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4D-2D equivalence for large-N Yang-Mills theory

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General string-theoretic considerations suggest that four-dimensional large-N gauge theories should have dual descriptions in terms of two-dimensional conformal field theories. However, for nonsupersymmetric confining theories such as pure Yang-Mills theory, a long-standing challenge has been to explicitly show that any such dual descriptions actually exist. In this paper, we consider the large-N limit of four-dimensional pure Yang-Mills theory compactified on a three-sphere in the solvable limit where the sphere radius is small compared to the strong length scale, and demonstrate that the confined-phase spectrum of this gauge theory coincides with the spectrum of an irrational two-dimensional conformal field theory.

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I. INTRODUCTION

Confining gauge theories in the large-N limit are believed to have dual descriptions as weakly-coupled string theories [1]. Since string theories have $2D$ worldsheet conformal field theory (CFT) descriptions, it is expected that confining $4D$ gauge theories may have alternative descriptions based on $2D$ CFTs. However, for nonsupersymmetric quantum field theories (QFTs) such as Yang-Mills (YM) theory, no concrete relation between large-N confining theories and $2D$ CFTs has ever been found.

In this paper we approach this problem by studying the large-N limit of $4D$ pure $SU(N)$ YM theory, formulated at temperature $T = \beta^{-1}$ and compactified on a three-sphere S^3 of radius R. One can thus view the theory as living on $S^3_R \times S^1_\beta$ with Euclidean metric signature. The virtues of this setting are two-fold. First, thanks to asymptotic freedom, if we take $\Lambda R \ll 1$ where Λ is the YM strong scale, then the 't Hooft coupling $\lambda \equiv g^2 N$ becomes small—i.e., $\lambda(1/R) \to 0$. As a result, the theory becomes solvable for any temperature $\beta \sim N^0$. Second, it is known [2] that large-N YM theory stays in the confined phase when $\beta/R \gtrsim 1$, even when $\lambda \to 0$. In this context "confinement" means that the system has an unbroken center symmetry and that its free energy scales as N^0. As sketched in Fig. 1, it is plausible that the physics of YM theory is smooth as a function of ΛR. Thus, the $\Lambda R \ll 1$ regime of the large-N confined phase represents a particularly tractable $4D$ starting point in our search for a dual $2D$ description.

Rather than attempt a string-theory construction of a $2D$ dual for large-N YM theory, we shall instead analyze the confined-phase spectrum of YM theory in the solvable $\Lambda R \ll 1$ limit. We work to the leading nontrivial order in the ΛR expansion, which turns out to be $(\Lambda R)^0$. Although this corresponds to $\lambda = 0$, the fact that the $\lambda = 0$ limit is nontrivial is one of the virtues of working with an S^3 compactification, as discussed above. Remarkably, at least in the $\lambda = 0$ limit, it turns out that a simple $2D$ CFT description emerges. Thus, in this limit, we conclude that the large-N confined-phase spectrum of $4D$ YM theory coincides with the spectrum of a $2D$ CFT. In the conclusions we briefly comment on possible relations between our result and string-theoretic expectations.

Specifically, recall that the complete spectrum of a QFT is encoded in its grand-canonical thermal partition function. We take $4D$ YM theory to be minimally coupled to the S^3 metric, so that the Kaluza-Klein energies on the three-sphere

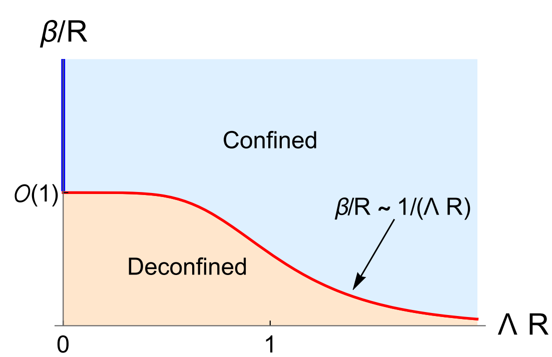

FIG. 1 (color online). A conjectured phase diagram for large-N YM theory on $S^3 \times S^1$. In the analytically tractable regime $\Lambda R \ll 1$, the deconfinement transition occurs at $\beta \sim R$, while for $\Lambda R \gg 1$, lattice studies have shown that it occurs at $\beta \sim 1/\Lambda$. This sketch illustrates the natural conjecture that these two limiting cases are smoothly connected. The results of this paper apply in the $\Lambda R \to 0$ region indicated by the blue line.

