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The Kepler Problem on Complex and Pseudo-Riemannian Manifolds

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Abstract

The motion of objects in the sky has captured the attention of scientists and mathematicians since classical times. The problem of determining their motion has been dubbed the Kepler problem, and has since been generalized into an abstract problem of dynamical systems. In particular, the question of whether a classical system produces closed and bounded orbits is of importance even to modern mathematical physics, since these systems can often be analysed by hand. The aforementioned question was originally studied by Bertrand in the context of celestial mechanics, and is therefore referred to as the Bertrand problem. We investigate the qualitative behaviour of solutions to the generalized Kepler problem, in which particles travel in an abstract space called a manifold. We find that although Bertrand's results do not generalize to even the most simple non-trivial example of a complex manifold, we can partially reduce the problem through the use of a function called a momentum map. We then study the generalized Kepler problem on pseudo-Riemannian surfaces of revolution. In this case we are able to demonstrate that a generalization of Bertrand's theorem holds, and we compute explicit expressions for the shape and period of the orbits. Furthermore, we compute a generalization of the Laplace-Runge-Lenz vector, which allows one to determine all solutions to the equations of motion in terms of three constant quantities.

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1 Introduction

The Bertrand problem is the century and a half old question: 'which classical mechanical systems have the property that all of their bounded orbits are closed?' [1] The Bertrand property is related to the integrability of a Hamiltonian system. In particular, a Hamiltonian system is Bertrand if it is maximally superintegrable, meaning it admits a maximal number of conserved quantities. It is known that the only Bertrand systems whose configuration space is $\mathbb{R}^3$ are the Kepler system, which classically describes the motion of a particle in a $1/r$ potential, and the simple harmonic oscillator. Both of these systems arise in some way from the Green's function of the Laplace-Beltrami operator of the configuration space. In general, the question of whether a system arising from such a Green's function is Bertrand is open for many manifolds. The problem has been entirely worked out in the case where the configuration space is a Riemannian three-manifold [4, 5, 2, 3]. It is also interesting to study superintegrable systems on surfaces [10, 8, 9, 6, 7]. We study the integrability of systems on a complex model space, the projective space $\mathbb{CP}^2$. The study of dynamics on complex projective space has been studied [12, 11] and is relevant to quantum superintegrable systems [13]. Later, we work out an explicit solution of the Kepler problem on pseudo-Riemannian surfaces of revolution [14], which have not been extensively studied in the context of the Bertrand problem [15].

1.1 Conventions

A variety of conventions are used in this thesis which may not be universally recognized. This section aims to clarify some of the choices in notation found herein. First of all, this thesis will use the Einstein summation convention.

Remark 1.1 (Einstein Convention)**.** Consider a mathematical expression containing two or more indexed quantities to be summed over, such as for example, $W_{bd} = \sum_{c=1}^{M} \sum_{a=1}^{N} X^c_{abd} Y^{ac}$. We will ignore the summation symbol whenever an index appears in both the superscripts and subscripts appearing in the expression. So the previous expression would instead be written $W_{bd} = \sum_{c=1}^{M} X^c_{abd} Y^{ac}$, where the index a is assumed to be summed over.

The following is a table of commonly used symbols and their names/descriptions.

Symbol	Description
M	Manifold of dimension n
$C^\infty(M)$	Space of smooth functions from M to $\mathbb{R}$
$L^2(M)$	Space of L^2-functions on M
TM	Tangent bundle of M
T^*M	Cotangent bundle of M

$\mathcal{T}^n_k(M)$	Tensor fields of valence (n,k) over TM
$\Lambda^k E = \mathrm{Alt}(\bigotimes_{i=1}^k E)$	Exterior power of a vector bundle E
$\Lambda^\bullet E = \bigoplus_{k=1}^n \Lambda^k E$	Exterior algebra of a vector bundle E
$\mathfrak{X}(M) = \Gamma(TM)$	$C^\infty(M)$-Module of vector fields on M
$\Omega^k(M) = \Gamma(\Lambda^k T^*M)$	$C^\infty(M)$-Module of differential k-forms on M
$\Omega^k_{\mathrm{cl}}(M)$	Closed differential k-forms on M
$\Omega^\bullet(M) = \bigoplus_{k=1}^n \Omega^k(M)$	Algebra of smooth differential forms on M
$\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$	Exterior derivative
$\wedge : \Omega^k(M) \times \Omega^\ell(M) \to \Omega^{k+\ell}(M)$	Exterior product
$\lrcorner : \mathfrak{X}(M) \times \Omega^k(M) \to \Omega^{k-1}(M)$	Interior product
$g \in \Gamma(T^*M \otimes T^*M)$	Riemannian metric on TM
$h \in \Gamma(T^{\mathbb{C}*}M \otimes \overline{T}^{\mathbb{C}*}M)$	Hermitian metric on $T^{\mathbb{C}}M$
$\sharp : \Omega^1(M) \to \mathfrak{X}(M)$	Sharp map induced by g (index raising)
$\flat : \mathfrak{X}(M) \to \Omega^1(M)$	Flat map induced by g (index lowering)
$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$	Levi-Civita connection of g
$\star : \Omega^k(M) \to \Omega^{n-k}(M)$	Hodge dual on M
$\mathrm{grad} f = \mathrm{d}f^\sharp$	Definition of the gradient

$\mathrm{div} X = \mathrm{tr}\,\nabla X$	Definition of the divergence
$\omega \in \Omega^2_{\mathrm{cl}}(M)$	Symplectic form

Table 1: List of commonly used mathematical symbols and expressions.

Here the Hodge star operator for a Riemannian metric g on an orientable manifold is the unique linear map $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ satisfying $\langle \alpha, \beta \rangle \mathrm{d}V_g = \star\alpha \wedge \beta$, where $\mathrm{d}V_g$ is the volume form associated to g.

1.2 Hamiltonian Mechanics of Point Particles

Mechanics is among the oldest disciplines of physics. Out of the many concepts introduced by mathematical physicists, the phase space first introduced to classical mechanics has become increasingly relevant to modern physics. While Hamiltonian mechanics originally described idealized point particles interacting in Euclidean space, the advent of symplectic and contact geometry in describing mechanical systems has allowed for the setting to vary drastically, permitting researchers to describe complex phenomena in quantum mechanics, statistical mechanics, and other applications. Additionally, the theory of classical fields allows one to extend Hamiltonian mechanics to an infinite-dimensional phase space, which can describe the motion of extended objects such as strings, deformable solids and fluids, as well as gravitational and electromagnetic fields. Much of the following section can be found in standard books on geometric mechanics and symplectic geometry such as those of Cannas da Silva and Arnold [16, 17].

A common approach to deriving Hamilton's equations is to begin with the Lagrangian formulation of mechanics. This is done using the calculus of variations, which is a way of defining the notion of a critical point for a functional. This framework can be used to reinterpret many important problems as optimization problems, including the problem of finding the extremal paths of particles on some manifold. A fully rigorous calculus of variations for functions on a Riemannian manifolds is defined by the variational bicomplex [18], which defines variations in terms of infinite jets. Calculus of variations can also be used to study the symmetries of a dynamical system, since there may be infinitely many critical points of a given functional related by some transformation [19].

We will focus here on the Hamiltonian mechanics of point particles interacting on a Riemannian manifold M. A point particle consists of two components: a positive real number m called the mass, and a curve $\gamma : I \to M$ called the trajectory. The trajectory describes the motion of the particle. For a system of N distinguishable particles with trajectories $\{\gamma_1, \ldots, \gamma_n\}$, the space of all possible configurations of the system is the manifold M^N. We wish to find a functional of these trajectories whose critical points model the trajectories of particles in the real world. According to experimental evidence, a suitable candidate functional appears to be the action functional, defined below.

Definition 1.1 (Action)**.** Consider a system of N ideal point particles, travelling along trajectories $\{x_1(t), \ldots, x_N(t) : t \in I \subseteq \mathbb{R}\}$ in a Riemannian manifold M of

dimension n with metric g. The action functional $S : C^\infty(I \times TM)^N \to \mathbb{R}$ of the system is defined by the following formula.

$$S[x_1, \ldots, x_N] = \int_I L(x_1(t), \ldots, x_N(t), \dot{x}_1(t), \ldots, \dot{x}_N(t), t)\, \mathrm{d}t \tag{1.1}$$

Here $L : TM^N \to \mathbb{R}$ is called the Lagrangian function. It is most common to consider Lagrangians of the form $L = K - V$. Such a Lagrangian is called separable. $K : TM \to \mathbb{R}$ denotes the kinetic energy of the system, which only explicitly depends on the velocities $(\dot{x}_1(t), \ldots, \dot{x}_N(t)) \in T_{x_1(t),\ldots,x_N(t)}M^N$. The function V is the potential energy of the system, which only depends on the positions $\{x_1(t), \ldots, x_N(t)\}$. We often write $S = \int_I L(x, \dot{x}, t)\, \mathrm{d}t$. The kinetic energy K can be interpreted as a weighted measure of how much motion is taking place in the system, while the potential energy V usually describes interactions between the particles as well as with any extraneous forces.

To determine the critical points of such a functional, we now turn to defining the variational derivative.

Definition 1.2 (Variational Derivative of Action)**.** Let I be an interval and let M be a smooth manifold. Consider a space of paths and their tangent vectors, $\mathcal{F} \subseteq C^\infty(I, TM)$. Define a family of functionals $S_L \in \mathcal{F}^*$ for each $L \in C^\infty(TM \times I)$, and let $\mathcal{F}_0 \subseteq L^2(I)$ be a suitable space of test functions. For each $\varphi \in \mathcal{F}_0$ consider the functional $D_\varphi S_L$ on $\mathcal{F}$ given by

$$(D_\varphi S_L)[f] = \lim_{\epsilon \to 0} \frac{S_L[f + \varepsilon\varphi] - S_L[f]}{\varepsilon}, \qquad f \in \mathcal{F} \tag{1.2}$$

This is called the Gateaux derivative of S_L in the direction of φ. If this limit is independent of $\varphi \in \mathcal{F}_0$, we say S_L is differentiable. The usual variational derivative is then defined by the formula

$$\left\langle \frac{\delta S}{\delta f}, \varphi \right\rangle = D_\varphi S_L[f]. \tag{1.3}$$

This definition takes some dissecting to fully understand. Let us explicitly calculate the functional $\frac{\delta S}{\delta f}$, given $S_L = \int_I L(x, \dot{x}, t)\,\mathrm{d}t$ as seen above, with $I = [a, b]$ and $M = \mathbb{R}$. Assume that L is analytic, and assume that φ and $\dot{\varphi}$ are bounded. Then,

$$\begin{aligned}
\left\langle \frac{\delta S}{\delta f}, \varphi \right\rangle &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\int_I L(x + \varepsilon\varphi, \dot{x} + \varepsilon\dot{\varphi}, t) - L(x, \dot{x}, t)\,\mathrm{d}t \right) \\
&= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\int_I \left[L(x, \dot{x}, t) + \frac{\partial L}{\partial x}\varepsilon\varphi + \frac{\partial L}{\partial \dot{x}}\varepsilon\dot{\varphi} + O(\varepsilon^2) \right] - L(x, \dot{x}, t)\,\mathrm{d}t \right) \\
&= \int_I \frac{\partial L}{\partial x}\varphi + \frac{\partial L}{\partial \dot{x}}\dot{\varphi}\,\mathrm{d}t \\
&= \int_I \left(\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} \right) \varphi\,\mathrm{d}t + \left. \frac{\partial L}{\partial \dot{x}}\dot{\varphi} \right|_a^b
\end{aligned}$$

Now, if we require from the beginning that the boundary terms vanish, we can directly read off the usual expression for the functional derivative as presented in physics textbooks. However, since infinite-dimensional vector spaces are not always isomorphic to their dual, we should be careful to note that $\frac{\delta S}{\delta f}$ only defines a function from I to $\mathbb{R}$ if $D_\varphi S_L[f]$ can be written in integral form as $\int \frac{\delta S}{\delta f}(L)\big|_t h(t)\mathrm{d}t$. In general, $\frac{\delta S}{\delta f}$ is a distribution, and we should interpret the expression $\langle \frac{\delta S}{\delta f}, \varphi \rangle$ as the natural pairing defined between any vector and an element of the dual space. In the case

where $\dot{\varphi}$ is taken to always vanish at the boundaries, it makes sense to write $\frac{\delta S}{\delta f}$ as a function, and we have

$$\frac{\delta S}{\delta f} = \frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}}. \tag{1.4}$$

If the boundary terms do not vanish, the functional derivative can only be interpreted in the sense of distributions, and the above equation would include instances of the Dirac delta distribution. The requirement that L is a critical point of S is given by the condition that $\frac{\delta S}{\delta f} = 0$. Therefore, we can see that for the simplest one-dimensional mechanical systems, the equations governing their motion are given by,

$$\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} \tag{1.5}$$

For a system of N particles in n dimensions, with trajectories given by $\{x_i^j(t) : i = 1, \ldots, N,\ j = 1, \ldots, n\}$ we have,

$$\frac{\partial L}{\partial x_i^j} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_i^j} \tag{1.6}$$

A very common model for physical systems is to take the kinetic energy to proportional to the Riemannian norm of some vector (this is clearly convex). For the remainder of the section we will consider Lagrangians of this form, where $L = K - V$.

Definition 1.3 (Separable Lagrangian)**.** If $L = K - V$ with $\frac{\partial K}{\partial x^i} = 0$ for all i and $\frac{\partial V}{\partial \dot{x}^i} = 0$ for all i, we say L is separable.

Separable Lagrangians are typically easier to study with numerical and analytic methods than nonseparable Lagrangians. Note that if L is of the form $K - V$, but

K depends on the position variables, then L is not separable. This happens, for instance, when the metric g depends on x^i. Therefore, many systems on curved manifolds are non-separable.

Definition 1.4 (Natural Kinetic Energy Function)**.** Let M be an n-dimensional manifold and let $\{x_1, \ldots, x_N\}$ be the trajectories of N particles on this manifold. The kinetic energy of the system is defined as,

$$K = \sum_{i=1}^{N} \frac{1}{2} m_i g(\dot{x}_i, \dot{x}_i) \tag{1.7}$$

Where $m_i, i = 1, \ldots, N$ are the masses of each particle.

We have so far only discussed the Lagrangian formalism for mechanics, in which we solve the second order system (1.6) for the trajectories x_i on M^N. However, with a particular change of variables we can convert this to a first order system for trajectories (x^i, p_i) on TM^N, which we dub the phase space of the system. This is the foundation of Hamiltonian mechanics. We will often call the curve $(x^i(t), p_i(t))$ in TM^N the trajectory of the particle in phase space.

Definition 1.5 (Legendre Transform)**.** Let M be a smooth manifold of dimension n, and let $\{x_i^j : i = 1, \ldots, N, j = 1, \ldots, n\}$ be coordinates on some patch $U \subseteq M^N$, and let L be a Lagrangian function. The Legendre transform of L with respect to the coordinates $\{x_i^j\}$ is defined as the following function,

$$H(x_1, \ldots, x_n, p_1, \ldots, p_n) = \sum_{i=1}^{N} \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{x}_i^j} \dot{x}_i^j - L(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n) \tag{1.8}$$

The Legendre transform of a well-behaved function defines a system of coordinates $\{(x_i^j, p_j^i)\}$ on T^*M^N where for all $x \in M^N$, if $v \in T_xM^N$ then $p = \left(\frac{\partial L}{\partial \dot{x}_i^j}\big|_{\dot{x}=v} \partial_i^j\right)^\flat$. In this sense the Legendre transform can be seen as an isomorphism between TM and T^*M. When L is of the form $K - V$ with K defined by (1.7) we see that this isomorphism is exactly the musical isomorphism determined by g. It is worth noting that the Legendre transform can be defined for any form of Lagrangian with non-vanishing second derivatives, not just those of the form $K - V$.

There is one small issue if we use the above function H as our definition for a Hamiltonian. The function is written in terms of coordinates on TM^N rather than T^*M^N. To fix this, we need to invert $\frac{\partial L}{\partial \dot{x}_i} = p_i$ to find an expression for $\dot{x}_i$, and substitute this into L. This can be tedious, but luckily if the Lagrangian is of $K-V$ type we can come up with a shortcut. Observe that

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{1}{2}\frac{\partial}{\partial \dot{x}_i} m_i g(\dot{x}_i, \dot{x}_i) = m_i \dot{x}_i^\flat.$$

So we have $H = 2K - (K - V) = K + V$.

Theorem 1.1. Let L be a Lagrangian function on TM^N. If L is of $K - V$ type, and $K = \frac{1}{2}\sum_i m_i g(\dot{x}_i, \dot{x}_i)$, then the Legendre transform of L is $H = \tilde{K} + V$ where $\tilde{K}$ is the pullback of K under the musical isomorphism induced by g. We often write $\tilde{K} = K$.

For our choice of kinetic energy function, the equivalent formula in terms of the

new variables is given as follows. Note that $\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial K}{\partial \dot{x}_i} = m_i \dot{x}_i^\flat$.

$$\begin{aligned} K &= \sum_i \frac{1}{2} m_i g(\dot{x}_i, \dot{x}_i) \\ &= \sum_i \frac{1}{2} m_i \dot{x}_i^\flat(\dot{x}_i) \\ &= \sum_i \frac{1}{2m_i} p_i(p_i^\sharp) \\ &= \sum_i \frac{1}{2m_i} g^{-1}(p_i, p_i) \end{aligned}$$

From this we see that $\frac{\partial H}{\partial p_i} = \frac{1}{m_i} p_i^\sharp = \dot{x}_i$ (since $\flat$ is the inverse of $\sharp$). This is the first of Hamilton's equations. Meanwhile, $\dot{p}_i = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}_i} = -\frac{\partial L}{\partial x_i} = -\frac{\partial H}{\partial x_i}$ by (1.6). This gives us the second of Hamilton's equations. A more general formulation of Hamilton's equations can be written in terms of the flow of a vector field called the Hamiltonian vector field.

Definition 1.6 (Symplectic Form)**.** Let M be a smooth manifold of dimension $m = Nn$, and let $\omega \in \Omega^2(M)$ be a closed two-form. This two-form is called a symplectic form if it is non-degenerate. That is, for all $X \in \mathfrak{X}(M)$, $\omega(X, Y) = 0$ for all $Y \in \mathfrak{X}(M)$ if and only if $X = 0$.

A smooth manifold endowed with a symplectic form is called a symplectic manifold. The dimension of a symplectic manifold is always even, and in fact if $M = T^*N$ for some N, then the following normal form characterization for ω exists.

Definition 1.7 (Darboux Coordinates)**.** Let M be a manifold of dimension n. Then

in any local trivialization $U \times \mathbb{R}^n \subseteq T^*M$, the standard symplectic form Ω is,

$$\Omega = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}p_i. \tag{1.9}$$

Where $\{x^i : i = 1, \ldots, n\}$ is a local system of coordinates on U and p_i are the corresponding coordinates on $\mathbb{R}^n$ with respect to $\mathrm{d}x^i$.

In general, in a small enough patch U on a symplectic manifold M with symplectic form ω, one can find a coordinate system on U so that ω is of the form (1.9).

Remark 1.2. Observe that the standard symplectic form Ω is exact. That is, $\Omega = \mathrm{d}\lambda$ where λ is defined as

$$\lambda = \sum_{i=1}^{n} x^i \mathrm{d}p_i \tag{1.10}$$

Definition 1.8 (Tautological One-Form)**.** Let x^i, p_i be defined as above. Then $\lambda = \sum_{i=1}^{n} x^i \mathrm{d}p_i$ is called the tautological one-form.

In physics, symplectic manifolds appear as the phase space of a system which conserves energy. That is, a dynamical system for which the Hamiltonian is constant in time along each trajectory. In general, a function of the coordinates x^i, p_i on $T*M$ which is constant along each trajectory is called an integral of motion for the system.

Definition 1.9 (Locally Hamiltonian Vector Field)**.** Let M be a symplectic manifold, let U be an open subset of M, and let $f \in C^\infty(T^*M)$. Then f defines a unique vector field X_f on T^*U called the Hamiltonian vector field associated to f, defined

so that

$$\mathrm{d}f = X_f \lrcorner \omega \tag{1.11}$$

Conversely, if $X \in \mathfrak{X}(T^*U)$ and $X \lrcorner \omega$ is exact, we say X is a locally Hamiltonian vector field. If $X \in \mathfrak{X}(T^*M)$ and $X \lrcorner \omega$ is exact on all of T^*M, then X is just called a Hamiltonian vector field.

It is also useful to define the following operation on functions, which makes computations using Hamiltonian vector fields somewhat simpler.

Definition 1.10 (Poisson Bracket)**.** Let $f, g \in C^\infty(T^*M)$. The Poisson bracket $\{f, g\}$ of f and g is defined as the following formula,

$$\{f, g\} = \omega(X_f, X_g) \tag{1.12}$$

In particular, in Darboux coordinates this becomes

$$\{f, g\} = \sum_{i=1}^{\dim M} \left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x^i} \frac{\partial f}{\partial p_i} \right) \tag{1.13}$$

Remark 1.3. Notice that $X_H f = \mathrm{d}f(X_H) = \omega(X_f, X_H) = \{f, H\} = \frac{\mathrm{d}f}{\mathrm{d}t}$.

