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Applications of Powerset Operators, Especially to Matroids

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Abstract

Let $\mathcal{V}$ denote a vector space over an arbitrary field with an inner product. For any collection $\mathcal{S}$ of vectors from $\mathcal{V}$ the collection of all vectors orthogonal to each vector in $\mathcal{S}$ is a subspace, denoted as $\mathcal{S}^{\perp_v}$ and called the *orthogonal complement* of $\mathcal{S}$. One of the fundamental theorems of vector space theory states that, $(\mathcal{S}^{\perp_v})^{\perp_v}$ is the subspace *spanned* by $\mathcal{S}$. Thus the "spanning" operator on the subsets of a vector space is the square of the "orthogonal complement" operator.

In matroid theory, the orthogonal complement of a matroid M is also well-defined and similarly results in another matroid. Although this new matroid is more commonly referred to as the 'dual matroid', denoted as M^*, and typically formed using a very different approach. There is an interesting relation between the circuits of a matroid M and the cocircuits of M (the circuits of its dual matroid M^*) which aligns much more closely to the orthogonal complement of a vector space.

We expand on this relation to define a powerset operator: $(\ )^*$. Given $\mathcal{S} \subseteq \mathcal{P}(E)$, we denote $\mathcal{S}^*$ to be the minimal sets of $\{X \subseteq E \colon X \text{ is nonempty}, |X \cap A| \neq 1 \text{ for each } A \in \mathcal{S}\}$. We call this powerset operator the **circuit duality operator**. Unlike the vector space orthogonal complement operator, this circuit duality operator may not behave as nicely when applied to collections that do not correspond to a matroid.

This thesis is an investigation into the development of tools and additional operators to help understand the collections of sets that result in a matroid under one or more applications of the circuit duality operator.

Applications of Powerset Operators, Especially to Matroids

by

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Chapter 1

Introduction

Matroids are the main objects of study in matroid theory. A matroid M over a finite ground set E can be represented by several different collections of subsets of E. There are several mappings that are fundamental in moving between the various collections describing a matroid. More generally, I'll refer to mappings between collections of subsets of E as 'powerset operators'.

One operator of particular interest is the mapping between the 'circuit' and 'cocircuit' collections of a matroid, which we call the 'circuit duality' operator. This thesis is an investigation into the consequences of applying these matroid operators to more arbitrary collections of sets, not just collections corresponding to matroids. A more focused goal is identifying the main features of collections of sets for which the application, or repeated applications, of the circuit duality operator results in a matroid 'spanned' by the collection; something akin to how two applications of the vector space orthogonal operator to any set of vectors results in a subspace spanned by the set.

A few of our main results are the following: We show that iterative applications of circuit duality becomes two periodic after finitely many steps (theorem 1 in section 3.2). We also give some interactions of our matroid powerset operators (proposition 3.2 in section 3.3.3). We give conditions for when applications of some of our matroid operators result in a collection that represents a matroid (theorems 2, 3 and 4 in sections 4.1.1, 4.1.2 and 4.2 respectively). We also give a family of collections that are stable under two applications of the circuit duality operator, but do not correspond to a matroid (theorem 6 in section 5.2).

Chapter 1 provides some background and preliminaries of matroid theory as well as introduces the many powerset operators used throughout this thesis. We also describe the matroid “spanning” problem and initial observations of our ‘circuit duality’ operator.

Chapter 2 includes a more in depth look into the inner workings of the circuit duality operator and list some basic properties of it.

Chapter 3 focuses on extending the other matroid operators into powerset operators and how they interact with one another. In section 3.2 we show that the sequence of iterative applications to an arbitrary collection of sets becomes two periodic after a finite number of applications.

Chapter 4 gives conditions for some of our matroid operators; that is, conditions for when a collection of subsets represents a matroid after the application of a powerset operator. We also give an evolving commuting diagram on various classes of collections with matroid operators between them.

Chapter 5 gives examples of collections that are non-matroidial even after applications of the circuit duality operator.

Chapter 6 introduces the notion of a ‘spanning subcollection’ and provides a characterization for the spanning subcollections of two particular uniform collections. We also investigate the effects of the circuit duality operator when applied to the collection of lines of a finite projective plane.

Appendix A lists the proofs of the operator properties from section 1.1.3 as well as some miscellaneous operator interactions and properties.

1.1 Preliminaries

1.1.1 The Beginnings of Matroid Theory

Matroids were originally introduced independently in 1935 by Hassler Whitney and Takeo Nakasawa, though Nakasawa's work was largely unknown for several decades. Matroid theory is an abstraction of linear algebra consisting of combinatorial objects, called matroids, whose structures generalize the notion of linear independence.

When Whitney first defined a matroid, he used the column vectors of a matrix over some field as his catalyst. Letting E be a finite set, he gave an axiomatic definition for when a collection of subsets of E were to be considered 'independent'. Any set not in this collection was considered 'dependent'. With the notion of independence defined, he then defined notions of rank and bases akin to the ones used in linear algebra. Whitney then gave an equivalent axiomatic system for the collections of bases of a matroid:

Definition 1. *A* ***matroid*** $M = (E, \mathcal{B})$ *consists of a finite set* E*, called the* ***ground set****, and a family* $\mathcal{B} = \mathcal{B}(M)$ *of subsets of* E *such that*

(B1) $\mathcal{B} \neq \emptyset$

(B2) *If* $B_1, B_2 \in \mathcal{B}$*, then* $|B_1| = |B_2|$*.*

(B3) *If* $B_1, B_2 \in \mathcal{B}$ *and* $x \in B_1 - B_2$*, then there is an element* $y \in B_2 - B_1$ *such that* $B_1 - \{x\} \cup \{y\} \in \mathcal{B}$*.*

Members of $\mathcal{B}(M)$ are called the bases of the matroid M. The family of independent sets of the matroid M, denoted by $\mathcal{I}(M)$, consists of all the subsets of the bases and the bases are the maximal independent sets under set-inclusion. Whitney noticed that the collection of complements of the bases of a matroid, $\mathcal{B}^* = \{E - B \colon B \in \mathcal{B}\}$, also satisfied the basis axioms. This new matroid was called the dual matroid and was denoted by M^*.

Later on, in 1942, William Tutte independently developed matroid theory from a very different perspective. Tutte began by working with subspaces of the vector space $\mathbb{F}^E$, the set of functions from the finite set E into the field $\mathbb{F}$. The support of a function f in this vector space is $\text{supp}(f) = \{x \in E \colon f(x) \neq 0\}$. Given a subspace $\mathcal{U} \subseteq \mathbb{F}^E$, Tutte considered the family, $\mathcal{S} = \sigma(\mathcal{U})$, of minimal nonempty sets in the collection of all supports of functions in $\mathcal{U}$. He called these sets 'cycles' and thought of them as the minimally dependent sets of a vector space.

Tutte then gave a set of axioms for these cycles which generated a vector space-like structure, which he called a 'net'. From these axioms, he then worked out the notion of 'independence' - a subset of E was 'independent' if it contained no cycle. Tutte then discovered that the collection of 'independent' sets of a net satisfied Whitney's definition for the independent sets of a matroid. Hence, Whitney's 'matroids' and Tutte's 'nets' were really one and the same. A cycle in a graph consists of the vertices and edges; the set of edges of a cycle is then an elementary circuit in the matroid of that graph. The circuit axioms of a matroid are as follows:

Definition 2. *A* ***matroid*** $M = (E, \mathcal{C})$ *consists of a finite set* E*, called the* ***ground set****, and a family* $\mathcal{C} = \mathcal{C}(M)$ *of subsets of* E *such that*

(C1) $\emptyset \notin \mathcal{C}$

(C2) *If* $C_1, C_2 \in \mathcal{C}$ *and* $C_1 \subseteq C_2$*, then* $C_1 = C_2$*.*

(C3) *If* $C_1, C_2 \in \mathcal{C}$ *with* $C_1 \neq C_2$ *and* $x \in C_1 \cap C_2$*, then* $C_3 \subseteq C_1 \cup C_2 - \{x\}$ *for some* $C_3 \in \mathcal{C}$*.*

Members of $\mathcal{C}(M)$ are called the circuits of the matroid M. The family of dependent sets of the matroid M are the supersets of the circuits and the circuits are the minimal dependent sets under set-inclusion.

Tutte's entry into duality came about in a very different manner than Whitney's. The vector space that Tutte had been working in has the usual inner product: for $f, g \in \mathbb{F}^E$, $f \cdot g = \sum_{x \in E} f(x)g(x)$. Thus, a subspace $\mathcal{U}$ has an orthogonal subspace $\mathcal{U}^{\perp_v} = \{g \in \mathbb{F}^E \colon f \cdot g = 0 \text{ for all } f \in \mathcal{U}\}$. Note that $\sum_{x \in E} f(x)g(x) = 0$ if and only if $\sum_{x \in \text{supp}(f) \cap \text{supp}(g)} f(x)g(x) = 0$;

which is not possible when $\operatorname{supp}(f) \cap \operatorname{supp}(g)$ is just a single element. Tutte noticed that the cycles of $\mathcal{U}^{\perp_v}$ could be constructed from the cycles of $\mathcal{U}$. And so he was able to prove that $\sigma(\mathcal{U}^{\perp_v})$ were the minimal nonempty sets of $\{X \subseteq E \colon |X \cap A| \neq 1 \text{ for each } A \in \mathcal{S}\}$.

The surprising fact is that Whitney's dual matroid and Tutte's orthogonal matroid are one and the same!

1.1.2 Definitions and Notation

Let E be a finite set and let $\mathcal{P}(E)$, called the **powerset** of E, denote the collection of all subsets of E. We'll refer to the elements of $\mathcal{P}(E)$ as **sets** and subsets of $\mathcal{P}(E)$ as **collections**. We also consider $\mathcal{P}(\mathcal{P}(E))$, the class of all collections of subsets of E which we denote with $\mathcal{P}^2(E)$ or simply $\mathcal{P}^2$ when E is understood. We'll refer to the elements of $\mathcal{P}^2$ as **collections** and subsets of $\mathcal{P}^2$ as **families** or **classes**.

We'll use $\mathcal{P}_k(E)$ to denote the collection of all subsets of E of size k, $\mathcal{P}_{\leq k}(E)$ to denote the collection of all subsets of E of size at most k, and $\mathcal{P}_{\geq k}(E)$ to denote the collection of all subsets of E of size at least k.

A **clutter** (also known as a *Sperner family*) is a collection of sets in which none of the sets contain any other. We denote the class of all clutters on E as $\mathcal{L}^2(E)$ or simply $\mathcal{L}^2$. The class of all circuit collections of a matroid with ground set E will be denoted by $\mathcal{C}^2(E)$ or simply $\mathcal{C}^2$ and the class of all basis collections of a matroid with ground set E will be denoted by $\mathcal{B}^2(E)$ or simply $\mathcal{B}^2$. Other particular classes of interest will be noted as they come up.

A function $\Box \colon \mathcal{P}^2 \to \mathcal{P}^2$ is called a **powerset operator**. We denote its application by exponentiation: $\Box(\mathcal{S}) = \mathcal{S}^{\Box}$. A collection $\mathcal{S}$ is called a **stable collection** of $\Box$ when $\mathcal{S} = \mathcal{S}^{\Box}$ and is called a **dual stable collection** of $\Box$ when $\mathcal{S} = (\mathcal{S}^{\Box})^{\Box}$. The family of stable collections for $\Box$ will be denoted by $\mathbb{S}(\Box)$ and the family of dual stable collections for $\Box$ will be denoted by $\mathbb{D}(\Box)$.

If M is a matroid, then we'll use $\mathcal{I}(M)$ to denote its collection of independent sets, $\mathcal{D}(M)$ to denote its collection of dependent sets, $\mathcal{B}(M)$ to denote its collection of bases, and $\mathcal{C}(M)$

to denote its collection of circuits.

1.1.3 Cast of Characters

We have two tools that have proven to be quite valuable throughout our findings. Although they are not powerset operators themselves, we present them here nonetheless: Let $\mathcal{S}$ and $\mathcal{T}$ be collections of subsets of a finite set E.

$$\textbf{Support:} \quad \operatorname{supp}(\mathcal{S}) := \{x \in E \colon x \in A \text{ for some } A \in \mathcal{S}\}$$

$$\textbf{Inner Pairwise Union:} \quad \mathcal{S} \overset{\times}{\cup} \mathcal{T} := \{A \cup B \colon A \in \mathcal{S} \text{ and } B \in \mathcal{T}\} \cup \mathcal{S} \cup \mathcal{T}$$

We have several powerset operators that we'll be studying throughout. We list them all now so that they may be referenced later. First up are the *simple* operators: Let $\mathcal{S}$ be a collection of subsets of a finite set E.

$$\textbf{Complementary:} \quad \mathcal{S}^{-} := \{E - A \colon A \in \mathcal{S}\}$$

$$\textbf{Complementation:} \quad \mathcal{S}^{c} := \{X \subseteq E \colon X \notin \mathcal{S}\}$$

$$\textbf{Minimality:} \quad \lfloor \mathcal{S} \rfloor := \{X \in \mathcal{S} \colon X \not\supset A \text{ for each } A \in \mathcal{S}\}$$

$$\textbf{Maximality:} \quad \lceil \mathcal{S} \rceil := \{X \in \mathcal{S} \colon X \not\subset A \text{ for each } A \in \mathcal{S}\}$$

$$\textbf{Superset Closure:} \quad \mathcal{S}^{\supseteq} := \{X \subseteq E \colon X \supseteq A \text{ for some } A \in \mathcal{S}\}$$

$$\textbf{Subset Closure:} \quad \mathcal{S}^{\subseteq} := \{X \subseteq E \colon X \subseteq A \text{ for some } A \in \mathcal{S}\}$$

$$\textbf{Union Closure:} \quad \mathcal{S}^{u} := \{X \subseteq E \colon X \text{ is a union of one or more sets in } \mathcal{S}\}$$

$$\textbf{Intersection Closure:} \quad \mathcal{S}^{i} := \{X \subseteq E \colon X \text{ is an intersection of one or more sets in } \mathcal{S}\}$$

$$\textbf{Meet:} \quad \mathcal{S}^{m} := \{X \subseteq E \colon |X \cap A| \neq 0 \text{ for each } A \in \mathcal{S}\}$$

$$\textbf{Orthogonality:} \quad \mathcal{S}^{\perp} := \{X \subseteq E \colon X \text{ is nonempty, } |X \cap A| \neq 1 \text{ for each } A \in \mathcal{S}\}$$

$$\textbf{cycles-to-independent:} \quad \mathcal{S}^{\mathcal{I}} := \{X \subseteq E \colon X \not\supseteq A \text{ for each } A \in \mathcal{S}\}$$

$$\textbf{bases-to-dependent:} \quad \mathcal{S}^{\mathcal{D}} := \{X \subseteq E \colon X \nsubseteq A \text{ for each } A \in \mathcal{S}\}$$

The next set of powerset operators are compositions of simple operators. We refer to them as *complex* operators:

$$\textbf{Blocking:} \quad \mathcal{S}^{b} := \lfloor \mathcal{S}^{m} \rfloor$$

$$\textbf{Circuit Duality:} \quad \mathcal{S}^{*} := \lfloor \mathcal{S}^{\perp} \rfloor$$

$$\textbf{cycles-to-bases:} \quad \mathcal{S}^{\mathcal{B}} := \lceil \mathcal{S}^{\mathcal{I}} \rceil$$

$$\textbf{bases-to-cycles:} \quad \mathcal{S}^{\mathcal{C}} := \lfloor \mathcal{S}^{\mathcal{D}} \rfloor$$

There are two particular clutters that we call trivial clutters: the empty clutter, denoted with $\emptyset$, and the clutter consisting of only the empty set, denoted with $\{\emptyset\}$. These clutters can be a bit finicky when it comes to our operators. For the most part, any clutter we consider throughout this paper will be a nontrivial clutter unless noted otherwise. That being said, we should observe how our operators behave on these trivial clutters for completeness.

First up is the empty collection $\emptyset$:

- $\text{supp}(\emptyset) = \emptyset^{-} = \lfloor \emptyset \rfloor = \lceil \emptyset \rceil = \emptyset^{\supseteq} = \emptyset^{\subseteq} = \emptyset^{u} = \emptyset^{i} = \emptyset$
- $\emptyset^{c} = \emptyset^{m} = \emptyset^{\mathcal{I}} = \emptyset^{\mathcal{D}} = \mathcal{P}(E)$
- $\emptyset^{\perp} = \mathcal{P}(E) - \{\emptyset\}$
- $\emptyset^{b} = \emptyset^{\mathcal{C}} = \{\emptyset\}$
- $\emptyset^{*} = \mathcal{P}_1(E)$
- $\emptyset^{\mathcal{B}} = \{E\}$

Now we consider $\{\emptyset\}$, the clutter consisting of just the empty set:

- $\text{supp}(\{\emptyset\}) = \{\emptyset\}^{m} = \{\emptyset\}^{b} = \{\emptyset\}^{\mathcal{I}} = \{\emptyset\}^{\mathcal{B}} = \emptyset$
- $\lfloor \{\emptyset\} \rfloor = \lceil \{\emptyset\} \rceil = \{\emptyset\}^{\subseteq} = \{\emptyset\}^{u} = \{\emptyset\}^{i} = \{\emptyset\}$
- $\{\emptyset\}^{c} = \{\emptyset\}^{\perp} = \{\emptyset\}^{\mathcal{D}} = \mathcal{P}(E) - \{\emptyset\}$

- $\{\emptyset\}^* = \{\emptyset\}^{\mathcal{C}} = \mathcal{P}_1(E)$

- $\{\emptyset\}^- = \{E\}$

- $\{\emptyset\}^{\supseteq} = \mathcal{P}(E)$

We now proceed with giving various properties of each of our powerset operators here. As to not smother the core findings of this thesis with elementary proofs of these basic properties, the proofs for each of the operators' properties will be listed in the appendix A.1.

Lemma 1.1 (Support Operator Properties). *Let $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(E)$.*

(a) $A \subseteq \operatorname{supp}(\mathcal{S})$ *for all* $A \in \mathcal{S}$

(b) $\operatorname{supp}(\mathcal{S}) = \bigcup_{A \in \mathcal{S}} A$

(c) $\mathcal{S} \subseteq \mathcal{T} \implies \operatorname{supp}(\mathcal{S}) \subseteq \operatorname{supp}(\mathcal{T})$

(d) $\operatorname{supp}(\mathcal{S} \cup \mathcal{T}) = \operatorname{supp}(\mathcal{S}) \cup \operatorname{supp}(\mathcal{T})$

(e) $\operatorname{supp}(\mathcal{S} \cap \mathcal{T}) \subseteq \operatorname{supp}(\mathcal{S}) \cap \operatorname{supp}(\mathcal{T})$ ***Reverse containment does not necessarily hold.***

Lemma 1.2 (Complementary Operator Properties). *Let $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(E)$.*

(a) $\mathcal{S}^{--} = \mathcal{S}$.

(b) $\mathcal{S} \subseteq \mathcal{T} \implies \mathcal{S}^- \subseteq \mathcal{T}^-$

(c) $(\mathcal{S} \cup \mathcal{T})^- = \mathcal{S}^- \cup \mathcal{T}^-$

(d) $(\mathcal{S} \cap \mathcal{T})^- = \mathcal{S}^- \cap \mathcal{T}^-$

(e) $|\mathcal{S}| = |\mathcal{S}^-|$

Lemma 1.3 (Complementation Operator Properties). *Let $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(E)$.*

(a) $\mathcal{S}$ *and* $\mathcal{S}^c$ *partition* $\mathcal{P}(E)$.

(b) $\mathcal{S}^{cc} = \mathcal{S}$

(c) $\mathcal{S} \subseteq \mathcal{T} \implies \mathcal{T}^c \subseteq \mathcal{S}^c$

(d) $(\mathcal{S} \cup \mathcal{T})^c = \mathcal{S}^c \cap \mathcal{T}^c$

(e) $(\mathcal{S} \cap \mathcal{T})^c = \mathcal{S}^c \cup \mathcal{T}^c$

Lemma 1.4 (Minimality Operator Properties)**.** *Let* $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(E)$*.*

(a) $\lfloor \mathcal{S} \rfloor \subseteq \mathcal{S}$

(b) $\lfloor \lfloor \mathcal{S} \rfloor \rfloor = \lfloor \mathcal{S} \rfloor$

(c) $\lfloor \mathcal{S} \cup \mathcal{T} \rfloor \subseteq \lfloor \mathcal{S} \rfloor \cup \lfloor \mathcal{T} \rfloor$

(d) $\emptyset \in \lfloor \mathcal{S} \rfloor \iff \lfloor \mathcal{S} \rfloor = \{\emptyset\} \iff \emptyset \in \mathcal{S}$

(e) If $\emptyset \notin \mathcal{S}$*, then for each* $A \in \mathcal{S}$*, there exists a nonempty* $A' \in \lfloor \mathcal{S} \rfloor$ *such that* $A' \subseteq A$*.*

(f) Minimality is not inclusion preserving nor inclusion reversing in general.

(g) $\lfloor \mathcal{S} \rfloor = \emptyset \iff \mathcal{S} = \emptyset$

(h) $\lfloor \mathcal{P}_{\geq k}(E) \rfloor = \mathcal{P}_k(E)$

Lemma 1.5 (Maximality Operator Properties)**.** *Let* $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(E)$*.*

(a) $\lceil \mathcal{S} \rceil \subseteq \mathcal{S}$

(b) $\lceil \lceil \mathcal{S} \rceil \rceil = \lceil \mathcal{S} \rceil$

(c) $\lceil \mathcal{S} \cup \mathcal{T} \rceil \subseteq \lceil \mathcal{S} \rceil \cup \lceil \mathcal{T} \rceil$

(d) $\emptyset \in \lceil \mathcal{S} \rceil \iff \lceil \mathcal{S} \rceil = \{\emptyset\} \iff \mathcal{S} = \{\emptyset\}$

(e) If $\emptyset \notin \mathcal{S}$*, then for each* $A \in \mathcal{S}$*, there exists a nonempty* $\hat{A} \in \lceil \mathcal{S} \rceil$ *such that* $A \subseteq \hat{A}$*.*

(f) Maximality is not inclusion preserving nor inclusion reversing in general.

(g) $\lceil \mathcal{S} \rceil = \emptyset \iff \mathcal{S} = \emptyset$

(h) $\lceil \mathcal{P}_{\leq k}(E) \rceil = \mathcal{P}_k(E)$

Lemma 1.6 (Superset Closure Operator Properties)**.** *Let* $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(E)$*. Let* $X, Y, A_k \subseteq E$ *for* $1 \leq k \leq n$*.*

(a) $\mathcal{S} \subseteq \mathcal{S}^{\supseteq}$

(b) $\mathcal{S} \subseteq \mathcal{T} \implies \mathcal{S}^{\supseteq} \subseteq \mathcal{T}^{\supseteq}$

(c) $\emptyset \in \mathcal{S}^{\supseteq} \iff \emptyset \in \mathcal{S} \iff \mathcal{S}^{\supseteq} = \mathcal{P}(E)$

(d) $\emptyset \neq \mathcal{S} \iff E \in \mathcal{S}^{\supseteq}$

(e) $(\mathcal{S}^{\supseteq})^{\supseteq} = \mathcal{S}^{\supseteq}$

(f) $(\mathcal{S} \cup \mathcal{T})^{\supseteq} = \mathcal{S}^{\supseteq} \cup \mathcal{T}^{\supseteq}$

(g) $(\mathcal{S} \cap \mathcal{T})^{\supseteq} \subseteq \mathcal{S}^{\supseteq} \cap \mathcal{T}^{\supseteq}$ ***Reverse containment does not necessarily hold.***

(h) $Y \not\subseteq X \implies \mathcal{P}(X) \cap \{Y\}^{\supseteq} = \emptyset$

(i) $\displaystyle\bigcap_{k=1}^{n} \{A_k\}^{\supseteq} = \left\{ \bigcup_{k=1}^{n} A_k \right\}^{\supseteq}$

Lemma 1.7 (Subset Closure Operator Properties)**.** *Let* $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(E)$*. Let* $X, Y, A_k \subseteq E$ *for* $1 \leq k \leq n$*.*

(a) $\mathcal{S} \subseteq \mathcal{S}^{\subseteq}$

(b) $\mathcal{S} \subseteq \mathcal{T} \implies \mathcal{S}^{\subseteq} \subseteq \mathcal{T}^{\subseteq}$

(c) $E \in \mathcal{S}^{\subseteq} \iff E \in \mathcal{S} \iff \mathcal{S}^{\subseteq} = \mathcal{P}(E)$

(d) $\emptyset \neq \mathcal{S} \iff \emptyset \in \mathcal{S}^{\subseteq}$

(e) $(\mathcal{S}^{\subseteq})^{\subseteq} = \mathcal{S}^{\subseteq}$

(f) $(\mathcal{S} \cup \mathcal{T})^{\subseteq} = \mathcal{S}^{\subseteq} \cup \mathcal{T}^{\subseteq}$

(g) $(\mathcal{S} \cap \mathcal{T})^{\subseteq} \subseteq \mathcal{S}^{\subseteq} \cap \mathcal{T}^{\subseteq}$ **Reverse containment does not necessarily hold.**

(h) $\{X\}^{\subseteq} = \mathcal{P}(X)$

(i) $\mathcal{S}^{\subseteq} = \bigcup_{k=1}^{n} \mathcal{P}(S_k),$ where $S_k \in \mathcal{S}$

(j) $\bigcap_{k=1}^{n} \{A_k\}^{\subseteq} \subseteq \left\{ \bigcup_{k=1}^{n} A_k \right\}^{\subseteq}$ **Reverse containment does not necessarily hold.**

Lemma 1.8 (Union Closure Operator Properties)**.** *Let* $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(E)$.

(a) $\mathcal{S} \subseteq \mathcal{S}^u$

(b) $\mathcal{S} \subseteq \mathcal{T} \implies \mathcal{S}^u \subseteq \mathcal{T}^u$

(c) $(\mathcal{S}^u)^u = \mathcal{S}^u$

(d) $(\mathcal{S} \cup \mathcal{T})^u = (\mathcal{S}^u \cup \mathcal{T}^u)^u$

(e) $(\mathcal{S} \cap \mathcal{T})^u \subseteq (\mathcal{S}^u \cap \mathcal{T}^u)$ **Reverse containment does not necessarily hold.**

(f) $(\mathcal{S} \cup \{\emptyset\})^u = \mathcal{S}^u \cup \{\emptyset\}$

(g) $\mathrm{supp}(\mathcal{S}) \in \mathcal{S}^u$

Lemma 1.9 (Intersection Closure Operator Properties)**.** *Let* $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(E)$.

(a) $\mathcal{S} \subseteq \mathcal{S}^i$

(b) $\mathcal{S} \subseteq \mathcal{T} \implies \mathcal{S}^i \subseteq \mathcal{T}^i$

(c) $(\mathcal{S}^i)^i = \mathcal{S}^i$

(d) $(\mathcal{S} \cup \mathcal{T})^i = (\mathcal{S}^i \cup \mathcal{T}^i)^i$

(e) $(\mathcal{S} \cap \mathcal{T})^i \subseteq (\mathcal{S}^i \cap \mathcal{T}^i)$ **Reverse containment does not necessarily hold.**

Lemma 1.10 (Meet Operator Properties)**.** *Let* $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(E)$.

(a) $\emptyset \in \mathcal{S} \iff \mathcal{S}^m = \emptyset$

(b) $\mathcal{S} \subseteq \mathcal{T} \implies \mathcal{T}^m \subseteq \mathcal{S}^m$

(c) $(\mathcal{S} \cup \mathcal{T})^m = \mathcal{S}^m \cap \mathcal{T}^m$

(d) $\mathcal{S}^m = \mathcal{T}^m \implies (\mathcal{S} \cup \mathcal{T})^m = \mathcal{S}^m$ ***Reverse direction does not necessarily hold.***

Lemma 1.11 (Orthogonality Operator Properties). *Let $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(E)$.*

(a) $\emptyset \notin \mathcal{S}^\perp$

(b) $\mathcal{S} \subseteq \{\emptyset\} \iff \mathcal{S}^\perp = \mathcal{P}(E) - \{\emptyset\}$

(c) $\mathcal{S} \subseteq \mathcal{T} \implies \mathcal{T}^\perp \subseteq \mathcal{S}^\perp$

(d) $(\mathcal{S} - \{\emptyset\})^\perp = \mathcal{S}^\perp = (\mathcal{S} \cup \{\emptyset\})^\perp$

(e) $(\mathcal{S} \cup \mathcal{T})^\perp = \mathcal{S}^\perp \cap \mathcal{T}^\perp$

Lemma 1.12 (cycles-to-independent Operator Properties). *Let $\mathcal{S} \subseteq \mathcal{P}(E)$.*

(a) $\mathcal{S}^\mathcal{I} = \mathcal{S}^{\supseteq c}$

(b) *$\mathcal{S}$ and $\mathcal{S}^\mathcal{I}$ are disjoint.*

(c) $\mathcal{S}^\mathcal{I} = \emptyset \iff \emptyset \in \mathcal{S}$

(d) $\emptyset \in \mathcal{S}^\mathcal{I} \iff \emptyset \notin \mathcal{S}$

Lemma 1.13 (bases-to-dependent Operator Properties). *Let $\mathcal{S} \subseteq \mathcal{P}(E)$.*

(a) $\mathcal{S}^\mathcal{D} = \mathcal{S}^{\subseteq c}$

(b) *$\mathcal{S}$ and $\mathcal{S}^\mathcal{D}$ are disjoint.*

(c) $\mathcal{S}^\mathcal{D} = \emptyset \iff E \in \mathcal{S}$

(d) $E \in \mathcal{S}^\mathcal{D} \iff E \notin \mathcal{S}$

Lemma 1.14 (Inner Pairwise Union Operator Properties)**.** *Let* $\mathcal{S}, \mathcal{T}, \mathcal{X}, \mathcal{Y} \subseteq \mathcal{P}(E)$*.*

(a) $\mathcal{S}, \mathcal{T} \subseteq \mathcal{S} \overset{\times}{\cup} \mathcal{T}$

(b) $\mathcal{S} \subseteq \mathcal{X}$ *and* $\mathcal{T} \subseteq \mathcal{Y} \implies \mathcal{S} \overset{\times}{\cup} \mathcal{T} \subseteq \mathcal{X} \overset{\times}{\cup} \mathcal{Y}$

(c) $(\mathcal{S} \overset{\times}{\cup} \mathcal{T}) \overset{\times}{\cup} \mathcal{U} = \mathcal{S} \overset{\times}{\cup} (\mathcal{T} \overset{\times}{\cup} \mathcal{U})$

1.2 The "Spanning" Problem for Matroids

Let $\mathcal{V}$ denote a vector space over an arbitrary field with an inner product. Two vectors u and v are said to be *orthogonal* when $u \cdot v = 0$. If $\mathcal{U}$ is a subspace of $\mathcal{V}$, then $\mathcal{U}^{\perp_v}$, the collection of all vectors orthogonal to each vector in $\mathcal{U}$, is also a subspace of $\mathcal{V}$ and is called the *orthogonal complement* of $\mathcal{U}$. One of the fundamental theorems of vector space theory states that, for any subspace $\mathcal{U} \subseteq \mathcal{V}$, we have $(\mathcal{U}^{\perp_v})^{\perp_v} = \mathcal{U}$. In proving this theorem, a slightly stronger result is proved, namely: for any collection $\mathcal{S}$ of vectors from $\mathcal{V}$, $(\mathcal{S}^{\perp_v})^{\perp_v}$ is the subspace *spanned* by $\mathcal{S}$. Thus the "spanning" operator on the subsets of a vector space is the square of the "orthogonal complement" operator.

In matroid theory, the orthogonal complement of a matroid M is also well-defined and similarly results in another matroid. Although this new matroid is more commonly referred to as the 'dual matroid', denoted as M^*, and typically formed using a very different approach. However, there is an interesting relation between the circuits of a matroid M and the cocircuits of M (the circuits of its dual matroid M^*) which aligns much more closely to the orthogonal complement of a vector space. In fact, the cocircuits of a matroid can be obtained from the collection of circuits of the matroid using this exact relation:

Proposition 1.1. [Oxl11, Proposition 2.1.23]. *Let M be a matroid having a ground set E. Then D is a circuit of M if and only if D is a minimal non-empty subset of E such that $|D \cap C^*| \neq 1$ for every cocircuit C^* of M.*

We expand the domain of this isomorphism between the circuit and cocircuits of a matroid to include all collections of subsets of E and define a powerset operator: $(\ \ )^*$. Given $\mathcal{S} \subseteq \mathcal{P}(E)$, we denote $\mathcal{S}^*$ to be the minimal sets of $\mathcal{S}^\perp = \{X \subseteq E \colon X \text{ is nonempty}, |X \cap A| \neq 1 \text{ for each } A \in \mathcal{S}\}$. We call this powerset operator the **circuit duality operator**. Unlike the vector space orthogonal complement operator, this circuit duality operator may not behave as nicely when applied to collections that do not correspond to a matroid.

This thesis is an investigation into the development of tools and additional operators to help understand the collections of sets that result in a matroid under one or more applications of the circuit duality operator.

1.2.1 An Overlap between Vector Spaces and Matroids

Let E be a finite set and let $\mathcal{P}(E)$ denote the collection of all subsets of E. There is a natural vector space over $\mathbb{Z}_2$ on $\mathcal{P}(E)$. Namely $(\mathcal{P}(E), \triangle)$, where $\triangle$ is defined to be the symmetric difference of two sets: $A \triangle B = (A \cup B) - (A \cap B)$. Given any subspace $\mathcal{V}$ of $(\mathcal{P}(E), \triangle)$, the collection of minimal nonempty sets in $\mathcal{V}$ form the circuits of a matroid. Matroids formed in this manner are called **binary matroids**.

There is also a natural inner product associated to the vector space $(\mathcal{P}(E), \triangle)$ which is given by $A \cdot B = |A \cap B| \pmod 2$. Thus, two sets are orthogonal if their intersection has even cardinality. And so orthogonality in this vector space directly defines a powerset operator on $\mathcal{P}(E)$: Given $\mathcal{S} \subseteq \mathcal{P}(E)$, we have

$$\mathcal{S}^{\perp_v} = \{X \subseteq E \colon |X \cap A| \equiv 0 \pmod 2 \text{ for each } A \in \mathcal{S}\}$$

Thus, for any $\mathcal{S} \subseteq \mathcal{P}(E)$, the orthogonal complement $\mathcal{S}^{\perp_v}$ is a subspace of $(\mathcal{P}(E), \triangle)$. And so its collection of minimal nonempty sets are the circuits of some binary matroid. We can then define a vector space 'duality' operator that is parallel to our circuit duality operator.

$$\mathcal{S}^{*_v} = \lfloor \mathcal{S}^{\perp_v} - \{\emptyset\} \rfloor$$

Just how closely these two operators are related becomes apparent when we define them directly:

$\mathcal{S}^*$ is the collection of minimal non-empty sets that intersect no set of $\mathcal{S}$ in **exactly one element**.

$\mathcal{S}^{*_v}$ is the collection of minimal non-empty sets that intersect no set of $\mathcal{S}$ in **an odd number of elements**.

For any collection $\mathcal{S} \subseteq \mathcal{P}(E)$, it follows that $(\mathcal{S}^{*_v})^{*_v}$ is the smallest binary matroid containing the sets in $\mathcal{S}$ as cycles (or dependent sets if they are not minimal in $\mathcal{S}$). This leads us to ask: Does our circuit duality operator follow similarly? Is $(\mathcal{S}^*)^*$ a matroid, for any $\mathcal{S} \subseteq \mathcal{P}(E)$? If not, what kind of structure is it? Specifically, given $\mathcal{C}$, the circuits of a matroid, which subcollections $\mathcal{S} \subseteq \mathcal{C}$ **span** $\mathcal{C}$ $[(\mathcal{S}^*)^* = \mathcal{C}]$?

1.3 A Rough First Look

Let's dive head first into the circuit duality operator and see it in action.

Definition 3. *Given a collection $\mathcal{S}$, we consider the collection of minimal nonempty sets (under set containment) that do not intersect any of the sets in the collection $\mathcal{S}$ in exactly one element, denoted as $\mathcal{S}^*$. We call this operator the* ***circuit duality operator****.*

1.3.1 Matroid Duality is Preserved

As per our discussion earlier in section 1.2, we have that the circuit duality operator preserves matroid duality. This was given more precisely as proposition 1.1. In particular, if $\mathcal{C}(M)$ is the collection of circuits of a matroid M, then applying the circuit duality operator yields $\mathcal{C}^*(M)$, the collection of cocircuits of M. It is well known in matroid theory [GM12, Proposition 3.18] that $\mathcal{C}^*(M) = \mathcal{C}(M^*)$ where M^* is the dual matroid to M.