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are given by $E_n = n/R$ in the $\lambda \to 0$ limit [2]. The partition function then takes the form

$$Z_{\mathrm{YM}}(\beta/R) = \sum_{n=0}^{\infty} d_n e^{-\beta E_n} = \sum_{n=0}^{\infty} d_n q^n \qquad (1)$$

where $q = e^{-\beta/R}$ and d_n counts the number of states with energy E_n. Our main result is the demonstration that the grand-canonical partition function Z_{YM} of Yang-Mills theory coincides with a chiral partition function of a $2D$ CFT:

$$Z_{\mathrm{YM}}(\tau) = Z_{\mathrm{2D}}(\tau). \qquad (2)$$

In writing Eq. (2), we have analytically continued q to $e^{2\pi i\tau}$ with $\tau \in \mathbb{H}$, the complex upper half-plane [3]. Thus $\mathrm{Im}\tau = \beta/(2\pi R)$. On the $2D$ CFT side of the equivalence, $\mathrm{Re}\tau$ has the standard interpretation of a chemical potential (in units of β) for rotations on the spatial circle of the torus $S^1_{2\pi R} \times S^1_\beta$. Determining the physical interpretation of $\mathrm{Re}\tau$ on the $4D$ gauge-theory side of Eq. (2) is an important matter for future work. We also emphasize that $Z_{\mathrm{YM}}(\tau)$ is *not* a modular-invariant function. Rather, our claim is that Z_{YM} is modular *covariant*, in the sense that it is built out of modular forms. Indeed, it is this feature which enables a match to the chiral-sector partition function of a $2D$ CFT, as in Eq. (2). We shall discuss this further in the conclusions.

II. THE 4*D* PARTITION FUNCTION

We begin by briefly explaining the computation of Z_{YM}, leaving a more leisurely exposition to Ref. [4]. To calculate the $4D$ partition function $Z_{\mathrm{YM}}(\tau)$, we take the large-N limit with Λ held fixed, which means taking the continuum limit after the large-N limit. We work on $S^3 \times S^1$ and assume that β and R are independent of N. Likewise, we do not consider states with energies $\gtrsim N$ because they lie beyond our UV cutoff. As is typical in studies of large-N theories, we work with the $U(N)$ version of YM theory rather than the $SU(N)$ version [5]. When $\Lambda R \to 0$, the microscopic degrees of freedom of YM theory reduce to an infinite collection of color-adjoint-valued harmonic oscillators. These oscillators are counted by the massless-vector partition function, which can be written as $z_v(\tau) = (6q^2 - 2q^3)/(1-q)^3$. The physical states are then determined by imposing the color Gauss law. In the $\lambda = 0$ confined phase, the physical single-particle states can be identified with single-trace operators, and their energies are proportional to their scaling dimensions. The counting problem for these states, and also for the multiparticle states, has been solved [2,6], and the resulting grand-canonical confined-phase partition function is given by

$$Z_{\mathrm{YM}}(\tau) = \prod_{n=1}^{\infty} \frac{1}{1 - z_v(q^n)} = \prod_{n=1}^{\infty} \frac{(1-q^n)^3}{1 - 3q^n - 3q^{2n} + q^{3n}} = 1 + 6q^2 + 16q^3 + 72q^4 + \cdots \qquad (3)$$

As expected in any confining large-N theory, we find that the d_n grow exponentially for large n. Thus, there are Hagedorn singularities in $Z_{\mathrm{YM}}(\tau)$. In Eq. (3), we find $d_n \sim e^{Cn}$ and $E_n \sim n$ for large n, with $C \equiv \log(2 + \sqrt{3}) \approx 1.317$. This contrasts with the behaviors $d_n \sim e^{\sqrt{n}}$ and $E_n \sim \sqrt{n}$ that would arise for a string theory with a flat target space. Of course, we are not in flat space: the spacetime curvature is $\sim 1/R$, which is of the same scale as the effective string tension $\alpha' \sim 1/R^2$ that follows from our spectrum. The scaling properties of d_n in Eq. (3) imply that the leading Hagedorn singularity of $Z_{\mathrm{YM}}(\beta)$ is at $\beta_H/R = C$, $\mathrm{Re}\tau = 0$, with subleading Hagedorn singularities accumulating along the line $\mathrm{Re}\tau = 0$ toward the point $\mathrm{Im}\tau = \infty$. Consequently, there will be a phase transition to a deconfined phase at β_H so long as $\mathrm{Re}\tau = 0$. This is discussed in detail in Refs. [2,7].

