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Exploring the Nature of the Co-emergence of Students' Representational Fluency and Functional Thinking

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Abstract

In this dissertation, I explore ways to support secondary school students' meaningful understanding of quadratic functions. Specifically, I investigate how students co-developed representational fluency (RF) and functional thinking (FT), when they gained meaningful understanding of quadratic functions. I also characterize students' co-emergence of RF and FT on each representation (e.g., a graph, a symbolic equation, and a table) and across multiple representations.

To accomplish these goals, I employed a design research methodology: a teaching experiment with eight Turkish-American secondary school students in an after-school context at a Turkish Community Center. I constructed the design principles and design elements for the study by networking two distinct domains of literature—representations and quantitative reasoning—to support students' meaningful learning. I conducted ongoing and retrospective analyses on the enhanced transcriptions of small- and whole-group interactions.

The analyses revealed a learning-ecology framework that supported secondary school students' meaningful understanding of quadratic functions. The learning-ecology framework consisted of three components: enacted task characteristics, teacher pedagogical moves, and socio-mathematical norms. Furthermore, the findings showed that students employed two types of reasoning when they created and connected representations of quantities and the relationships between them: static thinking and lateral thinking. Static thinking is recalling a learned fact to represent a quantitative relationship with no attention to how quantities covary on a representation, while lateral thinking is a creative way of thinking wherein students conceive of concrete representations of functions as an emergent quantitative relationship. The findings also showed that students' co-emergence of RF and FT can be operationalized into four levels starting from lesser sophisticated reasoning to greater sophisticated reasoning. Level 0 is a disconnection, level 1 is a partial connection, level 2 is a connection and level 3 is flexible a connection between students’ RF and FT. The dissertation informs teachers and the mathematics education community by (a) reporting and verifying the learning-ecology framework that supported students’ meaningful understanding of quadratic functions; and (b) characterizing students’ co-emergence of RF and FT within and across multiple representations.

EXPLORING THE NATURE OF THE CO-EMERGENCE OF STUDENTS' REPRESENTATIONAL FLUENCY AND FUNCTIONAL THINKING

By

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List of Acronyms

CCSSM	Common Core State Standard for Mathematics
FT	Functional Thinking
NCTM	National Council of Teachers of Mathematics
RF	Representational Fluency
QR	Quantitative Reasoning
TCC	Turkish Community Center
WODB	Which One Does not Belong

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Exploring the Nature of the Co-Emergence of Students' Representational Fluency and Functional Thinking

Learning quadratic functions includes making translations and connections among and within representations along with functional thinking; these are essential skills for students to develop in learning functions in general (NCTM, 2000). Functional thinking (FT) has been defined in general as a style of creative thinking about functions, creating patterns, and generalizing the functional relationships within concrete representations of functions (Blanton & Kaput, 2011; Stephens et al., 2017b). The National Council of Teachers of Mathematics (NCTM, 2000) emphasized both the use and connection of representations in making sense of functional relationships. Students should "create and use tabular, symbolic, graphical, and verbal representations and analyze and understand patterns, relations, and functions" (NCTM, 2000, p. 297) to be able to develop a robust understanding of functions.

Functions are one of the most complex and difficult topics for students to learn. Learning functions involves forms of representations, includes other complex topics (e.g., growth, limit, extrema, etc.), and integrates multiple subtopics of mathematics (Leinhardt, et al., 1990). In school curricula, the function concept heavily emphasizes linear and quadratic functions in order to prepare students for advanced mathematics (Dreyfus & Halevi, 1991). However, scholars have reported that students often have difficulty in developing robust understandings of functions in general (Carlson, et al., 2002; Moore, 2014; Thompson & Carlson, 2017) and quadratic functions in particular (Altindis & Fonger, 2019a; 2019b; Dreyfus & Halevi, 1991; Ellis & Grinstead, 2008; Even, 1998; Lobato et al., 2012; Wilkie, 2019; Zaslavsky, 1997; Zazkis et al., 2003).

Students' use of multiple representations and engagement in FT has been much emphasized in the mathematics education community for many years (NCTM, 2000, 2014).

Students are expected to develop a “deeper understanding of the ways in which changes in quantities can be represented mathematically” (NCTM, 2000, p. 305), as well as to create, connect, and translate among and within representations (Fonger, 2019) of quantities and their relationships in learning functions (Ellis, 2011). In other words, in order to develop a meaningful understanding of quadratic functions, the scholarship suggests that students need to reason about quantities and the relationships between them (Ellis, 2011) in creating, connecting, and translating among and within representations—which I refer to as representational fluency (RF) (Fonger, 2019).

I define “meaningful understanding” in this study as a student’s ability to create, interpret, invent, communicate about, and connect representations of functions within a flexible framework, including different approaches to reasoning about functions.

This study seeks to investigate a complex interrelation between students’ RF and FT. There may be some cognitive activities taking place when students engage in developing meaningful understanding. However, we in the mathematics education field still do not know ways of supporting this understanding (e.g., Ellis & Grinstead, 2008; Ellis, 2011). There are still unspecified elements about how students translate and connect among and within representations of covarying quantities of quadratic functions, as well as how they interpret and assign meaning to the concepts they are being taught.

This study explores ways to support students’ meaningful understanding of quadratic functions and ways to characterize the co-emergence of RF and FT in students’ thinking.

Statement of the Problem

Students often experience difficulties not only in developing robust understandings of quadratic functions as two quantities that covary simultaneously (Ellis & Grinstead, 2008; Ellis, 2011), but also in interpreting and translating concrete representations[1] of quadratic functions from one form to another (Even, 1998; Zaslavsky, 1997). While students are generally able to create translations and make connections within and among representations, they often experience difficulties transferring the underlying meaning from one representation to another (Adu-Gyamfi et al., 2012; Janvier, 1987a). For instance, students often experience difficulties in interpreting concrete representations of quadratic functions such as the following: (a) students might treat graphs as objects—pictorial entailments—rather than interpreting them as representing a relationship between two varying quantities (Moore & Thompson, 2015; Moschkovich, 1993; Zaslavsky, 1997); (b) students might articulate the parameters of quadratic functions in unsophisticated ways (Ellis & Grinstead, 2008; Even, 1998).

According to Dreyfus and Halevi (1991), "One of the central difficulties for students in the process of constructing their mental image of [quadratic] function is the establishment of the connection between the formula defining a function algebraically [e.g., $f(x) = 2x^2$] and its graphical representation" (p. 44). Other scholars have validated Dreyfus and Halevi's (1991) view by emphasizing not only connection and translation between graphical and algebraic representations, but also within and among graphic, symbolic, and tabular representations of quadratic functions with a sophisticated understanding (Borba & Confrey, 1996; Ellis & Grinstead, 2008; Even, 1998; Knuth, 2000; Zaslavsky, 1997; Zazkis et al., 2003).

In order to understand the problem further, it is important to note how students' RF and FT are connected to each other and to meaningful understanding. In the following section, I highlight studies that report that RF and FT are related and happen together.

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[1] Concrete representations are visualizations of mathematical concepts, including diagrams, symbols, graphs, and tables, all of which can be defined as ways of communicating and making sense of mathematical ideas (Dreyfus, 2002).

The Co-development of Students' Representational Fluency and Functional Thinking

In this section I articulate how RF and FT are related, and how in same cases the relationship between RF and FT remains ambiguous. This gives context to the current study, as a lack of RF skills in tandem with FT (Even, 1998) is a possible source of students' unsophisticated understanding and difficulties in learning quadratic functions. In other words, students' representational activity is intertwined with their FT in their development of a robust understanding of quadratic functions. Even (1998) states, "…linking representations are interrelated with another kind of knowledge and understanding, seeing the connection between the given equation and the related quadratic functions [the graph of quadratic function]" (p. 108). Based on the initial studies, we found that students' RF co-informs FT and vice versa (Fonger & Altindis, 2019; Altindis & Fonger, 2019). This finding is supported by other research as well (Even, 1998); including findings that students' FT co-informs students' RF (Ellis & Grinstead, 2008; Moore et al., 2013; Moore & Thompson, 2015). From this body of work, it is clear that there is a co-informing relationship between students' cognitive approaches to functions, how students think about quadratic functions—covariational reasoning[2] and correspondence reasoning[3]—and RF of quadratic functions.

Moore and his colleagues (2013) pointed out that students' representational practices are *rooted* in their covariational reasoning. Moore and Thompson (2015) validated that students' quantitative and covariational reasoning sets a groundwork for students' representational fluency: "We find that emergent shape thinking enables students to move among representations

[2]Covariational reasoning is "being able to move operationally from quantities or values, y_m to y_{m+1} coordinating with movement from quantities or values, x_m to x_{m+1}" (Confrey & Smith, 1994, p. 33). According to Thompson and his colleagues, covariational reasoning is being able to think about "two quantities' values varying" and understand that the two quantities are "varying simultaneously" (Thompson & Carlson, 2017, p. 425).

[3] Correspondence reasoning is understanding the relationship between the independent x and dependent y values by looking at the x and the y as coordinating dependent and independent values (Confrey & Smith, 1991; 1994; 1995).

while maintaining a subjective sense of invariance in the form of covarying quantities, thus supporting them in conceiving the ‘something’ that multiple representations are to represent.” (p. 788).

However, we do not know how covariational reasoning makes students move among representations, nor whether they are creating and connecting representations to represent covarying quantities. Hence, the nature of the intersection between students’ RF and FT is still ambiguous. This is a gap this study aims to address.

Supporting Students’ Co-emergence of Representational Fluency and Functional Thinking

Ellis and Grinstead (2008) argue not only for the need to focus on the translation of symbolic and graphical representations, but also for the need to shift the teaching of quadratic function to include a focus on reasoning with quantities, quantitative operations, and quantitative relationships. According to Ellis (2011), “One way to foster functional thinking is to leverage the power of the students’ capabilities to reason with quantities and relationships” (p. 215). In supporting students’ quantifications and understanding of quantitative operations, this approach leverages FT—in particular, covariational reasoning. Smith and Thompson (2007) suggest doing so by (a) forming an instructional sequence and (b) providing appropriate instructional support.

With that in mind, instructional activities include purposefully designed instructional sequences (Ellis, 2011; Ellis et al., 2015; Smith & Thompson, 2007). For instance, tasks’ characteristics should include features for reinforcing students’ FT within multiple representations, such as: making quantification visible to students, providing opportunities for measuring quantities, and providing subparts in the tasks to help students reflect on their thinking through translating and connecting among and within representations (Weber et al., 2014). In addition to providing appropriate instructional support, the instructional sequence can be designed using a variety of tools (e.g., GSP software, color tiles etc.) to provide appropriate support for students' developing RF and FT (Smith & Thompson, 2007).

Other researchers have built on this idea, pointing out that gaining a meaningful understanding of quadratic functions goes hand in hand with quantitative reasoning and representations (Fonger & Altindis, 2019). Moore and Thompson (2015) argued that teaching functions with a focus on quantitative and covariational reasoning encouraged students to think about concepts such as what makes a graph, what quantities form the graph, and how changing quantities affect the graph. They suggest that viewing the graph might help students to conceive of the graph as representing a relationship between changing quantities, rather than conceiving of the graph as an object. This is an example of how students might conceive of quadratic functions as describing two quantities which covary simultaneously on a graphical representation. Additional research is needed to elaborate on relevant support for the highlighted sophisticated learning. This study aims to address this gap.

Purpose and Aims of the Study

I situate this dissertation as an inquiry into how combining and coordinating various theories of quantitative reasoning (QR) (Thompson, 1994) and representations (Kaput, 1987; Dreyfus, 2002) might shed light on students' meaningful understanding of quadratic functions. I seek to advance the field of mathematics education by establishing an example that will enable researchers to design new practices and understand students' meaningful understanding of quadratic functions by networking local instructional theories (Gravemeijer & Cobb, 2006) in quantitative reasoning (Thompson, 1993) and representations (Kaput, 1987). This study aims to shed light on the following: (a) the nature of students' connections and translations among and within representations of quadratic functions in tandem with their FT; (b) ways to support students' co-emergence of RF and FT in learning quadratic functions; and (c) ways to bridge these two distinct yet related domains of literature—representations and QR—in supporting students' meaningful understanding of mathematics.

Research Questions

Two major questions guide this study:

1. What is the nature of the co-emergence of representational fluency and functional thinking among secondary school students as they develop a meaningful understanding of quadratic functions?
2. How do Turkish-American Muslim students' RF and FT co-develop as they develop a meaningful understanding of quadratic function in the context of a small-scale teaching experiment in an after-school setting?

Overview of Theoretical and Methodological Perspectives

In chapter 2, I will explore possible sources of students' difficulty in developing a meaningful understanding of quadratic functions in the existing literature. I will detail students' unsophisticated understandings of quadratic functions, how students conceive of graphs as objects, and how they provide unsophisticated interpretations for parameters of quadratic functions. I will investigate students' RF and FT, and the co-informing relationship between RF and FT in learning and teaching quadratic functions

In chapter 3, I will explore two distinct domains of the literature in detailing how the body of work on students' meaningful understanding is siloed into distinct groups—representations (e.g., Adu-Gyamfi et al.,2012; Bosse et al., 2012; Dreyfus, 2002; Fonger, 2019; Janvier, 1987; Kaput, 1987; Nitsch et al., 2015; Selling, 2016) and quantitative reasoning (e.g., Ellis, 2011; Ellis et al., 2015; Ellis & Grinstead, 2008; Confrey & Smith, 1994; Moore et al., 2013; Moore, 2014; Smith & Thompson, 2007; Thompson & Carlson, 2017; Thompson, 1994; Thompson, 2011). Representations set a groundwork for RF (Fonger, 2019), while quantitative reasoning sets a strong foundation for FT, in particular, covariational reasoning (Ellis, 2011). However, scholars typically have either focused on students' covariational thinking and paid little or less attention to students' RF, or vice versa. In this study, I networked the theories of quantitative reasoning (Thompson, 1993) and representations (Kaput, 1987b) as background theories for supporting students' meaningful understanding of quadratic functions (Simon 2009).

In chapter 4, I detail the methodology of my design-based research study, including the context and details of the small-scale teaching experiment. In this study, the method I utilized was creating a small-scale learning ecology—a teaching experiment (Steffe & Thompson, 2000). I created design conjectures informed by the affordances and influences of networking the theory of QR and the theory of representations (Kaput, 1987; Dreyfus, 2002; Thompson, 1994). My design conjectures included: (a) creating opportunities for students to construct mental images; (b) getting students to focus on quantitative operations rather than numerical operations; (c) emphasizing the role of concrete representations in quantitative processes; (d) grounding students' RF within a meaning of quantities; and (e) getting students to present the models of quantities in their minds via concrete representations. The design conjectures are also informed by three design elements: tasks and tools, norms, and teacher moves and prompts. The teaching experiment included eight teaching episodes, with each session lasting one hour. In the teaching experiment portion, the study included eight secondary school students from 8th, 9th and 10th grades, grouped into three groups. The study was conducted during the 2019–2020 school year in the CNY RISE Center. For data analyses, I networked analytical frameworks (Simon, 2009) for covariational and correspondence reasoning (Confrey & Smith, 1994; Thompson & Carlson, 2017), and RF (Fonger, 2019) for characterizing and supporting the co-emergent nature of students' RF in tandem with FT.

Definition of Key Terms

Functional thinking (FT) is a style of creative thinking about functions, creating patterns, and generalizing the functional relationships within concrete representations of functions (Blanton & Kaput, 2011; Stephens et al., 2017a). In this study, FT included two types of reasoning about functions: correspondence and covariational reasoning. *Correspondence reasoning* is understanding the relationship between the *x* and *y* values by looking at the *x* and the *y* as corresponding dependent and independent values or quantities (Confrey & Smith, 1991; 1994; 1995). *Covariational reasoning* is "being able to move operationally from y_m to y_{m+1} coordinating with movement from x_m to x_{m+1}" (Confrey & Smith, 1994, p. 33). According to Thompson and his colleagues, covariational reasoning is being able to think about "two quantities' values varying" and the two quantities "varying simultaneously" (Thompson & Carlson, 2017, p. 425).

Meaningful understanding is defined in this study to include a student's ability to create, interpret, invent, communicate, and connect representations of functions within a flexible framework, including different approaches to reasoning about functions.

Concrete (external) representations of functions are defined as visualizations of mathematical concepts, including diagrams, symbols, graphs, and tables, all of which can be defined as a way of communicating and making sense of mathematical ideas (Dreyfus, 2002).

Representational fluency (RF) is "the ability to create, interpret, translate between, and connect multiple representations—is a key to a meaningful understanding of mathematics" (Fonger, 2019, p. 1).

Thompson's theory of quantitative reasoning is based on Piaget's work on the mental images that a learner creates through their reasoning about quantities that covary (Piaget, 1967, as cited in Thompson, 1994). Thompson defines quantitative reasoning with three central tenets: *a quantity is in a mind, it is not in the world; quantification;* and *quantitative operations.* The three central tenets set a foundation for students' FT.

A connection within and among representations is defined as students' articulations of "an invariant feature of the mathematical object being represented across representational forms" (Fonger, 2019, p. 2).

The translation process, or cognitive process, involves transforming information concealed in a source representation into a targeted representation (Janvier, 1987b).

Interpretation of representations is any action that learners take in order to gain understanding or meaning from representation, or actions students may take while assessing the meaning of functions (Leinhardt et al.1990; Nitsch et al., 2015).

Creating, in the context of this study, is defined as a process of generating a part of or a whole representation when a function is not already provided (Bosse at al., 2012).

A teaching experiment is a small-scale version of a design-based research methodology in which a researcher takes a teaching role—a teacher-researcher—in exploring students' mathematical realities in a series of consecutive teaching sessions (Cobb & Steffe, 1983; Steffe & Thompson, 2000).

Chapter 2—Review of the Literature about Quadratic Functions

In this chapter, I define students' lesser sophisticated interpretations of quadratic functions as identified in the literature. In particular, I focus on two parts: (a) conceiving a graph as an object (a pictorial entailment) (e.g., Zaslavsky, 1997), and (b) interpreting parameters of quadratic functions in a lesser sophisticated manner (e.g., Even, 1998). I argue that one source of students' difficulty with developing a sophisticated understanding of quadratic functions originates from a lack of RF skills in tandem with FT. I also articulate connections between students' FT and RF in learning and teaching quadratic functions. I end this chapter by arguing that students' RF co-informs their FT, and vice versa, in the teaching and learning of quadratic functions.

Students' Unsophisticated Interpretations of Quadratic Functions

Scholars reported that students often develop an unsophisticated understanding of quadratic functions, such as: (a) conceiving a graph as an object (a pictorial entailment) (Ellis & Grinstead, 2008; Moschkovich et al., 1993; Zaslavsky, 1997); (b) only articulating the parameters of quadratic functions with an unsophisticated understanding (Borba & Confrey, 1996; Ellis & Grinstead, 2008; Even, 1998); (c) providing inappropriate generalization (Ellis & Grinstead, 2008; Wilkie, 2019); (d) conceiving of quadratic growth as exponential (Altindis & Fonger, 2018; 2019); and (e) depending heavily on algebraic representations, which limits the development of a robust understanding of quadratic functions (Ellis & Grinstead, 2008; Knuth, 2000). In order to address students' difficulty learning quadratic functions, studies have advocated building on knowledge of linear functions to learn quadratic functions (e.g., Movshovitz-Hadar, 1993); however, some studies contradicted this view (Ellis & Grinstead, 2008; Wilkie, 2019) by arguing that students might overgeneralize quadratic function based on their experience of learning linear function.

Seeing Graphs as Objects

One lesser sophisticated understanding of quadratic functions is that students may see a graph of a quadratic function as an object—referred to as seeing the graph as a *pictorial entailment*. In this case, students' understanding of quadratic functions is limited to the information that they can see on a graph—that is, they cannot see that the function on the graph has an infinite domain. (Leinhardt et al., 1990; Zaslavsky, 1997). For instance, if a student sees a graph that does not meet on the *y*-axis presented to her, then she may claim that the function does not have a *y*-intercept (Zaslavsky, 1997).

Moore and Thompson (2015) validated Zaslavsky's (1997) point by characterizing two ways students may conceive of a graph, which they referred to as *static shape thinking* and *emergent shape thinking*. They explain the two ways of thinking as such: *static shape thinking* does not involve quantitative reasoning, but students who think this way might infer things about quantities. Static shape thinking includes referring to the graph as "a piece of wire"— "graph as wire"—which means treating the graph as an object (Moore & Thompson, 2015, p. 785). On the other hand, *emergent shape thinking* is conceiving of a graph while thinking about what made this graph (thinking about the graph as two quantities that covary simultaneously; recording the relationship between two covarying quantities as a graph) and how quantities made this graph (using covariational reasoning). Furthermore, Moore and Thompson (2015) stated: "Students who are capable of thinking about graphs emergently gain insight into relationships between quantities that are more organic to the quantities and relationships" (p. 787).

Students' Difficulties Interpreting the Parameters of Quadratic Functions

A second major difficulty that students encounter when developing a meaningful understanding of function is understanding the role of the parameters[4] in $y = ax^2 + bx + c$. Dreyfus and Halevi (1990) explored the link between students' understanding of algebraic and graphical representations of quadratic functions; according to them, one of students' central difficulties was the articulation of parameters. Although students knew that the parameter *a* informed the opening of a parabola for the function, they could not articulate further. Furthermore, in a related study, students had difficulty differentiating between the parameters and slope of quadratic functions—conceiving the coefficient as a slope of quadratic functions (Ellis & Grinstead, 2008). And even teachers have difficulty articulating the parameters of quadratic functions (Even, 1998).

Ellis and Grinstead (2008) explored high school students' understanding of quadratic functions in the form of an algebraic equation ($y = ax^2 + bx + c$), and their findings revealed that students think of the parameter a as the slope of $y = ax^2 + bx + c$. They interpreted students' difficulty interpreting the parameters as coming from students' prior experience with linear functions. The idea that students might try to solve nonlinear function as if the function were linear was corroborated in other studies as well (Altindis & Fonger, 2019; Zaslavsky, 1997). This suggests that a heavy focus on a single representation (e.g., symbolic) creates difficulty for students in differentiating among the parameters of a quadratic function and its slope.

Hence, too much emphasis on one representation, a symbolic representation, without further support regarding what type of phenomenon this particular representation presents about

[4] In $y = ax^2 + bx + c$, the parameters of such a function are a, b, and c.

quadratic functions, might create limited student understanding, in particular, a limited understanding of the roles of a quadratic equation's parameters (Ellis, & Grinstead, 2008; Even, 1998).

Representational Fluency and Functional Thinking in Learning and Teaching Quadratic Functions

Scholars have reported an explicit connection between students' knowledge of representation and knowledge of FT in linking multiple representations (Ellis & Grinstead, 2008; Even, 1998; Knuth, 2000; Yerushalmy, 2006; Wilkie, 2019). For example, Even (1998) focused on exploring prospective secondary teachers' processes when connecting multiple representations and the functional approaches they used. Even explored 152 prospective secondary teachers' ability to link representations and how the process of linking intertwined with types of understanding about functions. Seven participants were chosen among the sample who could not find the solution to a quadratic function equation; then they were asked to use a graph to solve it. Two among the seven participants were able to find the solution to the quadratic equation in graphical form by linking the graph to the equation. However, more than half of the remaining prospective teachers still could not see the solution on a graphical representation. This finding brings up the question of what type of functional understanding is required when linking graphical representations to symbolic representations of quadratic functions.

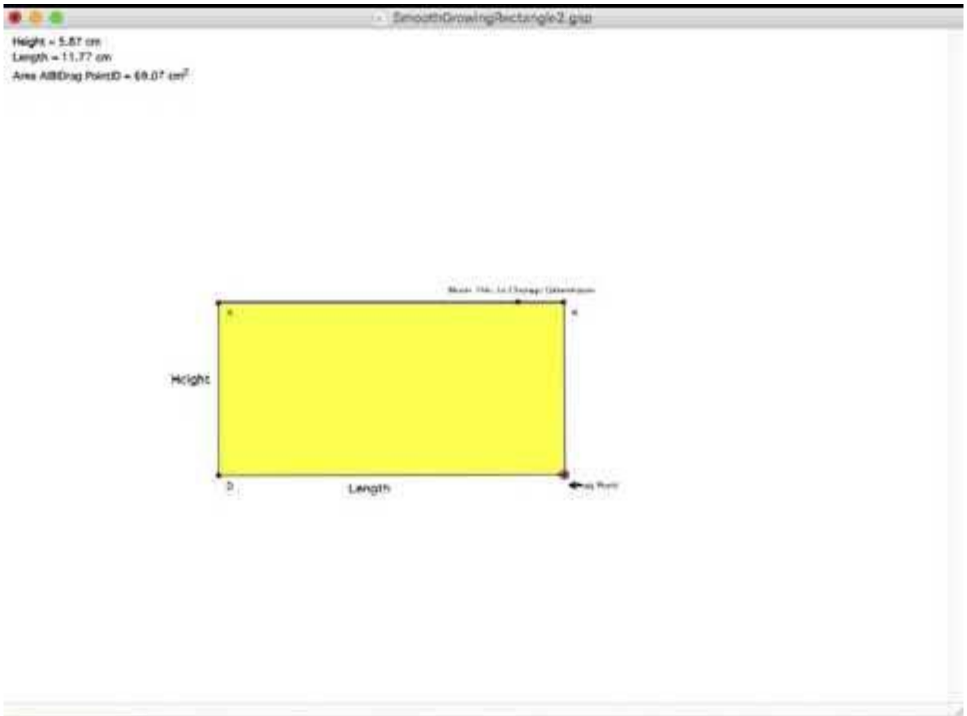
Even's (1998) findings indicated that the nature of the connection between students' representational knowledge and *pointwise* and *global* approaches to function is complicated; if students were able to make meaningful connections between representations, they were able to make these connections using a *pointwise* approach to function (thinking of a function as a set of discrete points) and a *global* approach to function (an approach to the behavior of a function). In other words, students' representational skills become sophisticated when the representational activities were grounded in functional thinking. Furthermore, students who employed the global approach (this approach is similar to covariational reasoning) to reasoning about multiple representations were more likely to create meaningful connections between representations than students who took a pointwise approach (this approach is similar to correspondence reasoning). Even (1998) focused only on linking between the pointwise and the global approaches, and connection and translation of representations.

Ellis and Grinstead (2008) furthered this line of research about merging RF and global and pointwise approaches to function and advised scholars to consider merging RF and quantitative reasoning. Ellis and Grinstead argued for the need not only to focus on translation between symbolic and graphical representations, but to shift learning function within quantitative reasoning toward representing functions as quantities and the relationship between them. This parallels the fact that Even (1998) reported that translating within and among multiple representations is not enough for prospective teachers to make sense of quadratic functions. Linking representations requires students to have functional understanding to be able to make sense of the functions, since functional understanding and RF are intertwined (Even, 1998)

Developing a sophisticated understanding of function also goes hand-in-hand with quantitative reasoning and representations (Fonger & Altindis, 2019; Thompson, 1994; Thompson & Carlson, 2017). For instance, let us imagine that students are solving a task situated in quantities, a growing rectangle task: when students drag the corner of the rectangle, the height, length and area of the rectangle increase (see Video 1).

Video 1

The Growing Rectangle Task



Note. The embedded video can also be found at this link: https://www.dropbox.com/s/4xwt4kxyl08sn02/TE_D3_GSP_Video-Growing-rectangle.mov?dl=0

In the task, students are asked to explain the relationship between the length, height, and area of a growing rectangle. Because the quantities—height, length and area—all increase, the students need to think about how a change in height would affect the length and area. The quantitative relationship between the height and the length is different than the relationship between height and the area, and these relationships can be visualized with representations. The relationship between height and length is a linear graph/straight line, which is connected to the symbolic representation $y = 2x$, while the relationship between height and area is a curve, which is connected to the symbolic representation $y = 2x^2$. Students may need to create and connect multiple representations to talk about and articulate quantities and quantitative relationships in this task—that is, to develop functional thinking while creating and interpreting representations.

A Co-informing Relationship Between Students' Representational Fluency and Functional Thinking

A covariational reasoning approach to quadratic functions can provide a foundation for students to understand the nature of quadratic growth and can support a meaningful transition to correspondence reasoning (Ellis, 2011). Ellis argues that covariational reasoning would naturally transition to correspondence reasoning through translation within and among representations. Ellis's study found the following:

> In both the linear and the quadratic case, the students made use of different representations (tabular, algebraic, and graphical) to describe and make sense of the quantitative situations involving gear ratios, speed, or growing rectangles. Since each representation was a way of describing the quantitative phenomena, rather than an instructor-introduced artifact divorced from any referents, the connections across the representations were natural ones that enabled seamless transitions (p. 233).

In Ellis's study, students created tables, graphs, and algebraic representations as they described and interpreted covarying quantities. Because each representation was a way of approaching those quantities, the students naturally made connections across representations. Since covariational reasoning entails complex cognitive activity in which students might need to engage in a sophisticated meaning-making process (Thompson & Carlson, 2017), a translation process within and among representations also involves cognitive activity (Janvier, 1987b). Therefore, RF co-emerges with FT, in particular, covariational reasoning.

When reasoning about quantities and the relationships between them, students' meaning-making process might involve both the use of representations and connecting multiple representations. One of the reasons for this is that representations (e.g., a tabular representation) might make functional relationships visible to students (Confrey & Smith, 1995). Related, Ellis (2011) and Moore, Paoletti, and Musgrave (2013) found that students' connection between multiple representations is grounded in covariational reasoning. They claimed that students' covariational reasoning fostered their understanding of the relationships between multiple representations. Moore et al. (2013) wrote that students engaged in covariational reasoning "to make sense of and conceive invariant relationships among multiple representations" (p. 472). Moore and his coauthors claimed that students' covariational reasoning fostered their understanding of the relationship between graphical and algebraic representations of quadratic functions on both Cartesian and Polar coordinate systems. Moore and Thompson (2015) made a similar argument: "We find that emergent shape thinking enables students to move among representations while maintaining a subjective sense of invariance in the form of covarying quantities, thus supporting them in conceiving the 'something' that multiple representations are to represent." (p. 788).

These studies report representations being used with covariational and correspondence reasoning, shedding light on important cognitive processes involved in representing mathematical ideas. However, in previous studies, not much attention was given to students' RF within and among representations of quadratic functions in tandem with evidence of covariational and correspondence reasoning. Furthermore, characterization of the intertwined nature of students' RF in tandem with FT remains an important area of inquiry (Ellis & Grinstead, 2008; Even, 1998; Dreyfus & Halevi, 1991).

This study aims to define ways to support students to co-develop RF and FT, and then define ways to characterize students' FT on each representation. Additionally, the study aims to characterize students' co-development of RF and FT as they create connections across representations to represent emergent quantitative relationships.

Chapter Summary

In chapter 2, I explored existing literature on quadratic functions. I reported that the literature suggests students may develop an unsophisticated interpretation of quadratic functions in the following two ways:

1. Students may conceive a graph of a quadratic function as an object—wherein students' understanding about the function is limited to what they see on the graph (Moore & Thompson, 2015; Zaslavsky, 1997).
2. They may have an unsophisticated understanding of parameters of quadratic functions—students may treat the parameters of quadratic functions as slopes (Ellis & Grinstead, 2008; Even, 1998).

I articulated that students' creation and translation of quadratic functions across representations requires understanding of the functions; this was first identified by Even (1998), who stated that students need pointwise and global approaches to functions (two specific forms of FT) in creating and translating among and within representations. Other scholars validated Even's points by providing evidence that students' representational activities are intertwined with their FT, in particular, covariational reasoning (Ellis & Grinstead, 2008; Moore et al., 2013). Additionally, I identified that while scholars indicated the importance of learning quadratic functions through reasoning about quantities and their relationships (Ellis, 2011), ways to support students as they create, connect, and translate the quantitative relationships among and within representations is still an open area of inquiry. The nature of students' co-development of RF and FT is still ambiguous.

In this study, my aim is not to identify whether the concept of multiple representations is "better than" FT; my intention instead is to network FT with the theory of multiple representations and understand them as different but complementary perspectives with which to explore and analyze the ways that students can meaningfully learn quadratic functions. In response to the difficulties students encounter in learning functions, I intend to follow the steps of studies which highlighted the need for teaching functions through quantitively rich context in tandem with flexibility in representations (Borba & Confrey, 1996; Ellis & Grinstead, 2008; Even, 1998). In the next chapter, I will explain how the literature led to specific design conjectures that might support students' meaningful understanding of quadratic functions.

Chapter 3—Background Theories and Theoretical Frameworks

I begin this chapter with an articulation of a theoretical orientation on what constitutes meaningful understanding. This study is informed by the background theories of representations and quantitative reasoning. In this chapter, I will detail historical development of the theory of representations (Kaput, 1987a; 1987b); within that, I will define concrete representations: graphs, tables, symbolic equations, and diagrams. I will articulate how the literature of representations informed the idea of representational fluency. I will define Thompson's theory of quantitative reasoning (1994), as well as FT, which is a way to organize students' cognitive approaches to classify students' conceptions of the meaning of functions. I will articulate how basic tenets of quantitative reasoning set a strong foundation for covariational reasoning and eventually FT; and I will finish the chapter with an argument that students' RF may co-inform FT in learning functions and vice versa.

A Theoretical Orientation on Meaningful Learning

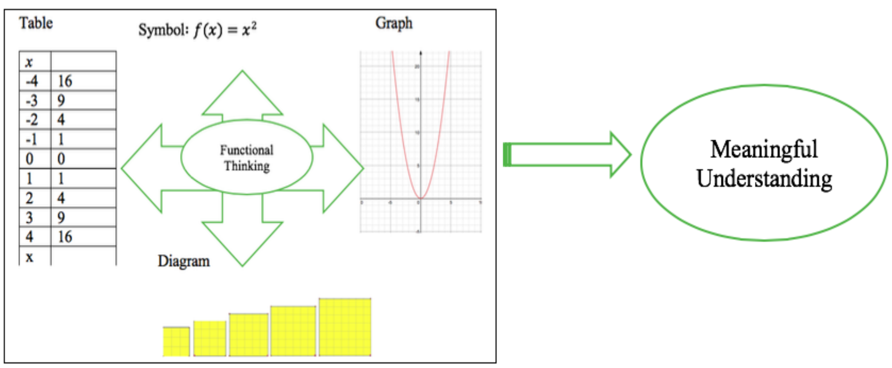
Understanding is "more than knowing or being skilled" (Dreyfus, 2002, p. 25). According to Voigt (1994), the mathematical meaning of understanding is an "individual sense-making process" and "development of mathematical knowledge" (p. 276). Sfard and Linchevski (1994) further posit that students' construction of meaning evolves with a skill of recognizing "abstract ideas hidden behind symbols." (p. 224). Meaningful learning is a process that occurs through using our senses by interacting, touching, seeing, and giving meaning to what we see, feel, and touch, then creating new images. In other words, meaningful learning is a process that results from the act of creation (Fonger & Altindis, 2019). With that in mind, meaningful learning of mathematics can be defined as creating, connecting, inventing, and translating within and among representations with a sophisticated interpretation of varying quantities.

Meaning-making might happen through students' interpretation of situations, conversations, symbols, and operations, all during various stages of learning (Lobato, 2014; Thompson & Saldanha, 2003; Vinner & Dreyfus, 1989; Voigt, 1994). Thompson (2016) stated that "the foremost characteristics of meanings is that they are invoked in the act of interpretation." (2016, p. 456). Fonger (2019) further defines meaningful learning as being able to *create, interpret*, *translate,* and *connect* representations of mathematical objects with a sophisticated level of understanding—that is, meaningful learning involves a high level of RF.

The process of meaning-making includes creating an image, connecting representations, comprehending, and explaining a mathematical idea within multiple perspectives (Dreyfus, 2002; Lobato et al., 2013). Dreyfus (2002) found that understanding happens when students engage in multiple mathematical activities. He further suggested that students taking time to reflect on the mathematical process might also improve their understanding of mathematical objects. In parallel with Dreyfus, Lobato and her coauthors (2013) explored ways to get students to meaningfully visualize the underlying concepts of mathematics by paying close attention to "the aspects of mathematical understanding" and "meaning, image, connection, way of comprehending the situation, and explanation" (2013, p. 26). Hence, I view the meaning-making processes as a foundation of conceptual understanding, and I believe that co-developing RF and FT is an essential portion of students' meaning-making processes.

According to Dreyfus (2002) and Lobato (2013), meaningful learning has multiple aspects, all of which require learners to create, invent, interpret, and comprehend within and among multiple representations of functions, reflect on their thinking, and explain their reasoning. Figure 1 represents the idea of developing a meaningful understanding of quadratic functions by creating, translating, and connecting among and within representations in tandem with FT. In Figure 1, each arrow represents students creating a connection across multiple representations when reasoning about quantitative relationships that covary across representations. The role of FT in the middle presents that the invariant feature across multiple representations is the quantitative relationship. In other words, each representation—table, graph, symbolic equation, and diagram—represents the quantitative relationship.

Figure 1
A Logic Model of Meaningful Learning



Note. A table, graph, symbol, diagram are types of representations.

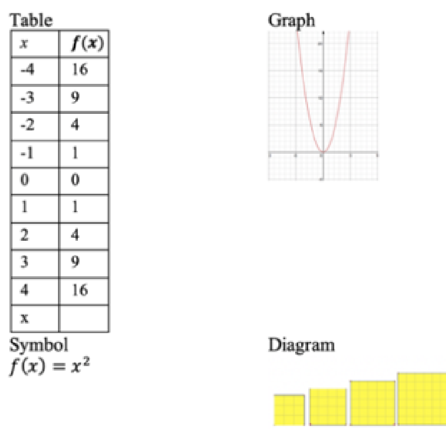
Theory of Representations

Representations have been a focus of the mathematics education research community for decades. Scholars have explored students' understanding of mathematics in regard to their representational activity, in particular, their translations between and among representations—creating, interpreting, and transforming representations (e.g., Adu-Gyamfi & Bosse, 2013; Janvier, 1987a; Movshovitz-Hadar, 1993). In general, the relationship between mathematics and representations is understood as cause and effect—as long as teaching and learning mathematics exists, representations and their role will exist within it. As Borba and Confrey (1996) write, "Mathematics does not exist independently of its representational forms; it exists through those forms" (p. 335). Furthermore, it has been conceived that *the idea of representation is continuous with mathematics itself*. Hence, as long as learning and teaching of mathematics are continuous, the inquiry into representations will exist (Kaput, 1987b).

There are four broad interacting types of representations: *cognitive and perceptual representations, explanatory representations, representations within mathematics, and external symbolic representations* (Dreyfus, 2002; Kaput, 1987a, p. 23). In this study, I will focus on external (concrete) representations. Throughout this study, the use of the word representation refers to the concrete representations of functions: graphs, tables, symbolic equations, and diagrams. Concrete representations, for functions, are defined as visualizations of mathematical concepts, including diagrams, symbols, graphs, and tables, and they are a way of communicating and making sense of mathematical ideas (Dreyfus, 2002); Figure 2 gives examples of a variety of concrete representations.

Figure 2

An Example of Concrete Representations of a Quadratic Function



Note. The figure represents an exemplar of concrete representations—a graph, a table, a diagram, and a symbolic equation of the same quadratic function.

Representations are viewed not only as representing mathematical constructs, but also as learners' activities. As stated by Von Glasersfeld (1987), representations are always "the results of someone's productive activity" (p. 217). Von Glasersfeld stressed the point that learning happens based on a learner's involvement and experience. Kaput (1987a;1987b) furthered this by stating the theoretical needs and practical concerns regarding the theory of representations. The practical concerns Kaput discussed centered around students' difficulties in translation and connection among and within multiple representations. These practical concerns will be elaborated in chapter three, section: *Supporting Students' Development of Representational Fluency*. With a theory for representations, Kaput (1987a) intended to shed light on developing students' ability to choose, compute, interpret, and translate among and within representations.

Teaching and Learning Functions via Concrete Representations

In this study, concrete representations play an important role by informing design principles and instructional support when teaching and learning quadratic functions (I will further articulate concrete representations' role in the design principles in chapter four, in the section: *Affordances and Influences of Networking QR and Representations*).

Dreyfus (2002) described four stages of the learning process in terms of concrete representations: "a) Using a single representation, b) using more than one representation in parallel, c) making links between representations, and d) integrating representations and flexibly switching between them" (2002, p. 39). In the current study, students are asked to represent quantities and their quantitative relationships as functions using multiple representations, specifically, by asking them to create representations, to use representations in parallel, and to integrate representations of the same function—that is, the same quantitative relationship (Dreyfus, 2002)—in order to help them create multiple mental images of quadratic functions

(Kaput, 1987a). Dreyfus's (2002) stages of understanding concrete representations inform the design principles and instructional supports; for further articulation of design principles, see chapter four, section: *Affordances and Influences of Networking Quantitative Reasoning and Representations.*

Dreyfus's four stages of learning with representations provide opportunities for students to select a representation(s) that lands on a productive learning activity, in part because students prefer different representations depending on the difficulty they experience in learning functions. In terms of representations, their study suggested that high-achieving students prefer problems presented via graphical representations, while low-achieving students prefer problems presented via tabular representations (Dreyfus & Eisenberg, 1981; Yerushalmy, 2006). If a problem is given to students which they have difficulty understanding, then they need to have the skill of creating another representation that is more meaningful for them.

However, most students do not recognize that multiple representations of an underlying mathematical idea embody the same information (Dufour-Janvier et al., 1987a). Students also do not see solving a problem with multiple representations as producing the same answer. They expect that the solution to different representations will produce different answers (Hitt, 1998). Dreyfus's four stages of learning with representations may enable students to conceive that varying concrete representations can represent the same underlying mathematical idea, and that they produce the same answer.

This study is intended not only to support students in creating, interpreting, and connecting representations, but also to help them to conceive of functions as quantities that change continuously and smoothly (Thompson & Carlson, 2017). I intend to support and characterize students' co-emergence of representational fluency—creating, connecting and translating among and within representations—and FT—a way of characterizing students' cognitive approaches to functions.

Quantitative Reasoning

Quantitative reasoning is not only a foundation for preparing students for advanced reasoning in calculus, but also a strong foundation for developing students' algebraic and covariational reasoning (Smith & Thompson, 2007; Ellis, 2011). In particular, QR is a foundation for reasoning with magnitudes which provides productive, coherent reasoning within magnitudes of quantities. Furthermore, quantitative reasoning is a foundation for covariational reasoning; covariational reasoning empowers students to see the invariant relationship between changing quantities (Thompson, 2011).

Thompson's theory of QR is based on Piaget's work on the mental images that learners create, or *mental constructions* (Thompson, 1994). Creation of mental constructions is a demanding process for students learning to conceptualize quantities, quantification, and relationships among quantities (Thompson, 2011). According to Piaget (1967), *images* are conceptualizations that people must create, not something that already exists in their understanding of functions or the world. Piaget (1967) theorizes that a given subject's mental operation of a function and their mental image of it are connected, and that the subject makes sense of an object by interacting with it. Following this logic, students might form an image of a function through reasoning about quantities that covary (Thompson, 1994). According to Thompson (1994), students' ability to build an image of changing quantities involves several layers: first, perceiving a change in one quantity; second, shifting into conceiving the two quantities as coordinated; and, finally, constructing an image of the two changing quantities as they covary simultaneously. These categories are based on Piaget's constructivist theory of learning (as cited in Thompson, 1994).

The *U.S. Common Core State Standard for Mathematics* (CCSSM) emphasizes quantities as numbers with units about which one can "reason abstractly and quantitively," (CCSSM, 2010, p. 6) but the CCSSM's definition of quantity is different from Thompson's definition. CCSSM defines quantities as *numbers with units*; Thompson, on the other hand, defines quantity as *a quality of an object* (CCSSM, 2010; Thompson, 1993). This study will use Thompson's definition of quantity—a quality of an object which is measurable (Thompson, 1993).

For instance, let us imagine a person is running while a trainer is watching the distance from the starting point with a stopwatch (shown in Figure 3). As the distance from starting point increases, the time also increases, so both the time and the distance are varying simultaneously. The trainer, who is watching the time and the running person, can visualize that the time and the distance are covarying simultaneously; therefore, the trainer is engaging in covariational reasoning.

Figure 3

A Visual Image of a Situation for Covariational Reasoning

Note. This figure is from Thompson and Carlson (2017, p. 426).

Setting a Strong Foundation for Functional Thinking Through the Central Tenets of Quantitative Reasoning

There are three central tenets of quantitative reasoning: quantity in mind (not in the real world), quantification, and quantitative operations, which inform the design principles of the present study. The central tenets of QR can set a strong groundwork for students' FT, covariational reasoning in particular (Ellis, 2011; Thompson, 2011). Ellis (2011) argued that "one way to foster FT is to leverage the power of students' capabilities to reason with quantities and their relationships" (p. 215).

Quantity in Mind. The concept of *quantity in mind* holds that quantities are mental constructions, and that the construction of mental images of quantities requires a great deal of effort for students (Thompson, 2011).