Moreover, applying the circuit duality operator twice results in the original collection of circuits: $\mathcal{C}^{**}(M) = \mathcal{C}(M^{**}) = \mathcal{C}(M)$ since $M = M^{**}$ for matroids. Hence the name: *circuit* duality operator.

Definition 4. *If a collection $\mathcal{X}$ represents the circuit collection of a matroid, then we'll say that $\mathcal{X}$ is* ***matroidial****.*

We'll talk more about matroidial collections throughout and give some conditions for when a collection is matroidial in chapter 4. Alternative methods for matroidial completions of clutters can be found in [Mar14].

1.3.2 Duality is Not Immediate

Example 1. *Let $\mathcal{S} = \{\{abd\}, \{ace\}, \{bce\}, \{cd\}, \{de\}\}$ be a "random" collection and let's repeatedly apply the circuit duality operator to it (see table 1.1). Here we are working over a five element ground set: $E = \{a, b, c, d, e\}$. (For simplicity we write 'abd' to represent the set $\{a, b, d\}$.)*

	Sets								
$\mathcal{S}$			abd		ace	bce	cd		de
$\mathcal{S}^*$					acde	bcde			
$\mathcal{S}^{**}$		abc	abd	abe			cd	ce	de
$\mathcal{S}^{***}$	ab				acde	bcde			
$\mathcal{S}^{****}$		abc	abd	abe			cd	ce	de

Table 1.1: Circuit duality computations for example 1.

In this example, we note that the second and fourth iterations of circuit duality yield the same collection. If we computed the fifth iteration, we would see that it would be equal to the third iteration. Moreover, any further iterations would simply alternate between these two collections: $\mathcal{S}^{**}$ and $\mathcal{S}^{***}$. It appears that the duality of our operator began to 'stabilize'.

Definition 5. *We say a collection is a* ***frame*** *when* $\mathcal{X} = \mathcal{X}^{**}$*.*

So in this example, $\mathcal{S}^{**}$ is a frame since $\mathcal{S}^{**} = \mathcal{S}^{****}$. Notably, if $\mathcal{X}$ is a frame, then $\mathcal{X}^{*}$ is also a frame. We denote the class of all frames as $\mathcal{R}^2$. We'll talk more about frames throughout this thesis. What we call 'frames' are referred to as 'semimatroids' in [Vad86].

We also note that matroidial collections are inherently frames by design:

Corollary 1 (Corollary to proposition 1.1)**.** *If* $\mathcal{S}$ *is matroidial, then* $\mathcal{S}$ *is a frame.*

This corollary gives us a containment of classes - the class of all circuit collections of a matroid is contained in the class of all frames; that is $\mathcal{C}^2 \subseteq \mathcal{R}^2$. In particular, $\mathcal{S}^{**}$ from the example above is matroidial since it satisfies the circuit axioms for a matroid (definition 2). In fact it is the **cycle matroid** of the graph on three vertices with two single edges a and b and a triple edge c, d, and e.

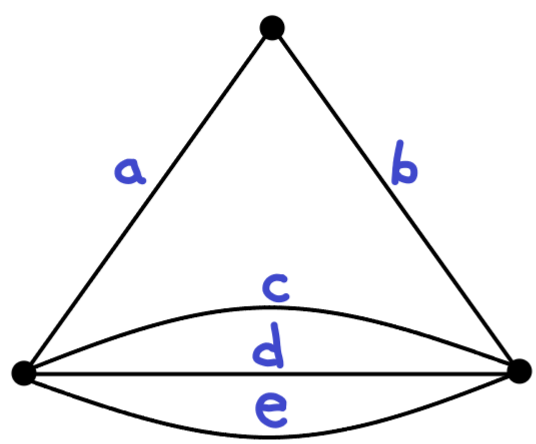


Figure 1.1: Graph corresponding to $\mathcal{S}^{**}$ in table 1.1.

Example 2. *Once again, let* $\mathcal{S}$*, given below, be a "random" collection and let's repeatedly apply the circuit duality operator to it. We are again working over a five element ground set:* $E = \{a, b, c, d, e\}$*.*

	Sets
$\mathcal{S}$	abc abd acd ade bce
$\mathcal{S}^*$	abcd abce abde acde bcde
$\mathcal{S}^{**}$	abc abd abe acd ace ade bcd bce bde cde
$\mathcal{S}^{***}$	abcd abce abde acde bcde

Table 1.2: Circuit duality computations for example 2.

In this example, we see that a collection gets repeated after three iterations - thus $\mathcal{S}^*$ is a frame. This collection is also matroidial - it corresponds to the **uniform matroid** $\mathcal{U}_{3,5}$ where the collection of circuits are all four element subsets of a five element ground set.

We remark that in the first example, our original collection $\mathcal{S}$ is not contained in the matroidial collection $\mathcal{S}^{**}$. But in the second example, we do have this containment.

Definition 6. *A subcollection $\mathcal{R} \subseteq \mathcal{S}$ is called a* ***spanning subcollection*** *(of $\mathcal{S}$) when $\mathcal{R}^* = \mathcal{S}^*$.*

Depending on the classification of $\mathcal{S}$ we can garner more information and ideal conditions for when circuit duality acts as a true matroid spanning operator: If $\mathcal{S}$ is a frame, then $\mathcal{R} \subseteq \mathcal{R}^{**}$ and $\mathcal{R}^{**}$ is a frame. If $\mathcal{S}$ is matroidial, then $\mathcal{R} \subseteq \mathcal{R}^{**}$ and $\mathcal{R}^{**}$ is matroidial.

Thus, we can see that the circuit duality operator can in fact generate matroidial collections. It can also maintain the containment of the original collection within the generated matroidial collection. It would seem that the circuit duality operator has potential to be a spanning operator for matroids, but extra conditions will certainly be required to do so.

1.3.3 Naturally Arising Questions

Following our observations and discussions of the two examples above, a few very natural questions come to mind:

Question 1. *If a collection is a frame, does it always correspond to the circuit collection of some matroid?*

If so, then we could potentially generate matroids from arbitrary collections using this circuit duality operator. However, we saw in the example above that it took a few iterations before we arrived at a pair of 'dual' collections. This begs the next natural question:

Question 2a. *Does the repeated application of the circuit duality operator always 'stabilize' (become 2-periodic)? That is, do we always converge to an alternating sequence of a pair of frames?*

Question 2b. *If so, is there an upper bound to the number of iterations needed before stability/duality (convergence) occurs?*

Additionally, we note that in the first example $\mathcal{S}$ only shares a few of the same sets as $\mathcal{S}^{**}$. Ideally we want $\mathcal{S}$ to be fully contained in $\mathcal{S}^{**}$ (like in the second example) and for $\mathcal{S}^{**}$ to be matroidial. So we ask:

Question 3a. *Under what conditions do we ensure that a collection is contained in its second dual?*

Question 3b. *If given any collection, what is the best relation between it and its second dual?*

Chapter 2

The Circuit Duality Operator

To better understand what the circuit duality operator is doing and discover its properties, we split it into the composition of the following two operators: the orthogonality operator and the minimality operator.

Orthogonality: $\mathcal{S}^{\perp} := \{X \subseteq E \colon X \text{ is nonempty}, |X \cap A| \neq 1 \text{ for each } A \in \mathcal{S}\}$

Minimality: $\lfloor \mathcal{S} \rfloor := \{X \in \mathcal{S} \colon X \not\supset A \text{ for each } A \in \mathcal{S}\}$

Circuit Duality: $\mathcal{S}^* = \lfloor \mathcal{S}^{\perp} \rfloor$

2.1 Core Operators: Orthogonality and Minimality

In section 1.1.3 we gave basic properties of all of our powerset operators. For now we'll focus on the minimality operator (lemma 1.4) and the orthogonality operator (lemma 1.11). These both have interactions with the support operator (lemma 1.1) as well as the inner pairwise union operator (lemma 1.14). In regards to the support operator, we say a collection is **fully supported** or **has full support** when $\text{supp}(\mathcal{S}) = E$.

2.1.1 Minimality, Support, and Inner Pairwise Unions

We'll first look at some collaboration that the minimality operator has with the supports and inner pairwise unions of collections:

Lemma 2.1. *Let* $\mathcal{S} \subseteq \mathcal{P}(E)$.

(a) $\emptyset \notin \mathcal{S} \cup \mathcal{T}$ *and* $\operatorname{supp}(\lfloor \mathcal{S} \rfloor) \cap \operatorname{supp}(\lfloor \mathcal{T} \rfloor) = \emptyset \implies \lfloor \mathcal{S} \cup \mathcal{T} \rfloor = \lfloor \mathcal{S} \rfloor \cup \lfloor \mathcal{T} \rfloor$

(b) $\operatorname{supp}(\mathcal{S} \overset{\times}{\cup} \mathcal{T}) = \operatorname{supp}(\mathcal{S} \cup \mathcal{T})$

(c) $\lfloor \mathcal{S} \overset{\times}{\cup} \mathcal{T} \rfloor = \lfloor \mathcal{S} \cup \mathcal{T} \rfloor$

Proof.

(a) Minimality properties always gives us one containment: $\lfloor \mathcal{S} \cup \mathcal{T} \rfloor \subseteq \lfloor \mathcal{S} \rfloor \cup \lfloor \mathcal{T} \rfloor$.

To show the reverse containment let $X \in \lfloor \mathcal{S} \rfloor \cup \lfloor \mathcal{T} \rfloor$. Without loss of generality let $X \in \lfloor \mathcal{S} \rfloor$. Then $X \in \mathcal{S}$ and $X \not\supset A$ for all $A \in \mathcal{S}$.

Suppose (by contradiction) that there exists $B \in \mathcal{T}$ such that $X \supset B$. Since $\emptyset \notin \mathcal{S} \cup \mathcal{T}$ then $\emptyset \notin \mathcal{T}$. Thus, there exists a nonempty $B' \in \lfloor \mathcal{T} \rfloor$ such that $X \supset B \supseteq B'$. Note that $B' \subseteq \operatorname{supp}(\lfloor \mathcal{T} \rfloor)$ and $X \subseteq \operatorname{supp}(\lfloor \mathcal{S} \rfloor)$. However, the two supports are disjoint and so we have $B' \cap X = \emptyset$. In particular, $X \not\supset B'$; a contradiction! Therefore, $X \not\supset B$ for all nonempty $B \in \mathcal{T}$.

Thus, $X \in \mathcal{S} \cup \mathcal{T}$ and $X \not\supset C$ for all $C \in (\mathcal{S} \cup \mathcal{T})$ which implies that $X \in \lfloor \mathcal{S} \cup \mathcal{T} \rfloor$. Hence, $\lfloor \mathcal{S} \rfloor \cup \lfloor \mathcal{T} \rfloor \subseteq \lfloor \mathcal{S} \cup \mathcal{T} \rfloor$. Therefore, $\lfloor \mathcal{S} \cup \mathcal{T} \rfloor = \lfloor \mathcal{S} \rfloor \cup \lfloor \mathcal{T} \rfloor$ as desired.

(b) ($\rightarrow$) Since $(\mathcal{S} \cup \mathcal{T}) \subseteq (\mathcal{S} \overset{\times}{\cup} \mathcal{T})$ and the support operator is inclusion preserving, then we have $\operatorname{supp}(\mathcal{S} \cup \mathcal{T}) \subseteq \operatorname{supp}(\mathcal{S} \overset{\times}{\cup} \mathcal{T})$.

($\leftarrow$) Let $x \in \operatorname{supp}(\mathcal{S} \overset{\times}{\cup} \mathcal{T})$, then $x \in D$ for some $D \in \mathcal{S} \overset{\times}{\cup} \mathcal{T}$. If $D \in \mathcal{S} \cup \mathcal{T}$ then $x \in \operatorname{supp}(\mathcal{S} \cup \mathcal{T})$ and we are done. If $D = A \cup B$, then $x \in A$ or $x \in B$. And so $x \in \operatorname{supp}(\mathcal{S})$ or $x \in \operatorname{supp}(\mathcal{T})$ which are both subsets of $\operatorname{supp}(\mathcal{S} \cup \mathcal{T})$. Hence, $x \in \operatorname{supp}(\mathcal{S} \cup \mathcal{T})$ and so $\operatorname{supp}(\mathcal{S} \overset{\times}{\cup} \mathcal{T}) \subseteq \operatorname{supp}(\mathcal{S} \cup \mathcal{T})$.

(c) $(\rightarrow)$ Let $X \in \lfloor \mathcal{S} \overset{\times}{\cup} \mathcal{T} \rfloor$, then $X \in \mathcal{S} \overset{\times}{\cup} \mathcal{T}$ and $X \not\supset D$ for all $D \in \mathcal{S} \overset{\times}{\cup} \mathcal{T}$. Since $\mathcal{S} \cup \mathcal{T} \subseteq \mathcal{S} \overset{\times}{\cup} \mathcal{T}$, then $X \not\supset C$ for all $C \in \mathcal{S} \cup \mathcal{T}$. In particular, X cannot be the union of two proper subsets, one from $\mathcal{S}$ and one from $\mathcal{T}$. Thus, we must have that $X \in \mathcal{S} \cup \mathcal{T}$ and therefore $X \in \lfloor \mathcal{S} \cup \mathcal{T} \rfloor$.

$(\leftarrow)$ Let $X \in \lfloor \mathcal{S} \cup \mathcal{T} \rfloor$. It follows that $X \subseteq \mathcal{S} \cup \mathcal{T} \subseteq \mathcal{S} \overset{\times}{\cup} \mathcal{T}$ and $X \not\supset C$ for all $C \in \mathcal{S} \cup \mathcal{T}$. Take any $D \in (\mathcal{S} \overset{\times}{\cup} \mathcal{T}) - (\mathcal{S} \cup \mathcal{T})$, then $D = A \cup B$ for some $A \in \mathcal{S}$ and $B \in \mathcal{T}$. Neither A nor B are proper subsets of X and so their union, D, cannot be a proper subset of X. Hence, $X \not\supset D$ for all $D \in \mathcal{S} \overset{\times}{\cup} \mathcal{T}$. Therefore, $X \in \lfloor \mathcal{S} \overset{\times}{\cup} \mathcal{T} \rfloor$.

□

2.1.2 Orthogonality Breakout View

The orthogonality operator is a bit more involved than minimality and has slightly more complex properties (lemma 1.11) like inclusion reversing, a De Morgan-like law on unions, and disregards the empty set as a member of the collection.

If we unwrap the set notation for $\mathcal{S}^\perp$ a little bit more, we see that it is really the union of three disjoint groups: sets that miss every set of $\mathcal{S}$, sets that meet some set(s) of $\mathcal{S}$ in two or more positions and miss the others, and sets that meet every set of $\mathcal{S}$ in two or more positions. More precisely, these groups are:

$$G_1 = \{X \in \mathcal{S}^\perp \colon |X \cap A| = 0 \text{ for each } A \in \mathcal{S}\}$$

$$G_2 = \{X \in \mathcal{S}^\perp \colon |X \cap A| = 0 \text{ and } |X \cap B| \geq 2 \text{ for some } A, B \in \mathcal{S}\}$$

$$G_3 = \{X \in \mathcal{S}^\perp \colon |X \cap A| \geq 2 \text{ for each } A \in \mathcal{S}\}$$

This more separated view allows us to make some observations about when any of these groups might be empty. Thus, making future calculations slightly easier.

Lemma 2.2. *Observe the following scenarios:*

(a) $G_1 = \emptyset \iff \mathcal{S}$ *is fully supported.*

(b) $G_3 = \emptyset \iff \mathcal{S}$ *contains a singleton or the empty set.*

(c) $\mathcal{S}^\perp$ *contains a singleton* $\iff \mathcal{S}$ *is not fully supported.*

(d) $\operatorname{supp}(\mathcal{S}) \in \mathcal{S}^\perp \iff \mathcal{S}$ *is singleton free.*

(e) $\{x\} \in \mathcal{S} \implies x \notin \operatorname{supp}(\mathcal{S}^\perp)$

Proof.

(a) $(\leftarrow)$ Let $\mathcal{S}$ be a fully supported collection and suppose (by contradiction) that $G_1 \neq \emptyset$. Then there exists $X \in \mathcal{P}(E) - \{\emptyset\}$ such that $|X \cap A| = 0$ for all $A \in \mathcal{S}$. However, this implies that $X \cap \operatorname{supp}(\mathcal{S}) = X \cap E = \emptyset$. Since $X \subseteq E$ (by definition), then it follows that $X = \emptyset$, a contradiction! Hence, $\mathcal{S}$ being fully supported implies $G_1 = \emptyset$.

$(\rightarrow)$ Let $G_1 = \emptyset$ and suppose (by contradiction) that $\mathcal{S}$ is not fully supported. In particular, $E - \operatorname{supp}(\mathcal{S}) \neq \emptyset$. Now let $X = E - \operatorname{supp}(\mathcal{S})$ and note that X is nonempty. It follows that $|X \cap A| = 0$ for all $A \in \mathcal{S}$ and so $X \in G_1$. Thus, $G_1 \neq \emptyset$, a contradiction! Hence, $G_1 = \emptyset$ implies $\mathcal{S}$ being fully supported.

(b) $(\leftarrow)$ Let $A \in \mathcal{S}$ and suppose that A is a singleton or the empty set. Note that for all $X \in \mathcal{P}(E) - \{\emptyset\}$ we have $|X \cap A| \leq |A| \leq 1$. Thus, in order for $X \in \mathcal{S}^\perp$, we must have $|X \cap A| = 0$. And so $X \notin G_3$ which further implies that $G_3 = \emptyset$.

$(\rightarrow)$ Let $G_3 = \emptyset$ and suppose (by contradiction) that $\mathcal{S}$ contains no singletons nor the empty set. In particular, $|A| \geq 2$ for all $A \in \mathcal{S}$. However, note that $|E \cap A| = |A| \geq 2$ for all $A \in \mathcal{S}$. Thus, $E \in G_3$ which implies $G_3 \neq \emptyset$, a contradiction! Hence, $G_3 = \emptyset$ implies $\mathcal{S}$ contains a singleton or the empty set. Thus, the if and only if holds.

(c) $(\leftarrow)$ If $\mathcal{S}$ is not fully supported, then $E - \operatorname{supp}(\mathcal{S}) \neq \emptyset$. Let $x \in E - \operatorname{supp}(\mathcal{S})$ and note that $x \notin A$ for all $A \in \mathcal{S}$. Hence, $|A \cap \{x\}| = 0 \neq 1$ for all $A \in \mathcal{S}$. Thus, $\{x\} \in \mathcal{S}^\perp$

and so $\mathcal{S}^\perp$ contains a singleton.

($\rightarrow$) If $\mathcal{S}^\perp$ contains a singleton, call it $\{x\}$, then for each $A \in \mathcal{S}$ we have that $|A \cap \{x\}| \neq 1$. But this implies that $|A \cap \{x\}| = 0$ for each $A \in \mathcal{S}$ and so $x \notin A$ for all $A \in \mathcal{S}$. Therefore, $x \notin \text{supp}(\mathcal{S})$. Hence, $\text{supp}(\mathcal{S}) \neq E$ and so $\mathcal{S}$ is not fully supported.

(d) First observe that $\text{supp}(\mathcal{S}) \in \mathcal{S}^\perp$ if and only if $|\text{supp}(\mathcal{S}) \cap A| \neq 1$ for all $A \in \mathcal{S}$. Furthermore, each set in $\mathcal{S}$ is a subset of the support and so $|\text{supp}(\mathcal{S}) \cap A| = |A|$. Thus, $|A| \neq 1$ if and only if A is not a singleton which holds if and only if $\mathcal{S}$ is singleton free.

(e) Suppose that $\mathcal{S}$ contains a singleton $\{x\}$. For each $A \in \mathcal{S}^\perp$ we have that $|A \cap \{x\}| \neq 1$. Moreover, $|A \cap \{x\}| \geq |\{x\}|$ which implies that $|A \cap \{x\}| = 0$. Thus, $x \notin A$ for all $A \in \mathcal{S}^\perp$ and therefore $x \notin \text{supp}(\mathcal{S}^\perp)$.

□

2.1.3 Orthogonality on Uniform Collections

A uniform collection is the collection of all subsets of E of a particular size. We'll use $\mathcal{P}_k(E)$ to denote a uniform collection consisting of sets of size k. Uniform collections will appear regularly throughout our investigations. We take some time now to see how the orthogonality operator deals with them. We first consider the peculiar case of the collections of all singletons of E.

Lemma 2.3. *Let E be a finite set. Then, $(\mathcal{P}_1(E))^\perp = \emptyset$.*

Proof. For each $x \in E$, $\{x\} \in \mathcal{P}_1(E)$. It follows from part (e) of lemma 2.2 that $x/ \notin \text{supp}((\mathcal{P}_1(E))^\perp)$. But then the support of $(\mathcal{P}_1(E))^\perp$ is empty which holds if and only if $(\mathcal{P}_1(E))^\perp = \emptyset$. □

We note that $\mathcal{P}_0(E) = \{\emptyset\}$. We've already seen in section 1.1.3 that $\{\emptyset\}^\perp = \mathcal{P}_{\geq 1}(E)$. We now consider the uniform collections consisting of sets sized greater than or equal to 2.

Lemma 2.4. *Let E be a finite set. For $2 \le k \le |E|$, we have $(\mathcal{P}_k(E))^\perp = \mathcal{P}_{\ge |E|-k+2}(E)$.*

Proof. Suppose (by way of contradiction) that $(\mathcal{P}_k(E))^\perp \not\subseteq \mathcal{P}_{\ge|E|-k+2}(E)$. It follows that there exists an $X \in \mathcal{P}_k(E)^\perp$ and such that $|X| < |E| - k + 2$. Thus, we have $|E - X| > |E| - (|E| - k + 2) = k - 2$. And so $|E - X| \ge k - 1$. Let $A \subseteq E - X$ such that $|A| = k - 1$ and let $x \in X$. It follows that $A \in \mathcal{P}_k(E)$. However, we also have $|X \cap (A \cup \{x\})| = |\{x\}| = 1$; a contradiction! Hence, $(\mathcal{P}_k(E))^\perp \subseteq \mathcal{P}_{\ge|E|-k+2}(E)$.

On the other hand, let $X \in \mathcal{P}_k(E)$ and $Y \in \mathcal{P}_{|E|-k+2}(E)$ and observe the following: $|E| \ge |X \cup Y| = |X| + |Y| - |X \cap Y| = k + (|E| - k + 2) - |X \cap Y|$. And so $|X \cap Y| \ge 2$, which now implies that $Y \in (\mathcal{P}_k(E))^\perp$. Thus, $\mathcal{P}_{\ge|E|-k+2}(E) \subseteq (\mathcal{P}_k(E))^\perp$ and therefore we have equality. □

2.1.4 Orthogonality on Small Supports

In the case where two or more ground sets are considered, we'll subscript our operators with the ground set that should be used. For example, $\mathcal{S}_F^\perp$ denotes the orthogonal collection of $\mathcal{S}$ with respect to subsets of the ground set F, while $\mathcal{S}_E^\perp$ denotes the orthogonal collection of $\mathcal{S}$ with respect to subsets of the ground set E.

Lemma 2.5 (Orthogonality over smaller supports)**.** *Let* $\mathrm{supp}(\mathcal{S}) \subseteq F \subseteq E$.

(a) $\mathcal{S}_F^\perp = \mathcal{S}_E^\perp \cap \mathcal{P}_{\ge 1}(F)$

(b) $\mathcal{S}_E^\perp = \mathcal{S}_F^\perp \overset{\times}{\cup} \mathcal{P}_{\ge 1}(E - F)$

Proof.

(a) We note that $X \in \mathcal{S}_F^\perp$ if and only if X is nonempty, $X \subseteq F \subseteq E$ and $|X \cap A| \ne 1$ for all $A \in \mathcal{S}$. This also holds if and only if $X \in \mathcal{S}_E^\perp \cap \mathcal{P}_{\ge 1}(F)$.

(b) $(\rightarrow)$ Let $X \in \mathcal{S}_E^\perp$. Then X is a nonempty subset of E and $|X \cap A| \ne 1$ for all $A \in \mathcal{S}$. If $X \cap F = \emptyset$, then $X \in \mathcal{P}_{\ge 1}(E - F) \subseteq \mathcal{S}_F^\perp \overset{\times}{\cup} \mathcal{P}_{\ge 1}(E - F)$ and we are done.

On the other hand, if $X \cap F \neq \emptyset$, then consider $X' = X \cap F$. Since $A \subseteq \text{supp}(\mathcal{S}) \subseteq F$, then it follows that

$$|X' \cap A| = |X \cap F \cap A| = |X \cap A| \neq 1$$

for all $A \in \mathcal{S}$. Hence, $X' \in \mathcal{S}_F^{\perp}$ and so

$$X = (X \cap F) \cup (X \cap (E - F)) = X' \cup (X \cap (E - F))$$

which implies that $X \in \mathcal{S}_F^{\perp} \overset{\times}{\cup} \mathcal{P}_{\geq 1}(E - F)$.

$(\leftarrow)$ Let $X \in \mathcal{S}_F^{\perp} \overset{\times}{\cup} \mathcal{P}_{\geq 1}(E - F)$. If $X \in \mathcal{S}_F^{\perp}$, then $X \in \mathcal{S}_E^{\perp}$ since $\mathcal{S}_F^{\perp} \subseteq \mathcal{S}_E^{\perp}$ from the previous part. If $X \in \mathcal{P}_{\geq 1}(E - F)$, then $\emptyset \subset X \subseteq E - F \subseteq E$ and $X \cap F = \emptyset$. Moreover, recall that for each $A \in \mathcal{S}$ we have $A \subseteq \text{supp}(\mathcal{S})$ and since $\text{supp}(\mathcal{S}) \subseteq F$, then it follows that $|X \cap A| = 0 \neq 1$ and so $X \in \mathcal{S}_E^{\perp}$.

If $X \notin \mathcal{S}_F^{\perp} \cup \mathcal{P}_{\geq 1}(E - F)$, then $X = Y \cup Z$ where $Y \in \mathcal{S}_F^{\perp}$ and $Z \in \mathcal{P}_{\geq 1}(E - F)$. Note that $\emptyset \subset Z \subseteq E - F \subseteq E - \text{supp}(\mathcal{S})$ and so $Z \cap A = \emptyset$ for all $A \in \mathcal{S}$. Thus, we have

$$|X \cap A| = |(Y \cup Z) \cap A| = |(Y \cap A) \cup (Z \cap A)| = |Y \cap A| \neq 1$$

for all $Y \in \mathcal{S}$. Hence, $X \in \mathcal{S}_E^{\perp}$.

□

2.2 Supplemental Operators: Union Closure and Superset Closure

To explore even more properties of orthogonality and minimality, we must introduce two more operators due to their strong synergies with the orthogonality and minimality operators respectively:

Union Closure: $\mathcal{S}^u := \{X \subseteq E \colon X \text{ is a union of one or more sets in } \mathcal{S}\}$

Superset Closure: $\mathcal{S}^{\supseteq} := \{X \subseteq E \colon X \supseteq A \text{ for some } A \in \mathcal{S}\}$

Basic properties of union closure (lemma 1.8) and superset closure (lemma 1.6) were introduced in section 1.1.3.

2.2.1 Interactions of Superset Closure and Minimality

First up are the relations between the superset closure and minimality operators. Part (c) of the following is given in [Oxl11, Lemma 2.1.22].

Proposition 2.1 (Minimality and Superset Closure Interactions)**.** *Let $\mathcal{S}, \mathcal{T}, \mathcal{Q} \subseteq \mathcal{P}(E)$.*

(a) $\lfloor \mathcal{S}^{\supseteq} \rfloor = \lfloor \mathcal{S} \rfloor$

(b) $\lfloor \mathcal{S} \rfloor^{\supseteq} = \mathcal{S}^{\supseteq}$

(c) $\lfloor \mathcal{S} \rfloor = \lfloor \mathcal{T} \rfloor \iff \mathcal{S} \subseteq \mathcal{T}^{\supseteq}$ *and* $\mathcal{T} \subseteq \mathcal{S}^{\supseteq}$

(d) $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{S}^{\supseteq} \implies \lfloor \mathcal{S} \rfloor = \lfloor \mathcal{T} \rfloor$

(e) $\lfloor \mathcal{S} \rfloor = \lfloor \mathcal{T} \rfloor$ *and* $\mathcal{S} \subseteq \mathcal{Q} \subseteq \mathcal{T} \implies \lfloor \mathcal{S} \rfloor = \lfloor \mathcal{Q} \rfloor = \lfloor \mathcal{T} \rfloor$

Proof.

(a) $(\leftarrow)$ Let $A' \in \lfloor \mathcal{S} \rfloor$. By definition $A' \in \mathcal{S}$ and $A' \not\supset A$ for all $A \in \mathcal{S}$. For all $B \supseteq A$, note that $A' \not\supset B$. Together, with the fact that $\mathcal{S} \subseteq \mathcal{S}^{\supseteq}$, it follows that $A' \in \lfloor \mathcal{S}^{\supseteq} \rfloor$. Thus, $\lfloor \mathcal{S} \rfloor \subseteq \lfloor \mathcal{S}^{\supseteq} \rfloor$.

$(\rightarrow)$ Let $A' \in \lfloor \mathcal{S}^{\supseteq} \rfloor$. By definition $A' \in \mathcal{S}^{\supseteq}$ and $A' \not\supset B$ for all $B \in \mathcal{S}^{\supseteq}$. In particular, $A' \not\supset A$ for all $A \in \mathcal{S}$. Moreover, we see that the only set in $\mathcal{S}^{\supseteq}$ that is contained in A' is A' itself. Thus, we must have $A' \in \mathcal{S}$. Therefore, $A' \in \lfloor \mathcal{S} \rfloor$ which now implies that $\lfloor \mathcal{S}^{\supseteq} \rfloor \subseteq \lfloor \mathcal{S} \rfloor$. And so, we have equality: $\lfloor \mathcal{S}^{\supseteq} \rfloor = \lfloor \mathcal{S} \rfloor$.

(b) ($\rightarrow$) We first note that $\lfloor \mathcal{S} \rfloor \subseteq \mathcal{S}$ and so $\lfloor \mathcal{S} \rfloor^{\supseteq} \subseteq \mathcal{S}^{\supseteq}$ since superset closure is inclusion preserving.

($\leftarrow$) We now let $B \in \mathcal{S}^{\supseteq}$. And so there exists $A \in \mathcal{S}$ such that $A \subseteq B$. Moreover, there exists $A' \in \lfloor \mathcal{S} \rfloor$ such that $A' \subseteq A$. Hence, $A' \subseteq B$ which now implies that $B \in \lfloor \mathcal{S} \rfloor^{\supseteq}$. Therefore, $\mathcal{S}^{\supseteq} \subseteq \lfloor \mathcal{S} \rfloor^{\supseteq}$ and so we have equality: $\lfloor \mathcal{S} \rfloor^{\supseteq} = \mathcal{S}^{\supseteq}$.

(c) ($\rightarrow$) Let $X \in \mathcal{S}$. If $X \in \lfloor \mathcal{S} \rfloor$, then $X \in \lfloor \mathcal{T} \rfloor \subseteq \mathcal{T} \subseteq \mathcal{T}^{\supseteq}$. If $X \notin \lfloor \mathcal{S} \rfloor$, then there exists $A' \in \lfloor \mathcal{S} \rfloor$ such that $A' \subset X$. Since $\lfloor \mathcal{S} \rfloor = \lfloor \mathcal{T} \rfloor$, then $A' \in \lfloor \mathcal{T} \rfloor \subseteq \mathcal{T}$. Hence, $X \in \mathcal{T}^{\supseteq}$. Following the same argument with $Y \in \mathcal{T}$ one can show that $Y \in \mathcal{S}^{\supseteq}$. Hence, we have shown that $\mathcal{S} \subseteq \mathcal{T}^{\supseteq}$ and $\mathcal{T} \subseteq \mathcal{S}^{\supseteq}$.

($\leftarrow$) Let $X \in \lfloor \mathcal{S} \rfloor$ then $X \in \mathcal{S}$ and $X \not\supset A$ for all $A \in \mathcal{S}$. Moreover, $X \not\supset A$ for all $A \in \mathcal{S}^{\supseteq}$. So in particular, $X \not\supset B$ for all $B \in \mathcal{T}$, since $\mathcal{T} \subseteq \mathcal{S}^{\supseteq}$. Additionally, $X \in \mathcal{T}^{\supseteq}$ since $\mathcal{S} \subseteq \mathcal{T}^{\supseteq}$. Thus, there exists $B \in \mathcal{T}$ such that $B \subseteq X$. However, since $X \not\supset B$ for all $B \in \mathcal{T}$, then we must have that $X = B$. Hence, $X \in \mathcal{T}$ and so $X \in \lfloor \mathcal{T} \rfloor$ by definition. Following the same argument with $Y \in \lfloor \mathcal{T} \rfloor$ one can show that $Y \in \lfloor \mathcal{S} \rfloor$. Hence, $\lfloor \mathcal{S} \rfloor = \lfloor \mathcal{T} \rfloor$.

(d) Since $\mathcal{S} \subseteq \mathcal{T}$, then $\mathcal{S} \subseteq \mathcal{T}^{\supseteq}$. By assumption, $\mathcal{T} \subseteq \mathcal{S}^{\supseteq}$, so by part (c) we have $\lfloor \mathcal{S} \rfloor = \lfloor \mathcal{T} \rfloor$.

(e) Since $\lfloor \mathcal{S} \rfloor = \lfloor \mathcal{T} \rfloor$, then we have $\mathcal{T} \subseteq \mathcal{S}^{\supseteq}$. And so $\mathcal{Q} \subseteq \mathcal{S}^{\supseteq}$. Since $\mathcal{S} \subseteq \mathcal{Q}$, then $\mathcal{S} \subseteq \mathcal{Q}^{\supseteq}$. Thus, by part (c) we get $\lfloor \mathcal{S} \rfloor = \lfloor \mathcal{Q} \rfloor$.

□

Although these operators completely nullify one another, as stated more precisely in parts (a) and (b) of the proposition, we also have necessary and sufficient conditions for when two collections have the same minimal collection. Additionally, we have a squeeze theorem condition for the minimality operator.

2.2.2 Interactions of Union Closure and Orthogonality

Next we look at some interesting interactions between the orthogonality and union closure operators:

Proposition 2.2 (Orthogonality and Union Closure Interactions). *Let $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(E)$.*

(a) $\mathcal{S}^{\perp} = \mathcal{S}^{\perp u} = \mathcal{S}^{u\perp}$

(b) $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{S}^{u} \implies \mathcal{S}^{\perp} = \mathcal{T}^{\perp}$

(c) $\mathcal{S}^{u} \subseteq \mathcal{S}^{\perp\perp}$

(d) $\mathcal{S}^{\perp} = \mathcal{S}^{\perp\perp\perp}$

(e) $\mathcal{S} \subseteq \lfloor \mathcal{S} \rfloor^{u} \implies \lfloor \mathcal{S} \rfloor^{\perp} = \mathcal{S}^{\perp}$

Proof.

(a) Since a collection is always contained in its union closure, then we have $\mathcal{S}^{\perp} \subseteq \mathcal{S}^{\perp u}$. Applying the orthogonality operator to $\mathcal{S} \subseteq \mathcal{S}^{u}$ gives us $\mathcal{S}^{u\perp} \subseteq \mathcal{S}^{\perp}$ since orthogonality is inclusion reversing. So by showing that $\mathcal{S}^{\perp u} \subseteq \mathcal{S}^{u\perp}$ we will get equality throughout.

Suppose (by contradiction) that $\mathcal{S}^{\perp u} \nsubseteq \mathcal{S}^{u\perp}$. Thus, there exists $A \in \mathcal{S}^{\perp u}$ and $B \in \mathcal{S}^{u}$ such that $|A \cap B| = 1$. Let x be this unique, common element. Note that $A = \bigcup_{j=1}^{n} A_j$ for some $A_j \in \mathcal{S}^{\perp}$ and $B = \bigcup_{k=1}^{m} B_k$ for some $B_k \in \mathcal{S}$. Thus, $x \in A_p \cap B_q$ for some $1 \leq p \leq n$ and $1 \leq q \leq m$. Moreover, note that $|A_j \cap B_k| \neq 1$ for all $j = 1, \ldots, n$ and $k = 1, \ldots, m$. It then follows that $|A_p \cap B_q| \geq 2$, so let $y \neq x$ be some element of $A_p \cap B_q$. This now implies that $|A \cap B| \geq 2$, a contradiction! Therefore, $\mathcal{S}^{\perp u} \subseteq \mathcal{S}^{u\perp}$ and so we have equality throughout.

