Reduced realizations and model reduction for switched linear systems

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Reduced realizations and model reduction for switched linear systems

a time-varying approach

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To my parents, wife and son

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Md Sumon Hossain
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Contents

1 Introduction

1.1 Background

Hybrid systems constitute a particular class of systems, whose behavior is modeled considering both discrete and continuous dynamics. In particular, such systems involve a class of linear/nonlinear systems and result from the interaction of continuous time subsystems with discrete events. More precisely, the internal variable of each system is regulated by a set of differential equations. Each of the separate systems are labeled as a discrete mode and the transitions between the discrete states may result in a jump in the continuous variable.

Switched systems are considered as a subclass of hybrid systems; a switched system is a dynamical system that consists of a finite number of subsystems or modes and a logical rule that orchestrates switching between these subsystems. The main property of switched systems is that these systems switch among a finite number of subsystems and the discrete events interacting with the subsystems are governed by a piecewise continuous function called the switching signal. However, one can classify switched systems based on the dynamics of their subsystems, for example continuous-time or discrete-time, linear or nonlinear and so on. Switched linear systems (SLSs) are dynamical systems formed by a collection of linear continuous state models which are switching among them according to a discrete signal.

During the last decades and because of the great number of applications, switched systems have been widely studied. In particular, switched systems have applications in control of aeronautical and mechanical systems, automotive industry, modelling of electronic switching systems and power converters. Considering time-dependent switching signals lead to time-varying systems, but linearity is *not* lost. Only when considering state dependent switching signals or when considering the switching signal as an input, one can arrive at a nonlinear system. In this thesis, switched linear systems are considered whose switching is depending on time. Therefore, the standard qualitative properties for standard systems can not be applied, but it is necessary to find specific tools for them. For a detailed characterization of switched systems, cf. [46, 76, 127].

With the ever-developing technologies in various engineering applications (microelectronics, micro-electro-mechanical systems, electromagnetism, fluid dynamics, control design, etc.), more and more mathematical systems with very large dimensions have to be simulated and solved. In some cases, these systems lead to analyzing large-scale and complex dynamical systems. Although, the computational speed and performance of the modern computers are increasing; simulation, optimization or real time controller design for such large-scale systems are still difficult due to extra memory requirements and additional computational complexity. Model order reduction (MOR) is a useful tool for dealing with such complexity, wherein one seeks a simpler model that can then be used as an efficient surrogate model to the original model, cf. [2, 3, 14, 107, 116] for motivations, applications, restrictions and techniques of MOR.

Differential-algebraic equations (DAEs) have become an important tool for modelling and simulation of constrained dynamical systems, such systems belong to a class of dynamical systems that are characterized by both algebraic and differential constraints; DAEs are also known as descriptor systems or singular systems. DAEs naturally occur when modelling linear electrical circuits, simple mechanical systems or, in general, linear systems with additional linear algebraic constraints. DAEs have been studied widely in both the engineering community, and in the numerical linear algebra community. DAEs have then been discovered to be suitable for modelling a vast variety of problem in economics [80, 81], demography [24], mechanical systems [23, 39], multibody dynamics [35, 115], electrical networks [5, 29, 34, 89], fluid mechanics [52, 79], chemical engineering [32, 91] and they are also particularly important in the simulation and design of very large-scale integrated circuits. The theory of DAEs is well developed and mature cf. [23, 24, 25, 29, 41, 68, 108, 110] and the references therein.

Switched DAEs arise when the system changes suddenly and the switching between the systems is induced by faults or an external switching rule, cf. [27, 28, 66, 75, 110, 141] for details. Switches or component faults induce jumps in certain state variables, and it is common to define additional jump maps based on physical arguments. In order to allow for jumps in the solution, the problem is embedded into a distributional solution framework. It turns out that general switched DAEs can have not only jumps in the solutions but also Dirac impulses and/or their derivatives. In particular, the dynamics of switched descriptor systems are determined by the switching among different modes, where each mode is characterized by a set of linear differential equations and algebraic constraints. A mathematical representation of this class of systems can be obtained in terms of switched linear differential-algebraic equations, cf. [22, 43, 86]. It can be shown that the classical distribution space discussed in [118] is not sufficient as a solution space for switched DAEs (the reason is that this space is too large). Furthermore, the problem is solved in [135] where the author proposes the space of *piecewise-smooth distributions* as an underlying solution space. Several modeling and control aspects related to switched DAE have been considered in the literature, e.g., [16, 19, 53, 77, 85, 132, 137, 138, 139, 149] and the references therein.

1.2 Realization theory

Realization theory is one of the main topics in control theory analysis, it involves finding (necessary and sufficient) conditions to construct a (preferably) unique minimal system which generates the specified input-output behavior of a certain class cf. [66, 143]. Moreover, realization theory provides a theoretical analysis for model reduction, system identification and filtering-observer design. In fact, realization theory and model reduction are closely related to each other with respect to the input–output behavior. Kalman, founder of modern control theory, originally championed the realization theory. In [44], the minimal state-space realization problem for (continuous) linear time invariant (LTI) systems was first stated by hidden pole-zero cancellation techniques and in [66], the input-output description reveals only the reachable and observable part of a dynamical system. Recently, many approaches have been discussed for the realization theory of hybrid/switched systems, e.g. in [10, 95, 96, 97, 98, 99, 100, 102, 146] and the references therein.

The realization problem for hybrid systems was first formulated in [54] without solution and the problem has evolved in many directions. Recently, in [95] the author combines the theory of rational formal power series with the classical automata theory to discuss the realization theory of hybrid systems. Unlike in the classical case, the author studies realizability of a family of input-output maps instead of a single input-output map. In particular, for most of the proposed methods, the switching signals are viewed as "input". Moreover, the authors in [140] have discussed the detection of temporal properties for piecewise-constant rank systems (PCR), indeed a PCR system differs from the switched linear system because the time-instants are *a priori fixed* and the linear system over each time interval is *time-varying*.

In contrast to the existing literature, this thesis is devoted to the cases where the switched system is considered as a piecewise-constant time-varying linear system, in particular, a minimal realization in general depends on the given switching signal. Several minimal realization approaches have been developed in the context of time-varying systems, and they are classified by constant rank systems, piecewise constant rank systems, global and intervalwise Kalman decomposition, cf. [31, 64, 66, 67, 140]. Most available methods have considered *differential* Kalman decomposition and some drawbacks are summarized in [64]. It is well known that realization theory of a system class relies on the reachability and observability of that system class. However, some unified approaches towards reachability and observability have been studied actively during the past couple of decades for switched systems as well as switched DAEs, cf. [17, 19, 70, 71, 72, 73, 130, 133].

To the authors' knowledge, there are few results available for realization theory of the system class considered in this thesis. Hence, it is essential to focus more on such classes of switched systems.

1.3 Model reduction

Model reduction has been studied for decades by researchers from multidisciplinary fields which include mathematicians, scientists and engineers from widely different communities. System-theoretic MOR approaches usually deal with a system under investigation described as a large-scale set of ordinary differential/difference equations (ODEs) or differential/difference-algebraic equations, whose dynamics are expressed in terms of a set of state variables.

The main objective for MOR is to derive some reduced order model (ROM) characterized by a significantly smaller number of states, and whose response approximates the original system response according to well defined criteria. In this setting, a reduction is required in order to replace the original large-scale description of individual components with an accurate and robust reduced model, so that a global system-level numerical simulation becomes feasible. That is, given a full order model, the objective of model reduction is to find a ROM such that the input-output behavior of these two models are close in some appropriate sense. Model reduction methods are quite well developed for linear systems as well as time-varying systems. Some well known methods are reviewed in the following which will be recalled later in this thesis.

Time-invariant systems

There are two well known model reduction methods for linear systems which are currently in use: singular value decomposition (SVD) based and Krylov subspace based methods, for an overview cf. [2, 15, 55, 145].

One of the most common approach of the SVD-based methods is the so-called *balanced truncation* (BT), which was first introduced by Mullis and Robert in [87, 88], and later in the systems and control literature by Moore in [84]. The key steps in this technique are the computation of the so-called *Hankel singular values* (HSVs) which are the eigenvalues of the balanced reachability and observability Gramians. Then, one can identify which states are difficult to control and difficult to observe (and vice versa). In particular, the states are ordered on the basis of their influence on the input-output behavior according to the singular values of the Hankel operator, which is the operator that maps past inputs energy to future outputs energy. These energy functions are quadratic functions characterized by the so-called reachability and observability Gramians. Then, by truncating the states corresponding to the smallest Hankel singular values, a reduced model is obtained which then approximates the given original system. Nowadays, balanced truncation is considered as a popular projection-based method for its simplicity; the construction is based on simple linear algebra decomposition's and there is no need to first choose a set of essential parameters. Furthermore, it has several desirable theoretical properties such as preservation of stability and availability of an error bound, cf. [14, 36, 45, 94]. The Gramian-based methods include optimal Hankel norm approximation [45], frequency weighted balanced truncation [36], singular perturbation approximation [78], dominant subspaces projection [93], dominant pole algorithm [111], positive real [55], bounded real [90].

Another important category for model reduction is the Krylov subspace based model reduction. Nowadays, moment matching using Krylov subspaces is one of the best choices for reduced order of large-scale systems, cf. [30, 37, 40, 105]. This approach consists in projecting the dynamical system onto Krylov subspaces, computed by an Arnoldi [38] or Lanczos process [7]. Then, the reduced system is obtained by matching the moments (and/or Markov parameters) of the original and reduced systems where the moments are the coefficients of the Taylor series expansion of the transfer function around a suitable expansion point. One very clear disadvantage of Krylov subspace methods is that the dimension of Krylov subspaces quickly becomes large, in many cases even prohibitively large. Unlike the SVD-based methods, stability of the reduced model constructed by Krylov methods is not guaranteed and no a prior error bound exists, cf. [4, 6, 63, 123].

Time-varying systems

Balanced truncation for linear time-varying systems has been developed in e.g., [61, 94, 121, 122, 142]. In [121] and [142], input-output balancing was discussed for various Gramians, and some necessary and sufficient conditions are given for input-output balancing. In particular, model reduction is considered there for exponentially stable and uniformly completely reachable and observable systems. In [142], three different types of time-varying balancing are proposed, so-called *fixed-interval*, *infinite-interval* and *sliding-interval* balancing, based on three different approaches for computing the time-varying Gramians. Furthermore, in [121] uniform and infinite-interval balanced realizations are investigated, and stability of the reduced system is studied in [122]. In this thesis, a switched system is considered as a special class of a time-varying system.

Differential-algebraic equations

Balanced truncation based model reduction technique for DAEs was first introduced in [82, 125] and then, it is studied extensively in [13, 82, 126]. In the context of DAEs, balanced truncation can be applied by decoupling the DAEs into the slow and fast subsystems, and then reduce them separately. In practice, it is not necessary to compute these subsystems explicitly since it is an expensive task and numerically ill-conditioned for large-scale systems. Instead, one can define two pairs of reachability/observability Gramians, and then by truncating the states of the balanced system corresponding to the small Hankel singular values, one can then approximate a reduced system for the given DAEs.

Switched linear systems

During the last few years, considerable attention has been dedicated to the problem of model reduction for switched linear systems and some techniques have been proposed. In the context of switched linear systems, almost, all model reduction techniques are made for an input-output approach. One idea of them is to reduce independently each subsystem alone, and construct desired reduced system by collecting all reduced subsystems. In that case, the switching signal is just reproduced into the reduced model, and then for the same input, both subsystems and corresponding reduced subsystems will have outputs which are very close with respect to a certain norm. Then, one can use the available best model reduction approach; balanced truncation or Hankel norm approximation. Another approach is to consider the whole system at once, in that case, the original switched system is equivalent to the special class of time-varying system. In particular, for each switching time interval the systems are given by the linear time-invariant system.

Recently, much work has been done in the area of model reduction for switched linear systems, the related work can be categorized as follows. In [92], a model reduction approach has been proposed for switched systems with autonomous switching which depends on continuous outputs. In [83], a simultaneous balancing transformation is proposed which is based on reachability/observability Gramians for each mode (in same dimension) and the assumption that all Gramians can simultaneously be transformed into a diagonal form. In [106], new reachability and observability Gramians in a bilinear framework have been introduced for linear switched systems and the reduced model is constructed by computing global projection matrices from the Gramians which satisfies generalized Lyapunov equations. An interesting model reduction method utilizing a so-called envelope system for switched linear systems is proposed in [117], which is based on the idea of embedding the solution behavior of the switched system into the solution behavior of a certain non-switched system; then standard model reduction techniques can be applied to the envelope system.

Generalized Gramians-based approaches are proposed in e.g., [11, 47, 51, 101, 103, 119, 120], those approaches are considered only for quadratically stable systems, and the Gramians are then the solutions of certain linear matrix inequalities (LMIs). Furthermore, an improvement of the underlying approaches are reviewed in e.g., [48, 49, 50, 114] by introducing new Gramians and its system-theoretical properties; again, certain stability assumptions are made.

None of above model reduction approaches consider the switched system as a piecewise-constant linear time-varying system, i.e. the question how to reduce a switched system for a given (and known) switching signal. Later, it will be seen that the specific mode sequences as well as the mode durations influence the size of the minimal realization, hence it is reasonable to conclude that in case of a known switching signal, the size of a reduced system which approximates the original input-output behavior sufficiently well, will also depend on the particular switching signals. Consequently, it can be shown that all of the above approaches will usually not result in the "best" reduced model for a specific switching signal. This motivates to study the minimal realization and model reduction for a class of switched systems with jumps.

Overall the motivation of this thesis is: Starting from non-minimal switched linear systems, new algorithms are developed to find a minimal realization which has the same response characteristics and capture some features of the original system. Later, balanced truncation is utilized to reduce the minimal system into a lower dimensional system that preserves the main characteristics of the original system.

1.4 Contribution of the thesis

So far, from the overview of the previous section, it can be concluded that realization theory and model reduction method for switched systems with known switching signal are not available in the existing literature. In particular, the time-varying nature of switched systems is not investigated.

The main objective of this thesis is thus to develop methods of realization theory as well as model reduction procedure of switched systems. In particular, this thesis is concerned about switched systems, given by ordinary differential equations with jumps, or differential-algebraic equations. Herein, two distinct parts are pursued. First, reduced realization of switched linear systems with known switching signal are developed by defining suitable reachable/unobservable subspaces. Secondly, model reduction techniques are discussed for both switched linear systems and switched DAEs as well. These contributions are discussed in Sections 1.4.1 and 1.4.2, respectively.

1.4.1 Reduced realization for switched linear systems

In the first part of the thesis (Chapters 3 and 4), realization theory is developed for switched linear systems with jumps where the switching signal is assumed to be fixed with known mode sequence. As mentioned above, switched linear systems are considered in this thesis as a piecewise-constant *time-varying* linear system. It is well known that a minimal realization for linear systems is just given by its reachable and observable parts. However, in the context of switched systems, this property is not true, even if each of the modes are reachable and observable independently.

In Chapter 3, switched linear systems with a single switch are considered. The problem of minimal realization is discussed and some results are presented. The key idea is to extend the reachable subspace of the second mode to include nonzero initial values (resulting from the first mode) and also extend the observable subspace of the first mode by taking information from the second mode into account. Moreover, the process of going from a non-minimal representation (with initial value zero) to a minimal one can be seen as removing "unreachable" and "unobservable" states. A minimal realization is obtained by first reducing the input-extended second mode, then reducing the output-extended of the first mode and finally, reduce the jump map between the modes. Furthermore, the method can be applied in arbitrary switching time duration wherein the overall time duration for the switched linear system is finite and given.

In Chapter 4, switched linear systems with mode-dependent state dimensions and/or state jumps are considered and a method is proposed to obtain a switched system of reduced size with identical input-output behavior. In particular, the method of Chapter 4 is very different and not directly comparable with the method used in Chapter 3. The time-varying reachable and unobservable subspaces for switched systems are reviewed and suitable extended reachable and restricted unobservable subspaces are defined. Then, a weak Kalman decomposition is introduced based on the extended reachable and restricted unobservable subspaces, and a reduced realization is obtained which has same input-output behavior as original systems. A key feature of the approach is that only the mode sequence of the switching signal needs to be known, not the exact switching times. However, the size of a minimal realization will in general depend on the mode durations, hence, it cannot be expected that the method always leads to minimal realization. Nevertheless, it is shown that the method is *optimal* in the sense that a repeated application does not lead to a further reduction. Furthermore, a practically relevant special case is highlighted, where minimality is achieved. Finally, motivated by an example it is conjectured that the reduced switched system has a minimal order for almost all switching time duration.

1.4.2 Model reduction for switched systems

Contrary to the previous section, the second part of the thesis deals with the development of model reduction techniques for switched systems where a coordinate transformation is investigated to find an approximation for the given switched system. As mentioned earlier, in this thesis switched systems are considered as a *special* class of time-varying systems, so it is expected that the coordinate transformation is also time-varying.

In Chapter 5, a time-varying approach is proposed to find a reduced model for switched linear systems. The key idea is first to approximate the piecewise-constant coefficient matrices of the switched linear system by continuously time-varying coefficient matrices and then apply available balanced truncation for (continuously) time-varying linear systems. The method provides a good reduced system which approximates the original switched system. It is shown that the original system and its approximation preserve some error bounds. However, the reduced system is not a simple switched system anymore, therefore, another method is investigated in Chapter 6.

In Chapter 6, a novel model reduction approach is investigated for switched linear systems with known mode duration. The propose method is based on a suitable definition of (time-varying) reachability and observability Gramians and it is shown that these Gramians satisfy precise interpretations in terms of input and output energy. Based on balancing the midpoint Gramians, a mode-wise model reduction is proposed to obtain a switched linear system of smaller size which then approximates the original system.

In Chapter 7, switched DAEs in continuous time are considered and the aim is to find a reduced realization as well as a lower dimensional model for switched DAEs. First, it is shown that the given switched DAE is input-output equivalent to a switched ODE with jumps and impulses which has input dependent jumps and Dirac impulses in the output. Under some conditions, these additional terms can be avoided to obtain a switched ODE with jumps and impulses. Secondly, a model reduction method via the techniques given in Chapter 6 is applied to obtain a good approximation and thereto, unreachable inconsistent states are removed first via the method discussed in Chapter 4.

In Chapter 8, a model reduction approach for singular linear switched systems (SLSSs) in discrete time is proposed. The result relies on the assumption that the given SLSS is jointly index-1. The key idea of the proposed method is to use first the one-step-map from [1] and find a time-varying system which has identical input-output behavior as the given SLSS. Then, available balanced truncation for time-varying systems in discrete time is adapted; moreover, the initial/final values given in Chapter 5 are also considered for the computation of forward reachability and backward observability Gramians.

1.5 Origins of the chapters

The following parts of this thesis are already published or submitted for publication. Chapter 3 has been presented at the European Control Conference (ECC 2021) and published in [59]. A brief version of Chapter 4 will be presented at the 10th International Conference on Mathematical Modelling (MATHMOD 2022) in Vienna, Austria, and this chapter has also been submitted for journal publication. A brief version of Chapter 5 has been presented at IFAC World Congress 2020 and published as [58]. Chapter 6 has been submitted for journal publication. Chapter 7 is currently under preparation for journal publication. Chapter 8 has been accepted to 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2022), Bayreuth, Germany. All publications are joint work with Stephan Trenn and Chapter 8 is also joint work with S. Sutrisno.

2 Preliminaries

2.1 Introduction

This chapter discusses the preliminaries required for the development of realization theory in Chapters 3, 4, and model reduction procedures in Chapters 5, 6, 7 and 8. The notation as used throughout this thesis is introduced in Section 2.2. Section 2.3 contains the system classes which are considered in this thesis. In Section 2.4, some important notions for reachability, observability and minimality are given.

2.2 General notation

In this section, some notation is given which will be used throughout the thesis. More specific notation that is used in one or only a few chapters will be defined within the chapters themselves.

The sets of natural, real, and complex numbers are defined by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. Let $\mathbb{R}^n$ ($\mathbb{C}^n$) denote the linear space of vectors with n real (complex) components. Moreover, the set of real (complex) $m \times n$ matrices is denoted by $\mathbb{R}^{m\times n}$ ($\mathbb{C}^{m\times n}$). Let S be a subspace of $\mathbb{R}^n$, then $S^\perp$ denotes the orthogonal complement of S in $\mathbb{R}^n$ with respect to the usual scalar inner product $\langle x, y\rangle = x^\top y$.

For a matrix A, $A^\top$ denotes the transpose of A. The image of a matrix $A \in \mathbb{R}^{m\times n}$ is denoted by $\operatorname{im} A$ and defined as

$$\operatorname{im} A = \left\{ Av \in \mathbb{R}^m \;\middle|\; v \in \mathbb{R}^n \right\} \subseteq \mathbb{R}^m.$$

The kernel of A is denoted by $\ker A$ and defined as

$$\ker A = \left\{ v \in \mathbb{R}^n \;\middle|\; Av = 0 \right\} \subseteq \mathbb{R}^n.$$

For some matrix $M \in \mathbb{R}^{m\times n}$ and set $\mathcal{M} \subseteq \mathbb{R}^n$, the image of $\mathcal{M}$ under M is $M\mathcal{M} := \{ Mx \mid x \in \mathcal{M} \}$ and for $\mathcal{M} \subseteq \mathbb{R}^m$ the preimage of $\mathcal{M}$ under M is $M^{-1}\mathcal{M} := \{ x \in \mathbb{R}^n \mid Mx \in \mathcal{M} \}$.

Let M, N be two subspaces of a vector space V, then the sum of the subspaces M and N is defined by

$$M + N = \left\{ m + n \;\middle|\; m \in M, n \in N \right\}.$$

Furthermore, if $M \cap N = \{0\}$, then the sum is called direct and is denoted as $M \oplus N$.

The class $\mathcal{L}_2^m(I)$ represents the class of functions $x : I \to \mathbb{R}^m$ for some $I \subseteq \mathbb{R}$ which are bounded in the $\mathcal{L}_2$ norm, denoted by $\|x\|_2$ and defined as $\|x(t)\|_2^2 := \int_I |x(t)|^2 \,\mathrm{d}t$; $|x|$ is used for absolute value. The notations $x(t^-)$ and $x(t^+)$ denote, respectively, the left- and right-sided limit of x at t, where it is implicitly assumed that they exist when used.

2.3 System classes

In this thesis, different system classes are considered in the scope of realization theory and model reduction. The system classes considered in this thesis are presented briefly in this section.

Linear time-invariant systems In continuous time, a *linear time-invariant* (LTI) system is a set of differential equations of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad t \in \mathbb{R}, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{2.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$. Here, A is the system (state) matrix, B the input (control) matrix, C the output matrix, and D the feed-forward matrix. The state $x(t)$ is in the n-dimensional state-space represented by a vector whose evolution in time follows a state vector trajectory. A time-invariant system is *asymptotically stable* if all the eigenvalues of the system matrix A have negative real parts. All solutions of the system are continuous and differentiable almost everywhere. Solutions depend on $x(0) = x_0$ and on $u(\cdot)$. Such systems are often denoted by (A, B, C, D).

Linear time-varying systems For a time-varying system the matrices A, B, C and D are varying over time, in contrast to a time-invariant system (where the matrices have constant entries). Such systems are the subject of Chapter 5 and have the state-space model

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad t \in \mathbb{R}, \\ y(t) &= C(t)x(t) + D(t)u(t), \end{aligned} \tag{2.2}$$

where $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, $C(t) \in \mathbb{R}^{p\times n}$, $D(t) \in \mathbb{R}^{p\times m}$, in particular, the state dimension is constant over time.

Generalized state-space system Generalized state-space systems (or descriptor systems, or differential-algebraic equations) are characterized by the equation

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \quad t \in \mathbb{R}, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{2.3}$$

where E and A not necessarily have to be square. However, only square systems are considered in this thesis.

If E is nonsingular, the generalized state-space system can be transformed into the state-space form (2.1). By convention, the system is in this case said to be of index zero.

If E is not invertible, the system containes some *algebraic* equations in addition to the differential equations. Such systems are the subject of Chapter 7, and the discrete time case is considered in Chapter 8.

Switched linear systems

Before introducing "general" switched linear systems, first the so-called switching signal needs to be introduced.

Assumption 2.1 (Switching signal)**.** The switching signal $\sigma : \mathbb{R} \to \mathbb{N}$ is right-continuous and in each compact interval there are only finitely many discontinuities.

If σ is constant on $(-\infty, s_0)$ for some $s_0 \in \mathbb{R}$, then the above standard assumptions guarantee that the set of all switching times of σ can be written as $\{\, s_k \in \mathbb{R} \mid k \in \mathbb{N} \,\}$ with $s_k < s_{k+1}$ for all $k \in \mathbb{N}$. Let $\tau_k := s_{k+1} - s_k$ denotes the mode duration of mode k. A switching signal $\sigma : [t_0, t_f) \to \{0, 1, \cdots, \mathtt{m}\}$ is said to be in *standard* form if

$$\sigma(t) = k \quad \text{for } t \in [s_k, s_{k+1}),$$

where $s_0 := t_0$ and $s_{\mathtt{m}+1} := t_f$.

For example, a switching signal $\sigma : \mathbb{R} \to \{0, 1, 2\}$ is illustrated in Figure 2.1 with switching time intervals $\tau_i, \quad i = 0, 1, 2$.

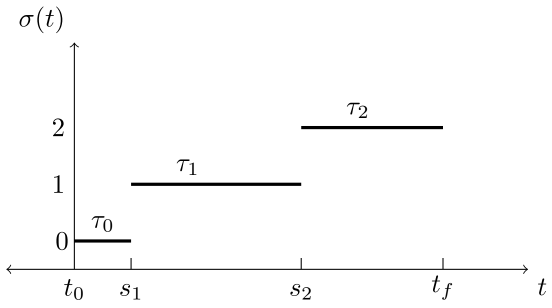

Figure 2.1: Switching signal $\sigma : \mathbb{R} \to \{0, 1, 2\}$ with intervals τ_0, τ_1, τ_2 between switches.

Definition 2.2 (Switched linear systems)**.** A switched linear system is a dynamical system given by

$$\Sigma_\sigma : \begin{cases} \dot{x}_k(t) = A_{\sigma(t)} x_k(t) + B_{\sigma(t)} u(t), & t \in (s_k, s_{k+1}), \\ x_k(s_k^+) = J_{\sigma(s_k^+),\sigma(s_k^-)} x_{k-1}(s_k^-), & k \in \mathbb{N}, \\ y(t) = C_{\sigma(t)} x_k(t) + D_{\sigma(t)} u(t), & t \in \mathbb{R}, \end{cases} \tag{2.4}$$

where $\sigma : \mathbb{R} \to \mathtt{M} = \{0, 1, 2, \ldots, \mathtt{m}\} \subseteq \mathbb{N}$ is the given switching signal with finitely many switching times $s_1 < s_2 < \cdots < s_\mathtt{m}$ in the bounded interval $[t_0, t_f)$ of interest and $x_k : (s_k, s_{k+1}) \to \mathbb{R}^{n_k}$ is the k-th piece of the state (whose dimension n_k may depend on the mode k).

For notational convenience, let $s_0 := t_0$, $s_{\mathtt{m}+1} := t_f$ and let the duration of mode k be denoted by $\tau_k := s_{k+1} - s_k$, $k \in \{0, 1, \ldots, \mathtt{m}\}$. In the context of realization theory and model reduction, it is common to assume that the system starts with a zero initial condition, i.e. set $x_{-1}(t_0^-) := 0$. The input and output are given by u and y, respectively.

For each mode $p \in \{0, 1, 2, \ldots, \mathtt{m}\}$, the system matrices A_p, B_p, C_p, D_p of appropriate size describe the (continuous) dynamics corresponding to the linear system active on the interval (s_k, s_{k+1}) where $\sigma(t) = p$. Furthermore, $J_{p,q} : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$ is the jump map from mode q to mode p. Note that due to the different space dimensions the introduction of a jump map is necessary; on the other hand, in case all state dimensions are equal, the consideration of a jump map is "optional" and leads to so called *impulsive* systems.

It will be assumed in the thesis that the switching signal is fixed, hence by suitable relabeling of the matrices, it is assumed that $\sigma(t) = k$ on (s_k, s_{k+1}). Consequently, one can simply write $J_k := J_{\sigma(s_k^+),\sigma(s_k^-)} = J_{k,k-1}$ and $\widehat{J}_k := \widehat{J}_{\sigma(s_k^+),\sigma(s_k^-)} = \widehat{J}_{k,k-1}$ in the following.

Furthermore, in some slight abuse of notation, one can speak in the following of the solution $x(\cdot)$ instead of the different solution pieces $x_k(\cdot)$. The solution of switched linear system (2.4) is given recursively by, for $t \in [s_k, s_{k+1})$ and $k = 1, \dots, \mathtt{m}$,

$$x(t) = e^{A_k(t-s_k)} J_k x(s_k^-) + \int_{s_k}^{t} e^{A_k(t-s)} B_k u(s) \, \mathrm{d}s, \tag{2.5}$$

and the output equation is given by

$$y(t) = C_k x(t) + D_k u(t), \; t \in [s_k, s_{k+1}), k = 0, 1, \dots, \mathtt{m}. \tag{2.6}$$

2.4 Reachability, observability and minimality

The notions of reachability and observability of state-space system are central topics in standard realization theory. Furthermore, reachability and observability as well as minimality of systems are considered in the scope of model reduction. It is well known in system theory for continuous time systems that the concepts of controllability/ reachability are dual to determinability / observability, but for switched systems this is not the case, c.f. [70].

2.4.1 Reachable and unobservable subspaces

Before introducing the reachable and unobservable subspaces, the following notation concerning A-invariant subspaces is introduced.

Definition 2.3. For $A \in \mathbb{R}^{n \times n}$ and a subspace $\mathcal{L} \subseteq \mathbb{R}^n$, let

$$\langle A \mid \mathcal{L} \rangle := \mathcal{L} + A\mathcal{L} + \cdots + A^{n-1}\mathcal{L}$$

be the smallest A-invariant subspace containing $\mathcal{L}$. Furthermore, let (here A^{-1} stands for the preimage, it is not assumed that A is invertible)

$$\langle \mathcal{L} \mid A \rangle := \mathcal{L} \cap A^{-1}\mathcal{L} \cap \cdots \cap A^{-(n-1)}\mathcal{L}$$

be the largest A-invariant subspace contained in $\mathcal{L}$.

Reachability investigates to what extent one can influence the state by a suitable choice of the control.

Definition 2.4. An LTI system (2.1) is said to be reachable in $[t_0, t_f]$, if for all $x_t \in \mathbb{R}^n$, there exists an input $u : [t_0, t_f] \to \mathbb{R}^m$ such that the system is driven from initial state $x(t_0) = 0$ to the final state $x(t_f) = x_f$.

Theorem 2.5 ([124]). *The system* (2.1) *is* reachable *if and only if*

$$\langle A \mid \operatorname{im} B\rangle = \operatorname{im} B + A(\operatorname{im} B) + \cdots + A^{n-1}(\operatorname{im} B) = \mathbb{R}^n.$$

In particular, the reachability matrix

$$\mathcal{C}(A,B) := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

is of full rank, i.e., $\operatorname{rank} \mathcal{C}(A,B) = n$ and so $\operatorname{im} \mathcal{C}(A,B) = \langle A \mid \operatorname{im} B\rangle = \mathbb{R}^n$.

Observability is the dual concept of reachability in the system-theoretic sense. Observability investigates to what extent it is possible to reconstruct the state x when the input u and the output y are known. One often can measure the output and prescribe the input, whereas the state variable is hidden.

Definition 2.6. An LTI system (2.1) is said to be observable in $[t_0, t_f]$, if for any given input u the initial state $x(t_0)$ can be uniquely determined from the observed output y on $[t_0, t_f]$.

Theorem 2.7 ([124]). *The system* (2.1) *is* observable *if and only if*

$$\langle \ker C \mid A\rangle = \{0\}.$$

Actually, the largest A-invariant subspace contained in $\ker C$ can also be written as

$$\langle \ker C \mid A\rangle = \ker \mathcal{O}(C,A),$$

where $\mathcal{O}(C,A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ is the observability matrix.

Note that similarity transformations do not affect reachability and observability.

2.4.2 Minimality

The notion of minimal realization is motivated by finding a system of smallest state-space dimension which has the same input-output map as a given system. For a given input-output map, minimality is characterized as follows.

Theorem 2.8 ([2]). *A state-space realization is minimal if and only if it is reachable and observable.*

Consider the linear system (2.1) with system matrices A, B, C, D. The well known Kalman decomposition (KD), first established by Kalman [66], gives a coordinate transformation

$$T = \begin{bmatrix} V_1 & V_2 & V_3 & V_4 \end{bmatrix},$$

where $\mathrm{im}\ V_2$ is the intersection of the reachable and unobservable subspace, $\mathrm{im}[V_1 \quad V_2]$ is the reachable subspace and $\mathrm{im}[V_1 \quad V_3]$ is the unobservable subspace. Then, the system matrices A, B, C, D can be transformed to the following block triangular form

$$(T^{-1}AT, T^{-1}B, CT, D) = \left(\begin{bmatrix} A^{11} & A^{12} & A^{13} & A^{14} \\ 0 & A^{22} & 0 & A^{24} \\ 0 & 0 & A^{33} & A^{34} \\ 0 & 0 & 0 & A^{44} \end{bmatrix}, \begin{bmatrix} B^1 \\ B^2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & C^2 & 0 & C^4 \end{bmatrix}, D \right),$$

where $\left(\begin{bmatrix} A^{11} & A^{12} \\ 0 & A^{22} \end{bmatrix}, \begin{bmatrix} B^1 \\ B^2 \end{bmatrix} \right)$ is reachable and $\left(\begin{bmatrix} A^{22} & A^{24} \\ 0 & A^{44} \end{bmatrix}, \begin{bmatrix} C^2 & C^4 \end{bmatrix} \right)$ is observable. In fact, a minimal realization is now given by (A^{22}, B^2, C^2, D).

2.4.3 Reachability and observability Gramians

Definition 2.9. Consider the system (2.1) with zero initial value on the interval $[t_0, t_f]$. Then, the reachability function $F_r^t : \mathbb{R}^n \to \mathbb{R}$ is defined by, for some $t \in [t_0, t_f]$

$$F_r^t(x_t) = \inf_u \int_{t_0}^{t} u(\tau)^\top u(\tau)\, \mathrm{d}\tau,$$

where $u : [t_0, t) \to \mathbb{R}^m$ is an input signal that steers the system from $x(t_0) = 0$ to $x(t) = x_t$.

Theorem 2.10 ([2, 84])**.** *Assume that the system* (2.1) *is reachable then*

$$P(t) := \int_{t_0}^{t} e^{A(\tau - t_0)} B B^\top e^{A^\top (\tau - t_0)}\, \mathrm{d}\tau$$

is positive definite, and the reachability function satisfies

$$F_r^t(x_t) = x_t^T P(t)^{-1} x_t.$$

Here, $P(t)$ is called the reachability Gramian.

This relation states that any state x_t that lies in an eigenspace of $P(t)^{-1}$ corresponding to large eigenvalues (which are equal to the eigenvectors of $P(t)$ with small eigenvalues) requires more input energy to control. The reachability function thus gives the least amount of energy needed to reach a certain state x_t.

Lemma 2.11 ([2]). *Consider the system* (2.1) *with the reachability Gramian $P(t)$ for some $t \in [t_0, t_f]$, the input*

$$u(t) := B^\top e^{A^\top (t_f - t)} P(t_f)^{-1}(x_f - e^{A(t_f - t_0)} x_0)$$

steer the system from $x(t_0) = x_0$ to $x(t_f) = x_f$ on the interval $[t_0, t_f]$ and that this is the input with minimal energy $\int_{t_0}^{t_f} u(\tau)^\top u(\tau) \mathrm{d}\tau = (x_f - e^{A(t_f - t_0)} x_0)^\top P(t_f)^{-1}(x_f - e^{A(t_f - t_0)} x_0)$ achieving this.

The observability function gives the output energy associated to some state x_t.

Definition 2.12. Consider a solution of the system (2.1) with zero input on the interval $[t, t_f]$, then the observability function $F_o^t : \mathbb{R}^n \to \mathbb{R}$ is defined as, for $t \in [t_0, t_f)$

$$F_o^t(x_t) = \int_t^{t_f} y(\tau)^\top y(\tau)\, \mathrm{d}\tau,$$

where $y(t)$ is the output of (2.1) for $x(t) = x_t$.

Theorem 2.13 ([2, 84]). *Consider a system* (2.1) *and let*

$$Q(t) := \int_t^{t_f} e^{A^\top (t_f - \tau)} C^\top C e^{A(t_f - \tau)}\, \mathrm{d}\tau$$

which is positive semidefinite. Then, the observability function is given by

$$F_o^t(x_t) = x_t^\top Q(t) x_t.$$

Here, $Q(t)$ is called the observability Gramian *and the system is observable if $Q(t)$ is positive definite.*

Similar to reachability, the state that lies along one of the eigenvectors of $Q(t)$ with small eigenvalues is difficult to observe. Therefore, the states that are difficult to control and observe are less important. In particular, the balancing-based MOR methods are based on identifying and truncating the less important states from the systems. This observation suggests that reduced order models may be obtained by eliminating those states that are difficult to reach or difficult to observe, for details c.f. [36, 45, 84, 148].

If all the eigenvalues of A in (2.1) have negative real part, it can be shown that for $t \to \infty$, the matrix $P(t)$ converges to a constant matrix P and, for $t \to -\infty$, the matrix $Q(t)$ converges to a constant matrix Q, and they satisfy the following algebraic Lyapunov equations

$$AP + PA^T + BB^T = 0, \tag{2.7}$$

$$A^T Q + QA + C^T C = 0. \tag{2.8}$$

Moreover, the Lyapunov equations can be solved by direct solvers cf. [8, 56] as well as iterative solvers cf. [14, 62].

Remark 2.14. In case of infinite time interval, asymptotic stability guarantees the existence (i.e., boundedness) of the Gramians whereas the property *positive definite* ensures reachability/observability of systems.

Clearly, the combination of the reachability and observability functions provides a characterization of the importance of the state components with respect to the input-output behavior. In general, it is not easy to identify importance of a state in arbitrary coordinates from an input-output perspective.

Part I

Reduced realization for switched systems

3 Reduced realization for switched linear systems with a single switch

3.1 Introduction

As discussed in Chapter 1, realization theory is a classical topic in the area of systems and control. In general, the aim of realization theory is to construct a state-space model from a given input-output behavior of a system.

Realization theory of switched systems has already been discussed in e.g., [10, 95, 97, 98, 99, 102] and the references therein. In particular, the cases of arbitrary and constrained switching are discussed where the switching signal is considered as an input. This consideration of the switching signal as an "input" is a common viewpoint in most of the existing works, i.e. it is not possible to use these results when trying to find a (minimal) realization for a given switching signal (or given mode sequence).

In contrast to much of the existing literature on switched systems, in this thesis, a switched linear system is viewed as a piecewise-constant time-varying linear system, in particular, a (minimal) realization in general depends on the specifically given switching signal. It is well known that finding a minimal realization (which can be interpreted as removing unobservable and unreachable states) is a first step towards model reduction (which furthermore reduces difficult to observe and difficult to reach states).

To be more specific, the switched linear system (2.4) (from Chapter 2) is considered with a single switch and the main goal is to find a reduced size switched system (for the same switching signal) of the form

$$\widehat{\Sigma}_\sigma : \begin{cases} \dot{\widehat{x}}_k(t) = \widehat{A}_{\sigma(t)}\widehat{x}_k(t) + \widehat{B}_{\sigma(t)}u(t), \ t \in (s_k, s_{k+1}), \\ \widehat{x}_k(s_k^+) = \widehat{J}_{\sigma(s_k^+),\sigma(s_k^-)}\widehat{x}_{k-1}(s_k^-), \\ y(t) = \widehat{C}_{\sigma(t)}\widehat{x}_k(t^+) + D_{\sigma(t)}u(t), \quad t \in [s_k, s_{k+1}), \end{cases} \tag{3.1}$$

which has the same input-output behavior as the original system (2.4).

It is well known that the feedthrough term $D_k u$ does not play any role in the analysis of realization theory as well as model reduction, hence in the following, this term is avoided.

This chapter is organized as follows. In Section 3.2, the problem formulation and preliminaries are given with the characterization of reachability and observability of SLSs. Section 3.3 discusses the main results with the proposed algorithm and some simulation results.

3.2 Problem setting

In this section, some notions and challenges related to reduced realization of switched linear systems of the form (2.4) are introduced. The formal definition of reduced realization is given at first.

Definition 3.1 (cf. [95]). Given the switched linear system (2.4), the *total dimension* is defined as

$$\dim \Sigma_\sigma := \sum_{k \in \mathtt{M}} n_k.$$

Furthermore, its input-output behavior is given by

$$\mathfrak{B}_\sigma^{io} := \left\{ (u, y) \;\middle|\; \begin{array}{l} \exists\, x_k : (s_k, s_{k+1}) \to \mathbb{R}^{n_k} \text{ satisfying (2.4)} \\ \text{and } x(t_0^-) = 0 \end{array} \right\}.$$

A switched linear system $\widehat{\Sigma}_\sigma$ with corresponding input-output behavior $\widehat{\mathfrak{B}}_\sigma^{io}$ is said to be a reduced realization of switched system Σ_σ if
1) $\mathfrak{B}_\sigma^{io} = \widehat{\mathfrak{B}}_\sigma^{io}$ and
2) $\dim \widehat{\Sigma}_\sigma \leqslant \dim \Sigma_\sigma$.

In the following, minimal realizations for the single switch case will also be discussed, which are reduced realization of smallest total dimension under all reduced realizations. It should be noted that, at this point it is not clear that the sequence of reduced state dimensions is unique for a minimal realization.

For non-switched linear systems, it is well known that a realization is minimal if, and only if it is reachable and observable. However, for switched linear systems with single switch case, this is not the case in general as the following example shows.