Quantification. Quantification is not the just process of students assigning numerical values to an attribute of an object; quantification is defined as "the process of conceptualizing an object and attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bi-linear, or multi-linear) with its unit." (Thompson, 2011, p. 37). Although the motive behind quantification is to measure a quantity, the quantification process includes (a) what it looks like to measure a quantity, (b) "what one measures to do so" and (c) "what a measure means after getting one" (Thompson, 2011, p. 38). For instance, students first conceive *the attribute of an object* (e.g., the height of a triangle) which could be measured. Then students think of a unit to measure the attribute, in this case the height of the triangle. A unit of the measure for the length, in this case, is centimeters. Finally, students conceive a relationship between a unit of measure—centimeters—and the measure of the length (e.g., 10 cm) as an attribute of the triangle.

Quantification of a Rate. Quantification of rate is a more complicated process than quantification of an attribute of a quantity (e.g., a length) (Johnson, 2015). *Quantification of a rate* involves units, which includes composed units (Johnson, 2015). Consider, for example, the relationship between the height and area of a triangle (see Video 1). When we increase the height of the triangle, the area of the triangle will increase. Let us imagine the ratio between the area and the height of the triangle, which are varying quantities. In order to quantify the ratio between height and area, we need to measure the rate of change between the height and area—that is, we need a unit to measure the relationship between height and area. We then need to relate the unit of measure to the measure of the attribute—relating the rate relationship between the height and the area to the rate of change of the area with respect to height.

Quantitative Operations. Quantitative operations are not the same as numeric operations; quantitative operations are the relationships among quantities. Numeric operations, on the other hand, are operations done within numerical relationships without conceiving of the meaning that those numbers present (Thompson, 2011). Quantitative operations involve operating within quantities and the relationships between these quantities (Thompson, 2011). Quantitative operations include creating new quantities by: (a) measuring things, (b) calculating ratios of quantities, or (c) operating quantities to create a new quantity (multiplicative comparisons can be created from a quantitative operation, but not from a numerical operation) (Smith & Thompson, 2007).

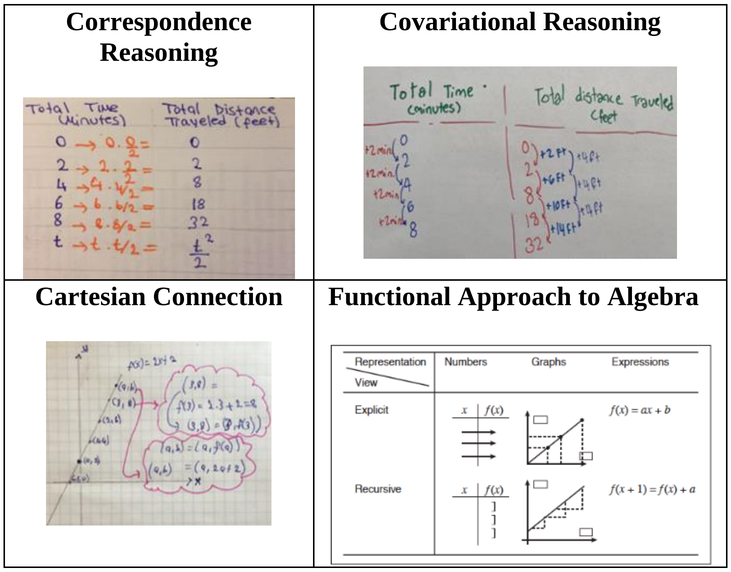
This theoretical foundation informed the design of the study. I chose the design principles and instructional supports according to the affordances of QR and representations, as follows: (a) creating opportunities for students to construct mental images of covarying quantities; (b) getting students to focus on quantitative operations rather than numerical operations; (c) emphasizing the role of concrete representations in quantitative processes; (d) grounding students' RF within the meaning of quantities; and (e) getting students to present the models of quantities in their minds via concrete representations. These design principles and how I operationalize them with RF and FT is elaborated in chapter four, in the section: *Affordances and Influences of Networking QR and Representations*.

Functional Thinking

Functional thinking (FT) is a process of "generalizing relationships between covarying quantities and representing and reasoning with those relationships through natural language, algebraic (symbolic) notation, tables, and graphs" (Blanton et al., 2015, p. 43). In the context of this study, FT is used to mean creating a generalized functional relationship between covarying variables by connecting, interpreting, and translating among and within concrete representations of functions (Dreyfus, 2002). In other words, the concept of FT is a way to organize the cognitive approach in order to characterize students' meanings of functions. FT in this study is framed to include: *a functional approach to algebra* (Yerushalmy, 2000; 2006), *correspondence* and *covariational reasoning* (Confrey & Smith, 1994; Thompson, 1994; Thompson & Carlson, 2017), and *Cartesian connection* (Moschkovich, 1993; Knuth, 2000). I conceptualize all these approaches under the umbrella of FT. I represent the four types of functional thinking on Figure 4.

Figure 4

Visual Images of the Four Types of Functional Thinking



Note. Figure 4 represents four visual images of FT; the image of a functional approach to algebra is from Yerushalmy (2000, p. 359).

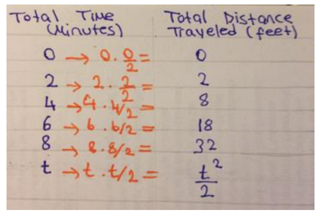
Correspondence Reasoning

Correspondence reasoning is defined as determining output (dependent) values as related to input (independent) values and identifying a symbolic equation describing the relationship between the dependent and independent values. (Confrey & Smith, 1991; 1994; 1995).

The correspondence perspective is an abstract definition of a function that focuses on a rule-style definition of the function. Correspondence reasoning is central to teaching and learning function in schools and colleges (Thompson & Carlson, 2017). In Figure 5, I provide a basic example of correspondence reasoning.

Figure 5

Correspondence Reasoning on a Quadratic Function



Note. This figure shows a correspondence approach to the time values, finding the corresponding distance values by creating a generalized symbolic equation on a tabular representation.

Covariational Reasoning

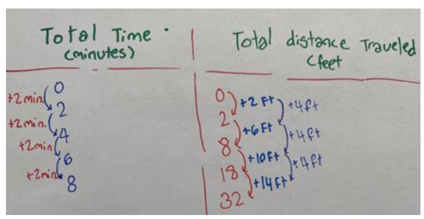
According to Confrey and Smith (1995), the covariation perspective can be defined as an understanding of the relationship of change between two or more quantities (e.g., the change in x and the change in y)—that is, "describing how one quantity varies in relation to another" (p. 79). Confrey and Smith (1994) did not ground their definition of covariation in quantitative reasoning per-se, but directly on radical constructivist logic (Confrey & Smith, 1991; 1994; Piaget, 2001; VonGlasersfeld, 1995). Confrey and Smith's (1994) definition: "A covariation approach, on the other hand, entails being able to move operationally from $y_m$to y_{m+1} coordinating with movement from $x_m$to x_{m+1}" (p. 33).

In parallel to Confrey and Smith (1994), according to Thompson and his colleagues, covariational reasoning is being able to think about "two quantities' values varying" and the two quantities "varying simultaneously" (Saldanha & Thompson, 1998; Thompson & Carlson, 2017, p. 425). Thompson and Carlson's (2017) definition of the covariation approach builds upon the foundation of Thompson's theory of quantitative reasoning, which itself is built upon the theory of radical constructivism (Piaget, 2001; VonGlasersfeld, 1995, as cited in Thompson & Carlson, 2017). Thompson's definition of covariation is students' understanding of relationships between quantitates that vary continuously.

In this study, I will use Thompson and Carlson's (2017) definition of covariational reasoning: being able to conceive of "two quantities' values varying," and the two quantities "varying simultaneously" (p. 425). Figure 6 presents covariational reasoning (Confrey & Smith, 1994) by coordinating the change of time variables (+2 minutes) with the change in distance and finding that the change of change of the distance variable (+4 feet) is a constant.

Figure 6

Covariational Reasoning on a Quadratic Function



Supporting Students' Development of Representational Fluency

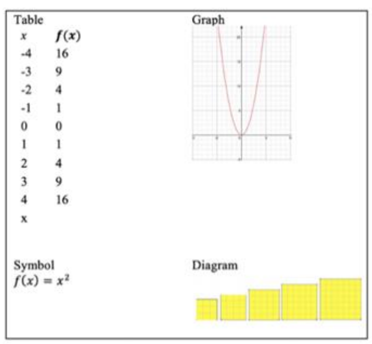
RF is an essential skill for students that can be developed by creating, interpreting, translating, and connecting among and within representations with a sophisticated understanding of the mathematical phenomenon (Fonger, 2019). I will define each of the discursive activities of creating, interpreting, connecting, and translating in the following section.

Creating, in the context of this study, is a process of generating a partial or whole representation when the function is not already provided (Bosse et al., 2012; Nitsch et al., 2015). *Interpretation* of representations is any action that learners take in order to gain understanding or meaning from a representation, or an action a student takes while assessing the meaning of functions (Leinhardt et al., 1990; Nitsch et al., 2015).

The *connection* within and among representations is defined as students' articulation of "an invariant feature of the mathematical object being represented across representational forms" (Fonger, 2019, p. 2). For instance, in the tabular representations shown in Figure 7, a student could see that the rate of rate of change of a quadratic function is constant by dividing the change in height ($f(x)$) values by the change in time (x) values of the height-versus-time function. The student creates a quadratic function graph and states that the rate of rate of change will be constant (see Figure 7). Hence, this particular student will be able to see that the rate of change of the rate of change of a quadratic function is constant on both the tabular and graphical representation of the quadratic function by connecting the representations.

Figure 7[5]

An Example of Concrete Representations of a Quadratic Function Representing the Height and Area of a Growing Square



Note. This figure is added in this section to help the reader to visualize quantities—the height and area of the growing square—and their quantitative relationship on the concrete representations: diagram, table, graph and symbolic equation.

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[5] Figure 7 is same as figure 2; I have added Figure 7 to help the reader to visualize the concrete representations I reference in the text.

The translation process, or cognitive process, involves *transforming information* concealed in a source representation into a targeted representation (Bosse et al., 2012; Janvier, 1987a). Translations of representations require the learner to have a "grasp" (meaningful interpretation) of each representation (Janvier, 1987a) This gives meaning to the source representation and allows students to interpret the same meaning in the form of the targeted representation (Lesh et. Al., 1987). For example, to translate the symbolic equation of a quadratic function, $y = x^2$, as a graph, students would need to interpret $y = x^2$ as a quadratic growth function with an origin on the y-intercept.

The difference between connection and translation is that connection involves further articulation of how the invariant feature of the mathematical phenomenon is represented across representations.

The Nature of Translation and Connection

When students create translations and connections among representations, they connect the invariant mathematical phenomena. Translation and connection of concrete representations is more than just mapping one representation to another representation of the same mathematical idea. It is being mindful of what is being translated and connected across representations. As highlighted by Adu-Gyamfi and his colleagues (2012), "It should be noted that it is not the representations that are translated but rather the ideas or constructs expressed in them" (p. 159). For instance, if using Figure 7, when paying attention to the representation with a series of diagrams, by viewing the height of the growing rectangle in the form of a diagram, students should be able to visualize the increase in the height. Then they can create a table to translate the growth in the height (x) to the table. In this case, the numbers on the table are not abstract numbers to students; rather, they represent the height of the rectangle, which keeps increasing. So, when the students translate, they are not translating the representation, but the idea behind it.

Difficulty in Translation and Connection. Janvier defines translation and connection with *directionality* (Janvier, 1987a). Mapping one representation to another representation requires cognitive activity (Janvier, 1987a); students have difficulty in translating different forms of representations depending on the demand for translation action (Adu-Gyamfi et al., 2012; Nitsch et al., 2015). For example, in Figure 7, students may translate the graph of the quadratic function into an algebraic equation, $f(x) = x^2$, which involves a certain degree of cognitive activity; the cognitive activity will be different when they translate the algebraic equation—$f(x) = x^2$—into a graph. Furthermore, students' cognitive approaches to a construct in a representation might be different for each representation, and they may also differ while moving from source to targeted representations based on students' functional understanding (Adu-Gyamfi & Bosse, 2013; Janvier, 1987).

The difficulties students may experience in translation and connection within and among representations might come from two sources: (a) the fact that each type of concrete representation requires different interpretations with varying difficulties, and (b) the fact that some of the translation and connection is harder because it requires deep conceptual understanding (Bosse et al., 2012). Although studies reported that students' difficulty in translation either depends on their inability to make the translation (Bosse et al., 2012) or the complexity of the function concept (Carlson et al., 2002), we are still left with uncertainty about how to support and characterize students' translation and connection among and within representations in tandem with their FT.

In summary, RF is not limited to creating, connecting, and translating among and within concrete representations; it is more about the mathematical phenomena being translated and connected among and within representations. Students who engage in greater sophisticated RF should have developed a meaning of the mathematical phenomena in both source and targeted representations (Janvier, 1987b; Lesh et al., 1987); the meaning students construct may co-inform FT. How students conceive of quantities and their relationships as represented by representations while they translate and connect among and within representations is still ambiguous. With this study, I intend to network FT and RF for the purpose of characterizing and supporting students' meaningful understanding of quadratic functions.

Theoretically Grounded Analytical Frameworks: Representational Fluency and Covariational Reasoning

For this study, I networked analytical frameworks as *lenses* (Simon, 2000) for analyzing RF (Fonger, 2019) and two types of FT—covariational (Thompson & Carlson, 2017) and correspondence reasoning (Comfrey & Smith, 1991)—in order to characterize the nature of students' co-development of FT and RF. In the above section, I defined these constructs. In this section, I detail a *networked* analytic lens.

The first analytic lens I networked is based on Carlson and her coauthors (2002) identification of covariational reasoning as "the cognitive activities involved in coordinating two varying quantities while attending to how they change about each other" (p. 354). They developed a framework of five mental actions that students engage in during covariational reasoning about the rate of change of covarying quantities. Later on, Castilla-Garsow (2012) further developed covariational reasoning by differentiating images of students' thinking as "chunky" or "smooth." In 2017, Thompson and Carlson revised their theory of covariational reasoning, expanding on Carlson and her colleagues' (2002) definition of the five mental actions involved in the covariational reasoning of rate concepts and integrating Castilla-Garsow's (2012) descriptions of chunky reasoning versus smooth and continuous reasoning. In the current study, I employed this revised version of the framework.

The second analytic lens I networked is based on Fonger's (2019) representational fluency framework. With this framework, RF can be measured in terms of the meaningfulness of students' learning, from lesser meaningfulness to higher meaningfulness. In Fonger's study (2019), each student approach to a problem was analyzed for meaningfulness on a scale of lesser to greater meaningfulness. Lesser meaningfulness includes *pre-structural understanding*, which is creating and interpreting one representation with unsophisticated thinking, and *multi-structural understanding*, which is creating or connecting multiple representations with unsophisticated thinking. Higher meaningfulness can then fall into the categories of *unistructural understanding*—creating and interpreting one representation with sophisticated understanding but not making connections— and *relational understanding*, which is creating, interpreting, and connecting multiple representations with sophisticated thinking.

Table 1 introduces the networked analytic lens adopted for this study. The left-most column "Level" lists the five levels in Thompson and Carlson's (2017) framework. The "Definition" column lists the definitions from their work. The "Verbal and Representational Activities" column elaborates my interpretation of the representational behaviors about quadratic functions in the growing rectangle context. Notice how the language of functional thinking and representational fluency is networked or combined.

Table 1

Networked Lens: Major Levels of Covariational Reasoning and Representational Fluency

Level	Definition	Verbal and Representational Activities in the Growing Rectangle Context
Smooth Continuous Covariation	"The person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly, moreover, continuously."	The student creates, connects, or interprets representations, thinking that rectangle area and height are continuously changing, and then they are varying smoothly, simultaneously, at every interval.
Chunky Continuous Covariation	"The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with a chunky continuous variation."	The student creates, connects, or interprets representations, thinking that the area of the rectangle is growing because the height is growing, conceiving that both area and height are varying at an interval. E.g., each time the height increases, the area also increases.
Coordination of Values	"The person coordinates the values of one variable (x) with the values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y)."	The student creates, connects, or interprets representations, conceiving of change in height and change in area as discrete points. E.g., when the height is two, the area is 12; when the height is three, the area is 27, which would then create a graph by lining up (2, 12), (3, 27) for height.
Gross Coordination of Values	"The person forms a gross image of quantities' values varying together, such as 'this quantity increases while that quantity decreases.' The person does not envision that individual values of quantities go together. Instead, they envision a loose, nonmultiplicative link between the overall changes in two quantities' values."	The student creates, connects, or interprets representations, conceiving that the area is increasing while the height is increasing, and they do not conceive that values of height and area are changing together.
Pre-Coordination of Values	"The person envisions the two variables' values varying, but asynchronously—one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects."	The student creates, connects, or interprets representations, conceiving that the height is changing, then the area is changing, but not conceiving of both area and height as changing together.
No Coordination	"The person has no image of variables varying together. The student focuses on one or another variable's variation with no coordination of values."	The student creates, connects, or interprets representations, thinking of only one quantity as varying, creating representations and interpreting a change in rectangle area or height without coordinating a change in both.

Note. This table represents definitions of major levels of covariational reasoning (Thompson & Carlson, 2017, p. 442) within the growing rectangle context (see Video 1.)

Chapter Summary

I started this chapter with a definition of meaningful learning as when students engage in interpreting, creating, reasoning, translating, and connecting among and within representations of a quadratic function while they conceive that quantities covary. I elaborated how students' concept of function depends on their prior experience in learning function, so there is no one "function concept" (Thompson & Carlson, 2017). I investigated two bodies of literature: broadly, QR and FT, and RF and representations.

I described how this study is informed by background theories: Kaput's representations (1987b) and Thompson's quantitative reasoning (1993). The theory of representations lays a groundwork for students' representational fluency— students' skills of creating, interpreting, integrating, connecting and translating among and with representations with a robust understanding of a mathematical concept (Fonger, 2019). I described how quantitative reasoning is a background theory which sets a strong foundation for covariational reasoning, and eventually FT (Ellis, 2011); and how FT—covariational and correspondence reasoning—is a way to organize the cognitive approach to characterizing students' meanings of functions. I closed this chapter by introducing theoretically grounded analytical frameworks for this study: representational fluency (Fonger, 2019) and covariational reasoning (Thompson & Carlson, 2017).

Chapter 4—Method

Design-Based Methodology

Historically, it has been reported that students have difficulty both creating and connecting multiple representations (e.g., Adu-Gyamfi, 2012) and developing a robust understanding of quadratic functions (Ellis & Grinstead, 2008) while reasoning about quantities and quantitative relationships (Carlson et al., 2002). Thus, the problem of practice is that students would not naturally co-develop representational fluency and functional thinking. In response to the problem, I conducted design-based research (Cobb et al., 2017) to investigate ways to reinforce students' understanding of quadratic functions. I conducted a teaching experiment (Steffe & Thompson, 2000) and created "a small-scale version of a learning ecology" (Cobb et al., 2003, p. 9). The teaching experiment included multiple teaching episodes, with each session lasting one hour.

Rationale for the Design-Based Methodology

Design-based methodology and the purpose of this study parallel one another. Design studies are intended to "develop a class of theories about the process of learning and the means that are designed to support that learning" (Schoenfeld, 2004, p. 10). The purpose of the study is to explore how combining and coordinating local instructional theories (Gravemeijer & Cobb, 2006) of representation (Kaput, 1987b) and quantitative reasoning (Thompson, 1993) might shed light on students' meaningful understanding of quadratic functions.

The teaching experiment, a form of design-based research, also provides opportunities for researchers to observe and have direct experience with students' mathematical reasoning (Cobb & Steffe, 1983; Steffe & Thompson, 2000). In the study, the intention of the teaching experiment is to better understand the nature of students' co-development of RF and FT when interacting with the teacher-researcher, tasks, and their peers. In other words, the intention is to make sense of the nature of students' construction of fluency in representations and FT in learning quadratic functions.

Another reason to use design-based methodology is that this methodology pushes the researcher to create a learning ecosystem, or *community of learners* (Brown, 1992; Cobb et al., 2003), which fits with the needs of the phenomena being explored (Brown, 1992). In creating the learning community, the researcher redefines teacher and student roles, which can be different than what they look like in a traditional classroom setting. In this study, the role of teachers may change dramatically from the traditional classroom. The instructor turns into a facilitator of learning, establishing themselves as a *responsive guide to students' discovery process* (Brown, 1992). The role of students in this study changes into constructors of knowledge and community members who take an active role in their learning.

Teaching Experiment

I conducted a teaching experiment (Steffe & Thompson, 2000). A teaching experiment provides opportunities for researchers to see, and have direct experience with, students' mathematical reasoning (Cobb & Steffe, 1983; Steffe & Thompson, 2000). With a teaching experiment, a researcher engages, interacts with, observes, and tries to understand students' understanding of mathematical concepts by looking at students' discussions, artifacts, written works, and ways of engaging with the mathematical tasks and tools.

In the present study, the teaching experiment methodology provided opportunities for me to test and revise my understanding of representational fluency and functional thinking. As a researcher, my engagement and interactions with the students, and witnessing the co-emergence of RF and FT in their thinking, provided me with insight into students' meaningful learning processes. To understand the nature of students' co-emergence of RF and FT, I paid attention to whatever the students said, created, and did regarding the quadratic functions; I then looked into students' interactions, explanations, and creations to create models of their thinking. Additionally, I explored what supported students' meaningful learning processes.

In sum, I have conducted a teaching experiment in order to better understand: the nature of students' co-development of representational fluency and functional thinking as they interacted with the teacher-researcher, tasks, and their peers, and what constitutes the development of a meaningful understanding of quadratic functions.

Chapter 4 Overview

The method chapter includes three phases: research design, experiment design, and data analysis. The first phase references the background theories and articulates the design of the instructional supports, such as mathematical activities, and the context for learning activities. The second phase, experiment design, includes a discussion of the teaching episodes, timelines, data collection, participants, and task-based interviews. The third phase, data analysis, consists of ongoing and retrospective analysis of the data gathered during the experiment (Cobb, 2000; Steffe & Thompson, 2000; Simon, 2000).

Phase 1: Research Design

In the first phase, I articulate five design principles—the affordances and influences of networking QR and representations. Then I introduce instructional supports, including instructional activities, and the context for learning activities.

Affordances and Influences of Networking Quantitative Reasoning and Representations

The present study is situated with a background in a theory of representations (Kaput, 1987a;1987b) and quantitative reasoning (Thompson, 1993). Recall from Chapter 3 that I have five design conjectures; in this section, I explain how I operationalize these design conjectures with explicit instructional support strategies. Table 2, below, shows both the design conjectures and the design elements, which are informed by the networking of both theories. The design elements (tasks, tools, norms, and teacher moves and prompts) will be elaborated on in the *Design of Instructional Supports* section.

Table 2

Conjecture Map of this Study

Design Principles	1. Creating opportunities for students to construct mental images 2. Getting students to focus on quantitative operations rather than numerical operations 3. Emphasizing the role of concrete representations in the quantification process 4. Grounding students' RF within the meaning of quantities 5. Getting students to present the models of quantities in their minds via concrete representations
Design Elements	Tasks Tools Norms Teacher moves and prompts

Creating Opportunities for Constructing Mental Images. In this section, I explain how I operationalize the first design principle. Students' mental operations regarding reasoning with quantities might be constructed through Dreyfus' four stages of learning[6] with representations (2002). While going through these stages, students should have opportunities to create, interpret, connect, and translate quantitative relationships across first within one representation, then later multiple representations. Furthermore, the cognitive operation of

[6] Dreyfus (2002) described four stages of the learning process in terms of concrete representations: "a) Using a single representation, b) using more than one representation in parallel, c) making links between representations, and d) integrating representations and flexibly switching between them" (2002, p. 39).

constructing mental images of quantities might become rich when students develop RF with quantitative reasoning.

For instance, in the first stage of learning, students interact with and present a representation (e.g., a table) of changing quantities. Then, in the second stage, students engage in constructing images of changing quantities by using two representations (a table and graph) in parallel. Lastly, in the third and fourth stages, students revisit and revise their image of the changing quantities and the invariant relationships among the quantities through translating and connecting among and within representations. In other words, the construction of students' images of changing quantities and the invariant relationships between quantities might be developed within RF, because students' RF might create opportunities for students to develop robust reasoning about quantities.

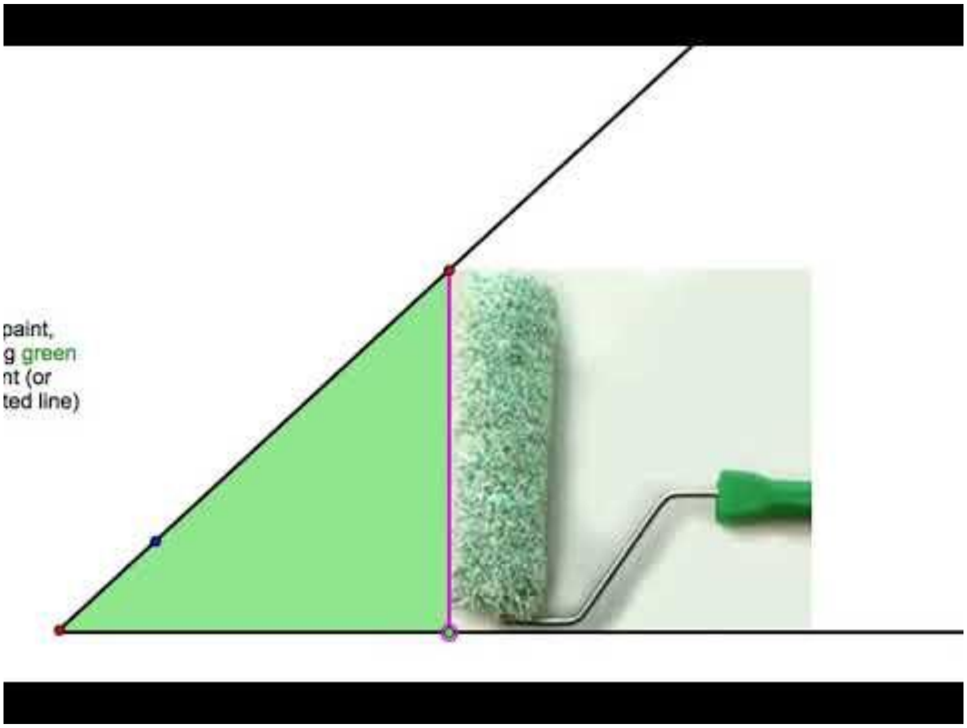
Focusing on Quantitative Operations Rather than Numerical Operations through Explicit Teacher-Researcher Prompts. In this section, I explain how I operationalize the second design principle by having students focus on quantitative operations rather than numerical operations via teacher-researcher prompts. The goal is to have students focus on the relationships among quantities rather than looking for a right answer (Weber et al., 2014). For example, in order to get students to *isolate the relevant quantities* (e.g., height, length, and area), the teacher researcher's prompt could be: *What quantities do you think contribute to the growth of the area? Explain why you picked those quantities and how you imagine measuring such quantities.* In order to get students to keep track of the growth of a rectangle, a teacher-researcher might ask students to think about how they can measure these variables. With that in mind, the teacher-researcher can ask questions such as*: How are these variables contributing to the growth in the area? How can you think about the growth in the area related to the height/length? How* *fast is the area growing?* The teacher-researcher can encourage students to create concrete representations to represent the relationship between the quantities. With this, the teacher-researcher may ask students to think about the magnitudes of the quantities when creating representations in order to push them to focus on quantitative operations rather than numerical operations.

QR reinforces students' ability to create an image of quantities, rather than rely on rote calculation, by looking at quantification within representations. For example, in the study, students are involved in the quantification of the length, area, and height of a triangle. Students watched a video where a green paint roller is scrolling to paint the wall; while the length of the paint roller increases, the area painted is also increasing.

In this example, the teacher-researcher may give students opportunities to think about, firstly, what it looks like to measure a height, area, or length; and secondly, the meaning of measuring the area, length, or height of the triangle. In doing so, the teacher-researcher may provide opportunities for the students to extend their thinking about how different measurements affect how they conceptualize different shapes (i.e., a height is a line in a triangle, while an area is the entirety of the triangle). In other words, what is the relationship between height and length—between quantities? The teacher-researcher may provide opportunities for the students to think what about it means to calculate the area by multiplying height and length, and furthermore, they may provide opportunities for students to think about *what a measure means after getting one* (Thompson, 2011).

Video 2

The Paint Roller Task Video



Note. This task was adapted from Ellis et al. (2015). The paint roller task video can also be found at this link: https://www.dropbox.com/s/sumv5wbh6kkajws/Paint-roller-Triangle%20copy.mov?dl=0

Representations' Role in the Quantification Process. In this section, I explain how I operationalize the third design principle: identifying the role of representations in a quantification process. Representations might provide instances for students to think and reason about quantities in the following ways: (a) how to represent quantities as a table, graph, symbolic equation, or diagram; (b) what is the relevant information about these quantities; and (c) what is the unit with which to measure of these quantities, or, what are the numerical values given to these quantities (Smith & Thompson, 2007).

Additionally, quantification might reinforce students' full comprehension of what is being represented, translated, and connected as quantities, as well as the relationship between quantities. The invariant relationship between quantities might be seen while students create, connect, and translate among representations. Students' representational activity might become meaningful if they reason about quantities, because reasoning with quantities might push students to engage in quantifying processes and quantitative operations (Thompson, 2011; Smith & Thompson, 2007).

Furthermore, when students reason with quantities, they might create another quantity. In most cases, the relationship between quantities can be found as a ratio—*multiplicative comparison* (Smith & Thompson, 2007). In this study, for example, students might create the area of a triangle as a quantity by multiplying the quantities of length and height of a triangle, and this created quantity (the area) would then have a measurable attribute.

Grounding Students' RF within the Meaning of Quantities. In this section, I explain how I operationalize the fourth design principle by grounding students' RF within quantities and the relationships between them. Students' use of quantitative reasoning might set a foundation for understanding through identifying and analyzing quantitative relationships (Smith & Thompson, 2007). Without quantitative meaning, students' representational activity might become ungrounded manipulations of numbers and operations within multiple representations.

There might also be a quantitative aspect of RF; the creation of representations in a quantitative context situates students' representational activity in the center of the process of quantification and understanding relationships between quantities, which might result in conceptual understanding (Smith & Thompson, 2007). Understanding the quantitative relationship might push students to think about how the relationship is changing. This an important notion, as it may explain how students make these important connections (Ellis & Grinstead, 2008; Moore et al., 2013). Encouraging students to consider the quantitative relationships within their representations rather than simply thinking about numbers or numerical values enables them to create, connect, and translate among and within representations of quantities. Through QR, students create non-numerical interpretations related to quantities (Smith & Thompson, 2007), and these interpretations become a foundation for students' RF.

Presenting Models of Quantities in Students' Minds via Concrete Representations. In this section, I explain how I operationalize the fifth design principle by examining students' models of quantities via representations. Students' conception of quantities takes place in the mind, not in the real world (Thompson, 2011); representing and connecting within concrete representations might provide a window for researchers to make sense of the models of quantities in the students' minds. Students' conceptualization of quantities, quantitative relationships, and quantitative operations is complex, and creating these conceptualizations requires more of the students in terms of cognitive engagement (Thompson, 2011). Teachers' conceptualization of quantities, quantitative relationships, and quantitative operations are different from students' conceptualization of quantities. In teaching quadratic functions during the current study, my intention was to create models of students' reasoning about quantities, quantitative relationships, and quantitative operations while they created, connected, and translated among and within representations of quadratic functions.

Design of Instructional Supports

The design of instructional components is flexible and can quickly respond to any changes that may be required by the researchers ongoing analysis of the iterative teaching episodes. In the study, the design of instructional supports is guided by the principles established in the two theories of quantitative reasoning (Thompson, 1993) and representations (Kaput, 1987b). During the practical implementation of instructional supports, as the TR, I could modify these components in order to fit the participants' need for meaningful learning. I take Simon's (1995) word on learning to heart: "Learning is likely to be fostered by challenging the learner's conception using a variety of contexts" (1995, p. 139). In an attempt to account for this, I intentionally allowed for changes to the study if the changes were likely to foster students' meaningful learning. The design of the instructional supports which I used in this study has two components: *mathematical activities* and *context for learning activities* (Simon, 1995). Mathematical activities include tasks, tasks' characteristics, and instructional sequences; context for learning activities includes instructional support via teacher-researcher prompts and the language used in the instructions.

Mathematical Activities. Mathematical activities during the study were guided by the principles established in the literature, specifically, the literature of quantitative reasoning and representation. In this section, I introduce the instructional activities, including *tasks, task characteristics* and *instructional sequences.*

Tasks. In the present study, I used three tasks: the paint roller task, the growing rectangle task, and the falling object task.

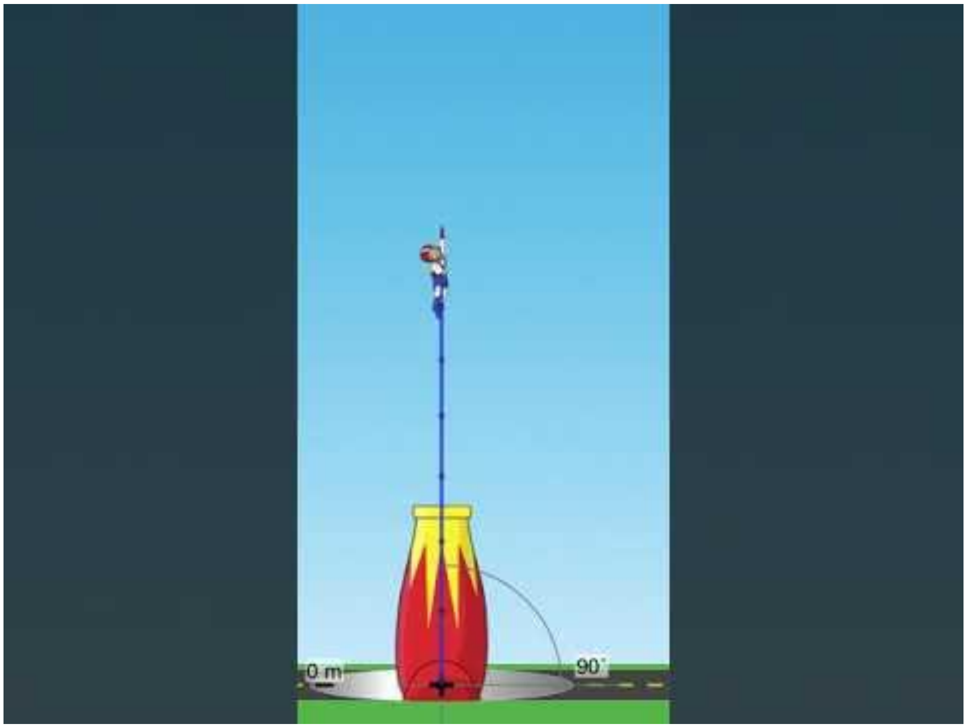
The Paint Roller Task and the Growing Rectangle Task. The paint roller task (Video 2) and the growing rectangle task (Video 1), the "Gamma tasks" were created by Amy Ellis and her colleagues (2011; 2015). Affordances of the Gamma tasks include supporting students' smooth covariational reasoning (Ellis et al., 2015), so these tasks might be powerful in exploring students' quantitative reasoning. These tasks include dynamic situations, diagrams, and videos that can help students see how a change in length affects a change in area by using color-coding that might help make the change in variables more visible to students (Johnson et al., 2017; Watson, 2015). For instance, creating a growing triangle can emphasize the understanding that two quantities covary.

The Falling Object Task. Although the aforementioned tasks are valuable and useful, I would also like to establish my identity through my own activity construction in the teaching experiment. With that in mind, I have modified the tasks on the *Projectile Motion* website's interactive simulations physics lab to create quadratic function tasks. A link to Projectile Motion: https://phet.colorado.edu/sims/html/projectile-motion/latest/projectile-motion_en.html.

The tasks emphasize an inquiry into the relationship between the height of a falling object and the time it takes to fall. Through these simulations, students may have the opportunity to throw the objects and to explore the relationship between the time passing and the height of the dropped object from the ground. The reasoning may start with magnitudes, then participants might have the opportunity to measure attributes of the falling objects. With these motion tasks, I intended to push students to focus on changing quantities, in particular, covariation between the height of the falling object and the time it takes to fall. I used this teaching experiment as an opportunity to develop my own tasks for supporting students' covariational reasoning. I have also created videos for the falling object task to provide students opportunities to focus on single simulations in creating and connecting representations of the quantities, which can emphasize the understanding that two quantities covary. See the inserted video for the falling object task, Video 3, where a cannon fires a person or a cannonball into the sky.

Video 3

The Falling Object Task Video



Note. You can find the the falling object task video at the following link: https://www.dropbox.com/s/fnsrmth5r9lr9c8/Rocket_Task.mov?dl=0.

Task Characteristics. There are five task characteristics that were purposefully designed for the study: (a) making quantification visible to students; (b) providing opportunities for measuring quantities; (c) providing subparts within the tasks to help students reflect on their thinking; (d) follow-up questions within the tasks; (e) notice and wonder structure.

First, to make quantification visible to students, tasks were situated in a quantitative context to encourage students to interact with quantifications (Weber et al., 2014). For instance, tasks located within a growing rectangle and triangle context create opportunities for students to engage in quantification and justification for the relationship among quantities (height, length, and area). Second, to provide opportunities for measuring quantities, the tasks were designed in such a way as to help students measure the quantities they identify. For instance, students may use a dynamic geometry software on a grid of one unit; this task characteristic may include student opportunities to measure the quantities. Third, tasks were designed with subparts to provide students space to explore and reflect on their thinking by using these subparts. For instance, a growing rectangle starts with 1x1, then grows gradually, so students may have various versions of the rectangle to go back and consider. Fourth, follow-up questions within tasks support students' ability to reason quantitively and revisit their thinking. Fifth, all these task characteristics were accompanied by concrete representations.

As a final task characteristic, each task was launched with a *notice* and *wonder* structure. The notice and wonder sections were designed for students to notice the quantities of a situation in motion. For instance, students watch a rocket thrown in the air, and they can see that the height of the rocket is changing with time. Alternatively, a paint roller is painting while the height of the roller is increasing. In the notice and wonder sections, students are asked to share their noticing and wondering; this portion of the task is designed to enable students to independently identify a focus question in each task, therefore generating a shared central question for the activity that is then used for further interaction. For example, in the paint roller task, the central question is about the relationship between height and area, and for the falling object task, the central questions is about the relationship between the height of the object and the time it takes to fall.

Instructional Sequence. Smith and Thompson (2007) highlighted two components to support students' quantitative reasoning (QR): (a) sequence of tasks and (b) appropriate support for students' QR. Appropriate support will be articulated in this chapter, in the section: *The*

Context of Learning Activities; here, I will discuss sequencing. I use the paint roller task, the growing rectangle task, and the falling object task to create an instructional task sequence for teaching quadratic functions within quantitative reasoning. See Appendix A for a sample of the lesson plan—the instructor version—from the instructional sequence.

The instructional sequence starts with the paint roller task, a growing triangle task created in dynamic geometry software, (Ellis et al., 2015; Ellis et al., 2018) and continues with the falling object task. Instructional activities are emphasized that require students to use covariational and correspondence reasoning by setting the tasks in order of the growing rectangle, or triangle, first, and then the falling object. For instance, the instructional sequence is designed for supporting and encouraging students to notice the quantities—the height, length, and area of the triangle—by identifying quantities and creating new quantities (e.g., area). Students are given a choice of which representations to create, and then they can sketch a graph to represent the relationship between quantities. In Table 3, below, I provide a breakdown of the instructional sequence per each day of the study.

The instructional sequence was designed with emphasis on Dreyfus's theory of representations (see the row for day 2 on Table 3). The participants are asked to create or extend the diagram, in this case, a given representation, then create a table from the diagrams they already have. Also, the diagram and table become parallel representations; then the next step requires participants to think graphically by drawing visual representations and making connections between a graph, table, and symbolic representation (see the rows for day 2 and day 3 on Table 3).

Table 3

Objectives and Tasks for Corresponding Days

Day	Objective	Task
Day 1	• Identifying length and area as quantities and as attributes of painting situations. • Estimating the relationship between height and area and sketching a graph for the connections. • Reasoning with the magnitudes of the quantities; realizing that the height of the paint roller, length of the triangle, and area of the triangle have magnitudes, and these magnitudes are changing in relation to one another.	The paint roller task
Day 2	• Realizing that quantities (height, length, and area) are measurable. • Creating a table and a graph to connect and reason about the relationship between the height and area of a triangle. • Creating parallel representations of quantities as a table and graph while reasoning about the change in the area concerning the difference in the length of the paint roller. • Creating connections and translations among and within tabular, symbolic, and graphical representations while reasoning about the change in the area as related to the change in the length of the paint roller.	The paint roller task
Day 3	• Recognizing that quantities (height, length, and area) are measurable. • Creating a table and a graph to connect and reason about the relationship between the height and area of the growing rectangle. • Creating connections and translations among and within tabular, symbolic, and graphical representations while reasoning about the change in the area in relation to the change in the length of the growing rectangle.	The growing rectangle task
Day 4	• Thinking with a magnitude of the quantities; realizing that the height of the rocket and the time have magnitudes, and these magnitudes are changing concerning one another. • Realizing that quantities (height, time, and range) are measurable. • Creating a table and a graph to connect and reason about the relationship between the height of the falling object and the time it takes to fall.	The falling object task
Day 5	• Creating connections and translations among and within tabular, symbolic, and graphical representations while reasoning about the change in time in relation to changes in the height of the falling object.	The falling object task
Day 6	• Exploring attributes of the quantitative relationship between the tasks via analyzing the quantitative relationships on multiple representations.	The paint roller task The growing rectangle task The falling object task
Day 7	• Exploring attributes of the quantitative relationship between the tasks via analyzing the quantitative relationships on multiple representations.	The paint roller task The growing rectangle task The falling object task
Day 8	• Comparing similarities and differences between quadratic and exponential functions.	The paint roller task The growing rectangle task The falling object task

Context for Learning Activities. In this section, I will provide details of the study's context for learning activities: instructional support via teacher-researcher prompts and language of instructions.

Instructional Support via Teacher-researcher Prompts. To provide appropriate support for students' facility in QR, I designed elements for the study that emphasize teachers' moves, prompts, and promoting actions for supporting students' mental construction of quantities and the relationships among quantities within multiple representations. In the study, the teacher-researcher used open-ended prompts to encourage students to identify and measure quantities and create opportunities for students to refine their concepts of changing quantities (Weber et al., 2014). Refer to Appendix B, where the teacher researcher's prompting questions for identification of quantities and reasoning within quantities are listed. Furthermore, the teacher-researcher designed supports that would push students to reason about the relationships between quantities by using a single representation, using two representations in parallel, and/or integrating and linking concrete representations (Dreyfus, 2002). In this way, the teacher-researcher might push students to reason, revise, and re-test their reasoning related to the relationships between quantities among and within multiple representations.

In addition to teachers' moves, prompts, and promoting actions, I aimed to develop norms that were centered on providing students with opportunities for quantitative reasoning. I set the expectations for the teaching experiment; since the participants had traditional classroom experiences, we renegotiated the classroom norms (Cobb, 2000). For instance, the negotiated classroom norms included rules for collaborating in a small- or whole-class discussions. I would ask students to come up with sets of norms they would like to propose to the classroom community, and I defined the classroom community as both teacher-researchers and students.

Language of Instruction. In this study, although the instruction language is English, I encouraged translanguaging, in particular, code-switching. Code-switching is "a well-governed process used as a communicative strategy to convey linguistics and social information" (Grosjean, 1999, p. 286, as cited in Moschkovich, 2007). For the purposes of this study, I define code-switching as using two languages in the same conversation (Chitera, 2009; Moschkovich, 2007; Setati, 2005). I view code-switching as a resource; it can enable participants to articulate, elaborate, repeat ideas, and add information in another language (Moschkovich, 2007); it can support the participants to develop a meaningful understanding of quadratic functions. In this teaching experiment, the students were bilingual and spoke Turkish and English. Hence, although the language of instruction, including on handouts or any written work, was English, I used code-switching between English and Turkish to communicate with the students about quadratic functions, and encouraged participants to use code-switching as well.

Phase 2: Experimenting

The second phase of the study is *experimenting*. This section includes descriptions of the teaching episodes, teaching location context, participants, the research team and the role of the teacher-researcher, a statement of positionality, timelines, data collection, and task-based interviews.

Teaching Episodes

The teaching experiment had eight episodes, each lasting for one hour. These episodes provided opportunities for me, as a teacher-researcher, to explore *students' mathematical constructions* (Cobb & Steffe, 1983), representational activities in tandem with FT. The teaching episodes were videotaped for retrospective analysis to better characterize and support students' co-development of RF and FT. In between each teaching episode, a research team revised and tested learning conjectures during ongoing analysis in relation to previous teaching episodes. The research team included four graduate students and my advisor Dr. Fonger. We revised and tested the designed instructional supports to better make sense of students' representational activity and FT.

Teaching Location Context

The venue for the teaching experiment was the Turkish Community Center (hereafter "the Center") in upstate New York. The duration of the project was eight instructional sessions that took place over approximately two weeks within the 2019–2020 school year. The Center hosts a weekend school for Turkish-American students who are interested in learning Turkish language and culture. The Center has several classrooms; each classroom has a whiteboard and 18–20 single chairs and tables.

