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RESEARCH ARTICLE

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Distributed dynamics for aggregative games: Robustness and privacy guarantees

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Abstract

This paper considers the problem of Nash equilibrium (NE) seeking in aggregative games, where the cost function of each player depends on an aggregate of all players' actions. We present a distributed continuous-time algorithm such that the actions of the players converge to NE by communicating to each other through a connected network. As agents may deviate from their optimal strategies dictated by the NE seeking protocol, we investigate robustness of the proposed algorithm against time-varying disturbances. In particular, we provide rigorous robustness guarantees by proving input-to-state stability (ISS) and $\mathcal{L}_2$-stability properties of the NE seeking dynamics. A major concern in communicative schemes among strategic agents is that their private information may be revealed to other agents or to a curious third party who can eavesdrop the communications. Motivated by this, we investigate privacy properties of the algorithm and identify to what extent privacy is preserved when all communicated variables are compromised. Finally, we demonstrate practical applications of our theoretical findings on two case studies; namely, on an energy consumption game and a coordinated charging of electric vehicles.

KEYWORDS

aggregative games, Nash equilibrium seeking, robustness, privacy

1 | INTRODUCTION

Game theory is the standard tool for studying the interaction behavior of self-interested agents/players and has attracted considerable attention due to its broad applications and technical challenges. An active research topic in this area concerns aggregative games that model a set of noncooperative agents aiming at minimizing their cost functions, while the action of each individual player is influenced by an aggregation of the actions of all the other players.[1] These games have appeared in a broad range of applications such as networked control systems,[2] demand-side management in smart grids,[3] charging control of plug-in electric vehicles,[4] and flow control of communication networks.[5] The common characteristic is that if noncooperative agents are left uncoordinated, their aggregate actions can negatively affect the shared architecture.

Abbreviations: NE, Nash equilibrium; ISS, input-to-state stability; VI, variational inequality; HVAC, heating ventilation air conditioning; PEV, plug-in electric vehicle; KKT, Karush–Kuhn–Tucker

In the context of noncooperative games, existence of a solution, Nash equilibrium (NE), and its computation have been extensively studied in the literature.[6] Earlier works considered the case where each agent has full access to the actions of all other agents, that is, all-to-all interactions.[6,7] However, recent works have attempted to relax this assumption due to computational and scalability issues. In this regard, the authors in References 8-11 presented distributed NE seeking algorithms where each player computes an estimation of the actions of all the other players by communicating to its neighbors. Although those algorithms are applicable to aggregative games, they are inefficient as they require that each player estimate the actions of all other players. In aggregative games, on the other hand, it is sufficient that each player estimates the aggregation term. This has led to various algorithms tailored for aggregative games, which can be classified as gather and broadcast[12-14] and distributed algorithms.[15-19] The former is based on the exchange of information with a central aggregator, whereas the latter relies on a peer-to-peer communication. This paper falls into the second category and presents a fully distributed NE seeking algorithm for aggregative games.

From a different perspective, distributed NE seeking algorithms for aggregative games can be divided into discrete-time[15-17,20,21] and continuous-time.[18,19,22,23] The discrete-time algorithms are based on gradient dynamics (synchronous algorithm in Reference 15), gossip technique (asynchronous algorithm in Reference 15), double-layer iterations,[16,20] forward-backward iteration,[17] and optimal response.[21] Tuning of the step sizes in these algorithms, however, is generally a hard task or it may require global information shared among all players. More specifically, diminishing step sizes are used in Reference 15, which typically slow down the convergence speed, and References 16,17,20 employ fixed step sizes, where global information is needed for selection of the step sizes. The continuous-time gradient-based algorithms in References 18,19,22,23, on the other hand, employ some tuning parameters shared among all players. In comparison to those works, we present a fully distributed NE seeking algorithm where the players need not to share any design parameters or their actions, and more importantly, equip our algorithm with rigorous robustness guarantees and study its privacy preserving properties as discussed below.

The players are not always rational in a game and deviation of their actions from a fully rational behavior is possible. Some examples are "stubborn players"[22,24] who do not fully obey the NE seeking dynamics, or "almost" rational players whose decisions are determined by their "bounded rationality".[25] Therefore, it is crucial that an NE seeking algorithm has suitable robustness properties. Additionally, having a robust algorithm is required when there exists uncertainty in a game. Robustness of an NE seeking algorithm with respect to slowly-varying channel gain in code division multiple access systems is studied in Reference 26. We refer the reader to References 27-29 for studies on robustness of gradient systems, saddle-point dynamics, and frequency regulation of power networks, respectively.

To investigate robustness, we add bounded time-varying disturbances to the dynamics of the algorithm, and show that the proposed distributed NE seeking algorithm is robust against such perturbations. We use input-to-state stability (ISS) as a notion of robustness, which examines whether the state trajectories of the system are bounded by a function of the perturbation.[30] In our robustness analysis, the main technical challenge is the existence of undamped communicating variables in the algorithm. We address this by including a sufficiently small cross-term in the ISS Lyapunov function. In addition to establishing ISS, we exploit $\mathcal{L}_2$-stability and explicitly analyze the effect of disturbances on convergence error to the NE of the game.

Generally speaking, NE seeking algorithms rely on communication either with a central aggregator[12] or among neighboring agents.[15] In the former, it is often assumed that the aggregator is trustworthy, whereas, in reality, private information can still be leaked by an aggregator either willingly or unwillingly. In the latter, private information can be revealed to other players through direct communication, or leaked to curious adversaries as a result of eavesdropping. More generally, for convincing strategic players to participate in any cooperative policy, privacy guarantees need to be put in place.

Motivated by the above concerns, we investigate the proposed distributed NE seeking algorithm from the viewpoint of privacy. Roughly speaking, privacy is preserved if private variables of the players cannot be uniquely reconstructed based on the available information on the structure of the algorithm, the class of cost functions, and communicated variables. To make sure this is the case, we will show that there are *replicas* of private quantities that are indistinguishable from the original ones in view of the available information. An alternative approach would be to use data perturbation techniques and rely on differential privacy.[31-33] The idea behind this technique is to add noise with appropriate statistical properties to the process under investigation in order to limit the ability of a curious party in estimating the private quantities of the system. Differential privacy is recently exploited in Reference 34, and a distributed NE seeking algorithm is proposed that preserves privacy of the player's objective function. A drawback of adding noise, however, is that the solution of the algorithm asymptotically diverges from the NE of the game. Our approach, on the contrary, uses an "observability" or "identifiability" principle, that is, private variables/quantities cannot be inferred from the information accessible to the curious adversary. We perform a worst-case privacy analysis by considering the scenario where the structure of the algorithm and all communicated variables are available to the adversary.

Contributions: The main contribution of the article is threefold. First, we present a fully distributed algorithm in continuous-time that obtains the NE in aggregative games. The algorithm does not require the players to share their actions or any design parameters; and we provide sufficient conditions for its convergence, which can be verified in a distributed manner. As the second contribution, by using ISS and $\mathcal{L}_2$-stability, we show that the algorithm is robust against bounded time-varying disturbances. Third, we demonstrate that the proposed algorithm preserves privacy of the players' private information against adversaries with full knowledge on all communicated variables and the structure of the algorithm. All features combined, this manuscript delivers a fully distributed NE seeking algorithm with formal robustness and privacy guarantees.

The rest of the article is organized as follows: Section 2 includes preliminaries and the problem formulation. In Section 3, a distributed NE seeking algorithm is proposed and its convergence analysis is provided. Robustness and privacy guarantees of the algorithm are established in Section 4. The rate of convergence and the $\mathcal{L}_2$-gain of the system are analytically provided. Two detailed case studies of an energy consumption game and coordinated charging of electric vehicles are provided in Section 5. The paper closes with conclusions in Section 6.

2 | NOTATIONS, PRELIMINARIES, AND PROBLEM FORMULATION

2.1 | Notations

We use $\mathbb{0}$ to denote a vector/matrix of all zeros, and $\mathbb{1}_n$ for the vector of all ones in $\mathbb{R}^n$. The identity matrix of size n is denoted by I_n. We omit the subscript whenever no confusion arises. The Kronecker product is denoted by $\otimes$. For given vectors $x_1, \dots, x_N \in \mathbb{R}^n$, we use the notation $\boldsymbol{x} := \mathrm{col}(x_1, \dots, x_N) = \left[x_1^\top, \dots, x_N^\top\right]^\top$ and $\boldsymbol{x}_{-i} := \mathrm{col}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$. Similarly, for given matrices $X \in \mathbb{R}^{m\times n}$ and $Y \in \mathbb{R}^{p\times n}$, we write $\mathrm{col}(X, Y) = [X^\top, Y^\top]^\top$. We use $\boldsymbol{A} := \mathrm{blockdiag}(A_1, \dots, A_N)$ to denote the block diagonal matrix constructed from the matrices $A_1, \dots, A_N$. The image and kernel of a matrix $A \in \mathbb{R}^{n\times m}$ are denoted by $\mathrm{im}A$ and $\ker A$, respectively. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. In addition, it is class $\mathcal{K}_\infty$ if $\alpha(s) \to \infty$ as $s \to \infty$. A function $F : \mathbb{R}^n \to \mathbb{R}^n$ is (strictly) monotone if $(x - y)^\top(F(x) - F(y)) \geq 0\, (> 0)$ for all $x \neq y \in \mathbb{R}^n$, and it is μ-strongly monotone if $(x - y)^\top(F(x) - F(y)) \geq \mu\|x - y\|^2$ for all $x, y \in \mathbb{R}^n$ and some $\mu \in \mathbb{R}_{>0}$. The space of piecewise continuous square-integrable functions that map $\mathbb{R}_{\geq 0}$ into $\mathbb{R}^n$ is denoted by $\mathcal{L}_2^n$. Equivalently, a piecewise continuous function $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ belongs to $\mathcal{L}_2^n$ if its $\mathcal{L}_2$ norm, defined as $\|x\|_{\mathcal{L}_2} := (\int_0^\infty x(t)^\top x(t)dt)^{\frac{1}{2}}$, is bounded. The extended space $\mathcal{L}_{2e}^n$ is defined as $\mathcal{L}_{2e}^n := \{x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n | x_\tau \in \mathcal{L}_2^n, \forall \tau \in \mathbb{R}_{\geq 0}\}$ where x_τ is a truncation of x such that $x_\tau(t) = x(t)$ for all $0 \leq t \leq \tau$ and $x_\tau(t) = 0$ for all $t > \tau$.

2.2 | Algebraic graph theory

Let $G_c = (\mathcal{I}, \mathcal{E})$ be an undirected graph that models the network of N agents with $\mathcal{I} = \{1, \dots, N\}$ being the node set associated to the agents, and $\mathcal{E}$ denoting the edge set. Each element of $\mathcal{E}$ is an unordered pair $\{i, j\}$ with $i, j \in \mathcal{I}$. The graph is connected if there is a path between every pair of nodes. The set of neighbors of agent i is $\mathcal{N}_i = \{j \in \mathcal{I} | \{i, j\} \in \mathcal{E}\}$. The Laplacian matrix of G_c is denoted by L with L_{ii} equal to the cardinality of $\mathcal{N}_i$, $L_{ij} = -1$ if $j \in \mathcal{N}_i$, and $L_{ij} = 0$ otherwise. The matrix L of an undirected graph is positive semidefinite and $\mathbb{1}_N \in \ker L$. If the graph is connected, L has exactly one zero eigenvalue, and $\mathrm{im}\mathbb{1}_N = \ker L$. The Moore–Penrose inverse of L is denoted by L^+.

2.3 | Projection and variational inequality

Given a closed convex set $\mathcal{S} \subseteq \mathbb{R}^n$, the projection of a point $v \in \mathbb{R}^n$ to $\mathcal{S}$ is denoted by $\mathrm{proj}_\mathcal{S}(v) := \arg\min_{y\in\mathcal{S}} \|y - v\|$. Given a point $x \in \mathcal{S}$, the normal cone of $\mathcal{S}$ at x is the set $\mathcal{N}_\mathcal{S}(x) := \{y \in \mathbb{R}^n | y^\top(z - x) \leq 0, \forall z \in \mathcal{S}\}$. The tangent cone of $\mathcal{S}$ at $x \in \mathcal{S}$ is denoted by $\mathcal{T}_\mathcal{S}(x) := \mathrm{cl}\left(\cup_{y\in\mathcal{S}} \cup_{h>0} h(y - x)\right)$ where $\mathrm{cl}(\cdot)$ denotes the closure of a set. For $v \in \mathbb{R}^n$ and $x \in \mathcal{S}$, the projection of v at x with respect to $\mathcal{S}$ is given by $\Pi_\mathcal{S}(x, v) := \lim_{h\to 0^+} \frac{1}{h}\left(\mathrm{proj}_\mathcal{S}(x + hv) - x\right)$, and it is equivalent to the projection of v to $\mathcal{T}_\mathcal{S}(x)$, that is, $\Pi_\mathcal{S}(x, v) = \mathrm{proj}_{\mathcal{T}_\mathcal{S}(x)}(v)$. By using Moreau's decomposition theorem (theorem 3.2.5 in

Reference 35), a vector $v \in \mathbb{R}^n$ can be decomposed as $v = \mathrm{proj}_{\mathcal{N}_S(x)}(v) + \mathrm{proj}_{\mathcal{T}_S(x)}(v)$ for any point $x \in \mathcal{S}$. Given a mapping $F : \mathcal{S} \to \mathbb{R}^n$, the variational inequality problem VI$(\mathcal{S}, F)$ is to find the point $\bar{x} \in \mathcal{S}$ such that $(x - \bar{x})^\top F(\bar{x}) \geq 0$ for all $x \in \mathcal{S}$.