Example 3.1. Consider a switched linear system with two modes

$$(A_0, B_0, C_0) = \left(\begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \right),$$
$$(A_1, B_1, C_1) = \left(\begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \right), \; J_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

and the single switching signal

$$\sigma(t) = \begin{cases} 0, & \text{on } (t_0, s_1), \\ 1, & \text{on } (s_1, t_f). \end{cases} \tag{3.2}$$

It is easily seen, that each mode is unreachable and unobservable. However, the switched system is reachable in the sense that each value $x(t_f^-) \in \mathbb{R}^3$ can be reached from zero by a suitable input and it is also observable in the sense that (for a vanishing input) only a zero initial value leads to a zero output.

On the other hand, the second state is unreachable in the 1st mode and unobservable in the 2nd mode. In particular, when starting with a zero initial value, for any input the value of the second state does not effect the output (because in the first mode it is identically zero and in the second mode the corresponding coefficient in the C-matrix is zero). Therefore, one can remove the second state without altering the input-output behavior.

Remark 3.2. The above definition of reduced realizations is not specifying any *method* how to obtain a reduced realization from a given switched system. In particular, it does not take into account constraints like the requirement that the reduced state is obtained via a uniform projection map (cf. [49, 50] in the context of model reduction). In general, a reduced realization can only be obtained by considering each mode individually (and by properly taking into account the effect from the other modes).

It is important to note that a naive approach to reduce each individual mode by removing unreachable and unobservable states will not work in general, this is illustrated with the following example.

Example 3.2. Consider a switched linear system with modes

$$(A_0, B_0, C_0) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix} \right),$$
$$(A_1, B_1, C_1) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix} \right),$$

with switching signal (3.2) and without jump.

It is clearly observed that the second state is not observable in the first mode and not reachable in the second mode, hence one may be tempted to remove this state to obtain an (unswitched) minimal realization given by

$$\dot{x}_1 = x_1 + u, \quad y = x_1. \tag{3.3}$$

However, it is easily seen that a non-zero input leads to a non-zero second state during the first mode, which then will effect the output of the second mode, i.e. system (3.3) obtained by simply reducing each mode individually does *not* have the same input-output behavior as the original switched system.

Another important challenge for obtaining a reduced realization is the fact, that even when one starts with a classical switched system (i.e. all states have the same dimensions and the jump map is the identity), a reduced realization may have different state-space dimensions and/or requires the definition of a jump map. This is illustrated with the following example.

Example 3.3. Consider a switched linear system with two modes

$$A_0 = A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_0 = B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$
$$C_0 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

with switching signal (3.2) and without jumps. It is easily seen that the first mode corresponds to a double integrator, while the second mode corresponds to a single integrator. Hence, a minimal realization is given by the following switched linear system with mode-dependent state dimensions:

on $[t_0, s_1)$:	on $[s_1, t_f)$:
$\dot{z}_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} z_0 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u,$	$\dot{z}_1 = 0 \cdot z_1 + u,$
$y = \begin{bmatrix} 0 & 1 \end{bmatrix} z_0,$	$y = z_1,$

with $z_1(s_1) = \begin{bmatrix} 1 & 0 \end{bmatrix} z_0(s_1)$.

The possible mode-dependence of a reduced realization is the main motivation to study switched system (2.4) with mode-dependent state dimensions and jumps, so that both systems (original system and the reduced realization) are from the same overall system class.

In Section 3.3, a method will be proposed which takes into account the effect of the different modes have on each other. Although, this method will not simply consider a minimal realization of each mode individually, the method of reducing a given (unswitched) linear system to a minimal one will play an important role and is recall first.

3.3 Minimal realization

The minimal realization of a linear system with non-zero initial values is given in Section 3.3.1 and the minimality for switched linear systems is discussed in Section 3.3.2.

3.3.1 Minimal realization for linear systems

Consider a linear system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0, \\ y(t) = Cx(t), \end{cases} \tag{3.4}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$.

As discussed in Section 2.4.2, the well known Kalman decomposition (KD) is a coordinate transformation $x = Tz$ which leads the system matrices to the following form

$$(T^{-1}AT, T^{-1}B, CT) = \left(\begin{bmatrix} A^{11} & A^{12} & A^{13} & A^{14} \\ 0 & A^{22} & 0 & A^{24} \\ 0 & 0 & A^{33} & A^{34} \\ 0 & 0 & 0 & A^{44} \end{bmatrix}, \begin{bmatrix} B^1 \\ B^2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & C^2 & 0 & C^4 \end{bmatrix} \right),$$

where $\left(\begin{bmatrix} A^{11} & A^{12} \\ 0 & A^{22} \end{bmatrix}, \begin{bmatrix} B^1 \\ B^2 \end{bmatrix} \right)$ is reachable and $\left(\begin{bmatrix} A^{22} & A^{24} \\ 0 & A^{44} \end{bmatrix}, \begin{bmatrix} C^2 & C^4 \end{bmatrix} \right)$ is observable. A minimal realization of (3.4) is now given by (A^{22}, B^2, C^2).

This method is based on the assumption that the initial value is zero. If arbitrary initial values are considered, it is easy to see that only the unobservable states can be removed. In the context of switched systems, the initial values for the second mode are neither zero nor completely arbitrary, but are constraint to the reachable subspace of the first mode, this motivates to find a minimal realization for the linear system

$$\Sigma^{\mathcal{X}_0} : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) \in \mathcal{X}_0, \\ y(t) = Cx(t), \end{cases} \tag{3.5}$$

where $\mathcal{X}_0 \subseteq \mathbb{R}^n$ is the subspace of relevant initial values.

Inspired by the fact, that the ODE $\dot{x} = Ax$ with initial condition $x(0) = x_0$ has the same solution as the impulsive ODE $\dot{x} = Ax + x_0\delta$, $x(0^-) = 0$ (see Appendix B.5), the following *input-extended* system corresponding to (3.5) (cf. [57] in the context of model reduction) is proposed as follows

$$\Sigma_e^{\mathcal{X}_0} : \begin{cases} \dot{x}_e(t) = Ax_e(t) + \begin{bmatrix} B & X_0 \end{bmatrix} \begin{bmatrix} u(t) \\ u_0(t) \end{bmatrix}, & x_e(0) = 0, \\ y_e(t) = Cx_e(t), \end{cases} \tag{3.6}$$

where $\operatorname{im} X_0 = \mathcal{X}_0$. Now apply the KD on the extended system and obtain a minimal extended realization

$$\widehat{\Sigma}_e^{\mathcal{X}_0} : \begin{cases} \dot{\widehat{x}}_e(t) = \widehat{A}_e \widehat{x}_e(t) + \begin{bmatrix} \widehat{B}_e & \widehat{X}_0 \end{bmatrix} \begin{bmatrix} u(t) \\ u_0(t) \end{bmatrix}, & \widehat{x}_e(0) = 0, \\ \widehat{y}_e(t) = \widehat{C}_e \widehat{x}_e(t), \end{cases} \tag{3.7}$$

and the corresponding minimal realization

$$\widehat{\Sigma}^{\mathcal{X}_0} : \begin{cases} \dot{\widehat{x}}(t) = \widehat{A}_e \widehat{x}(t) + \widehat{B}_e u(t), \quad \widehat{x}(0) = \widehat{x}_0 \in \operatorname{im} \widehat{X}_0, \\ \widehat{y}(t) = \widehat{C}_e \widehat{x}(t). \end{cases} \tag{3.8}$$

The properties of the reduced system $\widehat{\Sigma}^{\mathcal{X}_0}$ can be formalized in the following lemma.

Lemma 3.3. *Consider* $\Sigma^{\mathcal{X}_0}$ *as in* (3.5) *and* $\widehat{\Sigma}^{\mathcal{X}_0}$ *as in* (3.8) *obtained by first extending* (3.5) *to* (3.6)*, then reducing it via the KD to a minimal extended realization* (3.7) *and finally removing the extension. Then,* $\Sigma^{\mathcal{X}_0}$ *and* $\widehat{\Sigma}^{\mathcal{X}_0}$ *are input-output equivalent in the sense that for all trajectories* (x, u, y) *satisfying* (3.5)*, there exists* $\widehat{x}$ *such that* $(\widehat{x}, u, y)$ *satisfies* (3.8)*. Furthermore,* $\widehat{\Sigma}^{\mathcal{X}_0}$ *has the minimal state dimension under all systems which are input-output equivalent to* $\Sigma^{\mathcal{X}_0}$.

Proof. The output equation of $\Sigma^{\mathcal{X}_0}$ as in (3.5) with initial value $x(0) = x_0 \in \mathcal{X}_0$ is given by

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau. \tag{3.9}$$

Construct an input-extended system $\Sigma_e^{\mathcal{X}_0}$ as in (3.6), and let $x_0 = X_0 z_0$ for some z_0, where the columns of X_0 form the basis of $\mathcal{X}_0$. Then, the system (3.5) is input-output equivalent to system (3.6) with the input $u_0 = z_0\delta$, where δ denotes the Dirac delta distribution.

Again, the output equation of $\Sigma_e^{\mathcal{X}_0}$ as in (3.6), can be written by

$$\begin{aligned} y_e(t) &= \int_0^t Ce^{A(t-\tau)} \begin{bmatrix} B & X_0 \end{bmatrix} \begin{bmatrix} u(t) \\ u_0(t) \end{bmatrix} \mathrm{d}\tau \\ &= \int_0^t CTe^{T^{-1}AT(t-\tau)}T^{-1} \begin{bmatrix} B & X_0 \end{bmatrix} \begin{bmatrix} u(t) \\ u_0(t) \end{bmatrix} \mathrm{d}\tau, \end{aligned}$$

where T is the KD transformation matrix which transforms $(A, [B, X_0], C)$ into a KD, i.e.

$$T^{-1}AT = \begin{bmatrix} * & * & * & * \\ 0 & \widehat{A}_e & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix},$$

$$T^{-1}[B \quad X_0] = \begin{bmatrix} * & * \\ \widehat{B}_e & \widehat{X}_0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad CT = \begin{bmatrix} 0 & \widehat{C}_e & 0 & * \end{bmatrix}.$$

So utilizing $T^{-1}e^{A}T = e^{T^{-1}AT}$,

$$y_e(t) = \int_0^t \widehat{C}_e e^{\widehat{A}_e(t-\tau)} \begin{bmatrix} \widehat{B}_e & \widehat{X}_0 \end{bmatrix} \begin{bmatrix} u(t) \\ u_0(t) \end{bmatrix} \mathrm{d}\tau, \tag{3.10}$$

which is the output of minimal extended realization as in (3.7). Again, for $u_0 = z_0\delta$, equation (3.10) can be written as

$$y_e(t) = \widehat{C}_e e^{\widehat{A}_e t}\widehat{x}_0 + \int_0^t \widehat{C}_e e^{\widehat{A}_e(t-\tau)}\widehat{C}_e u(\tau)\,\mathrm{d}\tau, \tag{3.11}$$

where $\widehat{x}_0 = \widehat{X}_0 z_0$ and it represents the output equation for (3.8) with the solution $\widehat{x}$. Therefore, for any arbitrary trajectories (x, u, y) satisfying (3.5) there exists $\widehat{x}$ such that $(\widehat{x}, u, y)$ satisfies (3.8).

2nd part: By construction, it is true that $\widehat{\Sigma}_e^{\mathcal{X}_0}$ is the minimal realization of $\Sigma_e^{\mathcal{X}_0}$. Consider a system $\widetilde{\Sigma}^{\mathcal{X}_0}$ which is input-output equivalent to $\Sigma^{\mathcal{X}_0}$. Then, for all $x_0 \in \mathcal{X}_0$, there exists $\widetilde{x}_0 \in \widetilde{\mathcal{X}}_0$ such that for all input $u(\cdot)$,

$$y(\cdot, u, x_0) = \widetilde{y}(\cdot, u, \widetilde{x}_0).$$

Inspired by [57], an input-extended system $\widetilde{\Sigma}_e^{\mathcal{X}_0}$ as in (3.7) can be constructed for

$\widetilde{\Sigma}^{\mathcal{X}_0}$ such that

$$\widetilde{y}_e(\cdot, \begin{bmatrix} u \\ u_0 \end{bmatrix}, 0) = y_e(\cdot, \begin{bmatrix} u \\ u_0 \end{bmatrix}, 0), \ \forall u, u_0. \tag{3.12}$$

Then

$$\dim \widetilde{\Sigma}^{\mathcal{X}_0} = \dim \widetilde{\Sigma}_e^{\mathcal{X}_0} \geqslant \dim \widehat{\Sigma}_e^{\mathcal{X}_0} = \dim \widehat{\Sigma}^{\mathcal{X}_0},$$

because $\widehat{\Sigma}_e^{\mathcal{X}_0}$ has by construction the minimal state dimension under all (extended) systems satisfying (3.12), i.e., $\widehat{\Sigma}^{\mathcal{X}_0}$ has indeed the minimal state dimension under all systems which are input-output equivalent to $\Sigma^{\mathcal{X}_0}$. □

3.3.2 Minimal realization for switched linear systems: single switch

In this section, a method is proposed to find the minimal realization of switched linear systems

$$\begin{aligned} &(A_0, B_0, C_0), \quad \text{on } (t_0, s_1), \\ &(A_1, B_1, C_1), \quad \text{on } (s_1, t_f), \end{aligned} \tag{3.13}$$

where $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m_i}$, $C_i \in \mathbb{R}^{p_i \times n_i}$, $i \in \{0, 1\}$ with the single switching signal given by (3.2) and jump matrix is given by J_1.

The proposed technique consists of three main steps. First, a minimal realization of the second mode is constructed by taking into account the reachable subspace of the first mode. Second, a minimal realization of the first mode is found by taking into account the observable states of the second mode. The final step consists of defining the reduced jump map from first mode to second mode.

Step 1: Reduction of second mode

As discussed above, the second mode does in general not start with initial value zero, instead the values of $x(s_1^-)$ covers the whole reachable subspace

$$\mathcal{R}_0 = \operatorname{im}[B_0 \quad A_0 B_0 \ \cdots \ A_0^{n_0 - 1} B_0],$$

consequently $x(s_1^+) \in J_1 \mathcal{R}_0$.

Using the method described in Section 3.3.1, the second mode is extended to $(A_1, \overline{B}_1, C_1)$ with

$$\overline{B}_1 := [B_1 \quad J_1 R_0],$$

where $\operatorname{im} R_0 = \mathcal{R}_0$ and the reduction method is applied to obtain the reduced mode $(\widehat{A}_1, \widehat{B}_1, \widehat{C}_1)$ which is input-output equivalent with the second mode (under the assumption that the initial values for the second mode are determined by the reachable states of first mode).

Step 2: Reduction of first mode

Restricted to the first interval, all unobservable and unreachable states can be removed from the first mode without changing the input-output behavior. However, some unobservable, but reachable state from the first mode may become observable in the second mode and hence, the input of the first mode may indirectly influence the output of the second mode (via the initial value for the second mode). Therefore, the following general problem is arrived: Given a linear system (3.4) and a subspace $\mathcal{L} \subseteq \mathbb{R}^n$ of indirectly-observable states, find a minimal realization of (3.4) which does not "remove" indirectly-observable states.

To be more precise, assume that the reduction of (3.4) is achieved via a left projection Π_l and a right projection Π_r, i.e. $\Pi_l \Pi_r = I$ and the reduced system of (A, B, C) is given by

$$(\widehat{A}, \widehat{B}, \widehat{C}) = (\Pi_l A \Pi_r, \Pi_l B, C \Pi_r).$$

In particular, $\ker \Pi_l$ corresponds to the removed states. The condition that the subspace of indirectly-observed states $\mathcal{L}$ is not "removed" by the reduction procedure, can be formalized by the condition

$$\ker \Pi_l \cap \mathcal{L} = \{0\}.$$

Similar as in Step 1 where the input matrix is extended to enlarge the reachable subspace, a method is now proposed to extend the output matrix for extending the observable subspace, i.e. consider

$$\Sigma_e^{\mathcal{L}} : \begin{cases} \dot{x}_e(t) = A x_e(t) + B u(t), \quad x_e(0) = 0, \\ y_e(t) = \begin{bmatrix} C \\ L^\top \end{bmatrix} x_e(t), \end{cases} \tag{3.14}$$

where $\operatorname{im} L = \mathcal{L}$. Very similar as before, one can utilize the KD to obtain a minimal realization of (3.14) given by

$$\widehat{\Sigma}_e^{\mathcal{L}} : \begin{cases} \dot{\widehat{x}}_e(t) = \widehat{A}_e \widehat{x}_e(t) + \widehat{B}_e u(t), \quad \widehat{x}_e(0) = 0, \\ y_e(t) = \begin{bmatrix} \widehat{C}_e \\ \widehat{L}^\top \end{bmatrix} \widehat{x}_e(t). \end{cases} \tag{3.15}$$

Removing the additional rows in the output matrix, the proposed minimal realization of (3.4) is obtained which does not "remove" states from $\mathcal{L}$:

$$\widehat{\Sigma}^{\mathcal{L}} : \begin{cases} \dot{x}(t) = \widehat{A}_e \widehat{x}(t) + \widehat{B}_e u(t), \quad \widehat{x}(0) = 0, \\ \widehat{y}(t) = \widehat{C}_e \widehat{x}(t). \end{cases} \tag{3.16}$$

Now, investigate the properties of the reduced system $\widehat{\Sigma}^{\mathcal{L}}$.

Lemma 3.4. *The two systems* (3.4) *and* (3.16) *have the same input-output behavior. Furthermore, let* Π_l *and* Π_r *be the two projectors transforming* (3.4) *into* (3.16), *i.e.* $\Pi_l\Pi_r = I$, $\widehat{A}_e = \Pi_l A \Pi_r$, $\widehat{B}_e = \Pi_l B$ *and* $\widehat{C}_e = C\Pi_r$, *and assume* $\mathcal{L}$ *is contained in the reachable subspace, then*

$$\ker \Pi_l \cap \mathcal{L} = \{0\}.$$

Proof. The output equation of system (3.4) is

$$\begin{aligned} y(t) &= \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau \\ &= \int_0^t \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} C \\ L^\top \end{bmatrix} Te^{T^{-1}AT(t-\tau)}T^{-1}Bu(\tau)\,\mathrm{d}\tau, \end{aligned}$$

where T is the transformation for obtaining a KD for $(A, B, [C^\top \quad L]^\top)$, i.e.

$$T^{-1}AT = \begin{bmatrix} * & * & * & * \\ 0 & \widehat{A}_e & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} * \\ \widehat{B}_e \\ 0 \\ 0 \end{bmatrix},$$
$$\begin{bmatrix} C \\ L^\top \end{bmatrix} T = \begin{bmatrix} 0 & \widehat{C}_e & 0 & * \\ 0 & \widehat{L}^\top & 0 & * \end{bmatrix}.$$

Now, together with

$$\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} C \\ L^\top \end{bmatrix} T = \begin{bmatrix} 0 & \widehat{C}_e & 0 & * \end{bmatrix}, \quad T^{-1}e^{A}T = e^{T^{-1}AT},$$

the output equation is given by

$$y(t) = \int_0^t \widehat{C}_e e^{\widehat{A}_e(t-\tau)}\widehat{B}_e u(\tau)\,\mathrm{d}\tau = \widehat{y}(t),$$

where $\widehat{y}(t)$ is the output of minimal realization as in (3.16).

2nd part: Let the transformation matrices to obtain a KD of system (3.14) be

$$\widehat{T} = \begin{bmatrix} \widehat{V}_1 & \widehat{V}_2 & \widehat{V}_3 & \widehat{V}_4 \end{bmatrix}, \quad \widehat{T}^{-1} = \begin{bmatrix} \widehat{W}_1 \\ \widehat{W}_2 \\ \widehat{W}_3 \\ \widehat{W}_4 \end{bmatrix}.$$

Clearly, $\Pi_l = \widehat{W}_2$, $\Pi_r = \widehat{V}_2$, $\operatorname{im} L \subseteq \operatorname{im} \begin{bmatrix} C \\ L^\top \end{bmatrix}^\top =: (\ker C_e)^\perp$ and it follows that

$$\operatorname{im} L \subseteq \left(\ker \begin{bmatrix} C_e \\ C_e A \\ \vdots \\ C_e A^{n-1} \end{bmatrix} \right)^\perp = \left(\operatorname{im}[\widehat{V}_1 \quad \widehat{V}_2] \right)^\perp = \operatorname{im}[\widehat{W}_2^\top \quad \widehat{W}_4^\top].$$

Furthermore, by assumption $\mathcal{L}$ is contained in the reachable subspace, $\operatorname{im} L \subseteq \operatorname{im}[\widehat{V}_1 \quad \widehat{V}_2]$ (the reachable subspace). Hence,

$$\operatorname{im} L \subseteq \operatorname{im}[\widehat{V}_1 \quad \widehat{V}_2] \cap \operatorname{im}[\widehat{W}_2^\top \quad \widehat{W}_4^\top].$$

Consider an arbitrary $z \in \operatorname{im} L \cap \ker \Pi_l$. From $\Pi_r z = 0$ and

$$z \in \operatorname{im} L \subseteq \operatorname{im}[\widehat{V}_1 \quad \widehat{V}_2],$$

it follows that there exists z_1, z_2 such that $z = \widehat{V}_1 z_1 + \widehat{V}_2 z_2$, so

$$0 = \Pi_l z = \Pi_l \widehat{V}_1 z_1 + \Pi_l \widehat{V}_2 z_2 = z_2,$$

where $\Pi_l \widehat{V}_1 = 0$, $\Pi_l \widehat{V}_2 = I$. Therefore, $z = \widehat{V}_1 z_1$ and

$$z \in \operatorname{im} \widehat{V}_1 = \ker \begin{bmatrix} \widehat{W}_2 \\ \widehat{W}_3 \\ \widehat{W}_4 \end{bmatrix} \subseteq \ker \begin{bmatrix} \widehat{W}_2 \\ \widehat{W}_4 \end{bmatrix} = \left(\operatorname{im}[\widehat{W}_2^\top \quad \widehat{W}_4^\top] \right)^\perp.$$

Altogether,

$$z \in \operatorname{im}[\widehat{W}_2^\top \quad \widehat{W}_4^\top] \cap \left(\operatorname{im}[\widehat{W}_2^\top \quad \widehat{W}_4^\top] \right)^\perp = \{0\}.$$

This concludes that

$$\ker \Pi_l \cap \mathcal{L} = \{0\}.$$

The proof is now complete. □

The following statement is intuitively clear, however, the formal proof is not yet available and therefore it is formulated as an open conjecture.

Conjecture 3.5. Let $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ be a system which is input-output equivalent to (3.4) and which is obtained via a projection method with left-projector $\widetilde{\Pi}_l$. If $\ker \widetilde{\Pi}_l \cap \mathcal{L} = \{0\}$ then, the state dimension $\widetilde{n}$ of $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ is at least the state dimension of (3.16).

Step 3: Reduced jump map

Assume the mode (A_0, B_0, C_0) before the jump $J_1 : \mathbb{R}^{n_0} \to \mathbb{R}^{n_1}$ was reduced by a projection method with left- and right-projectors Π_l^0 and Π_r^0, and the mode (A_1, B_1, C_1) after the jump was reduced by left- and right-projectors Π_l^1 and Π_r^1. Then, the reduced jump $\widehat{J}_1 : \mathbb{R}^{\widehat{n}_0} \to \mathbb{R}^{\widehat{n}_1}$ is defined as

$$\widehat{J}_1 := \Pi_l^1 J_1 \Pi_r^0. \tag{3.17}$$

3.3.3 Algorithm and results

Combining all results from above sections, overall the algorithm is summarized as follows.

Step 1a. Compute the reachable subspace $\mathcal{R}_0 = \operatorname{im} R_0$ of the first subsystem (A_0, B_0, C_0) and extend the input matrix of the second mode to

$$\overline{B}_1 := \operatorname{im}[B_1 \quad J_1 R_0].$$

Step 1b. Calculate the KD of $(A_1, \overline{B}_1, C_1)$ with corresponding transformation matrix T_1 and, left- and right-projectors Π_l^1, Π_r^1 (i.e. the corresponding rows and columns of T_1^{-1} and T_1) and let

$$(\widehat{A}_1, \widehat{B}_1, \widehat{C}_1) = (\Pi_l^1 A \Pi_r^1, \Pi_l^1 B_1, C_1 \Pi_r^1).$$

Step 2a. Calculate the subspace $\mathcal{L}_0 = \mathcal{R}_0 \cap \mathcal{F}_0 =: \operatorname{im} L_0$ of additional observable states, where $\mathcal{F}_0 = \operatorname{im} F_0$ for some full column rank matrix $F_0 \in \mathbb{R}^{n_0 \times n_0^J}$ such that $J_1 F_0 = V_0^J$ for a full column rank matrix $V_0^J \in \mathbb{R}^{n_0 \times n_0^J}$ with $\operatorname{im} V_0^J := \operatorname{im} \Pi_r^1 \cap \operatorname{im} J_1$. Then extend the output matrix of the first mode as

$$\overline{C}_0 := \operatorname{im} \begin{bmatrix} C_0 \\ L_0^\top \end{bmatrix}.$$

Step 2b. Calculate the KD of $(A_0, B_0, \overline{C}_0)$ with corresponding transformation matrix T_0 and, left- and right-projectors Π_l^0, Π_r^0 (i.e. the corresponding rows and columns of T_0^{-1} and T_0) and let

$$(\widehat{A}_0, \widehat{B}_0, \widehat{C}_0) = (\Pi_l^0 A_0 \Pi_r^0, \Pi_l^0 B_0, C_0 \Pi_r^0).$$

Step 3. Calculate the reduced jump $\widehat{J}_1$ according to (3.17). The overall reduced switched system is then given by

$$\widehat{\Sigma}_\sigma : \begin{cases} \dot{\widehat{x}}_0 = \widehat{A}_0 \widehat{x}_0 + \widehat{B}_0 u, & \text{on } (t_0, s_1), \quad \widehat{x}_0(s_0^+) = 0, \\ \dot{\widehat{x}}_1 = \widehat{A}_1 \widehat{x}_1 + \widehat{B}_1 u, & \text{on } (s_1, t_f), \quad \widehat{x}_1(s_1^+) = \widehat{J}_1 \widehat{x}_0(s_1^-). \end{cases}$$

The steps of the algorithm are illustrated by the following example.

Example 3.4 (Example 3.1 revisited)**.** Recall the switched linear system in Example 3.1 and apply the reduction method to Example 3.1.

Step 1a. $R_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ (reachable subspace of first mode (A_0, B_0, C_0)).

Step 1b. Via the KD of the extended second mode $(A_1, [B_1 \quad J_1 R_0], C_1]$, the left- and right-projectors are obtained by

$$\Pi_l^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Pi_r^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and the corresponding reduced second mode

$$(\widehat{A}_1, \widehat{B}_1, \widehat{C}_1) = (\Pi_l^1 A_1 \Pi_r^1, \Pi_l^1 B_1, C_1 \Pi_r^1) = \left(\begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix} \right).$$

Step 2a. Calculate $\operatorname{im} J_1 \cap \Pi_r^1 = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} := \operatorname{im} V_0^J = J_1 \mathcal{F}_0$ with $\mathcal{F}_0 = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ and then $\operatorname{im} R_0 \cap \mathcal{F}_0 = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} =: \operatorname{im} L_0$.

Step 2b. Via the KD of the extended first mode $(A_0, B_0, [C_0^\top \quad L_0]^\top)$, the left- and right-projectors are obtained by

$$\Pi_l^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Pi_r^0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and the corresponding reduced first mode is given by

$$(\widehat{A}_0, \widehat{B}_0, \widehat{C}_0) = (\Pi_l^0 A_0 \Pi_r^0, \Pi_l^0 B_0, C_0 \Pi_r^0) = \left(\begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix} \right).$$

Step 3. The reduced jump map is given by

$$\widehat{J}_1 = \Pi_l^1 J_1 \Pi_r^0 = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}.$$

Figure 3.1 shows the output of the original and its reduced switched linear system for input $u(t) = 1$ and switching signal (3.2) with switching time $s_1 = 2$ and clearly both outputs coincide.

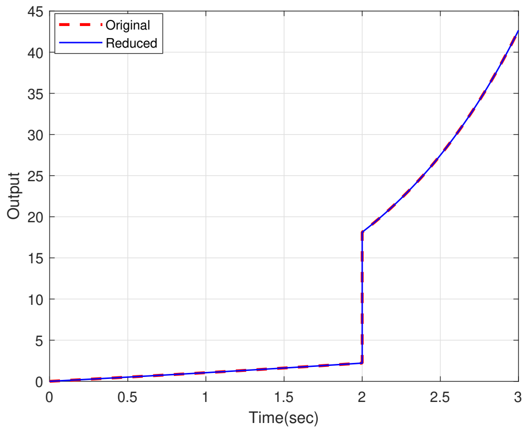

Figure 3.1: Outputs of the original system and its reduced system for Example 3.4.

Consider now Example 3.1 with the reversed switching signal

$$\sigma_1(t) = \begin{cases} 1, & \text{on } (t_0, s_1), \\ 0, & \text{on } (s_1, t_f), \end{cases} \tag{3.18}$$

and the same jump matrix J_1. The reduced realization is obtained as follows.

Step 1a. $R_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (reachable subspace of first mode (A_1, B_1, C_1).

Step 1b. Via the KD of the extended 2nd mode $(A_0, [B_0 \quad J_1 R_1], C_0]$, the left- and right-projectors are given by

$$\Pi_l^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Pi_r^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and the corresponding reduced second mode is given by

$$(\widehat{A}_0, \widehat{B}_0, \widehat{C}_0) = (\Pi_l^1 A_0 \Pi_r^1, \Pi_l^1 B_0, C_0 \Pi_r^1) = \left(\begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix} \right).$$

Step 2a. Calculate $\operatorname{im} J_1 \cap \Pi_r^1 = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} := \operatorname{im} V_1^J = J_1 \mathcal{F}_1$ with $\mathcal{F}_1 = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and then $\operatorname{im} R_1 \cap \mathcal{F}_1 = \operatorname{im} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} =: \operatorname{im} L_1$.

Step 2b. Via the KD of the extended first mode $(A_0, B_0, [C_0^\top \quad L_1]^\top)$, the left- and right-projectors are obtained by

$$\Pi_l^0 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad \Pi_r^0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and the corresponding reduced first mode is given by

$$(\widehat{A}_1, \widehat{B}_1, \widehat{C}_1) = (\Pi_l^0 A_1 \Pi_r^0, \Pi_l^0 B_1, C_1 \Pi_r^0) = (0.1, 1, 0).$$

Step 3. The reduced jump map is

$$\widehat{J}_1 = \Pi_l^1 J_1 \Pi_r^0 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Figure 3.2 shows the output of the original and its reduced system for input $u(t) = 1$ and switching signal (3.18) with switching time $s_1 = 2$ and clearly both outputs coincide.

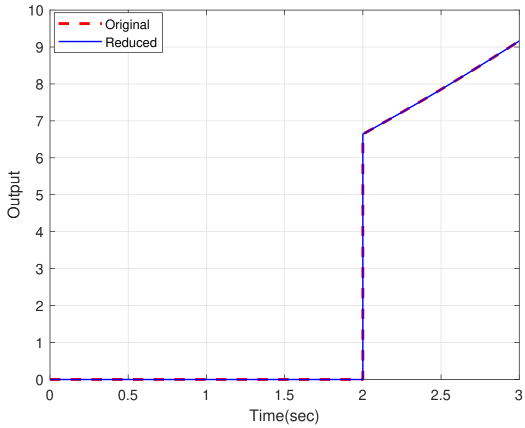

Figure 3.2: Outputs of the original system and its reduced system for Example 3.4 with reversed switching signal.

3.3.4 Correctness of algorithm

Theorem 3.6. *Consider the switched linear system Σ_σ with the single switch switching signal* (3.2) *and the reduced system $\widehat{\Sigma}_\sigma$ obtained via the above algorithm. Then, both systems are input-output equivalent in the sense of Definition 3.1.*

Proof. Consider the switched system Σ_σ as in (3.13) then the output equation is given by

$$y_\sigma(t) = \begin{cases} \displaystyle\int_{t_0}^{t} C_0 e^{A_0(t-\tau)} B_0 u(\tau)\,\mathrm{d}\tau, \ t \in [t_0, s_1), \\ \underbrace{C_1 e^{A_1(t-s_1)} J_1 x_0(s_1^-)}_{y_J(t)} + \displaystyle\int_{s_1}^{t} C_1 e^{A_1(t-\tau')} B_1 u(\tau')\,\mathrm{d}\tau', \ t \in [s_1, t_f). \end{cases}$$

From Lemma 3.4,

$$\int_{t_0}^{t} C_0 e^{A_0(t-\tau)} B_0 u(\tau)\,\mathrm{d}\tau = \int_{t_0}^{t} \widehat{C}_0 e^{\widehat{A}_0(t-\tau)} \widehat{B}_0 u(\tau)\,\mathrm{d}\tau.$$

Furthermore, from $x_0(s_1^-) = \Pi_r^0 \widehat{x}_0(s_1^-)$ and $C_1 T_1 e^{T_1^{-1} A_1 T_1} = [0 \ \ C_1 \Pi_r^1 e^{\Pi_l^1 A_1 \Pi_r^1} \ \ * \ \ *]$ and $T_1^{-1} J_1 \Pi_r^0 = \begin{bmatrix} * \\ \Pi_l^1 J_1 \Pi_r^0 \\ 0 \\ 0 \end{bmatrix}$, it follows that

$$\begin{aligned} y_J(t) &= C_1 e^{A_1(t-s_1)} J_1 \ x_0(s_1^-) \\ &= C_1 T_1 e^{T_1^{-1} A_1 T_1 (t-s_1)} T_1^{-1} J_1 \ \Pi_r^0 \ \widehat{x}_0(s_1^-) \\ &= C_1 \Pi_r^1 e^{\Pi_l^1 A_1 \Pi_r^1 (t-s_1)} \Pi_l^1 J_1 \ \Pi_r^0 \ \widehat{x}_0(s_1^-) \\ &= \widehat{C}_1 e^{\widehat{A}_1(t-s_1)} \widehat{J}_1 \ \widehat{x}_0(s_1^-) \\ &= \widehat{y}_J(t). \end{aligned}$$

Now from Lemma 3.3,

$$\int_{s_1}^{t} C_1 e^{A_1(t-\tau)} B_1 u(\tau)\,\mathrm{d}\tau = \int_{s_1}^{t} \widehat{C}_1 e^{\widehat{A}_1(t-\tau)} \widehat{B}_1 u(\tau)\,\mathrm{d}\tau.$$

The above results conclude that

$$y_\sigma(t) = \widehat{y}_\sigma(t) \quad \forall t,$$

where $\widehat{y}_\sigma(t)$ is the output of the reduced system $\widehat{\Sigma}_\sigma$. □

Theorem 3.7. *The reduced system $\widehat{\Sigma}_\sigma$ has minimal total dimension under all possible input-output equivalent system of Σ_σ, provided Conjecture 3.5 is true.*

Proof. Consider a system $\widetilde{\Sigma}_\sigma$ with modes $(\widetilde{A}_0, \widetilde{B}_0, \widetilde{C}_0)$, $(\widetilde{A}_1, \widetilde{B}_1, \widetilde{C}_1)$ and jump map $\widetilde{J}_1$ which is input-output equivalent to Σ_σ.

First of all, this implies that the second mode $(\widetilde{A}_1, \widetilde{B}_1, \widetilde{C}_1)$ with initial values $\widetilde{x}_0 \in \widetilde{J}_1 \widetilde{\mathcal{R}}_0$ ($\widetilde{\mathcal{R}}_0$ is the reachable subspace of 1st mode $(\widetilde{A}_0, \widetilde{B}_0, \widetilde{C}_0)$) is input-output equivalent with the original second mode (A_1, B_1, C_1) with initial value $x_0 \in J_1 \mathcal{R}_0$ (where $\mathcal{R}_0$ is the reachable subspace of the 1st mode (A_0, B_0, C_0)). Hence, Lemma 3.3 yields $\widetilde{n}_1 \geqslant \widehat{n}_1$.

Furthermore, input-output equivalence implies that for $\mathcal{L}_0$ as calculated in Step 2a, the left-projector $\widetilde{\Pi}_l^0$ used to obtain $\widetilde{A}_0$, has to satisfy $\ker \widetilde{\Pi}_l^0 \cap \mathcal{L}_0 = \{0\}$. Hence, Conjecture 3.5 implies $\widetilde{n}_0 \geqslant \widehat{n}_0$. Altogether,

$$\dim \widehat{\Sigma}_\sigma := (\widehat{n}_0 + \widehat{n}_0) \leqslant (\widetilde{n}_0 + \widetilde{n}_1) := \dim \widetilde{\Sigma}_\sigma.$$

Therefore, $\widehat{\Sigma}_\sigma$ has minimal total dimension under all possible input-output equivalent system of Σ_σ.

$\square$

4 Reduced realization for switched linear systems

4.1 Introduction

In Chapter 3, a reduced realization for switched linear systems is discussed for single switch case. In this chapter, a method is proposed for reduced realization of switched linear systems with general switching signals. Nonetheless, the results in this chapter differ from those in Chapter 3 in a way that the methods presented in this chapter do not directly rely on the system matrices but only the (unique) extended reachable / restricted unobservable subspaces. Without discussing realization theory, observability and reachability of switched systems have been studied in e.g., [71, 104, 127, 128, 134], the proposed approach is strongly inspired by these results.

Recall the switched linear system with general switching signals (as in (2.4)) of the form

$$\Sigma_\sigma : \begin{cases} \dot{x}_k(t) = A_{\sigma(t)}x_k(t) + B_{\sigma(t)}u(t), \ t \in (s_k, s_{k+1}) \\ x_k(s_k^+) = J_{\sigma(s_k^+),\sigma(s_k^-)}x_{k-1}(s_k^-), \quad k \in \mathbb{N} \\ y(t) = C_{\sigma(t)}x_k(t^+), \quad t \in [s_k, s_{k+1}), \end{cases} \tag{4.1}$$

and the reduced size (with the same switching signal σ) as in (3.1) of the form

$$\widehat{\Sigma}_\sigma : \begin{cases} \dot{\widehat{x}}_k(t) = \widehat{A}_{\sigma(t)}\widehat{x}_k(t) + \widehat{B}_{\sigma(t)}u(t), \ t \in (s_k, s_{k+1}) \\ \widehat{x}_k(s_k^+) = \widehat{J}_{\sigma(s_k^+),\sigma(s_k^-)}\widehat{x}_{k-1}(s_k^-), \quad k \in \mathbb{N} \\ y(t) = \widehat{C}_{\sigma(t)}\widehat{x}_k(t^+), \quad t \in [s_k, s_{k+1}), \end{cases} \tag{4.2}$$

which has the same input-output behavior as the original system Σ_σ.

As already highlighted in Chapter 3, a mode-wise reduction is not possible in general. Furthermore, Example 3.1 in Chapter 3 showed that a switched system which is reachable and observable, is not necessarily minimal. In order to obtain a reduced realization in the following, the notion of extended reachable and restricted unobservable subspaces will be introduced and utilized in the *weak*

Kalman decomposition.

This chapter is organized as follows. In Section 4.2, some preliminaries are given with the characterization of reachability and observability of SLSs and extended reachable / restricted unobservable subspaces are proposed. Section 4.3 discusses the main results and provides a reduction algorithm. Finally, some numerical results are shown in Section 4.4.

4.2 Preliminaries

As mentioned in Chapter 3, the well known Kalman decomposition is only valid for vanishing initial values; if arbitrary initial values are considered, only the unobservable part can be removed without altering the corresponding input-output behavior. In this section, an extended version of the Kalman decomposition, *weak Kalman decomposition* will be introduced which is crucial for further analysis of non-minimal switched systems with extended reachable and restricted unobservable subspaces. Later, the exact reachable/unobservable subspaces as well as the proposed extended reachable / restricted unobservable subspaces will be discussed. Some examples are given to show that overall reachability/observability of switched systems depend on the switching times.

4.2.1 Weak Kalman decomposition

In the context of switched systems all, apart from the first mode, will in general have non-trivial initial states but also not arbitrary initial states, which means that the classical KD cannot directly be used to obtain a reduced realization.

In addition to consider an extended reachable subspace for each mode (due to the partially nonzero initial state) also the local unobservable subspace may need to be restricted, due to the fact, that an unobservable state in the current mode may become observable in the future and hence cannot be removed without altering the overall input-output behavior of the switched system. This motivates to define a weak KD which takes into account an extended reachable subspace and restricted unobservable subspace.

Lemma 4.1 (Weak Kalman decomposition)**.** *Consider a classical LTI system* (A, B, C) *and let,* $\overline{\mathcal{R}} \supseteq \operatorname{im} B$ *and* $\underline{\mathcal{U}} \subseteq \ker C$ *be two A-invariant subspaces (an extended reachable subspace and a restricted unobservable subspace). For any coordinate transformation* $\overline{T} = \begin{bmatrix} \overline{V}^1 & \overline{V}^2 & \overline{V}^3 & \overline{V}^4 \end{bmatrix}$ *with* $\operatorname{im} \overline{V}^1 := \overline{\mathcal{R}} \cap \underline{\mathcal{U}}$*,* $\operatorname{im} \begin{bmatrix} \overline{V}^1 & \overline{V}^2 \end{bmatrix} := \overline{\mathcal{R}}$*,* $\operatorname{im} \begin{bmatrix} \overline{V}^1 & \overline{V}^3 \end{bmatrix} := \underline{\mathcal{U}}$*,* *leads to the following block triangular form*

$$(\overline{T}^{-1}A\overline{T}, \overline{T}^{-1}B, C\overline{T}) = \left(\begin{bmatrix} A^{11} & A^{12} & A^{13} & A^{14} \\ 0 & A^{22} & 0 & A^{24} \\ 0 & 0 & A^{33} & A^{34} \\ 0 & 0 & 0 & A^{44} \end{bmatrix}, \begin{bmatrix} B^1 \\ B^2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & C^2 & 0 & C^4 \end{bmatrix} \right). \tag{4.3}$$

In particular, $Ce^{At}B = C^2 e^{A^{22}t}B^2, \ \forall t \in \mathbb{R}$.

