Self-Consistent Crystalline Condensate in Chiral Gross-Neveu and Bogoliubov–de Gennes Systems

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We derive a new exact self-consistent crystalline condensate in the (1 + 1)-dimensional chiral Gross-Neveu model. This also yields a new exact crystalline solution for the one dimensional Bogoliubov–de Gennes equations and the Eilenberger equation of semiclassical superconductivity. We show that the functional gap equation can be reduced to a solvable nonlinear equation and discuss implications for the temperature-chemical potential phase diagram.

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Interacting fermion systems describe a wide range of physical phenomena, from particle physics, to solid state and atomic physics [1–4]. Important paradigms include the Peierls-Frohlich model of conduction [5], the Gorkov–Bogoliubov–de Gennes approach to superconductivity [6], and the Nambu–Jona-Lasinio (NJL) model of symmetry breaking in particle physics [7]. A (1 + 1)-dimensional version of the NJL model, the NJL_2 model [also known as the *chiral* Gross-Neveu model, χGN_2] has been widely studied as it exhibits asymptotic freedom, dynamical mass generation, and chiral symmetry breaking [8–11]. Surprisingly, the temperature-density phase diagram of this system is not yet fully understood. A gap equation analysis based on a homogeneous condensate suggests its phase diagram is the same as its discrete-chiral cousin, the original Gross-Neveu (GN_2) model [8], while recent work finds an inhomogeneous Larkin-Ovchinikov-Fulde-Ferrell (LOFF) helical complex condensate ("chiral spiral") below a critical temperature [12]. In fact, the phase diagram of the discrete-chiral version, the GN_2 model, has only recently been solved in the particle physics literature, analytically [13], and on the lattice [14]. There is a crystalline phase at low temperature and high density, and this phase is characterized by a periodically inhomogeneous (real) condensate that solves exactly the gap equation. This phase is not seen in the old phase diagram which was based on a uniform condensate [15]. Interestingly, this discrete-chiral GN_2 model (with vanishing bare fermion mass) is mathematically equivalent to several models in condensed matter physics: the real periodic condensate may be identified with a polaron crystal in conducting polymers [1,16,17], with a periodic pair potential in quasi 1D superconductors [18,19], and with the real order parameter for superconductors in a ferromagnetic field [20]. This system also is a paradigm of the phenomenon of fermion number fractionalization [21,22]. Variants of such models also apply to ultracold fermionic systems, for which there are interesting new theoretical and experimental developments [4,23,24].

In this Letter, we present an analogous *complex* crystalline condensate for the *chiral* GN system, the NJL_2 model. This condensate is an exact inhomogeneous solution to the gap equation, and also provides a new self-consistent solution to the Bogoliubov–de Gennes (BdG) [6] and Eilenberger [25] equations of superconductivity. Our solution may also be relevant for chiral superconductors and for incommensurate charge density waves in quasi 1D systems [26], which have chiral symmetry and an inherently complex order parameter.

Consider the massless NJL_2 model with Lagrangian

$$\mathcal{L} = \bar{\psi} i \not\partial \psi + \frac{g^2}{2}[(\bar{\psi}\psi)^2 + (\bar{\psi} i \gamma^5 \psi)^2], \tag{1}$$

which has a continuous chiral symmetry $\psi \to e^{i\gamma^5 \alpha}\psi$. We have suppressed summation over N flavors, which makes the semiclassical gap equation analysis exact in the $N \to \infty$ limit, a limit in which we can consistently discuss chiral symmetry breaking in 2D. The original GN_2 model [8], without the pseudoscalar interaction term $(\bar{\psi} i \gamma^5 \psi)^2$, has a discrete chiral symmetry $\psi \to \gamma^5 \psi$. There are two equivalent ways to find self-consistent static condensates. First, introduce bosonic condensate fields, $S \equiv \bar{\psi}\psi$ and $P \equiv \bar{\psi} i \gamma^5 \psi$, which we combine into a complex condensate field: $\Delta \equiv S - iP \equiv M e^{i\chi}$. Integrating out the fermion fields we obtain an effective action for the condensate as

$$S_{\rm eff} = -\frac{1}{2Ng^2}\int |\Delta|^2 - i \ln \det[i\not\partial - M e^{-i\gamma^5 \chi}]. \tag{2}$$

The corresponding (complex) gap equation is

$$\Delta(x) = -2iNg^2 \frac{\delta}{\delta \Delta(x)^*} \ln \det[i\not\partial - M(x) e^{-i\gamma^5 \chi(x)}]. \tag{3}$$

