Big|\, \sigma_1 < T, \dots, \sigma_{n-1} < T, \sigma_n > T \right)
$$

$$
\leq \sum_{n \geq 1} \mathbb{P}(N = n) \mathbb{P}\left(\sigma_n > \frac{t}{\kappa} - (n-1)T \,\Big|\, \sigma_n > T \right)
$$

$$
= \sum_{n \geq 1} \mathbb{P}(N = n) \mathbb{P}\left(\sigma_n > \frac{t}{\kappa} - nT \right)
$$

(6.15)

$$
= \sum_{n \geq 1} (1 - e^{-\kappa T})^{n-1} e^{-\kappa T} e^{-t + n\kappa T}
$$

$$
= e^{-t} \sum_{n \geq 1} (e^{\kappa T} - 1)^{n-1} = \frac{e^{-t}}{1 - (e^{\kappa T} - 1)},
$$

which concludes the proof. $\qquad\square$

Coming back to the proof of Th. 2.16, we first notice that if $\tau_\delta > t(\frac{1}{\kappa} + \frac{1}{\lambda})$, then $\ell_{\mathcal{R}}(\tau_\delta) > \frac{t}{\kappa}$ or $\ell_{\mathcal{X}\backslash\mathcal{R}}(\tau_\delta) > \frac{t}{\lambda}$. As a consequence, defining

$$
\begin{aligned}
A_{\mathcal{R}} &= \{\kappa\ell_{\mathcal{R}}(\tau_\delta) \vee \lambda\ell_{\mathcal{X}\backslash\mathcal{R}}(\tau_\delta) = \kappa\ell_{\mathcal{R}}(\tau_\delta) > t\} \\
A_{\mathcal{X}\backslash\mathcal{R}} &= \{\kappa\ell_{\mathcal{R}}(\tau_\delta) \vee \lambda\ell_{\mathcal{X}\backslash\mathcal{R}}(\tau_\delta) = \lambda\ell_{\mathcal{X}\backslash\mathcal{R}}(\tau_\delta) > t\}
\end{aligned}
$$

(6.16)

so that $\mathbb{P}(A_{\mathcal{R}}) + \mathbb{P}(A_{\mathcal{X}\backslash\mathcal{R}}) \leq 1$, we have

$$
\mathbb{P}_\nu(\tau_\delta > t(\tfrac{1}{\kappa} + \tfrac{1}{\lambda})) = \mathbb{P}_\nu(\tau_\delta > t(\tfrac{1}{\kappa} + \tfrac{1}{\lambda}) \mid A_{\mathcal{R}}) \mathbb{P}_\nu(A_{\mathcal{R}}) + \mathbb{P}_\nu(\tau_\delta > t(\tfrac{1}{\kappa} + \tfrac{1}{\lambda}) \mid A_{\mathcal{X}\backslash\mathcal{R}}) \mathbb{P}_\nu(A_{\mathcal{X}\backslash\mathcal{R}}).
$$

Using the independence between σ_κ, σ_λ and X, together with the previous lemma, we finally get

$$
\mathbb{P}_\nu\left(\tau_\delta > t\left(\tfrac{1}{\kappa} + \tfrac{1}{\lambda}\right) \right) \leq e^{-t}\left\{ \tfrac{1}{1-\xi} \right\} \left(\mathbb{P}_\nu(A_{\mathcal{R}}) + \mathbb{P}_\nu(A_{\mathcal{X}\backslash\mathcal{R}}) \right) \leq e^{-t}\left\{ \tfrac{1}{1-\xi} \right\}.
$$

6.5. **Proof of Theorem 2.18.** To prove the upper bound we consider the extended electrical network associated with $C^\lambda_\kappa(A, B)$ and follow the first steps of the proof of the upper bound in Theorem 2.9 (see subsection 5.1). We then reach, for some probability measure ν on $\mathcal{R}$, to

$$
1 + \kappa\mathbb{E}_\nu[\tau_{\mathcal{X}\backslash\mathcal{R},\lambda}] = \frac{\kappa\mu(\mathcal{R})}{C^\lambda_\kappa(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}.
$$

(6.17)

instead of equation (5.6). Using then, from Theorem 2.17, the analogous of equation (2.31), we obtain

$$
1 + \frac{\kappa}{\phi^*_{\mathcal{R},\lambda}}\left\{ 1 + \varepsilon^*_{\mathcal{R},\lambda} + \varepsilon^*_{\mathcal{R},\lambda} \ln \frac{1}{\varepsilon^*_{\mathcal{R},\lambda}\zeta_{\mathcal{R}}} \right\} \geq \frac{\kappa\mu(\mathcal{R})}{C^\lambda_\kappa(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})},
$$

(6.18)

or

$$\frac{1}{\phi_{\mathcal{R},\lambda}^*} \left\{ 1 + \varepsilon_{\mathcal{R},\lambda}^* + \varepsilon_{\mathcal{R},\lambda}^* \ln \frac{1}{\varepsilon_{\mathcal{R},\lambda}^* \zeta_{\mathcal{R}}} + \frac{\phi_{\mathcal{R},\lambda}^*}{\kappa} \right\} \geq \frac{\mu(\mathcal{R})}{C_\kappa^\lambda(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})} , \tag{6.19}$$

which gives the desired upper bound on $\phi_{\mathcal{R},\lambda}^*$.

The proof of the lower bound will be similar to the proof of the lower bound of Theorem 2.9, where we used a partial Poincaré inequality to control the mean exit time from $\mathcal{R}$. The difference here is that we will have to work on the whole space $\mathcal{X}$ and not only on $\mathcal{R}$. Since $\mu_{\mathcal{R},\lambda}^*$ is concentrated on $\mathcal{R}$, we will first compare its associated exit time $\tau_{\mathcal{X}\setminus\mathcal{R},\lambda}$ with the exit time of another quasi-stationary measure, $\tilde{\mu}_{\mathcal{X}}^*$, that spreads on the whole space $\mathcal{X}$. Then we will control $\tilde{\phi}_{\mathcal{X}}^*$, the escape rate from $\mathcal{X}$, with the spectral gap estimated in Theorem 2.10.

Let $\tilde{\mu}_{\mathcal{X}}^*$ be the quasi-stationary measure on $\mathcal{X}$ associated with the Markovian process $\tilde{X}$ on $\tilde{\mathcal{X}} = \mathcal{X} \cup \overline{\mathcal{X} \setminus \mathcal{R}}$ with generator $\tilde{\mathcal{L}}$ defined, for some $\tilde{\lambda} > 0$, by

$$(\tilde{\mathcal{L}}f)(\tilde{x}) = \begin{cases} (\mathcal{L}f)(x) & \text{if } \tilde{x} = x \in \mathcal{R} \\ (\mathcal{L}f)(x) + \lambda(f(\bar{x}) - f(x)) & \text{if } \tilde{x} = x \in \mathcal{X} \setminus \mathcal{R} \\ \tilde{\lambda}(f(\tilde{x}) - f(x)) & \text{if } \tilde{x} = \bar{x} \in \overline{\mathcal{X} \setminus \mathcal{R}} \end{cases} . \tag{6.20}$$

The associated escape rate $\tilde{\phi}_{\mathcal{X}}^*$ is, with obvious notation and for any probability measure ν on $\mathcal{X}$, the rate of exponential decay of $P_\nu(\tilde{\tau}_{\overline{\mathcal{X}\setminus\mathcal{R}}} > t)$ when t goes to infinity. Since

$$P_{\mu_{\mathcal{R},\lambda}^*} \left(\tilde{\tau}_{\overline{\mathcal{X}\setminus\mathcal{R}}} > t \right) \geq P_{\mu_{\mathcal{R},\lambda}^*} \left(\ell_{\mathcal{R}} \left(\tilde{\tau}_{\overline{\mathcal{X}\setminus\mathcal{R}}} \right) > t \right)$$
$$= P_{\mu_{\mathcal{R},\lambda}^*} \left(\tau_{\mathcal{X}\setminus\mathcal{R},\lambda} > t \right) = e^{-\phi_{\mathcal{R},\lambda}^* t} , \tag{6.21}$$

we have $\phi_{\mathcal{R},\lambda}^* \geq \tilde{\phi}_{\mathcal{X}}^*$.

We then have to estimate $\tilde{\phi}_{\mathcal{X}}^*$ and we do so by comparison with the spectral gap. By Lemma 3.1 applied with $\mathcal{R} = \mathcal{X}$ and the correct normalizations, and taking, with obvious notation, $f = \tilde{h}_{\mathcal{X}}^*$, which is indeed the minimizer in the variational principle satisfied by $\tilde{\phi}_{\mathcal{X}}^*$, we have

$$\tilde{\phi}_{\mathcal{X}}^* \geq \frac{\mathcal{D}(\tilde{h}_{\mathcal{X}}^*)}{\|\tilde{h}_{\mathcal{X}}^*\|^2} = \frac{\|\tilde{h}_{\mathcal{X}}^*\|^2 - 1}{\|\tilde{h}_{\mathcal{X}}^*\|^2} \frac{\mathcal{D}(\tilde{h}_{\mathcal{X}}^*)}{\mathrm{Var}_\mu(\tilde{h}_{\mathcal{X}}^*)} \geq \left(1 - \frac{1}{\|\tilde{h}_{\mathcal{X}}^*\|^2} \right) \gamma. \tag{6.22}$$

Now,

$$\|\tilde{h}_{\mathcal{X}}^*\|^2 \geq \sum_{x \in \mathcal{R}} \mu(x) \left(\frac{\tilde{\mu}_{\mathcal{X}}^*(x)}{\mu(x)} \right)^2 = \mu(\mathcal{R}) \sum_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) \left(\frac{\tilde{\mu}_{\mathcal{X}}^*(x)}{\mu(x)} \right)^2$$
$$\geq \mu(\mathcal{R}) \left(\sum_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) \frac{\tilde{\mu}_{\mathcal{X}}^*(x)}{\mu(x)} \right)^2 = \frac{1}{\mu(\mathcal{R})} (\tilde{\mu}_{\mathcal{X}}^*(\mathcal{R}))^2 . \tag{6.23}$$

Since the escape from $\mathcal{X}$ occurs at rate λ in each point of $\mathcal{X} \setminus \mathcal{R}$ and there are no direct connections between $\mathcal{R}$ and $\overline{\mathcal{X} \setminus \mathcal{R}}$, one has

$$\tilde{\mu}_{\mathcal{X}}^*(\mathcal{X} \setminus \mathcal{R}) \cdot \lambda = \tilde{\phi}_{\mathcal{X}}^* \leq \phi_{\mathcal{R},\lambda}^* \tag{6.24}$$

or

$$\tilde{\mu}_{\mathcal{X}}^*(\mathcal{R}) \geq \left\{ 1 - \frac{\phi_{\mathcal{R},\lambda}^*}{\lambda} \right\} . \tag{6.25}$$

From $\phi_{\mathcal{R},\lambda}^* \geq \tilde{\phi}_{\mathcal{X}}^*$, (6.22), Theorem 2.10 and (6.23) we obtain

$$\phi_{\mathcal{R},\lambda}^* \geq \left(1 - \frac{\mu(\mathcal{R})}{\left\{ 1 - \frac{\phi_{\mathcal{R},\lambda}^*}{\lambda} \right\}^2} \right) \frac{C_\kappa^\lambda(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}{\mu(\mathcal{R})(1 - \mu(\mathcal{R}))} \left\{ \frac{1}{1 + \max\left(\frac{\kappa + \phi_\kappa^*}{\gamma_{\mathcal{R}}}, \frac{\lambda + \phi_\kappa^*}{\gamma_{\mathcal{X}\setminus\mathcal{R}}} \right)} \right\} . \tag{6.26}$$

Developing the square, dropping a few terms and using the convexity of $x \mapsto 1/(1+x)$, this implies

$$\phi_{\mathcal{R},\lambda}^* \geq \frac{C_\kappa^\lambda(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}{\mu(\mathcal{R})} \left\{ \frac{1 - \mu(\mathcal{R}) - 2\phi_{\mathcal{R},\lambda}^*/\lambda}{1 - \mu(\mathcal{R})} \right\} \left\{ 1 - \max\left(\frac{\kappa + \phi_\kappa^\lambda}{\gamma_{\mathcal{R}}}, \frac{\lambda + \phi_\kappa^\lambda}{\gamma_{\mathcal{X}\setminus\mathcal{R}}}, \right) \right\} , \tag{6.27}$$

which is the desired result.