Proof. Since $\overline{\mathcal{R}} \cap \underline{\mathcal{U}} = \operatorname{im} \overline{V}^1$ is A-invariant there is a matrix A^{11} of appropriate size such that $A\overline{V}^1 = \overline{V}^1 A^{11}$. The A-invariance of $\overline{\mathcal{R}}$ implies that $A\overline{V}^2 \subseteq \operatorname{im} \begin{bmatrix} \overline{V}^1 & \overline{V}^2 \end{bmatrix}$, hence there exists A^{12}, A^{22} such that

$$A\overline{V}^2 = \overline{V}^1 A^{12} + \overline{V}^2 A^{22}.$$

Similarly, A-invariance of $\underline{\mathcal{U}}$ implies $A\overline{V}^3 \subseteq \operatorname{im} \begin{bmatrix} \overline{V}^1 & \overline{V}^3 \end{bmatrix}$, hence there exists A^{13}, A^{33} such that

$$A\overline{V}^3 = \overline{V}^1 A^{13} + \overline{V}^3 A^{33}.$$

Finally, $\operatorname{im} \begin{bmatrix} \overline{V}^1 & \overline{V}^2 & \overline{V}^3 & \overline{V}^4 \end{bmatrix} = \mathbb{R}^n$ implies existence of A^{14}, A^{24}, A^{34}, A^{44} such that

$$A\overline{V}^4 = \overline{V}^1 A^{14} + \overline{V}^2 A^{24} + \overline{V}^3 A^{34} + \overline{V}^4 A^{44}.$$

Combining all of the above,

$$A \begin{bmatrix} \overline{V}^1 & \overline{V}^2 & \overline{V}^3 & \overline{V}^4 \end{bmatrix} = \begin{bmatrix} \overline{V}^1 & \overline{V}^2 & \overline{V}^3 & \overline{V}^4 \end{bmatrix} \begin{bmatrix} A^{11} & A^{12} & A^{13} & A^{14} \\ 0 & A^{22} & 0 & A^{24} \\ 0 & 0 & A^{33} & A^{34} \\ 0 & 0 & 0 & A^{44} \end{bmatrix},$$

which shows that $\overline{T}^{-1}A\overline{T}$ has the desired block structure. Since $\operatorname{im} B \subseteq \overline{\mathcal{R}} = \operatorname{im} \begin{bmatrix} \overline{V}^1 & \overline{V}^2 \end{bmatrix}$, there exists B^1, B^2 such that

$$B = \overline{V}^1 B^1 + \overline{V}^2 B^2 = \begin{bmatrix} \overline{V}^1 & \overline{V}^2 & \overline{V}^3 & \overline{V}^4 \end{bmatrix} \begin{bmatrix} B^1 \\ B^2 \\ 0 \\ 0 \end{bmatrix},$$

from which the desired block structure of $\overline{T}^{-1}B$ follows. Finally, $\ker C \supseteq \underline{\mathcal{U}} = \operatorname{im} \begin{bmatrix} \overline{V}^1 & \overline{V}^3 \end{bmatrix}$ implies that $C[\overline{V}^1 \ \ \overline{V}^3] = \{0\}$, and hence, for $C^2 := C\overline{V}^2$ and $C^4 :=$

$C\overline{V}^4$, it follows that

$$C\overline{T} = C\begin{bmatrix}\overline{V}^1 & \overline{V}^2 & \overline{V}^3 & \overline{V}^4\end{bmatrix} = \begin{bmatrix}0 & C^2 & 0 & C^4\end{bmatrix}.$$

With these block structure, simple matrix multiplication leads to

$$Ce^{At}B = C^2 e^{A^{22}t}B^2, \forall\, t \in \mathbb{R}.$$

This completes the proof. □

It is well known that for a linear system (A, B, C), the reachable subspace $\mathcal{R}$ is given by $\mathcal{R} = \langle A \mid \operatorname{im} B\rangle$ and the unobservable subspace $\mathcal{U}$ is given by $\mathcal{U} = \langle \ker C \mid A\rangle$ as Definition 2.3.

Remark 4.2. Clearly, the choice $\overline{\mathcal{R}} = \mathcal{R}$ and $\underline{\mathcal{U}} = \mathcal{U}$ in Lemma 4.1 leads to the well known KD. Furthermore, any A-invariant subspace $\overline{\mathcal{R}} \supseteq \operatorname{im} B$ will be a superset of $\mathcal{R}$, because $\mathcal{R}$ is the smallest A-invariant subspace containing $\operatorname{im} B$; analogously, any A-invariant subspace $\underline{\mathcal{U}} \subseteq \ker C$ will be contained in $\mathcal{U}$. This motivation to call $\overline{\mathcal{R}} \supseteq \mathcal{R}$ an extended reachable subspace and $\underline{\mathcal{U}} \subseteq \mathcal{U}$ a restricted unobservable subspace in Lemma 4.1.

For a linear system (A, B, C) with given extended reachable subspace $\overline{\mathcal{R}}$ and restricted unobservable subspace $\underline{\mathcal{U}}$, the weak KD (4.3) immediately leads to the reduced system (A^{22}, B^2, C^2) which can be obtained from (A, B, C) by suitable left- and right-projections defined as follows.

Definition 4.3. For any coordinate transformation $\overline{T} = [\overline{V}^1\ \overline{V}^2\ \overline{V}^3\ \overline{V}^4]$ as in Lemma 4.1, let

$$\begin{bmatrix}\overline{W}^1\\ \overline{W}^2\\ \overline{W}^3\\ \overline{W}^4\end{bmatrix} := \overline{T}^{-1},$$

such that the sizes of $(\overline{W}^i)^\top$ matches the size of $\overline{V}^i$, $i = 1, 2, 3, 4$. Then, $\overline{W}^2$ and $\overline{V}^2$ are called the *weak KD left-projector* and *weak KD right-projector*, respectively.

By definition of the weak KD left- and right-projectors, $\overline{W}^2\overline{V}^2 = I$ and

$$(A^{22}, B^2, C^2) = (\overline{W}^2 A\overline{V}^2, \overline{W}^2 B, C\overline{V}^2).$$

4.2.2 Exact (time-varying) reachable subspace

The proposed reduction approach in this chapter relies on identifying suitable extended reachable and restricted unobservable subspaces for each mode of the switched system (4.1). Towards this goal, first provide expression for the exact (time-varying) reachable and unobservable subspaces for (4.1) in the following. Before doing so, it is briefly highlighted from Chapter 2 that the solution of (4.1) is given recursively by, for $t \in [s_k, s_{k+1})$ and $k = 1, \dots, \mathtt{m}$,

$$x(t) := e^{A_k(t-s_k)} J_k x(s_k^-) + \int_{s_k}^{t} e^{A_k(t-s)} B_k u(s)\, \mathrm{d}s, \tag{4.4}$$

and the output equation is given by

$$y(t) = C_k x(t),\ t \in [s_k, s_{k+1}),\ k = 0, 1, \dots, \mathtt{m}. \tag{4.5}$$

Now in the following, the formal definition of the reachable subspace of (4.1) is given.

Definition 4.4. The reachable subspace of the switched system (4.1) on time interval $[t_0, t)$ is defined by

$$\mathcal{R}^{\sigma}_{[t_0,t)} := \left\{\ x(t^-)\ \middle|\ \exists \text{ solution } (x,u) \text{ of (4.1) with } x(t_0^-) = 0\ \right\}.$$

The switched system (4.1) is called *reachable* (on (t_0, t_f)) if, and only if,

$$\mathcal{R}^{\sigma}_{[t_0,t_f)} = \mathbb{R}^{n_{\mathtt{m}}}.$$

To calculate the reachable subspaces of (4.1), the known reachability information from the previous modes needs to carry over appropriately to the current mode. Let $\mathcal{R}_k = \langle A_k \mid \operatorname{im} B_k \rangle$ be the local reachable subspace for mode k. It will be shown that the reachable subspace at the end of the k-th mode is defined by the following recursive equation, $k = 1, 2, \dots, \mathtt{m}$:

$$\begin{aligned} \mathcal{M}^{\sigma}_0 &:= \mathcal{R}_0, \\ \mathcal{M}^{\sigma}_k &:= \mathcal{R}_k + e^{A_k \tau_k} J_k \mathcal{M}^{\sigma}_{k-1}, \end{aligned} \tag{4.6}$$

where $\tau_k := s_{k+1} - s_k$ is the duration of mode k.

The intuition behind the sequence (4.6) is as follows. By starting with a zero initial value in the initial mode, clearly $\mathcal{R}^{\sigma}_{[t_0,s_1)} = \mathcal{R}_0$; continuing recursively, the reachable subspace at the end of mode k, is obtained by propagating forward the reachable subspace $\mathcal{M}^{\sigma}_{k-1}$ at the end of the previous mode, i.e. first jump via J_k and then propagate according to the matrix exponential (the time-evolution for a zero input). Finally, to take into account the effect of the input, the local reachable subspace of mode k is added. This intuition is formalized as follows.

Lemma 4.5 (Cf. [70]). *For all* $0 \leqslant k \leqslant \mathtt{m}$*,*

$$\mathcal{M}_k^\sigma = \mathcal{R}^\sigma_{[t_0, s_{k+1})}.$$

In particular, (4.1) *is reachable if, and only if* $\mathcal{M}_{\mathtt{m}}^\sigma = \mathbb{R}^{n_{\mathtt{m}}}$*.*

Proof. Clearly, $\mathcal{M}_0^\sigma = \mathcal{R}^\sigma_{[t_0, s_1)}$. Inductively, assume that for some $k \in \{1, 2, \ldots, \mathtt{m}\}$,

$$\mathcal{M}_{k-1}^\sigma = \mathcal{R}^\sigma_{[t_0, s_k)},$$

it will then be shown that $\mathcal{M}_k^\sigma = \mathcal{R}^\sigma_{[t_0, s_{k+1})}$.

Let $x_{k+1} \in \mathcal{M}_k^\sigma$, then there exists $x_k \in \mathcal{M}_{k-1}^\sigma$ and $x_u \in \mathcal{R}_k$ such that $x_{k+1} = e^{A_k \tau_k} J_k x_k + x_u$. From $\mathcal{M}_{k-1}^\sigma = \mathcal{R}^\sigma_{[t_0, s_k)}$, it follows that there exists a solution $(\widehat{x}, \widehat{u})$ on $[t_0, s_k)$ with $\widehat{x}(0^-) = 0$ and $\widehat{x}(s_k^-) = x_k$.

In view of (4.4), the extension of $(\widehat{x}, \widehat{u})$ on the interval $[t_0, s_{k+1})$ via $(\widehat{x}(t), \widehat{u}(t)) := (e^{A_k(t-s_k)} J_k x_k, 0)$ is a solution of (4.1) on the larger interval $[t_0, s_{k+1})$. Furthermore, there exists a solution $(\widetilde{x}, \widetilde{u})$ of mode k on (s_k, s_{k+1}) with $\widetilde{x}(s_k^+) = 0$ and $\widetilde{x}(s_{k+1}^-) = x_u$.

By setting, $(\widetilde{x}(t), \widetilde{u}(t)) = (0, 0)$ for all $t \in [t_0, s_k)$, it is easily seen that $(\widetilde{x}, \widetilde{u})$ is a solution of the switched system (4.1) on $[t_0, s_{k+1})$ with $\widetilde{x}(t_0^-) = 0$.

Altogether, by linearity it is true that $(x, u) := (\widehat{x}, \widehat{u}) + (\widetilde{x}, \widetilde{u})$ is a solution of (4.1) on $[t_0, s_{k+1})$ with $x(t_0^-) = 0$ and

$$x(s_{k+1}^-) = \widehat{x}(s_{k+1}^-) + \widetilde{x}(s_{k+1}^-) = e^{A_k \tau_k} J_k x_k + x_u = x_{k+1},$$

which implies that $x_{k+1} \in \mathcal{R}^\sigma_{[t_0, s_{k+1})}$. Hence,

$$\mathcal{M}_k^\sigma \subseteq \mathcal{R}^\sigma_{[t_0, s_{k+1})}.$$

To show the converse subspace relationship, let $x_{k+1} \in \mathcal{R}^\sigma_{[t_0, s_{k+1})}$, then there exists a solution (x, u) of (4.1) with $x(s_{k+1}) = x_{k+1}$.

From $x(s_k^-) \in \mathcal{R}^\sigma_{[t_0, s_k)} = \mathcal{M}_{k-1}^\sigma$ and

$$x_u := \int_{s_k}^{s_{k+1}} e^{A_k(s_{k+1}-s)} B_k u(s) \, \mathrm{d}s \in \mathcal{R}_k,$$

it follows immediately from (4.4) that $x_{k+1} = x(s_{k+1}) = e^{A_k \tau_k} J_k x(s_k^-) + x_u \in e^{A_k \tau_k} J_k \mathcal{M}_{k-1}^\sigma + \mathcal{R}_k = \mathcal{M}_k^\sigma$.

Now if the system (4.1) is reachable, then

$$\mathcal{R}^\sigma_{[t_0, s_{\mathtt{m}+1})} = \mathbb{R}^{n_{\mathtt{m}}},$$

and consequently,

$$\mathcal{M}_{\mathrm{m}}^{\sigma} = \mathbb{R}^{n_{\mathrm{m}}}.$$

This concludes the proof. □

From (4.6), it is clear that the reachable subspaces depend on the switching times (in fact, on the mode durations τ_k) and this dependency cannot be avoided in general as the following example shows. In particular, the overall reachability of the switched system (4.1) on (t_0, t_f) depends on the switching times and how long each mode is active.

Example 4.1 (Dependency on the switching times)**.** Consider the switched system (4.1) given by

$$A_0 = A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$
$$B_0 = B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

with $J_1 = J_2 = I$. It is noted that none of the pairs (A_i, B_i) are reachable. Consider the switching signal σ with the mode sequence $0 \to 1 \to 2$ and switching times s_1, s_2. Let $\{e_1, e_2\}$ denote the standard basis vectors for $\mathbb{R}^2$.

Clearly, $\mathcal{R}_0 = \mathcal{R}_2 := \operatorname{span}\{e_1\}$, $\quad \mathcal{R}_1 := \{0\}$, $\quad e^{A_1\tau} = \begin{bmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{bmatrix}$ and $e^{A_2\tau} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Hence

$$\begin{aligned}
\mathcal{M}_0^{\sigma} &= \mathcal{R}_0 = \operatorname{span}\{e_1\}, \\
\mathcal{M}_1^{\sigma} &= \mathcal{R}_1 + e^{A_1\tau_1} J_1 \mathcal{M}_0^{\sigma} = \operatorname{span}\left\{\begin{bmatrix} \cos\tau_1 \\ \sin\tau_1 \end{bmatrix}\right\}, \\
\mathcal{M}_2^{\sigma} &= \mathcal{R}_2 + e^{A_2\tau_2} J_2 \mathcal{M}_1^{\sigma} = \operatorname{span}\{e_1\} + \operatorname{span}\left\{\begin{bmatrix} \cos\tau_1 \\ \sin\tau_1 \end{bmatrix}\right\}.
\end{aligned}$$

If $\tau_1 = k\pi$ for any $k \in \mathbb{N}$ then $\mathcal{M}_2^{\sigma} = \operatorname{span}\{e_1\}$, otherwise, $\mathcal{M}_2^{\sigma} = \mathbb{R}^2$. This clearly shows that the overall reachability of a switched system depends on the switching times.

Note that, although $\mathcal{M}_k^{\sigma} \supseteq \mathcal{R}_k \supseteq \operatorname{im} B_k$, the subspace $\mathcal{M}_k^{\sigma}$ is not a suitable extended reachable subspace for the mode (A_k, B_k, C_k) in the sense of Lemma 4.1, because it is *not* A_k-invariant in general. Before addressing this problem in Section 4.2.4, recall first the "dual" subspace of the reachable subspaces: the unobservable subspaces.

4.2.3 Exact (time-varying) unobservable subspace

Definition 4.6. The unobservable subspace of the switched system (4.1) on time interval $[t, t_f)$ is defined by

$$\mathcal{U}^\sigma_{[t,t_f)} := \left\{ x(t^+) \;\middle|\; \exists \text{ solution } (x, u=0) \text{ such that } y=0 \text{ of (4.1) on } [t,t_f) \right\}.$$

The switched system (4.1) is called *observable* (on $[t_0, t_f)$) if, and only if,

$$\mathcal{U}^\sigma_{[t_0,t_f)} = \{0\}.$$

Similar as for the reachable subspaces, it is aimed to express the unobservable subspaces recursively. Starting from the last mode it is clear that the unobservable subspace is the same as the classical unobservable subspace $\mathcal{U}_{\mathtt{m}} = \langle \ker C_{\mathtt{m}} \mid A_{\mathtt{m}} \rangle$. Recursively, the unobservable subspace at switch number $k+1$ can now be propagated backwards in time by first taking the preimage under the jump J_{k+1} and then further propagating it back with the continuous flow of mode k, i.e. by $e^{-A_k \tau_k}$. Finally, this propagated subspace needs to be combined with the local unobservable subspace of mode k given by $\mathcal{U}_k = \langle \ker C_k \mid A_k \rangle$. This motivates the definition of the following sequence of subspaces, $k = \mathtt{m}-1, \mathtt{m}-2, \ldots, 0$:

$$\begin{aligned} \mathcal{N}^\sigma_{\mathtt{m}} &:= \mathcal{U}_{\mathtt{m}}, \\ \mathcal{N}^\sigma_k &:= \mathcal{U}_k \cap \left(e^{-A_k \tau_k} J^{-1}_{k+1} \mathcal{N}^\sigma_{k+1} \right). \end{aligned} \tag{4.7}$$

Lemma 4.7 (Cf. [70, 133])**.** *For all* $0 \leqslant k \leqslant \mathtt{m}$,

$$\mathcal{N}^\sigma_k = \mathcal{U}^\sigma_{[s_k,t_f)}.$$

In particular, (4.1) *is observable if, and only if* $\mathcal{N}^\sigma_0 = \{0\}$.

Proof. For $k = \mathtt{m}$, clearly $\mathcal{N}^\sigma_{\mathtt{m}} = \mathcal{U}^\sigma_{[s_{\mathtt{m}},t_f)}$. Inductively, assume now that for $k \in \{\mathtt{m}-1, \mathtt{m}-2, \ldots, 0\}$,

$$\mathcal{N}^\sigma_{k+1} = \mathcal{U}^\sigma_{[s_{k+1},t_f)},$$

and it will be shown that $\mathcal{N}^\sigma_k = \mathcal{U}^\sigma_{[s_k,t_f)}$.

Let $x_k \in \mathcal{N}^\sigma_k$, then $x_k \in \mathcal{U}_k$ and there exists $x_{k+1} \in \mathcal{N}^\sigma_{k+1} = \mathcal{U}^\sigma_{[s_{k+1},t_f)}$ such that $x_{k+1} = J_{k+1} e^{A_k \tau_k} x_k$. Consequently, the unique solution $(x, u=0)$ of (4.1) on $[s_k, t_f)$ with $x(s_k^+)$ satisfies $y = 0$ on $[s_k, s_{k+1})$ because $x_k \in \mathcal{U}_k$ and $y=0$ on $[s_{k+1}, t_f)$ because $x(s_{k+1}) = x_{k+1} \in \mathcal{U}^\sigma_{[s_{k+1},t_f)}$. This shows that $x_k \in \mathcal{U}^\sigma_{[s_k,t_f)}$.

Now, let $x_k \in \mathcal{U}^\sigma_{[s_k,t_f)}$, then the unique solution $(x, u=0)$ of (4.1) on $[s_k, t_f)$ with $x(s_k^+) = x_k$ has zero output. Consequently, $x_{k+1} := x(s_{k+1}^+) \in \mathcal{U}^\sigma_{[s_{k+1},t_f)} = \mathcal{N}^\sigma_{k+1}$. From $x_{k+1} = J_{k+1} e^{A_k \tau_k} x_k$, it follows that $x_k \in e^{-A_k \tau_k} J^{-1}_{k+1} \{x_{k+1}\} \subseteq e^{-A_k \tau_k} J^{-1}_{k+1} \mathcal{N}^\sigma_{k+1} = \mathcal{N}^\sigma_k$, which concludes the proof. $\square$

Similar as for reachability, observability of the switched system in general depends on the switching times. This is illustrated by considering again Example 4.1 with an additional output.

Example 4.2 (Dependency on the switching times)**.** Recall Example 4.1 with output submatrices

$$C_0 = C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

It is noted that none of the pairs (A_i, C_i) is observable.
Clearly, $\mathcal{U}_0 = \mathcal{U}_2 = \text{span}\{e_1\}$, $\quad \mathcal{U}_1 = \mathbb{R}^2$, $\quad e^{-A_1\tau} = \begin{bmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{bmatrix}$ and $e^{-A_2\tau} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Hence,

$$\begin{aligned} \mathcal{N}_2^\sigma &= \mathcal{U}_2 = \text{span}\{e_1\}, \\ \mathcal{N}_1^\sigma &= \mathcal{U}_1 \cap e^{-A_1\tau_1} J_2^{-1} \mathcal{N}_2^\sigma = \mathbb{R}^2 \cap \text{span}\left\{ \begin{bmatrix} \cos\tau_1 \\ -\sin\tau_1 \end{bmatrix} \right\}, \\ \mathcal{N}_0^\sigma &= \mathcal{U}_0 \cap e^{-A_0\tau_0} J_1^{-1} \mathcal{N}_1^\sigma = \text{span}\{e_1\} \cap \text{span}\left\{ \begin{bmatrix} \cos\tau_1 \\ -\sin\tau_1 \end{bmatrix} \right\}. \end{aligned}$$

If $\tau_1 = k\pi$ for any $k \in \mathbb{N}$, then $\mathcal{N}_0^\sigma := \text{span}\{e_1\}$, otherwise $\mathcal{N}_0^\sigma = \{0\}$. Therefore, the overall observability of (4.1) depends on the switching times.

Note that, similar to the reachable subspaces, although the unobservable subspaces $\mathcal{N}_k$ satisfy $\mathcal{N}_k \subseteq \mathcal{U}_k \subseteq \ker C$, they are *not* A_k-invariant and hence, they are not restricted unobservable subspaces in the sense of Lemma 4.1.

4.2.4 Extended reachable / restricted unobservable subspaces

So far, it is seen that the reachable and unobservable subspaces of (4.1) depend on the switching time. Even worse, when looking at the reachable / unobservable subspaces at a particular time $t \in (s_k, s_{k+1})$ between two switches, then it is easily seen that these subspaces in general also depend on the considered time t and a reduction method based on the exact reachable / unobservable subspaces will necessarily result in general time-varying coordinate transformations / projections (cf. Chapter 5) and would not lead to a reduced system of the desired form (4.2).

To circumvent this problem, suitable extended reachable and restricted unobservable subspaces are introduced for the switched system (4.1). The key idea is based on the fact that for any subspace $\mathcal{H} \subseteq \mathbb{R}^n$ and any matrix $A \in \mathbb{R}^{n\times n}$, the following subspace relationship holds for all t:

$$\langle \mathcal{H} \mid A\rangle \subseteq e^{At}\mathcal{H} \subseteq \langle A \mid \mathcal{H}\rangle. \tag{4.8}$$

By replacing the matrix exponential in the constructions of the reachable / unobservable subspaces by the corresponding A-invariant subspace, the following sequences (cf. [133] for the unobservable subspaces) can be derived:

$$\begin{aligned} \overline{\mathcal{R}}_0 &:= \mathcal{R}_0, \\ \overline{\mathcal{R}}_k &:= \mathcal{R}_k + \langle A_k \mid J_k\overline{\mathcal{R}}_{k-1}\rangle, \ k = 1, \dots, \mathtt{m}; \end{aligned} \tag{4.9}$$

$$\begin{aligned} \underline{\mathcal{U}}_{\mathtt{m}} &:= \mathcal{U}_{\mathtt{m}}, \\ \underline{\mathcal{U}}_k &:= \mathcal{U}_k \cap \langle J_{k+1}^{-1}\underline{\mathcal{U}}_{k+1} \mid A_k\rangle, \ k = \mathtt{m}-1, \dots, 0. \end{aligned} \tag{4.10}$$

In view of (4.8), it is easy to see that

$$\overline{\mathcal{R}}_k \supseteq \mathcal{M}_k^\sigma \supseteq \mathcal{R}_k \quad \text{and} \quad \underline{\mathcal{U}}_k \subseteq \mathcal{N}_k^\sigma \subseteq \mathcal{U}_k.$$

In particular, $\overline{\mathcal{R}}_{\mathtt{m}} = \mathbb{R}^{n_{\mathtt{m}}}$ and $\underline{\mathcal{U}}_0 = \{0\}$ respectively, are necessary conditions for reachability and unobservability of the overall switched system (4.1).

Finally, observe that by construction both $\overline{\mathcal{R}}_k$ and $\underline{\mathcal{U}}_k$ are A_k-invariant, i.e. they are extended reachable / restricted unobservable subspaces in the sense of Lemma 4.1 and now one can propose the main result about the reduction of switched systems of the form (4.1).

This section is concluded by highlighting an interesting special case, which is motivated by the following "application": Consider a large-scale network whose dynamics can be described by a linear ODE. The network can be controlled through several actuators at different locations and several sensors are distributed throughout the network. However, due to resource limitation at any given time only one or a limited number of actuators can be used and the data of only one or a limited number of sensors is available. This situation can be modelled by the following switched system (without jumps)

$$\begin{aligned} \dot{x} &= Ax + B_\sigma u, \\ y &= C_\sigma x, \end{aligned} \tag{4.11}$$

where the switching signal is determined by the schedule of the actuator and sensor usages.

In this scenario, it seems rather natural that the mode sequence is fixed a priori (e.g. to make sure that all sensors and actuators are equally used), while the time duration may depend on the actual measured outputs. For this setup the following result is given:

Proposition 4.8 (Constant A-case)**.** *Consider the switched linear system* (4.11) *with corresponding time-dependent reachable subspace $\mathcal{R}^\sigma_{[t_0,t)}$ and unobservable subspace $\mathcal{U}^\sigma_{[t,t_f)}$. Then, for all $t \in (s_k, s_{k+1})$,*

$$\mathcal{R}^\sigma_{[t_0,t)} = \overline{\mathcal{R}}_k \quad \textit{and} \quad \mathcal{U}^\sigma_{[t,t_f)} = \underline{\mathcal{U}}_k,$$

i.e. the time-varying reachable and unobservable subspaces are piecewise-constant and can be calculated recursively via (4.9) *and* (4.10).

Proof. Inductively, it is easily seen that $\mathcal{R}^\sigma_{[t_0,t)}$ and $\mathcal{U}^\sigma_{[t,t_f)}$ are A-invariant, from which the claim follows. □

4.3 Proposed reduction method

In the following, a method is proposed to compute a reduced realization (4.2) of (4.1) for a given switching signal with known mode sequence.

Step 1. Compute the sequence of extended reachable $\overline{\mathcal{R}}_0, \overline{\mathcal{R}}_1, \dots, \overline{\mathcal{R}}_\mathtt{m}$ and restricted unobservable subspaces $\underline{\mathcal{U}}_0, \underline{\mathcal{U}}_1, \dots, \underline{\mathcal{U}}_\mathtt{m}$ as in (4.9) and (4.10).
Step 2. Apply Lemma 4.1 to (A_k, B_k, C_k) with $(\overline{\mathcal{R}}_k, \underline{\mathcal{U}}_k)$ to compute the left- and right-projectors $\overline{W}^2_k, \overline{V}^2_k$, and let

$$\left(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k\right) = \left(\overline{W}^2_k A_k \overline{V}^2_k, \overline{W}^2_k B_k, C_k \overline{V}^2_k\right).$$

Step 3. Calculate the reduced jump map

$$\widehat{J}_k := \overline{W}^2_k J_k \overline{V}^2_{k-1}.$$

Before showing that the resulting reduced system (4.2) is indeed a realization of (4.1), first highlight an important connection between the solutions of both systems.

Lemma 4.9. *Consider the switched system Σ_σ as in* (4.1) *and the reduced system $\widehat{\Sigma}_\sigma$ as in* (4.2) *obtained by the left- and right-projectors $\overline{W}^2_{\sigma(\cdot)}, \overline{V}^2_{\sigma(\cdot)}$. If $x(\cdot)$ is a solution of Σ_σ then $\widehat{x}(\cdot) := \overline{W}^2_{\sigma(\cdot)} x(\cdot)$ is a solution of $\widehat{\Sigma}_\sigma$.*

Proof. Consider any time interval (s_k, s_{k+1}) between two switches, then for $t \in (s_k, s_{k+1})$,

$$\begin{aligned}\dot{\widehat{x}}(t) &= \overline{W}^2_k \dot{x} = \overline{W}^2_k A_k x(t) + \overline{W}^2_k B u(t)\\ &= [0 \;\; \widehat{A}_k \;\; 0 \;\; *] \overline{T}_k^{-1} x(t) + B^2_k u(t),\end{aligned}$$

where $\overline{T}_k = \begin{bmatrix}\overline{V}_k^1 & \overline{V}_k^2 & \overline{V}_k^3 & \overline{V}_k^4\end{bmatrix}$ is the coordinate transformation according to Lemma 4.1 for mode k. Since $x(t) \in \mathcal{R}^{\sigma}_{[t_0,t)} \subseteq \overline{\mathcal{R}}_k = \operatorname{im}[\overline{V}_k^1 \quad \overline{V}_k^2]$, it follows that $\overline{T}_k^{-1} x(t) = [* \ \ \widehat{x}(t)^\top \ \ 0 \ \ 0]^\top$ and hence, as claimed, for all $t \in (s_k, s_{k+1})$

$$\dot{\widehat{x}}(t) = \widehat{A}_k \widehat{x}(t) + \widehat{B}_k u(t).$$

In particular, due to unique solvability of linear ODEs, for any solutions x of Σ_σ and $\widehat{x}$ of $\widehat{\Sigma}_\sigma$ the following implication holds:

$$\overline{W}_k^2 x(s_k^+) = \widehat{x}(s_k^+) \implies \forall t \in (s_k, s_{k+1}) : \overline{W}_k^2 x(t) = \widehat{x}(t).$$

To show that $\widehat{x} = \overline{W}_\sigma^2 x$ is indeed a global solution of $\widehat{\Sigma}_\sigma$, it therefore remains to be shown that

$$\overline{W}_k^2 x(s_k^+) = \widehat{J}_k \overline{W}_{k-1}^2 x(s_k^-). \tag{4.12}$$

In fact,

$$\begin{aligned}\overline{W}_k^2 x(s_k^+) &= \overline{W}_k^2 J_k x(s_k^-) = \overline{W}_k^2 J_k \overline{T}_{k-1} \overline{T}_{k-1}^{-1} x(s_k^-) \\ &= \overline{W}_k^2 J_k \begin{bmatrix}\overline{V}_{k-1}^1 & \overline{V}_{k-1}^2 & \overline{V}_{k-1}^3 & \overline{V}_{k-1}^4\end{bmatrix} \begin{pmatrix} * \\ \overline{W}_{k-1}^2 x(s_k^-) \\ 0 \\ 0 \end{pmatrix}.\end{aligned}$$

From (4.10), it is easily seen that $J_k \, \underline{\mathcal{U}}_{k-1} \subseteq \underline{\mathcal{U}}_k$, hence

$$\begin{aligned}\operatorname{im} J_k \, \overline{V}_{k-1}^1 \subseteq \operatorname{im} J_k \ \begin{bmatrix}\overline{V}_{k-1}^1 & \overline{V}_{k-1}^3\end{bmatrix} &= J_k \, \underline{\mathcal{U}}_{k-1} \\ &\subseteq \underline{\mathcal{U}}_k = \operatorname{im} \begin{bmatrix}\overline{V}_k^1 & \overline{V}_k^3\end{bmatrix} \subseteq \ker \overline{W}_k^2,\end{aligned}$$

i.e. $\overline{W}_k^2 \, J_k \overline{V}_{k-1}^1 = 0$, from which it follows that

$$\overline{W}_k^2 x(s_k^+) = \overline{W}_k^2 J_k \overline{V}_{k-1}^2 \overline{W}_{k-1}^2 x(s_k^-)$$

as desired. □

As a consequence of the above and of the uniqueness of solutions, it follows that every solution $\widehat{x}$ of $\widehat{\Sigma}_\sigma$ with zero initial value and given input u satisfies $\widehat{x} = \overline{W}_\sigma^2 x$ where x is the solution of Σ_σ with zero initial value and the same input u.

Now it can be proved in the following that the corresponding outputs are indeed equal.

Theorem 4.10. *Consider the switched system Σ_σ as in* (4.1) *and the reduced system $\widehat{\Sigma}_\sigma$ as in* (4.2) *obtained by the above reduction method. Then, Σ_σ and $\widehat{\Sigma}_\sigma$ are input-output equivalent in the sense that for all inputs u the output y of* (4.1) *with initial condition $x(t_0^-) = 0$ equals the output $\widehat{y}$ of* (4.2) *with initial condition $\widehat{x}(t_0^-) = 0$.*

Proof. The output of Σ_σ on $[s_k, s_{k+1})$ is given by

$$\begin{aligned} y(t) &= C_k e^{A_k(t-s_k)} J_k x(s_k^-) + \int_{s_k}^{t} C_k e^{A_k(t-s)} B_k u(s) \, \mathrm{d}s \\ &=: y_{\mathrm{J}}(t) + y_{\mathrm{u}}(t). \end{aligned}$$

Inserting suitable identity matrices,

$$\begin{aligned} y_{\mathrm{J}} &= C_k \overline{T}_k e^{\overline{T}_k^{-1} A_k \overline{T}_k (t-s_k)} \overline{T}_k^{-1} J_k \overline{T}_{k-1} \overline{T}_{k-1}^{-1} x(s_k^-), \\ y_{\mathrm{u}}(t) &= \int_{s_k}^{t} C_k \overline{T}_k e^{\overline{T}_k^{-1} A_k \overline{T}_k (t-s)} \overline{T}_k^{-1} B_k u(s) \, \mathrm{d}s, \end{aligned}$$

where $\overline{T}_k = \begin{bmatrix} \overline{V}_k^1 & \overline{V}_k^2 & \overline{V}_k^3 & \overline{V}_k^4 \end{bmatrix}$ is the coordinate transformation according to Lemma 4.1 for mode k. The special block structure of the matrices $\overline{T}_k^{-1} A_k \overline{T}_k$, $\overline{T}_k^{-1} B_k$, $C_k \overline{T}_k$ implied by Lemma 4.1 immediately leads to

$$y_{\mathrm{u}}(t) = \int_{s_k}^{t} \widehat{C}_k e^{\widehat{A}_k(t-s)} \widehat{B}_k u(s) \, \mathrm{d}s.$$

Hence, for showing $\widehat{y}(t) = y(t) = y_{\mathrm{J}}(t) + y_{\mathrm{u}}(t)$, it remains to be shown that

$$y_{\mathrm{J}}(t) = \widehat{C}_k e^{\widehat{A}_k(t-s_k)} \widehat{J}_k \widehat{x}(s_k^-). \tag{4.13}$$

With similar arguments as used to establish (4.12) in Lemma 4.9, it can show that

$$\overline{T}_k^{-1} J_k \overline{T}_{k-1} \overline{T}_{k-1}^{-1} x(s_k^-) = \begin{pmatrix} * \\ \widehat{J}_{k,k-1} \overline{W}_k^2 x(s_k^-) \\ 0 \\ 0 \end{pmatrix}.$$

Using the already established fact in Lemma 4.9, that $\overline{W}_k^2 x(s_k^-) = \widehat{x}(s_k^-)$ together with the special block structures of $\overline{T}_k^{-1} A_k \overline{T}_k$, $\overline{T}_k^{-1} B_k$, $C_k \overline{T}_k$, it can be concluded that (4.13) holds. □

Remark 4.11 (Non-zero initial values). The proposed method can easily be adjusted to account for non-zero initial values. Assume $x(t_0^-) \in \mathcal{X}_0$ for some subspace

$\mathcal{X}_0 \subseteq \mathbb{R}^n$, then in (4.6) the initial definition just needs to be replaced by

$$\mathcal{M}_0^\sigma := \mathcal{R}_0 + e^{A_0\tau_0} J_0 \mathcal{X}_0,$$

and in (4.9), the initial subspace needs to be adjusted to

$$\overline{\mathcal{R}}_0 := \mathcal{R}_0 + \langle A_0 \mid J_0 \mathcal{X}_0 \rangle,$$

while the definition of the other subspaces remain unchanged.

A key feature of the proposed method is that it is independent of the actual switching times (or mode durations) and only requires knowledge of the mode sequence. The following example shows however that the size of a minimal realization depends on the mode durations, hence one cannot expect that the proposed method results in a minimal realization in general.

Example 4.3. Consider a switched system with modes

$$A_0 = A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C_0 = C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix},$$

with $J_1 = J_2 = I$. Assume the mode sequence $0 \to 1 \to 2$. Fix the switching time duration $\tau_1 = \pi/2$ for second mode. Then, the original solution x and the corresponding output y of each time interval can be characterized as follows:

$$t \in (t_0, s_1): \; x(t) = \begin{bmatrix} * \\ * \\ 0 \end{bmatrix}, \qquad y(t) = C_0 x(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} * \\ * \\ 0 \end{bmatrix},$$

$$t \in (s_1, s_1 + \tfrac{\pi}{2}): \; x(t) = \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \qquad y(t) = C_1 x(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} * \\ * \\ * \end{bmatrix},$$

$$x(s_2) = x(s_1 + \tfrac{\pi}{2}) = \begin{bmatrix} * \\ 0 \\ * \end{bmatrix},$$

$$t \in (s_2, t_f): \; x(t) = \begin{bmatrix} * \\ 0 \\ * \end{bmatrix}, \qquad y(t) = C_2 x(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} * \\ 0 \\ * \end{bmatrix}.$$

Clearly, the second and third state do not effect the output for this specific switching signal. In particular, it is easily seen that the overall input-output behavior is described by the (nonswitched) system $\dot{\widehat{x}} = u, \quad y = \widehat{x}$. However, if the proposed method is applied, the sequence of reachable and unobservable subspaces are given by

$$\begin{aligned} \mathcal{M}_1^\sigma &= \operatorname{im} B_0, & \mathcal{N}_0^\sigma &= \{0\}, \\ \mathcal{M}_2^\sigma &= \mathbb{R}^3, & \mathcal{N}_1^\sigma &= \{0\}, \\ \mathcal{M}_3^\sigma &= \mathbb{R}^3, & \mathcal{N}_2^\sigma &= \operatorname{span}\{e_3\}. \end{aligned}$$

Indeed, the sequences produce a reduced switched system with modes in dimensions 1, 3 and 2, respectively, instead of a one dimensional minimal system. Nevertheless, one should note that for $\tau_1 \neq k\pi/2$, the proposed method actually produces a minimal realization.

The previous example however indicates that the proposed method results in a minimal realization for *almost all* switching times. While it has not been possible to prove this conjecture, it is possible to show that the proposed method is *optimal* in the sense that a repeated application does not lead to a further reduction.

Theorem 4.12. *Consider the switched system Σ_σ and the reduced switched system $\widehat{\Sigma}_\sigma$ resulting from the proposed method. Let $\overline{\widehat{\mathcal{R}}}_{\sigma(\cdot)}$ and $\underline{\widehat{\mathcal{U}}}_{\sigma(\cdot)}$ be the sequences of extended reachable and restricted unobservable subspaces, respectively, of $\widehat{\Sigma}_\sigma$. Then*

$$\overline{\widehat{\mathcal{R}}}_{\sigma(\cdot)} = \mathbb{R}^{\widehat{n}_{\sigma(\cdot)}}, \quad \underline{\widehat{\mathcal{U}}}_{\sigma(\cdot)} = \{0\}.$$

In particular, the left- and right-projectors for a potential further reduction are given by identity matrices, i.e. no further reduction occurs.

Proof. The proposed methods yields for each mode k a coordinate transformation $\overline{T}_k$ such that (A_k, B_k, C_k) is transformed to

$$\left(\begin{bmatrix} A_k^{11} & A_k^{12} & A_k^{13} & A_k^{14} \\ 0 & \widehat{A}_k & 0 & A_k^{24} \\ 0 & 0 & A_k^{33} & A_k^{34} \\ 0 & 0 & 0 & A_k^{44} \end{bmatrix}, \begin{bmatrix} B_k^1 \\ \widehat{B}_k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & \widehat{C}_k & 0 & C_k^4 \end{bmatrix} \right), \tag{4.14}$$

where $(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k)$ is the input-output equivalent reduced system for mode k. By construction, the reachable and unobservable subspaces respectively, are given by

$$\overline{\mathcal{R}}_k = \overline{T}_k \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \underline{\mathcal{U}}_k = \overline{T}_k \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}.$$

Seeking a contradiction assume $\overline{\widehat{\mathcal{R}}}_k \subsetneq \mathbb{R}^{\widehat{n}_k}$ (Case I), or $\underline{\widehat{\mathcal{U}}}_k \neq \{0\}$ (Case II) for some k.