One of the main results of this Letter is that this gap equation can be reduced in an elementary manner to an ordinary differential equation, which moreover is soluble. A second approach to finding a self-consistent condensate is to solve the relativistic Hartree-Fock problem $H\psi = E\psi$, subject to the consistency condition $\langle \bar{\psi}\psi \rangle - i\langle \bar{\psi} i \gamma^5 \psi \rangle = -\Delta/g^2$, with single-particle Hamiltonian

$$H = -i\gamma^5 \frac{d}{dx} + \gamma^0 M(x) e^{-i\gamma^5 \chi(x)} = \begin{pmatrix} -i\frac{d}{dx} & \Delta(x) \\ \Delta^*(x) & i\frac{d}{dx} \end{pmatrix}. \tag{4}$$

This is the Bogoliubov–de Gennes (BdG), or Andreev, Hamiltonian, with $\Delta(x)$ playing the role of the order parameter [6]. We have chosen Dirac matrices $\gamma^0 = \sigma_1$, $\gamma^1 = -i\sigma_2$, and $\gamma^5 = \sigma_3$, to emphasize the natural complex combination $\Delta = S - iP$. The key object in our analysis is the Gork'ov Green's function, and, in particular, its coincident-point limit, the "diagonal resolvent":

$$R(x; E) \equiv \langle x| \frac{1}{H - E} |x\rangle. \tag{5}$$

This is a 2×2 matrix, and the spectral function is $\rho(E) = \frac{1}{\pi} \operatorname{Im} \operatorname{Tr}_{D,x} R(x; E + i\epsilon)$, where the trace is both a Dirac and spatial trace. Approximation methods, such as the gradient and semiclassical expansions, of $R(x; E)$ have been widely studied [27]. In one spatial dimension, $R(x; E)$ can be written in terms of two independent solutions $\psi_{1,2}$ to the Dirac or BdG equation $H\psi = E\psi$. That is, $R(x; E) = (\psi_1 \psi_2^T + \psi_2 \psi_1^T)\sigma_1/(2W)$, with Wronskian $W = -i\psi_1^T \sigma_2 \psi_2$. It follows immediately that $R(x; E)$ satisfies the following first order equation:

$$R' \gamma^5 = i[\gamma^5 (E - \gamma^0 M e^{-i\gamma^5 \chi}), R\gamma^5]. \tag{6}$$

In superconductivity, (6) is known as the Eilenberger equation [25], and in mathematical physics as the Dik'ii equation [28]. It is also straightforward to show that $R = R^\dagger$, $\det R = 1/4$, and $\operatorname{tr}(R\gamma^5) = 0$.

Our main observation is that the gap Eq. (3) motivates a self-consistent ansatz form of the 2×2 matrix $R(x; E)$, and when this is combined with the identity (6), the exact self-consistent condensate and associated resolvent are completely determined. To see this, note that the gap Eq. (3) can be viewed in two ways. First, write the log det term as

$$\int_{-\infty}^{\infty} dE \rho(E) \frac{1}{\beta} \ln(1 + e^{-\beta(E-\mu)}) \tag{7}$$

All dependence on $\Delta(x)$ resides in the spectral function $\rho(E)$, via $\operatorname{Tr} R$, and so the simplest solution to the gap Eq. (3) is for *diagonal* entries of the 2×2 matrix R [recall they are equal since $\operatorname{tr}(R\gamma^5) = 0$] to be linear in $|\Delta|^2$. On the other hand, we can also express the gap equation, by performing the functional derivatives, as

$$\Delta(x) = iNg^2 \operatorname{tr}_{D,E}[(\gamma^0 + \gamma^1) R(x; E)]. \tag{8}$$

This suggests that the *off-diagonal* entries of R be proportional to Δ and Δ^*. In fact, consistency between (6) and (8) introduces derivative terms, leading to

$$R(x; E) = \mathcal{N}(E) \begin{pmatrix} a(E) + |\Delta|^2 & b(E)\Delta - i\Delta' \\ b(E)\Delta^* + i\Delta'^* & a(E) + |\Delta|^2 \end{pmatrix}. \tag{9}$$

The gap equation is satisfied since we find $\operatorname{tr}_E \mathcal{N}(E) = 0$. With this ansatz, the diagonal entries of (6) are identically satisfied, while the off-diagonal entries imply that the condensate Δ satisfy the equation

$$\Delta'' - 2|\Delta|^2 \Delta + i(b - 2E)\Delta' - 2(a - Eb)\Delta = 0. \tag{10}$$