6.6. **Proof of Theorem 2.19.** By Theorem 2.15, for all x in $\mathcal{X}$,

$$\max_{y \in \mathcal{R}} \left| \frac{\mathbb{P}_x \left(X(\mathcal{T}) = y \,\middle|\, \sigma_\lambda > T^*_{\mathcal{X} \backslash \mathcal{R}, \kappa} \right)}{\mu^*_{\mathcal{X} \backslash \mathcal{R}, \kappa}(y)} - 1 \right| \leq \varepsilon^*_{\mathcal{X} \backslash \mathcal{R}, \kappa}. \tag{6.28}$$

Then

$$\begin{aligned}
\|\nu_x - \mu^*_{\mathcal{X} \backslash \mathcal{R}, \kappa}\|_{\mathrm{TV}} &\leq \frac{1}{2} \varepsilon^*_{\mathcal{X} \backslash \mathcal{R}, \kappa} + \mathbb{P} \left(\sigma_\lambda < T^*_{\mathcal{X} \backslash \mathcal{R}, \kappa} \right) \\
&= \frac{1}{2} \varepsilon^*_{\mathcal{X} \backslash \mathcal{R}, \kappa} + 1 - e^{-\lambda T^*_{\mathcal{X} \backslash \mathcal{R}, \kappa}} \\
&\leq \frac{1}{2} \varepsilon^*_{\mathcal{X} \backslash \mathcal{R}, \kappa} + \lambda T^*_{\mathcal{X} \backslash \mathcal{R}, \kappa}.
\end{aligned} \tag{6.29}$$

Also

$$\|\mu^*_{\mathcal{X} \backslash \mathcal{R}, \kappa} - \mu_{\mathcal{X} \backslash \mathcal{R}}\|_{\mathrm{TV}} \quad \leq \quad \frac{1}{2} \sqrt{\frac{\varepsilon^*_{\mathcal{X} \backslash \mathcal{R}}}{1 - \varepsilon^*_{\mathcal{X} \backslash \mathcal{R}}}}, \tag{6.30}$$

$$\|\mu_{\mathcal{X} \backslash \mathcal{R}} - \mu\|_{\mathrm{TV}} \quad = \quad \mu(\mathcal{R}), \tag{6.31}$$

and the upper on $\|\nu_x - \mu\|_{TV}$ follows from the triangular inequality.

Now, for all $t > 0$,

$$\|\mathbb{P}_x(X(t) = \cdot) - \mu\|_{\mathrm{TV}} \leq \|\nu_x - \mu\|_{\mathrm{TV}} + \mathbb{P}_x(\mathcal{T} > t) \tag{6.32}$$

and to prove our mixing time estimate, it is sufficient to show

$$\mathbb{P}_x \left(\mathcal{T} > t \right) \leq \frac{1}{2} \left(\frac{1}{2} + \eta \right) - \mu(\mathcal{R}) - \sqrt{\frac{\varepsilon^*_{\mathcal{X} \backslash \mathcal{R}}}{1 - \varepsilon^*_{\mathcal{X} \backslash \mathcal{R}}}} - \lambda T^*_{\mathcal{X} \backslash \mathcal{R}, 0} = \frac{1}{4} - \frac{\mu(\mathcal{R})}{2} \tag{6.33}$$

for

$$t \geq \frac{2}{\left(\frac{1}{2} - \mu(\mathcal{R}) \right) \phi^*_{\mathcal{R}, \lambda}} \left\{ 1 + \varepsilon^*_{\mathcal{R}, \lambda} + \varepsilon^*_{\mathcal{R}, \lambda} \ln \frac{1}{\varepsilon^*_{\mathcal{R}, \lambda} \zeta_R} + \frac{\phi^*_{\mathcal{R}, \lambda}}{\lambda} \right\}. \tag{6.34}$$

To obtain such an estimate we give an upper bound on the mean value of $\mathcal{T}$ and use Markov inequality. With $\mathcal{T}' = \sigma_\lambda \wedge \tau_\mathcal{R}$, we have, using (2.31) adapted to soft measures,

$$\begin{aligned}
\mathbb{E}_x[\mathcal{T}] &\leq \mathbb{E}[\sigma_\lambda] + \mathbb{E}_x \left[\mathbb{E}_{X(\mathcal{T}')}[\tau_{\mathcal{R}, \lambda}] \,\middle|\, X(\mathcal{T}') \in \mathcal{R} \right] \\
&\leq \frac{1}{\lambda} + \frac{1}{\phi^*_{\mathcal{R}, \lambda}} \left\{ 1 + \varepsilon^*_{\mathcal{R}, \lambda} + \varepsilon^*_{\mathcal{R}, \lambda} \ln \frac{1}{\varepsilon^*_{\mathcal{R}, \lambda} \zeta_\mathcal{R}} \right\} \\
&= \frac{1}{\phi^*_{\mathcal{R}, \lambda}} \left\{ 1 + \varepsilon^*_{\mathcal{R}, \lambda} + \varepsilon^*_{\mathcal{R}, \lambda} \ln \frac{1}{\varepsilon^*_{\mathcal{R}, \lambda} \zeta_\mathcal{R}} + \frac{\phi^*_{\mathcal{R}, \lambda}}{\lambda} \right\},
\end{aligned} \tag{6.35}$$

so that (6.34) implies (6.33).

7. Two examples

In this section we want to illustrate the analysis method of Section 2 with reference to toy models. We will recover known results for the Glauber dynamics of the Curie-Weiss model and give sharp asymptotics of its relaxation time, we will also study a variation on the so-called "n-dog" theme considered in [17] that illustrates the variety of scenarios one can encounter in proving our basic hypothesis on $\varepsilon^*_{\mathcal{R}}$ or controlling $T^*_{\mathcal{R}}$.

7.1. **Metastable behavior of the Curie-Weiss model.**

7.1.1. *Model, dynamics and one-dimensional representation.* Les us consider the Curie-Weiss model which is a mean-field spin system described by N spin variables, $\sigma = (\sigma_1, \ldots, \sigma_N) \in \mathcal{X} = \{-1, 1\}^N$, with Hamiltonian

$$H_{N,h}(\sigma) = -\frac{1}{2N} \sum_{i,j=1}^{N} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i \,, \tag{7.1}$$

where $h > 0$ is called the external field. The corresponding Gibbs probability measure on $\mathcal{X}$ is

$$\mu_{N,h,\beta}(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z_{N,h,\beta}}, \tag{7.2}$$

where $\beta > 0$ is the inverse of the temperature, and $Z_{N,h,\beta} = \sum_{\sigma \in \mathcal{X}} e^{-\beta H_{N,h}(\sigma)}$ is the normalizing factor called the partition function. To make the notation simpler, we set $H(\sigma) \equiv H_{N,h}(\sigma)$, $\mu(\sigma) \equiv \mu_{N,h,\beta}(\sigma)$ and $Z \equiv Z_{N,h,\beta}$.

For every $N \in \mathbb{N}$, and setting $[-1, 1]_N := \{-1, -1 + \frac{2}{N}, \ldots, 1\}$, let us define the total magnetization, $m_N : \mathcal{X} \mapsto [-1, 1]_N$, as

$$m_N(\sigma) \equiv \frac{1}{N} \sum_{i=1}^{N} \sigma_i \,. \tag{7.3}$$

Notice that it allows rewriting the Hamiltonian as a function of a one-dimensional parameter, i.e.

$$H(\sigma) = Nu(m_N(\sigma)) \,, \tag{7.4}$$

with $u(m) = -\frac{m^2}{2} - hm$, for $m \in [-1, 1]$. For simplicity, in the sequel we will identify functions defined on the discrete set $[-1, 1]_N$ with functions defined on $[-1, 1]$ by setting $f(m) \equiv f([2Nm]/2N)$.

For $m \in [-1, 1]$, let us consider the functions

$$f_N(m) = -\frac{1}{\beta N} \ln \sum_{\sigma \,:\, m_N(\sigma)=m} e^{-\beta H(\sigma)} \tag{7.5}$$

$$f(m) = \lim_{N \to \infty} f_N(m) \,. \tag{7.6}$$

A standard computation shows that

$$\begin{aligned} f_N(m) &= u(m) - \frac{1}{\beta} s(m) + \frac{1}{\beta N} \ln \left(\sqrt{\tfrac{(1-m)^2 \pi N}{2}} (1 + o(1)) \right) \\ &= f(m) + \frac{1}{\beta N} \ln \left(\sqrt{\tfrac{(1-m)^2 \pi N}{2}} (1 + o(1)) \right) \,, \end{aligned} \tag{7.7}$$

where $s(m) = -\left(\frac{1+m}{2} \ln \frac{1+m}{2} + \frac{1-m}{2} \ln \frac{1-m}{2} \right)$ is the entropy of Bernoulli random variables. Moreover, the critical points of the function $f(m)$ satisfy the equation

$$m = \tanh(\beta(m + h)) \tag{7.8}$$

and one gets, for $\beta > 1$ and $0 < h < \sqrt{\frac{\beta-1}{\beta}} + \frac{1}{\beta} \ln \left(\sqrt{\beta} + \sqrt{\beta - 1} \right)$, that the graph of $f(m)$ is given by a double-well with two minima $m_- < 0 < m_+$ and a maximum $m_0 < 0$.

We then consider the time evolution of this system provided by a heat-bath Glauber dynamics. This is a Markov chain on $\mathcal{X}$, denoted by $X = (X(t))_{t \geq 0}$, defined through the following (normalized) rates

$$\begin{cases} p(\sigma, \sigma^i) = \frac{1}{N} \frac{e^{-\beta H(\sigma^i)}}{e^{-\beta H(\sigma)} + e^{-\beta H(\sigma^i)}} \,, & \text{for } i = 1, \ldots, N \\ p(\sigma, \sigma) = 1 - \sum_{i=1}^{N} p(\sigma, \sigma^i) \\ p(\sigma, \sigma') = 0 \,, & \text{elsewhere} \end{cases} \tag{7.9}$$

where σ^i denotes the configuration obtained from σ by a spin-flip at the site i. An easy check shows that X is reversible w.r.t μ.

It turns out that the induced dynamics on the space $[-1, 1]_N$, $\bar{X}(t) := m_N(X(t))$, is also Markovian with transition rates

$$\begin{cases} \bar{p}(m, m \pm \frac{2}{N}) = \left(\frac{1 \mp m}{2} \right) \left(\frac{1 + \tanh(\beta \Delta_\pm)}{2} \right) \\ \bar{p}(m, m) = 1 - p(m, m + \frac{2}{N}) - p(m, m - \frac{2}{N}) \\ \bar{p}(m, m') = 0 \,, & \text{elsewhere} \end{cases} \,, \tag{7.10}$$

where $\Delta_\pm := \pm m \pm h + \frac{1}{N}$. Moreover, an easy check shows that the induced dynamics is reversible w.r.t. the probability measure $\bar{\mu}$ on $[-1, +1]_N$, given by

$$\bar{\mu}(m) := \frac{e^{-\beta N f_N(m)}}{Z_N} = \sum_{\sigma \,:\, m_N(\sigma) = m} \mu(\sigma)\,, \tag{7.11}$$

where f_N was defined in (7.7). When parameterized by m, the evolution of our system can be viewed as a one-dimensional random walk driven by a double-well potential. We thus consider the metastable region $\mathcal{R} \subset \mathcal{X}$ and the corresponding one-dimensional projection $\bar{\mathcal{R}} \subset [-1, +1]_N$:

$$\mathcal{R} := \{\sigma \in \mathcal{X} \,:\, m_N(\sigma) \le m_0\} \quad ; \quad \bar{\mathcal{R}} := \{m \in [-1, 1]_N \,:\, m \le m_0\}\,. \tag{7.12}$$

7.1.2. *Verifying hypotheses: First part.* In order to apply our Theorems (2.6) (and the related inequality (2.31)), we first have to verify the hypotheses and in particular provide a suitable upper bound on $\varepsilon_\mathcal{R}^* = \frac{\phi_\mathcal{R}^*}{\gamma_\mathcal{R}}$ and on $\zeta_\mathcal{R}^*$.

By Lemma 2.2 we get

$$\phi_\mathcal{R}^* \le \phi_\mathcal{R} = \mu_\mathcal{R}(e_\mathcal{R}) \le \mu_\mathcal{R}(\partial_-\mathcal{R}) = \frac{\bar{\mu}(m_0)}{\mu(\mathcal{R})} \le \exp\left(-\beta N \Gamma\right)(1 + o(1))\,, \tag{7.13}$$

where in the last inequality we set $\Gamma := f(m_0) - f(m_-)$ and used (7.7). The hypothesis on $\varepsilon_\mathcal{R}^*$ then follows immediately by applying the $(N \log N)^{-1}$ lower bound on $\gamma_\mathcal{R}$ that was derived in [39] by a very precise computation. Since we just need a rough control of this quantity, that will be then compared to $\phi_\mathcal{R}^*$, we provide here a new simpler argument that yields a bound of order $N^{-3/2}$.