2.4 | Aggregative games

We consider a set of players $\mathcal{I} = \{1, \dots, N\}$ where each player $i \in \mathcal{I}$ aims at minimizing a cost function $J_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by choosing its action variable x_i in a set $\mathcal{X}_i \subseteq \mathbb{R}^n$. The value of the cost function depends on x_i and an aggregation of all the other action variables. In particular, each player $i \in \mathcal{I}$ attempts to solve the following minimization problem

$$\begin{aligned} &\min_{x_i \in \mathcal{X}_i} J_i(x_i, s(\boldsymbol{x})) \\ &s(\boldsymbol{x}) := \frac{1}{N}\sum_{j \in \mathcal{I}} x_j = \frac{1}{N}\left(\mathbb{1}_N^\top \otimes I_n\right)\boldsymbol{x}, \end{aligned} \tag{1}$$

where $\boldsymbol{x} := \mathrm{col}(x_1, \dots, x_N)$ and $s(\boldsymbol{x})$ is the aggregation term. We use the compact notation $\mathcal{G}_{\mathrm{agg}} = (\mathcal{I}, (J_i)_{i\in\mathcal{I}}, (\mathcal{X}_i)_{i\in\mathcal{I}})$ to denote the aggregative game in (1). By definition, a point $\boldsymbol{x}^* := \mathrm{col}\left(x_1^*, \dots, x_N^*\right)$ is a Nash equilibrium (NE) of the game if

$$x_i^* \in \arg\min_{y \in \mathcal{X}_i} J_i\left(y, \frac{1}{N}y + \frac{1}{N}\sum_{j \neq i} x_j^*\right), \quad \forall\, i \in \mathcal{I}.$$

This means that at the NE, there is no player that can decrease its cost by unilaterally changing its action. We note that x_i^* depends on the optimal actions of all the other players, and therefore several coupled optimization problems need to be solved to obtain $\boldsymbol{x}^*$. Consequently, standard distributed optimization techniques cannot be used for solving this problem. In the next section, we derive local sufficient conditions for existence and uniqueness of NE and present a distributed algorithm that asymptotically converges to this point.

3 | DISTRIBUTED NE SEEKING DYNAMICS

First, we discuss some auxiliary results that are instrumental to prove convergence properties of the NE seeking algorithm proposed later in the section.

Assumption 1 (assumption 2(ii) in Reference 10). For all $i \in \mathcal{I}$, the action set $\mathcal{X}_i \subset \mathbb{R}^n$ is nonempty, convex, and compact, and the cost function J_i is C^1 in all its arguments. •

Let $\sigma_i \in \mathbb{R}^n$ be a local variable associated to each player $i \in \mathcal{I}$, with the cost function written as $J_i(x_i, \sigma_i)$, and define

$$f_i(x_i, \sigma_i) := \frac{\partial}{\partial x_i} J_i(x_i, \sigma_i) + \frac{1}{N}\frac{\partial}{\partial \sigma_i} J_i(x_i, \sigma_i). \tag{2}$$

It is easy to see that $\frac{\partial}{\partial x_i} J_i(x_i, s(\boldsymbol{x})) = f_i(x_i, s(\boldsymbol{x}))$. To proceed further, we need the following assumption:

Assumption 2. For all $i \in \mathcal{I}$, the mapping $x_i \mapsto f_i(x_i, \sigma_i)$ is μ_i-strongly monotone, and the mapping $\sigma_i \mapsto f_i(x_i, \sigma_i)$ is ℓ_i-Lipschitz continuous with $\mu_i > \ell_i$. •

The assumption above is a decentralized version of assumption 1 in Reference 12, and its conditions can be replaced by less conservative, yet more implicit, conditions; see Remark 2.

In game theory, the pesudo-gradient mapping defined as $\mathrm{col}\,((f_i(x_i, s(\boldsymbol{x})))_{i\in\mathcal{I}})$ plays a fundamental role in designing NE seeking algorithms. Motivated by this and the fact that the players may not have access to $s(\boldsymbol{x})$, we introduce the following mapping:

$$F(\boldsymbol{x}, \boldsymbol{\sigma}) := \begin{bmatrix} \boldsymbol{K}\,\mathrm{col}\,((f_i(x_i, \sigma_i))_{i\in\mathcal{I}}) \\ \boldsymbol{\sigma} - \boldsymbol{x} \end{bmatrix}, \tag{3}$$

where $\boldsymbol{K} := \text{blockdiag}(k_1 I_n, \dots, k_N I_n)$ with design parameters $k_i > 0$, and $\boldsymbol{\sigma} := \text{col}(\sigma_1, \dots, \sigma_N)$. In the literature,[17] the term $\text{col}\left((f_i(x_i, \sigma_i))_{i\in\mathcal{I}}\right)$ is referred to as "the extended pesudo-gradient mapping" where each player uses its local variable σ_i instead of the aggregation $s(\boldsymbol{x})$. The matrix $\boldsymbol{K}$ is added for additional flexibility and motivated by privacy reasons; see Remark 1. This mapping is further extended in (3) with the term $\boldsymbol{\sigma} - \boldsymbol{x}$ to construct an augmented map that is strongly monotone with respect to $(\boldsymbol{x}, \boldsymbol{\sigma})$. The latter property is instrumental in proving convergence of the algorithm (see Theorems 1 and 2). The following lemma summarizes some properties of (3).

Lemma 1. *Let Assumption 2 hold and choose k_i such that*

$$k_i \in \left(\frac{1}{\ell_i^2}\left(\sqrt{\mu_i} - \sqrt{\mu_i - \ell_i}\right)^2, \frac{1}{\ell_i^2}\left(\sqrt{\mu_i} + \sqrt{\mu_i - \ell_i}\right)^2 \right) \tag{4}$$

is satisfied for each $i \in \mathcal{I}$. Then,

(i) the map F in (3) is ϵ-strongly monotone.
(ii) the map $\boldsymbol{K}\,\text{col}\left((f_i(x_i, s(\boldsymbol{x})))_{i\in\mathcal{I}}\right)$ is ϵ-strongly monotone.

Proof. See Appendix A. ■

Remark 1. The condition in (4) can be equivalently expressed as $4k_i\mu_i - (k_i\ell_i + 1)^2 > 0$. Therefore, setting $k_i = 1$, for each i, returns the inequality $\sqrt{\mu_i} > (\ell_i + 1)/2$ which is a more restrictive condition than the one in Assumption 2, that is, $\mu_i > \ell_i$. Introducing the gain k_i yields a milder assumption and, as we will see later, contributes to the privacy of the proposed algorithm. •

We note that the results of the preceding lemma are sufficient for the existence and uniqueness of the NE. This is formally stated next.

Lemma 2. *Let Assumptions 1 and 2 be satisfied, then the aggregative game $\mathcal{G}_{\text{agg}} = (\mathcal{I}, (J_i)_{i\in\mathcal{I}}, (\mathcal{X}_i)_{i\in\mathcal{I}})$ with the cost function (1) has a unique NE $\boldsymbol{x}^* \in \mathcal{X}$ which is the solution of the variational inequality $VI(\mathcal{X}, \boldsymbol{K}\,\text{col}\left((f_i(x_i, s(\boldsymbol{x})))_{i\in\mathcal{I}}\right))$ with $\mathcal{X} := \prod_{i\in\mathcal{I}} \mathcal{X}_i$, the function $f_i(\cdot)$ defined as (2), and k_i selected as (4).*

Proof. See Appendix A. ■

Remark 2. To guarantee existence and uniqueness of the NE, it suffices that the pseudo-gradient mapping $\text{col}\left((f_i(x_i, s(\boldsymbol{x})))_{i\in\mathcal{I}}\right)$ is strongly monotone (Theorem 3(d) in Reference 36). Assumption 2, on the other hand, provides a decentralized condition that simultaneously guarantees existence of a unique NE and assists us later in proving convergence of our algorithm to the NE. We also note from the proof of Lemma 2 that Assumption 2 can be relaxed to any cost function $J_i(x_i, s(\boldsymbol{x}))$ that is strictly convex in x_i for all $\boldsymbol{x}_{-i} \in \mathcal{X}_{-i}$ (assumption 2(ii) in Reference 10) and results in strong monotonicity of the mapping $\text{col}(k_i f_i(x_i, \sigma_i), \sigma_i - x_i)$ for some $k_i > 0$. We will show how to use such a relaxation in the case studies discussed in Section 5. •

Having established existence and uniqueness of the NE, next we propose continuous-time distributed dynamics that converges to this points at the steady state. Motivated by privacy considerations, we assume that the players do not communicate their action variables x_i's, neither to the other players nor to a central unit. Instead, auxiliary variables will be communicated through a connected communication graph G_c. This motivates the following distributed NE seeking policy:

$$\begin{aligned}
\dot{x}_i(t) &= \Pi_{\mathcal{X}_i}\left(x_i(t), -k_i f_i\left(x_i(t), \sigma_i(t)\right)\right), \\
\dot{\sigma}_i(t) &= -\sigma_i(t) + x_i(t) - \sum_{j\in\mathcal{N}_i}\left(\psi_i(t) - \psi_j(t)\right), \\
\dot{\psi}_i(t) &= \sum_{j\in\mathcal{N}_i}\left(\sigma_i(t) - \sigma_j(t)\right),
\end{aligned} \tag{5}$$

where $i \in \mathcal{I}$, $\mathcal{N}_i$ denotes the set of neighbors of player i, and $\Pi_{\mathcal{X}_i}(x_i, \cdot)$ is the projection operator on to the tangent cone of $\mathcal{X}_i$ at the point $x_i \in \mathcal{X}_i$. We note that $\Pi_{\mathcal{X}_i}(x_i, -k_i f_i(x_i, \sigma_i)) = -k_i f_i(x_i, \sigma_i)$ at any point x_i in the interior of $\mathcal{X}_i$. At any boundary point of $\mathcal{X}_i$, the projection operator restricts the flow of $-k_i f_i(x_i, \sigma_i)$ such that any solution $x_i(t)$ of (5) remains in $\mathcal{X}_i$. It is worth mentioning that, at any point x_i, the value of the projection $\Pi_{\mathcal{X}_i}(x_i, \cdot)$ can be computed using lemma 2.1

in Reference 37.* Notice that the players only use the local parameter k_i, and communicate the variables σ_i and ψ_i. The variable σ_i is, in fact, a local estimation of $s(\boldsymbol{x})$, and the state components ψ_i's are defined to enforce consensus on σ_i's. Let $\boldsymbol{\psi} := \mathrm{col}(\psi_1, \dots, \psi_N)$ and L be the Laplacian matrix of the graph G_c. Then, the algorithm can be written in vector form as

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \Pi_{\mathcal{X}}\left(\boldsymbol{x}, -\boldsymbol{K}\,\mathrm{col}\left((f_i(x_i(t), \sigma_i(t)))_{i\in\mathcal{I}}\right)\right),\\ \dot{\boldsymbol{\sigma}}(t) &= -\boldsymbol{\sigma}(t) + \boldsymbol{x}(t) - (L\otimes I_n)\boldsymbol{\psi}(t),\\ \dot{\boldsymbol{\psi}}(t) &= (L\otimes I_n)\boldsymbol{\sigma}(t).\end{aligned} \tag{6}$$

Note that (6) is a discontinuous dynamical algorithm due to the projection operator. Therefore, we briefly discuss existence and uniqueness of solutions for this system. Consider the collective projected-vector form of the algorithm as follows

$$\mathrm{col}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{\psi}}) = \Pi_{\mathcal{X}\times\mathbb{R}^{nN}\times\mathbb{R}^{nN}}\left(\mathrm{col}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\psi}), -F_{\mathrm{ext}}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\psi})\right),$$

where

$$F_{\mathrm{ext}}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\psi}) := \begin{bmatrix} F(\boldsymbol{x}, \boldsymbol{\sigma}) + G\boldsymbol{\psi} \\ -(L\otimes I_n)\boldsymbol{\sigma} \end{bmatrix},$$

with $F(\boldsymbol{x}, \boldsymbol{\sigma})$ given by (3) and $G := \mathrm{col}(\mathbb{0}, L\otimes I_n)$. Using Assumption 1 and the fact that $\mathbb{R}^{nN}$ is a clopen set (closed-open set), the set $\mathcal{X}\times\mathbb{R}^{nN}\times\mathbb{R}^{nN}$ is closed and convex. It also follows from Lemma 1 that F_{ext} is continuous and monotone. Therefore, from theorem 1 in Reference 38, we conclude that for any initial condition $(\boldsymbol{x}(0), \boldsymbol{\sigma}(0), \boldsymbol{\psi}(0)) \in \mathcal{X}\times\mathbb{R}^{nN}\times\mathbb{R}^{nN}$, the system (6) has a unique solution which belongs to $\mathcal{X}\times\mathbb{R}^{nN}\times\mathbb{R}^{nN}$ on the time interval $[0, \infty)$.

We now characterize the equilibria of (6) and then proceed with the results concerning convergence. For clarity, note that Assumptions 1 and 2 are treated as standing assumptions and k_i's are selected as (4).

Proposition 1. *Let $\boldsymbol{x}^*$ be the NE of the game $\mathcal{G}_{\mathrm{agg}}$. Then, any equilibrium point of (6) is given by $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{\psi}}) = (\boldsymbol{x}^*, \mathbb{1}_N\otimes s(\boldsymbol{x}^*), \overline{\boldsymbol{\psi}})$ where $\overline{\boldsymbol{\psi}} \in \Psi$ with*

$$\Psi := \left\{\overline{\boldsymbol{\psi}} \in \mathbb{R}^{nN} \mid \overline{\boldsymbol{\psi}} = (L^+\otimes I_n)\boldsymbol{x}^* + \mathbb{1}_N\otimes\zeta,\ \zeta\in\mathbb{R}^n\right\}, \tag{7}$$

and L^+ is the Moore–Penrose inverse of L.

