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ON STABILITY, BOUNDEDNESS, AND SQUARE INTEGRABILITY  
OF SOLUTIONS OF CERTAIN THIRD ORDER NEUTRAL  
DIFFERENTIAL EQUATIONS

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*Abstract.* The authors establish some new sufficient conditions under which all solutions of a certain class of nonlinear neutral delay differential equations of the third order are stable, bounded, and square integrable. Illustrative examples are given to demonstrate the main results.

*Keywords:* stability; boundedness; square integrability; Lyapunov functional; neutral differential equation of third order

*MSC 2020:* 34K20, 34K40

## 1. INTRODUCTION

The study of qualitative behavior of solutions of nonlinear differential equations, such as their stability, boundedness, square integrability, etc., without explicitly determining the solutions has attracted the attention of researchers for many years going back to the pioneering work of Lyapunov (see [18]). The aim of this paper is to study the asymptotic stability of solutions to a class of third-order neutral equations of the form

$$(1.1) \quad (x''(t) + \Omega(x''(t-r)))' + \Psi(x(t))x''(t) + \Phi(x(t))x'(t) + h(x(t-\sigma)) = 0,$$

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and the boundedness and square integrability of solutions of the corresponding forced equation

$$(1.2) \quad (x''(t) + \Omega(x''(t-r)))' + \Psi(x(t))x''(t) + \Phi(x(t))x'(t) + h(x(t-\sigma)) = e(t),$$

for all  $t \geq t_1 \geq t_0 + \varrho$ , where  $\varrho = \max\{r, \sigma\}$ .

The asymptotic behavior of solutions of equations of the form of (1.1) with  $\Omega(x) = 0$  has been studied by many authors utilizing various methods. For example, in 1953, Simanov (see [29]) investigated the global stability of the zero solution of the equation

$$x''' + \psi(x, x')x'' + bx' + cx = 0,$$

where  $b$  and  $c$  are constants. Later, Ezeilo (see [8]) discussed the global stability of the zero solution of the equation of the form

$$x''' + \psi(x, x')x'' + \varphi(x') + g(x) = 0.$$

Swick in [30] studied the asymptotic behavior of solutions of the nonlinear differential equations

$$x''' + ax'' + g(x)x' + h(x) = e(t)$$

and

$$x''' + p(t)x'' + q(t)g(x') + h(x) = e(t).$$

Nakashima in [20] considered the perturbed versions of these equations. In 1972, Hara (see [14]) investigated the asymptotic behavior of solutions of differential equations of the form

$$x''' + a(t)x'' + b(t)g(x, x') + c(t)h(x) = p(t, x, x', x'')$$

and showed that all solutions are uniformly bounded and satisfy the conditions

$$x(t) \rightarrow 0, \quad x'(t) \rightarrow 0, \quad \text{and} \quad x''(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.$$

Hara in [15] also considered the third order equations

$$x''' + a(t)x + b(t)x' + c(t)x = p(t)$$

and

$$x''' + a(t)x'' + b(t)x' + c(t)h(x) = p(t, x, x', x'')$$

and established conditions under which all solutions of the above equations are uniformly bounded and tend to zero as  $t \rightarrow \infty$ .

More recently, Qian (see [23], [24]) and Omeike (see [21]) discussed the global stability and asymptotic behavior of solutions of the differential equations

$$x''' + \psi(x, x')x'' + f(x, x') = 0$$

and

$$x''' + \psi(x, x')x'' + f(x, x') = p(t, x, x', x'')$$

with particular attention paid to the boundedness of solutions. In 2007, Zhang and Si (see [36]) investigated the asymptotic stability of solutions of

$$x'''(t) + g(x'(t), x''(t)) + f(x(t), x'(t)) + h(x(t)) = 0.$$

By defining a Lyapunov functional, Tunç in [32] investigated the stability and boundedness of solutions to nonlinear third order differential equations with constant delay,  $r$ , of the form

$$\begin{aligned} x'''(t) + g(x(t), x'(t))x''(t) + f(x(t-r), x'(t-r)) + h(x(t-r)) \\ = p(t, x(t), x'(t), x(t-r), x'(t-r), x''(t)). \end{aligned}$$