This nonlinear Schrödinger equation (NLSE) is analytically soluble, and all previously known examples of self-consistent condensates in GN_2 and NJL_2 are special cases. Furthermore, the corresponding Dirac or BdG equation $H\psi = E\psi$ is also exactly soluble and has a spectrum consisting of a *single band* in the gap [29]. This provides an elementary explanation of the result from inverse scattering [9–11] that self-consistent ground state condensates are reflectionless (or finite-gap) with a single state (or band). For example, if we specialize to a real condensate, as is relevant for the [discrete-chiral] GN_2 model, the [rescaled] self-consistent solution is $\Delta(x) = \sqrt{\nu}\, \operatorname{sn}(x; \nu)$, which satisfies $\Delta'' - 2\Delta^3 = -(1 + \nu)\Delta$, where sn is a Jacobi elliptic function, with elliptic parameter $0 \le \nu \le 1$ [16–20]. The fermion spectrum has a single band in the gap, centered on $E = 0$, reflecting the charge-conjugation symmetry of the GN_2 system. As $\nu \to 1$, the period becomes infinite, and we obtain the famous kink, $\Delta(x) = \tanh(x)$, with a single bound state at $E = 0$. This midgap zero mode has many interesting physical consequences in polymer systems, and is the paradigm of the fractional fermion number phenomenon [21]. It is also worth noting that for the [discrete-chiral] GN_2 model with a bare fermion mass, the NLSE (10) acquires an inhomogeneous term, in which case the general solution is written as $\Delta(x) = \zeta(\gamma) + \zeta(x) - \zeta(x + \gamma)$, where ζ is the Weierstrass zeta function, and this represents a kink-antikink crystal (or bipolaron crystal in the polymer language [30]), which is a periodic generalization [13] of the Dashen-Hasslacher-Neveu (DHN) kink-antikink solution [9]: $\Delta(x) = \coth(b) + \tanh(x) - \tanh(x + b)$.

All these are *real* condensates, and are well known. The only previously known complex condensates are (i) the simple "chiral spiral" or LOFF solution $\Delta(x) = Ae^{iqx}$; and (ii) Shei's "twisted kink" solution [10], which we can express in complex form

$$\Delta(x) = e^{i\theta/2} \frac{\cosh(x \sin\frac{\theta}{2} - i\frac{\theta}{2})}{\cosh(x \sin\frac{\theta}{2})}. \tag{11}$$

θ is the angle through which the phase of the condensate rotates as x goes from $-\infty$ to $+\infty$. The single-particle fermion spectrum has a single bound state within the gap, located at $E = \cos(\frac{\theta}{2})$, as shown by the dashed line in Fig. 1. Consistency with the gap equation requires vanishing of the coefficient of Δ', which places conditions on parameters of the solution. For example, for Shei's solution the condition is that $\theta/(2\pi)$ is equal to the filling fraction of the bound level [10,11].

We point out a simple physical interpretation of this condition in terms of conserved currents (see also [31]). Both NJL_2 and GN_2 models have a conserved current $j^\mu = \bar{\psi}\gamma^\mu \psi$. Since $\langle j^\mu(x) \rangle = -iN \operatorname{tr}_{D,E}[\gamma^0 \gamma^\mu R(x)]$, we see that,

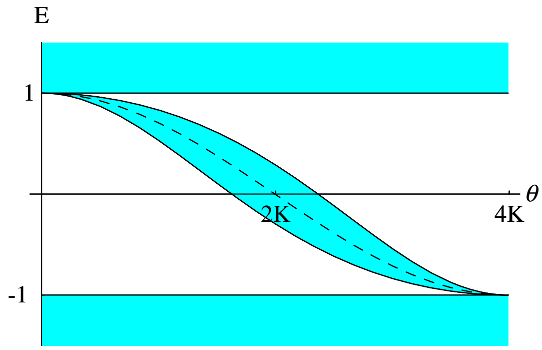

FIG. 1 (color online). Band spectrum of the BdG Hamiltonian (4) for the complex crystal condensate (13), as a function of the twist parameter θ. The dashed line denotes the infinite period limit [in this case $\nu \to 0$], which is Shei's solution (11). At $\theta = 2\mathbf{K}$, we recover the symmetric spectrum of the real kink crystal, relevant to the GN_2 system.