III. MODULAR SYMMETRIES

We now observe that the denominator in Eq. (3) can be factorized with roots that are inverses of each other:

$$1 - 3q^n - 3q^{2n} + q^{3n} = (1+q^n)(1-q^n z)(1-q^n/z) \qquad (4)$$

where $z = 2 + \sqrt{3}$. This pivotal algebraic observation was first made in Ref. [8] in the context of uncovering a subtle "temperature-reflection" symmetry for Z_{YM}. For our purposes, however, the key point is that this factorization allows Z_{YM} to be written as

$$Z_{\mathrm{YM}} = \prod_{n=1}^{\infty} \frac{(1-q^n)^3}{(1+q^n)(1-q^n z)(1-q^n z^{-1})}. \qquad (5)$$

This observation is very important because the structure of Eq. (5) matches the structure of the product representations of the Dedekind η-function and generalized Jacobi ϑ-functions. (In the related context of adjoint QCD, this was also noted in Ref. [9].) Specifically, the Dedekind η-function has the product representation $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$, while the generalized ϑ-function $\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\tau) \equiv \sum_{n\in\mathbb{Z}} q^{(n+\alpha)^2/2} e^{2\pi i n\beta}$ has a product representation of the form

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau) = q^{\alpha^2/2}\prod_{n=1}^{\infty}[(1-q^n) \times (1 + q^{n-\frac{1}{2}+\alpha}e^{2i\pi\beta})(1 + q^{n-\frac{1}{2}-\alpha}e^{-2i\pi\beta})]. \qquad (6)$$

Under the $S\colon \tau \to -1/\tau$ and $T\colon \tau \to \tau + 1$ generators of the modular group $SL(2,\mathbb{Z})$, we find $\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$ and $\eta(\tau+1) = e^{i\pi/12}\eta(\tau)$, while

$$S\colon\ \vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(-1/\tau) = \sqrt{-i\tau}\, e^{-2\pi i\alpha\beta}\,\vartheta\begin{bmatrix}-\beta\\ \alpha\end{bmatrix}(\tau),$$

$$T\colon\ \vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau+1) = e^{i\pi\alpha^2}\,\vartheta\begin{bmatrix}\alpha\\ \beta+\alpha+1/2\end{bmatrix}(\tau). \qquad (7)$$

Given these definitions, the structure of Eq. (5) allows us to rewrite the $4D$ partition function $Z_{\rm YM}$ as a finite product of Dedekind η-functions and Jacobi ϑ-functions:

$$Z_{\rm YM}(\tau) = \eta(\tau)^3 \left(\frac{-\sqrt{2}e^{-i\pi b}\eta(\tau)}{\vartheta[^{1/2}_{b+1/2}](\tau)} \right) \sqrt{\frac{2\eta(\tau)}{\vartheta_2(\tau)}} \qquad (8)$$

where $b = i\log(z)/2\pi \approx 0.21i$, where $\vartheta_2(\tau) \equiv \vartheta[^{1/2}_{0}](\tau)$, and where the identity $2\eta(2\tau)^2 = \eta(\tau)\vartheta_2(\tau)$ has been used in passing from Eq. (5) to Eq. (8). The fact that b is imaginary implies that $1/\vartheta[^{1/2}_{b+1/2}](\tau)$ has poles in the interior of the upper half plane, and is the reason the degeneracy factors d_n in Eq. (3) grow as $d_n \sim e^{Cn}$. The expression in Eq. (8)—and our interpretation of this expression in terms of specific $2D$ CFTs, as discussed below—are the key results of our paper, with many striking consequences.

IV. MODULARITY VERSUS DIMENSIONALITY

The first interesting implication of Eq. (8) becomes apparent upon realizing that it is extremely unusual for the partition function of a $4D$ theory to be expressible as a finite product of modular forms, as in Eq. (8). (See, e.g., Ref. [10] for an early discussion along these lines.) The large-$|\tau|$ behavior of a modular form is tied, through the S modular transformation, to its behavior near $|\tau| = 0$. For example, the Dedekind η-function has the large-$|\tau|$ expansion $\eta(\tau) = q^{1/24}(1 - q + \cdots)$; the S transformation then requires this function to behave at small $|\tau|$ as $\eta(\tau) \sim \exp[-i\pi/(12\tau)]/\sqrt{-i\tau}$. Similar statements can be made for the ϑ-functions. Thus, if a partition function can be written as a finite product of modular η-functions and ϑ-functions, then it must have the leading behavior

$$\lim_{\arg\tau\to\pi/2} [\log Z_{\rm modular}(\tau)|_{|\tau|\to 0}] \to \sigma R/\beta \qquad (9)$$