Ademola and Arawomo (see [1]) obtained criteria for uniform stability, uniform boundedness, and uniform ultimate boundedness of solutions for the more general third order nonlinear delay differential equation

$$x''' + f(x, x', x'')x'' + g(x(t-r(t)), x'(t-r(t))) + h(x(t-r(t))) = p(t, x, x', x'').$$

In [11], Graef et al. obtain sufficient conditions that guarantee the existence of square integrability and asymptotic stability of the zero solution of the non-autonomous third-order delay differential equation

$$(x(t) + \beta x(t-r))''' + a(t)(Q(x(t))x'(t))' + b(t)R(x(t))x'(t) + c(t)f(x(-r)) = 0,$$

and the boundedness and square integrability of solutions of the corresponding forced equation

$$(x(t) + \beta x(t-r))''' + a(t)(Q(x(t))x'(t))' + b(t)R(x(t))x'(t) + c(t)f(x(t-r)) = e(t),$$

which are different from equations (1.1) and (1.2) above.

Our motivation for the present work has come from the papers mentioned above as well as from many others in the literature. To the best of our knowledge, there do not appear to be any results of the type obtained in this paper for third order

neutral differential equations of the form (1.1) by using Lyapunov functionals. It is clear that equations (1.1) and (1.2) are different from those mentioned above due to the presence of the neutral term  $\Omega$  in our equations. Hence, our results include and generalize theirs. We note that many results concerning the theory of neutral functional differential equations are given in the monographs by Hale and Lunel (see [12], [13]). These equations find numerous applications in natural sciences and technology, but they are characterized by specific properties which make their study difficult in both concepts and techniques. For additional results related to those in this paper, we refer the reader to references [2]–[7], [9], [10], [16], [17], [19], [22], [25]–[28], [31], [33]–[35].

Here we assume that in equations (1.1) and (1.2) the functions  $\Phi, \Psi, h, \Omega: \mathbb{R} \rightarrow \mathbb{R}$  are continuous and  $h(0) = 0$ . In addition, it is also assumed that  $\Phi', \Psi', h'$ , and  $\Omega'$  exist and are continuous.

By a *solution* of (1.1) we mean a continuous function  $x: [t_x, \infty) \rightarrow \mathbb{R}$  for  $t_x \geq t_0 + \varrho$  such that  $x(t) \in C^2([t_x - \varrho, \infty), \mathbb{R})$ ,  $x(t)$  satisfies equation (1.1) on  $[t_x, \infty)$ , and  $Z(t) \in C^1([t_x, \infty), \mathbb{R})$ . Without further mention, we will assume throughout that every solution  $x(t)$  of (1.1) under consideration here is continuable to the right and nontrivial, i.e.,  $x(t)$  is defined on some ray  $[t_x, \infty)$  and  $\sup\{|x(t)|: t \geq T\} > 0$  for every  $T \geq t_x$ . Moreover, we tacitly assume that (1.1) possesses such solutions.

We set, for a solution  $x$  of (1.1),

$$\begin{aligned} Z(t) &= x''(t) + \Omega(x''(t-r)) \quad \forall t \geq t_0 + \varrho, \\ y(t) &= x'(t), \\ \Delta(t) &= x^2(t) + y^2(t) + Z^2(t), \end{aligned}$$

and

$$\Gamma(t) = x^2(t) + x'^2(t) + x''^2(t).$$

Equation (1.1) can be written as the system

$$(1.3) \quad \begin{cases} x'(t) = y(t), \\ y'(t) = z(t), \\ (z(t) + \Omega(z(t-r)))' = -\Psi(x(t))z(t) - \Phi(x(t))y(t) - h(x(t)) \\ \quad \quad \quad + \int_{t-\sigma}^t h'(x(s))y(s) \, ds. \end{cases}$$

It easy to see from (1.3) that

$$(1.4) \quad Z(t) = x''(t) + \Omega(x''(t-r)) = z(t) + \Omega(z(t-r)).$$

## 2. STABILITY

Before stating and proving our main results, we introduce the following hypotheses. Assume that there are positive constants  $d, \varphi_0, \varphi_1, \psi_0, \psi_1, n, \delta_0, \delta_1$ , and  $K$  such that the following conditions on the functions that appear in equation (1.1) are satisfied:

- (i)  $\psi_0 \leq \Psi(x) - d \leq \psi_1$  and  $\varphi_0 \leq \Phi(x) \leq \varphi_1$ ;
- (ii)  $|\Omega(x)| \leq K|x|$  for all  $x$ ;
- (iii)  $|h'(x)| \leq \delta_0$  for all  $x$ , and  $h(x)/x \geq \delta_1$  for all  $x \neq 0$ ;
- (iv)  $\int_{-\infty}^{\infty} (|\Psi'(u)| + |\Phi'(u)|) du \leq n$ .

