

Chern-Simons diffusion rate in strongly coupled $\mathcal{N} = 4$ SYM plasma in an external magnetic field

Gökçe Başar¹ and Dmitri E. Kharzeev^{1,2}

¹*Department of Physics and Astronomy, Stony Brook University, Stony Brook, New York 11794, USA*

²*Department of Physics, Brookhaven National Laboratory, Upton, New York 11973, USA*

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We calculate the Chern-Simons diffusion rate in a strongly coupled $\mathcal{N} = 4$ super Yang-Mills plasma in the presence of a constant external $U(1)_R$ magnetic flux via the holographic correspondence. Because of the strong interactions between the charged fields and non-Abelian gauge fields, the external Abelian magnetic field affects the thermal Yang-Mills dynamics and increases the diffusion rate, regardless of its strength. We obtain the analytic results for the Chern-Simons diffusion rate both in the weak and strong magnetic field limits. In the latter limit, we show that the diffusion rate scales as $B \times T^2$ and this can be understood as a result of a dynamical dimensional reduction.

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I. INTRODUCTION

The non-Abelian gauge fields possess a rich topological structure. The vacuum contains an infinite number of energy-degenerate sectors characterized by an integer: the Chern-Simons number N_{CS} that is a topological quantity determined by the global structure of the gauge fields. At zero temperature, the different Chern-Simons sectors are connected by quantum tunneling transitions (the instantons [1,2]); this leads to the picture of “ θ -vacuum” of non-Abelian gauge theories. At finite temperature, the gauge configurations that change the Chern-Simons number can also be activated thermally. We will, loosely speaking, refer to these configurations as “sphalerons” both at weak and at strong coupling, even though this term is usually reserved for describing the classical solutions at weak coupling [3,4]. As opposed to the tunneling processes (instantons), the sphaleron rate is not necessarily exponentially suppressed [5,6].

The change of Chern-Simons number in such a process is given in terms of the topological Pontryagin invariant

$$\Delta N_{\text{CS}} = \frac{g^2}{32\pi^2} \int d^4x F_{\mu\nu}^a \tilde{F}_a^{\mu\nu}(x) = \frac{g^2}{8\pi^2} \int d^4x \text{tr} \mathbf{E} \cdot \mathbf{B}, \quad (1)$$

where \mathbf{E} and \mathbf{B} are non-Abelian electric and magnetic fields, and g is the Yang-Mills coupling. The rate of change of Chern-Simons number is called the Chern-Simons diffusion rate Γ_{CS} . It is simply the probability of a Chern-Simons number-changing process to occur per unit volume and per unit time

$$\begin{aligned} \Gamma_{\text{CS}} &= \frac{\langle \Delta N_{\text{CS}}^2 \rangle}{Vt} \\ &= \int d^4x \left\langle \frac{g^2}{32\pi^2} F_{\mu\nu}^a \tilde{F}_a^{\mu\nu}(x) \frac{g^2}{32\pi^2} F_{\alpha\beta}^a \tilde{F}_a^{\alpha\beta}(0) \right\rangle. \end{aligned} \quad (2)$$

The weak coupling result for the diffusion rate for $SU(2)$ gauge theory is given by [7]

$$\Gamma_{\text{CS}} = \kappa' g^{10} \log(1/g^2) T^4 \quad (SU(2), \text{weak coupling}), \quad (3)$$

where κ' is a (numerically large) constant. This expression can be understood as a result of the Langevin—type dynamics of the non-Abelian gauge fields [7,8]. Since the sphaleron at the peak of the barrier separating Chern-Simons sectors is a purely magnetic field configuration, the factor $(g^2 T)^3$ in this expression can be understood as the inverse magnetic screening length that determines the characteristic inverse volume of the sphaleron, and $g^4 T \log(1/g)$ —as the typical inverse time scale of the process. Numerical calculations show that the constant $\kappa' \sim 10$ [8,9] for $SU(2)$.

The holographic AdS/CFT correspondence [10–12] makes it possible to compute the Chern-Simons diffusion rate in $\mathcal{N} = 4$ super Yang-Mills plasma at $N \rightarrow \infty$ in the strong coupling regime [13]. The strong coupling result is given by [13]

$$\Gamma_{\text{CS}} = \frac{(g^2 N)^2}{256\pi^3} T^4 \quad (\mathcal{N} = 4 \text{ SYM, strong coupling, large } N) \quad (4)$$

It is remarkable that compared to the weak coupling case, at strong coupling the diffusion rate is substantially enhanced. This means that Chern-Simons diffusion is not exclusively the property of semiclassical fields, as the (Minkowski boundary) dynamics at strong coupling has to be driven by quantum effects.