We first notice that the dynamics defined by (7.9) can be compared to a random walk on the hypercube. This suggests that a simple way to control the spectral gap is by mixing time estimates. To this aim we consider the dynamics reflected in $\mathcal{R}$, $X_\mathcal{R} = (X_\mathcal{R}(t))_{t \ge 0}$, and the related mixing time,

$$\tau_{mix,\mathcal{R}}\left(\tfrac{1}{4}\right) = \inf\{\max_{\sigma \in \mathcal{R}} \|\mathbb{P}_\sigma(X_\mathcal{R}(t) = \,\cdot\,) - \mu_\mathcal{R}\|_{TV} \le \tfrac{1}{4}\}\,. \tag{7.14}$$

In what follows we will denote by $c(\beta)$ a constant depending on β but independent of N, whose particular value may change from line to line. With the above notation it holds the following:

Proposition 7.1.

$$\tau_{mix,\mathcal{R}}\left(\tfrac{1}{4}\right) \le c(\beta) N^{\frac{3}{2}}\,. \tag{7.15}$$

Proof. The idea of the proof is based on coupling techniques, and we thus define the following coupling:

  a) We consider two independent dynamics $X_\mathcal{R}^\sigma$ and $X_\mathcal{R}^\eta$, with initial states $\sigma, \eta \in \mathcal{R}$, and let them run independently until they reach the same magnetization.
  b) We then run an analogous coupling to that defined in [39], where the only difference here is the reflection on $\mathcal{R}$. This coupling is defined for initial states $\sigma, \sigma' \in \mathcal{R}$ with $m_N(\sigma) = m_N(\sigma')$, and is defined in such a way so as to keep the magnetizations coupled along the dynamics.

Let $T_{\sigma,\eta}$ denote the coupling time for the global coupled dynamics $(X_\mathcal{R}^\sigma(t), X_\mathcal{R}^\eta(t))$, that is

$$T_{\sigma,\eta} = \inf\{t \ge 0 \,:\, X_\mathcal{R}^\sigma(t) = X_\mathcal{R}^\eta(t)\}\,. \tag{7.16}$$

To provide an estimate on $\tau_{mix,\mathcal{R}}\left(\tfrac{1}{4}\right)$, it is then enough to find t such that

$$\max_{\sigma,\eta \in \mathcal{R}} \mathbb{P}(T_{\sigma,\eta} > t) \le \tfrac{1}{4}\,. \tag{7.17}$$

The coupling time $T_{\sigma,\eta}$ can be controlled by first estimating the time to couple the magnetizations, and then the coupling time of the dynamics starting in configurations with equal magnetization.

Formally, let $(X_\mathcal{R}^m(t), X_\mathcal{R}^{m'}(t))$ be the induced coupled dynamics with initial states $m, m' \in \bar{\mathcal{R}}$, and define

$$\bar{T}_{m,m'} = \inf\{t \ge 0 \,:\, X_\mathcal{R}^m(t) = X_\mathcal{R}^{m'}(t)\}\,. \tag{7.18}$$

Then, for any $\sigma, \eta \in \mathcal{R}$,

$$
\mathbb{P}(T_{\sigma,\eta} > t) \leq \mathbb{P}\left(\max_{m,m' \in \mathcal{R}} \bar{T}_{m,m'} + \max_{\substack{\sigma,\sigma' \in \mathcal{R}: \\ m_N(\sigma) = m_N(\sigma')}} T_{\sigma,\sigma'} > t \right)
$$

$$
\leq \max_{m,m' \in \bar{\mathcal{R}}} \mathbb{P}(\bar{T}_{m,m'} > \tfrac{t}{2}) + \max_{\substack{\sigma,\sigma' \in \mathcal{R}: \\ m_N(\sigma) = m_N(\sigma')}} \mathbb{P}(T_{\sigma,\sigma'} > \tfrac{t}{2})
$$

(7.19)

Following the same argument of [39] (*see Lemma 2.9 and its proof*) it is easy to prove that, for any $\sigma, \sigma' \in \mathcal{R}$ such that $m_N(\sigma) = m_N(\sigma')$,

$$
\mathbb{P}(T_{\sigma,\sigma'} > c(\beta) N \log N) \leq \tfrac{1}{N}
$$

(7.20)

for a constant $c(\beta)$ depending on β but not in N.

To control the time $\bar{T}_{m,m'}$, let τ_m denote the stopping time in m for $X_{\bar{\mathcal{R}}}$. With some abuse of notation, let $\mathbb{E}_m$ denote the average over the dynamics $X_{\bar{\mathcal{R}}}$ with initial state $m \in \bar{\mathcal{R}}$ and define

$$
T := \max_{m \in \mathcal{R}} \mathbb{E}_m(\tau_{m_-}) = \max\left\{ \mathbb{E}_{-1}(\tau_{m_-}); \mathbb{E}_{m_0}(\tau_{m_-}) \right\}
$$

(7.21)

where the second equality is due to an obvious geometric fact. Then the following Lemmas hold.

Lemma 7.2. *With the above notation, it holds*

$$
T \leq c(\beta) N^{\frac{3}{2}} .
$$

(7.22)

Lemma 7.3. *For all $t \geq 40T$, it holds*

$$
\max_{m,m' \in \bar{\mathcal{R}}} \mathbb{P}(T_{m,m'} > t) = \mathbb{P}(T_{-1,m_0} > t) \leq \tfrac{1}{2} .
$$

(7.23)

Before proving the above Lemmas, let us conclude the proof of Prop. 7.1.
By inequalities (7.19)-(7.20) and Lemma 7.3, it follows that if $t = \max\{240T, N^{\frac{3}{2}}\}$, $N \geq 8$, and for any $\sigma, \eta \in \mathcal{R}$,

$$
\mathbb{P}(T_{\sigma,\eta} > t) \leq \mathbb{P}(\bar{T}_{-1,m_0} > 120T) + \max_{\substack{\sigma,\sigma' \in \mathcal{R}: \\ m_N(\sigma) = m_N(\sigma')}} \mathbb{P}(T_{\sigma,\sigma'} > N^{\frac{3}{2}})
$$

(7.24)

$$
\leq \frac{1}{8} + \frac{1}{N} \leq \frac{1}{4}
$$

By Lemma 7.2 the above inequality holds whenever $t > c(\beta) N^{\frac{3}{2}}$ and the statement of the Proposition follows. $\qquad\square$

We now come back to the proofs of the two Lemmas.

Proof of Lemma 7.2. Since $X_{\bar{\mathcal{R}}}$ is a one-dimensional dynamics, for any two states $x, y \in \bar{\mathcal{R}}$ we have the formula

$$
\mathbb{E}_x(\tau_y) = \frac{\bar{\mu}(V_{x,y})}{C(x,y)}
$$

(7.25)

where $V_{x,y}$ and $C(x,y)$ are, respectively, the equilibrium potential and the capacity between x and y. Moreover, if $x < y$, we have

$$
V_{x,y}(m) = \mathbb{P}_m(\tau_x < \tau_y) = \begin{cases} 1 & \text{if } m \leq x \\ 0 & \text{if } m \geq y \\ \frac{C(x,y)}{C(m,y)} & \text{if } x < m < y \end{cases} ,
$$

(7.26)

$$
C(x,y)^{-1} = \sum_{k=0}^{(y-x)\frac{N}{2}-1} \left(\bar{c}\left(x + \tfrac{2k}{N}, x + \tfrac{2(k+1)}{N} \right) \right)^{-1} ,
$$

(7.27)

with $\bar{c}(x,y) = \bar{\mu}(x)\bar{p}(x,y)$. Analogous formulas hold when $x > y$.

Remark. Since we are considering the dynamics reflecting in $\bar{\mathcal{R}}$, the classical version for the mean exit time would be $\mathbb{E}_x(\tau_y) = \frac{\bar{\mu}_{\bar{\mathcal{R}}}(V_{x,y})}{C_{\bar{\mathcal{R}}}(x,y)}$ rather than Eq. (7.25), where $C_{\bar{\mathcal{R}}}(x,y)$ is the capacity defined through conductances $\bar{c}(x,y) = \bar{\mu}_{\bar{\mathcal{R}}}(x)\bar{p}_{\bar{\mathcal{R}}}(x,y)$. However, it is easy to verify that for points $x, y \in \bar{\mathcal{R}}$ the two formulas are equivalent.

In Appendix B we will show that, if there are no local maxima of f_N in $[x, y]$,

$$C(x, y)^{-1} \leq c(\beta) \sqrt{N} Z_N \max_{z \in \{x,y\}} e^{\beta N f_N(z)} . \tag{7.28}$$

Putting together (7.25)-(7.28), and since $f(y) > f(m_-)$ for any $y \in [-1, m_-)$, we get

$$\mathbb{E}_{-1}(\tau_{m_-}) = \sum_{j=0}^{(m_-+1)\frac{N}{2}-1} \bar{\mu}_{\mathcal{R}}(-1 + \tfrac{2j}{N}) C(-1 + \tfrac{2j}{N}, m_-)^{-1} \leq c(\beta) N^{\frac{3}{2}} \tag{7.29}$$

Analogous computations can be done for $\mathbb{E}_{m_0}(\tau_{m_-})$, providing the same estimate. This concludes the proof of the Lemma. $\qquad \square$

Proof of Lemma 7.3. The first identity of (7.23) is obvious, due to the geometry of the problem.

We then focus on the two dynamics $X_{\mathcal{R}}^{-1}$ and $X_{\mathcal{R}}^{m_0}$, and define recursively the stopping times s_k, τ_k and s_k', for $k \geq 1$:

$$\begin{aligned}
s_1 &:= \inf_{t \geq 0} \{ X_{\mathcal{R}}^{-1}(t) = m_- \} \\
\tau_k &:= \inf_{t \geq s_k} \{ X_{\mathcal{R}}^{m_0}(t) = m_- \} \\
s_{k+1} &:= \inf_{t \geq \tau_k} \{ X_{\mathcal{R}}^{-1}(t) = m_- \} \\
s_k' &:= \sup_{t \leq \tau_k} \{ X_{\mathcal{R}}^{-1}(t) = m_- \}
\end{aligned} \tag{7.30}$$

Letting $\tau(t)$ denote the first clock ring after time t, we can define the event

$$A := \{ \exists k \leq 2 : s_k' = \tau_k \text{ or } X_{\mathcal{R}}^{-1}(s_k' + \tau(s_k')) > m_- \} . \tag{7.31}$$

Notice that, since s_k' is the time of the last visit in m_- of the dynamics $X_{\mathcal{R}}^{-1}$ before τ_k, and because $-1 < m_- < m_0$, the occurrence of the event $\{ X_{\mathcal{R}}^{-1}(s_k' + \tau(s_k')) > m_- \}$ implies that $T_{-1,m_0} < \tau_k$. In particular, we have $A \cup \{ \tau_2 \leq t \} \subset \{ T_{-1,m_0} \leq t \}$ and then

$$\mathbb{P}(T_{-1,m_0} > t) \leq \mathbb{P}(A^c) + \mathbb{P}(\tau_2 > t) . \tag{7.32}$$

From the definition (7.10) of rates $\bar{p}$, and using that $\frac{1-m_-}{2} > \frac{1+m_-}{2}$ together with the properties of the hyperbolic tangent, one can show that

$$\bar{p}(m_-, m_- - \tfrac{2}{N}) \leq \bar{p}(m_-, m_- + \tfrac{2}{N}) \iff \bar{p}(m_-, m_- - \tfrac{2}{N}) \leq \tfrac{1}{2}(1 - \bar{p}(m_-, m_-)) . \tag{7.33}$$

Thus

$$\begin{aligned}
\mathbb{P}(A^c) &\leq \mathbb{P}(X_{\mathcal{R}}^{-1}(s_k' + \tau(s_k')) < m_- , \text{ for } k = 1, 2) \\
&\leq \left(\mathbb{P}(X_{\mathcal{R}}^{-1}(t + \tau(t)) = m_- - \tfrac{2}{N} | X_{\mathcal{R}}^{-1}(t) = m_-, X_{\mathcal{R}}^{-1}(t + \tau(t)) \neq m_-) \right)^2 \\
&= \left(\frac{\bar{p}(m_-, m_- - \tfrac{2}{N})}{1 - \bar{p}(m_-, m_-)} \right)^2 \leq \frac{1}{4}
\end{aligned} \tag{7.34}$$