Participants

The participants were eight Turkish-American middle and high school students in the 8^{th}, 9^{th}, and 10^{th} grades from urban and suburban school districts. I recruited participants who were members of the Turkish community. Student participants did not receive monetary compensation for participating in this study. I read the consent procedure and scripts to students individually with the parents or legal guardians present. All students had an opportunity to ask questions and to take copies of the assent and consent forms. All the names used to refer to students in this paper are pseudonyms (Table 4). Prior to this study, the participants had taken Algebra 1 courses, and in Algebra 1, they might have learned about quadratic functions.

Table 4

Demographic of Participants and the Number of Days Participants Were Present

Name	Number of Days	Gender	Grade level	Location &Schools	Articulation Ability	Visual	Language Fluency
Mert	8	Male	Grade 8	Suburban	High	High	English
Asli	8	Female	Grade 10	Suburban	High	High	English
Yener	7	Male	Grade 8	Suburban	High	High	English
Tarik	7	Male	Grade 9	Urban	High	Moderate	English
Eren	7	Male	Grade 9	Urban	High	High	English
Salim	7	Male	Grade 10	Suburban	High	High	Turkish
Bahar	5	Female	Grade 10	Suburban	Moderate	Moderate	Turkish
Zerrin	4	Female	Grade 10	Suburban	High	Moderate	Turkish

Some of my participants are fluent in Turkish and know some English; some are fluent in English and know some Turkish (Table 4). For instance, Salim, who was one of the 10th-grade participants, has been in the US for three years, and before that he was in Turkey, which means he is more fluent in Turkish than English. Alternatively, Tarik was born in the US, and is able to read and write in Turkish; however, he uses English more frequently in daily life. I consider Tarik to be fluent in English, and Salim to be more fluent in Turkish. Moreover, Salim learned mathematics in Turkish for several years, so he might be more familiar with mathematical phrases in Turkish than English. For this reason, I encouraged participants to flexibly use code-switching between English and Turkish to provide languages as resources for learning about quadratic functions (Moschkovich, 2007).

The Roles of Teacher-Researchers and Participants

I defined the roles of students (Yackel & Cobb, 1986) and the teacher-researchers (Steffe & Thompson, 2000) to create a productive and robust learning community (Brown, 1992; McClain, 2002) that might help support students' meaningful learning through quantitative reasoning and representations. The role of the teacher-researcher was as a facilitator of learning, and the role of the student was as a constructor of knowledge. I used the National Council of

Teachers of Mathematics' Principles for Actions (NCTM, 2014) to guide my construction of the teacher-researchers' role and the participants' role.

My definition of the teacher-researcher's role is a facilitator of learning who asks questions, elicits students' thinking, and orients students' thinking toward one another (McDonald et al., 2013). My role in the study was the role of the teacher-researcher. I implemented the design by setting tasks, asking questions and giving participants thinking time, and supporting students in active learning (Stein et al.,2015). As one of the teacher-researchers, I paid close attention to the learning opportunities that emerged from students creating, interpreting, and translating multiple representations while flexibly using FT. During the learning process, I made sure to analyze my questioning patterns to avoid funneling questions.

In terms of the students' role, students were explicitly informed that they were in charge of their learning as well as their peers' learning, via explaining, arguing, and asking questions to their peers. Students were told that there was no right or wrong answer in solving the questions. Students were encouraged to take daily notes during the teaching experiment; this could be another role for the teachers and research team in the room—to encourage students to write down their thinking.

Positionality and Reflexivity

For the present study, the research employs *reflexivity in a moment*: a concept that can be described as being *fully conscious* of participants, culture, ideology, and political issues within all stages of the research process (Hesse-Biber & Piatelli, 2007). I identify as a Turkish Muslim woman who is an English language learner and native Turkish speaker. The participants in the study are Turkish-American Muslim students who speak both Turkish and English. In this case, I situate myself as an "insider" who has already established trust with the participants (Narayan, 1993). In designing and conducting the study in the center, I maintained my positionality with awareness of my participants' culture and identity, from forming research questions to designing, conducting, and writing about these participants. To avoid any biases in the present study, however, I invited "outsiders" to be involved in all the stages of the study as well, including designing, conducting, writing, and interpreting the data, while continually reflecting on it (Narayan, 1993).

Insider. I define my position in this culture as "insider" because I have known the participants for two to six years, and these relationships might have affected the participants' enrollment the study. As a member of the community, as a researcher, I have previously established trust between the participants and myself.

Prior to the study, there were several occasions where I was asked by the community leaders to talk about what it looks like to be a Turkish, practicing Muslim woman at a university. I had several personal conversations with these participants regarding my own experience and political issues in Turkey. Some of these participants are asylum-seekers in the U.S., and they face political oppressions in their home country. As someone going through the same experiences with them, this might create trust between the participants and me. I might also have a role model image or a mother image in the participants' minds, because my kids also go to the community center for language classes. I am not sure how these images would affect the participants' understanding of quadratic functions. I asked my advisor Dr. Fonger and other graduate students to be present as much as possible with me in all stages of the study, in particular the interviews and the teaching experiment. I also wrote memos and journals reflecting on my own identity concerning the participants. However, I developed ownership over the study while having a place for shared ideas.

These students might also not have felt comfortable having me around as the teacher figure due to my relationship with their parents, and as a member of the community. I might create pressure on them, forcing them to find a right answer for a problem in a way that was similar to a traditional mathematical classroom[7]. In order to avoid such stress, I sought to establish a classroom culture which valued ideas and reasoning rather than focusing upon a single right answer. As the teacher-researcher, I attempted to avoid hunting for a single correct answer by emphasizing students' thinking processes. However, as someone who has been in the mathematics field for 14 years, most of my experiences have focused on looking for a single right solution, rather than valuing students' reasoning even if it is not sophisticated. Because of this, my prior experience might have affected my decisions during the ongoing analysis. To avoid such a situation, I had research team meetings about testing and revising learning activities after every teaching episode and before the next one.

Timeline

The teaching experiment took place in the first two weeks of March 2020. Due to the Covid-19 pandemic, we ended up running some sessions on an adjusted schedule. The first two sessions were carried out on Tuesday and Saturday, one session each day; then, for the remaining sessions, we combined two sessions per day. The combined sessions took place on the same day, with a 15-minute break in between sessions; for example, sessions 5 and 6 were on the same day.

Data Collection and Data Storage

The data sources collected during the study included: classroom videos, pretests, video-recorded task-based interviews (Clement, 2000; Goldin, 2000), students' written work in small- and whole-group discussions, student journals, and teacher-researcher lesson plans (Figure 8).

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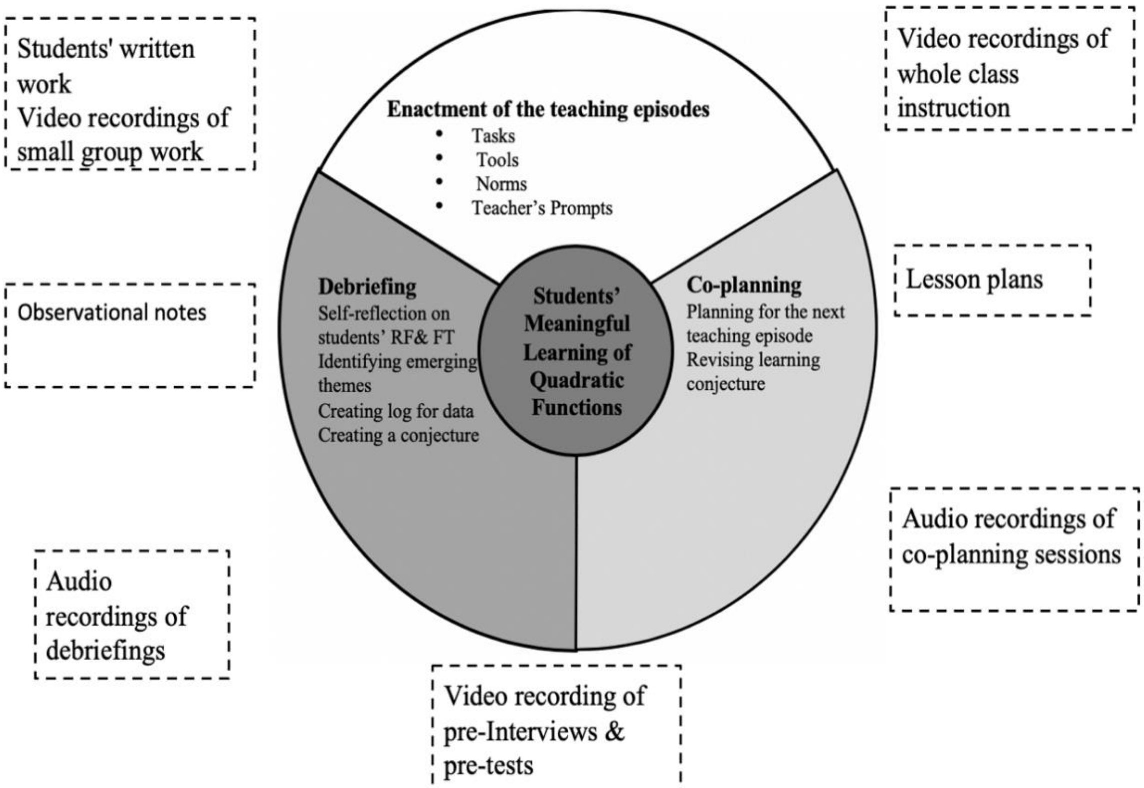
[7] Broadly speaking, in a traditional classroom, students' ways of thinking are seen as less valuable than getting a right answer.

Brown (1992) highlighted the importance of videotapes. Brown (1992) stated: “Tapes are invaluable for documenting conceptual change, in teachers as well as students, and they provide a database for discussion and reflective action on the part of teachers and researchers.” (p. 174). Following Brown's advice, I recorded videotapes for each small group, and as well as for each whole-group interaction. Then, to explore what changed in students’ co-development of RF and FT and what supported the co-development, I selected video recordings of whole and small groups with students’ corresponding written works. One of my analytical frameworks is RF (Fonger, 2019); for this, I needed to see which type of representations students were pointing at and how they interpreted representations while reasoning about quantities and quantitative relationships. Thus, video recording was a constructive way of documenting the data that fit with the analytical frameworks.

Video footage of the students’ interaction with tasks, tools, the teacher-researcher, and the team members was collected through a camera situated to the right corner of the classroom, facing the whiteboard. The participants worked in small groups. In order to capture students’ work in small groups, I videotaped each small-group interaction. The research team participated in several types of meetings of various lengths, which were also audiotaped. These meetings will be articulated in the *Ongoing Analysis* section. Pretests and interviews were videotaped. I was in charge of data collection; my job was to collect every piece of data and save it under the naming system: *TE#1_Day1_Datatype_Studentname_Date* e.g.: *TE1D1_Task_Amy_2019_15_4*. Note that not all the types of data that were collected in this study were discussed or utilized in this paper; the current analysis was aimed at answering the research questions, and not all of the data was required to accomplish this task. The data collected for this research may be shared with other researchers, teachers, and community members.

Figure 8

Data Collection



Note. This figure presents an iterative process of the teaching experiment, which will be articulated in the *Ongoing Analysis* section, with its corresponding data collection.

The Pretests and Structured-Task-Based Pre-Interviews. The**.** In this study, I conducted pretests and task-based pre-interviews with each participant before the teaching experiment. I designed the pretests and pre-interview protocol to gain a clearer perspective on the participants' prior understandings of quadratic functions. (See Appendix C1 for pretests, C2 for pre-interview protocol.).

The interviews were task-based, which was intended to help to make inferences about the participants' current understandings. This technique was intended to help me to develop conjectures, serve the research goal, and make inferences about participants' mathematical thinking (Goldin, 2000). Furthermore, I conducted the task-based interviews with a think-aloud protocol (Goldin, 2000) to better understand the meaning participants held about quadratic functions. Based on observations, I developed conjectures about participants' meaning-making processes about linking RF with FT, then asked questions to revise and re-test the conjectures.

I asked questions such as *Can you tell me what connection you see between the table and the graph? What do you think of these quantities? Tell me what you mean by saying/writing/drawing to present the relationship between the quantities. How did you think of solving this question, and how are these related to the quantities?* I gave participants enough time to think and solve, and then asked unscripted follow-up questions (Goldin, 2000). For instance, *Can you show me what you mean by that?* If the participant's response did not make sense to me, I asked clarifying questions, such as *Can you show me in another way/another representation?* If participants did not engage in linking multiple representations, I implemented "the guided use of heuristic questions" established by Goldin (2000, p. 523); I asked: *Do you see a pattern in the graph, table, or equation? Do you see any connection among these representations?*

Creating a Baseline to Serve for the Teaching Experiment with a Pretest and Pre-interview. The goal of the pretest and pre-interview was to establish a baseline for what ideas the participants were coming in with. I wanted to know what students knew and were able to do because, according to the Common Core New York State Curriculum for Mathematics, these participants should have been learning quadratic functions in Algebra 1 at grade 8.

Based on the pretest and pre-interview results and students' social interactions (e.g., who could work with whom better), I analyzed the students' levels of the different types of reasoning—correspondence reasoning or covariational reasoning with a representation—in order to choose small groups. I then created small groups by including academically heterogenous categories together as well as looking into social interactions among the grade levels these participants were in. These served as the criteria to form small groups for the teaching experiment. For instance, I grouped a student who used correspondence reasoning within graphical representation with another student who used coordinated change in quantities on a tabular representation. So, the small groups were heterogenous, including students with understanding of both RF and FT (covariational and correspondence reasoning). So, my understanding of the types of representations students employed to present quadratic relationships and what type of approach students used in reasoning about quantities informed the small groups. With that in mind, I provided opportunities for students to immerse themselves with multiple approaches to function in tandem with multiple representations.

Phase 3: Data Analysis

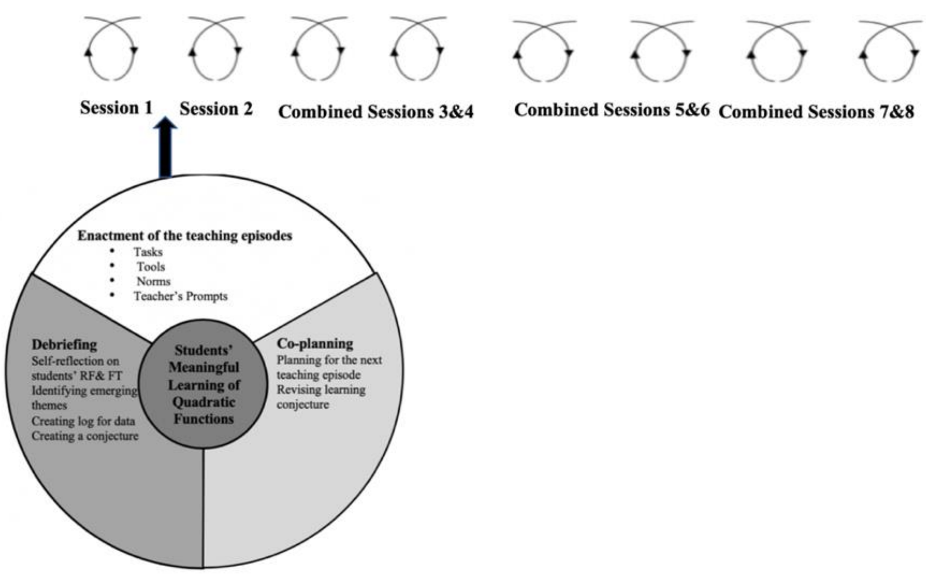
There are two types of analysis in design-based research: *ongoing* and *retrospective* analysis (Cobb, Jackson, & Dunlap, 2017). For the present study, the *ongoing analysis* took place while the teaching experiment was still in progress. The goal of the ongoing analysis was to support students' meaningful learning of quadratic functions through revising and creating new learning conjectures to support students' meaningful understanding of quadratic functions. On the other hand, *retrospective analyses* were conducted after the teaching experiment was completed (Cobb et al., 2017; Simon, 2000). Retrospective analyses were conducted to identify ways to support students' meaningful understanding of quadratic functions. Furthermore, retrospective analyses were conducted to characterize the co-emergence nature of students' RF and FT while they were developing a meaningful understanding of quadratic functions.

Ongoing Analysis

The ongoing analysis took place between teaching episodes (Cobb, 2000; Steffe & Thompson, 2000; Simon, 2000) and focused on networking the theories of FT and representational fluency. Together with the research team, I focused on observing students' flexibility in FT when they discussed and used QR and representations in tandem and how to support students' meaningful understanding of quadratic functions. The ongoing analysis took two forms: short debriefing sessions with the research team (Cobb et al., 2017), and co-planning. The visualization in Figure 9 represents the iterative process of this design research (Cobb et al., 2017) that occurred. In the next coming sub sections, I will provide information about debriefing and co-planning meetings. Figure 9 represents the debriefing and co-planning meetings the research team held before and after teaching episodes.

Figure 9

A Visual Image of Ongoing Analysis



During the short debriefing sessions, the research team made conjectures based on evidence from students' discursive activities while students were reasoning with quantities and creating, interpreting, connecting, and translating among and within representations. During the co-planning sessions the research team planned for the next teaching episodes by revising and testing the learning conjectures from the debriefing meeting and then forming new conjectures for the following teaching episode. I present the research team's meeting structure, goals, and timeline in Table 5.

Table 5

Research Team Meetings

	Time and Duration	Structure	Goal
Debriefing	A daily meeting after the teaching episode for 20–30 minutes	1. Self-reflection 2. Share out 3. Create a conjecture	1. Identifying emerging themes for the day's instruction 2. Creating new conjectures 3. Creating a written log for data
Co-planning	A weekly meeting before a teaching episode for 20–30 minutes	1. Revise conjectures according to observational notes and self-reflections 2. Plan for the next teaching episode	1. Testing and revising conjectures 2. Co-planning for the next teaching episodes

For both aspects of the ongoing analysis—debriefing and co-planning—I used Simon's (2000) *researcher's reflection-interaction cycle,* presented in Figure 10. According to this methodology, I, as a researcher, purposefully reflected on how students learned and how RF and FT were impacting their meaning-making. Then I created conjectures about the processes that led to meaningful learning, and I went on to test the conjectures to inquire whether the conjectures seemed to be supporting the participants. For example, I kept a daily researcher journal. I reflected on what went well and what did not go well during each session. I noticed that students were focused on naming the quadratic relationship as *quadratic* or *exponential;* they were not talking about the attributes of these functions. Thus, in reflecting and talking to the research team, we devised a solution to avoid naming functions, and instead to dive into attributes of those functions.

Figure 10

Researcher's Reflection and Interaction Cycle

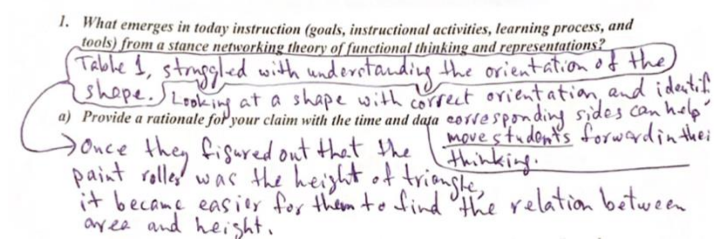
Note. Generated from Simon's researcher's reflection-interaction cycle (2000, p.239).

Debriefings. During the debriefing meetings, the research team first had independent writing time, then they shared their writing, and then they planned for the next teaching session. The debriefings started with self-reflection writing/sketching time, including probing questions and sharing with the team members. The research team was asked to write about emerging themes from the day's instruction—goals, instructional activities, learning processes, and tools in the context of FT and representations. The purpose of a writing session was to provide opportunities for the research team to generate conjectures regarding what supports students' meaningful understanding of quadratic functions, which was an open process.

Moreover, during the debriefings, team members reflected on new information they noticed about students' meaningful understanding of quadratic functions. We created new conjectures about students' meaningful learning and then modified, revised, and tested those conjectures with the upcoming teaching episodes. Furthermore, I posed probing questions to the research team about things that constituted students' meaningful learning of functions[8]. For example, in debriefing 1 (debriefing after day 1 of instruction), we reread and articulated the probing questions. I will present a probing question and an example response in the following figure. For Figure 11, an example of this part of the debriefing, the probing question was: "What emerged in today's instruction (goals, instructional activities, learning process, and tools) from a stance of networking theory of FT and representations? Provide a rationale for your claim with the time and data."

Figure 11

Waleed's[9] Reflection



Note. This figure presents a screenshot taken from one of the research team's notes about day 1 instruction.

Waleed, another member of the research team, wrote:

Table 1 [Eren and Salim's group] struggled the orientation of the shape. Once they figured out that the paint roller was the height of the triangle, it become easier for them to find the relationship between area and height. Looking at a shape with correct orientation and identify corresponding sides can help move students forward in their thinking.

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[8] For the probing questions, see Appendix D: Written Reflection Rubric

[9] Waleed and Kingsley are graduate students and members of the research team.

Although I was the teacher-researcher who would facilitate the whole-group interactions, the small groups were shared among the research team, and each of us acted as the teacher-researcher for a small group during small-group interactions. For instance, Waleed served as the teacher-researcher for Salim and Eren's group throughout the teaching experiment, while Kingsley followed Tarik, Mert, and Yener's learning experiences. The same teacher-researcher then wrote reflections for the same group of students throughout the teaching experiment.

The research team also took observational fieldnotes during each teaching episode. Observational fieldnotes were designed to show memorable, critical events—"aha moments"—concerning both FT and RF. The team members provided data excerpts with timestamps to support their claims so that I could refer back to the data during the retrospective analyses. These forms were designed to guide the team member to take observation notes, and those notes were subsequently used during the retrospective analyses.

Co-planning. During the co-planning meetings, the team members co-planned a lesson designed to support students' co-emergence of RF and FT. To do this, the research team and I analyzed students' daily handouts and journals to see any evidence of students co-developing RF and FT—if so, what was supporting the students' co-emergence of RF and FT? If not, what might support it the next day?

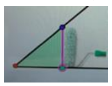
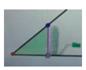
In co-planning meetings, we either met immediately after the teaching session or prior to the next teaching sessions. During the co-planning meeting, we analyzed students' work by looking at their written/drawn artifacts, and then we compared what we had planned and how that plan would fit with the students' needs as we saw things coming up during the sessions. I took extended notes and incorporated changes to the revised version of student handouts and instructor versions of the handouts. I made sure the research team had the finalized version of the instructor handouts before the teaching sessions.

I have summarized ongoing analysis in Table 6, including the themes of the debriefing and co-planning meetings, and each day's learning conjecture suggested by the research team.

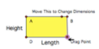
For example, in planning for day 2 based on students' work from day 1, none of the students' work showed that they exhibited a sophisticated understanding of how quantities were growing together—how a change in height for each time simultaneously affected the changing height (see the themes of ongoing analysis for day 1 on Table 6). In day 2 planning, we concluded that if we gave students numerical values for the quantities, they might just focus on finding an equation and ignore what those quantities represented (see Table 6 for the learning conjecture for the following day: first row and last column of Table 6).

Table 6

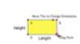
A Summary of Ongoing Analysis

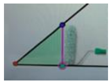
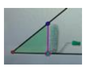
Day	Themes of Ongoing Analysis	Learning Conjecture for the Following Day
Day 1 **Focus Question:** What is the relationship between the length of the paint roller and the amount of the area covered? 	**Linking Representations.** Participants created representations (graphs, equations, tables) in parallel to indicate the relationship between length; however, there was not much articulation of how these two representations are connected. **Measuring the Magnitude of Length and Height and Creating a Unit to Measure Quantities.** Asli and Zerrin[10] were wondering if the base of the triangle and the length of the paint roller is isosceles. Asli was using a piece of white paper as a unit of measure to measure the base of the triangle and the length of the paint roller. In day one, students focused on identifying quantities of the length, height, and area of triangles. **Naming a Quantity—Debating on Whether naming it the Length of the Paint Roller or Height of the Triangle.** Students took the time to accept that the length of the paint roller was changing. They made a lengthy discussion about how to name it, the length of the paint roller, or the height of the triangle.	**Learning Conjecture for Day 2.** If we do not use numerical values for height and area, students may measure magnitudes of these attributes (height and area), and reason with these magnitudes. They may realize that the height of the paint roller, length of the triangle, and area of the triangle have magnitudes, and these magnitudes are changing in relation to one other. They may use things to measure, such as they can use a piece of paper to measure the magnitudes. Also, students may create a graph and make connections among length and area representations by identifying quantities and creating new quantities (e.g., area).
Day 2 **Focus Question:** What is the relationship between the length of the paint roller and the amount of the area covered? 	**Quadratic Functions Mean Parabola.** Students called the paint roller task a half-quadratic function because it did not have a negative domain. This made me wonder whether learning the parabola as the quadratic function graph becomes a constraint for students' meaning of the quadratic function. So, for the participants, if the quadratic function does not have the negative domain—the shape of a parabola—then it means this function is a half-quadratic function. **Creating and Connecting Multiple Representations with Unsophisticated Understanding.** Asli created a unit triangle in measuring and creating the quantities. She said she moved the unit triangle on the right and left to measure each base. So, she used a paper to measure if the unit triangle has the same base as the length of the paint roller. Then she concluded that the length of the paint roller (the height of the triangle) and the length of the paint roller's motion (the base of the triangle) had a one-to-one ratio. They grow equally. Then she generalized that the area of the triangle was ½ *x*^2. In another example, Asli created a quadratic equation and graphed that equation as an exponential graph and named it. The graph has a y-intercept. $A = \frac{1}{2}x^2$	**Learning Conjecture for Day 3–4.** If we let the participants engage in a growing rectangle context within a geometric sketchpad, they may gain a better understanding of quantities. If they see a rectangle is growing in both directions—the magic paint roller is growing both directions in painting the rectangle—this may push participants to think that the relationship between height and length is not similar to the relationship between the length and the area. Furthermore, if students create a table, a symbolic equation, and a graph to present change in the height of the rectangle in relation to the change in its area via the geometric sketchpad, their reasoning about height and area will become sophisticated. Students will use graphs, tables, and diagram representations to present the relationship between the height and area of a growing rectangle. For instance, Asli may see via the geometric sketchpad that the height and area of the rectangle are both 0, and this may invoke her to differentiate between exponential and quadratic growth

[10] All the participants' names are introduced in Table 4.

Day	Themes of Ongoing Analysis	Learning Conjecture for the Following Day
Combined Session Day 3 & 4 **Focus question**: How does the change in the height of a rectangle affect change in the area if presented on a graph, symbolic equation, and a table?  **Focus question:** What is the relationship between the height of the object and the time it takes to fall? 	**Quadratics Means a Parabola for Students**. Students have learned a particular way of classifying—exponential, quadratics (not quadratic) linear, or nonlinear—based on the representations. The quadratic function has to be a parabola, and it has to have both positive and negative domains. It has to have both parts of the curve. If it only has positive, then that doesn't make sense. That can't be quadratic. Quadratics means a parabola for students. **The Relationship between Height and Length is more Accessible than the Relationship between Height and Area.** Students have more sophistication on seeing coordination of values between height and length, but their reasoning between height and area is still a gross coordination of values. For instance, Tarik: "For every 1 cm the height increases the length increases by 2 cm." Mert: "The height increases by one. Therefore, as the length increasing by two, while height increases by one, that makes the area larger." **Linking two Representations' Graph and Table may not Impact Covariational Reasoning.** Students' use of reasoning about height and area may not develop in parallel with the use of multiple representations. For instance, we see the reasoning combined with a table and equation, but students still employ vague reasoning about quantities. In this situation, there is no attempt to make a connection between the table and the graph.	**Learning[11] Conjecture about How to Get Students to Move Away from Naming Functions and Focus on Characteristics of Functions.** Focusing on characteristics of quadratic functions and moving away from naming the functions via an activity: I will ask participants to write all the function names they know on a piece of paper, then throw that paper in the trash. Then I will tell them from now on, they cannot use any of the functions' names they know of, but they can use attributes of those functions to talk about them. Also, they can rename the functions based on the characteristics they see in the situations. The goal of this activity is to push students away from the naming of relationships between quantities as quadratic or exponential and focus more on attributes of the growth. **Learning Conjecture for Falling Object Task (Day 5–6).** If we encourage students to notice the quantities—the height of the falling object and the time it takes to fall—on the image, which is similar to the actual graph of the height and the time, this may invoke a sophisticated understanding of what it means to have a vertex. Furthermore, if we get students to create a representation (this will be students' choice) and then sketch a graph to represent, then they may develop a meaningful understanding of quadratic functions. The image of the falling object task is a good example to talk about a vertex of a quadratic function. Overall, in teaching episodes 4 and 5, students will be supported to identify the quantities in the falling object situation and represent the relationships among the quantities. And they may understand that the quantities (height, time, and range) are measurable. Students will develop a sophisticated understanding of quadratics by characterizing the connection between tables, graphs, and symbolic equations, while reasoning about the relationship between the height and the time in the falling object task.

[11] This learning conjecture was employed for the rest of the teaching episodes, Days 5–8

Day	Themes of Ongoing Analysis	Learning Conjecture for the Following Day
Combined Sessions Day 5 & 6 **Focus question:** What is the relationship between the height of the object and the time it takes to fall?  **Focus Question:** How is the relationship between the height and the time will be similar or different on the table, symbolic equation, or on a graph?	**In the Falling Object Task, for Students, the Path of the Ball is a Graph of the Height of the Ball and the Time it Takes to Fall.** It seems that the falling object task naturally lands on quadratic functions, as it starts with a visual, which is similar to the graph of a quadratic function. Yener employed chunky continuous covariational reasoning on the falling object task. He was quick to reason via a table. I have noticed instances where students engaged in correspondence and chunky covariational reasoning, but they were not sure how this reasoning would help them to create a vertex form of the equation. Students spent ample time looking for a symbolic equation, which made me wonder if the symbolic equation is the most efficient way for students to talk about functions. **Attributes of Students' Exploration of Quadratic Functions that Come out of the Falling Object Task: (**a) The vertex of the quadratic function in the falling object task is the maximum height the function can go; (b) variable versus unknown or coefficient of the quadratic functions; (c) articulating about the y-intercept of quadratic functions in the falling object situation; (d) the rate of change of a rate of change is constant; (e) articulating coefficients of the quadratic functions concerning quantities; and (f) articulating about the symmetrical nature of quadratic functions on both tables and graphs.	**Learning Conjecture for Day 7.** To push students to see how an exponential function has a y-intercept which cannot pass through the origin, and that is one way to differentiate it from quadratic functions, "which one does not belong" can be a good activity to evoke this understanding. **Which One Does not Belong (WODB).** Make a WODB activity presenting the three tasks (growing rectangle, triangle, and falling object) that the students already worked on, then add one exponential graph, and ask students to share why one of these figures does not belong here. In other words, asking students to point out a figure that does not belong, thereby explaining what the other three have in common. Furthermore, if each group presents the attributes of each situation and why they belong to this figure, while a single group is showing the other group by comparing and contrasting with the characteristics they have come up with, this may create learning opportunities for all. Some of the students were not there for all the tasks; we decided that asking students to remember the attributes of each task and how all the tasks were related to each other might provide learning opportunities through peer interaction.
Combined Sessions Day 7 & 8 Day 7–8 WODB    **Focus Questions:** How the quantitative relationships are similar or different for each task on a graph?	**Meaning of Positive Domain in Relation the Length—a Quantity**. There were discussions on why a graph of each situation starts on the first quadrant. And students talked about having a positive domain, and whether a quantity can be negative, and if the shape D cannot present quantities the same because it already has the quadrant, which means it is not a parabola. They used the word "double" to refer to the fact that each situation might have a parabola if they could have the length as negative.	**Learning Conjecture for Future Studies.** One of the characteristic features across the teaching episodes is that students get to reason, identify, and convince their peers that the quantities are changing together before they make connections or translations among and with quantities—understanding what the changes look like on a table, graph or diagram. This reasoning may be chunky continuous covariation or coordination of values, but it may still push students' meaning-making within representations. In this pilot study, I learned that introductions to the tasks at the beginning provided ample number of participants support to identify quantities, and the relationship between quantities, forming an important foundation for students' representational fluency.

Day	Themes of Ongoing Analysis	Learning Conjecture for the Following Day
Day 1 **Focus Question:** What is the relationship between the length of the paint roller and the amount of the area covered? 	**Linking Representations.** Participants created representations (graphs, equations, tables) in parallel to indicate the relationship between length; however, there was not much articulation of how these two representations are connected. **Measuring the Magnitude of Length and Height and Creating a Unit to Measure Quantities.** Asli and Zerrin[12] were wondering if the base of the triangle and the length of the paint roller is isosceles. Asli was using a piece of white paper as a unit of measure to measure the base of the triangle and the length of the paint roller. In day one, students focused on identifying quantities of the length, height, and area of triangles. **Naming a Quantity—Debating on Whether naming it the Length of the Paint Roller or Height of the Triangle.** Students took the time to accept that the length of the paint roller was changing. They made a lengthy discussion about how to name it, the length of the paint roller, or the height of the triangle.	**Learning Conjecture for Day 2.** If we do not use numerical values for height and area, students may measure magnitudes of these attributes (height and area), and reason with these magnitudes. They may realize that the height of the paint roller, length of the triangle, and area of the triangle have magnitudes, and these magnitudes are changing in relation to one other. They may use things to measure, such as they can use a piece of paper to measure the magnitudes. Also, students may create a graph and make connections among length and area representations by identifying quantities and creating new quantities (e.g., area).
Day 2 **Focus Question:** What is the relationship between the length of the paint roller and the amount of the area covered? 	**Quadratic Functions Mean Parabola.** Students called the paint roller task a half-quadratic function because it did not have a negative domain. This made me wonder whether learning the parabola as the quadratic function graph becomes a constraint for students' meaning of the quadratic function. So, for the participants, if the quadratic function does not have the negative domain—the shape of a parabola—then it means this function is a half-quadratic function. **Creating and Connecting Multiple Representations with Unsophisticated Understanding.** Asli created a unit triangle in measuring and creating the quantities. She said she moved the unit triangle on the right and left to measure each base. So, she used a paper to measure if the unit triangle has the same base as the length of the paint roller. Then she concluded that the length of the paint roller (the height of the triangle) and the length of the paint roller's motion (the base of the triangle) had a one-to-one ratio. They grow equally. Then she generalized that the area of the triangle was ½ *x*^2. In another example, Asli created a quadratic equation and graphed that equation as an exponential graph and named it. The graph has a y-intercept. $A = \frac{1}{2}x^2$	**Learning Conjecture for Day 3–4.** If we let the participants engage in a growing rectangle context within a geometric sketchpad, they may gain a better understanding of quantities. If they see a rectangle is growing in both directions—the magic paint roller is growing both directions in painting the rectangle—this may push participants to think that the relationship between height and length is not similar to the relationship between the length and the area. Furthermore, if students create a table, a symbolic equation, and a graph to present change in the height of the rectangle in relation to the change in its area via the geometric sketchpad, their reasoning about height and area will become sophisticated. Students will use graphs, tables, and diagram representations to present the relationship between the height and area of a growing rectangle. For instance, Asli may see via the geometric sketchpad that the height and area of the rectangle are both 0, and this may invoke her to differentiate between exponential and quadratic growth

[12] All the participants' names are introduced in Table 4.

In summary, the ongoing analysis using the researcher's reflection-interaction cycle (Simon, 2000) and involved both debriefings and co-planning sessions with the research team. During the debriefings**,** the research team reflected on the teaching sessions, whether the implementation of purposefully designed tasks was making use of multiple representations, connection between them, and quantitative reasoning accessible to the students. I invited each member of the team to interpret events that happened during the teaching episodes and whether or not these events may have contributed to students' meaningful learning. The reflections became a log of data which I could look back on to identify what supported students' RF and FT during retrospective analysis. During the co-planning sessions, the research team co-planned a lesson for the next teaching episode by analyzing students' handouts and journals to ascertain a sense of what might support students' meaningful understanding of quadratic functions.

Retrospective Analyses

After the teaching experiment, I conducted retrospective analyses, taking the data into account: lesson plans, the audio recordings of planning meetings, the audio recordings of daily analysis of students' work, the video recordings of small-group and whole-class instructions, written reflections, and students' journals and handouts. All audio and video recordings were turned into enhanced transcripts. I summarize research question 1 and 2 with the corresponding data in Table 7.

Table 7

Research Questions and Corresponding Data

Research Question	How do Turkish-American Muslim students' RF and FT co-develop as they develop a meaningful understanding of quadratic function in the context of a small-scale teaching experiment in an after-school setting?	What is the nature of students' co-emergence of RF and FT as secondary school students develop a meaningful understanding of quadratic functions?
Data Type	Lesson plans Video recordings of small-group and whole-class instructions Written reflections Students' journal and handouts	Students' journals Students' handouts Video recordings of small-group and whole-class instructions

In looking for ways ensure data was more manageable for coding, I selected two to three small groups of students for analysis. These are my selection categories for these groups of students:

- good class attendance—those who were present during all teaching episodes, and pre-interviews;
- gender—the groups that were representative of each gender;
- location—the groups that were comprised of those who were coming from both urban and suburban schools;
- articulation abilities—the groups included students who did articulate their thinking processes; and
- visuals—the groups included students who were using visuals to represent their thinking.

Based on the demographics of the participants introduced in Table 4, coupled with the selection categories listed above, I selected small groups which included Mert, Asli, Yener, Tarik, Eren, and Salim on which to conduct retrospective analyses. I represent the number of students who were present each day in Table 8.

Table 8

Group Members and Number of Students Who Were Present During Each Teaching Episode

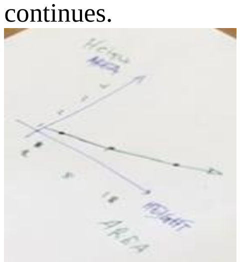
Groups	TE1 (n=6)	TE2 (n=4)	TE3 (n=6)	TE4 (n=5)	TE5 (n=8)	TE6 (n=8)	TE7 (n=8)	TE8 (n=8)
Group 1	Asli Bahar	N/A	Asli Yener	N/A	Asli Bahar Zerrin	Asli Bahar Zerrin	Asli Bahar Zerrin	Asli Bahar Zerrin
Group 2	Salim Eren	Mert Salim	Salim Eren	Mert Tarik Ekrem	Tarik Yener Mert	Tarik Yener Mert	Tarik Yener Mert	Tarik Yener Mert
Group 3	Mert Tarik	Asli Yener	Mert Tarik	Asli Yener	Eren Salim	N/A	Eren Salim	Eren Salim

Note. TE1 stands for teaching episode 1, and the small "n" stands for the number of students who were present during TE1.

Unit of analysis for answering Research Questions 1 and 2. Overall, the unit of analysis I used was the students' approaches and reasoning; students' discursive activities would be depicted as data. Table 9 illustrates an example of a fine-grained level of analysis. A single unit of analysis may consist of a line or two of students' dialogue, written work, or a representation, depending on the students' creation, interpretation, connection, reasoning, and translation of functions and concepts. This unit of analysis was chosen to help researchers measure the students' thinking about the process. With this table, I intend to give readers a clearer, more accurate sense of what my analyses look like and why or how those analyses might differ based on the data.

Table 9

Coding Example for Unit of Analysis

Coding with FT	Raw Data	Coding with RF	Unit of Analysis	Coder's Note
Correspondence reasoning	*Nicole*: Are you looking here at this diagram? *Student C*: Mhmm. *Nicole*: What will the area be in this one? *Student C*: Eighteen, I mean, eight.	Interpreting the diagram	Four lines of transcript	Interpreting the diagram by counting the height and the area of the triangle; Student C looks at the diagram and counts the length and area of the rectangle while he corresponds that length is two and the area is eight.
Coordinating the change in the height with the area	*Student C:* So, it's going up to eight. So now it's one going up two, and two goes up eight, and continues.  [Student C plots the points (2,8) and (3,18). He then connects all three points with a line.]	Creating a graph while interpreting the diagram.	Two lines of the transcript and a graph	Creating a graph to represent the relationship between height and area. The line looks like a curve but the student is not very sure how this is different than the line he drew for height and length. The line has an arrow on the end which means the growth in area and the height keep increasing. NOTE: he is using two representations in parallel; he uses the diagram and the graph as parallel representations.

Note. The data in the table is taken from an unpublished study Dr. Fonger and I conducted in 2018–2019 (Fonger & Altindis, 2019).

The Rationale for the Method of Analysis. It is worth noting that this study takes the *reflexivity* stance; neither a *psychological process* nor a *sociological process* is dominating the other in the analysis processes (Cobb & Whitenack, 1996). As noted by Cobb and Whitenack (1996), there is "a reflexive relationship between the children's mathematical activity and the social relationships they established" (p. 223). Cobb and Whitenack intended "to develop an interpretative perspective on a small group activity that brings both psychological and sociological to the fore" (1996, p. 223). This lens made it possible to analyze students' individual representational and functional activities and social relationships in small- and whole-group settings. Because students' RF might be constrained by how a small group of students interacts with one another, in certain situations, students might be satisfied with a single representation and decide not to create, connect and translate among multiple representations.

I conducted three rounds of analyses: (a) initial analysis, which includes only phase one; (b) episode-by-episode analysis, which includes phases two, three, and four, and (c) analysis of analyses, including phases five and six.

In the initial analysis, using phase one, I identified regularities in participants' interactions in small- and whole-group settings by creating enhanced transcripts of video and audio recordings, as well as extended memos. In the episode-by-episode analysis, I created the initial coding schema by coding the enhanced transcriptions of day 1 to day 8 using phase two. Then I re-coded to refute or agree with the codes or form the top-level codes—an emergent coding schema—using phase three. I then formed a developed coding schema—a learning-ecology framework—using phase four. In the analysis of analyses, I coded using the predetermined analytical frameworks of RF and FT—using phase five. Then I identified shifts in students' understanding of quadratic functions in relation to the supports students received during the teaching experiment and verified the learning-ecology framework with a guest coder by coding 25% of the data using phase six. I provide a table showing rounds of analyses with corresponding outcomes in Table 10. Next, I will elaborate each of the six phases of analysis.

Table 10

An Overview Table of the Method of Analysis

	Initial Analysis	Episode-by-Episode Analysis	Analysis of Analyses
Phase	**Phase one.** Identifying regularities and patterns in participants' and teachers' interactions in small-group and whole-class interactions.	**Phase two.** Creating an initial coding schema. **Phase three.** Creating an emergent coding schema. **Phase four.** Creating a developed coding schema—a learning-ecology framework	**Phase five.** Coding within analytical frameworks—RF and covariational reasoning. **Phase six.** Identifying shifts in students' understanding of quadratic functions, and coding with a guest coder within the learning-ecology framework.
Outcome	1. Enhanced transcripts of video and audio recordings 2. Extended Memos	1. An initial coding schema 2. An emergent coding schema 3. A learning-ecology framework	1. Coding within a predetermined framework 2. Coding within the learning-ecology framework 3. Verifying the learning-ecology framework

Initial Analysis: Identifying Regularities and Patterns in Participants' and Teacher-Researchers' Interaction in Small and Whole Groups Using Phase One. Initial analysis included phase one, where I created enhanced transcriptions of small- and whole-group interactions and extended memos.

Phase one. In phase one, I identified regularities and patterns in small- and whole-group interactions by creating enhanced transcripts and writing extended memos. I followed chronological order—transcribing from day 1 to day 8. Transcriptions included several processes. The first was rough transcription by a software; I used advanced speech recognition software Temi (https://www.temi.com/). Secondly, I cleaned up the targeted group's talk by running several rounds of watching, listening to the video, and comparing the oral with written text. I finalized the transcription by a final round of watching the video and comparing the transcriptions. I enhanced the transcriptions with triangulations (Bogden & Biklen, 2016), and matched lesson plans, transcriptions of audio recordings of planning meetings, audio recordings of daily analysis of students' work, video recordings of the small-group and whole-class instructions, and written reflections with corresponding teaching episodes. I also created extended memos.

Extended Memo Writing in Phase One. In my analysis for this study, I followed Saldana's (2009) method of free-writing memos and kept a researcher journal to create memos whenever I found it necessary. My memo-writing process began before coding; at that stage, I reflected on students' development of meaningful learning through flexibility in both representations and FT. Once I recorded a memo, I added a title to denote what story I wanted to tell with that memo, and then I added the memo to the data. I added a rationale section for each memo that relates it to the overarching goal, characterizing the co-emergence nature of students' RF and FT.

In both the process of transcriptions and enhancing, I have written memos. In each memo, I used a title that might or might not indicate a code or category for further analysis. In addition to that, I recorded a timestamp for each section of the memo and highlighted the time stamp. I have organized this information in Table 11.

In sum, the initial analysis included phase one, where I identified regularities and patterns in participants' and teacher-researcher' interactions in small groups. And I created enhanced transcripts of video and audio recordings and extended memos.