The Atiyah-Singer index theorem relates the change in N_{CS} to the chirality change in the fermionic sector—every Chern-Simons number-changing transition is accompanied by the flip of chirality. In thermal equilibrium, the sphalerons thus lead to the decay of any excess chiral charge $N_5 = \langle J_5^0 \rangle$ present in the medium. A linear response relation for a small chiral chemical potential leads to [8,9] the decay

$$\frac{dN_5}{dt} = -CN_5 \frac{\Gamma_{\text{CS}}}{T^3}, \quad (5)$$

where the constant C depends on the details of the fermionic sector.

These N_{CS} -changing transitions violate locally P and CP symmetries and lead to very interesting consequences in a number of various physical settings. In weak interactions, they are related to the violation of baryon plus lepton ($B + L$) number¹ and play a crucial role in electroweak baryogenesis scenarios [5,6,14–19]. In strong interactions, there exists a very strong constraint on the amount of global P and CP violation; this constraint originates mainly from the experimental upper bound for the neutron dipole moment [20]. However on theoretical grounds, the observed CP invariance is still lacking a conclusive explanation (“the strong CP problem”). On the other hand, the instantons and sphalerons provide explicit examples of the *local* fluctuations of topological charge leading, through the index theorem, to the local imbalance of chirality in QCD plasma. In a medium with high enough temperature, the fluctuations might lead to observable effects such as the “chiral magnetic effect” [21] which is an induction of an electric current by an external magnetic field in the presence of Chern-Simons number-changing processes.

In this paper we address the effect that an external Abelian magnetic field has on the topological fluctuations in the plasma. This question has been addressed in the past in several different contexts. The chiral magnetic effect is seen in various lattice simulations [22–24]. The effects of magnetic field on an instanton configuration is also studied on the lattice [25,26]. In [27] the Dirac spectrum of an instanton in the presence of magnetic field is analyzed and it is argued that in strong magnetic field regime some of the physical quantities such as the magnetic and electric dipole moments are dominated by the (near-) zero modes. In [28], it is shown that the diffusion rate due to an electroweak sphaleron in the Higgs phase increases with a presence of an external magnetic field.

Here we calculate the diffusion rate for strongly coupled $\mathcal{N} = 4$ plasma in the presence of an external magnetic field via holography. The dual gravity is characterized by a full solution of asymptotically AdS_5 , five-dimensional Einstein-Maxwell system with a constant magnetic flux. Since the solution takes into account all of the back-reaction of the magnetic field on the metric, this gravity description allows one to work with arbitrarily strong magnetic fields. This configuration was previously studied in [29] and, together with its various extensions, was pursued in a different context in [30].

The rest of the paper is organized as follows. In Sec. II, we summarize the magnetic brane solution presented in [29] which constitutes our dual metric. Then, in Sec. III, we calculate the diffusion rate and present the exact numerical result. Section IV, is devoted to the high-temperature and

low-temperature limits of this result, for which we obtain the analytic expressions. At high temperature we calculate the leading term to the zero magnetic field result (4) analytically by treating the magnetic field perturbatively. At low temperature, we use the (2 + 1)-dimensional Bañados, Teitelboim, Zanelli (BTZ) black hole solution to describe the dimensionally reduced metric and express the rate also analytically at this limit. We conclude the paper by arguing that the effect is negligible for heavy ion collisions and discuss a possible implication for a particular scenario of electroweak baryogenesis.

II. THE DUAL GEOMETRY

Let us briefly review the magnetic brane solution studied in [29] that constitutes the gravity dual of our problem. The basic setup is the five-dimensional Einstein-Maxwell theory with a negative cosmological constant. In the boundary gauge theory, the Abelian Maxwell field is associated with the $U(1)_R$ symmetry which the gauginos, the chiral multiplet fermions and scalars are charged under. The action also contains the five-dimensional Chern-Simons term that accounts for the $U(1)_R$ anomaly

$$S = -\frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left(R + F^{MN} F_{MN} - \frac{12}{l^2} \right) + \frac{1}{6\sqrt{3}\pi G_5} \int A \wedge F \wedge F + S_{\text{bdry}}. \quad (6)$$

The last term in (6) is the boundary term that is determined by imposing a sensible variational principle. The AdS radius l will be set to unity for the rest of the paper. It is also worthwhile to mention that the coefficient of the variation of the Chern-Simons term is fixed by the $U(1)_R$ anomaly of the gauge theory. This relates the field strength F in (6) to the physical field strength in the gauge theory \mathcal{F} as $\mathcal{F} = \sqrt{3}F$. We refer the reader to [29] for further details.