for a constant σ. This roughly amounts to the statement that $\log Z_{\rm modular} \sim T$ as $T \to \infty$. This is indeed the expected behavior for a $2D$ QFT. [In cases in which the order of limits in Eq. (9) may be important, the thermal interpretation of Eq. (9) is less straightforward.] However, the behavior described in Eq. (9) is certainly *not* the expected behavior for a $4D$ QFT, for which we generically expect [11]

$$\log Z_{\rm 4D} \sim \beta^{-3} \quad \text{as } \beta \to 0. \qquad (10)$$

For example, this is the behavior of a conformally coupled free massless scalar field on $S^3_R \times S^1_\beta$, for which

$$Z_{\rm s} = q^{1/240} \prod_{n\geq 1} (1 - q^n)^{-n^2} \qquad (11)$$

and

$$\log Z_{\rm s} \to \frac{\pi^4 R^3}{45}\beta^{-3} \quad \text{as } \beta \to 0. \qquad (12)$$

Similar results also emerge for free vector and fermion fields on $S^3 \times S^1$. Complexifying β and taking the small-$|\beta|$ limit as in the left side of Eq. (9) clearly cannot change these behaviors from that in Eq. (10) to that in Eq. (9) in generic free-field theories. Thus, in this sense, $4D$ QFTs whose partition functions can be written in terms of modular forms behave as if they were $2D$ QFTs, since they follow Eq. (9) rather than Eq. (10).

If we were to reverse the order of limits on the left side of Eq. (9) and take the $|\tau| \to 0$ limit with $\arg\tau = \pi/2$, pure YM theory would follow the scaling in Eq. (10). Such a limit cannot be studied from Eq. (3) due to the Hagedorn singularities, and the physics is governed by the deconfined phase [2]. For Yang-Mills theory, Eq. (9) is thus valid only with the order of limits indicated. We note that in other theories such as adjoint QCD with periodic boundary conditions for fermions, the Hagedorn singularities do not lie along $\arg\tau = \pi/2$ [9]; the two limits then commute and these theories exhibit $2D$ behavior in the sense of Eq. (9) irrespective of the order of limits [4].

V. VACUUM ENERGY

Another major consequence of Eq. (8) is that the modular properties of the η- and ϑ-functions fix the vacuum energy $E_{\rm YM}$ of our large-N YM theory to be zero.

To see this, we first recall that if we write the q-series expansion of a modular function $f(\tau)$ in the form $f = q^\Delta \sum_{n=0}^\infty a_n q^n$, then Δ can be thought of as the $2D$ vacuum energy. Its value is fixed by the modular properties of f and tied to the values of a_n. Were one to arbitrarily shift $\Delta \to \Delta + c$, the modular properties of $f(\tau)$ would be ruined because the S-transformation would map $q^c = e^{(2\pi i\tau)c}$ to $e^{(-2\pi i/\tau)c}$, thereby preventing $q^c f(\tau)$ from transforming as a modular form.

Next, we observe that the vacuum energy associated to the η-function is $1/24$, while $\vartheta[^a_b]$ has vacuum energy $a^2/2$. Summing the vacuum energies of the individual modular form in Eq. (8), we obtain a striking result:

$$E_{\rm YM} = 0. \qquad (13)$$

Indeed, this is the only value consistent with the q-expansion for $Z_{\rm YM}$ given in Eq. (3), provided that $E_{\rm YM}$ is calculated in a renormalization scheme which is consistent with the modular properties of $Z_{\rm YM}$ made evident in Eq. (8). This value, $E_{\rm YM} = 0$, also coincides with the result implied by T-reflection symmetry [8], and furthermore agrees with a direct evaluation of the sum over the confined-phase spectrum of finite-temperature large-N YM theory compactified on S^3, as performed in Ref. [12].

To understand this result within the framework of the existing literature, it is important to recall that the starting point for the analysis in this paper concerns the 't Hooft large-N limit, where N is sent to infinity with $\lambda(\mu_{\rm uv})$ and all other parameters held fixed, including the UV regularization scale $\mu_{\rm uv}$. This is the most natural thing to do in asymptotically free QFTs because it results in Λ being held fixed as $N \to \infty$, as one can see from the one-loop relation between the strong scale Λ and the 't Hooft coupling $\lambda(\mu_{\rm uv})$ at the UV regularization scale $\Lambda = \mu_{\rm uv} e^{-\frac{8\pi^2}{\beta_0 \lambda(\mu_{\rm uv})}}$, where $\beta_0 = 11/3$ is the one-loop beta-function coefficient. It is only within this understanding of the large-N limit that our expression for $Z_{\rm YM}$ in Eq. (3) is correct. Thus, when the confining-phase spectral data encoded in (3) is used to compute the vacuum energy—either by direct methods, as in Ref. [12], or by using the modular symmetries, as done here—the result of Eq. (13) is valid in the limit discussed above. In the literature there are other calculations of the Casimir energy for adjoint-matter gauge theories, both in the $\lambda = 0$ limit (as in, e.g., Ref. [2]) and using gauge-gravity duality for $\mathcal{N} = 4$ super-YM theory (as in Ref. [13]). These latter calculations yield the result $E_0 \sim N^2 \neq 0$. However these calculations assume an ordering for the large-μ and large-N limits which is opposite ours. There is thus no conflict with our results.