Proof. At any equilibrium point $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{\psi}})$, we have

$$\mathbb{0} = \Pi_{\mathcal{X}}\left(\overline{\boldsymbol{x}}, -\boldsymbol{K}\,\mathrm{col}\left((f_i(\overline{x}_i, \overline{\sigma}_i))_{i\in\mathcal{I}}\right)\right), \tag{8a}$$

$$\mathbb{0} = -\overline{\boldsymbol{\sigma}} + \overline{\boldsymbol{x}} - (L\otimes I_n)\overline{\boldsymbol{\psi}}, \tag{8b}$$

$$\mathbb{0} = (L\otimes I_n)\overline{\boldsymbol{\sigma}}. \tag{8c}$$

As the graph is connected, from (8c), we have $\overline{\boldsymbol{\sigma}} = \mathbb{1}_N\otimes\gamma$ for some $\gamma\in\mathbb{R}^n$. Therefore, (8b) becomes

$$\mathbb{0} = -(\mathbb{1}_N\otimes\gamma) + \overline{\boldsymbol{x}} - (L\otimes I_n)\overline{\boldsymbol{\psi}}.$$

Left-multiplying both sides by $\mathbb{1}_N^\top\otimes I_n$ gives $\gamma = \frac{1}{N}\left(\mathbb{1}_N^\top\otimes I_n\right)\overline{\boldsymbol{x}} = s(\overline{\boldsymbol{x}})$. This means that $\overline{\boldsymbol{\sigma}} = \mathbb{1}_N\otimes s(\overline{\boldsymbol{x}})$ and in turn, $\overline{\sigma}_i = s(\overline{\boldsymbol{x}})$. Now, we use (8a) and Moreau's decomposition theorem to get

$$\begin{aligned}\mathbb{0} &= \Pi_{\mathcal{X}}\left(\overline{\boldsymbol{x}}, -\boldsymbol{K}\,\mathrm{col}\left((f_i(\overline{x}_i, s(\overline{\boldsymbol{x}})))_{i\in\mathcal{I}}\right)\right)\\ &= -\boldsymbol{K}\,\mathrm{col}\left((f_i(\overline{x}_i, s(\overline{\boldsymbol{x}})))_{i\in\mathcal{I}}\right) - \mathrm{proj}_{\mathcal{N}_{\mathcal{X}}(\overline{\boldsymbol{x}})}\left(-\boldsymbol{K}\,\mathrm{col}\left((f_i(\overline{x}_i, s(\overline{\boldsymbol{x}})))_{i\in\mathcal{I}}\right)\right),\end{aligned}$$

where $\mathcal{N}_{\mathcal{X}}(\overline{\boldsymbol{x}})$ is the normal cone of $\mathcal{X}$ at $\overline{\boldsymbol{x}}\in\mathcal{X}$. Thus, we have $-\boldsymbol{K}\,\mathrm{col}\left((f_i(\overline{x}_i, s(\overline{\boldsymbol{x}})))_{i\in\mathcal{I}}\right)\in\mathcal{N}_{\mathcal{X}}(\overline{\boldsymbol{x}})$. In other words, $\overline{\boldsymbol{x}}$ is the solution of $\mathrm{VI}(\mathcal{X}, \boldsymbol{K}\,\mathrm{col}((f_i(x_i, s(\boldsymbol{x})))_{i\in\mathcal{I}}))$, and from Lemma 2, we conclude that $\overline{\boldsymbol{x}} = \boldsymbol{x}^*$ and $\overline{\boldsymbol{\sigma}} = \mathbb{1}_N\otimes s(\boldsymbol{x}^*)$. In addition, by substituting the obtained values and using (A4), equality (8b) yields

$$(L \otimes I_n)\overline{\boldsymbol{\psi}} = \boldsymbol{x}^* - \mathbb{1}_N \otimes s(\boldsymbol{x}^*) = (\Pi \otimes I_n)\boldsymbol{x}^*.$$

Noting that $\Pi = LL^+ = L^+L$, we conclude that $\overline{\boldsymbol{\psi}}$ belongs to the set Ψ given by (7). ∎

Proposition 1, shows that equilibria of (6) are crafted as desired, namely $\overline{\boldsymbol{x}}$ and $\overline{\boldsymbol{\sigma}}$ return the NE of the game and the aggregation value $s(\boldsymbol{x}^*)$, respectively. The next theorem establishes convergence of the solution of (6) to such an equilibrium.[†]

Theorem 1. *Consider the NE seeking algorithm (6) with initial condition* $(\boldsymbol{x}(0), \boldsymbol{\sigma}(0), \boldsymbol{\psi}(0)) \in \mathcal{X} \times \mathbb{R}^{nN} \times \mathbb{R}^{nN}$. *Then, the solution* $(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\psi})$ *converges to the equilibrium point* $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{\psi}}) = (\boldsymbol{x}^*, \mathbb{1}_N \otimes s(\boldsymbol{x}^*), \boldsymbol{\psi}^*)$ *where* $\boldsymbol{x}^*$ *is the unique NE of the aggregative game* $\mathcal{G}_{\text{agg}}$ *and*

$$\boldsymbol{\psi}^* = (L^+ \otimes I_n)\boldsymbol{x}^* + \frac{1}{N}\left(\mathbb{1}_N \mathbb{1}_N^\top \otimes I_n\right)\boldsymbol{\psi}(0). \tag{9}$$

Proof. Let $\xi := \text{col}(\boldsymbol{x}, \boldsymbol{\sigma})$, $\Lambda := \mathcal{X} \times \mathbb{R}^{nN}$, $G := \text{col}(\mathbb{0}, L \otimes I_n)$, and with a little abuse of the notation $F(\xi) := F(\boldsymbol{x}, \boldsymbol{\sigma})$. Then, we can rewrite (6) as follows

$$\begin{aligned} \dot{\xi} &= \Pi_\Lambda\left(\xi, -F(\xi) - G\boldsymbol{\psi}\right), \\ \dot{\boldsymbol{\psi}} &= G^\top \xi. \end{aligned} \tag{10}$$

Let $\tilde{\xi} := \xi - \overline{\xi}$ and $\tilde{\boldsymbol{\psi}} := \boldsymbol{\psi} - \overline{\boldsymbol{\psi}}$ where $\text{col}(\overline{\xi}, \overline{\boldsymbol{\psi}})$ is an equilibrium point of (10). Note that $\overline{\xi} = \text{col}(\overline{\boldsymbol{x}}, \overline{\boldsymbol{\sigma}})$, and by Proposition 1, we have $\overline{\boldsymbol{x}} = \boldsymbol{x}^*$, $\overline{\boldsymbol{\sigma}} = \mathbb{1}_N \otimes s(\boldsymbol{x}^*)$, and $\overline{\boldsymbol{\psi}} \in \Psi$ with Ψ given by (7). Considering the Lyapunov candidate $V(\tilde{\xi}, \tilde{\boldsymbol{\psi}}) := \frac{1}{2}\|\text{col}(\tilde{\xi}, \tilde{\boldsymbol{\psi}})\|^2$, we have

$$\begin{aligned} \dot{V} &:= \nabla V(\tilde{\xi}, \tilde{\boldsymbol{\psi}})^\top \text{col}(\dot{\xi}, \dot{\boldsymbol{\psi}}) \\ &= \tilde{\xi}^\top \Pi_\Lambda\left(\xi, -F(\xi) - G\boldsymbol{\psi}\right) + \tilde{\boldsymbol{\psi}}^\top G^\top \xi, \end{aligned}$$

where $\text{col}(\dot{\xi}, \dot{\boldsymbol{\psi}})$ stands for the right hand side of (10). By Moreau's decomposition theorem, we find that

$$\tilde{\xi}^\top \Pi_\Lambda\left(\xi, -F(\xi) - G\boldsymbol{\psi}\right) = \tilde{\xi}^\top \left(-F(\xi) - G\boldsymbol{\psi} - \text{proj}_{\mathcal{N}_\Lambda(\xi)}\left(-F(\xi) - G\boldsymbol{\psi}\right)\right).$$

Noting $\xi, \overline{\xi} \in \Lambda$, we have $-\tilde{\xi}^\top \text{proj}_{\mathcal{N}_\Lambda(\xi)}(-F(\xi) - G\boldsymbol{\psi}) \leq 0$, and $\dot{V}$ admits the following inequality

$$\dot{V} \leq -\tilde{\xi}^\top F(\xi) - \tilde{\xi}^\top G\boldsymbol{\psi} + \tilde{\boldsymbol{\psi}}^\top G^\top \xi. \tag{11}$$

Moreover, from (8) and Moreau's decomposition theorem we get

$$\begin{aligned} 0 &= \tilde{\xi}^\top \Pi_\Lambda\left(\overline{\xi}, -F(\overline{\xi}) - G\overline{\boldsymbol{\psi}}\right) \\ &= \tilde{\xi}^\top \left(-F(\overline{\xi}) - G\overline{\boldsymbol{\psi}} - \text{proj}_{\mathcal{N}_\Lambda(\overline{\xi})}\left(-F(\overline{\xi}) - G\overline{\boldsymbol{\psi}}\right)\right). \end{aligned}$$

Since $-\tilde{\xi}^\top \text{proj}_{\mathcal{N}_\Lambda(\overline{\xi})}(-F(\overline{\xi}) - G\overline{\boldsymbol{\psi}}) \geq 0$, we conclude that $\tilde{\xi}^\top(F(\overline{\xi}) + G\overline{\boldsymbol{\psi}}) \geq 0$. Consequently, we use (8c) and the ϵ-strongly monotonicity of $F(\xi)$ (Lemma 1(*i*)) to rewrite (11) as

$$\dot{V} \leq -\tilde{\xi}^\top \left(F(\xi) - F(\overline{\xi})\right) - \tilde{\xi}^\top G\tilde{\boldsymbol{\psi}} + \tilde{\boldsymbol{\psi}}^\top G^\top \xi \leq -\epsilon\|\tilde{\xi}\|^2,$$

It then follows from the invariance principle for discontinuous systems (proposition 2.1 in Reference 39) that $(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\psi})$ converges to $\Omega = \{(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\psi}) | \boldsymbol{x} = \overline{\boldsymbol{x}},\ \boldsymbol{\sigma} = \overline{\boldsymbol{\sigma}}, \boldsymbol{\psi} \in \Psi\}$ where Ψ is given by (7). Note that $(\mathbb{1}^\top \otimes I_n)\boldsymbol{\psi}(t)$ is a conserved quantity of the system and $\mathbb{1}^\top L^+ = 0$. Then, by (7), we find that the vector $\boldsymbol{\psi}$ converges to $\boldsymbol{\psi}^* = (L^+ \otimes I_n)\boldsymbol{x}^* + \frac{1}{N}\left(\mathbb{1}_N \mathbb{1}_N^\top \otimes I_n\right)\boldsymbol{\psi}(0)$, which completes the proof. ∎

Remark 3. The NE seeking algorithm (6) can be extended to strongly connected[‡] directed communication graphs as follows:

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \Pi_{\mathcal{X}}\left(\boldsymbol{x}, -\boldsymbol{K}\mathrm{col}\left((f_i(x_i(t), \sigma_i(t)))_{i\in\mathcal{I}}\right)\right),\\
\dot{\boldsymbol{\sigma}}(t) &= -\boldsymbol{\sigma}(t) + \boldsymbol{x}(t) - (B\otimes I_n)\boldsymbol{\psi}(t),\\
\dot{\boldsymbol{\psi}}(t) &= (B^\top \otimes I_n)\boldsymbol{\sigma}(t),
\end{aligned}$$

where the matrix B is the incidence matrix of the graph. Convergence of the above algorithm to the NE follows from arguments analogous to those used for Theorem 1. We note, however, that the extension is at the expense of assuming that each player i has access to the aggregated information of its neighboring edges. This is due to the presence of the term $(B\otimes I_n)\boldsymbol{\psi}(t)$ in the modified dynamics. •

4 | ROBUSTNESS AND PRIVACY ANALYSIS

In this section, we investigate robustness of the proposed algorithm with respect to additive time-varying disturbances and carry out privacy analysis. Throughout this section, we consider the case where $\mathcal{X}_i = \mathbb{R}^n$ and the cost function J_i is C^2 in all its arguments for all $i\in\mathcal{I}$, and Assumption 2 holds.

Then, the NE seeking dynamics (6) become smooth as follows

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= -\boldsymbol{K}\mathrm{col}\left((f_i(x_i(t), \sigma_i(t)))_{i\in\mathcal{I}}\right),\\
\dot{\boldsymbol{\sigma}}(t) &= -\boldsymbol{\sigma}(t) + \boldsymbol{x}(t) - (L\otimes I_n)\boldsymbol{\psi}(t),\\
\dot{\boldsymbol{\psi}}(t) &= (L \otimes I_n)\boldsymbol{\sigma}(t).
\end{aligned}\tag{12}$$

This enables us to provide robustness guarantees using the notions of input-to-state stability (ISS) and $\mathcal{L}_2$-stability. Moreover, we remark that to our best knowledge, there is no theory for ISS of projected dynamical systems, which led us to work with $\mathbb{R}^n$ rather than $\mathcal{X}_i$.

4.1 | Robustness analysis

We analyze robustness of the dynamical algorithm (12) against additive perturbations. The perturbations can capture possible deviation of the players from a fully rational behavior or a deliberate addition of noise to improve privacy.

Let $\boldsymbol{\xi} := \mathrm{col}(\boldsymbol{x},\boldsymbol{\sigma})$, $G := \mathrm{col}(\mathbb{0}, L\otimes I_n)$, and with some abuse of the notation $F(\boldsymbol{\xi}) := F(\boldsymbol{x},\boldsymbol{\sigma})$ with $F(\boldsymbol{x},\boldsymbol{\sigma})$ given by (3). Then, we can compactly rewrite (12) with the disturbance $\boldsymbol{\nu} : \mathbb{R}_{\geq 0} \to\in \mathbb{R}^{2nN}$ as follows

$$\begin{aligned}
\dot{\boldsymbol{\xi}}(t) &= -F(\boldsymbol{\xi}(t)) - G\boldsymbol{\psi}(t) + \boldsymbol{\nu}(t),\\
\dot{\boldsymbol{\psi}}(t) &= G^\top\boldsymbol{\xi}(t).
\end{aligned}\tag{13}$$

To analyze performance of the above algorithm, we resort to the notion of input-to-state stability (ISS).[40] The next theorem presents our main results in this regard.