Let us now introduce a constant magnetic flux in the x_3 direction

$$F = B dx^1 \wedge dx^2. \quad (7)$$

The constant flux satisfies the Maxwell equations trivially. For the Einstein equations

$$R_{MN} = 4g_{MN} + \frac{1}{3} F^{AB} F_{AB} g_{MN} - 2F_{MA} F_N^A \quad (8)$$

we start with the general form of the five-dimensional metric

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + e^{2V(r)}(dx_1^2 + dx_2^2) + e^{2W(r)}dx_3^2, \quad (9)$$

and seek asymptotically AdS_5 solutions with a horizon to study the problem at nonzero temperature, in accordance with the AdS/CFT correspondence. Unfortunately, there is no known analytical solution to the Einstein equations

¹We will refer to the violation of $(B + L)$ simply as baryon number violation from now on.

satisfying these properties. However it is possible to solve them numerically. Notice that the metric components U , V and W are functions of a single variable r . Therefore the Einstein Eqs. (8) are a set of coupled ordinary differential equations in r . Following [29], we rescale the coordinates such that the horizon is at $r = 1$ (i.e. $U(1) = 0$), and $U'(1) = 1$, $V(1) = W(1) = 0$. The remaining two initial conditions $V'(1)$ and $W'(1)$ are encoded in the Einstein equations. Then, given the initial conditions, we integrate the Einstein equations from the horizon $r = 1$ to the boundary $r \rightarrow \infty$ numerically. The result is that the solutions indeed are asymptotically AdS_5

$$U(r) \rightarrow r^2, \quad e^{2V(r)} \rightarrow v r^2, \quad e^{2W(r)} \rightarrow w r^2 \quad \text{as } r \rightarrow \infty \quad (10)$$

Here v and w are the functions of the magnetic field strength B . To get the correct AdS_5 limit on the boundary, we should rescale the x_1 , x_2 and x_3 coordinates

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + \frac{e^{2V(r)}}{v}(dx_1^2 + dx_2^2) + \frac{e^{2W(r)}}{w}dx_3^2 \quad (11)$$

This rescaling also affects the form of the physical magnetic flux

$$F = \frac{B}{v} dx_1 \wedge dx_2 \quad (12)$$

It is also useful to define the physical magnetic field strength in the gauge theory \mathcal{B} :

$$\mathcal{B} = \sqrt{3} \frac{B}{v}. \quad (13)$$

III. THE CHERN-SIMONS DIFFUSION RATE

The diffusion rate (2) is nothing but the zero-frequency and wavelength limit of the symmetrized Wightman correlator of the topological charge

$$G^{\text{sym}}(\omega, \vec{k}) \equiv \frac{1}{2} \int d^4x e^{-ikx} \left\langle \left[\frac{1}{4} F_{\mu\nu}^a \tilde{F}_a^{\mu\nu}(x), \frac{1}{4} F_{\mu\nu}^a \tilde{F}_a^{\mu\nu}(0) \right] \right\rangle. \quad (14)$$

The fluctuation-dissipation theorem relates the symmetrized Wightman correlator to the imaginary part of the Green's function

$$G^{\text{sym}}(\omega, \vec{k}) = -\coth\left(\frac{\omega}{2T}\right) \text{Im}[G^R(\omega, \vec{k})] \\ \approx -\frac{2T}{\omega} \text{Im}[G^R(\omega, \vec{k})], \quad (15)$$

where the retarded Green's function is

$$G^R(\omega, \vec{k}) \equiv -i \int d^4x e^{-ikx} \theta(t) \left\langle \left[\frac{1}{4} F_{\mu\nu}^a \tilde{F}_a^{\mu\nu}(x), \frac{1}{4} F_{\mu\nu}^a \tilde{F}_a^{\mu\nu}(0) \right] \right\rangle. \quad (16)$$

Hence, to extract the diffusion rate all we need to know is the zero-momentum and small-frequency limit of the retarded Green's function

$$\Gamma_{\text{CS}} = -\left(\frac{g^2}{8\pi^2}\right)^2 \lim_{\omega \rightarrow 0} \frac{2T}{\omega} \text{Im}[G^R(\omega, \vec{k} = 0)]. \quad (17)$$

The imaginary part of the retarded Green's function is associated with the thermal dissipation and therefore is a T -odd function. As a result the leading term in the small-frequency expansion has to be linear in ω , with the coefficient that controls the diffusion rate (17).