(b) Since $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{S}^{u}$, then applying orthogonality yields $\mathcal{S}^{u\perp} \subseteq \mathcal{T}^{\perp} \subseteq \mathcal{S}^{\perp}$. Now by part (a), we have $\mathcal{S}^{\perp} \subseteq \mathcal{T}^{\perp} \subseteq \mathcal{S}^{\perp}$. Hence, $\mathcal{S}^{\perp} = \mathcal{T}^{\perp}$.

(c) Let $A \in \mathcal{S}^u$. By part (a) we know that $\mathcal{S}^{u\perp} = \mathcal{S}^\perp$. So for each $X \in \mathcal{S}^\perp$ we have that $|X \cap Y| \neq 1$ for all $Y \in \mathcal{S}^u$. In particular, $|X \cap A| \neq 1$ for all $X \in \mathcal{S}^\perp$. Hence, $A \in \mathcal{S}^{\perp\perp}$. Thus, we have $\mathcal{S}^u \subseteq \mathcal{S}^{\perp\perp}$.

(d) From part (c) we have $\mathcal{S} \subseteq \mathcal{S}^u \subseteq \mathcal{S}^{\perp\perp}$. Then applying the orthogonality operator yields $\mathcal{S}^{\perp\perp\perp} \subseteq \mathcal{S}^{u\perp}$. On the other hand, apply part (c) to $\mathcal{S}^\perp$ which gives us $\mathcal{S}^{\perp u} \subseteq \mathcal{S}^{\perp\perp\perp}$. Now by part (a), we have that $\mathcal{S}^\perp$ contains and is contained in $\mathcal{S}^{\perp\perp\perp}$. Hence, we have equality: $\mathcal{S}^\perp = \mathcal{S}^{\perp\perp\perp}$.

(e) Note that we have $\lfloor \mathcal{S} \rfloor \subseteq \mathcal{S} \subseteq \lfloor \mathcal{S} \rfloor^u$. Applying orthogonality then gives us that $\lfloor \mathcal{S} \rfloor^{u\perp} \subseteq \mathcal{S}^\perp \subseteq \lfloor \mathcal{S} \rfloor^\perp$. Part (a) then implies the first and last collections and equal and so we have equality throughout: $\mathcal{S}^\perp = \lfloor \mathcal{S} \rfloor^\perp$.

□

These observations give us conditions for when two collections have the same orthogonal collection. We also see that the orthogonal collection is closed under unions. Part (e) of the proposition will be of particular interest when further discussing frames.

Interestingly enough, the first orthogonal collection equaling the third orthogonal collection for any $\mathcal{S} \subseteq \mathcal{P}(E)$ is rather surprising fact. Possibly even more surprising is that we saw in example example 1 that this may not be true for the circuit duality operator despite its proximity to the orthogonality operator. That is; $\mathcal{S}^*$ need not be equal to $\mathcal{S}^{***}$ despite $\mathcal{S}^\perp$ always equaling $\mathcal{S}^{\perp\perp\perp}$.

2.3 Circuit Duality Operator Observations

We now have several observations about the circuit duality operator that follow immediately:

Proposition 2.3 (Circuit Duality)**.** *Let* $\mathcal{S} \subseteq \mathcal{P}(E)$ *and* $\operatorname{supp}(\mathcal{S}) \subseteq F \subseteq E$.

(a) $(\mathcal{S} - \{\emptyset\})^* = \mathcal{S}^* = (\mathcal{S} \cup \{\emptyset\})^*$

(b) $\mathcal{S}^{\perp} \subseteq \mathcal{T}^{\perp} \subseteq \mathcal{S}^{\perp\supseteq} \implies \mathcal{S}^{*} = \mathcal{T}^{*}$

(c) $\mathcal{S} \subseteq \mathcal{Q} \subseteq \mathcal{T}$ *and* $\mathcal{S}^{*} = \mathcal{T}^{*} \implies \mathcal{S}^{*} = \mathcal{Q}^{*} = \mathcal{T}^{*}$

(d) $\mathcal{S}^{*}_{E} = \mathcal{S}^{*}_{F} \cup \mathcal{P}_{1}(E - F)$

(e) $\mathcal{S}^{**}_{E} = \mathcal{S}^{**}_{F}$

(f) $\{x\} \in \mathcal{S} \implies \mathcal{S}^{*}_{E} = (\mathcal{S} - \{\{x\}\})^{*}_{E-\{x\}}$

(g) $\{x\} \in \mathcal{S} \implies \mathcal{S}^{**}_{E} = (\mathcal{S} - \{\{x\}\})^{**}_{E-\{x\}} \cup \{\{x\}\}$

(h) $(\mathcal{P}_{1}(E))^{*} = \emptyset$

(i) *For* $2 \leq k \leq |E|$, *we have* $(\mathcal{P}_{k}(E))^{*} = \mathcal{P}_{|E|-k+2}(E)$

(j) $\mathcal{P}_{k}(E)$ *is a frame for* $1 \leq k \leq |E|$.

Proof.

(a) This follows from part (d) of lemma 1.11: $(\mathcal{S} - \{\emptyset\})^{\perp} = \mathcal{S}^{\perp} = (\mathcal{S} \cup \{\emptyset\})^{\perp}$. Applying minimality then yields $\lfloor(\mathcal{S} - \{\emptyset\})^{\perp}\rfloor = \lfloor\mathcal{S}^{\perp}\rfloor = \lfloor(\mathcal{S} \cup \{\emptyset\})^{\perp}\rfloor$ which by definition gives $(\mathcal{S} - \{\emptyset\})^{*} = \mathcal{S}^{*} = (\mathcal{S} \cup \{\emptyset\})^{*}$.

(b) This follows from part (d) of proposition 2.1: $\mathcal{S}^{\perp} \subseteq \mathcal{T}^{\perp} \subseteq \mathcal{S}^{\perp\supseteq} \implies \lfloor\mathcal{S}^{\perp}\rfloor = \lfloor\mathcal{T}^{\perp}\rfloor$ which by definition gives $\mathcal{S}^{*} = \mathcal{T}^{*}$.

(c) Since the orthogonality operator is inclusion reversing then $\mathcal{S} \subseteq \mathcal{Q} \subseteq \mathcal{T} \implies \mathcal{T}^{\perp} \subseteq \mathcal{Q}^{\perp} \subseteq \mathcal{S}^{\perp}$. By hypothesis and the definition of circuit duality, we have that $\lfloor\mathcal{S}^{\perp}\rfloor = \lfloor\mathcal{T}^{\perp}\rfloor$. Thus, by part (e) of proposition 2.1 we get that $\lfloor\mathcal{S}^{\perp}\rfloor = \lfloor\mathcal{Q}^{\perp}\rfloor = \lfloor\mathcal{T}^{\perp}\rfloor$ which by definition gives $\mathcal{S}^{*} = \mathcal{Q}^{*} = \mathcal{T}^{*}$.

(d) From part (b) of lemma 2.5 we have $\mathcal{S}^{\perp}_{E} = \mathcal{S}^{\perp}_{F} \overset{\times}{\cup} \mathcal{P}_{\geq 1}(E - F)$. Note that by definition, the empty set is not in any of the collections. Moreover, note that $\mathcal{S}^{\perp}_{F}$ and $\mathcal{P}_{\geq 1}(E - F)$ have disjoint supports. We now apply minimality to both sides of the equation:

$\lfloor \mathcal{S}_E^{\perp} \rfloor = \lfloor \mathcal{S}_F^{\perp} \overset{\times}{\cup} \mathcal{P}_{\geq 1}(E - F) \rfloor$. And so by parts (a) and (c) of lemma 2.1 it follows that $\mathcal{S}_E^* = \mathcal{S}_F^* \cup \mathcal{P}_1(E - F)$.

(e) From part (d) we have $\mathcal{S}_E^* = \mathcal{S}_F^* \cup \mathcal{P}_1(E - F)$ and so applying orthogonality yields $(\mathcal{S}_E^*)_E^{\perp} = (\mathcal{S}_F^*)_E^{\perp} \cap (\mathcal{P}_1(E - F))_E^{\perp}$. By part (b) of lemma 2.5 we get

$$(\mathcal{S}_E^*)_E^{\perp} = \Big((\mathcal{S}_F^*)_F^{\perp} \overset{\times}{\cup} \mathcal{P}_{\geq 1}(E - F) \Big) \cap \Big((\mathcal{P}_1(E - F))_{E-F}^{\perp} \overset{\times}{\cup} \mathcal{P}_{\geq 1}(F) \Big)$$

by lemma 2.3 we have that $(\mathcal{P}_1(E - F))_{E-F}^{\perp} = \emptyset$ and so the right term is simply $\mathcal{P}_{\geq 1}(F)$. Thus, we have

$$(\mathcal{S}_E^*)_E^{\perp} = \Big((\mathcal{S}_F^*)_F^{\perp} \overset{\times}{\cup} \mathcal{P}_{\geq 1}(E - F) \Big) \cap \mathcal{P}_{\geq 1}(F)$$

Note that sets in the inner pairwise union term either contain an element of $E - F$ or do not. In the latter case, the set is an element of $(\mathcal{S}_F^*)_F^{\perp}$. Since sets in $\mathcal{P}_{\geq 1}(F)$ contain no elements of $E - F$ then the intersection reduces to

$$(\mathcal{S}_E^*)_E^{\perp} = (\mathcal{S}_F^*)_F^{\perp} \cap \mathcal{P}_{\geq 1}(F)$$

Moreover, note that $(\mathcal{S}_F^*)_F^{\perp}$ is a subset of $\mathcal{P}_{\geq 1}(F)$ and so the right hand side is simply $(\mathcal{S}_F^*)_F^{\perp}$. Hence, applying minimality yields $\mathcal{S}_E^{**} = \mathcal{S}_F^{**}$.

(f) Observe that $\mathcal{S} = (\mathcal{S} - \{\{x\}\}) \cup \{\{x\}\}$ and so applying orthogonality yields $\mathcal{S}^{\perp} = (\mathcal{S} - \{\{x\}\})^{\perp} \cap \{\{x\}\}^{\perp}$. Note that $\{\{x\}\}^{\perp} = \mathcal{P}_{\geq 1}(E - \{x\})$ and so now by part (b) of lemma 2.5 we have $\mathcal{S}_E^{\perp} = \Big((\mathcal{S} - \{\{x\}\})_{E-\{x\}}^{\perp} \overset{\times}{\cup} \mathcal{P}_{\geq 1}(\{x\}) \Big) \cap \mathcal{P}_{\geq 1}(E - \{x\})$. Note that the only sets in the inner pairwise union that do not contain x are the sets in $(\mathcal{S} - \{\{x\}\})_{E-\{x\}}^{\perp}$ and so the right hand side reduces down to $(\mathcal{S} - \{\{x\}\})_{E-\{x\}}^{\perp} \cap \mathcal{P}_{\geq 1}(E - \{x\})$. Moreover, $(\mathcal{S} - \{\{x\}\})_{E-\{x\}}^{\perp}$ is a subset of $\mathcal{P}_{\geq 1}(E - \{x\})$ and so we have $\mathcal{S}_E^{\perp} = (\mathcal{S} - \{\{x\}\})_{E-\{x\}}^{\perp}$. Now taking minimal sets yields $\mathcal{S}_E^* = (\mathcal{S} - \{\{x\}\})_{E-\{x\}}^*$.

(g) From part (f) we have $\mathcal{S}_E^* = (\mathcal{S} - \{\{x\}\})_{E-\{x\}}^*$, so applying orthogonality yields $(\mathcal{S}_E^*)_E^{\perp} =$

$((\mathcal{S} - \{\{x\}\})^*_{E-\{x\}})^\perp_E$. Since $\{x\} in \mathcal{S}$, then by part (e) of lemma 2.2 we have that $x \notin \text{supp}(\mathcal{S}^*_E)$ and so $x \notin \text{supp}((\mathcal{S} - \{\{x\}\})^*_{E-\{x\}})$. Then by part (b) of lemma 2.5 we have $(\mathcal{S}^*_E)^\perp_E = ((\mathcal{S} - \{\{x\}\})^*_{E-\{x\}})^\perp_{E-\{x\}} \overset{\times}{\cup} \mathcal{P}_1(\{x\})$. Note that neither collection of the inner pairwise union contains the empty set and they have disjoint supports. Thus, by parts (a) and (c) of lemma 2.1 tell us that applying minimality yields $\lfloor(\mathcal{S}^*_E)^\perp_E\rfloor = \lfloor((\mathcal{S} - \{\{x\}\})^*_{E-\{x\}})^\perp_{E-\{x\}}\rfloor \cup \lfloor\mathcal{P}_1(\{x\})\rfloor$ which simplifies to $\mathcal{S}^{**}_E = (\mathcal{S} - \{\{x\}\})^{**}_{E-\{x\}} \cup \{\{x\}\}$.

(h) By lemma 1.11, we have $(\mathcal{P}_1(E))^\perp = \emptyset$ and so applying minimality yields $(\mathcal{P}_1(E))^* = \emptyset$.

(i) By lemma 1.11, we have $(\mathcal{P}_k(E))^\perp = \mathcal{P}_{\geq |E|-k+2}(E)$ then applying minimality yields $(\mathcal{P}_k(E))^* = \mathcal{P}_{|E|-k+2}(E)$.

(j) First consider the case where $k = 1$: From part (h), we have $(\mathcal{P}_1(E))^{**} = (\emptyset)^* = \mathcal{P}_1(E)$. Hence, $\mathcal{P}_1(E)$ is a frame.

Now suppose that $2 \leq k \leq |E|$: From part (i), we have $(\mathcal{P}_k(E))^{**} = (\mathcal{P}_{|E|-k+2}(E))^* = \mathcal{P}_{|E|-(|E|-k+2)+2}(E) = \mathcal{P}_k(E)$. Hence, $\mathcal{P}_k(E)$ is a frame.

□

From this proposition, we learned of a few generalities that we can take when it comes to finding potential frames. Part (g) tells us that singletons do not affect the dual stability of our circuit duality operator. And so we can restrict our search to singleton free collections. Part (e) tells us that the second dual of a collection over the full ground set E is the same as if taken over a smaller ground set. Thus, we'll simply take the smallest ground set we can - the support of the collection. And so we can restrict our search to fully supported collections. Part (a) tells us that circuit duality is indifferent when it comes to the empty set. Since the empty set is never in the dual collection, then we can restrict our search to collections that do not contain the empty set as an element.

A few of the more interesting results are given here as a separate proposition. Most notably, part (a) of the following proposition answers question 3b of section 1.3.3.

Proposition 2.4.

(a) $\mathcal{S} \subseteq \mathcal{S}^{*\perp}$

(b) $\lfloor \mathcal{S} \rfloor = \mathcal{S}^{**} \iff \mathcal{S}^{*\perp} \subseteq \mathcal{S}^{\supseteq}$

(c) If $\mathcal{S}$ *is a clutter, then* $\mathcal{S}$ *is a frame if and only if* $\mathcal{S}^{*\perp} \subseteq \mathcal{S}^{\supseteq}$.

(d) If $\mathcal{S}$ *is fully supported and* $A \in \mathcal{S}$ *is size two, then* $A \in \mathcal{S}^{**}$.

Remark: Though the proof for part (c) relies on a result given later (lemma 3.4 in section 3.2) we leave the result and the proof of part (c) here rather than awkwardly bring it up later on in a less appropriate setting.

Proof.

(a) Start with $\mathcal{S}^{\perp}$ and note that $\mathcal{S}^{*} = \lfloor \mathcal{S}^{\perp} \rfloor \subseteq \mathcal{S}^{\perp}$ since minimality is an intensive operator. Then apply orthogonality: the inclusion reversion yields $\mathcal{S}^{\perp\perp} \subseteq \mathcal{S}^{*\perp}$. Since union closure is extensive and part (c) of proposition 2.2 we have $\mathcal{S} \subseteq \mathcal{S}^{u} \subseteq \mathcal{S}^{\perp\perp} \subseteq \mathcal{S}^{*\perp}$.

(b) ($\leftarrow$) Suppose that $\mathcal{S}^{*\perp} \subseteq \mathcal{S}^{\supseteq}$. From part (a) and the extensive property of superset closure we have $\mathcal{S} \subseteq \mathcal{S}^{*\perp} \subseteq \mathcal{S}^{*\perp\supseteq}$. Thus, by part (c) of proposition 2.1 we get that $\lfloor \mathcal{S} \rfloor = \mathcal{S}^{**}$.

($\rightarrow$) Since $\lfloor \mathcal{S} \rfloor = \mathcal{S}^{**}$, then by part (c) of proposition 2.1, we have that $\mathcal{S} \subseteq \mathcal{S}^{*\perp\supseteq}$ and $\mathcal{S}^{*\perp} \subseteq \mathcal{S}^{\supseteq}$.

(c) This is immediate from part (b) since $\mathcal{S}$ is a clutter if and only if $\lfloor \mathcal{S} \rfloor = \mathcal{S}$.

(d) Let $\mathcal{S}$ be a fully supported collection and let $A \in \mathcal{S}$ such that $|A| = 2$. From part (a) we know that $A \in \mathcal{S}^{*\perp}$. Since $\mathcal{S}$ is fully supported, then $|A \cap B| \geq 2$ for some $B \in \mathcal{S}^{*}$. Since $|A| = 2$, then $|A \cap B| = 2$ which further implies that $A \subseteq B$. Now let W be a proper nonempty subset of A, then $|W| = 1$ and $W \subseteq B$. Thus, $|W \cap B| = 1$ which implies that $W \notin \mathcal{S}^{*\perp}$. Hence, $A \in \lfloor \mathcal{S}^{*\perp} \rfloor = \mathcal{S}^{**}$.

□

We further remark that since the orthogonal collection is closed under unions, and union closure is inclusion preserving, then part (a) from above could be improved to $\mathcal{S}^u \subseteq \mathcal{S}^{*\perp}$ for free.

Chapter 3

Matroid Operators

Before we can say more about the circuit duality operator we need to look at the properties of some other powerset operators that play well with circuit duality. Some of the matroid isomorphisms involving circuits are: circuit duality, blocking, cycles-to-bases, bases-to-cycles, and the complementary operator. Except for the complementary operator, each matroid operator is a composition of simple operators. So in order to better understand each of our matroid operators, we'll split them into their composite operators and study their interactions.

Complementary: $\mathcal{S}^- := \{E - A \colon A \in \mathcal{S}\}$

Complementation: $\mathcal{S}^c := \{X \subseteq E \colon X \notin \mathcal{S}\}$

Minimality: $\lfloor \mathcal{S} \rfloor := \{X \in \mathcal{S} \colon X \not\supset A \text{ for each } A \in \mathcal{S}\}$

Maximality: $\lceil \mathcal{S} \rceil := \{X \in \mathcal{S} \colon X \not\subset A \text{ for each } A \in \mathcal{S}\}$

Superset Closure: $\mathcal{S}^{\supseteq} := \{X \subseteq E \colon X \supseteq A \text{ for some } A \in \mathcal{S}\}$

Subset Closure: $\mathcal{S}^{\subseteq} := \{X \subseteq E \colon X \subseteq A \text{ for some } A \in \mathcal{S}\}$

Meet: $\mathcal{S}^m := \{X \subseteq E \colon |X \cap A| \neq 0 \text{ for each } A \in \mathcal{S}\}$

$$\textbf{cycles-to-independent:} \quad \mathcal{S}^{\mathcal{I}} := \{X \subseteq E \colon X \not\supseteq A \text{ for each } A \in \mathcal{S}\}$$

$$\textbf{bases-to-dependent:} \quad \mathcal{S}^{\mathcal{D}} := \{X \subseteq E \colon X \not\subseteq A \text{ for each } A \in \mathcal{S}\}$$

And so our matroid operators have the following compositions.

$$\textbf{Blocking:} \quad \mathcal{S}^{b} := \lfloor \mathcal{S}^{m} \rfloor$$

$$\textbf{bases-to-cycles:} \quad \mathcal{S}^{\mathcal{C}} := \lfloor \mathcal{S}^{\mathcal{D}} \rfloor$$

$$\textbf{cycles-to-bases:} \quad \mathcal{S}^{\mathcal{B}} := \lceil \mathcal{S}^{\mathcal{I}} \rceil$$

3.1 Composite Operators

In this section we will focus on the more interesting interactions between the composite operators. For readers with a particularly strong curiosity, we list a few additional operator interactions in appendix A.2. For basic properties of each composite operator we refer to reader to section 1.1.3. Specifically, the basic properties for each of the following can be found there: the complementary operator (lemma 1.2), the complementation operator (lemma 1.3), the minimality operator (lemma 1.4), the maximality operator (lemma 1.5), the superset closure operator (lemma 1.6), the subset closure operator (lemma 1.7), the meet operator (lemma 1.10), the cycles-to-independent operator (lemma 1.12), and the bases-to-dependent operator (lemma 1.13).

3.1.1 Simple Matroid Operator Decompositions

We'll first look at the cycles-to-independent and bases-to-dependent operators, which have a rather close relation to superset closure and subset closure respectively. These observations were initially given as part (a) of lemma 1.12 and lemma 1.13 respectively, but we list them again now for emphasis. Both proofs are listed in appendix A.1.

$$\mathcal{S}^{\mathcal{I}} = \mathcal{S}^{\supseteq c} \qquad \text{and} \qquad \mathcal{S}^{\mathcal{D}} = \mathcal{S}^{\subseteq c}$$

So it turns out that the cycles-to-independent and bases-to-dependent operators are a composition of simple operators themselves; consisting of superset closure, subset closure, and complementation. Although they should be considered 'complex' operators per our original definition, we'll continue to refer to them as simple operators or sometimes simple *matroid* operators for extra emphasis when appropriate.

3.1.2 Interactions of Maximality and Subset Closure

Similar to how superset closure and minimality go hand in hand, subset closure and maximality also have a close relationship.

Lemma 3.1 (Maximality and Subset Closure Interactions)**.** *Let* $\mathcal{S}, \mathcal{T}, \mathcal{Q} \subseteq \mathcal{P}(E)$*.*

(a) $\lceil \mathcal{S}^{\subseteq} \rceil = \lceil \mathcal{S} \rceil$

(b) $\lceil \mathcal{S} \rceil^{\subseteq} = \mathcal{S}^{\subseteq}$

(c) $\lceil \mathcal{S} \rceil = \lceil \mathcal{T} \rceil \iff \mathcal{S} \subseteq \mathcal{T}^{\subseteq}$ *and* $\mathcal{T} \subseteq \mathcal{S}^{\subseteq}$

(d) $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{S}^{\subseteq} \implies \lceil \mathcal{S} \rceil = \lceil \mathcal{T} \rceil$

(e) $\lceil \mathcal{S} \rceil = \lceil \mathcal{T} \rceil$ *and* $\mathcal{S} \subseteq \mathcal{Q} \subseteq \mathcal{T} \implies \lceil \mathcal{S} \rceil = \lceil \mathcal{Q} \rceil = \lceil \mathcal{T} \rceil$

Proof.

(a) $(\leftarrow)$ Let $\hat{A} \in \lceil \mathcal{S} \rceil$. By definition $\hat{A} \in \mathcal{S}$ and $\hat{A} \not\subset A$ for all $A \in \mathcal{S}$. For all $B \subseteq A$, note that $\hat{A} \not\subset B$. Together, with the fact that $\mathcal{S} \subseteq \mathcal{S}^{\subseteq}$, it follows that $\hat{A} \in \lceil \mathcal{S}^{\subseteq} \rceil$. Thus, $\lceil \mathcal{S} \rceil \subseteq \lceil \mathcal{S}^{\subseteq} \rceil$.

$(\rightarrow)$ Let $\hat{A} \in \lceil \mathcal{S}^{\subseteq} \rceil$. By definition $\hat{A} \in \mathcal{S}^{\subseteq}$ and $\hat{A} \not\subset B$ for all $B \in \mathcal{S}^{\subseteq}$. In particular, $\hat{A} \not\subset A$ for all $A \in \mathcal{S}$. Moreover, we see that the only set in $\mathcal{S}^{\subseteq}$ that is contained in $\hat{A}$ is $\hat{A}$ itself. Thus, we must have $\hat{A} \in \mathcal{S}$. Therefore, $\hat{A} \in \lceil \mathcal{S} \rceil$ which now implies that $\lceil \mathcal{S}^{\subseteq} \rceil \subseteq \lceil \mathcal{S} \rceil$. And so, we have equality: $\lceil \mathcal{S}^{\subseteq} \rceil = \lceil \mathcal{S} \rceil$.

(b) $(\rightarrow)$ We first note that $\lceil \mathcal{S} \rceil \subseteq \mathcal{S}$ and so $\lceil \mathcal{S} \rceil^{\subseteq} \subseteq \mathcal{S}^{\subseteq}$ since subset closure is inclusion preserving.

$(\leftarrow)$ We now let $B \in \mathcal{S}^{\subseteq}$. And so there exists $A \in \mathcal{S}$ such that $B \subseteq A$. Moreover, there exists $\hat{A} \in \lceil \mathcal{S} \rceil$ such that $A \subseteq \hat{A}$. Hence, $B \subseteq \hat{A}$ which now implies that $B \in \lceil \mathcal{S} \rceil^{\subseteq}$. Therefore, $\mathcal{S}^{\subseteq} \subseteq \lceil \mathcal{S} \rceil^{\subseteq}$ and so we have equality: $\lceil \mathcal{S} \rceil^{\subseteq} = \mathcal{S}^{\subseteq}$.

(c) $(\rightarrow)$ Let $X \in \mathcal{S}$. If $X \in \lceil \mathcal{S} \rceil$, then $X \in \lceil \mathcal{T} \rceil \subseteq \mathcal{T} \subseteq \mathcal{T}^{\subseteq}$. If $X \notin \lceil \mathcal{S} \rceil$, then there exists $\hat{A} \in \lceil \mathcal{S} \rceil$ such that $X \subset \hat{A}$. Since $\lceil \mathcal{S} \rceil = \lceil \mathcal{T} \rceil$, then $\hat{A} \in \lceil \mathcal{T} \rceil \subseteq \mathcal{T}$. Hence, $X \in \mathcal{T}^{\subseteq}$. Following the same argument with $Y \in \mathcal{T}$ one can show that $Y \in \mathcal{S}^{\subseteq}$. Hence, we have shown that $\mathcal{S} \subseteq \mathcal{T}^{\subseteq}$ and $\mathcal{T} \subseteq \mathcal{S}^{\subseteq}$.

$(\leftarrow)$ Let $X \in \lceil \mathcal{S} \rceil$ then $X \in \mathcal{S}$ and $X \not\subset A$ for all $A \in \mathcal{S}$. Moreover, $X \not\subset A$ for all $A \in \mathcal{S}^{\subseteq}$. So in particular, $X \not\subset B$ for all $B \in \mathcal{T}$, since $\mathcal{T} \subseteq \mathcal{S}^{\subseteq}$. Additionally, $X \in \mathcal{T}^{\subseteq}$ since $\mathcal{S} \subseteq \mathcal{T}^{\subseteq}$. Thus, there exists $B \in \mathcal{T}$ such that $X \subseteq B$. However, since $X \not\subset B$ for all $B \in \mathcal{T}$, then we must have that $X = B$. Hence, $X \in \mathcal{T}$ and so $X \in \lceil \mathcal{T} \rceil$ by definition. Following the same argument with $Y \in \lceil \mathcal{T} \rceil$ one can show that $Y \in \lceil \mathcal{S} \rceil$. Hence, $\lceil \mathcal{S} \rceil = \lceil \mathcal{T} \rceil$.

(d) Since $\mathcal{S} \subseteq \mathcal{T}$, then $\mathcal{S} \subseteq \mathcal{T}^{\subseteq}$. By assumption, $\mathcal{T} \subseteq \mathcal{S}^{\subseteq}$, so by part (c) we have $\lceil \mathcal{S} \rceil = \lceil \mathcal{T} \rceil$.

(e) Since $\lceil \mathcal{S} \rceil = \lceil \mathcal{T} \rceil$, then we have $\mathcal{T} \subseteq \mathcal{S}^{\subseteq}$. And so $\mathcal{Q} \subseteq \mathcal{S}^{\subseteq}$. Since $\mathcal{S} \subseteq \mathcal{Q}$, then $\mathcal{S} \subseteq \mathcal{Q}^{\subseteq}$. Thus, by part (c) we get $\lceil \mathcal{S} \rceil = \lceil \mathcal{Q} \rceil$.

□

3.1.3 Interactions between Meet, Minimality, and Superset Closure

The meet operator has some close ties to superset closure and therefore minimality.

Lemma 3.2 (Meet, Minimality, and Superset Closure Interactions)**.** *Let* $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(E)$.

(a) $\lfloor \mathcal{S} \rfloor^m = \mathcal{S}^m$

(b) $\mathcal{S}^m = \mathcal{S}^{m\supseteq} = \mathcal{S}^{\supseteq m}$

(c) $\mathcal{S}^{mm} = \mathcal{S}^{\supseteq}$

(d) $\lfloor \mathcal{S} \rfloor = \lfloor \mathcal{T} \rfloor \iff \mathcal{S}^m = \mathcal{T}^m$

Proof.

(a) We first note that when $\emptyset \in \mathcal{S}$ we have $\emptyset \in \lfloor \mathcal{S} \rfloor$. And so both meet collections are empty since they both contain the empty set. Hence, equality holds. Additionally, if $\mathcal{S} = \emptyset$, then $\lfloor \mathcal{S} \rfloor = \emptyset = \mathcal{S}$. Thus, equality holds in this case as well. So without loss of generality, we may further suppose that $\mathcal{S}$ is nonempty and does not contain the empty set.

Since the meet operator is inclusion reversing and $\lfloor \mathcal{S} \rfloor \subseteq \mathcal{S}$, then we have $\mathcal{S}^m \subseteq \lfloor \mathcal{S} \rfloor^m$.

We now aim to show the reverse inclusion: let $M \in \lfloor \mathcal{S} \rfloor^m$. It follows that $|M \cap A'| \neq 0$ for all $A' \in \lfloor \mathcal{S} \rfloor$. Note that for all $A \in \mathcal{S}$ there exists $A' \in \lfloor \mathcal{S} \rfloor$ such that $A' \subseteq A$. And so $M \cap A' \subseteq M \cap A$ which implies that $0 \neq |M \cap A'| \leq |M \cap A|$. Hence, $M \in \mathcal{S}^m$.

(b) First, note that from proposition 2.1 we have $\lfloor \mathcal{S}^{\supseteq} \rfloor = \lfloor \mathcal{S} \rfloor$. Thus, applying the meet operator yields $\lfloor \mathcal{S}^{\supseteq} \rfloor^m = \lfloor \mathcal{S} \rfloor^m$. And so by part (a) we get that $\mathcal{S}^{\supseteq m} = \mathcal{S}^m$.

Secondly, note that $\mathcal{S} \subseteq \mathcal{S}^{\supseteq}$ for all $\mathcal{S}$ and so we have $\mathcal{S}^m \subseteq \mathcal{S}^{m\supseteq}$. Now let $A \in \mathcal{S}^{m\supseteq}$. It follows that there exists $M \in \mathcal{S}^m$ such that $M \subseteq A$. Therefore, $0 \neq |M \cap X| \leq |A \cap X|$ for all $X \in \mathcal{S}$. And so $A \in \mathcal{S}^m$. Thus, we have equality throughout: $\mathcal{S}^m = \mathcal{S}^{m\supseteq} = \mathcal{S}^{\supseteq m}$.

(c) $(\leftarrow)$ Let $A \in \mathcal{S}$ and note that every set in $\mathcal{S}^m$ meets A. Changing perspectives now, we see this also implies that A meets every set in $\mathcal{S}^m$. Thus, $A \in \mathcal{S}^{mm}$ by definition which now implies that $\mathcal{S} \subseteq \mathcal{S}^{mm}$. And so by the inclusion preserving property of the superset closure operator, and by part (b), we have that $\mathcal{S}^{\supseteq} \subseteq \mathcal{S}^{mm\supseteq} = \mathcal{S}^{mm}$.

($\rightarrow$) We'll show this direction by proving that the complement holds: $\mathcal{S}^{\supseteq c} \subseteq \mathcal{S}^{mmc}$. Which holds if and only if $\mathcal{S}^{mm} \subseteq \mathcal{S}^{\supseteq}$. Let $A \notin \mathcal{S}^{\supseteq}$. Then A contains no set of $\mathcal{S}$ which implies that $E - A$ meets every set of $\mathcal{S}$. And so $E - A \in \mathcal{S}^m$. Now note that $A \cap (E - A) = \emptyset$ which implies that $A \notin \mathcal{S}^{mm}$. Therefore, we have that $\mathcal{S}^{\supseteq c} \subseteq \mathcal{S}^{mmc}$ and so by complementation we have $\mathcal{S}^{mm} \subseteq \mathcal{S}^{\supseteq}$. Hence, equality.

(d) ($\rightarrow$) Applying the meet operator to $\lfloor \mathcal{S} \rfloor = \lfloor \mathcal{T} \rfloor$ yields $\lfloor \mathcal{S} \rfloor^m = \lfloor \mathcal{T} \rfloor^m$. Now by part (a) we get $\mathcal{S}^m = \mathcal{T}^m$.

($\leftarrow$) Applying the meet operator to $\mathcal{S}^m = \mathcal{T}^m$ yields $\mathcal{S}^{mm} = \mathcal{T}^{mm}$. So by part (c) we get $\mathcal{S}^{\supseteq} = \mathcal{T}^{\supseteq}$. Now by part (c) of proposition 2.1 we get $\lfloor \mathcal{S} \rfloor = \lfloor \mathcal{T} \rfloor$.

□

3.1.4 Complementary Operator Effects

The complementary operator has some interesting interactions as well. Acting as a sort of switch, it flips an operator back and forth between its 'complementary' or 'dual' operator.

Lemma 3.3 (Complementary Operator Effects)**.** *Let* $\mathcal{S} \subseteq \mathcal{P}(E)$.

(a) $\lfloor \mathcal{S} \rfloor^- = \lceil \mathcal{S}^- \rceil$

(b) $\lfloor \mathcal{S}^- \rfloor = \lceil \mathcal{S} \rceil^-$

(c) $\mathcal{S}^{\supseteq -} = \mathcal{S}^{-\subseteq}$

(d) $\mathcal{S}^{-\supseteq} = \mathcal{S}^{\subseteq -}$

Proof.

(a) Observe the following: $W \in \lfloor \mathcal{S} \rfloor^- \iff E - W \in \lfloor \mathcal{S} \rfloor \iff E - W \in \mathcal{S}$ and $E - W \not\supset A$ for all $A \in \mathcal{S} \iff W \in \mathcal{S}^-$ and $W \not\subset E - A$ for all $E - A \in \mathcal{S}^- \iff W \in \lceil \mathcal{S}^- \rceil$.

(b) Applying the previous result to the collection $\mathcal{S}^-$ and then applying the complementary operator yields $\lfloor \mathcal{S}^- \rfloor^{--} = \lceil \mathcal{S}^{--} \rceil^-$. Since the complementary is self-inverse, then we have $\lfloor \mathcal{S}^- \rfloor = \lceil \mathcal{S} \rceil^-$.

(c) Observe the following: $W \in \mathcal{S}^{\supseteq -} \iff E - W \in \mathcal{S}^{\supseteq} \iff E - W \supseteq A$ for some $A \in \mathcal{S} \iff W \subseteq E - A$ for some $E - A \in \mathcal{S}^- \iff W \in \mathcal{S}^{-\subseteq}$.

(d) Applying the previous result to the collection $\mathcal{S}^-$ and then applying the complementary operator yields $\mathcal{S}^{-\supseteq --} = \mathcal{S}^{--\subseteq -}$. Since the complementary operator is self-inverse, we then have $\mathcal{S}^{-\supseteq} = \mathcal{S}^{\subseteq -}$.

□

3.2 Iterative Applications of Circuit Duality Converges to a Frame

With the official introduction of both minimality and maximality, we are able to take a short detour to give a quick characterization of clutters. This clutter criterion is the last tool we need to prove that iterative applications of the circuit duality operator always converges to a frame.