For ease of exposition we adopt the following notation:

$$\begin{aligned}\theta_1(t) &= \Psi'(x(t))x'(t), \\ \theta_2(t) &= \Phi'(x(t))x'(t), \\ \omega(t) &= |\theta_1(t)| + |\theta_2(t)|.\end{aligned}$$

**Theorem 2.1.** *In addition to conditions (i)–(iv), assume that there are positive constants  $A, B, C$ , and  $\varepsilon$  such that:*

- (v)  $-d\delta_1 + \frac{1}{2}(\delta_0 K + d\psi_1) = -A$ ;
- (vi)  $\delta_0 - d\varphi_0 + \frac{1}{2}(d + K(d + \varphi_1)) = -B$ ;
- (vii)  $(d - \psi_0) + \frac{1}{2}d(\psi_1 + 1) + \frac{1}{2}K(3d + \varphi_1 + 2\delta_0 + 2\psi_1) + \varepsilon = -C$ ;
- (viii)  $\delta_0/\varphi_0 < d < \min\{\frac{1}{3}\psi_0, \frac{1}{2}\varphi_0\}$ .

*Then, the zero solution of (1.3) is uniformly asymptotically stable provided that*

$$\sigma < \frac{2}{\delta_0} \min\left\{\frac{A}{d}, \frac{B}{(3d + 1 + K)}, C, \frac{\varepsilon}{K}\right\}.$$

**Proof.** The proof of this theorem depends on properties of the continuously differentiable functional  $W(t, x_t, y_t, z_t) = W$  defined by

$$(2.1) \quad W = V e^{-\eta^{-1} \int_{t_1}^t \omega(s) ds},$$

where

$$\begin{aligned}V &= V(t, x_t, y_t, z_t) = V_1 + V_2 + V_3, \\ V_1 &= d \int_0^x h(u) du + h(x)y + \frac{1}{2}\Phi(x)y^2, \\ V_2 &= \frac{1}{2}(dy + Z)^2 + \frac{d}{2}(\Psi(x) - d)y^2 + dxZ + \frac{d}{2}\Phi(x)x^2, \\ V_3 &= \mu \int_{-\sigma}^0 \int_{t+s}^t y^2(u) du ds + \gamma \int_{t-r}^t z^2(s) ds.\end{aligned}$$

Here,  $\gamma, \mu$  and  $\eta$  are positive constants to be specified later in the proof.

Since  $h(0) = 0$ , it is easy to verify that

$$2 \int_0^x h'(u)h(u) \, du = h^2(x),$$

and since  $|h'(x)| \leq \delta_0$ , we see that  $|h(x)| \leq \delta_0|x|$ .

From conditions (i), (iii), and (viii),

$$\begin{aligned} V_1 &= d \int_0^x h(u) \, du + \frac{\Phi(x)}{2} \left( y + \frac{h(x)}{\Phi(x)} \right)^2 - \frac{1}{2\Phi(x)} h^2(x) \\ &\geq d \int_0^x h(u) \, du - \frac{1}{\varphi_0} \int_0^x h'(u)h(u) \, du \geq \int_0^x \left( d - \frac{\delta_0}{\varphi_0} \right) h(u) \, du \geq \left( d - \frac{\delta_0}{\varphi_0} \right) \frac{\delta_1}{2} x^2. \end{aligned}$$