We compute the retarded Green's function G^R of the topological charge by using the AdS/CFT duality where the gravity side is described by the metric (11). The duality relates the operator $\frac{1}{4} F_{\mu\nu}^a \tilde{F}_a^{\mu\nu}$ in $\mathcal{N} = 4$ super Yang-Mills with the Ramond-Ramond scalar C_0 which is the axion in the 10-dimensional type IIB supergravity theory. In the absence of the $U(1)_R$ Maxwell field, the axion is nothing but a free massless scalar in the consistent truncation to the five-dimensional effective theory after Kaluza-Klein reduction of the S^5 . To the second-order in fluctuations in the fields, it remains as a free scalar even with the presence of a background $U(1)_R$ flux. To see this let us analyze the axion part of the 10-dimensional type IIB supergravity action in the string frame

$$S_{\text{axion}} \propto \int \sqrt{-G} (|dC_0|^2 + |C_2 - C_0 \wedge dB|^2), \quad (18)$$

where C_0 is the axion, C_2 and B are antisymmetric Ramond-Ramond and NS-NS 2 forms, respectively. In the gauge theory, the $U(1)_R$ symmetry is realized as the diagonal $U(1)_R$ of the Cartan subgroup $U(1)_R^3$ of $SU(4)_R$. In the supergravity picture, this corresponds to wrapping around the three angles of the S^5 simultaneously by a special choice of truncation parameters. The Kaluza-Klein reduction can be written as [31,32]

$$ds^2 = g_{MN} dx^M dx^N + \sum_{i=1}^3 d\mu_i^2 + \mu_i^2 \left(d\phi_i + \frac{2}{\sqrt{3}} A_\mu dx^\mu \right)^2 \quad (19)$$

where g_{MN} is the five-dimensional AdS metric, μ_i 's are two independent S^5 parameters with the constraint $\sum_{i=1}^3 \mu_i^2 = 1$ and ϕ_i 's are the remaining three angles of S^5 . The Kaluza-Klein reduction of this metric is shown [31,32] to be a consistent truncation that generates the action (6). To see that the Maxwell sector does not affect the axion sector it is sufficient to observe that the only place where the $U(1)_R$ field might show up in the action (18) is the 10-dimensional metric determinant $\sqrt{-G}$. However this is not possible because the gauge field can be thought of as a reparametrization of the angles ϕ_i of S^5 under which G is invariant. From the gauge theory perspective, there cannot be any gauge-dependent term in the action; therefore G has to be independent of A_μ , which is in fact true. Also the NS-NS 2 form B can be turned off consistently in the compactification. As a result, the axion field C_0 is a massless scalar in the five-dimensional Einstein-Maxwell theory.

The associated Klein-Gordon equation for a generic metric of the form (9) is not analytically soluble, since we do not even know the metric explicitly. However it can be expressed in terms of the functions U , V and W that determine the metric at the linear order in frequency, which is the limit we are interested in. The dissipative processes that lead to an imaginary part in the retarded Green's function are associated with the in-falling boundary conditions for the dual scalar field near the black brane horizon. We shall study the Klein-Gordon equation in Eddington-Finkelstein coordinates where the in-falling boundary conditions near the horizon appear naturally. Let us define

$$t_* = t + \int_{\infty}^r \frac{r'}{U(r')} ; \quad (20)$$

the metric (9) with this coordinate transformation becomes

$$ds^2 = -U(r)dt_*^2 + 2drdt_* + e^{2V(r)}(dx_1^2 + dx_2^2) + e^{2W(r)}dx_3^2 \quad (21)$$

Note that the frequency remains unchanged: $\omega_* \sim i\partial_{t_*} = i\partial_t \sim \omega$. Furthermore, the wave functions are of the form $e^{-i\omega t_* + i\vec{k}\cdot\vec{x}} \phi_{\omega, \vec{k}}(r)$. The mode function $\phi_{\omega, \vec{k}}(r)$ should have regular near horizon behavior. Near the horizon, $t_* \approx t + \frac{1}{U'(r_h)} \ln|r - r_h|$ up to a constant, where $r = r_h$ is the horizon. The Hawking temperature for a general $U(r)$ is given by the relation: $U'(r_h) = 4\pi T$. Therefore the wave function has the expected in-falling behavior $e^{-i\omega t} |r - r_h|^{-i(\omega/4\pi T)}$ which arises naturally in the Eddington-Finkelstein coordinates.

The Klein-Gordon equation $\partial_{\mu}(\sqrt{-g}g^{\mu\nu}\partial_{\nu}\phi) = 0$ with the metric (21) reads

$$\phi''_{\omega, \vec{k}} + (f(r) + \ln U(r))' \phi'_{\omega, \vec{k}} - \frac{i\omega}{U(r)}(2\phi'_{\omega, \vec{k}} + f'(r)\phi_{\omega, \vec{k}}) - \frac{1}{U(r)}(e^{-2V(r)}\vec{k}_{\perp}^2 + e^{-2W(r)}\vec{k}_{\parallel}^2)\phi_{\omega, \vec{k}} = 0. \quad (22)$$