Lemma 3.4 (Clutter Conditions)**.** *Let $\mathcal{S} \subseteq \mathcal{P}(E)$.*

(a) $\mathcal{S}$ is a clutter if and only if $\lfloor \mathcal{S} \rfloor = \lceil \mathcal{S} \rceil = \mathcal{S}$

(b) $\lfloor \mathcal{S} \rfloor \subseteq \lceil \mathcal{S} \rceil \iff \lfloor \mathcal{S} \rfloor = \lceil \mathcal{S} \rceil \iff \lceil \mathcal{S} \rceil \subseteq \lfloor \mathcal{S} \rfloor$

Proof.

(a) Note that $\lfloor \mathcal{S} \rfloor \subseteq \mathcal{S}$ and $\lceil \mathcal{S} \rceil \subseteq \mathcal{S}$ for all $\mathcal{S} \subseteq \mathcal{P}(E)$.

($\rightarrow$) Since $\mathcal{S}$ is a clutter, then no set of $\mathcal{S}$ contains any other. Thus, each set of $\mathcal{S}$ is minimal with respect to set containment. On the other hand, if no set contains any other then no set can be contained by any other. Thus, each set of $\mathcal{S}$ is maximal with respect to set containment. Therefore, $\mathcal{S} \subseteq \lfloor \mathcal{S} \rfloor$ and $\mathcal{S} \subseteq \lceil \mathcal{S} \rceil$. And so $\lfloor \mathcal{S} \rfloor = \lceil \mathcal{S} \rceil = \mathcal{S}$.

($\leftarrow$) We note that the conclusion holds for the trivial clutters: $\lfloor \emptyset \rfloor = \emptyset = \lceil \emptyset \rceil$ and $\lfloor \{\emptyset\} \rfloor = \{\emptyset\} = \lceil \{\emptyset\} \rceil$.

Suppose that $\lfloor \mathcal{S} \rfloor = \lceil \mathcal{S} \rceil = \mathcal{S}$ and further suppose (by way of contradiction) that $\mathcal{S}$ is not a clutter. It follows that there exist distinct sets A and B in $\mathcal{S}$ such that $A \subset B$. Moreover, note that $\emptyset \notin \mathcal{S}$. If it were, then $\lfloor \mathcal{S} \rfloor = \{\emptyset\}$ which forces $\mathcal{S} = \{\emptyset\}$ in order to maintain the equality. However, this is a trivial clutter and we are supposing that $\mathcal{S}$ is not a clutter.

It then follows that there exists nonempty $A' \in \lfloor \mathcal{S} \rfloor$ such that $A' \subseteq A$. Additionally, we also have a nonempty $\hat{B} \in \lceil \mathcal{S} \rceil$ such that $B \subseteq \hat{B}$. Thus, we have $A' \subset \hat{B}$ which implies that $A' \in \lfloor \mathcal{S} \rfloor - \lceil \mathcal{S} \rceil$ and $\hat{B} \in \lceil \mathcal{S} \rceil - \lfloor \mathcal{S} \rfloor$. However, this gives us $\lfloor \mathcal{S} \rfloor \neq \lceil \mathcal{S} \rceil$; a contradiction! Therefore, $\mathcal{S}$ must be a clutter.

(b) Clearly $\lceil \mathcal{S} \rceil = \lfloor \mathcal{S} \rfloor$ implies both $\lfloor \mathcal{S} \rfloor \subseteq \lceil \mathcal{S} \rceil$ and $\lceil \mathcal{S} \rceil \subseteq \lfloor \mathcal{S} \rfloor$. Thus, we only need to show that $\lfloor \mathcal{S} \rfloor \subseteq \lceil \mathcal{S} \rceil$ and $\lceil \mathcal{S} \rceil \subseteq \lfloor \mathcal{S} \rfloor$ each individually imply equality.

Suppose that $\lfloor \mathcal{S} \rfloor \subseteq \lceil \mathcal{S} \rceil$ and further suppose (by way of contradiction) that this containment is proper. It then follows that these minimal and maximal collections do not contain the empty set. Else we get that $\{\emptyset\} = \lceil \mathcal{S} \rceil = \lfloor \mathcal{S} \rfloor$, but we are supposing that the minimal collection is properly contained in the maximal collection.

Moreover, we get that there exists $\hat{A} \in \lceil \mathcal{S} \rceil - \lfloor \mathcal{S} \rfloor$. Thus, there exists nonempty $A \in \mathcal{S}$ such that $A \subset \hat{A}$. Furthermore, there exists nonempty $A' \in \lfloor \mathcal{S} \rfloor$ such that $A' \subseteq A$. But then we have $A' \subset \hat{A}$ and so $A' \notin \lceil \mathcal{S} \rceil$; a contradiction to $\lfloor \mathcal{S} \rfloor \subseteq \lceil \mathcal{S} \rceil$. Hence, we must have $\lfloor \mathcal{S} \rfloor = \lceil \mathcal{S} \rceil$.

Similarly, suppose that $\lceil \mathcal{S} \rceil \subseteq \lfloor \mathcal{S} \rfloor$ and further suppose (by way of contradiction) that this containment is proper. Again, these minimal and maximal collections do not contain the empty set.

Furthermore, we get that there exists $A' \in \lfloor \mathcal{S} \rfloor - \lceil \mathcal{S} \rceil$. Thus, there exists nonempty $A \in \mathcal{S}$ such that $A' \subset A$. Thus, there exists nonempty $\hat{A} \in \lceil \mathcal{S} \rceil$ such that $A \subseteq \hat{A}$. But now we have $A' \subset \hat{A}$ which implies that $\hat{A} \notin \lfloor \mathcal{S} \rfloor$; a contradiction to $\lceil \mathcal{S} \rceil \subseteq \lfloor \mathcal{S} \rfloor$. Hence, we must have $\lfloor \mathcal{S} \rfloor = \lceil \mathcal{S} \rceil$.

□

We are now ready to tackle one of our initial questions from section 1.3.3. Let us introduce a relation on collections: For $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(E)$, we write $\mathcal{S} \to \mathcal{T}$ when $\mathcal{S}^{\mathcal{I}} \supseteq \mathcal{T}^{\mathcal{I}}$. Note that when $\mathcal{S}$ is a clutter we have that $\mathcal{S}$ is uniquely determined by $\mathcal{S}^{\mathcal{I}}$:

Lemma 3.5. *The powerset operator* $(\ \ )^{\mathcal{I}}$ *is injective when restricted to clutters.*

Proof. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be clutters of E such that $\mathcal{S}_1^{\mathcal{I}} = \mathcal{S}_2^{\mathcal{I}}$. By part (a) of lemma 1.12 that $\mathcal{S}_1^{\supseteq c} = \mathcal{S}_2^{\supseteq c}$. Since the complementation operator is self-inverse, then we have $\mathcal{S}_1^{\supseteq} = \mathcal{S}_2^{\supseteq}$. We now apply the minimality operator to yield $\lfloor \mathcal{S}_1^{\supseteq} \rfloor = \lfloor \mathcal{S}_2^{\supseteq} \rfloor$. It then follows from proposition 2.1 that $\lfloor \mathcal{S}_1 \rfloor = \lfloor \mathcal{S}_2 \rfloor$. But since each are clutters, then by lemma 3.4 we have $\mathcal{S}_1 = \mathcal{S}_2$. Hence, the cycles-to-independent operator is injective on clutters. □

Lemma 3.6. *The relation* $\to$ *is reflexive, transitive, and anti-symmetric when restricted to clutters.*

Proof. Reflexivity is clear:

$$\mathcal{S} \to \mathcal{S} \iff \mathcal{S}^{\mathcal{I}} \supseteq \mathcal{S}^{\mathcal{I}}$$

Transitivity is also easy enough to see: Suppose that $\mathcal{S} \to \mathcal{T}$ and $\mathcal{T} \to \mathcal{Q}$, then we have $\mathcal{S}^{\mathcal{I}} \supseteq \mathcal{T}^{\mathcal{I}}$ and $\mathcal{T}^{\mathcal{I}} \supseteq \mathcal{Q}^{\mathcal{I}}$. Thus, $\mathcal{S}^{\mathcal{I}} \supseteq \mathcal{Q}^{\mathcal{I}}$ which implies that $\mathcal{S} \to \mathcal{Q}$.

Now for anti-symmetry: Suppose that $\mathcal{S} \to \mathcal{T}$ and $\mathcal{T} \to \mathcal{S}$, then we have $\mathcal{S}^{\mathcal{I}} = \mathcal{T}^{\mathcal{I}}$. Since $\mathcal{S}, \mathcal{T} \in \mathcal{L}^2$, then both $\mathcal{S}$ and $\mathcal{T}$ are uniquely determined by $\mathcal{S}^{\mathcal{I}}$ and $\mathcal{T}^{\mathcal{I}}$. Hence, $\mathcal{S} = \mathcal{T}$. □

Lemma 3.7. *If* $\mathcal{S}$ *is a clutter, then* $\mathcal{S} \to \mathcal{S}^{**}$.

Proof. Let $X \in (\mathcal{S}^{**})^{\mathcal{I}}$ and suppose (by contradiction) that $X \notin \mathcal{S}^{\mathcal{I}}$. That is, there exists $D \in \mathcal{S}$ such that $X \supseteq D$. Note that by definition, $|D \cap C| \neq 1$ for all $C \in \mathcal{S}^*$. In particular, this means that $D \in \mathcal{S}^{*\perp}$. Thus, there exists $D' \in \mathcal{S}^{**}$ such that $D \supseteq D'$. However, this now implies that $X \supseteq D \supseteq D'$ which contradicts the fact that $X \in (\mathcal{S}^{**})^{\mathcal{I}}$. Hence, we must have $X \in \mathcal{S}^{\mathcal{I}}$. Therefore, $\mathcal{S}^{\mathcal{I}} \supseteq (\mathcal{S}^{**})^{\mathcal{I}}$ and so $\mathcal{S} \to \mathcal{S}^{**}$. □

Notation: We'll use the notation of $\mathcal{S}^{*^{(m)}}$ to denote m applications of the circuit duality operator $(\quad)^*$ on the collection $\mathcal{S}$.

The following result coincides with a more general result of [Vad86, Proposition 2.2].

Theorem 1. *Let $\mathcal{S} \subseteq \mathcal{P}(E)$. There exists $n \geq 0$ such that $\mathcal{S}^{*^{(m)}} = \mathcal{S}^{*^{(m+2)}}$ for all $m \geq n$ but $\mathcal{S}^{*^{(m)}} \neq \mathcal{S}^{*^{(m+2)}}$ for all $m < n$.*

Proof. Let $\mathcal{S} \subseteq \mathcal{P}(E)$. If $\mathcal{S}$ is not a clutter, then note that $\mathcal{S}^*$ is a clutter. So without loss of generality, we may further suppose that $\mathcal{S}$ is a clutter.

By lemma 3.7 we have the sequence of relations

$$\mathcal{S} \to \mathcal{S}^{**} \to \mathcal{S}^{*^{(4)}} \to \ldots \to \mathcal{S}^{*^{(m)}} \to \ldots$$

Thus, we have the sequence of nested subsets:

$$\mathcal{S}^{\mathcal{I}} \supseteq (\mathcal{S}^{**})^{\mathcal{I}} \supseteq (\mathcal{S}^{*^{(4)}})^{\mathcal{I}} \supseteq \ldots \supseteq (\mathcal{S}^{*^{(m)}})^{\mathcal{I}} \supseteq \ldots$$

Note that this sequence is bounded below by the empty set and bounded above by $\mathcal{S}^{\mathcal{I}}$. Hence, for $m > |\mathcal{S}^{\mathcal{I}}|$ (which is finite) we must have equality between at least two of the collections. Let n be the smallest of the superscripts where this equality holds. At this equality we have $\mathcal{S}^{*^{(n)}} \to \mathcal{S}^{*^{(n+2)}}$ and $\mathcal{S}^{*^{(n+2)}} \to \mathcal{S}^{*^{(n)}}$. Thus, by the anti-symmetry of $\to$ we have that $\mathcal{S}^{*^{(n)}} = \mathcal{S}^{*^{(n+2)}}$. □

Thus, we have answered question 2a with a resounding yes. Moreover, the proof gives a *very* rough upper bound of $|\mathcal{S}^{\mathcal{I}}|$ in response to question 2b. A similar question was asked in [Vad86, section 5] in which the author gave some partial results. In every example that I have looked at I have never encountered a m larger than 2. Thus, we propose the following conjecture:

Conjecture 1. *Let $\mathcal{S} \subseteq \mathcal{P}(E)$. Then $\mathcal{S}^{*^{(2)}} = \mathcal{S}^{*^{(4)}}$.*

3.3 Matroid Operator Observations and Properties

3.3.1 Simple Matroid Operator Interactions

We now aim to see how the simple matroid operators interact with the other simple operators.

Lemma 3.8 (Simple Matroid Operator Interactions)**.** *Let $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(E)$.*

(a) $\lfloor \mathcal{S} \rfloor^{\mathcal{I}} = \mathcal{S}^{\mathcal{I}} = \mathcal{S}^{\supseteq\mathcal{I}} = \mathcal{S}^{\mathcal{I}\subseteq}$

(b) $\lfloor \mathcal{S} \rfloor \subseteq \mathcal{T} \subseteq \mathcal{S}^{\supseteq} \implies \mathcal{S}^{\mathcal{I}} = \mathcal{T}^{\mathcal{I}}$

(c) $\lceil \mathcal{S} \rceil^{\mathcal{D}} = \mathcal{S}^{\mathcal{D}} = \mathcal{S}^{\subseteq\mathcal{D}} = \mathcal{S}^{\mathcal{D}\supseteq}$

(d) $\lceil \mathcal{S} \rceil \subseteq \mathcal{T} \subseteq \mathcal{S}^{\subseteq} \implies \mathcal{S}^{\mathcal{D}} = \mathcal{T}^{\mathcal{D}}$

Proof.

(a) Note that $\lfloor \mathcal{S} \rfloor^{\supseteq} = \mathcal{S}^{\supseteq} = \mathcal{S}^{\supseteq\supseteq}$ and so taking complements of each yield the same collection; thus $\lfloor \mathcal{S} \rfloor^{\mathcal{I}} = \mathcal{S}^{\mathcal{I}} = \mathcal{S}^{\supseteq\mathcal{I}}$. Note that $\mathcal{S}^{\mathcal{I}} \subseteq \mathcal{S}^{\mathcal{I}\subseteq}$ since subset closure is inclusion preserving. Let $W \in \mathcal{S}^{\mathcal{I}\subseteq}$. Then $W \subseteq I$ for some $I \in \mathcal{S}^{\mathcal{I}}$. Note that $A \not\subseteq I$ for all $A \in \mathcal{S}$. It then follows that $A \not\subseteq W$ for all $A \in \mathcal{S}$. Hence, $W \in \mathcal{S}^{\mathcal{I}}$ and therefore $\mathcal{S}^{\mathcal{I}\subseteq} = \mathcal{S}^{\mathcal{I}}$ which gives us equality throughout.

(b) Since the superset closure operator is inclusion preserving, then $\lfloor \mathcal{S} \rfloor \subseteq \mathcal{T} \subseteq \mathcal{S}^{\supseteq} \implies \lfloor \mathcal{S} \rfloor^{\supseteq} \subseteq \mathcal{T}^{\supseteq} \subseteq \mathcal{S}^{\supseteq\supseteq}$. Note that that first and last collections are equal, hence we have equality throughout. In particular, $\mathcal{S}^{\supseteq} = \mathcal{T}^{\supseteq}$ and so taking complements yields $\mathcal{S}^{\mathcal{I}} = \mathcal{T}^{\mathcal{I}}$.

(c) Note that $\lceil \mathcal{S} \rceil^{\subseteq} = \mathcal{S}^{\subseteq} = \mathcal{S}^{\subseteq\subseteq}$ and so taking complements of each yield the same collection; thus $\lceil \mathcal{S} \rceil^{\mathcal{D}} = \mathcal{S}^{\mathcal{D}} = \mathcal{S}^{\subseteq\mathcal{D}}$. Note that $\mathcal{S}^{\mathcal{D}} \subseteq \mathcal{S}^{\mathcal{D}\supseteq}$ since superset closure is inclusion preserving. Let $W \in \mathcal{S}^{\mathcal{D}\supseteq}$. Then $D \subseteq W$ for some $D \in \mathcal{S}^{\mathcal{D}}$. Note that $D \nsubseteq A$ for all $A \in \mathcal{S}$. It then follows that $W \nsubseteq A$ for all $A \in \mathcal{S}$. Hence, $W \in \mathcal{S}^{\mathcal{D}}$ and therefore $\mathcal{S}^{\mathcal{D}\supseteq} = \mathcal{S}^{\mathcal{D}}$ which gives us equality throughout.

(d) Since the subset closure operator is inclusion preserving, then $\lceil \mathcal{S} \rceil \subseteq \mathcal{T} \subseteq \mathcal{S}^{\subseteq} \implies \lceil \mathcal{S} \rceil^{\subseteq} \subseteq \mathcal{T}^{\subseteq} \subseteq \mathcal{S}^{\subseteq\subseteq}$ Note that that first and last collections are equal, hence we have equality throughout. In particular, $\mathcal{S}^{\subseteq} = \mathcal{T}^{\subseteq}$ and so taking complements yields $\mathcal{S}^{\mathcal{D}} = \mathcal{T}^{\mathcal{D}}$.

□

Of particular interest are the relations between the cycles-to-dependent, bases-to-dependent, complementary, and meet operators.

Proposition 3.1. *Let $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(E)$.*

(a) $\mathcal{S}^{\mathcal{I}} = \mathcal{S}^{m-}$

(b) $\mathcal{S}^{\mathcal{D}} = \mathcal{S}^{-m}$

(c) $\mathcal{S}^{\mathcal{I}\mathcal{D}} = \mathcal{S}^{\supseteq}$ *and* $\mathcal{S}^{\mathcal{D}\mathcal{I}} = \mathcal{S}^{\subseteq}$

Proof.

(a) Observe the following: $I \in \mathcal{S}^{\mathcal{I}} \iff I \nsupseteq A$ for all $A \in \mathcal{S} \iff (E - I) \cap A \neq \emptyset$ for all $A \in \mathcal{S} \iff I \in \mathcal{S}^{m-}$.

(b) Observe the following: $D \in \mathcal{S}^{\mathcal{D}} \iff D \nsubseteq A$ for all $A \in \mathcal{S} \iff D \cap (E - A) \neq \emptyset$ for all $A \in \mathcal{S} \iff D \in \mathcal{S}^{-m}$.

(c) From parts (a) and (b) above, the self-inversion of the complementary operator, and part (c) of lemma 3.2 it follows that

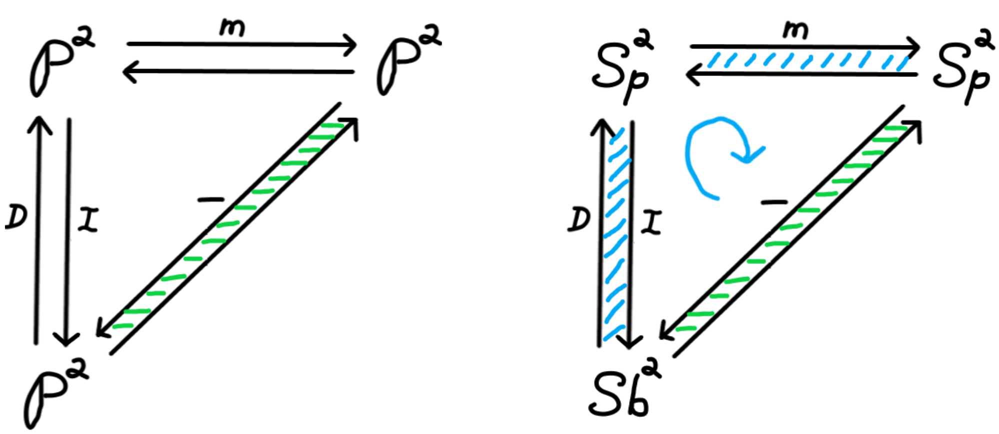
$$\mathcal{S}^{\mathcal{I}\mathcal{D}} = \mathcal{S}^{m--m} = \mathcal{S}^{mm} = \mathcal{S}^{\supseteq}$$

Additionally, from part (c) above, the self-inversion of the complementary operator, the idempotence of the subset closure, and part (d) of lemma 3.3 we have

$$\mathcal{S}^{\mathcal{D}\mathcal{I}} = \mathcal{S}^{\subseteq\mathcal{D}\mathcal{I}} = \mathcal{S}^{\subseteq-mm-} = \mathcal{S}^{\subseteq-\supseteq-} = \mathcal{S}^{\subseteq\subseteq--} = \mathcal{S}^{\subseteq}$$

□

From this proposition, parts (a) and (c) of lemma 3.8, and part (b) of lemma 3.2, we see that the meet, cycles-to-independent, bases-to-dependent, and complementary operators all commute on/between superset closed and subset closed collections. Thus, we can give a commuting diagram:



(a) All collections. (b) Restricted classes.

Figure 3.1: Commuting diagrams for the meet, cycles-to-independent, bases-to-dependent, and complementary operators.

In the left diagram, we consider all collections of subsets of a finite set represented by the class $\mathcal{P}^2$. By restricting the classes in the diagram to the class of all superset closed collections, denoted $\mathcal{S}p^2$, and the class of all subset closed collections, denoted $\mathcal{S}b^2$, then the diagram becomes a fully commuting diagram as shown in the right diagram.

3.3.2 Matroid Operator Interactions with Simple Operators

We now have several observations about our matroid operators that follow from our composite operator interactions:

Lemma 3.9 (cycles-to-bases and bases-to-cycles)**.** *Let* $\mathcal{S} \subseteq \mathcal{P}(E)$*.*

(a) $\mathcal{S}^{\mathcal{I}} = \mathcal{S}^{\mathcal{B}\subseteq}$

(b) $\mathcal{S}^{\mathcal{D}} = \mathcal{S}^{\mathcal{C}\supseteq}$

(c) $\lfloor \mathcal{S} \rfloor^{\mathcal{B}} = \mathcal{S}^{\mathcal{B}} = \mathcal{S}^{\supseteq\mathcal{B}}$

(d) $\lfloor \mathcal{S} \rfloor \subseteq \mathcal{T} \subseteq \mathcal{S}^{\supseteq} \implies \mathcal{S}^{\mathcal{B}} = \mathcal{T}^{\mathcal{B}}$

(e) $\lceil \mathcal{S} \rceil^{\mathcal{C}} = \mathcal{S}^{\mathcal{C}} = \mathcal{S}^{\subseteq\mathcal{C}}$

(f) $\lceil \mathcal{S} \rceil \subseteq \mathcal{T} \subseteq \mathcal{S}^{\subseteq} \implies \mathcal{S}^{\mathcal{C}} = \mathcal{T}^{\mathcal{C}}$

Proof.

(a) This is immediate from part (b) of lemma 3.2 and part (a) of lemma 3.8: $\lceil \mathcal{S}^{\mathcal{I}} \rceil = \mathcal{S}^{\mathcal{B}} \implies \lceil \mathcal{S}^{\mathcal{I}} \rceil^{\subseteq} = \mathcal{S}^{\mathcal{I}\subseteq} = \mathcal{S}^{\mathcal{I}} = \mathcal{S}^{\mathcal{B}\subseteq}$.

(b) This is immediate from part (b) of proposition 2.1 and part (c) of lemma 3.8: $\lfloor \mathcal{S}^{\mathcal{D}} \rfloor = \mathcal{S}^{\mathcal{C}} \implies \lfloor \mathcal{S}^{\mathcal{D}} \rfloor^{\supseteq} = \mathcal{S}^{\mathcal{D}\supseteq} = \mathcal{S}^{\mathcal{D}} = \mathcal{S}^{\mathcal{C}\supseteq}$.

(c) This follows immediately from part (a) of lemma 3.8: $\lfloor S \rfloor^{\mathcal{B}} = \lceil \lfloor \mathcal{S} \rfloor^{\mathcal{I}} \rceil = \lceil \mathcal{S}^{\mathcal{I}} \rceil = \mathcal{S}^{\mathcal{B}}$ and $\mathcal{S}^{\supseteq\mathcal{B}} = \lceil \mathcal{S}^{\supseteq\mathcal{I}} \rceil = \lceil \mathcal{S}^{\mathcal{I}} \rceil = \mathcal{S}^{\mathcal{B}}$.

(d) This follows immediately from part (b) of lemma 3.8: $\lfloor \mathcal{S} \rfloor \subseteq \mathcal{T} \subseteq \mathcal{S}^{\supseteq} \implies \mathcal{S}^{\mathcal{I}} = \mathcal{T}^{\mathcal{I}}$. Applying the maximality operator then yields $\mathcal{S}^{\mathcal{B}} = \mathcal{T}^{\mathcal{B}}$.

(e) This follows immediately from part (c) of lemma 3.8: $\lceil S \rceil^{\mathcal{C}} = \lfloor \lceil \mathcal{S} \rceil^{\mathcal{D}} \rfloor = \lfloor \mathcal{S}^{\mathcal{D}} \rfloor = \mathcal{S}^{\mathcal{C}}$ and $\mathcal{S}^{\subseteq \mathcal{C}} = \lfloor \mathcal{S}^{\subseteq \mathcal{D}} \rfloor = \lfloor \mathcal{S}^{\mathcal{D}} \rfloor = \mathcal{S}^{\mathcal{C}}$.

(f) This follows immediately from part (d) of lemma 3.8: $\lceil \mathcal{S} \rceil \subseteq \mathcal{T} \subseteq \mathcal{S}^{\subseteq} \implies \mathcal{S}^{\mathcal{D}} = \mathcal{T}^{\mathcal{D}}$. Applying the minimality operator then yields $\mathcal{S}^{\mathcal{C}} = \mathcal{T}^{\mathcal{C}}$.

□

Part (a) of the next observation was first presented in [EF70].

Lemma 3.10 (Blocking)**.** *Let* $\mathcal{S} \subseteq \mathcal{P}(E)$.

(a) $\mathcal{S}^{bb} = \lfloor \mathcal{S} \rfloor$

(b) $\lfloor \mathcal{S} \rfloor^{b} = \mathcal{S}^{b}$

(c) $\mathcal{S}^{b} = \mathcal{T}^{b} \iff \mathcal{S}^{m} = \mathcal{T}^{m}$

(d) $\mathcal{S}^{b} = \mathcal{T}^{b} \implies (\mathcal{S} \cup \mathcal{T})^{b} = \mathcal{S}^{b}$ ***Reverse direction does not necessarily hold.***

Proof.

(a) This follows from parts (a) and (c) of lemma 3.2 as well as part (a) of proposition 2.1: $\mathcal{S}^{bb} = \lfloor \lfloor \mathcal{S}^{m} \rfloor^{m} \rfloor = \lfloor \mathcal{S}^{mm} \rfloor = \lfloor \mathcal{S}^{\supseteq} \rfloor = \lfloor \mathcal{S} \rfloor$.

(b) This follows from part (a) of lemma 3.2: $\lfloor \mathcal{S} \rfloor^{b} = \lfloor \lfloor \mathcal{S} \rfloor^{m} \rfloor = \lfloor \mathcal{S}^{m} \rfloor = \mathcal{S}^{b}$.

(c) From part (c) of proposition 2.1 and part (b) of lemma 3.2 we have the following observation: $\mathcal{S}^{b} = \mathcal{T}^{b} \iff \mathcal{S}^{m} \subseteq \mathcal{T}^{m\supseteq} = \mathcal{T}^{m}$ and $\mathcal{T}^{m} \subseteq \mathcal{S}^{m\supseteq} = \mathcal{S}^{m} \iff \mathcal{S}^{m} = \mathcal{T}^{m}$.

(d) Suppose that $\mathcal{S}^{b} = \mathcal{T}^{b}$. By part (c) we get $\mathcal{S}^{m} = \mathcal{T}^{m}$. It then follows from part (d) of lemma 1.10 that $\mathcal{S}^{m} = (\mathcal{S} \cup \mathcal{T})^{m}$. Applying minimality then results in $(\mathcal{S} \cup \mathcal{T})^{b} = \mathcal{S}^{b}$.

Note: The reverse implication does not necessarily hold: Let $\mathcal{S} = \{\{a\}, \{b\}\}$ and $\mathcal{T} = \{\{a\}\}$ and $E = \{a, b\}$. It then follows that $(\mathcal{S} \cup \mathcal{T})^{b} = \mathcal{S}^{b} = \{\{ab\}\}$, however $\mathcal{T}^{b} = \{\{a\}\}$ which is incomparable to $\mathcal{S}^{b}$.

□

3.3.3 Matroid Operator Interactions with One Another

With the interesting relations between meet and cycles-to-independent and bases-to-dependent of proposition 3.1 we garnish some nice relations between blocking and cycles-to-bases and bases-to-cycles. Part (a) of the next observation coincides with [Vad86, Proposition 1.2]. **Remark:** There is actually a typo in Vaderlind's paper, the "$b_0(M)$" should really be "$\widetilde{b_0(M)}$".

Proposition 3.2 (Blocking and Matroid Operators)**.** *Let* $\mathcal{S} \subseteq \mathcal{P}(E)$*.*

(a) $\mathcal{S}^{b-} = \mathcal{S}^{\mathcal{B}}$

(b) $\mathcal{S}^{-b} = \mathcal{S}^{\mathcal{C}}$

(c) $\mathcal{S}^{\mathcal{CB}} = \lceil \mathcal{S} \rceil$ *and* $\mathcal{S}^{\mathcal{BC}} = \lfloor \mathcal{S} \rfloor$

Proof.

(a) From parts (a) of lemma 3.3 and (a) of proposition 3.1, it follows that $\mathcal{S}^{b-} = \lfloor \mathcal{S}^{m} \rfloor^{-} = \lceil \mathcal{S}^{m-} \rceil = \lceil \mathcal{S}^{\mathcal{I}} \rceil = \mathcal{S}^{\mathcal{B}}$.

(b) From part (b) of proposition 3.1, it follows that $\mathcal{S}^{-b} = \lfloor \mathcal{S}^{-m} \rfloor = \lfloor \mathcal{S}^{\mathcal{D}} \rfloor = \mathcal{S}^{\mathcal{C}}$.

(c) From part (a) of lemma 3.10, parts (a) and (b) of this result, the self-inversion of the complementary operator, and part (a) of lemma 3.3 it follows that

$$\mathcal{S}^{\mathcal{CB}} = \mathcal{S}^{b--b} = \mathcal{S}^{bb} = \lfloor \mathcal{S} \rfloor \qquad \text{and} \qquad \mathcal{S}^{\mathcal{BC}} = \mathcal{S}^{-bb-} = \lfloor \mathcal{S}^{-} \rfloor^{-} = \lceil \mathcal{S}^{--} \rceil = \lceil \mathcal{S} \rceil$$

□

With how similar this proposition is to proposition 3.1 you might expect a similar looking diagram and you would be correct. However, we will also be incorporating the circuit duality operator into our diagram and so we delay exploring commutations until section 4.3.

3.3.4 Stable and Dual Stable Families of Operators

Now that we have some results about our operators, we can give some of the stable and dual stable families for them. This first grouping consists of simple operators. The class of all union closed collections is denoted by $\mathcal{U}^2$.

	$-$	c	$\lfloor\ \rfloor$	$\lceil\ \rceil$	$\supseteq$	$\subseteq$	u	m	$\perp$
$\mathbb{S}(\square)$			$\mathcal{L}^2$	$\mathcal{L}^2$	$\mathcal{S}p^2$	$\mathcal{S}b^2$	$\mathcal{U}^2$		
$\mathbb{D}(\square)$	$\mathcal{P}^2$	$\mathcal{P}^2$						$\mathcal{S}p^2$	

Table 3.1: Stable families for simple operators.

This second grouping consists of matroid operators or combinations of matroid operators.

	b	$*$	$\mathcal{ID}$	$\mathcal{DI}$	$\mathcal{BC}$	$\mathcal{CB}$
$\mathbb{S}(\square)$			$\mathcal{S}p^2$	$\mathcal{S}b^2$	$\mathcal{L}^2$	$\mathcal{L}^2$
$\mathbb{D}(\square)$	$\mathcal{L}^2$	$\mathcal{R}^2$				

Table 3.2: Stable families for matroid operators.

Chapter 4

Matroidial Conditions and Interactions with Circuit Duality

4.1 Matroidial Conditions for the Matroid Operators

Although these matroid operators map between the various collections of a matroid, just like with the circuit duality operator, they can really be applied to any collection - not just collections associated with matroids. So it is no surprise that they can also be used to generate matroids, albeit not iteratively, but under some other conditions.

4.1.1 cycles-to-bases Results

The following proposition and theorem generalizes a known result in matroid theory [GM12, Proposition 4.7, part (1)].

Proposition 4.1. *Let* $\mathcal{S} \subseteq \mathcal{P}(E) - \{\emptyset\}$ *be a nonempty collection. For any* $B \in \mathcal{S}^{\mathcal{B}}$ *and any* $x \in E - B$*, there exists* $A_x \in \lfloor \mathcal{S} \rfloor$ *such that* $x \in A_x \subseteq B \cup \{x\}$*.*

Proof. Let $\mathcal{S} \subseteq \mathcal{P}(E) - \{\emptyset\}$ be a nonempty collection. It follows from part (d) of lemma 1.6, parts (a) and (c) of lemma 1.12, part (g) of lemma 1.5, and parts (d) and (g) of lemma 1.4

that $\mathcal{S}^{\mathcal{B}}$ is nonempty and contains only proper subsets of E, and that $\lfloor \mathcal{S} \rfloor$ is nonempty and contains only nonempty sets.

Let $B \in \mathcal{S}^{\mathcal{B}}$ and $x \in E - B$ be given. Note that by definition B is maximal with respect to containing no set of $\mathcal{S}$. Thus, for all $x \in E - B$, $B \cup \{x\}$ contains some $W_x \in \mathcal{S}$. Moreover, for all $W_x \subseteq B \cup \{x\}$ we must have $x \in W_x$. Lastly, there exists $A_x \in \lfloor \mathcal{S} \rfloor$ such that $x \in A_x \subseteq W_x$. If not, then $A_x \subseteq B$, a contradiction to $B \in \mathcal{S}^{\mathcal{B}}$. Thus, the result holds. □

Theorem 2. *Let $\mathcal{S} \subseteq \mathcal{P}(E) - \{\emptyset\}$ be a nonempty collection. Then, $\mathcal{S}^{\mathcal{B}}$ is collection of bases of a matroid (or equivalently $\lfloor \mathcal{S} \rfloor$ is the collection of circuits of a matroid) if and only if for each $B \in \mathcal{S}^{\mathcal{B}}$ and $x \in E - B$, there exists a unique $C_x \in \lfloor \mathcal{S} \rfloor$ such that $x \in C_x \subseteq B \cup \{x\}$.*

Proof. Let $\mathcal{S} \subseteq \mathcal{P}(E) - \{\emptyset\}$ be a nonempty collection. It follows from part (d) of lemma 1.6, parts (a) and (c) of lemma 1.12, part (g) of lemma 1.5, and parts (d) and (g) of lemma 1.4 that $\mathcal{S}^{\mathcal{B}}$ is nonempty and contains only proper subsets of E, and that $\lfloor \mathcal{S} \rfloor$ is nonempty and contains only nonempty sets.

($\rightarrow$) Suppose that $\mathcal{S}^{\mathcal{B}}$ is the collection of bases for some matroid. It then follows that $\lfloor \mathcal{S} \rfloor = \mathcal{S}^{\mathcal{BC}}$ is the collection of circuits for the same matroid. Let $B \in \mathcal{S}^{\mathcal{B}}$ and $x \in E - B$. By the previous proposition we have that there exists $C_x \in \lfloor \mathcal{S} \rfloor$ such that $x \in C_x \subseteq B \cup \{x\}$. Thus, we have existence.

Now suppose (by way of contradiction) that we also have $A_x \in \lfloor \mathcal{S} \rfloor - \{C_x\}$ such that $x \in A_x \subseteq B \cup \{x\}$. Since $\lfloor \mathcal{S} \rfloor$ follows the circuit axioms for matroids, then by (C3) of the circuit axioms we have that there exists $D \in \lfloor \mathcal{S} \rfloor$ such that $D \subseteq (C_x \cup A_x) - \{x\}$. However, this now implies that B (a basis) contains D (a cycle) since $(C_x \cup A_x) - \{x\} \subseteq B$; a contradiction! Therefore, C_x is the unique set in $\lfloor \mathcal{S} \rfloor$ such that $x \in C_x \subseteq B \cup \{x\}$.