In order to estimate $\mathbb{P}(\tau_2 > t)$, we divide the interval $[0, t]$ in $k = \lfloor \frac{t}{8T} \rfloor$ subintervals of length $8T$, where T was defined in (7.21). The event $\{ \tau_2 > t \}$ is then included in the event that, in at least $k - 2$ subintervals, at most one of the process has arrivals in m_-. On each interval, this happens with probability bounded above by

$$\mathbb{P}_{-1}(\tau_{m_-} > 8T) + \mathbb{P}_{m_0}(\tau_{m_-} > 8T) \leq \frac{1}{8T}(\mathbb{E}_{-1}(\tau_{m_-}) + \mathbb{E}_{m_0}(\tau_{m_-})) \leq \frac{1}{4}, \tag{7.35}$$

by Markov's inequality, and then

$$\mathbb{P}(\tau_2 > t) \leq \binom{k-2}{k} \left(\frac{1}{4} \right)^{k-2} \leq 2^{-k+3} . \tag{7.36}$$

The statement follows taking $t \geq 40T$, so that $k \geq 5$ and $\mathbb{P}(\tau_2 > t) \leq \frac{1}{4}$, and finally, from (7.32) and the previous estimates,

$$\mathbb{P}(T_{-1,m_0} > t) \leq \frac{1}{2} . \tag{7.37}$$

$\qquad \square$

Coming back to the hypotheses on $\varepsilon_{\mathcal{R}}^*$, from the well known inequality $\gamma_{\mathcal{R}}^{-1} \leq \tau_{mix,\mathcal{R}}(\frac{1}{4})$, and by (7.13) and (7.15), we obtain

$$\varepsilon_{\mathcal{R}}^* = \frac{\phi_{\mathcal{R}}^*}{\gamma_{\mathcal{R}}} \leq c(\beta) N^{\frac{3}{2}} \exp\left(-\beta N\Gamma\right)(1 + o(1)), \tag{7.38}$$

which goes to 0 for any N large enough, and thus satisfies the hypothesis of our main theorems. Moreover, from Lemma 2.5 and the trivial bounds $\Delta_{\mathcal{R}} \geq N$ and $D_{\mathcal{R}} \geq c(\beta)$, we get

$$\zeta_{\mathcal{R}}^* \geq e^{-\Delta_{\mathcal{R}} D_{\mathcal{R}}} \geq e^{-Nc(\beta)} \tag{7.39}$$

By inequality 2.28 and the previous estimates, this implies that for N large enough the condition $\phi_{\mathcal{R}}^* \cdot T_{\mathcal{R}}^* \ll 1$ is satisfied.

7.1.3. *Asymptotic law of the exit time.* Applying Th. 2.6 and by the related inequality (2.31), we get that in the limit $N \to \infty$ and for all distributions ν on $\mathcal{R}$,

$$\begin{cases} \mathbb{E}_\nu(\tau_{\mathcal{X}\backslash\mathcal{R}}) \leq \phi_{\mathcal{R}}^{*-1}(1 + o(1)) \\ \mathbb{E}_\nu(\tau_{\mathcal{X}\backslash\mathcal{R}}) \geq (1 - \pi_{\mathcal{R}}(\nu))\phi_{\mathcal{R}}^{*-1}(1 + o(1)) \end{cases} \tag{7.40}$$

and for all $t > 0$,

$$\mathbb{P}_\nu(\phi_{\mathcal{R}}^* \tau_{\mathcal{X}\backslash\mathcal{R}} > t) = (1 - \pi_{\mathcal{R}}(\nu))e^{-t}(1 + o(1)). \tag{7.41}$$

In particular, for $\nu = \mu_{\mathcal{R}}$,

$$\mathbb{E}_{\mu_{\mathcal{R}}}(\tau_{\mathcal{X}\backslash\mathcal{R}}) = \phi_{\mathcal{R}}^{*-1}(1 + o(1)) \tag{7.42}$$

$$\mathbb{P}_{\mu_{\mathcal{R}}}(\phi_{\mathcal{R}}^* \cdot \tau_{\mathcal{X}\backslash\mathcal{R}} > t) = e^{-t}(1 + o(1)), \quad \forall t \geq 0. \tag{7.43}$$

The next step concerns the estimation of $\phi_{\mathcal{R}}^*$. By Th. 2.9, assuming that N is large enough to have $\varepsilon_{\mathcal{R}}^* + \phi_{\mathcal{R}}^* \cdot T_{\mathcal{R}}^* \ll 1$, and choosing k such that $\phi_{\mathcal{R}}^* \ll k \ll \gamma_{\mathcal{R}}$, we have

$$\phi_{\mathcal{R}}^* = \frac{C_k(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})}{\mu(\mathcal{R})}. \tag{7.44}$$

In order to estimate $C_k(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})$, we use its two variational characterizations, (2.33) and (2.36), with suitable test functions. The one-dimensional nature of the model suggests that the capacities of the dynamics over $\mathcal{X}$ could be well approximated by the analogous quantities computed for the induced dynamics over $[-1, 1]_N$. This has the advantage that the equilibrium potential of the one-dimensional chain, namely the minimizer in (2.33) for $k, \lambda = \infty$, can be explicitly given.

Following this idea, for the upper bound we consider the test function $V(\sigma) := V_{m_-, m_0}(m_N(\sigma))$, where V_{m_-, m_0} is the function defined in (7.26). In other words, $V(\sigma)$ is the equilibrium potential associated to the one-dimensional chain, with boundary conditions $V(m_-) = 1$ and $V(m_0) = 0$. Explicitly, for $m \in [-1, 1]_N$,

$$V_{m_-, m_0}(m) = \mathbb{P}_m(\tau_{m_-} < \tau_{m_0}) = \begin{cases} 1 & \text{if } m \leq m_- \\ 0 & \text{if } m \geq m_0 \\ \frac{C(m_-, m_0)}{C(m, m_0)} & \text{otherwise} \end{cases}, \tag{7.45}$$

where $C(x, y)^{-1} = \sum_{k=0}^{(y-x)\frac{N}{2}-1} \left(\bar{c}\left(x + \frac{2k}{N}, x + \frac{2(k+1)}{N}\right)\right)^{-1}$ and $\bar{c}(x, y) = \bar{\mu}(x)\bar{p}(x, y)$.

Then we have

$$C_k(\mathcal{R}, \mathcal{X}\backslash\mathcal{R}) \leq \mathcal{D}(V) + k \sum_{\sigma \in \mathcal{R}} \mu(\sigma)(V(\sigma) - 1)^2$$

$$= C(m_-, m_0) + k \sum_{m=m_-}^{m_0} \frac{e^{-\beta N f_N(m)}}{Z_N} \left(\frac{C(m_-, m_0)}{C(m_-, m)}\right)^2 \tag{7.46}$$

$$\leq C(m_-, m_0) + kc(\beta)NC(m_-, m_0),$$

where the last inequality is due to (7.28) which is derived in Appendix B. Since $k \ll \gamma_{\mathcal{R}} \leq c(\beta)N^{-3/2}$, it holds

$$C_k(\mathcal{R}, \mathcal{X}\backslash\mathcal{R}) \leq C(m_-, m_0)(1 + o(1)). \tag{7.47}$$

Similarly, for the lower bound on $C_k(\mathcal{R}, \mathcal{X}\setminus\mathcal{R})$ we consider a unitary test flow ψ which is constant on all couples (σ, σ') of given magnetization. Specifically we set $\psi(\sigma, \sigma') := \Psi(m_N(\sigma), m_N(\sigma'))$ and define

$$\begin{cases} \Psi(m, m + \frac{2}{N}) = \left(S(m)\frac{(1-m)N}{2}\right)^{-1} & \forall\, m \in [m_-, m_0]_N\,, \\ \Psi(m, m') = 0 & \text{otherwise} \end{cases} \tag{7.48}$$

with $S(m) = |\{\sigma\,:\,m_N(\sigma) = m\}|$. With this definition, the flow Ψ is the unitary flow from m_- to m_0 that realized the minimum in the Thompson principle for the one-dimensional chain. Inserting the test flow in (2.36), we then have

$$\begin{aligned} C_k(\mathcal{R}, \mathcal{X}\setminus\mathcal{R})^{-1} &\leq \mathcal{D}(\psi) + \frac{\mu(\mathcal{R})}{k}S(m_-)\frac{e^{-\beta N u(m_-)}}{\mu(\mathcal{R})Z_N}\left(\frac{Z_N \cdot e^{-\beta N u(m_-)}}{S(m_-)}\right)^2 \\ &= \sum_{m=m_-}^{m_0} \frac{Z_N \cdot e^{\beta N f_N(m)}}{\bar{p}(m, m+\frac{2}{N})} + \frac{1}{k}Z_N \cdot e^{\beta N f_N(m_-)} \\ &= C(m_-, m_0) + \frac{1}{k}\mu(m_-)^{-1}\,. \end{aligned} \tag{7.49}$$

Since $k^{-1} \ll (\phi_{\mathcal{R}}^*)^{-1} \leq \mu(m_-)C_k(\mathcal{R}, \mathcal{X}\setminus\mathcal{R})^{-1}$ (by (7.44)), we get

$$C_k(\mathcal{R}, \mathcal{X}\setminus\mathcal{R}) \geq C(m_-, m_0)(1 + o(1))\,. \tag{7.50}$$

From (7.44)) and with the above estimate, we then have

$$\phi_{\mathcal{R}}^* = \frac{C(m_-, m_0)}{\mu(\mathcal{R})}(1 + o(1))\,. \tag{7.51}$$

Finally, $\mu(\mathcal{R})$ and the capacity $C(m_-, m_0)$ defined in (7.27) can be both evaluated for large N (see Appendix B), providing the following asymptotic expressions:

$$C(m_-, m_0) = \frac{\sqrt{(1-m_0^2)\beta|f''(m_0)|}}{\pi N}\frac{e^{-\beta N f(m_0)}}{Z_N}(1 + o(1)) \tag{7.52}$$

$$\mu(\mathcal{R}) = \frac{e^{-\beta N f(m_-)}}{Z_N \cdot \sqrt{\beta f''(m_-)(1-m_-^2)}}(1 + o(1))\,. \tag{7.53}$$

Altogether, under the same hypotheses of before, we have

$$\mathbb{E}_{\mu_{\mathcal{R}}}(\tau_{\mathcal{X}\setminus\mathcal{R}}) = \frac{\pi N \cdot e^{\beta N(f(m_0) - f(m_-))}}{\beta\sqrt{|f''(m_0)|f''(m_-)(1-m_0^2)(1-m_-^2)}}(1 + o(1)) \tag{7.54}$$

7.1.4. *Verifying hypotheses: Second part.* To move to the second part of the analysis, which goes from Th. 2.10 to Th. 2.19, we first have to estimate the quantities $\phi_{\mathcal{X}\setminus\mathcal{R}}^*$, $\gamma_{\mathcal{X}\setminus\mathcal{R}}$ and $\varepsilon_{\mathcal{X}\setminus\mathcal{R}}^*$ related to the dynamics over $\mathcal{X}\setminus\mathcal{R}$. As for $\phi_{\mathcal{R}}^*$, we can find easily a rough (but sufficient) upper bound over $\phi_{\mathcal{X}\setminus\mathcal{R}}^*$ by Lemma 2.2. By trivial estimates we get

$$\phi_{\mathcal{X}\setminus\mathcal{R}}^* \leq \phi_{\mathcal{X}\setminus\mathcal{R}} = \mu_{\mathcal{X}\setminus\mathcal{R}}(e_{\mathcal{X}\setminus\mathcal{R}}) \leq \mu_{\mathcal{X}\setminus\mathcal{R}}(\partial_+\mathcal{R}) = \frac{\bar{\mu}(m_0 + \frac{2}{N})}{\mu(\mathcal{X}\setminus\mathcal{R})} \leq \exp\left(-\beta N\Gamma'\right)(1 + o(1))\,, \tag{7.55}$$

where in the last inequality we set $\Gamma' := f(m_0 + \frac{2}{N}) - f(m_+)$ and used (7.7).