Here we defined $e^{f(r)} \equiv \sqrt{-g} = e^{2V(r)+W(r)}$ and $'$ is short for ∂_r . We now take $\vec{k} = 0$ and solve (22) for small frequencies. To do this, we expand $\phi_{\omega}(r)$ to linear order in ω for $\omega \ll 1^2$

$$\phi_{\omega, \vec{k}=0} = \phi_0(r) + i\omega\phi_1(r) + \mathcal{O}(\omega^2). \quad (23)$$

The Klein-Gordon equation is now expanded in the same fashion as

²Here there is a slight abuse of notation. By writing $\omega \ll 1$ we actually mean $\frac{\omega}{4\pi T} \ll 1$. But it is always possible to work with dimensionless frequency by rescaling t_* and r which we are doing by setting $U'(1) = 1$ (i.e., $4\pi T = 1$).

$$\phi_0'' + (f + \ln U)' \phi_0' - i\omega \left(\frac{2}{U} \phi_0' + \frac{f'}{U} \phi_0 - \phi_1'' - (f + \ln U)' \phi_1' \right) = 0, \quad (24)$$

which should be solved order by order in ω .

The $\mathcal{O}(\omega^0)$ order equation has the solution

$$\phi_0(r) = c_1 + c_2 \int^r \frac{dr'}{\sqrt{-g(r')}U(r')}. \quad (25)$$

The second term is divergent near the horizon since $U(r_h) = 0$. In order $\phi_0(r)$ to be well-behaved near the horizon, c_2 must vanish. We also set $c_1 = 1$. This choice translates into normalizing the homogeneous axion source in the boundary theory to unity in the AdS/CFT dictionary.

After plugging $\phi_0(r) = 1$ into (24), the $\mathcal{O}(\omega^1)$ equation becomes

$$\phi_1'' + (f + \ln U)' \phi_1' = \frac{f'}{U}, \quad (26)$$

which can be solved as

$$\phi_1(r) = \int_{\infty}^r \frac{dr'}{U(r')} (1 - e^{f(r_h)-f(r')}) = \int_{\infty}^r \frac{dr'}{U(r')} \left(1 - \frac{\sqrt{-g(r_h)}}{\sqrt{-g(r')}} \right). \quad (27)$$

Here the constant $\sqrt{-g(r_h)}$ is chosen to assure the regularity of the solution near the horizon. The other integration constant is determined by setting the limit of the integral such that ϕ_1 vanishes near the boundary and does not effect the normalization of the source.

As a result the solution of the Klein-Gordon Eq. (22) in the zero-momentum, small-frequency limit is:

$$\begin{aligned} \phi_{\omega, \vec{k}=0} &= 1 + i\omega \int_{\infty}^r \frac{dr'}{U(r')} \left(1 - \frac{\sqrt{-g(r_h)}}{\sqrt{-g(r')}} \right) + \mathcal{O}(\omega^2) \\ &= 1 + i\omega \int_{\infty}^r \frac{dr'}{U(r')} (1 - e^{2V(r_h)+W(r_h)-2V(r')-W(r')}) \\ &\quad + \mathcal{O}(\omega^2). \end{aligned} \quad (28)$$

We now expand (28) near the boundary. Recall that our metric converges to the AdS_5 metric $r^{-2}dr^2 + r^2dx^{\mu}dx_{\mu}$ near the boundary $r \rightarrow \infty$. Therefore $\sqrt{-g} \rightarrow r^3$, $U(r) \rightarrow r^2$ and

$$\phi_{\omega, \vec{k}=0}(r) \approx 1 - \frac{i\omega}{r} + \frac{i\omega\sqrt{-g(r_h)}}{4r^4} + \dots, \quad (29)$$

which reproduces the expected boundary behavior of a massless scalar. The first term is the non-normalizable mode that corresponds to the homogeneous axion source $\delta\Phi$ normalized to unity. The coefficient of the r^{-4} term is: $\delta\langle \frac{1}{4}F_{\mu\nu}^a \tilde{F}_a^{\mu\nu} \rangle$, that is the response of the system to the axion source $\delta\Phi$. Putting back all the constants it is found as

$$\delta\left\langle \frac{1}{4}F_{\mu\nu}^a \tilde{F}_a^{\mu\nu} \right\rangle = (2\Delta_{F\tilde{F}} - 4)\kappa \frac{i\omega\sqrt{-g(r_h)}}{4} = i\kappa\omega\sqrt{-g(r_h)}. \quad (30)$$

Here $\Delta_{F\bar{F}} = 4$ is the scaling dimension of the topological charge operator. We can easily calculate the retarded Green's function in the zero-momentum, small-frequency limit through the linear response relation (Kubo's formula)

$$\delta \left\langle \frac{1}{4} F_{\mu\nu}^a \tilde{F}_a^{\mu\nu}(\omega) \right\rangle = G^R(\omega) \delta\Phi. \quad (31)$$