($\leftarrow$) Suppose that for each $B \in \mathcal{S}^{\mathcal{B}}$ and $x \in E - B$, there exists a unique $C_x \in \lfloor \mathcal{S} \rfloor$ such that $x \in C_x \subseteq B \cup \{x\}$. We first noted that since $\emptyset \notin \mathcal{S}$, then $\mathcal{S}^{\mathcal{B}} \neq \emptyset$; thus (B1) of the basis axioms for matroids is satisfied.

Next we'll show that all sets in $\mathcal{S}^{\mathcal{B}}$ have the same cardinality: By way of contradiction, let $B_1, B_2 \in \mathcal{S}^{\mathcal{B}}$ such that $|B_1| < |B_2|$ and that $|B_1 \cap B_2|$ is maximal with respect to these conditions. Since $\mathcal{S}^{\mathcal{B}}$ is a clutter by definition, then $B_1 - B_2$ is nonempty, so let $x \in B_1 - B_2$. By hypothesis, let $A_x \in \lfloor \mathcal{S} \rfloor$ be the unique set such that $x \in A_x \subseteq B_2 \cup \{x\}$. Since $A_x \nsubseteq B_1$ by definition of $\mathcal{S}^{\mathcal{B}}$, then let $y \in A_x - B_1$ and consider the set $B_2 \cup \{x\} - \{y\}$.

Since A_x is the only set in $\lfloor \mathcal{S} \rfloor$ that contains x and is contained in $B_2 \cup \{x\}$, then there are no sets in $\lfloor \mathcal{S} \rfloor$ that are contained in $B_2 \cup \{x\} - \{y\}$. It follows that $(B_2 \cup \{x\} - \{y\}) \in \lfloor \mathcal{S} \rfloor^{\mathcal{I}} = \mathcal{S}^{\mathcal{I}}$ and so there exists $B_3 \in \mathcal{S}^{\mathcal{B}}$ such that $B_2 \cup \{x\} - \{y\} \subseteq B_3$.

We now observe the following inequality:

$$|B_1| < |B_2| = |B_2 \cup \{x\} - \{y\}| \leq |B_3|$$

Moreover, since $x \in B_1$ and $y \notin B_1$, we have

$$|B_1 \cap B_2| < |B_1 \cap (B_2 \cup \{x\} - \{y\})| = |B_1 \cap B_2| + 1 \leq |B_1 \cap B_3|$$

which contradicts the maximality of $|B_1 \cap B_2|$. Hence, all sets in $\mathcal{S}^{\mathcal{B}}$ must have the same cardinality. This satisfies (B2) of the basis axioms for matroids.

Furthermore, this now implies (B3) of the basis axioms for matroids: We follow the same construction to get $B_2 \cup \{x\} - \{y\} \subseteq B_3$, but now the cardinality condition forces $|B_2| = |B_2 \cup \{x\} - \{y\}| = |B_3|$. This then implies equality: $B_2 \cup \{x\} - \{y\} = B_3$. And so $B_2 \cup \{x\} - \{y\} \in \mathcal{S}^{\mathcal{B}}$ which finally satisfies (B3) of the basis axioms for matroids. □

In the special case when $\mathcal{S}$ is a nontrivial clutter, we now have a matroidial characterization.

Corollary 2. *Let $\mathcal{S} \subseteq \mathcal{P}(E) - \{\emptyset\}$ be a clutter. Then $\mathcal{S}$ is the collection of circuits of a matroid if and only if $\mathcal{S}^{\mathcal{B}}$ satisfies the theorem above.*

4.1.2 bases-to-cycles Results

We now attempt to shift our focus towards the bases-to-circuits operator and find an analogous result, though our efforts are unfortunately less fruitful. The first result essentially says that for each 'circuit' in $\mathcal{S}^{\mathcal{C}}$, if you remove an element from it, then there exists a 'basis' $B \in \mathcal{S}^{\mathcal{CB}} = \lceil \mathcal{S} \rceil$ that contains the now 'independent' set.

Proposition 4.2. *Let $\mathcal{S} \subseteq \mathcal{P}(E) - \{E\}$ be a nonempty collection. For any $C \in \mathcal{S}^{\mathcal{C}}$ and any $x \in C$, there exists $B_x \in \lceil \mathcal{S} \rceil$ such that $x \notin B_x$ and $B_x \cap C = C - \{x\}$.*

Proof. Let $\mathcal{S} \subseteq \mathcal{P}(E) - \{E\}$ be a nonempty collection. It follows from part (d) of lemma 1.7, parts (a) and (c) of lemma 1.13, part (g) of lemma 1.4, and parts (d) and (g) of lemma 1.5 that $\mathcal{S}^{\mathcal{C}}$ is nonempty and contains only nonempty sets, and that $\lceil \mathcal{S} \rceil$ is nonempty and contains only proper subsets of E.

Let $C \in \mathcal{S}^{\mathcal{C}}$ and $x \in C$ be given. Since C is not contained in any of set of $\mathcal{S}$ minimally, then $C - \{x\}$ is contained in some set $W_x \in \mathcal{S}$ and so $C - \{x\}$ is contained in some set $B_x \in \lceil \mathcal{S} \rceil$. Moreover, we must have that $x \notin B_x$, else we have $C \subseteq B_x$ which contradicts $C \in \mathcal{S}^{\mathcal{C}}$. It follows that $(C - \{x\}) \cap B_x = C - \{x\} \subseteq C \cap B_x$. On the other hand, we have $C \cap B_x \subseteq C$ and so $C \cap B_x - \{x\} = C \cap B_x \subseteq C - \{x\}$. Therefore, we have equality. □

This next results says that if your collection is the circuits of a matroid, then for each circuit in $\mathcal{S}^{\mathcal{C}}$, you can remove any element. This produces independent set which can then be built up to a basis, no matter which element was removed.

Theorem 3. *Let $\mathcal{S} \subseteq \mathcal{P}(E) - \{E\}$. If $\mathcal{S}^{\mathcal{C}}$ is the collection of circuits of a matroid (or equivalently $\lceil \mathcal{S} \rceil$ is the collection of bases of a matroid), then for each $C \in \mathcal{S}^{\mathcal{C}}$, there exists $T \subseteq E - C$ such that for all $y \in C$ we have $(C \cup T) - \{y\} \in \lceil \mathcal{S} \rceil$.*

Proof. Let $\mathcal{S} \subseteq \mathcal{P}(E) - \{E\}$. It follows that $\mathcal{S}^{\mathcal{C}}$ is nonempty.

Suppose that $\mathcal{S}^{\mathcal{C}}$ is the collection of circuits for some matroid. It then follows that $\lceil \mathcal{S} \rceil = \mathcal{S}^{\mathcal{CB}}$ is the collection of bases for the same matroid. Thus, $\lceil \mathcal{S} \rceil$ is nonempty which implies $\mathcal{S}$ is nonempty. Therefore, $\mathcal{S}^{\mathcal{C}}$ does not contain the empty set.

Let $C \in \mathcal{S}^{\mathcal{C}}$ and $x \in C$. By the previous proposition we have that there exists $B_x \in \lceil \mathcal{S} \rceil$ such that $x \notin B_x$ and $B_x \cap C = C - \{x\}$. Note that $C - \{x\} \subseteq B_x$. Define $T = B_x - C \subseteq E - C$ and observe the following:

$$C \cup T = ((C - \{x\}) \cup \{x\}) \cup T = (B_x \cap C) \cup \{x\} \cup T = B_x \cup \{x\}$$

Now let $y \in C$. If $y = x$, then

$$(C \cup T) - \{y\} = (C \cup T) - \{x\} = (B_x \cup \{x\}) - \{x\} = B_x$$

which is in $\lceil \mathcal{S} \rceil$. So suppose $y \neq x$ and again observe that

$$(C \cup T) - \{y\} = (B_x \cup \{x\}) - \{y\}$$

Again, by the previous proposition, there exists $B_y \in \lceil \mathcal{S} \rceil$ such that $y \notin B_y$ and $B_y \cap C = C - \{y\}$. Moreover, since $C - \{x\} \subseteq B_x$ and $C - \{y\} \subseteq B_y$, then we must have $x \in B_y - B_x$ and $y \in B_x - B_y$. by hypothesis, $\lceil \mathcal{S} \rceil$ is the collection of bases of a matroid, and so it follows from the basis exchange axiom that $(B_x \cup \{x\}) - \{y\}$ is contained in $\lceil \mathcal{S} \rceil$. Hence, $T \subseteq E - C$ and $(C \cup T) - \{y\}$ is contained in $\lceil \mathcal{S} \rceil$ for all $y \in C$, as desired. □

Conjecture 2. *The converse to theorem 3 holds.*

4.2 Interactions with the Circuit Duality Operator

We now proceed to see how our recently introduced matroid operators interact with the circuit duality operator:

Lemma 4.1. *Let $\mathcal{S} \subseteq \mathcal{P}(E)$. Then $\mathcal{S}^b \subseteq \mathcal{S}^{*\mathcal{I}}$.*

Proof. We first note that when $\mathcal{S}$ is the empty collection, the result holds:

$$\emptyset^b = \{\emptyset\} = \{\{x\} \colon x \in E\}^{\mathcal{I}} = \emptyset^{*\mathcal{I}}$$

Secondly, if $\mathcal{S}$ contains the empty set, the result also holds since the blocking collection of $\mathcal{S}$ is empty, by part (a) of lemma 1.10 and part (g) of lemma 1.4. Thus, without loss of generality, we may further assume that $\mathcal{S} \subseteq \mathcal{P}(E) - \{\emptyset\}$ is a nonempty collection.

From part (a) of proposition 3.2 and the involution of the complementary operator we have that $\mathcal{S}^b = \mathcal{S}^{\mathcal{B}-}$. Let $B \in \mathcal{S}^{\mathcal{B}}$ and suppose (by way of contradiction) that $E - B \notin \mathcal{S}^{*\mathcal{I}}$. It then follows that there exists $T \in \mathcal{S}^*$ such that $T \subseteq E - B$. Let $x \in T$, then by proposition 4.1 there exists $A_x \in \lfloor \mathcal{S} \rfloor \subseteq \mathcal{S}$ such that $x \in A_x \subseteq B \cup \{x\}$. However, we then have that $|A_x \cap T| = |\{x\}| = 1$; a contradiction since $T \in \mathcal{S}^*$. Hence, $E - B \in \mathcal{S}^{*\mathcal{I}}$ and so $\mathcal{S}^b \subseteq \mathcal{S}^{*\mathcal{I}}$. ☐

Theorem 4. *Let $\mathcal{S} \subseteq \mathcal{P}(E) - \{\emptyset\}$. If $\mathcal{S}^b \subseteq \mathcal{S}^{*\mathcal{B}}$, then $\lfloor \mathcal{S} \rfloor$ is the set of circuits of a matroid.*

Proof. We first note that when $\mathcal{S}$ is the empty collection, we had from the previous lemma that $\mathcal{S}^b = \mathcal{S}^{*\mathcal{I}}$. Applying maximality then yields $\mathcal{S}^b = \mathcal{S}^{*\mathcal{B}}$. Moreover, $\mathcal{S} = \emptyset$ is the collection of circuits of the free matroid - every subset of E is an independent collection and so there are no circuits. Thus, we may further assume that $\mathcal{S}$ is a nonempty collection.

Let $\mathcal{S} \subseteq \mathcal{P}(E) - \{\emptyset\}$ be a nonempty collection such that $\mathcal{S}^b \subseteq \mathcal{S}^{*\mathcal{B}}$ and suppose (by way of contradiction) that $\lfloor \mathcal{S} \rfloor$ is not the set of circuits of a matroid. Since $\lfloor \mathcal{S} \rfloor^{\mathcal{B}} = \mathcal{S}^{\mathcal{B}}$ and $\lfloor \mathcal{S} \rfloor = \mathcal{S}^{\mathcal{B}\mathcal{C}}$, then $\mathcal{S}^{\mathcal{B}}$ is not the set of bases of a matroid.

Let $B \in \mathcal{S}^{\mathcal{B}}$ and $x \in E - B$. It then follows from the contrapositive of theorem 2 that there exist distinct $A_x, A'_x \in \mathcal{S}$ such that $x \in A_x, A'_x \subseteq B \cup \{x\}$. Without loss of generality, let $y \in A_x - A'_x \subseteq B$. From part (a) of proposition 3.2 and the involution of the complementary operator we have that $\mathcal{S}^b = \mathcal{S}^{\mathcal{B}-}$. So by hypothesis, $E - B \in \mathcal{S}^{*\mathcal{B}}$ and so by proposition 4.1 there exists $T_y \in \mathcal{S}^*$ such that $y \in T_y \subseteq (E - B) \cup \{y\}$. However, we now have that $|T_y \cap A_x| = |\{y\}|$ or $|\{x, y\}|$. But in the latter case, we have $|T_y \cap A'_x| = |\{x\}|$.

And so in either case, it follows that $T_y \notin \mathcal{S}^*$; a contradiction. Hence, we must have that $\lfloor \mathcal{S} \rfloor$ is the set of circuits of a matroid. □

Remark: The converse to this theorem does not necessarily hold. Take $\mathcal{S} = \{\{ab\}, \{abcd\}\}$ on the ground set $E = \{a, b, c, d\}$. It follows that $\mathcal{S}^b = \{\{a\}, \{b\}\}$ while $\mathcal{S}^{*\mathcal{B}} = \{\{ac\}, \{ad\}, \{bc\}, \{bd\}\}$ which are incomparable.

However, the converse might hold when $\mathcal{S} \subseteq \lfloor \mathcal{S} \rfloor^u$, since by part (e) of proposition 2.2 we get that $\mathcal{S}^* = \lfloor \mathcal{S} \rfloor^*$ in this case.

Conjecture 3. *Let $\mathcal{S} \subseteq \mathcal{P}(E)$. If $\lfloor \mathcal{S} \rfloor$ is the collection of circuits of a matroid and $\mathcal{S} \subseteq \lfloor \mathcal{S} \rfloor^u$, then $\mathcal{S}^b \subseteq \mathcal{S}^{*\mathcal{B}}$.*

Conjecture 4. *Let $\mathcal{S} \subseteq \mathcal{P}(E)$. If $\mathcal{S}^b \subseteq \mathcal{S}^{*\mathcal{B}}$, then $\mathcal{S}^b = \mathcal{S}^{*\mathcal{B}}$.*

Lemma 4.2. *The following are equivalent:*

(a) $\mathcal{S}^b = \mathcal{S}^{*\mathcal{B}}$

(b) $\mathcal{S}^* = \mathcal{S}^{b\mathcal{C}}$

(c) $\mathcal{S}^* = \mathcal{S}^{\mathcal{B}b}$

(d) $\mathcal{S}^{\mathcal{B}} = \mathcal{S}^{*b}$

Proof. (a $\iff$ b) Suppose that $\mathcal{S}^b = \mathcal{S}^{*\mathcal{B}}$. Applying $(\ )^{\mathcal{C}}$ yields $\mathcal{S}^{b\mathcal{C}} = \mathcal{S}^{*\mathcal{B}\mathcal{C}}$. Now by part (c) of proposition 3.2 it follows

$$\mathcal{S}^{b\mathcal{C}} = \mathcal{S}^{*\mathcal{B}\mathcal{C}} = \lfloor \mathcal{S}^* \rfloor = \mathcal{S}^*$$

Now suppose that $\mathcal{S}^* = \mathcal{S}^{b\mathcal{C}}$. Applying $(\ )^{\mathcal{B}}$ yields $\mathcal{S}^{*\mathcal{B}} = \mathcal{S}^{b\mathcal{C}\mathcal{B}}$. Now by part (c) of proposition 3.2 it follows

$$\mathcal{S}^{*\mathcal{B}} = \mathcal{S}^{b\mathcal{C}\mathcal{B}} = \lceil \mathcal{S}^b \rceil = \mathcal{S}^b$$

(b $\iff$ c) From parts (a) and (b) of proposition 3.2 observe that:

$$\mathcal{S}^{b\mathcal{C}} = \mathcal{S}^{b-b} = \mathcal{S}^{\mathcal{B}b}$$

Thus, it is clear that (b) $\iff$ (c).

(c $\iff$ d) Suppose that $\mathcal{S}^* = \mathcal{S}^{\mathcal{B}b}$. Applying $(\ )^b$ yields $\mathcal{S}^{*b} = \mathcal{S}^{\mathcal{B}bb}$. Now by part (a) of lemma 3.10 it follows

$$\mathcal{S}^{*b} = \mathcal{S}^{\mathcal{B}bb} = \lfloor \mathcal{S}^{\mathcal{B}} \rfloor = \mathcal{S}^{\mathcal{B}}$$

Now suppose that $\mathcal{S}^{\mathcal{B}} = \mathcal{S}^{*b}$. Applying $(\ )^b$ yields $\mathcal{S}^{\mathcal{B}b} = \mathcal{S}^{*bb}$. Now by part (a) of lemma 3.10 it follows

$$\mathcal{S}^{\mathcal{B}b} = \mathcal{S}^{*bb} = \lfloor \mathcal{S}^{*} \rfloor = \mathcal{S}^{*}$$

□

Theorem 5. *Let $\mathcal{S} \subseteq \mathcal{P}(E)$. If $\mathcal{S}^b = \mathcal{S}^{*\mathcal{B}}$, then $\lfloor \mathcal{S} \rfloor = \mathcal{S}^{**}$.*

Proof. From the previous lemma and parts (b) and (c) of proposition 3.2, we have

$$\mathcal{S}^{**} = \mathcal{S}^{*\mathcal{B}b} = \mathcal{S}^{b\mathcal{C}\mathcal{B}b} = \lceil \mathcal{S}^b \rceil^b = \mathcal{S}^{bb} = \lfloor \mathcal{S} \rfloor$$

□

Corollary 3. *Let $\mathcal{S} \subseteq \mathcal{P}(E)$. If $\mathcal{S}^b = \mathcal{S}^{*\mathcal{B}}$ and $\mathcal{S}$ is a clutter, then $\mathcal{S}$ is a frame.*

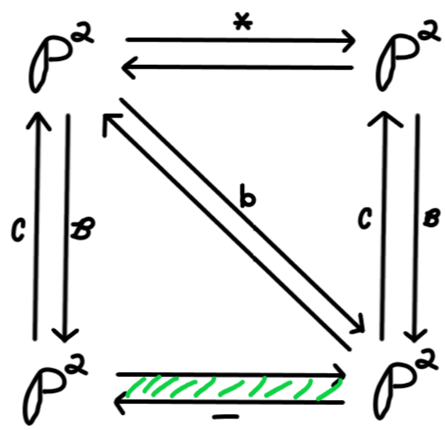
Remark: The converse of this corollary fails; that is, $\mathcal{S}$ being a frame does **not** imply that $\mathcal{S}^b = \mathcal{S}^{*\mathcal{B}}$. Consider $\mathcal{S} = \{bcd, abc, abd, acd, acx\}$ and note that $\mathcal{S}^b \neq \mathcal{S}^{*\mathcal{B}}$.

Lemma 4.3. *Let $\mathcal{S} \subseteq \mathcal{P}(E)$. If $\mathcal{S}$ is the collection of bases of a matroid, then $\mathcal{S}^{\mathcal{C}} = \mathcal{S}^{b*}$ and $\mathcal{S}^b = \mathcal{S}^{\mathcal{C}*}$.*

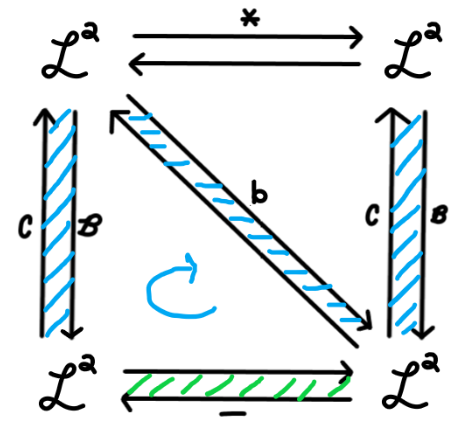
Proof. Since $\mathcal{S}$ is the collection of bases of a matroid, then $\mathcal{S}^{\mathcal{C}}$ is the collection of circuits of the matroid and $\mathcal{S}^d$ is the collection of cocircuits of the matroid. It then follows that $\mathcal{S}^{\mathcal{C}*}$ is the collection of cocircuits of the matroid and $\mathcal{S}^{b*}$ is the collection of circuits of the matroid. Since the circuit and cocircuit collections of a matroid are unique to that matroid, then we must have $\mathcal{S}^{\mathcal{C}} = \mathcal{S}^{b*}$ and $\mathcal{S}^{b} = \mathcal{S}^{\mathcal{C}*}$. □

4.3 A Commuting Diagram Emerges

Based on our findings above, we see that these matroidial operators seem to play fairly well with one another, but the collections that they play well on get more restrictive when incorporating more commutations. We introduce a diagram to illustrate these compatibilities (see fig. 4.1a). Initially, we'll consider all collections of subsets of a finite ground set. Recall the denotation of this class as $\mathcal{P}^2$.



(a) All collections.

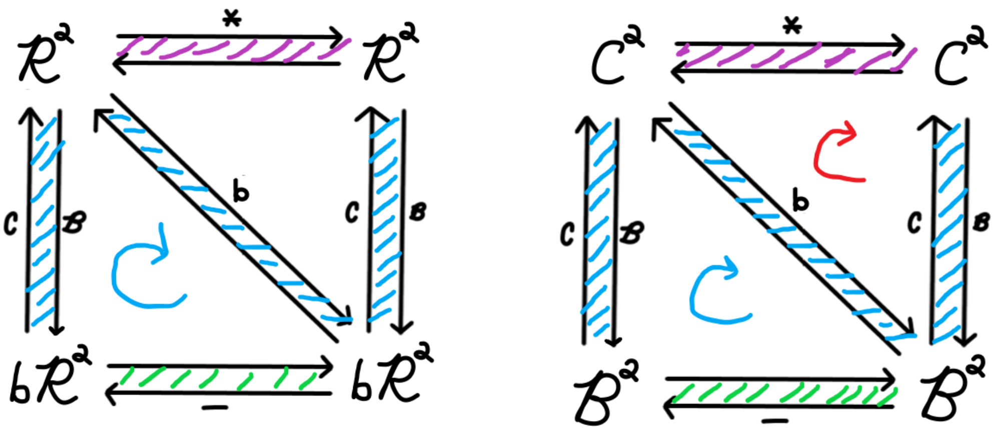


(b) All clutters.

Figure 4.1: First two restrictions of the commuting diagram for the matroid operators.

Restricting our collections down to $\mathcal{L}^2$, the class of all clutters, we saw from proposition 3.2 that a few new commutations come into play (see fig. 4.1b).

The next commutation that we can incorporate is for circuit duality, though we'll need to restrict clutters to two classes: frames and bracings (see fig. 4.2a). We say a collection $\mathcal{X}$ is a **brace** or a **bracing** when $\mathcal{X}^b$ is a frame. Recall that $\mathcal{R}^2$ denotes the class of all frames, so we'll efficiently denote the class of all bracings as $b\mathcal{R}^2$. Note that a bracing is a clutter since the blocking operator maps to clutters and that there is a one-to-one relation between



(a) Frames and bracings. (b) Circuits and bases.

Figure 4.2: Second two restrictions of the commuting diagram for the matroid operators.

the class of frames and the class of bracings since the blocking operator is an involution on clutters.

Lastly, we want to see when the upper triangle commutes. We know that it commutes for matroidial collections (since these operators are the isomorphisms between (co)circuits and (co)bases collections). Do these operators only commute for matroidial collections? Our results from this section are all pointing towards a yes. Fortunately, we are not being misled despite our results not quite carrying us across the finish line (yet).

It was shown in [Vad86, Proposition 2.3] that a clutter is a collection of circuits of a matroid if and only if the clutter was a fixed point of the map $(\ )^{b\mathcal{C}*}$. Then in [CFM91, Corollary 5.4], it was shown that a clutter is a collection of bases of a matroid if and only if the clutter was a fixed point of the map $(\ )^{b*\mathcal{B}}$. This leads to the final fully commuting diagram (fig. 4.2b). Recall that $\mathcal{C}^2$ denotes the class of all collections of circuits of a matroid and $\mathcal{B}^2$ denotes the class of all collections of bases of a matroid.

Chapter 5

Non-Matroidial Frames

We now shift our focus to frames. We saw in section 3.2 that repeated applications of the circuit duality operator always resulted in a frame. Additionally, we saw that matroidial collections are inherently frames. Thus, we had the containment of classes: $\mathcal{C}^2 \subseteq \mathcal{R}^2$. Back in section 1.3.3 question 1 asked "If a collection is a frame, does it always correspond to the circuit collection of some matroid?" The answer (unfortunately spoiled by the title of this chapter) is no! No, it does not!

In this section we give two examples of non-matroidial frames. Furthermore, we give a breakdown of each of their structures; both of which are built upon the backs of matroidial collections. Additionally, each non-matroidial frame provides a definitive example for the proper containment of the class of all matroidial collections within the class of all frames: $\mathcal{C}^2 \subset \mathcal{R}^2$.

5.1 Mutated Binary Frames

Recall the mention of a binary matroid from section 1.2.1. These matroids were formed from the subspaces of $(\mathcal{P}(E), \triangle)$ where $\triangle$ is the symmetric difference of two sets. However, binary matroids have many characterizations [Oxl11, Theorem 9.1.2]. We give one of particular use to us: If for every pair of circuits of a matroid, their symmetric difference is also a circuit, then the matroid is a binary matroid.

Example 3. *Consider the collection* $\mathcal{S} = \{\{abc\}, \{cde\}, \{abde\}, \{ace\}\}$ *and its dual* $\mathcal{S}^*$*. The ground set for this particular collection is* $E = \{a, b, c, d, e\}$*.*

	Sets					
$\mathcal{S}$	abc			cde	abde	ace
$\mathcal{S}^*$		acd	bce		abde	ace
$\mathcal{S}^{**}$	abc			cde	abde	ace

Table 5.1: Circuit duality computations for example 3.

Clearly, the collection is a frame. However, this frame is non-matroidial - it fails the circuit axioms of a matroid. Specifically, the pair of sets $(\{abc\}, \{ace\})$ fails condition (C3) of the circuit axioms (definition 2). Their intersection is $\{ac\}$ while their union is $\{abce\}$. However, there is no set in $\mathcal{S}$ that is a subset of $\{abe\} = \{abce\} - \{c\}$ nor $\{bce\} = \{abce\} - \{a\}$. Thus, $\mathcal{S}$ is not the collection of circuits of some matroid. We note that the pair of sets $(\{cde\}, \{ace\})$ also fail the third circuit axiom.

A keen observer might note that this particular frame does contain a matroidial subcollection. Specifically, the subcollection $\{\{abc\}, \{cde\}, \{abde\}\}$ has the form $\{A, B, A \triangle B\}$, which is a binary matroid of size three - one of the smallest binary matroids.

5.1.1 Construction

Through numerous computer checks, we've seen that taking $\mathcal{M} = \{A, B, A \triangle B\}$, a binary matroid of size three, and adding a set M to it so that certain conditions are satisfied results in a non-matroidial frame. We phrase this more precisely as the following conjecture:

Conjecture 5. *Let* E *be a finite set and let* A *and* B *be proper subsets of* E *such that the following properties hold:*

(a) $A - B$*,* $B - A$*, and* $A \cap B$ *are all nonempty.*

(b) At least two of the three above sets contain two or more elements.

Let $\mathcal{S} = \{A, B, A \triangle B, M\}$ such that M has nonempty intersection with $A - B$, $B - A$, and $A \cap B$; at most one of which is fully contained in M. Then $\mathcal{S}$ is a non-matroidial frame.

The non-matroidial property is easy enough to see: the circuit axioms fail with the pair (M, A) as well as the pair (M, B).

We make a few notes about the required conditions:

- If any of $A - B$, $B - A$, or $A \cap B$ are empty, then we get that $\mathcal{S}$ is no longer a clutter. And so, $\mathcal{S}$ cannot be a frame.

- If at least two of the three above sets contain only a single element, then M fully contains at least two of the sets, which then implies that M contains at least one of A, B, or $A \triangle B$. This results in $\mathcal{S}$ not being a clutter and therefore not being a frame.

This extra set M seems to infect each of the cells of the Venn diagram for A and B by sharing at least one element of each cell. But its presence within $\mathcal{S}$ does not appear to disrupt the structure needed to be a frame.

5.2 Tethered Uniform Frames

Recall the brief mention of a uniform matroid from example 2. A uniform matroid $\mathcal{U}_{k,n}$ is a matroid on a ground set of size n that has $\mathcal{P}_k(E)$ as its collections of bases and $\mathcal{P}_{k+1}(E)$ as its collection of circuits.

Example 4. *Consider $\mathcal{S} = \{\{abc\}, \{abd\}, \{acd\}, \{bcd\}, \{abx\}\}$ and its dual collection $\mathcal{S}^*$. The ground set for this particular collection is $E = \{a, b, c, d, x\}$.*

This collection is also a non-matroidial frame - failing the third condition of the circuit axioms. Specifically, it fails with the pair of sets $(\{abx\}, \{abd\})$: their intersection is $\{ab\}$ while their union is $\{abdx\}$. However, there is no set in $\mathcal{S}$ that is a subset of either $\{bdx\}$

	Sets				
$\mathcal{S}$	abc	abd	acd	bcd	abx
$\mathcal{S}^*$	abc	abd	acdx	bcdx	
$\mathcal{S}^{**}$	abc	abd	acd	bcd	abx

Table 5.2: Circuit duality computations for example 4.

or $\{adx\}$. Thus, $\mathcal{S}$ is not the collection of circuits of some matroid. We note that the pair of sets $(\{adx\}, \{abc\})$ also fails the third circuit axiom.

Again, a sharp eye might notice that this particular frame contains the subcollection $\{\{abc\}, \{abd\}, \{acd\}, \{bcd\}\}$ - which corresponds to the circuit collection of the uniform matroid $\mathcal{U}_{2,4}$.

This peculiar set $\{a, b, x\}$ 'tethers' itself to each circuit in our uniform collection by sharing at least one element. It also contains a foreign element x which is not in the support of the uniform collection; it seems to act as a filler element in the first dual. The resulting 'tethered' collection is a non-matroidial frame.

5.2.1 Construction

A more general construction for tethered uniform frames is as follows:

Theorem 6. *Let F be a finite set of size $n \geq 4$. Let F_i be a subset of F of size i such that $2 \leq i \leq n-2$. Take*

$$\mathcal{S} = \mathcal{P}_{n-1}(F) \cup \{F_i \cup \{x\}\}$$

Then $\mathcal{S}$ is a non-matroidial frame; that is, $\mathcal{S} = \mathcal{S}^{**}$ and $\mathcal{S}$ is not the collection of circuits of a matroid.

We'll prove this result through a series of lemmas. But first, some comments about the sizing requirements:

- We need F to be at least 4 in order to ensure that sets in $\mathcal{P}_{n-1}(F)$ are at least size 3 which further ensures that $(\mathcal{P}_{n-1}(F))^{\perp} = \mathcal{P}_{\geq n-(n-1)+2}(F) = \mathcal{P}_{\geq 3}(F)$.

- We want $i \geq 2$ to ensure that $F_i \cup \{x\}$ is at least a triple for similar reasons as above.

- We want $i \leq n-2$ to ensure that $\mathcal{S}$ remains a clutter: if $i = n-1$, then $F_i \in \mathcal{P}_{n-1}(F)$ and so $F_i \subset F_i \cup \{x\}$ which means $\mathcal{S}$ would not be a clutter. Hence, it could not be a frame.

Lemma 5.1. *$\mathcal{S}$ is non-matroidial.*

Proof. Let $C_1 = F_i \cup \{x\}$ and take C_2 to be any set in $\mathcal{P}_{n-1}(F)$ that contains F_i. It follows that the pair (C_1, C_2) fails the third circuit axiom: Note that $C_1 \cup C_2 = F_i \cup \{x\} \cup K$ such that $K \in \mathcal{P}_{n-i-1}(F - F_i)$. Then for any $y \in C_1 \cap C_2$ we have $(C_1 \cup C_2) - \{y\} = (F_i \cup \{x\} \cup K) - \{y\}$ By construction, the only set in $\mathcal{S}$ that contains x is $F_i \cup \{x\}$, but this is not contained in $(F_i \cup \{x\} \cup K) - \{y\}$ since $y \in F_i$. On the other hand, any set in $\mathcal{S}$ that does not contain x has size $n-1$. However, subsets of $(F_i \cup \{x\} \cup K) - \{y\}$ that do not contain x have a maximum size of $n-2$. Thus, there is no set in $\mathcal{S}$ that is contained in $(C_1 \cup C_2) - \{y\}$ for any $y \in C_1 \cap C_2$. Therefore, $\mathcal{S}$ is not the collection of cycles of a matroid. $\square$

Lemma 5.2. *The first dual of $\mathcal{S}$ is the following:*

$$\mathcal{S}^* = \left\{ \begin{array}{ll} A \cup \{x\}, & |A \cap F_i| = 1 \\ A, & \textit{else} \end{array} \right\} \textit{where } A \in \mathcal{P}_3(F)$$

Proof. Let $\mathcal{N} = \left\{ \begin{array}{ll} A \cup \{x\}, & |A \cap F_i| = 1 \\ A, & \text{else} \end{array} \right\}$ where $A \in \mathcal{P}_3(F)$. We first note that by construction $\mathcal{N}$ is a clutter. We now aim to show that $\mathcal{N} \subseteq \mathcal{S}^{\perp} \subseteq \mathcal{N}^{\supseteq}$. It would then follow by part (c) of proposition 2.1 that $\lfloor \mathcal{N} \rfloor = \lfloor \mathcal{S}^{\perp} \rfloor$, but since $\mathcal{N}$ is a clutter, then we get that $\mathcal{N} = \mathcal{S}^*$.

We show the first containment now: let $W \in \mathcal{N}$. Suppose that $x \notin W$. Then by construction, $W \in \mathcal{P}_3(F)$ which now implies that $W \in (\mathcal{P}_{n-1}(F))^{\perp}$. Moreover, $|W \cap (F_i \cup \{x\})| = |W \cap F_i| \neq 1$ by construction. So then $W \in \{F_i \cup \{x\}\}^{\perp}$ which implies that $W \in (\mathcal{P}_{n-1}(F))^{\perp} \cap \{F_i \cup \{x\}\}^{\perp} = \mathcal{S}^{\perp}$.

Now suppose that $x \in W$. It then follows by construction that $W = A \cup \{x\}$ for some $A \in \mathcal{P}_3(F)$ such that $|A \cap F_i| = 1$. Thus, $|W \cap (F_i \cup \{x\})| = 2 \neq 1$ and so $W \in \{F_i \cup \{x\}\}^{\perp}$. Note that $x \notin B$ for any $B \in \mathcal{P}_{n-1}(F)$ and so $|W \cap B| = |A \cap B| \neq 1$ for all $B \in \mathcal{P}_{n-1}(F)$ since $A \in \mathcal{P}_3(F) \subseteq (\mathcal{P}_{n-1}(F))^{\perp}$. Hence, $W \in (\mathcal{P}_{n-1}(F))^{\perp}$ and therefore $W \in \mathcal{S}^{\perp}$. In conclusion, $\mathcal{N} \subseteq \mathcal{S}^{\perp}$.

We now show the second containment: let $T \in \mathcal{S}^{\perp}$. We'll consider cases: $|T \cap F_i| = 0$, $|T \cap F_i| = 1$, and $|T \cap F_i| \geq 2$.

(Case: $|T \cap F_i| = 0$) Since $T \in \mathcal{S}^{\perp}$, then $T \in \{F_i \cup \{x\}\}^{\perp}$. So by assumption we have $|T \cap (F_i \cup \{x\})| = |T \cap \{x\}| \neq 1$, which implies that $x \notin T$. Moreover, note that $T \in (\mathcal{P}_{n-1}(F))^{\perp} = \mathcal{P}_{\geq 3}(F)$. Thus, there exists $W \in \mathcal{P}_3(F)$ such that $W \subseteq T$. Moreover, $|W \cap F_i| = 0$ since $|W \cap F_i| \leq |T \cap F_i| = 0$ which implies that $W \in \mathcal{N}$. Hence, $T \in \mathcal{N}^{\supseteq}$.