Using the Poisson bracket we can give a proper definition of an integrable system.

Definition 1.11 (Integrable System)**.** Consider a Hamiltonian system on some configuration space M of dimension n. Suppose there are n conserved quantities $f_1, \ldots, f_n : T^*M \to \mathbb{R}$ so that $\mathrm{d}f_1, \ldots, \mathrm{d}f_n$ are linearly independent at each $(x, p) \in T^*M$, and so that $\{f_i, f_j\} = 0$ for all i, j. Then the system is said to be completely integrable.

An integrable system has the remarkable property that it reduces to a much simpler system of equations on the level sets of the conserved quantities $f_1, \dots f_n$. This is made precise by the Arnold-Liouville theorem.

Theorem 1.2 (Arnold-Liouville)**.** Consider a completely integrable system on an n dimensional manifold M with n conserved quantities $f_1, \dots f_n$. Let $M_f = \{(x,p) \in T^*M : f_i(x,p) = c_i, i = 1, \dots n\}$, where for each i, c_i is a constant real number. Then the following statements hold.

1. If M_f is compact and connected then it is diffeomorphic to the torus $T^n = (S^1)^n$, and in a neighbourhood of M_f we may construct a system of local coordinates called action-angle variables $I_1, \dots I_n, \phi_1, \dots \phi_n$, where $I_1 = f_1(x,p), \dots I_n = f_n(x,p)$ and $\phi_1, \dots \phi_n \in [0, 2\pi)$.
2. The equations of motion become

$$\dot{I}_k = 0, \qquad \dot{\phi}_k = \omega_k(I_1, \dots I_n), \qquad k = 1, \dots n \tag{1.14}$$

   Where ω_k is a smooth function of the conserved quantities. Hence the equations for ϕ_i are independent and can be solved separately.

In general, the theorem can be extended to the case where the system has more than n conserved quantities. From this, it is found that if the system is maximally superintegrable then M_f is one dimensional and therefore diffeomorphic to a circle, meaning the orbits are closed.

In Darboux coordinates, we can write a particle's trajectory in phase space as

(x^i, p_i) as above, and arrive at Hamilton's equations as usual.

Theorem 1.3 (Hamilton's Equations)**.** For a Hamiltonian system with $H = K+V$, the equations of motion $\frac{\mathrm{d}}{\mathrm{d}t}\big(x^i, p_i\big) = X_H((x^i, p_i))$ simplify to the following.

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x^i} \tag{1.15}$$

Proof. Simply compute $\{x^i, H\} = \frac{\partial H}{\partial p_i} - 0$, and $\{p_i, H\} = -\frac{\partial H}{\partial x^i}$.

Now we will describe symmetries of Hamiltonian systems. Let G be a connected Lie group and let $A : G \times M \to M$ be a smooth Lie group action preserving the Hamiltonian H the symplectic form ω. If we write $A_\phi : M \to M$ for the action of some group element ϕ, then for each $\phi \in G$, $A_\phi^*\omega|_q = \omega|_q$ for all $q \in M$. For brevity we write $A_\phi(q) = \phi \cdot q$. For a compact, simply connected Lie group G, any group element ϕ can be written as the Riemannian exponential of some Lie algebra element $\xi \in \mathfrak{g} = \mathrm{Lie}(G)$. This means that the action $A_\phi(q)$ can be written as $\exp(t\xi) \cdot q$. The induced action on T^*M is called the cotangent lift of A, and is given by $\tilde{A}_\phi(q, p) = (\phi \cdot q, \phi^{-1} \cdot p)$. So the action in general is given by $(q, p) \mapsto (\exp(t\xi) \cdot q, \exp(-t\xi) \cdot p)$. We can also write the action $A_{\exp(t\xi)}$ on T^*M parametrized by t as a flow operator associated to some vector field $X_\xi \in \mathfrak{X}(T^*M)$, so that $\exp(t\xi) \cdot (q, p) = \exp_{(q,p)}(tX_\xi)$. We call this vector field the infinitesimal generator of ϕ.

Suppose that for all $\phi \in G$ we have $X_{H \circ A_\phi} = X_H$. Then the functions H and $H \circ A_\phi$ determine the same Hamiltonian system. This means that for such a

Lie group, each solution to Hamilton's equations determines a family of solutions induced by the action of A. This family of solutions forms what is known as an invariant set for the Hamiltonian system.

To characterize the invariant subsets, we can construct a function $\mu : T^*M \to \mathfrak{g}^*$ called the momentum map for A, which has the property that for each $\xi \in \mathfrak{g}$, if X_ξ is the infinitesimal generator associated to ξ then,

$$\mathrm{d}\langle \mu(x,p), \xi \rangle = X_\xi \lrcorner \, \Omega|_{(x,p)} \text{ for all } (x,p) \in T^*M \tag{1.16}$$

So if $\xi_1, \ldots, \xi_k$ are a complete set of generators for $\mathfrak{g}$, we can find μ by solving the system of ODEs determined by $\mathrm{d}\langle \mu(x,p), \xi_i \rangle$ for $i = 1, \ldots, k$. Note here that if $\omega = \Omega$ is the standard symplectic form, then we have $\mathrm{d}\langle \mu, \xi \rangle = X_\xi \lrcorner \mathrm{d}\lambda = -\,\mathrm{d}(X_\xi \lrcorner \, \lambda)$. So we have $\langle \mu, \xi \rangle = X_\xi \lrcorner \, \lambda$. This provides a straightforward way of computing the moment map.

Definition 1.12 (Momentum Map)**.** Let $\mu : T^*M \to \mathfrak{g}^*$ be a differentiable function satisfying (1.16). Suppose that μ is G-equivariant. That is, for all $\phi \in G$, $\mathrm{ad}^*(A_\phi)\mu|_{(x,p)} = \mu|_{A_\phi(x,p)}$. Then μ is said to be a momentum map for the action A.

The image of the momentum map is an element of the dual of the Lie algebra. The quantities $\langle \mu, \xi \rangle$ are to be interpreted as the 'conserved' quantities of the system, as it can be shown that $\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0$ where $\frac{\mathrm{d}}{\mathrm{d}t}$ is defined with respect to the Hamiltonian flow determined by H. The level sets of μ therefore determine some invariant sets of the Hamiltonian system.

1.3 Complex Geometry

Complex geometry concerns manifolds with a so-called complex structure. This is a special kind of linear map related to the manifold which allows one to use complex coordinates $z : U \subseteq M \to \mathbb{C}^n$ rather than real coordinates $x : V \subseteq M \to \mathbb{R}^{2n}$. For a self-contained definition, one can begin with a topological space which is locally homeomorphic to $\mathbb{C}^n$, and perform the usual construction for manifolds except with the condition that the transition functions are holomorphic. Instead of a smooth structure, the resulting structure is called a complex structure. Much of this material is discussed in textbooks on complex geometry or related subjects [22, 20, 21].

Definition 1.13 (Complex Manifold)**.** Let M be a Hausdorff and second-countable topological space. Suppose there exists $n \in \mathbb{N}$ so that M is locally homeomorphic to $\mathbb{C}^n$. Each local homeomorphism $\varphi_U : U \to \mathbb{C}^n$ is called a chart. Let C be an open cover of M with the property that for each $U \in C$, $U \cong \mathbb{C}^n$, and let U and V be elements of this cover. If for each pair of charts, $\varphi_U, \varphi_V : M \to \mathbb{C}^n$, the transition function $\varphi_U \circ \varphi_V^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic, we say the collection Φ_C of charts is a complex atlas (or holomorphic atlas) for M, and we say M is an n-dimensional complex manifold when endowed with this structure.

We can define the tangent bundle of a complex manifold in a similar way. We will begin by constructing the real tangent bundle. Let $\{z^1, \ldots, z^n\}$ be coordinates for M in a patch U. For each $i = 1, \ldots, n$ we write $z_i = x^i + iy^i$. Given

the real coordinates $\{x^1, y^1, \ldots, x^n, y^n\}$, we can define the tangent bundle of the underlying $2n$-dimensional real manifold. A basis for each fibre is then given by $\{\partial_{x^1}, \partial_{y^1}, \ldots, \partial_{x^n}, \partial_{y^n}\}$, which extend to coordinate vector fields locally. Given these, we can define complex vector fields corresponding to the holomorphic and antiholomorphic derivative operators ∂_{z^i} and $\partial_{\overline{z}^i}$.

Definition 1.14 (Complexified Tangent Bundle)**.** We define the complex tangent bundle by $T^{\mathbb{C}}M = TM \otimes \mathbb{C}$.

Definition 1.15 (Wirtinger Derivatives)**.** Let M be an n-dimensional complex manifold and let $\{x^1, y^1, \ldots, x^n, y^n\}$ be real coordinates on $U \subseteq M$. Define the following sections of $T^{\mathbb{C}}M|_U$:

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - i\frac{\partial}{\partial y^i}\right), \qquad \frac{\partial}{\partial \overline{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + i\frac{\partial}{\partial y^i}\right) \tag{1.17}$$

Together, the smooth vector fields $\{\partial_{z^1}, \partial_{\overline{z}^1}, \ldots, \partial_{z^n}, \partial_{\overline{z}^n}\}$ form a coordinate frame for $T^{\mathbb{C}}M|_U$. These are called the Wirtinger derivatives.

In the coordinate chart U defined above, we can convert our $2n$ real coordinates $\{x^i, y^i\}$ into n complex coordinates z^i defined so that $x^i = \frac{1}{2}(z^i + \overline{z}^i)$ and $y^i = \frac{1}{2i}(z^i - \overline{z}^i)$. This allows us to generalize many formulas from real differential geometry. When viewed as a real vector bundle, the complexified tangent bundle has a natural bundle endomorphism J called a complex structure, which has the property that $J\partial_{x_i} = \partial_{y_i}$ and $J\partial_{y_i} = -\partial_{x_i}$. In complex coordinates, we then have $J\partial_{z_i} = i\partial_{z_i}$ and $J\partial_{\overline{z}_i} = -i\partial_{\overline{z}_i}$.

Definition 1.16 (Holomorphic/Antiholomorphic Tangent Bundle)**.** Let M be an n-dimensional complex manifold with complex structure J. The holomorphic tangent bundle of M is defined to be the $+i$ eigenbundle of J, and is denoted by $T^{(1,0)}M$. Similarly, the antiholomorphic tangent bundle is defined as the $-i$ eigenbundle of J, and is denoted $T^{(0,1)}M$.

Definition 1.17 (Degree of Complex Differential Form)**.** Let M be a complex manifold. A degree (p, q) differential form is a section of $\Lambda^p T^{(1,0)*}M \wedge \Lambda^q T^{(0,1)*}M$.

The direct sum decomposition of $T^{\mathbb{C}}M$ also allows to decompose the exterior derivative into a component acting on holomorphic forms and a component acting on antiholomorphic forms.

Definition 1.18 (Dolbeault Operators)**.** Let d denote the exterior derivative on $\Lambda^\bullet T^{\mathbb{C}*}M$. For a differential form specified in local coordinates by $\omega = \omega_{IJ}\mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J$, this is given by $\mathrm{d}\omega = \frac{\partial \omega_{IJ}}{\partial z^k}\mathrm{d}z^k \wedge \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J + \frac{\partial \omega_{IJ}}{\partial \overline{z}^k}\mathrm{d}\overline{z}^k \wedge \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J$, with I and J being multi-indices denoting the holomorphic and antiholomorphic components of ω. The holomorphic Dolbealt operator ∂ is then defined in coordinates by

$$\partial\omega = \frac{\partial \omega_{IJ}}{\partial z^k}\mathrm{d}z^k \wedge \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J, \tag{1.18}$$

and the antiholomorphic Dolbeault operator $\overline{\partial}$ by

$$\overline{\partial}\omega = \frac{\partial \omega_{IJ}}{\partial \overline{z}^k}\mathrm{d}\overline{z}^k \wedge \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J \tag{1.19}$$

The exterior derivative then satisfies the identity $\mathrm{d} = \partial + \overline{\partial}$.

Recall that for a complex vector space, the corresponding notion of a positive definite inner product is a sesquilinear product. This motivates the following definition.

Definition 1.19 (Hermitian Metric)**.** Let M be an n-dimensional complex manifold. A Hermitian metric on M is a section h of $T^{\mathbb{C}*}M \otimes \overline{T^{\mathbb{C}*}M}$ satisfying the following properties.

1. For all $v, w \in T_z^{\mathbb{C}}M$, we have $h(v,w) = \overline{h(w,v)}$
2. For all $v \neq 0 \in T_z^{\mathbb{C}}M$ we have $h(v,v) > 0$

A complex manifold M with a Hermitian metric is often called a Hermitian manifold. It is the complex generalization of the notion of a Riemannian manifold. Some remarkable properties of Hermitian metrics are as follows.

Theorem 1.4. Let M be a complex manifold with Hermitian metric h. The bilinear form defined by $g = \frac{1}{2}(h + \overline{h}) = \operatorname{Re} h$ defines a Riemannian metric on TM. Additionally, the bilinear form $\omega = \frac{1}{2i}(h - \overline{h}) = \operatorname{Im} h$ is a skew-symmetric degree (1,1) differential form, and is referred to as the fundamental form of M.

A particularly well-behaved subclass of Hermitian manifolds are the Kähler manifolds, defined by the following rule.

Definition 1.20 (Kähler Manifold)**.** Let M be a Hermitian manifold with metric h. Then M is called Kähler if $\mathrm{d}\omega = 0$, where ω is the fundamental form associated with h.

Definition 1.21 (Hermitian Laplace-Beltrami Operators)**.** Let M be a Hermitian manifold with metric h. Recall that the exterior derivative on $T^{\mathbb{C}*}M$ decomposes into $\mathrm{d} = \partial + \overline{\partial}$, where $\partial, \overline{\partial}$ are the holomorphic and antiholomorphic Dolbeault operators. We similarly define the codifferentials $\partial^\star = -\star\,\partial\star$ and $\overline{\partial}^\star = -\star\,\overline{\partial}\star$, where $\star$ is the Hodge star operator associated to h. With these, we can define the holomorphic and antiholomorphic Laplace-Beltrami operators,

$$\Delta^\partial = \partial\partial^\star + \partial^\star\partial, \qquad \Delta^{\overline{\partial}} = \overline{\partial}\overline{\partial}^\star + \overline{\partial}^\star\overline{\partial} \tag{1.20}$$

Theorem 1.5. If M is a Kähler manifold, the holomorphic and antiholomorphic Laplace-Beltrami operators defined by a Hermitian metric h agree, and are related to the Riemannian Laplace-Beltrami operator defined by $g = \mathrm{Re}(h)$ according to the formula [23],

$$\Delta^g = 2\Delta^\partial = 2\Delta^{\overline{\partial}} \tag{1.21}$$

Proof. Expand ∂ and $\overline{\partial}$ in terms of real coordinates and then simplify. See the lecture notes by Moroianu [23].

This fact is what allows us to generalize the Kepler problem to Kähler manifolds in complex coordinates, as the Green's function of the Hermitian Laplace-Beltrami operator will agree with the Riemannian one.

Consider a Hermitian metric on $T^{\mathbb{C}}M$. Recall that we can construct a Riemannian metric on TM from the Hermitian metric on $T^{\mathbb{C}}M$. Since the Hamiltonian is real, the existence of this Riemannian metric allows us to relate the complex

Hamiltonian dynamics on M to the underlying real dynamics.

Theorem 1.6. Let M be a complex manifold with Hermitian metric h and corresponding Riemannian metric $g = \mathrm{Re}(h)$. Consider a system of N point particles with masses $\{m_1, \ldots, m_N\}$, with trajectories $\{z_1(t), \ldots, z_N(t) \mid t \in \mathbb{R}^+\}$. The kinetic energy function is given by,

$$K = \sum_{i=1}^{N} \frac{1}{2m_i} g^{-1}(p_{x_i}, p_{x_i}) = \sum_{i=1}^{N} \frac{1}{2m_i} h^{-1}(p_{z_i}, p_{z_i}). \tag{1.22}$$

Additionally, in order to define Hamiltonian mechanics on Kähler manifolds we need to know the expression for the standard symplectic form in complex coordinates. Recall that the standard symplectic form is given by $\Omega = \mathrm{d}\lambda$, where $\lambda = \sum_{i=1}^{n} x^i \mathrm{d}p_{x^i}$. For a Kähler manifold there are $2n$ real coordinates (x^i, y^i), so we can write

$$\begin{aligned}
\lambda &= \sum_{i=1}^{n} (x^i \mathrm{d}p_{x^i} + y^i \mathrm{d}p_{y^i}) \\
&= \frac{1}{4} \sum_{i=1}^{n} ((z^i + \overline{z}^i)\, \mathrm{d}(p_{z^i} + p_{\overline{z}^i}) - (z^i - \overline{z}^i)\, \mathrm{d}(p_{z^i} - p_{\overline{z}^i})) \\
&= \frac{1}{2} \sum_{i=1}^{n} (z^i \mathrm{d}p_{\overline{z}^i} + \overline{z}^i \mathrm{d}p_{z^i})
\end{aligned}$$

So the standard symplectic form is

$$\Omega = \frac{1}{2} \sum_{i=1}^{n} (\mathrm{d}z^i \wedge \mathrm{d}p_{\overline{z}^i} + \mathrm{d}\overline{z}^i \wedge \mathrm{d}p_{z^i}) \tag{1.23}$$

Note that the symplectic form Ω and the fundamental form ω both define symplectic structures, but are otherwise unrelated. When considering the base space M as the

configuration space, not the phase space, it is more natural to use Ω when defining Hamiltonian systems, since the fundamental form is not assumed to be related to the dynamics. Using this as our definition for the symplectic form, we arrive at the following versions of Hamilton's equations.

Theorem 1.7. Let M be a complex manifold and let H be a Hamiltonian function on M. Then Hamilton's equations take the following form,

$$\dot{z}^i = 2\frac{\partial H}{\partial p_{\overline{z}^i}} \qquad \dot{p}_{z^i} = -2\frac{\partial H}{\partial \overline{z}^i} \tag{1.24}$$

When studying symmetries it is also essential to compute the equation for the momentum map in complex coordinates. Suppose that G is a connected Lie group acting on M. Then suppose that $\xi \in \mathrm{Lie}(G)$, and let X_ξ be given in real coordinates by

$$X_\xi = X_\xi^i \partial_{x^i} + Y_\xi^i \partial_{y^i} - X_\xi^i \partial_{p_{x^i}} - Y_\xi^i \partial_{y^i} \tag{1.25}$$

Then in complex coordinates we have

$$X_\xi = \frac{1}{2}(X_\xi^i + iY_\xi^i)\partial_{z^i} + \frac{1}{2}(X_\xi^i - iY_\xi^i)\partial_{\overline{z}^i} - \frac{1}{2}(X_\xi^i + iY_\xi^i)\partial_{p_{z^i}} - \frac{1}{2}(X_\xi^i - iY_\xi^i)\partial_{p_{\overline{z}}^i} \tag{1.26}$$

Using this formula we can easily find expressions for momentum maps on complex manifolds given the required expression in real coordinates. An example we will be interested in is the complex projective space.

Definition 1.22 (Complex Projective Space)**.** Consider the collection of lines through the origin in $\mathbb{C}^n$. That is, the set $\mathbb{CP}^{n-1} = \{[\lambda z, \lambda \in \mathbb{C}] : z \in \mathbb{C}^n\}$. This set is called

the complex projective space, and has the structure of a complex manifold. A useful representation of this manifold is as a quotient of $\mathbb{C}^n \setminus \{0\}$ by the equivalence relation $z \sim w \iff \exists \lambda \in \mathbb{C}$ s.t. $z = \lambda w$.

Since a line in $\mathbb{C}^n$ is entirely determined by a single nonzero complex vector $(Z_0, \ldots, Z_n)$, we often denote the line using so-called homogenous coordinates, $Z = [Z_0 : \ldots : Z_n]$, where we use colons rather than commas to indicate that only the ratios between the components matter. To see why this is true, observe that if Z_i is nonzero, then we can define $\tilde{\lambda} = \lambda / Z_i$, and then $[\tilde{\lambda} z, \lambda \in \mathbb{C}]$ and $[\lambda z, \lambda \in \mathbb{C}]$ are equal as sets.