Case I: For $k = 0$, it can see that from $\overline{\mathcal{R}}_0 = \mathcal{R}_0$, the pair $(\widehat{A}_0, \widehat{B}_0)$ must be reachable and hence, $\overline{\widehat{\mathcal{R}}}_0 = \widehat{\mathcal{R}}_0 = \mathbb{R}^{\widehat{n}_0}$. Assume now inductively that for some k,

$$\overline{\widehat{\mathcal{R}}}_{k-1} = \mathbb{R}^{\widehat{n}_{k-1}} \text{ and } \overline{\widehat{\mathcal{R}}}_k \subsetneq \mathbb{R}^{\widehat{n}_k}.$$

Since $\overline{\widehat{\mathcal{R}}}_k$ is $\widehat{A}_k$-invariant and contains $\operatorname{im} \widehat{B}_k$, choose a coordinate transformation $\overline{\widehat{T}}_k$ such that $(\widehat{A}_k, \widehat{B}_k)$ is transformed to

$$\left(\begin{bmatrix} \widehat{A}_k^1 & * \\ 0 & \widehat{A}_k^2 \end{bmatrix}, \begin{bmatrix} \widehat{B}_k^1 \\ 0 \end{bmatrix} \right), \tag{4.15}$$

and $\operatorname{im} \overline{\widehat{T}}_k \begin{bmatrix} I \\ 0 \end{bmatrix} = \overline{\widehat{\mathcal{R}}}_k$. By adjusting the original coordinate transformation $\overline{T}_k$, assume in the following that $(\widehat{A}_k, \widehat{B}_k)$ is actually equal to (4.15). In particular,

$$\operatorname{im} \begin{bmatrix} I \\ 0 \end{bmatrix} = \overline{\widehat{\mathcal{R}}}_k = \widehat{\mathcal{R}}_k + \langle \widehat{A}_k \mid \widehat{J}_k \overline{\widehat{\mathcal{R}}}_{k-1} \rangle.$$

Since $\widehat{\mathcal{R}}_k = \langle \widehat{A}_k \mid \widehat{B}_k \rangle \subseteq \operatorname{im} \begin{bmatrix} I \\ 0 \end{bmatrix}$, it follows that

$$\operatorname{im} \begin{bmatrix} I \\ 0 \end{bmatrix} \supseteq \langle \widehat{A}_k \mid \widehat{J}_k \overline{\widehat{\mathcal{R}}}_{k-1} \rangle = \langle \widehat{A}_k \mid \operatorname{im} \widehat{J}_k \rangle \supseteq \operatorname{im} \widehat{J}_k.$$

Therefore, (A_k, B_k, J_k) is actually transformed to

$$\left(\begin{bmatrix} * & * & * & * \\ 0 & \begin{bmatrix} \widehat{A}_k^1 & * \\ 0 & \widehat{A}_k^2 \end{bmatrix} & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}, \begin{bmatrix} * \\ \begin{bmatrix} \widehat{B}_k^1 \\ 0 \end{bmatrix} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} * \\ \begin{bmatrix} J_k^1 \\ 0 \end{bmatrix} \\ 0 \\ 0 \end{bmatrix} \right).$$

From this one can arrive at the following contradiction:

$$\operatorname{im} \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \overline{\mathcal{R}}_k = \mathcal{R}_k + \langle A_k \mid J_k \overline{R}_{k-1} \rangle \subseteq \operatorname{im} \begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, it has inductively been shown that $\overline{\widehat{\mathcal{R}}}_k = \mathbb{R}^{\widehat{n}_k}$, for all modes k.

Case II: Assume $\widehat{\underline{\mathcal{U}}}_k \neq \{0\}$. Analogously as in Case I, the contradiction

$$\underline{\mathcal{U}}_k \neq \operatorname{im} \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix},$$

arises, thus the details are omitted. □

For the special case of constant A-matrices, the proposed method does in fact result in a minimal realization.

Corollary 4.13. *Consider the switched system* (4.11) *with mode-independent A-matrix. Then, the reduced switched system obtained via the proposed reduction method is minimal.*

Proof. This is a simple consequence from Proposition 4.8, because in any mode a smaller reduced model would necessarily remove some reachable and observable states and hence cannot lead to the same input-output behavior. □

4.4 Numerical results

This section demonstrates the operation of the proposed reduction method for the switched linear system. The proposed method is illustrated by means of numerical examples. The source code for the numerical examples is available from [60].

Example 4.4. Consider a switched linear system with modes:

$$(A_0, B_0, C_0) = \left(\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \right),$$

$$(A_1, B_1, C_1) = \left(\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \right),$$

$$(A_2, B_2, C_2) = \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \right),$$

$$J_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}.$$

Assume the mode sequence $0 \to 1 \to 2$. Now apply the proposed reduction method and the reduced realization can be obtained as follows.

Step 1. Here, $\mathcal{R}_0 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathcal{R}_1 = \begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}$, $\mathcal{R}_2 = \begin{bmatrix}1&0\\0&1\\0&1\end{bmatrix}$, $\mathcal{U}_0 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathcal{U}_1 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$, $\mathcal{U}_2 = \begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}$. Now the sequence of reachable and unobservable subspaces is computed:

$$\overline{\mathcal{R}}_0 = \mathcal{R}_0 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \overline{\mathcal{R}}_1 = \mathcal{R}_1 + \langle A_1 \mid J_1\overline{\mathcal{R}}_0\rangle = \begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix},$$

$$\overline{\mathcal{R}}_2 = \mathcal{R}_2 + \langle A_2 \mid J_2\overline{\mathcal{R}}_1\rangle = \mathbb{R}^3,$$

$$\underline{\mathcal{U}}_2 = \mathcal{U}_2 = \begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}, \quad \underline{\mathcal{U}}_1 = \mathcal{U}_1 \cap \langle J_2^{-1}\,\underline{\mathcal{U}}_2 \mid A_1\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix},$$

$$\underline{\mathcal{U}}_0 = \mathcal{U}_0 \cap \langle J_1^{-1}\,\underline{\mathcal{U}}_1 \mid A_0\rangle = \{0\}.$$

Step 2. Via the proposed method, the sequence of left- and right-projectors are obtained by

$$(\overline{W}_0^2, \overline{V}_0^2) = \left(\begin{bmatrix}1\\-2\\-2\end{bmatrix}^\top, \begin{bmatrix}1\\0\\0\end{bmatrix}\right),$$

$$(\overline{W}_1^2, \overline{V}_1^2) = \left(\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}^\top, \begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}\right),$$

$$(\overline{W}_2^2, \overline{V}_2^2) = \left(\begin{bmatrix}0\\-1\\0\end{bmatrix}^\top, \begin{bmatrix}0\\-1\\-1\end{bmatrix}\right).$$

The reduced switched system is given by

$$(\widehat{A}_0, \widehat{B}_0, \widehat{C}_0) = (\overline{W}_0^2 A_0 \overline{V}_0^2, \overline{W}_0^2 B_0, C_0 \overline{V}_0^2) = (2, 1, 1)\,,$$
$$(\widehat{A}_1, \widehat{B}_1, \widehat{C}_1) = \left(\overline{W}_1^2 A_1 \overline{V}_1^2, \overline{W}_1^2 B_1, C_1 \overline{V}_1^2\right) = \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \end{bmatrix}\right),$$
$$(\widehat{A}_2, \widehat{B}_2, \widehat{C}_2) = \left(\overline{W}_2^2 A_2 \overline{V}_2^2, \overline{W}_2^2 B_2, C_2 \overline{V}_2^2\right) = (1, -1, -1)\,.$$

Step 3. The reduced jump maps are given by

$$\widehat{J}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \widehat{J}_2 = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

Figure 4.1 shows the output of the original and its reduced system for input $u(t) = 1$ with switching times $s_1 = 2$ and $s_2 = 5$ over $[0, 6]$ and clearly both outputs coincide.

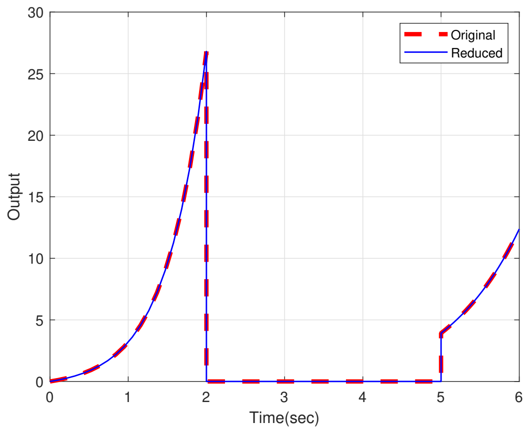

Figure 4.1: Outputs of the original system and the proposed reduced system.

4.5 Discussion

In this chapter, a method is proposed for obtaining a reduced realization for switched linear systems with jumps and mode-dependent state dimensions; the switching signal is assumed to be fixed with known mode sequence. The proposed reduction method is independent of the switching times and hence in principle also applicable for state dependent switched systems if a certain mode sequence is known a-priori. The proposed reduction method is based on a weak Kalman decomposition of each mode by defining suitable extended reachable and restricted unobservable subspaces. It is conjectured that the proposed method results in a minimal realization for almost all switching times, however, a definite answer to this question is still ongoing research. It cannot be expected that the proposed method will result in a minimal realization for *all* switching times, an example is provided for which the dimension of the minimal realization depends on the specific switching times.

So far, it is assumed that all subspace related operations (intersections, sums, pre-images) can be carried out with exact arithmetics, however, for large-scale systems and/or for systems with numerical coefficient matrices the involved subspace calculations are in general ill-posed. A suitable adaption of the proposed algorithm utilizing e.g. the singular value decomposition to carry out the subspace calculations approximately is a topic of future research.

Part II

Model reduction for switched systems

5 Model reduction (time-varying) for switched linear systems

5.1 Introduction

In Chapters 3 and 4, realization theory is discussed for switched linear systems. This chapter investigates the model reduction method for switched linear systems by considering the overall system as a time-varying system.

As mentioned in the introduction (Chapter 1), model reduction techniques turn out to be an important tool in the context of the simulation of various applications and problems. In the last decades, switched systems gained much interest as a modelling framework with applications, cf. [76, 127]. MOR of switched systems is highly relevant for large-scale applications in the systems and control community. It is well known that the key purpose of MOR is to find a lower order approximation of a dynamical system which can be used in simulation and optimization instead of the original system.

Consider the switched linear system (as in (2.4) without state jumps) of the form

$$\Sigma_\sigma : \begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad t \in \mathbb{R}, \\ y(t) = C_{\sigma(t)}x(t), \end{cases} \tag{5.1}$$

where $\sigma : \mathbb{R} \to \mathtt{M} = \{0, 1, 2, \dots, \mathtt{m}\}$ with finitely many switching times $s_1 < s_2 < s_3 < \cdots < s_\mathtt{m}$ in the bounded interval $[t_0, t_f)$ of interest. The system matrices are $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{p\times n}$, where $i \in \mathtt{M}$ and n is the number of state variables, called the order of the subsystems. It is noted that all states have the same dimension. Here, the k-th mode is active in the interval $[s_k, s_{k+1})$, $k = 0, 1, \dots, \mathtt{m}$ and the duration of k-th mode is $\tau_k = s_{k+1} - s_k$. The input function u is assumed to be piecewise continuous and bounded. For notational convenience, let $s_0 := t_0$, $s_{\mathtt{m}+1} := t_f$ of length $\tau_k := s_{k+1} - s_k$, $k \in \{0, 1, 2, \dots, \mathtt{m}\}$.

As discussed in Chapter 1, there are some approaches on MOR of switched systems e.g., [11, 49, 83, 92, 101, 103, 106, 117, 119, 120]. However, it is clear from the previous chapter that the (time-variant) switching signal recognises switched systems as a special *time-varying* system. In contrast to these existing works, the main gap is to find a (time-varying) model reduction depending on a given (known) switching signal. While some results are available on model reduction for general linear time-varying systems, e.g., [61, 94, 121, 122, 142], they usually assume at least continuity of the coefficient matrices (which of course is not satisfied for switched systems). This limitation provides indeed an interesting topic to analysis model reduction of switched systems in a time-varying nature which will be discussed here.

The chapter is organized as follows. In Section 5.2, the research problem is illustrated with a simple example, why a naive model reduction approach is not working. In Section 5.3, balanced truncation is reviewed for general time-varying systems. In Section 5.4, model reduction of switched linear systems is discussed by proposing an approximating time-varying system. Finally, Section 5.5 contains some numerical results.

5.2 Problem statement

As mentioned in Chapter 1, the key idea of model reduction is to represent a complex dynamical system by a much simpler one, this may refer to many different techniques. Moreover, only projection-based method, known as balanced truncation is presented here which is considered as a prominent projection-based model reduction method (cf. [14]).

In projection-based technique, the original state variables x is approximated by a vector $\widehat{x}$ so that the system (5.1) can be approximated by the reduced system

$$\widehat{\Sigma}_\sigma : \begin{cases} \dot{\widehat{x}}(t) = \widehat{A}_{\sigma(t)}\widehat{x}(t) + \widehat{B}_{\sigma(t)}u(t), \quad t \in \mathbb{R}, \\ \widehat{y}(t) = \widehat{C}_{\sigma(t)}\widehat{x}(t), \end{cases} \tag{5.2}$$

where the system matrices are $\widehat{A}_i \in \mathbb{R}^{r\times r}$, $\widehat{B}_i \in \mathbb{R}^{r\times m}$, $\widehat{C}_i \in \mathbb{R}^{p\times r}$, for $i \in \mathtt{M}$ and $r \ll n$.

A question arises: Is the balanced truncation approach applicable to switched systems by reducing each individual subsystems independently? The following simple example shows that the naive idea of balanced truncation does not work in general.

Example 5.1. Consider the switched linear system (5.1) with system matrices

$$A_0 = A_1 = \begin{bmatrix} -0.5 & 0.01 \\ 0.01 & -0.5 \end{bmatrix}, \; B_0 = C_0^\top = \begin{bmatrix} 0.001 \\ 1 \end{bmatrix},$$
$$B_1 = C_1^\top = \begin{bmatrix} 1 \\ 0.001 \end{bmatrix},$$

and the switching signal is given by

$$\sigma(t) = \begin{cases} 0 & \text{on } [0,1), \\ 1 & \text{on } [1,2). \end{cases} \tag{5.3}$$

Applying standard balanced truncation for linear systems on the two intervals $[0,1)$ and $[1,2)$ separately, the same one-dimensional reduced order model can be obtained by

$$\begin{aligned} \dot{\tilde{x}}(t) &= -0.5\,\tilde{x}(t) + u(t), \\ \tilde{y}(t) &= \tilde{x}(t). \end{aligned}$$

Figure 5.1 shows the output of the original and reduced system with the input $u(t) = (\sin(5t) + 0.05)e^{-.5t}$. Clearly, the output of the reduced model does not match the original output after the switch, although each individual mode is approximated sufficiently well with a small (known) error bound.

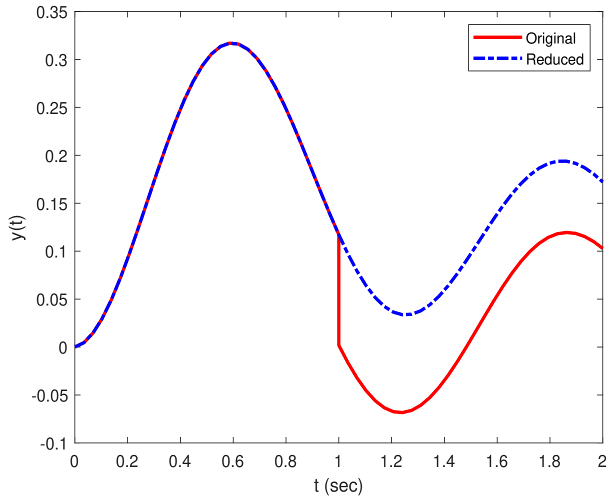

Figure 5.1: Outputs of the original system and 1st order reduced system for Example 5.1.

The above example shows, that a piecewise balanced truncation method will not result in good approximations of a switched system in general. The underlying problem is the time-varying nature of the switched system, so a method will be proposed to use a time-varying balanced truncation method for the switched system (5.1).

5.3 Balanced truncation for time-varying systems

In this section, balanced truncation of general time-varying systems on a finite time interval is discussed. Consider the linear time-varying system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t),\ t \in [t_0, t_f], \\ y(t) &= C(t)x(t), \end{aligned} \tag{5.4}$$

with time-varying (continuous) system matrices $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$.

5.3.1 Reachability and observability Gramians

It is well known that balanced truncation relies on reachability and observability Gramians. This section deals with balancing of general time-varying systems, so the time-varying reachability and observability Gramians are reviewed first. Inspired by [113, 121, 142], the time-varying reachability and observability Gramians are defined as follows.

Definition 5.1. Consider the linear time-varying system (5.4) with system matrices $A(\cdot), B(\cdot)$ and $C(\cdot)$. The time-varying reachability and observability Gramians at $t \in [t_0, t_f]$ are defined as

$$P(t) := P_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)B(\tau)^\top \Phi(t,\tau)^\top \,\mathrm{d}\tau, \tag{5.5}$$

$$Q(t) := Q_f + \int_{t}^{t_f} \Phi(\tau,t)^\top C(\tau)^\top C(\tau)\Phi(\tau,t)\,\mathrm{d}\tau, \tag{5.6}$$

where P_0, Q_f are some symmetric positive semidefinite matrices and $\Phi(t,\tau)$ is the transition matrix of the system, satisfying for all t and τ,

$$\frac{\partial}{\partial t}\Phi(t,\tau) = A(t)\Phi(t,\tau),$$

with the initial condition $\Phi(\tau,\tau) = I$.

It can be shown that these Gramians are the solution of the following differential Lyapunov equations:

$$\dot{P}(t) = A(t)P(t) + P(t)A(t)^\top + B(t)B(t)^\top, \tag{5.7}$$

$$\dot{Q}(t) = -(A(t)^\top Q(t) + Q(t)A(t) + C(t)^\top C(t)), \tag{5.8}$$

with initial/final conditions $P(t_0) = P_0,\ \ Q(t_f) = Q_f$.

However, all existing works have considered zero initial/final Gramians and then the analysis is only carried out for the compact time interval (t_0, t_f), cf.

[74, 121, 142]. Here, the choice for non-zero initial/final Gramians is made due to the fixed time interval $[t_0, t_f]$. In the case $P_0 = 0$, $Q_f = 0$, the reachability Gramian will be zero at the starting point and gradually builds up while the observability Gramian decreases and will be zero at the final time. Those points are therefore singular points for the balancing transformation and will being unbounded as t approaches to t_0 or t_f. This motivates to consider nonzero initial/final Gramians.

Remark 5.2. If the dynamical system is considered on the whole time axis (as is usually for the time-invariant case, then P_0 and Q_f are chosen such that $P(-\infty) = 0$ and $Q(\infty) = 0$. This however makes it necessary to assume that the linear system is asymptotically stable. This chapter is only interested in the behavior on the finite interval $[t_0, t_f]$, which basically arbitrarily assign the values for P_0 and Q_f; in other words, one can choose arbitrarily how the system behaves outside the interval of interest as long as it is exponentially stable. Moreover, the choice of the initial/final Gramians is crucial in the sense that they play an important role for the magnitude of the time-varying Gramians. In the context of time-varying case, two versions can be proposed for the initial/final Gramians. One choice would be to assume that the first mode is active in whole past i.e., $(-\infty, t_0]$ and the Gramian of the first mode is considered as the initial reachability Gramian. Similarly, by assuming that the last mode is active in whole future i.e., $[t_f, \infty)$ and the Gramian of the last mode is considered as the final value for observability Gramian. However, in this case the computation of infinite Gramians are only possible for stable modes, so it is more restricted for the considered systems. On the other hand, a second choice could be the identity matrix which would not affect the direction of the states which are difficult to control and difficult to observe. By scaling the identity matrix with a smaller magnitude, one can restrict the influence of these artificial initial/final Gramians relatively to the time-varying Gramians and also for the bounded time-varying coordinate transformation matrices.

Note that, $P(t)$ and $Q(t)$ are both symmetric and positive semidefinite for all $t \in [t_0, t_f]$. It is assumed that the input-output balancing with respect to the reachability and observability Gramians is defined over specific time intervals. Hence, no assumption is needed with regard to the stability of the system.

Remark 5.3. The pair $(A(\cdot), B(\cdot))$ is completely reachable on $[t_0, t_f]$ if, and only if P_0 is positive definite, and the pair $(A(\cdot), C(\cdot))$ is completely observable on $[t_0, t_f]$ if, and only if Q_f is positive definite. In the literature [142], on time-varying Gramians $P(\cdot), Q(\cdot)$, the notions of boundedly completely reachable/observable is defined with following inequalities:

$$0 < \alpha I \leqslant P(t) \leqslant \bar{\alpha} I < \infty, \quad \forall\, t \in \mathbb{R},$$

$$0 < \beta I \leqslant Q(t) \leqslant \bar{\beta} I < \infty, \quad \forall\, t \in \mathbb{R},$$

for any $0 < \alpha < \bar{\alpha} < \infty$ and $0 < \beta < \bar{\beta} < \infty$. However, these are equivalent to the notions of the finite time interval if P_0 and Q_f are positive definite.

5.3.2 Balancing and model reduction

As mentioned earlier, the balancing of a system is accomplished by a transformation of the state vector. In time-varying systems, such a transformation is also time-varying and it is common to restrict the class of allowed coordinate transformations so called Lyapunov transformations.

Definition 5.4. The mapping $T : [t_0, t_f] \to \mathbb{R}^{n\times n}$ is called a Lyapunov transformation (or, short, Lyapunov) iff $T(t)$, $T(t)^{-1}$ and $\dot{T}(t)$ are well defined and bounded on $[t_0, t_f]$.

Now it highlights that bounded reachability/observability ensures that the Gramians $P(\cdot)$ and $Q(\cdot)$ are Lyapunov transformations.

Lemma 5.5. *If P_0 and Q_f are positive definite then $P(\cdot)$ and $Q(\cdot)$ are Lyapunov.*

Definition 5.6. The linear time-varying system (5.4) is called *balanced* on $[t_0, t_f]$ if there exists positive definite P_0, Q_f such that

$$P(t) = Q(t) = \Xi(t), \ \forall\, t \in [t_0, t_f],$$

where $\Xi(t)$ is a diagonal matrix.

Theorem 5.7. *For every time-varying system* (5.4) *on $[t_0, t_f]$ with P_0 and Q_f which are positive definite, there exists a Lyapunov transformation T such that the transformed system is balanced on $[t_0, t_f]$ and the transformed system matrices are given by*

$$\begin{aligned}
\overline{A}(t) &:= T(t)^{-1}(A(t)T(t) - \dot{T}(t)),\\
\overline{B}(t) &:= T(t)^{-1}B(t),\\
\overline{C}(t) &:= C(t)T(t).
\end{aligned}$$

Proof. Consider P_0 and Q_f are positive definite then $P(t)$ and $Q(t)$ are Lyapunov for all $t \in [t_0, t_f]$. Consider the Cholesky decomposition of the reachability and observability Gramians:

$$P(t) = R(t)R(t)^\top, \quad Q(t) = L(t)L(T)^\top,$$

and the singular value decomposition

$$R^\top(t)L(t) := U(t)^\top \Xi(t) V(t)^\top,$$

where $\Xi(t)$ is a diagonal matrix containing the Hankel singular values (HSVs) and let, the HSVs are ordered decreasingly, i.e., $\xi_1(t) \geqslant \xi_2(t) \geqslant \cdots \geqslant \xi_n(t) > 0$, $t \in [t_0, t_f]$.

It is shown in [142] that for any continuous system, there exists a *continuously differentiable* nonsingular transformation matrix $T(t)$, $t \in [t_0, t_f]$, defined as

$$\begin{aligned} T(t) &= R(t)U(t)\Xi(t)^{-1/2}, \\ T(t)^{-1} &= \Xi(t)^{-1/2}V(t)^\top L(t)^\top. \end{aligned}$$

Under these transformation, it is easily seen that the reachability and observability Gramians transform to

$$\begin{aligned} \overline{P}(t) &= T(t)^{-1}P(t)T(t)^{-\top} = \Xi(t), \\ \overline{Q}(t) &= T(t)^\top Q(t)T(t) = \Xi(t), \end{aligned}$$

with $\overline{P}_0 = T(t_0)^{-1}P_0T(t_0)^{-\top}$ and $\overline{Q}_f = T(t_f)^{-1}Q_fT(t_f)^{-\top}$.

In particular,

$$\overline{P}(t)\overline{Q}(t) = T(t)^{-1}P(t)Q(t)T(t).$$

Then, the input-output equivalent *balanced* system of (5.4) is obtained by

$$\begin{aligned} \dot{\overline{x}}(t) &= \overline{A}(t)\overline{x}(t) + \overline{B}(t)u(t), \\ y(t) &= \overline{C}(t)\overline{x}(t), \end{aligned}$$

where the transformed system matrices are given by

$$\begin{aligned} \overline{A}(t) &:= T(t)^{-1}(A(t)T(t) - \dot{T}(t)), \\ \overline{B}(t) &:= T(t)^{-1}B(t), \\ \overline{C}(t) &:= C(t)T(t). \end{aligned}$$

Hence, the transformed system is balanced with the Gramians $\overline{P}(t)$, $\overline{Q}(t)$. □

For balanced system, one can decide which HSVs are important and so the singular value decomposition can be divided into two parts:

$$R^\top(t)L(t) := \begin{bmatrix} \widehat{U}(t) & \widetilde{U}(t) \end{bmatrix} \begin{bmatrix} \widehat{\Xi}(t) & 0 \\ 0 & \widetilde{\Xi}(t) \end{bmatrix} \begin{bmatrix} \widehat{V}(t) & \widetilde{V}(t) \end{bmatrix}^\top,$$

where all HSVs in $\widehat{\Xi}(t)$ are larger than all HSVs in $\widetilde{\Xi}(t)$.

According to the partition, an input-output equivalent (balanced) system has the form

$$\begin{aligned}\begin{bmatrix}\dot{\widehat{x}}(t)\\ \dot{\widetilde{x}}(t)\end{bmatrix} &= \begin{bmatrix}\widehat{A}(t) & \bar{A}_{12}(t)\\ \bar{A}_{21}(t) & \widetilde{A}(t)\end{bmatrix}\begin{bmatrix}\widehat{x}(t)\\ \widetilde{x}(t)\end{bmatrix} + \begin{bmatrix}\widehat{B}(t)\\ \widetilde{B}(t)\end{bmatrix}u(t),\\ y(t) &= \begin{bmatrix}\widehat{C}(t) & \widetilde{C}(t)\end{bmatrix}\begin{bmatrix}\widehat{x}(t)\\ \widetilde{x}(t)\end{bmatrix},\end{aligned} \tag{5.9}$$

where $\widehat{x}(t) \in \mathbb{R}^r$, $\widehat{A}(t) \in \mathbb{R}^{r\times r}$, $\widehat{B} \in \mathbb{R}^{r\times m}$, $\widehat{C} \in \mathbb{R}^{p\times r}$, $r < n$ is the dimension of $\widehat{\Xi}$.

By deleting the lower part, the reduced (and balanced) system of (5.9) is given by

$$\begin{aligned}\dot{\widehat{x}}(t) &= \widehat{A}(t)\widehat{x}(t) + \widehat{B}(t)u(t),\\ \widehat{y}(t) &= \widehat{C}(t)\widehat{x}(t).\end{aligned} \tag{5.10}$$

The error bound for time-varying system (5.4) can be computed as follows.

Theorem 5.8. *([113, Thm 1]). Assume the system* (5.4) *has a balanced reduced realization* (5.10) *on* $[t_0, t_f]$ *with*

$$\Xi(\cdot) = \operatorname{diag}(\widehat{\Xi}(\cdot), \widetilde{\Xi}(\cdot)),$$

where each singular value $\xi_i(\cdot)$, $i = r+1, \dots, n$ *in* $\widetilde{\Xi}(\cdot)$ *is either nonincreasing or nondecreasing over time. Then, the ROM is balanced by* $\widehat{\Xi}(\cdot) = \operatorname{diag}(\xi_1(\cdot), \xi_2(\cdot), \cdots, \xi_r(\cdot))$ *and*

$$\|y - \widehat{y}\|_{L_2} \leqslant 2\left(\sum_{k=r+1}^{n} \sup_{t\in[t_0,t_f]} \xi_k(t)\right)\|u\|_{L_2}.$$

The overall procedure for the reduced system is summarized by the following algorithm.

Step 1. Compute the Cholesky decomposition

$$P(t) = R(t)R(t)^\top, \quad Q(t) = L(t)L(T)^\top.$$

Step 2. Calculate the singular value decomposition

$$R^\top(t)L(t) = \begin{bmatrix}\widehat{U}(t) & \widetilde{U}(t)\end{bmatrix}\begin{bmatrix}\widehat{\Xi}(t) & 0\\ 0 & \widetilde{\Xi}(t)\end{bmatrix}\begin{bmatrix}\widehat{V}(t) & \widetilde{V}(t)\end{bmatrix}^\top,$$

where $\widehat{\Xi}(t) = \operatorname{diag}(\xi_1(t), \dots, \xi_r(t))$ and $\widetilde{\Xi}(t) = \operatorname{diag}(\xi_{r+1}(t), \dots, \xi_n(t))$.

Step 3. Compute the reduced system (5.10) with

$$\widehat{A}(t) = \Pi_l(t)(A(t)\Pi_r(t) - \dot{\Pi}_r(t)),$$
$$\widehat{B}(t) = \Pi_l(t)B(t), \quad \widehat{C}(t) = C(t)\Pi_r(t),$$

where $\Pi_l(t) = \widehat{\Xi}(t)^{-1/2}\widehat{V}(t)^\top \widehat{L}(t)^\top$ and $\Pi_r(t) = \widehat{R}(t)\widehat{U}(t)\widehat{\Xi}(t)^{-1/2}$.

5.4 Balanced truncation for switched systems via approximations

As mentioned earlier, the available balanced truncation for time-varying systems assume that the coefficient matrices are at least continuous. However, the switched system (5.1) has *discontinuous* coefficient matrices.

5.4.1 Approximated switched systems

In order to still be able to use the existing methods, the following approximation of the switched system (5.1) is proposed by the following (continuously) time-varying systems:

$$\Sigma_\varepsilon : \begin{cases} \dot{x}_\varepsilon(t) = A_\varepsilon(t)x_\varepsilon(t) + B_\varepsilon(t)u(t), \quad x_\varepsilon(t_0) = 0, \\ y_\varepsilon(t) = C_\varepsilon(t)x_\varepsilon(t), \end{cases} \tag{5.11}$$

where $A_\varepsilon(\cdot)$, $B_\varepsilon(\cdot)$ and $C_\varepsilon(\cdot)$ are defined as follows, for sufficiently small $\varepsilon > 0$,

$$(A_\varepsilon(t), B_\varepsilon(t), C_\varepsilon(t)) := (A_0, B_0, C_0), \quad : t \in [t_0, s_1),$$

$$A_\varepsilon(t) = \begin{cases} A_{i-1} + \frac{t-s_i}{\varepsilon}(A_i - A_{i-1}) & : t \in [s_i, s_i+\varepsilon], \\ A_i & : t \in (s_i+\varepsilon, s_{i+1}], \end{cases}$$

$$B_\varepsilon(t) = \begin{cases} B_{i-1} + \frac{t-s_i}{\varepsilon}(B_i - B_{i-1}) & : t \in [s_i, s_i+\varepsilon], \\ B_i & : t \in (s_i+\varepsilon, s_{i+1}], \end{cases}$$

$$C_\varepsilon(t) = \begin{cases} C_{i-1} + \frac{t-s_i}{\varepsilon}(C_i - C_{i-1}) & : t \in [s_i, s_i+\varepsilon], \\ C_i & : t \in (s_i+\varepsilon, s_{i+1}], \end{cases}$$

where $i = 1, \dots, \mathtt{m}$.

Remark 5.9. It is clear that the coefficient matrices in time-varying system (5.11) are bounded and continuous even for small ε, since the term $\frac{t-s_i}{\varepsilon} \in (0,1)$, $\forall$ $t \in (s_i, s_i+\varepsilon)$. Furthermore, the coefficient matrices are differentiable almost everywhere, however, the derivatives grow proportional to $1/\varepsilon$.

The solution of the time-varying state equations in system (5.11) is given by

$$x_\varepsilon(t) = \Phi_\varepsilon(t,t_0)x_\varepsilon(t_0) + \int_{t_0}^{t} \Phi_\varepsilon(t,\tau)B_\varepsilon(\tau)u(\tau)\,\mathrm{d}\tau,$$

where $\Phi_\varepsilon(t,\tau)$ is known as state transition matrix.

The error of the approximation can be derived as follows.

Theorem 5.10. *Consider the system* (5.1) *and its approximation* (5.11) *with the same input u, and let x and x_ε be the corresponding solutions. Then, there exist $\overline{\varepsilon} > 0$ and a constant $c > 0$ such that for all $\varepsilon \in (0, \overline{\varepsilon})$*

$$\|x_\varepsilon(t) - x(t)\| < c\,\varepsilon,\ \forall t \in [t_0, t_f). \tag{5.12}$$

Proof. Clearly, for $t \in [t_0, s_1)$, $\|x_\varepsilon(t) - x(t)\| = 0$. Assume that the statement is true on $[s_{k-1}, s_k)$, then for some c_{k-1}

$$\|x_\varepsilon(t) - x(t)\| < c_{k-1}\,\varepsilon,\ \forall t \in [s_{k-1}, s_k]. \tag{5.13}$$

Now for $\varepsilon > 0$,

$$\begin{aligned}
\|x_\varepsilon(s_k+\varepsilon) - x(s_k+\varepsilon)\| &= \|\Phi_\varepsilon(s_k+\varepsilon, s_k)x_\varepsilon(s_k) - \Phi(s_k+\varepsilon, s_k)x(s_k)\| \\
&\qquad + \int_{s_k}^{s_k+\varepsilon} \|\Phi_\varepsilon(s_k+\varepsilon,\tau) - \Phi(s_k+\varepsilon,\tau)\|\|u\|\,\mathrm{d}\tau \\
&\leqslant \|\Phi_\varepsilon(s_k+\varepsilon, s_k)\|\|x_\varepsilon(s_k) - x(s_k)\| + \|\Phi_\varepsilon(s_k+\varepsilon, s_k) - \Phi(s_k+\varepsilon, s_k)\|\|x(s_k)\| \\
&\quad + \varepsilon\|\Phi_\varepsilon(s_k+\varepsilon, s_k) - \Phi(s_k+\varepsilon, s_k)\|\|u\|.
\end{aligned}$$

From Taylor's theorem, $\Phi_\varepsilon(s_k+\varepsilon, s_k) = I + O(\varepsilon)$, $\|\Phi_\varepsilon(s_k+\varepsilon, s_k)\| \leqslant c_1$, and hence, $\|\Phi_\varepsilon(s_k+\varepsilon, s_k) - \Phi(s_k+\varepsilon, s_k)\| \leqslant c_2\varepsilon$. Again, $\|x(s_k)\| \leqslant (s_k - t_0)\|\Phi(s_k, t_0)\|\|u\| = \widehat{c}_k$. Now putting all together, it follows that

$$\|x_\varepsilon(s_k+\varepsilon) - x(s_k+\varepsilon)\| \leqslant c_1 c_{k-1}\varepsilon + c_2\widehat{c}_k\varepsilon + c_2 c_3\varepsilon^2 = \widetilde{c}_k\varepsilon,$$

where $\widetilde{c}_k = c_1 c_{k-1} + c_2\widehat{c}_k + c_2 c_3\varepsilon$. Inductively, the proof can now be concluded. It is clearly seen that the constant c in (5.12) depends on the input u, on the length of the interval $[t_0, t_f)$ and on the (magnitude of the) matrices A_i and B_i.

□

Remark 5.11. A similar bound for the output does *not* hold for *all* $t \in [t_0, t_f)$, in general the output of (5.1) is discontinuous (because the C-matrix switches) while the output of (5.11) is continuous. Nevertheless, away from the switching times (where $C_\varepsilon(t) = C_{\sigma(t)}$) the error bound (5.12) trivially carries over to a corresponding error bound for the outputs.

The main goal is now to derive a reduce model for the approximate system (5.11) of the form

$$\widehat{\Sigma}_\varepsilon : \begin{cases} \dot{\widehat{x}}_\varepsilon(t) = \widehat{A}_\varepsilon(t)\widehat{x}_\varepsilon(t) + \widehat{B}_\varepsilon(t)u(t), \quad \widehat{x}(t_0) = 0, \\ \widehat{y}_\varepsilon(t) = \widehat{C}_\varepsilon(t)\widehat{x}_\varepsilon(t). \end{cases} \tag{5.14}$$

5.4.2 Computational aspects of Gramians

In literature, there exist several different approaches for computing the Gramians for linear time-varying systems, e.g., in [65, 113, 121, 122, 142]. The Gramians can be computed either by solving the differential Lyapunov equations (5.7), (5.8) (see [12]), or by explicitly computing the transition matrix and integration. For the latter approach the following property of the Gramians can be exploited.

Lemma 5.12. *Given the time steps $s_0 < s_1 < \cdots < s_k$, $k \in \mathbb{N}$, the Gramians defined in (5.5) and (5.6) at s_i can be calculated recursively as follows*

$$\begin{aligned} P_\varepsilon(s_i) &= \Phi_\varepsilon(s_i, s_{i-1})P_\varepsilon(s_{i-1})\Phi_\varepsilon^\top(s_i, s_{i-1}) \\ &\quad + \int_{s_{i-1}}^{s_i} \Phi_\varepsilon(s_i,\tau)B_\varepsilon(\tau)B_\varepsilon^\top(\tau)\Phi_\varepsilon^\top(s_i,\tau)\,\mathrm{d}\tau, \\ Q_\varepsilon(s_i) &= \Phi_\varepsilon^\top(s_{i+1}, s_i)Q_\varepsilon(s_{i+1})\Phi_\varepsilon(s_{i+1}, s_i) \\ &\quad + \int_{s_i}^{s_{i+1}} \Phi_\varepsilon^\top(\tau, s_i)C_\varepsilon^\top(\tau)C_\varepsilon(\tau)\Phi_\varepsilon(\tau, s_i)\,\mathrm{d}\tau, \end{aligned}$$

for $i = 1, 2, \dots, k$.

Proof. This is simple consequence from the definition and the property of the transition matrix, in particular,

$$\Phi_\varepsilon(s_i,\tau) = \Phi_\varepsilon(s_i, s_{i-1})\Phi_\varepsilon(s_{i-1},\tau),$$

for any $i = 1, 2, \dots, k$ and $\tau \in \mathbb{R}$. □

Remark 5.13. From Lemma 5.12, it is clear that computation of Gramians could be expensive, however, this computation will speed up for the special (piecewise constant) structure of the considered switched linear system.

In the following, balanced truncation given in Section 5.3.2 can be applied to the system (5.11) with $(A_\varepsilon(t), B_\varepsilon(t), C_\varepsilon(t))$ and the balancing transformation matrices are computed using the Gramians.

Lemma 5.14. *Assume P_0 and Q_f are positive definite. Then, there exists a Lyapunov transformation $T_\varepsilon : [t_0, t_f] \to \mathbb{R}^{n\times n}$ such that for all $t \in [t_0, t_f]$,*

$$T_\varepsilon(t)^{-1} P_\varepsilon(t) T_\varepsilon(t)^{-\top} = T_\varepsilon(t)^\top Q_\varepsilon(t) T_\varepsilon(t) = \Xi_\varepsilon(t),$$

for a diagonal matrix $\Xi_\varepsilon(t)$. In fact,

$$T_\varepsilon(t) = R_\varepsilon(t) U_\varepsilon(t) \Xi_\varepsilon(t)^{-1/2},$$

$$T_\varepsilon(t)^{-1} = \Xi_\varepsilon(t)^{-1/2} V_\varepsilon(t)^\top L_\varepsilon(t)^\top,$$

where $U_\varepsilon(t)\Xi_\varepsilon(t)V_\varepsilon(t)^\top$ is the singular value decomposition of $R_\varepsilon(t)^\top L_\varepsilon(t)$, and where $R_\varepsilon(t)R_\varepsilon(t)^\top = P_\varepsilon(t)$ and $L_\varepsilon(t)L_\varepsilon(t)^\top = Q_\varepsilon(t)$ are the Cholesky decompositions of P_ε and Q_ε, respectively.

Proof. First observe that from positive definiteness of P_0 and Q_f, it follows that $P_\varepsilon(t)$ and $Q_\varepsilon(t)$ are positive definite for all t, which ensures that $P_\varepsilon(t)Q_\varepsilon(t)$ is invertible, hence $\Xi_\varepsilon(t)^{-1/2}$ is well defined.

Furthermore, it can be shown that all involved matrices to define T_ε are Lyapunov. From Theorem 5.7,

$$\begin{aligned} \overline{P}_\varepsilon(t) &:= T_\varepsilon(t)^{-1} P_\varepsilon(t) T_\varepsilon(t)^{-\top} = \Xi_\varepsilon(t), \\ \overline{Q}_\varepsilon(t) &:= T_\varepsilon(t)^\top Q_\varepsilon(t) T_\varepsilon(t) = \Xi_\varepsilon(t). \end{aligned}$$

□

Corollary 5.15. *Consider the system* (5.1) *and its time-varying approximation* (5.11) *(with sufficiently small ε). With the balanced form in Lemma 5.14, one can find a reduced system as in* (5.14) *which approximates the original system* (5.1).

5.5 Numerical results

In this section, two examples are presented to illustrate the proposed method. First, recall the Example 5.1 and then a switched linear system is considered with two modes where the system matrices are arbitrarily generated.

Example 5.2 (Example 5.1 revisited)**.** Recall the switched system from Example 5.1 and approximate it as in (5.11) with $\varepsilon = 0.1$ and $\varepsilon = 10^{-3}$. Figure 5.2 shows the output of the approximation compared to the output of the original switched system and it is clearly visible that indeed (5.11) is a good approximation of (5.1) for sufficiently small ε.

Next from Lemma 5.14, apply the proposed reduction technique. It is expected that a first order reduced (time-varying) system approximates the original switched system well. Figure 5.3 shows that for the given input as Example 5.1 the output of the first order reduced system is a good approximation of the output of the original switched system.

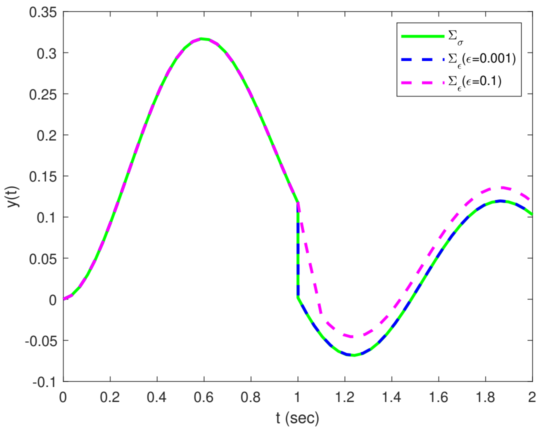

Figure 5.2: Outputs of the original system and its approximation with $\varepsilon = 0.1$ and $\varepsilon = 0.001$ for Example 5.2.