(Case: $|T \cap F_i| = 1$) Since $T \in \mathcal{S}^{\perp}$, then $T \in \{F_i \cup \{x\}\}^{\perp}$. So by assumption we have $1 \neq |T \cap (F_i \cup \{x\})| \geq |T \cap F_i| = 1$, which implies that $x \in T$. Moreover, $x \notin A$ for all $A \in \mathcal{P}_{n-1}(F)$ which now implies that $1 \neq |T \cap A| = |T - \{x\} \cap A|$ for all $A \in \mathcal{P}_{n-1}(F)$. Thus, $T - \{x\} \in (\mathcal{P}_{n-1}(F))^{\perp} = \mathcal{P}_{\geq 3}(F)$. Thus, there exists $V \in \mathcal{P}_3(F)$ such that $V \subseteq T - \{x\}$ and $|V \cap F_i| = 1$ since $2 \leq n - i = |F - F_i|$. Therefore, take $W = V \cup \{x\}$. Then $W \subseteq T$ and $W \in \mathcal{N}$ and so $T \in \mathcal{N}^{\supseteq}$.

(Case: $|T \cap F_i| \geq 2$) Let $S = \{a, b\}$ such that $a, b \in T \cap F_i$, then $S \subseteq T$ and $|S \cap F_i| = 2$. Suppose (by way of contradiction) that $S = T$, then $F - \{a\} \in \mathcal{P}_{n-1}(F)$ and $|T \cap (F - \{a\})| = 1$ which implies that $T \notin (\mathcal{P}_{n-1}(F))^{\perp} \subseteq \mathcal{S}^{\perp}$; a contradiction! Hence, S is a proper subset of T, which implies that $|T| \geq 3$. Suppose (by way of contradiction) $|T \cap (F - S)| = 0$, then $T = \{a, b, x\}$ since $|T| \geq 3$. However, the same contradiction as above comes up: $F - \{a\} \in \mathcal{P}_{n-1}(F)$ and $|T \cap (F - \{a\})| = 1$ which implies that

$T \notin (\mathcal{P}_{n-1}(F))^{\perp} \subseteq \mathcal{S}^{\perp}$; a contradiction! Hence, there exists $c \in T \cap (F - S)$ and so it follows that $\{a, b, c\} \in \mathcal{P}_3(F)$ is a subset of T and $|\{a, b, c\} \cap F_i| \neq 1$. Therefore, $\{a, b, c\} \in \mathcal{N}$ which now implies that $T \in \mathcal{N}^{\supseteq}$.

Therefore, $\mathcal{S}^{\perp} \subseteq \mathcal{N}^{\supseteq}$ and so we have $\mathcal{N} \subseteq \mathcal{S}^{\perp} \subseteq \mathcal{N}^{\supseteq}$. Thus, by part (c) of proposition 2.1 we have $\lfloor \mathcal{N} \rfloor = \lfloor \mathcal{S}^{\perp} \rfloor$ and since $\mathcal{N}$ is a clutter, then $\mathcal{N} = \mathcal{S}^*$, as desired. □

Lemma 5.3. $\mathcal{S}$ *is a frame.*

Proof. We first note that by construction $\mathcal{S}$ is a clutter. We now aim to show that $\mathcal{S} \subseteq \mathcal{S}^{*\perp} \subseteq \mathcal{S}^{\supseteq}$. It would then follow by a minimality lemma that $\lfloor \mathcal{S} \rfloor = \lfloor \mathcal{S}^{*\perp} \rfloor$, but since $\mathcal{S}$ is a clutter, then we get that $\mathcal{S} = \mathcal{S}^{**}$. Recall that we always have the first containment by part (a) of proposition 2.4: $\mathcal{S} \subseteq \mathcal{S}^{*\perp}$.

We now aim to show the second containment: Let $K \in \mathcal{S}^{*\perp}$ and suppose that $x \notin K$. For each $A \in \mathcal{P}_3(F)$ either $A \in \mathcal{S}^*$ or $(A \cup \{x\}) \in \mathcal{S}^*$. It then follows that $|K \cap A| = |K \cap (A \cup \{x\})| \neq 1$. Thus, $K \in (\mathcal{P}_3(F))^{\perp} = \mathcal{P}_{\geq n-1}(F)$ and so there exists $W \in \mathcal{P}_{n-1}(F) \subseteq \mathcal{S}$ such that $W \subseteq K$ which now implies that $K \in \mathcal{S}^{\supseteq}$.

Now suppose that $x \in K$ and that $|K| \geq n$. Then $|K - \{x\}| \geq n - 1$ and $K - \{x\} \subseteq F$. Thus, there exists $W \in \mathcal{P}_{n-1}(F) \subseteq \mathcal{S}$ such that $W \subseteq K$. Hence, $K \in \mathcal{S}^{\subseteq}$.

Lastly, suppose that $x \in K$ and that $|K| \leq n-1$. Then $|K - \{x\}| \leq n-2$. Suppose (by way of contradiction) that $F_i \not\subseteq K$. Let $y \in F_i - K$. Since $i \leq n-2$, then $|F - F_i| \geq 2$ and so there exists $A_y \in \mathcal{P}_3(F)$ such that $A_y \cap F_i = \{y\}$ and $|A_y \cap (F - F_i)| = 2$. It follows that $A_y \cup \{x\} \in \mathcal{S}^*$ and so $|K \cap (A_y \cup \{x\})| \neq 1$. Since $x \in K - A_y$, then we have $|K \cap A_y| \neq 0$. Moreover, since $y \notin K$ then we get that $K \cap (F - F_i) \neq \emptyset$. (We'll use this later on.)

Since $|K| \leq n - 1$, then $|K - \{x\}| \leq n - 2$ which implies that $|F - K| \geq 2$. So let $a, b \in F - K$. Suppose (by way of contradiction) that $a, b \notin F_i$, then $a, b \neq y$. And so $\{a, b, y\} \in \mathcal{P}_3(F)$ and $|\{a, b, y\} \cap F_i| = 1$ which implies that $\{a, b, x, y\} \in \mathcal{S}^*$. However, $|K \cap \{a, b, x, y\}| = |\{x\}| = 1$ and so $K \notin \mathcal{S}^{*\perp}$; a contradiction. Hence, we must have either $a \in F_i$ or $b \in F_i$.

Without loss of generality, let $a \in F_i$. Suppose (by way of contradiction) that $a \neq y$.

Since $K \cap (F - F_i) \neq \emptyset$ (from earlier) let $z \in K \cap (F - F_i)$. Then $\{a, y, z\} \in \mathcal{S}^*$ since $|\{a, y, z\} \cap F_i| \neq 1$. However, $|K \cap \{a, y, z\}| = |\{z\}| = 1$ and so $K \notin \mathcal{S}^{*\perp}$; a contradiction. Hence, we must have $a = y$.

Now suppose (by way of contradiction) that $b \in F_i$ as well, then $\{b, y, z\} \in \mathcal{S}^*$ since $|\{b, y, z\} \cap F_i| \neq 1$. Similarly, $|K \cap \{b, y, z\}| = |\{z\}| = 1$ and so $K \notin \mathcal{S}^{*\perp}$; a contradiction. Hence, we must have $b \notin F_i$.

Suppose (by way of contradiction) that $K \cap F_i = \emptyset$, then since $|F_i| \geq 2$ there exists $w \in F_i - K$ such that $w \neq y$. It then follows that $\{w, y, z\} \in \mathcal{S}^*$ since $|\{w, y, z\} \cap F_i| \neq 1$. However, $|K \cap \{w, y, z\}| = |\{z\}| = 1$ and so $K \notin \mathcal{S}^{*\perp}$; a contradiction. Hence, we must have $K \cap F_i \neq \emptyset$. So let $v \in K \cap F_i$, then $\{b, v, y\} \in \mathcal{S}^*$ since $|\{b, v, y\} \cap F_i| \neq 1$. However, $|K \cap \{b, v, y\}| = |\{v\}| = 1$ and so $K \notin \mathcal{S}^{*\perp}$; a contradiction. Since we've exhausted all of our options, then our original assumption must be false; that is we must have $F_i \subseteq K$.

It now follows that $F_i \cup \{x\} \subseteq K$, which now implies that $K \in \mathcal{S}^{\supseteq}$. Hence, we now have $\mathcal{S} \subseteq \mathcal{S}^{*\perp} \subseteq \mathcal{S}^{\supseteq}$. It then follows by part (c) of proposition 2.1 that $\lfloor \mathcal{S} \rfloor = \lfloor \mathcal{S}^{*\perp} \rfloor$, but since $\mathcal{S}$ is a clutter, then we get that $\mathcal{S} = \mathcal{S}^{**}$, as desired. □

5.3 Further Generalizations

Although we haven't explored generalizing mutated binary frames, we do believe that this is certainly room for it. Tethered uniform collections, however have received much more of our attention.

Large Tethering Sets

We note that the construction above yields a whole family of non-matroidial frames. Numerous computer/computation checks have confirmed the next conjecture, but a proof has not yet been completed. We think much of the previous proof can be adapted to the more generalized result.

Conjecture 6. *Let F be a finite set of size $n \geq 4$. Let F_i be a subset of F of size i such that $2 \leq i \leq n-2$. Let M be a disjoint set from F. Take*

$$\mathcal{S} = \mathcal{P}_{n-1}(F) \cup \left(\bigcup_{\alpha \in M} \{F_i \cup \{\alpha\}\} \right) \cup \mathcal{P}_2(M)$$

Then the first dual of $\mathcal{S}$ is the following:

$$\mathcal{S}^* = \left\{ \begin{array}{ll} A \cup M, & |A \cap F_i| = 1 \\ A, & else \end{array} \right\} \text{ where } A \in \mathcal{P}_3(F)$$

Moreover, $\mathcal{S}$ is a non-matroidial frame; that is $\mathcal{S} = \mathcal{S}^{**}$ and $\mathcal{S}$ is not the collection of circuits of a matroid.

Tethered k-Uniform Frames

Though we do have a proven construction for a non-matroidial frame, it limits your sets to be of size $n-1$, where $n = |F|$. The hope is to generalize these tethered uniform frames to be composed of sets of size k where $2 \leq k \leq n-1$. Exhaustive computation on small ground sets reinforce this conjecture.

Multi-Tethered Uniform Frames

Additionally, it might be possible to add multiple 'tethering' sets to the uniform collection $\mathcal{P}_{n-1}(E)$ and form a non-matroidial frame. Again, we think much of the previous proof can be salvaged for both of these generalizations, but it is not as clear that these constructions will attain our goals.

5.3.1 Additional Questions

Now that we've seen a family of non-matroidial frames (tethered uniform frames) and a potential second family (mutated binary frames), a few more questions come to mind.

The families of frames that we constructed were built from uniform matroids and binary matroids respectively; hence the naming of the families. So we ask the following:

Question 4a. *Can every matroidial collection be modified to build a non-matroidial frame?*

Question 4b. *Is every non-matroidial frame a modified matroidial collection?*

5.3.2 Family Inheritance

When we first defined what a frame is, we noted that matroidial collections are frames. we just saw in examples 3 and 4 the existence of non-matroidial frames. By definition, frames are clutters since they consist of minimal sets of an orthogonal collection but in example 1 we unknowingly showed (at the time) an example of a clutter that was not a frame. Hence, we have the following relation between a these classes:

Proposition 5.1. *We have the following class containments:* $\mathcal{P}^2 \supset \mathcal{L}^2 \supset \mathcal{R}^2 \supset \mathcal{C}^2$

This sequence of containments gives us some insight into frames: the circuit axioms are too strong of a condition for a collection to be a frame. Specifically, the third circuit axiom ensures we have a frame, but it is overkill. And axioms (C1) and (C2) alone are not enough to ensure you have a frame, though they are necessary conditions to be a frame.

Chapter 6

Spanning Subcollections

Back in section section 1.3.2 we introduced the definition of a *spanning subcollection*, which we list again now.

Definition. *A subcollection $\mathcal{R} \subseteq \mathcal{S}$ is called a* ***spanning subcollection*** *(of $\mathcal{S}$) if $\mathcal{R}^* = \mathcal{S}^*$.*

Recall from our original discussion that if $\mathcal{S}$ is a frame, then $\mathcal{R} \subseteq \mathcal{R}^{**}$. And if $\mathcal{S}$ is a matroidial frame, then $\mathcal{R}$ is a subcollection that can be used to build said matroid by applying the circuit duality operator.

6.0.1 Closure Under Unions

We now present an interesting property of the circuit duality operator which provides some additional properties of spanning subcollections.

Proposition 6.1. *Let $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(E)$. If $\mathcal{S}^* = \mathcal{T}^*$, then $(\mathcal{S} \cup \mathcal{T})^* = \mathcal{S}^*$.*

Proof. First observe that $\mathcal{S} \subseteq (\mathcal{S} \cup \mathcal{T})$ which then implies $(\mathcal{S} \cup \mathcal{T})^\perp = (\mathcal{S}^\perp \cap \mathcal{T}^\perp) \subseteq \mathcal{S}^\perp \subseteq \mathcal{S}^{\perp\supseteq}$.

Let $W_0 \in \mathcal{S}^\perp$. Since $\mathcal{S}^* = \mathcal{T}^*$, then by part (c) of proposition 2.1 we have that $\mathcal{S}^\perp \subseteq \mathcal{T}^{\perp\supseteq}$ and $\mathcal{T}^\perp \subseteq \mathcal{S}^{\perp\supseteq}$. It then follows that there exists $W_1 \in \mathcal{T}^\perp$ such that $W_0 \supseteq W_1$. Similarly, there exists $W_2 \in \mathcal{S}^\perp$ such that $W_0 \supseteq W_1 \supseteq W_2$. Continuing this process builds a nested chain of sets

$$W_0 \supseteq W_1 \supseteq W_2 \supseteq \dots$$

Since our ground set if finite, then W_0 is finite and so we must have equality occur somewhere in the chain; that is $W_k = W_{k+1}$ for some non-negative integer k. It then follows that $W_k \in \mathcal{S}^\perp$ and $W_k \in \mathcal{T}^\perp$ and so $W_k \in \mathcal{S}^\perp \cap \mathcal{T}^\perp = (\mathcal{S} \cup \mathcal{T})^\perp$. Since $W_0 \supseteq W_k$, then we have that $W_0 \in (\mathcal{S} \cup \mathcal{T})^{\perp\supseteq}$. Hence, $\mathcal{S}^\perp \subseteq (\mathcal{S} \cup \mathcal{T})^{\perp\supseteq}$.

Therefore, by part (c) of proposition 2.1 we have

$$\mathcal{S}^* = \lfloor \mathcal{S}^\perp \rfloor = \lfloor (\mathcal{S} \cup \mathcal{T})^\perp \rfloor = (\mathcal{S} \cup \mathcal{T})^*$$

□

This proposition gives some insight into the spanning subcollections of a particular collection: the family of all spanning subcollections of a fixed collection $\mathcal{S}$ is closed under unions.

6.1 Uniform Collections

We say a collection is **separable** when the collection can be written as a disjoint union of nonempty subcollections with disjoint nonempty supports. We say a collection is **connected** when it is not separable.

6.1.1 Spanning Criterion for $\mathcal{P}_2(E)$

Theorem 7. *Let $\mathcal{R}$ be any subcollection of $\mathcal{P}_2(E)$, then*

$$\mathcal{R} \textit{ is connected and fully supported} \iff \mathcal{R}^\perp = \mathcal{P}_2(E)^\perp = \{E\}$$

Proof. $(\rightarrow)$ Let $\mathcal{R}$ be a connected and fully support subcollection of $\mathcal{P}_2(E)$. Let $X \in \mathcal{R}^\perp$.

Since $\mathcal{R}$ is fully supported, then by part (a) of lemma 2.2 $|X \cap \{a, b\}| = 2$ for some $\{a, b\} \in \mathcal{R}$. I then claim that $|X \cap \{x, y\}| = 2$ for all $\{x, y\} \in \mathcal{R}$: Suppose (by contradiction) that there exists a pair of elements $\{r, s\} \in \mathcal{R}$ such that $|X \cap \{r, s\}| = 0$.

Note that by the connectedness property of $\mathcal{R}$ there must exist a chain of pairs $\{k_0, k_1\}$, $\{k_1, k_2\}$, ..., $\{k_{n-1}, k_n\}$, all of which are sets in $\mathcal{R}$, such that (without loss of generality) $r = k_0$ and $a = k_n$. If no such chain exists, then $\mathcal{R}$ can be partitioned in to the subcollection of all pairs that can chain to $\{r, s\}$ and the disjoint subcollection of all pairs that cannot be chained to $\{r, s\}$ - thus, contradicting the connectedness of $\mathcal{R}$.

Note that $\{k_0, k_1, \ldots, k_n\} \cap X \neq \emptyset$ since $k_n = a$. Let i be the smallest index such that $k_{i-1} \notin X$ but $k_i \in X$. It then follows that $|X \cap \{k_{i-1}, k_i\}| = 1$, which contradicts $X \in \mathcal{R}^{\perp}$. Hence, we must have $|X \cap \{x, y\}| = 2$ for all $\{x, y\} \in \mathcal{R}$. Thus, $\{x, y\} \subseteq X$ for all $\{x, y\} \in \mathcal{R}$. Therefore,

$$\bigcup_{\{a,b\} \in \mathcal{R}} \{a, b\} = \operatorname{supp}(\mathcal{R}) = E \subseteq X$$

Hence, $X = E$ and so $\mathcal{R}^{\perp} = \{E\} = \mathcal{P}_2(E)^{\perp}$. Therefore, $\mathcal{R}$ being connected and fully supported implies $\mathcal{R}^{\perp} = \mathcal{P}_2(E)^{\perp} = \{E\}$.

($\leftarrow$) Now let $\mathcal{R}^{\perp} = \mathcal{P}_2(E)^{\perp} = \{E\}$. Suppose (by way of contradiction) that $\mathcal{R}$ is not fully supported. Then by part (c) of lemma 2.2 $\mathcal{R}^{\perp}$ contains a singleton. Hence, $\mathcal{R}^{\perp} \neq \mathcal{P}_2(E)^{\perp} = \{E\}$; a contradiction!

Now suppose (by way of contradiction) that $\mathcal{R}$ is not connected. Then $\mathcal{R}$ is the disjoint union of some nonempty subcollections $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}_2(E)$ such that $\mathcal{S}$ and $\mathcal{T}$ have disjoint supports. Note that both $\mathcal{S}$ and $\mathcal{T}$ are not fully supported. And so the singletons of $E - \operatorname{supp}(\mathcal{S}) = \operatorname{supp}(\mathcal{T})$ are contained in $\mathcal{S}^{\perp}$ and the singletons of $E - \operatorname{supp}(\mathcal{T}) = \operatorname{supp}(\mathcal{S})$ are contained in $\mathcal{T}^{\perp}$. Hence, by part (a) of proposition 2.2 the union closures of the sets of singletons are contained in the respective orthogonal collections; in particular, $\operatorname{supp}(\mathcal{T}) \in \mathcal{S}^{\perp}$ and $\operatorname{supp}(\mathcal{S}) \in \mathcal{T}^{\perp}$.

Furthermore, neither subcollection contains a singleton since both are subcollections of

$\mathcal{P}_2(E)$. Thus, by part (d) of lemma 2.2 $\text{supp}(\mathcal{S}) \in \mathcal{S}^\perp$ and $\text{supp}(\mathcal{T}) \in \mathcal{T}^\perp$. Therefore, we have $\{\text{supp}(\mathcal{S}), \text{supp}(\mathcal{T})\} \subseteq \mathcal{R}^\perp$ since $\mathcal{R}^\perp = (\mathcal{S} \cup \mathcal{T})^\perp = \mathcal{S}^\perp \cap \mathcal{T}^\perp$. However, it now follows that $\mathcal{R}^\perp \neq \mathcal{P}_2(E)^\perp = \{E\}$; a contradiction!

Therefore, $\mathcal{R}^\perp = \mathcal{P}_2(E)^\perp = \{E\}$ implies $\mathcal{R}$ is connected and fully supported. □

Corollary 4. *Let $\mathcal{R}$ be any subcollection of $\mathcal{P}_2(E)$, then*

$$\mathcal{R} \textit{ is connected and fully supported } \iff \mathcal{R}^* = \mathcal{P}_2(E)^* = \{E\}$$

Proof. ($\rightarrow$) From the theorem above, we get that $\mathcal{R}^\perp = \mathcal{P}_2(E)^\perp$. So then applying minimality yields $\mathcal{R}^* = \mathcal{P}_2(E)^* = \{E\}$.

($\leftarrow$) Let $\mathcal{R}^* = \mathcal{P}_2(E)^* = \{E\}$. Suppose (by way of contradiction) that $\mathcal{R}$ is not fully supported and connected. From the contrapositive of the theorem above we get that $\mathcal{R}^\perp \neq \mathcal{P}_2(E)^\perp = \{E\}$. It then follows that $\mathcal{R}^\perp$ contains X, some proper nonempty subset of E. But then by minimality, $E \notin \mathcal{R}^*$ and so $\mathcal{R}^* \neq \{E\} = \mathcal{P}_2(E)$; a contradiction! □

6.1.2 Spanning Criterion for $\mathcal{P}_{n-1}(E)$

Theorem 8. *Let $\mathcal{R}$ be any subcollection of $\mathcal{P}_{n-1}(E)$, then*

$$|\mathcal{R}| \geq n-1 \iff \mathcal{R}^\perp = \mathcal{P}_{n-1}(E)^\perp = \mathcal{P}_{\geq 3}(E)$$

Proof. ($\rightarrow$) Suppose that $|\mathcal{R}| \geq n-1$ and further suppose (by way of contradiction) that $\mathcal{R}^\perp \neq \mathcal{P}_{n-1}(E)^\perp$. Since $\mathcal{R} \subseteq \mathcal{P}_{n-1}(E)$, it then follows that there exists $W \in \mathcal{R}^\perp - \mathcal{P}_{n-1}(E)^\perp$. Moreover, since $\mathcal{P}_{n-1}(E)^\perp = \mathcal{P}_{\geq 3}(E)$, then $|W| \leq 2$. Additionally, since $|\mathcal{R}| \geq n-1$ and $\mathcal{R} \subseteq \mathcal{P}_{n-1}(E)$, then $\mathcal{R}$ is fully supported. Thus, by the contrapositive of part (c) of lemma 2.2 we must have $|W| = 2$ and so $W = \{x, y\}$ for some $x, y, \in E$.

Note that $E - \{x\}$ and $E - \{y\}$ are both sets in $\mathcal{P}_{n-1}(E)$. Moreover, there are $\binom{n}{n-1} = n$ sets in $\mathcal{P}_{n-1}(E)$. Since $|\mathcal{R}| \geq n-1$, then $\mathcal{R}$ must contain either $E - \{x\}$ or $E - \{y\}$. However, it follows that $|W \cap (E - \{x\})| = |\{y\}| = 1$ and $|W \cap (E - \{y\})| = |\{x\}| = 1$. In either case, $W \notin \mathcal{R}^{\perp}$; a contradiction.

($\leftarrow$) Suppose that $\mathcal{R}^{\perp} = \mathcal{P}_{n-1}(E)^{\perp} = \mathcal{P}_{\geq 3}(E)$ and further suppose (by way of contradiction) that $|\mathcal{R}| \leq n-2$. Let $E - \{x\}$ and $E - \{y\}$ be two of the sets in $\mathcal{P}_{n-2}(E) - \mathcal{R}$. Note that $E - \{x\}$ and $E - \{y\}$ are the only two sets in $\mathcal{P}_{n-1}(E)$ that do not contain both x and y. Thus, every set in $\mathcal{R}$ must contain both x and y. Hence, $|\{x, y\} \cap A| = |\{x, y\}| = 2 \neq 1$ for all $A \in \mathcal{R}$. Thus, $\{x, y\} \in \mathcal{R}^{\perp}$. However, $\{x, y\} \notin \mathcal{P}_{\geq 3}(E) = \mathcal{P}_{n-1}(E)^{\perp}$ which implies that $\mathcal{R}^{\perp} \neq \mathcal{P}_{n-1}(E)^{\perp} = \mathcal{P}_{\geq 3}(E)$; a contradiction! □

Corollary 5. *Let $\mathcal{R}$ be any subcollection of $\mathcal{P}_{n-1}(E)$, then*

$$|\mathcal{R}| \geq n - 1 \iff \mathcal{R}^* = \mathcal{P}_{n-1}(E)^* = \mathcal{P}_3(E)$$

Proof. ($\rightarrow$) Suppose that $|\mathcal{R}| \geq n - 1$. From the theorem above, we get that $\mathcal{R}^{\perp} = \mathcal{P}_{n-1}(E)^{\perp} = \mathcal{P}_{\geq 3}(E)$. So applying minimality yields $\mathcal{R}^* = \mathcal{P}_{n-1}(E)^* = \mathcal{P}_3(E)$. Hence, $|\mathcal{R}| \geq n - 1 \implies \mathcal{R}^* = \mathcal{P}_{n-1}(E)^* = \mathcal{P}_3(E)$.

($\leftarrow$) Suppose that $\mathcal{R}^* = \mathcal{P}_{n-1}(E)^* = \mathcal{P}_3(E)$ and further suppose (by way of contradiction) that $|\mathcal{R}| \leq n - 2$. Then by the contrapositive of the above theorem, we get that $\mathcal{R}^{\perp} \neq \mathcal{P}_{n-1}(E)^{\perp} = \mathcal{P}_{\geq 3}(E)$. It then follows that there exists an $X \in \mathcal{R}^{\perp} - \mathcal{P}_{\geq 3}(E)$. Moreover, $|X| = 2$ since $\mathcal{R}$ is fully supported. And so, by minimality, $X \in \mathcal{R}^*$. However, $X \notin \mathcal{P}_3(E)$ which implies that $\mathcal{R}^* \neq \mathcal{P}_3(E) = \mathcal{P}_{n-1}(E)$; a contradiction! Therefore, $\mathcal{R}^* = \mathcal{P}_{n-1}(E)^* = \mathcal{P}_3(E) \implies |\mathcal{R}| \geq n - 1$. □

6.2 Projective Planes of Small Order

A finite field of order $q = p^n$, where p is prime, produces a projective plane of order q with $q^2 + q + 1$ points and lines. Moreover, each line contains $q + 1$ points and each point is the intersection of $q + 1$ lines. The smallest projective plane is of order 2 and is usually referred to as the Fano Plane. The next two planes of small order are of order 3 and 4. These three planes contain 7, 13, and 21 lines and points respectively.

It is well known in matroid theory that a projective plane gives a matroid using the point-line incidence geometry method. (See [GM12] for a nice introductory explanation and [Oxl11] for a deeper dive.) In particular, the collection of circuits of the matroid, corresponding to the projective plane, are precisely the sets of three colinear points and the sets of four points, no three of which are colinear.

6.2.1 Initial Findings

Let $\mathcal{S}$ be the collection of lines of any projective plane of finite order. We have the following results for projective planes of order 2, 3, and 4: the set of symmetric differences of any two lines is contained in $\mathcal{S}^*$, $\mathcal{S}$ is contained in the second dual $\mathcal{S}^{**}$, and lastly $\mathcal{S}^*$ is matroidial.

Moreover, the matroid corresponding to $\mathcal{S}^{**}$ is different than the one obtained using the finite geometry method. Only in the case of the projective plane of order 2 (the Fano plane), does the matroid obtained from the finite geometry coincide with the matroid obtained from the circuit duality operator.

6.2.2 Generalized Results

Based on our findings with the projective planes of order 2, 3, and 4, we attempted to extract the observations as generalizations. We were successful with two so far. Remark: $l_1 \triangle l_2$ denotes the symmetric difference of the lines l_1 and l_2, which is defined to be the union of the points on the lines minus the intersection point of the lines.

Lemma 6.1. *Let $\mathcal{S}$ be the lines of a projective plane. Then the collection*

$$\mathcal{A} = \{l_1 \triangle l_2 : \quad l_1, l_2 \in \mathcal{S}\}$$

is contained in $\mathcal{S}^*$.

Proof. Let $A \in \mathcal{A}$, then $A = l_1 \triangle l_2 = (l_1 \cup l_2) - (l_1 \cap l_2)$ for some $l_1, l_2 \in \mathcal{S}$. Let $l \in \mathcal{S}$. Suppose that $l = l_1$ or $l = l_2$. Without loss of generality, let $l = l_1$. It follows that $|A \cap l| = |l_1 - l_2| = (q+1) - 1 = q \neq 1$. Now suppose that l is distinct from l_1 and l_2. If $l \cap l_1 = l \cap l_2$, then $|A \cap l| = 0 \neq 1$. If $l \cap l_1 \neq l \cap l_2$, then $|A \cap l| = 2 \neq 1$. In any case, we have that $|A \cap l| \neq 1$ and so $A \in \mathcal{S}^\perp$.

Now let B be a nonempty subset of A. Then $A - B$ is nonempty. Suppose that B has empty intersection with either l_1 or l_2. Without loss of generality, take $B \cap l_1$ to be empty. Then $B \cap l_2$ must be nonempty, so let $y \in B \cap l_2$. Let $x \in A \cap l_1$ and note that there is a unique line $l_{xy} \in \mathcal{S}$ that contains both x and y. However, it now follows that $|B \cap l_{xy}| = |\{y\}| = 1$ and so $B \notin \mathcal{S}^\perp$.

Now suppose that B has nonempty intersection with both l_1 and l_2. Let $x \in A - B$ and without loss of generality, let x lie on l_1. Then let $y \in B \cap l_2$. Again, there exists a unique line $l_{xy} \in \mathcal{S}$ that contains both x and y. And again, it follows that $|B \cap l_{xy}| = |\{y\}| = 1$, implying that $B \notin \mathcal{S}^\perp$.

Hence, no proper subsets of A are in the orthogonal collection for $\mathcal{S}$ which implies that A is minimal within $\mathcal{S}^\perp$. Thus, $A \in \lfloor \mathcal{S}^\perp \rfloor = \mathcal{S}^*$ and therefore $\mathcal{A} \subseteq \mathcal{S}^*$ as desired. □

Lemma 6.2. *Let $\mathcal{S}$ be the lines of a projective plane. Then $\mathcal{S} \subseteq \mathcal{S}^{**}$.*

Proof. Note that we already have $\mathcal{S} \subseteq \mathcal{S}^{*\perp}$ for any collection.

Let $l \in \mathcal{S}$ and let m be a proper nonempty subset of l. Let $x \in l - m$ and $y \in l \cap m$. Note that l is the only line that contains both x and y. Let l_x be a line different from l that contains x and l_y be a line different from l that contains y. Now observe that $|m \cap (l_x \triangle l_y)| = |\{y\}| = 1$. By the previous proposition, we know that $l_x \triangle l_y \in \mathcal{S}^*$ and so $m \notin \mathcal{S}^{*\perp}$.

Hence, no proper subsets of l are in the orthogonal collection for $\mathcal{S}^*$ which implies that l is minimal within $\mathcal{S}^{*\perp}$. Thus, $l \in \lfloor \mathcal{S}^{*\perp} \rfloor = \mathcal{S}^{**}$ and therefore $\mathcal{S} \subseteq \mathcal{S}^{**}$ as desired. □

The next result is one from Vaderlind [Vad86, Lemma 5.2], but helps us give a nice result concerning the lines of a projective plane. We present it in the language we've grown fond of in this thesis.

Lemma 6.3. [Vad86, Lemma 5.2] *Let $\mathcal{S}$ be a clutter. If $\mathcal{S} \subseteq \mathcal{S}^{**}$, then $\mathcal{S}^*$ is a frame.*

Proposition 6.2. *Let $\mathcal{S}$ be the lines of a projective plane. Then $\mathcal{S}^*$ is a frame.*

Proof. This follows immediately from lemmas 6.2 and 6.3 since the collection of lines of a projective plane is a clutter. □

Other generalizations are currently being worked on and the hope is to show that $\mathcal{S}^*$ is matroidial. For now, we leave it as a conjecture:

Conjecture 7. *Let $\mathcal{S}$ be the lines of a projective plane. Then $\mathcal{S}^*$ is matroidial.*

Appendix A

Appendix

A.1 Proofs of Operator Properties from Preliminaries

The Support Operator

Proof of lemma 1.1.

(a) Let $A \in \mathcal{S}$. For any $x \in A$, we have $x \in \text{supp}(\mathcal{S})$ by definition. Hence, $A \subseteq \text{supp}(\mathcal{S})$ for all $A \in \mathcal{S}$.

(b) $(\leftarrow)$ By part (a) we have that each set in $\mathcal{S}$ is contained in the support. And so their union is also contained in the support.

$(\rightarrow)$ Let $x \in \text{supp}(\mathcal{S})$, then $x \in A$ for some $A \in \mathcal{S}$. It then follows that $x \in \bigcup_{A \in \mathcal{S}} A$ which then implies that $\text{supp}(\mathcal{S}) \subseteq \bigcup_{a \in \mathcal{S}} A$. Therefore, we now have equality.

(c) Let $x \in \text{supp}(\mathcal{S})$, then $x \in A$ for some $A \in \mathcal{S}$. Since $\mathcal{S} \subseteq \mathcal{T}$, then $x \in A$ for some $A \in \mathcal{T}$. Thus, $x \in \text{supp}(\mathcal{T})$ which now implies that $\text{supp}(\mathcal{S}) \subseteq \text{supp}(\mathcal{T})$.

(d) $(\rightarrow)$ Let $x \in \text{supp}(\mathcal{S} \cup \mathcal{T})$, then $x \in A$ for some $A \in \mathcal{S}$ or $x \in B$ for some $B \in \mathcal{T}$. If $x \in A$, then $x \in \text{supp}(\mathcal{S})$. On the other hand, if $x \in B$, then $x \in \text{supp}(\mathcal{T})$. In either case, $x \in \text{supp}(\mathcal{S}) \cup \text{supp}(\mathcal{T})$.

($\leftarrow$) The reverse containment holds since $\mathcal{S}, \mathcal{T} \subseteq \mathcal{S} \cup \mathcal{T}$ and part (c) tells us that the support operator is inclusion preserving. Thus, $\text{supp}(\mathcal{S}), \text{supp}(\mathcal{T}) \subseteq \text{supp}(\mathcal{S} \cup \mathcal{T})$ which implies that $\text{supp}(\mathcal{S}) \cup \text{supp}(\mathcal{T}) \subseteq \text{supp}(\mathcal{S} \cup \mathcal{T})$. Hence, we have equality.

(e) ($\rightarrow$) Let $x \in \text{supp}(\mathcal{S} \cap \mathcal{T})$, then $x \in C$ for some $C \in \mathcal{S} \cap \mathcal{T} \subseteq \mathcal{S}, \mathcal{T}$. It follows that $x \in \text{supp}(\mathcal{S}) \cap \text{supp}(\mathcal{T})$.

Note: The reverse direction does not necessarily hold: Take $\mathcal{S} = \{\{ab\}\}$ and $\mathcal{T} = \{\{bc\}\}$, then $\text{supp}(\mathcal{S}) \cap \text{supp}(\mathcal{T}) = \{b\}$ while $\text{supp}(\mathcal{S} \cap \mathcal{T}) = \emptyset$.

□

The Complementary Operator

Proof of lemma 1.2.

(a) $A \in \mathcal{S}^{--}$ if and only if $E - A \in \mathcal{S}^{-}$ if and only if $E - (E - A) = A \in \mathcal{S}$.

(b) Let $A \in \mathcal{S}^{-}$, then $E - A \in \mathcal{S} \subseteq \mathcal{T}$. Thus, $A \in \mathcal{T}^{-}$ and therefore $\mathcal{S}^{-} \subseteq \mathcal{T}^{-}$.

(c) $A \in (\mathcal{S} \cup \mathcal{T})^{-}$ if and only if $E - A \in (\mathcal{S} \cup \mathcal{T})$. We note that $E - A \in \mathcal{S}$ if and only if $A \in \mathcal{S}^{-}$. Similarly, $E - A \in \mathcal{T}$ if and only if $A \in \mathcal{T}^{-}$. In either case, $A \in \mathcal{S}^{-} \cup \mathcal{T}^{-}$.

(d) $A \in (\mathcal{S} \cap \mathcal{T})^{-}$ if and only if $E - A \in (\mathcal{S} \cap \mathcal{T})$ if and only if $E - A \in \mathcal{S}$ and $E - A \in \mathcal{T}$. This holds if and only if $A \in \mathcal{S}^{-}$ and $A \in \mathcal{T}^{-}$ if and only if $A \in \mathcal{S}^{-} \cap \mathcal{T}^{-}$.