Also, using (i) and (viii) we have

$$\begin{aligned} V_2 &= \frac{1}{4}(Z^2 + 4dxZ + 2d\Phi(x)x^2) + \frac{1}{4}(Z^2 + 4dyZ + 2d(\Psi(x) - d)y^2) + \frac{1}{2}d^2y^2 \\ &= \frac{1}{8}(Z + 2dx)^2 + \frac{1}{4}d\Phi(x) \left( x + \frac{1}{\Phi(x)}Z \right)^2 + (\Phi(x) - 2d) \left( \frac{1}{4}dx^2 + \frac{1}{8\Phi(x)}Z^2 \right) \\ &\quad + \frac{1}{8}(Z + 2dy)^2 + \frac{1}{4}d(\Psi(x) - d) \left( y + \frac{1}{\Psi(x) - d}Z \right)^2 \\ &\quad + (\Psi(x) - 3d) \left( \frac{dy^2}{4} + \frac{Z^2}{8(\Psi(x) - d)} \right) + \frac{1}{2}d^2y^2 \\ &\geq (\varphi_0 - 2d) \left( \frac{d}{4}x^2 + \frac{1}{8\varphi_1}Z^2 \right) + (\psi_0 - 3d) \left( \frac{d}{4}y^2 + \frac{1}{8\psi_1}Z^2 \right) \\ &\geq k_0(x^2 + Z^2) + k_1(y^2 + Z^2), \end{aligned}$$

where  $k_0 = \frac{1}{4}(\varphi_0 - 2d) \min\{d, 1/(2\varphi_1)\}$ , and  $k_1 = \frac{1}{4}(\psi_0 - 3d) \min\{d, 1/(2\psi_1)\}$ .

Hence, there exists a positive constant  $\lambda_0$ , small enough so that

$$(2.2) \quad V \geq \lambda_0 \Delta(t).$$

Therefore, from (2.2) and (2.1), we obtain

$$(2.3) \quad W \geq \lambda_1(x^2(t) + y^2(t) + Z^2(t)) = \lambda_1 \Delta(t),$$

where  $\lambda_1 = \lambda_0 e^{-n/\eta}$ .

To obtain an upper estimate on  $V$ , note that by using Schwarz's inequality, we have

$$V_1 \leq \frac{\delta_0}{2}(d + \delta_0)x^2 + \frac{1}{2}(1 + \varphi_1)y^2,$$

and

$$V_2 \leq \frac{d}{2}(1 + \varphi_1)x^2 + \frac{d}{2}(1 + d + \psi_1)y^2 + \left( d + \frac{1}{2} \right) Z^2.$$

Thus,

$$(2.4) \quad V \leq \lambda_2 \Delta(t) + \mu \int_{-\sigma}^0 \int_{t+s}^t y^2(u) \, du \, ds + \gamma \int_{t-r}^t z^2(s) \, ds,$$

where

$$\lambda_2 = \max \left\{ \frac{\delta_0}{2}(d + \delta_0) + \frac{d}{2}(1 + \varphi_1), \frac{1}{2}(1 + \varphi_1) + \frac{d}{2}(1 + d + \psi_1), \left(d + \frac{1}{2}\right) \right\}.$$

Since

$$(2.5) \quad e^{-n/\eta} < e^{-\eta^{-1} \int_{t_1}^t \omega(s) \, ds} < 1,$$

from (2.2), (2.4) and (2.1), we see that

$$\lambda_1 \Delta(t) \leq W \leq V \leq \lambda_2 \Delta(t) + \mu \int_{-\sigma}^0 \int_{t+s}^t y^2(u) \, du \, ds + \gamma \int_{t-r}^t z^2(s) \, ds,$$

where  $\lambda_1 = \lambda_0 e^{-n/\eta}$ .

Now taking the derivative of  $V$  along the trajectories of system (1.3), we obtain

$$V'_{(1.3)} = U_1 + U_2 + U_3 + \frac{d}{2} \theta_1(t) y^2 + \frac{1}{2} \theta_2(t) (y^2 + dx^2)$$

where

$$\begin{aligned} U_1 &= -dxh(x(t)) + (h'(x) - d\Phi(x) + \mu\sigma)y^2(t) + (d - \Psi(x(t)) + \gamma)z^2(t), \\ U_2 &= dyz(t) - d\Psi(x(t))xz(t) - \gamma z^2(t-r) \\ &\quad + (d - \Phi(x))y - h(x(t)) + (d - \Psi(x(t)))z(t)\Omega(z(t-r)), \end{aligned}$$

and

$$U_3 = (dx + dy + Z) \int_{t-\sigma}^t h'(x(s))y(s) \, ds - \mu \int_{t-\sigma}^t y^2(s) \, ds.$$

