Preference Games and Sink Equilibria

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June, Oliver Biggar

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Abstract

In this thesis we present a new foundation for game theory. Our model for a game is defined precisely by the *response graph*, a natural object underlying all classical games. We call this model a *preference game*. Preference games generalise classical games in that all classical games have an associated preference game. Preference games also have a natural choice of solution concept: the *sink equilibria*, which are the sink strongly connected components of this graph. These exist in all games, and generalise pure Nash equilibria. We argue that preference games and sink equilibria form a predictive and well-founded model of strategic interaction that both clarifies problems from classical game theory and presents new solutions and predictions.

Our approach is three-pronged. First, we show preference games are **axiomatically well-motivated** and so are broadly-applicable models of strategic interaction. We obtain these games by weakening the classical formulation of game theory with two axioms: *ordinality*, asserting the model requires only that players know a discrete order over the outcomes; and *relevance*, asserting that our model depends only on preferences over outcomes players can choose between. We give a new discussion of the Prisoner's Dilemma, where we show that the paradox can be rephrased as a consequence of Nash equilibria satisfying the relevance axiom, while Pareto efficiency satisfies only the ordinality axiom. We show further that preference games implicitly capture in their graph structure the equivalences caused by renaming players or strategies, while expressing these equivalences in classical games is cumbersome.

Second, we show preference games give **new insight into strategic interaction**. We examine two-player games, with a focus on zero-sum and potential games. Despite both being classically defined in terms of utility functions, we show that their strategic structure can be easily understood through preference games, which make clear the duality between these two classes. We use this to prove a new theorem of classical game theory: in every two-player game, every non-iteratively-dominated strategy takes part in a 2x2 subgame with the preference structure of Matching Pennies or 2x2 Coordination. As a consequence, any two-player game sharing a response graph with both a zero-sum game and a potential game is dominance-solvable. The proofs are combinatorial.

Thirdly, we show **preference games are compatible with game dynamics**, while classical games and mixed Nash equilibria are not. Game dynamics—the mathematical model of strategic adjustment used in evolutionary game theory—cannot generically converge to mixed Nash equilibria. By contrast, pure Nash equilibria are *attracting fixed-points*; the natural generalisation of this dynamic concept is the *sink chain components*, a topological object defined by the Fundamental Theorem of Dynamical Systems. While these do not exist in general, we **prove an open problem** establishing their existence under the *replicator dynamic*, the best-known game dynamic. We prove that sink chain components under the replicator *always contain sink equilibria*, and we conjecture that this relationship is always one-to-one. Thus we obtain the surprising result that the complex, sometimes chaotic, behaviour of the replicator dynamic is governed in the long run by the response graph of the game, with the outcome determined by a simple combinatorial object—sink equilibria. Sink equilibria also describe the long-run outcomes of all discrete dynamics defined by Markov chains on the response graph. We conclude with a number of open problems that emerge from preference games.