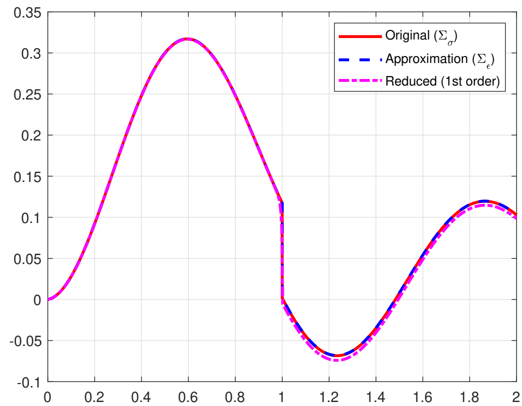

Figure 5.3: Outputs of the original system, approximation with $\varepsilon = 0.001$, and 1st order reduced system for Example 5.2.

Example 5.3. Consider a randomly generated switched linear system with modes

$$A_0 = \begin{bmatrix} -0.74 & 0.3 & 0.2 & -0.01 & -0.06 \\ 0.965 & -1.43 & -0.5 & 0.8 & -0.26 \\ 0.922 & -0.0487 & -0.44 & 0.03 & 0.054 \\ -0.98 & 0.28 & 0.31 & -0.764 & 0.07 \\ -0.634 & -1.26 & 0.534 & 0.662 & -0.48 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 2 & 1.4 & 1.1 & -0.06 & 0.08 \end{bmatrix}^T,$$

$$C_0 = \begin{bmatrix} 2.5 & 2 & 1.6 & 0.02 & -0.03 \end{bmatrix},$$

$$A_1 = A_0 - 0.5I_5 (I \text{ denotes identity matrix}),$$

$$B_1 = \begin{bmatrix} 2.5 & 1.8 & 0.3 & 0.6 & -1 \end{bmatrix}^T,$$

$$C_1 = \begin{bmatrix} 1.5 & 1.4 & 0.7 & 0.1 & 0.2 \end{bmatrix}, \varepsilon = 10^{-3}.$$

Consider $u(t) = (\sin(5t) + 0.05)e^{-0.5t}$ and the switching signal $\sigma : [0, 6] \to \{0, 1\}$

$$\sigma(t) = \begin{cases} 0 & : t \in [0, 1) \cup [2, 4), \\ 1 & : t \in [1, 2) \cup [4, 6]. \end{cases}$$

In Figure 5.4, it is seen that the first singular value is significantly larger than the others (by taking $P_\varepsilon(0) = Q_\varepsilon(6) = 0.2I$). Hence, by applying the proposed balanced truncation method and by truncating the small four singular values one can obtain a first order reduced model. Figure 5.5 displays the output of the original switched system, its full order approximation with $\varepsilon = 0.001$ and the first order reduced system which shows that they are nicely matching.

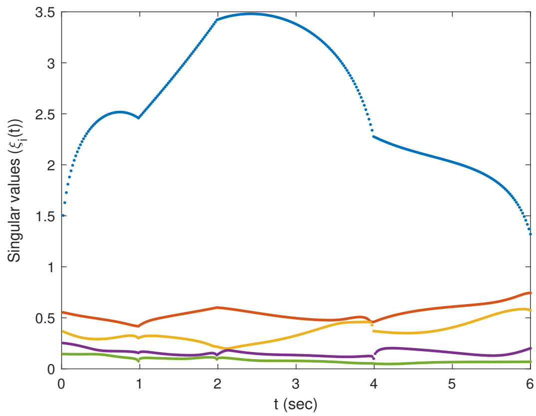

Figure 5.4: Hankel singular values of pointwise Gramians for Example 5.3.

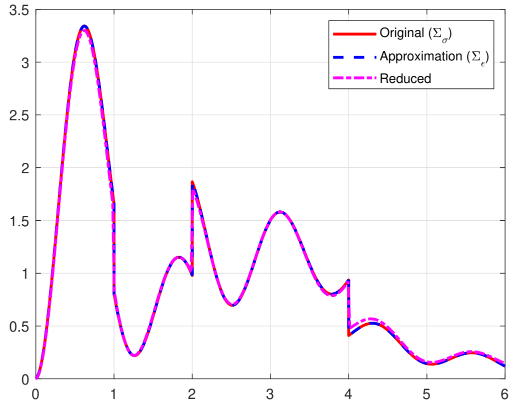

Figure 5.5: Outputs of the original system, approximation with $\varepsilon = 0.001$, and 1st order reduced system for Example 5.3.

Note that from Figure 5.4, it is apparent that the truncated Hankel singular values do not satisfy the assumption in Theorem 5.8, hence, currently to make a general statement about the error bounds is not possible for this example.

5.6 Discussion

In this chapter, a time-varying approach have been presented for proposing a reduced order approximation of switched linear systems. The key idea is to approximate the discontinuous switched system by a continuously time-varying system and use available balanced truncation methods for time-varying linear systems. Some error bounds are also proposed. Two numerical examples illustrate the applicability and good performance.

The overall error bound is composed of two error bounds: One between the switched system and its (full order) approximation and another error bound comes from the time-varying balanced truncation. For the former quantitative bounds are not yet available and for the latter it is not clear yet, how smaller values for ε effect the error bound. Furthermore, calculation for the time-varying Gramians may be computationally infeasible for higher-order systems and it has to be investigated whether efficient approximation methods can be derived. Finally, it is noted that the reduced model is not a simple switched system anymore. Hence, the model complexity significantly increases and an alternative approach will be presented in Chapter 6.

6 Model reduction for switched linear systems

6.1 Introduction

As discussed in Chapters 1 and 5, several approaches of model reduction for switched linear systems have been proposed, e.g., [11, 48, 49, 50, 101, 103, 119, 120]. It is mentioned that none of the existing model reduction approaches consider the switched system as a piecewise-constant linear time-varying system for a given switching signal. Consequently, all of the existing methods which do not consider a known switching signal will usually not results in the *best* reduced switched system for a specific switching signal. Moreover, a reduced realization of switched systems is proposed in Chapter 4 and, observed that the minimal realization depends on the specific mode sequences as well as the mode durations.

In Chapter 5, a model reduction method is discussed for switched linear systems where the switched system is considered as a (fully) time-varying system. The main idea was to first approximate the given time-varying system by assuming continuity of the coefficient matrices with a sufficiently small tolerance and then apply balanced truncation for the time-varying system. However, even when relaxing the continuity assumptions (e.g. by approximating the switched system with a continuous piecewise linear system) the resulting reduced system is fully time-varying and not piecewise-constant as usually desired. Furthermore, the studies on reduced realization of switched systems also showed that the reduced switched system will in general have mode-dependent state dimension and it is necessary to consider jump maps between the states of the different modes.

Likewise, in the case of model reduction it is reasonable to aim for a reduced switched system with mode-dependent state dimension and to consider jumps of the states at the switches. In order to stay within the same system class, recall the switched linear system (2.4) with a given switching signal

$$\Sigma_\sigma : \begin{cases} \dot{x}_k(t) = A_{\sigma(t)} x_k(t) + B_{\sigma(t)} u(t), \ t \in (s_k, s_{k+1}), \\ x_k(s_k^+) = J_{\sigma(s_k^+),\sigma(s_k^-)} x_{k-1}(s_k^-), \quad k \in \mathtt{M}, \\ y(t) = C_{\sigma(t)} x_k(t^+), \quad t \in [s_k, s_{k+1}), \end{cases} \tag{6.1}$$

and the main aim of this chapter is to obtain a lower dimensional SLS (with the same switching signal σ) of the form

$$\widehat{\Sigma}_\sigma : \begin{cases} \dot{\widehat{x}}_k(t) = \widehat{A}_{\sigma(t)} \widehat{x}_k(t) + \widehat{B}_{\sigma(t)} u(t), \ t \in (s_k, s_{k+1}), \\ \widehat{x}_k(s_k^+) = \widehat{J}_{\sigma(s_k^+),\sigma(s_k^-)} \widehat{x}_{k-1}(s_k^-), \quad k \in \mathtt{M}, \\ \widehat{y}(t) = \widehat{C}_{\sigma(t)} \widehat{x}_k(t^+), \quad t \in [s_k, s_{k+1}), \end{cases} \tag{6.2}$$

where $\widehat{x}_k \in \mathbb{R}^{r_k}$, $r_k \ll n_k$ and the output $\widehat{y}$ of the reduced system is similar to the output y of the original system (for the same input).

This chapter is organized as follows. In Section 6.2, the problem formulation is reviewed with a simple example, then the definition and characterization of the time-varying reachable and unobservable subspaces of the switched linear system (6.1) are recalled. In Section 6.3, the time-varying reachability and observability Gramians are proposed and their relationship to the corresponding reachable and unobservable subspaces is highlighted; furthermore, a precise connection to input- and output energy is proven. Then, in Section 6.4 a mode-wise midpoint balanced truncation method is proposed to obtain a reduced model which disregards simultaneously difficult to reach and difficult to observe states. It also provides a discussion about the numerical implementation of the algorithm and its feasibility for large-scale systems. Finally, in Section 6.5 some numerical experiments are provided which illustrate the effectiveness of the approach.

6.2 Preliminaries

In this section, some problem setting and preliminaries for switched systems are given.

From the Example 5.1, it is clear that the naive idea of balanced truncation does not work in general. There are two underlying main reason why mode-wise balanced truncation will in general not work for switched systems:
1) Balancing each mode individually results in a mode-dependent coordinate transformation, so even without reducing the state dimension, the resulting switched system will not preserve the input-output behavior *unless an additional state-jump is introduced to take into account the mode-dependent coordinate transformations*;

2) A mode-wise reduction removes difficult to observe and difficult to reach states in each mode, however, *a difficult to observe state in one mode may be easily observable in another mode*, hence in order to approximately preserve the input-output behavior, one should not be removed such a state.

In order to resolve the second issue, the (time-varying) reachable and unobservable subspaces of switched systems will be recalled first. Recall the formal definition of the reachable subspace (Definition 4.4) and unobservable subspace (Definition 4.6) on the intervals $[t_0, t)$ and $[t, t_f)$, respectively.

Definition 6.1. The reachable subspace of the switched system (6.1) on time interval $[t_0, t)$ is

$$\mathcal{R}_{[t_0,t)} := \left\{ x(t^-) \;\middle|\; \begin{array}{l} \exists \text{ solution } (x,u) \text{ of (6.1)} \\ \text{with } x(t_0^-) = 0 \end{array} \right\}.$$

The switched system (6.1) is called *reachable* (on (t_0, t_f)) if, and only if,

$$\mathcal{R}_{[t_0,t_f)} = \mathbb{R}^{n_{\mathtt{m}}}.$$

Definition 6.2. The unobservable subspace of the switched system (6.1) on time interval $[t, t_f)$ is

$$\mathcal{U}_{[t,t_f)} := \left\{ x(t^+) \;\middle|\; \begin{array}{l} \exists \text{ solution } (x, u=0) \text{ such that} \\ y = 0 \text{ of (6.1) on } [t, t_f) \end{array} \right\}.$$

The switched system (6.1) is called *observable* (on $[t_0, t_f)$) if, and only if,

$$\mathcal{U}_{[t_0,t_f)} = \{0\}.$$

In [70], it is shown that the exact (time-varying) reachable and unobservable subspaces can be calculated by the following recursive formulas. The reachable subspaces are given by, for $k = 1, 2, \ldots, \mathtt{m}$,

$$\begin{aligned} \mathcal{R}_{[t_0,t)} &= \mathcal{R}_0, && t \in (t_0, s_1], \\ \mathcal{R}_{[t_0,t)} &= e^{A_k \tau_k} J_k \mathcal{R}_{[t_0,s_k)} + \mathcal{R}_k, && t \in (s_k, s_{k+1}], \end{aligned} \tag{6.3}$$

and the unobservable subspaces are given by, for $k = \mathtt{m}-1, \mathtt{m}-2, \ldots, 0$,

$$\begin{aligned} \mathcal{U}_{[t,t_f)} &= \mathcal{U}_{\mathtt{m}}, && t \in [s_{\mathtt{m}}, t_f), \\ \mathcal{U}_{[t,t_f)} &= \left(e^{-A_k(s_{k+1}-t)} J_{k+1}^{-1} \mathcal{U}_{[s_{k+1},t_f)} \right) \cap \mathcal{U}_k, && t \in [s_k, s_{k+1}). \end{aligned} \tag{6.4}$$

Here, $\mathcal{R}_k := \langle A_k \mid \operatorname{im} B_k \rangle$ and $\mathcal{U}_k := \langle \ker C_k \mid A_k \rangle$ are the classical local reachable and unobservable subspaces of mode k, and J_{k+1}^{-1} stands for the preimage (the jump maps are not assumed to be invertible and are rectangular in general anyway).

It should be noted that these subspaces do not contain quantitative information about how easy/difficult it is to reach a reachable state or observe an observable state. Consequently, while these subspaces are quite helpful to derive a reduced realization, they cannot be used directly to obtain reduced model which discards difficult to reach and difficult to observe states. To quantify reachability and observability, it is necessary to introduce suitable Gramians.

6.3 Exact (time-varying) Gramians for switched linear systems

The proposed reduction method is a generalization of the well established balanced truncation method and therefore relies on a suitable definition of Gramians which then need to be balanced.

6.3.1 Reachability Gramian

Consider the system (6.1) with the mode (A_k, B_k, C_k) and the k-th switching time duration $\tau_k := s_{k+1} - s_k$. Moreover, it is seen from Section 4.2.2 that the reachable subspaces of the overall switched systems depend on the switching time durations. This motivates to propose following recursive definition for the reachability Gramians of (6.1), for $k = 1, 2, \dots, \mathtt{m}$:

$$
\begin{aligned}
\mathcal{P}_0^\sigma(t) &:= P_0(t), && t \in [t_0, s_1), \\
\mathcal{P}_k^\sigma(t) &:= e^{A_k(t-s_k)} J_k \mathcal{P}_{k-1}^\sigma(s_k^-) J_k^\top e^{A_k^\top (t-s_k)} + P_k(t), && t \in [s_k, s_{k+1}),
\end{aligned}
\tag{6.5}
$$

where $P_k(t) = \int_{s_k}^t e^{A_k(\tau - s_k)} B_k B_k^\top e^{A_k^\top(\tau - s_k)} \, \mathrm{d}\tau$ is the classical reachability Gramian of mode k on the interval (s_k, t).

The intuition behind the sequence is as follows: Since the first system starts with zero initial value, the reachability Gramian in the first mode is the classical reachability Gramian of the first mode. Continuing recursively, the Gramian between the switching time s_k and s_{k+1} is obtained by propagating forward the Gramian just before switch k in time, i.e., first jump via J_k and then propagating according to the matrix exponential and finally, take into account the classical reachability Gramian for mode k.

First it will be shown that the reachability Gramian $\mathcal{P}_k^\sigma(t)$ indeed spans the reachable subspace $\mathcal{R}_{[t_0,t)}$.

Lemma 6.3. *For all $k = 0, 1, \ldots, \mathtt{m}$ and $t \in (s_k, s_{k+1}]$,*

$$\operatorname{im} \mathcal{P}_k^\sigma(t^-) = \mathcal{R}_{[t_0,t)}.$$

In particular, (6.1) *is reachable on $[t_0, t)$ i.e., $\mathcal{R}_{[t_0,t)} = \mathbb{R}^{n_k}$ for $t \in (s_k, s_{k+1}]$ if, and only if, $\mathcal{P}_k^\sigma(t^-)$ is positive definite (and hence nonsingular).*

Proof. It is well known (see e.g. [2]) that $\mathcal{R}_{[t_0,t)} = \mathcal{R}_0 = P_0(t) = \operatorname{im} \mathcal{P}_0^\sigma(t^-)$ for all $t \in (t_0, s_1]$. Proceeding inductively, assume that for some $k \leqslant \mathtt{m}$,

$$\operatorname{im} \mathcal{P}_{k-1}^\sigma(s_k^-) = \mathcal{R}_{[t_0,s_k)}.$$

From (6.3), the reachable subspace at $t \in (s_k, s_{k+1}]$ is given by

$$\mathcal{R}_{[t_0,t)} = e^{A_k(t-s_k)} J_k \mathcal{R}_{[t_0,s_k)} + \mathcal{R}_k. \tag{6.6}$$

Furthermore, $\mathcal{P}_{k-1}^\sigma(s_k^-)$ is by construction symmetric and positive semidefinite, hence $\mathcal{P}_{k-1}^\sigma(s_k^-)^{1/2}$ is well defined and $\operatorname{im} \mathcal{P}_{k-1}^\sigma(s_k^-) = \operatorname{im} \mathcal{P}_{k-1}^\sigma(s_k^-)^{1/2}$ and, therefore,

$$\operatorname{im} e^{A_k(t-s_k)} J_k (\mathcal{P}_{k-1}^\sigma(s_k^-))^{\frac{1}{2}} = e^{A_k(t-s_k)} J_k \mathcal{R}_{[t_0,s_k)}.$$

Note that for any matrix M, $\operatorname{im} M = \operatorname{im}(MM^\top)$ and consequently

$$\operatorname{im} e^{A_k(t-s_k)} J_k (\mathcal{P}_{k-1}^\sigma(s_k^-))^{\frac{1}{2}} = \operatorname{im} e^{A_k(t-s_k)} J_k \mathcal{P}_{k-1}^\sigma(s_k^-) J_k^\top e^{A_k^\top(t-s_k)}.$$

Together with $\mathcal{R}_k = \operatorname{im} P_k(t)$ and the general fact that $\operatorname{im}(M_1 + M_2) = \operatorname{im} M_1 + \operatorname{im} M_2$ for any two symmetric positive semidefinite matrices M_1 and M_2 of the same size, one can now conclude from (6.6) the desired subspace equation

$$\begin{aligned}\operatorname{im} \mathcal{P}_k^\sigma(t^-) &= \operatorname{im} e^{A_k(t-s_k)} J_k \mathcal{P}_{k-1}^\sigma(s_k^-) J_k^\top e^{A_k^\top(t-s_k)} + \operatorname{im} P_k(t) \\ &= \mathcal{R}_{[t_0,t)}.\end{aligned}$$

As already highlighted above, by definition $\mathcal{P}_k^\sigma(t^-)$ is symmetric and positive semidefinite. Consequently, positive-definiteness of $\mathcal{P}_k^\sigma(t^-)$ is equivalent to invertibility which in turn is equivalent to $\operatorname{im} \mathcal{P}_k^\sigma(t^-) = \mathbb{R}^{n_k}$. □

Remark 6.4 (Singularity of reachability Gramian). If the switched system (6.1) is reachable for all $t \in (t_0, t_f)$, i.e. $\mathcal{R}_{[t_0,t)} = \mathbb{R}^{n_k}$ for $t \in (s_k, s_{k+1}]$, Lemma 6.3 implies that the reachability Gramians $\mathcal{P}_k^\sigma(t^-)$ are all nonsingular. However, the right limits at the switching times $\mathcal{P}_k^\sigma(s_k^+) = J_k \mathcal{P}_{k-1}^\sigma(s_k^-) J_k^\top$ are in general singular, because a full row rank assumption has not been made on the jump map J_k (which in fact cannot hold if $n_k > n_{k-1}$). In particular, any time-varying coordinate transformation defined in terms of the singular $\mathcal{P}_k^\sigma(t^-)$ will result in an unbounded behavior to the right of each switching time (unless the corresponding jump map J_k has full row rank).

One of the main theoretical results will now be presented which is the connection of the reachability Gramian with minimal input energy required to reach a given final state in a given time.

Theorem 6.5 (Reachability Gramian and input energy)**.** *Consider the switched system* (6.1) *with zero initial value and given switching signal. For some $t \in (s_k, s_{k+1}) \subseteq (t_0, t_f)$ assume that the corresponding reachability Gramian $\mathcal{P}_k^\sigma(t)$ as well as the classical Gramians $\mathcal{P}_k(t)$ and all previous Gramians $\mathcal{P}_j^\sigma(s_{j+1}^-)$, $j = 0, 1, \dots, k-1$ are positive definite. Then, for all $x_t \in \mathbb{R}^{n_k}$*

$$\min_u \int_{t_0}^t u(\tau)^\top u(\tau)\,\mathrm{d}\tau = x_t^\top \mathcal{P}_k^\sigma(t^-)^{-1} x_t,$$

where the minimum is taken over all $u : [t_0, t) \to \mathbb{R}^m$ which result in a solution of (6.1) *with $x(t^-) = x_t$. In other words, the directions of eigenvectors of $\mathcal{P}_k^\sigma(t^-)$ corresponding to the smallest eigenvalues require the most energy to be reached from zero.*

Proof. For $t \in (t_0, s_1]$, Lemma 2.11 (with $x_0 = 0$) already shows the claim of the theorem. Proceeding inductively, assume that the claim is shown for $t = s_k$, and it will be then shown for $t \in (s_k, s_{k+1}]$. In particular, the minimal input energy to reach any z from zero on the interval $[t_0, s_k)$ is given by $z^\top P_{k-1}^\sigma(s_k^-) z$ and from Lemma 2.11, it follows that the minimal input energy to reach any x_t from $J_k z$ is given by $(x_t - e^{A_k(t-s_k)} J_k z)^\top P_k(t)^{-1} (x_t - e^{A_k(t-s_k)} J_k z)$, where $P_k(t) := \int_{s_k}^t e^{A_k(\tau - s_k)} B_k B_k^\top e^{A_k^\top(\tau - s_k)} \mathrm{d}\tau$. Clearly, to reach x_t from zero on the time interval $[t_0, t)$ with minimal energy, one needs to find z^* which minimize the sum of the minimal energy to reach z on $[t_0, s_k)$ and the minimal energy to reach x_t from $J_k z$ on $[s_k, t)$, i.e. it needs to be shown that

$$\begin{aligned}&\min_z \left(z^\top P_{k-1}^\sigma(s_k^-)^{-1} z + (x_t - e^{A_k(t-s_k)} J_k z)^\top P_k(t)^{-1} (x_t - e^{A_k(t-s_k)} J_k z) \right) \\ &\quad = x_t^\top P_k^\sigma(t)^{-1} x_t.\end{aligned}$$

The above minimization has the form

$$\min_z (z^\top M z - 2c^\top z + \alpha),$$

with $M = P_{k-1}^\sigma(s_k^-)^{-1} + J_k^\top e^{A_k^\top(t-s_k)} P_k(t)^{-1} e^{A_k(t-s_k)} J_k$, $c^\top = x_t^\top P_k(t)^{-1} e^{A_k(t-s_k)} J_k$, $\alpha = x_t^\top P_k(t)^{-1} x_t$. Note that, M is the sum of a symmetric positive definite matrix $P_{k-1}^\sigma(s_k^-)^{-1}$ and a symmetric positive semidefinite matrix and hence, it is itself a symmetric positive definite matrix. Consequently, the unique minimizer is given by $z^* := M^{-1} c$ and the minimal value is then $\alpha - c^\top M^{-1} c$. Hence, it remains to be shown that

$$\begin{aligned} P_k(t)^{-1} - P_k(t)^{-1} e^{A_k(t-s_k)} J_k M^{-1} J_k^\top e^{A_k^\top(t-s_k)} P_k(t)^{-1} &= P_k^\sigma(t)^{-1} \\ &= \Big(e^{A_k(t-s_k)} J_k \mathcal{P}_{k-1}^\sigma(s_k^-) J_k^\top e^{A_k^\top(t-s_k)} + P_k(t)\Big)^{-1}. \end{aligned} \tag{6.7}$$

Recall the well known Woodbury matrix identity which states that for any $G_0 \in \mathbb{R}^{n_0\times n_0}$, $G_1 \in \mathbb{R}^{n_1\times n_1}$ invertible and $F_1 \in \mathbb{R}^{n_1\times n_0}$,

$$(F_1 G_0 F_1^\top + G_1)^{-1} = G_1^{-1} - G_1^{-1} F_1 (G_0^{-1} + F_1^\top G_1^{-1} F_1)^{-1} F_1^\top G_1^{-1}.$$

With $G_0 = P_{k-1}^\sigma(s_k^-)$, $G_1 = P_k(t)$, $F_1 = e^{A_k(t-s_k)} J_k$, this identity equals exactly the desired relationship (6.7). □

Before presenting the main results, the observability Gramian of switched system is discussed as follows.

6.3.2 Observability Gramian

The following recursive definition is proposed for the (time-varying) observability Gramian of the switched system (6.1):

$$\begin{aligned} \mathcal{Q}_{\mathtt{m}}^\sigma(t) &:= Q_{\mathtt{m}}(t), & t &\in [s_{\mathtt{m}}, t_f), \\ \mathcal{Q}_k^\sigma(t) &:= e^{A_k^\top(s_{k+1}-t)} J_{k+1}^\top \mathcal{Q}_{k+1}^\sigma(s_{k+1}^+) J_{k+1} e^{A_k(s_{k+1}-t)} + Q_k(t), & t &\in [s_k, s_{k+1}), \end{aligned} \tag{6.8}$$

where $Q_k(t) := \int_t^{s_{k+1}} e^{A_k^\top(s_{k+1}-\tau)} C_k^\top C_k e^{A_k(s_{k+1}-\tau)}\, \mathrm{d}\tau$ is the classical observability Gramian of mode k on the interval (t, s_{k+1}).

The intuition for this definition is that starting from the time-limited observability Gramian of the last mode, the observability Gramian for the interval (t, t_f) is composed of the Gramian for $k+1$-st mode on (s_{k+1}, t_f) which is propagated backwards in time under the jump J_{k+1} together with the matrix exponential of k-th mode and the classical observability Gramian of mode k on the interval (t, s_{k+1}). It will now be shown that the kernel of the observability Gramian is indeed the unobservable subspace of the switched system.

Lemma 6.6. *For all $k = \mathtt{m}, \dots, 0$ and $t \in [s_k, s_{k+1})$,*

$$\mathcal{U}_{[t,t_f)} = \ker \mathcal{Q}_k^\sigma(t^+).$$

In particular, (6.1) *is observable on $[t, t_f)$ i.e., $\mathcal{U}_{[t,t_f)} = \{0\}$ for $t \in [s_k, s_{k+1})$ if, and only if, $\mathcal{Q}_k^\sigma(t^+)$ is positive definite.*

Proof. It is well known (see e.g. [2, Prop. 4.10]) that $\mathcal{U}_{[t,t_f)} = \mathcal{U}_{\mathtt{m}} = Q_{\mathtt{m}}(t) = \ker \mathcal{Q}_{\mathtt{m}}^\sigma(t^+)$ for all $t \in [s_{\mathtt{m}}, t_f)$. Proceeding inductively, assume now that for some

$k < \mathtt{m}$,

$$\mathcal{U}_{[s_{k+1},t_f)} = \ker \mathcal{Q}^{\sigma}_{k+1}(s^+_{k+1}),$$

and it will be shown that for $t \in [s_k, s_{k+1})$,

$$\mathcal{U}_{[t,t_f)} = \ker \mathcal{Q}^{\sigma}_{k}(t^+).$$

First observe that $\mathcal{Q}^{\sigma}_{k+1}(s^+_{k+1})^{1/2}$ is well defined because $\mathcal{Q}^{\sigma}_{k+1}(s^+_{k+1})$ is symmetric and positive semidefinite, furthermore, $\ker \mathcal{Q}^{\sigma}_{k+1}(s^+_{k+1}) = \ker \mathcal{Q}^{\sigma}_{k+1}(s^+_{k+1})^{1/2}$. Together with the general property $M^{-1} \ker N = \ker NM$ for arbitrary suitable matrices M and N, it follows that

$$e^{-A_k(s_{k+1}-t)} J^{-1}_{k+1} \ker \mathcal{Q}^{\sigma}_{k+1}(s^+_{k+1}) = \ker \mathcal{Q}^{\sigma}_{k+1}(s^+_{k+1})^{1/2} J_{k+1} e^{A_k(s_{k+1}-t)}.$$

Utilizing further that for any matrix M, it holds that $\ker M = \ker M^\top M$, then $\mathcal{U}_{[t,t_f)}$ is equal to

$$\ker \left(e^{A_k^\top(s_{k+1}-t)} J^\top_{k+1} \mathcal{Q}^{\sigma}_{k+1}(s^+_{k+1}) J_{k+1} e^{A_k(s_{k+1}-t)} \right) \cap \mathcal{U}_k.$$

Since $\mathcal{U}_k = \ker Q_k(t)$ for all $t \in [s_k, s_{k+1})$, the claim follows from the fact that $\ker(M_1 + M_2) = \ker(M_1) \cap \ker(M_2)$ for any two positive semidefinite matrices M_1 and M_2. Now positive-definiteness of $\mathcal{Q}^{\sigma}_{k}(t^+)$ is equivalent to invertibility which in turn is equivalent to $\ker \mathcal{Q}^{\sigma}_{k}(t^+) = \{0\}$. This completes the proof. $\square$

Before stating the relationship between the observability Gramian and the output energy, similar to Remark 6.4 a remark about the singularity of the observability Gramian is given in the following.

Remark 6.7 (Singularity of observability Gramian). By Lemma 6.6, the observability Gramian $\mathcal{Q}^{\sigma}_{k}(t^+)$ is nonsingular for all $t \in [s_k, s_{k+1}) \subseteq [t_0, t_f)$ if, the switched system is observable for all $t \in [t_0, t_f)$, i.e. $\mathcal{U}_{[t,t_f)} = \{0\}$. However, the left limit at the switching times $\mathcal{Q}^{\sigma}_{k}(s^-_{k+1}) = J^\top_{k+1} \mathcal{Q}^{\sigma}_{k+1}(s^+_{k+1}) J_{k+1}$ is in general singular, because any full column rank assumption is not made on the jump matrix J_{k+1} (which in fact cannot be satisfied if $n_k > n_{k+1}$).

Theorem 6.8. *(Observability Gramian and output energy). Consider a solution of the switched system* (6.1) *with zero input on the interval $[t, t_f)$ for $t \in [s_k, s_{k+1})$ and corresponding observability Gramian $\mathcal{Q}^{\sigma}_{k}(t)$. Then, the corresponding output satisfies*

$$\int_t^{t_f} y(\tau)^\top y(\tau)\,\mathrm{d}\tau = x(t^+)^\top \mathcal{Q}^{\sigma}_{k}(t^+) x(t^+). \tag{6.9}$$

In other words, states values at time t which are in the direction of an eigenvector corresponding to the smallest eigenvalue of $\mathcal{Q}^{\sigma}_{k}(t^+)$ produce only a small amount of output *energy and are therefore hard to observe.*

Proof. For $t \in [s_{\mathtt{m}}, t_f)$, $y(\tau) = e^{A_{\mathtt{m}}(\tau - t)}x(t^+)$ and hence,

$$\int_t^{t_f} y(\tau)^\top y(\tau)\,\mathrm{d}\tau = x(t^+)^\top Q_{\mathtt{m}}(t)x(t^+) = \mathcal{Q}_{\mathtt{m}}^\sigma(t^+).$$

Proceeding inductively, assume now that for some $k = \mathtt{m}-1, \mathtt{m}-2, \dots, 0$,

$$\int_{s_{k+1}}^{t_f} y(\tau)^\top y(\tau)\,\mathrm{d}\tau = x(s_{k+1}^+)^\top \mathcal{Q}_{k+1}^\sigma(s_{k+1}^+)x(s_{k+1}^+) \tag{6.10}$$

and it will then be shown that for all $t \in [s_k, s_{k+1})$ the equation (6.9) holds for any solution $x(\cdot)$ of (6.1) with zero input on $[t, t_f)$. Then,

$$\int_t^{t_f} y(\tau)^\top y(\tau)\,\mathrm{d}\tau = \int_t^{s_{k+1}} y(\tau)^\top y(\tau)\,\mathrm{d}\tau + \int_{s_{k+1}}^{t_f} y(\tau)^\top y(\tau)\,\mathrm{d}\tau.$$

For $\tau \in [t, s_{k+1})$, $y(\tau) = e^{A_k(\tau - t)}x(t^+)$ and hence

$$\int_t^{s_{k+1}} y(\tau)^\top y(\tau)\,\mathrm{d}\tau = x(t^+)^\top Q_k(t)x(t^+).$$

From (6.10) together with $x(s_{k+1}^+) = J_{k+1}e^{A_k(s_{k+1}-t)}x(t^+)$, it can be concluded that $\int_{s_{k+1}}^{t_f} y(\tau)^\top y(\tau)\mathrm{d}\tau$ is equal to

$$x(t^+)^\top e^{A_k^\top(s_{k+1}-t)}J_{k+1}^\top \mathcal{Q}_{k+1}^\sigma(s_{k+1}^+)J_{k+1}e^{A_k(s_{k+1}-t)}x(t^+).$$

Altogether, it arrives at (6.9). □

6.4 Midpoint balanced truncation

6.4.1 Motivation and algorithm

For any specific time $t \in (s_k, s_{k+1}) \subseteq (t_0, t_f)$, the corresponding reachability Gramian $\mathcal{P}_k^\sigma(t^-)$ and observability Gramian $\mathcal{Q}_k^\sigma(t^+)$ give precise quantitative information about which state direction is difficult to reach from zero on the interval $[t_0, t)$ and which state direction is difficult to observe from the output on $[t, t_f)$. While in the reduced realization context, the property of unreachability or unobservability is to some degree independent from the actual mode duration, this is not the case for the quantitative measure of reachability or observability.

In fact, the values of the integrals in the definitions of the Gramians explicitly depend on the mode duration and due to the positive semidefinite nature of the involved matrices, the magnitude of the Gramians will increase with an increased mode duration. This implies that it is in principle not possible to have a good model reduction method for switched systems which is independent of the mode duration.

In addition to the dependence of the Gramians on the mode duration, the Gramians also depend on the time $t \in (s_k, s_{k+1})$ within a given mode. Since the goal is to obtain a switched linear system of the form (6.2) as a reduced model, each mode needs to be reduced in a time-invariant fashion and it needs to be decided which information of the time-varying Gramians is utilized for the reduction method. In the following, it is proposed to take the Gramians evaluated at the midpoints

$$g_k = \frac{s_k + s_{k+1}}{2}, \; k = 0, 1, \dots, \mathtt{m},$$

of each mode as the basis of the model reduction, i.e. the following midpoint Gramians are considered:

$$\begin{aligned} \overline{\mathcal{P}}_k^\sigma &:= \mathcal{P}_k^\sigma(g_k), \\ \overline{\mathcal{Q}}_k^\sigma &:= \mathcal{Q}_k^\sigma(g_k). \end{aligned}$$

In particular, the implicit assumption is made that a state-direction which is difficult to reach/observe at the midpoint of a mode is also difficult to reach/observe in the whole time interval (s_k, s_{k+1}) in which mode k is active.

The key intuition behind the midpoint Gramians is that mode k needs to be active for a while so that one can really see which states are easy to reach in this mode (i.e. the corresponding reachability Gramian is changed sufficiently). The same applies to the observability Gramian, but there it is needed to stay long enough in mode k. So, in this sense, the middle point is 'optimal' as any other choice would make relative reachability and observability properties smaller compared to the already calculated reachability/observability properties from the past/future. Another aspect to consider is the potential singularity of the Gramians around the switching times as pointed out in Remarks 6.4 and 6.7, which indicates that balancing each mode based on Gramians close to the switching times would lead to coordinate transformations which are close to being singular; again this motivates to choose the midpoint Gramians because they are "farthest away" from potential singularities.

As a first step of the model reduction, it is first necessary to identify states which are simultaneously difficult to reach and difficult to observe (quantified via the midterm Gramians). This can be achieved by constructing a mode-dependent coordinate transformation $\widetilde{x}_k = T_k x_k$ in such a way that the corresponding midpoint Gramians (w.r.t. to the new coordinates) become equal and diagonal (i.e. balanced). Before continuing the discussion, it is highlighted how a mode-wise coordination transformation effects the form of the switched system and the corresponding Gramians.

Lemma 6.9. *Consider the switched system* (6.1) *and a mode-wise coordinate transformation* $\widetilde{x}_k = T_k x_k$ *for a family of invertible matrices* $T_k \in \mathbb{R}^{n_k \times n_k}$, $k \in \mathtt{M}$*. Then, the input-output behavior of* (6.1) *with zero initial value is equal to the input-output behavior of*

$$\widetilde{\Sigma}_\sigma : \begin{cases} \dot{\widetilde{x}}_k(t) = \widetilde{A}_k \widetilde{x}_k(t) + \widetilde{B}_k u(t),\ t \in (s_k, s_{k+1}), \\ \widetilde{x}_k(s_k^+) = \widetilde{J}_k \widetilde{x}_{k-1}(s_k^-), \quad k \in \mathtt{M}, \\ y(t) = \widetilde{C}_k \widetilde{x}_k(t^+), \quad t \in [s_k, s_{k+1}), \end{cases} \tag{6.11}$$

where

$$\widetilde{A}_k = T_k A_k T_k^{-1}, \quad \widetilde{B}_k = T_k B_k, \quad \widetilde{C}_k = C_k T_k^{-1},$$

and

$$\widetilde{J}_k = T_k J_k T_{k-1}^{-1}.$$

Furthermore, the corresponding Gramians satisfy

$$\widetilde{\mathcal{P}}_k^\sigma(t) = T_k \mathcal{P}_k^\sigma(t) T_k^\top, \quad \widetilde{\mathcal{Q}}_k^\sigma(t) = T_k^{-\top} \mathcal{Q}_k^\sigma(t) T_k^{-1},$$

in particular, the eigenvalues of $\mathcal{P}_k^\sigma(t)\mathcal{Q}_k^\sigma(t)$ *are invariant under mode-wise coordinate transformations.*

Proof. This can easily be verified inductively. □

Remark 6.10 (Necessity of jumps). It should be noted that a mode-wise coordinate transformation applied to a switched linear system without jumps necessarily introduced jumps of the form $\widetilde{J}_k = T_k T_{k-1}^{-1}$.

The following lemma is a well known result and shows how a balancing coordinate transformation can be found.

Lemma 6.11 ([84])**.** *For* $P, Q \in \mathbb{R}^{n \times n}$ *symmetric and positive definite, there exists invertible* $T \in \mathbb{R}^{n \times n}$ *and a diagonal positive definite matrix* $\Xi \in \mathbb{R}^{n \times n}$ *such that*

$$TPT^\top = \Xi = T^{-\top} Q T^{-1}.$$

In fact, $T = \Xi^{-1/2} V^\top L^\top$ *and* $T^{-1} = RU\Xi^{-1/2}$ *where* $P = RR^\top$ *and* $Q = LL^\top$ *is a Cholesky decomposition and* $R^\top L = U \Xi V^\top$ *is a singular value decomposition.*

The idea is now to carry out a mode-wise balancing of the original switched system (6.1) based on the midpoint Gramians to obtain the transformed switched system (6.11) whose corresponding midpoint Gramians are equal and diagonal. Then, one can remove all state-components (in the balanced coordinate system) corresponding to sufficiently small entries in the diagonal balanced midpoint Gramians to obtain a reduced system which will have a similar input-output behavior because only those state-components have been removed which are simultaneously difficult to reach and difficult to observe.

The overall midpoint balanced truncation method is summarized in Algorithm 1. Note that, the algorithm will only be able to run successfully if each midterm Gramian is nonsingular, which in view of Lemmas 6.3 and 6.6 implies that the switched system is reachable and observable at each midpoint of each mode. If this condition is not satisfied, it is possible to first eliminate unreachable and unobservable states via the method proposed in Chapter 4.

Algorithm 1: Midpoint balanced truncation

Data: Modes (A_k, B_k, C_k, J_k), $k = 0, 1, \dots, \mathtt{m}$, switching times s_k, $k = 0, \dots, \mathtt{m}+1$, reduction threshold ε_k or desired reduction size $r_k \leqslant n_k$, $k = 0, 1, \dots, \mathtt{m}$.
Result: Reduced modes $(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k, \widehat{J}_k)$, $k = 0, \dots, \mathtt{m}$.

```
Compute the sequence of midpoint reachability Gramians P̄^σ_0, P̄^σ_1, ..., P̄^σ_m.
Compute the sequence of midpoint observability Gramians Q̄^σ_m, Q̄^σ_{m−1}, ...,
  Q̄^σ_0.
for k = 0, ..., m do
    if P̄^σ_k and Q̄^σ_k nonsingular then
        Compute Cholesky decompositions P̄^σ_k =: R_k R_k^⊤ and Q̄^σ_k =: L_k L_k^⊤.
        Compute singular value decomposition R_k^⊤ L_k =: U_k Ξ_k V_k^⊤ with
          decreasing diagonal entries in Ξ_k.
        In case threshold ε_k is given: choose maximal r_k ⩽ n_k such that
          r_k-th entry of Ξ_k is bigger than ε_k.
        Calculate transformation matrices T_k := Ξ^{−1/2} V_k^⊤ L_k^⊤ and
          T_k^{−1} := R_k U_k Ξ^{−1/2}.
        Define left-projector Π^l_k as the first r_k rows of T_k and the
          right-projector Π^r_k as the first r_k columns of T_k^{−1}.
        Compute: Â_k := Π^l_k A_k Π^r_k, B̂_k := Π^l_k B_k, Ĉ_k := C_k Π^r_k.
        if k > 0 then
            Compute: Ĵ_k := Π^l_k J_k Π^r_{k−1}.
        end
    end
    else
        Abort: Midterm Gramians not positive definite, no balanced
          truncation possible, apply reduced realization algorithm first.
    end
end
```

6.4.2 Numerical aspects

The main motivation for model reduction is usually that the state dimensions of the original system is very large so that running (many) simulations or designing feedback controllers is not feasible. Hence, it is necessary to critically reflect whether the proposed reduction method is in fact feasible for large-scale systems. Clearly, the calculations of the midpoint Gramians (lines 1 and 2) in Algorithm 1 are by far the most expensive part of the whole method, followed by the Cholesky decompositions (line 5) and the singular value decomposition (line 6). Since the latter are also used in classical balanced truncation methods, there are already many efficient implementations available and they will not be further discussed here.

In the following, only the calculation of the reachability Gramians will be discussed, because the calculation of the observability Gramians is just a "transposed" version thereof.

In order to obtain the midpoint reachability Gramians, one needs efficient methods for 1) the calculation of the classical reachability Gramians $P_k(g_k)$ and $P_k(s_{k+1})$ for each mode k on the intervals $[s_k, g_k)$ and $[s_k, s_{k+1})$; and 2) the left- and right-multiplication action of the matrix exponential $e^{A_k(g_k-s_k)} = e^{A_k\tau_k/2}$ on the already calculated matrix $J_k\mathcal{P}^\sigma_{k-1}(s_k^-)J_k^\top$. Note that, $e^{A_k(s_{k+1}-s_k)} = e^{A_k\tau_k} = (e^{A_k\tau_k/2})^2$ which can be utilized in the calculation of the matrix $e^{A_k\tau_k}J_k\mathcal{P}^\sigma_{k-1}(s_k^-)J_k^\top e^{A_k^\top\tau_k}$.