An atlas for $\mathbb{CP}^n$ can be constructed as follows. Observe that if one of Z_i is nonzero, we can choose a fixed representative of the line $[Z_0 : \ldots : Z_i : \ldots : Z_n]$ by dividing through by Z_i, giving us the homogenous representation $[Z_0/Z_i : \ldots : 1 : \ldots : Z_n/Z_i]$. The non-fixed coordinates are denoted $z_j = Z_j/Z_i, j = 0, \ldots, i-1$, $z_j = Z_{j+1}/Z_i, j = i, \ldots, n-1$, and are referred to as affine coordinates. Let C_i be the (closed) subset of $\mathbb{CP}^n$ consisting of lines $W = [W_0 : \ldots : W_n]$ with $W_i = 0$. Then the coordinates $(z_0, \ldots, z_{n-1})$ define a complex chart $\mathbb{CP}^n \setminus C_i \cong \mathbb{C}^n$. This procedure therefore gives us an atlas for $\mathbb{CP}^n$ consisting of n charts. Not only is $\mathbb{CP}^n$ a complex manifold, it is also Kähler.

Definition 1.23 (Fubini-Study Metric)**.** The Hermitian metric on $\mathbb{CP}^n$, given as the quotient metric of the standard inner product on $\mathbb{C}^{n+1}$, is called the Fubini-Study

metric, and takes the following form in affine coordinates,

$$h = \left(\frac{(1+|z|^2)\delta_{ij} - \overline{z}_i z_j}{(1+|z|^2)^2} \right) \mathrm{d}z^i \otimes \mathrm{d}\overline{z}^j, \tag{1.27}$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

1.4 The Kepler Problem

The Kepler problem is the classic problem of predicting the motion of idealized massive objects obeying a form of the Newtonian law of universal attraction. In $M = \mathbb{R}^3$, this law takes the form,

$$\Delta^g \Phi = 4\pi k \rho, \qquad \lim_{\|\mathbf{x}\| \to \infty} \Phi(\mathbf{x}) = 0, \tag{1.28}$$

where ρ is the mass density of the system of particles, k is a constant[1], and Φ is the potential energy function. Here Δ is the usual Laplacian operator. This determines the equations of motion via the Hamiltonian $H = K + \Phi$. Since we are taking the particles to be points, the above equation will contain so-called delta functions on the right hand side. We should therefore write this in weak form, which allows for ρ to be interpreted as a distribution.

$$\forall\, v \in W^{1,2}(\mathbb{R}^3), \qquad \int_M (\nabla\Phi \cdot \nabla v + \rho v)\, \mathrm{d}V = 0 \tag{1.29}$$

Here, $W^{1,2}(\mathbb{R}^3)$ is a Sobolev space, defined as the following subset of $L^2(\mathbb{R}^3)$.

$$W^{1,2}(\mathbb{R}^3) = \{ f \in L^2(\mathbb{R}^3) \ : \ \|f\|^2 + \|\nabla f\|^2 < \infty \} \tag{1.30}$$

[1]We avoid the use of the constant G, since we also use G to denote the Green's function

This equation can then be solved using the method of Green's functions. For each $\mathbf{y}$ in M, we solve the above partial differential equation for an idealized point mass located at $\mathbf{y}$. The mass density is interpreted as a distribution, and can be thought of as $\delta_{\mathbf{y}}$, where $\delta_{\mathbf{y}} \in (L^2(\mathbb{R}^3))^*$ is the Dirac delta distribution defined by $\delta_{\mathbf{y}}[f] = f(\mathbf{y})$. This gives us the following integral equation,

$$\forall \mathbf{y} \in \mathbb{R}^3, v \in W^{1,2}(\mathbb{R}^3), \qquad \int_{\mathbb{R}^3} \nabla_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) \cdot \nabla_{\mathbf{x}} v(\mathbf{x}) \, \mathrm{d}V = 4\pi \tag{1.31}$$

We can solve this using a straightforward application of Fourier transforms, resulting in

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{4\pi \|\mathbf{x} - \mathbf{y}\|} \tag{1.32}$$

The solution $G(\mathbf{x}, \mathbf{y})$ is called the Green's function for the partial differential equation, and has the property that $\int_{\mathbb{R}^3} G(\mathbf{x}, \mathbf{y}) \rho(\mathbf{y}) \mathrm{d}V = \Phi(\mathbf{x})$ is a solution to (1.29). If ρ is smooth, then Φ is a strong solution to (2.2). More importantly, it defines the potential energy of a particle in the vicinity of a point mass. It is useful to note that the Green's function for the Poisson equation is symmetric: $G(x, y) = G(y, x)$. We can easily generalize this to a system of many particles by adding together the potential energies.

Definition 1.24 (Kepler Potential in $\mathbb{R}^n$)**.** The potential energy of a system of particles, each located at positions $x_1, \ldots, x_N$ and with masses $m_1, \ldots, m_N$, interacting

via a Kepler potential is given by the following function.

$$V(x_1, \ldots, x_N) = k \sum_{i=1}^{N} \sum_{j<i} m_i m_j G(x_i, x_j) \tag{1.33}$$

With $k \in \mathbb{R}$ a constant as above.

The total Hamiltonian for such a system is

$$H(x, p) = \sum_{i=1}^{N} \frac{1}{2m_i} \|p_i\|^2 + k \sum_{i=1}^{N} \sum_{j<i} m_i m_j G(x_i, x_j). \tag{1.34}$$

This gives rise to Hamilton's equations, which are

$$\dot{x}_i = \frac{1}{m_i} p_i, \tag{1.35}$$

$$\dot{p}_i = -m_i k \sum_{j \neq i} m_j \frac{\partial}{\partial x_j} G(x_i, x_j). \tag{1.36}$$

This abstract formulation lets us generalize the Kepler problem to arbitrary Riemannian and Hermitian manifolds by replacing the ordinary Laplacian with the Laplace-Beltrami Operator.

There are various different versions of the Kepler problem. The simplest is the restricted two-body problem, in which we study the motion of a point particle around a fixed mass. We will ignore collisions, since the potential is often singular at the location of the fixed mass. This reduces the configuration space of the problem to just $M \setminus \{p\}$. There is a choice we must therefore make when calculating the dynamics on a manifold. Do we compute the Green's function for the Laplace-Beltrami operator on M, or on $M \setminus \{p\}$? They are not the same manifold. In what follows we assume that the dynamics are defined using the Laplace-Beltrami operator

on M, which gives us what is sometimes called the intrinsic Kepler problem. It has been shown that when M is a compact surface the Green's function for $M \setminus \{p\}$ is related to the Green's function on M by a straightforward expression [6]. However, the dynamics on the two manifolds may not match each other completely.

Definition 1.25 (Restricted Two-Body Kepler Problem)**.** Consider a point particle of mass m located at a position $x \in \mathbb{R}^3$, orbiting a fixed particle of mass m_0 located at a position $y \in \mathbb{R}^3$. The dynamics of this problem are determined by the following Hamiltonian.

$$H = \frac{1}{2m}p^2 + mm_0kG(x, y) \tag{1.37}$$

A major problem is to determine which manifolds yield Green's function potentials whose orbits are closed. If the Green's function $G(x, y)$ satisfies $G(x, y) = V(d(x, y))$ for some V, with $d(x, y)$ the Riemannian distance function [24], we call G a central potential around the point x. In the restricted two-body case, a change of variables to $r = d(x, y)$ is often possible. Central potentials for which the solutions to Hamilton's equations are closed are called closing potentials.

Definition 1.26 (Closing Potential)**.** Consider a Hamiltonian system of the form $H = K(x, p) + V(x)$, with $x \in M$ and $p \in T_xM$. The function V is called a closing potential if all bound solutions to Hamilton's equation are closed and periodic.

When M is a compact rank-one symmetric space, a Euclidean half-space, or any Riemannian 3-manifold, all closing potentials have been classified [1, 4, 2, 5, 7]. On these manifolds, any closing potential either takes the form $V(d(x, y)) = kG(x, y)$

or $V(d(x,y)) = kG(x,y)^{-2}$. The former is referred to as a Kepler potential, while the latter is referred to as an oscillator potential (as it reduces to the potential of the simple harmonic oscillator in the case $M = \mathbb{R}^n$). It is known that such a closing potential results in a maximally superintegrable system. That is, a system which is not only integrable in the Louville sense, but which has a maximal number of additional conserved quantities that determine the solutions completely. For instance, the restricted two-body Kepler problem in $\mathbb{R}^3$ has a Hamiltonian of the form

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) - \frac{k}{4\pi r},$$

where we have taken the fixed mass to be located at the origin. This Hamiltonian yields four ordinary conserved quantities: the Hamiltonian itself and the angular momenta associated with the infinitesimal generators of the rotation group SO(3). Since we are working in $\mathbb{R}^3$, the angular momenta can be computed as the components of $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ where we have associated $T_{\mathbf{r}}\mathbb{R}^3$ with $\mathbb{R}^3$, and where $\mathbf{p} = (p_x, p_y, p_z)$. However, this problem acquires an additional three conserved quantities, which form the so-called Laplace-Runge-Lenz vector $\mathbf{A} = \mathbf{p} \times \mathbf{L} - \frac{mk}{r}\mathbf{r}$. This conserved quantity does not arise due to symmetries coming from the isometry group of $\mathbb{R}^3$. Instead, it is a result of a so-called hidden symmetry. A hidden symmetry does not necessarily arise due to the action of a group G on the configuration space.

One interpretation of hidden symmetries is in terms of Killing tensors. If we consider the infinitesimal generators associated to each of the ordinary conserved

quantities, most of them arise as generators of the isometry group of the configuration space, and so they are Killing vectors for the metric g. We then widen our view to include higher order Killing tensors. That is, tensor fields $K \in \mathcal{T}^n_k(M)$ satisfying $[K, g]_S = 0$ where $[\cdot, \cdot]_S$ is the Schouten bracket [24]. Using these we can sometimes find additional conserved quantities. Typically we study rank-two Killing tensors, whose associated conserved quantities are quadratic polynomials in the momenta p of the form $K|_x(p, p) + W(x)$ for some function W [25, 26]. Similarly, if we instead consider the oscillator potential $V(r) = kr^2$ we find that there is an additional set of conserved quantities generated by a Killing tensor called the Fradkin tensor. This leads to an equivalent of the Laplace-Runge-Lenz vector for the harmonic oscillator system. Hidden conserved quantities can also be found using the Hamilton-Jacobi formalism. Let $H(x, p)$, with $(x, p) \in T^*M$, be the Hamiltonian of some system. The Hamilton-Jacobi equation is a partial differential equation of the form $H(x, \frac{\partial S}{\partial x}) = E$, where $S(x)$ is an unknown function to be solved for. The solution S then allows one to recover the momenta p as functions of time, and via Hamilton's equations the positions as well. The condition which S must satisfy in order for this method to yield solutions to Hamilton's equations is given by Jacobi's theorem [17]. If the partial differential equation is separable, we may take the ansatz $S(x, p) = S_1(x^1) + \ldots S_n(x^n)$, and the resulting system of equations is of the form $f_1(x, \frac{\partial S_1}{\partial x^1}) = \lambda_1, \ldots, f_n(x, \frac{\partial S_n}{\partial x^n}) = \lambda_n$, where each function $f_1, \ldots, f_n$ is conserved, as we can see from the fact that the separation parameters $\lambda_1, \ldots, \lambda_n$ are constant by

definition [27].

2 Integrability of the Kepler Problem

2.1 The Intrinsic Kepler Problem on Complex Projective Space

Definition 2.1 (Riemannian Laplace-Beltrami Operator)**.** Let M be a Riemannian manifold of dimension n with metric g. The Laplace-Beltrami operator Δ^g is equivalently defined by the following formulas,

1. $\Delta^g f = \mathrm{div}(\mathrm{grad} f)$
2. $\Delta^g f = (\mathrm{d}\mathrm{d}^\star + \mathrm{d}^\star\mathrm{d})f$

Here the codifferential $\mathrm{d}^\star : \Omega^\bullet(M) \to \Omega^\bullet(M)$ is defined by $\mathrm{d}^\star|_{\Omega^k(M)} = (-1)^k \star \mathrm{d} \star$ for each $k \in \{0, \ldots, n\}$. In local coordinates these formulas are written as follows.

$$\Delta^g f = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j f \right) \tag{2.1}$$

Definition 2.2 (Poisson Equation)**.** Let M be a Riemannian or Kähler manifold with Riemannian metric g. The Poisson equation on M is,

$$\Delta^g f = 4\pi k \rho \tag{2.2}$$

Where ρ and f are interpreted in the sense of distributions.

The solution to a general Poisson equation can usually found using the method of Greens' functions, provided ρ is well behaved. Typically we take ρ to be of compact support. The Green's function for such a problem is a function $G : M^2 \to \mathbb{R}$ so that

$G(x, y)$ solves the partial differential equation

$$\Delta^g|_x G(x, y) = 4\pi k\delta(y - x) \tag{2.3}$$

Theorem 2.1. Let G be the Green's function for the general Poisson equation (2.2). Then,

$$f(x) = \int_M 4\pi k\rho(y)G(x, y)\,\mathrm{d}V_g(y) \tag{2.4}$$

is a solution to (2.2).

However, if we are working on a compact manifold there is no boundary data to specify a unique Green's function. The boundary conditions are then replaced by some constraint designed to remove the extra degrees of freedom from the problem. A common constraint for the Poisson equation on a compact domain with no boundary is the Gauss constraint.

Definition 2.3 (Gauss Constraint)**.** Let M be a compact Riemannian manifold. The Gauss constraint is the following contraint on the solution to the Poisson equation,

$$\int_M f\mathrm{d}V_g = 0 \tag{2.5}$$

With the Gauss constraint, we can completely determine all weak solutions to the Poisson equation in terms of a Green's function.

We extend a theorem of Szábo on the existence of radially symmetric harmonic functions in order to investigate the existence of radially symmetric potential energy

functions.

Theorem 2.2. Let M be a simply connected manifold with Riemannian metric g and associated Laplace-Beltrami operator Δ^g. Then the Green's function G is radially symmetric if and only if M is isometric to either a compact rank-one symmetric space, $\mathbb{R}^n$, or the half-space $\mathbb{H}^n$ for some n.

Proof. We begin by considering the solution to (2.2) in a Riemannian normal neighbourhood U of x. Let $r(y) = d_g(x, y)$ be the radial distance function defined in the normal neighbourhood, and suppose G is radially symmetric. Then there exists $f \in C^2(\mathbb{R}_+, \mathbb{R})$ so that $G(x, y) = f(r(y))$ and so that f satisfies the equation

$$\Delta_y^g \, f \circ r(y) = -V^{-1} + \delta_x \tag{2.6}$$

Recall that when applied to a composition of functions, the Laplacian yields the formula $\Delta^g(\phi \circ \psi) = -(\phi'' \circ \psi)\|\nabla\psi\|^2 + (\phi' \circ \psi)\Delta^g\psi$. Due to the definition of the Riemannian distance function, we can immediately see that $\|\nabla r\|^2 = 1$. Similarly, it is known from Riemannian geometry that $\Delta^g r = -\operatorname{Tr}\big(\nabla^2 r\big) = -\operatorname{Tr}(\nabla N)$, with N being the normal vector to ∂U. Recalling that $-\nabla N$ is the shape operator, we recognize this as an expression for the mean curvature H of ∂U. We have $\Delta^g r = H$. Considering Riemannian polar coordinates r, θ at x, we arrive at the following equation for f.

$$f''(r) + f'(r)H(r, \theta) = V^{-1} - \delta_x \tag{2.7}$$

This equation is only well defined if $f''(r)$ is independent of θ, which happens if and only if H is radially symmetric. But it is known that a manifold M has radially symmetric mean curvature if and only if it is locally harmonic [28], which means M is a compact rank-one spherically symmetric space.

Because we are interested in compact rank-one spherically symmetric Kähler manifolds, the obvious place to start is $\mathbb{CP}^2$.

2.1.1 The Green's Function Potential on Complex Projective Space

The complex projective plane admits a number of useful symmetries. Particularly, the isometry group of $\mathbb{CP}^2$ with the usual Fubini-Study metric is the special unitary group SU(3). The point particle dynamics on this manifold can then be simplified using Hamiltonian reduction. The Hamiltonian reductions of $(\mathbb{CP}^2)^2$ and $(\mathbb{CP}^2)^3$ with respect to this action have been studied extensively [11, 12], as well as the implications for point particle and point vortex dynamics.

In this section we will be interested in the restricted two-body problem on $\mathbb{CP}^2$ as a toy example of a complex Hamiltonian system. The first step is to derive the related Green's function. To do this we can exploit the symmetries of $\mathbb{CP}^2$. Recall that the injectivity radius $\text{inj}(M)$ of a manifold is defined as the largest value of $d(x, y)$ so that x and y are connected by a minimizing geodesic. A Blaschke manifold is defined as follows.

Definition 2.4 (Blaschke Manifold)**.** Let M be a closed Riemannian manifold.

Then M is said to be Blaschke if diameter$(M) = \text{inj}(M)$.

One also needs the volume density function in order to calculate the Green's function. This is defined by,

$$\omega_x(y) = \sqrt{\det h|_{\exp_x^{-1}(y)}} \tag{2.8}$$

Definition 2.5 (Locally Harmonic Manifold)**.** Let M be a Riemannian manifold and let $\omega_x(y)$ be the volume density on M. We say M is locally harmonic at x if for all $\varepsilon > 0$ there exists a function $\Omega_x : [0, \varepsilon) \to \mathbb{R}$ so that $\omega_x(y) = \Omega_x(d(x, y))$ for all $y \in B_\varepsilon(x)$. If this property holds at all x, M is called locally harmonic.

Theorem 2.3. [29, Theorem 2.17] Let M be an n-dimensional locally harmonic Blaschke manifold. The Green's function for the Laplacian on M is given by $G(x, y) = \phi(d(x, y))$, where

$$\phi'(r) = -\frac{1}{\text{Vol}(M)} \frac{\int_r^{\text{inj}(M)} v(t)\,\mathrm{d}t}{v(r)}, \tag{2.9}$$

and $v(r) = \text{Vol}(S^{n-1}) r^{n-1} \Omega(r)$.

For $\mathbb{CP}^2$, the distance function is given by the great circle distance,

$$d(z, w) = \arccos\left(\sqrt{\frac{Z_\alpha \overline{W}^\alpha W_\beta \overline{Z}^\beta}{|Z|^2 |W|^2}}\right), \tag{2.10}$$

where Z and W are any homogenous coordinate representatives of z and w. The volume density function is determined by $\Omega(r) = 8 \sin^3 r \cos r$ [29]. We therefore

calculate the Green's function on $\mathbb{CP}^2$ by solving (2.9), yielding

$$G(z,w) = \frac{1}{8V}\left(\frac{1}{1-\frac{Z_\alpha \overline{W}^\alpha W_\beta \overline{Z}^\beta}{|Z|^2|W|^2}} - \ln\left(1 - \frac{Z_\alpha \overline{W}^\alpha W_\beta \overline{Z}^\beta}{|Z|^2|W|^2}\right)\right) \tag{2.11}$$

The restricted two-body problem has one particle fixed. We can choose this point arbitrarily, so we will take it to be $W = [1:0:0]$. Then the total Hamiltonian for a single particle at position $Z = [1 : z^1 : z^2]$ is,

$$\begin{aligned} H = &\frac{1}{2m_1(|z|^2+1)^2}\left((1+|z^1|^2)|p_{z1}|^2 + (1+|z^2|^2)|p_{z2}|^2 + z_1\overline{z}^2 p_{z^2}p_{\overline{z}^1} + z^2\overline{z}^1 p_{z^1}p_{\overline{z}^2}\right) \\ &+ \frac{km_1m_2}{4\pi^2}\left(\frac{1}{|z|^2} + \ln\left(1+|z|^{-2}\right)\right) \end{aligned} \tag{2.12}$$

The Green's function is known to give rise to Bertrand systems in the three-dimensional case [2]. Therefore it is the natural potential energy function to choose when beginning to study integrable systems on a general manifold.

To investigate the integrability of this system, it is useful to understand the symmetries of the Hamiltonian. Using a large enough symmetry group, we hope to construct a reduced space called the symplectic reduction.