Theorem 2. *Consider the NE seeking algorithm (13) with initial condition* $(\boldsymbol{\xi}(0),\boldsymbol{\psi}(0)) \in \mathbb{R}^{2nN}\times\mathbb{R}^{nN}$. *Suppose the disturbance* $t\mapsto \boldsymbol{\nu}(t)$ *is piecewise continuous and bounded for all* $t\in[0,\infty)$, *and assume that there exists some positive constant* γ_i *such that* $\|\nabla f_i(x_i,\sigma_i)\| \leq \gamma_i$ *for all* $x_i, \sigma_i\in\mathbb{R}^n$ *and* $i\in\mathcal{I}$. *Let* $\boldsymbol{\xi}^* := \mathrm{col}(\boldsymbol{x}^*, \mathbb{1}_N\otimes s(\boldsymbol{x}^*))$ *and* $\boldsymbol{\psi}^*$ *be given by (9). Then, the corresponding solution of (13) satisfies*

$$\|\mathrm{col}(\tilde{\boldsymbol{\xi}}(t), \tilde{\boldsymbol{\psi}}(t))\| \leq \beta_0 e^{-\lambda t}\|\mathrm{col}(\tilde{\boldsymbol{\xi}}(0), \tilde{\boldsymbol{\psi}}(0))\| + \beta_1 \sup_{0\leq\tau\leq t}\|\boldsymbol{\nu}(\tau)\|, \quad \forall t\geq 0,\tag{14}$$

where $\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi}-\boldsymbol{\xi}^*$, $\tilde{\boldsymbol{\psi}} = \boldsymbol{\psi}-\boldsymbol{\psi}^*$, *and* $\beta_0,\beta_1,\lambda\in\mathbb{R}_{>0}$.

Proof. Note that $\boldsymbol{\psi}^* \in \Psi$ with Ψ given by (7), thus it follows from Proposition 1 that the pair $(\boldsymbol{\xi}^*, \boldsymbol{\psi}^*)$ satisfies

$$\mathbb{0} = -F(\boldsymbol{\xi}^*) - G\boldsymbol{\psi}^*, \tag{15a}$$

$$\mathbb{0} = G^\top \boldsymbol{\xi}^*. \tag{15b}$$

Consider the solution $(\boldsymbol{\xi}(t), \boldsymbol{\psi}(t))$ of (13) initialized at $(\boldsymbol{\xi}(0), \boldsymbol{\psi}(0))$. Using the above equalities, we notice that this solution satisfies

$$\begin{aligned} \dot{\tilde{\boldsymbol{\xi}}} &= -(F(\boldsymbol{\xi}) - F(\boldsymbol{\xi}^*)) - G\tilde{\boldsymbol{\psi}} + \boldsymbol{v}, \\ \dot{\tilde{\boldsymbol{\psi}}} &= G^\top \tilde{\boldsymbol{\xi}}, \end{aligned} \tag{16}$$

where $\tilde{\boldsymbol{\xi}}$ and $\tilde{\boldsymbol{\psi}}$ are given in the statement of the theorem. Define $\boldsymbol{\Pi} := \Pi \otimes I_n$ where $\Pi = I - \frac{1}{N}\mathbb{1}_N\mathbb{1}_N^\top$. We next show that $\tilde{\boldsymbol{\psi}}(t) \in \operatorname{im} \boldsymbol{\Pi}$ for all $t \geq 0$. Recall the definition of $\boldsymbol{\psi}^*$ given by (9), it then follows from $\tilde{\boldsymbol{\psi}}(0) = \boldsymbol{\psi}(0) - \boldsymbol{\psi}^*$ that $(\mathbb{1}_N^\top \otimes I_n)\tilde{\boldsymbol{\psi}}(0) = \mathbb{0}$. Thus, bearing in mind that $(\mathbb{1}_N^\top \otimes I_n)\tilde{\boldsymbol{\psi}}(t)$ is a conserved quantity of the dynamics (16), we obtain $\mathbb{1}_N \otimes (\mathbb{1}_N^\top \otimes I_n)\tilde{\boldsymbol{\psi}}(t) = \mathbb{0}$ for all $t \geq 0$. The latter can be written as $(\mathbb{1}_N\mathbb{1}_N^\top \otimes I_n)\tilde{\boldsymbol{\psi}}(t) = \mathbb{0}$, which results in $\boldsymbol{\Pi}\tilde{\boldsymbol{\psi}}(t) = \tilde{\boldsymbol{\psi}}(t)$, that is, $\tilde{\boldsymbol{\psi}}(t) \in \operatorname{im} \boldsymbol{\Pi}$.

In view of the ISS result of theorem 4.19 in Reference 41, while bearing in mind that $(\tilde{\boldsymbol{\xi}}(t), \tilde{\boldsymbol{\psi}}(t)) \in \mathbb{R}^{2nN} \times \operatorname{im}\boldsymbol{\Pi}$ for all $t \geq 0$, it suffices to find a Lyapunov function satisfying a suitable dissipation inequality. Namely, to find a continuously differentiable function $V : \mathbb{R}^{2nN} \times \operatorname{im} \boldsymbol{\Pi} \to \mathbb{R}$, for which there exist class $\mathcal{K}_\infty$ functions α_1, α_2, a class $\mathcal{K}$ function ρ, and a continuous positive definite function $W(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\psi}})$ such that

$$\alpha_1(\|\operatorname{col}(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\psi}})\|) \leq V(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\psi}}) \leq \alpha_2(\|\operatorname{col}(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\psi}})\|), \tag{17}$$

$$\frac{\partial V}{\partial \tilde{\boldsymbol{\xi}}}^\top \dot{\tilde{\boldsymbol{\xi}}} + \frac{\partial V}{\partial \tilde{\boldsymbol{\psi}}}^\top \dot{\tilde{\boldsymbol{\psi}}} \leq -W(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\psi}}), \ \forall \ \|\operatorname{col}(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\psi}})\| \geq \rho(\|\boldsymbol{v}\|) > 0, \tag{18}$$

for all $(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\psi}}) \in \mathbb{R}^{2nN} \times \operatorname{im} \boldsymbol{\Pi}$ and $\boldsymbol{v} \in \mathbb{R}^{2nN}$. Choose the Lyapunov candidate $V(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\psi}}) := \frac{1}{2}\operatorname{col}(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\psi}})^\top P_0 \ \operatorname{col}(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\psi}})$ where

$$P_0 := \begin{bmatrix} I & \kappa G \\ \kappa G^\top & I \end{bmatrix}, \quad \kappa > 0.$$

By considering $\alpha_1(\|\operatorname{col}(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\psi}})\|) = \alpha_1\|\operatorname{col}(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\psi}})\|^2$ with some $\alpha_1 > 0$, the first inequality of (17) is satisfied if and only if $P_0 - 2\alpha_1 I \succcurlyeq 0$ which is equivalent to

$$1 - 2\alpha_1 > 0, \qquad (1 - 2\alpha_1)^2 I - \kappa^2 GG^\top \succcurlyeq 0.$$

Let $\kappa_1 := 1/\lambda_{\max}(L)$ where $\lambda_{\max}(L)$ is the maximum eigenvalue of L, then the above inequalities are satisfied for $\kappa \in (0, \kappa_1)$ and $\alpha_1 = (1 - \kappa\lambda_{\max}(L))/2$. For the second inequality of (17), an analogous argument can be used to obtain $\alpha_2(\|\operatorname{col}(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\psi}})\|) = \alpha_2\|\operatorname{col}(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\psi}})\|^2$ with $\alpha_2 = (1 + \kappa\lambda_{\max}(L))/2$.

We take the derivative of V along (16) and use the ϵ-strong monotonicity of $F(\boldsymbol{\xi})$ to obtain

$$\frac{\partial V}{\partial \tilde{\boldsymbol{\xi}}}^\top \dot{\tilde{\boldsymbol{\xi}}} + \frac{\partial V}{\partial \tilde{\boldsymbol{\psi}}}^\top \dot{\tilde{\boldsymbol{\psi}}} \leq -\epsilon\|\tilde{\boldsymbol{\xi}}\|^2 + \kappa\|G^\top\tilde{\boldsymbol{\xi}}\|^2 - \kappa\tilde{\boldsymbol{\psi}}^\top G^\top(F(\boldsymbol{\xi}) - F(\boldsymbol{\xi}^*)) - \kappa\|G\tilde{\boldsymbol{\psi}}\|^2 + (\tilde{\boldsymbol{\xi}} + \kappa G\tilde{\boldsymbol{\psi}})^\top \boldsymbol{v}. \tag{19}$$

Define $U(\boldsymbol{\xi}, \boldsymbol{\xi}^*) := \int_0^1 \nabla F(\boldsymbol{\xi}^* + h(\boldsymbol{\xi} - \boldsymbol{\xi}^*))\, dh$. Then, by the fundamental theorem of calculus, we have $F(\boldsymbol{\xi}) - F(\boldsymbol{\xi}^*) = U(\boldsymbol{\xi}, \boldsymbol{\xi}^*)\tilde{\boldsymbol{\xi}}$. Consequently, the equation (19) becomes

$$\frac{\partial V}{\partial \tilde{\boldsymbol{\xi}}}^\top \dot{\tilde{\boldsymbol{\xi}}} + \frac{\partial V}{\partial \tilde{\boldsymbol{\psi}}}^\top \dot{\tilde{\boldsymbol{\psi}}} \leq -\frac{\epsilon}{2}\|\tilde{\boldsymbol{\xi}}\|^2 - \frac{\kappa}{2}\|G\tilde{\boldsymbol{\psi}}\|^2 - \operatorname{col}(\tilde{\boldsymbol{\xi}}, G\tilde{\boldsymbol{\psi}})^\top P(\boldsymbol{\xi}, \boldsymbol{\xi}^*)\ \operatorname{col}(\tilde{\boldsymbol{\xi}}, G\tilde{\boldsymbol{\psi}}) + \operatorname{col}(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\psi}})^\top R\ \boldsymbol{v}, \tag{20}$$

where

$$P(\boldsymbol{\xi}, \boldsymbol{\xi}^*) := \frac{1}{2}\begin{bmatrix} \epsilon I - 2\kappa GG^\top & \kappa U(\boldsymbol{\xi}, \boldsymbol{\xi}^*)^\top \\ \kappa U(\boldsymbol{\xi}, \boldsymbol{\xi}^*) & \kappa I \end{bmatrix}, \quad R := \begin{bmatrix} I \\ \kappa G^\top \end{bmatrix}.$$

Clearly, the matrix P is positive definite if and only if

$$\kappa > 0, \quad \epsilon I - 2\kappa GG^\top - \kappa U(\xi, \xi^*)^\top U(\xi, \xi^*) \succ \mathbb{0},$$

for all $\xi \in \mathbb{R}^{2nN}$. By using $\|\nabla f_i(x_i, \sigma_i)\| \leq \gamma_i$, it is straightforward to investigate that $\|U(\cdot,\cdot)\|^2 \leq \overline{\gamma}^2 + 2$ where $\overline{\gamma} := \max_{i \in \mathcal{I}}(\gamma_i k_i)$. We then conclude that $P \succ \mathbb{0}$ if $\kappa \in (0, \kappa_2)$ with $\kappa_2 := \epsilon/(\overline{\gamma}^2 + 2 + 2\lambda_{\max}(L)^2)$. Moreover, note that for all $\tilde{\boldsymbol{\psi}} \in \operatorname{im} \boldsymbol{\Pi}$, we have

$$\|G\tilde{\boldsymbol{\psi}}\| \geq \lambda_{\min}(L)\|\tilde{\boldsymbol{\psi}}\|,$$

where $\lambda_{\min}(L)$ is the smallest nonzero eigenvalue of L. Thus, we deduce from (20) that for any $\kappa \in (0, \kappa_2)$ and all $(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\psi}}) \in \mathbb{R}^{2nN} \times \operatorname{im} \boldsymbol{\Pi}$, the following inequalities hold:

$$\begin{aligned}\frac{\partial V}{\partial \tilde{\xi}}^\top \dot{\tilde{\xi}} + \frac{\partial V}{\partial \tilde{\boldsymbol{\psi}}}^\top \dot{\tilde{\boldsymbol{\psi}}} &\leq -\frac{\epsilon}{2}\|\tilde{\xi}\|^2 - \frac{\kappa}{2}\lambda_{\min}(L)^2\|\tilde{\boldsymbol{\psi}}\|^2 + \operatorname{col}(\tilde{\xi}, \tilde{\boldsymbol{\psi}})^\top R\, \boldsymbol{v} \\ &\leq -\frac{\kappa}{2}\lambda_{\min}(L)^2\|\operatorname{col}(\tilde{\xi}, \tilde{\boldsymbol{\psi}})\|^2 + \|R\|\|\operatorname{col}(\tilde{\xi}, \tilde{\boldsymbol{\psi}})\|\|\boldsymbol{v}\|,\end{aligned}$$

where $\|R\| = \sqrt{1 + \kappa^2\lambda_{\max}(L)^2}$. Hence, (18) is obtained by setting $W(\tilde{\xi}, \tilde{\boldsymbol{\psi}}) = \alpha_3\|\operatorname{col}(\tilde{\xi}, \tilde{\boldsymbol{\psi}})\|^2$ and $\rho(\|\boldsymbol{v}\|) = \alpha_4\|\boldsymbol{v}\|$ with $\alpha_3 = \frac{\kappa}{4}\lambda_{\min}(L)^2$ and $\alpha_4 = \frac{4}{\kappa\lambda_{\min}(L)^2}\|R\|$. Consequently, (17) and (18) are satisfied for any $0 < \kappa < \min\{\kappa_1, \kappa_2\}$, and it follows from theorem 4.19 in Reference 41 that the inequality (14) is satisfied with

$$\beta_0 = \sqrt{\frac{\alpha_2}{\alpha_1}}, \quad \lambda = \frac{\alpha_3}{2\alpha_2}, \quad \beta_1 = \alpha_4\sqrt{\frac{\alpha_2}{\alpha_1}}. \tag{21}$$