Table 11

A Sample Organization of Memos for Initial Analysis, Phase One

Title of Code or Category	Data Source	Time	Note
Encouraging students to create multiple representations can be classified as support	TE_D2_Asli _Yener	12–15 minutes	At the beginning of this conversation, Asli's thinking was vague, and Nigar asked Asli to sketch a diagram. Then, based on the sketch, Asli created a table. After the process of creating the drawing and table, her articulation about the length of the paint roller and the area covered became solid.
Peers teaching and learning	TE_D2_Asli _Yener	3–5 minutes	Yener was not present on day 1; he came for day 2. That means it was the first time for Yener to see the paint roller task. So, NA asked Asli to explain the task to Yener. This was encouraging Asli to articulate what the task is. In explaining the task, Asli gave him an overview of what she thinks of the task. Such interactions might be promising for future analyses. In other words, asking Asli to explain the task to Yener is a way of positioning Asli as competent and creating a collaboration for future interactions.

Episode-by-Episode Analysis: Developing a Learning-Ecology Framework Using Phases Two, Three, and Four. To create a developed coding schema, I conducted phases two, three and four. In phase two, I created an initial coding schema by coding the enhanced transcriptions of the small- and whole-group interactions. In phase three, I narrated the coding segments by refuting, revising, and redefining the codes from the initial coding schema (developed in phase two) to form an emergent coding schema (developed phase three). Finally, in phase four, I redefined, refuted, and revisited the categories of the emergent coding schema to form a developed coding schema (i.e., a learning-ecology framework).

Phase two: Creating an Initial Coding Schema. During this phase, I coded the enhanced transcriptions of small- and whole-group interactions that I had created in phase one. I coded the enhanced transcriptions in chronological order, from day 1 to day 8. In this process, I created an initial coding schema to verify it by re-coding and narrating the coded segments. I have extended the "narration" of the code and the code's definition so that the framework better captured the data. I made two kinds of changes regarding re-coding and narrating the coded segments: coded segments may have been changed to different codes, and the initial coding schema was updated. The narration process helped me develop and refine the definitions and descriptions of the codes. For instance, there were initially two codes: *asking students to be specific* and *asking students for further explanations*; after narrating and defining, I combined these codes.

Delineation of How Subcodes Were Created. Some of the subcodes were generated from the extended memos and the researchers' journals. For instance, asking questions to visualize or sketch was the theme of a memo related to teacher-researcher promoting actions. And *asking students to visualize and sketch* become a subcode. I also used the design elements as subcodes. After creating the initial coding schema, I created subcodes from the design elements and regularities and patterns identified during initial analysis.

Co-occurrence in the Data. A single sentence, or multiple sentences, can be coded with several codes, which means codes may overlap. In other words, I get to code a single sentence with several codes. For example, I have coded this sentence as *a quadratic equation*, as well as *identifying quantities*, and also *making sense of peers thinking*: "Look, look, that is x [length of the rectangle] and this $x+2$ [height of the rectangle], then height times length, x squared plus 2x. Bu [this is] quadratic." In this example, students identified quantities, created a new quantity—area—then created a symbolic equation; and all these are happening in small-group interactions.

Phase Three: Creating an Emergent Coding Schema with the Process of Revising, Redefining, or Refuting Initial Chronological Categories. In creating an *emergent coding schema*, I redefined some of the codes, narrated coded segments, added new subcodes under the top-level codes, and refuted some of the existing codes. For instance, a top-level code, *teacher-*

researcher prompts and moves did not have a code that defined *probing for continuous covariational reasoning*. *Probing for continuous covariational reasoning* defines instances where researchers ask questions or make a pedagogical move in supporting students' continuous covariational reasoning. In developing the coding schema in phase three, I added the code, *probing for continuous covariational reasoning* to the initial coding schema. I also redefined some of the codes by adding subcodes underneath. For instance, the code in the initial coding schema was *clarification-asking for clarification*; however, in this code, there are instances wherein the researcher asks for attributes of functions, so I added a new subcode called *encouraging to focus attributes. Encouraging to focus attributes* defines cases in which researchers encouraged students to focus on attributes of the function rather than naming the function as quadratic or exponential. Lastly, I refuted some of the existing codes; for example, *identifying quantities in growing rectangle situations*. *Identifying quantities in growing rectangle situations* code described that students were identifying quantities only in a growing rectangle situation. This code overlapped with *identifying quantities*, which describes instances where students identify quantities in any case. So, I refuted the code *identifying quantities in growing rectangle situations* in the process of creating an emergent coding schema.

Investigating what supported students' meaningful understanding of quadratic functions required me as a researcher to look across codes and categories informing emergent coding schema (Cobb & Whitenack, 1996; Glaser & Strauss 1967). I coded and narrated what supported students' learning of quadratic functions. That means I looked across the codes and categories on the topic of learning, and then I compared my findings to the codes and categories on the topic of teaching. With the process of redefining, refuting, and revising the existing codes, I formed an emergent coding schema.

Phase Four: Creating a Developed Coding Schema—a Learning-Ecology Framework. Using the emergent coding schema (developed as an outcome of phase three), I revisited and coded the teaching experiment data and small-group and whole-class interactions to explore what supported or constrained students' meaningful understanding of quadratic functions. Then I revised, redefined, or refuted the chronological emergent categories. To form the main categories for a learning-ecology framework with axial coding (Strauss, 1987), I further revised, redefined, and refuted top-level codes by looking for similarities and intersections among codes. In other words, I looked at how the categories in the emergent coding schema were related to the concept of supporting students' meaningful understanding of quadratic functions.

For instance, for the code *focus question,* the term *focus question* is coding for researchers using prompts for focus questions in small-group interaction, for facilitating whole-group interaction around the focus question, for forming questions to serve as a foundation for the focus questions in students' handouts, and for answering focus question in the journal. I have noticed conflict between making *focus question* the main category or making it a subcategory under other categories. To decide, I revisited each emergent category to merge, refute, or redefine it. To develop the main categories of a learning ecology that supported a meaningful understanding of quadratic functions, I explored the leading codes' chain. I then exported the coded segment to a word document. I then interpreted each of the coded pieces to see whether it fit or refuted the top-level code. For instance, *agree-disagree* is a subcode categorized under the main category *orienting students thinking one another.* To include a code in a category, I interpreted each coded segment, answering how the subcode serves the main category and why this a good fit for such categorization.

In sum, with the episode-by-episode round of coding, I created an initial coding schema (phase two), an emergent coding schema (phase three), and then the developed coding schema (phase four). I also described and narrated the codes. Through the process of creating an initial coding schema, emergent coding schema, and developed coding schema, the episode-by-episode analysis yields a learning-ecology framework.

Analysis of Analyses: Coding the Predetermined Framework and Verifying the Learning-Ecology Framework Using Phase Five and Six. In phase five, I coded in terms of the analytical frameworks—RF and FT. In phase six, to verify the learning-ecology framework, I identified shifts in students' understanding of quadratic functions in relation to the supports the students received during the teaching experiment. I invited a guest coder to code 25% of the enhanced transcription of the small- and whole-group interactions. We set coders' agreements. The agreements proved that the learning-ecology framework supported students in the identified shifts in understanding.

Phase Five: Coding with Predetermined Frameworks. In phase five, I coded with predetermined frameworks: RF (Fonger, 2019) and FT—covariational reasoning (Thompson & Carlson, 2017) and correspondence reasoning (Confrey & Smith 1994). I coded the data with RF and FT frameworks to see whether the main categories came from these frameworks' overlap with one another. I coded with the two frameworks—RF and FT—in random order. To do this, I divided the data into small chunks, separating them by representation (e.g., tables) and coded them in random order. First, I coded for RF, then selected another excerpt and coded that group of data with FT. Randomly choosing portions to code during phase five ensured that the data did not arrive in sequential order, so that I had less bias in seeing how these two theoretical frameworks might overlap or how they might differ in characterizing the co-emergence of students' RF and FT.

Phase Six: Verifying the Learning-Ecology Framework. I coded with the learning-ecology framework, and verified the learning-ecology framework by identifying shifts in students' RF and FT and establishing coders' agreements by having a guest coder code 25% of the data set.

Identifying Shifts in Students' RF and FT. I have identified four shifts in students' RF and FT in total, one shift per student. In the process of identifying these shifts, I listed types of reasoning about quantities for each participant, from less-sophisticated reasoning to more significant, sophisticated reasoning. I color-coded each participant's name and where they engaged in levels of covariational reasoning. Then I counted the amount of each participant's reasoning—in that the covariational framework levels quantify how many times each participant engaged in each type of reasoning. Based on the quantification, I created a table to show the different kinds of reasoning and how many times each participant engaged in each.

Based on each participant's number of levels in covariational reasoning (Table 12), I selected the participants who had multiple instances of reasoning across levels. For instance, Yener, Mert, and Eren had four levels of reasoning, and Asli had three levels of reasoning, while Zerrin and Tarik had two levels of reasoning and Salim had one level of reasoning. Hence, I chose for this phase to focus on Yener, Asli, Mert, and Eren. I have selected these four students because they were consistent with movement from lesser sophisticated reasoning to greater sophisticated reasoning. To further focus the analysis, I looked for variation in the kinds of reasoning these four participants exhibited each day. Along with looking for variation in covariational reasoning, I identified several shifts in the students' RF.

Table 12

The Participants' Covariational Reasoning

	Chunky Continuous Covariation	Chunky Continuous Second Covariation—the change in change	Coordination of Values	Gross Coordination of Values
Yener	9	10	25	10
Mert	2	1	6	39
Asli	1	N/A	6	7
Tarik	1	N/A	N/A	11
Zerrin	N/A	N/A	1	4
Eren	4	1	6	21
Salim	N/A	N/A	N/A	15

Note. Table 12 presents the participants' names and the number of times they *each* engaged in the various kinds of reasoning they employed throughout the teaching experiment.

I explored students' FT and RF shifts by breaking the enhanced transcripts into chunks, starting with the lines before the less-sophisticated reasoning and moving to lines with more significant sophisticated reasoning. For instance, I looked at the lines coded with gross coordination of values and then moved to a chunky continuous variation of values and RF. I identified participants who showed up at both levels. For example, Yener had both gross covariational reasoning FT and transposition—RF (line 33 on the transcript) and chunky continuous covariational reasoning with multidirectional connections—RF (line 108 on the same transcript). Then I invited a guest coder to code the enhanced transcriptions to reflect the emergent learning-ecology categories—the supports—and whether the existing categories were there prior to, during, and/or after Yener's shift in thinking took place.

Establishing Coders' Agreement. With the guest coder, we verified the learning-ecology framework by coding with the learning-ecology framework separately and setting a coder agreement. We coded 25% of the enhanced transcription of small-whole data sets in particular. We focused on either of the developed main categories of the learning ecology prior to, during, and after the identified shift took place. In other words, we verified the learning-ecology framework by comparing the identified shifts in students' reasoning and looking for instructional support relative to those shifts. And then we reconciled the coding decisions by working together to ask questions, highlight what was common and what was uncommon in the codes, and provide evidence for the codes.

We looked into disagreements among the codes, when the actual codes conflicted with each other, and tried to reconcile those disagreements. When this happened, we set a coder agreement by redefining the code together. When we redefined the codes, we also extended the code definition to ensure that the code defined a broad meaning. In other words, we either developed the descriptions of the codes, or we clarified our definitions by rewriting them. When making definition changes, we revisited all the coded segments on that category to make sure that the coded segment matched with the updated definition. We strengthened and verified the learning-ecology framework by talking through and establishing these coders' agreements.

To sum up the retrospective analysis portion of the study, I used Cobb and Whitenack's (1996) techniques, which drew from Glaser and Strauss's (1967) constant comparison method. After three rounds of analyses: initial analysis, episode-by-episode analysis, and analysis of analyses—I established a verified learning-ecology framework and a characterization of students' development of RF and FT.

Trustworthiness

The practices I used to maintain the trustworthiness for this study included reporting phases of analysis, and conducting and establishing a coder's agreement. Requiring researchers to report the process of analysis in each phase is a well-detailed technique to ensure trustworthiness in design studies (Cobb et al. 2017). With that in mind, I reported all the phases of the data analysis process systematically, along with the corresponding evidence from the data and the learning conjectures the researchers made about students' co-development of RF and FT. The ultimate way to establish trustworthiness of analysis is inviting a guest coder and setting a reconciling coding agreement among coders. This was established in retrospective analysis phase 6.

Chapter Summary

In chapter 4, I articulated the use of a design-based methodology for this study to test and investigate the development of learning processes (Brown, 1992; Cobb et al., 2017). I articulated how the design conjectures were informed by the affordances and influences of networking the theories of quantitative reasoning (Thompson, 1994) and representations (Kaput, 1987b) (e.g., getting students to present the models of quantities in their minds via concrete representations). I provided the details and context of the small-scale teaching experiment and the research team.

I detailed my method of analysis in answering the research questions. There were two types of analysis. Ongoing analysis took place between teaching episodes. For retrospective analysis, I conducted several rounds of analyses. To answer research question one, I used the Cobb and Whitenack (1996) method of analyzing data; they used Glaser and Strauss's (1967) constant comparison method to analyze data sets from small- and whole-class interactions. To answer research question two, I networked two analytical frameworks as lenses for analyzing (Simon, 2000) using RF (Fonger, 2019) and FT—covariational reasoning (Thompson & Carlson, 2017) and correspondence reasoning (Confrey & Smith, 1994). I finished this chapter by detailing he trustworthiness of this study.

Chapter 5—Results and Findings

In this chapter, I present my findings by addressing the following research questions:

1. What is the nature of the co-emergence of representational fluency and functional thinking among secondary school students as they develop a meaningful understanding of quadratic functions?
2. How can secondary school students be supported to develop a meaningful understanding of quadratic functions?

This chapter includes two parts. In the first part, I characterize the co-emergence of RF and FT among students for each representation (a table, a graph and a symbolic equation), and across multiple representations, as they created and connected representations to present quantitative relationships of quadratic functions. In the second part, I define and verify a learning-ecology framework that articulates supports for students' meaningful understanding of quadratic functions.

Part 1: Characterizing Students' Co-emergence of RF and FT in Learning About Quadratic Function

Recall research question one: What is the nature of the co-emergence of RF and FT among secondary school students as they develop a meaningful understanding of quadratic functions? In response to my research question one, I in this section, I articulate two main findings. First, I report the results and findings that emphasized ways to characterize students' reasoning about quantities and quantitative relationships on each of these representations: a table, a graph, and a symbolic equation. Second, I operationalize students' co-emergence of RF and FT into four levels, based on students' ability to create and connect multiple representations of quantitative relationships.

Finding 1: Students' Reasoning about Quantities in Concrete Representations

For students' reasoning about quantities in concrete representations I found two types of thinking: lateral thinking and static thinking. Lateral thinking is the co-development of RF and FT. Static thinking, on the other hand, is the disconnection of RF and FT. A disconnection between RF and FT is defined as instances where students create representations without realizing that the representations present covarying quantitative relationships. In another words, with static thinking, students are able to solve the problem using representations, but they conceive the representations as objects. I define and exemplify both types of thinking in the following section by focusing on each of the main representations. I start with the table representation, then later move to the graph and symbolic representations.

Students' Reasoning About Quantities Within Table Representations. Students' thinking about quantities within the table representation entailed two types of reasoning: tabular static thinking and tabular lateral thinking. Broadly speaking, tabular static thinking is when students create a table based on a symbolic equation or learned facts without attention to what that table represents. As I define in this study, lateral thinking is a creative way of thinking or reasoning about covarying quantities to solve a problem using concrete representations, which includes conceiving of quantities as covarying quantities on a table. An overview of these characterizations is given in Table 13; in the following sections I elaborate each construct in turn, with examples of the two types of reasoning for each.

Table 13

Overview of Students' Reasoning About Quantities Within Table Representations

Tabular Static Thinking	Tabular Lateral Thinking
(1) Sets of learned rules about quantities without coordination (2) Points on the table as a string of numbers	(1) Determining the vertex of a quadratic function by conceiving quantities as covarying quantities on a table (2) Recognizing that quantitative relationships can be generalizable as well as interchangeable

Tabular Static Thinking. Students' tabular static thinking entails two forms of lesser sophisticated reasoning about quantities: sets of learned rules about quantities without coordination, and points as a string of numbers.

The first way that students approach a table with tabular static thinking is through a set of learned rules. In this way of reasoning, students are employing a learned fact to create a table to present quantities and the relationships between them, but they are conceiving of the quantities only as string of numbers with units. In other words, tabular static thinking is about applying a particular rule to find what should be the next pairs of numbers on a table.

With tabular static thinking, the quantities are not understood to be covarying; rather, students use a quantity to find the corresponding quantity by applying a set of learned facts or formulas.

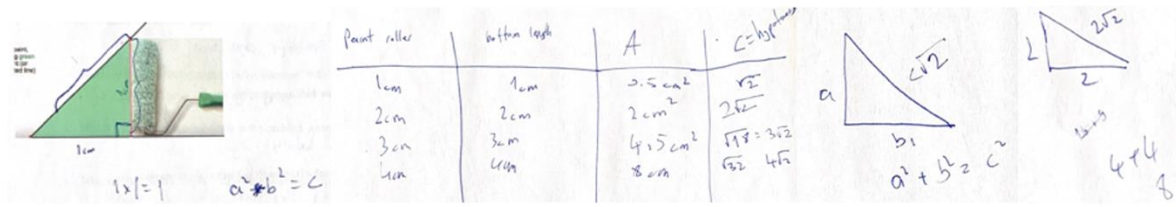
The below vignette is taken from Mert and Salim's small-group interactions, when they were exploring the relationship between the length of the paint roller and the amount of the area it covered. Consider the paint roller task and the following vignette:

1 Mert: Relationship between the length of the paint roller and the amount of the area...

2 Salim: We can find the hypotenuse, too. Look hepsi ayni [they are all same] [Figure 12 (a)].

Figure 12

(a) Salim's Pythagorean Theorem, (b) and (c) Salim's Table for Length, Height, and Area, and (d) Salim's Computation for Finding the Hypotenuse



(a) (b) (c) (d)

3 Mert: What is that [$a^2 + b^2 = c$]?

4 Salim: To find this side [the hypotenuse] let's complete this table [Figure 12 (c)].

5 Salim: When it is 1 cm, 1 cm, 0.5 cm^2, 2 cm, 2 cm, 2 cm^2, 3 cm, 3 cm, 4.5 cm^2, 4 cm, 4 cm, 8 cm^2.

6 Mert: Oh, it is like this [copies the same table].

7 Salim: Soyle yapiyorsun. [You are doing like this.] This is a, this b and this is c.

8 Mert: OK.

9 Salim: So, if this is one, this is one, then the line is square root of two.

10 Mert: We did not use the radical.

11 Salim: If this was two, two, then what is it now? Eight, right?

12 Mert: I do not know. How did you get that?

13 Salim: A squared plus squared is equal to c squared. Hypotenuse is the c. This part [pointing at the computer screen]. Let's find the hypotenuse for each [he adds the last column on the table below, Figure 12 (c)].

In this example, Salim used a learned fact—the Pythagorean formula—to create a table that represented the height, length, and hypotenuse of a triangle, as well as its area (lines 4–6). Salim used the Pythagorean formula for finding the quantities (lines 4–8) without coordinating that the change in height was related to the change in length or area. Although Salim used the Pythagorean formula appropriately to fill out the table, he could not see that the table represented quantities with magnitudes. Hence, for Salim, the quantities on the table were a string of numbers created by plugging values into the Pythagorean equation—a learned fact.

When we look at the conversation between Salim and Mert, Mert stated that they needed to explore the relationship between the paint roller's length and the area of the rectangle (line 1). Salim pointed out that the triangle's height and base were the same; they could use the Pythagorean theorem to find the hypotenuse (lines 2 and 4). Although he wrote 1 cm, 1 cm for the sides of the triangle, he did not focus on how the height increase affected the growth on the base of the triangle or its area. Salim wrote $c^2 = a^2 + b^2$ [Figure 12 (d)] but Mert did not understand how the rule helped identify the relationship between the paint roller's length and the area (line 3). Through the $c^2 = a^2 + b^2 \ and \ A = \frac{lh}{2}$ symbolic equations, Salim completed the table (line 5). Salim explained the computation and found the area and the length of the triangle (line 13). Still, his thinking centered on mapping the paint roller's area and length—as correspondence reasoning. His reasoning depended on the known rule of A$= \frac{lh}{2}$ (area of a triangle) [see Figure 12 (c)]. For Salim and Mert, the table represented numerals which could be found with learned facts. This example illustrates how static thinking about a table is when students conceive of quantities as numerals created by a learned fact—a form of correspondence reasoning.

The second way of approaching a table with tabular static thinking entails conceiving of values as strings of numbers, and mapping or relating each quantity to create new numerals. As we saw with the above vignette, Salim conceived of quantities with no attention to magnitude of each quantity (line 5). He said: “When it is $1\text{ cm}, 1\text{ cm}, 0.5\text{ cm}^2,\ 2\text{ cm},\ 2\text{ cm},\ 2\text{ cm}^2,$ $3\text{ cm}, 3\text{ cm}, 4.5\text{ cm}^2, 4\text{ cm}, 4\text{ cm}, 8\text{ cm}^2$.” Even if Salim wrote the units in the table, when he articulated, he referenced the quantities on the table as string of numbers. As Salim said: “If this was 2, 2, then what is it now? Eight, right?” I interpret that in this case, for Salim, there was not much difference between a height of 2 cm and an area of 2 cm^2; the quantities were a string of numbers with no magnitudes (line 11). Although Salim used units of measurement—cm, cm^2—for height, length, and area, he still conceived of these quantities as a string of numbers, rather than a magnitude of height or area. Thus, I conclude that Salim conceived of quantities on the table as numeric generations. I present a summary of the constructs of tabular static thinking across sets of learned rules and points as a string of numbers in Table 14.

Table 14

Tabular Static Thinking

Aspect of Tabular Static Thinking	Definition	Example
(1) Sets of learned rules	Creating a table with a learned rule to represent quantities	Mert: What is that $[a^2 + b^2 = c]$? Salim: To find this side [the hypotenuse] let’s complete this table [Figure 12 (c)].
(2) Points as a string of numbers	Approaching quantities as a string of numbers or numeric generations.	Salim: If this was two, two, then what is it now? Eight, right?

***Tabular Lateral Thinking*.** The second, more sophisticated type of reasoning on a table, tabular lateral thinking, has two aspects: (a) determining the vertex of a quadratic function by conceiving of covarying quantities on a table, and (b) recognizing that quantitative relationships can be generalizable as well as interchangeable.

The first aspect of tabular lateral thinking that I found during the study was that students' tabular lateral thinking pushed them to determine the vertex of a quantitative relationship on a table by reasoning about covarying quantities. They were able to define the vertex of the quantitative relationship as the highest magnitude the quantities could be. Students were able to recognize a vertex point on a table by (a) coordinating the change of change in height (i.e., the second change) with the change in the time—interrelatedness 2—and (b) coordinating the first change in height with the first change in time—interrelatedness 1. Furthermore, students were able to connect interrelatedness 1 and 2 (see Table 15) to identify the vertex points of quantitative relationships.

Table 15

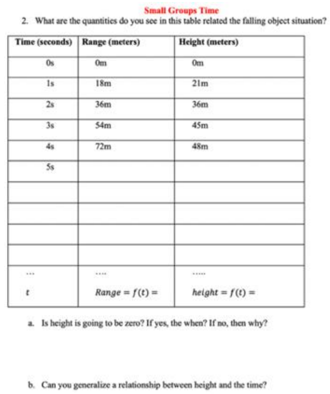
Definitions of Interrelatedness 1 and 2

	Definition	Example
Interrelatedness 1	A coordination of the first change on one quantity with the first change in another quantity.	Asli: "As the height increases by one unit, the length increases two units."
Interrelatedness 2	A coordination of the first change in one quantity with the second change in another quantity.	Yener: "We found that that amount of the area, it changes per height change was four. So, it would change from two to eight. And then when it went from eight to 18 and change time just for more than six 18 to 32 it changed from 10 to 14, which adds a difference of four. So, adds four each time to it."

As an example of students determining the vertex of an equation through covarying quantities, the vignette below is taken from Mert, Yener, and Tarik's small-group interactions when they were exploring the relationship between the height, range, and time in the falling object task (Figure 13).

Figure 13

The Falling Object Task and the Corresponding Table



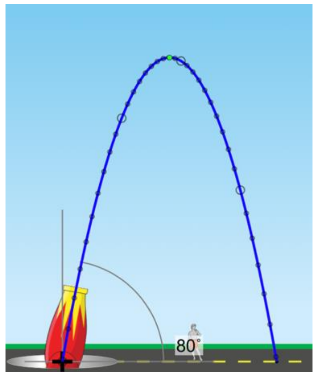
Small Groups Time

2. What are the quantities do you see in this table related the falling object situation?

Time (seconds)	Range (meters)	Height (meters)
0s	0m	0m
1s	18m	21m
2s	36m	36m
3s	54m	45m
4s	72m	48m
5s		
… t	…. $Range = f(t) =$	….. $height = f(t) =$

a. Is height is going to be zero? If yes, the when? If no, then why?

b. Can you generalize a relationship between height and the time?



Note. This figure presents a screenshot from the falling object task's video and the corresponding table.

14 Yener: I am pretty sure that is the peak, and then it starts going down.

15 Mert: That's the 48 peak?

16 Yener: I think so. I mean, I don't think, because change changes six every time. It can change six meters.

17 NA: When you say peak? What do you mean by the peak?

18 Yener: The maximum. How high the ball goes in general.

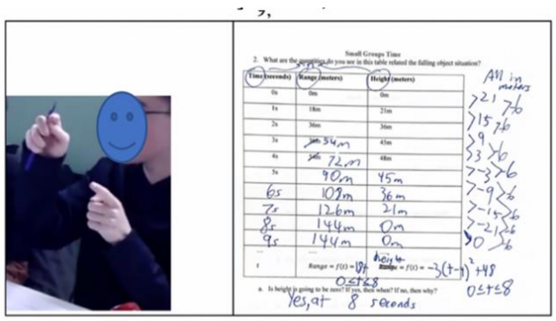
19 Mert: Wait, wait. How was the ball, like, is 21 meters. Oh, yeah. OK.

20 Tarik: Oh, yeah.

21 Yener: Because it gets shorter, the distance of those points [showing with his hand on Figure 14 (a)], so, 21, 15, uhm [he adds values on Figure 14 (b)] this is nine.

Figure 14

(a) Yener's Gesture and (b) Yener's Table of the Falling Object Task



(a) (b)

Note. This figure presents (a) Yener's gesture to show how the increments get shorter when moving away from the vertex and (b) his table to show this logic.

Yener stated that when the time is four seconds and height is 48 meters, that point must be the peak. He said, "I am pretty sure that is the peak, and then it starts going down." Yener, Mert and Tarik attempted to coordinate the first change with the second change: the second change in height, compared to the first change in time, decreases six meters per second—interrelatedness 2 (line 14-21). Yener recognized that the height reaches the maximum at four seconds, the parabola's vertex: "The maximum. How high the ball goes in general?" Yener also noticed that the distance in between increments of height gets smaller for each increment of the height when getting closer to the maximum height, or the vertex, of the relationship (line 21). He stated, "Because it gets shorter the distance of those points [showing with his hand, Figure 14 (a)], so, 21, 15, uhm [he adds the values on the table] this is nine [Figure 14 (b)]."

In creating a table of this quantitative relationship—the height and time of the falling object—students might interchange between coordinating the first change in time with the second change in height as they create and interpret the quantitative relationships on a table (line 14) [Figure 14 (b)]. In other words, when students create and analyze a single representation with tabular lateral thinking, they build on or reference the quantitative relationships in the representation.

For instance, when Yener and his peers were making a table, they referred to the height and the time to complete the table (lines 16 and 19). At the same time, they defined the quadratic function's vertex on the table with quantitative relationships. They also noticed that when the increments of quantities get closer to the vertex [they refer to the vertex as the maximum height the cannonball can reach], for every one second on x-axis, the increments of quantities represented on a y-axis get smaller; when they get away from the vertex, the increments get larger (line 21). As we see with this example, the students reasoned and made sense of a quantitative relationship while completing the task within a given representation.

In the vignette above, having students create a table to present a quantitative relationship pushed their thinking to identify the relationship's vertex as the maximum the cannonball could reach (line 16). They also placed the maximum point as the midpoint and generalized that the distance between points gets larger (line 21). When the points move away from the maximum (the vertex), compared to when the quantities on the y-axis get closer to the maximum/vertex, the distance between points gets smaller. Furthermore, reasoning that the amount of change in height decreased six meters for every second made Yener see that the maximum height must be 48 meters (16). He calculated the first change, in height, as $21\text{ m}, 15\text{ m}, 9\text{ m}$ and 3 m; then he recognized that the difference, the first change is decreasing by six meters for each one second, because $15\text{ m} - 21\text{ m} = -6\text{ m}, 9\text{ m} - 15\text{ m} = -6\text{ m}, 3\text{ m} - 9\text{ m} = -6\text{ m}$ for every one second [Figure 14 (b)]. This reasoning helped Yener to identify that the peak the cannonball can reach must be 48 m, because the amount of the increase of increase of height—that is, the second change in the height—is six meters, which will make the first change -3 m when the time is 3 seconds, so the difference between 45 m and 48 m is − 3 m (lines 16–21). And the height must be 45 m, so that means 48 m must be the maximum point (the vertex), and the distance in between the height for each single second gets larger; for instance, when the height values are away from the maximum, the change in height is 21 m, as opposed to when the height values are closer to the vertex, where the change is 3 m for every second.

Thus, fluency in interrelatedness 1 and 2 regarding the quantitative relationship enabled Yener to identify the vertex points on the table and complete the given representation. This is evidence to suggest that students' FT affected their RF by allowing them to identify the vertex as the maximum height on the table, and also helped them to create a generalization about the change in the height for each second when the distance in between the height for each single second is closer to the vertex compared to when they are further away from the vertex on a tabular representation.

My findings also suggested a second way students engaged in tabular lateral thinking. This could be found in students' ability to recognize that quantitative relationships can be generalized and interchanged. During the study, students switched back and forth between how they coordinated the changes among quantities on a table. With the growing rectangle task, for instance, they coordinated the height with length, and then switched back and coordinated the length with the height. Students created a table to reason about the height and length of the growing rectangle. Then they interpreted that for every one-centimeter increase in height, the length increases by two centimeters. Using the same table, the students could swap their reasoning and say that for every two-centimeter increase in length, there is a one-centimeter increase in height. Thus, creating and interpreting tables representing quantities might enable students to reason interchangeably about quantities and their relationships. In other words, tabular lateral thinking allows students to coordinate the change from height to length and vice versa.

Consider the growing rectangle task and the following vignette[13] in relation to lines 22 to 25.

22 NA: How that table [Figure 15] is helping you to see the relationship between the height and area?

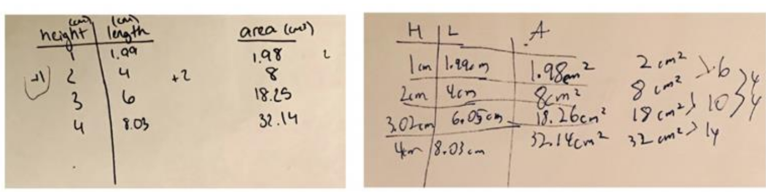
23 Asli: It just helps me visualize how to like as the height is increasing by one the length increases by two because you can clearly see the difference [Figure 15 (a)].

24 NA: Two what? One what?

25 Asli: 2 cm, for each 1 cm that height increases, the length increases by 2 cm or vice versa for each 2 cm length increases, the height increases by 1 cm.

Figure 15

a) Asli's Table and (b) Yener's Table for the Height, Length and Area of the Growing Rectangle



(a) (b)

Students created a table for the height, length, and area of the growing rectangle to see a pattern or a relationship between quantities [Figure 15 (a) and (b)]. As Asli stated, "*We try to record some height and length values, maybe to see a relationship."* Seeing all the magnitudes of the quantities on the table enables students to describe the relationships between them (lines 23

[13] This vignette will be cited in the section *Supporting Students' Co-Emergence of Representational Fluency and Functional Thinking*, in part two of this chapter.

and 25); thus, the tabular representation may provide opportunities for students to visualize how quantitative relationships are covarying. For instance, NA asked Asli how that table helped her see the relationship between the height and the area (line 22). Asli stated that it helped her to visualize the relationship (line 23): “It just helps me visualize how, like, as the height is increasing by one, the length increases by two cause you can clearly see the difference [tracing with her pen on the table, Figure 15 (a)].

When the change in length depended on a change in height (line 25), the students showed that they could also interchange the quantities and interpret that every two-centimeter change in length depended on the one-centimeter change in height. As Asli stated, “For each 1 cm that height increases, the length increases by 2 centimeters or vice versa for each 2 cm length increases, the height increases by 1 cm.” Hence, tabular lateral thinking enabled students to interchangeably coordinate quantities.

In sum, in the study, I found that tabular lateral thinking enables students to conceive that quantities covary when they create and interpret a table to present the quantitative relationships. Furthermore, students were able to identify the vertex point of quadratic function via tabular lateral thinking. This kind of thinking supported students’ ability to recognize that the distance between coordinate points gets smaller when they approach the vertex on the table, and that the change of the quantities (on the y-axis) gets larger when the points move away from the vertex point. With this type of thinking, students notice that the points are not merely a string of numbers; instead, there is a pattern between them. The students conceive that there is a pattern, and that quantitative relationships are generalizable and interchangeable. Thus, it is essential for students’ robust understanding of quadratic relationships that they recognize quantities with their magnitudes on a table and conceive quadratic relationships as covarying quantities; tabular lateral thinking can help them achieve this. I summarize the construct of tabular lateral thinking with Table 16.

Table 16

Tabular Lateral Thinking

Aspect of Tabular Static Thinking	Definition	Example
(1) Determining the vertex through covarying quantities	Identifying a vertex of a quadratic function on a tabular representation by coordinating the first change in one quantity with the first or second change in another quantity	Yener: "I am pretty sure that is the peak, and then it starts going down." Yener: "I think so. I mean, because change changes six every time. It can change six meters."
(2) Recognizing quantitative relationship as generalizable and interchangeable	Recognizing that there is a pattern between quantities and interchangeably coordinating the changes among the quantities	Asli: "For each 1 cm that height increases, the length increases by 2 centimeters or vice versa for each 2 cm length increases, the height increases by 1 cm."

Students' Reasoning About Quantities Within Symbolic Representations. During the study, I found that students' reasoning about quantities within symbolic representations entailed two types of reasoning: algebraic static thinking and algebraic lateral thinking. Algebraic static thinking about symbolic equations occurred when students used a known formula for the area of triangle or rectangle, $A = \frac{hxb}{2}$, or $A = hxl$, to generate a symbolic equation to represent a quantitative relationship. At this level of thinking, students used the formula ($A = hxl$) to create symbolic equations, such as $y = x * 2x = 2x^2$, without thinking about how the quantities (e.g., the height and area) might covary together, nor the meaning for the coefficient in the symbolic equation. In contrast, algebraic lateral thinking occurred when students conceived of two quantities covarying together as they created and interpreted symbolic equations. For example, students made a connection between the coefficients of symbolic equations and the covarying change of the quantities.

I give an overview of these characterizations of students' reasoning in Table 17.

Table 17

Overview of Students' Reasoning about Quantities within the Symbolic Representation

Algebraic Static Thinking	Algebraic Lateral Thinking
(1) Creating a symbolic equation from the known formula: $A = \frac{hxb}{2},\ or\ A = hxl$, with no attention to covarying quantities	(1) Redefining a symbolic equation within covarying quantities and determining the domain and range of the function within a quantitative context (2) Making a connection between the coefficient of the symbolic quadratic equation and the covarying quantities (3) Switching flexibly between correspondence and covariational reasoning

Algebraic Static Thinking. Students' algebraic static thinking about quantitative relationships includes creating a symbolic equitation with a correspondence reasoning. As we will see with the following vignette, I found that students with this kind of thinking could create a symbolic equation and a ratio between length and area; however, they could not relate how the length and area of the growing triangle task were related to one another. Furthermore, they were not always able to explain the origin and how the relationship passes the origin on a symbolic equation.

The following vignette is taken from Mert and Salim's small-group interactions when they were investigating the relationship between the length of the paint roller and the area it covered.

26 Mert: Like, it starts from the zero. And it is going to go forever.

27 WR[14]: Why do you think it starts with the zero?

28 Salim: According to the formula [pointing at the formula in Figure 16 (a)], if one of them is zero, the area has to be zero.

29 Mert: If the length is zero, area will be zero. How do you visualize? When the length…

30 Salim: Ratio yapalim. [We make a ratio.] [He draws a triangle with x:$\frac{x^2}{2}$, Figure 16 (b).]

31 Mert: OK. How are we going to make the ratio? Bu length me olmasi gerekiyor yoksa area mi? [Is this supposed to be length or the area?] [He points to the x.]

32 Salim: This is the length since they're both equal. O yuzden ikisi de aynisi oluyor. [They are both the same length times the other length, divided by two, is the area.]

33 Mert: Why is that x square?

34 Salim: Because the x times x is x^2.

35 Mert: But then wait, don't you, like, is not it like a division symbol?

36 Salim: Kind of…

37 Mert: You don't need to divide x by that.

38 Salim: O ratio. [That is the ratio.]

[14] WR is one of the TR: Waleed's initials.

Figure 16

(a) Salim's Formula of the Area of the Triangle and (b) Salim's Ratio between Length and Area for the Paint Roller Task

(a) (b)

Mert stated that height and area start from zero (line 26). Then, when the teacher-researcher asked why (line 27), Salim showed that the knowledge was coming from the symbolic formula he knew (line 28). Salim's thinking involved plugging the numbers into the formula; when the height was zero, the area would be zero, or vice versa. Salim used what he knew about a triangle area formula and applied the formula to argue that when the height was zero, the area would be zero. Salim used x to represent the length, and $\frac{x^2}{2}$ to define the area and the ratio between them (line 30)—a form of correspondence reasoning. Salim drew back from what he knew about the triangle's area and created a symbolic ratio, the magnitudes of which were not known.

For Salim and Mert, it was not clear what the ratio represented; there were expressions, but what each expression meant was not clear (lines 31–38). As I interpret the above vignette, the length being zero was not meaningful for Salim, because the symbolic equation was static and did not represent covarying quantities. Hence, static thinking might involve knowing an equation to solve the problem, but the meaning of what the equation represents and how the variables or quantities in the equation are related is still a mystery to the students.

In sum, the findings have shown that when students used algebraic static thinking, they were creating a symbolic equation from the known formula, such as $A = \frac{hxb}{2}$, or $A = hxl$, without understanding that the equation represented a quantitative relationship. They were able to solve a problem with the known formula but lacked an understanding of the quantities and their underlying relationships. In other words, reliance on a formula moved students away from reasoning about quantities.

I end this section with a table summarizing the constructs of algebraic static thinking, Table 18.

Table 18

Algebraic Static Thinking

Aspect of Algebraic Static Thinking	Definition	Example
(1) Creating a symbolic equation	Creating a symbolic equation from a learned rule to represent quantities with correspondence reasoning	Salim: This is the length since they're both equal. O yuzden ikisi de aynisi oluyor [$\frac{L*h}{2} = A, y = \frac{x^2}{2}$]. [They are both the same length times the other length, divided by two, is the area.]

Algebraic Lateral Thinking. The second, more sophisticated form of students' reasoning on a symbolic equation is algebraic lateral thinking. Algebraic lateral thinking entails three constructs: (a) redefining a symbolic equation with covariation; (b) making a connection between the coefficient and covarying quantities; and (c) switching flexibly between covariational and correspondence reasoning.

The first type of algebraic lateral thinking the study findings suggest is conceiving that the symbolic equation represents the covarying relationship of the range and the time of the falling object. During the study, this type of thinking became a source of support for students to make sense of the symbolic equation and its domain. As we will see in the following example, students coordinated the change in the range and the time of the falling object to create a symbolic equation. They used the same reasoning to make sense of the domain of the relationship. The following vignette is taken from Mert, Yener, and Tarik's small-group interactions, when they were exploring the relationship between height, time and range in the falling object task. The falling object task was introduced with a table, and at the very end of the table, the students were prompted to create a symbolic equation to present the relationship between height, time and range (see Figure 13). Consider the falling object task and the following vignette:

39 Tarik: Can we make rules?

40 Yener: So, the range is…

41 Mert: OK, so the range is…

42 Yener: A time $[t]$ times 18.

43 Mert: Oh yeah, a time $[t]$ times 18.

44 Yener: $18t$.

45 Tarik: What?

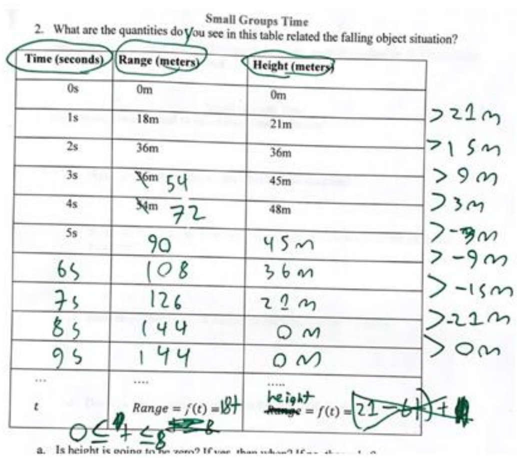
46 Yener: Range is 18*t*. Cause range increase by 18 each time, right? Oh, it stops. Wait, so it would be from, and then at the end, you would write like…

47 Mert: Oh, domain thing like…

48 Yener: Like, from here and then, like, is less than or equal to x is less than equal to *t*.

Figure 17

Mert's Table and Symbolic Equation to Present the Relationship Between the Height, Time, and Range on the Falling Object Task.



Small Groups Time

2. What are the quantities do you see in this table related the falling object situation?

Time (seconds)	Range (meters)	Height (meters)	
0s	0m	0m	
1s	18m	21m	> 21m
2s	36m	36m	> 15m
3s	~~36m~~ 54	45m	> 9m
4s	~~54m~~ 72	48m	> 3m
5s	90	45m	> -3m
6s	108	36m	> -9m
7s	126	22m	> -15m
8s	144	0m	> -22m
9s	144	0m	> 0m
…	….	…..	
t	Range = f(t) = 18t	height = f(t) = (21 − 6t)t	

$0 \leq t \leq 8$

a. Is height is going to be zero? If yes, then when? If no, then why?

49 Mert: t is less than or equal to eight, is that it? Less than or equal to eight.

50 Yener: From zero to eight.

51 Mert: Oh, OK, mine [t less than and equal to 8] is basically is the same.

52 Yener: No, cause that's just t. Then, if you do equal less than eight, then it comes negatives. If you just say less than eight, it counts negatives. You have to make an end at zero. You have to say zero 'cause you've said just less than eight.

53 Mert: I see what you mean.

In the above vignette, Mert, Yener, and Tarik were prompted to write a symbolic equation when the time is t. Tarik suggested that they should make a symbolic equation, "a rule" (line 39). Yener and Mert created a symbolic equation presenting the relationships between height and time and between time and range for the falling object task (lines 42–44).

They had the relationships on the table (Figure 17), and they reasoned with chunky, continuous covariational reasoning. Yener said, “range is $18t$. Cause range increase by 18 each time, right,” and he wrote $f(t) = 18t$.

As we see in the above vignette, Yener and his group conceived the symbolic equation as a lateral quantitative relationship rather than a static symbolic equation. Mert defined the domain of $f(t) = 18t$ as t as the time which is $t \leq 8$ (line 49) and he said, “t is less than or equal to eight, is that it? Less than or equal to eight.”

Yener said, “No, cause that’s just t. Then, if you do equal less than eight, then it comes negatives. If you just say less than eight, it counts negatives. You have to make end at zero. You have to say zero because you’ve said just less than eight.” According to his reasoning, t was not just a static symbol for Yener, it is a quantity—the time the cannonball takes to fall—so his objection was that time cannot be negative (line 52). Then Mert agreed that *t* could not be a negative number (line 53). He said, “I see what you mean,” and as he said that he canceled out $t \leq 8$ and wrote $0 \leq t \leq 8$]. Yener saw that if Mert defined that time is less than or equal to eight, then the time would keep going to negative, which did not make sense to him. He said, “You have to make end at zero,” which suggests that, in Yener’s thinking, the symbolic equation, $f(t) = 18t$, and the expression of the domain, $t \leq 8$, were connected with lateral thinking.

Hence, these students employed algebraic lateral thinking because they created symbolic representations of a quantitative relationship (range and the time) while understanding that the symbols represented quantities and covarying quantitative relationships. This vignette provides evidence that RF and FT’s co-emergence becomes a source of support to help students make sense of linear relationships as covarying quantities when dealing with a symbolic equation and its domains. With algebraic lateral thinking, the symbolic equation $f(t) = 18t$ is an emergent relationship between quantities expressed in algebraic symbols, and its domain represents quantities.

The second form of algebraic lateral thinking suggested by the findings was observed when students associated the coefficient of the symbolic quadratic equation with the covarying quantities. Students made sense of a quadratic equation's leading coefficient—"a" in $y = ax^2$—by coordinating the change in one quantity with the change in another quantity. The following vignette is taken from Asli and Yener's small-group interactions when they were exploring the relationship between the height and area in the growing rectangle task. As we will see, Asli related the quadratic function's coefficient to covarying quantities.