To get a lower bound over $\gamma_{\mathcal{X}\setminus\mathcal{R}}$, as for $\gamma_{\mathcal{R}}$ we proceed by first estimating the mixing time of the dynamics reflected in $\mathcal{X}\setminus\mathcal{R}$, $X_{\mathcal{X}\setminus\mathcal{R}} = (X_{\mathcal{X}\setminus\mathcal{R}}(t))_{t\geq 0}$. With obvious notation, it holds the following:

Proposition 7.4.

$$\tau_{mix, \mathcal{X}\setminus\mathcal{R}}\left(\tfrac{1}{4}\right) \leq c(\beta)N^{\frac{3}{2}}\,. \tag{7.56}$$

Proof. The proof is the same as for Prop. 7.1, and can write down just replacing $\mathcal{R}$ with $\mathcal{X}\setminus\mathcal{R}$, the states $-1, m_-$ and m_0 respectively with $m+, -1$ and $m_0 + \frac{2}{N}$ and the time T defined in (7.21) with

$$T' = \max_{m\in\mathcal{X}\setminus\mathcal{R}} \mathbb{E}_m(\tau_{m_+}) = \max\left\{\mathbb{E}_{+1}(\tau_{m_+}); \mathbb{E}_{m_0+\frac{2}{N}}(\tau_{m_+})\right\}\,.$$

$\square$

As a consequence of (7.55) and Prop. 7.4, we obtain

$$\varepsilon^*_{\mathcal{X}\backslash\mathcal{R}} = \frac{\phi^*_{\mathcal{X}\backslash\mathcal{R}}}{\gamma_{\mathcal{X}\backslash\mathcal{R}}} \leq c(\beta)N^{\frac{3}{2}}\exp\left(-\beta N\Gamma'\right)(1+o(1)), \tag{7.57}$$

and also

$$\frac{\phi^*_{\mathcal{R}}}{\gamma_{\mathcal{X}\backslash\mathcal{R}}} \leq c(\beta)N^{\frac{3}{2}}\exp\left(-\beta N\Gamma\right)(1+o(1)), \tag{7.58}$$

which are both $\ll 1$ for any N large enough.

7.1.5. *Relaxation, transition and mixing times.* From inequalities (7.57) and (7.58), we can choose k, λ in Theorems 2.10-2.19, such that $\phi_{\mathcal{R}} \ll k \ll \gamma_{\mathcal{R}}$ and $\phi_{\mathcal{R}} + \phi_{\mathcal{X}\backslash\mathcal{R}} \ll \lambda \ll \gamma_{\mathcal{X}\backslash\mathcal{R}}$, and then get matching upper and lower bound over on the relaxation time γ and the mean transition time. Explicitly, by Th. 2.10, 2.17 and 2.18, it holds that in the limit $N \to \infty$ and for k, λ such that $\phi^*_{\mathcal{R}} \ll k \ll \gamma_{\mathcal{R}}$ and $\max\{\phi^*_{\mathcal{R}}, \phi^*_{\mathcal{X}\backslash\mathcal{R}}\} \ll \lambda \ll \gamma_{\mathcal{X}\backslash\mathcal{R}}$,

   i)
$$\gamma^{-1} = \frac{\mu(\mathcal{R})\mu(\mathcal{X}\backslash\mathcal{R})}{C^\lambda_k(\mathcal{R},\mathcal{X}\backslash\mathcal{R})}(1+o(1)) \tag{7.59}$$

   ii) For all distribution ν over $\mathcal{R}$,
$$\begin{cases} \mathbb{E}_\nu(\tau_{\mathcal{X}\backslash\mathcal{R},\lambda}) \leq \phi^{*\,-1}_{\mathcal{R},\lambda}(1+o(1)) \\ \mathbb{E}_\nu(\tau_{\mathcal{X}\backslash\mathcal{R},\lambda}) \geq (1-\pi_{\mathcal{R}}(\nu))\phi^{*\,-1}_{\mathcal{R},\lambda}(1+o(1)) \end{cases} \tag{7.60}$$

   and for all $t > 0$,
$$\mathbb{P}_\nu(\phi^*_{\mathcal{R},\lambda}\tau_{\mathcal{X}\backslash\mathcal{R},\lambda} > t) = (1-\pi_{\mathcal{R}}(\nu))e^{-t}(1+o(1)). \tag{7.61}$$

   In particular, for $\nu = \mu_{\mathcal{R}}$,
$$\mathbb{E}_{\mu_{\mathcal{R}}}(\tau_{\mathcal{X}\backslash\mathcal{R},\lambda}) = \phi^{*\,-1}_{\mathcal{R},\lambda}(1+o(1)) \tag{7.62}$$

$$\mathbb{P}_{\mu_{\mathcal{R}}}(\phi^*_{\mathcal{R},\lambda}\tau_{\mathcal{X}\backslash\mathcal{R},\lambda} > t) = e^{-t}(1+o(1)), \quad \forall t \geq 0. \tag{7.63}$$

   iii)
$$\phi^*_{\mathcal{R},\lambda} = \frac{C^\lambda_k(\mathcal{R},\mathcal{X}\backslash\mathcal{R})}{\mu(\mathcal{R})}(1+o(1)) \tag{7.64}$$

To provide quantitative estimates on the relaxation and transition time, it thus remains to estimate the capacity $C^\lambda_k(\mathcal{R},\mathcal{X}\backslash\mathcal{R})$. As for $C_k(\mathcal{R},\mathcal{X}\backslash\mathcal{R})$, we make use of the variational characterizations (2.33) and (2.36), with suitable test functions. The functions that we consider are extensions of those defined for $C_k(\mathcal{R},\mathcal{X}\backslash\mathcal{R})$, in the sense that they are defined similarly but on a bigger support.

Explicitly, let $\widetilde{V}(\sigma) := V_{m_-,m_+}(m_N(\sigma))$, with V_{m_-,m_+} defined in (7.26). Plugging $\widetilde{V}$ into (2.33), we obtain the upper bound

$$C^\lambda_k(\mathcal{R},\mathcal{X}\backslash\mathcal{R}) \leq \mathcal{D}(\widetilde{V}) + k\sum_{\sigma\in\mathcal{R}}\mu(\sigma)(\widetilde{V}(\sigma)-1)^2 + \lambda\sum_{\sigma\in\mathcal{X}\backslash\mathcal{R}}\mu(\sigma)(\widetilde{V}(\sigma))^2 \tag{7.65}$$

Since $\widetilde{V}$ is defined as the equilibrium potential of the one-dimensional chain, with boundary condition $\widetilde{V}(m_-) = 1$ and $\widetilde{V}(m_+) = 0$, we have that $\mathcal{D}(\widetilde{V}) = C(m_-,m_+)$. Using inequality (7.28) and choosing $k, \lambda \ll \gamma_{\mathcal{X}\backslash\mathcal{R}} \leq c(\beta)N^{-\frac{3}{2}}$, the second and third terms of (7.65) are bounded as

$$k\sum_{\sigma\in\mathcal{R}}\mu(\sigma)(\widetilde{V}(\sigma)-1)^2 = k\sum_{m=m_-}^{m_0}\frac{e^{-\beta N f_N(m)}}{Z_N}\left(\frac{C(m_-,m_+)}{C(m_-,m-\frac{2}{N})}\right)^2$$

$$\leq kc(\beta)N^{\frac{3}{2}}\left(C(m_-,m_+)\right)^2 Z_N\, e^{\beta N f_N(m_0)}$$

$$\leq c(\beta)\left(C(m_-,m_+)\right)^2 Z_N\, e^{\beta N f_N(m_0)}$$

$$\lambda\sum_{\sigma\in\mathcal{X}\backslash\mathcal{R}}\mu(\sigma)(\widetilde{V}(\sigma))^2 = \lambda\sum_{m=m_0}^{m^+}\frac{e^{-\beta N f_N(m)}}{Z_N}\left(\frac{C(m_-,m_+)}{C(m,m_+)}\right)^2$$

$$\leq \lambda c(\beta)N^{\frac{3}{2}}\left(C(m_-,m_+)\right)^2 Z_N\, e^{\beta N f_N(m_0)}$$

$$\leq c(\beta)\left(C(m_-,m_+)\right)^2 Z_N\, e^{\beta N f_N(m_0)}$$

In Appendix B, the capacity $C(m_-, m_+)$ is evaluated for large N and the following formula is obtained

$$C(m_-, m_+) = \frac{\sqrt{(1-m_0^2)\beta|f''(m_0)|}}{2\pi N} \frac{e^{-\beta N f(m_0)}}{Z_N}(1 + o(1)).\tag{7.66}$$

This implies that the second and third terms above are $o(C(m_-, m_+))$ and then

$$C_k^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R}) \leq C(m_-, m_+)(1 + o(1)).\tag{7.67}$$

For the lower bound we consider a test unitary flow $\widetilde{\psi}(\sigma, \sigma') := \widetilde{\Psi}(m_N(\sigma), m_N(\sigma'))$ with

$$\begin{cases} \Psi(m, m + \frac{2}{N}) = \left(S(m)\frac{(1-m)N}{2}\right)^{-1} & \forall m \in [m_-, m_+]_N, \\ \Psi(m, m') = 0 & \text{otherwise} \end{cases}.\tag{7.68}$$

Inserting the test flow in (2.36), we then have

$$\begin{aligned} C_k^\lambda(\mathcal{R}, \mathcal{X}\backslash\mathcal{R})^{-1} &\leq \mathcal{D}(\widetilde{\psi}) + \frac{\mu(\mathcal{R})}{k}S(m_-)\frac{e^{-\beta N u(m_-)}}{\mu(\mathcal{R})Z_N}\left(\frac{Z_N \cdot e^{-\beta N u(m_-)}}{S(m_-)}\right)^2 \\ &\quad + \frac{\mu(\mathcal{X}\backslash\mathcal{R})}{\lambda}S(m_+)\frac{e^{-\beta N u(m_+)}}{\mu(\mathcal{X}\backslash\mathcal{R})Z_N}\left(\frac{Z_N \cdot e^{-\beta N u(m_+)}}{S(m_+)}\right)^2 \\ &= \sum_{m=m_-}^{m_0}\frac{Z_N \cdot e^{\beta N f_N(m)}}{\bar{p}(m, m + \frac{2}{N})} + \frac{1}{k}Z_N \cdot e^{\beta N f_N(m_-)} + \frac{1}{\lambda}Z_N \cdot e^{\beta N f_N(m_+)} \\ &\leq C(m_-, m_+)^{-1}(1 + o(1)), \end{aligned}\tag{7.69}$$

where in the last step we used that $k^{-1} \ll \phi_\mathcal{R}^{*-1} = \mu(m_-)C(m_-, m_0)^{-1}$, $\lambda^{-1} \ll \phi_{\mathcal{X}\backslash\mathcal{R}}^{*-1} = \mu(m_+)C(m_0, m_+)^{-1}$, and the fact that $C(m_-, m_0), C(m_0, m_+)$ and $C(m_-, m_+)$ are all of order $N^{-1}e^{-\beta N f(m_0)}$ (see Appendix B). From (7.64) and with the above estimates, we then have

$$\phi_{\mathcal{R},\lambda}^* = \frac{C(m_-, m_+)}{\mu(\mathcal{R})}(1 + o(1)).\tag{7.70}$$

Finally, if we choose λ so that $\lambda T_{\mathcal{X}\backslash\mathcal{R}}^* \ll 1$, for example $\lambda \ll N^{-5/2}$, then we can apply Th. 2.19 and get an upper bound on the mixing time of the same order of the transition and relaxation times. Altogether, in the limit $N \to \infty$ and for k, λ such that $e^{-\beta N\Gamma} \ll k \ll N^{-3/2}$ and $e^{-\beta N\Gamma}\} \ll \lambda \ll N^{-5/2}$, it holds

i)

$$\begin{aligned} \gamma^{-1} &= \mathbb{E}_{\mu_\mathcal{R}}(\tau_{\mathcal{X}\backslash\mathcal{R},\lambda})(1 + o(1)) \\ &= \frac{2\pi N \cdot e^{\beta N(f(m_0) - f(m_-))}}{\beta\sqrt{|f''(m_0)|f''(m_-)(1-m_0^2)(1-m_-^2)}}(1 + o(1)) \end{aligned}\tag{7.71}$$

(ii)

$$\tau_{mix}(\tfrac{1}{4}) \leq 4\gamma^{-1}(1 + o(1))\tag{7.72}$$

Remark. Notice that in the Curie-Weiss model the mean exit time and the mean transition time differ asymptotically only by a factor 2 (see Eqs. (7.54)and (7.71)). This is a slight difference but one that clarifies the different rule of the exit time from the transition time. Notice also that by the well-known bound $\tau_{mix}(\frac{1}{4}) \geq \gamma^{-1}$, the second result shows that the mixing time and the relaxation time are of the same order, which is $N \cdot e^{\beta N(f(m_0) - f(m_-))}$.