VI. CFT INTERPRETATION

The striking modular structure of Eq. (8) suggests that the spectrum of our $4D$ YM theory coincides with that of a chiral (e.g., left- or right-moving) $2D$ CFT. This motivates the central question we shall now explore for the rest of this paper: what is the $2D$ CFT which gives rise to Eq. (8), and thus gives a $2D$ description of $4D$ YM theory in the large-N limit?

Unfortunately, we will not be able to give a complete answer to this question. The reason ultimately has to do with the fact that many distinct CFTs can have coincident spectra without being equivalent. They may differ, for example, in their correlation functions. In general, the most important aspects of a given $2D$ CFT are governed by its central charge (conformal anomaly) c and its spectrum of operator conformal dimensions $h_i, i = 1, \ldots, n$, where n is the number of so-called "primary" fields in the CFT. Along with the explicit traces over states, knowledge of c and the h_i's goes a long way in nailing down relevant aspects of the CFT such as its selection rules and correlation functions. But partition functions are only sensitive to the combinations $h_i^{(\rm eff)} \equiv h_i - c/24$, rather than the values of c and h_i individually. Consequently, without additional assumptions about the CFT in question (such as the assumption of unitarity, which would additionally tell us that $\min\{h_i\} = 0$), this represents a fundamental limitation on our ability to specify a unique CFT.

We will therefore answer a different but related question: do there exist any $2D$ CFTs to which our large-N YM theory is *isospectral*? Remarkably, we shall show that at least one such $2D$ CFT indeed exists. To see this, we first recall that a free $c = 1$ scalar CFT has a chiral spectrum whose trace is given by $1/\eta(\tau)$, while the $\mathbb{Z}_2$ orbifold of this CFT has a chiral sector whose trace is $(2\eta(\tau)/\vartheta_2(\tau))^{1/2}$. Furthermore, the direct product of two copies of the $c = -26$ bc ghost CFT has a chiral spectrum whose trace is given by $\eta(\tau)^4$. Perhaps the most challenging to interpret is the remaining factor in Eq. (8), specifically

$$\frac{-\sqrt{2} e^{-i\pi b} \eta(\tau)}{\vartheta\left[{1/2 \atop b+1/2}\right](\tau)}. \tag{14}$$

However, this can be identified as the trace of the chiral (e.g., left-moving) states in the vacuum sector of the $c = 2$ bosonic $\beta\gamma$ ghost CFT recently explored in Ref. [14]. This is a logarithmic CFT [15], and it has a $U(1)$ conserved charge. Thus the vacuum-sector chiral partition function of the $c = 2$ $\beta\gamma$ CFT depends on the choice of a complex fugacity $z = e^{+\mu\beta}$. To match with our expressions for YM theory, we set $\mu\beta = 2\pi i b = -\log(2 + \sqrt{3})$.

Putting this together, we therefore conclude that the expression in Eq. (8) can be viewed as the trace over the chiral spectrum of a theory which is the direct product of five known CFTs, one of which is irrational. This then justifies the central claim of this paper in Eq. (2): there is indeed an irrational $2D$ CFT which is isospectral to the finite-temperature large-N $4D$ YM compactified on S^3 in the $\Lambda R \to 0$ limit.

Aside from explaining our observations concerning $E_{\rm YM}$ and the small-$|\tau|$ behavior of $Z_{\rm YM}$, the equivalence in Eq. (2) has an intriguing further implication. Two-dimensional CFTs have infinite-dimensional symmetries which always include the Virasoro symmetry. The result in Eq. (2) then strongly suggests that the spectrum-generating algebra of large-N YM theory includes a Virasoro algebra in the $\lambda = 0$ limit. It would be very interesting to find the explicit realization of this Virasoro symmetry algebra within YM theory.

VII. PRIMARY OPERATOR SPECTRUM

We now collect information concerning the spectrum of conformal dimensions $h_i^{(\rm eff)}$ corresponding to the primary fields of this tensor-product CFT. Our approach proceeds by determining the diagonal modular-invariant associated with the expression in Eq. (8), and then computing the eigenvalues of the modular T operator to extract $h_i^{(\rm eff)}$.