Thus we obtain G^R in the leading linear frequency limit

$$G^R(\omega, \vec{k} = 0) = \frac{\delta \left\langle \frac{1}{4} F_{\mu\nu}^a \tilde{F}_a^{\mu\nu}(\omega) \right\rangle}{\delta\Phi} = i\kappa\omega\sqrt{-g(r_h)} + \mathcal{O}(\omega^2). \quad (32)$$

Using the properly scaled metric (11) it reduces to

$$G^R(\omega, \vec{k} = 0) = \frac{i\omega\kappa}{v\sqrt{w}} + \mathcal{O}(\omega^2). \quad (33)$$

Recall that the diffusion rate is related to the retarded Green's function as

$$\begin{aligned} \Gamma_{\text{CS}} &= - \left(\frac{g^2}{8\pi^2} \right)^2 \lim_{\omega \rightarrow 0} \frac{2T}{\omega} \text{Im}[G^R(\omega, \vec{k} = 0)] \\ &= \left(\frac{g^2}{8\pi^2} \right)^2 \frac{N^2}{8\pi^2} \frac{2T}{v\sqrt{w}}. \end{aligned} \quad (34)$$

In the last step we have used the holographic relation between the gravitational and gauge couplings: $\kappa = -\frac{1}{16\pi G_5} = -\frac{N^2}{8\pi^2}$ where G_5 is the five-dimensional Einstein constant. Recall that the functions v and w are obtained in the units $U(1) = 0$, $U'(1) = 1$. In these units $T = 1/4\pi$ and due to conformal symmetry we can choose our physical parameter to be the dimensionless magnetic field strength \mathcal{B}/T^2 . We then express the diffusion rate in terms of this dimensionless ratio as follows:

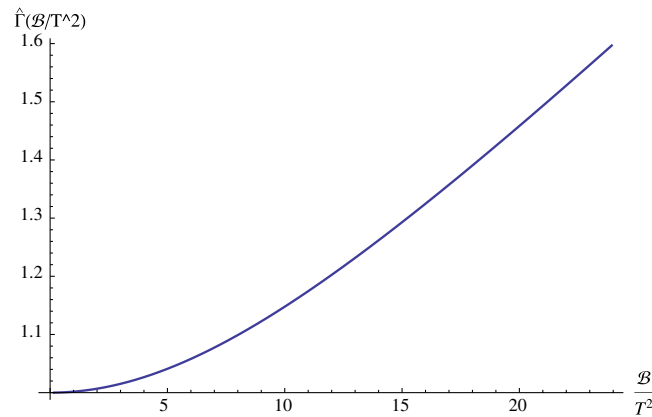


FIG. 1 (color online). The Chern-Simons diffusion rate $\hat{\Gamma} = \Gamma(\mathcal{B}, T)/\Gamma_0(T)$, normalized by the zero magnetic flux value $\Gamma_0 = \frac{\lambda^2}{256\pi^3} T^4$ as a function of the dimensionless magnetic field \mathcal{B}/T^2 . The diffusion rate is monotonously increasing with the magnetic field strength and has the asymptotic behavior $\Gamma \sim \mathcal{B}T^2$ for $\mathcal{B} \gg T^2$.

$$\hat{\Gamma}(\mathcal{B}/T^2) = \frac{\Gamma(\mathcal{B}, T)}{\Gamma_0} = \frac{2^6}{v\sqrt{w}} \quad \Gamma_0 = \frac{(g^2 N)^2}{256\pi^3} T^4. \quad (35)$$

This quantity measures the ratio of the diffusion rate in a medium with a magnetic flux to the one in a medium without a magnetic flux, Γ_0 . From the plot (Fig. 1), we see that $\hat{\Gamma}(\mathcal{B}/T^2)$ is a monotonously increasing function of the magnetic field strength. We now investigate the high- and low-temperature limits of the diffusion rate.

IV. HIGH- AND LOW-TEMPERATURE LIMITS

A. Weak magnetic field (high- T) limit

In the limit where $\mathcal{B} \ll T^2$, the magnetic field can be treated as a perturbation to the T^4 scaling of the diffusion rate. Because of the R -charge symmetry of the gauge theory, we expect the perturbation series to start at the order \mathcal{B}^2 . Also we can argue that we do not expect the result to depend on the direction of the magnetic field at the leading order and the analyticity of the perturbation series dictates the \mathcal{B}^2 dependence. It is also clear from the gravity side that the perturbative corrections to the Einstein Eqs. (8) will be at the order \mathcal{B}^2 since the magnetic field enters the equations in a gauge invariant way. Let us perturb the Einstein Eqs. (8) around the AdS_5 black brane solution

$$\begin{aligned} U(r) &= r^2 - \frac{r_H^4}{r^2} + \mathcal{B}^2 u_1(r) & V(r) &= \log(r) + \mathcal{B}^2 v_1(r) \\ W(r) &= \log(r) + \mathcal{B}^2 w_1(r), \end{aligned} \quad (36)$$

where $r_H = \pi T$. To extract the diffusion rate, we only need to know $\sqrt{-g}$ at the perturbed horizon. Let us call the perturbed horizon r_p , such that $U(r_p) = 0$. By expanding to first order in \mathcal{B}^2 we get

$$\sqrt{-g(r_p)} = e^{2V(r)+W(r)} = r_p^3 (1 + \mathcal{B}^2 (2v_1(r_p) + w_1(r_p))). \quad (37)$$