■

Remark 4. [§]In the disturbance-free case, that is, $v(\cdot) = 0$, the treatment in Theorem 2 proves *exponential* convergence of the proposed algorithm with the rate of convergence given by (21). Similarly, in the case of general games and by considering suitable assumptions on the pesudo-gradient mapping, the presented NE seeking algorithm in Reference 10 is exponentially stable (see theorems 1 and 2 in Reference 10). Therefore, it is also ISS with respect to additive time-varying disturbances as a result of lemma 4.6 in Reference 41. However, that algorithm is fundamentally different than ours, which makes the analysis dissimilar. Specifically, the consensus term in Reference 10 appears as damping on the relative state variables, which contributes to the exponential convergence property. For our presented algorithm, the consensus action appears as cross terms, resulting in the presence of undamped communicating variables $\boldsymbol{\psi}$ in (13). To overcome this technical difficulty, we included a sufficiently small cross-term in the ISS Lyapunov function. •

Remark 5. The assumption of the boundedness of $\|\nabla f_i(x_i, \sigma_i)\|$ can be relaxed at the expense of establishing ISS in a local sense. In particular, let $\|\nabla f_i(x_i, \sigma_i)\| < \gamma_i$ for all (x_i, σ_i) that belong to a compact set $\|\operatorname{col}(x_i - x_i^*, \sigma_i - s(\boldsymbol{x}^*))\| < r_i$ with some $r_i > 0$. Then, analogous to the proof of Theorem 2, the matrix P in (20) is positive definite for any $\kappa \in (0, \kappa_2)$ and $\|\tilde{\xi}\| < r$ with $r := \min_{i \in \mathcal{I}} r_i$. Therefore, (17) and (18) are satisfied for any $0 < \kappa < \min\{\kappa_1, \kappa_2\}$ and for all $\|\operatorname{col}(\tilde{\xi}, \tilde{\boldsymbol{\phi}})\| < r$. By using theorem 4.18 and exercise 4.60 in Reference 41, we conclude that (14) is satisfied for $\|\operatorname{col}(\tilde{\xi}(0), \tilde{\boldsymbol{\psi}}(0))\| < r\sqrt{\alpha_1/\alpha_2}$ and $\sup_{t \geq 0}\|\boldsymbol{v}(t)\| < (r/\alpha_4)\sqrt{\alpha_1/\alpha_2}$. •

The established ISS property provides stability guarantees for *all* state variables in the presence of disturbance. As the objective of considering algorithm (13) is NE computation, it is advantageous to explicitly analyze the effect of disturbance on $\boldsymbol{\xi} = \operatorname{col}(\boldsymbol{x}, \boldsymbol{\sigma})$. We pursue this by using the notion of $\mathcal{L}_2$-stability (definition 5.1 in Reference 41). Let the performance output $\boldsymbol{y}$ be defined as

$$\boldsymbol{y} := \xi - \xi^*, \tag{22}$$

where $\boldsymbol{\xi}^* := \operatorname{col}(\boldsymbol{x}^*, \mathbb{1}_N \otimes s(\boldsymbol{x}^*))$. Then, the dynamics (13) is $\mathcal{L}_2$-stable from input $\boldsymbol{v}$ to output $\boldsymbol{y}$ with the $\mathcal{L}_2$-gain less than or equal to $\delta \in \mathbb{R}_{>0}$ if for any $(\boldsymbol{\xi}(0), \boldsymbol{\psi}(0)) \in \mathbb{R}^{2nN} \times \mathbb{R}^{nN}$, there exists some constant $\beta \geq 0$ such that

$$\|\boldsymbol{y}_\tau\|_{\mathcal{L}_2} \leq \delta\|\boldsymbol{v}_\tau\|_{\mathcal{L}_2} + \beta,$$

for all $\boldsymbol{v} \in \mathcal{L}_{2e}^{2nN}$ and $\tau \in \mathbb{R}_{\geq 0}$ (see Section 2.1 for related definitions).

Theorem 3. *Let the disturbance signal $t \mapsto \boldsymbol{v}(t)$ belong to the extended space $\mathcal{L}_{2e}^{2nN}$. Then, the dynamics (13) with the performance output $\boldsymbol{y}$ given by (22) is $\mathcal{L}_2$-stable from input $\boldsymbol{v}$ to output $\boldsymbol{y}$ with its $\mathcal{L}_2$-gain satisfying*

$$\mathcal{L}_2\text{-gain} \leq \frac{1}{\min_{i\in\mathcal{I}} \epsilon_i}, \tag{23}$$

where $\epsilon_i = \frac{1}{2}\left(k_i\mu_i + 1 - \sqrt{(k_i\mu_i - 1)^2 + (k_i\ell_i + 1)^2}\right)$.

Proof. Consider the error dynamics (16), and let $V(\tilde{\xi}, \tilde{\psi}) := \frac{1}{2}\|\text{col}(\tilde{\xi}, \tilde{\psi})\|^2$. Then, we obtain

$$\dot{V} = -\tilde{\xi}^\top(F(\xi) - F(\xi^*)) + \tilde{\xi}^\top \boldsymbol{v} \leq -\epsilon\|\tilde{\xi}\|^2 + \tilde{\xi}^\top \boldsymbol{v}\ ,$$

where the inequality follows from the ϵ-strong monotonicity of $F(\xi)$. Noting $\boldsymbol{y} = \tilde{\xi}$, the above inequality yields

$$\dot{V} \leq \boldsymbol{y}^\top\boldsymbol{v} - \epsilon\boldsymbol{y}^\top\boldsymbol{y}\ .$$

By adding and subtracting $\frac{1}{2\epsilon}\boldsymbol{v}^\top\boldsymbol{v} + \frac{\epsilon}{2}\boldsymbol{y}^\top\boldsymbol{y}$ from the right hand side, we obtain

$$\begin{aligned}\dot{V} &\leq -\frac{1}{2\epsilon}(\boldsymbol{v} - \epsilon\boldsymbol{y})^\top(\boldsymbol{v} - \epsilon\boldsymbol{y}) + \frac{1}{2\epsilon}\boldsymbol{v}^\top\boldsymbol{v} - \frac{\epsilon}{2}\boldsymbol{y}^\top\boldsymbol{y}\\ &\leq \frac{1}{2\epsilon}\boldsymbol{v}^\top\boldsymbol{v} - \frac{\epsilon}{2}\boldsymbol{y}^\top\boldsymbol{y} = \frac{\epsilon}{2}\left(\frac{1}{\epsilon^2}\boldsymbol{v}^\top\boldsymbol{v} - \boldsymbol{y}^\top\boldsymbol{y}\right).\end{aligned}$$

Hence, the storage function $\frac{1}{\epsilon}V$ certifies $\mathcal{L}_2$-stability of the system with $\mathcal{L}_2$-gain $\leq \frac{1}{\epsilon}$ (see chapter 8 of Reference 42). Bearing in mind (A2) in the proof of Lemma 1, the parameter ϵ can be explicitly chosen as $\epsilon = \min_{i\in\mathcal{I}} \epsilon_i$ where ϵ_i is the smallest eigenvalue of the matrix

$$\begin{bmatrix} k_i\mu_i & -\frac{(k_i\ell_i+1)}{2} \\ -\frac{(k_i\ell_i+1)}{2} & 1 \end{bmatrix}.$$

The proof concludes by computing this minimum eigenvalue. ■

Theorem 3 characterizes the $\mathcal{L}_2$-stability of the algorithm in terms of the design parameters k_i's and the parameters of the individual cost functions. Aiming at minimizing the effect of noise on the performance output $\boldsymbol{y} = \xi$, one can view the right hand side of (23) as a function of k_i's satisfying (4), and seek for its minimizer. By direct computation, this minimizer is obtained as

$$k_i^* := \frac{\mu_i^2 - \ell_i^2 + 2\mu_i\ell_i}{\ell_i(\mu_i^2 + \ell_i^2)}.$$

4.2 | Privacy analysis

Here, we turn our attention to privacy properties of the presented NE seeking algorithm. The general idea is that privacy is preserved if a curious party cannot uniquely reconstruct the actual private variables/quantities of a player. A curious party can be one of the players or an external adversary. The adversary model is honest-but-curious meaning that it does not interfere with the implementation of the algorithm but rather tries to infer the private quantities of interest based on available information and/or eavesdropping.

In order to estimate the privacy-sensitive quantities of the players, a curious adversary generally needs to employ the accessible information and implement a *reverse engineering* or an *identification* mechanism. The adversary is consequently more likely to succeed when the cost functions of the players and the game dynamics are less complex. Motivated by this fact and to provide more explicit results, we perform a "worst-case" privacy analysis by considering cost functions that result in linear NE seeking dynamics. Note that parameter identification is much more probable for the adversary when the dynamics are linear. As such, we restrict our attention to cost functions given by

$$J_i(x_i, s(\boldsymbol{x})) := x_i^\top Q_i x_i + (D_i\ s(\boldsymbol{x}) + d_i)^\top x_i,$$

where $Q_i = Q_i^\top \in \mathbb{R}^{n\times n}$, $Q_i \succ \mathbb{0}$, $D_i \in \mathbb{R}^{n\times n}$, and $d_i \in \mathbb{R}^n$. Note that in this case, Assumption 2 reduces to

$$\lambda_{\min}\left(2Q_i + \frac{1}{2N}(D_i + D_i^\top)\right) > \|D_i\|. \tag{24}$$

Let

$$\begin{aligned} \boldsymbol{A} &:= \text{blockdiag}\left(2Q_i + \frac{1}{N}D_i^\top\right),\ \boldsymbol{D} := \text{blockdiag}(D_i), \\ \boldsymbol{d} &:= \text{col}(d_i), \quad \forall i \in \mathcal{I}. \end{aligned} \tag{25}$$

Then, (12) reduces to

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}_q \boldsymbol{z}(t) + \boldsymbol{D}_q, \tag{26}$$

where

$$\begin{aligned} \boldsymbol{A}_q &:= \begin{bmatrix} -\boldsymbol{KA} & -\boldsymbol{KD} & \mathbb{0} \\ I & -I & -(L\otimes I_n) \\ \mathbb{0} & (L\otimes I_n) & \mathbb{0} \end{bmatrix}, \\ \boldsymbol{z} &:= \text{col}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\psi}), \quad \boldsymbol{D}_q := \text{col}(-\boldsymbol{Kd}, \mathbb{0}, \mathbb{0}). \end{aligned} \tag{27}$$

The cost parameters Q_i, D_i, and d_i associated to each player i will be treated as *private information*. Note that the design parameter k_i and the action of each player $x_i(t)$ are not readily accessible to the other players. On the contrary, both $\sigma_i(t)$ and $\psi_i(t)$ are communicated to other agents. Therefore, the latter information is accessible to other players due to direct communication, or to an adversary as a result of eavesdropping. To pursue a worst-case privacy analysis, we consider the scenario where all communicated variables $(\boldsymbol{\sigma}(t), \boldsymbol{\psi}(t))$ are subject to eavesdropping and the Laplacian matrix L is completely known to the adversary as *side knowledge*. Moreover, the goal and structure of the algorithm are considered public, that is, accessible to any curious party. Now, we consider the following definition:

Definition 1. Privacy of a player $i \in \mathcal{I}$ is preserved if its private information, namely the triple (Q_i, D_i, d_i), is not uniquely identifiable from the available information to the adversary. In addition, we say that an algorithm preserves privacy if privacy is preserved for all players. •

Note that the privacy property in Definition 1 is valid even if $N-1$ players collude to obtain private information of one specific player. The following result establishes the privacy preservation property of the presented algorithm.

Theorem 4. *The NE seeking algorithm (26) preserves privacy in the sense of Definition 1.*[¶]

Proof. Recall that the structure of the algorithm and the Laplacian matrix L is known to the adversary. Such knowledge can be embedded in the following *replica* of (26):

$$\begin{aligned} \dot{\boldsymbol{z}}'(t) &= \boldsymbol{A}_q' \boldsymbol{z}'(t) + \boldsymbol{D}_q', \\ \boldsymbol{A}_q' &:= \begin{bmatrix} -\boldsymbol{K}'\boldsymbol{A}' & -\boldsymbol{K}'\boldsymbol{D}' & \mathbb{0} \\ I & -I & -(L\otimes I_n) \\ \mathbb{0} & (L\otimes I_n) & \mathbb{0} \end{bmatrix}, \\ \boldsymbol{z}' &:= \text{col}(\boldsymbol{x}', \boldsymbol{\sigma}', \boldsymbol{\psi}'), \boldsymbol{D}_q' := \text{col}(-\boldsymbol{K}'\boldsymbol{d}', \mathbb{0}, \mathbb{0}), \end{aligned} \tag{28}$$

where the vectors and matrices with "prime" are defined analogously to the ones without in (27). On top of that the adversary has access to $(\boldsymbol{\sigma}(t), \boldsymbol{\psi}(t))$ via eavesdropping. To establish the proof, we need to show that there exists a triple $(Q_i', D_i', d_i') \neq (Q_i, D_i, d_i)$ which is consistent with the replica dynamics (28) as well as the eavesdropped information $(\boldsymbol{\sigma}(t), \boldsymbol{\psi}(t)) = (\boldsymbol{\sigma}'(t), \boldsymbol{\psi}'(t))$. This would show that (Q_i, D_i, d_i) is not uniquely identifiable from the knowledge of the adversary.