7.2. The wasp graph. Given three positive real numbers r_a, r_t and r_w and a positive integer n, we set $l_a = \lfloor nr_a \rfloor$, $l_t = \lfloor nr_t \rfloor$ and $l_w = \lfloor nr_w \rfloor$. We then consider two cubic lattices with vertices indexed by $\{0, \ldots, l_a\}^3$ and $\{0, \ldots, l_t\}^3$, four copies of the square lattice with vertices indexed by $\{0, \ldots, l_w\}^2$ and we attached them together by identifying some corners as in Figure 1, forming then the "wasp graph" with its four "wings" and its "abdomen" attached to its central "thorax". We finally place ourself in the regime $n \gg 1$ and consider the random walk with constant fixed rate α between nearest-neighbour, with $\alpha \leq 1/6$ to satisfy our hypothesis (2.2).

Without wings and with $r_a = r_t = 1$ our wasp would be the three-dimensional "n-dog" of [17] and we would have a relaxation time and mixing time of order n^3. We will reprove this result by using our (κ, λ)-capacities, actually considering the same kind of flows as those used in [17] but,
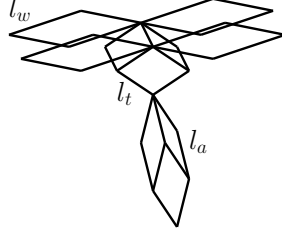
FIGURE 1. A wasp without a head, and maybe misplaced wings.

as we will see, with some more flexibility in building such flows. We will also show that, as one could expect, adding the wings will not change the spectral gap and mixing time asymptotics. This is, in particular, to illustrate how one can recursively apply Theorem (2.10): $\gamma_\mathcal{R}$ can be estimated by applying the theorem to the restricted dynamics itself. The last reason why we introduced this toy model that combines two and three-dimensional graphs is that it illustrates some limits of our result: while using the three-dimensional parts of our graph we will be able to estimate easily the mixing time of our random walk and prove asymptotic exponential laws for exit and transition times, the two-dimensional "wing pair restricted" random walk is associated with a too slowly decreasing $\varepsilon_\mathcal{R}^*$ to control more than the relaxation time: we are in the regime $\varepsilon_\mathcal{R}^* \ll 1$ but *outside* the regime $\phi_\mathcal{R}^* T_\mathcal{R}^* \ll 1$.

To fix some notation, let us call $\mathcal{R}_t$ the cubic lattice $\{0, \ldots, l_t\}^3$ and $\mathcal{R}_a$ this other cubic lattice from which one corner is removed to have a partition of the vertices $\mathcal{X}_b = \mathcal{R}_t \cup \mathcal{R}_a$ that describe the wasp body obtained after remotion of the wings. In the same way we call $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{R}_4$ the four square lattices from which one point has been removed to obtain a partition of $\mathcal{R} = \mathcal{R}_t \cup \bigcup_{i=1}^4 \mathcal{R}_i$ that is the front part of the wasp. We then have a partition of the whole graph $\mathcal{X} = \mathcal{R} \cup \mathcal{R}_a$

Let us start with the study of the $\mathcal{X}_b$-restricted random walk. We will write ϕ_t^*, γ_t and ε_t^* instead of $\phi_{\mathcal{R}_t}^*$, $\gamma_{\mathcal{R}_t}$ and $\varepsilon_{\mathcal{R}_t}^*$. From Lemma 2.2 we have, with obvious notation, $\phi_t^* \leq \phi_t = 3\alpha/(1 + l_t)^3$. As far as γ_t is concerned we can use the following lemma.

Lemma 7.5. *For $d \geq 1$, if γ_d is the spectral gap of the random walk on the d-dimensional lattice $\{0, \ldots, l\}^d$ with nearest-neighbour jump rate α, then*

$$\frac{1}{\gamma_d} \leq \frac{dl(l+1)}{2\alpha/e} \, . \tag{7.73}$$

In addition, if we call 0 the all 0 coordinate vertex and γ_d' the spectral gap of restricted random walk on $\{0, \ldots, l\}^d \setminus \{0\}$, then

$$\frac{1}{\gamma_d'} \leq \frac{d(l+1)^2}{2\alpha^{d+1}/e} \, . \tag{7.74}$$

Proof. The estimate (7.73) is obtained by the standard coordinate by coordinate coupling for the lazy version of the original random walk. The same coupling can be used for the random walk on the graph with one removed corner to bring each coordinate of two lazy random walks at distance 1 at most. Since one can then build another coupling making them meet in d steps at most with probability α^d at least, and start the coupling again from the beginning if they do not, the mean meeting time of these lazy random walks is bounded from above by

$$\frac{1}{\alpha^d}\left(\frac{dl(l+1)}{2\alpha} + d\right) \leq \frac{d(l+1)^2}{2\alpha^{d+1}} \, . \tag{7.75}$$

Markov's inequality makes then possible to bound the mixing time of the lazy walk, from which one deduces (7.74) for the original walk. $\qquad\square$

The first part of the lemma together with the previous estimate on ϕ_t^* gives then

$$\varepsilon_t^* \leq \frac{3l_t(l_t+1)}{2\alpha/e}\frac{3\alpha}{(l_t+1)^3} \leq \frac{9e}{1+l_t} \, . \tag{7.76}$$

To prove that l_t^3 is the correct order of the exit time from $\mathcal{R}_t$ we apply Theorem 2.9 to estimate ϕ_t^* from below and then just have to build a unitary flow from $\mathcal{R}_a$ to $\mathcal{R}_t$ to estimate from below a (κ, λ)-capacity with $\lambda = +\infty$. We send a flow of strength 1 to the junction corner (this will be modified when working with $\lambda < +\infty$) and have to absorb it in $\mathcal{R}_t$. Since we know the probabilistic meaning of the optimal flow and κ has, heuristically, to be small enough to be close to local equilibrium at absorption, we should absorb a fraction of order $1/(1+l_t)^3$ of this unitary flow in each vertex of $\mathcal{R}_t$. But we are not constrained to realize this exactly and this is where we have some flexibility that helps in computation. Since we also know from the electrical network picture that the optimal flow should in some sense be radially distributed, we build our flow as the mean of a random simple flow with some spherical symmetry. Let us first explain how two build a certain random path ξ. We begin by choosing a point Q with positive coordinates in the origin centered euclidean ball of radius $(1 + l_t)$ according to the normalized Lebesgue measure. This point belongs to some unitary cube with integer coordinates corners and we call Q' the corner with the smallest coordinates. We then approximate the radius $[0, Q]$ by a coordinate non-decreasing path that starts from 0, is only made of edges along the unitary cubes crossed by $[0, Q]$, and ends in Q'. Such a path is in particular a shortest path on the lattice that links 0 with Q' and the fact that is exists can be shown by shown by recurrence on the dimension and by using projections along coordinate axes. For such a random path ξ (since Q is random) we define a flow ψ_ξ by $\psi_\xi(x, y) = \mathbb{1}_{\{(x,y) \in \xi\}} - \mathbb{1}_{\{(y,x) \in \xi\}}$. We finally use as test flow in Thomson's principle the associated mean flow, that is the flow ψ such that, for x and y nearest neighbours with $\|y\|_2 > \|x\|_2$, $\psi(x, y) = \mathbb{P}((x, y) \in \xi)$. Since the distance between these approximating paths and their associated radius is smaller than $\sqrt{3}$, the chosen point has to be in cone of half angle α, with $\sin \alpha = \sqrt{3}/\|y\|_2$ for y to be used in the approximating path. It follows that

$$\psi(x, y) \leq \frac{1}{\frac{1}{8} \frac{4\pi(1+l_t)^3}{3}} \frac{2\pi}{3} (1 + l_t)^3 (1 - \cos \alpha) = 4 \left(1 - \sqrt{1 - \frac{3}{\|y\|_2^2}} \right) \leq \frac{12}{\|y\|_\infty^2} \,. \qquad (7.77)$$

Also, for all $x \in \mathcal{R}_t$, we have $\operatorname{div}_x \Psi = \mathbb{P}(Q' = x)$ and

$$\mathbb{E}_{\mu_{R_t}} \left[\left(\frac{\operatorname{div}\Psi}{\mu_{\mathcal{R}_t}} \right)^2 \right] = \sum_{x \in \mathcal{R}_t} \frac{1}{(1+l_t)^3} (1 + l_t)^6 \mathbb{P}^2 (Q' = x)$$

$$= \sum_{x \in \mathcal{R}_t} (1 + l_t)^3 \frac{\operatorname{Vol}(C(x) \cap B_{1/8})^2}{\left(\frac{1}{8} \frac{4}{3} \pi (1 + l_t)^3 \right)^2} \qquad (7.78)$$

where $C(x)$ is the unitary cube with x as smallest coordinate coordinate corner, $B_{1/8}$ is the positive coordinate part of the ball of radius $(1 + l_t)$ and Vol stands for the Lebesgue measure. It follows that

$$\mathbb{E}_{\mu_{R_t}} \left[\left(\frac{\operatorname{div}\Psi}{\mu_{\mathcal{R}_t}} \right)^2 \right] \leq \sum_{x \in \mathcal{R}_t} \frac{1}{(1+l_t)^3} \frac{\operatorname{Vol}(C(x) \cap B_{1/8})}{\left(\frac{1}{8} \frac{4}{3} \pi \right)^2} = \frac{6}{\pi} \qquad (7.79)$$

Thomson's principle gives then, with μ_b the uniform measure on $\mathcal{X}_b$, $\kappa > 0$ and $C_{\kappa,b}(R_t, R_a)$ the κ-capacity between R_t and R_a that is computed relatively to the restricted random walk in $\mathcal{X}_b$,

$$\frac{\mu_b(\mathcal{R}_t)}{C_{\kappa,b}(\mathcal{R}_t, \mathcal{R}_a)} \leq \frac{(1+l_t)^3}{\alpha} + \sum_{k=0}^{l_t-1} 3 \frac{(1+l_t)^3}{\alpha} \left[3(1+k)^2 - 3(1+k) + 1 \right] \frac{144}{(1+k)^4} + \frac{1}{\kappa} \frac{1}{8} \frac{6}{\pi}$$

$$\leq \frac{(1+l_t)^3}{\alpha} \left(1 + 432 \sum_{k \geq 1} \frac{3}{k^2} \right) + \frac{6}{\kappa\pi} \qquad (7.80)$$

$$\leq 2161 \frac{(1+l_t)^3}{\alpha} + \frac{6}{\kappa\pi} \,.$$

Choosing $1/\kappa \ll n^3$ this shows that $l_t^3 = \lfloor r_t n \rfloor^3$ is the correct order for the exit time.

To estimate transition and relaxation time with the same tools, we have to estimate (κ, λ)-capacities with finite λ. Using not only randomly chosen sinks but randomly chosen sources also,

we can define a mean flow as previously to prove, with obvious notation,

$$
\begin{aligned}
\frac{\mu_b\left(\mathcal{R}_t\right)\mu_b\left(\mathcal{R}_a\right)}{C_{\kappa,b}^\lambda\left(\mathcal{R}_t,\mathcal{R}_a\right)} \leq{} & \mu_b\left(\mathcal{R}_a\right)\sum_{k=0}^{l_t-1}3\frac{(1+l_t)^3}{\alpha}\left[3(1+k)^2-3(1+k)+1\right]\frac{144}{(1+k)^4} \\
& +\mu_b\left(\mathcal{R}_t\right)\sum_{k=0}^{l_a-1}3\frac{(1+l_a)^3}{\alpha}\left[3(1+k)^2-3(1+k)+1\right]\frac{144}{(1+k)^4} \\
& +\mu_b\left(\mathcal{R}_a\right)\frac{6}{\kappa\pi}+\mu_b\left(\mathcal{R}_t\right)\frac{6}{\lambda\pi} \\
\leq{} & 2160\left(\mu_b\left(\mathcal{R}_a\right)\frac{(1+l_t)^3}{\alpha}+\mu_b\left(\mathcal{R}_t\right)\frac{(1+l_a)^3}{\alpha}\right)+\frac{6}{\kappa\pi}+\frac{6}{\lambda\pi}\,,
\end{aligned}
\tag{7.81}
$$

to get, by choosing also $1/\lambda\ll n^3$,

$$
\frac{\mu_b\left(\mathcal{R}_t\right)\mu_b\left(\mathcal{R}_a\right)}{C_{\kappa,b}^\lambda\left(\mathcal{R}_t,\mathcal{R}_a\right)}\leq 2160\frac{2r_t^3 r_a^3}{r_t^3+r_a^3}\frac{n^3}{\alpha}+o(n^3)\,.
\tag{7.82}
$$

By Theorem 2.10, using (7.74) and choosing $\kappa,\lambda\ll 1/n^2$, this gives an upper bound on the relaxation time with the same asymptotics. Theorem 2.18 also provides a similar upper bound on the mean transition time. Going to lower bounds on $1/\gamma_b=1/\gamma_{\mathcal{X}_b}$ and $1/\phi_{t,\lambda}^*$ one could estimate (κ,λ)-capacities through Dirichlet principle, but it is better to recall that $\mu_b(\mathcal{R}_t)/C_{\kappa,b}^\lambda(\mathcal{R}_t,\mathcal{R}_a)\geq (1-\epsilon)/\phi_{t,\lambda}^*$ for any ϵ and large enough n and $1/\phi_{t,\lambda}^*\geq 1/\phi_t^*\geq(1+l_t)^3/(3\alpha)$.