54 Asli: Well, this works, too.

55 NA: What works?

56 Asli: I wrote 2h squared [Figure 18]. Basically, the same thing. It is just distributed.

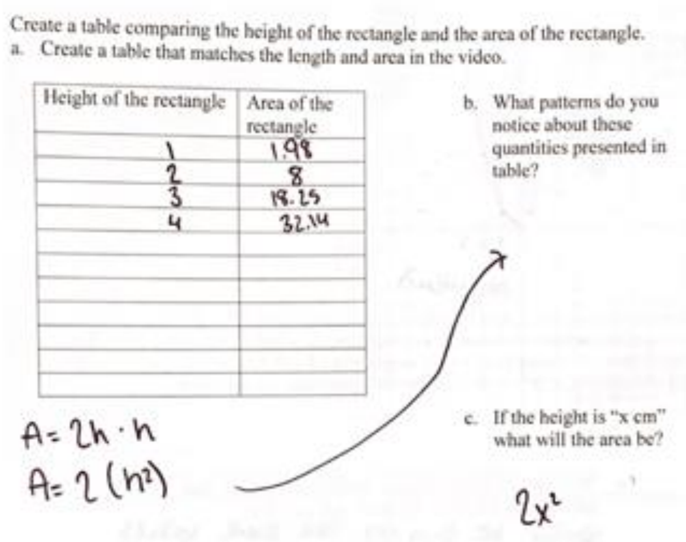
57 NF: Oh, OK. What is the two? What do you mean [referring to the coefficient above]?

58 Asli: As since the height is [inaudible], the area of the rectangle is height times length and since the length is 2h. As the height increases by one unit, the length increases two units, so that will make h as 2h squared.

Asli noticed that $2h^2$ is the same as h x $2h$. Her reasoning is that for every one unit of height there is a two-unit length increase (line 58); that is why the area must be $2h^2$. The teacher-researcher, NF, probed Asli to explain why $2h^2$ should be same as h x $2h$ (line 57). Asli defined the area formula's coefficient with reasoning about coordinating a change in height for one unit with a change in length for two units.

Figure 18

Asli's Table and Symbolic Equation


Create a table comparing the height of the rectangle and the area of the rectangle.
a. Create a table that matches the length and area in the video.

Height of the rectangle	Area of the rectangle
1	1.98
2	8
3	18.25
4	32.14

b. What patterns do you notice about these quantities presented in table?

c. If the height is "x cm" what will the area be?

$A = 2h \cdot h$

$A = 2(h^2)$

$2x^2$

As we see with the above example, the symbolic equation's coefficient relates to the coordination of change between height and length. Asli said, "As since the height is [inaudible], the area of the rectangle is height times length and since the length is *2h.* As the height increases by one unit, the length increases two units, so that will make *h* as *2h* squared." As we see here, in Asli's thinking, the co-emergence of coordination of values (FT) and multi-connectional (RF) reasoning enabled her to make sense of the coefficient on the symbolic equation of a quadratic function. Hence, for Asli, the symbolic equation's coefficient represents that for every one-unit increase in height there is a two-unit rise in length. So, the symbolic equation's coefficient, $2h^2$, is related to the change in height in relation to change in length.

A third form of students' algebraic lateral thinking suggested by the findings was observed in how students engaged flexibly in covariational and correspondence reasoning. Students employed both correspondence and covariational reasoning on a symbolic representation to reason about quantities. The following vignette is taken from Eren and Salim's small-group interaction when they were investigating the relationship between the height and area of the growing rectangle. As we will see with the vignette, students employed both covariational and correspondence reasoning when reasoning about changing quantities on a symbolic equation.

59 Eren: OK. So, we found out that the equation for the rectangle is, for the area is *y* equals two *x* squared. Height as *x*, and then length is two *x* [$y = 2x^2$] [Figure 19].

60 Salim: We multiplied them.

61 Eren: Then, every time the height increases by one, the length is twice of that. And when you multiply that the areas, big numbers it will be.

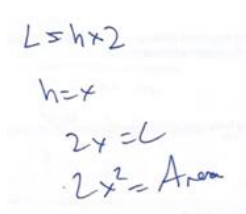
62 NA: So, so we want to be specific. What do you mean by numbers?

63 Eren: Um, so, like, it grows by a larger amount each time. So, when the height is one, it's two, the area is two. And when the height is two, the area is eight, so it grows by six. Well, one the height, the height is three. The area's 18 grows by 10 the next time and it too…

64 NA: So, when you say the height is two, two what?

Figure 19

Salim and Eren's Symbolic Equation



65 Eren: 2 cm.

66 NA: What about the area?

67 Eren: Cm squared.

68 NA: OK. What does that tell you?

69 Salim: Squared, uhm, height times two, height square times two, that is why it grows quadratics.

70 NA: What does that two represent for you?

71 Salim: It represents the length.

72 NA: Length?

73 Eren: Oh, the two represents that number being twice the size of the height.

In this interaction, Eren described the height of the rectangle as x and the length of the rectangle as $2x$ (line 59). Then Eren and Salim multiplied these two to create the symbolic equation (line 60). Eren started by thinking with covariation on both quantities, the height and the length: *"*Then every time the height increases by one, the length is twice of that. And when you multiply that the areas, big numbers it will be.*"* In this example, Eren coordinated a change in height with a change in length as he was reasoning about a quantitative relationship on a symbolic equation (lines 60 and 63). Note that, in his reasoning, correspondence and covariational reasoning about the height with the area of the rectangle were intertwined (line 62). He said: *"*Um, so, like, it grows by a larger amount each time. So, when the height is one, it's two, the area is two. And when the height is two, the area is eight, so it grows by six. Well, one the height, the height is three. The areas 18 grows by 10 the next time and it to.*"* As we see here, Eren's thinking about the height and area of the growing rectangle switched to corresponding reasoning from covariational reasoning at the very beginning. When he matched height values with the area, he soon realized the area was also growing. He coordinated the change in height with the changing area to reason that the height and the area covary together on the symbolic equation (line 61). To understand how the symbolic equation might become meaningful for students, NA asked Eren: “So, when you say the height is two, two what.” In response, Eren first stated the height is cm, and the length is twice that. Eren then stated, *“*Oh, the two represents that number being twice the size*.”*

As we see with this vignette, students might use corresponding reasoning and covariational reasoning together; the flexibility in such reasoning might be a resource to help students articulate the coefficient of the symbolic equations. Furthermore, algebraic lateral thinking might enable students to flexibly employ covariational and correspondence reasoning.

In sum, I have highlighted that, when using algebraic lateral thinking on the symbolic equation of quantitative relationships, students were able to do the following. First, using algebraic lateral thinking, students could redefine and make sense of a symbolic equation within covarying quantities (e.g., $f(t) = 18t$ being defined as “range is *18t*, cause range increase by 18 [meters] each time*”*). They were able to determine the domain of the function within a quantitative context, (e.g., a domain cannot be negative because it represents a quantity, and a negative quantity does not exist). Second, via algebraic lateral thinking, students were able to notice that each letter on the equation represents a quantity, and all the quantities are related to each other. Furthermore, students were able to create a connection between the coefficient of the symbolic quadratic equation $A = 2h^2$ and the covarying quantities, because they interpreted that the increase in the height of one unit covaried with a two-unit increase in the length. They concluded that the coefficient 2 in the symbolic equation emerged from the relationship between the height and the length of the growing rectangle. Finally, when using algebraic lateral thinking, students were able to flexibly switch between correspondence reasoning and covariational reasoning on the symbolic equation. Hence, algebraic lateral thinking is conceiving of a symbolic equation as presenting an emergent quantitative relationship, and the coefficient in the symbolic equation is related the coordination of change between the quantities. I summarize the constructs of algebraic lateral thinking in Table 19.

Table 19

Algebraic Lateral Thinking

Aspect of Algebraic Lateral Thinking	Definition	Example
(1) Redefining a symbolic equation with covariation	Redefining and making sense of a symbolic equation as covarying quantities with the domain and the range of the equation within a quantitative context	Mert: "[$f(t) = 18t, t \leq 8$] range is 18*t*, cause range increase by 18 [meters] each time." Yener: "No, cause that's just *t*. Then, if you do equal less than eight, then it comes negatives. If you just say less than eight, it counts negatives. You have to make end at zero. You have to say zero because you've said just less than eight."
(2) Making a connection between the coefficient and covarying quantities	Seeing that a coefficient of the symbolic equation emerges from coordination of the change among quantities	Asli: "[A=$2h^2, h$ x$2h$ or $2hxh = 2h^2$] As since the height is [inaudible], the area of the rectangle is height times length and since the length is 2*h*. As the height increases by one unit, the length increases two units, so that will make *h* as 2*h* squared."
(3) Switching flexibly between covariation and correspondence reasoning.	Flexibly switching between covariational and correspondence reasoning while approaching a symbolic equation.	Eren: "[$y = 2x^2$] Um, so, like, it grows by a larger amount each time. So, when the height is one, it's two, the area is two. And when the height is two, the area is eight, so it grows by six. Well, one the height, the height is three. The area's 18 grows by 10 the next time and it to."

Students' Reasoning About Quantities Within Graphical Representations. The findings regarding students' reasoning about quantities within the graphical representations entailed two types of reasoning: graphical static thinking and graphical lateral thinking. Graphical static thinking entails students expressing that they understood graphs as corresponding to the independent values and the dependent values. Graphical lateral thinking, on the other hand, is when students are imagining graphs as representing covarying quantities, where the quantitative relationships covary together on the graphs. An overview of these characterizations of students' reasoning is given in Table 20; in the following sections, I elaborate on each construct in turn, with examples.

Table 20

An Overview of Students' Reasoning About Quantities Within Graphical Representations

Graphical Static Thinking	Graphical Lateral Thinking
(1) Mapping dependent and independent values of quantities as a form of correspondence reasoning	(1) Imagining a graph as a motion or change that keeps increasing or decreasing (2) Generalizing the change on a graph by referencing the vertex point (3) Identifying the vertex as a symmetry line on the graph (4) Identifying the second change for quadratic growth as constant

***Graphical Static Thinking*.** I found during the study that students' static thinking on a graph might involve presenting quantities on the x-axis and y-axis by mapping the independent and dependent variables. When using this type of thinking, students' image of a graph is mapping the independent and dependent values without attention to each increment. Students perceive the graph as a *pictorial entailment* (Zaslavsky, 1997), with no attention to how the quantities behave on the graph. The below vignette is taken from a small-group interaction when Yener was exploring the height and area of the growing rectangle.

74 NA: How do you draw this graph?

75 Yener: Uh, so, the height is the x-axis in the areas, the y-axis. Oh, wait, yeah. Uh,

when the height is one, uh, the area would be three. So, I put a dot on 1, 3, and then, when I was two, the area would be 12, so I put a dot on 2, 12, and then, when the height was three in the area of be 27, I'll put a dot up there, and then four would be 48 and then five and be seen the five.

76 NA: OK. Is this a straight line, the graph you have [pointing at the graph on Figure 20]?

77 Yener: I think so.

78 NA: OK. Well, tell me, why you think so?

79 Yener: There must be a constant pattern between height and area.

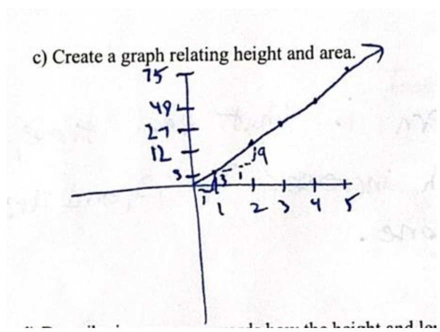
80 NA: Tell me what you mean by the constant pattern?

81 Yener: I do not know.

We see from the following vignette that Yener created a graph (Figure 20) by mapping quantities on the x and y coordinates without looking at the size of each increment between values; this type of thinking represents more correspondence reasoning (line 75). He drew a graph based on the fact that he knew area equals height times length; his reasoning seemed to be that the area would be different because he multiplied the height and the length to make the area. As we see here, he created a graph based on what he knew—the fact that the area is the height times the length. Even if he had mapped the quantities onto the graph, his image of the graph was of a map that charts the values of height with corresponding areas, with no attention given to the increments of the height (x-axis) and the area (y-axis). As we see, he matched a height of two to an area of 12, a height of three to an area of 27, and a height of four to an area of 48 (line 75). Although he was purposeful about carefully mapping the independent and dependent variables accurately onto the graph, he was not paying attention to each increment on the graph. The increment for the coordinates of (2, 12) is same as the (3, 27). In other words, he mapped the independent and the dependent variables with no attention to the size of each incremental increase in the area—the dependent variable (line 75). Thus, the line Yener made to represent the area and height was linear. He engaged in corresponding the values, or thinking of them as pairs of coordinate values.

Figure 20

Yener's Graph of the Growing Rectangle



In this example of graphical static thinking, Yener's thinking about this graph as representing quantitative relationships was focused on what he saw as mapping the numerical values of the height with the area. This supports the finding that, when using static thinking about the graph of a quantitative relationship, students mapped the quantity on the independent *x*-axis with the quantity on the dependent y-axis.

I have summarized the aspects of graphical lateral thinking, with definitions and corresponding examples, on Table 21.

Table 21

Graphical Static Thinking

Aspect of Graphical Static Thinking	Definition	Example
Mapping dependent and independent values of quantities without thinking about covarying quantities	Mapping the quantity on the independent x-axis with the quantity on the dependent y-axis without attention to how the two quantities are changing together for the same increments on the x-axis.	Yener: Uh, so, the height is the x-axis in the areas, the y-axis. Oh, wait, yeah. Uh, when the height is one, uh, the area would be three. So, I put a dot on 1, 3, and then, when I was two, the area would be 12, so I put a dot on 2, 12, and then, when the height was three in the area of be 27, I'll put a dot up there, and then four would be 48 and then five and be seen the five.

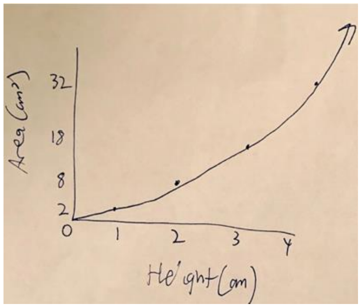
Graphical Lateral Thinking. The second form of students' reasoning on a graph suggested by the study**,** graphical lateral thinking, entails four constructs: (a) imagining a graph as a motion of increase or decrease; (b) generalizing the change on a graph and the vertex; (c) identifying the vertex and symmetry; and (d) identifying the second change for quadratic growth as constant.

The first aspect of graphical lateral thinking is defined as when students' image of a graph is associated with motion and/or change. With this kind of thinking, even if students didn't state a specific amount of change of both quantities on a graph, they still thought of the graphs as either increasing or decreasing. The graph is not a pictorial entailment; rather, it has a motion. For example, Asli and Yener interpreted their graph as the amount of change in the growing rectangle; the change in the area covaried with the change in the height on the graph (see Figure 21). As Yener said: "We found that that amount of the area, it changes per height change was four. So, it would change from two to eight. And then when it went from eight to 18 and change time just four more than six 18 to 32 it changed from 10 to 14, which adds a difference of four.

So, adds four each time to it.*"* As we see with this example, Yener had an image of the emerging quantities on the graph. In other words, Yener coordinated the change in the growing rectangle's height with the change in its area. Such reasoning is evidence that Yener perceived an image of the graph in his mind which represented covarying quantities.

Figure 21

Yener's Graph of the Height and Area of the Growing Rectangle



The second aspect of graphical lateral thinking is defined as when students generalize about the emergent quantitative relationships on a graph by referencing the vertex. During the study, students engaging in this kind of reasoning understood that if the coordinate points get closer to the vertex, the distance between the dependent quantities gets smaller. If the quantitative relationship on the graph moves away from the vertex, the magnitude among the quantities gets larger—the distance in *y*-values (vertically) gets larger. The following vignette is taken from whole-group interactions, when Mert, Yener, and Tarik were presenting on the relationship between the height of the falling object and the time it takes to fall.

82 Yener: The farther away the time is from the vertex's time for the more, the greater distance of the height between the points…

83 NA: Do you want to visualize for us here? Just quickly…

84 Yener: So, this is the vertex of four. So, for the time as the *x* value, this is a time of three. There won't be as much of an increase in height. So, it's just, like, let's see, here to here. This is way bigger. So, up here, uh, if this is all four and this is three, uh, this distance between these points vertically or the height-wise, uh, is less near the, uhm, is down here where there's two and this is one. It's way bigger cause it's away from the vertex. [He draws on the whiteboard, Figure 22].

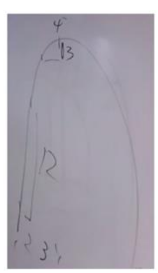
85 NA: So, you were saying, can just say one more time. You were saying…

86 Yener: So, uh, the farther away the x value are that from the vertex's x value, the greater the distance between the y value points vertically will be (see Figure 22).

Yener made a statement regarding how the distance between height values (the *y*-axis on Figure 22) gets larger when quantities on the *y*-axis move away from the vertex point, for the same values of time (the *x*-axis on his drawing) (line 82). He made a conjecture about the height change when the quantities were either near or far from the vertex points. As we see from his statement, his image of the graph was an emergent quantitative relationship that covaried on the graph (lines 82 and 86).

Figure 22

Yener's Graph of the Falling Object on the Whiteboard



Yener understood that for the same change in the *x*-axis—the time—the height values got smaller when they approached the vertex (line 84). He generalized how the height and the time of the falling object covaried on the graph by saying "so, uh, the farther away the x value are that from the vertex's *x* value, the greater the distance between the *y* value points vertically will be." So, lateral thinking on a graph might help students generalize that, with the same increment of change on the *x*-axis, the quantity on the *y*-axis increases noticeably more when it is away from the vertex point than when it is near the vertex point, as we saw with Yener in the example above.

In the third aspect of graphical lateral thinking found in the study, students conceived that the vertex point was where a symmetry line for the quantitative relationship passes through the graph: the symmetry line passes through the vertex point vertically and cuts the quantitative relationship in half. In other words, graphical lateral thinking enabled students to identify the vertex as a symmetry line on the graph that cuts the emergent quantitative relationship into two identical pieces; the amount of change on opposite sides of the symmetry line is the same, and, in the falling object task, as one increases the other one decreases with the same magnitude. The vignette below is taken from a whole-class interaction when Yener presented to the group.

87 Yener: Because for quadratics near the vertex, you get the points to become closer together, because it becomes less linear. Kind of.

88 NA: What you mean by less linear?

89 Yener: The top. The top of the quadratic is more in curved than the, like, one of the legs. It started slowing down how much the y changes, and then it starts speeding up again. Check the other thing except, if you flip it then cut it in half, it would be like the other thing. OK… [He cuts the graph at the vertex, see Figure 23].

Figure 23

Yener's Graph of the Falling Object with the Symmetry Line

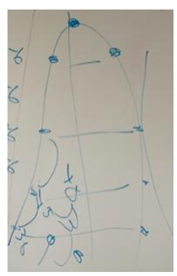
As we see here, Yener noticed that the height of the falling object gets further away from the vertex, and the vertex is the symmetry line that divides the graph's equal height (line 87). By "equal height," I mean that students imagined that height values on the *y*-axis were identical on both sides of the graph (line 89). As we see, Yener placed small vertical lines on the graph to show that their values were identical (Figure 23). And the vertex is the symmetry line, with the change in height reflected on the other side of the line due to the increase and decrease in the falling object's height. When utilizing lateral thinking, students' co-emergence of RF and FT enabled them to coordinate that the change between the height and the time of the falling object on the left and the right side of the vertex has the same magnitude; while one decreases, the other increases by the same amount (line 89).

The fourth aspect of graphical lateral thinking that the findings suggested was that students recognized that the coordination of the first change in a quantity with the second change in another quantity is a constant across the vertex. Therefore, they understood that the rate of rate of change in a quadratic function is a constant.

The following example is taken from a whole-class interaction when Yener was presenting to the group on the relationship between the height of the falling object and the time it took to fall. Yener stated that the second change in height was coordinated with the first change in the time and concluded that the change was six per second. Yener: "Here to here [placing the dots on the graph's horizontal vertex symmetry line (Figure 24) to show how the change is identical across the symmetry line]. For example, from here to here, let's say it was from 36 to 40, 36 to 45 uh, the difference between these two numbers would be nine. Yes. And then, if from 45 the let say this is 48, the difference between these would be three. The between these is a six. Yes. Which is how much it keeps changing. Negative six each time." With graphical lateral thinking, students conceived of the graph as an emerging quantitative relationship where they could identify the second constant difference between quantities.

Figure 24

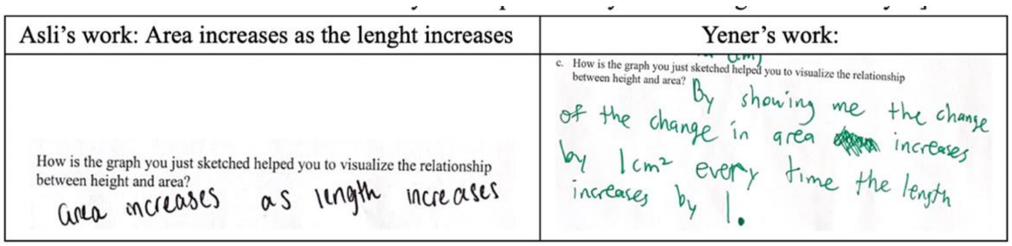
Yener's Graph of the Falling Object Presenting the Second Constant Difference



A graph of a quantitative relationship helped students to identify the second constant difference, and furthermore, it helped them to identify the emergent relationship between height and area on a graph. For example, when Yener and Asli were asked how the graph helped them see the relationship, they responded by coordinating the first change in one quantity (height) and the second change in another quantity (the area of a rectangle) with a constant growth rate. (See Yener and Asli's responses on Figure 25.)

Figure 25

Asli and Yener's Written Answers to the Question: "How is the Graph You Just Sketched Helped You Visualize the Relationship between Height and Area?"



Yener wrote that creating the graph helped him: "By showing me the change of the change in area increases by 1 cm squared every time the length [the length of the paintbrush] increased by one." Asli said that the graph helped her to visualize that "area increase as length increases."

In sum, graphical lateral thinking might reinforce students' ability to recognize an image with a graph of a quantitative relationship as covarying wherein the relationship keeps increasing or decreasing as a set of covarying quantities. Graphical lateral thinking enabled students make sense of quadratic functions in the ways listed below:

1. Students developed an image of motion, or change, in graphs representing quantitative relationships, as the quantities covary on the graph.
2. Students could generalize that, for the same increment of quantity on the x-axis, the change in between quantities on the y-axis gets larger when points are further away from the vertex compared to when the points are closer to the vertex.
3. Students understood that the vertex of a quantitative relationship is either the maximum or minimum magnitude that quantity on y-axis can become.

4. Students understood that the vertex is always the symmetry point where a vertical symmetry line could pass through the graph of a quantitative relationship.
5. Students could perceive that the relationship between the first change in the height of a rectangle or triangle and the second change in area is always constant, with a growing linear relationship. In other words, the change of change for quadratic growth is constant.

I summarize students' graphical lateral thinking in Table 22 below.

Table 22

Graphical Lateral Thinking

Aspect of Graphical Lateral Thinking	Definition	Example
Imagining a graph as a motion of increase or decrease	Developing an image of a graph as a set of emergent quantitative relationships	Yener: We found that that amount of the area, it changes per height change was four. So, it would change from two to eight. And then when it went from eight to 18 and change time just for more than six 18 to 32 it changed from 10 to 14, which adds a difference of four. So, adds four each time to it.
Generalizing the change on a graph and the vertex	Characterizing the emergent relationship between quantities on a graph by generalizing the change in quantity.	Yener: So, uh, the farther away the x value are that from the vertex's x value, the greater the distance between the y value points vertically will be (Figure 22).
Identifying the vertex and symmetry	Identifying the vertex point, where the symmetry line passes through, by cutting the quantitative relationship in half, and naming the vertex as the minimum or the maximum point the quantitative relationships could reach.	Yener: The top. The top of the quadratic is more in curved than the, like, one of the legs. It started slowing down how much the y changes, and then it starts speeding up again. Check the other thing except, if you flip it then cut it in half, it would be like the other thing. OK… [He cuts the graph at the vertex, see Figure 23].
Identifying the second change for quadratic growth as constant.	Identifying the second change for quadratic growth as constant—interrelatedness 2—and understanding that the relationship between the first change in length and the second change in area is constant.	Yener: By showing me the change of the change in area increases by 1 cm squared every time the length [the length of the paintbrush] increased by one. (See Figure 25).

Finding 2: Levels of Connections Between Students' Representational Fluency and Functional Thinking

Level 0: Disconnection Between Students' Representational Fluency and Functional Thinking. A disconnection between students' RF and FT is defined as when students are able to create multiple representations to present a quantitative relationship, but they don't perceive that the relationship co-emerges across multiple representations. In other words, students think of the quantitative relationship as static or nonemergent.

A disconnection between students' RF and FT creates multiple representations in parallel when engaging in a static way of thinking about quantitative relationships. Students might create representations in parallel, but they don't interpret that the representations present a quantitative relationship that covaries. In other words, students might solve a problem using two or more representations, but they lack the ability to explain what each representation presents. They engage in static thinking to solve the problem using at least two representations; they justify each representation using other types of representations in solving the task.

The following vignette is taken from Salim and Mert's small-group interactions with the growing triangle task—that is, the paint roller task—when they were investigating the relationship between the length of the paint roller and the area it covered.

90 Salim: We have a specific graph [Figure 26 (a)]. This is the question [$y = \frac{x^2}{2}$].

91 NA: Where is your area?

92 Salim: y is the area.

93 Salim: And the x is the length.

94 NA: So, what is the…? So, this is the symbolic equation. How is the relationship looks like in this equation and on this graph?

95 Salim: When we graph this [$y = \frac{x^2}{2}$], we get the graph.

96 NA: The information you have here explains graph or the equation. [Pointing at his written work “The relationship between the length of the paint roller and the area covered is quadratics.”] [See Figure 26 (c).]

97 Salim: It explains the graph.

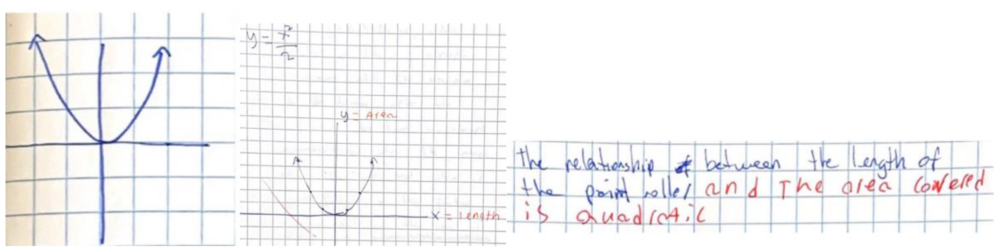
98 NA: Tell me how?

99 Salim: Because the graph is quadratics and the relationship, uhm…

100 Mert: Quadratics.

Figure 26

Salim’s (a) Graph, (b) Symbolic Equation, and (c) Written Response About the Relationship Between the Length and Area of the Rectangle



(a) (b) (c)

101 NA: What is your proof of being a quadratic?

102 Salim: The equation.

Mert and Salim constructed a parabola and a symbolic equation, claiming that the relationship between the height and the area of a triangle was a quadratic function [Figure 26 (a) and (b)]. They stated that their symbolic equation showed that the relationship between the length of a triangle and the area is quadratic, and when they graphed it, it made a parabola (lines 95 and 97). Salim was careful that the area could not be negative; however, he stated that a quadratic function graph must be a parabola because he had $y = \frac{x^2}{2}$. And according to the equation, the graph had a negative domain (line 90). In this case of disconnection between RF and FT, Salim had specific thinking about what a graph of $y = \frac{x^2}{2}$ should look like—a parabola. Salim and Mert claimed that they even created a particular graph (line 90). Still, the graph they created was a canonical graph of a quadratic function rather than a graph of the relationship between the height and the area of the triangle in the task (lines 97–102). Salim wrote, "The relationship between the length of the paint roller and the area covered is quadratics." [See Figure 26 (c).] Although Salim wrote that the relationship between the length of the paint roller and the area covered was quadratic, he drew a canonical graph to present the equation $y = \frac{x^2}{2}$ (line, 95). As we see with Figure 26 (a), the graph had a negative length, and in line 95 Salim stated the graph presented the equation. This evidence supports my claim that here there was a disconnection between Salim's RF and FT.

As we learn from this vignette, students might create a symbolic equation and a graph of it, but they might not see that the graph is presenting quantitative relationships (lines 97–102). This indicates that there may be a disconnection between students' representational skills and their FT. This data indicates that creating a symbolic representation and graphing that equation may not be meaningful for students if they cannot see that representations present two quantities with positive domain. Because students created a canonical graph and symbolic equation without specifying the domain of the equation, that shows they might not have been making sense of what the two representations presented and how they were connected—a disconnection between their RF and FT.

Level 1: A Partial Connection between Representational Fluency and Functional Thinking. The second level of my characterization of students' RF and FT, a partial connection between RF and FT, can be defined as students being able to conceive that quantities have co-emerging relationships on a single representation with chunky continuous covariational reasoning. At this level, students can create a single representation to present an emerging quantitative relationship—interrelatedness 2[15]. However, they are not able to carry interrelatedness 2 over to representations other than the source representation—they have difficulty making a connection among the source and the targeted representation.

At this level, students' thinking switches back and forth between interrelatedness 1[16] and interrelatedness 2 as they create and connect concrete representations to present emergent quantitative relationships. Students may create a symbolic equation or a graph representing a quantitative relationship. However, they may not be able to differentiate the type of interrelatedness they are presenting with these representations. At this level, students can differentiate between interrelatedness 1 and 2 on tabular representations by making a conjecture to generalize the relationship between the quantities. However, there should be a distinction between interrelatedness 1 and 2 when they create and connect these relationships on a symbolic equation, and this distinction is absent at level 2.

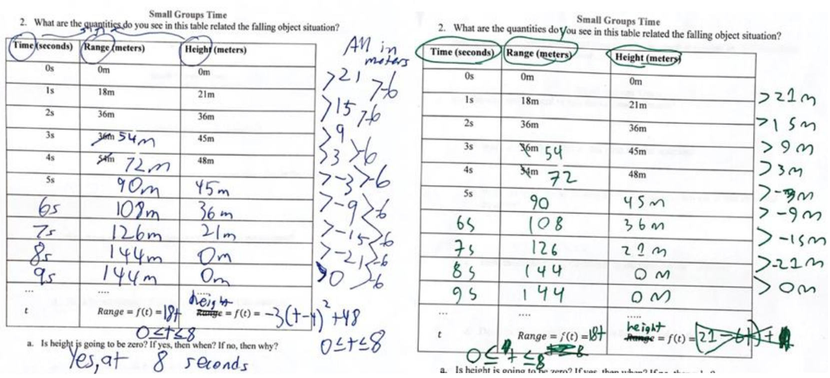
The following vignette is taken from Mert, Tarik, and Yener's small-group interactions when they were investigating the relationship between the height of the falling object and the time it took to fall on a partially completed table [see the partially completed table in Figure 27 (a) and (b)]. Before the following vignette, Mert, Yener, and Tarik had agreed on the relationship

[15] Interrelatedness 2: a coordination of the first change in one quantity with the second change in another quantity.
[16] Interrelatedness 1: a coordination of the first change on one quantity with the first change in another quantity.

between the height and time—that for each second, the height was increasing six meters on the table. Yener said, “And then, if from 45 the let say this is 48, the difference between these would be three. The between these is a six. Yes. Which is how much it keeps changing. Negative six each time.” In the following vignette, the students were creating a symbolic equation to present the height [height $= f(t)$] when the time is t.

Figure 27

(a) Yener’s Table and (b) Mert’s Table for the Height, Range, and Time of the Falling Object



(a) (b)

103 Yener: OK, and height wouldn’t be…

104 Mert: It’s going to be, oh that’s a difficult one.

105 Yener: You put 21 somewhere. I don’t know.

106 NA: What do you mean by “put 21 somewhere”?

107 Mert: Let me just write this one.

108 Yener: Cause it starts out in the change starts and ends at 21, 21 [pointing at the first change in height on table Figure 27 (a)].

109 Mert: So, like, it decreases by six each time. So, it's going to be. I think it's going to be 21 minus 6t $[21 - 6t]$ [Figure 27 (b)].

110 Yener: Yeah, that's good.

111 NA: Why don't you write here?

112 Tarik: But that's not going to be right because, at one point, it's going to reach heights. How are we going to do it?

113 Yener: Yeah. it would be, uhm…It would be zero, because that won't be, you have to add, like, a quantity that makes it height.

114 Tarik: We can make a rule.

115 Yener: That's, like, that's the change. The change. Like, this thing [pointing at the second change in the height on Figure 27 (a)].

116 Tarik: It has to put one of these in so that once it reaches a one number, then you have to stop, and then it has to decrease.

117 Mert: Well, no, this doesn't work.

118 Tarik: Exactly, why? Why does it get works? But because it doesn't reach the maximum. Look, if what will happen if we calculate this. What did you say?

119 Mert: 21 minus 6t.

120 Tarik: Trying to. It's just going down.

121 Mert: I got this. Is it going to be this? [He adds 9; $[21 - 6t + 9]$ on Figure 27 (b).]

122 Yener: No.

123 Tarik: What is that +9?

124 Yener: Because is this how this one becomes 36, I think you have to add 48.

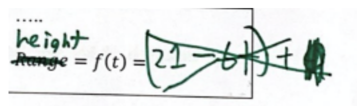
125 Mert: I don't know what we're supposed to. I am kind of confused. It is going to be

like… [he crosses out his equation, Figure 27].

126 Yener: I don't know ’cause you have to somehow add a number.

Figure 28

Mert’s Symbolic Equation of the Height when the Time is t



In this small-group interaction, Tarik, Yener, and Mert were able to identify the vertex point on the table, and they determined that every time the time increases by one second, the change in the height was a decrease of six meters. They generalized the interrelatedness 2 by as for every one second, the height’s increase decreases by six meters (line 109). When they were asked to create a symbolic equation using the table, they switched back and forth between whether the symbolic equation represented interrelatedness 1 or 2.

Yener suggested that the equation had to have a 21 because the height increased 21 meters at the beginning (line 105). He stated, “Cause it starts out in the change starts and ends at 21, 21 [pointing at the first change in height on Figure 27 (a)].” For Yener, since the change in height on the table started and ended with a change of 21 meters, the equation had to have a 21 in it (line 108). When Yener referred to interrelatedness 1 in his thinking by stating “*You put 21 somewhere* [in the equation],” the first change in height was coordinated with the first change in time. Tarik, Yener, and Mert looked for an equation to define the height—$f(t)$—when the time was t. There was no clear distinction between whether they were trying to represent interrelatedness 1 or 2 on the symbolic equation they were trying to make. Yener stated the symbolic equation had to have 21 (line 105) and Mert built on it by saying, “So, like, it decreases by six each time. So, it's going to be… I think it's going to be 21 minus 6t [$21 - 6t$]" (line 109). Mert's thinking represented interrelatedness 2—the coordination of the first change in the time with the second change in height. Since the relationship was that the height's increase decreases for every six meters, Mert perceived as a constant rate of change. Tarik disagreed with this equation (line 112). Tarik noticed that the relationship was between the time and height, reasoning that $f(t) = (21 - 6t)$ would not reach the maximum (line 119). Tarik was still looking for a symbolic equation called a "rule" to present interrelatedness 1 (line 114). He disagreed with Mert and Yener by saying, "But that's not going to be right because, at one point, it's going to reach [the maximum] heights. How are we going to do it?" He furthered his claim by saying that the equation did not work, "because it doesn't reach the maximum." So, for Tarik, the equation $f(t) = (21 - 6t)$, did not represent interrelatedness 1—the coordination of the first change in both quantities (the height and the time). Yener, however, noticed that the equation presented interrelatedness 2, between the second change in the height and the first change in the time. He stated that "That's, like, that's the change. The change, like, this thing [pointing at the second change in the height on figure 27 (a)]."

As we notice here, the students had difficulty differentiating types of interrelatedness between height and time. When they generalized, their basic reasoning made sense—"The amount the height changes decreased by six for every second"—but representing the interrelatedness of the time and the height across multiple representational forms was a challenge for them. For Tarik, the symbolic equation showed that the relationship had a maximum point because he was looking to present interrelatedness 1 (lines 116–121). As Tarik said, "It has to put one of these in so that once it reaches a one number, then you have to stop, and then it has to decrease."

Since interrelatedness 1 is a curve, it increases by an “uneven” rate, while interrelatedness 2 is linear and increases by a constant rate. As we see here, the students had difficulty differentiating what the symbolic equation presented in terms of interrelatedness. As we see with Yener’s group, chunky continuous second covariational reasoning might land on a symbolic equation similar to the symbolic equation’s derivative equation for a quantitative relationship. For Mert, the equation $f(t) = (21 - 6t)$ might have represented the relationship between the second change in the height and with the first change in the time—interrelatedness 2.

At this level, students’ lateral thinking about a quantitative relationship on a table might go in two directions: interrelatedness 1 and 2. When students create and connect to the symbolic equation, they have difficulty distinguishing whether the symbolic equation represents interrelatedness 1 or 2. Even if students were engaged in coordinating that for every second, the amount the height’s increase decreases by six meters (interrelatedness 2), it was challenging for them to see how this relationship would look on a symbolic equation. As we see, Mert created $f(t) = (21 - 6t) + 9$ to present the relationship between the height and time. As we notice, he wrote the symbolic equation for the relationship as a linear function with a negative six slope which is decreasing by negative six each time—which represents interrelatedness 2.

Hence, to students at this level, chunky continuous second covariational reasoning is more visible on a tabular representation than a symbolic equation. Students might need to be supported to differentiate between interrelatedness 1 and 2 on a symbolic equation. And there is a connection between them: the symbolic equation for interrelatedness 1 is a quadratic relationship, and interrelatedness 2 is the derivative of that quantitative relationship—a linear relationship. Since the coordination of quantities determines symbolic equations, an understanding of a quantitative relationship brings richness to students' thinking about symbolic equations.

In sum, at level 1, with a partial connection between their RF and their FT, recognizing a connection between a table and symbolic equation of quantitative relationships might be challenging for students. Students have difficulty differentiating whether interrelatedness 1 or 2 is being represented with the symbolic equation because the symbolic equation for interrelatedness 1 is a quadratic relationship, and interrelatedness 2 is the derivative of that quantitative relationship—a linear relationship. Although students at this level had difficulty in creating a symbolic equation of interrelatedness 2, they were able to flexibly move between interrelatedness 1 and 2 in creating and connecting a table and symbolic equation for the quantitative relationship.

Level 2: A Connection between Representational Fluency and Functional Thinking. The next level I identified to characterize students' connection between RF and FT is a connection between students' RF and FT. Level 2 is defined as when students conceive that quantities have co-emerging relationships on a single representation—i.e. chunky continuous covariational reasoning—and they are able to create two representations to present the emergent quantitative relationship's interrelatedness 2, and carry over the interrelatedness 2 from the source to the targeted representations.

Understanding the connection between the table and graph with chunky continuous second covariational reasoning—interrelatedness 2— enabled students to visualize the relationship as a negative linear graph—a derivative of a quadratic function with a negative leading coefficient. Eren engaged in interrelatedness 2 between height and time on the falling object task by making the connection between the table and graph [Figure 29 (a), (b), and (c)]. This reasoning enabled Eren to envision that the graph would be a negative linear graph.

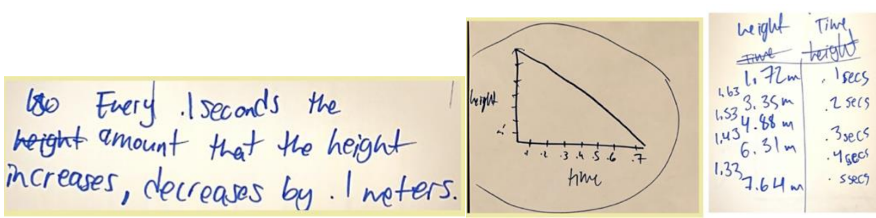
Consider the falling object task and vignette below:

127 Eren: OK, so let me just uhm every point one second. The height of the amount that the height increases decrease by 0.1 meters. Every 0.1 seconds, the amount that the height increases decrease by one point second or all the cases but one meter [Figure 29 (a)].

128 Eren: So, it's a negative linear graph [Figure 29 (b)]

Figure 29

Eren's (a) Written Statement, (b) Graph, and (c) Table of the Relationship between the Falling Object's Height and the Amount of Time it Takes to Fall.



(a) (b) (c)

Eren, Tarik, and Mert created a table and reasoned that for every 0.1 second, the height increase decreased by 0.1 meters (interrelatedness 2) (line 127). Eren said, *"*Okay, so let me just, uhm, every point one second. The height the amount that the height increases decreases by 0.1 meters for every 0.1 seconds. The amount that the height increases decrease by one point second or all the cases but one meter.*"* Although Eren coordinated the change in height with the first change in time in portions of 0.1 second on the table, he had difficulty shifting this reasoning onto a graphical representation. When he was asked to visualize the reasoning on a graph, he imagined the first change in time with the second change in height; he then concluded that the graph should be linear with a negative slope (128)—he said, "so it is a negative linear graph." In other words, Eren was graphing his reasoning "the height the amount that the height increases decreases by 0.1 meters for every 0.1 seconds*"* as a line with negative slope—a derivative on the interrelatedness 1.

As we see here, Eren's understanding of interrelatedness 2 on a tabular representation [line 127, Figure 29 (c)] helped him to see that the graph represented that the second rate of change in height was a negative linear slope, because the amount of increase in the height was decreasing 0.1 meter for every 0.1 second. With this example, Eren created the first derivative of a quadratic function graph to present the relationship between the first change in time and the second change in height [Figure 29 (b)]. Although Eren had not learned about derivatives or a graph of the first derivative of quadratic relationships, his reasoning about interrelatedness 2 between tables and graphs allowed him to create the first derivative graph of quadratic relationships (lines 127 and 128). Hence, Eren's level 2 connection between RF and RT was helping him to create and connect the table and the graph of interrelatedness 2.

Level 3: Flexible Connections between Representational Fluency and Functional Thinking. The fourth level of connection between students' RF and RT that I identified in the study was a flexible connection between RF and FT. This can be defined as when students understand that quantities have a co-emerging relationship and can flexibly switch between interrelatedness 1 and 2. At this level, they are able to create two or more representations to present interrelatedness 1 and 2, and they can flexibly switch back and forth between the targeted representation and the source representation with a clear understanding of interrelatedness 1 and

2. The following vignette is from a small-group interaction between Yener and Asli. They were investigating the height and area of the growing rectangle. To prove students' flexible connection between RF and FT, I will refer to Asli and Yener's vignette.

129 Yener: How much the area changing each time. Uhm the change, in the amount the area changes will be constant for each time. So, this time it changes by six, the next time it changes by 10, which is four more than six, next time it changes 14, which is four more than 10. So, it keeps increasing like that. The change in the area will be four each time. [Pointing at the table on Figure 30 (a).]

130 NF: Why do you think it keeps going up by four?

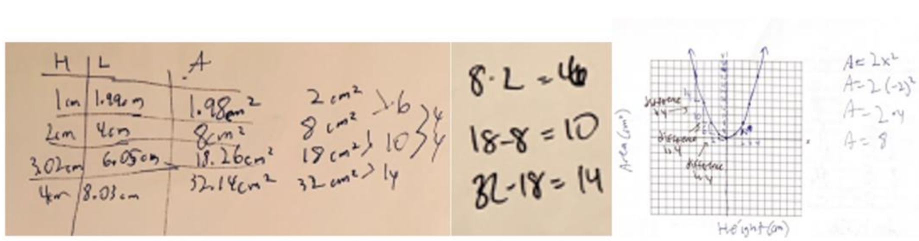
131 Asli: Because it works for this, I guess.

132 NF: How is this you just talked is related to the way the area is changing is the same? How can you see that in the graph? How is this related to the graph? [NF points at Figure 30 (b).]

133 Asli: Because it means that you need to calculate.

Figure 30

(a) Yener and Asli's Table Presenting the Height and Area of the Growing Rectangle; (b) Yener's Graphs Showing How the Area Increases



(a) (b)

134 Yener: The distance between this point and this point will be a number and then this point between this point will be a number that is 4 cm more than this number. The numbers between these will be…uhm.

135 NF: Are you making a match? Are making exact match? Do you see the fours in your graph? Or this is the six times four?

136 Yener: Because this would be, oh, wait. So, if you look at the points here, there would be, this is between these two points, the first two will be two, and this is between this one and the one over there will be six, which is four more than two, and it kept going all the 10, then 14, and it keeps going increases four by each time [Pointing at the graph Figure 30 (b).]

