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UNIFORMLY CONVEX SPIRAL FUNCTIONS
AND UNIFORMLY SPIRALLIKE FUNCTIONS ASSOCIATED
WITH PASCAL DISTRIBUTION SERIES

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Abstract. The aim of this paper is to find the necessary and sufficient conditions and inclusion relations for Pascal distribution series to be in the classes $\mathcal{SP}_p(\alpha, \beta)$ and $\mathcal{UCV}_p(\alpha, \beta)$ of uniformly spirallike functions. Further, we consider an integral operator related to Pascal distribution series. Several corollaries and consequences of the main results are also considered.

Keywords: analytic function; Hadamard product; uniformly spirallike function; Pascal distribution series

MSC 2020: 30C45

1. INTRODUCTION AND DEFINITIONS

Let \mathcal{A} denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$. Further, let \mathcal{T} be a subclass of \mathcal{A} consisting of functions of the form

$$(1.2) \quad f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in \mathbb{U}.$$

A function $f \in \mathcal{A}$ is spirallike if

$$\Re\left(e^{-i\alpha} \frac{zf'(z)}{f(z)}\right) > 0$$

for some α with $|\alpha| < \frac{1}{2}\pi$ and for all $z \in \mathbb{U}$. Also $f(z)$ is convex spirallike if $zf'(z)$ is spirallike.

In [28], Selvaraj and Geetha introduced the following subclasses of uniformly spirallike and convex spirallike functions.

Definition 1.1. A function f of the form (1.1) is said to be in the class $\mathcal{SP}_p(\alpha, \beta)$ if it satisfies the following condition:

$$\Re\left(e^{-i\alpha} \frac{zf'(z)}{f(z)}\right) > \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta, \quad |\alpha| < \frac{\pi}{2}, 0 \leq \beta < 1$$

and $f \in \mathcal{UCV}_p(\alpha, \beta)$ if and only if $zf'(z) \in \mathcal{SP}_p(\alpha, \beta)$.

We write

$$\mathcal{TSP}_p(\alpha, \beta) = \mathcal{SP}_p(\alpha, \beta) \cap \mathcal{T} \quad \text{and} \quad \mathcal{UCT}_p(\alpha, \beta) = \mathcal{UCV}_p(\alpha, \beta) \cap \mathcal{T}.$$

In particular, we note that $\mathcal{SP}_p(\alpha, 0) = \mathcal{SP}_p(\alpha)$ and $\mathcal{UCV}_p(\alpha, 0) = \mathcal{UCV}_p(\alpha)$, the classes of uniformly spirallike and uniformly convex spirallike functions were introduced by Ravichandran et al. (see [25]).

For $\alpha = 0$, the classes $\mathcal{UCV}_p(\alpha)$ and $\mathcal{SP}_p(\alpha)$, respectively, reduces to the classes \mathcal{UCV} and \mathcal{SP} introduced and studied by Ronning (see [27]). For more interesting developments of some related subclasses of uniformly spirallike and uniformly convex spirallike functions, the readers may be referred to the works of Frasin (see [7], [8]), Goodman (see [15], [16]), Al-Hawary and Frasin (see [1]), Kanas and Wiśniowska (see [17], [18]) and Ronning (see [26], [27]).

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^\tau(A, B)$, $\tau \in \mathbb{C} \setminus \{0\}$, $-1 \leq B < A \leq 1$, if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]} \right| < 1, \quad z \in \mathbb{U}.$$

This class was introduced by Dixit and Pal, see [3].

A variable X is said to be *Pascal distribution* if it takes the values $0, 1, 2, 3, \dots$ with probabilities $(1 - q)^m$, $\frac{1}{1!}qm(1 - q)^m$, $\frac{1}{2!}q^2m(m + 1)(1 - q)^m$, $\frac{1}{3!}q^3m(m + 1) \times (m + 2)(1 - q)^m, \dots$, respectively, where q and m are called the parameters, and thus

$$P(X = k) = \binom{k + m - 1}{m - 1} q^k (1 - q)^m, \quad k = 0, 1, 2, 3, \dots$$