As far as exponential asymptotic laws and mixing time asymptotics are concerned, our results depend on our ability, with obvious notation, to control ζ_t^* and show that $\varepsilon_t^*\ln(1/\zeta_t^*)$ goes to zero and ensure $\phi_t^* T_t^*\ll 1$. This cannot be achieved by using Lemma 2.5, since $\varepsilon_t^*/\gamma_t=\phi_t^*/\gamma_t^2\gg 1$ and $\varepsilon_t^* D_t$ is of order 1. (Estimates provided by (2.23) and (2.22) would actually be enough in dimension four and five respectively.) We are, however, in the special case where (2.26) holds and proves, since ϕ_t^* and ϕ_t are of the same order, that $\varepsilon_t^*\ln(1/\zeta_t^*)\ll 1$. This proves local thermalization on time scale $n^2\ln n$ and exponential asymptotic laws immediately follow. This also proves that the mixing time goes like n^3 as soon as $r_t\neq r_a$.

We prove now that these asymptotics on relaxation, transition, exit and mixing times are still valid on the full wasp graph, wings included. To do so we note that our previous flow used to estimate (κ,λ)-capacities between $\mathcal{R}_t$ and $\mathcal{R}_a$ in $\mathcal{X}_b$ can still be used to estimate (κ,λ)-capacities between $\mathcal{R}$ (wings included) and $\mathcal{R}_a$ in the full space $\mathcal{X}$. The key point now is to control $\gamma_{\mathcal{R}}$. If our wasp had only one wing $\mathcal{R}_1$, we could have use Theorem 2.10 directly on $\mathcal{R}=\mathcal{R}_1\cup(\mathcal{R}\setminus\mathcal{R}_1)$. We will use instead Lemma 2.11 and, anyway, will have to estimate (κ,λ)-capacities between $\mathcal{R}_1$ and $\mathcal{R}_t$ and compare it with $\gamma_1=\gamma_{\mathcal{R}_1}$ and $\gamma_t=\gamma_{\mathcal{R}_t}$.

Let us start by estimating $\phi_1^*=\phi_{\mathcal{R}_1}^*$. In this two-dimensional case the easy bound $\phi_1^*\leq\phi_1$ is not a good one. We then use the variational principle satisfied by ϕ_1^* (see Lemma 2.2) with the same kind of test function we would have use to estimate $C_\kappa(\mathcal{R}_1,\mathcal{R}_t)$. With $V(x)=(\ln(\|x\|_\infty))/(1+\ln l_1)$ for $x\in\mathcal{R}_1$, we have, with obvious notation,

$$
\begin{aligned}
\mathcal{D}_1(V) &=\sum_{k=0}^{l_1-1}2(k+1)\frac{\alpha}{(1+l_1)^2}\left[\frac{\ln(k+2)-\ln(k+1)}{1+\ln l_1}\right]^2 \\
&\leq\frac{2\alpha}{(1+\ln l_1)^2(1+l_1)^2}\sum_{k=0}^{l_1-1}\frac{1}{k+1} \\
&\leq\frac{2\alpha}{(1+\ln l_1)^2(1+l_1)^2}(1+\ln l_1)=\frac{2\alpha}{(1+\ln l_1)(1+l_1)^2}
\end{aligned}
\tag{7.83}
$$

We also have

$$
\begin{aligned}
\mu_1(V^2) &=\sum_{k=0}^{l_1}(2k+1)\frac{1}{(1+l_1)^2}\frac{\ln^2(1+k)}{(1+\ln l_1)^2} \\
&\geq\frac{1}{(1+l_1)^2(1+\ln l_1)^2}\int_1^{1+l_1}(2x-1)\ln^2 x\,dx \\
&\geq\frac{1}{3}\qquad\text{for }l_1\geq 20.
\end{aligned}
\tag{7.84}
$$

Since $V_{|\partial_- \mathcal{R}_t} \equiv 0$ it follows that $\phi_1^* \leq 6\alpha/((1+l_1)^2(1+\ln l_1))$ for $l_1 \geq 20$, and ε_1^* decreases at least like $1/\ln l_1$.

To see that we have found the right order for ϕ_1^* we estimate the κ-capacity between $\mathcal{R}_1$ and $\mathcal{R}_t$ by using the same kind of flow as previously. For nearest neighbours x and y with $\|y\|_2 > \|x\|_2$ such a flow ψ satisfies, with $\sin\alpha \leq \sqrt{2}/\|y\|_2$ and $\alpha \leq \pi/4$, so that $\sin\alpha \geq \alpha/\sqrt{2}$

$$\psi(x,y) \leq \frac{1}{\frac{1}{4}\pi l_1^2}\alpha l_1^2 \leq \frac{4}{\pi}\sqrt{2}\frac{\sqrt{2}}{\|y\|_2} \leq \frac{8}{\pi\|y\|_\infty} \,. \tag{7.85}$$

Thomson principle then gives

$$\begin{aligned}
\frac{\mu_\mathcal{R}(\mathcal{R}_1)}{C_\kappa(\mathcal{R}_1,\mathcal{R}_t)} &\leq \frac{(1+l_1)^2}{\alpha} + \sum_{k=0}^{l_1-1} 2(2k+1)\frac{(1+l_1)^2}{\alpha}\left(\frac{8}{\pi(k+1)}\right)^2 + \frac{1}{\kappa}\frac{1}{\frac{1}{4}\pi} \\
&\leq \frac{(1+l_1)^2}{\alpha}\left(1 + \frac{256}{\pi^2}\sum_{k=0}^{l_1-1}\frac{1}{k+1}\right) + \frac{4}{\kappa\pi} \\
&\leq \frac{(1+l_1)^2}{\alpha}\left(1 + 26(1+\ln l_1)\right) + \frac{4}{\kappa\pi} \,,
\end{aligned} \tag{7.86}$$

which proves, choosing $1/\kappa \ll n^2\ln n$ that $1/\phi_1^*$ is of order $n^2\ln n$.

Combining these two- and three-dimensional flows we get, denoting by $C_{\kappa,\mathcal{R}}^\lambda(\cdot,\cdot)$ the (κ,λ)-capacity that is computed relatively to the random walk restricted in $\mathcal{R}$.

$$\begin{aligned}
\frac{\mu_\mathcal{R}(\mathcal{R}_1)\mu_\mathcal{R}(\mathcal{R}_t)}{C_{\kappa,\mathcal{R}}^\lambda(\mathcal{R}_1,\mathcal{R}_t)} &\leq 26\mu_\mathcal{R}(\mathcal{R}_t)\frac{(1+l_1)^2(1+\ln l_1)}{\alpha} + 2160\mu_\mathcal{R}(\mathcal{R}_1)\frac{(1+l_t)^3}{\alpha} \\
&\quad + \mu_\mathcal{R}(\mathcal{R}_t)\frac{4}{\kappa\pi} + \mu_\mathcal{R}(\mathcal{R}_1)\frac{\pi}{6\lambda} \,.
\end{aligned} \tag{7.87}$$

With $1/\kappa \ll n^2\ln n$ as previously and $1/\lambda \ll n^3$, since $\mu_\mathcal{R}(R_t)$ is of order 1 and $\mu_\mathcal{R}(R_1)$ is of order $1/n$, this leads to

$$\frac{\mu_\mathcal{R}(\mathcal{R}_1)\mu_\mathcal{R}(\mathcal{R}_t)}{C_{\kappa,\mathcal{R}}^\lambda(\mathcal{R}_1,\mathcal{R}_t)} \leq 26c_1^2\frac{n^2\ln n}{\alpha} + O(n^2) \tag{7.88}$$

and, choosing also $1/\kappa, 1/\lambda \gg n^2$, one has in the same way, using the previous test function V in Dirichlet principle,

$$\frac{\mu_\mathcal{R}(\mathcal{R}_1)\mu_\mathcal{R}(\mathcal{R}_t)}{C_{\kappa,\mathcal{R}}^\lambda(\mathcal{R}_1,\mathcal{R}_t)} \geq \frac{1}{2}c_1^2\frac{n^2\ln n}{\alpha} + O(n^2) \,. \tag{7.89}$$

From Lemma 2.11, (7.73) and (7.74) it follows that $1/\gamma_\mathcal{R} = o(n^3)$ and the results obtained for the wasp without wings holds with the wings also.

We note however that when applying the previous two-dimensional computation to the random walk restricted to a pair of wings, we obtain similarly a good spectral gap control but we are not able to show the asymptotic exponential law or derive mixing time estimates, since, in this case $T_{\mathcal{R}_1}^*$ and $1/\phi_1^*$ are of the same order.

APPENDIX A. ESTIMATING $\zeta_\mathcal{R}^*$

A.1. **Crude and very crude estimates.** We prove here lemma 2.5.

Proof of i). One has, for all x in $\mathcal{R}$ and $t > 0$,

$$\mu_\mathcal{R}^*(x) = \mathbb{P}_{\mu_\mathcal{R}^*}\left(X(t) = x \,\big|\, \tau_{\mathcal{X}\setminus\mathcal{R}} > t\right) \geq \mathbb{P}_{\mu_\mathcal{R}^*}\left(X(t) = x, \tau_{\mathcal{X}\setminus\mathcal{R}} > t\right) \,. \tag{A.1}$$

By the natural coupling between X and $X_\mathcal{R}$ up to time $\tau_{\mathcal{X}\setminus\mathcal{R}}$ and stochastic domination of $\tau_{\mathcal{X}\setminus\mathcal{R}}$ by an exponential random variable with parameter $\alpha_\mathcal{R}$ that is independent from X_R, it follows

$$\mu_\mathcal{R}^*(x) \geq \mathbb{P}_{\mu_\mathcal{R}^*}\left(X_\mathcal{R}(t) = x\right)e^{-\alpha_\mathcal{R} t} \,. \tag{A.2}$$

By Cauchy-Schwarz inequality, Proposition 2.1, and the standard trick to control $\ell_\infty(\mu_\mathcal{R})$ norms with $\ell_2(\mu_\mathcal{R})$ norms (the same we used in the proof of Theorem 2.4) we get

$$\mu_\mathcal{R}^*(x) \geq \left(1 - e^{-\gamma_\mathcal{R} t}\sqrt{\frac{\varepsilon_\mathcal{R}^*}{(1-\varepsilon_\mathcal{R}^*)\mu_\mathcal{R}(x)}}\right)e^{-\alpha_\mathcal{R} t}\mu_\mathcal{R}(x) \,. \tag{A.3}$$

To make this bound useful, we notice that the term inside the bracket is larger than or equal to $1/2$ if

$$t \geq t_0 := \frac{1}{2\gamma_{\mathcal{R}}} \ln\left(\frac{4\varepsilon_{\mathcal{R}}^*}{(1-\varepsilon_{\mathcal{R}}^*)\mu_{\mathcal{R}}(x)}\right)$$

If $t_0 > 0$ for all $x \in \mathcal{R}$, that is if $\frac{4\varepsilon_{\mathcal{R}}^*}{(1-\varepsilon_{\mathcal{R}}^*)\zeta_{\mathcal{R}}} > 1$, then we can plug in its value in (A.3) and get, by definition of $\zeta_{\mathcal{R}}^*$,

$$\zeta_{\mathcal{R}}^* \geq \min_{x \in \mathcal{R}} \frac{\mu_{\mathcal{R}}(x)}{4}\left(\sqrt{\frac{4\varepsilon_{\mathcal{R}}^*}{(1-\varepsilon_{\mathcal{R}}^*)\mu_{\mathcal{R}}(x)}}\right)^{-\frac{2\alpha_{\mathcal{R}}}{\gamma_{\mathcal{R}}}} \geq \frac{\zeta_{\mathcal{R}}}{4}\left(\sqrt{\frac{4\varepsilon_{\mathcal{R}}^*}{(1-\varepsilon_{\mathcal{R}}^*)\zeta_{\mathcal{R}}}}\right)^{-\frac{2\alpha_{\mathcal{R}}}{\gamma_{\mathcal{R}}}}. \tag{A.4}$$

On the other hand, if $\frac{4\varepsilon_{\mathcal{R}}^*}{(1-\varepsilon_{\mathcal{R}}^*)\zeta_{\mathcal{R}}} \leq 1$, we can just take the value $t = 0$ in (A.3) and get

$$\zeta_{\mathcal{R}}^* \geq \min_{x \in \mathcal{R}}\left(1 - \sqrt{\frac{\varepsilon_{\mathcal{R}}^*}{(1-\varepsilon_{\mathcal{R}}^*)\mu_{\mathcal{R}}(x)}}\right)^2 \mu_{\mathcal{R}}(x) \geq \frac{\zeta_{\mathcal{R}}}{4}. \tag{A.5}$$

Taking the logarithm of $1/\zeta_{\mathcal{R}}^*$ and putting things together, we obtain the stated inequality.