2.1.2 Special Unitary Group Symmetry

Let $C_1 = \{[0 : Z_2 : Z_3] : Z_2, Z_3 \in \mathbb{C}\} \cup [1:0:0]$. There is an SU(2) action on $T^{\mathbb{C}*}(\mathbb{CP}^2 \setminus C_1)$ which leaves this Hamiltonian invariant. It is induced by the following action of SU(2) on $\mathbb{CP}^2$. Let $U \in \mathrm{SU}(2)$, with matrix representation given

by

$$[U] = \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1$$

The formula for the SU(2) action on $\mathbb{CP}^2$ is,

$$U \cdot [z^1 : z^2 : Z_3] = [z^1 : \alpha z^2 - \overline{\beta} Z_3 : \beta z^2 + \overline{\alpha} Z_3] \tag{2.13}$$

We can see that this is independent of z^1, which means there is a single fixed point at $[1 : 0 : 0]$. Luckily, in the aforementioned restricted two-body problem the Green's function has a singularity at $|(z^1, z^2)| = 0$, meaning the group action is free on the entire domain of H, which is all we need to define dynamics. If we lift this to an action on $T^{\mathbb{C}*}(\mathbb{CP}^2 \setminus C_1)$, the induced action on a covector p is from the cotangent lift of U, which pulls back p by $U^{-1} = U^\dagger$. So in affine coordinates the total action is given by,

$$U \cdot ((z^1, z^2), (p_1, p_2)) = ([U](z^1, z^2), [U]^\dagger (p_1, p_2)) \tag{2.14}$$

To compute the momentum map of this action, the quickest way is to define $\mathfrak{su}(2)$ as being generated by the imaginary unit quaternions. That is,

$$\mathfrak{su}(2) = \{\alpha i + \beta j + \gamma k : \alpha, \beta, \gamma \in \mathbb{R},\, ijk = -1,\, i^2 = j^2 = k^2 = -1\}. \tag{2.15}$$

The fundamental representation of $\mathbb{H}$ on $\mathbb{C}^2$ then provides us the following matrices.

$$[i] = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \qquad [j] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad [k] = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \tag{2.16}$$

In real coordinates, the first infinitesimal generator is,

$$X_i = x_1\frac{\partial}{\partial y_2} - y_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial y_1} - y_2\frac{\partial}{\partial x_1} - p_{x_1}\frac{\partial}{\partial p_{y_2}} + p_{y_1}\frac{\partial}{\partial p_{x_2}} - p_{x_2}\frac{\partial}{\partial p_{y_1}} + p_{y_2}\frac{\partial}{\partial p_{x_1}}$$

This calculation simplifies a bit in complex coordinates. In these coordinates, the associated infinitesimal generators are (up to an irrelevant normalization factor)

$$\begin{aligned} X_i &= \mathrm{Im}\left(z^2\partial_{z^1} + z^1\partial_{z^2} + p_{z^2}\partial_{p_{z^1}} + p_{z^1}\partial_{p_{z^2}}\right) \\ X_j &= \mathrm{Im}\left(z^2\partial_{z^1} - z^1\partial_{z^2} + p_{z^2}\partial_{p_{z^1}} - p_{z^1}\partial_{p_{z^2}}\right) \\ X_k &= \mathrm{Im}\left(z^1\partial_{z^1} - z^2\partial_{z^2} - p_{z^1}\partial_{p_{z^1}} + p_{z^2}\partial_{p_{z^2}}\right) \end{aligned} \tag{2.17}$$

Finally, we compute the momentum map associated with the SU(2) action by computing $X_\xi \lrcorner\, \lambda$.

Theorem 2.4 (Momentum map for SU(2) on $T^*\,\mathbb{CP}^2$)**.** The map $\mu : T^{\mathbb{C}\,*}(\mathbb{CP}^2 \setminus C_1) \to \mathfrak{su}^*(2)$ defined by

$$\mu(z^1, z^2, p_{z^1}, p_{z^2}) = \frac{1}{2}\begin{bmatrix} \mathrm{Im}(z^2 p_{\bar{z}^1} + z^1 p_{\bar{z}^2}) \\ \mathrm{Im}(z^2 p_{\bar{z}^1} - z^1 p_{\bar{z}^2}) \\ \mathrm{Im}(z^1 p_{\bar{z}^1} - z^2 p_{\bar{z}^2}) \end{bmatrix} \tag{2.18}$$

is a momentum map for the action (2.13).

Proof. We compute for example,

$$X_i \lrcorner\, \lambda = \frac{1}{2i}\left(z^2\partial_{z^1} + z^1\partial_{z^2} + p_{z^2}\partial_{p_{z^1}} + p_{z^1}\partial_{p_{z^2}}\right. \tag{2.19}$$

$$\left. -\overline{z}^2\partial_{\bar{z}^1} - \overline{z}^1\partial_{\bar{z}^2} - p_{\bar{z}^2}\partial_{p_{\bar{z}^1}} - p_{\bar{z}^1}\partial_{p_{\bar{z}^2}}\right) \lrcorner\, \frac{1}{2}\sum_{i=1}(z^i \mathrm{d}p_{\bar{z}^i} + \overline{z}^i \mathrm{d}p_{z^i}) \tag{2.20}$$

$$= \frac{1}{4i}(\overline{z}^2 p_{z^1} + \overline{z}^1 p_{z^2} - z^1 p_{\bar{z}^2} - z^2 p_{\bar{z}^2}) \tag{2.21}$$

$$= \frac{1}{2}\,\mathrm{Im}(z^2 p_{\bar{z}^1} + z^1 p_{\bar{z}^2}) \tag{2.22}$$

The rest are similar. The inverse image of each group orbit, $\mu^{-1}(\mathcal{O}_{(z^1, z^2, p_{z^1}, p_{z^2})})$, is an invariant set for the Hamiltonian system. Unfortunately, progress in understanding the equations of the level sets algebraically is difficult. In the future it could be useful to bound the maximum value of $|z|$ attained by the orbit using these equations, given some initial values for the momenta p_{z^1}, p_{z^2}.

2.1.3 Torus Group Symmetry

The T^2 torus action on $\mathbb{CP}^2$ has a well known momentum map, whose image in $\mathfrak{g}^*$ is known as the Delzant polytope of $\mathbb{CP}^2$. However, we are interested in an

action induced on the cotangent bundle. This is an action on an 8 real-dimensional manifold, so the momentum map doesn't completely reduce the system. The action on $\mathbb{CP}^2$ is given by

$$(\theta, \varphi) \cdot [z^1 : z^2 : Z_3] = [z^1 : e^{i\theta} z^2 : e^{i\varphi} Z_3] \tag{2.23}$$

This induces an action on $T^{\mathbb{C}*}(\mathbb{CP}^2 \setminus C_1)$ via the cotangent lift. The generators of this action are just $\mathrm{diag}(i, 0, -i, 0)$ and $\mathrm{diag}(0, i, 0, -i)$, which give us the infinitesimal generators,

$$\begin{aligned} X_\theta &= \mathrm{Im}\left(z^1 \frac{\partial}{\partial \overline{z}^1} - p_{z^1} \frac{\partial}{\partial p_{\overline{z}^1}} \right) \\ X_\varphi &= \mathrm{Im}\left(z^2 \frac{\partial}{\partial \overline{z}^2} - p_{z_2} \frac{\partial}{\partial p_{\overline{z}^1}} \right) \end{aligned} \tag{2.24}$$

Theorem 2.5 (Momentum map for T^2 on $T^*\,\mathbb{CP}^2$)**.** A momentum map for the action induced on $T^{\mathbb{C}*}(\mathbb{CP}^2 \setminus C_1)$ by (2.23) is given by,

$$\mu(z^1, z^2, p_{z^1}, p_{z^2}) = \frac{1}{2} \begin{bmatrix} \mathrm{Im}(z^1 p_{z^1}) \\ \\ \mathrm{Im}(z^2 p_{z^2}) \end{bmatrix}. \tag{2.25}$$

Notice that the SU(2) action considered in the previous section coincides with this action if and only if $\theta = -\varphi$, since elements of SU(2) must have unit determinant. This means their combined actions reduce to an action of $\mathrm{SU}(2) \times S^1 = \mathrm{U}(2)$ on phase space.

Theorem 2.6 (Momentum map for U(2) on $T^*\,\mathbb{CP}^2$)**.** A momentum map for the action of U(2) on $T^{\mathbb{C}\,*}(\mathbb{CP}^2 \setminus C_1)$ is given by,

$$\mu(z^1, z^2, p_{z^1}, p_{z^2}) = \frac{1}{2}\begin{bmatrix} \mathrm{Im}(z^2 p_{\bar{z}^1} + z^1 p_{\bar{z}^2}) \\ \mathrm{Im}(z^2 p_{\bar{z}^1} - z^1 p_{\bar{z}^2}) \\ \mathrm{Im}(z^1 p_{\bar{z}^1} - z^2 p_{\bar{z}^2}) \\ \mathrm{Im}(z^1 p_{z^1} + z^2 p_{z^2}) \end{bmatrix} \tag{2.26}$$

Proof. The Lie algebra $\mathfrak{u}(2)$ is generated by the imaginary quaternions as well as $i\,\mathbb{1}$. The infinitesimal generator associated with $i\,\mathbb{1}$ is exactly the sum of the vector fields generated by the two generators of T^2. Therefore, the component of the momentum map associated with that generator is the sum of the two components of the T^2 momentum map.

Since U(2) is the stabilizer of $[1:0:0]$ in SU(3), which is the isometry group of $\mathbb{CP}^2$, we can see that all of the usual symmetries have been exhausted. Thus it may be possible to show that no more integrals of motion exist. Numerical evidence suggests that some orbits are not constrained to a two-dimensional real submanifold of $\mathbb{CP}^2$, so it is likely that this is the case.

2.1.4 Numerical Solutions

Numerical solutions to the system of equations are given on the following pages. The numerical solutions were calculated using the implicit partitioned Runge-Kutta method with Gauss coefficients provided by Mathematica 13.0 [30], which is the

recommended method for some non-separable Hamiltonian systems. Indeed, we found that the energy did not drift significantly in the numerical solutions.

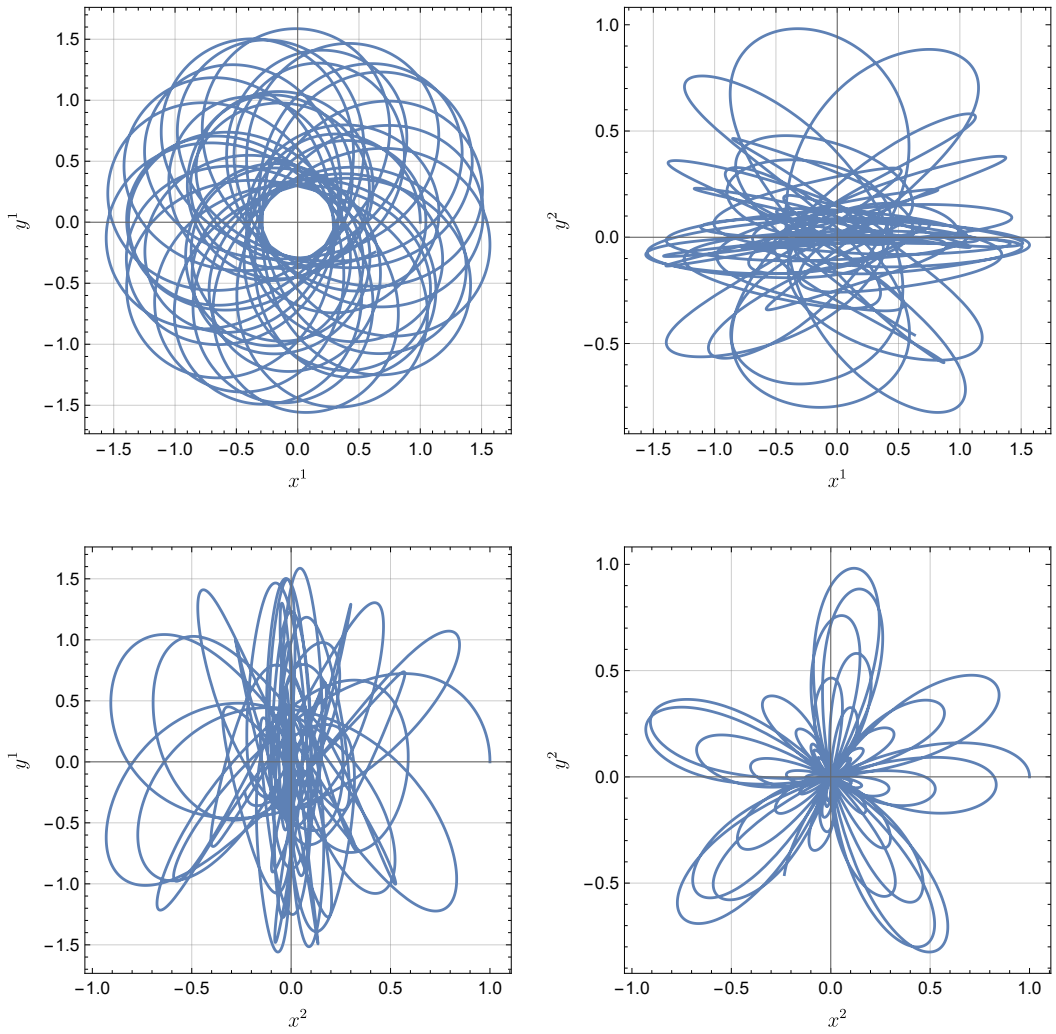

Figure 1: Particle trajectory in $\mathbb{CP}^2$, given the initial conditions $z^1 = 1, z^2 = 1, p_{z^1} = i, p_{z^2} = 0$ for $0 \leq t \leq 40$.

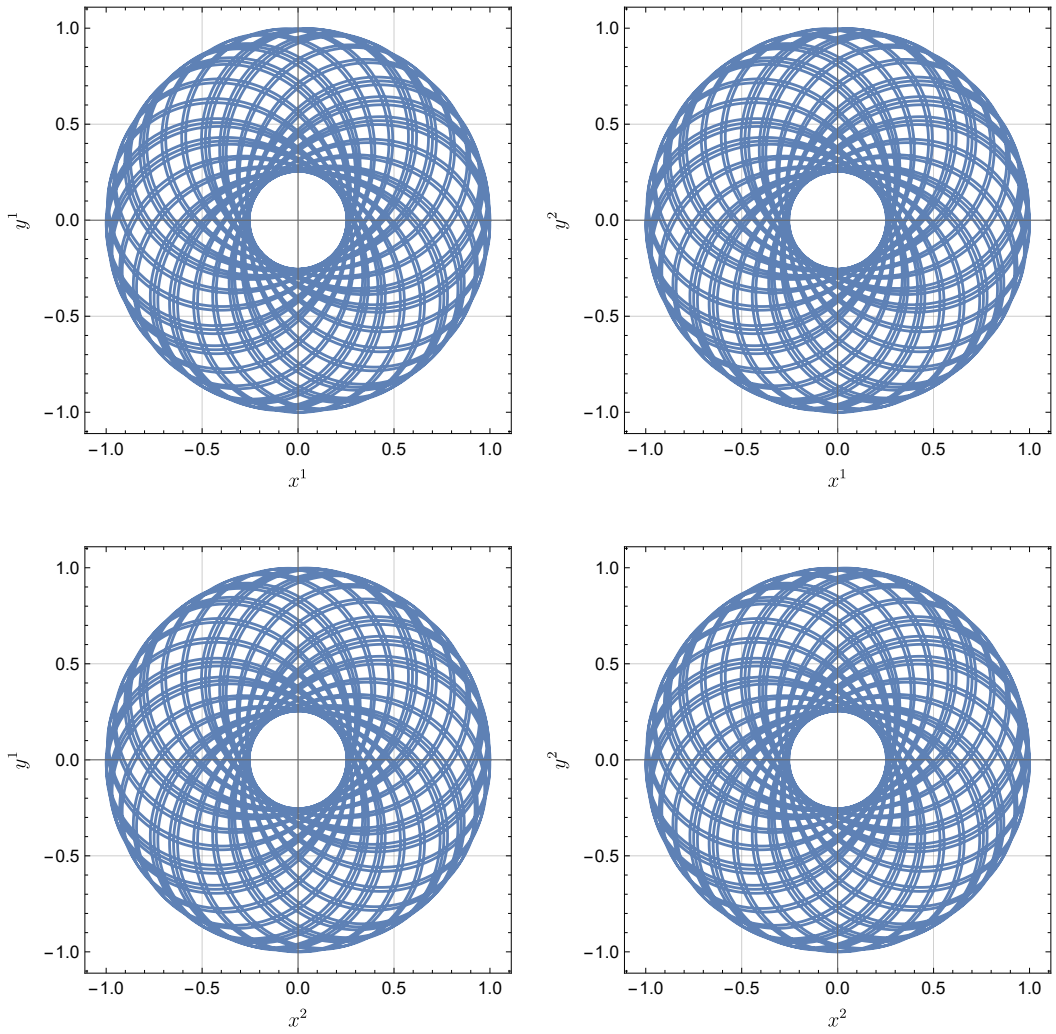

Figure 2: Particle trajectory in $\mathbb{CP}^2$, given the initial conditions $z^1 = 1, z^2 = 1, p_{z^1} = i, p_{z^2} = i$ for $0 \leq t \leq 40$.

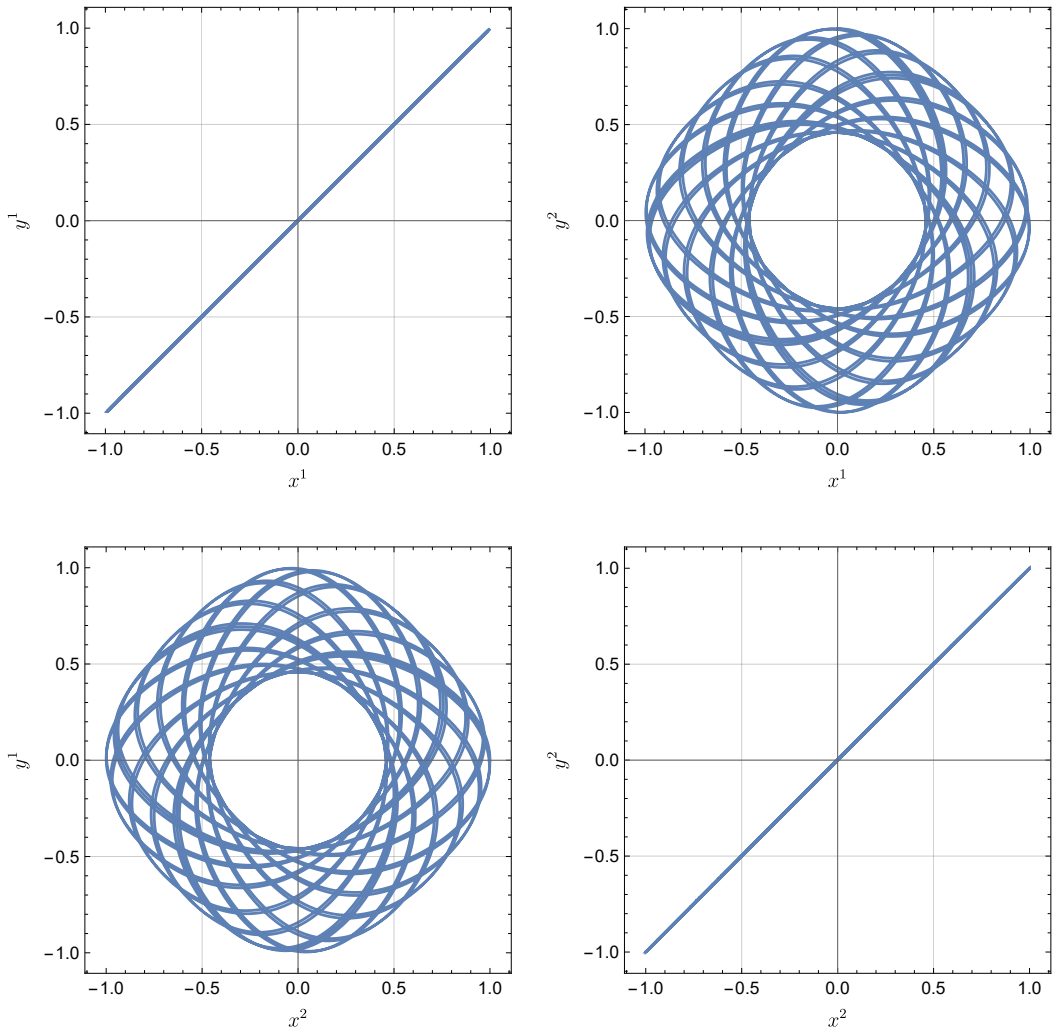

Figure 3: Particle trajectory in $\mathbb{CP}^2$, given the initial conditions $z^1 = 0, z^2 = 1 + i, p_{z^1} = 1 + i, p_{z^2} = 0$ for $0 \leq t \leq 40$.