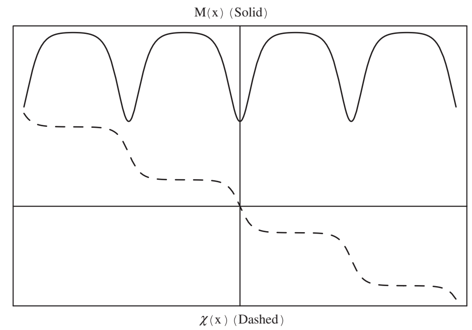

FIG. 2. The amplitude $M(x)$ [solid line] and phase $\chi(x)$ [dashed line] of $\Delta = Me^{i\chi}$ in (13), over several periods. The amplitude is periodic while the phase rotates by an angle 2φ each period.

for a static condensate, current conservation follows trivially since $\langle j^1(x)\rangle = 0$; this is the physical origin of the identity $\mathrm{tr}_D(\gamma^5 R) = 0$. The NJL_2 model also has a conserved axial current, $j_5^\mu = \bar\psi\gamma^\mu\gamma^5\psi$, and the Eilenberger equation (6) implies

$$\partial_\mu\langle j_5^\mu\rangle \equiv -iN\mathrm{tr}(R') = 2S(x)\langle\bar\psi i\gamma^5\psi\rangle - 2P(x)\langle\bar\psi\psi\rangle. \tag{12}$$

Axial current conservation then follows from the gap equation, using precisely the same condition on the coefficient of Δ'.

We have found a nontrivial crystalline solution to (10)

$$\Delta(x) = A\frac{\sigma(iAx + \theta/2 - \mathbf{K})}{\sigma(iAx - \mathbf{K})\sigma(\theta/2)}\exp\{iAx[-\zeta(\theta/2) + \mathrm{cs}(\theta/2)] + \theta\eta/2 + i\mathrm{am}(\theta/2)\}. \tag{13}$$

Here, σ and ζ are Weierstrass sigma and zeta functions, $A(\theta) = 2\mathrm{sd}(\theta/4)\mathrm{cn}(\theta/4)$ and $\eta = \zeta(\mathbf{K})$, with $\mathbf{K}(\nu)$ the elliptic half-period. For this condensate, both the amplitude and the phase are x-dependent, as shown in Figs. 2 and 3. The essential parameters of the solution (13) are: (i) the parameter θ which [via (15) below] characterizes the chiral twist of $\Delta(x)$ over one period; and (ii) the elliptic parameter ν which, together with θ, determines the crystal period. The spinor solutions of the Dirac or BdG equation can also be expressed explicitly in terms of elliptic functions, and one can perform the Hartree-Fock analysis, as in the GN_2 system, to prove self-consistency of the crystalline condensate [29]. The spectrum is that of a Dirac particle with a *single* band in the gap, as shown in Fig. 1. However, unlike the real case where the band lies symmetrically in the center of the gap, here the band is offset. Indeed, the band edges are given by: $E_1 = -1$, $E_2 = -1 + 2\mathrm{cn}^2(\theta/4;\nu)$, $E_3 = -1 + 2\mathrm{cd}^2(\theta/4;\nu)$, $E_4 = +1$, as shown in Fig. 1. This spectrum is a band version of the Shei spectrum, reducing to the Shei solution in the infinite period limit. It is also a deformation of the kink crystal spectrum of the [discrete-chiral] GN_2 system, reducing to that case when $\theta = 2\mathbf{K}$. We find the *exact* diagonal resolvent (9) with

$$\mathcal{N}(E) = \frac{1}{4}\frac{1}{\sqrt{(E^2-1)(E-E_2)(E-E_3)}}, \tag{14}$$

$b(E) = 2E - (E_2 + E_3)$, and $a(E) = 2(E+1)\times[E - 1 - (E_2+E_3)/2] + 1 + (E_2+E_3) - (E_2-E_3)^2/4$.

Under a shift through one period $L = 2\mathbf{K}'/A$ of the crystal, the BdG Hamiltonian is invariant up to a global chiral rotation through an angle φ:

$$H(x+L) = e^{i\gamma^5\varphi}H(x)e^{-i\gamma^5\varphi},$$
$$\varphi = \mathbf{K}'[-\zeta(\theta/2) + \mathrm{cs}(\theta/2) - i\theta\zeta(i\mathbf{K}')/(2\mathbf{K}')]. \tag{15}$$

The solutions to $H\psi = E\psi$ acquire a chiral rotation and a Bloch phase under a period shift, $\psi(x+L) = e^{ikL}e^{i\varphi\gamma^5}\psi(x)$. The [relativistic] Bloch momentum k is related to the spectral function by $\rho(E) = dk/dE$.