From the linearized (in \mathcal{B}^2) Einstein equations we obtain the following relations:

$$u_1(r) = -\frac{2}{9r^2} (1 + 3 \log(r/r_p)) \quad 2v_1'(r) + w_1'(r) = 0 \quad (38)$$

These corrections should vanish as $r \rightarrow \infty$ since we should have asymptotically AdS_5 behavior. This condition sets the constant $2v_1(r) + w_1(r) = 0$ leading to

$$\sqrt{g(r_p)} = r_H^3 \left(1 + \frac{1}{6} \frac{\mathcal{B}^2}{r_H^4} \right) = \pi^3 T^3 \left(1 + \frac{1}{6\pi^4} \frac{\mathcal{B}^2}{T^4} \right). \quad (39)$$

As a result, to leading order in \mathcal{B}^2 we get

$$\Gamma_{\text{CS}} = \frac{(g^2 N)^2}{256\pi^3} T^4 \left(1 + \frac{1}{6\pi^4} \frac{\mathcal{B}^2}{T^4} + \mathcal{O}\left(\frac{\mathcal{B}^4}{T^8}\right) \right). \quad (40)$$

Comparing with the exact numerical result (35) we have found that this approximation differs from the exact result only by 2.5% for $\mathcal{B} \sim 10T^2$. This suggests that a proper expansion parameter is probably $\mathcal{B}^2/(\pi T)^4$.

B. Strong magnetic field (low- T) limit and BTZ black hole

For strong magnetic field $B \gg T^2$, v and w have the limits

$$\lim_{B \rightarrow \infty} \frac{1}{v\sqrt{w}} = \frac{B}{6v(4\pi T)^2} = \frac{\mathcal{B}}{96\sqrt{3}\pi^2 T^2}. \quad (41)$$

In the last step we plugged in the physical magnetic field strength $\mathcal{B} = \sqrt{3}B/v$. Therefore in the presence of a strong magnetic field the diffusion rate is

$$\Gamma(\mathcal{B}, T) = \frac{(g^2 N)^2}{384\sqrt{3}\pi^5} \mathcal{B} T^2, \quad \mathcal{B} \gg T^2. \quad (42)$$

It is possible to obtain the same result in a way where the physics is more transparent. In [29], it was shown that there is another solution to Einstein-Maxwell configuration which is a product of a $(2+1)$ -dimensional BTZ black hole in the (r, t, x_3) directions and a flat surface in the transverse plane

$$ds^2 = -3(r^2 - r_h^2)dt^2 + \frac{dr^2}{3(r^2 - r_h^2)} + 3r^2 dx_3^2 + \frac{B}{\sqrt{3}}(dx_1^2 + dx_2^2). \quad (43)$$

The Hawking temperature is $T = 3r_h/2\pi$. This is an appropriate description of the dual gauge theory in the strong magnetic field limit. When $\sqrt{B} \gg T$, the fields occupy the lowest Landau levels and transitions to higher Landau orbits are suppressed by the factor $e^{-\sqrt{B}/T}$. As a result the motion in the transverse plane is frozen and the only remaining degree of freedom is along the longitudinal direction x_3 where the motion is not affected by the magnetic field. The overall factor of B in front of the transverse plane corresponds to the density of the lowest Landau levels in the gauge theory. The geometry (43) is therefore merely the holographic manifestation of the dimensional reduction of the system in the presence of a strong magnetic field. Using $\sqrt{-g} = r_h B$ and the expression (32) for the retarded Green's function we obtain

$$G^R(\omega, \vec{k} = 0) = i\omega\kappa B r_h = i\omega\kappa \frac{2\pi T \mathcal{B}}{3\sqrt{3}} + \mathcal{O}(\omega^2). \quad (44)$$

Note that the expression (32) is valid only for metrics that are asymptotically AdS_5 since the retarded Green's function is read off from the coefficient of r^{-4} in the expansion of ϕ near the boundary. Here in the dimensionally reduced calculation, we assume that the magnetic field strength is very close to UV scale of the gauge theory so that the transition from BTZ to AdS_5 occurs close to the UV boundary and far away from the horizon in the dual

picture. Therefore, the BTZ approximation is good near the horizon and we can use the BTZ metric for $\sqrt{-g(r_h)}$.