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Chapter 1

Introduction

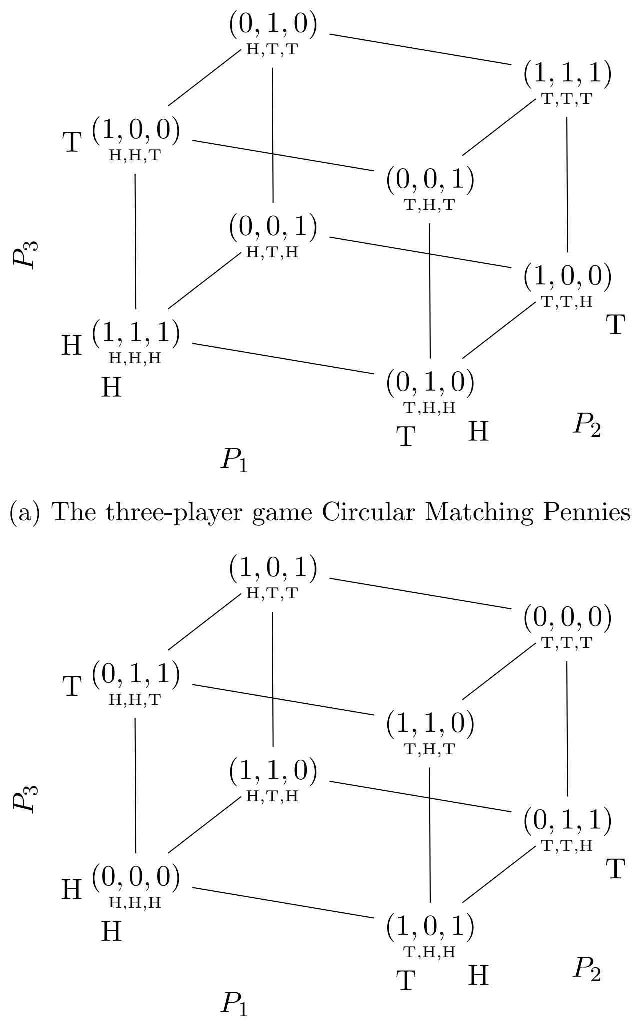
1.1 Modelling Strategic Interaction

When constructing a mathematical model, we encounter a *Goldilocks problem*: a model that is too detailed may be intractable to solve, or may not isolate the concept studied by the modeller, while one that is too general may be unable to give useful predictions. Constructing a model that is 'just right' is an art; the goal is to pick out precisely the assumptions to gain insight without losing applicability.

As game theory users, we wish to analyse—and ideally, predict—the outcomes of strategic interactions. To do this, we must answer a key question: how much do we know about the strategic interactions we hope to analyse? This question has grown in importance as game theory has overflowed its original applications in economics and mathematics and claimed an important place in computer science, biology and sociology (Myerson, 1997; Fudenberg and Tirole, 1991; Roughgarden, 2010).

In this thesis we propose a general model of a game we call a *preference game*. We argue that preference games isolate the intuitive notion of strategic interaction (Chapter 3). By making few assumptions of the modeller, preference games can be applied to a range of strategic interactions. Despite this generality, preference games give useful predictions (Chapter 4)—sometimes better predictions than classical game theory (Chapter 5).

To be concrete, we will consider two specific games, depicted in Figure 1.1. Our first example is a three-player game called 'Circular Matching Pennies' (Sandholm, 2010), which works as follows. The players sits in a circle and place a penny on the table simultaneously, choosing to have either Heads or Tails showing. Each player aims to *match* the choice of the player to their right. Specifically, we assume they are rewarded with a positive payoff for matching, say, with value 1, and 0 payoff for not matching. This is shown in Figure 1.1a.



(a) The three-player game Circular Matching Pennies

(b) The three-player game Circular Mismatching Pennies

Figure 1.1: The payoff tables for the Circular Matching and Mismatching Pennies games (described in the text). Each player (P_1, P_2 and P_3) has two strategies: Heads (H) or Tails (T), and receives a payoff of either 1 or 0. The players are numbered anticlockwise.

Our second example is a very similar game, which is called three-player 'Circular Mismatching Pennies'. Again each player places a penny on the table, but now the aim of each player is for their choice to *not* match that of the player to their right. If so, they receive a payoff of value 1, and otherwise they receive 0 payoff. This is shown in Figure 1.1b.

We proceed by analysing these games through the lens of classical game theory, the key idea of which is identification of a game's Nash equilibria (Nash, 1951). These are an assignment of strategies for each player where no player can improve their payoff by a unilaterally changing their strategy. Both of these games have an identical *mixed Nash equilibrium* (Definition 2.2.5) where all players play both Heads and Tails with equal probability. However, Matching Pennies has two additional *pure Nash equilibria* (Definition 2.2.3), which correspond to the strategy profiles (Heads, Heads, Heads) and (Tails, Tails, Tails).

The Circular Matching Pennies game showcases the general differences between pure and mixed Nash equilibria. As an example, the pure Nash equilibrium (Heads, Heads, Heads) is '*stable*' in two intuitive senses[1]:

1. (*Model stability*): If the payoffs in our game are perturbed by some infinitesimal quantity, this profile remains a pure Nash equilibrium.
2. (*Strategic stability*): Even knowing that another player may play Tails with some infinitesimal probability, each player's best choice is still to play Heads.

Concerningly, the mixed Nash equilibrium in Circular Matching Pennies—and mixed Nash equilibria in general—is not stable in either of the senses described above. First, infinitesimal changes in the payoffs will generally move the equilibrium (the mixed equilibrium is not *model stable*). Second, if a player plays a non-equilibrium strategy, such as playing Tails infinitesimally more often than Heads, than the player to their left maximises their payoff by playing Heads all of the time! This is a general property of mixed Nash equilibria, as the following theorem asserts.