Conditions (i) and (iii) imply that

$$U_1 \leq -d\delta_1 x^2 + (\delta_0 - d\varphi_0 + \mu\sigma)y^2(t) + (d - \psi_0 + \gamma)z^2(t).$$

Using Schwarz's inequality again, together with conditions (i)–(iii), we obtain

$$\begin{aligned} U_2 &\leq \frac{1}{2}(\delta_0 K + d\psi_1)x^2(t) + \frac{1}{2}(d + K(d + \varphi_1))y^2 \\ &\quad + \frac{1}{2}(d\psi_1 + d + K(d + \psi_1) + \delta_0 K)z^2(t) \\ &\quad + \left(\frac{K}{2}(2d + \varphi_1 + \delta_0 + \psi_1) - \gamma\right)z^2(t-r). \end{aligned}$$



Notice that

$$\begin{aligned} dx(t) \int_{t-\sigma}^t h'(x(s))y(s) \, ds &\leq \frac{d\delta_0}{2}\sigma x^2(t) + \frac{d\delta_0}{2} \int_{t-\sigma}^t y^2(s) \, ds, \\ dy(t) \int_{t-\sigma}^t h'(x(s))y(s) \, ds &\leq \frac{d\delta_0}{2}\sigma y^2(t) + \frac{d\delta_0}{2} \int_{t-\sigma}^t y^2(s) \, ds, \\ z(t) \int_{t-\sigma}^t h'(x(s))y(s) \, ds &\leq \frac{\delta_0}{2}\sigma z^2(t) + \frac{\delta_0}{2} \int_{t-\sigma}^t y^2(s) \, ds, \\ \Omega(z(t-r)) \int_{t-\sigma}^t h'(x(s))y(s) \, ds &\leq \frac{\delta_0 K}{2}\sigma z^2(t-r) + \frac{\delta_0 K}{2} \int_{t-\sigma}^t y^2(s) \, ds. \end{aligned}$$

With some rearrangement of terms and using the estimates above, we can easily obtain

$$\begin{aligned} V'_{(1.3)} &\leq \left(-d\delta_1 + \frac{1}{2}(\delta_0 K + d\psi_1) + \frac{d\delta_0}{2}\sigma\right)x^2(t) \\ &\quad + \left(\delta_0 - d\varphi_0 + \frac{1}{2}(d + K(d + \varphi_1)) + \left(\mu + \frac{d\delta_0}{2}\right)\sigma\right)y^2(t) \\ &\quad + \left(d - \psi_0 + \frac{1}{2}(d\psi_1 + d + K(d + \psi_1 + \delta_0)) + \gamma + \frac{\delta_0}{2}\sigma\right)z^2(t) \\ &\quad + \left(\frac{K}{2}(2d + \varphi_1 + \delta_0 + \psi_1) - \gamma + \frac{K\delta_0}{2}\sigma\right)z^2(t-r) \\ &\quad + \frac{d}{2}\theta_1(t)y^2 + \frac{1}{2}(dx^2 + y^2)\theta_2(t) + \left(\frac{\delta_0}{2}(2d + 1 + K) - \mu\right) \int_{t-\sigma}^t y^2(s) \, ds. \end{aligned}$$

If we now choose

$$\frac{\delta_0}{2}(2d + 1 + K) = \mu \quad \text{and} \quad \frac{K}{2}(2d + \varphi_1 + \delta_0 + \psi_1) + \varepsilon = \gamma,$$

then

$$\begin{aligned} V'_{(1.3)} &\leq \lambda_3\omega(t)(x^2(t) + y^2(t)) + \left(-A + \frac{d\delta_0}{2}\sigma\right)x^2(t) \\ &\quad + \left(-B + \frac{\delta_0}{2}(3d + 1 + K)\sigma\right)y^2(t) + \left(-C + \frac{\delta_0\sigma}{2}\right)z^2(t) \\ &\quad + \left(-\varepsilon + \frac{\delta_0 K}{2}\sigma\right)z^2(t-r), \end{aligned}$$

where  $\lambda_3 = \frac{1}{2}(d + 1)$ . Now

$$(2.6) \quad Z^2 = z^2 + \Omega^2(z(t-r)) + 2z\Omega(z(t-r)),$$

so applying Schwarz's inequality and condition (ii) gives

$$(2.7) \quad Z^2 \leq 2(z^2 + \Omega^2(z(t-r))) \leq 2(z^2 + K^2 z^2(t-r)).$$

If we now take

$$\sigma < \frac{2}{\delta_0} \min \left\{ \frac{A}{d}, \frac{B}{(3d+1+K)}, C, \frac{\varepsilon}{K} \right\},$$