137 NF: You are showing 2, and 6, 10 and 14, the distance?

138 Yener: Yeah, they all have equal distances. This is the point.

When Asli and Yener created and interpreted quantitative relationships on a table [Figure 30 (a)] and a graph [Figure 30 (b)], their reasoning pushed them to make connections among the graphs and tables (lines 129 and 136). At the same time, they saw the invariant feature that for every 1 cm increase in height, the area increased 4 cm^2—interrelatedness 2. Yener stated that every 1 cm rise in height resulted in the area increasing 4 cm^2 on the table and graph (lines 129–136). Yener made connections between the table and graph to present the emergent quantitative relationships. We see that Yener also recognized that for every 1 cm increase in height, the change of change (i.e., the second difference) in the area was 4 cm^2 on both the table and the graph (line 136).

For Yener and Asli, at level 3 thinking, the table and the graph were no longer static symbols; instead, the table and graph represented the growth relationship between the height and the area of the growing rectangle. As we see in the vignette, Yener said, "How much the area changing each time? Uhm, the change, in the amount the area changes will be constant for each time. So, this time it changes by six, the next time it changes by 10, which is four more than six, next time it changes 14, which is four more than 10. So, it keeps increasing like that. The change in the area will be four each time. [Pointing at the table on Figure 30 (a)]." Yener noticed that the second difference in area is a constant, $4\ \text{cm}^2$ for every 1 cm increase in height on the table—interrelatedness 2. Consequently, Yener interpreted the table as a growth relationship between the height and the second difference in the area, instead representing numerals on the table. Note that when Yener was asked to explain "what increases by four each time?" (lines 130 and 132), he knowingly switched back and forth between interrelatedness 1 and 2—coordinating the change in height with the first change in the area. Eventually, he articulated how he drew the $4\ \text{cm}^2$, a constant increase in the area (line 136). This excerpt shows that Yener had fluency in his thinking about interrelatedness 1 and 2—covariational reasoning—and switched back and forth between coordination of values and chunky second covariational reasoning. Hence, if students have level 3 fluency in interrelatedness 1 and 2, they might engage in reasoning types that fit with the nature of the representations they intend to present for the quantitative relationships.

Yener explained how he concluded the second constant difference in the area by interpreting the table with coordination of height and area, so he had fluency in interrelatedness 1 and 2 (line, 136). Then Yener was prompted by NF to articulate a $4\ \text{cm}^2$ change in area for each increment of time on the graph; NF asked, "How is this you just talked is related to the way the area is changing is the same? How can you see that in the graph? How is this related to the graph? [NF pointing at the graph in Figure 30 (b)]" This question prompted Yener to notice that the constant increase in the area is a distance between the coordinate points of the graph. He said, "Because this would be, oh, wait. So, if you look at the points here, there would be, this is between these two points, the first two will be two, and this is between these one and the one over these will be six, which is four more than two, and it kept going all the 10, then 14, and it keeps going increases four by each time [pointing at the graph in Figure 30 (b)]. As we see here, Yener identified the constant increase in the graph as the "difference is four" in between coordinate points on the *y*-axis [see Figure 30 (b)]. He saw the same growth on the graph in height and area, so to him the graph represented the growth—emergent shape thinking (Moore & Thompson, 2015).

Hence, at this level, when students approach a graph and a table as representing a quantitative relationship that grows, they can interpret the table and graph as growth rather than seeing them as static shapes or symbols of quadratic functions. On top of this, students' reasoning about quantitative relationships by creating and analyzing a table and a graph might push them to gain fluency in interrelatedness 1 and 2. In other words, students at this level switch back and forth between interrelatedness 1 and 2 to explain the quantitative relationships on a table and graph. An overview of my findings characterizing students' co-emergence of RF and FT is given in Table 23.

Table 23

Table Summary of Levels of Students' Co-emergence of Representational Fluency and Functional Thinking

Level	Students' Mental Image of a Quantitative Relationship	Representational Activities
Level 0	Conceiving that quantities are static non-emerging—no coordination among the quantities	Creating multiple representations with no connections to present static quantitative relationships—use of multiple representations in parallel to present a static quantitative relationship
Level 1	Conceiving that quantities have co-emerging relationships on a single representation—chunky continuous covariational reasoning	Creating a single representation to present an emerging quantitative relationship (interrelatedness 2), but no ability to carry interrelatedness 2 over to the targeted representation—no connection between source and targeted representation
Level 2	Conceiving that quantities have co-emerging relationships on a single representation—chunky continuous covariational reasoning	Creating two representations to present an emergent quantitative relationships interrelatedness 2, and carry over the interrelatedness 2 on the source and targeted representation
Level 3	Conceiving that quantities have co-emerging relationship, and flexibly switching between interrelatedness 1 and 2	Creating two or more representations to present interrelatedness 1 and 2, and flexible switched back and forth between targeted and source representations with a flexibility of interrelatedness 1and 2.

Part 2: Supporting Secondary School Students in Developing a Meaningful Understanding of Quadratic Functions

A main finding of this study, and the answer to my second research question, is a learning-ecology framework that articulates supports that help students to develop a meaningful understanding of quadratic functions. The learning-ecology framework included three main categories: (a) teacher pedagogical moves; (b) socio-mathematical norms; and (c) enacted task characteristics. Since these categories do not constitute distinctly separate layers in developing students' meaningful understanding of quadratic functions, I refer to them as a "learning-ecology[17] framework." Such terminology helps to draw attention to the interdependent nature of each learning component. With that in mind, the development of a meaningful understanding of quadratic function among students during the study did not occur along a linear path; instead, it required intertwined layers of supports working in a nonlinear fashion.

In this chapter, I define *students' meaningful understanding of quadratic functions* as instances in which students co-develop RF and FT while learning about quadratic functions. Subsequently, I introduce and verify the learning-ecology framework by identifying shifts in students' RF and FT when the framework is present. Four shifts were identified in students' RF and FT; I define and summarize each of them here in part one.

What Counts as "Support:" the Learning-Ecology Framework

I define the *support* that students received during the study as a learning-ecology framework that helped students' meaningful understanding of quadratic functions. The learning-ecology framework consisted of three intertwined components: teacher pedagogical moves, socio-mathematical norms, and enacted task characteristics. I present the three components of the learning-ecology framework in Table 24.

[17] The word ecology is borrowed from biology; it defines the relationship between organisms and their surroundings.

Table 24

The Learning-Ecology Framework

Enacted Task Characteristics	Socio-mathematical Norms	Teacher Pedagogical Moves
Setting an infrastructure for students' QR • Identifying changing attributes of tasks • Coordinating changes among quantities • Generalization Launching students' RF • Visualization • Creating and making connections among representations	Peer pressure for justification • Comparison between graphs of quadratic functions and exponential functions • Realization of a limited knowledge of quadratic functions • Justification via the question "where is your reasoning?" • Skepticism about how two quantities are related • Justification of whether a quantitative relationship is linear or nonlinear Peer approval	Supporting students' co-emergence of RF and FT • Teacher pedagogical moves to support creating a representation of quantitative relationships • Teacher pedagogical moves to support connections among representations of quantitative relationships Creating a foundation for FT • Probing students to identify the attributes of an object or a situation • Probing for a unit to measure an object's attributes • Probing for the coordination of change between quantities • Encouraging students to justify their reasoning about the relationship between quantities • Probing for continuous covariational reasoning

Enacted Task Characteristics. The first component of the learning-ecology framework is *enacted task characteristics*. I define enacted task characteristics as the instances in which students are given opportunities to articulate, talk about, answer, and/or discuss quantitative relationships within tables, graphs, and symbolic equations during small- and whole-group interactions (King, 2011; Stein et al., 2007). In other words, acted task characteristics are statements and questions about a problem or a set of problems that encourage students to articulate, talk about, discuss, and/or create representations to present quantitative relationships. Enacted task characteristics are a form of instructional support; I have divided the characteristics into clusters of those promoting students' QR and those promoting students' RF.

Setting an Infrastructure for Students' Quantitative Thinking. There are three types of enacted task characteristics that fall under the umbrella of setting an infrastructure for students'

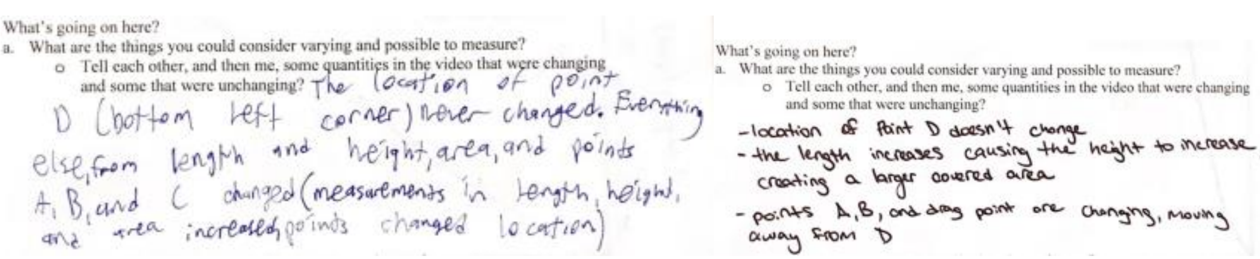
QR: (a) identifying changing attributes of the tasks or situations, (b) coordinating the change among quantities, and (c) making generalizations about quantitative relationships.

The first of the enacted task characteristics is asking students to identify attributes of a situation or their tasks—identifying relevant quantities, and units to measure the quantities. Students were requested or prompted to identify quantities by looking at the attributes of the task and identifying relevant quantities. After tracing appropriate quantities within the task context, they were prompted to think about a unit to measure the quantities.

In the following vignette, Asli and Yener watched a video (see Video 1) featuring a growing rectangle being sketched via a dynamic geometry software. Student handouts were structured so that students were asked to think and talk to each other about varying quantities and possible ways to measure those quantities. The task was structured to ask students to identify varying quantities; for example, the question in Figure 31: “What are the things you could consider varying and possible to measure?”

Figure 31

(a) Yener’s and (b) Asli’s Ideas About Varying Quantities of the Growing Rectangle



(a) (b)

See the vignette below, which is the conversation students had in responding to the question on the task: "What are the things you could consider varying and possible to measure?"

139 Asli: Location of point D does not change.

140 Yener: Yeah. [Figure 31 (a) shows Yener's written answer: The location of point D (bottom left corner) never changed. Everything else, from the length and the height, area and the points A, B, and C changed (measurements in length, height, and area increased, points changed location)].

141 NF: Can you talk to each other?

142 Asli: We just wrote down when we talked about before we got the paper. [Figure 31 (b).]

Asli and Yener identified the corners of the rectangle; D was not changing (line 139–140). Asli referred to it as the D's location; Yener stated that D is at the "bottom left corner," not changing (Figure 31). They agreed that everything else is changing on the task. Asli noticed that "the length increases causing the height to increase, creating a larger covered area" (see Figure 31 [b]). Asli also recognized that the corners of the rectangle are changing, so she wrote "Points A, B, and drag points are changing, moving away from D." Yener agreed with Asli that A, B, and C changed. Length, height, and area changed as well. Yener recognized that the change in height, length, and the area increases when the locations of A, B, and C (corners of the rectangle) change (line 140). Hence, I drew a conclusion that creating a foundation for students' QR might involve getting students to determine what is changing or varying in a dynamic task context. The tasks' structure, along with necessary tools, supports students in identifying varying relevant quantities. Students begin to recognize which quantities are constant, which are variable, and how to measure them.

The second enacted task characteristic is the coordination of change among quantities: probing, asking, or reinforcing students to coordinate changes among quantities. The tasks were structured to ask students how a change in one quantity affects the change in another in order to get students to coordinate the change between quantities. For example, one of the enacted task characteristics is asking students: "How does the change in height affect change in area?" In the following vignette, Asli and Yener were investigating the relationship between the height, length, and area of the growing rectangle task.

143 Yener: How does change in height is affect the change in area? If the height changes, the length changes.

144 Asli: The change in height increases the area covered. Because it contributes to the formula to get the area.

145 Yener: When the height changes, the area changes. Here is the area changes too.

146 NF: Can you be more specific? About how the height changes, the length changes. This also be an area.

147 Asli: When the length increasing the heights increases.

148 Yener: Increase uhm. I think they might increase by the same amount. Yeah, they probably started over different, and then they increased amount each time the height and length.

149 Yener: Oh, I found this when height changes by two, length changes by three. That means that is constant.

150 Asli: Okay. So, what I wrote is the change in height increases the area covered because it contributes to the formula necessary to calculate the area [Figure 32 (a)].

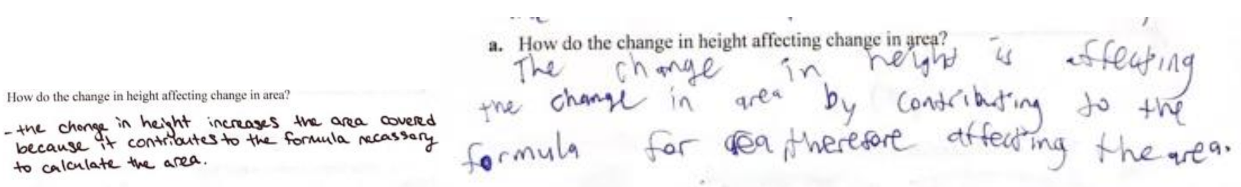
151 Yener: Mine is same thing with height is affecting the change. [Figure 32 (b); he

wrote: “The change in height is affecting the change in area by contributing to the formula for area therefore affecting the area.”]

For this type of enacted task characteristic, students are asked to see how the change in one quantity affects the change in another quantity (Figure 32). These questions (e.g., how does change in height affect the change in area?) form a foundation upon which students can engage in the coordination of change in quantities. For instance, Yener read the question (line 143): “How does change in height is affect the change in area?” Then he coordinated height with the length such that if the height changes (line 145), the length changes. Asli built on Yener’s reasoning by stating (line 144) “The change in height increases the area covered.”

Figure 32

(a) Asli’s and (b) Yener’s Response to “How do the change in height affecting change in area?”



Yener and Asli engaged in the task jointly; Yener agreed with Asli’s statement, which encouraged Asli to justify her statement (line 144). She said, “Because it contributes to the formula to get the area.” Asli’s justification is about the corresponding reasoning. Yener said: “Increase, Uhmm. I think they might increase by the same amount, Yeah, they probably started over different and then they increased amount each time the height and length.” Yener noticed that the height and length of the growing rectangle started with a different amount that changed in magnitude or amount each time (line 148). Then Yener said: "Oh, I found this when height changes by two, length changes by three. That means that is constant." Asli read her written responses: "Okay. So, what I wrote is the change in height increases the area covered because it contributes to the formula necessary to calculate the area" (line 150).

In responding to the task characteristics, students not only respond the questions on the tasks, but they also attempt to justify their responses[18]. As we saw from Asli, she was reading her answer and also justifying it (line 150). Furthermore, Yener read his response by comparing and contrasting his answer for the same question with Asli's (line 151).

Observing the results of this student exchange, we can infer that this student ability to reason about relevant quantities and coordinating changes in quantities develops when they are prompted to consider how a change in one quantity affected change in another quantity. In other words, asking students about how a change in one quantity may affect the change in another can be an effective way to support healthy peer deliberation and the development of more advanced reasoning.

Lastly, enacted task characteristics involved structuring tasks to ask students to generalize the relationship between quantities. In terms of this study, a generalization is a form of support that pushes students to think about a pattern representing the relationship between quantities (e.g., the length of the paint roller and its area). With enacted task characteristics, students were asked to answer the same focus questions[19] in small- and whole-group settings in their handouts and had individual writing time for answering the same problem in their journal. The below vignette is taken from a whole-group interaction, when students explored the relationship between length of the paint roller and the area covered by the paint roller. Enacted task

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[18] Students' interactions will be further discussed in the section *Socio-mathematical Norms* in chapter 5.
[19] For example, *"What is the relationship between the length of the paint roller and the amount of the area being covered?"*

characteristics were structured with a focus question to provide opportunities for the students to look for a pattern about the quantitative relationships.

And in the vignette below, the students were exploring the focus question: “What is the relationship between the length of the paint roller and the amount of the area being covered?” The focus question is designed to prompt students to coordinate a change in the length of the paint roller and a change in the area it covered. In other words, the question itself states that there is a relationship between the length of the paint roller and the area covered, which pushes students to generalize about the relationship.

Consider the vignette below:

152 NA: So, we will present the focus question [“What is the relationship between the length of the paint roller and the amount of the area being covered?”]. I will ask this group to present first. Yener. Ready.

153 Yener: I did not finish everything. But I have my answer.

154 NA: Okay. So, when someone is presenting, we want to ask questions, and we want to compare their thinking with ours—what they have on there. All right?

155 Yener: Wait. So, I just answer the focus question?

156 NA:Okay. Yeah. We are just answering the focus questions. But we are providing some evidence for our thinking.

157 Asli: Do you want to start first?

158 Yener: Okay, I'll do it first.

159 Yener: So, the focus question is, what’s the relation between the length of the paint roller and the amount of area covered? And my answer is that every time the length increases by one centimeter, the amount the area changes by or the change in the

change of area, it increases by 1 centimeter.

As we see with above vignette, the teacher-researcher stated that as a classroom community, the students were trying to answer the focus question, which was about generalizing the relationship between quantities (line 152). Subsequently, the students' attention was directed to the relationship between the growing triangles length and area (line 155). The paint roller task creates a growing triangle, the students' attention is directed to how the area growing related to its length. As we see, the teacher-researcher asked Asli and Yener if they could present, and when they agreed to present, she restated that as a community, they were trying to answer the focus question (line 152–154). Yener confirms that they were just answering the focus question by saying, "Wait. So, I just answer the focused question" (line 155). The teacher-researcher oriented Yener toward answering the focus question and providing evidence to the claim they made in answering the focus question (line 154). Yener read the question (Figure 33): "What is the relationship between the length of the paint roller and amount of the area being covered?" and answered it by saying, "And my answer is that every time the length increases by one centimeter, the amount the area changes by or the change in the change of area, it increases by 1 centimeter" (line 159).

I drew a conclusion that having students answer the same focus questions about covarying quantities in social (small- and whole-group settings) and individual contexts (journals and individual handouts during writing time) might provide students with opportunities to articulate their thinking to a more sophisticated understanding of their reasoning. And the process of answering the focus question on the task is also a form of generalizing the quantitative relationships.

Figure 33

A Focus Question for the Paint Roller Task

The focus Question: What is the relationship between the length of the paint roller and amount of the area covered?

To use this enacted task characteristics, the students' handouts and journals center on a focus question. For example, "What is the relationship between the length of the paint roller and the amount of the area being covered?" Students' handouts are designed to aid students in answering the focus question. Additionally, the teacher-researcher's prompts in whole- and small-group settings, along with students' journals, center on answering the same focus questions. Enacted task characteristics are a form of support in small- and whole-group settings where students are encouraged to generalize quantitative relationships.

In this example, we see that enacted task characteristics are asking students to generalize the relationship by getting students to answer the focus question in small- and whole-group settings, centered around identifying a pattern between quantities. Thus, enacted task characteristics are pushing students to generalize a relationship between quantities. Below, I provide Table 25 as a summary table for setting an infrastructure for students' QR.

Table 25

A Summary Table for Setting an Infrastructure for Students' Quantitative Reasoning

Enacted Task Characteristics	Definition	Example
Identifying Changing Attributes of Tasks	Task characteristics that craft opportunities for students to identify varying quantities by looking at the attributes of the task and identifying relevant changing quantities.	Posing, stating or asking students—"What are the things you could consider varying possible to measure?" Asli: "The length increases causing the height to increase, creating a larger covered area."
Coordinating Change among Quantities	Task characteristics which set opportunities for students to understand a coordination of change among quantities by probing, asking about, or reinforcing when students talk about quantitative relationships.	Probing, asking, or reinforcing students to engage in coordination of change among quantities—"How does the change in height affect the change in area?" Asli: "Okay. So, what I wrote is the change in height increases the area covered because it contributes to the formula necessary to calculate the area."
Generalization	Task characteristics that create opportunities for generalization by asking students to generalize the relationship between quantities by answering the focus question.	Posing a focus question to reinforce students to explore a pattern about quantitative relationships—"What is the relationship between the length of the paint roller and the amount of the area being covered?" Yener: "And my answer is that every time the length increases by one centimeter, the amount the area changes by or the change in the change of area, it increases by 1 centimeter."

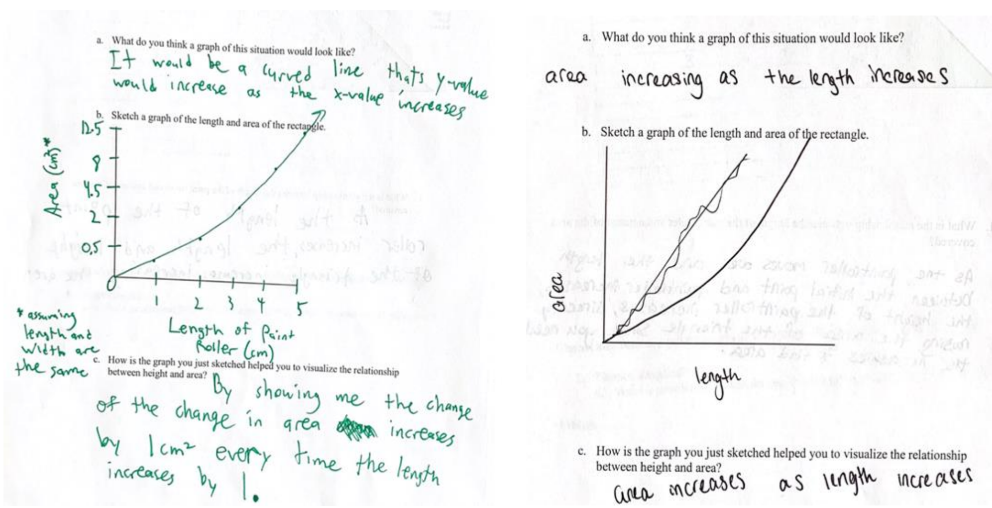
Launching Students' Representational Fluency. In this section, I will provide results about the second main form of enacted task characteristics—launching students' representational fluency—including (a) visualization, and (b) creating and connecting representations to present quantitative relationships.

Visualization-draw-sketch tasks are the first kind of enacted task characteristic to support students' representational fluency; they are designed to build on students' intuitive sense of visualization without using a coordinate grid to support their visualization. The purposefully designed sequence of tasks featured "unstructured" visualization without a grid before introducing a formal Cartesian grid to support students' RF and FT. Students were always given a set of perpendicular lines without tick marks or grid lines.

For instance, Yener and Asli watched the paint roller task, and they were asked to think graphically about the quantitative relationship between the length of the paint roller and the amount of the area covered, both without a grid and on a grid. In the vignette below, Asli and Yener were answering the same questions on the handout (Figure 34). The enacted task characteristics asked Yener and Asli: *"*What do you think a graph of this situation would look like*?"* Such probing helped them engage in answering the question, then they read and articulated their answers to each other, and they got a chance to agree or disagree with their peers' answers.

Figure 34

(a) Yener's Graph and (b) Asli's Graph of the Length and Area of the Paint Roller Task without a Grid



(a) (b)

Asli described her reasoning this way: "So, for (a), I said the area is increasing while the length is increasing" [Figure 34 (a)]. Yener responded with "Well, I would agree with it. What it be uhm…"

As we see in Figure 34, the task statement is asking about the relationship between the length of the paint roller and amount of the area covered without a grid. The question: "What do you think a graph of this situation would look like?" requires unstructured visualization.

Yener noticed that the relationship would be curve, and he wrote: "It would be a curve line that's y-value would increase as the x-value increases." Asli wrote, "Area increasing while the length is increasing," and in response to Asli, Yener stated that he would agree that the paint roller's area increases while the length is increasing.

I found that enacted task characteristics such as asking Yener and Asli to articulate their thoughts about their visualizations, supported their reasoning about quantitative relationships on a graph. When Yener responded to the question "How is the graph you just sketched helped you to visualize the relationship between length and area?" he wrote: "By showing me the change of the change in area increases by $1\ cm^2$ every time the length increases by one."

So, as one of the enacted task characteristics, is visualization with and without grids supported students' RF when it included asking students to think about a graph without a grid; the enacted task characteristics probes students to agree or disagree with each other's thinking and articulate further.

The second type of enacted task characteristics for launching students' RF includes enacted task characteristics that reinforce students' reasoning as they make connections between representations (e.g., a table and a graph of quantities) while they visualize and generalize about quantitative relationships. Encouraging their generalizations about changing quantities and the connections between representations of quantitative relationships in joint writing is a form of support that might develop students' co-emergence of RF and FT. The vignette below is taken from Eren and Salim’s small-group interactions, when they explored the relationship between the length of the paint roller and the area covered by the paint roller.

160 Salim: As the height of the triangle increases.

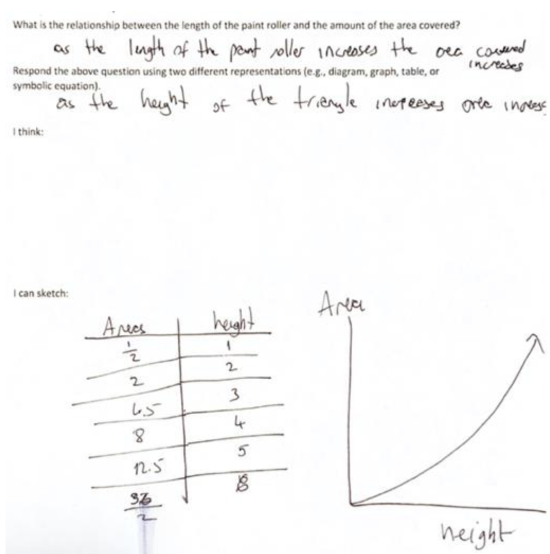
161 Eren: Hmm. No, look, look, [he points at the question] ‘Respond the above question using two different representations (e.g., diagram, graph, table, or symbolic equations)’ [on Figure 35] diagram, table, or symbolic equation.

162 Salim: It is the same thing. [Referring to the diagram being the same as this table.]

163 Eren: This is not.

Figure 35

Salim and Eren’s Journal from Day 1



164 Salim: The symbolic equation. Al sana [here you go], height times length divided by two. What are you thinking? Um Hmm. Ne kullanalim? [What should we use?] Graph mi yapalim? [Should we make a graph?]

165 Eren: We should draw the graph. And then we should write a table.

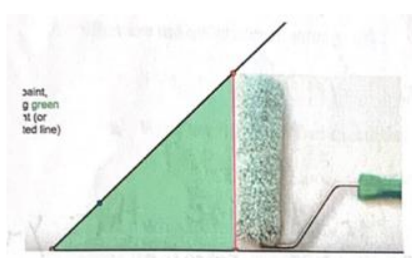
166 Salim: Triangle is mantikli [Triangle makes sense] (see Figure 35)

167 Eren: Triangle is not a, not a diagram or a table or a graph. We just draw the graph and the table.

In a small group, Eren and Salim articulated the type of representations needed to show the relationship between the length of the paint roller and the area covered. They argued about Figure 36; it seemed Salim named Figure 36 as a table (line 162), while Eren was skeptical in naming the figure as a table (line 163). Salim was not very clear what representations they should pick to present (line 164). They talked about the table, graph and triangle. For Eren, the triangle was not a diagram (line 167), and Salim did not seem to differentiate between the screenshot of the video and a table (line 167, see Figure 36). They both agreed on creating a graph to represent the relationship between height and area.

Figure 36

The Figure Salim Named as a Table



Salim said: "triangle is mantikli;" the wording is half English and half Turkish. The students' source of difficulty may come from the language itself. Still, I want to highlight that here, enacted task characteristics—journal prompting statements and questions—pushed the students to select for themselves what representation to use (see Figure 36). In Salim and Eren's case, a graph was chosen to talk about quantitative relationships.

When Salim and Eren were prompted, "What is the relationship between the length of the paint roller and the amount of the area covered?" they wrote: "As the length of the paint roller increases the area covered increases" (see Figure 36). Alongside that, they wrote: "As the height of the triangle increases area increases." Eren's thinking shifted from thinking about the paint roller to seeing the area covered as a triangle and referring to the paint roller's length as the paint roller's height.

The magnitude of the change in both quantities was not specified in their wording. As such, asking Eren and Salim to create multiple representations to represent the relationship may have supported them in identifying magnitudes of the change in covarying quantities in each quantity—coordination of growth. The students' graph and their table of the height and the area of the growing triangle, along with their reasoning, "as the length of the paint roller increased the area covered increases," indicated that Eren and Salim coordinated the length of the paint roller with the amount of area covered by the paint roller in parallel with a table and a graph without further explanations of how the relationship grows with a specific magnitude. I concluded that enacted task characteristics that encourage students to select a type of representation may constitute a source of support for students in creating a representation, which is more meaningful for them in presenting quantitative relationships. Below, in Table 26, I provide a summary of enacted task characteristics that support launching students' RF.

Table 26

A Summary Table for Launching Students' Representational Fluency

Enacted Task Characteristics	Definition	Example
Visualization-draw-sketch	Tasks were purposeful in sequencing "unstructured" visualization without a grid before introducing a formal Cartesian grid as a design principle to support students' RF and FT—thinking graphically with and without a grid.	How do you visualize the relationship between the area painted and the length of the paint roller? Asli: "So, for (a), I said the area is increasing while the length is increasing [Figure 34 (a), she responds the above question]." Yener: "Well, I would agree with it. What it be uhm…" [Yener wrote:] "It would be curve line that's *y*-value would increase as the *x*-value increases" [Figure 34 (b)]"
Creating and making connections among representations	Creating and connecting among representations while reasoning about quantities and quantitative relationships.	"What are the height and the area mean in this graph? How is that relationship similar or different on the graph and the table?" Eren: "No, look, look, [he points at the question above] 'Respond the above question using two different representations (e.g., diagram, graph, table, or symbolic equations)' [on Figure 35] diagram, table, or symbolic equation." Salim: "The symbolic equation. Al sana [here you go], height times length divided by two. What are you thinking? Um Hmm. Ne kullanalim? [What should we use?] Graph mi yapalim? [Should we make a graph?]"

Socio-mathematical Norms. The second major finding (i.e., category) of the learning-ecology framework is specific socio-mathematical norms that support students' meaningful understanding of quadratic functions. In small- and whole-group settings, students are put in charge of their learning by explaining, arguing, being skeptical, and asking their peers questions (Yackel & Cobb, 1996). The findings suggested two main socio-mathematical norms that played a role in supporting students' meaningful understanding: (a) peer pressure for justifications and (b) peer approval.

Peer Pressure for Justification. *Justification* in this present study is defined as students' attempts to explain why and how to each other. The justification students provide is not evaluated for validity or invalidity. Instead, justification is intended as support for all students; when asked why and how, most students try to articulate why their thinking should be valid.

Peer pressure for justification is defined as a situation beginning when students explain their thinking or claim the relationship of quantities. Their peers might or might not fully understand their statement. Still, they do contradict their peers' reasoning. In situations like this, students pressure their peers to articulate their decisions more clearly by asking them to justify why and how—challenging peers' thinking by asking them to explain why, and/or giving contradictory examples to their classmates' statements.

In this section, I will provide my findings of socio-mathematical norms related to peer pressure for justification. The types of understanding and reasoning that were created by peer pressure for justification included: (a) comparison between graphs of quadratic functions and exponential functions; (b) realization of limited knowledge of quadratic functions; (c) justification via the question "where is your reasoning?" (d) skepticism about how two quantities are related; and finally, (e) justification of whether a quantitative relationship is linear or nonlinear.

The first form of a peer pressure for justification is when students push each other to compare between graphs of quadratic functions and graphs of exponential functions. This kind of peer pressure may create instances for students to engage in creating, comparing, and contrasting graphs of exponential versus quadratic functions, especially when asked why and how by their peers. Via peer pressure for justification in small- and whole-group settings, students may be positioned as being responsible for justifying their thinking, while their peers push them to explain why and how. Examples of challenging peers' thinking included students asking their peers to explain why they thought the relationship between height and area is a quadratic relationship or offering contradictory examples of why the relationship was not exponential.

In the following vignette, Salim and Mert were exploring the relationship between the length of the paint roller and the amount of the area being covered in a small-group setting. They watched the paint roller task video.

168 NA: So, does exponential touches or not touches to zero?

169 Salim: Yeah, it does.

170 Mert: I am confused. It does?

171 NA: Show me where touches to zero.

172 Salim: Where it touches? I said, so quadratics.

173 Mert: Wait, then it [the relationship between the height of the triangle and the area, in the paint roller situation], can be exponential?

174 Salim: it cannot be.

175 Mert: Why can't it be?

176 Salim: I said so.

177 Mert: Very good explanation. Look like this. [He sketches the graph on Figure 37 (a).]

Figure 37

(a) Mert's Sketch of Exponential Function, (b) Salim's Sketch of Quadratic Function, and (c) Salim's Symbolic Equation of the Relationship of the Length and Area of the Triangle

(a) (b) (c)

178 Salim: Look, if it is x squared, then it cannot be exponential.

179 Mert: Why can it be?

180 Salim: Nasil acikliycam bunu? [How am I going to explain this?]

181 Waleed[20]: What if we graph both as quadratic and exponential, then we think about the difference.

182 Salim: Yeah, let's do that.

183 Mert: I know what it is, U shape

184 Salim: U, but this side [negative x values] does not mean anything. It is only this side. But still quadratics [he sketches the graph shown in Figure 37 (b)].

185 Mert: What is the point of this line [the half of the parabola crossed out by black pen, see Figure 37 (b)]?

186 Salim: It just, um. It is just how.

187 Mert: Then, that line exists, right?

188 Salim: Yeah.

189 Met: But it cannot exist because it is negative. The negative area is not a thing

190 Salim: It can exist because it is an equation [Figure 37 (c)]. It is going to exist anyway.

Salim stated that exponential function touched the origin, then NA repeated the statement (line 168). Salim also stated that the relationship between length and area is constitutive of quadratic growth (line 172). Skeptical of this explanation, Mert (line 174) asserted that if an exponential function can go through the origin, then the relationship between the length of the

[20] Although I will articulate more about this in the teacher pedagogical moves section, I would like to draw attention to line 64, where the teacher-researcher is prompting students to create graphs of quadratic and exponential functions for justifying and comparing within two parallel representations.

paintbrush and area constitutes exponential growth (lines 170-175). He was challenging Salim to identify the relationship and a reason for the distinction between exponential and quadratic functions graphs (line 177).

Although Salim was in 10th grade and Mert was in 8th grade, Mert was still pushing Salim to explain why the relationship was quadratic and how it differed from exponential growth if the exponential function passed through the origin (line 56–60). In order to question Salim's reasoning, Mert sketched an exponential function graph that was so close to the origin that it canceled out the negative domain, which substantiated his argument about how the paint roller's length and the painted area's graph would look if it passed through the origin. Mert constructed a contradictory example to challenge Salim's statement (line 177–179).

In response, Salim clarified his explanation, stating "I said so" in response to the concept that "if it is *x* squared, then it cannot be exponential" (line 178). Mert's interpretation of quadratic functions was a parabola (line 83). Salim cut out the negative domain and formed a parabola, which contradicted Mert (line 183–184) on Figure 37 (b), a fact that Mert interpreted such that a quadratic function with half a parabola and length and height cannot have a negative domain (line 173 and line 183). Asking or explaining why and how in small-group interactions benefited both Mert and Salim. Mert's conception of quadratic functions became visible to Salim, while Salim's perception of having exponential functions pass through zero became more discernable.

The vignette highlights several points related to socio-mathematical norms. Salim and Mert used two representations—a graph of the quadratic function and a graph of the exponential function—in parallel. Salim perceived that if a function had a symbolic equation of x^2, it must be a quadratic function (line 178). For Salim, the symbolic equation on Figure 37 (c) had to have a negative domain because it showed a full parabola. Thus, even if quantities could not be negative, the function of height and area had to present a parabola. Salim named the relationship as a quadratic function based on a canonical symbolic equation. He used a graph and a symbolic equation in parallel to argue that the relationship was a quadratic function. Salim's reasoning was too vague for Mert. Salim did not accept the explanation that simply using a symbolic equation [Figure 37 (c)] and graph [Figure 37 (b)] sufficiently substantiated that the relationship represented was a quadratic function. In contrast, Mert perceived that the graph represented the relationship between the length of the paint roller and the area covered. Therefore, Mert pressured Salim for a justification about when the graph had a positive and negative domain, saying that graph could not represent quantitative relationships since quantities did not exist in the negative domain.

This vignette suggests that social interactions such as peer pressure for justification among small and whole groups fostered more sophisticated student reasoning and understanding. As we observe in the above vignette, students benefited from asking each other to explain why and how when they were trying to explain the differences between the graphs of quadratic functions and the graphs of exponential functions. Students pressured each other to justify how quadratic function graphs and exponential function graphs were similar to and different from one another, even though both graphs represent a quantitative relationship. In the vignette, both Salim and Mert experienced peer pressure that compelled them to explain their thinking. These peer interventions constitute a form of support that results in a more meaningful understanding for both students of what quadratic functions represent—the quantitative relationship which is on the positive domain of the graph and how it compares to the exponential function graph.

The second form of peer pressure for justification helps set a groundwork for students to recognize what they know or don't know about quadratic functions. Socio-mathematical norms that put pressure on students to justify their reasoning might move students away from vague explanations, such as naming a quantitative relationship as quadratic, to more complicated explanations of how the quantitative relationship could be presented as a graph and a symbolic equation. I found that socio-mathematical norms could help students to recognize that while they knew what a quadratic function looked like, they might not know what it meant.

Eren and Salim explored the relationship between the height, length, and area of the growing rectangle. The students elaborated on the graphs and the symbolic equations of quadratic functions in parallel. Consider the growing rectangle task and the vignette below:

191 Salim: Quadratics. Last time we did this, it [the relationship between the height and area of a growing triangle] was quadratics.

192 Eren: Why?

193 Salim: Because the equation is quadratic.

194 Eren: But why?

195 Salim: It comes out to be quadratics

196 Eren: Why?

197 Salim: Because when you graph, it looks quadratics.

198 NA: What do you mean by that?

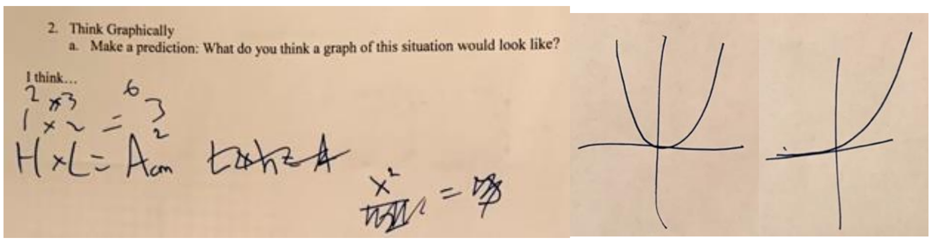
199 Eren: Yes, explain.

200 Salim: That is height time length is equal A. No, it is height times length. x squared divided by 2 [$\frac{x^2}{2}$] [Figure 38 (a) & (b)]; when you graph this, it came out quadratics

201 Salim: Quadratics. It is height time length cm squared. What do you think?

Figure 38

(a) Salim's Symbolic Equation, (b) Salim's Graph of that Equation, and (c) Salim's Graph of an Exponential Function



(a) (b) (c)

202 Eren: I do not know what quadratic means.

203 Salim: Here [he draws a parabola—(Figure 38 (b)]

204 Eren: I know what it looks like.

205 Salim: This is exponential. [Salim's drawing on Figure 38 (c)].

206 Eren: I know what it looks like, but I forgot what it meant. What does quadratic mean?

207 Salim: This kind of graph.

As we see in the vignette, Salim made a connection back to the earlier growing triangle task—the paint roller task (line 191). He saw the relationship between the height of the triangle and its area as similar to the height and the area of a rectangle. He reminded Eren that they had done this before, and it was a quadratics function. Eren probed Salim's thoughts by asking "Why*?"* Salim's response about the symbolic equation registered his recognition that the quadratic function had an equation of x^2(line 193). Salim's justification did not convince Eren, and he continued to probe Salim by insisting he explain why the relationship was quadratic (line 194). Salim justified his position by explaining that when he created a symbolic equation representing the height and the area, it became an x squared, and the graph of the equation $y = x^2$ was a parabola [see Figure 38 (a) and (b) and lines 195–200]. According to Salim, that was why the relationship between the height and area of the growing rectangle was quadratic.

Eren and Salim continued with their conversation about what it meant to be quadratic. Salim stated that area is the multiplication of the rectangle's length and height, which is in centimeters squared (line 201). He asked, "What do you think?" Eren's response was that he knew what a quadratic function looked like, but that he did not understand what it meant to be quadratic: "I know what it looks like, but I forgot what it meant." These socio-mathematical norms facilitated the students' ability to redefine the concepts for themselves by talking about quadratic functions and what it meant to be quadratic (line 206).

In small-group settings like this one, students pushed each other to justify the quantitative relationship—the relationship between the length of the paint roller and the area covered. The justification involved using two representations, a graph and a symbolic equation, in parallel as the students proved their reasoning, which also supported the students' RF.

As we see above, even if students could not articulate what it meant to be a quadratic function that represents quantitative relationships, they acknowledged to their peers that they knew what a quadratic function looked like, but they didn't know what it meant. The peer pressure for justification helped students to realize what they knew and didn't know about quadratic functions. These realizations are an important part of the learning process because they might push students to wonder about what it means to be a quadratic function; additionally, having students re-voice what they know and don't know might set a groundwork for further exploration.

The third form of peer pressure for justification in socio-mathematical norms was when the students asked about their peers' reasoning when their peers made a statement about quantitative relationships. Students pressured each other to justify their statements, or guessed by asking each other "where is your reasoning?" In the following vignette, Mert and Tarik pressure each other to explain their reasoning about a quantitative relationship.

Mert and Tarik watched the growing rectangle task video, and they were investigating the question, "What is the relationship between the height and the area of a rectangle?" Mert and Tarik talked about explaining what it means to have an "uneven rate." Through peer pressure, Mert reasoned that if the rate among quantities is not constant, then a graph of these quantities should be a curve. See the interaction below and consider the growing rectangle task:

208 Mert: Oh, OK. So, OK, I'll go, I guess. OK. The height increases by one each time the area increases that are uneven rate. And because of that, it is, I'm not sure. It's either exponential or quadratics.

209 Tarik: It is exponential.

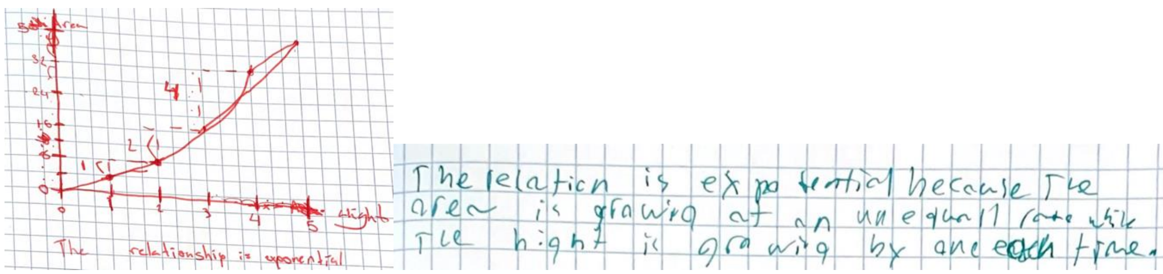
210 Mert: because it's, it can be quite don't because if you multiply, wait, no, I don't know. It's an exponential, I guess.

211 Tarik: Where's your reasoning?

212 Mert: Oh, my reasoning is because it is, it is growing at an unequal rate and the source like curving and when it's occurring, it's an exponential thing. Exponential function [see Figure 39 (b)].

Figure 39

(a) Tarik's Graph for the Paint Roller Task, and (b) Mert's Written Artifacts for the Area for the Paint Roller Task.



(a) (b)

Note. Mert wrote in Figure 39 (b): "The relationship is exponential because the area is growing at an unequal rate while the height is growing by one each time."

In the above conversation, Mert and Tarik reasoned about the relationship between quantities in a scenario in which the rate of growth between quantities—the height and the area of the growing rectangle task—is not constant, so the graph curves. Mert observed that the height increased 1 cm each time, and the area was increasing with "uneven rate," indicating that the relationship was either exponential or quadratic (line 209). Tarik named it as exponential (line 210), but Mert was skeptical. Subsequently, Tarik interrogated Mert's thinking by asking, "Where is your reasoning?" (line 211).

Questioning a peer's reasoning was a form of instructional support that emerged from small-group engagement. Given his peer's inquiry, Mert explained his reasoning, stating, "It is growing at an unequal rate and the source like curving and when it's occurring." When Tarik pressed Mert to articulate his reasoning, he supported Mert in deducing that the area and height had unequal rates of growth; consequently, these two quantities should be curving when they increase. Granted, Mert and Tarik did not define or articulate how the area and the height of the rectangle curved and the ratio between them, but they did establish a norm in their small group that when they state a claim, they need to explain the reasoning behind the claim. If the explanation is not provided, then one of the peers should raise the question: "Where is your reasoning?"

The fourth form of peer pressure for justification is skepticism about a peer's explanation or statement. Being skeptical about a peer's explanations or reasoning is about disagreeing with the other students' thinking and pushing them to be specific how the quantities are related. And peer pressure for being specific about how two quantities are related is a form of support that might create a foundation for covariational reasoning; as we see in this study, students' skepticism and peer pressure encouraged their peers to be specific about the relationships between quantities.