We begin by defining the quantities

$$T_{m,n} \equiv \frac{-\sqrt{2} e^{-i\pi b n} \eta(\tau)^4}{\vartheta\left[{mb+1/2 \atop nb+1/2}\right](\tau)} \left(\frac{2\eta(\tau)}{\vartheta\left[{P(m)/2 \atop P(n)/2}\right](\tau)}\right)^{1/2}, \tag{15}$$

where $\{m, n\}$ are relatively prime integers (a relationship which we shall henceforth denote $m \perp n$), and $P(k) \equiv \frac{1}{2}(1 + (-1)^k)$, $k \in \mathbb{Z}$. Thus $P(k) = 0$, 1 for odd or even k, respectively. The set $\{T_{m,n}\}$ is a basis for a vector space over the field $\mathbb{C}$ with two key properties: it contains the "seed term" in Eq. (8), and it is the minimal set which is closed under the action of the $SL(2, \mathbb{Z})$ modular group.

The first property follows by noting that $T_{0,1}(\tau)$ coincides with Eq. (8). The verification of the second property proceeds in two steps. First, it can be shown that, up to overall phases and extraneous factors of $\sqrt{-i\tau}$, the S and T modular transformations map $T_{m,n}$ to $T_{-n,m}$ and $T_{m,n+m}$, respectively. Second, we observe that if $\{m,n\}$ are relatively prime, then $\{-n,m\}$ and $\{m,n+m\}$ are also relatively prime. Since all modular transformations can be generated by sequences of S and T, it then follows that the full modular "orbit" of our seed term $T_{0,1}$ is contained within the set of coprime integers $\{m,n\}$. Indeed, it is also possible to demonstrate [4] that the modular orbit actually covers *all* coprimes.

As a result, the minimal "diagonal" modular-invariant generated from Eq. (8) is given by

$$Z_{\rm diagonal} = ({\rm Im}\tau)^{3/2}\sum_{m\perp n}|T_{m,n}|^2. \tag{16}$$

The appearance of the factor of $({\rm Im}\tau)^{3/2}$ is standard when combining holomorphic and antiholomorphic components, such as our $T_{m,n}$ factors, each of which has modular weight $k=3/2$. It also ensures that $Z_{\rm diagonal}$ is fully modular invariant. Moreover, it can be verified numerically that the infinite sum in Eq. (16) converges except for an isolated set of points corresponding to the Hagedorn singularities. The numerical values of $Z_{\rm diagonal}$ on the interior of the unit-q disk are shown in Fig. 2.

In order to extract the spectrum of effective conformal dimensions $h_i^{(\rm eff)}$, we now rewrite $Z_{\rm diagonal}$ in a basis of eigenfunctions of the modular $T\colon \tau\to\tau+1$ operator. We do this because such eigenfunctions $\chi(\tau)$ will have eigenvalues $\exp[2\pi i h_i^{(\rm eff)}]$ under T, allowing us to read off

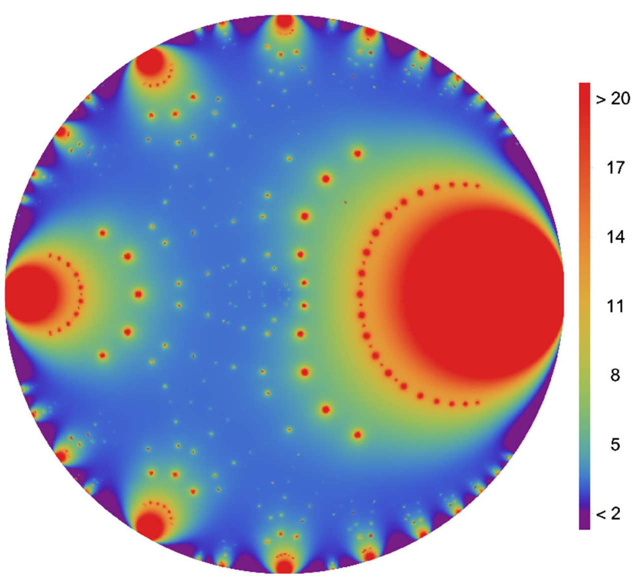

FIG. 2 (color online). The numerical values of Eq. (16) with $|m|$, $|n|\le 10$, plotted within the unit-q disk.