The calculation of the classical reachability Gramian $\int_0^t e^{A\tau}BB^\top e^{A^\top\tau}\mathtt{d}\tau$ for a given matrices A, B and a time duration t has already been addressed in the context of time-limited balanced truncation [42], with further investigations in [33, 69, 109]. In particular, it can be shown that the reachability Gramian $P(t_1) = \int_{t_0}^{t_1} e^{A(\tau-t_0)}BB^\top e^{A^\top(\tau-t_0)}\,\mathrm{d}\tau$ for $\dot{x} = Ax + Bu$ considered on the interval $(t_0, t_1]$ is the solution of the Lyapunov equation

$$AP + PA^\top + BB^\top - e^{A(t_1-t_0)}BB^\top e^{A^\top(t_1-t_0)} = 0.$$

Hence, the calculation of $P_k(g_k)$ and $P_k(s_{k+1})$ reduces to the ability to efficiently calculate the matrix exponential and the ability to efficiently solve a Lyapunov equation. These are standard numerical tasks and efficient implementations exist for example in Matlab. The overall calculation of the midpoint reachability Gramian is summarized in Algorithm 2.

For switched systems (6.1) with state dimensions up to one thousand, the matrix exponentials and the solution of the Lyapunov equations can be obtained on a current laptop within seconds with the standard Matlab functions `expm` and `lyap`, so the proposed method is already feasible for many practical problems without any further code optimization and sophisticated approximation techniques. How to adapt the approach to very large-scale systems (state dimensions in the order of millions) is a numerical challenge but outside the scope of this chapter.

Algorithm 2: Computation of midpoint reachability Gramians

Data: Modes (A_k, B_k, J_k), $k = 0, 1, \dots, \mathtt{m}$ and mode durations τ_k, $k = 0, \dots, \mathtt{m}$.
Result: Midpoint Gramians $\overline{\mathcal{P}}_k^\sigma$, $k = 0, \dots, \mathtt{m}$.
Initialization: $P_0^\sigma = 0$.
for $k = 0, 1 \dots, m$ **do**
  Calculate: $F_{k,1/2} := e^{A_k \tau_k/2}$.
  Obtain $P_{k,1/2}$ as solution of the Lyapunov equation
  $A_k P_{k,1/2} + P_{k,1/2} A_k^\top + B_k B_k^\top - F_{k,1/2} B_k B_k^\top F_{k,1/2}^\top = 0.$
  Calculate: $\overline{\mathcal{P}}_k^\sigma := F_{k,1/2} J_k P_{k-1}^\sigma J_k^\top F_{k,1/2}^\top + P_{k,1/2}$.
  Calculate $F_k := (F_{k,1/2})^2$.
  Obtain P_k as solution of the Lyapunov equation
  $A_k P_k + P_k A_k^\top + B_k B_k^\top - F_k B_k B_k^\top F_k^\top = 0.$
  if $k < m$ **then**
    Calculate $P_{k+1}^\sigma := F_k J_k P_{k-1}^\sigma J_k^\top F_k^\top + P_k$.
  end
end

6.5 Numerical results

In the following, some academic examples are presented to illustrate the proposed method. First, the Example 5.1 is revisited, which was used to motivate that a naive mode-wise balanced truncation is not resulting in a suitable reduced system.

Example 6.1 (Example 5.1 revisited)**.** Consider the switched linear system from Example 5.1 with system matrices

$$A_0 = A_1 = \begin{bmatrix} -0.5 & 0.01 \\ 0.01 & -0.5 \end{bmatrix}, \; B_0 = C_0^\top = \begin{bmatrix} 0.001 \\ 1 \end{bmatrix},$$
$$B_1 = C_1^\top = \begin{bmatrix} 1 \\ 0.001 \end{bmatrix},$$

and a switching signal $\sigma : [0, 2)$ given by $\sigma(t) = 0$ on $[0, 1)$ and $\sigma(t) = 1$ on $[1, 2)$.

The proposed method is applied to obtain a reduced system, the steps are given as follows.

Step 1. Following Algorithm 2, the midpoint Gramians of the time intervals $[0, 1)$

and $[1, 2)$ are calculated:

$$\overline{\mathcal{P}}_0^\sigma = \begin{bmatrix} 10^{-5} & 0.0013 \\ 0.0013 & 0.3935 \end{bmatrix},$$
$$\overline{\mathcal{P}}_1^\sigma = \begin{bmatrix} 0.3935 & 0.0052 \\ 0.0052 & 0.3834 \end{bmatrix},$$
$$\overline{\mathcal{Q}}_0^\sigma = \begin{bmatrix} 0.3834 & 0.0052 \\ 0.0052 & 0.3935 \end{bmatrix},$$
$$\overline{\mathcal{Q}}_1^\sigma = \begin{bmatrix} 0.3935 & 0.0013 \\ 0.0013 & 10^{-5} \end{bmatrix}.$$

Step 2. The corresponding balanced Gramians are

$$\Xi_0 = \Xi_1 = \begin{bmatrix} 0.3935 & 0 \\ 0 & 0.0006 \end{bmatrix}.$$

Since the last diagonal entry is significantly smaller than the first, one can reduce both modes to first order without significantly influencing the input-output behavior.

Step 3. The calculated left- and right-projectors according to Algorithm 1 are obtained by

$$(\overline{\Pi}_0^l, \overline{\Pi}_0^r) = \left(\begin{bmatrix} 0.0164 \\ 1 \end{bmatrix}^\top, \begin{bmatrix} 0.0033 \\ 0.9999 \end{bmatrix} \right),$$
$$(\overline{\Pi}_1^l, \overline{\Pi}_1^r) = \left(\begin{bmatrix} -0.9999 \\ -0.0033 \end{bmatrix}^\top, \begin{bmatrix} -1 \\ -0.0164 \end{bmatrix} \right).$$

Step 4. The corresponding first order reduced switched system is then given by

$$(\widehat{A}_0, \widehat{B}_0, \widehat{C}_0) = \left(\overline{\Pi}_0^l A_0 \overline{\Pi}_0^r, \overline{\Pi}_0^l B_0, C_0 \overline{\Pi}_0^r \right) = (-0.4998, 1, 1),$$
$$(\widehat{A}_1, \widehat{B}_1, \widehat{C}_1) = \left(\overline{\Pi}_1^l A_1 \overline{\Pi}_1^r, \overline{\Pi}_1^l B_1, C_1 \overline{\Pi}_1^r \right) = (-0.4998, -1, -1),$$

and the jump matrix is given by $\widehat{J}_1 = -0.0066$.

Figure 6.1 depicts the output of the original switched system and its 1st order approximation with input $u(t) = (\sin(5t) + 0.05)e^{-.5t}$. Clearly, both outputs match nicely. The related relative errors between the two outputs are depicted in Figure 6.2, which shows that the relative error is less then 0.5%.

In the following, a larger system is considered with more switches and also mode-dependent state dimensions.

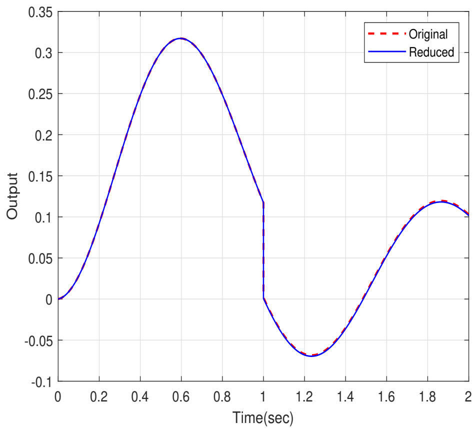

Figure 6.1: Outputs of the original system and the proposed 1st order system.

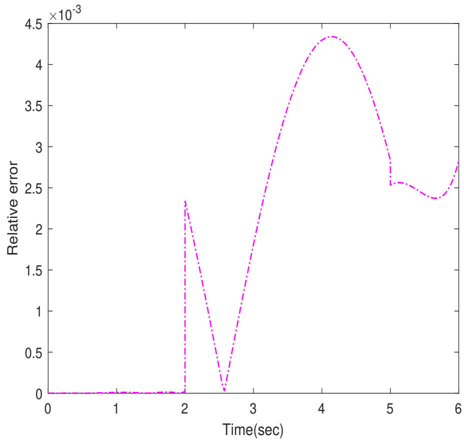

Figure 6.2: Relative errors of the original system and the proposed 1st order system.

Example 6.2. Consider a switched linear system with modes:

$$(A_0, B_0, C_0) = \left(\begin{bmatrix} 0.2 & 0.1 & 0.01 & 0.02 \\ 0.02 & 0.1 & 0.2 & 0.01 \\ 0.3 & 0.02 & 0.5 & 0.01 \\ 0.04 & 0.1 & 0.01 & 0.6 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0.7 \\ 1 \\ 0.01 \end{bmatrix}^\top \right),$$

$$(A_1, B_1, C_1) = \left(\begin{bmatrix} -0.2 & 0.01 & 0 \\ 0.1 & 0.1 & 0.2 \\ 0 & 0.1 & -0.3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0.2 \\ -0.02 \end{bmatrix}, \begin{bmatrix} 0.1 \\ 0.01 \\ 0.004 \end{bmatrix}^\top \right),$$

$$(A_2, B_2, C_2) = \left(\begin{bmatrix} 0.8 & 0.1 & 0 & -0.1 & 0.01 \\ 0.07 & 0.5 & 0 & 0.1 & 0 \\ 0.1 & 0.2 & 0.3 & 0.01 & 0 \\ 0.1 & 0 & 0 & 0.1 & 0.01 \\ 0 & 0 & 0.1 & 0 & 0.4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -0.2 \\ 0.1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0.2 \\ 0.1 \\ 0.2 \end{bmatrix}^\top \right),$$

$$J_1 = \begin{bmatrix} 0.3 & 1 & 0 & 0 \\ 0.1 & 0.2 & 0.1 & -1 \\ 0 & 0.1 & 0 & 1 \end{bmatrix},$$

$$J_2 = \begin{bmatrix} 1 & 0.1 & 0 \\ 0.02 & -0.2 & 0.1 \\ 0 & 0.01 & 0.1 \\ 0.1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

and a switching signal given by $\sigma(t) = 0$ on $[0, 2)$, $\sigma(t) = 1$ on $[2, 3)$ and $\sigma(t) = 2$ on $[3, 5)$. A reduced system is obtained as follows:
Step 1. Following Algorithm 2, the midpoint Gramians are calculated for the time intervals $[0, 2)$, $[2, 3)$ and $[3, 5)$:

$$\overline{\mathcal{P}}_0^\sigma = \begin{bmatrix} 5.7006 & 7.0498 & -5.1347 & 3.8804 \\ 7.0498 & 8.8090 & -6.3863 & 4.7298 \\ -5.1347 & -6.3863 & 4.6396 & -3.4675 \\ 3.8804 & 4.7298 & -3.4675 & 2.6942 \end{bmatrix},$$

$$\overline{\mathcal{P}}_1^\sigma = \begin{bmatrix} 23.6495 & -11.9148 & 2.6197 \\ -11.9148 & 8.4388 & -1.4232 \\ 2.6197 & -1.4232 & 0.2991 \end{bmatrix},$$

$$\overline{\mathcal{P}}_2^\sigma = \begin{bmatrix} 91.0695 & 19.9927 & 0.9495 & 14.3635 & 5.0961 \\ 19.9927 & 10.1823 & -1.8749 & 2.2150 & 1.0600 \\ 0.9495 & -1.8749 & 0.8944 & 0.5362 & 0.0800 \\ 14.3635 & 2.2150 & 0.5362 & 2.4461 & 0.8301 \\ 5.0961 & 1.0600 & 0.0800 & 0.8301 & 0.3060 \end{bmatrix},$$

$$\overline{\mathcal{Q}}_0^\sigma = \begin{bmatrix} 14.2249 & 7.2736 & 6.3552 & -2.8392 \\ 7.2736 & 10.4901 & 4.4036 & -7.4309 \\ 6.3552 & 4.4036 & 3.0521 & -2.2824 \\ -2.8392 & -7.4309 & -2.2824 & 5.8998 \end{bmatrix},$$

$$\overline{\mathcal{Q}}_1^\sigma = \begin{bmatrix} 7.9665 & 3.6209 & -1.0210 \\ 3.6209 & 1.6646 & -0.4353 \\ -1.0210 & -0.4353 & 0.1814 \end{bmatrix},$$

$$\overline{\mathcal{Q}}_2^\sigma = \begin{bmatrix} 2.1060 & -3.6360 & 0.3586 & -0.1436 & 0.3697 \\ -3.6360 & 6.3082 & -0.6224 & 0.2310 & -0.6410 \\ 0.3586 & -0.6224 & 0.0614 & -0.0226 & 0.0632 \\ -0.1436 & 0.2310 & -0.0226 & 0.0193 & -0.0237 \\ 0.3697 & -0.6410 & 0.0632 & -0.0237 & 0.0651 \end{bmatrix}.$$

Step 2. The corresponding balanced Gramians are

$$\Xi_0 = \begin{bmatrix} 10.3447 & 0 & 0 & 0 \\ 0 & 1.0422 & 0 & 0 \\ 0 & 0 & 0.0020 & 0 \\ 0 & 0 & 0 & 10^{-6} \end{bmatrix},$$

$$\Xi_1 = \begin{bmatrix} 10.5854 & 0 & 0 \\ 0 & 0.2399 & 0 \\ 0 & 0 & 0.0063 \end{bmatrix},$$

$$\Xi_2 = \begin{bmatrix} 10.6192 & 0 & 0 & 0 & 0 \\ 0 & 0.5082 & 0 & 0 & 0 \\ 0 & 0 & 0.0014 & 0 & \\ 0 & 0 & 0 & 10^{-5} & 0 \\ 0 & 0 & 0 & 0 & 10^{-10} \end{bmatrix}.$$

With a truncation threshold of 0.1, the reduced dimensions is two for all three modes.

Step 3. The calculated left- and right-projectors according to Algorithm 1 are obtained by

$$(\overline{\Pi}_0^l, \overline{\Pi}_0^r) = \left(\begin{bmatrix} -1.1180 & 1.0923 \\ -0.8135 & -1.8677 \\ -0.5399 & 0.0753 \\ 0.4345 & 1.9427 \end{bmatrix}^\top, \begin{bmatrix} -0.7395 & 0.2032 \\ -0.9227 & -0.0423 \\ 0.6694 & -0.0653 \\ -0.4972 & 0.3624 \end{bmatrix} \right),$$

$$(\overline{\Pi}_1^l, \overline{\Pi}_1^r) = \left(\begin{bmatrix} -0.8674 & -0.0995 \\ -0.3935 & -0.3247 \\ 0.1122 & -0.4054 \end{bmatrix}^\top, \begin{bmatrix} -1.4672 & 1.8930 \\ 0.6475 & -4.0771 \\ -0.1586 & 0.3347 \end{bmatrix} \right),$$

$$(\overline{\Pi}_2^l, \overline{\Pi}_2^r) = \left(\begin{bmatrix} -0.4453 & -0.0207 \\ 0.7694 & -0.2092 \\ -0.0759 & 0.0226 \\ 0.0301 & 0.1375 \\ -0.0782 & 0.0182 \end{bmatrix}^\top, \begin{bmatrix} -2.3741 & -7.8328 \\ -0.0888 & -4.4533 \\ -0.1811 & 0.9209 \\ -0.4449 & -0.7821 \\ -0.1374 & -0.4051 \end{bmatrix} \right).$$

Step 4. Applying the left- and right-projectors according to Algorithm 1, the reduced switched system (6.2) is given by

$$
\begin{aligned}
(\widehat{A}_0, \widehat{B}_0, \widehat{C}_0) &= \left(\overline{\Pi}_0^l A_0 \overline{\Pi}_0^r, \overline{\Pi}_0^l B_0, C_0 \overline{\Pi}_0^r\right) \\
&= \left(\begin{bmatrix} 0.0264 & 0.0389 \\ -1.1032 & 0.4960 \end{bmatrix}, \begin{bmatrix} -3.1623 \\ -1.6263 \end{bmatrix}, \begin{bmatrix} -2.2001 \\ 0.5182 \end{bmatrix}^\top \right), \\
(\widehat{A}_1, \widehat{B}_1, \widehat{C}_1) &= \left(\overline{\Pi}_1^l A_1 \overline{\Pi}_1^r, \overline{\Pi}_1^l B_1, C_1 \overline{\Pi}_1^r\right) \\
&= \left(\begin{bmatrix} -0.2028 & 0.3664 \\ 0.0385 & 0.2969 \end{bmatrix}, \begin{bmatrix} -0.9483 \\ -0.1563 \end{bmatrix}, \begin{bmatrix} -0.1466 \\ 0.1619 \end{bmatrix}^\top \right), \\
(\widehat{A}_2, \widehat{B}_2, \widehat{C}_2) &= \left(\overline{\Pi}_2^l A_2 \overline{\Pi}_2^r, \overline{\Pi}_2^l B_2, C_2 \overline{\Pi}_2^r\right) \\
&= \left(\begin{bmatrix} 0.6373 & 0.7872 \\ 0.0500 & 0.6001 \end{bmatrix}, \begin{bmatrix} 1.1555 \\ -0.4875 \end{bmatrix}, \begin{bmatrix} -2.3047 \\ 1.0988 \end{bmatrix}^\top \right), \\
\widehat{J}_1 = \overline{\Pi}_1^l J_1 \overline{\Pi}_0^r &= \begin{bmatrix} 0.8566 & 0.1279 \\ 0.0722 & 0.1011 \end{bmatrix}, \\
\widehat{J}_2 = \overline{\Pi}_2^l J_2 \overline{\Pi}_1^r &= \begin{bmatrix} 0.4940 & 0.0110 \\ 0.0205 & -0.1383 \end{bmatrix}.
\end{aligned}
$$

Figure 6.3 depicts the output of the original switched system and its approximation for the input $u(t) = 0.5\sin(0.5t)$. Clearly, both outputs match nicely. The related relative errors between the two outputs are depicted in Figure 6.4, which shows that the relative error is less then 0.5%.

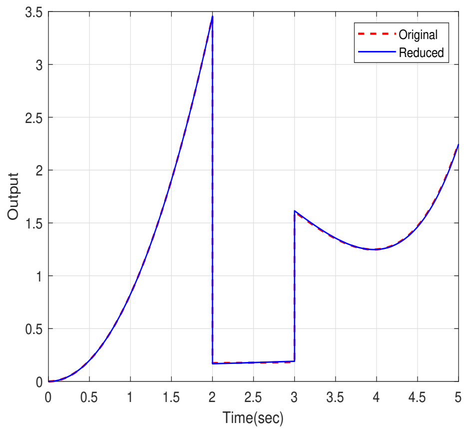

Figure 6.3: Outputs of the original system and the proposed reduced system with truncation threshold 0.1.

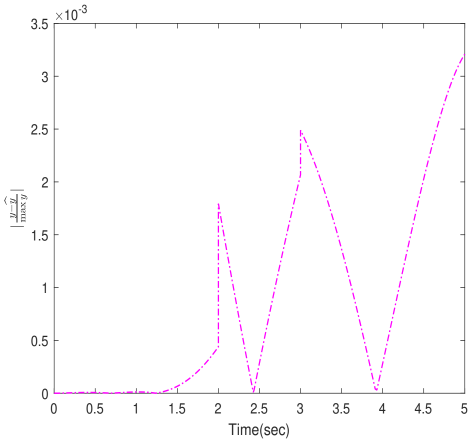

Figure 6.4: Relative errors of the original system and the proposed reduced system with truncation threshold 0.1.

Next, another reduced system is computed by considering the larger threshold 0.25, then the dimensions of the reduced modes will be 2, 1 and 2, respectively. Figure 6.5 shows that the input-output behavior and the related relative errors between the two outputs are depicted in Figure 6.6, which still results in a good approximation (but the relative error is already significantly larger).

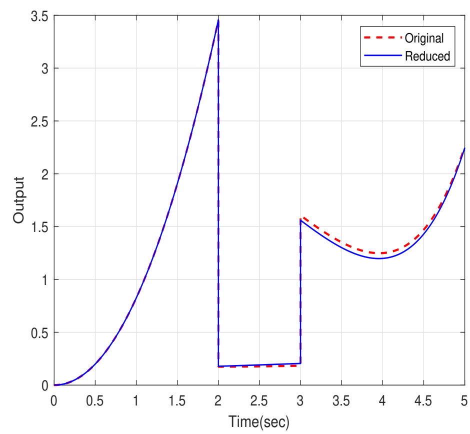

Figure 6.5: Outputs of the original system and the proposed reduced system with truncation threshold 0.25.

Finally, consider another larger threshold 1.5, then the dimension of each reduced mode will be one. The input-output behavior is depicted in Figure 6.7 which does not result in a good approximation especially in the last mode.

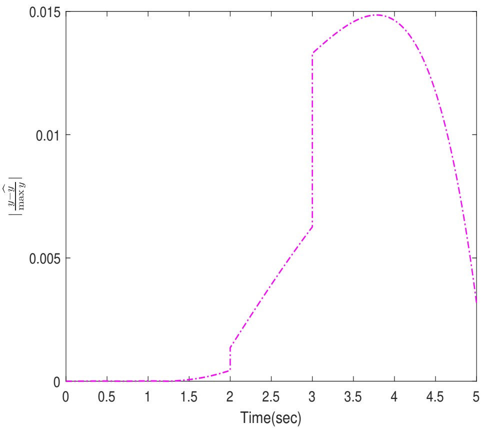

Figure 6.6: Relative errors of the original system and the proposed reduced system with truncation threshold 0.25.

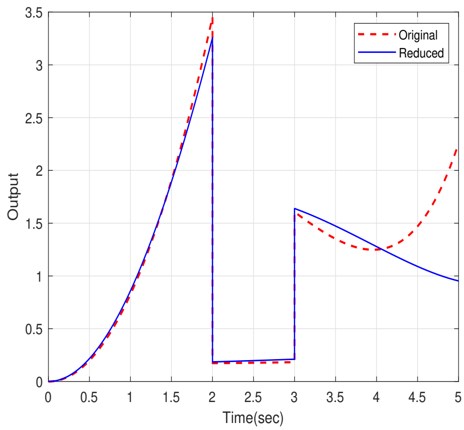

Figure 6.7: Outputs of the original system and the proposed reduced system with truncation threshold 1.5.

The examples show that there is clear relationship between the size of the removed eigenvalues of the balanced Gramians and the error between the output of the original and reduced system. However, it is not clear whether an explicit error bound similar to the classical balanced truncation method can be obtained.

6.6 Discussion

In this chapter, a reduction method is proposed for switched linear systems. First, suitable reachability and observability Gramians are defined such that they provide precise quantitative information about how difficult to reach/observe a state is at a specified time. Based on this information, a mode-wise midpoint balanced truncation method is proposed which results in a reduced switched system whose input-output behavior is similar to the original one. Some numerical issues are discussed and for moderately large sized original systems the proposed method is applicable, while for very large-scale systems (e.g. millions of state variables) the proposed method is not directly applicable and further adjustments are necessary. Furthermore, no error bound is given at the moment, although due to the proven energy interpretation the method results in a very good approximation of the input-output behavior.

7 Model reduction for switched DAEs

7.1 Introduction

As mentioned in the Introduction (in Chapter 1), DAEs naturally occur for modelling and simulation in many applications such as multi-body systems, circuit simulation, control theory, fluid dynamics, and many other areas. Many practical systems nowadays exhibit impulsive dynamical behaviors due to abrupt changes at certain instants during the dynamical processes.

Switched DAEs arise when the system dynamics undergo sudden structural changes and the dynamics of each mode are algebraically constrained. In particular, switched DAEs occur when dynamic behaviors change suddenly by failure or an external (given) switching signal, cf. [66, 75, 110, 141] for details.

Consider the switched DAEs of the form

$$\begin{aligned} E_{\sigma(t)}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad x(t_0) = 0, \\ y(t) &= C_{\sigma(t)}x(t), \end{aligned} \tag{7.1}$$

where $\sigma : \mathbb{R} \to \mathtt{M}$ and, x, u, y denote respectively, the state, input and output. Throughout this chapter, regularity (as forthcoming Definition 7.1) of the matrix pairs (E_i, A_i) will be assumed. The active mode is defined by the quadruple (E_k, A_k, B_k, C_k) in appropriate dimension. If $E_k = I$ for all $k \in \mathtt{M}$, system (7.1) is a switched linear system (studied in the previous Chapters).

In general, E_k is not assumed to be invertible, which means that in addition to differential equations the state x has to satisfy certain algebraic constraints. At a switching instant the algebraic constraints before the switch and the algebraic constraints after the switch do not match in general, i.e., the state variable has to jump in order to satisfy the algebraic constraints after the switch. These induced jumps are a major difference to switched ODE without jumps. Another major difference is the possible presence of Dirac impulses in the state variable x in response to a state jump, see [136] for details.

Clearly, switched DAEs allow jumps and impulsive behaviors in the solution at switching times, moreover, a distributional solution theory for switched DAEs is well investigated in [135]. As was pointed out in this work, switched DAEs are particularly challenging as they give rise to behaviors that are found neither in regular switched systems, nor in LTI DAEs. Despite these difficulties, many properties of such systems are now understood, cf. [43, 70, 71, 77, 112, 129, 131, 134, 136, 146, 147]. To the author's best knowledge, model reduction of such system is first investigated in this chapter.

The chapter is organized as follows. In Section 7.2, some preliminaries for linear DAEs are given. In Section 7.3, it is shown that a switched DAE can be written as a switched ODE with jumps and impulses and it will be proven that both systems have the same input-output behavior. Some conditions are provided here such that the switched ODE has no input dependent jump and no Dirac presents in output. In Section 7.4, the model reduction procedure is discussed. The reduced realization is computed first by removing the unreachable and unobservable states via the method discussed in Chapter 4 and then a reduced system is computed via the method proposed in Chapter 6. Finally, these results are illustrated by means of an example in Section 7.5.

7.2 DAE preliminaries

In this section, some notations and properties are discussed for the (non-)switched DAE

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{7.2}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$.

As discussed in Chapter 1, the algebraic constraints may cause that the solutions of an initial value problem are no longer unique or that there does not exist solutions at all. Furthermore, the inhomogeneity has to be *consistent* with the DAEs in order for solutions to exist. A comprehensive representation of the solution theory of general LTI DAEs along with possible distributional solutions based on the theory is developed in [135], see Appendix B for a concise recapitulation.

7.2.1 Regularity and quasi-Weierstrass form

Regularity of a matrix pair is closely related to the solution behavior of the corresponding DAEs. In particular, regularity is necessary and sufficient for the property that for every sufficiently smooth inhomogeneity the DAE is solvable and the solution is unique for every consistent initial value. To show sufficiency, one can return to the problem of finding an appropriate canonical form, which can be derived on the basis of the Jordan canonical form of a single matrix cf. [68].

Definition 7.1 (Regularity)**.** A matrix pair (E, A) is called *regular* if, and only if, $\det(Es - A) \in \mathbb{R}[s]$ is not the zero polynomial. Furthermore, the DAE $E\dot{x}(t) = Ax(t) + Bu(t)$ and the switched DAE (7.1) will be called regular, if and only if, the matrix pairs (E, A), (E_k, A_k), $k \in \mathbb{N}$ are regular, otherwise they are said to be singular.

Following theorem for *Quasi-Weierstrass Form* (QWF) is an important characterization of regularity.

Theorem 7.2 (Quasi-Weierstrass Form, cf. [18, 41])**.** *A matrix pair* (E, A) *is regular if and only if, there exists invertible matrices* $S, T \in \mathbb{R}^{n\times n}$ *such that*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{7.3}$$

where $N \in \mathbb{R}^{n_N \times n_N}$ *is nilpotent,*[1] $J \in \mathbb{R}^{n_J \times n_J}$ $(n_J := n - n_N)$ *and* I *stands for an identity matrix of appropriate size. The decoupling* (7.3) *is called quasi-Weierstrass form.*

Under the regularity assumption, for every regular matrix pair (E, A) there exists unique subspaces $\mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^n$ with $\mathcal{V} \oplus \mathcal{W} = \mathbb{R}^n$ such that for any choice of full column rank matrices V, W with $\operatorname{im} V = \mathcal{V}$ and $\operatorname{im} W = \mathcal{W}$, the nonsingular matrices $T = [V \ \ W]$ and $S = [EV \ \ AW]^{-1}$ transform into a decoupled DAE according to (7.3) with an ODE part

$$\dot{v} = Jv + B_v u,$$

and a nilpotent DAE part

$$N\dot{w} = w + B_w u.$$

Note that, the first subsystem is called slow subsystem and the second subsystem is known as fast subsystem. The matrices J and N depend on the specific coordinates chosen for $\mathcal{V}$ and $\mathcal{W}$, but in the original coordinates in $\mathbb{R}^n$ these dynamics are of course independent of the coordinates chosen for $\mathcal{V}$ and $\mathcal{W}$ (for details see [29]). In the original coordinates, the dynamics can be described by defining following projector and selectors.

[1] Recall that a matrix $N \in \mathbb{R}^{n\times n}$ is called nilpotent, when there is a $\nu \in \mathbb{N}$ such that $N^\nu = 0$; the smallest number ν with this property is called nilpotency index of N.

Definition 7.3. Consider a regular matrix pair (E, A) and its QWF (7.3) by choosing transformation matrices S, T. Then, the *consistency projector* is

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

the *differential selector* is

$$\Pi^{\text{diff}}_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S,$$

the *impulse selector* is

$$\Pi^{\text{imp}}_{(E,A)} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S.$$

Note that the projectors do not depend on the specific choice of S and T (see [135, Section 4.2.2]). In general, it is obvious that the differential projector and impulse selectors are not projectors because they are not idempotent [2]. Furthermore, the consistency projector is the unique projector onto $\mathcal{V}$ along $\mathcal{W}$. These projector and selectors are crucial for providing an explicit solution formula for DAEs, in particular, the specific choice of QWF is not necessary but it suffices to find the unique subspace decomposition $\mathcal{V} \oplus \mathcal{W} = \mathbb{R}^n$.

7.2.2 QWF via Wong sequences

The two subspaces $\mathcal{V}$ and $\mathcal{W}$ can efficiently be calculated by a direct approach via the so called Wong sequences (first given in [144]) which are defined as follows:

$$\begin{aligned} \mathcal{V}^0 &:= \mathbb{R}^n, & \mathcal{V}^{i+1} &:= A^{-1}(E\mathcal{V}^i), \quad i \in \mathbb{N}, \\ \mathcal{W}^0 &:= \{0\}, & \mathcal{V}^{j+1} &:= E^{-1}(A\mathcal{W}^j), \quad j \in \mathbb{N}, \end{aligned} \tag{7.4}$$

where $M^{-1}(\mathcal{N}) := \{x \in \mathbb{R}^n \mid Mx \in \mathcal{N}\}$ denotes the preimage of a linear subspace $\mathcal{N} \subseteq \mathbb{R}^n$ of a matrix $M \in \mathbb{R}^{n \times n}$. After finitely many steps the sequences in (7.4) converge and the limits are given by

$$\mathcal{V}^* := \bigcap_{i \in \mathbb{N}} \mathcal{V}^i, \quad \mathcal{W}^* := \bigcup_{i \in \mathbb{N}} \mathcal{W}^i.$$

Furthermore, it is easily seen that

$$\ker A \subseteq \mathcal{V}^*, \quad \ker E \subseteq \mathcal{W}^*.$$

Note that the QWF can be formulated by the Wong sequences as follows.

[2] A matrix Π is called idempotent if, and only if, $\Pi^2 = \Pi$.

Theorem 7.4. *(QWF via Wong sequences [135, Thm. 4.2.4]). Consider the regular matrix pair* (E, A) *with corresponding Wong limits* $\mathcal{V}^*$ *and* $\mathcal{W}^*$. *For any full rank matrices* V *and* W *such that* $\operatorname{im} V = \mathcal{V}^*$ *and* $\operatorname{im} W = \mathcal{W}^*$, *the invertible matrices* $T := [V \quad W]$, $S := [EV \quad AW]^{-1}$ *transform* (E, A) *into QWF* (7.3).

7.2.3 Explicit solution formula

Assume that the regular DAEs (7.2) is decoupled into two subsystems

$$\begin{aligned} \dot{v}(t) &= Jv(t) + B_v u(t), \\ N\dot{w}(t) &= w(t) + B_w u(t), \end{aligned} \tag{7.5}$$

where N is nilpotent whose nilpotent index is denoted by ν.

It is clear that the ODE $\dot{v}(t) = Jv(t) + B_v u(t)$ has a unique solution

$$v(t) = e^{Jt}v_0 + \int_0^t e^{J(t-s)} B_v u(s)\, \mathrm{d}s,$$

for any integrable input $u(t)$ and any given initial value v_0.

The nilpotent subsystem $N\dot{w}(t) = w(t) + B_v u(t)$ has a unique solution

$$w(t) = -\sum_{i=0}^{\nu-1} N^i B_w u^{(i)}(t),$$

with nilpotency index $\nu \in \mathbb{N}$ and $u(t)$ is ν-times piecewise continuously differentiable. It is necessary that the input function $u(t)$ is sufficiently smooth and the initial value w_0 satisfies

$$w_0 = -\sum_{i=0}^{\nu-1} N^i B_w u^{(i)}(0).$$

For any scalar $\varepsilon > 0$, the properties of $u(\tau)$, $0 \leqslant \tau \leqslant t - \varepsilon$ have no contribution to $w(t)$. This shows an interesting phenomenon between the substates v and w: $v(t)$ represents a cumulative effect of $u(\tau)$, $0 \leqslant \tau \leqslant t$, with no relation to $u(t)$ while $w(t)$ responses so rapidly that it instantaneously reflects the properties of $u(t)$ at time t. This is why they are called the slow and fast subsystems respectively.

The solution formula discusses above so far, depends on the non-unique system matrices T, J, N, B_v, B_w, and is therefore not coordinate free. In order to find a coordinate free, geometric solution formula define as follows:

$$\begin{aligned}
A^{\mathrm{diff}} &:= \Pi^{\mathrm{diff}}_{(E,A)} A, \quad & B^{\mathrm{diff}} &:= \Pi^{\mathrm{diff}}_{(E,A)} B, \quad & C^{\mathrm{diff}} &:= C\Pi_{(E,A)}, \\
E^{\mathrm{imp}} &:= \Pi^{\mathrm{imp}}_{(E,A)} E, \quad & B^{\mathrm{imp}} &:= \Pi^{\mathrm{imp}}_{(E,A)} B, \quad & C^{\mathrm{imp}} &:= C(I - \Pi_{(E,A)}).
\end{aligned} \tag{7.6}$$

The solution formula based on the projected matrices is formalized as follows.

Theorem 7.5. *(Explicit solution formula [136, Thm. 6.4.4]). Consider the triplet (E, A, B) from* (7.2) *with regular matrix pair (E, A) and corresponding projectors $\Pi_{(E,A)}$, $\Pi^{\mathrm{diff}}_{(E,A)}$ and $\Pi^{\mathrm{imp}}_{(E,A)}$ as Definition 7.3. Calculate A^{diff}, B^{diff}, E^{imp}, B^{imp} as in* (7.6)*, then for a given smooth input u, all solutions of the DAE* (7.2) *satisfy*

$$x(t) = e^{A^{\mathrm{diff}} t}\Pi_{(E,A)} x(0) + \int_0^t e^{A^{\mathrm{diff}}(t-s)} B^{\mathrm{diff}} u(s)\,\mathrm{d}s - \sum_{i=0}^{n-1} (E^{\mathrm{imp}})^i B^{\mathrm{imp}} u^{(i)}(t). \tag{7.7}$$

Remark 7.6. For a regular matrix pair (E, A) with QWF (7.3) and the decomposition $\mathcal{V} \oplus \mathcal{W} = \mathbb{R}^n$, it is seen that

$$A^{\mathrm{diff}} := \Pi^{\mathrm{diff}}_{(E,A)} A = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} SA = T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

$$E^{\mathrm{imp}} := \Pi^{\mathrm{imp}}_{(E,A)} E = T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} SA = T \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} T^{-1}.$$

Moreover, $\operatorname{im} A^{\mathrm{diff}} \subseteq \mathcal{V} = \operatorname{im} \Pi_{(E,A)} = \operatorname{im} \Pi^{\mathrm{diff}}_{(E,A)}$ and $\operatorname{im} E^{\mathrm{imp}} \subseteq \mathcal{W} = \operatorname{im} \Pi^{\mathrm{imp}}_{(E,A)}$. Consequently, the solution x given by (7.7) can be decomposed as

$$x = x^{\mathrm{diff}} \oplus x^{\mathrm{imp}},$$

with $x^{\mathrm{diff}}(t) \in \mathcal{V}$ and $x^{\mathrm{imp}}(t) \in \mathcal{W}$ for all $t \in \mathbb{R}$, and they are the unique solutions of

$$\begin{aligned}
\dot{x}^{\mathrm{diff}} &= A^{\mathrm{diff}} x^{\mathrm{diff}} + B^{\mathrm{diff}} u, \quad x^{\mathrm{diff}}(0) = \Pi_{(E,A)} x(0), \\
E^{\mathrm{imp}} \dot{x}^{\mathrm{imp}} &= x^{\mathrm{imp}} + B^{\mathrm{imp}} u.
\end{aligned}$$

Furthermore, the following equivalence holds

$$E\dot{x} = Ax \iff \dot{x} = A^{\mathrm{diff}} x \text{ and } x(0) \in \operatorname{im} \Pi_{(E,A)}.$$

Finally, observe that

$$\begin{aligned} \operatorname{im} A^{\text{diff}} &\subseteq \mathcal{V}^*, & \operatorname{im} B^{\text{diff}} &\subseteq \mathcal{V}^*, \\ \operatorname{im} E^{\text{imp}} &\subseteq \mathcal{W}^*, & \operatorname{im} B^{\text{imp}} &\subseteq \mathcal{W}^*. \end{aligned}$$

An important feature for DAEs is the so called consistency space, defined as follows.

Definition 7.7. The consistency space is defined by

$$\mathcal{V}_{(E,A)} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ smooth solution } (x,u) \text{ of} \\ E\dot{x} = Ax \text{ and } x(t_0^-) = x_0 \end{array} \right\}.$$

Definition 7.8. The augmented consistency space is defined by

$$\mathcal{V}_{(E,A,B)} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ smooth solution } (x,u) \text{ of} \\ E\dot{x} = Ax + Bu \text{ and } x(t_0^-) = x_0 \end{array} \right\}.$$

Lemma 7.9. *([17, Corollary 4.5]). Consider the regular DAE* (7.2) *then,*

$$\mathcal{V}_{(E,A)} = \mathcal{V}^* = \operatorname{im} \Pi_{(E,A)} \text{ and } \mathcal{V}_{(E,A,B)} = \mathcal{V}^* \oplus \langle E^{\text{imp}} \mid \operatorname{im} B^{\text{imp}} \rangle.$$

7.2.4 Initial trajectory problems

The inconsistent initial value can only occur, when it is assumed that the DAE (7.2) does not hold in the past but that the past trajectory was governed by something else or given. It also arises in the context of switched DAEs. In fact, for given input u and the initial value at some $t_0 \in \mathbb{R}$, the solution satisfies

$$x(t_0) = \Pi_{(E,A)} x(t_0) - \underbrace{\sum_{i=0}^{n-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(t)}_{=:-x_{\mathrm{u},t_0}},$$

then $x(t_0)$ is restricted to the affine subspace

$$\operatorname{im} \Pi_{(E,A)} + x_{\mathrm{u},t_0}. \tag{7.8}$$

For switched DAEs (7.1), these mode-dependent affine subspaces are not equal. In that case, the study of inconsistent initial values come into play. To understand the problem, an *initial trajectory problem* (ITP) corresponding to (E, A, B) is constructed by

$$\begin{aligned} x_{(-\infty,0)} &= x^0_{(-\infty,0)}, \\ (E\dot{x})_{[0,\infty)} &= (Ax + Bu)_{[0,\infty)}, \end{aligned} \tag{7.9}$$

where x^0 is some given (past) trajectory. In general, the trajectory x^0 will not evolve within the affine subspace (7.8), while the solution x will have to evolve in the subspace (7.8) for positive times.

Consequently, any solution x will exhibit a jump in response to an inconsistent initial value, but the derivative of this jump appears as a Dirac impulse in the expression $E\dot{x}$. The restriction to the open interval $(0,\infty)$ would neglect this Dirac impulse, the restriction to the closed interval $[0,\infty)$ keeps the Dirac impulse in the second equation of (7.9) and hence, the past trajectory can influence the future. Furthermore, distributional solutions are necessary for which a restriction to intervals is well defined. The space of piecewise-smooth distribution $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ (see Appendix B) satisfies these requirements and it is possible to prove the following existence and uniqueness result for the ITP (7.9) embedded into this special distributional solution space.

Theorem 7.10. *(Existence and uniqueness of ITP solutions, [136, Thm. 6.5.1]). Consider the ITP* (7.9) *with regular matrix* (E,A)*. Then, for all initial trajectories* $x^0 \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ *and all inputs* $u \in \mathbb{D}^m_{\mathrm{pw}\mathcal{C}^\infty}$*, there exist a unique* $x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ *which satisfies the ITP* (7.9).

7.3 Equivalent solution for switched DAEs

The main objective of this section is to find a switched ODE whose solution represents the solution of switched DAEs.