Plugging this result for G^R into the expression for the diffusion rate [first line of (35)] leads to the correct strong magnetic field result (42). This result can be interpreted as follows: Γ_{CS} is a transport coefficient and its dynamics are governed by the IR sector of the theory. In the R -charged sector, the IR dynamics is dominated by the lowest Landau level fermions. The contribution of scalars to IR is suppressed exponentially since they have no zero modes as opposed to fermions. The holographic dimensional reduction mechanism explained above suggests that the strong interactions between the dimensionally reduced R -charged sector and R -neutral non-Abelian gauge sector effectively induces a dimensional reduction for the gauge field in the IR, even though it does not see the magnetic field at the level of Lagrangian. Therefore the Chern-Simons number-changing transitions occur in one spatial dimension which brings a factor of T^2 to the rate on dimensional grounds. This should be multiplied by the density of lowest Landau levels in transverse plane which scales as B/π .

V. CONCLUSION AND DISCUSSIONS

In the strongly coupled $\mathcal{N} = 4$ theory, the existence of an external $U(1)_R$ magnetic field increases the Chern-Simons diffusion rate in comparison to the zero magnetic field case. This modification is due to the strong interactions between the R -charged sector of the theory and the R -neutral Yang-Mills fields that are realized through the alteration of the dual gravity by the magnetic field. Furthermore in the strong magnetic field limit the Chern-Simons number transitions occur in the dimensionally reduced system and the rate thus scales as $B \times T^2$. The absence of the interaction between the $U(1)_R$ and Yang-Mills fields makes the $U(1)_R$ magnetic field a reasonable proxy for a "real" $U(1)$ magnetic field. We thus expect a similar behavior in a strongly coupled, non-SUSY gauge theory with a $U(1)$ magnetic field.

We would like to point out an interesting analogy between the electroweak sphaleron in a magnetic field and our result. The electroweak sphaleron in the Higgs phase (Klinkhamer-Manton sphaleron) [4] is spherically symmetric for zero Weinberg angle. However a nonzero Weinberg angle reduces the symmetry into an axial one [33,34]. Furthermore, the deformed sphaleron develops a magnetic moment. Because of this dipole moment, an external magnetic field decreases the energy of the sphaleron which lowers the barrier between the topologically inequivalent vacua [28,35]. This puts some constraints on the electroweak baryogenesis scenario with a primordial magnetic field since a lower barrier means topological transitions are more frequent and they wash out any initial baryon number asymmetry.

Our result suggests that an external magnetic field raises the rate of topological transitions also in strongly coupled

regime. In this case the energy of the sphaleron is altered by the magnetic field due to strong interactions. Furthermore, the spherical symmetry is reduced to axial symmetry through dimensional reduction since there is a residual $O(2)$ symmetry in the transverse plane. These effects of the magnetic field are, of course, due to a different mechanism than the Kinkhamer-Manton sphaleron case above. However, it is noteworthy that even though the mechanisms are quite different, it seems a coupling to an extra $U(1)$ field affects the non-Abelian sector in a similar fashion for both cases.

There are electroweak baryogenesis scenarios where there exist CP -violating processes in the scalar (Higgs) sector [14–17]. These processes can be described by an effective inhomogeneous axion field where chiral quarks are created locally on these inhomogeneities. These inhomogeneities correspond to the walls of the bubbles of broken phase nucleated inside the symmetric phase. As the created net chirality propagates into the symmetric phase it triggers the electroweak sphaleron transitions in the symmetric phase which ultimately creates the baryon number. However, the strong sphaleron transitions damp the net chirality and therefore reduce the baryon number creation [9,18]. Taking into account the existence of a primordial magnetic field, based on our result we can conclude that it would further decrease the rate of baryon

number generation if the relevant strong interactions are in the strongly coupled regime. Therefore, it seems that an existence of a primordial magnetic field works against the baryon number generation.

Finally, let us pass from the big bang to the little bang: the heavy ion collisions. The background magnetic field created in these collisions is typically of the order $T \sim \sqrt{B} \sim m_\pi$ [21,36]. Our result suggests that a magnetic field of this strength has a negligible effect on the topological charge diffusion. To be more precise

$$\frac{\Gamma(\mathcal{B}, T) - \Gamma(\mathcal{B}, 0)}{\Gamma(\mathcal{B}, 0)} \approx \frac{1}{6\pi^4} \approx 0.0017 \quad (45)$$

for $\mathcal{B} = T^2$. Therefore the magnetic field changes the rate only by 0.17%. This means that the quantitative estimates of the chiral magnetic effect in heavy ion collisions can be safely done by using the sphaleron rate in the absence of magnetic flux.

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