To conclude, we discuss briefly the implications of this self-consistent solution to the gap equation for the (T, μ) phase diagram of the NJL_2 model. Recall that the corresponding real condensate characterizes the inhomogeneous crystalline phase of the GN_2 system [13]. For the NJL_2 model, the Ginzburg-Landau (GL) approach shows that the "chiral spiral" phase identified in [12] has a richer struc-

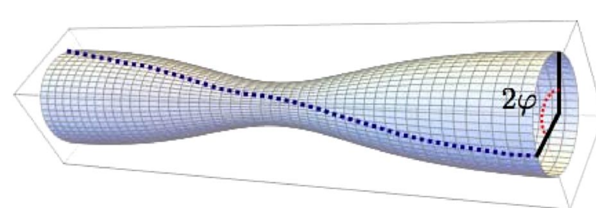

FIG. 3 (color online). Crystalline complex kink (13) plotted as a function of x over one period. The cross section denotes $\Delta(x) = S(x) - iP(x)$, and indicates a net rotation through the twist parameter 2φ over one period.

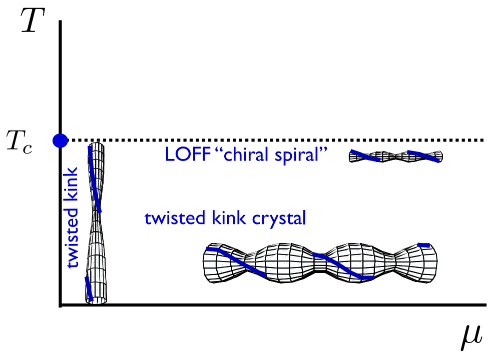

FIG. 4 (color online). Phase diagram of the chiral NJL_2 model, from a Ginzburg-Landau analysis based on a crystalline condensate of the form in (13). Along the T axis (below T_c) the condensate is of the form of Shei's twisted kink (11), while on the $T = T_c$ line, the condensate reduces to the form of the LOFF chiral spiral [12]. The condensate is depicted as in Fig. 3.

ture, characterized by a crystalline complex condensate of the form (13). Near the tricritical point of the massless NJL_2 system, the GL effective Lagrangian is

$$\mathcal{L}_{GL} = c_0 + c_2|\Delta|^2 + c_3\mathrm{Im}[\Delta(\Delta')^*] + c_4[|\Delta|^4 + |\Delta'|^2] + c_5\mathrm{Im}[(\Delta'' - 3|\Delta|^2\Delta)(\Delta')^*] + c_6\{2|\Delta|^6 + 8|\Delta|^2|\Delta'|^2 + 2\mathrm{Re}[(\Delta')^2(\Delta^*)^2] + |\Delta''|^2\}. \quad (16)$$

Here, the coefficients c_n are known functions of T and μ [32]. (This GL approach has been used previously in [32] to describe the phase diagram of the massive and massless NJL_2 models, in the vicinity of the tricritical point. For the *massive* NJL_2 model no (complex) exact solution to the gap equation is known, so [32] is the current state-of-the-art for the massive model.) In the GN_2 model, which has a real condensate, there is a tricritical point at $T = 0.3183$, $\mu = 0.6082$, given by the point $c_2 = c_4 = 0$. In the [chiral] NJL_2 model, which has a complex condensate, there is a tricritical point at $T = 0.5669$, $\mu = 0$; given by $c_2 = c_3 = 0$. To search for possible crystalline phases near the tricritical point, we keep terms up to c_4 and study the effective equation of motion

$$c_4\Delta'' - ic_3\Delta' - (c_2 + 2c_4|\Delta|^2)\Delta = 0. \quad (17)$$

Note that this has precisely the same form as the NLSE (10) found from the gap equation and the Eilenberger equation. Thus, we can use our solution (13) as a variational ansatz and compute the free energy. We have found that the resulting free energy is lower than that of the LOFF-form "chiral spiral" variational ansatz $\Delta(x) = Ae^{iqx}$, for $T < T_c$. Noting that our ansatz reduces to the chiral spiral in the perturbative limit (just as the GN_2 self-consistent crystal condensate reduces to the LOFF form $\Delta(x) = A\sin(qx)$ in this limit [13]), we are led to the phase diagram shown in Fig. 4. For a detailed discussion of the phase diagram, based on the *full* free energy (beyond the GL approximation), using our exact spectral data, see [29].

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