Proof of ii). The first inequality is obvious from the definition of $\zeta_{\mathcal{R}}^*$. Let $\hat{X}$ denote the discrete version of X like in Section 3.2 and let $N(t)$ be the number of rings up to time t. Then, for $z \in \mathcal{R}$, we have

$$\begin{aligned}
\mu_{\mathcal{R}}^*(z) &= \lim_{t\to\infty} \mathbb{P}_x(X(t) = z \,|\, \tau_{\mathcal{X}\backslash\mathcal{R}} > t) \\
&= \lim_{t\to\infty} \sum_{k\geq 0} \mathbb{P}_x(\hat{X}(k) = z \,|\, \hat{\tau}_{\mathcal{X}\backslash\mathcal{R}} > k)\mathbb{P}(N(t) = k) \\
&\geq \lim_{t\to\infty} \sum_{k\geq 0} \mathbb{P}_x(\hat{X}(k+D_{\mathcal{R}}) = z \,|\, \hat{\tau}_{\mathcal{X}\backslash\mathcal{R}} > k+D_{\mathcal{R}})\mathbb{P}(N(t) = k+D_{\mathcal{R}}) \\
&= \lim_{t\to\infty} \sum_{k\geq 0}\sum_{y\in\mathcal{R}} \mathbb{P}_x(\hat{X}(k) = y \,|\, \hat{\tau}_{\mathcal{X}\backslash\mathcal{R}} > k)\mathbb{P}_y(\hat{X}(D_{\mathcal{R}}) = z \,|\, \hat{\tau}_{\mathcal{X}\backslash\mathcal{R}} > D_{\mathcal{R}}) \\
&\qquad\qquad \times \mathbb{P}(N(t) = k+D_{\mathcal{R}}),
\end{aligned} \tag{A.6}$$

where we used the notation $\hat{\tau}_{\mathcal{X}\backslash\mathcal{R}}$ for the hitting time of the chain $\hat{X}$ on $\mathcal{X}\backslash\mathcal{R}$. Since for all $y \in \mathcal{R}$ we have

$$\mathbb{P}_y(\hat{X}(D_{\mathcal{R}}) = z \,|\, \hat{\tau}_{\mathcal{X}\backslash\mathcal{R}} > D_{\mathcal{R}}) \geq \mathbb{P}_y(\hat{X}(D_{\mathcal{R}}) = z\,,\, \hat{\tau}_{\mathcal{X}\backslash\mathcal{R}} > D_{\mathcal{R}}) \geq e^{-\Delta_{\mathcal{R}} D_{\mathcal{R}}},$$

we get

$$\begin{aligned}
\mu_{\mathcal{R}}^*(z) &\geq e^{-\Delta_{\mathcal{R}} D_{\mathcal{R}}} \lim_{t\to\infty} \sum_{k\geq 0}\sum_{y\in\mathcal{R}} \mathbb{P}_x(\hat{X}(k) = y \,|\, \hat{\tau}_{\mathcal{X}\backslash\mathcal{R}} > k)\mathbb{P}(N(t) = k+D_{\mathcal{R}}) \\
&= e^{-\Delta_{\mathcal{R}} D_{\mathcal{R}}} \lim_{t\to\infty} \mathbb{P}(N(t) \geq D_{\mathcal{R}}) = e^{-\Delta_{\mathcal{R}} D_{\mathcal{R}}}.
\end{aligned} \tag{A.7}$$

A.2. **Superharmonicity of $h_{\mathcal{R}}^*$.** To prove that $h_{\mathcal{R}}^*$ is a super-harmonic function, notice that, for all $x \in \mathcal{R}$,

$$\begin{aligned}
(\mathcal{L}h_{\mathcal{R}}^*)(x) &= -h_{\mathcal{R}}^*(x) + \sum_{y\in\mathcal{X}} p(x,y)h_{\mathcal{R}}^*(y) \\
&= -h_{\mathcal{R}}^*(x) + \sum_{y\in\mathcal{R}} p(x,y)\frac{\mu_{\mathcal{R}}^*(y)}{\mu_{\mathcal{R}}(y)} \\
&= -h_{\mathcal{R}}^*(x) + \sum_{y\in\mathcal{R}} p_{\mathcal{R}}^*(x,y)\frac{\mu_{\mathcal{R}}^*(y)}{\mu_{\mathcal{R}}(y)} \\
&= -h_{\mathcal{R}}^*(x) + \sum_{y\in\mathcal{R}} p_{\mathcal{R}}^*(y,x)\frac{\mu_{\mathcal{R}}^*(y)}{\mu_{\mathcal{R}}(x)} \\
&= -\phi_{\mathcal{R}}^* h_{\mathcal{R}}^*(x) \leq 0,
\end{aligned} \tag{A.8}$$

where in the last two lines we used the reversibility of $p_{\mathcal{R}}^*$ w.r.t. $\mu_{\mathcal{R}}$ and that $\mu_{\mathcal{R}}^* p_{\mathcal{R}}^* = (1-\phi_{\mathcal{R}}^*)\mu_{\mathcal{R}}^*$.

Appendix B. Computation of relevant quantities in the Curie-Weiss model

Here we provide some accurate estimates over relevant quantities in the characterization of the metastable behavior for the Curie-Weiss model.

B.1. **Measure of the metastable set.** Here we prove formula (7.53) which provides the asymptotic expression of $\mu(\mathcal{R})$. By definition on μ and $\mathcal{R}$ and using (7.7), we have

$$
\begin{aligned}
Z_N \cdot \bar{\mu}(\mathcal{R}) &= \sum_{k=(-1-m_-)\frac{N}{2}}^{(m_0-m_-)\frac{N}{2}} e^{-\beta N f_N(m_-+\frac{2k}{N})} \\
&= \sqrt{\frac{2}{N}} \frac{1}{\sqrt{\pi(1-m_-^2)}} \sum_{k=(-1-m_-)\frac{N}{2}}^{(m_0-m_-)\frac{N}{2}} e^{-\beta N f(m_-+\frac{2k}{N})}(1+o(1)) \\
&= \sqrt{\frac{2}{N}} \frac{1}{\sqrt{\pi(1-m_-^2)}} e^{-\beta N f(m_-)} \sum_{k=-\lceil N^{\frac{2}{3}}\rceil}^{\lfloor N^{\frac{2}{3}}\rfloor} e^{-\frac{\beta N f''(m_-)}{2}(\frac{2k}{N})^2}(1+o(1))
\end{aligned}
\tag{B.1}
$$

where in the last step we use Taylor approximation and observe that

$$
\sum_{|k|\geq\lceil N^{\frac{2}{3}}\rceil} e^{-\frac{\beta N f''(m_-)}{2}(\frac{2k}{N})^2} \leq N e^{-c(\beta)N^{\frac{1}{3}}}.
$$

Approximating the sum in (B.1) with an integral, we finally get

$$
\begin{aligned}
Z_N \cdot \bar{\mu}(\mathcal{R}) &= \frac{1}{\sqrt{\pi(1-m_-^2)}} e^{-\beta N f(m_-)} \int_{\mathbb{R}} e^{-\beta f''(m_-)x^2}dx(1+o(1)) \\
&= \frac{1}{\sqrt{\beta f''(m_-)(1-m_-^2)}} e^{-\beta N f(m_-)}(1+o(1)).
\end{aligned}
\tag{B.2}
$$

B.2. **Capacities between points in the macroscopic scale.** Here we provide an asymptotic expression for the capacity between points in the one-dimensional dynamics with transition rates (7.10), that is the dynamics induced on $[-1,1]_N$ by the Curie-Weiss heath-bath dynamics.

As recalled in Section 7.1, for points $x < y \in [-1,1]_N$ it holds

$$
C(x,y)^{-1} = \sum_{k=0}^{(y-x)\frac{N}{2}-1} \left(\bar{c}\left(x+\frac{2k}{N}, x+\frac{2(k+1)}{N}\right)\right)^{-1},
$$

with $\bar{c}(x,y) = \bar{\mu}(x)\bar{p}(x,y)$. In the following, we will first provide an asymptotic approximation for $C(x,y)^{-1}$ when $m_0 \notin [x,y]$, and then compute the asymptotic formulas of $C(m_-,m_0)^{-1}$, $C(m_0,m_+)^{-1}$ and $C(m_-,m_0)^{-1}$.

If $m_0 \notin [x,y]$, we can assume w.l.o.g. that $f(x) > f(z)$ for all $z \in (x,y)$. Bounding below the rates $\bar{p}$ with a positive constant $c(\beta)$ and from (7.7), we get

$$
\begin{aligned}
C(x,y)^{-1} &\leq c(\beta)\sqrt{N}Z_N \sum_{k=0}^{(y-x)\frac{N}{2}-1} e^{\beta N f(x+\frac{2k}{N})} \\
&\leq c(\beta)\sqrt{N}Z_N e^{\beta N f(x)} \sum_{k=0}^{(y-x)\frac{N}{2}-1} e^{-\beta|f'(\xi_k)|2k} \\
&\leq c(\beta)\sqrt{N}Z_N e^{\beta N f(x)}
\end{aligned}
\tag{B.3}
$$

where in the second line we used $f(x+\frac{2k}{N}) - f(x) = -|f'(\xi_k)|\frac{2k}{N}$ for some $\xi_k \in (x, x+\frac{2k}{N})$ and that there exists a constant $c > 0$ such that $|f(\xi_k)| > c$ uniformly in k. If instead $f(y) > f(z)$ for all $z \in (x,y)$, then is enough to switch x and y and run the argument above, as $C(x,y) = C(y,x)$. Altogether, this provides inequality (7.28).

To compute $C(m_-, m_0)$, we first notice that, since m_0 is a critical point of f, we can write

$$\tanh(\beta \Delta_\pm(x + \tfrac{2k}{N})) = \pm \tanh(\beta(m_0 + h))(1 + o(1)) = \pm m_0(1 + o(1))$$

and then get the approximation $\bar{p}(x \pm \tfrac{2k}{N}, x \pm \tfrac{2(k+1)}{N}) = \frac{(1 - m_0^2)}{4}(1 + o(1))$. Proceeding as for the computation of $\bar{\mu}(\mathcal{R})$, we have

$$
\begin{aligned}
C(m_-, m_0)^{-1} &= \frac{4}{(1 - m_0^2)} Z_N \sum_{k=0}^{(m_0 - m_-)\frac{N}{2} - 1} e^{\beta N f_N(m_0 - \frac{2k}{N})}(1 + o(1)) \\
&= 2\sqrt{\frac{2N\pi}{(1 - m_0^2)}} Z_N e^{\beta N f(m_0)} \sum_{k=0}^{\lfloor N^{\frac{2}{3}} \rfloor} e^{-\frac{\beta N |f''(m_0)|}{2}(\frac{2k}{N})^2}(1 + o(1)) \qquad \text{(B.4)} \\
&= 2N \sqrt{\frac{\pi}{(1 - m_0^2)}} Z_N e^{\beta N f(m_0)} \int_0^{+\infty} e^{-\beta |f''(m_0)| x^2} dx \, (1 + o(1)) \\
&= \frac{\pi N}{\sqrt{(1 - m_0^2)\beta |f''(m_0)|}} Z_N e^{\beta N f(m_0)}(1 + o(1)).
\end{aligned}
$$

This provides formula (7.52).

Similarly we can compute $C(m_0, m_+)^{-1}$ and $C(m_-, m_+)^{-1}$. In the first case we let the sum over k of (B.4) run from $(m_0 - m_+)\frac{N}{2}$ to 0, and then get the same result as for $C(m_-, m_0)^{-1}$. When computing $C(m_-, m_+)^{-1}$, we let the sum over k run from $(m_0 - m_+)\frac{N}{2}$ to $(m_0 - m_-)\frac{N}{2} - 1$. We then approximate the sum by an integral over all $\mathbb{R}$ that finally produces an extra factor of 2 with respect to (B.4). This yields formula (7.66).

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