Very recently, El-Deeb et al. (see [6]) introduced a power series whose coefficients are probabilities of Pascal distribution

$$(1.3) \quad \Psi_q^m(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1} (1-q)^m z^n, \quad z \in \mathbb{U},$$

where $m \geq 1$, $0 \leq q \leq 1$ and we note that by ratio test, the radius of convergence of above series is infinity. We also define the series

$$(1.4) \quad \Phi_q^m(z) = 2z - \Psi_q^m(z) = z - \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1} (1-q)^m z^n, \quad z \in \mathbb{U}.$$

Now, we consider the linear operator

$$\mathcal{I}_q^m(z): \mathcal{A} \rightarrow \mathcal{A}$$

defined by the convolution or Hadamard product

$$(1.5) \quad \mathcal{I}_q^m f(z) = \Psi_q^m(z) * f(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1} (1-q)^m a_n z^n, \quad z \in \mathbb{U}.$$

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions by using hypergeometric functions (see for example [2], [12], [19], [29], [30]) and by the recent investigations (see for example [4], [5], [9]–[11], [13], [14], [20]–[24]), in the present paper we determine the necessary and sufficient conditions for $\Phi_q^m(z)$ to be in our classes $\mathcal{TS}\mathcal{P}_p(\alpha, \beta)$ and $\mathcal{UCT}_p(\alpha, \beta)$ and connections of these subclasses with $\mathcal{R}^\tau(A, B)$. Finally, we give conditions for the integral operator $\mathcal{G}_q^m f(z) = \int_0^z \Psi_q^m(t) t^{-1} dt$ belonging to the above classes.

To establish our main results, we need the following lemmas.

Lemma 1.2 ([28]). *A function f of the form (1.2) is in $\mathcal{TS}\mathcal{P}_p(\alpha, \beta)$ if and only if it satisfies*

$$(1.6) \quad \sum_{n=2}^{\infty} (2n - \cos \alpha - \beta) |a_n| \leq \cos \alpha - \beta, \quad |\alpha| < \frac{\pi}{2}, \quad 0 \leq \beta < 1.$$

In particular, when $\beta = 0$, we obtain a necessary and sufficient condition for a function f of the form (1.2) to be in the class $\mathcal{TS}\mathcal{P}_p(\alpha)$ as

$$(1.7) \quad \sum_{n=2}^{\infty} (2n - \cos \alpha) |a_n| \leq \cos \alpha, \quad |\alpha| < \frac{\pi}{2}.$$

Lemma 1.3 ([28]). *A function f of the form (1.2) is in $\mathcal{UCT}_p(\alpha, \beta)$ if and only if it satisfies*

$$(1.8) \quad \sum_{n=2}^{\infty} n(2n - \cos \alpha - \beta)|a_n| \leq \cos \alpha - \beta, \quad |\alpha| < \frac{\pi}{2}, \quad 0 \leq \beta < 1.$$

In particular, when $\beta = 0$, we obtain a necessary and sufficient condition for a function f of the form (1.2) to be in the class $\mathcal{UCT}_p(\alpha)$ as

$$(1.9) \quad \sum_{n=2}^{\infty} n(2n - \cos \alpha)|a_n| \leq \cos \alpha, \quad |\alpha| < \frac{\pi}{2}.$$

Lemma 1.4 ([3]). *If $f \in \mathcal{R}^\tau(A, B)$ is of the form (1.1), then*

$$|a_n| \leq (A - B) \frac{|\tau|}{n}, \quad n \in \mathbb{N} \setminus \{1\}.$$

The result is sharp.

2. THE NECESSARY AND SUFFICIENT CONDITIONS

For convenience, in the sequel we use the following identities for $m \geq 1$ and $0 \leq q < 1$:

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} q^n &= \frac{1}{(1-q)^m}, & \sum_{n=0}^{\infty} \binom{n+m-2}{m-2} q^n &= \frac{1}{(1-q)^{m-1}}, \\ \sum_{n=0}^{\infty} \binom{n+m}{m} q^n &= \frac{1}{(1-q)^{m+1}}, & \sum_{n=0}^{\infty} \binom{n+m+1}{m+1} q^n &= \frac{1}{(1-q)^{m+2}}. \end{aligned}$$