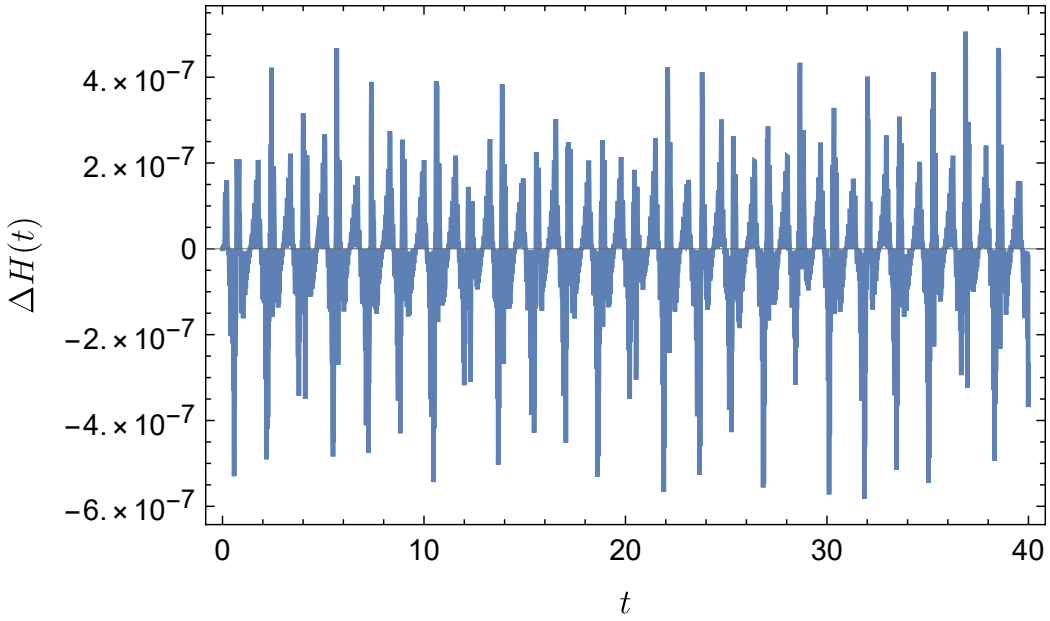

Figure 4: Energy drift for a numerical solution to the $\mathbb{CP}^2$ system, given the initial conditions $z^1 = 2, z^2 = 0, p_{z^1} = 0, p_{z^2} = 0.01i$ for $0 \leq t \leq 40$.

2.2 Kepler Problem on Surfaces of Revolution in Minkowski Space

2.2.1 Background

The Kepler problem on surfaces of revolution in $\mathbb{R}^3$ given the usual Riemannian metric is well studied [8, 6, 9, 31, 7]. The problem has also been introduced to the pseudo-Riemannian case and a characterization of Bertrand surfaces of revolution in $\mathbb{R}^{2,1}$ with a time-like axis of revolution is known [15]. However, many computations which have been performed in the Riemannian case have yet to be extended to the pseudo-Riemannian case. The setting is defined as follows.

Definition 2.6 (Lorentz-Minkowski Space)**.** The Lorentz-Minkowski Space $\mathbb{R}^{2,1}$ is the vector space $\mathbb{R}^3 = \{(x, y, t)\}$ equipped with the metric

$$\eta = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}t^2$$

Definition 2.7 (Space-like/Time-like/Null Submanifolds of $\mathbb{R}^{2,1}$)**.** A pseudo-Riemannian submanifold of $\mathbb{R}^{2,1}$ is a submanifold M of $\mathbb{R}^3$ so that the Minkowski metric η restricts to a pseudo-Riemannian metric on $TM \subseteq T\,\mathbb{R}^3$. A null submanifold of $\mathbb{R}^{2,1}$ is a submanifold of M so that η restricts to a degenerate bilinear form on TM. Some types of submanifold are as follows.

1. Time-like if for all $v \in T_pM$, $\eta(v, v) < 0$ unless $v = 0$
2. Space-like if for all $v \in T_pM$, $\eta(v, v) > 0$ unless $v = 0$
3. Null if there exists some non-vanishing vector field $v \in \mathfrak{X}(M)$ so that $\eta(v, v) = 0$ for all $p \in M$

To construct rotational surfaces in Minkowski space we construct the orbit of the image of a differentiable curve $\gamma : I \to \mathbb{R}^{2,1}$ under a subgroup of the isometry group,

$$\mathrm{SO}(2, 1) = \{A \in M_{3\times 3}(\mathbb{R}) : A^tA = \mathbb{1}, \det A = 1\}. \tag{2.27}$$

Here A^t is the metric-dual operator defined by the equation

$$\eta(A\mathbf{x}, \mathbf{y}) = \eta(\mathbf{x}, A^t\mathbf{y}) \;\; \forall \mathbf{x}, \mathbf{y} \in T_p\,\mathbb{R}^{2,1}\,.$$

The isometry group has three important one-parameter subgroups, a one parameter group of rotations in the $x - y$ plane, S^1, the group of hyperbolic translations,

$\mathrm{SO}(1,1)$, and a group of null rotations. The surfaces generated by the hyperbolic translations are called rotational surfaces, to emphasize that they are not the same kind of surface as a usual surface of revolution. We will focus on surfaces generated by the S^1 action. The axis of rotation can be taken to be a null, space-like, or time-like one-dimensional submanifold of $\mathbb{R}^{2,1}$.

Definition 2.8 (Surfaces of Revolution with a Space-like/Time-like/Null Axis of Revolution)**.** We begin with a planar curve $(a(s), b(s))$ in either the $x-y$ or $y-t$ plane and consider the orbit of the curve under the action of a subgroup of the isometry group. Any rotational surfaces can be parametrized by either $s \in (a,b)$ and $\theta \in S^1$ in the time-like case, or $\theta \in \mathbb{R}$ in the space-like or null cases. This results in one of the following surfaces. [14]

1. Surfaces of revolution with a space-like axis can be written in the form

$$(x, y, t) = (a(s), b(s)\sinh\theta, b(s)\cosh\theta) \tag{2.28}$$

2. Surfaces of revolution with a time-like axis be written in the form

$$(x, y, t) = (a(s)\cos\theta, a(s)\sin\theta, b(s)) \tag{2.29}$$

3. Surfaces of revolution with a null axis can be written in the form

$$(x, y, t) = \left(-\theta(a(s)-b(s)), a(s) - \frac{\theta^2}{2}(a(s)-b(s)), b(s) - \frac{\theta^2}{2}(a(s)-b(s))\right) \tag{2.30}$$

Where we take the two functions $a(s), b(s)$ to satisfy $a'(s)^2 - b'(s)^2 > 0$ as in

[14].

Theorem 2.7. The metric on a pseudo-Riemannian surface of revolution is,

1. for a surface with a space-like axis:

$$\eta|_M = (a'(s)^2 - b'(s)^2)\mathrm{d}s^2 + a(s)^2\mathrm{d}\theta^2 \tag{2.31}$$

2. for a surface with a time-like axis:

$$\eta|_M = (a'(s)^2 - b'(s)^2)\mathrm{d}s^2 - b(s)^2\mathrm{d}\theta^2 \tag{2.32}$$

3. for a surface with a null axis:

$$\eta|_M = (a'(s)^2 - b'(s)^2)\mathrm{d}s^2 + (a(s) - b(s))^2\mathrm{d}\theta^2 \tag{2.33}$$

Proof. The space-like and time-like cases are straightforward. For the null case, we have

$$\begin{aligned}
\eta|_M &= \theta^2(a'(s) - b'(s))^2\mathrm{d}s^2 + (a(s) - b(s))^2\mathrm{d}\theta^2 \\
&\quad + \theta^2(a(s) - b(s))^2\mathrm{d}\theta^2 + (a'(s) - \frac{\theta^2}{2}(a'(s) - b'(s)))^2\mathrm{d}s^2 \\
&\quad - (b'(s) - \frac{\theta^2}{2}(a'(s) - b'(s)))^2\mathrm{d}s^2 - \theta^2(a(s) - b(s))^2\mathrm{d}\theta^2 \\
&= (a'(s)^2 - b'(s)^2)\mathrm{d}s^2 + (a(s) - b(s))^2\mathrm{d}\theta\cdot
\end{aligned}$$

2.2.2 Surfaces with a Time-Like Axis

From here on any surface of revolution with a time-like axis will be referred to as S_{timelike}. The Lagrangian of a particle on S_{timelike} is the same as the case of a

Riemannian manifold, just with a different metric signature. It is,

$$L = \frac{1}{2}m\left((a'(s)^2 - b'(s)^2)\dot{s}^2 - b(s)^2\dot{\theta}^2\right) - V(s). \tag{2.34}$$

From this we derive the equations of motion in Lagrangian form. Beginning with the Euler-Lagrange equation $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$, we get

$$(a'(s)^2 - b'(s)^2)\ddot{s} + (a(s)a'(s) - b(s)b'(s))\dot{s}^2 + b(s)\dot{\theta}^2 + \frac{1}{s}\frac{\partial V}{\partial s} = 0 \tag{2.35}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(mb(s)^2\dot{\theta}\right) = 0 \tag{2.36}$$

We immediately get two conserved quantities H and $p_\theta = mb(s)^2\dot{\theta}$ which are in involution (i.e. $\{H, p_\theta\} = 0$). Since the system is two-dimensional, the presence of two conserved quantities means the system is Louville integrable. It is also useful to mention a particular change of coordinates which sometimes simplifies the above metric. Observe that $a'(s)^2 - b'(s)^2 > 0$ wherever the metric is well defined, since we assume the signature is invariant. Let us take $u(s) = \int \sqrt{a'(s)^2 - b'(s)^2}\mathrm{d}s$. Set $f(u_0)^2 = b(u^{-1}(u_0))^2$. Then the metric is of the form,

$$\eta|_M = \mathrm{d}u^2 - f(u)^2\mathrm{d}\theta^2 \tag{2.37}$$

Zagryadskii [7, 15] finds that in order for two closing potentials (Kepler-like and oscillator-like) to exist on such a surface, there should exist coordinates where the metric is of the form

$$\eta_1|_M = \mathrm{diag}\left(\frac{\mu^2}{(s^2 - c)^2}, \frac{1}{s^2 - c}\right) \tag{2.38}$$

However, if one only requires the existence of a single closing potential, then it was

found that one can use a more general metric of the form,

$$\eta_2|_M = \mathrm{diag}\left(\frac{\mu^2}{(s^2 - c - ts^{-2})^2}, \frac{1}{s^2 - c - ts^{-2}}\right). \tag{2.39}$$

Different values for $c, \mu, t \in \mathbb{R}$ result in different possible potential energy functions. This metric can be put into the form (2.37) with an appropriate change of variables. Let us first justify why this choice of metric indeed results in Bertrand surfaces. We begin with the Lagrangian on an arbitrary surface, using the simplified metric (2.37).

$$L = \frac{1}{2}m\left(\dot{u}^2 - f(u)^2\dot{\theta}^2\right) - V(u) \tag{2.40}$$

The conserved momentum is $p_\theta = mf(u)^2\dot{\theta}$. So the effective potential is,

$$V_{\text{eff}}(u) = -\frac{1}{2mf(u)^2}p_\theta^2 - V(u) \tag{2.41}$$

Therefore if we ignore θ as a dynamical variable this system is effectively the same as a one-dimensional system on $\mathbb{R}$ with potential $V_{\text{eff}}(u)$, so the proof used in the classical case applies. Santoprete [3] showed that in the Riemannian case where we have the metric $\mathrm{d}u^2 + f(u)^2\mathrm{d}\theta^2$, the function f must satisfy (2.42) below.

$$\beta^4 - 5(-f''f + (f')^2)\beta^2 - 5f''(f')^2f + 4(f'')^2f^2 - 3f'''f'f^2 + 4(f')^4 = 0 \tag{2.42}$$

In the pseudo-Riemannian case, the metric is therefore the same except with $f \mapsto if$.

$$-\beta^4 - 5(-f''f + (f')^2)\beta^2 - 5f''(f')^2f + 4(f'')^2f^2 - 3f'''f'f^2 + 4(f')^4 = 0 \tag{2.43}$$

For the case where the constant $t = 0$, the metric (2.39) can be written in the form

(2.37) where we have,

$$u = -\frac{\mu}{\sqrt{c}} \operatorname{arctanh}\left(\frac{s}{\sqrt{c}}\right) + D, \tag{2.44}$$

$$f_{\eta_2}(u) = \frac{1}{C}\cosh\left(\frac{C\beta}{2}(u+K)\right). \tag{2.45}$$

For the case where $t \neq 0$, a tractable expression for $f(u)$ can not be obtained. Luckily, the coordinates (s, θ) lead to a more analytically tractable orbit equation as we will show in a following section.

2.2.3 Numerical Solutions

It is useful when performing analytic calculations to perform numerical experiments to guide one's intuition. Numerical methods for time-independent Hamiltonian systems are generally required to conserve energy to some degree. That is, the total energy H of the system should not drift too much as the system is evolved numerically. This property is satisfied by a certain class of numerical methods called symplectic methods.

Suppose we have a dynamical system defined by some system of ordinary differential equations on a manifold M. The fundamental solution to the system is then a flow operator Φ_t on M which we would like to approximate numerically. A numerical method for calculating Φ_t is a discrete dynamical system, i.e. a map $\varphi_\delta : M \to M$, which takes a point $p \in M$ and approximates the output of $\Phi_\delta(p)$ for some $\delta \in \mathbb{R}$. Recursively applying φ_δ to an initial point p_0 results in a sequence of points $p_n, n \in \mathbb{N}$ which approximate an orbit of Φ_t. For a Hamiltonian system, we

would like the sequence of points p_n to lie close to a level curve of the Hamiltonian H, and ideally to level sets of any other conserved quantities. The method φ_δ is said to be a symplectic method if φ_δ preserves the symplectic form. That is, if $\varphi_\delta^*\omega|_{q,p} = \omega|_{\varphi_\delta(q,p)}$. It is a well known feature of symplectic methods that when H is a separable Hamiltonian, $|H(q_n, p_n) - H(q_0, p_0)|$ is bounded by some ΔE and often quasi-periodic in n. If the value of ΔE is small enough, the method will result in accurate solutions to Hamilton's equations.

In general, symplectic methods are a robust way to bound how much a method violates conservation of energy. Unfortunately, non-separable Hamiltonian systems are not well behaved even when solved using ordinary symplectic methods, such as the Störmer-Verlet methods. Some efforts have been made to make the usual methods perform better when solving non-separable systems, such as partitioned Runge-Kutta methods. We found the partitioned Runge-Kutta method to be sufficient in simulating the non-superintegrable $\mathbb{CP}^2$ Kepler problem (2.12). However, in this case the specialized methods such as the methods of Pihajoki [32] and Tao [33] are required to achieve sufficient numerical stability.

The method of Pihajoki extends an earlier algorithm for solving Hamiltonian systems called the auxillary velocity algorithm to the case of non-separable Hamiltonians. This extended method lifts a non-separable Hamiltonian system from T^*M (with coordinates (q, p) to $(T^*M)^2$ (with coordinates $(q, p, \tilde{q}, \tilde{p})$) by defining the extended Hamiltonian $\tilde{H}(q, p, \tilde{q}, \tilde{p}) = H(q, \tilde{p}) + H(\tilde{q}, p)$. We label $H_1 = H(q, \tilde{p})$ and

$H_2 = H(\tilde{q}, p)$. Hamilton's equations then read

$$\dot{\tilde{q}} = \frac{\partial H_1}{\partial p}, \qquad \dot{p} = -\frac{\partial H_1}{\partial \tilde{q}}, \qquad \dot{q} = \frac{\partial H_2}{\partial \tilde{p}}, \qquad \dot{\tilde{p}} = -\frac{\partial H_2}{\partial q} \tag{2.46}$$

The solution to Hamilton's equations is abstractly given by the flow map $\Phi_t = \exp\big(tX_{\tilde{H}}\big)$. The idea of Pihajoki's method is then to approximate Φ_t with an alternating composition of flow maps derived from H_1 and H_2. For this we need the Baker-Campbell-Hausdorff formula,

$$\exp(\delta A)\exp(\delta B) = \exp\bigg(\delta(A+B) + \frac{\delta^2}{2}[A,B] + O(\delta^3)\bigg), \tag{2.47}$$

from which Pihajoki derives the formula

$$\exp\big(\delta X_{\tilde{H}}\big) = \exp\bigg(\frac{\delta}{2}X_{H_1}\bigg)\exp(\delta X_{H_2})\exp\bigg(\frac{\delta}{2}X_{H_1}\bigg) + O(\delta^3). \tag{2.48}$$

This is an example of a type of leapfrog method called a splitting method [34]. Expanding the exponential maps to first order in δ then yields the following three-

step method.

$$
\begin{aligned}
q_{n+1/2} &= q_n + \frac{\delta}{2}\frac{\partial H_2}{\partial p}(\tilde{q}_n, p_n) \\
\tilde{q}_{n+1/2} &= \tilde{q}_n + \frac{\delta}{2}\frac{\partial H_1}{\partial \tilde{p}}(q_n, \tilde{p}_n) \\
\tilde{p}_{n+1/2} &= \tilde{p}_n - \frac{\delta}{2}\frac{\partial H_2}{\partial q}(\tilde{q}_n, p_n) \\
p_{n+1/2} &= p_n - \frac{\delta}{2}\frac{\partial H_1}{\partial \tilde{q}}(q_n, \tilde{p}_n) \\
\tilde{q}_{n+1} &= \tilde{q}_n + \delta\frac{\partial H_1}{\partial \tilde{p}}(q_{n+1/2}, \tilde{p}_{n+1/2}) \\
q_{n+1} &= q_n + \delta\frac{\partial H_2}{\partial p}(\tilde{q}_{n+1/2}, p_{n+1/2}) \\
p_{n+1} &= p_n - \delta\frac{\partial H_1}{\partial q}(q_{n+1/2}, \tilde{p}_{n+1/2}) \\
\tilde{p}_{n+1} &= \tilde{p}_n - \delta\frac{\partial H_2}{\partial \tilde{q}}(\tilde{q}_{n+1/2}, p_{n+1/2}) \\
q_{n+1} &= q_{n+1/2} + \frac{\delta}{2}\frac{\partial H_2}{\partial p}(\tilde{q}_{n+1}, p_{n+1}) \\
\tilde{q}_{n+1} &= \tilde{q}_{n+1/2} + \frac{\delta}{2}\frac{\partial H_1}{\partial \tilde{p}}(q_{n+1}, \tilde{p}_{n+1}) \\
\tilde{p}_{n+1} &= \tilde{p}_{n+1/2} - \frac{\delta}{2}\frac{\partial H_2}{\partial \tilde{q}}(\tilde{q}_{n+1}, p_{n+1}) \\
p_{n+1} &= p_{n+1/2} - \frac{\delta}{2}\frac{\partial H_1}{\partial q}(q_{n+1}, \tilde{p}_{n+1})
\end{aligned}
\tag{2.49}
$$

This kind of method is called a leapfrog method due to the intermediate steps $q_{n+1/2}, p_{n+1/2}$. If an exact expression for the partial derivatives is known, this method is very computationally efficient.

One can show that the numerical solution is an exact solution of the related

Hamiltonian systems defined by,

$$H_{212} = H_1 + H_2 - \frac{1}{24}\delta^2(2\{\{H_2, H_1\}, H_1\} - \{\{H_1, H_2\}, H_2\}) + O(\delta^3) \tag{2.50}$$

$$H_{121} = H_1 + H_2 - \frac{1}{24}\delta^2(2\{\{H_1, H_2\}, H_2\} - \{\{H_2, H_1\}, H_1\}) + O(\delta^3), \tag{2.51}$$

where $\delta > 0$ is the time-step. By subtracting $\tilde{H}$ from each of these related Hamiltonians, we see that this algorithm conserves energy to order $O(\delta^2)$. The author demonstrated comparable performance to partitioned Runge-Kutta methods. Unfortunately, this method has the problem that the two resulting solutions can diverge from one another over a large enough range of time. An improved method was developed by Tao, which adds an additional term to the extended Hamiltonian given by

$$H_C = \frac{\omega}{2}\left(\|q - \tilde{q}\|^2 + \|p - \tilde{p}\|^2\right). \tag{2.52}$$

This term introduces what Tao refers to as phase space mixing. The modified method is then given by the following extended Hamiltonian,

$$\hat{H} = H_1(q, \tilde{p}) + H_2(\tilde{q}, p) + H_C(q, p, \tilde{q}, \tilde{p}) \tag{2.53}$$

The numerical method is then defined by discretizing the following exponential map,

$$\exp\left(\delta X_{\hat{H}}\right) = \exp\left(\frac{\delta}{2}X_{H_1}\right)\exp\left(\frac{\delta}{2}X_{H_2}\right)\exp(\delta X_{H_C})\exp\left(\frac{\delta}{2}X_{H_2}\right)\exp\left(\frac{\delta}{2}X_{H_1}\right) + O(\delta^2). \tag{2.54}$$

This gives us a five-step method. The only difference between this map and the one

defined by Pihajoki is the central term, whose discretization is given by

$$
\begin{aligned}
q &\mapsto \frac{1}{2}(q+\tilde{q}) + \cos(2\delta\omega)(q-\tilde{q}) + \sin(2\delta\omega)(p-\tilde{p}) \\
p &\mapsto \frac{1}{2}(p+\tilde{p}) - \sin(2\delta\omega)(q-\tilde{q}) + \cos(2\delta\omega)(p-\tilde{p}) \\
\tilde{q} &\mapsto \frac{1}{2}(q+\tilde{q}) - \cos(2\delta\omega)(q-\tilde{q}) - \sin(2\delta\omega)(p-\tilde{p}) \\
\tilde{p} &\mapsto \frac{1}{2}(p+\tilde{p}) + \sin(2\delta\omega)(q-\tilde{q}) - \cos(2\delta\omega)(p-\tilde{p})
\end{aligned}
$$

Numerical results demonstrate that for appropriate values of ω, the difference between the two solutions is quasi-periodic for small time scales. The above method will be denoted by φ_δ^2, as the error is second order in δ. A higher order method can be constructed by recursively computing $\varphi_\delta^\ell = \varphi_{\gamma\delta}^{\ell-2} \circ \varphi_{(1-2\gamma)\delta}^{\ell-2} \circ \varphi_{\gamma\delta}^{\ell-2}$, with $\gamma = 1/(2 - 2^{\ell+1})$. This results in a general ℓ-step algorithm of higher order. For integrable systems (such as the one under study), the general method has an error bound of order $O(T\delta^\ell\omega)$ for $t < T \sim O(\min(\delta^{-\ell}\omega^{-1}, \sqrt{\omega}))$. The algorithm described was implemented in Python 3 [35] using the just-in-time compiler package provided by Numba [36]. The main time-stepper function was based on the function provided in the supplementary materials of the preprint 'Nonseparable Symplectic Neural Networks' by Xiong et al. [37]. The results of numerical simulations imply that the system does indeed have closed orbits. Furthermore, the orbits are very close to epicycloids in shape.