Theorem 1.1.1 (Strategic instability of mixed Nash Equilibria)**.** *Let x be a mixed Nash equilibrium in a strict game, and let P be any player playing a mixed strategy. In* every neighbourhood *of x there is a strategy profile x' where P receives strictly larger payoff by playing a pure strategy.*

That is, even arbitrarily close to a mixed Nash equilibrium, there are points where any player wishes to move away from the equilibrium—in fact, as far away as possible, because they would prefer to play a pure strategy. Because this theorem holds for any mixed Nash equilibrium, it also holds for any of the various refinements of the Nash concept, including proper equilibria, perfect equilibria, trembling-hand perfect equilibria and evolutionarily stable strategies (Myerson, 1978). This problem is unavoidable; no solution concept between pure and mixed Nash can ever be stable in this sense.

[1] This can be seen as a Corollary of Lemma 5.2.23.

There is another big problem with mixed Nash equilibria, and this one is computational: it is generally intractable to compute, or even approximate, a Nash equilibrium (Daskalakis et al., 2009; Daskalakis, 2013). Pure Nash equilibria, on the other hand, can be computed in linear time.

This is bad news from the perspective of prediction of behaviour. In general games, neither us, as analysts, nor the players themselves can compute an equilibrium strategy. Even in a small game like Circular Matching Pennies where the equilibrium can be computed by brute force, a precise description of the equilibrium requires knowing the payoffs to arbitrary precision, because mixed Nash equilibria are not *model stable*. We need such a precise description because the equilibrium is not *strategically stable*; if players play infinitesimally far from equilibrium, the other players would prefer to play non-equilibrium strategies. Thus without arbitrary computing power and payoffs which are known to arbitrary accuracy, the mixed Nash equilibrium is not a reasonable prediction of behaviour.

In Circular Matching Pennies, this suggests that the pure Nash equilibria, by virtue of being *efficiently computable*, *model stable* and *strategically stable*, are a better predictor of the 'outcome' of the game than the mixed Nash equilibrium. As an apt physical analogy, the outcome of this game is like a coin toss, coming down on either (Heads, Heads, Heads) or (Tails, Tails, Tails). The mixed Nash equilibria is analogous to the coin standing on its side—technically possible, but never a reasonable prediction. In Circular Mismatching Pennies, though, we cannot conclude that pure Nash equilibria are the correct prediction—there are no pure Nash equilibria, and so in classical game theory we seemingly have no choice but to accept the unique mixed Nash equilibrium as our outcome, or else say nothing at all. Can we do any better?

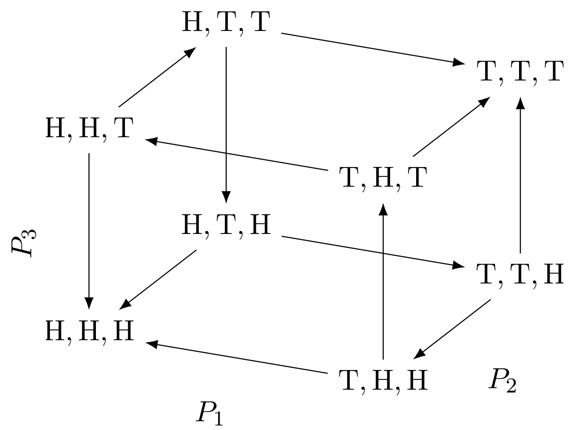
The classical model of Circular Mismatching Pennies does not give a useful prediction. Our solution is to revise the model. It appears that a modelling problem occurred when we translated the English-language description of the game—in terms of the choices each player preferred to make—into a specific numerical payoff. Thus, instead of introducing assumptions and data to explain the solution, we try the opposite approach: reduce the model of a game to its bare essentials, which are each player's *preference order over their strategies*[2]. We will use this model to construct a well-founded and predictive notion of outcome.

1.2 Preference Games

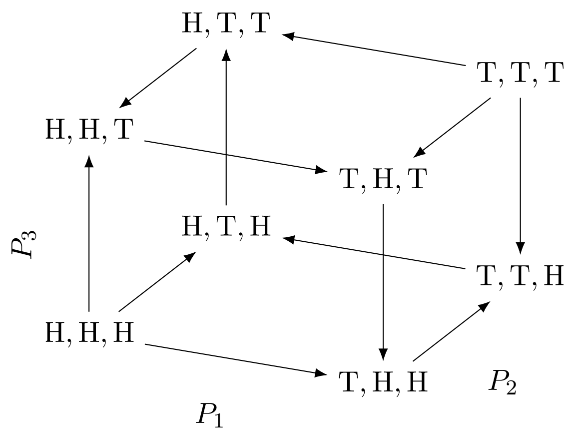
In Figure 1.2 we present two directed graphs associated with Circular Matching and Mismatching Pennies respectively. The nodes of these graphs are the strategy profiles[3] of each game, where we have now marked an *arc* between strategy profiles if they differ only in the strategy of one player, with the direction of the arc indicating which strategy

[2]A *preference order* is a *weak order*. See Section 2.1.