then we can write

$$\begin{aligned} (2.8) \quad V'_{(1.3)} &\leq \lambda_3 \omega(t)(x^2(t) + y^2(t)) - \alpha_1 x^2(t) - \alpha_2 y^2(t) - \alpha_3 z^2(t) - \alpha_4 z^2(t-r), \\ &\leq \lambda_3 \omega(t)(x^2(t) + y^2(t)) - \alpha_1 x^2(t) - \alpha_1 y^2(t) - \alpha_5 (z^2(t) + K^2 z^2(t-r)) \\ &\leq \lambda_3 \omega(t)(x^2(t) + y^2(t)) - \alpha_1 x^2(t) - \alpha_2 y^2(t) - \frac{\alpha_5}{2} Z^2, \end{aligned}$$

where

$$\begin{aligned} \alpha_1 &= A - \frac{\delta_0}{2} \sigma, \quad \alpha_2 = B - \frac{\delta_0}{2} (3d+1+K) \sigma, \quad \alpha_3 = C - \frac{\delta_0 \sigma}{2}, \\ \alpha_4 &= \varepsilon - \frac{\delta_0 K}{2} \sigma, \quad \alpha_5 = \min \left\{ \alpha_3, \frac{\alpha_4}{K^2} \right\}, \end{aligned}$$

and

$$\alpha_i > 0 \quad \text{for } i = 1, 2, \dots, 5.$$

Hence,

$$V'_{(1.3)} \leq \lambda_3 \omega(t)(x^2(t) + y^2(t)) - \lambda_4 \Delta(t),$$

where  $\lambda_4 = \min\{\alpha_1, \alpha_2, \frac{1}{2}\alpha_5\}$ . Therefore, from (2.1) and (2.2), we have

$$\begin{aligned} W'_{(1.3)} &= \left( V' - \frac{1}{\eta} \omega(t) V \right) e^{-\eta^{-1} \int_{t_1}^t \omega(s) ds} \\ &\leq \left( \lambda_3 \omega(t)(x^2(t) + y^2(t)) - \lambda_4 \Delta(t) - \frac{\lambda_0}{\eta} \omega(t) \Delta(t) \right) e^{-\eta^{-1} \int_{t_1}^t \omega(s) ds}. \end{aligned}$$

By taking  $\eta = \lambda_0 / \lambda_3 = 2\lambda_0 / (d+1)$ , we obtain

$$W'_{(1.3)} \leq -\lambda_5 \Delta(t),$$

where  $\lambda_5 = \lambda_4 e^{-n/\eta}$ .

The uniform asymptotic stability of the zero solution of (1.1) follows immediately.  $\square$

### 3. BOUNDEDNESS

Our main theorem in this section is for the forced equation (1.2). The corresponding system becomes

$$(3.1) \quad \begin{cases} x'(t) = y(t), \\ y'(t) = z(t), \\ (z(t) + \Omega(z(t-r)))' = -\Psi(x(t))z(t) - \Phi(x(t))y(t) - h(x(t)) \\ \qquad \qquad \qquad + \int_{t-\sigma}^t h'(x(s))y(s) ds + e(t). \end{cases}$$

**Theorem 3.1.** *In addition to the conditions of Theorem 2.1, assume there exists a positive constant  $e_1$  such that*

$$(ix) \quad \int_{t_0}^t |e(s)| ds < e_1 \text{ for all } t \geq t_0.$$

*Then there exists a positive constant  $N$  such that any solution of (3.1) satisfies*

$$(3.2) \quad |x(t)| \leq N, \quad |y(t)| \leq N, \quad \text{and} \quad |Z(t)| \leq N.$$

*Proof.* On differentiating (2.1) along the solutions of system (3.1), we obtain

$$W'_{(3.1)} \leq -\lambda_5 \Delta(t) + |e(t)|(|dx(t)| + d|y(t)| + |Z(t)|)e^{-\eta^{-1} \int_{t_1}^t \omega(s) ds}.$$