We proceed with the proof by defining $\boldsymbol{y} := \mathrm{col}(\boldsymbol{\sigma}, \boldsymbol{\psi})$, thus we have

$$\boldsymbol{y}(t) = \boldsymbol{C}_q \boldsymbol{z}(t), \qquad \boldsymbol{C}_q := \begin{bmatrix} \mathbb{0} & I & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & I \end{bmatrix}.$$

It also follows that the condition $(\boldsymbol{\sigma}'(t), \boldsymbol{\psi}'(t)) = (\boldsymbol{\sigma}(t), \boldsymbol{\psi}(t))$ becomes

$$\boldsymbol{y}(t) = \boldsymbol{C}_q \boldsymbol{z}(t) = \boldsymbol{C}_q \boldsymbol{z}'(t), \quad \forall t \geq 0. \tag{29}$$

Since (26) and (28) are linear dynamics under constant inputs, it follows from analogous arguments to proposition 1 in Reference 43 that the condition (29) is satisfied if and only if there exists $\boldsymbol{z}'(0)$ such that

$$\begin{aligned} \boldsymbol{C}_q \boldsymbol{A}_q^k \boldsymbol{z}(0) &= \boldsymbol{C}_q \boldsymbol{A}_q'^k \boldsymbol{z}'(0), \\ \boldsymbol{C}_q \boldsymbol{A}_q^k \boldsymbol{D}_q &= \boldsymbol{C}_q \boldsymbol{A}_q'^k \boldsymbol{D}_q', \end{aligned} \quad \forall k \in \mathbb{N} \cup \{0\}. \tag{30}$$

Verifying (30) for $k = 0$ results in

$$\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}'(0), \quad \boldsymbol{\psi}(0) = \boldsymbol{\psi}'(0),$$

as expected. For $k = 1$, we use the above equations and obtain

$$\boldsymbol{x}(0) = \boldsymbol{x}'(0), \quad \boldsymbol{K}\boldsymbol{d} = \boldsymbol{K}'\boldsymbol{d}'.$$

By continuing this process, we see that the condition (30) becomes

$$\begin{aligned} (\boldsymbol{\sigma}(0), \boldsymbol{\psi}(0)) &= (\boldsymbol{\sigma}'(0), \boldsymbol{\psi}'(0)), & (\boldsymbol{K}\boldsymbol{A})^k \boldsymbol{K}\boldsymbol{D} &= (\boldsymbol{K}'\boldsymbol{A}')^k \boldsymbol{K}'\boldsymbol{D}', \\ (\boldsymbol{K}\boldsymbol{A})^k \boldsymbol{x}(0) &= (\boldsymbol{K}'\boldsymbol{A}')^k \boldsymbol{x}'(0), & (\boldsymbol{K}\boldsymbol{A})^k \boldsymbol{K}\boldsymbol{d} &= (\boldsymbol{K}'\boldsymbol{A}')^k \boldsymbol{K}'\boldsymbol{d}', \end{aligned}$$

for all $k \in \mathbb{N} \cup \{0\}$. This implies that

$$\begin{aligned} \boldsymbol{z}(0) &= \boldsymbol{z}'(0), & \boldsymbol{K}\boldsymbol{D} &= \boldsymbol{K}'\boldsymbol{D}', \\ \boldsymbol{K}\boldsymbol{A} &= \boldsymbol{K}'\boldsymbol{A}', & \boldsymbol{K}\boldsymbol{d} &= \boldsymbol{K}'\boldsymbol{d}'. \end{aligned}$$

We further deduce from the definitions given by (25) that

$$k_i Q_i = k_i' Q_i', \quad k_i D_i = k_i' D_i', \quad k_i d_i = k_i' d_i', \quad \forall i \in \mathcal{I}. \tag{31}$$

Consequently, since k_i is unknown to an adversary, it cannot uniquely reconstruct the privacy-sensitive triple (Q_i, D_i, d_i) of player i. Namely, there exists $(Q_i', D_i', d_i') \neq (Q_i, D_i, d_i)$ that is indistinguishable from (Q_i, D_i, d_i) based on the knowledge accessible to the adversary. This completes the proof. ■

Remark 6. The proof above highlights the fact that the design parameter k_i plays the role of a "secret", and privacy of player i is preserved against strong adversaries with access to the algorithm structure, network topology (Laplacian matrix), as well as all communicated variables. However, we see from (31) that the ratios between the cost parameters Q_i, D_i, and d_i, that is, $Q_i^{-1} D_i$ and $Q_i^{-1} d_i$, can be reconstructed by such a strong adversary. To avoid reconstruction of such ratios, we can leverage the ISS property of the algorithm, established in Theorem 2, and modify (26) as follows:

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}_q \boldsymbol{z}(t) + \boldsymbol{D}_q + \boldsymbol{u}(t), \tag{32}$$

where $\boldsymbol{u}(t) = \mathrm{col}(\boldsymbol{v}(t), \mathbb{0})$ with $t \mapsto \boldsymbol{v}(t)$ being piecewise continuous and bounded for all $t \in [0, \infty)$. Thanks to the established ISS property, the NE of the game can be exactly computed as long as $\lim_{t \to \infty} \boldsymbol{v}(t) = \mathbb{0}$. The added value in terms of privacy is that parameter identification cannot be pursued in transient time by the adversary due to the presence of

$\boldsymbol{v}(t)$ which acts as a deterministic noise. We note, however, that addition of $\boldsymbol{v}(t)$ degrades the transient performance and convergence rate of the algorithm. We also remark that the following relation among the privacy sensitive parameters (Q_i, D_i, d_i) would still be revealed at steady-state as $\boldsymbol{v}(t)$ vanishes[#]:

$$\mathbb{0} = \left(2Q_i + \frac{1}{N}D_i^\top\right)x_i^* + D_i s(\boldsymbol{x}^*) + d_i. \tag{33}$$

For similar ideas in the context of preserving privacy in average consensus see References 44 and 45. •

5 | CASE STUDIES

In this section, we consider two illustrative case studies that are formulated as aggregative games.

5.1 | Energy consumption game

This case study considers the energy consumption problem of consumers equipped with heating ventilation air conditioning (HVAC) systems in smart grids. As proposed in Reference 46, this problem can be formulated into a noncooperative game where each consumer i chooses its energy consumption such that the following cost function is minimized

$$J_i(x_i, s(\boldsymbol{x})) = \theta\gamma^2(x_i - \hat{x}_i)^2 + (aNs(\boldsymbol{x}) + b)x_i,$$

where $\theta, \gamma, a \in \mathbb{R}_{>0}$ are respectively the cost, thermal, and price-elasticity coefficients, $x_i \in \mathcal{X}_i$ is the energy consumption and $b \in \mathbb{R}_{>0}$ is its corresponding basic price, $\hat{x}_i \in \mathcal{X}_i$ is the required energy consumption for maintaining the target indoor temperature, and $Ns(\boldsymbol{x}) = \sum_{j\in\mathcal{I}} x_j$ is the total energy consumption. The action set $\mathcal{X}_i \subset \mathbb{R}$ is defined as

$$\mathcal{X}_i := \left\{x_i \in \mathbb{R} \,|\, x_i \in [\underline{x}_i, \overline{x}_i]\right\},$$

where $\underline{x}_i, \overline{x}_i \in \mathbb{R}_{>0}$ are the minimum and maximum acceptable energy consumption, respectively, with $\underline{x}_i < \overline{x}_i$. In this game, Assumption 2 is satisfied if $a < 2\theta\gamma^2/(N-1)$ for $N > 1$. Using Remark 2, this condition can be further relaxed by finding some $k_i > 0$ such that the mapping $\mathrm{col}(k_i f_i(x_i, \sigma_i), \sigma_i - x_i)$ with $f_i(x_i, \sigma_i) = (2\theta\gamma^2 + a)x_i + aN\sigma_i - 2\theta\gamma^2\hat{x}_i + b$ is strongly monotone. By performing the calculations, we obtain that for all $a > 0$ and $N \geq 1$, the mapping is strongly monotone if

$$k_i \in \left(\frac{1}{(aN)^2}\left(\sqrt{\mu_i} - \sqrt{\mu_i + aN}\right)^2, \frac{1}{(aN)^2}\left(\sqrt{\mu_i} + \sqrt{\mu_i + aN}\right)^2\right), \tag{34}$$

where $\mu = 2\theta\gamma^2 + a$. Therefore, we guarantee convergence to the NE without any restrictions on the cost parameters.

We consider $N = 5$ players in this game, that is, $\mathcal{I} = \{1, \dots, 5\}$, with $\theta\gamma^2$ normalized to one, $a = 0.04$, $b = 5(\$/(\mathrm{kWh}))$, $\mathrm{col}((\hat{x}_i)_{i\in\mathcal{I}}) = \mathrm{col}(50, 55, 60, 65, 70)(\mathrm{kWh})$, $\mathrm{col}((\overline{x}_i)_{i\in\mathcal{I}}) = \mathrm{col}(60, 66, 72, 78, 84)(\mathrm{kWh})$, and $\mathrm{col}((\underline{x}_i)_{i\in\mathcal{I}}) = \mathrm{col}(40, 44, 46, 52, 56)(\mathrm{kWh})$.[22] For this set of values, the unique NE of the game is computed as $\boldsymbol{x}^* = \mathrm{col}(41.5, 46.4, 51.3, 56.2, 61.1)(\mathrm{kWh})$ (see section VI-C in Reference 22).

To implement the NE seeking algorithm, the players are assumed to communicate through the connected undirected graph depicted in Figure 1. Each player i arbitrarily chooses the design parameter k_i in the interval (34) as $\mathrm{col}((k_i)_{i\in\mathcal{I}}) = \mathrm{col}(6.2, 0.8, 7.2, 3.5, 0.7)$. The initial conditions $\sigma_i(0)$ and $\psi_i(0)$ are chosen randomly, and $x_i(0) \in \mathcal{X}_i$ is selected as $x_i(0) = 0.5(\overline{x}_i + \underline{x}_i)$. Figure 2 depicts the resulting action variables of the players and demonstrates their convergence to the NE of the game.

Next, we consider the case where the action set is $\mathcal{X}_i = \mathbb{R}$ and illustrate privacy and robustness properties of the dynamics.

Privacy: Here, we demonstrate that an external adversary cannot uniquely reconstruct the private information of the players. Analogous to (28), we consider a replica of the NE seeking algorithm with the parameters $\theta'\gamma'^2 = 0.25$, $a' = 0.01$,

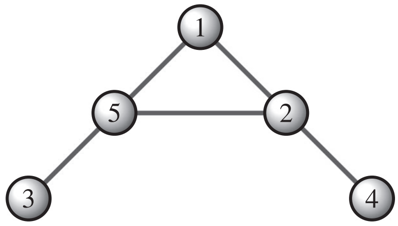

FIGURE 1 Communication graph in energy consumption game

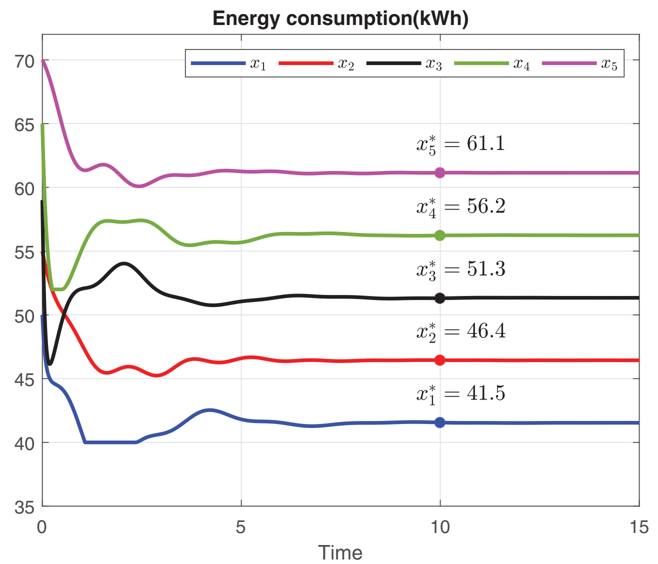

FIGURE 2 Action variables of consumers equipped with HVAC systems

$b' = 7(\$/(\text{kWh}))$, $\text{col}((\hat{x}_i')_{i\in\mathcal{I}}) = \text{col}(61.5, 66.5, 71.5, 76.5, 81.5)(\text{kWh})$, and $\text{col}((k_i')_{i\in\mathcal{I}}) = \text{col}(25.1, 3.1, 28.9, 14.2, 2.8)$.[‖] Note that these parameters are different from the true cost parameters of the players. We can see from Figure 3 that the eavesdropped information by the adversary $(\boldsymbol{\sigma}(t), \boldsymbol{\psi}(t))$ is consistent with $(\boldsymbol{\sigma}'(t), \boldsymbol{\psi}'(t))$ for all times. This means that the private information of the players is not identifiable and the NE seeking algorithm preserves privacy.

Robustness: We illustrate robustness of the algorithm by adding bounded disturbances to the dynamics. The disturbance vector $\boldsymbol{v} : \mathbb{R}_{\geq 0} \to \mathbb{R}^{10}$ is added according to (13), and it prevents an adversary from using parameter identification techniques during transient time. We consider $\lim_{t\to\infty} \boldsymbol{v}(t) = \mathbb{0}$ to maintain the NE of the game in the steady state (see Remark 6). Specifically, five elements of $\boldsymbol{v}(t)$ are considered as vanishing uniformly distributed random numbers in the interval $[-20, 20]$ with sampling time 0.1(s). The other five elements are sinusoidal signals with frequencies between 5 to 25(rad/s), and their time-varying amplitudes start from 10 and 20 and converge to zero. As can be seen from Figure 4, the action variables remain bounded and converge to the NE of the game, which is consistent with our analysis. Note that the presence of disturbances decreases the convergence rate and degrades the transient performance of the algorithm.