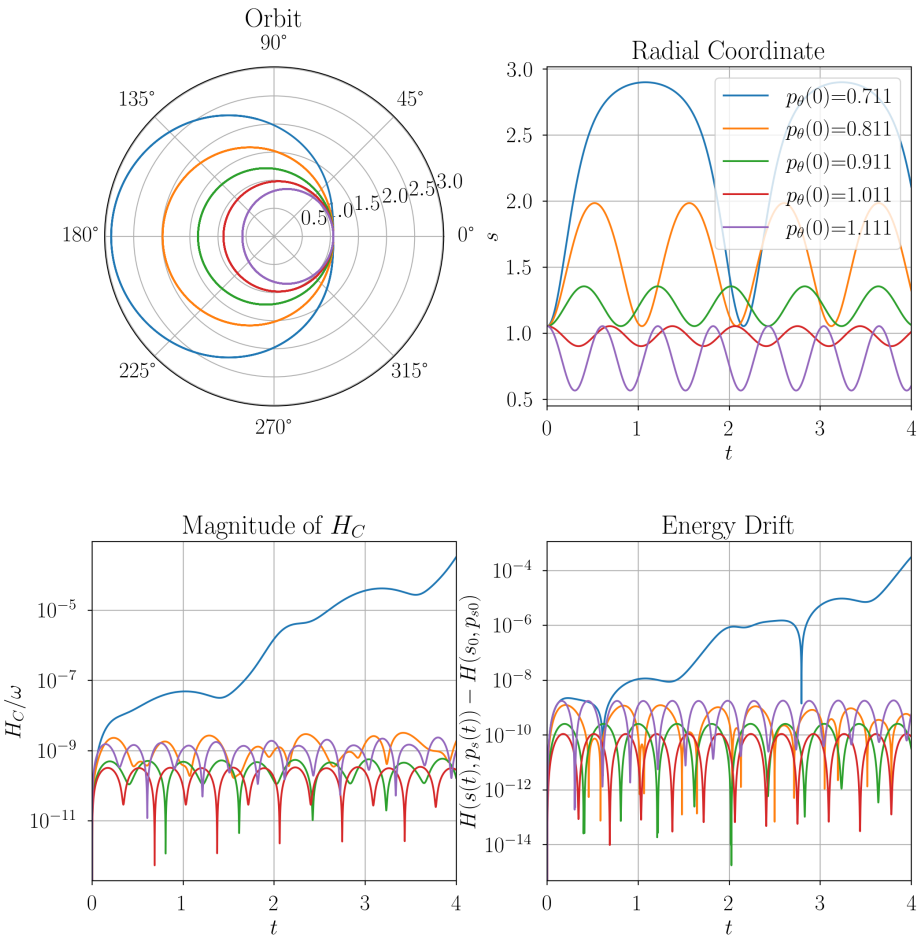

Figure 5: Orbits of the Hamiltonian system on S_{timelike} with $m = 1, \mu = 1, c = 10, t = 0, A = -1, B = 0, \beta = 1,\ s_0 = \sqrt{c}/3,\ \dot{s}_0 = 0$.

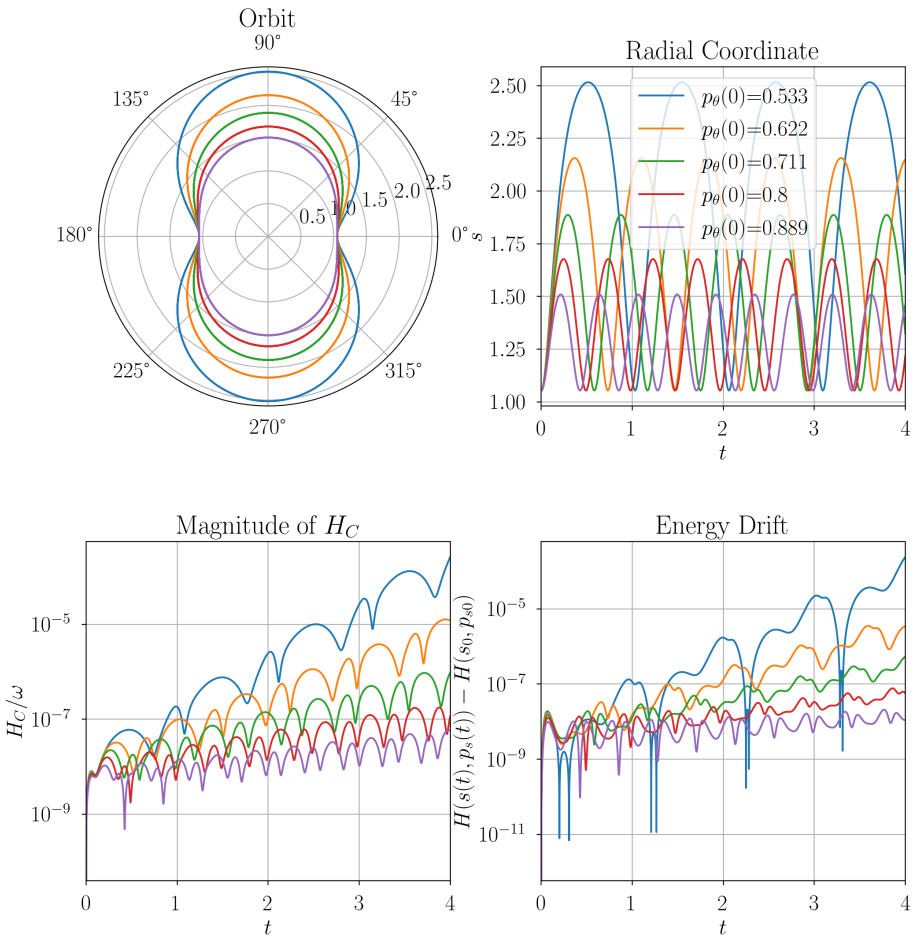

Figure 6: Orbits of the Hamiltonian system on S_{timelike} with $m = 1, \mu = 1, c = 10, t = 0, A = 1, B = 0, \beta = -2$, $s_0 = \sqrt{c}/3$, $\dot{s}_0 = 0$.

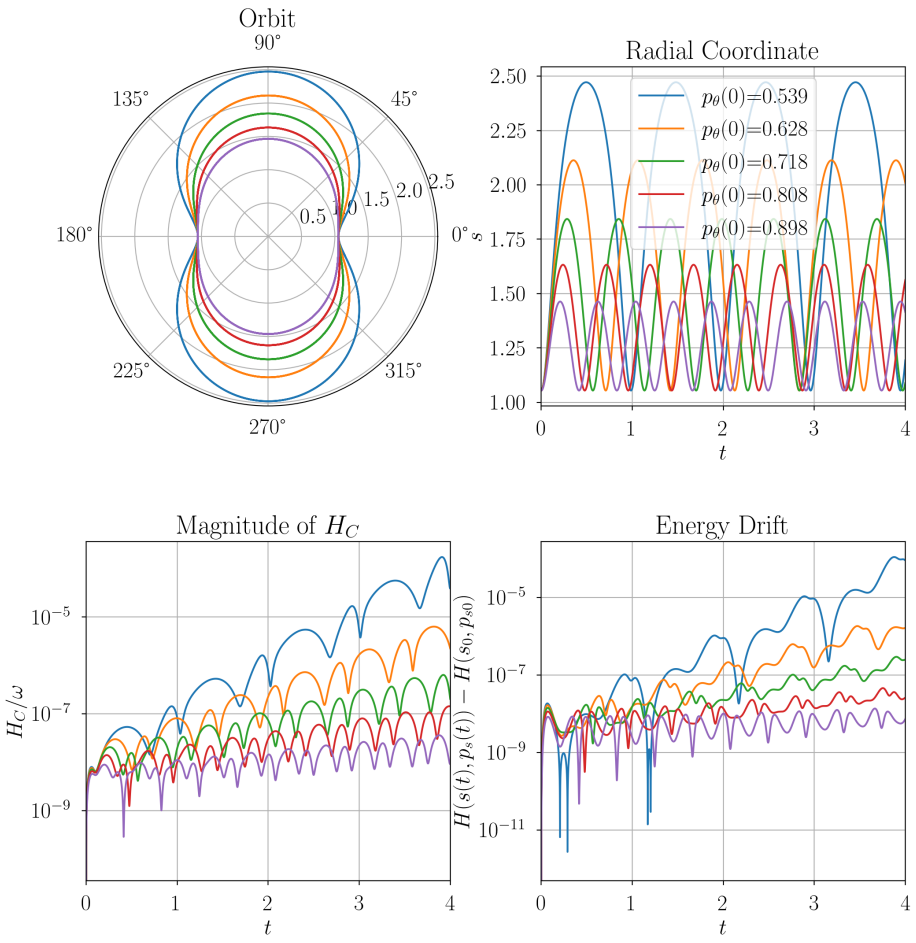

Figure 7: Orbits of the Hamiltonian system on S_{timelike} with $m = 1, \mu = 1, c = 10, t = 0.1, A = 1, B = 0, \beta = -2$, $s_0 = \sqrt{c}/3$, $\dot{s}_0 = 0$.

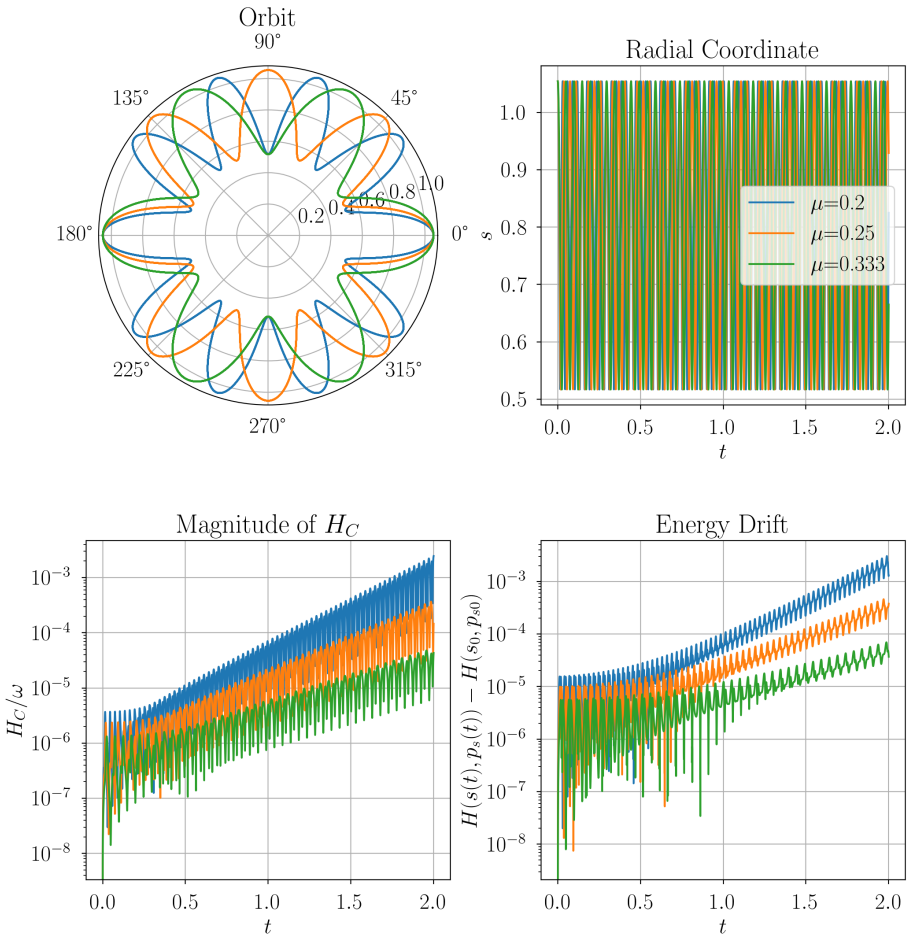

Figure 8: Orbits of the Hamiltonian system on S_{timelike} with $\mu = 1/5, 1/4, 1/3$ and $m = 1, c = 10, t = 0.1, A = 1, B = 0, \beta = -2$, $s_0 = \sqrt{c}/3$, $\dot{s}_0 = 0$, $\dot{\theta}_0 = 0.25$. This plot demonstrates how the number of lobes change with varying μ.

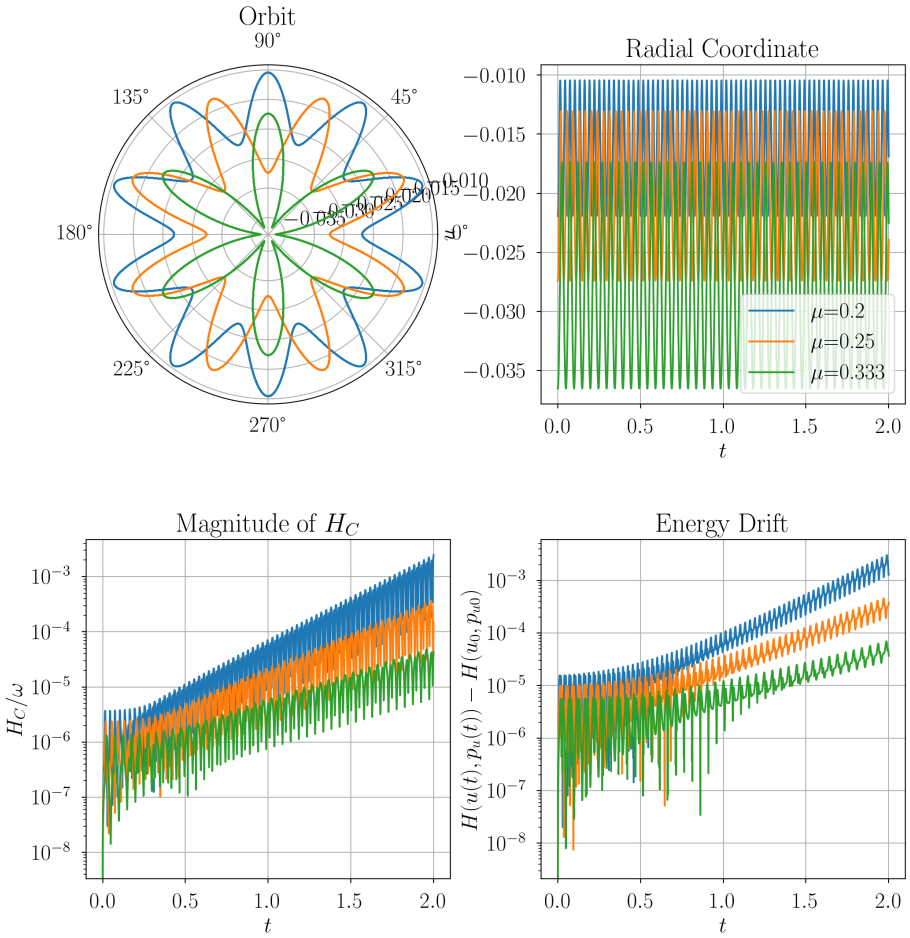

Figure 9: Orbits of the Hamiltonian system on S_{timelike} with $\mu = 1/5, 1/4, 1/3$ and $m = 1, c = 10, t = 0.1, A = 1, B = 0, \beta = -2,\ s_0 = \sqrt{c}/3,\ \dot{s}_0 = 0,\ \dot{\theta}_0 = 0.25$. The orbits here are shown in the (u, θ) coordinate system which converts the metric into the form (2.37).

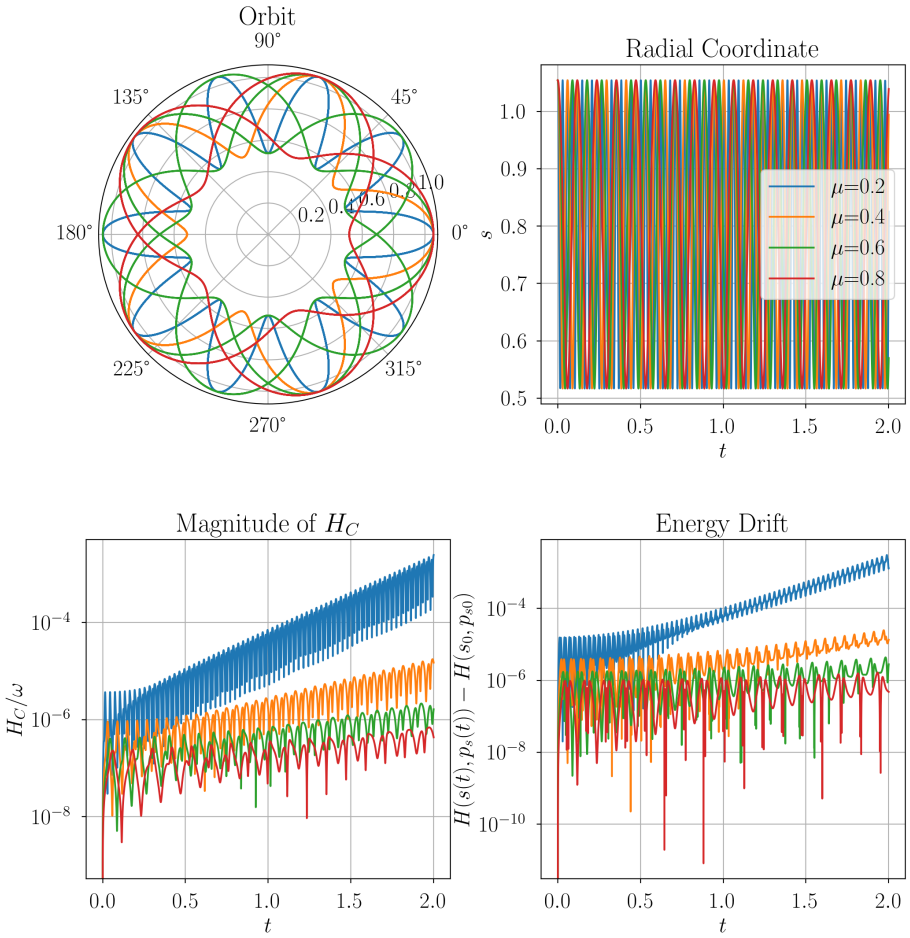

Figure 10: Orbits of the Hamiltonian system on S_{timelike} with $\mu = 1/5, 2/5, 3/5, 4/5$ and $m = 1, c = 10, t = 0.1, A = 1, B = 0, \beta = -2$, $s_0 = \sqrt{c}/3$, $\dot{s}_0 = 0$, $\dot{\theta}_0 = 0.25$. This plot demonstrates how the number of lobes is only determined by the denominator of μ in reduced form.