Now, from (2.5) and applying the inequality  $|u| \leq u^2 + 1$ , we find that

$$W'_{(3.1)} \leq -\lambda_5 \Delta(t) + \lambda_6 |e(t)|(\Delta(t) + 3),$$

where  $\lambda_6 = \max\{d, 1\}$ . In view of (2.3), the above estimates imply that

$$(3.3) \quad W'_{(3.1)} \leq -\lambda_5 \Delta(t) + \frac{\lambda_6}{\lambda_1} |e(t)|W(t) + 3\lambda_6 |e(t)|.$$

An integration of (3.3) from  $t_1$  to  $t$  gives

$$W(t) \leq e_2 + \frac{\lambda_6}{\lambda_1} \int_{t_1}^t W(s)|e(s)| ds$$

for some positive constant  $e_2$ . An application of Gronwall's inequality shows that  $W(t)$  is bounded, and the conclusion of the theorem follows immediately.  $\square$

#### 4. SQUARE INTEGRABILITY

Our next result concerns the square integrability of the solutions of equation (1.2).

**Theorem 4.1.** *If all the conditions of Theorem 3.1 are satisfied, then for a solution  $x$  of (1.2)*

$$\int_{t_0}^{\infty} \Gamma(s) \, ds < \infty.$$

*Proof.* From (2.8), we have

$$(4.1) \quad \begin{aligned} V'_{(1.3)} &\leq \lambda_3 \omega(t)(x^2(t) + y^2(t)) - \alpha_1 x^2(t) - \alpha_2 y^2(t) - \alpha_3 z^2(t) \\ &\leq \lambda_3 \omega(t)(x^2(t) + y^2(t)) - \beta_1 \Gamma(t), \end{aligned}$$

where  $\beta_1 = \min\{\alpha_1, \alpha_2, \alpha_3\}$ . Therefore, from (2.1), (2.2), and (4.1), we have

$$\begin{aligned} W'_{(1.3)} &= \left( V'_{(1.3)} - \frac{1}{\eta} \omega(t)V \right) e^{-\eta^{-1} \int_{t_1}^t \omega(s) \, ds} \\ &\leq \left( \lambda_3 \omega(t)(x^2(t) + y^2(t)) - \beta_1 \Gamma(t) - \frac{\lambda_0}{\eta} \omega(t)\Delta(t) \right) e^{-\eta^{-1} \int_{t_1}^t \omega(s) \, ds}. \end{aligned}$$

Now

$$W'_{(3.1)} = W'_{(1.3)} + e(t)(dx(t) + dy(t) + Z(t))e^{-\eta^{-1} \int_{t_1}^t \omega(s) \, ds},$$

and since  $\lambda_0/\eta = \lambda_3$ ,

$$(4.2) \quad \begin{aligned} W'_{(3.1)} &\leq -\beta_2 \Gamma(t) + |e(t)|(d|x(t)| + d|y(t)| + |Z(t)|)e^{-\eta^{-1} \int_{t_1}^t \omega(s) \, ds} \\ &\leq -\beta_2 \Gamma(t) + \left( \frac{\lambda_6}{\lambda_1} W(t) + 3\lambda_6 \right) |e(t)|, \end{aligned}$$

where  $\beta_2 = \beta_1 e^{-n\lambda_3/\lambda_0}$ . Define  $H(t)$  by

$$(4.3) \quad H(t) = W(t) + \tau \int_{t_1}^t \Gamma(s) \, ds \quad \forall t \geq t_1,$$

where  $\tau > 0$  is a constant to be specified later. Differentiating  $H$  and using (4.2), we obtain

$$H'(t) \leq (\tau - \beta_2)\Gamma(t) + \left( \frac{\lambda_6}{\lambda_1} W(t) + 3\lambda_6 \right) |e(t)|.$$

Choosing  $\tau - \beta_2 < 0$ , then from the boundedness of  $W(t)$ ,

$$(4.4) \quad H'(t) \leq \lambda_7 |e(t)|$$

for some  $\lambda_7 > 0$ . Integrating (4.4) from  $t_1$  to  $t$ , and using condition (ix), we see that  $H(t)$  is bounded. In view of (4.3), this implies that

$$\int_{t_1}^{\infty} \Gamma(s) \, ds$$

is bounded, which is what we wished to show.  $\square$

**Remark 4.2.** Notice that by Theorem 4.1,

$$\int_{t_1}^{\infty} (x^2(s) + y^2(s) + z^2(s)) \, ds < \infty$$

and consequently

$$\int_{t_1}^{\infty} Z^2(s) \, ds < \infty,$$

i.e., the solutions of system (3.1) are square integrable.