the values of $h_i^{(\rm eff)}$ (mod 1). Fortunately, constructing eigenfunctions of the T-operator from linear combinations of the $T_{m,n}$'s in Eq. (15) is relatively straightforward. Since

$$T_{m,n}(\tau+1) = e^{\pi i\{[1-P(m)]/8+m^2|b|^2\}}T_{m,n+m}(\tau), \tag{17}$$

we see that any linear combination which includes $T_{m,n}$ must also include $T_{m,n+m}$, $T_{m,n+2m}$, and indeed all $T_{m,n+km}$ where $k\in\mathbb{Z}$. Our T-invariant linear combinations can therefore be indexed by an arbitrary integer m and a second integer $\ell\perp m$ obeying $0\le\ell<|m|$. Hence T-eigenfunctions can be constructed analogously to Bloch eigenfunctions, by summing over all components $T_{m,\ell+km}$ with $k\in\mathbb{Z}$ with a Bloch phase $\alpha\in[0,1)\subset\mathbb{R}$:

$$\chi_{m,\ell,\alpha} = \sum_{k\in\mathbb{Z}} e^{2\pi i\alpha k}T_{m,\ell+mk}. \tag{18}$$

It then follows that

$$\chi_{m,\ell,\alpha}(\tau+1) = e^{2\pi i h^{(\rm eff)}_{m,\ell,\alpha}}\chi_{m,\ell,\alpha}(\tau), \tag{19}$$

where

$$h^{(\rm eff)}_{m,\ell,\alpha} = \frac{1}{2}\left[\frac{1-P(m)}{8}+m^2|b|^2\right]-\alpha. \tag{20}$$

One might wonder whether $\{\chi_{m,\ell,\alpha}\}$ is the complete set of T-eigenfunctions. However, we have verified this by checking that summing over $\chi_{m,\ell,\alpha}$ reproduces Eq. (16):

$$Z_{\rm diagonal} = ({\rm Im}\tau)^{3/2}\sum_{m\in\mathbb{Z}}\sum_{\substack{0\le\ell<|m|\\ \ell\perp m}}\int_0^1 d\alpha|\chi_{m,\ell,\alpha}|^2. \tag{21}$$

This confirms that Eq. (20) is the desired set of effective conformal dimensions (mod 1) of the primary operators in our CFT. The fact that these dimensions depend on α—a continuous real variable—confirms that we are dealing with an *irrational* CFT [16]. Our observations are also consistent with the 2D logarithmic CFT interpretation discussed above, since it is known that logarithmic CFTs typically have a continuously infinite number of primary operators [17].

VIII. OUTLOOK

We have presented evidence that the confined phase of finite-temperature 4D nonsupersymmetric large-N pure Yang-Mills theory compactified on a three-sphere of radius R has a remarkable modular structure, as exposed by Eq. (8). This has many interesting consequences, such as the fact that this 4D gauge theory is isospectral to an irrational 2D CFT in the $\Lambda R\to 0$ limit, as summarized in Eq. (2). Moreover, as we shall demonstrate in a separate paper [4], modularity in the sense of Eq. (8) and isospectrality to

$2D$ irrational CFTs as in Eq. (2) turn out to be generic properties of large-N confined-phase gauge theories with adjoint massless matter in the $\lambda \to 0$ limit. In Ref. [4] we shall also show that this structure is present in the large-N limit of the $\mathcal{N} = 4$ superconformal index.

As briefly mentioned above, Z_{YM} is not a modular-invariant function. This is clear from the fact that Z_{YM} [or equivalently $T_{0,1}(\tau)$] is but the seed for a modular orbit; if Z_{YM} had been truly modular invariant, no such extended orbit would have arisen. This can also be understood in terms of the thermodynamics of these YM theories. If Z_{YM} had been truly modular invariant, the thermodynamic behaviors for high and low temperatures would have been essentially identical. However, this is impossible for many reasons, not the least of which is the existence of a deconfinement transition and associated Hagedorn singularities.

Rather, as we have seen, Z_{YM} is modular *covariant*, in the sense that it is built out of modular forms. This is what enables a match between Z_{YM} and the chiral-sector partition function of a $2D$ CFT. As will be discussed in detail in Ref. [4], the fact that confined-phase large-N partition functions are modular covariant but not modular invariant appears to be generic in the $\lambda \to 0$ limit, and holds even in situations lacking a deconfinement phase transition or Hagedorn singularities. Indeed, such situations arise in certain theories with adjoint fermions with periodic boundary conditions [4,9,18] where the partition function has a $(-1)^F$ twist. These twisted partition functions have a modular structure which is completely analogous to what we observe for YM theory, but because of the lack of Hagedorn instabilities, the modular structure has direct implications for the twisted thermodynamics for such theories.