The below vignette is taken from Eren and Mert's small-group interactions when they engaged in exploring the falling object task with the focus question, "What is the relationship between the height of the falling object and the time it takes to fall?"

Consider the falling object task and the vignette below:

213 Mert: It's fallen down, it goes up to 60 16.02 meters up which 1.1 seconds

214 Eren: But you're not answering the question.

215 Mert: This drops down.

216 NA: Think about it. Think about how time affects the height of the falling object.

217 Mert: Um, it, um…

218 Eren: How does the time affected the object? But it goes, read the questions. How does time affect the height of the falling object?

219 Mert: It makes the increase in decrease, increases.

220 Eren: You’re not answering the question.

221 Mert: Yes, I am.

222 Eren: How does the time affect, the time?

223 Mert: Affects height by increasing and decreasing.

224 Eren: How does the time affect the height of the falling object?

225 Mert: More time, uhm as time increases…

226 Eren: It’s not really like how is the time...

227 Mert: The time affects the height.

228 Eren: But how.

In the small-group interaction, Mert stated that the object went up 16.02 m for 1.1 seconds (line 213). Mert’s reasoning did not involve any variation among the time and the height of the falling object, and Eren objected to Mert that he was not answering the question (line 214). Mert responded by saying the object falls down (line 215). Eren became skeptical about Mert’s answer to the question and he pressured Mert by reading the question (line 218): “How does the time affect the height of the falling object?” Eren pressured Mert to be specific about the relationship by reading the question aloud. In response, Mert indicated that he only saw the change in height. As such, he stated that height is increasing and decreasing (line 219). Still, Eren urged Mert to note the change in height and in the time: “You are not answering the question… How does the time affect the time?” Eren’s questions were assertions that a change in height was affected by a change in time. Mert still did not see that time was increasing. Nonetheless, Eren did not accept the height and time as two quantities that were varying separately (line 224). Eren insisted to Mert that the changes in height and time were relevant, contending that they covary (lines 222–226). Once Mert finally accepted Eren's perception of the correlation between time and height (line 227), Eren still pressed him to explain how time affects the height by saying "but how."

Students being skeptical of peer responses about how two quantities are related may aid the ability of the students involved to conceive that two quantities are related and that a change in one affects the change in the other. Therefore, students exerting peer pressure on other students to be specific about how two quantities are related can be a form of support in developing the students' initial thinking about covarying quantities. Thus, socio-mathematical norms supported students' quantitative reasoning.

The fifth form of peer pressure for justification of a statement is another form of support in learning about quadratic functions that represent covarying quantities. Peer pressure played an important role in supporting students' ability to name a quantitative relationship as linear or nonlinear by getting peers to discuss how the two quantities covaried together.

Consider the following vignette and student artifacts and the falling object task:

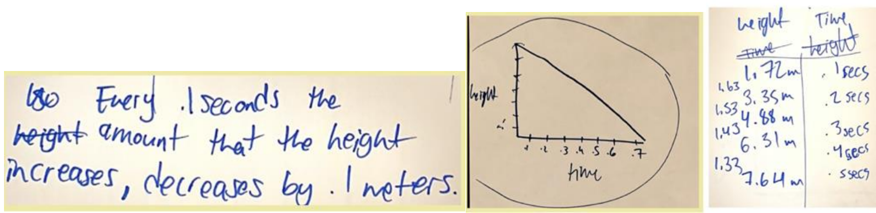
229 Eren: OK, so let me just uhm every point one second. The height of the amount that the height increases decrease by 0.1 meters. Every 0.1 seconds, the amount that the height increases decrease by one point second or all the cases but one meter [Figure 41 (a)].

230 Eren: So, it's a negative linear graph [Figure 40 (b)]

231 NA: Negative linear graph. Why?

Figure 40[21]

Eren's (a) Written Statement, (b) Graph, and (c) Table of the Relationship between the Falling Object's Height and the Amount of Time it Takes to Fall.



(a) (b) (c)

232 Eren: Because I put, actually if I put time over here.

233 Mert: What?

234 Eren: It would be a normal linear graph.

235 Mert: No, wait, is it, it's not going to go. It's shape as it goes on. It's going to start going down, down, down. When we were like going over there, it's going to go down eventually.

236 Eren: Eventually. According to this not yet, but if I go past the one point 81 seconds point. Just graph for this. I want to graph for that. It's going to be. It's going to be linear. [Figure 40 (a)]

237 Mert: No each time. It is not.

238 Eren: Look look. 1.62.

239 Mert: But doesn't increase and decrease at the same rate.

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[21] Figure 40 is same as figure 29; I have added figure 40 to help the reader to visualize the concrete representations I reference in the text.

240 Eren: Yes, it does. Look at it. I'm literally showing [Figure 40 (b) and (c)]

241 Mert: No. Every time it's not going to by one. It's going by 1.6 and 1.3, 1.43 and 1.33.

242 Eren: Sure, sure. OK, it makes sense. Now that I think about, yeah. OK. That makes more sense. So, as the time increases, the height. It's not linear. I figured that out.

In this vignette, Eren interpreted the height in relation to the time on the table as a coordination of the values of the second difference in height with the first difference of time values (line 229). He interpreted that the amount that the height increased also amounted to a decrease of 0.1 m for every 0.1 seconds. He identified a linear relationship between the second change in height, coordinated with the first change in time (lines 230 & 234). In graphing such a relationship, Eren concluded that the relationship was linear since the increase in height decreased by 0.1 m for every 0.1 seconds [Figure 40 (b)]. However, Mert did not agree about calling the relationship linear, and he pressured Eren to justify his reasoning. He saw the cannonball as going down each time. He did not see the second difference in height. Mert argued with Eren that the equation was not going to be linear; he also noted that the relationship between the height and time was not growing at the same rate, which is why it could not constitute linear growth (lines 235–239). As a result of Mert's observation, Eren began to notice a variation in the amount that the height increased every time the time increase was not constant (line 242). In other words, the level of increase was not constant. As such, Eren responded: "Sure, sure. OK, it makes sense. Now that I think about, yeah. OK. That makes more sense. So as the time increases, the height. It's not linear. I figured that out."

After some probing by his peer, Eren realized that the coordinated change between height and time in the falling object task was not linear. So Mert's pressure to justify the rate between the height and the time helped Eren to realize he was coordinating the first change in height with the second change in time, which is not a linear increase.

I summarize the five types of peer pressure for justification described above in Table 27.

Table 27

A Summary of Socio-mathematical Norms: Peer Pressure for Justification

Type of Learning via Peer Pressure for Justification	Example
Comparison between graphs of quadratic and exponential functions	Salim: Look if it is x squared, then it cannot be exponential. Mert: Why can it be?
Realization of a limited understanding of quadratic functions—knowing what they look like but not what they mean	Salim: Quadratics. It is height time length cm squared. What do you think? Eren: I do not know what quadratic means. Eren: I know what it looks like, but I forgot what it meant. What does quadratic mean?
Justification via the question: “Where is your reasoning?”	Mert: Because it’s, it can be quite don’t because if you multiply, wait, no, I don’t know. It’s an exponential, I guess. Tarik: Where's your reasoning?
Skepticism about how quantities are related	Mert: It makes the increase in decrease, increases. Eren: You're not answering the question. Mert: Yes, I am. Eren: How does the time affect the height of the falling object?
Justification of whether a quantitative relationship is linear or nonlinear	Mert: But doesn’t increase and decrease at the same rate. Eren: Yes, it does. Look at it. I’m literally showing. Mert: No. Every time it’s not going to by one. It’s going by 1.6 and 1.3, 1.43 and 1.33. Eren: Sure, sure. OK, it makes sense. Now that I think about, yeah. OK. That makes more sense. So, as the time increases, the height. It’s not linear. I figured that out.

Peer Approval. Peer approval is the second component of socio-mathematical norms that facilitated students’ meaningful learning. In this study, I define peer approval as a response or statement on which students agree with one another. And peer approval can aid students in making sense of quantitative relationships when learning about quadratic functions. When students’ statements get approved by their peers, that might motivate them to further justify their statement. Peer approval is a form of socio-mathematical norms that further encourages students to be specific about how quantities will change.

Consider the brief exchange below between Eren and Salim as they discussed the falling object task.

243 Eren: No, the range is not good. The range is just going to continue increasing by 3. The range is just going to continue with that.

244 Salim: That is true.

245 Eren: Because the ball cannot go back. It's always going to increase. So, the height is going to decrease, 36, 21, Yes. OK. Yeah. Yeah. The height is going to be zero.

When students made a statement that got approved by other peers in the small group, it often motivated them to extend or further their explanations. As we observe above, when Eren noted that the range was incorrect and would increase, Salim voiced his approval of Eren's observation by saying, "That is true." His peer's approval invited Eren to unpack his observation by explaining why the range would continue to increase: "Because the ball cannot go back." Furthermore, Eren provided a contradictory example to make his statement stronger. He said: "So, the height is going to decrease, 36, 21, Yes. OK. Yeah. Yeah. The height is going to be zero."

As the above situation supports, when students make a statement (line 243), and their peers approve the statement (line 244), they may gain confidence and provide complementary evidence of their thinking to make their case stronger (line 245). Hence, gaining peer approval for their claims may push students to further develop their explanations in small-group settings.

In Table 28, I provide a summary table for the elements of students' socio-mathematical norms that I discussed above.

Table 28

A Summary of Socio-mathematical Norms: Peer Approval

Socio-mathematical Norm	Definition	Example
Peer Approval	Peer approval is defined as when a statement made by a group member gets approved by the rest of the group.	Eren: No, the range is not good. The range is just going to continue increasing by three. The range is just going to continue with that. Salim: That is true.

Teacher Pedagogical Moves. The last component of the learning-ecology framework is teacher pedagogical moves. I define teacher pedagogical moves as questions or statements a teacher raises to encourage or elicit student reasoning about quantities and their representations. My findings suggest that effective teacher pedagogical moves cluster around: (a) supporting students' co-emergence of RF and FT, and (b) creating a foundation for students' FT.

Supporting Students' Co-emergence of Representational Fluency and Functional Thinking. The first type of teacher pedagogical moves are those clustered around characterizations of student reasoning, including creating, translating, and connecting representations when reasoning about quantities and quantitative relationships. I found that students' co-emergence of RF and FT was supported via specific teacher pedagogical moves when the context involved quantitatively rich sets of tasks within small- and whole-group settings. Teacher pedagogical moves that supported students' co-development of RF and FT included two main elements: (a) creating representations of quantitative relationships; and (b) connecting representations to reason about quantities and quantitative relationships. In the following section, I will introduce collections of teacher pedagogical moves that aided students in these two activities.

First, I found that supporting students in their creation and interpretation of representations involved several sets of teacher pedagogical moves. Exactly what sets depended on the features of the particular type of representation and students' experience with such representations. This cluster of teacher pedagogical moves includes: (a) prompting students for generalizations about quantitative relationships; (b) pushing students to record data about changing quantities; (c) encouraging students to create a table for the data; and (d) asking students how the table will help them to see the relationship.

Consider the following vignette, in which Asli and Yener were investigating the height, length, and area of the growing rectangle.

246 Asli: I am looking at height and length when the height so like from, it took in the time when the height increased by 1 cm, length increased by 2 cm. And I am going to just look at the values.

247 NF: Does that always happen?

248 Asli: Yes, I was going to check.

249 Yener: At one point when this change three, and it was changing by four [he wrote 3 cm, 4 cm].

250 NF: Maybe you could record some of the values that you're paying attention to. You kind of collect that data to compare.

251 Asli: Where does it start?

252 Yener: it does not constant, be 0.2 cm away, 0.2 cm different change little bit.

253 Asli: Can we get the piece of paper? We try to record some height and length values, maybe to see a relationship.

254 NA: How are you going to record it? Are you making a table?

255 Asli: It makes sense because we just had to guess numbers.

256 NA: How that table [Figure 41] is helping you to see the relationship between the height and area?

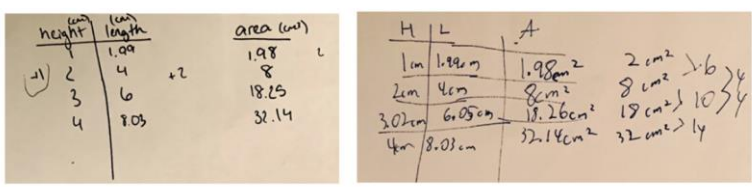
257 Asli: It just helps me visualize how to like as the height is increasing by one the length increases by two because you can clearly see the difference [Figure 41 (a)].

258 NA: Two what? One what?

259 Asli: 2 cm, for each 1 cm that height increases, the length increases by 2 cm or vice versa for each 2 cm length increases, the height increases by 1 cm.

Figure 41[22]

(a) Asli's Table and (b) Yener's Table for the Height, Length and Area of the Growing Rectangle



(a) (b)

This vignette provides evidence of four teacher pedagogical moves that are relevant to supporting students in creating and interpreting within a table when reasoning about quantities and their relationships. First, when students noticed how two quantities changed within a single point on a diagram (line 246), the teacher pedagogical move urged them to generalize by asking if that always works for other points (line 247). Second, when students started talking about

[22] Figure 41 is same as figure 15; I have added figure 41 to help the reader to visualize the concrete representations I reference in the text

changes in points (line 249), the teacher pedagogical moves encouraged them to record the data (line 250). Third, when the students loosely recorded the magnitude of the quantities, as in line 249, when Yener wrote quantities without labeling them as height or length, the teacher pedagogical moves suggested that students record their data in a table (lines 250–253). Consequently, both Asli and Yener created a table clearly delineating the height, length, and area of the growing rectangle (Figure 41). Finally, to support students' quantification, a teacher pedagogical move asked students how the table they created would help them detect the relationship between quantities (line 256).

I found that prompting students to create and interpret quantities on a diagram and asking them how to represent the relationship in a table drew upon a set of teacher pedagogical moves. This set of teacher pedagogical moves is crucial, because it may have impacted a shift in Asli's reasoning. Using a diagram, she was attempting to coordinate a change in height and length on a single point (line 246). Later on, she said: "for each 1 cm that height increases, the length increases by 2 cm or vice versa for each 2 cm length increases, the height increases by 1 cm." Her explanation became more sophisticated; not only did she interpret how the height and length of the growing rectangle were related, but she also flipped the quantities to argue how they were related interchangeably to a diagram and table. This development in Asli's explanation provides evidence of a shift in the level of sophistication of her thinking, a process of reasoning development that the teacher pedagogical moves facilitated.

Note that with this claim, I am not arguing that the shift in student reasoning occurred because of teacher pedagogical moves; instead, I assert that teacher pedagogical moves helped to guide that shift, which was then reinforced in small-group discussions that engaged in the quantitatively productive tasks.

The second type of teacher pedagogical move that is helpful in encouraging students' co-emergence of RF and RT are those moves that support connections among representations. I found that asking students what the graph of the situation might look like aided them in connecting a symbolic equation and graphs of the quadratic function while engaging in reasoning about quantities and their relationships. I found that asking students what the graph of a vertex-form quadratic equation was might help them to make sense of the terms and variables in the equation and identify the vertex point, both on the symbolic equation and the graph of the equation.

In the below vignette, Yener and Asli were investigating the relationship between height and time in the falling object task. They used the PHeT simulation to measure the height and time of the falling object. Then they created a symbolic equation, $f(x) = (x - 2.07)^2 + 21.38$.

260 Yener: The vertex forms. So, the vertex for this is uh, this, uh, this 2.07. [He writes $f(x) = (x - 2.07)^2 + 21.38$]. [Figure 42 (a)]

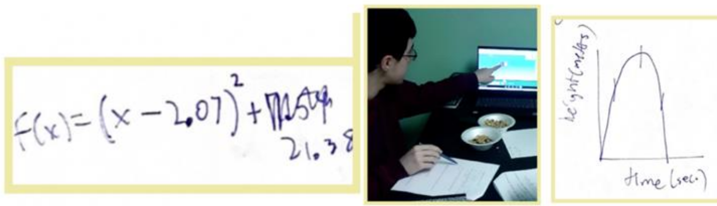
261 NA: So, if you think of a graph of this, how does it going to look like?

262 Yener: [He is pointing at screen and vertex form above with one hand, and the other hand is holding a pen and pointing at the vertex point in the equation, Figure 42 (b).] This, I guess this is the vertex form and here this 2.07.

263 Yener. So, this would be a time, this height. [He sketches the graph below and places a vertical line on the vertex.] [Figure 42 (c)].

Figure 42

(a) Yener's Symbolic Equation, (b) Image at the Vertex on the Screen and the Equation on the Paper, and (c) Yener's Graph of the Height and the Time on the Falling Object Task.



(a) (b) (c)

Yener created a symbolic equation based on the time's numerical values and the height from the screen of the PhET simulation. He called it a vertex form; he wrote $f(x) = (x - 2.07)^2 + 21.38$ (line 260). He interpreted the simulation as a path of the cannonball, which is essentially a graph of the time and the falling cannonball's height [Figure 42 (c)]. The simulation allowed Yener to measure individual points by using the simulation's components of measuring height, time, and range.

Yener interpreted the simulation as a falling object graph, identifying time as the *x*-axis and height as the *y*-axis [Figure 42 (c)]. To help Yener to identify the vertex on both the equation and the graph, NA asked Yener how this situation would look on a graph. As we see, the question does not emphasize any specific type of graph, but rather the graph of the given situation. So, the teacher pedagogical move itself is prompting Yener to create a graph to show how the vertex looks on the graph and how that is connected to the symbolic equation's vertex form (line 262 and 263). In other words, asking Yener about the graph of this particular situation might have created a foundation for him to establish a connection to the given symbolic equation, the simulation graph, and a graph of his choice—Yener sketched Figure 42 (c). Note that Yener identified the vertex points on the computer screen and on the equation, and then shaded them on his sketched graph.

This vignette provides evidence that one way to prompt students to make a connection among and within a graph and a symbolic equation while reasoning about quantities is to ask them to imagine what the graph of the equation would look like in that situation. From this vignette, I concluded that asking students about graphs of quantitative relationships might set a groundwork for students to create connection between the symbolic equations and the graphs of quantitative relationships.

Teacher pedagogical moves play an important role in prompting students to make a connection between a table and a graph when reasoning about quantitative relationships. In particular, when students only see the quantitative relationship on a single representation, they may not feel the need to look at the relationship from another viewpoint and articulate how the quantitative relationship would be different. With that in mind, teacher pedagogical moves at this stage can push students to articulate how the quantitative relationship would look on a graph and a table. Therefore, teacher pedagogical moves are a form of support that can reinforce students to further articulate quantitative relationships by making a connection between tables and graphs. I refer to Asli and Yener's vignette[23] (lines 129–138) in the section earlier in chapter 5. The section is called *Level 3: Flexible Connection Between Representational Fluency and Functional Thinking.*

Yener described that each time the area was increasing by four by coordinating 1 cm with the height and the change of change of area on a tabular representation, he said: "How much the area changing each time. Uhm the change, in the amount the area changes will be constant for

[23] This vignette is cited in the section *Level 3: Flexible Connections between Representational Fluency and Functional Thinking*, at chapter 5, part 1 from line 129-138.

each time. So, this time it changes by six, the next time it changes by 10, which is four more than six, next time it changes 14, which is four more than 10. So, it keeps increasing like that. The change in the area will be four each time. [Pointing at the table on Figure 30 (a).]. Yener saw the relationship between height and area as constant, which increased by four cm^2(line 129). Then NF asked him how the relationship would look like on a graphical representation (line 132), she said: “How is this you just talked is related to the way the area is changing is the same? How can you see that in the graph? How is this related to the graph? [NF points at graph on Figure 30 (b).] This was a form of teacher pedagogical moves which helped Yener to recognize the same quantitative relationship on graphical representations as distance (line 136 and 138). Yener said: “Because this would be, oh, wait. So, if you look at the points here, there would be, this is between these two points, the first two will be 2, and this is between this one and the one over there will be six, which is four more than two, and it kept going all the 10, then 14, and it keeps going increases four by each time [Pointing at the graph below Figure 30 (b).] Yeah, they all have equal distances.” With this vignette, we noticed one more time that teacher pedagogical moves might aid students to make connection between graphs and tables of quantitative relationships. I present a summary of the teacher pedagogical moves discussed in the above section in Table 29.

Table 29

A Summary of Teacher Pedagogical Moves for Supporting Students' Co-emergence of Representational Fluency and Functional Thinking

Teacher Pedagogical Move	Example
Teacher pedagogical moves to support creating a representation of reasoning about quantities	NF: Maybe you could record some of the values that you are paying attention to. You kind of collect data to compare. NA: How are you going to record it? Are you making a table? NA: How that table is helping you to see the relationship between the height and area?
Teacher pedagogical moves to support connections among representations of quantitative relationships	NA: So, if you think of a graph of this, how does it going to look like? NF: How is this you just talked is related to the way the area is changing is the same? How can you see that in the graph? How is this [table] related to the graph?

Creating a Foundation for Functional Thinking. In this section, I will articulate the second component of effective teacher pedagogical moves for supporting students' co-emergence of RF and FT: creating a foundation for students' FT. Teacher pedagogical moves in this cluster include: (a) probing students to identify the attributes of an object or situation; (b) probing students for a unit to measure an attribute of an object; (c) probing students for coordination among quantities; (d) encouraging students to justify their reasoning about the relationship between quantities; and (e) probing students for continuous covariational reasoning.

The first main category of teacher pedagogical moves for creating a foundation for students' FT is probing students to notice and recognize objects' or tasks' characteristics. This includes asking students to: (a) isolate relevant quantities; or (b) visualize or sketch the appropriate quantities. I will provide evidence for these two points in the following section.

First, I found that teacher pedagogical moves reinforced students' ability to identify attributes of a task, and the initial step was to isolate relevant quantities. When students were introduced to a new task, the first thing they were prompted by a teacher pedagogical move to do was to identify the attributes of the situation. As we see in the below example, the students were asked to identify quantities in the falling object task. The falling object task included several attributes, and it was crucial for the students to identify the relevant quantities to focus on. The below vignette is from a whole-class interaction, wherein the falling object task was introduced for the first time.

264 NA: So, we have, what are the quantities we have.

265 Zerrin: We have the diameter of the cannonball. We have the diameter of cannonball, and then you have the power, like the speed of the cannonball.

266 NA: Okay. Speed.

267 Zerrin: And then a more holistic perspective. We had the weather conditions, which doesn't make sense here. Okay.

268 Salim: Diameter of staff?

269 NA: Okay. So, if we say the range, and I saw you guys wrote the height.

270 Zerrin: Oh yeah, the air resistance.

271 NA: And then what? The time you guys were talking about the time.

272 Tarik: Yes.

273 NA: Okay. This is, this is somehow physics. Zerrin and Salim. We are going to focus on only the range, height and time as the variables.

NA asked students to list all the types of quantities they noticed in the task (line 264). Students saw the diameter of the cannonball, speed, etc. (lines 265–270). Then NA directed the students to focus on the relevant quantities—the height, the range, and the time (line 273). This is an informal way of directing students to focus on the relevant quantities and placing irrelevant quantities to the side. In this conversation, we learned that in isolating relevant quantities, students can first be asked to list all the quantities they see in the situation. Then they can be prompted to focus on the appropriate quantities and the relationship between the quantities (line, 273). Asking students to list all the quantities they notice in the task is a way of helping them to navigate which quantities are relevant.

The second teacher pedagogical moves that supported students in identifying the attributes of a situation involved asking students to visualize or sketch the relevant quantities. Since, as we recall, quantities are in students' minds, not in the real world (Thompson, 1994) asking students to visualize or sketch is one way to see what they have in their minds that may be related to the relevant quantities.

On the falling object task, the range is the horizontal distance between the point at which the cannonball leaves and the point at which it lands. The time, on the other hand, is how long the cannonball remains in the air. The time can also be visualized as a horizontal path, but students should be able to differentiate between these two quantities. One way to help students to recognize the relevant units and the relationships between them is to have the students draw a visual diagram and then explain what each element of the diagram represents.

The following vignette is taken from a whole-group interaction, when the falling object task was introduced; students were investigating the question "What is the relationship between the height of the falling object and the time it takes to fall?" After NA stated to the whole class that they would be investigating quantities—the range, the time, and the height—she asked students to visualize the relevant quantities by sketching on the whiteboard.

274 NA: If the cannonball is here, where is the range the height? Who is going to visualize it? [She makes one of the black dots seen on Mert's sketch, Figure 43].

Figure 43

Mert's Sketch of Relevant Quantities—the Height, Range, and Time for the Falling Object Task

275 Tarik: Um, the range is on how long it takes for the cannonball. Are you talking about the range or the time?

276 NA: We are talking about the range.

277 Mert: The range is goanna be. Uh, can I show?

278 NA: Sure.

279 Mert: Okay. I'm going to show it in a different color. It's going to be like this [Figure 43].

280 Mert: This is the range now. Because cannonball was like there at the end, but now it's right now. Yep. Okay. Maybe just the height is also this line.

In the falling object exercise, NA asked students to visualize the height, the range, and the time. NA placed a black dot on the cannonball's path (Figure 43) then asked to students to show the height, range, and time on the sketched path of the cannonball (line 274). Tarik defined the range as "range is on how long it takes for the cannonball," then realized he was describing the time (line 275). Mert built on Tarik's realization, drawing the range as the horizontal distance and height as the vertical distance (lines 277 and 279, Figure 43). Mert selected a point along the path of the falling cannonball. He used a black marker to represent time and green to sketch the range. Since Tarik had some confusion about the range and time, the teacher pedagogical move requesting Mert to sketch might have helped him to differentiate between the two.

From the above vignette, we learned that asking, or probing, students to visualize, sketch or draw appropriate quantities is a form of instructional support that helps students identify the attributes of an object. This teacher pedagogical move—asking Mert to create a visual diagram for height, range, and time—was a way of supporting him and his peers in gaining a shared understanding of these quantities. Since the *x*-axis on the sketch could represent both the range and the time, depending on how you approached the question, it was essential to have students visualize both the range and the time before they proceeded with coordinating the change in range with the change in time.

The second category of teacher pedagogical move that supports creating a foundation for students' FT is asking them for a unit to measure the quantities. One of the ways to support students' reasoning about quantities and their relationships is asking students to perceive units for certain quantities. As we see from the below example, students might initially focus on only the numerical values of the quantities, and they might not see that these numerical values represent the magnitudes of the quantities. So, via teacher pedagogical moves, we might help students to notice the magnitudes of the quantities.

The below vignette is taken from Salim and Eren's small-group interactions, when they were investigating the relationship between the height and the area in the growing rectangle task.

281 NA: [Pointing at the table in Figure 44 (a)] What do these numbers mean?

282 Salim: That is the height, one.

283 NA: One what?

284 Salim: Cm

285 NA: These are what?

286 Salim: Cm squared.

287 NA: How are they similar or different from each other? How that centimeter is different than cm squared.

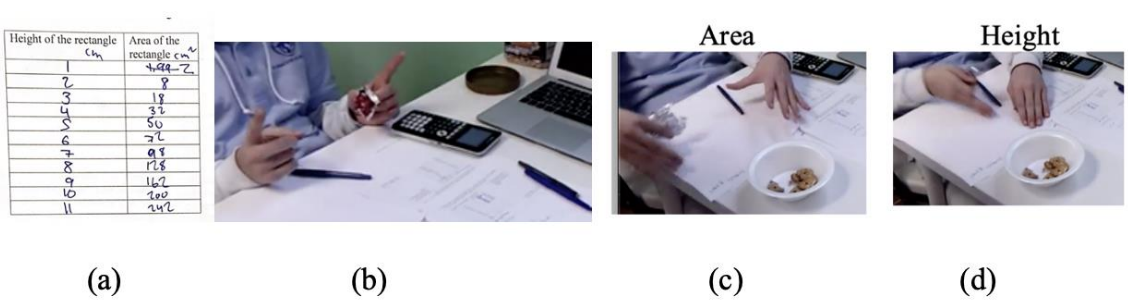
288 Eren: Cm squared means the area, and cm means Uhm I do not know [holding his fingers as a line, to show the height as a centimeter; see screenshot, Figure 44 (b)].

289 NA: Let’s say you will teach someone younger; how are you going to teach someone what are the area and height?

290 Salim: Area is the amount of place covered. Now you explain.

Figure 44

(a) Salim and Eren’s table; (b) Eren’s Gesture Indicating Height as a Centimeter; (c) Eren’s Gesture Indicating Area as “Amount of It Covered;” and (d) Eren’s Gesture to Represent the Height, in Centimeters, as the Side of the Paper on his Desk.



(a) (b) (c) (d)

291 Eren: You are doing good. Yeah, amount of it covered [showing with his hand, see the screenshot, Figure 44 (c)], the height I how long is one of the sides [tracing his hand on the side of a paper, see the screenshot, Figure 44 (d)].

292 Salim: How long is the side.

Salim recorded the height and area of a rectangle on a table without assigning its units [Figure 44 (a)]. Then the teacher-researcher, NA, asked what the numbers represented (line 281). Salim stated that the numbers were representing the height (line 282); however, Salim did not state any unit for the height. Then NA asked for the unit by saying “one what.” This is one of the teacher pedagogical moves which directs students to describe a unit for the quantities (lines 283 and 285). Salim affirmed that the units represented were centimeters and centimeters squared (lines 284 and 286). Once Salim and Eren explained their units of measurement, NA asked them to compare and contrast them, saying, “How are they similar or different from each other? How that centimeters are different than cm squared?” This is a way to help a student to articulate what area versus height means (line 287–291).

As we see in the above example, creating a foundation for functional thinking helps students articulate and select units for measuring magnitudes of quantities. In the above example, Eren talked about the area’s magnitude as the amount of area covered (line 291). He said, “cm squared means the area” while gesturing to the figure on the surface of a paper [see Figure 44 (c)]. For Eren and Salim, the unit for measuring area should have been centimeters squared, in comparison to the unit for measuring height, which should have been centimeters (lines 288–291). Eren and Salim conceived height as a quantity that was a line—or a side: as we see above, Eren said, “height is how long is one of the sides,” and he used his finger to trace on the side of the paper he had on the desk to show it [Figure 44 (d)]. Therefore, Eren and Salim used centimeters to measure the height.

As we observed above, these two students identified the area as the amount covered, and the height as a side of it. The activity provided the teacher-researcher with an opportunity to gauge what the students understood and thought about units of quantity and how they compare to one another. Asking students to assess what area versus height means may involve using these quantities by comparing the units they have selected to identify the magnitude of height in comparison to area. Hence, prompting students to articulate about the unit for measuring quantities is a form of support a teacher can offer in assisting them to reason about quantities, which ultimately builds a foundation for their ability to reason about the relationship between those quantities. Thus, probing students for a unit to measure an object's attributes is a form of instructional support that sets a foundation for functional thinking via teacher pedagogical moves.

The third category of TPM that sets a foundation for students' FT is probing students to engage in coordination of the change among quantities. Probing students for their understanding of coordinating the changes among quantities included: (a) asking students how quantities are related or (b) asking students to generalize the quantitative relationships. I will use the following vignette to articulate these two points.

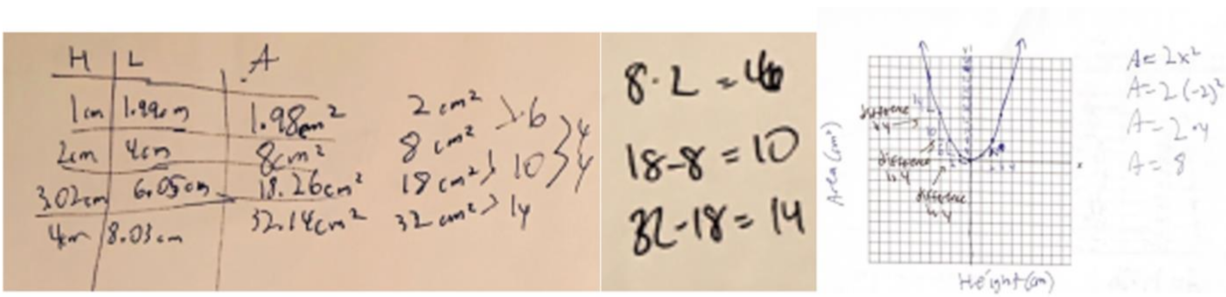
The vignette is taken from Asli and Yener's small-group interactions, when they were investigating the relationships between the height, length, and area of the growing rectangle task. Asli and Yener watched the video of the growing rectangle; NF, as the teacher-researcher, reinforced Asli and Yener to notice changing quantities and coordinate the change among them. Consider the growing rectangle task and the following conversation:

293 NF: Is that making sense? What I am asking? You were starting to do over here. What are these numbers? [Pointing at area values 2, 8, 18, and 32 on Figure 45 (a).]

294 Yener: I just try to round these numbers.

Figure 45[24]

(a) Yener's Table of the Height, Length, and Area; (b) Asli's Written Work; and (c) Asli's Graph for the Growing Rectangle Task



(a) (b) (c)

295 NF: Oh, 2, 8, 18, 32, I guess my question is as the height increasing by one, how is the area increasing?

296 Asli: So, like, so the changes in the height is always one, for the area is like, for example, 8–2, or 18–8, which is 10. It is like not constant. [Figure 45 (b).]

297 NF: But that is only two. How do you know it is not constant?

As we see in this vignette, NF pointed at the number's area values: 2, 8, 18, and 32 (line 293). With this, NF asked Yener and Asli what those numbers on the table on Figure 45

[24] Figure 45 is same as figure 30; I have added figure 45 to help the reader to visualize the concrete representations I reference in the text

represented. Yener rounded those numbers to the nearest whole numbers (line 294). To get Yener and Asli's attention on the relationship of change between the height and the area, NF said: "Oh, 2, 8, 18, 32, I guess my question is as the height increasing by one, how is the area increasing." In response, Asli explained that the change in height was always 1 cm, but that the area was changing in a varying amount (line 296). Asli said, "So like, so the changes in the height is always one, for the area is like, for example, 8–2, or 18–8, which is 10. It is like not constant." NF then asked Asli to be specific about the change in the area; she also inquired about how they could determine it was not constant by just looking at several area values: "But that is only two [8–2 and 18–8]; how do you know it is not constant?" (Line 296.) As we see, teacher pedagogical moves here helped turn Asli and Yener's attention to how the height and the area are changing together, and how the height and area of the growing rectangle are related. This is an example of teacher pedagogical moves setting a foundation for students' FT.

The above conversation continued, with Yener making a statement that the relationship between height and the area was increasing by four each time:

298 Yener: it [the area of the rectangle] increases by four each time, the changes in the area.

299 NF: What increases by four each time?

300 Yener: How much the area changing each time. Uhm the change, in the amount the area changes will be constant for each time. So, this time it changes by six, the next time it changes by 10, which is four more than six, next time it changes 14, which is four more than 10. So, it keeps increasing like that. The change in the area will be four each time. [Pointing at the table on Figure 45 (a).]

301 NF: Why do you think it keeps going up by four?

Yener made a conjecture about the change in the area: "it increases by four each time, the changes in the area." Inviting Yener and Asli to make conjectures about the coordination between the change in area and in height is a form of teacher pedagogical move. Furthermore, asking students to articulate about quantitative relationships, as NF did in the line 299, is another form of teacher pedagogical move that supports students. As we see during the conversation, due to a teacher pedagogical move, Yener explained how he reached the generalization of the area increasing by four each time. It was not only Yener who benefited from such a conversation by articulating his thinking; Asli also benefited from hearing when NF asked what was changing 4 cm^2 each time and Yener's answer (lines 299–300).

One way to build an eventual foundation for FT is by first probing students for the coordination of change between quantities; getting students to recognize that quantities are changing together and that they are related. I found that pushing students to conceive that quantities are related, and that they covary, might include several steps of probing. This may amount to asking students if a single quantity is changing and how it is related to other quantities (line 295). As a follow-up, a teacher may ask students about the quantities interchangeably; that is, they might ask how quantity A changed with respect to quantity B, then inquire about how a change in quantity B is related to a change in quantity A. When students notice the change, the teacher should then press them to be explicit about the amount of those changes. Lastly, when students recognize covarying quantities on several points, they should also be asked to generalize how the two quantities are related overall.

The fourth category of teacher pedagogical moves that sets a foundation for students' FT is encouraging students to justify their reasoning about quantities and the relationships between them. In the vignette[25] (lines 293–297), Yener saw the change of change in the area on the graph as increasing 4 cm^2 each time on his table [Figure 45 (a)]. NF then encouraged him to justify it on a graph; because he saw a 4 cm increase on a table in each instance (line 298), he said: "it increases by four each time." In order to prompt him to think about his justification, NF asked what increased four each time and what that increase looked like on a graph (line 299). She said, "How is this you just talked is related to the way the area is changing is the same? How can you see that in the graph? How is this related to the graph?" while pointing at the graph in Figure 45 (b). Yener identified the increase in distance of 4 cm by saying (line 134), "The distance between this point and this point will be a number and then this point between this point will be a number that is 4 cm more than this number." As we see above, teacher pedagogical moves that get students to explain their reasoning about the relationship between the height and area on a table and a graph are a form of instructional support which helps to construct a foundation for sophisticated understanding.

The fifth category of teacher pedagogical move that sets a foundation for students' FT is probing students to develop continuous covariational reasoning. Forms of instructional support to help students develop continuous covariational reasoning include shrinking and enlarging portions of values and having them visualize change and points. The below vignette is taken from Mert and Yener's small-group interactions. Mert and Yener were investigating the relationship between height, time, and range on the falling object task. The conversation between Mert and Yener focused on what the changes and points were, an essential discussion for determining whether or not there was change happening on the points. This necessary

[25] This vignette is also cited in the section *Level 3: Flexible Connections between Representational Fluency and Functional Thinking,* at chapter 5, part 1 from line 129-138.

conversation assisted the students with beginning to build a foundation for continuous covariational reasoning.

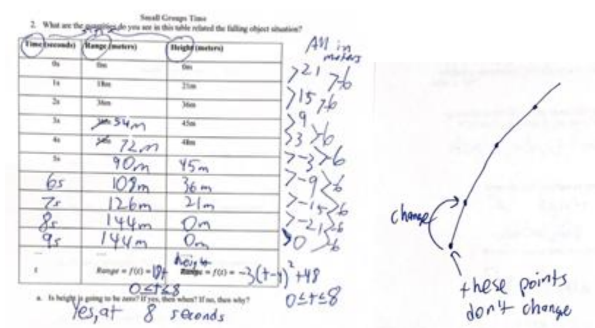
302 Yener: look, the point is not the change.

303 NA: What is the change?

304 Yener: It changes like between two points. From one point to others, it changes. These are points they don't change. [Pointing at 0, 1, 2 seconds, on the table in Figure 46 (a)].

Figure 46

(a) Yener's Table of the Range, Height, and the Time, and (b) Yener's Visualization of the Change and the Points



(a) (b)

Note. This represents (a) Yener's table when he first referred to the points not changing the time on this table, and (b) Yener's sketch identifying the change and points.

305 NA: If you take this table and break down the seconds as 0.0001, 0.0002, what will happen?

306 Yener: It still the points. One the to other is the change. It changes from one point to

the other.

307 Mert: Yeah. So like. Yeah, there are numbers between the things it doesn't tell, like the 0 seconds 21 meters instantly. There are some points in between them, in this. It only shows by ones.

308 NA: Okay, I have 0, 0.0001 to 0.0002, and 1 second. Where are the changes happening here [Figure 47]? [She writes the 0s, .0001s, and .0002s for Figure 47.]

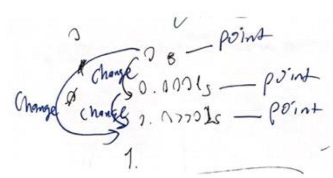
309 Yener: It will go from 0 to 0.0001 seconds, and then 0.0001 to 0.0002 seconds. [He visualizes the points by drawing arrows and identifying the points versus change, see Figure 47].

310 NA: So 0 to 0.0001 changes. Do you want to write for me? Like we're trying to differentiate between the points and the change.

311 Yener: There is also changes between these two [0 to 1] seconds.

Figure 47

Yener's Sketch of the Point and Change



Note. This figure represents numbers written by NA [0s, 0.0001s, 0.00002s, and 1] with a black pen, while Yener wrote verbs and arrows with a dark blue pen [change, point, and arrows].

Yener explained that the points were not changing, but that the changes were taking place in between points. He said that it changes "from one point to others; it changes. these are points they don't change.” To encourage student reasoning about whether or not height and time were changing between portions of points or not on the points, NA created small portions of points, like 0.0001, 0.0002, etc., generating a table to ask about changes in and between the points. Yener still insisted that it changed from one point to another. He said, “it still the points. One the to other is the change. It changes from one point to the other.”

For Yener, whatever portion of time values one had, the change happened in those portions. Then NA drew much smaller portions (Figure 47) than the portions in Yener’s table (see Figure 47): 0 to 0.0001 and 0.0001 to 0.0002. As a follow-up, Yener made arrows showing the change [Figure 47 (b)]. When NA asked questions, she encouraged Yener and Mert to visualize and write down their statements to elicit deeper thinking (lines 303–208). NA was explicitly asking them to differentiate between and at the points (line 310). Interpreting NA’s request, Yener responded that “The point is like at that where the ball is[26]*.”* His response constitutes evidence that, with the use of a teacher pedagogical move involving smaller and larger increments of time, Yener’s interpretation of change versus points might have shifted towards where the cannonball was at that point. Hence, a teacher can help students create a small portion of points on the table in order to visualize the points and the changes. Furthermore, getting students to write their thinking is a form of instructional support that may build a foundation for continuous covariational reasoning. I present a summary of the teacher pedagogical moves discussed in the above sections in Table 30.

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[26] Yener stated *“The point is like at that where the ball is”* at a later time in the vignette, thus I cited without showing it in the excerpt.

Table 30

A Summary of Teacher Pedagogical Moves for Creating a Foundation for Students' Functional Thinking

Teacher Pedagogical Move	Example
Probing students to identify the attributes of an object or situation	NA: So, we have, what are the quantities we have in the falling object situation? NA: Let's say you will teach someone younger; how are you going to teach someone what are the area and height?
Probing for a unit to measure an object's attributes	NA: How are they similar or different from each other? How that centimeters are different than cm squared?
Probing for the coordination of change between quantities	NF: Oh, 2, 8 18, 32, I guess my question is as the height increasing by one, how is the area increasing?
Encouraging students to justify their reasoning about the relationship between quantities	NF: Why do you think it keeps going up by four?
Probing for continuous covariational reasoning	NA: Okay, I have 0, 0.0001 to 0.0002, and 1 second. Where are the changes happening here? [She writes the 0, .0001, .0002.]

What Counts as "Meaningful Understanding of Quadratic Functions"

In order to understand how to support students in developing a meaningful understanding of quadratic function, it is also useful to define what this entails. For the purpose of this study, *a meaningful understanding of quadratic function* includes a student's ability to create, interpret, invent, communicate, and connect representations of quadratic functions within a flexible framework, including different approaches to reasoning about functions. In the study, a meaningful understanding of quadratic functions includes co-developing RF and FT in learning about quadratic functions. Specifically, developing a meaningful understanding includes shifting from less-sophisticated FT and RF (e.g., no coordination of values and pre-structural fluency) to significant sophisticated RF and FT (e.g., chunky continuous second covariational reasoning—

covariational reasoning and relational fluency—RF). I will discuss these concepts in more detail below.