[3]A choice of strategy for each player. See Chapter 2 for formal definitions.



(a) The response graph of Circular Matching Pennies



(b) The response graph of Circular Mismatching Pennies

Figure 1.2: The response graphs of Circular Matching and Mismatching Pennies, corresponding to the payoff tables in Figure 1.1.

is preferred by that player. This graph is known as the *response graph* (Papadimitriou and Piliouras, 2019). While the payoffs are not depicted, we can still identify the pure Nash equilibria: they are the nodes where *all arcs are inward*.

In this thesis we pose the following question: what would the theory of games look like if a game were defined *solely by its response graph?* We call such a game a **preference game**.

Let us recall our descriptions of Circular Matching and Mismatching Pennies. We defined these games by specifying whether each player aimed to match or mismatch the choice of the player to their right. This information defined, for a fixed choice of strategies for the other players, the preference order in which each player preferred their own strategies. It was only then that we chose specific values (1 or 0) for the payoffs which induced these preference orders. However, *these values had no meaning* beyond instantiating the preference order.

Observe that the preference orders for each player for fixed choices of the other players is precisely the information that defines the response graphs in Figure 1.2 (Theorem 3.3.8), and so is precisely the information that defines a preference game; this is the motivation behind the name "preference game". Thus, the Circular Matching and Mismatching Pennies games *are naturally modelled as preference games*. In order to apply classical game theory, we had to define our orders by real-numbered payoffs. This implies, for instance, that changing the payoff to Player 2 for matching Player 3 from 1 to 1.1 would produce a different game, an observation that is *fundamentally misleading*. For an example of how, see Section 3.3.2.

Because payoffs are used in the classical definition of a game (Definition 2.2.1), one cannot 'reach the start line' in classical game theory without numerical payoffs. This leads to situations such as the above where 'arbitrary' numbers are required. Preference games allow us to model a strategic interaction without assuming payoffs are known.

1.2.1 Sink equilibria

In our 'preference game theory', what is the solution concept for a game? Pure Nash equilibria seem to be a plausible proposition, but they have a clear drawback: they do not exist in all games. Mixed Nash equilibria are no longer a viable choice because we cannot describe them without payoffs.

Instead we propose the simplest answer from graph theory. In the response graph, pure Nash equilibria are always *sink strongly connected components* (Definition 2.1). For succinctness, we shall refer to these simply as *sinks*. In fact pure NEs are exactly the sinks that are singletons. We therefore propose the following solution concept: the **sinks of the response graph**. These are guaranteed to exist in all games, and they coincide with pure Nash equilibria when they are singletons. In algorithmic game theory, these have been called *sink equilibria* (Goemans et al., 2005)[4] where they have been studied

[4]Sink equilibria, as defined in Goemans et al. (2005), are not identical to what we have defined here.

in the context of the Price of Anarchy (Roughgarden, 2005).

It seems at first that this idea is too simple, and that preference game theory cannot be powerful enough to provide useful insights, especially given that the core theorems of game theory (such as the minimax theorem and Nash's existence theorem) cannot even be stated, let alone proven. Instead we show in this thesis that preference game theory and the solution concept of a sink equilibrium is both a well-motivated and predictive theory that is compatible with ideas from ordinal utility theory and evolutionary and algorithmic game theory.

We aim to demonstrate that

1. Preference games are a **better conceptual model** of strategic interaction,
2. preference games **give new insight into game theory**, and
3. sink equilibria are a **better dynamic solution concept than Nash equilibria**.

These three ideas are the respective foci of Chapters 3, 4 and 5.

1.2.2 Evolutionary game theory

Evolutionary game theory (Hofbauer and Sigmund, 2003) is a subfield of game theory which focuses on the dynamic adjustment procedures that players use to modify their strategies over time. This area is of increasing interest in computer science (Roughgarden, 2010) because of its close connection with *machine learning*, in particular *online learning*. It is known in this field that agents who learn their strategy over time generically do not converge to mixed Nash equilibria (Sandholm, 2010), which agrees with the discussion of stability we gave above.

Evolutionary games are typically presented as a *dynamical system* (Chapter 2) defined by a differential or difference equation in strategy space. The path traced out by a mixed strategy profile over time is called a *trajectory*. It appears that dynamics on games depend heavily on payoffs and so one might expect that they would not relate closely to preference games. However we show in Chapter 5 that the 'long-run' outcomes of two plausible dynamics on games, while generally unrelated to Nash equilibria, are determined by the *sink equilibria*, giving a surprising connection with preference games.