7.3.1 Switched ODEs with jumps and impulses

The switched DAEs (7.1) can be expressed as the switched ODE with jumps and Dirac impulses of the form

$$\boxed{\begin{aligned} \dot{z} &= A_k^{\mathrm{diff}} z + B_k^{\mathrm{diff}} u, \quad \text{on } (s_k, s_{k+1}), \\ z(s_k^+) &= \Pi_k(z(s_k^-) + U_k^-), \quad z(t_0^-) = 0, \\ w &= C_k^{\mathrm{diff}} z + D_k U, \quad \text{on } (s_k, s_{k+1}), \end{aligned}} \tag{7.10a}$$

$$\boxed{w[s_k] = -\sum_{i=0}^{n-1} C_k (E_k^{\mathrm{imp}})^{i+1} z(s_k^-)\delta_{s_k}^{(i)} + U_k^{\mathrm{imp}},} \tag{7.10b}$$

where

$$U_k^- := \sum_{j=0}^{n-1} (E_{k-1}^{\mathrm{imp}})^j B_{k-1}^{\mathrm{imp}} u^{(j)}(s_k^-),$$

$$D_k := -C_k[E_k^{\mathrm{imp}} B_k^{\mathrm{imp}}, \dots, (E_k^{\mathrm{imp}})^{(n-1)} B_k^{\mathrm{imp}}],$$

$$U := \left[u^\top, \dot{u}^\top, \dots, u^{(n-1)^\top}\right]^\top,$$

$$U_k^{\mathrm{imp}} := \sum_{i=1}^{n-1} C_k (E_k^{\mathrm{imp}})^i \left(U_k^- \delta_{s_k}^{(i)} - B_k^{\mathrm{imp}} (u_k)^{(i)}[s_k]\right),$$

with $u_k := u_{[s_k, s_{k+1})}$.

In the following, it will be seen that the switched ODE (7.10) with jumps and impulses gives a solution formula for the switched DAE (7.1).

Theorem 7.11 (Connection between (7.1) and (7.10))**.** *The switched DAE* (7.1) *with regular matrix pair* (E_k, A_k) *has the same input-output behavior as the switched ODE* (7.10) *with jumps and output-impulses.*

Proof. To simplify the notation, assume in this proof that the input u is smooth on each interval (s_k, s_{k+1}), however, the following arguments remain valid even for the general case that u is a piecewise-smooth distribution.

Let x denote the solution of the switched DAEs (7.1) and z the solution of the ODE (7.10) for the same input u. The proof can be divided into three steps as follows.

Step 1: It is first shown that $z = \Pi_k x$ on (s_k, s_{k+1}) where Π_k is the consistency projector for mode k. From Theorem 7.5, the solution of switched DAEs (7.1) is given by, for $t \in (s_k, s_{k+1})$

$$x(t) = e^{A_k^{\mathrm{diff}}(t-s_k)} \Pi_k x(s_k^-) + \int_{s_k}^t e^{A_k^{\mathrm{diff}}(t-\tau)} B_k^{\mathrm{diff}} u(\tau)\, \mathrm{d}\tau - \sum_{i=0}^{n-1} (E_k^{\mathrm{imp}})^i B_k^{\mathrm{imp}} u^{(i)}(t), \tag{7.11}$$

and it is easily seen that $(I - \Pi_k)x(t) = -\sum_{i=0}^{n-1} (E_k^{\mathrm{imp}})^i B_k^{\mathrm{imp}} u^{(i)}(t)$.

For $k = 0$ (on (t_0, s_1)), the solution is given by, for $t \in (t_0, s_1)$

$$\begin{aligned} x(t) &= \int_{t_0}^t e^{A_0^{\mathrm{diff}}(t-\tau)} B_0^{\mathrm{diff}} u(\tau)\, \mathrm{d}\tau - \sum_{i=0}^{n-1} (E_0^{\mathrm{imp}})^i B_0^{\mathrm{imp}} u^{(i)}(t) \\ &= z(t) + (I - \Pi_0) x(t). \end{aligned}$$

This implies that $z(t) = \Pi_0 x(t)$. Inductively, assume that for $t \in (s_k, s_{k+1})$, $z(t) = \Pi_k x(t)$ and $z(s_{k+1}^-) = \Pi_k x(s_{k+1}^-)$. Now for $t \in (s_{k+1}, s_{k+2})$,

$$\begin{aligned} x(t) = e^{A_{k+1}^{\mathrm{diff}}(t-s_{k+1})} \Pi_{k+1} x(s_{k+1}^-) &+ \int_{s_{k+1}}^t e^{A_{k+1}^{\mathrm{diff}}(t-\tau)} B_{k+1}^{\mathrm{diff}} u(\tau)\, \mathrm{d}\tau \\ &- \sum_{i=0}^{n-1} (E_{k+1}^{\mathrm{imp}})^i B_{k+1}^{\mathrm{imp}} u^{(i)}(t), \end{aligned}$$

and for (7.10), it follows that

$$z(t) = e^{A_{k+1}^{\text{diff}}(t-s_{k+1})} z(s_{k+1}^+) + \int_{s_{k+1}}^t e^{A_{k+1}^{\text{diff}}(t-\tau)} B_{k+1}^{\text{diff}} u(\tau)\, \mathrm{d}\tau.$$

Furthermore,

$$\Pi_{k+1} x(t) = e^{A_{k+1}^{\text{diff}}(t-s_{k+1})} \Pi_{k+1} x(s_{k+1}^-) + \int_{s_{k+1}}^t e^{A_{k+1}^{\text{diff}}(t-\tau)} B_{k+1}^{\text{diff}} u(\tau)\, \mathrm{d}\tau,$$

and from $U_{k+1}^- = (I - \Pi_k) x(s_{k+1}^-)$, it follows that

$$z(s_{k+1}^+) = \Pi_{k+1}(z(s_{k+1}^-) + U_{k+1}^-) = \Pi_{k+1} x(s_{k+1}^-).$$

Consequently, it follows that $z(t) = \Pi_{k+1} x(t)$. Therefore, the statement is true for $k+1$ and hence, for all $k \in \{0, 1, \ldots, \mathtt{m}\}$,

$$z(t) = \Pi_k x(t),\ t \in (s_k, s_{k+1}).$$

Step 2: It is shown that the corresponding outputs on (s_k, s_{k+1}) are equal. The output equation on (s_k, s_{k+1}) is given by, $t \in (s_k, s_{k+1})$

$$\begin{aligned}
y(t) &= C_k x(t) \\
&= C_k(\Pi_k \Pi_k + I - \Pi_k) x(t) \\
&= C_k^{\text{diff}} z(t) - C_k \sum_{i=1}^{n-1} (E_k^{\text{imp}})^i B_k^{\text{imp}} u^{(i)}(t) \\
&= w(t).
\end{aligned}$$

Step 3: Finally, it is shown that the impulse parts of the outputs are equal. The impulse solution with Dirac impulses and jumps at the switching times $t = s_k$ is given by (see Appendix B.8):

$$x[s_k] = -\sum_{i=0}^{n-1} (E_k^{\text{imp}})^{i+1} x(s_k^-) \delta_{s_k}^{(i)} - \sum_{i=1}^{n-1} (E_k^{\text{imp}})^i B_k^{\text{imp}} (u_k)^{(i)}[s_k],$$

where $u_k := u_{[s_k, s_{k+1})}$. Hence, the impulsive output is given by

$$\begin{aligned}
y[s_k] &= C_k x[s_k] \\
&= -\sum_{i=0}^{n-1} C_k (E_k^{\text{imp}})^{i+1} x(s_k^-) \delta_{s_k}^{(i)} - \sum_{i=1}^{n-1} C_k (E_k^{\text{imp}})^i B_k^{\text{imp}} (u_k)^{(i)}[s_k]
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i=0}^{n-1} C_k (E_k^{\mathrm{imp}})^{i+1} \left(z(s_k^-) - \sum_{j=0}^{n-1} (E_{k-1}^{\mathrm{imp}})^j B_{k-1}^{\mathrm{imp}} u^{(j)}(s_k{}^-) \right) \delta_{s_k}^{(i)} - \\
&\qquad\qquad\qquad\qquad \sum_{i=1}^{n-1} C_k (E_k^{\mathrm{imp}})^i B_k^{\mathrm{imp}} (u_k)^{(i)}[s_k] \\
&= -\sum_{i=0}^{n-1} C_k (E_k^{\mathrm{imp}})^{i+1} z(s_k^-) \delta_{s_k}^{(i)} + \sum_{i=1}^{n-1} C_k (E_k^{\mathrm{imp}})^i \left(U_k^- \delta_{s_k}^{(i)} - B_k^{\mathrm{imp}} (u_k)^{(i)}[s_k] \right) \\
&= w[s_k],
\end{aligned}
$$

where $U_k^- := \sum_{j=0}^{n-1} (E_{k-1}^{\mathrm{imp}})^j B_{k-1}^{\mathrm{imp}} u^{(j)}(s_k^-)$.

This completes the proof. □

As mentioned earlier, a method is proposed for reduced realization of switched ODEs with jumps in Chapter 4, and Chapter 6 discusses model reduction method for the reduced realization of switched ODE with jumps. In this chapter, it will be shown how the results of previous chapters can be used and then it is important to highlight the main differences between the switched ODE (7.10) with jumps and impulses, and switched ODE (2.4) with jumps. These differences can be distinguished into three cases.

Case I: Additional input dependent jump ($U_k^- \neq 0$) may occur in (7.10a), e.g., for mode k

$$z(s_k^+) = \Pi_k (z(s_k^-) + U_k^-).$$

Case II: Input dependent feedthrough term $D_k U \neq 0$ can appear in the output of (7.10a) as

$$w = C_k^{\mathrm{diff}} z + D_k U.$$

Case III: Additional Dirac impulse in the output (7.10b) induced by an inconsistent state before the switch as well as by the input

$$w[s_k] = -\sum_{i=0}^{n-1} C_k (E_k^{\mathrm{imp}})^{i+1} z(s_k^-) \delta_{s_k}^{(i)} + U_k^{\mathrm{imp}}.$$

In the context of reduced realization and model reduction (discussed in Chapters 4 and 6, respectively), these differences can be handled under some assumptions which will be discussed as follows.

7.3.2 Solution for input free jump

From equation (7.10a), it is visible that an additional input dependent term

$$\Pi_k U_k^- = \Pi_k \sum_{j=0}^{n-1} (E_{k-1}^{\text{imp}})^j B_{k-1}^{\text{imp}} u^{(j)}(s_k^-),$$

appears in the state jumps which is a significant difference to switched ODE with jumps. Currently, it is not possible to handle this input dependent term in jumps. Therefore, the following result is important for further analysis.

Theorem 7.12. *The switched system* (7.10a) *has no input dependent jump if, and only if,*

$$\Pi_k (E_{k-1}^{\text{imp}})^j B_{k-1}^{\text{imp}} = 0, \ \forall j \geqslant 0.$$

In particular, a sufficient condition for an input independent jump rule of (7.10a) *is*

$$\Pi_k \Pi_{k-1}^{\text{imp}} = 0, \forall k \in \{1, \dots, m\}.$$

Proof. The solution of (7.10a) has an input independent jump rule if, and only if, $\Pi_k U_k^- = 0$, i.e.

$$\Pi_k \sum_{j=0}^{n-1} (E_{k-1}^{\text{imp}})^j B_{k-1}^{\text{imp}} u^{(j)}(s_k^-) = 0,$$

which is true if, and only if,

$$\Pi_k (E_{k-1}^{\text{imp}})^j B_{k-1}^{\text{imp}} = 0, \ \forall j \geqslant 0. \tag{7.12}$$

Furthermore, since $E_{k-1}^{\text{imp}} := \Pi_{k-1}^{\text{imp}} E_{k-1}$ and $B_{k-1}^{\text{imp}} := \Pi_{k-1}^{\text{imp}} B_{k-1}$, the condition (7.12) is satisfied if

$$\Pi_k \Pi_{k-1}^{\text{imp}} = 0.$$

□

7.3.3 State dependent Dirac impulse free output

The impulse output from (7.10b) is given by

$$w[s_k] = -\sum_{i=0}^{n-1} C_k (E_k^{\text{imp}})^{i+1} z(s_k^-) \delta_{s_k}^{(i)} + U_k^{\text{imp}}. \tag{7.13}$$

The presence of Dirac impulses in response to inconsistency initial values reduces the unobservable subspaces, hence the methods used in previous chapters cannot be directly used. Therefore, following result gives a necessary condition for impulse free output (there could still be impulse coming from U_k^{imp}, but these do not influence the observability properties at all).

Theorem 7.13. *The output of system* (7.10) *has no Dirac impulse induced by inconsistent initial values if the following sufficient condition holds*

$$C_k(E_k^{\text{imp}})^i\overline{\mathcal{R}}_{k-1} = \{0\},\ i = 1, 2, \cdots .$$

Here, $\overline{\mathcal{R}}_{k-1}$ *is the extended reachable subspace (introduced in Section 4.2.4) for mode* $k-1$ *of the switched system* (7.10a).

Proof. It is clear that the output equation (7.13) will not contain Dirac impulses induced by an inconsistent value $z(s_k^-)$ if

$$C_k(E_k^{\text{imp}})^{i+1}z(s_k^-) = 0,\ i = 0, 1, 2, \cdots .$$

From Chapter 4, $z(s_k^-) \in \mathcal{R}_{[t_0,s_k)} \subseteq \overline{\mathcal{R}}_{k-1}$,

$$C_k(E_k^{\text{imp}})^{i+1}\overline{\mathcal{R}}_{k-1} = \{0\}, \quad i = 0, 1, 2, \cdots ,$$

which gives a sufficient condition. □

It is noted that for the case when $C_k(E_k^{\text{imp}})^i\overline{\mathcal{R}}_{k-1} \neq \{0\}$, $i = 1, 2, \ldots,$ the unobservable subspaces for the mode k need to be adjusted. Anyway, this chapter will not pursue this case.

7.4 Model reduction

In this section, a model reduction method will be discussed for the switched DAE (7.1). It is well known that the additional feedthrough term (which does not depend on state variable) in the output equation does not affect the analysis of realization theory and model reduction techniques. Furthermore, the additional impulsive output depending on the input only is also not effecting the reduced realization and model reduction.

The overall procedure can be summarized in three main steps as follows:

Step 1: Under the assumptions in Theorems 7.12 and 7.13, a switched DAE of the form (7.1) or, switched ODE as in (7.10) is transformed to following switched ODE with jumps and impulses:

$$\begin{aligned} \dot{z} &= A_k^{\text{diff}}z + B_k^{\text{diff}}u, \text{ on } (s_k, s_{k+1}), \\ z(s_k^+) &= \Pi_k z(s_k^-), \quad z(t_0^-) = 0, \\ w &= C_k^{\text{diff}}z + D_kU, \quad \text{ on } (s_k, s_{k+1}), \end{aligned} \tag{7.14}$$

together with the additional impulsive output

$$w[s_k] = U_k^{\mathrm{imp}}. \tag{7.15}$$

Step 2: From the system (7.14), the unreachable and unobservable states can be removed via the method proposed in Chapter 4 with the corresponding left- and right-projectors $(\widehat{W}_k^2, \widehat{V}_k^2)$, $k = \{0, 1, \dots, \mathtt{m}\}$. Then, the reduced realization is given by

$$\begin{aligned} \dot{\widehat{z}} &= \widehat{A}_k^{\mathrm{diff}} \widehat{z} + \widehat{B}_k^{\mathrm{diff}} u, \quad \text{on } (s_k, s_{k+1}), \\ \widehat{z}(s_k^+) &= \mathcal{J}_k \widehat{z}(s_k^-), \quad \widehat{z}(t_0^-) = 0, \\ w &= \widehat{C}_k^{\mathrm{diff}} \widehat{z} + D_k U, \quad \text{on } (s_k, s_{k+1}), \end{aligned} \tag{7.16}$$

with unaltered impulsive output (7.15), and $\widehat{A}_k^{\mathrm{diff}} = \widehat{W}_k^2 \, A_k^{\mathrm{diff}} \, \widehat{V}_k^2$, $\widehat{B}_k^{\mathrm{diff}} = \widehat{W}_k^2 \, B_k^{\mathrm{diff}}$, $\widehat{C}_k^{\mathrm{diff}} = C_k^{\mathrm{diff}} \, \widehat{V}_k^2$, and $\mathcal{J}_k = \widehat{W}_k^2 \Pi_k \widehat{V}_{k-1}^2$. Then, the system (7.16) has the same input-output behavior as (7.10) and (7.14).

Step 3: Finally, under the assumption that the midpoint Gramians of the system (7.16) are nonsingular, a reduced model can be obtained from system (7.16) via the algorithm given in Chapter 6. Let the corresponding left- and right-projectors be given by $(\overline{\Pi}_k^l, \overline{\Pi}_k^r)$, $k = \{0, 1, \dots, \mathtt{m}\}$. Then, the reduced system is

$$\begin{aligned} \dot{h} &= \widehat{\mathcal{A}}_k^{\mathrm{diff}} h + \widehat{\mathcal{B}}_k^{\mathrm{diff}} u, \quad \text{on } (s_k, s_{k+1}), \\ h(s_k^+) &= \widehat{\mathcal{J}}_k h(s_k^-), \quad h(t_0^-) = 0, \\ \widehat{w} &= \widehat{\mathcal{C}}_k^{\mathrm{diff}} h + D_k U, \quad \text{on } (s_k, s_{k+1}), \end{aligned} \tag{7.17}$$

where $\widehat{\mathcal{A}}_k^{\mathrm{diff}} = \overline{\Pi}_k^l \, \widehat{A}_k^{\mathrm{diff}} \, \overline{\Pi}_k^r$, $\widehat{\mathcal{B}}_k^{\mathrm{diff}} = \overline{\Pi}_k^l \, \widehat{B}_k^{\mathrm{diff}}$, $\widehat{\mathcal{C}}_k^{\mathrm{diff}} = \widehat{C}_k^{\mathrm{diff}} \, \overline{\Pi}_k^r$ and $\widehat{\mathcal{J}}_k = \overline{\Pi}_k^l \, \mathcal{J}_k \, \overline{\Pi}_{k-1}^r$ and with unaltered impulsive output (7.15).

7.5 Numerical results

This section demonstrates the steps of the proposed reduction techniques for a switched DAE. The proposed method is illustrated by a numerical example.

Example 7.1. Consider a switched DAE of the form (7.1) with two modes

$$(E_0, A_0, B_0, C_0) = \left(\begin{bmatrix} 1&1&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0&0&0&0 \\ -1&1&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&-1&0&0&0 \\ 0&0&0&0&0&-1 \\ 0&0&0&0&1&0 \end{bmatrix}, \begin{bmatrix} 0.7 \\ 0.2 \\ -0.01 \\ 0.1 \\ 0.1 \\ 0.2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0.2 \\ 1 \\ 0.02 \\ -0.1 \\ 0.2 \end{bmatrix}^\top \right),$$

$$(E_1, A_1, B_1, C_1) = \left(\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} -1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&-1\\ 0&0&0&0&1&0 \end{bmatrix}, \begin{bmatrix} 0\\ 0.1\\ 0.02\\ 0.01\\ -1\\ 0.1 \end{bmatrix}, \begin{bmatrix} 0\\ 0.05\\ 0.01\\ 0.2\\ 1\\ -2 \end{bmatrix}^\top \right),$$

and the switching signal is given by $\sigma(t) = 0$ on $[0, 2)$, and $\sigma(t) = 1$ on $[2, 4)$.

After some matrix calculation, the matrices for QWF (7.3) of the given switched DAE are obtained by

$$(S_0, T_0, J_0, N_0) = \left(\begin{bmatrix} 1&-1&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&-1&0 \end{bmatrix}, \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}, \begin{bmatrix} 2&-1&0&0\\ -1&1&0&0\\ 0&0&0&1\\ 0&0&-1&0 \end{bmatrix}, \begin{bmatrix} 0&1\\ 0&0 \end{bmatrix} \right),$$

$$(S_1, T_1, J_1, N_1) = \left(\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&-1&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&-1&0 \end{bmatrix}, \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}, \begin{bmatrix} -1&0&0\\ 0&0&1\\ 0&-1&-1 \end{bmatrix}, \begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&0&0 \end{bmatrix} \right).$$

Clearly, the 1st mode has index-2 because, $N_0 \neq 0$, $N_0^2 = 0$, and the 2nd mode has index-1 because $N_1 = 0$. All three projectors and selectors (7.6) are given by

$$(\Pi_0, \Pi_0^{\text{diff}}, \Pi_0^{\text{imp}}) = \left(\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&-1&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&-1&0 \end{bmatrix} \right),$$

$$(\Pi_1, \Pi_1^{\text{diff}}, \Pi_1^{\text{imp}}) = \left(\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&-1&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&-1&0 \end{bmatrix} \right),$$

and then, the projected matrices are given by

$$(E_0^{\text{imp}}, A_0^{\text{diff}}, B_0^{\text{diff}}, B_0^{\text{imp}}, C_0^{\text{diff}}, C_0^{\text{imp}}) =$$
$$\left(\begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0 \end{bmatrix}, \begin{bmatrix} 2&-1&0&0&0&0\\ -1&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix} \begin{bmatrix} 0.5\\ 0.2\\ -0.01\\ 0.1\\ 0\\ 0 \end{bmatrix}, \begin{bmatrix} 0\\ 0\\ 0\\ 0\\ 0.2\\ -0.1 \end{bmatrix}, \begin{bmatrix} 1\\ 0.2\\ 1\\ 0.02\\ 0\\ 0 \end{bmatrix}^\top, \begin{bmatrix} 0\\ 0\\ 0\\ 0\\ -0.1\\ 0.2 \end{bmatrix}^\top \right),$$

$$(E_1^{\text{imp}}, A_1^{\text{diff}}, B_1^{\text{diff}}, B_1^{\text{imp}}, C_1^{\text{diff}}, C_1^{\text{imp}}) =$$
$$\left(\begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} -1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&-1&-1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0\\ 0.1\\ -0.08\\ 0\\ 0\\ 0 \end{bmatrix}, \begin{bmatrix} 0\\ 0\\ 0\\ 0.01\\ 0.1\\ 1 \end{bmatrix}, \begin{bmatrix} 0\\ 0.05\\ 0.01\\ 0\\ 0\\ 0 \end{bmatrix}^\top, \begin{bmatrix} 0\\ 0\\ 0\\ 0.2\\ 1\\ -2 \end{bmatrix}^\top \right).$$

Step 1: Clearly, $\Pi_1 \Pi_0^{\text{imp}} = 0$ (sufficient condition), $\Pi_1 E_0^{\text{imp}} B_0^{\text{imp}} = 0$ and $C_1 E_1^{\text{imp}} = 0$. From Theorem 7.12 and 7.13, the switched ODE as in (7.14) which is input-output equivalent to the given switched DAE, is constructed by

$$\begin{aligned} &\left(A_0^{\text{diff}}, B_0^{\text{diff}}, C_0^{\text{diff}}\right), \\ &\left(A_1^{\text{diff}}, B_1^{\text{diff}}, C_1^{\text{diff}}\right), \end{aligned} \tag{7.18}$$

with the jump matrix Π_1 and feedthrough terms D_0U and D_1U are calculated from the impulsive matrices $(C_i^{\text{imp}}, E_i^{\text{imp}}, B_i^{\text{imp}})$, $i \in \{0, 1\}$ and the derivatives of the input. Clearly, the additional Dirac impulse term $U_1^{\text{imp}} = 0$.
Step 2: From the method discussed in Chapter 4 for reduced realization, the pair of left- and right-projections is obtained by

$$(\widehat{W}_0^2, \widehat{V}_0^2) = \left(\begin{bmatrix} 0 & 0 & 0.5 & 0 \\ 0 & 0 & -1 & -1 \\ -1 & 1 & 0 & -1 \\ 1 & -2 & -0.5 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^\top, \begin{bmatrix} 1 & -1 & 2 & -2 \\ 1 & 0 & 0 & -1 \\ -2 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right),$$

$$(\widehat{W}_1^2, \widehat{V}_1^2) = \left(\begin{bmatrix} 0 & 0 \\ 0.5 & -1 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^\top, \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ -2 & -1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right).$$

Then, the reduced realization is given by

$$\begin{aligned} (\widehat{A}_0^{\text{diff}}, \widehat{B}_0^{\text{diff}}, \widehat{C}_0^{\text{diff}}) &= (\widehat{W}_0^2\ A_0^{\text{diff}}\ \widehat{V}_0^2, \widehat{W}_0^2\ B_0^{\text{diff}}, C_0^{\text{diff}}\ \widehat{V}_0^2) \\ &= \left(\begin{bmatrix} 3 & 2 & 0 & 0 \\ -5 & -3 & 0 & 0 \\ -0.5 & -2.5 & 4 & -2.5 \\ 3 & 1 & 2 & -1 \end{bmatrix}, \begin{bmatrix} 0.11 \\ -0.21 \\ 0 \\ -0.09 \end{bmatrix}, \begin{bmatrix} -0.82 \\ -2.02 \\ 2 \\ -2.2 \end{bmatrix}^\top \right), \\ (\widehat{A}_1^{\text{diff}}, \widehat{B}_1^{\text{diff}}, \widehat{C}_1^{\text{diff}}) &= (\widehat{W}_1^2\ A_1^{\text{diff}}\ \widehat{V}_1^2, \widehat{W}_1^2\ B_1^{\text{diff}}, C_1^{\text{diff}}\ \widehat{V}_1^2) \\ &= \left(\begin{bmatrix} -1 & -0.5 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0.05 \\ -0.02 \end{bmatrix}, \begin{bmatrix} 0.08 \\ -0.01 \end{bmatrix}^\top \right), \end{aligned}$$

and the jump matrix is calculated as

$$\mathcal{J}_1 = \overline{W}_1^2 \Pi_1 \overline{V}_0^2 = \begin{bmatrix} 0.5 & 0 & 0 & -0.5 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

Figure 7.1 shows the output of the original system and the proposed reduced realization for input $u(t) = \sin(t)$ with switching time $s_1 = 2$ and clearly both outputs coincide.

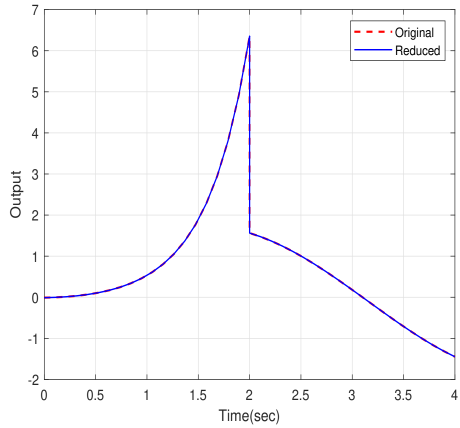

Figure 7.1: Outputs of the original system and the reduced realization.

Step 3: Consider the reduced realization

$$\left(\widehat{A}_0^{\text{diff}}, \widehat{B}_0^{\text{diff}}, \widehat{C}_0^{\text{diff}}\right),$$
$$\left(\widehat{A}_1^{\text{diff}}, \widehat{B}_1^{\text{diff}}, \widehat{C}_1^{\text{diff}}\right),$$

and the jump matrix is given by $\mathcal{J}_1$. Now the model reduction method via Chapter 6 is applied. Following Algorithm 2, the midpoint Gramians are given by

$$P_0 = \begin{bmatrix} 0.0040 & -0.0090 & 0.0206 & 0.0037 \\ -0.0090 & 0.0219 & -0.1085 & -0.0352 \\ 0.0206 & -0.1085 & 2.3316 & 0.9919 \\ 0.0037 & -0.0352 & 0.9919 & 0.4298 \end{bmatrix},$$
$$P_1 = \begin{bmatrix} 18.8692 & -7.0257 \\ -7.0257 & 2.6184 \end{bmatrix},$$
$$Q_0 = \begin{bmatrix} 1.1030 & -3.3867 & 6.0882 & -4.5096 \\ -3.3867 & 28.8824 & -44.7221 & 35.9538 \\ 6.0882 & -44.7221 & 70.2726 & -56.0252 \\ -4.5096 & 35.9538 & -56.0252 & 44.8814 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 0.0018 & -0.0007 \\ -0.0007 & 0.0005 \end{bmatrix}.$$

The corresponding balanced Gramians are obtained by

$$\Xi_0 = \begin{bmatrix} 8.9481 & 0 & 0 & 0 \\ 0 & 0.1168 & 0 & 0 \\ 0 & 0 & 0.0015 & 0 \\ 0 & 0 & 0 & 10^{-7} \end{bmatrix},$$
$$\Xi_1 = \begin{bmatrix} 0.2136 & 0 \\ 0 & 0.0007 \end{bmatrix}.$$

With a truncation threshold of 0.1, the dimension of the reduced modes will be 2 and 1, respectively. Then, the left- and right-projectors are obtained by

$$(\overline{\Pi}_0^l, \overline{\Pi}_0^r) = \left(\begin{bmatrix} -0.2496 & 2.1017 \\ 1.7767 & 2.3227 \\ -2.8013 & -0.6869 \\ 2.2290 & 1.9020 \end{bmatrix}^\top, \begin{bmatrix} -0.0074 & -0.1669 \\ 0.0298 & 0.3396 \\ -0.5050 & 0.6531 \\ -0.2106 & 0.5313 \end{bmatrix} \right),$$
$$(\overline{\Pi}_1^l, \overline{\Pi}_1^r) = \left(\begin{bmatrix} -0.0920 \\ 0.0386 \end{bmatrix}^\top, \begin{bmatrix} -9.3980 \\ 3.4997 \end{bmatrix} \right).$$

Applying the left- and right-projectors according to Algorithm 1, the reduced switched system is given by

$$\begin{aligned}
(\widehat{\mathcal{A}}_0^{\text{diff}}, \widehat{\mathcal{B}}_0^{\text{diff}}, \widehat{\mathcal{C}}_0^{\text{diff}}) &= \left(\overline{\Pi}_0^l\, \widehat{A}_0^{\text{diff}}\, \overline{\Pi}_0^r, \overline{\Pi}_0^l\, \widehat{B}_0^{\text{diff}}, \widehat{C}_0^{\text{diff}}\, \overline{\Pi}_0^r \right) \\
&= \left(\begin{bmatrix} 2.5147 & -0.4561 \\ -0.4745 & 0.7582 \end{bmatrix}, \begin{bmatrix} -0.6012 \\ -0.4277 \end{bmatrix}, \begin{bmatrix} -0.6008 \\ -0.4120 \end{bmatrix}^\top \right), \\
(\widehat{\mathcal{A}}_1^{\text{diff}}, \widehat{\mathcal{B}}_1^{\text{diff}}, \widehat{\mathcal{C}}_1^{\text{diff}}) &= \left(\overline{\Pi}_1^l\, \widehat{A}_1^{\text{diff}}\, \overline{\Pi}_1^r, \overline{\Pi}_1^l\, \widehat{B}_1^{\text{diff}}, \widehat{C}_1^{\text{diff}}\, \overline{\Pi}_1^r \right) \\
&= (-1.4290, -0.0054, -0.7868).
\end{aligned}$$

The reduced jump map is given by

$$\widehat{\mathcal{J}}_1 = \overline{\Pi}_1^l\, \mathcal{J}_1\, \overline{\Pi}_0^r = \begin{bmatrix} -0.0166 & 0.0593 \end{bmatrix}.$$

Figure 7.2 depicts the output of the original switched DAE and its approximation for the input $u(t) = 0.5\sin(0.5t)$. Clearly, both outputs match nicely. The related relative errors between the two outputs are depicted in Figure 7.3, which shows that the relative error is less then 1%.

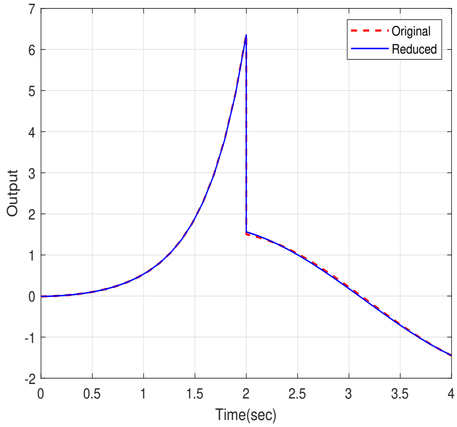

Figure 7.2: Outputs of the original system and the proposed reduced system.

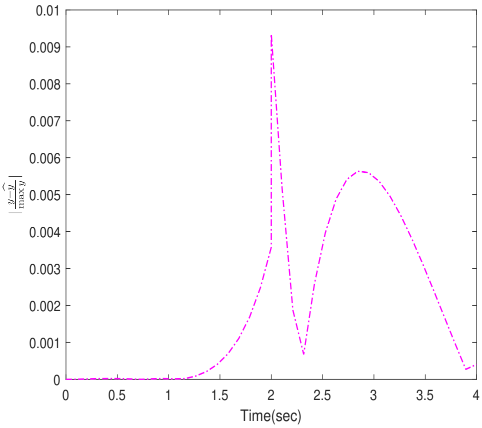

Figure 7.3: Relative errors of the original system and the proposed reduced system.

7.6 Discussion

In this chapter, a reduction method for switched DAEs is proposed. First, a formula is derived which shows that the switched DAEs can be transformed to a switched ODE with jumps and impulses, which preserves same input-output behavior. The proposed formula contains input dependent jumps, and Dirac impulses may occur in output at the switching times. However, these additional terms can be eliminated by assuming some conditions. In particular, due to the algebraic constraints the model reduction method from Chapter 6 cannot be used directly, instead the inconsistent states which are unreachable need to be removed first with the method from Chapter 4. At this moment, it is not proven how to deal with the input dependent jumps and Dirac impulses coming from the states.

8 Model reduction for singular linear switched systems in discrete time

8.1 Introduction

In Chapter 7, a model reduction method for switched DAEs is presented in continuous-time. This chapter is devoted to model reduction of singular linear switched systems in discrete time. The key motivation of this chapter comes from [1] where authors have proposed a solution formula for SLSSs which is time-varying system in discrete time. Loosely speaking, the dynamics of a SLSS in discrete time with certain property can be recovered from a time-varying system.

Consider the SLSSs in discrete time of the form

$$\begin{aligned} E_{\sigma(k)}x(k+1) &= A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k), \\ y(k) &= C_{\sigma(k)}x(k),\ k \in \mathtt{M}, \end{aligned} \tag{8.1}$$

where $x(k) \in \mathbb{R}^n$ is the state at time $k \in \mathtt{M}$ and $\sigma : \mathbb{N} \to \mathtt{M} = \{0, 1, 2, \dots, \mathtt{m}\}$, $\mathtt{m} \in \mathbb{N}$, is the switching signal with the switching times $0 < s_1 < s_2 < \dots < s_{\mathtt{m}}$ in the bounded interval $[k_0, k_f) := \{k_0, k_0 + 1, \dots, k_f - 1\}$ of interest.

The system matrices are $E_i, A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{p\times n}$, where $i \in \mathtt{M}$. The matrices E_i are in general singular, which is related to the presence of (mode-dependent) algebraic constraints. Assume that the i-th mode is active in the interval $[s_i, s_{i+1})$, for $i = 0, 1, \dots, \mathtt{m}$ (where $s_0 := 0$) and define the duration of the i-th mode as $\tau_i = s_{i+1} - s_i - 1$. This chapter is concerned about the input-output behavior of (8.1) so assume in the following that $x(0) = 0$.

There are already some existing results on MOR for switched systems for discrete time case, e.g. [9, 10, 21, 120]. However, in contrast to the existing literature, here, the system (8.1) is viewed as a *time-varying* linear system. In particular, the reduction generally depends on the specifically given switching signal and results in a time-varying reduced model. The concept of balanced realization to time-varying systems (in continuous-time) is discussed in [26, 74, 121, 142] and provides some inspiration for this work; in particular, for the proposed balanced truncation of switched linear systems in Chapter 5. The resulting reduced model is not a switched system anymore, but fully time-varying, which is not very practicable to the continuous time case. However, in discrete time, such a fully time-varying reduced model may still be feasible in practical applications and is a motivation to apply this time-varying balanced truncation to SLSS (8.1).

This introduction concludes with an example, which illustrates the fact that a straight forward mode-wise reduction to obtain a reduced switched system does not work in general.

Example 8.1. Consider an (ordinary) switched linear system in discrete time composed by two modes with

$$A_0 = A_1 = \begin{bmatrix} -1 & 0.01 & 0.001 \\ 0.01 & -1 & 0.001 \\ 0.001 & 0.0002 & -1 \end{bmatrix},$$
$$B_0 = C_0^\top = \begin{bmatrix} 3 \\ 0.1 \\ 0.001 \end{bmatrix}, \quad B_1 = C_1^\top = \begin{bmatrix} 0.001 \\ 0.01 \\ 3 \end{bmatrix}.$$

The switching signal is given by $\sigma(t) = 0$ on $t \in [0, 4)$ and $\sigma(t) = 1$ on $[4, 7]$.

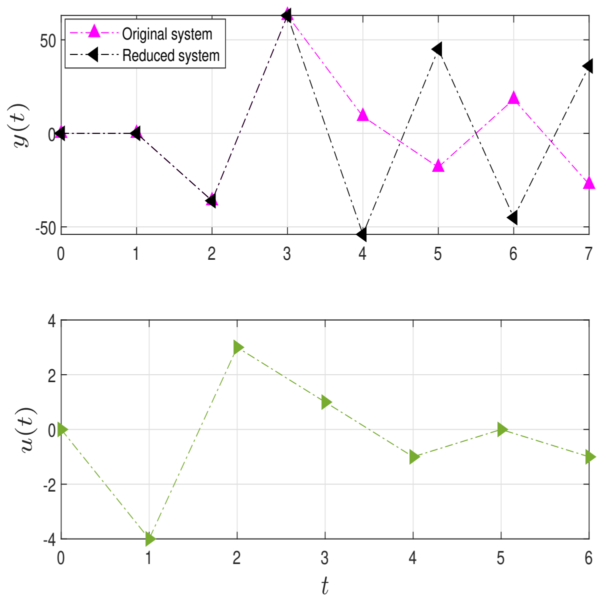

Figure 8.1: Comparison between outputs of original system and its reduced system.

It is easily seen that on each of the intervals $[0,4)$ and $[4,7]$, the following one dimensional reduced order model is a good approximation for the input-output behavior of the individual modes

$$\begin{aligned}\widehat{x}(k+1) &= -\widehat{x}(k) + 3u(k),\\ \widehat{y}(k) &= 3\,\widehat{x}(k).\end{aligned}$$

However, when considering the overall switched system, the input-output behavior of the reduced system is not a good matching with the original input-output behavior, see Figure 8.1.

The remaining chapter is structured as follows. In Section 8.2. some results are presented for the so-called one-step-map which play an important role for an explicit solution formula for SLSS (8.1). Section 8.2.2 provides the computation procedure of time-varying balanced realizations in discrete time. In Section 8.3, time-varying balanced truncation method for SLSS (8.1) is discussed. Finally, some numerical results are presented in Section 8.4.

8.2 Preliminaries

In this section, it is shown that the solutions of a SLSS can equivalently expressed in terms of a time-varying system.

8.2.1 Equivalent solution of SLSSs and time-varying systems

For the existence and uniqueness of the solution of the SLSS (8.1), the following assumption is needed, cf. [1].

Assumption 8.1. The SLSS (8.1) is *jointly index-1*, i.e.

$$\mathcal{S}_i \oplus \ker E_j = \mathbb{R}^n, \;\; \forall i,j \in \mathtt{M},$$

where $\mathcal{S}_i = A_i^{-1}(\operatorname{im} E_i)$.

Under the jointly index-1 assumption, the solution of (8.1) with $x(0)=0$ exists. This solution is unique and satisfies the following lemma.

Lemma 8.2 ([1]). *Assume the SLSS* (8.1) *is jointly index*-1. *For a given switching signal* σ, *there exist corresponding matrices* $\widetilde{A}_k$, $\widetilde{B}_k$ *and* $\widetilde{F}_k$, *such that all solutions of* (8.1) *with* $x(0)=0$ *satisfy*

$$x(k+1) = \widetilde{A}_k x(k) + \widetilde{B}_k u(k) + \widetilde{F}_k u(k+1), \quad k \in \mathtt{M}. \tag{8.2}$$

Proof. Let $\sigma(-1) := \sigma(0)$ and, for $k \in \mathtt{M}$,

$$\widetilde{A}_k := V_{\sigma(k)} \begin{bmatrix} \bar{A}^1_{\sigma(k),\sigma(k-1)} & 0 \\ -\bar{A}^2_{\sigma(k+1),\sigma(k)}\bar{A}^1_{\sigma(k),\sigma(k-1)} & 0 \end{bmatrix} V^{-1}_{\sigma(k-1)}, \tag{8.3a}$$

$$\widetilde{B}_k := V_{\sigma(k)} \begin{bmatrix} \bar{B}^1_{\sigma(k),\sigma(k-1)} \\ -\bar{A}^2_{\sigma(k+1),\sigma(k)}\bar{B}^1_{\sigma(k),\sigma(k-1)} \end{bmatrix}, \tag{8.3b}$$

$$\widetilde{F}_k := V_{\sigma(k)} \begin{bmatrix} 0 \\ -\bar{B}^2_{\sigma(k+1),\sigma(k)} \end{bmatrix}, \tag{8.3c}$$

where

$$\begin{bmatrix} \bar{A}^1_{i,j} & 0 \\ -\bar{A}^2_{i,j} & I_{n_2} \end{bmatrix} = V_i^{-1} G_{i,j}^{-1} A_i V_j, \quad \begin{bmatrix} \bar{B}^1_{i,j} \\ \bar{B}^2_{i,j} \end{bmatrix} = V_i^{-1} G_{i,j}^{-1} B_i,$$

$$G_{i,j} = E_i + A_i Q_{i,j}, \quad Q_{i,j} = V_j \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} V_i^{-1},$$

here $V_i := [g_i^1 \ \dots \ g_i^{n_1} \ h_i^{n_1+1} \ \dots \ h_i^n]$ such that $g_i^1, \dots, g_i^{n_1}$ and $h_i^{n_1+1}, \dots, h_i^n$ are the bases of $\mathcal{S}_i$ and $\ker E_i$, respectively, for $i = 1, \dots, n$. The remaining proof is similar to the proof of [1, Thm. 5.1] and is therefore omitted. □

Remark 8.3. The one-step-map from $x(k)$ to $x(k+1)$ depends on the modes at time $k-1$, k and $k+1$. This implies that the allowed space of consistent initial values also depends on the choice of $\sigma(-1)$, here, it will be assumed that $\sigma(-1) = \sigma(0)$. As pointed out in [1, Rem. 5.2], the effect of a different choice of $\sigma(-1)$ is not yet fully understood and is still under investigation; nevertheless, since the initial condition $x(0) = 0$, this is of no further concern here. This is considered as one of the assumptions considered in this study.