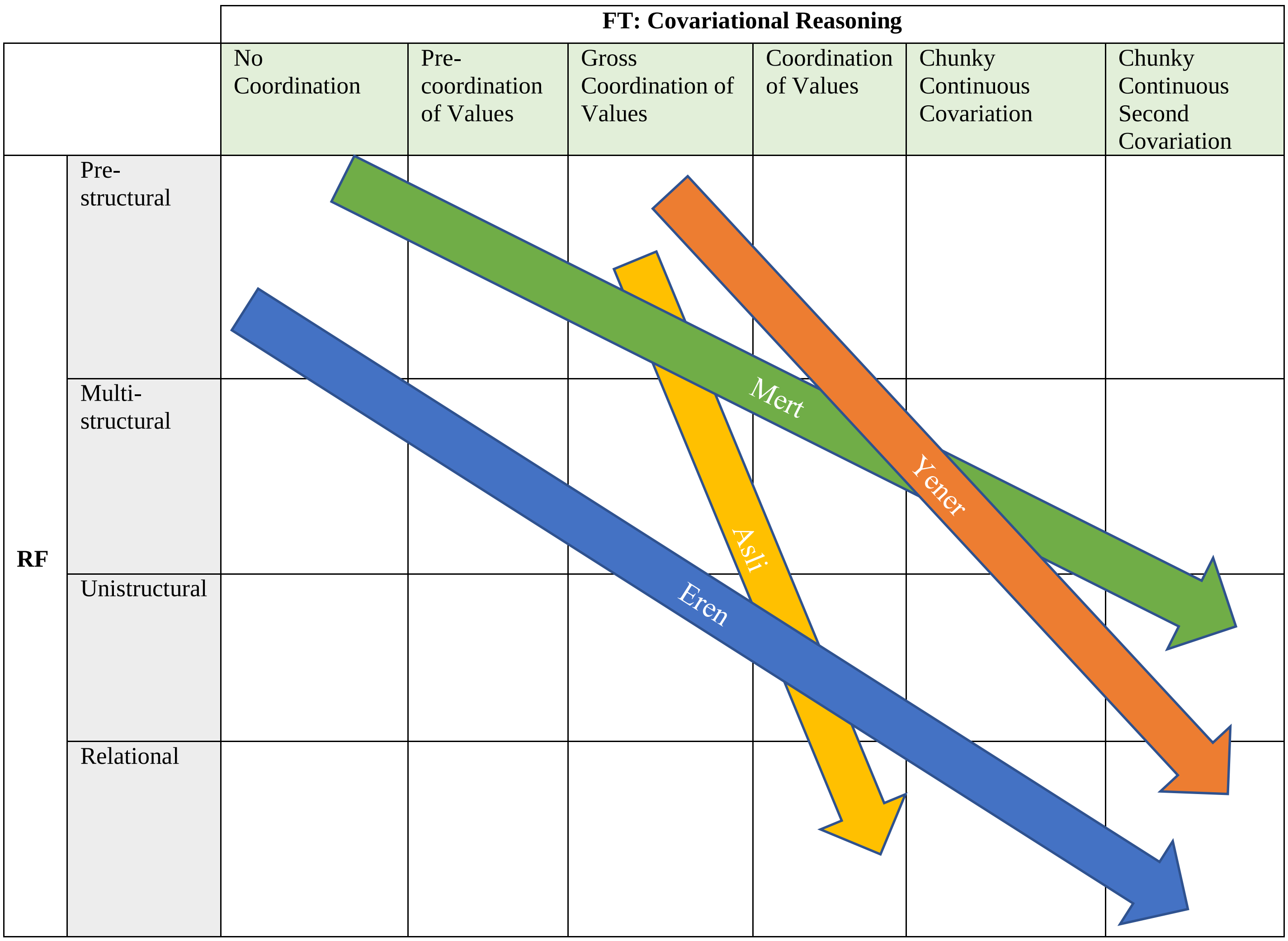
Identified Shifts in Students' Meaningful Understanding of Quadratic Functions. A shift in students' thinking is a transition between functional thinking levels—covariational reasoning (Thompson & Carlson, 2016), and representational fluency (Fonger, 2019). For instance, when a student first thinks of two quantities that covary as a gross coordination with pre-structural fluency, then shifts to coordinating the values of these quantities with relational fluency.

To verify the learning-ecology framework, I identified shifts in students' meaningful understanding that occurred during the study. These shifts were formed as students co-developed FT and RF. I have identified four shifts in the RF and FT of four participants: Eren, Yener, Mert, and Asli. Each of these shifts took place on four different days. For instance, Eren's RF and FT shifted from no coordination of the time and the height of the falling object, with pre-structural fluency, to chunky continuous second covariational reasoning—FT and representational fluency, or RF.

Figure 48 presents shifts in the thinking of Eren, Yener, Mert, and Asli, from lesser sophisticated covariational reasoning and lesser meaningful fluency to more significant sophisticated covariational reasoning and RF. The arrows on the chart show the nuances of their developing understanding. Each arrow represents one student's shift during one teaching session.

Figure 48

Identified Shifts in Students' Representational Fluency and Functional Thinking



The four arrows as a collection represent shifts that occurred on four different days during three different tasks. This does not mean there were no other shifts in students' thinking during the sessions; however, in this study, I have selected the four most salient shifts and individuals. Hence, I focused my analysis on each of these four students as they developed a meaningful understanding.

My objective in representing this variety of examples as four different shifts was to provide evidence of shifts in understanding which were supported by the learning-ecology framework, showing that the learning-ecology framework impacted students' meaningful understanding of quadratic functions. This section will define shift one, as an example, and provide a summary table for the rest of the shifts.

Eren's covariational reasoning and representational fluency shifted from *no coordination of values* and *pre-structural* fluency *to chunky continuous second covariational reasoning* and *unidirectional translations* (Fonger, 2019; Thompson & Carlson, 2017).

The shift occurred within the context of the falling object task—the PhET simulation video and the simulation itself, with the focus question, "What is the relationship between the height of the falling object and the time it takes to fall?"

Eren's initial functional thinking and representational fluency could be summed up by his phrase*:* "it goes up and down." During shift one, Eren watched a video and simulated throwing the object several times in the PhET. Eren's initial approach to the falling object task as "it goes up and then down" indicated no attempt to explain how the time and the height of the object were related. Furthermore, he did not attempt to create another representation, and looked at the simulations without a further attempt to coordinate the time and the height on the falling object task. He simply said: "there's no answer; this is an open-ended question."

During this task, Eren and his group were supported by the learning-ecology framework: enacted task characteristics, socio-mathematical norms, and teacher pedagogical moves. First, enacted task characteristics supported Eren as he learned about quadratic functions. Eren engaged in purposefully designed tasks; the enacted task characteristics of these tasks included questions centered on coordination among quantities, identifying attributes of the tasks, and identifying units for measurement of the quantities (see lines 274–280). Furthermore, the enacted task characteristics prompted Eren to engage in generalization of quadratic relationships. For example, Eren read the statement from the falling object task: “How does the time affect the height?” Another example for generalization is the enacted task characteristics of answering the focus question. Eren said: “Okay answer the question. How does the title affect, how does the time affect the height of the falling object?”

Second, Eren’s thinking was reinforced by socio-mathematical norms. Eren and his peers developed meaningful understanding by agreeing with each other’s solutions, getting approval from their peers, building on each other’s thinking, and taking responsibility for teaching their peers (see line 213–228). At the same time, they communicated their skepticism about peer responses by criticizing each other’s answers and pressuring their peers to be specific about how the height and the time it took the cannonball to fall were related (see lines 229–242). Pressuring for further justification, for example, Eren responded to Mert: “So you don't have the evidence, the evidence you are giving. It doesn’t make sense.”

Lastly, teacher pedagogical moves supported Eren’s development of a meaningful understanding of quadratic functions. The teacher pedagogical moves were clustered around supporting Eren’s co-development of RF and FT: asking him to create a table and then a graph to present the relationship of height and time. Then Eren was prompted to make a connection between the graph and the table when considering the relationship between the height and the time. Eren’s thinking was also supported via teacher pedagogical moves prompting him to visualize-draw-sketch his statements. For example, NA asked Eren: “Let’s think about graph now. Just sketch the graph. How is the time and height going to look like on a graph?” Through teacher pedagogical moves, Eren was encouraged to justify his reasoning about the relationship.

In addition to other types, a set of teacher pedagogical moves clustered around creating a foundation for Eren's functional thinking by probing for coordination among quantities, prompting Eren to conceive that the height, range, and time were changing together. Additionally, the teacher pedagogical move included asking Eren how the quantities were related, then getting him to generalize the relationship, and probing him to be specific about the change in the quantities—height, time, and range. For example, NA: *"*So can you find the relationships between the time and the height?" As another example, to prompt students to identify relevant quantities, NA asked Eren's group to identify the relevant quantities of the height, the range, and the time the cannonball takes to fall, causing the students to think of these quantities as measurable. NA: "What are the quantities you see in the falling object situation?*"* The unit of measurement for the height and the range were meters; the unit of measurement for the time were seconds. NA asked Eren, "Do you want to measure the height, and the time?" Eren used the PHeT simulations and measured; when measuring, he said: "Okay. When the time is half a second. So high. It's 7.64 on the time is one to go high. Highest 12 point 82 when the time is 1.81, 81 seconds. The height is 60.02 and then it drops down. You see the relationship?"

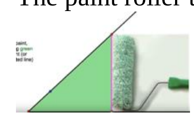
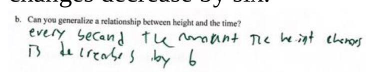
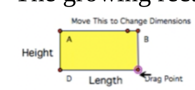
In sum, during this session, small- and whole-group interactions, along with teacher pedagogical moves and enacted task characteristics, formed a learning-ecology framework that helped Eren to develop a meaningful understanding of quadratic functions. In other words, Eren co-developed functional thinking and representational fluency in learning about quadratic functions.

As we see, at the very beginning, Eren indicated that the relationship between the height and time of the falling object was an open-ended problem; all he understood was "it goes up and down" (see first row of Table 31). Through the learning-ecology framework, he engaged in reasoning with chunky continuous second covariational reasoning within unidirectional translations (see the first row and the third column of Table 31). He recognized that for every 0.1 second change in time, the amount of the increase decreased by 0.1 meter. Eren attempted to coordinate a change in time with the change of change in height. So, he saw that the second change in the height changed by 0.1 meter when the time changed for 0.1 seconds. Eren said: "Okay, so let me just uhm every point one second. The height the amount that the height increases decrease by 0.1 meters. Sir. Every 0.1 seconds. The amount that the height increases decrease by one point second or all the cases but one meter." Eren created a table and graph in parallel to present that the change of change in height was 0.1 meters, without making the connection from table to graph explicit [see Figure 29 (a), (b), & (c)]. The learning-ecology framework supported him as he made this shift from a lesser sophisticated level of understanding to a greater sophisticated reasoning.

Next, I will use a summary table (Table 31) to represent the four identified exemplar shifts in students' reasoning, including Eren's shift discussed above; I will define the shifts, then I will highlight students' initial RF and FT, then students' co-development of RF and FT with corresponding tasks throughout the study.

Table 31

Definitions of the Identified Shifts in Students' Co-Development of Representational Fluency and Functional Thinking

Shift	Student's Initial FT and RF	Student's Co-Developed FT and RF	Task and focus question
Shift 1: Student's covariational reasoning and representational fluency shifted from no coordination of quantities and pre-structural representational fluency to chunky continuous second covariational reasoning and unidirectional translation—RF.	**No coordination and pre-structural fluency** Eren: "there's no answer; this is an open-ended question." "It goes up and down."	**Chunky continuous second covariational reasoning within unidirectional translations**: Eren: "Okay, so let me just uhm every point one second. The height the amount that the height increases decrease by 0.1 meters. Every 0.1 seconds. The amount that the height increases decrease by one point second or all the cases but one meter."	The falling object task:  What is the relationship between the height of the falling object and the time it takes to fall?
Shift 2: Student's RF and FT shifted from gross coordination of values and pre-structural representational fluency to chunky continuous second covariational reasoning within multidirectional connections	**Gross coordination of height and area and pre-structural fluency** Yener: "As the height of the paint roller increases, the length and height of the triangle increases, increasing the area"	**Chunky continuous second covariational reasoning within multidirectional connections** Yener: "A change of the change of the area increases by 1 cm squared each time when length increase 1cm each time."	The paint roller task:  What is the relationship between the length of the paint roller and amount of the area covered?
Shift 3: Student's FT and RF shifted from no coordination of values and pre-structural representational fluency to chunky continuous second covariational reasoning within unistructural fluency.	**No coordination and pre-structural fluency** Mert: "You see, after two seconds, uhm three seconds, it reaches the height, and it does not go on further."	**Chunky continuous second covariational reasoning within unistructural fluency** Mert: "Yeah. Every second the amount the height changes decrease by six."  	The falling object task:  What is the relationship between the height of the falling object and the time it takes to fall?
Shift 4: Student's RF and FT shifted from gross coordination of values and pre-structural representational fluency to coordination of values and multidirectional connections**.**	**Gross coordination of values and pre-structural fluence** Asli: "When the length increasing the heights increases"	**Coordination of values and multidirectional connections** Asli: "As since the height is [inaudible], the area of the rectangle is height times length, and since the length is $2h$ As the height increases by one unit, the length increases two units, so that will make has $2h$ squared."	The growing rectangle task  How does the change in the height of a rectangle affect the change in the area if presented on a graph, symbolic equation, and a table?

Chapter 6—Discussion

The prior literature has reported students' lesser sophisticated interpretations of quadratic functions, such as conceiving of graphs as objects (Moschkovich et al., 1993; Zaslavsky, 1997). In response to students' limited understandings and interpretations of quadratic function, in the current study, I employed a design-based research methodology (Cobb et al., 2017) in documenting and detailing the theories and design in terms of their contribution to supporting students' meaningful learning of quadratic functions. Recall that the design principles were informed by the theories of quantitative reasoning (Thompson, 1994) and representations (Kaput, 1987a; 1987b; Dreyfus, 2000). I conducted a teaching experiment with eight Turkish-American middle and high school students (Grades 8–10) and conducted both ongoing and retrospective analyses. The analyses centered on answering two research questions:

1. What is the nature of the co-emergence of RF and FT among secondary school students as they develop a meaningful understanding of quadratic functions?
2. How can secondary school students be supported to develop a meaningful understanding of quadratic functions?

Summary of Main Findings

To answer research question one, I characterized both students' co-emergence of RF and FT and students' disconnection between RF and FT, both on each representation and across multiple representations. I operationalized two levels of reasoning about quantitative relationships, static and lateral thinking, on each type of representation: a table, an algebraic, and a graph. I finished part one by operationalizing students' co-emergence of RF and FT into four levels: level 0, disconnection; level 1, partial connection; level 2, connection; and level 3, flexible connections between RF and FT.

To answer research question two I introduced the learning-ecology framework that the findings of the study suggested, which is made up of support for students that includes teacher pedagogical moves, enacted task characteristics, and small- and whole-group dynamics. Then I defined a meaningful understanding of quadratic function as students' co-development of RF and FT when learning about quadratic functions. Then, in order to verify the learning-ecology framework, I highlighted the four most salient shifts in students' RF and FT using examples of students in the study who co-developed RF and FT while being supported by teacher pedagogical moves, enacted task characteristics, and small- and whole-group dynamics. Lastly, I provided evidence that the learning-ecology framework was present when students' thinking shifted from initial to co-developed RF and FT. I closed this section with a summary of the shifts in students' thinking.

In this section, I will provide a summary of the conclusions related to the findings. I will focus on the three key findings that emerged from this study: (a) characterization of students' RF and FT on each representation, (b) connection of students' RF and FT across multiple representations, and (c) the learning-ecology framework. In the following I will discuss each these areas with its relation to the existing literature. I will articulate how each of these key findings mirror and advance the existing literature.

Students' Reasoning About Quantitative Relationships on each Representation: Lateral and Static Thinking

In the coming paragraphs, I will articulate how lateral and static thinking relates to the existing literature.

Tabular Lateral Thinking and Its Relation to Existing Literature

According to Wilkie (2019), students who employed correspondence reasoning on a tabular representation are more likely to create algebraic equations for representing growing patterns. Wilkie (2019) explored how students conceive of quadratic functions in connection to multiple representations. Wilkie recruited 12 high school students and conducted task-based interviews with each participant. While Wilkie reported that students benefited from approaching tabular representation with correspondence reasoning, the present study showed the opposite.

The findings of this study indicated that when students employed covariational reasoning on a tabular representation—tabular lateral thinking—they were able to create algebraic equations to present quantitative relationships. Furthermore, with tabular lateral thinking, they were able to make sense of the vertex of a quadratic function and flexibly switch back and forth between interrelatedness 1 and 2 in reasoning about quantities. Thus, current findings provide evidence that covariational reasoning on a tabular representation might help students to create algebraic equations as well as to make sense of what is being represented on the table.

Lobato and her coauthors (2012) explored quadratic functions in the context of speed, distance, and acceleration. The participants were 24 eighth-grade students. Lobato and coauthors found that covariational reasoning on tabular representations enabled students to identify the rate of rate of change as a constant. While these findings mirror what Lobato and her coauthors (2012) found, the current study also suggests that students benefited from flexibly switching back and forth between interrelatedness 1 and 2 on a tabular representation when learning about quadratic functions. In other words, the significance of these findings is that tabular lateral thinking enabled students to simultaneously engage in interrelatedness 1 and 2 as they performed tasks of creating symbolic equations and identifying the vertex point on a tabular representation.

Algebraic Lateral Thinking and Its Relation to Existing Literature

Zaslavsky (1998) studied 800 secondary school students and explored obstacles students face in understanding quadratic functions. She reported that students don't conceive of the coefficient of quadratic functions as a point when the coefficient has a value of zero. These types of inappropriate interpretation of the parameters of quadratic functions were also identified by Even (1998). Even explored 152 pre-service mathematics teachers' flexibility in moving from one representation to other and reported that the pre-service teachers had difficulty making sense of the parameters of quadratic functions.

In answer to these historically identified obstacles, several studies have subsequently found that students have difficulty understanding the parameters of quadratic functions when they take the general form of the algebraic equation $y = ax^2 + bx + c$ (Borba & Confrey, 1996; Ellis & Grinstead, 2008; Zaslavsky, 1997). The findings of this dissertation suggest that students might associate the coefficients of quadratic functions with quantitative relationships. Recall that in lines 56–58, Asli was able to see that the coefficient of $y = 2x^2$is 2 cm, and that it relates to the relationship between height and length in the context of the growing rectangle. With that in mind, the present findings inform us that students might benefit from conceiving algebraic equation representations of quadratic function as a quantitative relationship that covaries on the algebraic equation. In other words, students were able to better understand the parameters of the quadratic function by coordinating the change in height with the change in length, operating length and height to create a symbolic equation of the new quantity, and associating the coefficient with the covariation among the quantities.

The current findings showed that students with algebraic lateral thinking can create a symbolic equation that represents covarying quantities, and this might reinforce the students' ability to see the symbols on the equation as associated with covarying quantities, and to see that coefficients have a meaning within the quantitative relationship. These findings proved that support for students' meaningful understanding of quadratic functions can be rooted in the combination of students' RF and FT. As we recall from the prior literature (Ellis & Grinstead, 2008), students conceived of the leading coefficient as the slope of quadratic functions. Building on that, the current findings provide evidence that students' meaningful understanding of the coefficients of quadratic function can be supported by networking the theory of QR and the theory of representations.

Graphical Lateral Thinking and Its Relation to Existing Literature

Historically, the literature suggests that students face challenges when they create and interpret graphical representations of functions. First, in their literature review, Leinhardt and coathors (1990) documented students' difficulty creating and interpreting graphical representations. Then Zaslavsky (1997) reported that students conceive of graphical representations as *pictorial entailments,* meaning that students treat the graph as what they see rather than imagining that the graph still continues even if the image does not show it. Zaslavsky explored 800 secondary school students' interpretation of quadratic function graphs. In parallel to Zaslavsky, Oehrtman and coauthors (2008) reported that undergraduate students have difficulty in interpreting and creating a graph of a function that models a quantitative relationship in certain situations (e.g., graphing height versus volume of an uncanonical shape when the object was being filled with water). Building on that, Moore and Thompson (2015) reported in a conference proceeding that students conceive of the graph of a function as a *piece of wire*; they named this type of understanding *static shape thinking*. The findings of this dissertation mirror the prior literature about the limited interpretation of graphical representation by showing that if a quadratic function has a vertex at the origin with a positive leading coefficient and it does not have a negative domain, then students may name the graph as "half-quadratic" because they have a static image of a parabola associated with quadratic function.

This kind of thinking might be problematic for students, especially students who have a static image of a parabola, as it means such students might not look at the domain or range of a function, and they may only perceive a quadratic function as a static picture of a parabola. Such thinking might limit students' robust understanding of quadratic functions. Additionally, with graphical static thinking, students conceive of all nonlinear growths as exponential growths, even if the graph represents a quantitative relationship which has a starting point from the origin (Altindis & Fonger. 2019).

In a prior study with Dr. Fonger, I recruited five pre-service secondary teachers and conducted task-based interviews. The task was situated in the context of the growing rectangle, and the participants were asked to create multiple representations of the relationship between the height and area of the rectangle. We found that the five participants identified nonlinear functions as exponential functions without further inquiry (Altindis & Fonger, 2019). The participants conceived of non-linear growth as exponential growth.

Scholars have agreed that students should be able to create and interpret graphs to represent a dynamic function situation (e.g., Oehrtman et al., 2008). In response to such needs, Moore and Thompson (2015) introduced *emergent shape thinking*, which is related to students conceiving of a graph as an emergent quantitative relationship that covaries. Moore and Thompson reported that middle school and preservice teachers perceived graphs as emergent quantitative relationships. The present findings corroborate and extend Moore and Thompson's point by showing that students were able to generalize the emerging nature of the quantitative relationship on a graph.

With such findings, the present study advances the field of mathematics education by (a) showing that the emergent quantitative relationships include a generalizable pattern, and (b) reporting that students with graphical lateral thinking might be able to coordinate the change in a quantitative relationship on multiple increments of a graph. The findings also showed that students noticed that, on a parabola of a quadratic relationship, as the amount of one increment is decreasing, the relationship is decreasing by the same increment. Furthermore, students with graphical lateral thinking could notice that the change of change that is increasing or decreasing on the left and right of the parabola is constant for the same increments on the x-axis.

Note the difference in terminology, Moore and Thompson called this type of reasoning emergent shape thinking, while I have called it graphical lateral thinking. I wanted to emphasize that lateral thinking has a more general meaning than emergent shape thinking; while Moore and Thompson have a definition that is general to the whole representation of graph, my conception of graphical lateral thinking is specific to each point on the graph as well as the whole of the graph. Such thinking not only characterizes students' process of reasoning about the graph, but also highlights students' reasoning for each point on the graph in contrast to how students approach each increment of the graph (specifically, how the steepness changes when we move closer or further away from the vertex of quadratic functions on the graph).

A Model of Networking Theories: Connections Between Students' Representational Fluency and Functional Thinking

In this study, I networked the theories of quantitative reasoning and representation. These findings modeled creating a rich web of multiple theories that sheds light on students' meaningful understanding of quadratic functions. Networking theories is defined as a *diversity of approaches* or *ways of making theories interact* (Kidron et al., 2019). Networking theoretical perspectives has been popular for several decades, in the mathematics education community in particular (Bikner-Ahsbash & Prediger, 2010; Dreyfus, 2010; Kidron et al., 2019). However, practically, it has not been clear in the field how to network theories with a design methodology and how to make sense of the findings via multiple analytical lenses. This study sets a model of how to network theories using Thompson's theory of quantitative reasoning and the theory of representations (Dreyfus, 2002; Kaput 1987a, 1987b) as an example for: (a) strategically combining and coordinating theories as complements to support students' meaningful understanding of quadratic functions; (b) how to employ multiple analytical frameworks—in this case, functional thinking (Confrey & Smith, 1994; Thompson, & Carlson, 2017) and representational fluency (Fonger, 2019)—to characterize students' meaningful understanding of quadratic functions.

In the literature, Even (1998) reported that there is an intertwined relationship between students' representational activity and their reasoning about functions. The findings of this dissertation advanced Even's argument by characterizing the intertwined nature of RF and FT into four distinct categories, levels 0 to 3. With these findings, I characterized students' co-emergence of RF and FT across multiple representations by organizing from the lesser sophisticated emergence of RF and FT to the more sophisticated co-emergence of RF and FT.

In sum, the significance of these findings to the field of mathematics education is in demonstrating: (a) that students co-developed RF and FT, and that the co-development reinforced their meaningful understanding of functions; (b) that students' RF and FT co-inform one another—RF sets a foundation for FT and vice versa; and (c) a model for characterizing students' co-development of RF and FT from level 0 to level 3. The current findings support the conclusion that students' fluency in both RF and FT enables them to engage with the complicated nature of mathematical phenomena (in this case, quadratic functions), and that this fluency becomes a resource for students to develop a meaningful understanding of mathematics.

The Learning-Ecology Framework

The findings indicated that students' meaningful understanding of quadratic functions can be supported with a mechanism: the learning-ecology framework. This mechanism was made from tasks, tools, teaching actions, socio-mathematical norms, and probing questions centered on the theories of quantitative reasoning (Thompson, 1994), and representations (Kaput, 1987a;1987b). The findings suggested the three main components of the learning-ecology framework: enacted task characteristics, socio-mathematical norms, and teacher pedagogical moves.

The findings of the current study empirically proved that the learning-ecology framework supported students' meaningful understanding of quadratic functions. The findings also verified the learning-ecology framework by identifying four specific shifts in students' thinking when they were learning about quadratic functions during the study. My evidence for the effectiveness of the learning-ecology framework comes from the positive shifts I identified in students' co-development of RF and FT, as well as productive shifts in students' reasoning process when using one or more representations in learning about quadratic functions. The shifts showed that students' co-development of RF and FT transformed from lesser sophisticated reasoning to greater sophisticated reasoning when they completed tasks and worked within the learning-ecology framework. Recall, for example, that Eren's thinking shifted from lesser sophisticated thinking to grater sophisticated reasoning via the learning-ecology framework. In Figure 48, I represented the four identified shifts in students' co-development of RF and FT, which verify the effectiveness of the framework.

To develop the learning-ecology framework, I drew on the work of other researchers, who suggested that a productive learning climate (Brown, 1992) includes a teacher's role as a facilitator of learning to provide appropriate support for students' meaningful learning regarding quantities (Smith & Thompson, 2007) by eliciting students' thinking and orienting their thinking toward one another (McDonald et al., 2013). The students' role in a productive learning climate centers on explaining their thinking, asking questions, providing a justification, being skeptical with peers' explanation (Yackel & Cobb, 1986), and engaging with purposefully sequenced tasks (Smith & Thompson, 2007). Building on that, my findings suggest that students' meaningful understanding of quadratic functions can be rooted in the learning-ecology framework. In the coming paragraph, I will discuss each component of the learning-ecology framework and its relation to the prior literature.

Enacted Task Characteristics

The findings from this study parallel prior literature that posits that the design of enacted tasks' characteristics can be a form of instructional support in learning and teaching about mathematics (King, 2011; Stein et al., 2007). While prior literature focused on making quantities visible to students (e.g., Johnson et al., 2017), this study advances the prior literature by suggesting that designing tasks with prompts, statements, or questions that redirect students' attention toward recognizing a coordination among quantities can provide effective support for students' meaningful learning.

The findings indicated that, for both launching RF and supporting QR, specific characteristics of purposefully designed tasks created opportunities for students to gain a robust understanding of quadratic functions. The study showed evidence of enacted task characteristics that would emphasize students creating and making connections across representations to present quantities and quantitative relationships within the tasks. Furthermore, certain enacted task characteristics pushed students to engage in quantifying processes and quantitative operations (Thompson, 2011; Smith & Thompson, 2007), in tandem with Dreyfus's (2002) four stages of learning with multiple representations. In other words, the findings suggest that via enacted task characteristics, students developed images of emergent quantitative relationships by creating and connecting between graphs, tables, and symbolic equations to present the emergent quantitative relationships in the tasks.

In sum, certain purposefully designed task characteristics might support students to co-develop RF and FT because these characteristics, discussed above, provide opportunities for students to talk, articulate, discuss, and create and connect concrete representations to represent emergent quantitative relationships as they learn about quadratic functions.

Socio-Mathematical Norms

In general, when students are participating in activities within a small- and whole-group structure, social norms help to shape the students' explanations, their reasoning, and their ability to make sense of other explanations (Cobb & Yackel, 1995). Social norms are *joint social constructions*—collectively constructed by the whole class community—and cannot depend on a teacher or students alone (Cobb, 2000). Along with these definitions of social norms from the prior literature, the findings of this study proved the essential role of social norms and socio-mathematical norms in shaping how students learn about quadratic functions. As stated earlier, the participants in the study were Turkish-American students, and these findings build on the literature about social and socio-mathematical norms by reporting unique small- and whole group interactions that took place in a community center, with a researcher who was an insider and had established trust with the participants as a member of community via having the same language and culture. The participants and the leading teacher-researcher had a common culture and language. As we recall, the findings showed that in this setting, the students were skeptical about their peers' work, and they pressured their peers to justify their reasoning. The group dynamics between peers are unique to these findings: because the participants were members of the same community, that might have created opportunities for them to be critical about each other's work and feel safe when pressuring one another for justification.

As we saw, the teacher-researcher's role is as a facilitator of the classroom learning community, and during the study, the students recognized this role. Also, students were able to see themselves as constructors of socio-mathematical norms when learning about quadratic functions. These findings corroborate or confirm previous research in suggesting that the process of building up socio-mathematical norms in small- and whole group interactions impacted students' learning by developing students' intellectual autonomy, enabling the students to become aware of when and how to contribute to the mathematics classroom and what counts as a mathematically correct solution, thinking, or reasoning (Yackel & Cobb, 1996). Cobb and Yackel (1995) define intellectual autonomy as "...students' awareness of and willingness to draw on their intellectual capabilities when making mathematical decisions and judgment" (p. 9), that is, when participating in the mathematics classroom. As we recall from lines 208–212, Mert and Tarik were in change of their own learning; they contributed to each other's thinking and eventually learned more about quadratic functions by asking "where is your reasoning?"

The findings of the current study confirm Yackel and Cobb's (1996) conception that justification and explanation develop teachers' and students' *taken-as-shared meaning of* *mathematics*. Yackel and Cobb (1996) wrote that, "the development of individual reasoning and sense-making processes cannot be separated from their participation in the interactive constitution of taken-as-shared mathematical meaning" (p. 460). This concept which was furthered by Stephan (2003), who presented that the robust relationship between individual learning and social process impacts the *taken-as-shared learning of the community*. Stephan stated that "…students' development cannot be adequately explained in cognitive terms alone; social and cultural processes must be acknowledged when explaining mathematical development" (p. 28). The findings of this dissertation advance Stephan's point by relating students' meaningful understanding of quadratic functions to students' social and cultural dynamics.

The uniqueness of the current findings is that they highlight how an insider's established trust with students might set a space for students to engage in critical thinking about each other's reasoning. These findings also further Yackel and Cobb's (1996) definition of "taken-as-shared;" recall that Eren stated that he knew what a quadratic function looked but he did not know what it meant. The taken-as-shared definition of a quadratic function is a parabola; as we recall from Salim's graph (Figure 38), Eren pushed Salim to critically examine what it meant for a function to be quadratic function. These interactions indicate that in certain situations, students started questioning the meaning of knowledge that was taken-as-shared.

Teacher Pedagogical Moves

The findings of this study also endeavor to complement and advance the existing literature on teacher moves and promoting actions for supporting students' understanding of quadratic functions as a part of the learning-ecology framework. Previous researchers characterized teachers' talk moves (Michaels & O'Connor, 2015) and discourse actions (Candela et al., 2020) in eliciting and supporting students' reasoning in a general mathematics classroom. The current findings corroborate those findings, yet add texture to our understanding of how such teacher pedagogical moves can support students' co-development of RF and FT. This validates certain sets of teacher pedagogical moves that promote students' sophisticated understanding of quadratic functions as covarying quantities.

These findings are unique because I networked two distinct literatures to discover sets of common pedagogical moves that can help teachers specifically in supporting students' co-development of RF and FT; the findings set an example of identifying pedagogical moves that support students' meaningful understanding of quadratic functions. When identifying teacher pedagogical moves that created a foundation for students' FT, I drew on the work of other researchers who (a) identified the central tenants of quantitative reasoning—quantification and quantitative operations (Thompson, 2011)—and (b) explored ways of supporting students' understanding of quantities and quantitative operations (Smith &Thompson, 2007). I took as a baseline that students' skills in reasoning about quantities and their relationships help to foster students' FT (Ellis, 2011).

Overall, these findings on teacher pedagogical moves furthered existing literature by pointing to specific sets of teacher pedagogical moves that were empirically proven during this study to be affective in teaching students about quadratic functions within a quantitative context. One of the key contributions of this study was that these sets of empirically proven teacher pedagogical moves may help create a foundation for students' development of continuous covariational reasoning. Most of the prior literature stated that students should develop continuous covariational reasoning as part of a sophisticated understanding of functions (Carlson et al., 2002; Moore; 2014); however, empirically proven sets of teacher pedagogical moves that might support this development had not been explored. Therefore, the current findings set an example by identifying specific teacher pedagogical moves that support students' continuous covariational reasoning.

A Role of Language and Culture in Learning About Quadratic Functions

In the current study, I identify myself as an insider who established trust with participants (Narayan, 1993). As the researcher and participants, we share the culture and language, which may have played a critical role in this study. I have known the participants for four to seven years. I saw them grow up in the same community and neighborhood. My role in participants' minds might vary; they know me as an "auntie" or a Turkish-American practicing Muslim, who speaks the same language they do, laughs at the same jokes, and shares the same culture. I see that all the values and experiences I share with the participants may have impacted the students' meaningful learning of quadratic functions.

An example, I emphasized and encouraged students to engage in code-switching in English and Turkish. This is partially because some of the participants had more formal mathematics in Turkish than in English; some of the participants learned mathematics in Turkish until 7th grade. Then they switched to English for 8th to 10th grades. Knowing their background, I code-switched with students, constantly used the terms in both Turkish and English. When they used formal mathematics terms both in Turkish and English in the same conversation, I understood what they were trying to say.

In sum, culture and language may have impacted students' meaningful understanding of quadratic functions during this study. However, to measure how this dynamic might support students' learning, I would need specific analytical tools.

Limitations and Suggestions for Future Studies

Limitations

There were several limitations to the current study, including (a) having few team members and a large amount of data; and (b) an inconsistent timeline between teaching episodes. I will discuss these limitations in this section, as well as presenting opportunities that the current study suggests for further research.

One of the limitations was having such a large amount of data. Design studies usually require the work of a team to analyze that data. Although I had a research team, I believe that my team size was on the small side, and that this project would better be conducted with a larger group of 20 to 30 people to gain a complete understanding of students' meaningful learning by supporting and characterizing students' co-emergence of RF and FT. With an extended team size, this research could also be conducted with the perspectives of multiple experts from the two distinct yet related areas of literature that I networked, quantitative reasoning and representations.

The timeline was also a limitation to my study; the time between teaching episodes depended on the availability of the research team, Covid-19 contingencies, and the participants, rather than a thorough ongoing analysis of previous teaching episodes.

Suggestions for Further Studies

The research site was a community center, rather than a school setting, which may have positively affected students' motivation in solving questions. This study piloted ways to connect students' culture, identity, and language as an "insider." This study should be conducted in school settings by bringing students' culture and identity into the process of learning about functions in the context of the networked theories of QR and representations. With that in mind, I recommend conducting this study in school settings with teacher playing the role of the teacher-researcher; such a study might yield robust pedagogical moves for school settings.

The current study showed that students have difficulty differentiating between interrelatedness 1 and 2 when they make a connection across multiple representations; we need further study to explore ways to ease such difficulty, which might include supporting students to make a connection with interrelatedness 1, and then interrelatedness 2, then seeing how each emergent quantitative relationship is related across multiple representations. In particular, we need studies that explore how and why a connection between a table and symbolic equation of quantitative relationships might pose a challenge for students trying to differentiate whether interrelatedness 1 or 2 is being represented with a symbolic equation. This is especially true because the symbolic equation for the interrelatedness 1 is a quadratic relationship, and interrelatedness 2 is the derivative of the quadratic relationship. Hence, we need studies to explore whether, if students create a connection between multiple representations along with flexibility between interrelatedness 1 and 2, such thinking would yield a foundation for the meaningful understanding of a derivative.

Lastly, although this study shed light on the ways to merge two distinct yet related areas of literature, quantitative reasoning and representations, there is also a need to see how networked theories could support other families of functions.

Implications of the Study

There are several implications of the study: implications for teachers of mathematics and implications for curriculum writers.

Implications for Teachers

There is a gap, or a disconnection, between research and practice in teaching and learning mathematics (Silver & Lunsford, 2007). Therefore, the findings of this study can inform mathematics teachers as to how to better prepare secondary or high school students for advanced mathematics via networking the theories of quantitative reasoning and representations. These findings offer an example for teachers to see how students' understanding of quadratic functions can be supported through teacher pedagogical moves, enacted task characteristics, and socio-mathematical norms. With the teaching experiment, along with the learning-ecology framework, the findings provide opportunities for teachers to learn theory-guided designs in teaching and learning about quadratic functions.

These findings inform teachers that students' learning or sense-making does not progress on a linear path. Therefore, the components that support learning should build off of one another: a learning-ecology framework that takes into account teachers' moves, prompts and promoting actions, socio-mathematical norms, and enacted task characteristics.

This study can guide teachers toward empirically proven sets of teacher pedagogical moves that support students' meaningful understanding of functions when students engage in quantitatively rich tasks within small- and whole-group settings. The reported findings also showcase specific enacted task characteristics that support students' meaningful understanding by allowing students to talk, articulate, create, interpret, connect, and communicate about multiple representations when presenting emergent quantitative relationships.

Furthermore, this study can help teachers by highlighting socio-mathematical norms which can be used to support students as and encourage them to criticize, justify, articulate, and express skepticism about peers' explanations when learning about quadratic functions.

Finally, learning-ecology frameworks can be used to teach other function families as well by generalizing that a function family should be taught by getting students to co-develop RF and FT.

Implications for Curriculum Writers

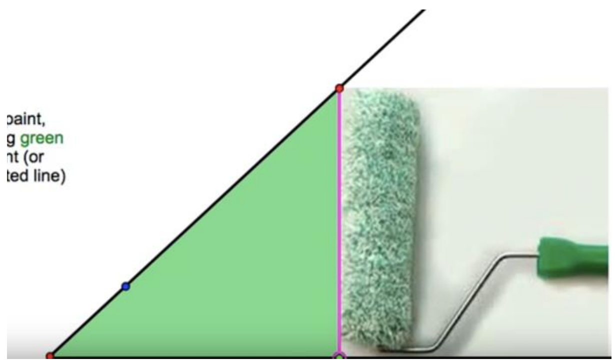
The findings of this study can help curriculum developers create a curriculum that emphasizes designing tasks, tools, and curriculum materials that center on quantitative reasoning (Smith & Thompson, 2007; Thompson, 1994) and representations (Dreyfus, 2002; Kaput, 1987a;1987b). With that in mind, curriculum writers can design curricula that provide opportunities for students to represent emergent quantitative relationships by creating and connecting multiple representations. In other words, the curriculum materials should create opportunities for students to articulate, talk about, and identify quantities, and to create and connect multiple representations, when presenting emergent quantitative relationships. Furthermore, the curriculum materials should set up opportunities for students to coordinate among quantities, make generalizations, estimate, justify, and visualize an emergent quantitative relationship with multiple representations. The findings of this study would help curriculum developers to develop a curriculum which emphasizes meaningful understanding of quadratic functions via the specific, empirically tested elements of the learning-ecology framework.

Appendix A: A Sample of Lesson Plan

Learning Goal: Encouraging students to notice the quantities: the height of paint roller, and the length and the area of the triangle, by identifying quantities, creating new quantities (e.g., area) creating a representation (this will be students' choice), and then sketching a graph to represent to a quantitative relationship. Overall, in this teaching episode, students will be supported to identify the quantities for paint-roller task situation and represent the relationship among the quantities.

The Focus Question: What is the relationship between the length of the paint roller and the area covered?

Research Goal: Exploring quantities in *students' minds*: how the height of the paint roller exists in students' minds, how students engage in the *quantification process,* and what the attributes of the paint roller and the painted area are in students' minds. What is the unit of measure of the height and area of the triangle in students' minds? What are quantitative operations students engage with? How do you support students' quantitative operations? What representations are students likely to start using?



I am going to let you watch a video; I want you record any notices and wonders you come across. Let us imagine there is a magic paint roller that sweeps out paint as you see on the screen. What did you notice and wonder about the magic paint roller?
What do you notice and wonder when you watch the video? Fill out the chart below.

I notice	I wonder

Quantification: Describing Situations, Attributes of a Situation, Quantities as Quantities

1. Are these quantities (length, area, and height) measurable?
    - What can we measure in this situation?
    - What are the units for these quantities?
2. Are the height and area related? How? Why?
3. What is the relationship between quantities (area and the length)?
4. How is the change in height of the paint roller affecting the change in the area?
5. How big will the painted area be if the length of the paint roller is big or small? Why?

Quantitative reasoning and creating a representation of the quantities of length and area

1. **Estimate:** What is the relationship between the length of the paint roller and amount of the area covered?
    a. What is your estimation in regard to the relationship between height and area?
    b. What information do you need in order to find out how close your estimation was (the main questions)?
    c. How are you going to use the information?
    d. Do you think the area painted and the length of the paint roller are related? Why? Why not?
    e. What is the relationship between the area painted and the length of the paint roller?
    f. Draw a picture of this situation. How do you visualize the relationship between the area painted and the length of the paint roller?

Appendix B: The Teacher-Researcher's Prompting Questions

Identification of Quantities and reasoning within Quantities

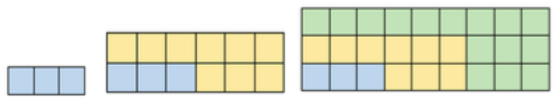
1. What are the quantities?
2. What are the important quantities in this problem?
3. Why are these quantities important in this problem context?
4. What do the quantities mean in this problem?
    a. Give one important quantity
    b. And what is the importance of that quantity in the world of this quantity?
    c. What is another important quantity in this problem? Why?

. Another important quantity in the problem?

5. Any more important quantities? And why are they important?
6. Can you create or come up with another quantity using these quantities? What does the new quantity mean in this problem?
7. How are these quantities related?
8. What is the new quantity in this problem?
9. What is the relationship between these quantities?
10. What information do you have here?
11. Can you make another quantity by using these quantities you identified? Why is the quantity you created important?
12. What is the meaning of the quantity you just made? Why do you think that quantity is important in this problem? What are the quantities representing? And how are these quantities being related?
13. Are these quantities proportional? And how are they proportional?

Appendix C 1: Pretest

1. Draw the next rectangle in the pattern (modified from Ellis, 2011)



a) Do you see a pattern? Explain.

b) Represent the height, length, and area in a table.

Height		Area

c) Create a graph relating height and area.

d) Describe in your own words how the height and length are related.

e) Use what you know about the rectangle to write an equation relating the height and area.

f) What is the connection between the symbolic equation and the graph you created to show the height and area?

Appendix C2: Pre-Task-Based Interview Protocol

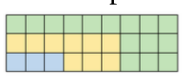
Questions:
***** Before I start asking you questions, I wanted to say that I am not here to evaluate what you did is right or wrong. I am interested in knowing what you think of these solutions. Remember there is no right or wrong way of thinking about these questions.**

Goal: Characterize students' quantitative reasoning and RF in examining quadratic growth in discrete growing rectangle task
Researcher Prompts:

1. Which pattern did you see? Can you explain the pattern?
    a. What images do you have in mind related to the pattern?
    b. Could you draw the pattern?

If participants could not see any pattern, ask students to create a diagram, either with tiles or drawing a diagram. **(their drawing**)

Quantitative Reasoning and Representational Fluency
Diagram. Set up: Use algebra tiles to make the first three iterations below. Create the next rectangle in the pattern.



Quantification: Identifying quantities on a table and graph

2. What are the quantities you see on this diagram?
    a. Where do you see length?
    b. Where do you see height?
    c. Can you make another quantity by using these quantities you identified?
    d. Where do you see area?
3. Can you create a table?
    a. Can you extend the table for more values?
    b. What are the quantities?
    c. What do the quantities mean in this growing rectangle problem?
    d. Can you create or come up with another quantity using these quantities?
    e. How are the height and length related?
    f. How did you find the area?
    g. How are the length and area related?
    h. How are height and area related?

Connection between diagram and table within quantities

4. How is your table related to your diagram?
    a. What are the quantities in this table and diagram?
    b. What are the units for those quantities? Can you show me on the table and diagram?
    c. How are these quantities related on the table and the diagram?
    d. Describe the connections between the table and the diagram.
    e. What do you see on the diagram?

f. What are the quantities you see on the diagram?
g. How do you use the quantities to create the table?
h. How do you create the table?
i. How do you use diagram to create the table?
j. How is what you built (in tiles/drawing) connected to your table?

Graph: Connections among graphs, one created without a grid the other one created with a grid.

5. What would this situation look like in a graph? **(papers with no grid o it, no graph papers!)**
   a. Where is height in the graph?
   b. Where is area in the graph?
   c. How are the area and height related?
   d. If you plotted points from your table, where would they be on that graph?
   e. **(Offer grid paper)**
   f. What do the quantities (length, area, height) mean in this graph?
   g. What does area mean in relation to height**?**

Reasoning in symbolic equation

6. What rule would get you from any height to the area?
   a. Can you generalize the relationship between height and area?
   b. Can you write an equation for height and area?
   c. What does the area mean in this equation?
   d. What does the height mean in this equation?
   e. How are the area and height related in in this equation?

Connection among and within a graph and a symbolic equation with quantitative reasoning

7. Can you explain your thinking in connecting the graph with the equation?
   a. Where are the height and area on a symbolic representation?
   b. What do you see in connecting the graph and the symbolic equation?
   c. How are the height and the area related on a symbolic equation?
   d. How are the height and area related on a graph?
8. What is the best representation to explain the relationship between height and area?
   a. Why?
   b. How did you decide that?
9. Which one—table, graph, diagram, or symbolic equation—makes more sense to you about the relationship between height and area?
   a. Why? How did you think about this relationship?

Appendix D: Teacher-Researcher Written Reflection Rubric

Date, Time:
Name:

1. ***What emerged in today's instruction (goals, instructional activities, learning process, and tools) from a stance of networking the theories of FT and representations?***

a) ***Provide a rationale for your claim with the time and data***

2. ***What new information am I acquiring about students' meaningful understanding of function that might challenge our stance on networking theories?***

a) ***Provide a rationale from the data***

3. **Write a burning question about things that constitute students' meaningful learning of function.**

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