(e) For each $A \in \mathcal{S}$, we have $E - A \in \mathcal{S}^{-}$. Since the complement of any set is unique, then we have a one-to-one correspondence between $\mathcal{S}$ and $\mathcal{S}^{-}$. Hence, they have the same cardinality.

□

The Complementation Operator

Proof of lemma 1.3. These all follow from basic set theory: here $\mathcal{P}(E)$ is our universal set and $\mathcal{S}$ is an arbitrary subset of $\mathcal{P}(E)$. □

The Minimality Operator

Proof of lemma 1.4.

(a) Sets in $\lfloor \mathcal{S} \rfloor$ must be members of $\mathcal{S}$ by definition.

(b) $(\rightarrow)$ This is immediate from part (a) and applying minimality to $\lfloor \mathcal{S} \rfloor$.

$(\leftarrow)$ Let $A' \in \lfloor \mathcal{S} \rfloor$, then $A' \in \mathcal{S}$ and $A' \not\supset A$ for all $A \in \mathcal{S}$. In particular, $A' \not\supset A$ for all $A \in \lfloor \mathcal{S} \rfloor$. Hence, $A' \in \lfloor \lfloor \mathcal{S} \rfloor \rfloor$.

(c) Let $C' \in \lfloor \mathcal{S} \cup \mathcal{T} \rfloor$. It then follows that $C' \in \mathcal{S} \cup \mathcal{T}$ and $C' \not\supset C$ for all $C \in (\mathcal{S} \cup \mathcal{T})$. If $C' \in \mathcal{S}$, then we have $C' \not\supset A$ for all $A \in \mathcal{S}$. Thus, $C' \in \lfloor \mathcal{S} \rfloor$.

On the other hand, if $C' \in \mathcal{T}$, then we have $C' \not\supset B$ for all $B \in \mathcal{T}$. Thus, $C' \in \lfloor \mathcal{T} \rfloor$. In either case, we have $C' \in \lfloor \mathcal{S} \rfloor \cup \lfloor \mathcal{T} \rfloor$. Therefore, $\lfloor \mathcal{S} \cup \mathcal{T} \rfloor \subseteq \lfloor \mathcal{S} \rfloor \cup \lfloor \mathcal{T} \rfloor$ as desired.

(d) (First $\iff$) Clearly $\{\emptyset\} = \lfloor \mathcal{S} \rfloor$ implies $\emptyset \in \lfloor \mathcal{S} \rfloor$. On the other hand, $\emptyset \in \lfloor \mathcal{S} \rfloor \subseteq \mathcal{S}$ implies that every nonempty set in $\mathcal{S}$ properly contains the empty set. Thus, $\lfloor \mathcal{S} \rfloor$ cannot contain any nonempty sets and therefore $\lfloor \mathcal{S} \rfloor = \{\emptyset\}$. We now prove the second $\iff$.

$(\rightarrow)$ Suppose $\lfloor \mathcal{S} \rfloor = \{\emptyset\}$. Since $\lfloor \mathcal{S} \rfloor \subseteq \mathcal{S}$ by part (a) then it follows that $\emptyset \in \mathcal{S}$.

$(\leftarrow)$ Suppose that $\emptyset \in \mathcal{S}$. The empty set is a proper subset of every nonempty set, hence no nonempty sets of $\mathcal{S}$ are contained in $\lfloor \mathcal{S} \rfloor$. Moreover, no set properly contains the empty set, and so $\emptyset \in \lfloor \mathcal{S} \rfloor$ which now implies that $\lfloor \mathcal{S} \rfloor = \{\emptyset\}$.

(e) Suppose that $\emptyset \notin \mathcal{S}$, then $\mathcal{S}$ contains only nonempty sets. Let $A \in \mathcal{S}$ and then let $\mathcal{A} = \{X \in \mathcal{S} \colon X \subset A\}$. If $\mathcal{A}$ is empty, then $A \in \lfloor \mathcal{S} \rfloor$ by definition and our conclusion holds since $A \subseteq A$.

If $\mathcal{A}$ is nonempty, then let $A' \in \mathcal{A}$ be of minimum size. Suppose (by way of contradiction) that there exists $B \in \mathcal{S}$ such that $B \subset A'$. It follows that $|B| < |A'|$ and $B \in \mathcal{A}$;

a contradiction to the minimum size requirement of A'. Hence, no set in $\mathcal{S}$ is properly contained in A'. And so $A' \in \lfloor \mathcal{S} \rfloor$ and our conclusion holds.

(f) Consider the collections $\mathcal{S} = \{ab, bc\} \subseteq \{ab, bc, b, abc\} = \mathcal{T}$. Note that $\lfloor \mathcal{S} \rfloor = \mathcal{S}$ but $\lfloor \mathcal{T} \rfloor = \{b\}$ which is incomparable to $\lfloor \mathcal{S} \rfloor$.

(g) $(\leftarrow)$ This is clear since $\lfloor \emptyset \rfloor = \emptyset$.

$(\rightarrow)$ Suppose (by way of contradiction) that $\lfloor \mathcal{S} \rfloor = \emptyset$ but $\mathcal{S}$ is not empty. Then there exists $A \in \mathcal{S}$. Let $\mathcal{A} = \{B \in \mathcal{S} \colon A \supset B\}$. If $\mathcal{A}$ is empty then $A \not\supset B$ of any $B \in \mathcal{S}$ and therefore $A \in \lfloor \mathcal{S} \rfloor$ by definition; a contradiction.

If $\mathcal{A}$ is nonempty, then let $B' \in \mathcal{A}$ such that $B' \not\supset B$ for any $B \in \mathcal{A}$. Moreover, note that $B' \not\supset C$ for all $C \in \mathcal{S}$. Else, $B' \supset B''$ some $B'' \in \mathcal{S}$ which implies that $A \supset B''$ and so $B'' \in \mathcal{A}$. But this contradicts how we chose B'. It then follows that $B' \in \lfloor \mathcal{S} \rfloor$ by definition; a contradiction. Thus, we must have $\lfloor \mathcal{S} \rfloor$ is empty.

(h) $(\leftarrow)$ Let $A \in \mathcal{P}_k(E)$, then $|A| = k$ and so $|A| \geq k$ which implies $A \in \mathcal{P}_{\geq k}(E)$. Moreover, any proper subsets of A must have size strictly less than k. Thus, $A \not\supset B$ for all $B \in \mathcal{P}_{\geq k}(E)$ since they all have size greater than or equal to k. Hence, $A \in \lfloor \mathcal{P}_{\geq k}(E) \rfloor$.

$(\rightarrow)$ Let $A' \in \lfloor \mathcal{P}_{\geq k}(E) \rfloor$, then $|A'| \geq k$ and $A' \not\supset B$ for all $B \in \mathcal{P}_{\geq k}(E)$. In particular, note that for any set C of size k, any proper superset of C must have size strictly greater than k. Thus, $|A'| \leq k$ which implies that $|A'| = k$. Hence, $A' \in \mathcal{P}_k(E)$.

□

The Maximality Operator

Proof of lemma 1.5.

(a) Sets in $\lceil \mathcal{S} \rceil$ must be members of $\mathcal{S}$ by definition.

(b) $(\rightarrow)$ This is immediate from part (a) and applying maximality to $\lceil \mathcal{S} \rceil$.

$(\leftarrow)$ Let $\hat{A} \in \lceil \mathcal{S} \rceil$, then $\hat{A} \in \mathcal{S}$ and $\hat{A} \not\subset A$ for all $A \in \mathcal{S}$. In particular, $\hat{A} \not\subset A$ for all $A \in \lceil \mathcal{S} \rceil$. Hence, $\hat{A} \in \lceil \lceil \mathcal{S} \rceil \rceil$.

(c) Let $\hat{C} \in \lceil \mathcal{S} \cup \mathcal{T} \rceil$. It then follows that $\hat{C} \in \mathcal{S} \cup \mathcal{T}$ and $\hat{C} \not\subset C$ for all $C \in (\mathcal{S} \cup \mathcal{T})$. If $\hat{C} \in \mathcal{S}$, then we have $\hat{C} \not\subset A$ for all $A \in \mathcal{S}$. Thus, $\hat{C} \in \lceil \mathcal{S} \rceil$.

On the other hand, if $\hat{C} \in \mathcal{T}$, then we have $\hat{C} \not\subset B$ for all $B \in \mathcal{T}$. Thus, $\hat{C} \in \lceil \mathcal{T} \rceil$. In either case, we have $\hat{C} \in \lceil \mathcal{S} \rceil \cup \lceil \mathcal{T} \rceil$. Therefore, $\lceil \mathcal{S} \cup \mathcal{T} \rceil \subseteq \lceil \mathcal{S} \rceil \cup \lceil \mathcal{T} \rceil$ as desired.

(d) (First $\iff$) Clearly $\{\emptyset\} = \lceil \mathcal{S} \rceil$ implies $\emptyset \in \lceil \mathcal{S} \rceil$. On the other hand, $\emptyset \in \lceil \mathcal{S} \rceil \subseteq \mathcal{S}$ implies that every nonempty set in $\mathcal{S}$ properly contains the empty set. Thus, $\lceil \mathcal{S} \rceil$ cannot contain any nonempty sets and therefore $\lceil \mathcal{S} \rceil = \{\emptyset\}$. We now prove the second $\iff$.

$(\rightarrow)$ Suppose $\lceil \mathcal{S} \rceil = \{\emptyset\}$. Since $\lceil \mathcal{S} \rceil \subseteq \mathcal{S}$ then it follows that $\emptyset \in \mathcal{S}$. Note that the empty set is a proper subset of every nonempty set. However, $\lceil \mathcal{S} \rceil = \{\emptyset\}$ tells us that $\mathcal{S}$ cannot contain any nonempty sets since $\emptyset$ is not a proper subset of any sets in $\mathcal{S}$. And so $\mathcal{S} = \{\emptyset\}$.

$(\leftarrow)$ This is immediate from our observations about the clutter $\{\emptyset\}$.

(e) Suppose that $\emptyset \notin \mathcal{S}$, then $\mathcal{S}$ contains only nonempty sets. Let $A \in \mathcal{S}$ and then let $\mathcal{A} = \{X \in \mathcal{S} \colon X \supset A\}$. If $\mathcal{A}$ is empty, then $A \in \lceil \mathcal{S} \rceil$ by definition and our conclusion holds since $A \supseteq A$.

If $\mathcal{A}$ is nonempty, then let $\hat{A} \in \mathcal{A}$ be of maximum size. Suppose (by way of contradiction) that there exists $B \in \mathcal{S}$ such that $B \supset \hat{A}$. It follows that $|B| > |\hat{A}|$ and $B \in \mathcal{A}$; a contradiction to the maximum size requirement of $\hat{A}$. Hence, no set in $\mathcal{S}$ properly contains $\hat{A}$. And so $\hat{A} \in \lceil \mathcal{S} \rceil$ and our conclusion holds.

(f) Consider the collections $\mathcal{S} = \{ab, bc\} \subseteq \{ab, bc, b, abc\} = \mathcal{T}$. Note that $\lceil \mathcal{S} \rceil = \mathcal{S}$ but $\lceil \mathcal{T} \rceil = \{abc\}$ which is incomparable to $\lceil \mathcal{S} \rceil$.

(g) $(\leftarrow)$ This is clear since $\lceil \emptyset \rceil = \emptyset$.

$(\rightarrow)$ Suppose (by way of contradiction) that $\lceil \mathcal{S} \rceil = \emptyset$ but $\mathcal{S}$ is not empty. Then there exists $A \in \mathcal{S}$. Let $\mathcal{A} = \{B \in \mathcal{S} \colon A \subset B\}$. If $\mathcal{A}$ is empty then $A \not\subset B$ of any $B \in \mathcal{S}$ and therefore $A \in \lceil \mathcal{S} \rceil$ by definition; a contradiction.

If $\mathcal{A}$ is nonempty, then let $B' \in \mathcal{A}$ such that $B' \not\subset B$ for any $B \in \mathcal{A}$. Moreover, note that $B' \not\subset C$ for all $C \in \mathcal{S}$. Else, $B' \subset B''$ some $B'' \in \mathcal{S}$ which implies that $A \subset B''$ and so $B'' \in \mathcal{A}$. But this contradicts how we chose B'. It then follows that $B' \in \lceil \mathcal{S} \rceil$ by definition; a contradiction. Thus, we must have $\lceil \mathcal{S} \rceil$ is empty.

(h) $(\leftarrow)$ Let $A \in \mathcal{P}_k(E)$, then $|A| = k$ and so $|A| \leq k$ which implies $A \in \mathcal{P}_{\leq k}(E)$. Moreover, any proper supersets of A must have size strictly greater than k. Thus, $A \not\subset B$ for all $B \in \mathcal{P}_{\leq k}(E)$ since they all have size less than or equal to k. Hence, $A \in \lceil \mathcal{P}_{\leq k}(E) \rceil$.

$(\rightarrow)$ Let $\hat{A} \in \lceil \mathcal{P}_{\leq k}(E) \rceil$, then $|\hat{A}| \leq k$ and $\hat{A} \not\subset B$ for all $B \in \mathcal{P}_{\leq k}(E)$. In particular, note that for any set C of size k, any proper subset of C must have size strictly less than k. Thus, $|\hat{A}| \geq k$ which implies that $|\hat{A}| = k$. Hence, $\hat{A} \in \mathcal{P}_k(E)$.

□

The Superset Closure Operator

Proof of lemma 1.6.

(a) Note that every set is a subset of itself. Hence, $\mathcal{S} \subseteq \mathcal{S}^{\supseteq}$.

(b) Let $X \in \mathcal{S}^{\supseteq}$, then $X \supseteq A$ for some $A \in \mathcal{S}$. But $\mathcal{S} \subseteq \mathcal{T}$ and so $X \supseteq A$ for some $A \in \mathcal{T}$. Thus, $X \in \mathcal{T}^{\supseteq}$ and therefore $\mathcal{S}^{\supseteq} \subseteq \mathcal{T}^{\supseteq}$.

(c) Suppose that $\emptyset \in \mathcal{S}^{\supseteq}$, then $\emptyset \supseteq A$ some $A \in \mathcal{S}$. But the only set that contains the empty set is the empty set itself. Thus, A must be the empty set and so we have $\emptyset \in \mathcal{S}$. Hence, $\emptyset \in \mathcal{S}^{\supseteq} \implies \emptyset \in \mathcal{S}$.

Suppose that $\emptyset \in \mathcal{S}$. Since every set contains the empty set, then every subset of E contains a set in $\mathcal{S}$. Thus, $\mathcal{S}^{\supseteq} = \mathcal{P}(E)$. Hence, $\emptyset \in \mathcal{S} \implies \mathcal{S}^{\supseteq} = \mathcal{P}(E)$.

Clearly, $\mathcal{S}^{\supseteq} = \mathcal{P}(E) \implies \emptyset \in \mathcal{S}^{\supseteq}$. Therefore, each scenario implies one another.

(d) $(\rightarrow)$ Since E is a superset of every subset of E and $\mathcal{S}$ contains at least one subset of E, then $E \in \mathcal{S}^{\supseteq}$.

$(\leftarrow)$ Suppose that $E \in \mathcal{S}^{\supseteq}$, then $E \supseteq A$ for some $A \in \mathcal{S}$. Hence, $\mathcal{S}$ is nonempty.

(e) $(\leftarrow)$ Since the superset closure operator is inclusion preserving, then $\mathcal{S} \subseteq \mathcal{S}^{\supseteq}$ implies $\mathcal{S}^{\supseteq} \subseteq (\mathcal{S}^{\supseteq})^{\supseteq}$.

$(\rightarrow)$ Let $X \in (\mathcal{S}^{\supseteq})^{\supseteq}$, then $X \supseteq A$ for some $A \in \mathcal{S}^{\supseteq}$. but $A \supseteq B$ for some $B \in \mathcal{S}$. Hence, $X \supseteq B$ for some $B \in \mathcal{S}$. Thus, $X \in \mathcal{S}^{\supseteq}$ and therefore $(\mathcal{S}^{\supseteq})^{\supseteq} \subseteq \mathcal{S}^{\supseteq}$.

(f) $(\rightarrow)$ Let $X \in (\mathcal{S} \cup \mathcal{T})^{\supseteq}$, then $X \supseteq A$ for some $A \in (\mathcal{S} \cup \mathcal{T})$. Without loss of generality let $A \in \mathcal{S}$. This implies that $X \in \mathcal{S}^{\supseteq}$ and therefore, $X \in \mathcal{S}^{\supseteq} \cup \mathcal{T}^{\supseteq}$. Hence, $(\mathcal{S} \cup \mathcal{T})^{\supseteq} \subseteq \mathcal{S}^{\supseteq} \cup \mathcal{T}^{\supseteq}$.

$(\leftarrow)$ Let $X \in \mathcal{S}^{\supseteq} \cup \mathcal{T}^{\supseteq}$. Without loss of generality take $X \in \mathcal{S}^{\supseteq}$. Then $X \supseteq A$ for some $A \in \mathcal{S}$. Thus, $X \supseteq A$ for some $A \in \mathcal{S} \cup \mathcal{T}$. Hence, $X \in (\mathcal{S} \cup \mathcal{T})^{\supseteq}$ and therefore $\mathcal{S}^{\supseteq} \cup \mathcal{T}^{\supseteq} \subseteq (\mathcal{S} \cup \mathcal{T})^{\supseteq}$ which now implies equality.

(g) Note that $(\mathcal{S} \cap \mathcal{T})$ is a subset of both $\mathcal{S}$ and $\mathcal{T}$. Since the superset closure operator is inclusion preserving, then we have that $(\mathcal{S} \cap \mathcal{T})^{\supseteq}$ is a subset of both $\mathcal{S}^{\supseteq}$ and $\mathcal{T}^{\supseteq}$. Hence, $(\mathcal{S} \cap \mathcal{T})^{\supseteq} \subseteq \mathcal{S}^{\supseteq} \cap \mathcal{T}^{\supseteq}$.

Note: the reverse containment may not hold. E.g. Let $\mathcal{S} = \{A\}$ and $\mathcal{T} = \{B\}$, then the left hand side is empty while the right hand side contains E.

(h) Note that every set in $\mathcal{P}(X)$ is a subset of X. On the other hand, every set in $\{Y\}^{\supseteq}$ contains Y. Since $Y \not\subseteq X$, then no superset of Y is contained in $\mathcal{P}(X)$. Moreover, no subset of X contains Y and so no subset of X is an element of $\{Y\}^{\supseteq}$. Therefore, $\mathcal{P}(X) \cap \{Y\}^{\supseteq} = \emptyset$.

(i) $(\rightarrow)$ Let $X \in \bigcap_{k=1}^{n}\{A_k\}^{\supseteq}$, then $X \in \{A_k\}^{\supseteq}$ for all $k = 1, \ldots, n$. Thus, $X \supseteq A_k$ for all k and so $X \supseteq (\bigcup_{k=1}^{n} A_k)$. Therefore, $X \in \{\bigcup_{k=1}^{n} A_k\}^{\supseteq}$ which implies $\bigcap_{k=1}^{n}\{A_k\}^{\supseteq} \subseteq \{\bigcup_{k=1}^{n} A_k\}^{\supseteq}$.

$(\leftarrow)$ Let $X \in \{\bigcup_{k=1}^{n} A_k\}^{\supseteq}$. It follows that $X \supseteq (\bigcup_{k=1}^{n} A_k)$. In particular, $X \supseteq (\bigcup_{k=1}^{n} A_k) \supseteq A_k$ for each $k = 1, \ldots, n$. Hence, $X \in \{A_k\}^{\supseteq}$ for each k and therefore $X \in \bigcap_{k=1}^{n}\{A_k\}^{\supseteq}$. Hence, $\{\bigcup_{k=1}^{n} A_k\}^{\supseteq} \subseteq \bigcap_{k=1}^{n}\{A_k\}^{\supseteq}$ which now implies equality.

□

The Subset Closure Operator

Proof of lemma 1.7.

(a) Note that every set is a subset of itself. Hence, $\mathcal{S} \subseteq \mathcal{S}^{\subseteq}$.

(b) Let $X \in \mathcal{S}^{\subseteq}$, then $X \subseteq A$ for some $A \in \mathcal{S}$. Since $\mathcal{S} \subseteq \mathcal{T}$, then $X \subseteq A$ for some $A \in \mathcal{T}$. Thus, $X \in \mathcal{T}^{\subseteq}$ and therefore $\mathcal{S}^{\subseteq} \subseteq \mathcal{T}^{\subseteq}$.

(c) Suppose that $E \in \mathcal{S}^{\subseteq}$, then $E \subseteq A$ some $A \in \mathcal{S}$. But the only set that contains the empty set is the empty set itself. Thus, A must be the empty set and so we have $E \in \mathcal{S}$. Hence, $E \in \mathcal{S}^{\subseteq} \implies E \in \mathcal{S}$.

Suppose that $E \in \mathcal{S}$. Since every set contains the empty set, then every subset of E contains a set in $\mathcal{S}$. Thus, $\mathcal{S}^{\subseteq} = \mathcal{P}(E)$. Hence, $E \in \mathcal{S} \implies \mathcal{S}^{\subseteq} = \mathcal{P}(E)$.

Clearly, $\mathcal{S}^{\subseteq} = \mathcal{P}(E) \implies E \in \mathcal{S}^{\subseteq}$. Therefore, each scenario implies one another.

(d) $(\rightarrow)$ Since $\mathcal{S}$ is nonempty, then let $A \in \mathcal{S}$. The empty set is a subset of A and so $\emptyset \in \mathcal{S}^{\subseteq}$.

$(\leftarrow)$ Suppose that $\emptyset \in \mathcal{S}^{\subseteq}$, then $\emptyset \subseteq A$ for some $A \in \mathcal{S}$. Hence, $\mathcal{S}$ is nonempty.

(e) $(\leftarrow)$ Since the subset closure operator is inclusion preserving, then $\mathcal{S} \subseteq \mathcal{S}^{\subseteq}$ implies $\mathcal{S}^{\subseteq} \subseteq (\mathcal{S}^{\subseteq})^{\subseteq}$.

($\rightarrow$) Let $X \in (\mathcal{S}^{\subseteq})^{\subseteq}$, then $X \subseteq A$ for some $A \in \mathcal{S}^{\subseteq}$. but $A \subseteq B$ for some $B \in \mathcal{S}$. Hence, $X \subseteq B$ for some $B \in \mathcal{S}$. Thus, $X \in \mathcal{S}^{\subseteq}$ and therefore $(\mathcal{S}^{\subseteq})^{\subseteq} \subseteq \mathcal{S}^{\subseteq}$.

(f) ($\rightarrow$) Let $X \in (\mathcal{S} \cup \mathcal{T})^{\subseteq}$, then there exists $A \in (\mathcal{S} \cup \mathcal{T})$ such that $X \subseteq A$. Without loss of generality let $A \in \mathcal{S}$. Then $X \subseteq A$ for some $A \in \mathcal{S}$ and so $X \in \mathcal{S}^{\subseteq}$. Therefore, $X \in \mathcal{S}^{\subseteq} \cup \mathcal{T}^{\subseteq}$. Hence, $(\mathcal{S} \cup \mathcal{T})^{\subseteq} \subseteq \mathcal{S}^{\subseteq} \cup \mathcal{T}^{\subseteq}$.

($\leftarrow$) Let $X \in \mathcal{S}^{\subseteq} \cup \mathcal{T}^{\subseteq}$. Without loss of generality, take $X \in \mathcal{S}^{\subseteq}$. Then $X \subseteq A$ for some $A \in \mathcal{S}$. And so $X \subseteq A$ for some $A \in \mathcal{S} \cup \mathcal{T}$. Hence, $X \in (\mathcal{S} \cup \mathcal{T})^{\subseteq}$ and therefore $\mathcal{S}^{\subseteq} \cup \mathcal{T}^{\subseteq} \subseteq (\mathcal{S} \cup \mathcal{T})^{\subseteq}$.

(g) Note that $(\mathcal{S} \cap \mathcal{T})$ is a subset of both $\mathcal{S}$ and $\mathcal{T}$. Since the superset closure operator is inclusion preserving, then we have that $(\mathcal{S} \cap \mathcal{T})^{\subseteq}$ is a subset of both $\mathcal{S}^{\subseteq}$ and $\mathcal{T}^{\subseteq}$. Hence, $(\mathcal{S} \cap \mathcal{T})^{\subseteq} \subseteq \mathcal{S}^{\subseteq} \cap \mathcal{T}^{\subseteq}$.

Note: the reverse containment may not hold. E.g. Let $\mathcal{S} = \{A\}$ and $\mathcal{T} = \{B\}$, then the left hand side is empty while the right hand side contains $\emptyset$.

(h) This is immediate from the definition.

(i) This follows from the subset closure operator being additive over unions, how the operator behaves on collections containing a single set, and the finiteness of our collections: Note that $\mathcal{S} = \bigcup_{S_k \in \mathcal{S}} \{S_k\}$. And so $\mathcal{S}^{\subseteq} = \left(\bigcup_{S_k \in \mathcal{S}} \{S_k\}\right)^{\subseteq} = \bigcup_{S_k \in \mathcal{S}} \left(\{S_k\}^{\subseteq}\right) = \bigcup_{S_k \in \mathcal{S}} \mathcal{P}(S_k)$.

(j) Let $X \in \bigcap_{k=1}^{n} \{A_k\}^{\subseteq}$, then $X \in \{A_k\}^{\subseteq}$ for all $k = 1, \ldots, n$. Thus, $X \subseteq A_k$ for all k and so $X \subseteq (\bigcup_{k=1}^{n} A_k)$. Therefore, $X \in \{\bigcup_{k=1}^{n} A_k\}^{\subseteq}$ which implies $\bigcap_{k=1}^{n} \{A_k\}^{\subseteq} \subseteq \{\bigcup_{k=1}^{n} A_k\}^{\subseteq}$.

Note: the reverse containment may not hold. E.g. Let $A_1 = \{a\}$ and $A_2 = \{b\}$, then the left hand side is $\{\emptyset\}$ while the right hand side is $\mathcal{P}(\{a, b\})$.

$\square$

The Union Closure Operator

Proof of lemma 1.8.

(a) Let $A \in \mathcal{S}$. Note that $A = A \cup A$, hence $A \in \mathcal{S}^u$.

(b) Let $X \in \mathcal{S}^u$, then $X = \bigcup_{k=1}^{n} S_k$ where $S_k \in \mathcal{S} \subseteq \mathcal{T}$. Hence, $X \in \mathcal{T}^u$ and therefore $\mathcal{S}^u \subseteq \mathcal{T}^u$.

(c) $(\leftarrow)$ Since $\mathcal{S} \subseteq \mathcal{S}^u$ and that the union closure operator is inclusion preserving, then we get that $\mathcal{S}^u \subseteq \mathcal{S}^{uu}$.

$(\rightarrow)$ Let $X \in \mathcal{S}^{uu}$, then $X = \bigcup_{k=1}^{n} B_k$ where $B_k \in \mathcal{S}^u$. But each $B_k = \bigcup_{l=1}^{m} A_{kl}$ where $A_{kl} \in \mathcal{S}$. Thus, $X = \bigcup_{k=1}^{n} (\bigcup_{l=1}^{m} A_{kl})$ where $A_{kl} \in \mathcal{S}$ which implies that $X \in \mathcal{S}^u$. Therefore, $\mathcal{S}^{uu} \subseteq \mathcal{S}^u$.

(d) $(\leftarrow)$ Note that $\mathcal{S}$ and $\mathcal{T}$ are both subsets of $(\mathcal{S} \cup \mathcal{T})$, then by the inclusion preserving property of the union closure operator, we get that $\mathcal{S}^u$ and $\mathcal{T}^u$ are both subsets of $(\mathcal{S} \cup \mathcal{T})^u$. Hence, $\mathcal{S}^u \cup \mathcal{T}^u \subseteq (\mathcal{S} \cup \mathcal{T})^u$. Now by the inclusion preserving and idempotent properties of the union closure operator, we have $(\mathcal{S}^u \cup \mathcal{T}^u)^u \subseteq (\mathcal{S} \cup \mathcal{T})^u$.

$(\rightarrow)$ Let $X \in (\mathcal{S} \cup \mathcal{T})^u$, then $X = \bigcup_{h=1}^{n} C_h$ where $C_h \in \mathcal{S} \cup \mathcal{T}$. For each C_h, it is an element of either $\mathcal{S}$ or $\mathcal{T}$, so then we'll split our big union into two sub-unions: one union over the sets in $\mathcal{S}$ and the other over the union of sets in $\mathcal{T}$.

$$X = \bigcup_{h=1}^{n} C_h = \left(\bigcup_{j=1}^{m} A_j \right) \cup \left(\bigcup_{k=1}^{l} B_k \right), \text{ where } A_j \in \mathcal{S} \text{ and } B_k \in \mathcal{T}$$

We now let $A = \bigcup_{j=1}^{m} A_j$ and $B = \bigcup_{k=1}^{l} B_k$. Thus, $A \in \mathcal{S}^u$ and $B \in \mathcal{T}^u$. Therefore, $X \in (\mathcal{S}^u \cup \mathcal{T}^u)^u$.

(e) Note that $(\mathcal{S} \cap \mathcal{T})$ is a subset of both $\mathcal{S}$ and $\mathcal{T}$, then by the inclusion preserving property of the union closure operator, we get that $(\mathcal{S} \cap \mathcal{T})^u$ is a subset of both $\mathcal{S}^u$ and $\mathcal{T}^u$. Hence, $(\mathcal{S} \cap \mathcal{T})^u \subseteq \mathcal{S}^u \cap \mathcal{T}^u$.

Note: The reverse containment does not necessarily hold. Take $\mathcal{S} = \{\{a\}, \{bc\}\}$ and $\mathcal{T} = \{\{ab\}, \{c\}\}$, then $\mathcal{S}^u = \{\{a\}, \{bc\}, \{abc\}\}$ and $\mathcal{T}^u = \{\{ab\}, \{c\}, \{abc\}\}$. Moreover, $(\mathcal{S} \cap \mathcal{T}) = \emptyset$ and so $(\mathcal{S} \cap \mathcal{T})^u = \emptyset^u = \emptyset$ which is a proper subset of $\{\{abc\}\} = \mathcal{S}^u \cap \mathcal{T}^u$.

(f) $(\rightarrow)$ Let $A \in (\mathcal{S} \cup \{\emptyset\})^u$. If A is empty, then $A \in \{\emptyset\} \subseteq \mathcal{S}^u \cup \{\emptyset\}$.

If A is nonempty, then $A = \bigcup_{A_k \in \mathcal{S} \cup \{\emptyset\}} A_k$. Since $A_k = A_k \cup \emptyset$ for all A_k, then $A = \bigcup_{A_k \in \mathcal{S}} A_k$ which implies that $A \in \mathcal{S}^u \subseteq \mathcal{S}^u \cup \{\emptyset\}$.

In either case, $A \in \mathcal{S}^u \cup \{\emptyset\}$ and so $(\mathcal{S} \cup \{\emptyset\})^u \subseteq \mathcal{S}^u \cup \{\emptyset\}$.

$(\leftarrow)$ Let $A \in \mathcal{S}^u \cup \{\emptyset\}$. If A is empty, then $A \in \{\emptyset\} \subseteq (\mathcal{S} \cup \{\emptyset\}) \subseteq (\mathcal{S} \cup \{\emptyset\})^u$.

If A is nonempty, then $A \in \mathcal{S}^u$ and so $A = \bigcup_{A_k \in \mathcal{S}} A_k = \left(\bigcup_{A_k \in \mathcal{S}} A_k\right) \cup \emptyset = \bigcup_{A_k \in \mathcal{S} \cup \{\emptyset\}} A_k$. Thus, $A \in (\mathcal{S} \cup \{\emptyset\})^u$. And so $\mathcal{S}^u \cup \{\emptyset\} \subseteq (\mathcal{S} \cup \{\emptyset\})^u$ which now implies equality.

(g) For each $x \in \operatorname{supp}(\mathcal{S})$ we have that $x \in A_x$ some $A_x \in \mathcal{S}$. Note that $\operatorname{supp}(\mathcal{S}) = \bigcup_{x \in \operatorname{supp}(\mathcal{S})} \{x\} \subseteq \bigcup_{x \in \operatorname{supp}(\mathcal{S})} A_x \subseteq \operatorname{supp}(\mathcal{S})$. Hence we have equality throughout. In particular, $\operatorname{supp}(\mathcal{S}) = \bigcup_{x \in \operatorname{supp}(\mathcal{S})} A_x$ and so $\operatorname{supp}(\mathcal{S}) \in \mathcal{S}^u$.

□

The Intersection Closure Operator

Proof of lemma 1.9.

(a) Let $A \in \mathcal{S}$. Note that $A = A \cap A$, hence $A \in \mathcal{S}^i$.

(b) Let $X \in \mathcal{S}^i$, then $X = \bigcap_{k=1}^n S_k$ where $S_k \in \mathcal{S} \subseteq \mathcal{T}$. Hence, $X \in \mathcal{T}^i$ and therefore $\mathcal{S}^i \subseteq \mathcal{T}^i$.

(c) $(\leftarrow)$ Since $\mathcal{S} \subseteq \mathcal{S}^i$ and that the intersection closure operator is inclusion preserving, then we get that $\mathcal{S}^i \subseteq \mathcal{S}^{ii}$.

$(\rightarrow)$ Let $X \in \mathcal{S}^{ii}$, then $X = \bigcap_{j=1}^n B_j$ where $B_j \in \mathcal{S}^i$. But each $B_j = \bigcap_{k=1}^m A_{jk}$ where $A_{jk} \in \mathcal{S}$. Thus, $X = \bigcap_{j=1}^n (\bigcap_{k=1}^m A_{jk})$ where $A_{jk} \in \mathcal{S}$ which implies that $X \in \mathcal{S}^i$. Therefore, $\mathcal{S}^{ii} \subseteq \mathcal{S}^i$.

(d) $(\leftarrow)$ Note that $\mathcal{S}$ and $\mathcal{T}$ are both subsets of $(\mathcal{S} \cup \mathcal{T})$, then by the inclusion preserving property of the intersection closure operator, we get that $\mathcal{S}^i$ and $\mathcal{T}^i$ are both subsets of $(\mathcal{S} \cup \mathcal{T})^i$. Hence, $\mathcal{S}^i \cup \mathcal{T}^i \subseteq (\mathcal{S} \cup \mathcal{T})^i$. Now by the inclusion preserving and idempotent properties of the intersection closure operator, we have $(\mathcal{S}^i \cup \mathcal{T}^i)^i \subseteq (\mathcal{S} \cup \mathcal{T})^i$.

$(\rightarrow)$ Let $X \in (\mathcal{S} \cup \mathcal{T})^i$, then $X = \bigcap_{h=1}^{n} C_h$ where $C_h \in \mathcal{S} \cup \mathcal{T}$. For each C_h, it is an element of either $\mathcal{S}$ or $\mathcal{T}$, so then we'll split our big intersection into two sub-intersections: one intersection over the sets in $\mathcal{S}$ and the other over the intersection of sets in $\mathcal{T}$.

$$X = \bigcap_{h=1}^{n} C_h = \left(\bigcap_{j=1}^{m} A_j \right) \cap \left(\bigcap_{k=1}^{l} B_k \right), \text{ where } A_j \in \mathcal{S} \text{ and } B_k \in \mathcal{T}$$

We now let $A = \bigcap_{j=1}^{m} A_j$ and $B = \bigcap_{k=1}^{l} B_k$. Thus, $A \in \mathcal{S}^i$ and $B \in \mathcal{T}^i$. Therefore, $X \in (\mathcal{S}^i \cup \mathcal{T}^i)^i$.

(e) Note that $(\mathcal{S} \cap \mathcal{T})$ is a subset of both $\mathcal{S}$ and $\mathcal{T}$, then by the inclusion preserving property of the intersection closure operator, we get that $(\mathcal{S} \cap \mathcal{T})^i$ is a subset of both $\mathcal{S}^i$ and $\mathcal{T}^i$. Hence, $(\mathcal{S} \cap \mathcal{T})^i \subseteq \mathcal{S}^i \cap \mathcal{T}^i$.