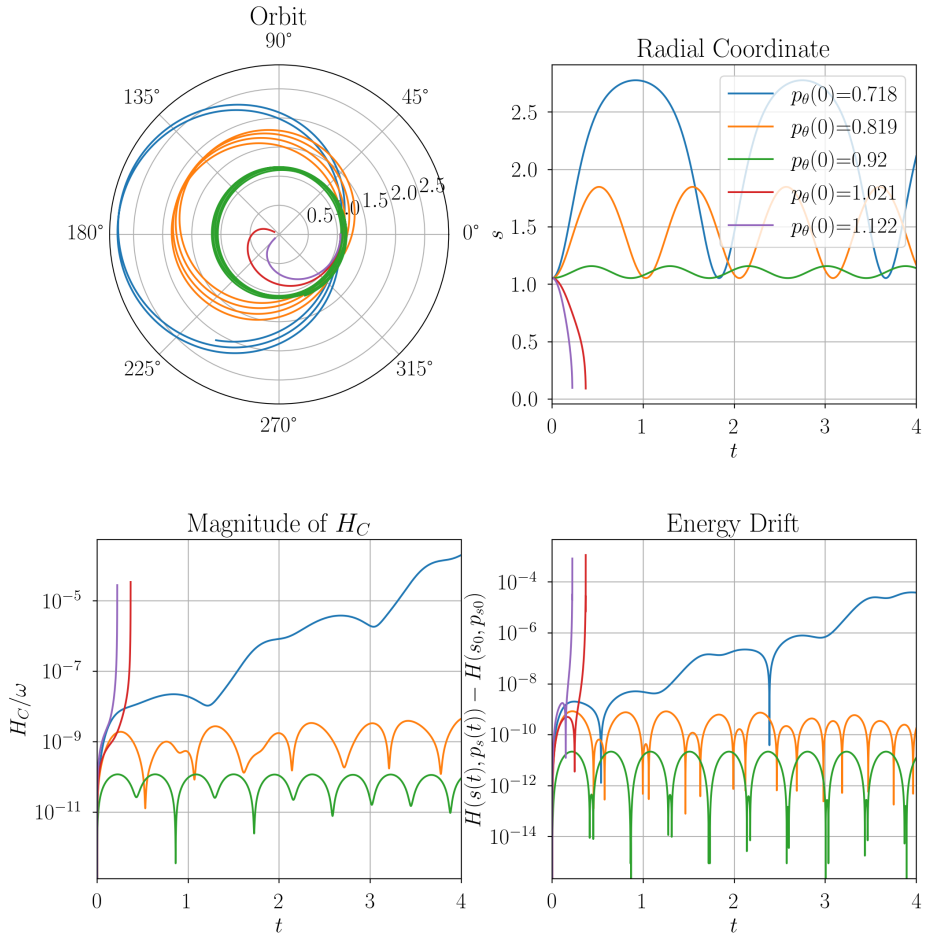

Figure 11: Orbits of the Hamiltonian system on S_{timelike} with $m = 1, \mu = 1, c = 10, t = 0.1, A = -1, B = 0, \beta = 1$, $s_0 = \sqrt{c}/3$, $\dot{s}_0 = 0$. This plot demonstrates how orbits with $t \neq 0$ do not close when $\beta = 1$. Additionally, the system exhibits numerical instabilities for some initial conditions.

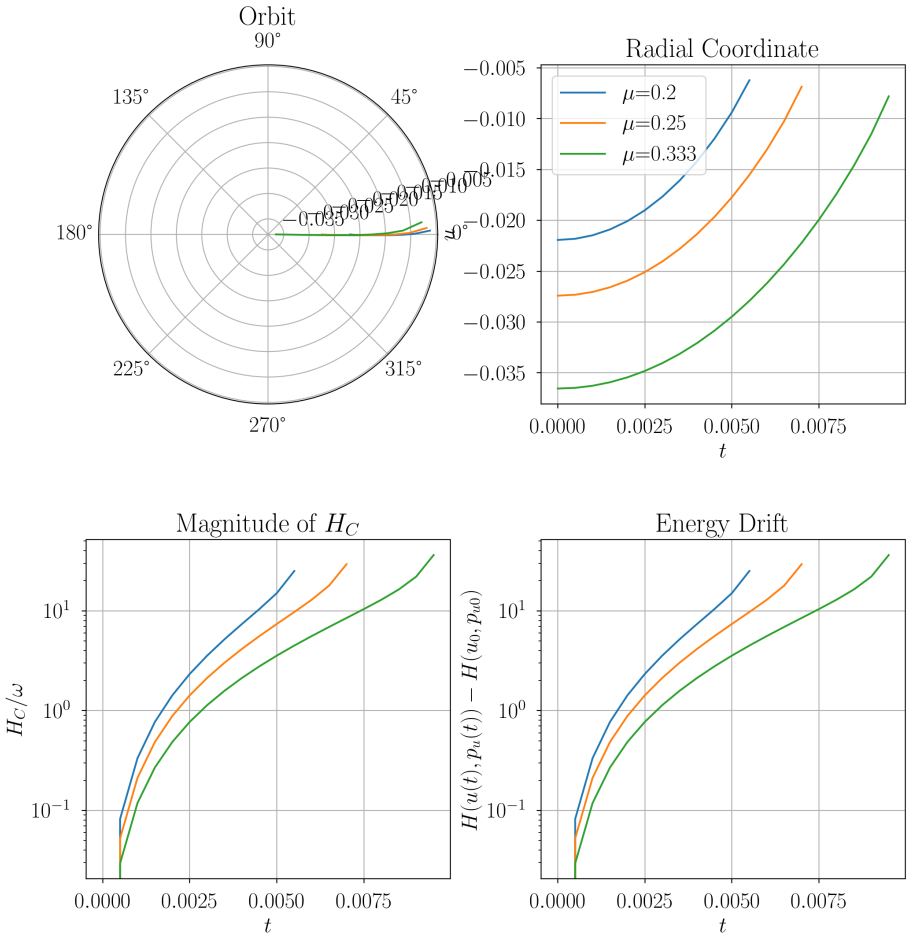

Figure 12: Orbits of the Hamiltonian system on S_{timelike} with $\mu = 1/5, 1/4, 1/3$ and $m = 1, c = 10, t = 0.1, A = 1, B = 0, \beta = -2$, $s_0 = \sqrt{c}/3$, $\dot{s}_0 = 0$, $\dot{\theta}_0 = 0.25$, created using an ordinary Runge Kutta method. This plot demonstrates how ordinary methods fail to conserve energy and are generally unstable.

2.2.4 Exact Expressions for the Orbit

Let us write down the Lagrangian corresponding to the metric in terms of s.

$$L = \frac{1}{2}m\left(\frac{\mu^2}{(s^2 - c - t/s^2)^2}\dot{s}^2 + \frac{1}{s^2 - c - t/s^2}\dot{\theta}^2\right) - V(s) \tag{2.55}$$

From this we derive the Euler-Lagrange equations.

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m}{s^2 - c - t/s^2}\dot{\theta}\right) = \frac{\mathrm{d}}{\mathrm{d}t}p_\theta = 0 \\ \frac{\mu^2}{(s^2 - c - t/s^2)^2}\ddot{s} + \frac{2\mu^2(t/s^3 + s))}{(s^2 - c - t/s^2)^3}\dot{s}^2 - \frac{p_\theta^2(t + s^4)}{m^2 s^3} + \frac{1}{m}\frac{\partial V}{\partial s} = 0\end{gathered} \tag{2.56}$$

Equivalently, we can write the system in Hamiltonian form as,

$$H = \frac{1}{2m}\left(\mu^{-2}\left(s^2 - c - \frac{t}{s^2}\right)^2 p_s^2 + \left(s^2 - c - \frac{t}{s^2}\right)p_\theta^2\right) + V(s), \tag{2.57}$$

where Hamilton's equations become,

$$\begin{aligned}\dot{s} &= \frac{1}{m\mu^2}\left(s^2 - c - \frac{t}{s^2}\right)^2 p_s, \\ \dot{\theta} &= -\frac{1}{m}\left(s^2 - c - \frac{t}{s^2}\right)p_\theta, \\ \dot{p}_s &= -\frac{2}{m\mu^2}\left(s + \frac{t}{s^3}\right)\left(s^2 - c - \frac{t}{s^2}\right)p_s^2 - \frac{1}{m}\left(s + \frac{t}{s^3}\right)p_\theta^2 - \frac{\partial V}{\partial s}, \\ \dot{p}_\theta &= 0.\end{aligned} \tag{2.58}$$

We can also write the equation for s in a form which decouples the $\dot{s}$ and $\dot{p}_s$ equations, since H and p_θ are constants of motion and depend only on initial conditions.

$$\dot{s}^2 = \frac{1}{m^2\mu^4}\left(s^2 - c - \frac{t}{s^2}\right)^2\left(2mH - p_\theta^2\left(s^2 - c - \frac{t}{s^2}\right) - 2mV(s)\right) \tag{2.59}$$

From here, if we assume that s and θ are periodic we can apply the inverse function theorem to get $\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}^{-1} = \frac{\mathrm{d}s}{\mathrm{d}\theta}$. This gives us the orbit equation,

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2 = \frac{1}{\mu^4}\left(\frac{2m(H-V)}{p_\theta^2} - \left(s^2 - c - \frac{t}{s^2}\right)\right). \tag{2.60}$$

We also solve the equations of motion numerically using the explicit non-separable symplectic integrator introduced by Tao [33], which extends the work of Pihajoki [32]. A more detailed description of this method can be found in the next section, along with plots of the numerical solutions. The numerical data suggests that these orbits are epitrochoids, which are a generalization of the well known epicycloid. A large class of epitrochoids can be written as polar curves of the form,

$$r = a + b\cos(\Omega(\theta - \theta_0)) \tag{2.61}$$

We begin our investigation by noting that this polar curve satisfies the ODE,

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = \Omega^2(b^2 - (r-a)^2). \tag{2.62}$$

We can show that this equation matches the orbit equation (2.60) exactly in the case $\beta = 1, t = 0$, while in the case $\beta = -2$ we find that the orbit equation is solved by the square root of (2.61).

Theorem 2.8. Given a pseudo-Riemannian surface of revolution with time-like axis in $\mathbb{R}^{2,1}$ with the metric (2.39) and $\mu \in \mathbb{Q}$, if $t = 0$ then both the Kepler and oscillator potentials admit closed and bounded orbits. In the Kepler case, these orbits are epitrochoids transformed by a square root function. In the oscillator case,

the orbits are simple epitrochoids. If $t \neq 0$ then only the Kepler potential admits closed and bounded orbits.

Proof. Let us begin with the more complicated case: for $\beta = -2$ we can rearrange the orbit equation as follows.

$$s^2 \left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2 = \frac{1}{\mu^2}\left(\frac{t}{m} - \frac{A}{p_\theta^2} + \left(\frac{2(H-B)}{p_\theta^2} + \frac{c}{m}\right)s^2 - \frac{1}{m}s^4\right). \tag{2.63}$$

We then substitute $y = s^2$. Then, $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 2s\frac{\mathrm{d}s}{\mathrm{d}\theta}$, so $\frac{1}{4}\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 = s^2\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2$. After completing the square, this reduces the orbit equation to the equation of an epitrochoid,

$$\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 = -\frac{1}{4\mu^4}\left(\left(y + \frac{m(H-B)}{p_\theta^2} + \frac{c}{2}\right)^2 - t + \frac{mA}{p_\theta^2} - \frac{1}{4}\left(\frac{2m(H-B)}{p_\theta^2} + c\right)^2\right). \tag{2.64}$$

With

$$a = \frac{m(H-B)}{p_\theta^2} + \frac{c}{2}, \tag{2.65}$$

$$b = \sqrt{\frac{mA}{p_\theta^2} - t - \frac{1}{4}\left(\frac{2m(H-B)}{p_\theta^2} + c\right)^2}, \tag{2.66}$$

$$\Omega = \frac{1}{2\mu}. \tag{2.67}$$

From this we see that for $\mu = \frac{p}{q} \in \mathbb{Q}$ with q, p coprime, the number of lobes in the orbit is $2q$. In the case $\beta = 1$ we get

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2 = \frac{1}{\mu^2}\left(\frac{2m(H-B)}{p_\theta^2} + c - \frac{m^2A^2}{2p_\theta^4} - \left(s - \frac{mA}{p_\theta^2}\right)^2\right). \tag{2.68}$$

So the orbital parameters are

$$a = -\frac{mA}{p_\theta^2}, \tag{2.69}$$

$$b = \frac{2m(H-B)}{p_\theta^2} + c - \frac{m^2A^2}{2p_\theta^4}, \tag{2.70}$$

$$\Omega = \frac{1}{\mu}. \tag{2.71}$$

This completes the proof. Additionally, we can see that in order for the orbits to close we require $\mu \in \mathbb{Q}$, as found by Zagryadskii. One sees that for $\mu = \frac{p}{q} \in \mathbb{Q}$ with q, p coprime, the number of lobes is q.

Theorem 2.9. For $\beta = 1$, $t = 0$, the orbital period of a periodic orbit is given by,

$$T = \frac{m\eta}{2(a^2-c)p_\theta\Omega\Delta}\left(\frac{\log\left(2\eta - k_2^-\right) - \log\left(2\eta + k_2^-\right)}{k_1^-} + \frac{\log\left(2\eta - k_2^+\right) - \log\left(2\eta + k_2^+\right)}{k_1^+}\right) \tag{2.72}$$

with constants $k_1^\pm, k_2^\pm, \Delta, \eta$ and ζ defined below.

Proof. The period of a bound orbit is given by,

$$T = \int_0^{2\pi} \frac{\mathrm{d}\theta}{\dot{\theta}} \tag{2.73}$$

Substituting our known expression for $\dot{\theta}$ in terms of $s(\theta)$, this becomes,

$$T = \frac{m}{p_\theta(a^2-c)} \int_0^{2\pi} \frac{\mathrm{d}\theta}{(a + b\cos(\Omega(\theta - \theta_0)))^2 - c} \tag{2.74}$$

We make the following substitutions to simplify the integral:

$$\eta = \frac{b^2}{a^2 - c}, \tag{2.75}$$

$$\zeta = \frac{2ab}{a^2 - c}, \tag{2.76}$$

$$u = \cos(\Omega(\theta - \theta_0)). \tag{2.77}$$

This transforms the integral into the following form,

$$T = \frac{m}{2p_\theta \Omega(a^2 - c)} \int_{-1}^{1} \frac{\mathrm{d}u}{(\eta u^2 + \zeta u + 1)\sqrt{1 - u^2}}, \tag{2.78}$$

for which an analytic expression is known [38]. The indefinite integral is

$$\begin{aligned}
\mathcal{T} = \frac{m\eta}{2(a^2 - c)p_\theta \Omega \Delta} \Big(& \log\Big(\sqrt{1 - u^2}k_1^- - uk_2^- + 2\eta\Big)/k_1^- \\
& - \log\Big(\sqrt{1 - u^2}k_1^- + uk_2^- + 2\eta\Big)/k_1^- \\
& + \log\Big(\sqrt{1 - u^2}k_1^+ + uk_2^+ + 2\eta\Big)/k_1^+ \\
& - \log\Big(\sqrt{1 - u^2}k_1^+ - uk_2^+ + 2\eta\Big)/k_1^+ \Big),
\end{aligned}$$

with $\Delta = \sqrt{\zeta^2 - 4\eta}$ and

$$k_1^\pm = \sqrt{4\eta^2 + 2\zeta(\Delta \pm \zeta) + 4\eta} + 2\eta, \tag{2.79}$$

$$k_2^\pm = \zeta \pm \Delta. \tag{2.80}$$

Substituting in the bounds gives,

$$T = \frac{m\eta}{2(a^2-c)p_\theta \Omega \Delta}\left(\frac{\log\big(2\eta - k_2^-\big) - \log\big(2\eta + k_2^-\big)}{k_1^-} + \frac{\log\big(2\eta - k_2^+\big) - \log\big(2\eta + k_2^+\big)}{k_1^+}\right)$$

Which is what we wanted to show.

Theorem 2.10. For $\beta = -2$, $t \in \mathbb{R}$, the period of an orbit is given by,

$$T = \frac{-m}{2p_\theta}\left(\ell_3^+ \log\left(\left(\frac{2b\Delta' + \ell_2^+}{2b\Delta' - \ell_2^+}\right)\left(\frac{\Delta' - 2(b+a) + c}{\Delta' - 2(a-b) + c}\right)\right) - \ell_3^- \log\left(\left(\frac{2b\Delta' + \ell_2^-}{2b\Delta' - \ell_2^-}\right)\left(\frac{\Delta' + 2(b+a) - c}{\Delta' + 2(a-b) - c}\right)\right)\right) \tag{2.81}$$

with constants $\ell_1^\pm, \ell_2^\pm, \ell_3^\pm, \Delta'$ defined below.

Proof. Following the same procedure as above, except without the need to define new parameters η and ζ, we find that the integral becomes,

$$T = \frac{-m}{2p_\theta}\int_{-1}^{1} \frac{\mathrm{d}u}{(bu + a - c - t/(a+bu))\sqrt{1-u^2}}.$$

The indefinite integral is known [38],

$$\begin{aligned}\mathcal{T} = \frac{-m}{2p_\theta}\Big(&\ell_3^+ \log\Big(\sqrt{2}\Delta'\ell_1^+\sqrt{1-u^2} + \ell_2^+ u + 2b\Delta'\Big) \\ &-\ell_3^- \log\Big(\sqrt{2}\Delta'\ell_1^-\sqrt{1-u^2} + \ell_2^- u + 2b\Delta'\Big) \\ &+\ell_3^+ \log\big(-2bu - 2a + \Delta' + c\big) - \ell_3^- \log\big(2bu + 2a + \Delta' - c\big)\Big),\end{aligned}$$

with $\Delta' = \sqrt{c^2 + 4t}$ and

$$\ell_1^{\pm} = 2(b^2 + ac - a^2 - t) \pm (c - 2a)\Delta',$$

$$\ell_2^{\pm} = (2a - c)\Delta' \pm \Delta'^2,$$

$$\ell_3^{\pm} = \frac{\Delta' \pm c}{2(a^2 + a(c \pm \Delta') + b^2 - t) - c\Delta' - c^2}.$$

The result is then,

$$T = \frac{-m}{2p_\theta}\left(\ell_3^+ \log\left(\left(\frac{2b\Delta' + \ell_2^+}{2b\Delta' - \ell_2^+}\right)\left(\frac{\Delta' - 2(b+a) + c}{\Delta' - 2(a-b) + c}\right)\right)\right.$$
$$\left. -\ell_3^- \log\left(\left(\frac{2b\Delta' + \ell_2^-}{2b\Delta' - \ell_2^-}\right)\left(\frac{\Delta' + 2(b+a) - c}{\Delta' + 2(a-b) - c}\right)\right)\right).$$

As required.

2.2.5 Laplace-Runge-Lenz Vector

In the case of a Riemannian or pseudo-Riemannian surface of constant curvature it is known that the Kepler problem admits two additional conserved quantities which form the components of the Laplace-Runge-Lenz vector [25].

As we have shown, the Kepler problem on a pseudo-Riemannian surface of revolution with metric (2.39) admits closed bounded orbits. This implies the existence of one additional conserved quantity.

The components of this conserved quantity are typically assumed to be quadratic in the momenta, and have the general form

$$I = K^{ab}(q)p_a p_b + W(q) \tag{2.82}$$

For such a quantity to be conserved, it is sufficient that K_{ab} is any Killing tensor, while W must satisfy some partial differential equation in the coordinates [25]. The conserved quantity associated with the Killing tensor $K = \frac{1}{2}mg$ is simply the Hamiltonian itself, while any other pair of conserved quantities of the above form is known as a Laplace-Runge-Lenz vector.

One approach to generating such conserved quantities is to find a local coordinate frame of Killing vectors for the metric, and then in these coordinates the Killing tensor is constant. However, the metric (2.39) admits only one Killing vector ∂_θ.

An alternative approach would be to study the Hamilton-Jacobi equation for this system. As described in the introduction, the Hamilton-Jacobi equation is a partial differential equation whose solution allows one to recover the solutions to Hamilton's equations by taking appropriate partial derivatives. Conserved quantities can be constructed by performing a separation of variables on the Hamilton-Jacobi equation, each constant of separation being a conserved quantity. In the case of the Kepler problem on $\mathbb{R}^3$ it is known that the Laplace-Runge-Lenz vector is a result of the separability of the Hamilton-Jacobi equation in cylindrical parabolic coordinates, while the remaining conserved quantities arise by separation in Cartesian and spherical coordinates. Unfortunately this approach requires a hypothesis as to which coordinate system to use, and is intractable for the current problem. If one were to compute a valence two Killing tensor directly, then such a coordinate system could be computed. In any case, we must compute the components of K by

hand.