5.2 | Coordinated charging of electric vehicles

Here, we consider the problem of coordinated charging for a population $\mathcal{I} = \{1, \dots, N\}$ of plug-in electric vehicles (PEVs) over a charging horizon $\mathcal{T} = \{1, \dots, n\}$.[4,47] Let $x_i = \text{col}\left((x_i^k)_{k\in\mathcal{T}}\right)$ where $x_i^k \in \mathbb{R}$ is the charging control of player $i \in \mathcal{I}$ at time $k \in \mathcal{T}$. Then, each player i is aimed at choosing $x_i \in \mathcal{X}_i$ and minimizing its cost function

$$J_i(x_i, s(\boldsymbol{x})) = \sum_{k\in\mathcal{T}} \left(a(d^k + Ns(\boldsymbol{x}^k)) + b\right) x_i^k + \sum_{k\in\mathcal{T}} \left(q_i(x_i^k)^2 + c_i x_i^k\right),$$

where $a, b \in \mathbb{R}_{>0}$ respectively are the price-elasticity coefficient and the basic price, d^k and $Ns(\boldsymbol{x}^k) = \sum_{j\in\mathcal{I}} x_j^k$ respectively are the total non-PEV demand and the total PEV demand at time k, and $q_i, c_i \in \mathbb{R}_{>0}$. In the cost function, the first term is

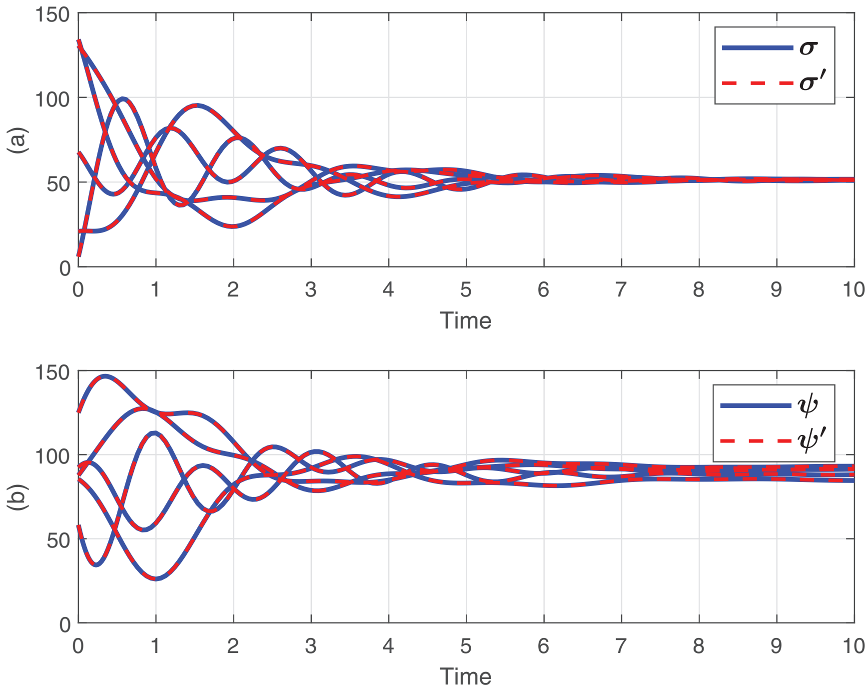

FIGURE 3 Eavesdropped information $(\boldsymbol{\sigma}, \boldsymbol{\psi})$ and its corresponding replica $(\boldsymbol{\sigma}', \boldsymbol{\psi}')$

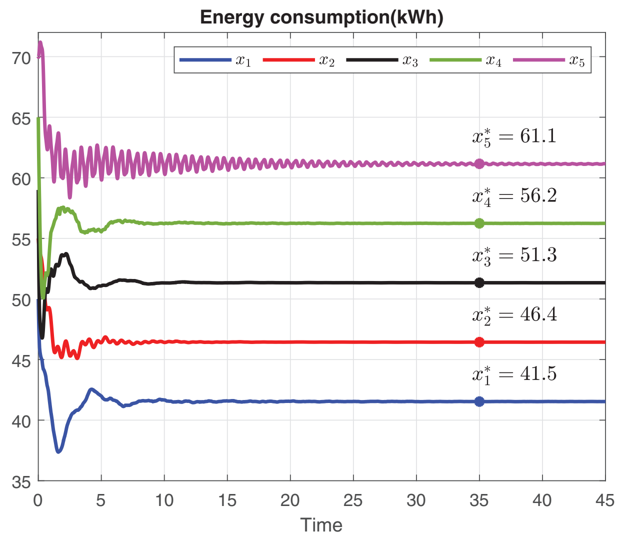

FIGURE 4 Action variables of consumers equipped with HVAC systems in the presence of disturbances

the electricity bill of player i, and the quadratic term models its battery degradation cost.[47] For each vehicle, the charging rate x_i^d is bounded as $0 \le x_i^d \le \bar{x}_i$ and its summation for all $k \in \mathcal{T}$ should be equal to the required energy of the agent defined as γ_i. Therefore, the constraint set of x_i is $\mathcal{X}_i = \mathcal{X}_i^1 \cap \mathcal{X}_i^2$ where[**]

$$\mathcal{X}_i^1 := \{x_i \in \mathbb{R}^n | x_i^k \in [0, \bar{x}_i], \forall k \in \mathcal{T}\},$$
$$\mathcal{X}_i^2 := \left\{x_i \in \mathbb{R}^n \,|\, \sum_{k \in \mathcal{T}} x_i^k = \gamma_i\right\}. \tag{35}$$

In practice, it is assumed that $n\bar{x}_i \ge \gamma_i$ to grantee that $\mathcal{X}_i$ is nonempty. The goal is compute the NE and schedule charging strategies for the entire horizon, and in this regard, a gather and broadcast algorithm is presented in Reference 47 which guarantees convergence when $q_i > aN$ (see theorem 3.1 in Reference 47).

Note that $f_i(x_i, \sigma_i) = (2q_i + a)x_i + aN\sigma_i + ad + (b + c_i)\mathbb{1}_n$ with $d = \text{col}\left((d^k)_{k\in\mathcal{T}}\right)$; therefore, the mapping col $(k_i f_i(x_i, \sigma_i), \sigma_i - x_i)$ is strongly monotone when k_i satisfies

$$k_i \in \left(\frac{1}{(aN)^2}\left(\sqrt{\mu_i} - \sqrt{\mu_i + aN}\right)^2, \frac{1}{(aN)^2}\left(\sqrt{\mu_i} + \sqrt{\mu_i + aN}\right)^2 \right),$$

where $\mu_i = 2q_i + a$. Thus convergence to the NE is guaranteed without requiring $q_i > aN$.

To compute the NE, each player i can implement (5); however, since $\mathcal{X}_i$ is the intersection of two sets, namely, $\mathcal{X}_i^1$ and $\mathcal{X}_i^2$, it is not easy to find a closed-form expression for the projection operator $\Pi_{\mathcal{X}_i}(x_i, \cdot)$. To overcome this challenge, we use the fact that the solution $x_i(t)$ of the NE dynamics does not need to belong to $\mathcal{X}_i$ for all $t \geq 0$, yet it should converge to the NE inside this set. This allows us to treat $x_i \in \mathcal{X}_i^2$ as a "soft constraint" and modify (6) as follows:

$$\begin{aligned}
\dot{x}_i &= \Pi_{\mathcal{X}_i^1}(x_i, -k_i f_i(x_i, \sigma_i) - \mathbb{1}_n \lambda_i), \\
\dot{\sigma}_i &= -\sigma_i + x_i - \sum_{j\in\mathcal{N}_i} \left(\psi_i - \psi_j\right), \\
\dot{\psi}_i &= \sum_{j\in\mathcal{N}_i} \left(\sigma_i - \sigma_j\right), \\
\dot{\lambda}_i &= \mathbb{1}_n^\top x_i - \gamma_i,
\end{aligned} \tag{36}$$

where $\lambda_i \in \mathbb{R}$ is the Lagrangian multiplier. Note that the projection in the x_i-component is solely based on $\mathcal{X}_i^1$, thus its closed-form expression can be obtained from lemma 2.1 in Reference 37. The variable λ_i is included for convergence to the set $\mathcal{X}_i^2$. A supplementary discussion on the convergence of the above algorithm to the NE is provided in Appendix B.

A population of $N = 100$ players, that can communicate by a connected undirected graph, are considered in this game. The charging horizon is from 12:00 am on one day to 12:00 am on the next day. In order to generate the numerical parameters, we consider some nominal values and randomize them similar to Reference 13. In the price function, $a = 3.8 \times 10^{-3}$ and $b = 0.06(\$/(\text{kWh}))$ are considered. The parameters of the quadratic functions are uniformly distributed random numbers as $q_i \sim \{0.004\} + [-0.001,\ 0.001]$ and $c_i \sim \{0.075\} + [-0.01,\ 0.01]$. In order to generate γ_i's, inspired by Reference,[47] we assume that the battery capacity sizes of the PEVs are $\Phi_i \sim \{30\} + [-5,\ 5](\text{kWh})$, also their initial states of charge (SOC_{i_0}) satisfy a Gaussian distribution with the mean 0.5 and the variance 0.1, and the final states of charge (SOC_{i_f}) are equal to 0.95; thus, $\gamma_i = \Phi_i(\text{SOC}_{i_f} - \text{SOC}_{i_0})$. In addition, the maximum admissible charging controls are set to $\bar{x}_i \sim \{10\} + [-2,\ 2](\text{kWh})$.

For each player $i \in \mathcal{I}$, the design parameter of the algorithm is $k_i = (2(2q_i + a) + aN)/(aN)^2$, the initial condition of the action variable is chosen as $x_i(0) = (\gamma_i/n)\mathbb{1}_n \in \mathcal{X}_i$, and $\sigma_i(0)$, $\psi_i(0)$, and $\lambda_i(0)$ are selected randomly. Figure 5 illustrates the total non-PEV demand d over the charging horizon as well as the total demand at the equilibrium $d + \sum_{i\in\mathcal{I}} x_i^*$. As can be seen, the PEVs shifted their charging intervals to the nighttime, which minimizes their effects on the grid, and as explained in Reference 4, the NE has the desired "valley filling" property.

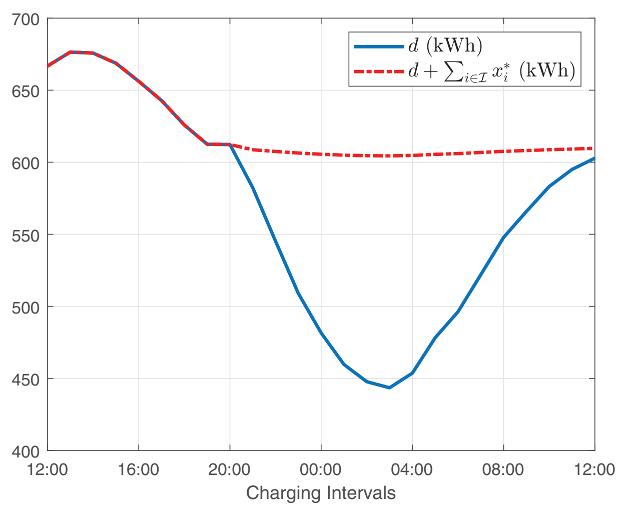

FIGURE 5 Total non-PEV demand d and its summation with total-PEV demand at the equilibrium $d + \sum_{i\in\mathcal{I}} x_i^*$

6 | CONCLUSIONS

By employing the structure of aggregative games, we presented a distributed NE seeking algorithm and provided sufficient conditions for convergence to the NE. Raised by practical concerns about deviation from the NE seeking dynamics due to irrationality of the players, we proved robustness of the proposed algorithm against disturbances in the sense of ISS and $\mathcal{L}_2$-stability. Moreover, we have studied privacy guarantees of the algorithm by showing that private information of the players cannot be uniquely reconstructed even if all communicated variables are accessed by an adversary. Extension of the results to games with coupling constraints as well as time-varying communication graphs is left for future work. Another research question is using aggregative game dynamics as controllers to steer a physical system. Examples of the latter in Cournot and Bertrand competitions can be found in References 2 and 48, respectively.

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

ENDNOTES

*Finding a closed-form expression of $\Pi_{\mathcal{X}_i}(x_i, \cdot)$ can be challenging in certain cases depending on the structure of the set $\mathcal{X}_i$. In Section 5.2, we present a reformulation of (5) that circumvents this challenge in the context of coordinated charging of electric vehicles.

†Note that the system (6) has no isolated equilibrium (see Equation (7)); nevertheless, convergence to a point within the set of equilibria can be guaranteed as stated in the theorem.

‡A directed graph is strongly connected if there is a path between every pair of nodes.

§The authors thank Sergio Grammatico for pointing out this connection.

¶The information available to the adversary is provided in the paragraph preceding Definition 1.

#This coincides with the fact that the NE satisfies the relation $\frac{\partial}{\partial x_i} J_i(x_i^*, s(\boldsymbol{x}^*)) = \mathbb{0}$, which in the case of unconstrained linear-quadratic games takes the form in (33).

‖There exist infinite number of parameters satisfying (31) and these values are chosen solely for the sake of presentation.

**The constraint set $\mathcal{X}_i^1$ in (35) implies that the PEVs charge their batteries over the entire horizon $\mathcal{T}$. In case a PEV, say i, would like to charge during a shorter horizon $\mathcal{T}_i \subseteq \mathcal{T}$, the set $\mathcal{X}_i^1$ modifies to $\overline{\mathcal{X}}_i^1 := \left\{x_i \in \mathbb{R}^n | x_i^k \in [0, \overline{x}_i], \forall k \in \mathcal{T}_i \text{ and } x_i^k = 0, \forall k \in \mathcal{T} \setminus \mathcal{T}_i\right\}$.

††In fact ϵ_i can be taken as the smallest eigenvalue of (A2).