Note: The reverse containment does not necessarily hold. Take $\mathcal{S} = \{\{ac\}, \{bc\}\}$ and $\mathcal{T} = \{\{ac\}, \{cd\}\}$, then $\mathcal{S}^i = \{\{ac\}, \{bc\}, \{c\}\}$ and $\mathcal{T}^i = \{\{ac\}, \{cd\}, \{c\}\}$. Moreover, $(\mathcal{S} \cap \mathcal{T}) = \{\{ac\}\}$ and so $(\mathcal{S} \cap \mathcal{T})^i = \{\{ac\}\}^i = \{\{ac\}\}$ which is a proper subset of $\{\{ac\}, \{c\}\} = \mathcal{S}^i \cap \mathcal{T}^i$.

□

The Meet Operator

Proof of lemma 1.10.

(a) $(\rightarrow)$ Suppose that $\mathcal{S}$ contains the empty set. Note that the intersection of any set with the empty set is empty and therefore has intersection size zero. Hence, every subset of E has an intersection size of zero with a set in $\mathcal{S}$. Therefore, no subset of E is contained in $\mathcal{S}^m$ and so $\mathcal{S}^m = \emptyset$.

($\leftarrow$) Suppose that $\mathcal{S}^m$ is empty. This implies that every subset of E has an empty intersection with some set in $\mathcal{S}$. In particular, E has an empty intersection with some set in $\mathcal{S}$. But $E \cap A = A$ for any subset of E. Thus, $|E \cap A| = |A| = 0$ implies that $A = \emptyset$. Therefore, $\mathcal{S}$ contains the empty set.

(b) Let $X \in \mathcal{T}^m$, then $|X \cap A| \neq 0$ for all $A \in \mathcal{T}$. In particular, $|X \cap A| \neq 0$ for all $A \in \mathcal{S} \subseteq \mathcal{T}$. Thus, $X \in \mathcal{S}^m$. Therefore, $\mathcal{T}^m \subseteq \mathcal{S}^m$.

(c) ($\rightarrow$) Note that both $\mathcal{S}$ and $\mathcal{T}$ are subsets of $\mathcal{S} \cup \mathcal{T}$. Then by the inclusion reversing property of the meet operator, we get that $(\mathcal{S} \cup \mathcal{T})^m$ is a subset of both $\mathcal{S}^m$ and $\mathcal{T}^m$. Hence, $(\mathcal{S} \cup \mathcal{T})^m \subseteq \mathcal{S}^m \cap \mathcal{T}^m$.

($\leftarrow$) Let $X \in \mathcal{S}^m \cap \mathcal{T}^m$ and let $C \in \mathcal{S} \cup \mathcal{T}$. If $C \in \mathcal{S}$, then $|X \cap C| \neq 0$ since $X \in \mathcal{S}^m$. On the other hand, if $C \in \mathcal{T}$, then $|X \cap C| \neq 0$ since $X \in \mathcal{T}^m$. Hence, $X \in (\mathcal{S} \cup \mathcal{T})^m$.

(d) Suppose that $\mathcal{S}^m = \mathcal{T}^m$. From our hypothesis and the meet operator's inclusion reversing property we have $(\mathcal{S} \cup \mathcal{T})^m = \mathcal{S}^m \cap \mathcal{T}^m = \mathcal{S}^m$.

Note: The reverse implication does not necessarily hold. The best we can get is $\mathcal{S}^m \subseteq \mathcal{T}^m$ which can be a proper containment: Let $\mathcal{S} = \{\{a\}, \{b\}\}$ and $\mathcal{T} = \{\{a\}\}$ and $E = \{a, b\}$. It then follows that $(\mathcal{S} \cup \mathcal{T})^m = \mathcal{S}^m = \{\{ab\}\}$, however $\mathcal{T}^m = \{\{a\}, \{ab\}\}$ which properly contains $\mathcal{S}^m$.

□

The Orthogonality Operator

Proof of lemma 1.11.

(a) This is immediate from the definition.

(b) ($\rightarrow$) We saw above how the orthogonality operator handles the trivial clutters: $\emptyset^{\perp} = \{\emptyset\}^{\perp} = \mathcal{P}(E) - \{\emptyset\}$.

($\leftarrow$) Suppose that $\mathcal{S}^{\perp} = \mathcal{P}(E) - \{\emptyset\}$. Then in particular every singleton of $\mathcal{P}(E)$ is orthogonal to every set in $\mathcal{S}$. This implies that each set in $\mathcal{S}$ has size zero, but the only set of size zero is the empty set. So if $\mathcal{S}$ contains any sets, it can only contain the empty set. Hence, $\mathcal{S} \subseteq \{\emptyset\}$.

(c) Let $X \in \mathcal{T}^{\perp}$, then $|X \cap B| \neq 1$ for all $B \in \mathcal{T}$. Since $\mathcal{S} \subseteq \mathcal{T}$, then in particular $|X \cap A| \neq 1$ for all $A \in \mathcal{S}$. Thus, $X \in \mathcal{S}^{\perp}$.

(d) Note the following inclusion: $(\mathcal{S} - \{\emptyset\}) \subseteq \mathcal{S} \subseteq (\mathcal{S} \cup \{\emptyset\})$. Thus, by the inclusion reversing property of the orthogonality operator we have $(\mathcal{S}-\{\emptyset\})^{\perp} \supseteq \mathcal{S}^{\perp} \supseteq (\mathcal{S}\cup\{\emptyset\})^{\perp}$. Thus, showing that $(\mathcal{S} - \{\emptyset\})^{\perp} \subseteq (\mathcal{S} \cup \{\emptyset\})^{\perp}$ gives us equality throughout.

Let $X \in (\mathcal{S} - \{\emptyset\})^{\perp}$, then $|X \cap A| \neq 1$ for all $A \in \mathcal{S} - \{\emptyset\}$. Note that $|X \cap \emptyset| = 0 \neq 1$. Thus, $X \in (\mathcal{S} \cup \{\emptyset\})^{\perp}$ and therefore $(\mathcal{S} - \{\emptyset\})^{\perp} \subseteq (\mathcal{S} \cup \{\emptyset\})^{\perp}$. Hence, we have equality throughout.

(e) ($\rightarrow$) Note that both $\mathcal{S}$ and $\mathcal{T}$ are subsets of $\mathcal{S} \cup \mathcal{T}$. Then by the inclusion reversing property of the orthogonality operator, we get that $(\mathcal{S} \cup \mathcal{T})^{\perp}$ is a subset of both $\mathcal{S}^{\perp}$ and $\mathcal{T}^{\perp}$. Hence, $(\mathcal{S} \cup \mathcal{T})^{\perp} \subseteq \mathcal{S}^{\perp} \cap \mathcal{T}^{\perp}$.

($\leftarrow$) Let $X \in \mathcal{S}^{\perp} \cap \mathcal{T}^{\perp}$. Then $|X \cap A| \neq 1$ for all $A \in \mathcal{S}$ and $|X \cap B| \neq 1$ for all $B \in \mathcal{T}$. Let $C \in \mathcal{S} \cup \mathcal{T}$. If $C \in \mathcal{S}$, then $|X \cap C| \neq 1$ since $X \in \mathcal{S}^{\perp}$. On the other hand, if $C \in \mathcal{T}$, then $|X \cap C| \neq 1$ since $X \in \mathcal{T}^{\perp}$. Thus, $|X \cap C| \neq 1$ for all $C \in \mathcal{S} \cup \mathcal{T}$ and so $X \in (\mathcal{S} \cup \mathcal{T})^{\perp}$. Therefore, we have equality.

□

The cycles-to-bases Operator

Proof of lemma 1.12.

(a) $\mathcal{S}^{\mathcal{I}}$ is the collection of sets that contain no sets of $\mathcal{S}$. The collection of sets that contain a set of $\mathcal{S}$ is the superset closure of $\mathcal{S}$. Any set not in the superset closure of $\mathcal{S}$ must then be in $\mathcal{S}^{\mathcal{I}}$. Thus, these collections are complementary; that is $\mathcal{S}^{\mathcal{I}} = \mathcal{S}^{\supseteq c}$.

(b) Since $\mathcal{S} \subseteq \mathcal{S}^{\supseteq}$ then $(\mathcal{S} \cap \mathcal{S}^{\mathcal{I}}) \subseteq (\mathcal{S}^{\supseteq} \cap \mathcal{S}^{\supseteq c}) = \emptyset$. Hence, $\mathcal{S}$ and $\mathcal{S}^{\mathcal{I}}$ are disjoint.

(c) We first note that $\mathcal{S}^{\mathcal{I}} = \emptyset \iff \mathcal{S}^{\supseteq} = \mathcal{P}(E)$. In particular, $\emptyset \in \mathcal{S}^{\supseteq}$. Moreover, we also have $\emptyset \in \mathcal{S}^{\supseteq} \iff \emptyset \in \mathcal{S}$.

(d) This follows directly from $\mathcal{S}$ and $\mathcal{S}^{\mathcal{I}}$ being disjoint.

□

The bases-to-dependent Operator

Proof of lemma 1.13.

(a) $\mathcal{S}^{\mathcal{D}}$ is the collection of sets that are not contained in any set of $\mathcal{S}$. The collection of sets that are contained in a set of $\mathcal{S}$ is the subset closure of $\mathcal{S}$. Any set not in the subset closure of $\mathcal{S}$ must then be in $\mathcal{S}^{\mathcal{D}}$. Thus, these collections are complementary; that is $\mathcal{S}^{\mathcal{D}} = \mathcal{S}^{\subseteq c}$.

(b) Since $\mathcal{S} \subseteq \mathcal{S}^{\subseteq}$ then $(\mathcal{S} \cap \mathcal{S}^{\mathcal{D}}) \subseteq (\mathcal{S}^{\subseteq} \cap \mathcal{S}^{\subseteq c}) = \emptyset$. Hence, $\mathcal{S}$ and $\mathcal{S}^{\mathcal{D}}$ are disjoint.

(c) We first note that $\mathcal{S}^{\mathcal{D}} = \emptyset \iff \mathcal{S}^{\subseteq} = \mathcal{P}(E)$. In particular, $E \in \mathcal{S}^{\subseteq}$. Moreover, we also have $E \in \mathcal{S}^{\subseteq} \iff E \in \mathcal{S}$.

(d) This follows directly from $\mathcal{S}$ and $\mathcal{S}^{\mathcal{D}}$ being disjoint.

□

The Inner Pairwise Union Operator

Proof of lemma 1.14.

(a) This is immediate from the definition.

(b) Let $X \in (\mathcal{S} \overset{\times}{\cup} \mathcal{T})$. If $X \in \mathcal{S}$, then $X \in \mathcal{X} \subseteq (\mathcal{X} \overset{\times}{\cup} \mathcal{Y})$. If $X \in \mathcal{T}$, then $X \in \mathcal{Y} \subseteq (\mathcal{X} \overset{\times}{\cup} \mathcal{Y})$. If $X \in (\mathcal{S} \overset{\times}{\cup} \mathcal{T}) \setminus (\mathcal{S} \cup \mathcal{T})$, then $X = A \cup B$ for some $A \in \mathcal{S} \subseteq \mathcal{X}$ and some $B \in \mathcal{T} \subseteq \mathcal{Y}$. Hence, $X \in (\mathcal{X} \overset{\times}{\cup} \mathcal{Y})$. In any case, the containment holds.

(c) Let $X \in (\mathcal{S} \overset{\times}{\cup} \mathcal{T}) \overset{\times}{\cup} \mathcal{U}$, then we have

$$X = (A \cup B) \cup C = A \cup (B \cup C)$$

for some $A \in \mathcal{S}$, $B \in \mathcal{T}$, and $C \in \mathcal{U}$. Hence, $X \in \mathcal{S} \overset{\times}{\cup} (\mathcal{T} \overset{\times}{\cup} \mathcal{U})$.

The reverse containment is the same argument as above but in reverse.

□

A.2 Additional Operator Interactions

A.2.1 Simple Operators

Some operators have rather straightforward interactions, regardless of the collection that is being evaluated.

Miscellaneous Results

Lemma A.1. *Let* $\mathcal{S} \subseteq \mathcal{P}(E)$.

(a) $\{\emptyset, E\} \subseteq \mathcal{S} \iff \mathcal{S}^{\supseteq} = \mathcal{S}^{\subseteq} = \mathcal{P}(E)$

(b) $\emptyset \neq \mathcal{S} \implies \mathcal{S}^{\subseteq\supseteq} = \mathcal{S}^{\supseteq\subseteq} = \mathcal{P}(E)$

(c) $\emptyset \neq \mathcal{S} \implies \mathcal{S}^{\subseteq u} = \mathcal{P}(\mathrm{supp}(\mathcal{S})) = \mathcal{S}^{u\subseteq}$

(d) $\emptyset \neq \mathcal{S} \implies \lfloor \mathcal{S}^{\subseteq} \rfloor = \{\emptyset\}$

(e) $\emptyset \neq \mathcal{S} \implies \lceil \mathcal{S}^{\supseteq} \rceil = \{E\}$

(f) $\emptyset \neq \mathcal{S} \implies \lceil \mathcal{S}^{u} \rceil = \{\text{supp}(\mathcal{S})\}$

(g) $\mathcal{S}^{u} \subseteq \mathcal{S}^{\supseteq}$ ***Reverse containment does not necessarily hold.***

(h) $\mathcal{S}^{i} \subseteq \mathcal{S}^{\subseteq}$ ***Reverse containment does not necessarily hold.***

Proof.

(a) $(\rightarrow)$ Since $\emptyset \in \mathcal{S}$ then $\mathcal{S}^{\supseteq} = \mathcal{P}(E)$ and since $E \in \mathcal{S}$ then $\mathcal{S}^{\subseteq} = \mathcal{P}(E)$. Hence, $\mathcal{S}^{\supseteq} = \mathcal{S}^{\subseteq} = \mathcal{P}(E)$.

$(\leftarrow)$ Since $\mathcal{S}^{\supseteq} = \mathcal{P}(E)$, then it contains the empty set and therefore $\mathcal{S}$ contains the empty set. On the other hand, since $\mathcal{S}^{\subseteq} = \mathcal{P}(E)$ it contains E and therefore $\mathcal{S}$ contains E. Hence, $\{\emptyset, E\} \subseteq \mathcal{S}$.

(b) Since $\mathcal{S}$ is nonempty, then $\emptyset \in \mathcal{S}^{\subseteq}$ and $E \in \mathcal{S}^{\supseteq}$. It then follows that $(\mathcal{S}^{\subseteq})^{\supseteq} = (\mathcal{S}^{\supseteq})^{\subseteq} = \mathcal{P}(E)$.

(c) $(\rightarrow)$ Let $X \in (\mathcal{S}^{\subseteq})^{u}$, then $X = \bigcup_{k=1}^{n} A_k$ for some $A_k \in \mathcal{S}^{\subseteq}$. Observe the following:

$$X = \bigcup_{k=1}^{n} A_k \subseteq \bigcup_{A \in \mathcal{S}^{\subseteq}} A = \text{supp}(\mathcal{S}^{\subseteq}) = \text{supp}(\mathcal{S})$$

Thus, $X \in \mathcal{P}(\text{supp}(\mathcal{S}))$ and therefore $(\mathcal{S}^{\subseteq})^{u} \subseteq \mathcal{P}(\text{supp}(\mathcal{S}))$.

$(\leftarrow)$ Note that $\{\emptyset\} \cup \{\{x\} \colon x \in \text{supp}(\mathcal{S})\} \subseteq \mathcal{S}^{\subseteq}$. Thus, we have the following relation:

$$\begin{aligned}(\{\emptyset\} \cup \{\{x\} \colon x \in \text{supp}(\mathcal{S})\})^{u} &= (\{\emptyset\}^{u} \cup \{\{x\} \colon x \in \text{supp}(\mathcal{S})\}^{u})^{u} \\ &= (\{\emptyset\} \cup \mathcal{P}_{\geq 1}(\text{supp}(\mathcal{S})))^{u} \\ &= (\mathcal{P}(\text{supp}(\mathcal{S})))^{u} \\ &= \mathcal{P}(\text{supp}(\mathcal{S}))\end{aligned}$$

Thus, by the inclusion preservation of union closure, we have $\mathcal{P}(\text{supp}(\mathcal{S})) \subseteq (\mathcal{S}^{\subseteq})^u$ and therefore equality.

. . . .

Note that $\text{supp}(\mathcal{S}) \in \mathcal{S}^u$ since $\text{supp}(\mathcal{S}) = \bigcup_{A \in \mathcal{S}} A$. Moreover, every set in $\mathcal{S}^u$ is a subset of $\text{supp}(\mathcal{S})$. Thus, $(\mathcal{S}^u)^{\subseteq} = \mathcal{P}(\text{supp}(\mathcal{S}))$.

(d) Since $\mathcal{S}$ is nonempty, then $\emptyset \in \mathcal{S}^{\subseteq}$. It then follows that $\lfloor \mathcal{S}^{\subseteq} \rfloor = \{\emptyset\}$.

(e) Since $\mathcal{S}$ is nonempty, then $E \in \mathcal{S}^{\subseteq}$. It then follows that $\lceil \mathcal{S}^{\supseteq} \rceil = \{E\}$ since every set in $\mathcal{S}$ is a subset of E and E is the only set that contains itself.

(f) Note that $\text{supp}(\mathcal{S}) \in \mathcal{S}^u$ since $\text{supp}(\mathcal{S}) = \bigcup_{A \in \mathcal{S}} A$. Moreover, $\text{supp}(\mathcal{S})$ is not a proper subset for any $A \in \mathcal{S}^{\supseteq}$. Thus, $\lceil \mathcal{S}^u \rceil = \{\text{supp}(\mathcal{S})\}$.

(g) Let $X \in \mathcal{S}^u$, then $X = \bigcup_{k=1}^{n} A_k$ where $A_k \in \mathcal{S}$. Thus, $X \supseteq A_k$ for each $A_k \in \mathcal{S}$. Hence, $X \in \mathcal{S}^{\supseteq}$.

Note: The reverse containment does not necessarily hold. Take $\mathcal{S} = \{\{ab\}, \{cd\}\}$, then $\mathcal{S}^u = \{\{ab\}, \{cd\}, \{abcd\}\}$ which is a proper subset of
$\mathcal{S}^{\supseteq} = \{\{ab\}, \{cd\}, \{abc\}, \{abd\}, \{acd\}, \{bcd\}, \{abcd\}\}$.

(h) Let $X \in \mathcal{S}^i$, then $X = \bigcap_{k=1}^{n} A_k$ where $A_k \in \mathcal{S}$. Thus, $X \subseteq A_k$ for each $A_k \in \mathcal{S}$. Hence, $X \in \mathcal{S}^{\subseteq}$.

Note: The reverse containment does not necessarily hold. Take $\mathcal{S} = \{\{ab\}, \{bc\}\}$, then $\mathcal{S} = \{\{ab\}, \{bc\}, \{b\}\}$ which is a proper subset of $\mathcal{S}^{\subseteq} = \{\{ab\}, \{bc\}, \{a\}, \{b\}, \{c\}, \emptyset\}$.

□

Support Interactions

Lemma A.2. *Let $\mathcal{S} \subseteq \mathcal{P}(E)$.*

(a) $\text{supp}(\mathcal{S}) = \text{supp}(\mathcal{S}^{\subseteq}) = \text{supp}(\mathcal{S}^u) = \text{supp}(\mathcal{S}^i)$

(b) $\emptyset \neq \mathcal{S} \implies \operatorname{supp}(\mathcal{S}^{\supseteq}) = E$

(c) $\emptyset \notin \mathcal{S} \cup \mathcal{T}$ *and* $\operatorname{supp}(\lceil \mathcal{S} \rceil) \cap \operatorname{supp}(\lceil \mathcal{T} \rceil) = \emptyset \implies \lceil \mathcal{S} \cup \mathcal{T} \rceil = \lceil \mathcal{S} \rceil \cup \lceil \mathcal{T} \rceil$

Proof.

(a) By the inclusion preservation of the support operator, we have that $\operatorname{supp}(\mathcal{S})$ is a subset of $\operatorname{supp}(\mathcal{S}^{\subseteq})$, $\operatorname{supp}(\mathcal{S}^{u})$, and $\operatorname{supp}(\mathcal{S}^{i})$. Thus, we only need to show each reverse inclusion to attain equality.

($\operatorname{supp}(\mathcal{S}^{\subseteq}) \subseteq \operatorname{supp}(\mathcal{S})$) Note that for all $C \in \mathcal{S}^{\subseteq}$ we have $C \subseteq A$ for some $A \in \mathcal{S}$. Thus,

$$\operatorname{supp}(\mathcal{S}^{\subseteq}) = \bigcup_{C \in \mathcal{S}^{\subseteq}} C \subseteq \bigcup_{A \in \mathcal{S}} A = \operatorname{supp}(\mathcal{S})$$

($\operatorname{supp}(\mathcal{S}^{u}) \subseteq \operatorname{supp}(\mathcal{S})$) Note that for all $C \in \mathcal{S}^{u}$ we have $C = \bigcup_{k=1}^{n} A_k$ for some $A_k \in \mathcal{S}$. Thus,

$$\operatorname{supp}(\mathcal{S}^{u}) = \bigcup_{j=1}^{m} C_j = \bigcup_{j=1}^{m} \left(\bigcup_{k=1}^{n} A_{jk} \right) \subseteq \bigcup_{A \in \mathcal{S}} A = \operatorname{supp}(\mathcal{S})$$

($\operatorname{supp}(\mathcal{S}^{i}) \subseteq \operatorname{supp}(\mathcal{S})$) Note that for all $C \in \mathcal{S}^{i}$ we have $C = \bigcap_{k=1}^{n} A_k$ for some $A_k \in \mathcal{S}$. In particular, $C \subseteq A_k$ for each $k = 1, \ldots, n$. Thus,

$$\operatorname{supp}(\mathcal{S}^{i}) = \bigcup_{j=1}^{m} C_j \subseteq \bigcup_{j=1}^{m} \left(\bigcup_{k=1}^{n} A_{jk} \right) \subseteq \bigcup_{A \in \mathcal{S}} A = \operatorname{supp}(\mathcal{S})$$

(b) Since $\mathcal{S}$ is nonempty, then $\{E\}$ is in $\mathcal{S}^{\supseteq}$. And so we have $E \subseteq \operatorname{supp}(\mathcal{S}) \subseteq E$. Hence, equality.

(c) Maximality properties always gives us one containment: $\lceil \mathcal{S} \cup \mathcal{T} \rceil \subseteq \lceil \mathcal{S} \rceil \cup \lceil \mathcal{T} \rceil$.

To show the reverse containment let $X \in \lceil \mathcal{S} \rceil \cup \lceil \mathcal{T} \rceil$. Without loss of generality let $X \in \lceil \mathcal{S} \rceil$. So then $X \in \mathcal{S}$ and $X \not\subset A$ for all $A \in \mathcal{S}$. Moreover, X is nonempty since $\emptyset \notin \mathcal{S} \cup \mathcal{T}$.

Suppose (by contradiction) that there exists $B \in \mathcal{T}$ such that $X \subset B$. Thus, there exists a nonempty $\hat{B} \in \lceil \mathcal{T} \rceil$ such that $X \subset B \subseteq \hat{B}$. Note that $\hat{B} \subseteq \text{supp}(\lceil \mathcal{T} \rceil)$ and $X \subseteq \text{supp}(\lceil \mathcal{S} \rceil)$. However, the two supports are disjoint and so we have $X \cap \hat{B} = \emptyset$. In particular, $X \not\subset \hat{B}$; a contradiction! Therefore, $X \not\subset B$ for all $B \in \mathcal{T}$.

Thus, $X \in \mathcal{S} \cup \mathcal{T}$ and $X \not\subset C$ for all $C \in (\mathcal{S} \cup \mathcal{T})$ which implies that $X \in \lceil \mathcal{S} \cup \mathcal{T} \rceil$. Hence, $\lceil \mathcal{S} \rceil \cup \lceil \mathcal{T} \rceil \subseteq \lceil \mathcal{S} \cup \mathcal{T} \rceil$. Therefore, $\lceil \mathcal{S} \cup \mathcal{T} \rceil = \lceil \mathcal{S} \rceil \cup \lceil \mathcal{T} \rceil$ as desired.

□

Additional Complementary Interactions

Lemma A.3. *Let* $\mathcal{S} \subseteq \mathcal{P}(E)$.

(a) $\mathcal{S}^{u-} = \mathcal{S}^{-i}$

(b) $\mathcal{S}^{-u} = \mathcal{S}^{i-}$

Proof.

(a) Observe the following: $W \in \mathcal{S}^{u-} \iff E - W \in \mathcal{S}^{u} \iff E - W = \bigcup_{k=1}^{n} A_k$ for some $A_k \in \mathcal{S} \iff W = \bigcap_{k=1}^{n}(E - A_k)$ for some $(E - A_k) \in \mathcal{S}^{-} \iff W \in \mathcal{S}^{-i}$.

(b) Applying the previous result to the collection $\mathcal{S}^{-}$ and then applying the complementary operator yields $\mathcal{S}^{-u--} = \mathcal{S}^{--i-}$. Since the complementary is self-inverse, then we have $\mathcal{S}^{-u} = \mathcal{S}^{i-}$.

□

A.2.2 Simple Operator Synergies

Minimality and Union Closure Interactions

With union closure being a weaker version of the superset closure operator, it has some nice interactions with the minimality operator that follow from proposition 2.1.

Lemma A.4. *Let* $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(E)$.

(a) $\mathcal{S} \subseteq \mathcal{T}^u$ *and* $\mathcal{T} \subseteq \mathcal{S}^u \implies \lfloor \mathcal{S} \rfloor = \lfloor \mathcal{T} \rfloor$

(b) $\mathcal{S} \subseteq \lfloor \mathcal{S} \rfloor^u$ *and* $\mathcal{T} \subseteq \lfloor \mathcal{T} \rfloor^u$ *and* $\lfloor \mathcal{S} \rfloor = \lfloor \mathcal{T} \rfloor \implies \mathcal{S} \subseteq \mathcal{T}^u$ *and* $\mathcal{T} \subseteq \mathcal{S}^u$

(c) $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{S}^u \implies \lfloor \mathcal{S} \rfloor = \lfloor \mathcal{T} \rfloor$

Proof.

(a) By assumption we have $\mathcal{S} \subseteq \mathcal{T}^u \subseteq \mathcal{T}^{\subseteq}$ and $\mathcal{T} \subseteq \mathcal{S}^u \subseteq \mathcal{S}^{\subseteq}$. Therefore, we have $\lfloor \mathcal{S} \rfloor = \lfloor \mathcal{T} \rfloor$.

(b) Let $X \in \mathcal{S}$. If $X \in \lfloor \mathcal{S} \rfloor$, then $X \in \lfloor \mathcal{T} \rfloor \subseteq \mathcal{T} \subseteq \mathcal{T}^u$. If $X \notin \lfloor \mathcal{S} \rfloor$, observe the following: Since $\mathcal{S} \subseteq \lfloor \mathcal{S} \rfloor^u$, then $X = \bigcup_{k=1}^{n} W'_k$ where $W'_k \in \lfloor \mathcal{S} \rfloor$. Note that for each k, $W'_k \in \lfloor \mathcal{S} \rfloor$ and $W'_k \subset X$. Since $\lfloor \mathcal{S} \rfloor = \lfloor \mathcal{T} \rfloor$, then $W'_k \in \lfloor \mathcal{T} \rfloor \subseteq \mathcal{T}$. Hence, $X \in \mathcal{T}^u$ and therefore $\mathcal{S} \subseteq \mathcal{T}^u$.

Following the same argument as above with $Y \in \mathcal{T}$ one can show that $Y \in \mathcal{S}^u$. Hence, we have shown that $\mathcal{S} \subseteq \mathcal{T}^u$ and $\mathcal{T} \subseteq \mathcal{S}^u$.

(c) Since $\mathcal{S} \subseteq \mathcal{T}$, then $\mathcal{S} \subseteq \mathcal{T}^u$. By assumption, $\mathcal{T} \subseteq \mathcal{S}^u$, and so we have $\lfloor \mathcal{S} \rfloor = \lfloor \mathcal{T} \rfloor$.

□

Maximality and Intersection Closure Interactions

Similarly, intersection closure is a weaker version of the subset closure operator, and so it also plays nicely with the maximality operator following from lemma 3.1.

Lemma A.5. *Let* $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(E)$.

(a) $\mathcal{S} \subseteq \mathcal{T}^i$ *and* $\mathcal{T} \subseteq \mathcal{S}^i \implies \lceil \mathcal{S} \rceil = \lceil \mathcal{T} \rceil$

(b) $\mathcal{S} \subseteq \lceil \mathcal{S} \rceil^i$ *and* $\mathcal{T} \subseteq \lceil \mathcal{T} \rceil^i$ *and* $\lceil \mathcal{S} \rceil = \lceil \mathcal{T} \rceil \implies \mathcal{S} \subseteq \mathcal{T}^i$ *and* $\mathcal{T} \subseteq \mathcal{S}^i$

(c) $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{S}^i \implies \lceil \mathcal{S} \rceil = \lceil \mathcal{T} \rceil$

Proof.

(a) By assumption we have $\mathcal{S} \subseteq \mathcal{T}^i \subseteq \mathcal{T}^{\subseteq}$ and $\mathcal{T} \subseteq \mathcal{S}^i \subseteq \mathcal{S}^{\subseteq}$. Therefore, we have $\lceil \mathcal{S} \rceil = \lceil \mathcal{T} \rceil$.

(b) Let $X \in \mathcal{S}$. If $X \in \lceil \mathcal{S} \rceil$, then $X \in \lceil \mathcal{T} \rceil \subseteq \mathcal{T} \subseteq \mathcal{T}^i$. If $X \notin \lceil \mathcal{S} \rceil$, observe the following: Since $\mathcal{S} \subseteq \lceil \mathcal{S} \rceil^i$, then $X = \bigcap_{k=1}^{n} \hat{W}_k$ where $\hat{W}_k \in \lceil \mathcal{S} \rceil$. Note that for each k we have $\hat{W}_k \in \lceil \mathcal{S} \rceil$ and $X \subset \hat{W}_k$. Since $\lceil \mathcal{S} \rceil = \lceil \mathcal{T} \rceil$, then $W_k \in \lceil \mathcal{T} \rceil \subseteq \mathcal{T}$ for each k. Hence, $X \in \mathcal{T}^i$ and therefore $\mathcal{S} \subseteq \mathcal{T}^i$.

Following the same argument as above with $Y \in \mathcal{T}$ one can show that $Y \in \mathcal{S}^i$. Hence, we have shown that $\mathcal{S} \subseteq \mathcal{T}^i$ and $\mathcal{T} \subseteq \mathcal{S}^i$.

(c) Since $\mathcal{S} \subseteq \mathcal{T}$, then $\mathcal{S} \subseteq \mathcal{T}^i$. By assumption, $\mathcal{T} \subseteq \mathcal{S}^i$, so then we have $\lceil \mathcal{S} \rceil = \lceil \mathcal{T} \rceil$.

□

A.2.3 Matroid Operators

Lemma A.6. *Let* $\mathcal{S} \subseteq \mathcal{P}(E)$.

(a) $\mathcal{S}^{\mathcal{B}} = \emptyset \iff \emptyset \in \mathcal{S}$ *or* $\{x\} \in \mathcal{S}$ *for each* $x \in E$.

(b) $\mathcal{S}^{\mathcal{D}} \neq \{\emptyset\}$

(c) $\mathcal{S}^{\mathcal{C}} = \emptyset \iff E \in \mathcal{S}$

(d) $\mathcal{S}^{\mathcal{B}} = \emptyset \iff \emptyset \in \mathcal{S}$

Proof.

(a) $(\rightarrow)$ By part (d) of the minimality observations we have that $\mathcal{S}^{\mathcal{B}} = \emptyset \iff \mathcal{S}^{\mathcal{I}} \subseteq \{\emptyset\}$. Suppose that $\mathcal{S}^{\mathcal{I}} = \emptyset$. Then $\mathcal{S}^{\supseteq} = \mathcal{P}(E)$ which implies that $\emptyset \in \mathcal{S}$ by part (e) of the superset closure observations.

Now suppose that $\mathcal{S}^{\mathcal{I}} = \{\emptyset\}$. Then $\mathcal{S}^{\supseteq} = \mathcal{P}(E) - \{\emptyset\}$ which then implies that $\{x\} \in \mathcal{S}$ for each $x \in E$. Since they are each the only nonempty sets that contain themselves; thus in order to be in $\mathcal{S}^{\supseteq}$, they must have already be contained in $\mathcal{S}$.

Therefore, either $\emptyset \in \mathcal{S}$ or $\{x\} \in \mathcal{S}$ for each $x \in E$.

(b) Part (f) of the subset closure observations says that $\emptyset \in \mathcal{S}^{\subseteq}$ for all nonempty collections $\mathcal{S}$. Thus, $\emptyset \notin \mathcal{S}^{\mathcal{D}}$ for all nonempty $\mathcal{S}$ by part (a) of the bases-to-dependent observations. In the case where $\mathcal{S} = \emptyset$, we then have $\emptyset^{\subseteq} = \emptyset$ which implies that $\mathcal{S}^{\mathcal{D}} = \mathcal{P}(E)$. So for any collection $\mathcal{S}$, we have $\mathcal{S}^{\mathcal{D}} \neq \{\emptyset\}$.

(c) This follows from part (g) of the maximality observations and part (c) of the bases-to-dependent observations: $\mathcal{S}^{\mathcal{C}} = \emptyset \iff \mathcal{S}^{\mathcal{D}} = \emptyset \iff E \in \mathcal{S}$.

(d) This follows from part (g) of the minimality observations and part (c) of the cycles-to-independent observations: $\mathcal{S}^{\mathcal{B}} = \emptyset \iff \mathcal{S}^{\mathcal{I}} = \emptyset \iff \emptyset \in \mathcal{S}$.

(e) This is immediate from part (b) of the maximality and subset closure interactions and part (a) of the matroid interactions: $\lceil \mathcal{S}^{\mathcal{I}} \rceil = \mathcal{S}^{\mathcal{B}} \implies \lceil \mathcal{S}^{\mathcal{I}} \rceil^{\subseteq} = \mathcal{S}^{\mathcal{I}\subseteq} = \mathcal{S}^{\mathcal{I}} = \mathcal{S}^{\mathcal{B}\subseteq}$.

□

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Education

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Teaching Experience

Instructor of Record

Syracuse University

- Math 397 – Calculus III Spring 2020 (partially online), Fall 2020 (hybrid), Spring 2021 (hybrid), Fall 2021
- Math 295 – Calculus I Fall 2018, Spring 2019, Fall 2019, Summer 2021 (online), Spring 2022

University of Vermont

- Math 20 – Calculus II Spring 2015, Fall 2015
- Math 19 – Calculus I Spring 2016
- Math 10 – Pre-Calculus Fall 2014

University of Hartford

- Math 240 – Calculus of Several Variables Summer 2015, Summer 2016, Summer 2017, Summer 2018
- Math 112 – A Short Course in Calculus Summer 2014, Summer 2015, Summer 2016

Graduate Teaching Assistant

Syracuse University

- Math 296 – Calculus II Fall 2016, Fall 2017, Spring 2018
- Math 284 – Business Calculus Spring 2017

Professional Presentations

Duality on Collections of Subsets November 2020
Binghamton University Graduate Conference in Algebra and Topology (BUGCAT)

The 'Matroid' Duality Operator September 2021
Syracuse MGO Colloquium

Math Graduate Activities

Association for Women in Mathematics (AWM), SU Student Chapter Spring 2018 – present
Secretary of the SU Student Chapter Fall 2020 – Spring 2021

- Career Panel : Helped plan and execute a career panel consisting of: a postdoctoral researcher in industry, two postdoctoral researchers in academia, a senior program manager in outreach, an assistant professor, and a research mathematician in government.

Directed Reading Program Spring 2020 – present
Program Coordinator Spring 2021 – Spring 2022

- Redefined the standards for both undergraduate and graduate participation with the intention of increasing overall participation in the program.
- Reviewed program applicants, guided graduate mentors, and organized mentor pairings.

Graduate Mentor

- *Combinatorics: Pólya Enumeration Theory* Spring 2021
- *Matroid Theory: Matroid Duality and the Greedy Algorithm* Spring 2021
- *Matroid Theory: Introductory Matroid Theory* Fall 2020
- *Graph Theory: Network Flows and the Augmenting Flow Algorithm* Spring 2020 – Fall 2020
- *Graph Theory: Planar Graphs* Spring 2020

Future Professoriate Program Fall 2019 – Spring 2021
Structured professional development experience for aspiring faculty.

Honors and Awards

Outstanding TA Award Spring 2021
Merit based award for teaching assistants at Syracuse University

Certificate in University Teaching Spring 2021
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