By simple calculations we derive the following relations:

$$\begin{aligned} \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1} &= \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} q^n - 1 = \frac{1}{(1-q)^m} - 1, \\ \sum_{n=2}^{\infty} (n-1) \binom{n+m-2}{m-1} q^{n-1} &= qm \sum_{n=0}^{\infty} \binom{n+m}{m} q^n = \frac{qm}{(1-q)^{m+1}}, \end{aligned}$$

and

$$\begin{aligned} \sum_{n=3}^{\infty} (n-1)(n-2) \binom{n+m-2}{m-1} q^{n-1} &= q^2 m(m+1) \sum_{n=0}^{\infty} \binom{n+m+1}{m+1} q^n \\ &= \frac{q^2 m(m+1)}{(1-q)^{m+2}}. \end{aligned}$$

Unless otherwise mentioned, we shall assume in this paper that $|\alpha| < \frac{1}{2}\pi$, $0 \leq \beta < 1$ and $0 \leq q < 1$. First we obtain the necessary and sufficient conditions for $\Phi_q^m(z)$ to be in $\mathcal{TS}\mathcal{P}_p(\alpha, \beta)$.

Theorem 2.1. *If $m \geq 1$, then $\Phi_q^m(z) \in \mathcal{TS}\mathcal{P}_p(\alpha, \beta)$ if and only if*

$$(2.1) \quad \frac{2qm}{1-q} + (2 - \cos \alpha - \beta)(1 - (1-q)^m) \leq \cos \alpha - \beta.$$

Proof. Since

$$(2.2) \quad \Phi_q^m(z) = z - \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1} (1-q)^m z^n$$

in view of Lemma 1.2, it suffices to show that

$$(2.3) \quad \sum_{n=2}^{\infty} (2n - \cos \alpha - \beta) \binom{n+m-2}{m-1} q^{n-1} (1-q)^m \leq \cos \alpha - \beta.$$

Writing

$$n = (n-1) + 1$$

in (2.3) we have

$$\begin{aligned} & \sum_{n=2}^{\infty} (2n - \cos \alpha - \beta) \binom{n+m-2}{m-1} q^{n-1} (1-q)^m \\ &= \sum_{n=2}^{\infty} (2(n-1) + 2 - \cos \alpha - \beta) \binom{n+m-2}{m-1} q^{n-1} (1-q)^m \\ &= 2 \sum_{n=2}^{\infty} (n-1) \binom{n+m-2}{m-1} q^{n-1} (1-q)^m \\ &\quad + (2 - \cos \alpha - \beta) \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1} (1-q)^m \\ &= 2qm(1-q)^m \sum_{n=0}^{\infty} \binom{n+m}{m} q^n \\ &\quad + (2 - \cos \alpha - \beta)(1-q)^m \left(\sum_{n=0}^{\infty} \binom{n+m-1}{m-1} q^n - 1 \right) \\ &= \frac{2qm}{1-q} + (2 - \cos \alpha - \beta)(1 - (1-q)^m). \end{aligned}$$

But this last expression is bounded above by $\cos \alpha - \beta$ if and only if (2.1) holds. \square

Theorem 2.2. *If $m \geq 1$, then $\Phi_q^m(z) \in \mathcal{UCT}_p(\alpha, \beta)$ if and only if*

$$(2.4) \quad \frac{2q^2m(m+1)}{(1-q)^2} + (6 - \cos \alpha - \beta) \frac{qm}{1-q} + (2 - \cos \alpha - \beta)(1 - (1-q)^m) \leq \cos \alpha - \beta.$$

Proof. In view of Lemma 1.3, we must show that

$$(2.5) \quad \sum_{n=2}^{\infty} n(2n - \cos \alpha - \beta) \binom{n+m-2}{m-1} q^{n-1} (1-q)^m \leq \cos \alpha - \beta.$$