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APPENDIX A. PROOFS OF THE LEMMAS

Proof of Lemma 1. (i) The mapping F is ϵ-strongly monotone if

$$\mathrm{col}\left(\boldsymbol{x}-\boldsymbol{x}',\boldsymbol{\sigma}-\boldsymbol{\sigma}'\right)^\top\left(F(\boldsymbol{x},\boldsymbol{\sigma})-F(\boldsymbol{x}',\boldsymbol{\sigma}')\right)\geq\epsilon\|\boldsymbol{x}-\boldsymbol{x}'\|^2+\epsilon\|\boldsymbol{\sigma}-\boldsymbol{\sigma}'\|^2, \tag{A1}$$

for all $\boldsymbol{x},\boldsymbol{x}',\boldsymbol{\sigma},\boldsymbol{\sigma}'\in\mathbb{R}^{nN}$. By adding and subtracting $(\boldsymbol{x}-\boldsymbol{x}')^\top\boldsymbol{K}\mathrm{col}\left((f_i(x_i',\sigma_i))_{i\in\mathcal{I}}\right)$ from the left hand side and using (3) and Assumption 2, we have

$$\mathrm{col}\left(\boldsymbol{x}-\boldsymbol{x}',\boldsymbol{\sigma}-\boldsymbol{\sigma}'\right)^\top\left(F(\boldsymbol{x},\boldsymbol{\sigma})-F(\boldsymbol{x}',\boldsymbol{\sigma}')\right)\geq\sum_{i\in\mathcal{I}}k_i\mu_i\|x_i-x_i'\|^2-\sum_{i\in\mathcal{I}}(k_i\ell_i+1)\|x_i-x_i'\|\|\sigma_i-\sigma_i'\|+\sum_{i\in\mathcal{I}}\|\sigma_i-\sigma_i'\|^2.$$

As a result, to establish the inequality in (A1), it is sufficient to define $\epsilon:=\min_{i\in\mathcal{I}}\epsilon_i$ where $\epsilon_i>0$ satisfies

$$k_i\mu_i\|x_i-x_i'\|^2-(k_i\ell_i+1)\|x_i-x_i'\|\|\sigma_i-\sigma_i'\|+\|\sigma_i-\sigma_i'\|^2\geq\epsilon_i\|x_i-x_i'\|^2+\epsilon_i\|\sigma_i-\sigma_i'\|^2.$$

Clearly, such ϵ_i exists providing that[††]

$$\begin{bmatrix} k_i\mu_i & -\frac{(k_i\ell_i+1)}{2}\\ -\frac{(k_i\ell_i+1)}{2} & 1\end{bmatrix}>\mathbb{0}. \tag{A2}$$

This conditions holds if and only if $k_i>0$ satisfies $4k_i\mu_i-(k_i\ell_i+1)^2>0$, which is equivalent to (4).

(ii) Let $\boldsymbol{\sigma}=\mathbb{1}_N\otimes s(\boldsymbol{x})$ and $\boldsymbol{\sigma}'=\mathbb{1}_N\otimes s(\boldsymbol{x}')$. By using the definition of $s(\boldsymbol{x})$ we get $\boldsymbol{\sigma}-\boldsymbol{\sigma}'=\mathbb{1}_N\otimes s(\boldsymbol{x}-\boldsymbol{x}')$. Hence, inequality (A1), proven in part (i), becomes

$$\begin{aligned}(\boldsymbol{x}-\boldsymbol{x}')^\top\boldsymbol{K}\,\mathrm{col}\left((f_i(x_i,s(\boldsymbol{x}))-f_i(x_i',s(\boldsymbol{x}')))_{i\in\mathcal{I}}\right)+(\mathbb{1}_N\otimes s(\boldsymbol{x}-\boldsymbol{x}'))^\top\left((\mathbb{1}_N\otimes s(\boldsymbol{x}-\boldsymbol{x}'))-(\boldsymbol{x}-\boldsymbol{x}')\right)&\\ \geq\epsilon\|\boldsymbol{x}-\boldsymbol{x}'\|^2+\epsilon\|\mathbb{1}_N\otimes s(\boldsymbol{x}-\boldsymbol{x}')\|^2,&\end{aligned} \tag{A3}$$

where we used the definition of F given by (3). Let

$$\Pi:=I-\frac{1}{N}\mathbb{1}_N\mathbb{1}_N^\top. \tag{A4}$$

Then, we employ $\mathbb{1}_N\otimes s(\boldsymbol{x}-\boldsymbol{x}')=\frac{1}{N}\left(\mathbb{1}_N\mathbb{1}_N^\top\otimes I_n\right)(\boldsymbol{x}-\boldsymbol{x}')$ to obtain

$$(\mathbb{1}_N\otimes s(\boldsymbol{x}-\boldsymbol{x}'))-(\boldsymbol{x}-\boldsymbol{x}')=-(\Pi\otimes I_n)(\boldsymbol{x}-\boldsymbol{x}'),$$

Therefore, the second term on the left hand side of (A3) is zero as $\mathbb{1}_N^\top\Pi=0$, and the proof is complete. ■

Proof of Lemma 2. Under Assumption 1, it follows from theorem 4.3 in Reference 6 that the game admits an NE if $J_i(x_i,s(\boldsymbol{x}))$ is strictly convex in x_i for all $\boldsymbol{x}_{-i}\in\mathcal{X}_{-i}=\prod_{j\neq i}\mathcal{X}_j$. For this, it suffices $f_i(x_i,s(\boldsymbol{x}))$ to be η_i-strongly monotone in x_i, that is,

$$(x_i-x_i')^\top\left(f_i\left(x_i,\frac{1}{N}x_i+\frac{1}{N}\sum_{j\neq i}x_j\right)-f_i\left(x_i',\frac{1}{N}x_i'+\frac{1}{N}\sum_{j\neq i}x_j\right)\right)\geq\eta_i\|x_i-x_i'\|^2,$$

for all $x_i, x_i' \in \mathcal{X}_i$, $\boldsymbol{x}_{-i} \in \mathcal{X}_{-i}$, and some $\eta_i > 0$. By adding and subtracting $(x_i - x_i')^\top f_i(x_i', s(\boldsymbol{x}))$ from the left hand side and using Assumption 2, it is straightforward to show that the above inequity is satisfied with $\eta_i := \mu_i - \frac{\ell_i}{N}$. Hence, the game has an NE, namely $\overline{\boldsymbol{x}} \in \mathcal{X}$, that is a solution of the variational inequality $\mathrm{VI}(\mathcal{X}, \mathrm{col}\,((f_i(x_i, s(\boldsymbol{x})))_{i\in\mathcal{I}}))$ (proposition 1.4.2 in Reference 7). Moreover, since $\boldsymbol{K}\,\mathrm{col}\,((f_i(x_i, s(\boldsymbol{x})))_{i\in\mathcal{I}})$ is strongly monotone (Lemma 1(*ii*)) and $\mathcal{X}$ is closed and convex, the variational inequality $\mathrm{VI}(\mathcal{X}, \boldsymbol{K}\,\mathrm{col}\,((f_i(x_i, s(\boldsymbol{x})))_{i\in\mathcal{I}}))$ has a unique solution $\boldsymbol{x}' \in \mathcal{X}$ (theorem 2.3.3 in Reference 7). Lastly, we need to show that $\overline{\boldsymbol{x}}$ is unique and equal to $\boldsymbol{x}'$.

Clearly, we have

$$(\boldsymbol{x} - \overline{\boldsymbol{x}})^\top \mathrm{col}\left((f_i(\overline{x}_i, s(\overline{\boldsymbol{x}})))_{i\in\mathcal{I}}\right) \geq 0, \quad \boldsymbol{x} \in \mathcal{X},$$

which can be rewritten as

$$\sum_{i\in\mathcal{I}} (x_i - \overline{x}_i)^\top f_i(\overline{x}_i, s(\overline{\boldsymbol{x}})) \geq 0, \quad \boldsymbol{x} \in \mathcal{X}.$$

For a given $j \in \mathcal{I}$, set $x_i = \overline{x}_i$ for all $i \in \mathcal{I} \setminus \{j\}$. Then, by using $k_j > 0$, the above inequality yields

$$k_j (x_j - \overline{x}_j)^\top f_j(\overline{x}_j, s(\overline{\boldsymbol{x}})) \geq 0, \quad \forall x_j \in \mathcal{X}_j.$$

By performing the same procedure for the other components of $\overline{\boldsymbol{x}}$ and rewriting all obtained inequalities into the vector form, we can see that $\overline{\boldsymbol{x}}$ is the solution of $\mathrm{VI}(\mathcal{X}, \boldsymbol{K}\,\mathrm{col}\,((f_i(x_i, s(\boldsymbol{x})))_{i\in\mathcal{I}}))$, that is, $\overline{\boldsymbol{x}} = \boldsymbol{x}'$. Consequently, since $\overline{\boldsymbol{x}}$ is an arbitrary solution and $\boldsymbol{x}'$ is unique, both variational inequality problems have an identical solution, which concludes the proof. ■

APPENDIX B. ON CONVERGENCE OF THE MODIFIED NE SEEKING ALGORITHM (36)

Consider the algorithm (36) in vector form as follows:

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \Pi_{\mathcal{X}^1}\left(\boldsymbol{x}, -\boldsymbol{K}\,\mathrm{col}\,((f_i(x_i, \sigma_i))_{i\in\mathcal{I}}) - (I_N \otimes \mathbb{1}_n)\lambda\right), \\
\dot{\boldsymbol{\sigma}} &= -\boldsymbol{\sigma} + \boldsymbol{x} - (L \otimes I_n)\boldsymbol{\psi}, \\
\dot{\boldsymbol{\psi}} &= (L \otimes I_n)\boldsymbol{\sigma}, \\
\dot{\lambda} &= (I_N \otimes \mathbb{1}_n^\top)\boldsymbol{x} - \boldsymbol{\gamma},
\end{aligned} \tag{B1}$$

where $\lambda = \mathrm{col}((\lambda_i)_{i\in\mathcal{I}})$, $\boldsymbol{\gamma} = \mathrm{col}((\gamma_i)_{i\in\mathcal{I}})$, and $\mathcal{X}^1 = \prod_{i\in\mathcal{I}} \mathcal{X}_i^1$ with $\mathcal{X}_i^1$ defined in (35). Similar to (6), we can guarantee that for any initial condition $(\boldsymbol{x}(0), \boldsymbol{\sigma}(0), \boldsymbol{\psi}(0), \lambda(0)) \in \mathcal{X}^1 \times \mathbb{R}^{nN} \times \mathbb{R}^{nN} \times \mathbb{R}^N$, the solution of (B1) is unique and belongs to $\mathcal{X}^1 \times \mathbb{R}^{nN} \times \mathbb{R}^{nN} \times \mathbb{R}^N$ for all $t \geq 0$. We claim that such a solution converges to an equilibrium corresponding to the NE of the game. To prove this, consider an equilibrium point $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{\psi}}, \overline{\lambda})$, then we have $\overline{\boldsymbol{\sigma}} = \mathbb{1}_N \otimes s(\overline{\boldsymbol{x}})$, $(L \otimes I_n)\overline{\boldsymbol{\psi}} = (\Pi \otimes I_n)\overline{\boldsymbol{x}}$, and

$$\mathbb{0} = \Pi_{\mathcal{X}^1}\left(\overline{\boldsymbol{x}}, -\boldsymbol{K}\mathrm{col}\left((f_i(\overline{x}_i, s(\overline{\boldsymbol{x}})))_{i\in\mathcal{I}}\right) - (I_N \otimes \mathbb{1}_n)\overline{\lambda}\right), \tag{B2}$$

$$\mathbb{0} = (I_N \otimes \mathbb{1}_n^\top)\overline{\boldsymbol{x}} - \boldsymbol{\gamma}. \tag{B3}$$

The second equality implies that $\overline{\boldsymbol{x}} \in \mathcal{X}^2 = \prod_{i\in\mathcal{I}} \mathcal{X}_i^2$. Employing Moreau's decomposition theorem and (B2), we can perform an analogous analysis to the proof of Proposition 1 and conclude that $\overline{\boldsymbol{x}}$ is the solution of $\mathrm{VI}(\mathcal{X}^1, \boldsymbol{K}\,\mathrm{col}\,((f_i(x_i, s(\boldsymbol{x})))_{i\in\mathcal{I}}) + (I_N \otimes \mathbb{1}_n)\overline{\lambda})$. This means that $\overline{\boldsymbol{x}}$ is also the solution of the following optimization problem (see equation 1.2.1 in Reference 7):

$$\min_{y\in\mathcal{X}^1} y^\top (\boldsymbol{K}\mathrm{col}\left((f_i(\overline{x}_i, s(\overline{\boldsymbol{x}})))_{i\in\mathcal{I}}\right) + (I_N \otimes \mathbb{1}_n)\overline{\lambda}).$$

Next we use the definition of $\mathcal{X}^1$ and write the KKT conditions corresponding to this optimization problem. Let $g_i^k(\boldsymbol{x}) := \mathrm{col}(x_i^k - \overline{x}_i, -x_i^k)$, $g_i(\boldsymbol{x}) := \mathrm{col}\left((g_i^k(\boldsymbol{x}))_{k\in\mathcal{T}}\right)$, and $g(\boldsymbol{x}) := \mathrm{col}\,((g_i(\boldsymbol{x}))_{i\in\mathcal{I}})$; then we see that $g(\boldsymbol{x}) \leq 0$ represents the set $\mathcal{X}^1$.

Therefore, there exists $\boldsymbol{\mu} \in \mathbb{R}^{2nN}$ such that the following KKT conditions hold

$$\begin{aligned}
&\mathbb{0} = \boldsymbol{K}\mathrm{col}\left((f_i(\overline{x}_i, s(\overline{\boldsymbol{x}})))_{i\in\mathcal{I}}\right) + (I_N \otimes \mathbb{1}_n)\overline{\lambda} + \frac{\partial g}{\partial \boldsymbol{x}}(\overline{\boldsymbol{x}})^\top \boldsymbol{\mu},\\
&0 \leq \boldsymbol{\mu} \perp g(\boldsymbol{x}) \leq 0.
\end{aligned}$$

Considering the above equations together with (B3), we conclude from proposition 1.3.4(b) in Reference 7 that $\overline{\boldsymbol{x}}$ is the solution of VI$(\mathcal{X}, \boldsymbol{K}\,\mathrm{col}\,((f_i(x_i, s(\boldsymbol{x})))_{i\in\mathcal{I}}))$, and in turn, it is the NE of the game (Lemma 2). Convergence analysis of the algorithm is similar to Theorem 1.