We conclude this paper with an example to illustrate our results.

**Example 4.3.** Consider the third order neutral delay differential equation

$$(4.5) \quad \left( x''(t) + \frac{1}{100} \frac{x''(t-r)}{1 + e^{x''(t-r)}} \right)' + \left( \frac{1}{10 + x^2} + 3.3 \right) x'' + \left( \frac{\cos x}{4 + x^2} + 6.25 \right) x' + \left( 2x(t - \frac{1}{10}) + \frac{x(t - \frac{1}{10})}{1 + |x(t - \frac{1}{10})|} \right) = \frac{1}{1 + t^2}.$$

Taking  $d = 0.7$ , we see that

$$\psi_0 = 2.6 = 3.3 - 0.7 \leq \Psi(x) - d = \frac{1}{10 + x^2} + 3.3 - 0.7 \leq \frac{1}{10} + 3.3 - 0.7 = 2.7 = \psi_1.$$

We also have

$$\varphi_0 = 6 \leq \Phi(x) = \frac{\cos x}{4 + x^2} + 6.25 \leq 6.5 = \varphi_1$$

and

$$|\Omega(x)| = \frac{1}{100} \left| \frac{x}{1 + e^x} \right| < \frac{1}{100} |x| = K|x|.$$

Now

$$h(x) = 2x + \frac{x}{1 + |x|},$$

so  $h(0) = 0$ ,

$$\frac{h(x)}{x} \geq 2 = \delta_1 \quad \text{for } x \neq 0, \quad \text{and} \quad |h'(x)| = \left| 2 + \frac{1}{(1 + |x|)^2} \right| \leq 3 = \delta_0.$$

Simple calculations show that

$$\int_{-\infty}^{\infty} |\Psi'(u)| \, du = \int_{-\infty}^{\infty} \left| \frac{-2u}{(10+u^2)^2} \right| \, du = 2 \int_0^{\infty} \frac{2u}{(10+u^2)^2} \, du = \frac{1}{5}$$

and

$$\int_{-\infty}^{\infty} |\Phi'(u)| \, du \leq \int_{-\infty}^{\infty} \left( \left| \frac{\sin u}{4+u^2} \right| + \left| \frac{2u \cos u}{(4+u^2)^2} \right| \right) \, du \leq \pi.$$

Hence, conditions (i)–(iv) hold.

If we take  $\varepsilon = K = \frac{1}{100}$ , it is easy to see that

$$-d\delta_1 + \frac{1}{2}(\delta_0 K + d\psi_1) = -0.44 = -A,$$

$$\delta_0 - d\varphi_0 + \frac{1}{2}(d + K(d + \varphi_1)) = -0.814 = -B,$$

$$(d - \psi_0) + \frac{d}{2}(\psi_1 + 1) + \frac{K}{2}(3d + \varphi_1 + 2\delta_0 + 2\psi_1) + \varepsilon = -0.495 = -C,$$

$$\frac{\delta_0}{\varphi_0} = \frac{3}{6} = 0.5 < 0.7 = d < \min\left\{\frac{\psi_0}{3}, \frac{\varphi_0}{2}\right\} = \min\left\{\frac{2.6}{3}, 3\right\} \approx 0.867,$$

and so (v)–(viii) hold. Clearly,  $e(t) = 1/(1+t^2)$  satisfies

$$\int_0^t |e(s)| \, ds < \infty \quad \forall t \geq t_0,$$

so (ix) holds. Finally, if

$$\sigma = 0.1 < \frac{2}{\delta_0} \min\left\{\frac{A}{d}, \frac{B}{(3d+1+K)}, C, \frac{\varepsilon}{K}\right\} \approx 0.1745,$$

then all the conditions of Theorems 2.1, 3.1, and 4.1 hold, so all solutions of equation (4.5) are bounded,  $x$ ,  $x'$ , and  $x''$  are square integrable, and if  $e(t) \equiv 0$ , then the zero solution of (4.5) is uniformly asymptotically stable.

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