Writing

$$n = (n-1) + 1$$

and

$$n^2 = (n-1)(n-2) + 3(n-1) + 1$$

in (2.5) we get

$$\begin{aligned} & \sum_{n=2}^{\infty} n(2n - \cos \alpha - \beta) \binom{n+m-2}{m-1} q^{n-1} (1-q)^m \\ &= \sum_{n=2}^{\infty} (2(n-1)(n-2) + 6(n-1)) \binom{n+m-2}{m-1} q^{n-1} (1-q)^m \\ & \quad + \sum_{n=2}^{\infty} (n-1)(-\cos \alpha - \beta) \binom{n+m-2}{m-1} q^{n-1} (1-q)^m \\ & \quad + \sum_{n=2}^{\infty} (2 - \cos \alpha - \beta) \binom{n+m-2}{m-1} q^{n-1} (1-q)^m \\ &= 2q^2m(m+1)(1-q)^m \sum_{n=0}^{\infty} \binom{n+m+1}{m+1} q^n \\ & \quad + (6 - \cos \alpha - \beta)qm(1-q)^m \sum_{n=0}^{\infty} \binom{n+m}{m} q^n \\ & \quad + (2 - \cos \alpha - \beta)(1-q)^m \left(\sum_{n=0}^{\infty} \binom{n+m-1}{m-1} q^n - 1 \right) \\ &= 2q^2m(m+1)(1-q)^m \frac{1}{(1-q)^{m+2}} + (6 - \cos \alpha - \beta)qm(1-q)^m \frac{1}{(1-q)^{m+1}} \\ & \quad + (2 - \cos \alpha - \beta)(1-q)^m \left(\frac{1}{(1-q)^m} - 1 \right) \\ &= \frac{2q^2m(m+1)}{(1-q)^2} + (6 - \cos \alpha - \beta) \frac{qm}{1-q} + (2 - \cos \alpha - \beta)(1 - (1-q)^m). \end{aligned}$$

Therefore we see that the last expression is bounded above by $\cos \alpha - \beta$ if (2.4) is satisfied. \square

3. INCLUSION PROPERTIES

Making use of Lemma 1.4, we will study the action of the Pascal distribution series on the class $\mathcal{TSP}_p(\alpha, \beta)$.

Theorem 3.1. *Let $m > 1$ and $f \in \mathcal{R}^\tau(A, B)$. Then $I_q^m f(z) \in \mathcal{TSP}_p(\alpha, \beta)$ if*

$$(3.1) \quad (A-B)|\tau| \left(2(1-(1-q)^m) - \frac{\cos \alpha + \beta}{q(m-1)} ((1-q) - (1-q)^m - q(m-1)(1-q)^m) \right) \leq \cos \alpha - \beta.$$

Proof. In view of Lemma 1.2, it suffices to show that

$$\sum_{n=2}^{\infty} (2n - \cos \alpha - \beta) \binom{n+m-2}{m-1} q^{n-1} (1-q)^m |a_n| \leq \cos \alpha - \beta.$$

Since $f \in \mathcal{R}^\tau(A, B)$, then by Lemma 1.4, we have

$$(3.2) \quad |a_n| \leq \frac{(A-B)|\tau|}{n}.$$

Thus, we have

$$\begin{aligned} & \sum_{n=2}^{\infty} (2n - \cos \alpha - \beta) \binom{n+m-2}{m-1} q^{n-1} (1-q)^m |a_n| \\ & \leq (A-B)|\tau| \left(\sum_{n=2}^{\infty} \frac{1}{n} (2n - \cos \alpha - \beta) \binom{n+m-2}{m-1} q^{n-1} (1-q)^m \right) \\ & = (A-B)|\tau| (1-q)^m \\ & \quad \times \left(2 \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1} - (\cos \alpha + \beta) \sum_{n=2}^{\infty} \frac{1}{n} \binom{n+m-2}{m-1} q^{n-1} \right) \\ & = (A-B)|\tau| (1-q)^m \left(2 \left(\sum_{n=0}^{\infty} \binom{n+m-1}{m-1} q^n - 1 \right) \right. \\ & \quad \left. - \frac{\cos \alpha + \beta}{q(m-1)} \left(\sum_{n=0}^{\infty} \binom{n+m-2}{m-2} q^n - 1 - (m-1)q \right) \right) \\ & = (A-B)|\tau| \\ & \quad \times \left(2(1-(1-q)^m) - \frac{\cos \alpha + \beta}{q(m-1)} ((1-q) - (1-q)^m - q(m-1)(1-q)^m) \right). \end{aligned}$$

But this last expression is bounded by $\cos \alpha - \beta$ if (3.1) holds. This completes the proof of Theorem 3.1. □

Applying Lemma 1.3 and using the same technique as in the proof of Theorem 3.1, we have the following result.