But the implications of modular structure go way beyond constraints on possible thermodynamics—modular structure also greatly constrains the spectrum of the corresponding quantum field theory. For example, we have shown in this paper that the YM partition function is modular covariant, and specifically has the structure of a meromorphic modular form of weight $k = 3/2$. This then amounts to a powerful statement about the symmetries governing the spectrum of YM theory. In particular, we have seen that the behavior of Z_{YM} at small $|\tau|$ is typical of a $2D$ rather than $4D$ theory. We have also seen that the large-$|\tau|$ behavior of Z_{YM} is that of a theory with a vanishing vacuum energy. Both of these highly unusual features are a result of the constraints on the spectrum following from the modular covariance of Z_{YM}. Perhaps most dramatically, the modular structure allowed us to exhibit a spectral equivalence between $4D$ YM theory at large N in the $\Lambda R \to 0$ limit, and the chiral sector of a particular irrational $2D$ CFT, as summarized in Eq. (2). Indeed, as demonstrated in Ref. [4], relations such as this will continue to hold even for theories with matter.

Since the spectrum-generating algebras of $2D$ CFTs always contain at least the Virasoro algebra, these observations imply that the spectrum of confining large-N theories is organized by a hidden Virasoro algebra in the $\Lambda R \to 0$ limit. Of course, the $\Lambda R \to 0$ limit is a free limit, and it is well known that free theories always have infinite-dimensional symmetries. However the striking point—at least for confining four-dimensional large-N gauge theories in the free limit—is that these (spectrum-generating) symmetries turn out to be of a two-dimensional nature. Coupled with our expectation that large-N Yang-Mills theory has conserved higher-spin currents in the $\lambda = 0$ limit [19], these observations suggest that the $2D$ CFT will have a W-symmetry [20]. As discussed in Ref. [4], examination of the behavior of the characters of our $2D$ CFTs indeed provides some hints that these $2D$ CFTs have enhanced symmetries at the special points in their parameter space where their chiral-sector spectra coincide with $4D$ gauge theories.

It is not clear how easily the $4D$-$2D$ relation that we found, as summarized through Eq. (2), fits with standard string-theoretic expectations. From a string-theoretic perspective one might have expected that it would be the *single-trace* partition function—which can be thought of as representing the fluctuations of a single string—that would have a simple $2D$ CFT description, assuming one is possible. It is less clear why the grand-canonical partition function Z_{YM}, which takes into account all multitrace states and hence represents the fluctuations of an ensemble of many strings, should have a $2D$ CFT description. From this perspective, our result in Eq. (2)—and the analogous relations that we shall find in Ref. [4] for other, adjoint-matter gauge theories—are even more remarkable.

Our results suggest a large number of interesting topics for future research. Obviously, it would be very interesting to understand whether Eq. (2) has an explanation within string theory, perhaps by making contact with the ideas in, e.g., Refs. [21]. It is also important to understand whether our large-N $4D$-$2D$ spectral equivalence extends to correlation functions, and to explore how it is related to other known $4D$-$2D$ relations, such as those discussed in Refs. [22]. Note that unlike the $4D$-$2D$ relations discussed in the context of supersymmetric indices (which by construction focus on a subset of states of the $4D$ theory), our $4D$-$2D$ relation in Eq. (2) involves the full thermal partition function and hence concerns the entire finite-energy spectrum of the large-N $4D$ theory.

Another interesting direction would be to develop an understanding of the modular structure of expressions like Eq. (8) directly from a $4D$ point of view, perhaps by making use of ideas from, e.g., Refs. [23]. Given recent progress in the understanding of the bulk duals of $2D$ CFTs (see, e.g., Ref. [24]), it is tempting to wonder whether our results may help to uncover the bulk dual of YM theory and of other nonsupersymmetric $4D$ adjoint-matter theories. It would also be interesting to understand the extent to which the continuous spectrum of primary operators in the $2D$ theory suggested by our analysis has an interpretation in 4D YM theory.

Finally, it is natural to wonder what may happen to the modular properties we have found when we consider corrections away from the free limit in Yang-Mills theory. To explore this question directly from the 4D QFT side, one would want analytic expressions for the thermal partition function away from the $\lambda = 0$ limit. This is a challenge even in the most favorable case of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, but perhaps it can be handled using integrability techniques. Alternatively, if one could establish that the equivalence of the $\lambda = 0$ theory extends to the generating functional of correlation functions, then it might be possible to approach this question from the 2D side of the relation by identifying the 't Hooft-coupling deformation of the 4D theory with a classically marginal deformation of the 2D CFT.

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