Motivated by Lemma 8.2, consider the following time-varying surrogate system for (8.1) with a given switching signal σ:

$$\begin{aligned} x(k+1) &= \widetilde{A}_k x(k) + \begin{bmatrix} \widetilde{B}_k & \widetilde{F}_k \end{bmatrix} \widetilde{u}(k), \\ y(k) &= C_k x(k), \ k \in \mathtt{M}, \end{aligned} \tag{8.4}$$

where $x(0) = 0$, $\widetilde{u}(k) = \begin{bmatrix} u(k) \\ u(k+1) \end{bmatrix}$, $C_k := C_{\sigma(k)}$ and $\widetilde{A}_k$, $\widetilde{B}_k$, $\widetilde{F}_k$ are given by (8.3). Writing $\tilde{u} = \begin{bmatrix} I \\ \mathcal{T}_1 \end{bmatrix} u$, where $\mathcal{T}_1\{u\}(k) := u(k+1)$ denotes the time-shift operator, it is clear that (8.1) and (8.4) have the same input-output behavior.

Note that the solution of a jointly index-1 SLSS (8.1) does not exist for any initial value $x(0) \in \mathbb{R}^n$. In fact, the consistency space of jointly index-1 (8.1), under the assumption $\sigma(-1) = \sigma(0)$, is $\operatorname{im} V_{\sigma(0)} \begin{bmatrix} I & 0 \\ -\widehat{A}^2_{\sigma(0),\sigma(0)} & \widehat{B}^2_{\sigma(0),\sigma(0)} \end{bmatrix}$. This has some implications on the relationship between system (8.1) and (8.4) in terms of observability and reachability. Here, observability means that if the input and output are identically zero on $[k_0, k_f)$, the state has to be zero as well. On the other hand, reachability means that for each $x_f \in \mathbb{R}^n$, there exists an input such that the corresponding solution satisfies $x(k_f - 1) = x_f$. Clearly, reachability of (8.1) implies a reachable time-varying surrogate system (8.4), whereas observability of (8.1) does not imply the observability of its surrogate system (8.4). However, observability of (8.4) implies that (8.1) is also observable.

The main goal is to find a reduced size time-varying system for the time-varying system (8.4) of the form

$$\begin{aligned} \widehat{x}(k+1) &= \widehat{A}_k \widehat{x}(k) + \begin{bmatrix} \widehat{B}_k & \widehat{F}_k \end{bmatrix} \begin{bmatrix} u(k) \\ u(k+1) \end{bmatrix}, \\ \widehat{y}(k) &= \widehat{C}_k \widehat{x}(k),\ k \in \mathtt{M}, \end{aligned} \tag{8.5}$$

with reduced system matrices $\widehat{A}_i \in \mathbb{R}^{\widehat{n}\times\widehat{n}}$, $\widehat{B}_i, \widehat{F}_i \in \mathbb{R}^{\widehat{n}\times m}$, $\widehat{C}_i \in \mathbb{R}^{p\times\widehat{n}}$ and $\widehat{n} \ll n$, such that $\widehat{y} \approx y$ for input u. Due to the input-output equivalence between (8.1) and (8.4), the reduced system (8.5) will also be a good surrogate model for the original switched system (8.1).

8.2.2 Balanced truncation for time-varying systems

First the balanced truncation of time-varying systems in discrete time is reviewed, for details see [26, 121, 142]. Consider a time-varying system in discrete time of the form

$$\begin{aligned} x(k+1) &= A_k x(k) + B_k u(k), \quad k \in [k_0, k_f), \\ y(k) &= C_k x(k), \end{aligned} \tag{8.6}$$

with corresponding system matrices A_k, B_k, C_k, $k \in [k_0, k_f)$.

8.2.3 Time-varying Gramians

Reachability and observability Gramians play an important role in balancing-based theory.

Definition 8.4. The time-varying reachability and observability Gramians of (8.6) are defined recursively by, for $k \in [k_0, k_f)$

$$P(k) = A_{k-1} P(k-1) A_{k-1}^\top + B_{k-1} B_{k-1}^\top, \tag{8.7}$$

$$Q(k) = A_k^\top Q(k+1) A_k + C_k^\top C_k, \tag{8.8}$$

with some positive semidefinite initial/final values $P(k_0) = P_0$ and $Q(k_f) = Q_f$, the intuition for these initial/final values are given in Remark 5.2.

Note that, the reachability Gramian is constructed *forward* in time, while the observability Gramians evolve *backward* constructions. It is clear that $P(k)$ and $Q(k)$ are both symmetric and positive semidefinite for all $k \in [k_0, k_f]$. Due to the finiteness of the interval $[k_0, k_f)$, no assumption is needed with regard to stability of the system.

8.2.4 Balanced realization

As mentioned in Chapter 5, the system (8.6) will be balanced if the Gramians are equal.

Definition 8.5. The system (8.6) is *balanced* on time interval $[k_0, k_f]$ if there exist positive definite matrices P_0, Q_f such that

$$P(k) = Q(k) = \Xi(k), \ \forall \ k \in [k_0, k_f],$$

where $\Xi(k)$ is a diagonal matrix at time k.

From [26] and Theorem 5.7, one can find a time-varying coordinate transformation as $\overline{x}(k) = T(k)x(k)$ with positive definite P_0, Q_f such that the system (8.6) is balanced. Under these transformation, it is easily seen that the corresponding Gramians satisfy

$$\begin{aligned}\overline{P}(k) &= T(k)P(k)T(k)^\top,\\ \overline{Q}(k) &= T(k)^{-\top}Q(k)T(k)^{-1},\end{aligned}$$

where $\overline{P}_0 = T(k_0)P_0T(k_0)^\top$, and $\overline{Q}_f = T(k_f)^{-\top}Q_fT(k_f)^{-1}$. In particular,

$$\overline{P}(k)\overline{Q}(k) = T(k)P(k)Q(k)T(k)^{-1}.$$

Then, the input-output equivalent *balanced* system of (8.6) is given by

$$\begin{aligned}\overline{x}(k+1) &= \overline{A}_k\overline{x}(k) + \overline{B}_k u(k),\\ y(k) &= \overline{C}_k x(k),\end{aligned}$$

where the transformed system matrices are obtained by $\overline{A}_k := T(k+1)A_kT(k)^{-1}$, $\overline{B}_k := T(k+1)B_k$, $\overline{C}_k := C_kT(k)^{-1}$.

8.3 Model reduction

Consider the time-varying system (8.4). Assume that the corresponding time-varying reachability and observability Gramians for $(\widetilde{A}_k, [\widetilde{B}_k, \widetilde{F}_k], \widetilde{C}_k)$, are given by $\widetilde{P}(k)$ and $\widetilde{Q}(k)$ respectively, with some initial / final Gramians $\widetilde{P}_0, \widetilde{Q}_f$. Now an assumption is needed for model reduction methods.

Assumption 8.6. Assume a transformation $\widetilde{T}$ such that the balanced Gramians of (8.4) are obtained by

$$\widetilde{T}(k)\widetilde{P}(k)\widetilde{T}(k)^\top = \widetilde{T}(k)^{-\top}\widetilde{Q}(k)\widetilde{T}(k)^{-1} = \widetilde{\Xi}(k),$$

and let, the (uniformly) partitioned form $\widetilde{\Xi}(k) = \begin{bmatrix} \widehat{\Xi}(k) & 0 \\ 0 & \overline{\Xi}(k) \end{bmatrix}$ where all diagonal entries in $\overline{\Xi}(k)$ are significantly smaller than those in $\widehat{\Xi}(k)$ and $\widehat{\Xi}(k) \in \mathbb{R}^{\widehat{n}\times\widehat{n}}$.

With the Assumption 8.6, consider the singular value decomposition

$$\widetilde{R}(k)^\top\widetilde{L}(k) = \begin{bmatrix} \widehat{U}(k) & \overline{U}(k) \end{bmatrix} \begin{bmatrix} \widehat{\Xi}(k) & 0 \\ 0 & \overline{\Xi}(k) \end{bmatrix} \begin{bmatrix} \widehat{V}(k) & \overline{V}(k) \end{bmatrix}^\top,$$

where $\widetilde{R}(k)\widetilde{R}(k)^\top = \widetilde{P}(k)$ and $\widetilde{L}(k)\widetilde{L}(k)^\top = \widetilde{Q}(k)$ are obtained by a Cholesky decomposition. Then, the reduced system of the form (8.5) can be obtained by

$$\begin{aligned} \widehat{A}_k &:= \widehat{\Pi}^l(k+1)\widetilde{A}_k\widehat{\Pi}^r(k), \\ \begin{bmatrix} \widehat{B}_k & \widehat{F}_k \end{bmatrix} &:= \widehat{\Pi}^l(k+1)\begin{bmatrix} \widetilde{B}_k & \widetilde{F}_k \end{bmatrix}, \\ \widehat{C}_k &:= \widetilde{C}_k\widehat{\Pi}^r(k). \end{aligned}$$

where the left- and right-projectors are calculated as

$$\begin{aligned} \widehat{\Pi}^l(k) &:= \widehat{\Xi}(k)^{-1/2}\widehat{V}(k)^\top\widetilde{L}(k)^\top \in \mathbb{R}^{\widehat{n}\times n}, \\ \widehat{\Pi}^r(k) &:= \widetilde{R}(k)\widehat{U}(k)\widehat{\Xi}(k)^{-1/2} \in \mathbb{R}^{n\times\widehat{n}}. \end{aligned}$$

8.4 Numerical results

This section illustrates the proposed method by providing an example.

Example 8.2. Consider a SLSS (8.1) with two modes

$$(E_0, A_0, B_0, C_0) = \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.02 \\ 2 \\ 1 \\ 0.2 \end{bmatrix}, \begin{bmatrix} -0.1 \\ 0.1 \\ 0.1 \\ 2 \end{bmatrix}^\top \right),$$

$$(E_1, A_1, B_1, C_1) = \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.01 \\ 2 \\ 0.5 \\ 0.1 \end{bmatrix}, C_0 \right),$$

Consider a switching signal $\sigma : [0, 9) \to \{0, 1\}$,

$$\sigma(k) = \begin{cases} 0 & : k \in [0, 4) \cup [7, 9), \\ 1 & : k \in [4, 7). \end{cases}$$

It can easily be verified that the pairs (E_0, A_0) and (E_1, A_1) are jointly index-1. Hence, by Lemma 8.2, the time-varying system (8.4) is obtained with the following system matrices

$$(\widetilde{A}_k, \widetilde{B}_k) = \begin{cases} \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0.02 \\ 1.98 \\ 1 \\ 0 \end{bmatrix} \right) & : k = 0, 1, 2, 3, 7, 8, \\ \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0.01 \\ 2 \\ 0.5 \\ 0 \end{bmatrix} \right) & : k = 4, 5, 6, \end{cases}$$

$$\widetilde{F}_k = \begin{cases} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.2 \end{bmatrix} & : k = 0, 1, 2, 6, 7, 8, \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.1 \end{bmatrix} & : k = 3, 4, 5. \end{cases}$$

The corresponding reachability and observability Gramians are calculated respectively, $\widetilde{P}(k)$ and $\widetilde{Q}(k)$ for $k \in [0, 9)$ with initial/final values $\widetilde{P}_0 = 0.002I$ and $\widetilde{Q}_f = 0.002I$. The corresponding HSVs are depicted in Figure 8.2 and it is apparent that the last two HSVs are significantly smaller than the first two. Hence, a two dimensional reduced system is obtained which approximates the time-varying system (8.4) and hence, the original SLSS.

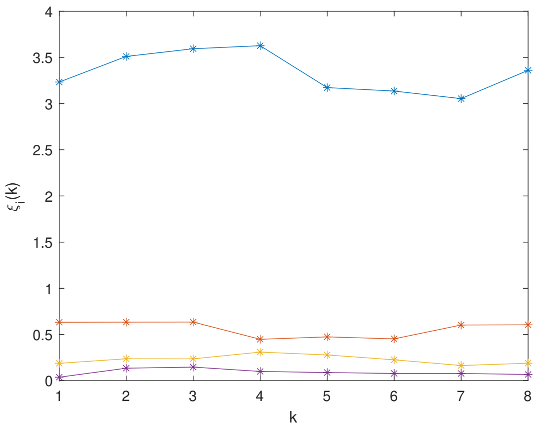

Figure 8.2: Hankel singular values of balanced Gramians at each time instance.

The computed two dimensional reduced systems at each time steps are given by $(\widehat{A}_k, [\widehat{B}_k, \widehat{F}_k], \widehat{C}_k) =$

$$\left(\begin{bmatrix} 0.9206 & -0.0051 \\ -0.0107 & 0.0012 \end{bmatrix} \begin{bmatrix} -1.8615 & 0.0046 \\ -0.0535 & 0.6305 \end{bmatrix} \begin{bmatrix} -0.1410 \\ -0.6334 \end{bmatrix}^\top\right),$$

$$\left(\begin{bmatrix} 0.9761 & -0.0071 \\ -0.0058 & -0.0076 \end{bmatrix} \begin{bmatrix} -1.0832 & 0.0074 \\ -0.0603 & 0.6287 \end{bmatrix} \begin{bmatrix} -0.2387 \\ -0.6332 \end{bmatrix}^\top\right),$$

$$\left(\begin{bmatrix} 0.9887 & -0.0116 \\ -0.0027 & -0.0071 \end{bmatrix} \begin{bmatrix} -0.7265 & 0.0117 \\ -0.0445 & 0.8861 \end{bmatrix} \begin{bmatrix} -0.3859 \\ -0.4449 \end{bmatrix}^\top\right),$$

$$\left(\begin{bmatrix} 1.1336 & -0.0155 \\ -0.4762 & -0.0398 \end{bmatrix} \begin{bmatrix} -0.3934 & 0.0143 \\ -0.0311 & 0.4016 \end{bmatrix} \begin{bmatrix} -0.7193 \\ -0.4515 \end{bmatrix}^\top\right),$$

$$\left(\begin{bmatrix} 0.9856 & -0.0335 \\ -0.0019 & 0.1912 \end{bmatrix} \begin{bmatrix} -0.2249 & 0.0206 \\ -0.0536 & 0.4158 \end{bmatrix} \begin{bmatrix} -1.0114 \\ -0.4262 \end{bmatrix}^\top\right),$$

$$\left(\begin{bmatrix} 0.9719 & -0.0475 \\ -0.0036 & 0.0701 \end{bmatrix} \begin{bmatrix} -0.1850 & 0.0286 \\ -0.0047 & 0.2986 \end{bmatrix} \begin{bmatrix} -1.3323 \\ -0.5423 \end{bmatrix}^\top\right),$$

$$\left(\begin{bmatrix} 0.8014 & -0.0770 \\ -0.8119 & 0.0689 \end{bmatrix} \begin{bmatrix} -0.1249 & 0.0523 \\ 0.1196 & -0.5938 \end{bmatrix} \begin{bmatrix} -1.4777 \\ 0.5432 \end{bmatrix}^\top \right),$$

$$\left(\begin{bmatrix} 0.9471 & 0.0834 \\ -0.0047 & 0.0125 \end{bmatrix} \begin{bmatrix} -0.1026 & 0.1118 \\ -0.0261 & 0.4245 \end{bmatrix} \begin{bmatrix} -2.8391 \\ -0.1943 \end{bmatrix}^\top \right).$$

Consider randomly generated input $u(\cdot)$ with $u(0) = 0$, and the input-output behavior is calculated for the system (8.4) and its reduced system with relative errors. Figures 8.3 displays the outputs, the input signal, and the relative errors for original system and the proposed two dimensional reduced system. Clearly, both outputs match nicely and the relative error is less then 6%.

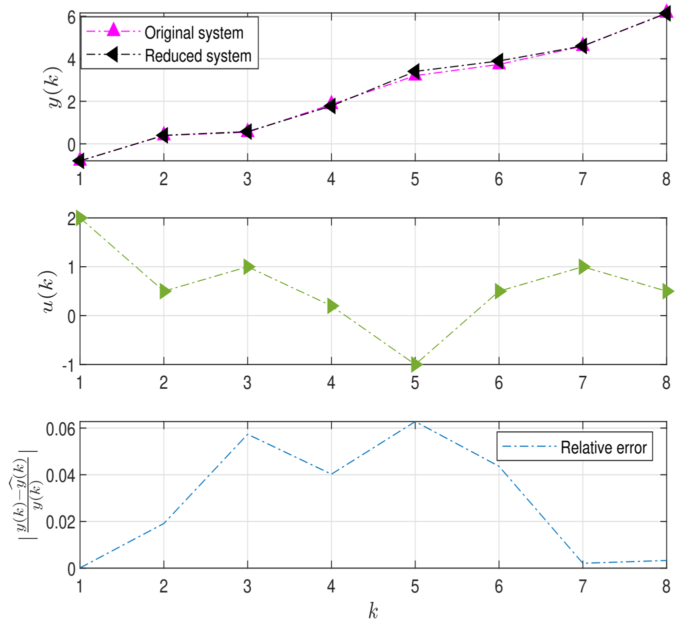

Figure 8.3: Outputs and relative errors of original system (8.4) and the proposed 2nd order approximation.

Next, another initial /final value of the Gramians is considered by increasing the magnitudes as $\widetilde{P}_0 = 0.5I$ and $\widetilde{Q}_f = 0.5I$. With the same input sequence as in Figure 8.3, the input-output behavior with the relative error is depicted in Figure 8.4, which shows that the choice of the initial /final values of Gramians plays an important role in the error analysis. Therefore, it is concluded that taking small magnitude with identity matrix could be the best choice for the initial /final values of the Gramians.

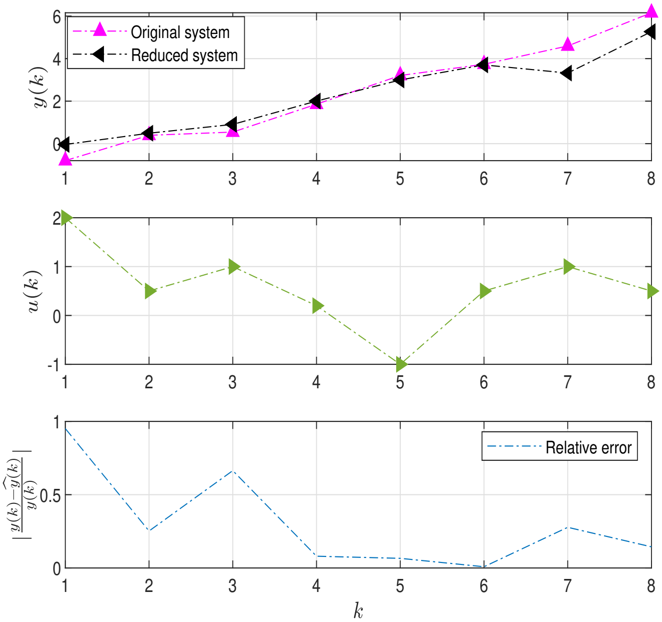

Figure 8.4: Outputs and relative errors of the original system (8.4) and the proposed 2nd order approximation with initial/final values $\widetilde{P}_0 = 0.5I$, $\widetilde{Q}_f = 0.5I$.

8.5 Discussion

In this chapter, a model reduction method for index-1 singular linear switched systems is presented. The key novelty is the viewpoint of the SLSS as a piecewise-constant time-varying system. First, an input-extended time-varying system is presented which has identical input-output behavior as the original index-1 SLSS. Then, a time-varying reduced system is proposed by applying balanced truncation for time-varying system which approximates the original SLSS. For the computation of Gramians in discrete time, the initial / final Gramians are also considered. An example is given for the performance of the reduced system.

9 Conclusions

9.1 Conclusions

In this thesis, the reduced realization and model reduction for switched systems have been considered. It has been mentioned in introduction (Chapter 1) that existing realization methods generally do not consider the switched system as a piecewise-constant time-varying linear system, they allow the switching signal as an input. It has been seen that a reduced realization in general depends on the specifically known switching signal. Therefore, the main objective of this thesis was the construction of reduced realization and model reduction procedures on this system class. Later, the proposed methods have been used for underlying problem of switched DAEs and finally, model reduction for singular switched system in discrete time has been investigated.

The main contributions of the thesis can be summarized as follows:

- ◇ In Chapter 3, a method for reduced realization of switched linear systems with mode-dependent state dimensions has been proposed and the technique has been investigated only for single switch case. The key novelty is the viewpoint of switched systems as a piecewise-constant time-varying system. A reduced realization is obtained by first constructing an input-extended version of the second mode, and an output-extended version of the first mode and then, removing unreachable and unobservable states and finally, reduce the jump map between the modes. More specifically, the proposed technique consists two main steps; first, a minimal realization of the second mode is constructed by taking into account the reachable subspace of the first mode. Secondly, a minimal realization of the first mode is calculated by taking into account the observable states of the second mode. The method is basically relies on the Kalman decomposition of each mode and on finding the reachable and observable states for each mode individually.

In Chapter 4, a reduced realization procedure has been proposed for switched linear systems with fixed switching signal and known mode sequence; the results is extended for general switching signals. It has clearly been shown that the reachable and unobservable subspaces of the given switched system depend on the switching time duration which results in a reduced (fully) time-varying systems. To circumvent this problem, suitable extended reachable and restricted unobservable subspaces have been defined according to the known mode sequences and a sequence of corresponding projectors is obtained. To do so, a weak Kalman decomposition based on extended reachable / restricted unobservable subspaces is introduced which allows the proposed extended reachable / restricted unobservable subspaces, and which then can be used to remove certain unreachable and unobservable states for each mode. The computed reduced switched system then preserves the same input-output behavior as the original system. A conjecture has been stated that the resulting reduced system has minimal size for almost all switching times and it is not possible to reduce it further. An example is provided to claim that, in general the dimension of the minimal realization depends on the specific switching times.

- ◇ In Chapter 5, a model reduction procedure has been proposed for switched linear systems where each mode has the same state dimension. The key feature is the time-varying nature of switched systems. The methods has two main steps; first, a continuously time-varying system is proposed which approximates the (discontinuous) switched system by allowing some errors and secondly, available balanced truncation for time-varying linear systems is applied to find a lower order reduced system. Moreover, balanced truncation is reviewed in finite time horizon with initial/final condition of the Gramians. Finally, it has been shown that the reduced system approximates the original switched linear system; some error bounds are also proposed.

  In Chapter 6, a model reduction method has been investigated for switched linear systems with known mode duration. It has been shown that the time-varying reachability/observability Gramians of a switched linear system depend on the mode duration and they provide quantitative measure for the states at a specified time. It has also been proved that the proposed Gramians provide a quantitative measure for the input-output energy. A midpoint based balanced truncation has been investigated for a balanced system with similar input-output behavior as the given system. Moreover, these Gramians matrices span the original reachable/observable subspaces for switched systems. The computation of Gramians and some numerical issues have also been discussed so that the proposed method is applicable for moderately large-scale system.

In Chapter 7, switched DAEs have been studied, and a switched ODE with jumps and impulses is constructed which has identical input-output behavior as the original switched DAE. Under some assumptions, a reduced realization can be computed for the proposed switched ODE with jumps via the method given in Chapter 4, and the computed reduced system has the same input-output behavior as the original switched DAE. Finally, the model reduction method from Chapter 6 has been applied to obtain a good approximation of the input-output behavior. Numerical example has been illustrated the results of the two reduction processes.

In Chapter 8, singular linear switched systems are considered in discrete time. It has been shown that under some assumptions, certain time-varying system has the same input-output behavior as the given singular switched system. Balanced truncation has been proposed for the time-varying system to obtain a lower dimensional time-varying system which then approximates the original singular system. There, the reachability and observability Gramians are also computed by introducing initial/final values. Finally, an example demonstrates the proposed reduction techniques.

9.2 Future work

In what follows, some ideas are provided for future work. In Chapters 3 and 4, reduced realization is gained insight with some results for continuous time switched systems. These results are also of interest for the discrete time case. Furthermore, the minimality is not proven yet, and is still on ongoing research. Since all operations in the reduced realization method are calculated with exact arithmetic, the computational efficiency of the proposed method in large-scale setting is not clear at the moment, and further research is necessary.

Mode-wise balanced truncation method is developed in Chapter 6 by defining suitable Gramians; no error bound is proven, although, the proposed Gramians have a precise energy interpretation. Finding a suitable error bound is important topic for future research. Numerical issues for computation of Gramians is discussed there, while for very large-scale setting the proposed method is not applicable without further investigation.

Model reduction for switched DAEs is investigated by making the assumptions that the DAE does not produce state dependent Dirac impulses in output at switching time and that the jump map is independent from the input. It is of interest to extend the existing model reduction methods such that these assumptions are not necessary anymore.

A Some basics on linear algebra

In this appendix, a self-contained introduction is presented to some control theory tools that are useful in this thesis.

Lemma A.1 ([20]). *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping, and S_1 and S_2 are subspaces of $\mathbb{R}^n$. Then, the following statements holds:*

(i) $(\ker A)^\perp = \operatorname{im} A^\top$ *and* $(\operatorname{im} A)^\perp = \ker A^\top$,

(ii) $(S_1 + S_2)^\perp = S_1^\perp \cap S_2^\perp$ *and* $(S_1 \cap S_2)^\perp = S_1^\perp + S_2^\perp$.

Definition A.2. A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is *positive semidefinite* if $z^\top Az \geqslant 0$ for all $z \in \mathbb{R}^n$. Again, if $z^\top Az > 0$ for all $z \neq 0$, then A is called *positive definite*.

Definition A.3 (Cholesky decomposition)**.** Let $A \in \mathbb{R}^{n\times n}$ be a positive definite matrix. The *Cholesky decomposition* is given by

$$A = R^\top R,$$

where R is lower triangular with positive diagonal elements.

Theorem A.4 ([2]). *For every matrix A of rank r, there exists orthogonal matrices U and V such that*

$$A = U\Xi V^\top = \sum_{i=1}^{r} u_i \xi_i v_i,$$

where $U = [u_1 \dots u_r \cdots u_m]$ *and* $V = [v_1 \dots v_r \cdots v_n]$, *and* $\Xi = \operatorname{diag}(\xi_1 \dots \xi_r \; 0 \dots 0)$ *contains the singular values of A.*

B Distribution theory

B.1 Review of classical distribution theory

In this section, some well known results for the analysis of switched DAEs are reviewed from [135].

Let $\mathcal{C}^\infty := \{\ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is arbitrarily often differentiable }\}$ be the space of *smooth functions*. The *support* of $f \in \mathcal{C}^\infty$ is given by $\operatorname{supp} f := \operatorname{cl}\{\ x \in \mathbb{R} \mid f(x) \neq 0\ \}$ where $\operatorname{cl} M$ is the closure of the set $M \subseteq \mathbb{R}$. The space of *test functions* $\mathcal{C}_0^\infty : \mathbb{R} \to \mathbb{R}$ is given by

$$\mathcal{C}_0^\infty := \{\ \varphi \in \mathcal{C}^\infty \mid \operatorname{supp}\varphi \text{ is bounded }\}.$$

Definition B.1 (Distributions)**.** The space of *distributions*, denoted by $\mathbb{D}$, is then the dual of the space of test functions, i.e.,

$$\mathbb{D} := \{\ D : \mathcal{C}_0^\infty \to \mathbb{R} \mid D \text{ is linear and continuous }\}.$$

Here, continuity is with respect to a certain topology on the space of test functions.

The main two properties of distributions are 1) that they can be interpreted as generalized functions, and 2) that they are arbitrarily often differentiable. To be more precise, let $L_{1,\mathrm{loc}}$ be the space of locally integrable functions, then

$$L_{1,\mathrm{loc}} \to \mathbb{D}, f \to f_{\mathbb{D}} := \varphi \mapsto \int_{\mathbb{R}} f\varphi,$$

is well defined (i.e. $f_{\mathbb{D}}$ is indeed a distribution) and an injective homomorphism. For $i \geqslant 1$, the i-th derivative of an arbitrary distribution $D \in \mathbb{D}$ is given by

$$D^{(i)}(\varphi) := D^{(i-1)}(\varphi'),\ \varphi \in \mathcal{C}_0^\infty,$$

where $D^{(0)}(\varphi) := -D(\varphi)$. Distributions can be multiplied with smooth functions:

$$(\alpha D)(\varphi) := D(\alpha\varphi),\ \alpha \in \mathcal{C}^\infty, D \in \mathbb{D}, \varphi \in \mathcal{C}_0^\infty.$$

The simplest distribution which is not induced by a function is the Dirac impulse. The *Dirac impulse* at $t \in \mathbb{R}$ is given by

$$\delta_t : \mathcal{C}_0^\infty \to \mathbb{R},\ \varphi \mapsto \delta_t(\varphi) := \varphi(t).$$

The Dirac impulse is the distributional derivative of the so called *Heaviside function* $\mathbb{1}_{[0,\infty)}$, i.e.

$$\forall t \in \mathbb{R}:\ \delta_t = \frac{\mathrm{d}_\mathbb{D}}{\mathrm{dt}}(\mathbb{1}_{[t,\infty)})_\mathbb{D}.$$

B.2 Piecewise-smooth distributions

Switched DAEs of the form $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$, $y = Cx$ do not have classical solutions as the consistency spaces of different modes do not have to coincide and then jumps or Dirac impulses might occur. However, it can be shown that the space of distributions is not a suitable solution space as it is "too large". In general, it is impossible to define restrictions to intervals and multiplications for distributions, for details see [135, Thm 2.2.2]. To overcome this problem, the space of piecewise-smooth distributions is introduced in [135].

Definition B.2 (Piecewise-smooth functions, [135])**.** The space of *piecewise-smooth functions* is defined by

$$\mathcal{C}^\infty_{\mathrm{pw}} := \left\{ \sum_{i\in\mathbb{Z}} \mathbb{1}_{[t_i,t_{i+1})}\alpha_i \ \middle|\ \begin{array}{c} (\alpha_i)_{i\in\mathbb{Z}} \in (\mathcal{C}^\infty)^\mathbb{Z}, \{\, t_i \in \mathbb{R} \mid i \in \mathbb{Z} \,\} \\ \text{locally finite with } t_i < t_{i+1},\, i \in \mathbb{Z} \end{array} \right\}.$$

Definition B.3 (Piecewise-smooth distributions, [135])**.** The space of *piecewise-smooth distributions* is given by

$$\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} := \left\{ f_\mathbb{D} + \sum_{t\in T} D_t \ \middle|\ f \in \mathcal{C}^\infty_{\mathrm{pw}}, D_t \in \operatorname{span}\{\delta_t, \delta_t', \cdots\} \right\},$$

where $T \subset \mathbb{R}$ is locally finite, and $\operatorname{span}\{\delta_t, \delta_t', \cdots\}$ denotes the set of all finite linear combinations of the Dirac impulse at $t \in \mathbb{R}$ and its derivatives. The set $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ is a subspace of $\mathbb{D}$ and closed under differentiation and restriction to intervals.

B.3 Distributional solution

Definition B.4 (Classical solution)**.** A classical solution of DAE $E\dot{x}(t) = Ax(t) + f(t)$ is any differential function $x \in \mathcal{C}^1(\mathbb{R} \to \mathbb{R}^n)$ such that $E\dot{x}(t) = Ax(t) + f(t)$ holds for all $t \in \mathbb{R}$.

Lemma B.5. *The ODE $\dot{x} = Ax + Bu$ with initial condition $x(0) = x_0$ has the same solution as the impulsive ODE $\dot{x} = Ax + Bu + x_0\delta,\ x(0^-) = 0$ on $(0, \infty)$.*

Proof. The (distributional) solution of system $\dot{x} = Ax + Bu + \delta_0 x_0$ with $x(0^-) = 0$ is given by

$$x = e^{A\cdot} \int_{0-} e^{-A\cdot}(Bu + \delta_0 x_0),$$

where $H =: \int_{0-} D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ for $D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$, is the distributional antiderivative operator such that $H' = D$ and $H(0^-) = 0$. Since $e^{-A0} = I$, it follows that

$$e^{A\cdot} \int_{0-} e^{-A\cdot}\delta_0 x_0 = e^{A\cdot} x_0 \mathbb{1}_{[0,\infty)},$$

which is the solution of system $\dot{x} = Ax + x_0\delta$ with $x(0^-) = 0$. □

Lemma B.6 ([136]). *For $(E, A, B) = (I, A, B)$, the unique solution of the corresponding ODE-ITP* (7.9) *is given by*

$$x = x^0_{(-\infty,0)} + \left(\left(e^{A\cdot} x^0(0^-)\right)_{\mathbb{D}} + e^{A\cdot} \int_{0-} e^{A\cdot} Bu\right)_{[0,\infty)}.$$

Lemma B.7 ([136]). *For $(E, A, B) = (N, I, B)$ with nilpotent N, the unique solution of the corresponding DAE-ITP* (7.9) *is given by*

$$x = x^0_{(-\infty,0)} - \sum_{i=0}^{\nu-2} N^{i+1} x^0(0^-)\delta^{(i)} - \sum_{i=0}^{\nu-1} N^i B \frac{d}{dt}^i (u_{[0,\infty)}),$$

where $1 \leqslant \nu \leqslant n$ is the nilpotency index of N and $\delta^{(i)}$ denotes the i-th derivative of the Dirac impulse δ.

Lemma B.8 (Solution of switched DAEs, [136]). *Consider the regular switched DAE $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$ with switching signal $\sigma : \mathbb{R} \to \mathbb{N}$. Then, for every $u \in \mathbb{D}^u_{\text{pw}\mathcal{C}^\infty}$, there exists a unique (distributional) solution $x \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$ satisfies the following properties: for any $t \in (s_k, s_{k+1})$*

$$x(t^+) = e^{A_k^{\text{diff}}(t-s_k)} \Pi_k x(s_k^-) + \left(e^{A_k^{\text{diff}}\cdot} \int_{0-} e^{A_k^{-\text{diff}}\cdot} B_k^{\text{diff}} u\right)$$
$$- \sum_{i=0}^{\nu-1} (E_k^{imp})^i B_k^{\text{imp}} u^{(i)},$$

and the impulsive part at the switching times s_k is given by

$$x[s_k] = \left(e^{A_k^{\text{diff}}\cdot}\int_{0^-} e^{A_k^{-\,\text{diff}}\cdot} B_k^{\text{diff}} u\right)[s_k] - \sum_{i=0}^{\nu-2}(E_k^{\text{imp}})^{i+1} x(s_k^-)\delta_{s_k}^{(i)}$$
$$- \sum_{i=0}^{\nu-1}(E_k^{\text{imp}})^{i} B_k^{\text{imp}} (u_k)^{(i)}[s_k],$$

where $(u_k)^{(i)}[s_k] := \sum_{j=0}^{i-1} u^{(i-j-1)}(s_k^+)\delta^{(j)} - u_k[s_k]^{(i)}$, $u_k := u_{[s_k, s_{k+1})}$, $\delta^{(i)}$ denotes the i-th derivative of the Dirac impulse δ.

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Summary

This thesis addresses several problems related to reduced models of switched systems. In particular, three switched system classes are considered; switched linear systems as well as switched differential-algebraic equations in continuous time, and singular linear switched systems in discrete time. In general, the study aims to find a reduced model for a given switched system with fixed switching signal and known mode sequence. The proposed solutions studied in this thesis can be distinguish into two main parts.

Part I deals with reduced realization for switched linear systems with known mode sequence. First, some results are given for the single switch case. The key feature of the method is to obtain a reduced realization by removing certain states from an input-extended second mode and output-extended first mode, and this can be done by applying Kalman decomposition. The reduced system preserves the same input-output behavior as original systems. Secondly, a technique is proposed for reduced realization for switched linear systems with general switching signals. It is shown that the reachable and unobservable subspaces of a switched system are time-varying and depend on the switching time duration. A weak Kalman decomposition based on extended reachable / restricted unobservable subspaces is introduced to overcome the problem of time variance. Based on this weak Kalman decomposition, it is possible to remove certain unreachable and unobservable states for each mode. The proposed reduced system has the same input-output behavior as original switched systems. Finally, it is conjectured that the proposed reduced system has a smallest order for almost all switching time duration.

Contrary to Part I, Part II deals with model reduction techniques for switched linear systems and switched differential-algebraic equations. First, the discontinuity property in the time-varying nature of switched systems is approximated by a continuous time-varying system by introducing some errors. Thereto, balanced truncation for time-varying system is applied to obtain a reduced system which then approximates the original switched system. For large-scale systems, the overall computation for the reduced system may be computationally infeasible. Later, another method is proposed for a known mode sequence and the reduced system preserves then the system classes. The quantitative information for each mode

is carried out by defining suitable Gramians and, these Gramians are exploited at midpoint of the given switching time duration. Finally, balanced truncation leads to a mode-wise reduction. The proposed method provides a good model with suitable thresholds for the given switched system. Moreover, the proposed method is applicable in a moderately large-scale setting.

A model reduction method for switched differential-algebraic equations in continuous time is proposed. A switched linear system with jumps and impulses is constructed which has the identical input-output behavior as original systems. The resulting switched linear system contains then two additional features; state dependent Dirac impulses at switching time instant and input dependent jumps. Then, assuming some conditions, a reduced realization can be obtained via the method proposed in Chapter 4 which preserves the same input-output behavior as the original switched differential-algebraic equation. Later, an approximation is obtained via the method given in Chapter 6.

Finally, a model reduction approach for singular linear switched systems in discrete time is studied. It is shown that the solution of the given singular system can be recovered by an input-extended time-varying system. Then, balanced truncation for time-varying systems in discrete time is implemented to obtain an approximation. The choice of initial/final values of the reachability and observability Gramians are also investigated.

Summarizing, several methods are developed in this thesis for reduced realization and model reduction for switched linear systems as well as singular switched systems.

Samenvatting

Dit proefschrift behandelt verschillende problemen gerelateerd aan gereduceerde modellen van zogenaamde switched systems. In het bijzonder worden drie klassen switched systems beschouwd: switched linear systems, switched differential-algebraic equations in continue tijd, en singular switched systems in discrete tijd. In het algemeen is het doel van de studie om een gereduceerd model te vinden voor een bepaald switched system met een vast switching signal en een bekende volgorde van modi. De voorgestelde oplossingen die in dit proefschrift worden bestudeerd kunnen worden onderscheiden in twee hoofdonderdelen.

Deel I behandelt gereduceerde realisatie voor switched linear systems met bekende volgorde van modi. Eerst worden enkele resultaten gegeven voor het geval van een enkele switch. Het belangrijkste kenmerk van de methode is om een gereduceerde realisatie te verkrijgen door bepaalde toestanden te verwijderen uit een input-extended tweede modus en output-extended eerste modus. Dit wordt gedaan door Kalmandecompositie toe te passen. Het gereduceerde systeem behoudt hetzelfde ingangs-uitgangsgedrag als het originele systeem. Ten tweede wordt een techniek voorgesteld voor gereduceerde realisatie voor switched linear systems met algemene switching signals. Er wordt aangetoond dat de bereikbare en niet-waarneembare deelruimten van een switched system in de tijd variëren en afhankelijk zijn van de duur van de switching time. Een zwakke Kalmandecompositie op basis van uitgebreide bereikbare / beperkte niet-waarneembare deelruimten wordt geïntroduceerd om het probleem van tijdsvariantie op te lossen. Op basis van deze zwakke Kalmandecompositie, is het mogelijk om bepaalde onbereikbare en niet-waarneembare toestanden voor elke modus te verwijderen. Het voorgestelde gereduceerde systeem heeft hetzelfde ingangs-uitgangsgedrag als het originele switched system. Ten slotte wordt verondersteld dat het voorgestelde gereduceerde systeem de kleinste orde heeft voor bijna alle duur van switching times.

In tegenstelling tot deel I, behandelt deel II modelreductietechnieken voor switched linear systems en switched differential-algebraic equations. Ten eerste wordt de discontinuïteitseigenschap van tijdsvariante switched systems benaderd door een continu tijdsvariant systeem door enkele fouten te introduceren. Daartoe

wordt gebalanceerde afkapping voor het tijdsvariant systeem toegepast om een gereduceerd systeem te verkrijgen dat dan het oorspronkelijke switched system benadert. Voor grootschalige systemen kan de totale berekening voor het gereduceerde systeem rekenkundig onhaalbaar zijn. Later wordt een andere methode voorgesteld voor een bekende volgorde van modi, en het gereduceerde systeem behoudt dan de systeemklassen. De kwantitatieve informatie voor elke modus wordt uitgevoerd door geschikte Gramianen te definiëren en deze Gramianen worden gebruikt halverwege de gegeven duur van de switching time. Ten slotte leidt gebalanceerde afkapping tot een modusgewijze reductie. De voorgestelde methode biedt een goed model met geschikte drempelwaarden voor het gegeven switched system. Bovendien is de voorgestelde methode toepasbaar in een gematigd grootschalige setting.

Een modelreductiemethode voor switched differential-algebraic equations in continue tijd wordt voorgesteld. Er wordt een switched linear system met sprongen en impulsen geconstrueerd dat hetzelfde ingangs-uitgangsgedrag heeft als het originele systeem. Het resulterende switched linear system bevat dan twee extra functies: toestandsafhankelijke Dirac impulsen op switching times en ingangsafhankelijke sprongen. Vervolgens kan, onder bepaalde voorwaarden, een gereduceerde realisatie worden verkregen via de methode voorgesteld in Hoofdstuk 4 die hetzelfde ingang-uitgangsgedrag behoudt als de originele switched differential-algebraic equation. Later wordt een benadering verkregen via de methode gegeven in Hoofdstuk 6.

Ten slotte wordt een modelreductiemethode voor singular linear switched systems in discrete tijd bestudeerd. Er wordt aangetoond dat de oplossing van het gegeven singular system kan worden teruggevonden door een input-extended tijdsvariant systeem. Vervolgens wordt gebalanceerde afkapping voor tijdsvariante systemen in discrete tijd geïmplementeerd om een benadering te verkrijgen. De keuze van begin-/eindwaarden van de bereikbaarheids- en de waarneembaarheidsgramianen worden ook onderzocht.

Samenvattend zijn er in dit proefschrift verschillende methoden ontwikkeld voor gereduceerde realisatie en modelreductie voor zowel switched linear systems als singular switched systems.