Theorem 3.2. *Let $m \geq 1$ and $f \in \mathcal{R}^\tau(A, B)$. Then $\mathcal{I}_q^m f(z) \in \mathcal{UCT}_p(\alpha, \beta)$ if*

$$(3.3) \quad (A - B)|\tau| \left(\frac{2qm}{1-q} + (2 - \cos \alpha - \beta)(1 - (1-q)^m) \right) \leq \cos \alpha - \beta.$$

4. AN INTEGRAL OPERATOR

Theorem 4.1. *If $m \geq 1$, then the integral operator*

$$(4.1) \quad \mathcal{G}_q^m f(z) = \int_0^z \frac{\Psi_q^m(t)}{t} dt$$

is in the class $\mathcal{UCT}_p(\alpha, \beta)$ if and only if inequality (2.1) is satisfied.

Proof. Since

$$\mathcal{G}_q^m f(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1} (1-q)^m \frac{z^n}{n},$$

by Lemma 1.3 we only need to show that

$$\sum_{n=2}^{\infty} n(2n - \cos \alpha - \beta) \frac{1}{n} \binom{n+m-2}{m-1} q^{n-1} (1-q)^m \leq \cos \alpha - \beta,$$

or equivalently,

$$(4.2) \quad \sum_{n=2}^{\infty} (2n - \cos \alpha - \beta) \binom{n+m-2}{m-1} q^{n-1} (1-q)^m \leq \cos \alpha - \beta.$$

The remaining part of the proof of Theorem 4.1 is similar to that of Theorem 2.1, and so we omit the details. □

Theorem 4.2. *If $m > 1$, then the integral operator $\mathcal{G}_q^m f(z)$ given by (4.1) is in the class $\mathcal{TSP}_p(\alpha, \beta)$ if and only if*

$$2(1 - (1-q)^m) - \frac{\cos \alpha + \beta}{q(m-1)} ((1-q) - (1-q)^m - q(m-1)(1-q)^m) \leq \cos \alpha - \beta.$$

Proof. The proof of Theorem 4.2 is similar to the proof of Theorem 4.1, so we omitted the proof of Theorem 4.2. □

5. COROLLARIES AND CONSEQUENCES

By specializing the parameter $\beta = 0$ in Theorems 2.1–4.2, we obtain the following corollaries.

Corollary 5.1. *If $m \geq 1$, then $\Phi_q^m(z) \in \mathcal{TS}\mathcal{P}_p(\alpha)$ if and only if*

$$(5.1) \quad \frac{2qm}{1-q} + (2 - \cos \alpha)(1 - (1-q)^m) \leq \cos \alpha.$$

Corollary 5.2. *If $m \geq 1$, then $\Phi_q^m(z) \in \mathcal{UCT}_p(\alpha)$ if and only if*

$$(5.2) \quad \frac{2q^2m(m+1)}{(1-q)^2} + (6 - \cos \alpha)\frac{qm}{1-q} + (2 - \cos \alpha)(1 - (1-q)^m) \leq \cos \alpha.$$

Corollary 5.3. *Let $m > 1$ and $f \in \mathcal{R}^\tau(A, B)$. Then $\mathcal{I}_q^m f(z) \in \mathcal{TS}\mathcal{P}_p(\alpha)$ if*

$$(5.3) \quad (A-B)|\tau| \left(2(1 - (1-q)^m) - \frac{\cos \alpha}{q(m-1)} ((1-q) - (1-q)^m - q(m-1)(1-q)^m) \right) \leq \cos \alpha.$$

Corollary 5.4. *Let $m \geq 1$ and $f \in \mathcal{R}^\tau(A, B)$. Then $\mathcal{I}_q^m f(z) \in \mathcal{UCT}_p(\alpha)$ if*

$$(5.4) \quad (A-B)|\tau| \left(\frac{2qm}{1-q} + (2 - \cos \alpha)(1 - (1-q)^m) \right) \leq \cos \alpha.$$

Corollary 5.5. *If $m \geq 1$, then the integral operator $\mathcal{G}_q^m f(z)$ given by (4.1) is in the class $\mathcal{UCT}_p(\alpha)$ if and only if inequality (5.1) is satisfied.*

Corollary 5.6. *If $m > 1$, then the integral operator $\mathcal{G}_q^m f(z)$ given by (4.1) is in the class $\mathcal{TS}\mathcal{P}_p(\alpha)$ if and only if*

$$2(1 - (1-q)^m) - \frac{\cos \alpha}{q(m-1)} ((1-q) - (1-q)^m - q(m-1)(1-q)^m) \leq \cos \alpha.$$

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