We therefore assume the form (2.82) holds, and directly compute the Poisson bracket $\{H, I\}$. By collecting like powers of the momenta, one can define a system of partial differential equations whose solution yields I. This approach has been used to compute a Laplace-Runge-Lenz vector for constant-curvature Riemannian surfaces of revolution [10]. We will attempt this approach for $t = 0, \beta = 1$, which is the most tractable version of the problem. Consider a conserved quantity of the form

$$I = K^{ss}(s,\theta)p_s^2 + K^{\theta\theta}(s,\theta)p_\theta^2 + 2K^{s\theta}(s,\theta)p_s p_\theta + W(s,\theta) \tag{2.83}$$

Then we require that $\{H, W\} = 0$, and that

$$\{K^{ss}(s,\theta)p_s^2 + K^{\theta\theta}(s,\theta)p_\theta^2 + 2K^{s\theta}(s,\theta)p_s p_\theta, H\} = 0 \tag{2.84}$$

Recall that Poisson bracket is defined by (1.13),

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q^i} = \frac{\partial f}{\partial s}\frac{\partial g}{\partial p_s} - \frac{\partial f}{\partial p_s}\frac{\partial g}{\partial s} + \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial p_\theta} - \frac{\partial f}{\partial p_\theta}\frac{\partial g}{\partial \theta}.$$

Let us write $H = (\mu^{-2}h(s)^2 p_s^2 + h(s)p_\theta^2)/2m + V(s)$, with $h(s) = s^2 - c - t/s^2$. We

get,

$$\begin{aligned}
\{K^{ss}p_s^2, h(s)^2p_s^2\} &= 2\frac{\partial K^{ss}}{\partial s}h(s)^2p_s^3 - 4K^{ss}h(s)h'(s)p_s^3 \\
\{K^{ss}p_s^2, h(s)p_\theta^2\} &= -2K^{ss}h'(s)p_sp_\theta^2 + 2\frac{\partial K^{ss}}{\partial \theta}h(s)p_\theta p_s^2 \\
\{K^{ss}p_s^2, V(s)\} &= -2\frac{\partial V}{\partial s}K^{ss}p_s \\
\{K^{\theta\theta}p_\theta^2, h(s)^2p_s^2\} &= 2\frac{\partial K^{\theta\theta}}{\partial s}h(s)^2p_sp_\theta^2 \\
\{K^{\theta\theta}p_\theta^2, h(s)p_\theta^2\} &= 2\frac{\partial K^{\theta\theta}}{\partial \theta}h(s)p_\theta^3 \\
\{K^{\theta\theta}p_\theta^2, V(s)\} &= 0 \\
\{K^{s\theta}p_sp_\theta, h(s)^2p_s^2\} &= 2\frac{\partial K^{s\theta}}{\partial s}h(s)^2p_s^2p_\theta - 2K^{s\theta}h(s)h'(s)p_s^2p_\theta \\
\{K^{s\theta}p_sp_\theta, h(s)p_\theta^2\} &= -K^{s\theta}h'(s)p_\theta^3 + 2\frac{\partial K^{s\theta}}{\partial \theta}h(s)p_sp_\theta^2 \\
\{K^{s\theta}p_sp_\theta, V(s)\} &= -\frac{\partial V}{\partial s}K^{s\theta}p_\theta \\
\{W(s,\theta), h(s)^2p_s^2\} &= 2\frac{\partial W}{\partial s}h(s)^2p_s \\
\{W(s,\theta), h(s)p_\theta^2\} &= 2\frac{\partial W}{\partial \theta}h(s)p_\theta
\end{aligned}$$

Collecting like terms, we get the following equations,

$$\begin{aligned}
\frac{\partial K^{ss}}{\partial s}h(s)^2 - 2K^{ss}h(s)h'(s) &= 0, \\
\frac{\partial K^{\theta\theta}}{\partial \theta}h(s) - K^{s\theta}h'(s) &= 0, \\
2\frac{\partial K^{s\theta}}{\partial \theta}h(s) - K^{ss}h'(s) + \frac{1}{\mu^2}\frac{\partial K^{\theta\theta}}{\partial s}h(s)^2 &= 0, \\
\mu^2\frac{\partial K^{ss}}{\partial \theta}h(s) + 2\frac{\partial K^{s\theta}}{\partial s}h(s)^2 - 2K^{s\theta}h(s)h'(s) &= 0,
\end{aligned} \tag{2.85}$$

as well as some equations for V and W

$$\frac{\partial W}{\partial s}h(s)^2 + 2m\mu^2\frac{\partial V}{\partial s}K^{ss} = 0, \qquad \frac{\partial W}{\partial \theta}h(s) + 2m\frac{\partial V}{\partial s}K^{s\theta} = 0 \tag{2.86}$$

We rearrange the first four equations as,

$$\begin{aligned}
\frac{\partial K^{\theta\theta}}{\partial \theta} &= \frac{h'(s)}{h(s)}K^{s\theta} \\
\frac{\partial K^{\theta\theta}}{\partial s} &= -2\mu^2\frac{1}{h(s)}\frac{\partial K^{s\theta}}{\partial \theta} + \mu^2 K^{ss}\frac{h'(s)}{h(s)^2} \\
\frac{\partial K^{s\theta}}{\partial s} &= \frac{h'(s)}{h(s)}K^{s\theta} - \mu^2\frac{1}{h(s)}\frac{\partial K^{ss}}{\partial \theta} \\
\frac{\partial K^{ss}}{\partial s} &= 2\frac{h'(s)}{h(s)}K^{ss}
\end{aligned} \tag{2.87}$$

So $K^{ss}(s,\theta) = C_1(\theta)h(s)^2$. If $\beta = 1$ then we must have the following consistency condition from (2.86),

$$-\mu^2 C_1'(\theta) = \frac{1}{h(s)}\left(\frac{\partial K^{s\theta}}{\partial s} - \frac{h'(s)}{h(s)}K^{s\theta}\right)$$

Meanwhile, the last equation of (2.85) requires that

$$\mu^2 C_1'(\theta)h(s)^3 + 2h(s)^2\left(\frac{\partial K^{s\theta}}{\partial s} - \frac{h'(s)}{h(s)}K^{s\theta}\right) = 0$$

So we must have $C_1'(\theta) = -C_1'(\theta)$, or $C_1'(\theta) = 0$. Therefore in this case we can set $K^{ss} = 0$ without loss of generality, since any terms in the conserved quantity of the form $h(s)^2 p_s^2$ can be absorbed into the Hamiltonian, which only requires a corresponding modification of $K^{\theta\theta}$ and W which would be reflected by removing K^{ss} from the above equations.

Now let us solve the system of equations completely. In the simplest case we have $t = 0$ and $\beta = 1$, so $V(s) = As + B$, and $h(s) = s^2 - c$. From the first equation of (2.85) we have $K^{s\theta} = K_1(\theta)(s^2 - c)$. Cross differentiating the second and third equations of (2.85) then gives us

$$\frac{\partial^2 K^{s\theta}}{\partial \theta^2} = -\frac{1}{2\mu^2}(s^2 - c)\left(\frac{2s}{s^2 - c}\frac{\partial K^{s\theta}}{\partial s} + \frac{2}{s^2 - c}K^{s\theta} - \frac{4s^2}{(s^2 - c)^2}K^{s\theta}\right),$$

Which reduces to the simple form

$$\frac{\partial^2 K^{s\theta}}{\partial \theta^2} = -\frac{1}{\mu^2}K^{s\theta}.$$

So,

$$(s^2 - c)K_1''(\theta) = -\frac{1}{\mu^2}K_1(s^2 - c).$$

This is solved by,

$$K_1(\theta) = E_1 \cos(\theta/\mu) + E_2 \sin(\theta/\mu).$$

Plugging this into the second equation of (2.85) gives,

$$\frac{\partial K^{\theta\theta}}{\partial \theta} = 2sK_1, \tag{2.88}$$

which is solved by,

$$K^{\theta\theta} = 2\mu s(E_1 \sin(\theta/\mu) - E_2 \cos(\theta/\mu)) + K_3(s). \tag{2.89}$$

Here $K_3'(s)$ vanishes since $\frac{\partial K^{\theta\theta}}{\partial s} = -\frac{1}{h}2\mu^2\frac{\partial K^{s\theta}}{\partial \theta}$. Since K_3 is a constant we can ignore

it as p_θ is known to be conserved. Finally we solve for W. We have,

$$\frac{\partial W}{\partial \theta} = -\frac{2mA}{s^2-c}\left((s^2-c)(E_1\cos(\theta/\mu)+E_2\sin(\theta/\mu))\right), \qquad \frac{\partial W}{\partial s}=0, \tag{2.90}$$

so,

$$W(\theta) = -2mA\mu(E_2\cos(\theta/\mu)-E_1\sin(\theta/\mu)). \tag{2.91}$$

All in all we have found two conserved quantities corresponding to the choices $E_1 = 0, E_2 \neq 0$ and $E_2 = 0, E_1 \neq 0$. These are the components of the Laplace-Runge-Lenz vector of our system.

$$I_1 = (s^2-c)\cos\left(\frac{\theta}{\mu}\right)p_s p_\theta + (\mu s p_\theta^2 - mA\mu)\sin\left(\frac{\theta}{\mu}\right), \tag{2.92}$$

$$I_2 = (s^2-c)\sin\left(\frac{\theta}{\mu}\right)p_s p_\theta - (\mu s p_\theta^2 - mA\mu)\cos\left(\frac{\theta}{\mu}\right). \tag{2.93}$$

This is quite similar to the form of the Laplace-Runge-Lenz vector components for the Riemannian case [10], which were computed in a coordinate system where the metric is of the form $\mathrm{d}u^2 + f(u)\mathrm{d}\theta^2$ while our metric is instead of the form $\mu^{-2}h(s)^{-2}\mathrm{d}s^2 + h(s)^{-1}\mathrm{d}\theta^2$. Observe that if we square these two quantities and add them, we find

$$I_1^2 + I_2^2 = 4(s^2-c)^2 p_s^2 p_\theta^2 + (2sp_\theta^2 + 2mA)^2\mu^2,$$

which reduces to,

$$I_1^2 + I_2^2 = 8m\mu^2(H-B)p_\theta^2 + 4\mu^2 c p_\theta^4 + 4m^2A^2\mu^2. \tag{2.94}$$

So we can see that there are three independent conserved quantities for the system.

This provides direct proof that the system (2.56) is superintegrable in the $\beta = 1$ case.

Now let us attempt to solve the case when $\beta = -2$. In this case, t does not necessarily have to be zero. Equations (2.86) become,

$$\frac{\partial W}{\partial s}h(s)^2 = -4mA\mu^2\frac{1}{s^3}K^{ss}, \qquad \frac{\partial W}{\partial \theta}h(s) = -4mA\frac{1}{s^3}K^{s\theta}. \tag{2.95}$$

This results in the following consistency condition

$$-\frac{\mu^2}{2h(s)^2s^3}\frac{\partial K^{ss}}{\partial \theta} = \frac{1}{h(s)s^4}K^{s\theta}. \tag{2.96}$$

From the first equation of (2.85) we again get

$$K^{ss}(s,\theta) = C_1(\theta)h(s)^2.$$

So $K^{s\theta} = -\frac{1}{2}s\mu^2h(s)C_1'(\theta)$. We then get,

$$2\frac{\partial K^{\theta\theta}}{\partial \theta} + \mu^2 sh'(s)C_1'(\theta) = 0 \tag{2.97}$$

Which means that,

$$K^{\theta\theta}(s,\theta) = -\frac{1}{2}\mu^2 sh'(s)C_1(\theta) + K(s) \tag{2.98}$$

The additional factor $K(s)$ is then required to be of the form $Kh(s)$. So this term can also be absorbed into W. Substituting in $h(s) = s^2 - c - t/s^2$ then yields the

following remaining equations,

$$\begin{aligned} C_1''(\theta) &= -\frac{4}{\mu^2}C_1(\theta), \\ \frac{\partial W}{\partial s} &= \frac{4Am\mu^2 C_1(\theta)}{s^3}, \\ \frac{\partial W}{\partial \theta} &= -\frac{2A\mu^2 m C_1'(\theta)}{s^2} \end{aligned} \tag{2.99}$$

So we have $C_1(\theta) = F_1\cos(2\theta/\mu) + F_2\sin(2\theta/\mu)$, and $W(s,\theta) = -2A\mu^2 m s^{-2}C_1(\theta)$. Putting this together, we find that there are two quadratic conserved quantities of the form

$$\begin{aligned} \tilde{I}_1 = &\left(s^2 - c - \frac{t}{s^2}\right)^2 \cos\left(\frac{2\theta}{\mu}\right)p_s^2 + \mu s\left(s^2 - c - \frac{t}{s^2}\right)\sin\left(\frac{2\theta}{\mu}\right)p_s p_\theta \\ &- \mu^2 s\left(s + \frac{t}{s^3}\right)\cos\left(\frac{2\theta}{\mu}\right)p_\theta^2 - 2A\mu^2 m\frac{1}{s^2}\cos\left(\frac{2\theta}{\mu}\right), \end{aligned} \tag{2.100}$$

$$\begin{aligned} \tilde{I}_2 = &\left(s^2 - c - \frac{t}{s^2}\right)^2 \sin\left(\frac{2\theta}{\mu}\right)p_s^2 - \mu s\left(s^2 - c - \frac{t}{s^2}\right)\cos\left(\frac{2\theta}{\mu}\right)p_s p_\theta \\ &- \mu^2 s\left(s + \frac{t}{s^3}\right)\sin\left(\frac{2\theta}{\mu}\right)p_\theta^2 - 2A\mu^2 m\frac{1}{s^2}\sin\left(\frac{2\theta}{\mu}\right). \end{aligned} \tag{2.101}$$

Due to the factors of $1/s^2$, the expression relating $\tilde{I}_1, \tilde{I}_2, H$ and p_θ is intractable. In conclusion, we have the following main result.

Theorem 2.11. The intrinsic Kepler/oscillator Hamiltonian system defined by system (2.58) admits 3 independent conserved quantities, making it superintegrable. Furthermore, the additional conserved quantity is determined by a generalized Laplace-Runge-Lenz vector $(I_1, I_2) \in C^\infty(T^*M, \mathbb{R}^2)$. In the case where $\beta = 1$ the

components are given by (2.93),

$$I_1 = (s^2 - c)\cos\left(\frac{\theta}{\mu}\right)p_s p_\theta + (\mu s p_\theta^2 - mA\mu)\sin\left(\frac{\theta}{\mu}\right),$$
$$I_2 = (s^2 - c)\sin\left(\frac{\theta}{\mu}\right)p_s p_\theta - (\mu s p_\theta^2 - mA\mu)\cos\left(\frac{\theta}{\mu}\right),$$

while in the case $\beta = -2$, the components are given by (2.101),

$$\begin{aligned}\tilde{I}_1 = &\left(s^2 - c - \frac{t}{s^2}\right)^2 \cos\left(\frac{2\theta}{\mu}\right)p_s^2 + 2\mu s\left(s^2 - c - \frac{t}{s^2}\right)\sin\left(\frac{2\theta}{\mu}\right)p_s p_\theta \\ &- \mu^2 s\left(s + \frac{t}{s^3}\right)\cos\left(\frac{2\theta}{\mu}\right)p_\theta^2 - 2A\mu^2 m\frac{1}{s^2}\cos\left(\frac{2\theta}{\mu}\right), \\ \tilde{I}_2 = &\left(s^2 - c - \frac{t}{s^2}\right)^2 \sin\left(\frac{2\theta}{\mu}\right)p_s^2 - 2\mu s\left(s^2 - c - \frac{t}{s^2}\right)\cos\left(\frac{2\theta}{\mu}\right)p_s p_\theta \\ &- \mu^2 s\left(s + \frac{t}{s^3}\right)\sin\left(\frac{2\theta}{\mu}\right)p_\theta^2 - 2A\mu^2 m\frac{1}{s^2}\sin\left(\frac{2\theta}{\mu}\right).\end{aligned}$$

3 Conclusion

The Bertrand problem involves finding potential energy functions whose corresponding Hamiltonian systems yield closed orbits. This has been well studied on Riemannian model spaces and surfaces of revolution. Extending these results to Riemannian homogenous spaces as well as complex manifolds would result in a much larger class of manifolds being understood. Additionally, a classification on pseudo-Riemannian spaces is known but major calculations are not stated clearly in the literature. We have provided an explicit solution to the orbit equation for any Bertrand pseudo-Riemannian surface of revolution. Additionally, we provide explicit expressions for the orbital period and the Laplace-Runge-Lenz vector for

the system. An expression relating the components of the Laplace-Runge-Lenz vector to the Hamiltonian and angular momentum was calculated in the case where the potential energy is of the form $V = As + B$. However, an equivalent expression for the case where $V = As^{-2} + B$ has not been computed, and further work is required to construct such an expression. In the future, it would be useful to work backwards from our expression for the Laplace-Runge-Lenz vector in order to construct a coordinate system on this surface which separates the Hamilton-Jacobi equation. It should also be investigated whether there exist analogues of Kepler's second and third laws for this system.

List of Definitions

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Appendix I: Python Code

The following is a snippet of python code including the definition of the Hamiltonian system as well as the symplectic integrator constructed by Tao [33].

```
import numpy as np
from numba import jit, njit


@njit # Use just-in-time compiler for numerics
def Equation(q,p,params):
"""
Computes derivatives of q and p using
Hamilton's equations (written out explicitly)


Inputs:
    q : Array<float,2>
    p : Array<float,2>
    params : Tuple<float,7>


outputs :
    qdot, pdot : Tuple<Array<float,2>>
"""
    mu,c,a,A,B,beta,m=params
    s       = q[0]
    p_s     = p[0]
```

```
    theta     = q[1]
    p_theta   = p[1]
    sdot      = p_s*(1/(m*mu*mu))*(s*s-c-a/(s*s))*(s*s-c-a/(s*s))
    p_sdot    = (2/(m*mu*mu))*(s+a/(s*s*s))*(s*s-c-a/(s*s))*p_s*p_s +\
          (1/m)*(s+a/(s*s*s))*p_theta*p_theta + A*beta*np.abs(s)**(beta-1)
    thetadot = (1/m)*(s*s-c-a/(s*s))*p_theta
    return np.array([p_sdot,0.0]), np.array([sdot,thetadot])


@njit
def Nonsep_SymInt_2step(q, p, x, y, dt, n_steps,params, w):
"""
Compute n_steps steps of Tao's second order method.


Inputs:
    q : Array<float>
    p : Array<float>
    x : Array<float>
    y : Array<float>
    dt : float
    ---- The parameter delta in the thesis
    n_steps : int
    params : Tuple<float>
    w : float
```

```
    ——— The parameter omega in Tao (2016)

Outputs:

    qn, pn, xn, yn : Tuple<Array<float>>
"""
    h = dt / n_steps
    for i_step in range(int(n_steps)):
        x1, p1 = Equation(q, y, params)
        p      = p - x1 * h * 0.5
        x      = x + p1 * h * 0.5
        q1, y1 = Equation(x, p, params)
        q      = q + y1 * h * 0.5
        y      = y - q1 * h * 0.5
        q1     = 0.5 * (q - x)
        p1     = 0.5 * (p - y)
        x1     = np.cos(2 * w * h) * q1 + np.sin(2 * w * h) * p1
        y1     = -np.sin(2 * w * h) * q1 + np.cos(2 * w * h) * p1
        q1     = 0.5 * (q + x)
        p1     = 0.5 * (p + y)
        q      = q1 + x1
        p      = p1 + y1
        x      = q1 - x1
        y      = p1 - y1
        q1, y1 = Equation(x, p, params)
```

```
        q       = q + y1 * h * 0.5
        y       = y - q1 * h * 0.5
        x1, p1 = Equation(qn, yn, params)
        p       = p - x1 * h * 0.5
        x       = x + p1 * h * 0.5

    return q, p, x, y


@jit
def Nonsep_SymInt_4step(q,p,x,y,dt,n_steps,params,w):
"""
Fourth order four-step method.
"""
    gamma         = 1/(2-2**(1/(5)))
    q1,p1,x1,y1 = Nonsep_SymInt_2step(q,p,x,y,dt*gamma,n_steps,params,w)
    q2,p2,x2,y2 = Nonsep_SymInt_2step(q1,p1,x1,y1,
        dt*(1-2*gamma),n_steps,params,w)
    return Nonsep_SymInt_2step(q2,p2,x2,y2,
        dt*gamma,n_steps,params,w)


@jit
def Nonsep_SymInt_6step(q,p,x,y,dt,n_steps,params,w):
"""
Sixth order six-step method.
```

```
"""
    gamma          = 1/(2-2**(1/(7)))
    q1,p1,x1,y1 = Nonsep_SymInt_4step(q,p,x,y,dt*gamma,n_steps,params)
    q2,p2,x2,y2 = Nonsep_SymInt_4step(q1,p1,x1,y1,
        dt*(1-2*gamma),n_steps,params,w)
    return Nonsep_SymInt_4step(q2,p2,x2,y2,
        dt*gamma,n_steps,params,w)

def Iterate_Solver(q0,p0,tmin,tmax,dt,n_steps,params,w):
"""
Generates a full numerical solution.

Inputs:
    q0 : Array<float>
    p0 : Array<float>
    tmin : float
    tmax : float
    dt : float
    n_steps : int
    params : Tuple<float>
    w : float

Outputs:
    s_out : Array<float>
```

```
    ---- s values of solution
    p_s_out : Array<float>
    ---- p_s values of solution
    theta_out : Array<float>
    ---- theta values of solution
    hamiltonian_err : List<float>
    ---- energy drift
    t_vals : List<float>
    ---- t values for each datapoint
"""
    t       = tmin
    q       = q0
    p       = p0
    x       = q
    y       = p
    data    = [[*q0,*p0]]
    hamiltonian_err = [0]
    t_vals = [tmin]
    while t<=tmax:
        print("[{} %]\r".format(round(100*t/tmax,4)),end='')
        q,p,x,y = Nonsep_SymInt_6step(q,p,x,y,dt,n_steps,params,w)
        data.append([q[0],q[1],p[0],p[1]])
        hamiltonian_err.append(np.abs(q[0]-x[0])**2 \
            +np.abs(p[0]-y[0])**2)
```

```
        t += dt
        t_vals.append(t)
    print('\n')
    s_out = np.array([d[0] for d in data])
    theta_out = np.array([d[1] for d in data])
    p_s_out =np.array([d[2] for d in data])

    return s_out, p_s_out, theta_out, hamiltonian_err, t_vals
```