As an example, we turn again to Circular Matching and Mismatching Pennies, this time depicted in Figure 1.3. We plot in the mixed strategy space the trajectories of the *replicator dynamic* (Hofbauer and Sigmund, 2003) from a variety of initial mixed

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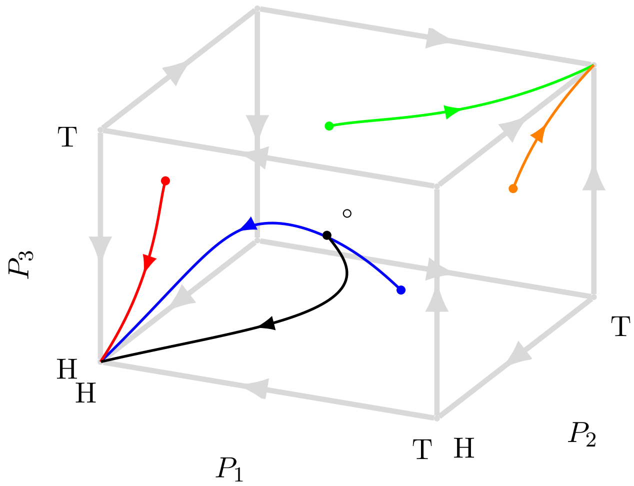
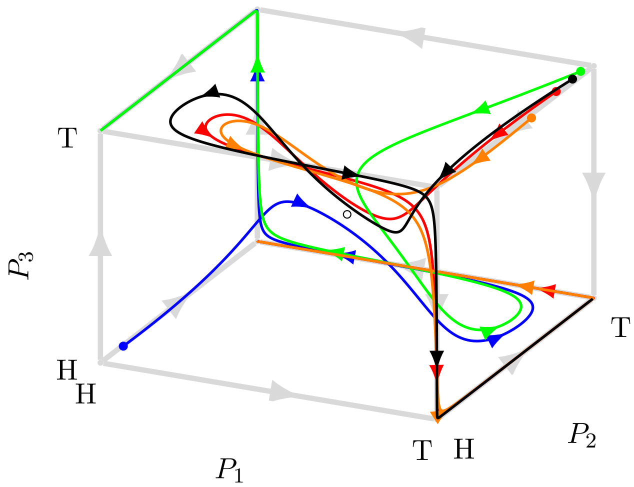
We use sink equilibria to refer to the sinks of the *response graph*, whereas in Goemans et al. (2005) they refer to the sinks of the *best-response graph*. The best-response graph is a subgraph of the response graph where there is one arc per player out of any profile, each leading to the comparable profile that is the best response for that player. We find in this thesis that the response graph, rather than best-response graph, seems to capture more important properties of the game, and many of our results do not hold for best-response graphs. However the general relationship between these models is a problem for further research (Chapter 7).



(a) The trajectories of the replicator dynamic in Circular Matching Pennies. The starting points are $(0.8, 0.2, 0.4)$, $(0.1, 0.2, 0.7)$, $(0.4, 0.6, 0.8)$, $(0.9, 0.7, 0.6)$ and $(0.3, 0.8, 0.2)$, where the value is the probability of playing Heads.



(b) The trajectories of the replicator dynamic in Circular Mismatching Pennies. The starting points are $(0.04, 0.06, 0.05)$, $(0.93, 0.91, 0.92)$, $(0.97, 0.98, 0.98)$, $(0.88, 0.86, 0.82)$ and $(0.96, 0.95, 0.96)$, where the value is the probability of playing Heads. This diagram is inspired by Figure 9.5 of Sandholm (2010).

Figure 1.3: The trajectories of the replicator dynamic (Hofbauer and Sigmund, 2003) on the strategy space of the Circular Matching and Mismatching Pennies games. In each case we choose five starting points. In both games, almost all starting points converge to the sink equilibria. The mixed Nash equilibrium in both games at $(0.5, 0.5, 0.5)$ is represented by a hollow circle.

strategy profiles. The replicator dynamic is a famous dynamic used in evolutionary game theory (Sandholm, 2010). Additionally, it is also significant for learning theory because it is the continuous-time limit of the *Multiplicative-Weights Update* algorithm, a well-known online learning algorithm (Arora et al., 2012). In Circular Matching Pennies we find that, from almost all starting points, the players converge to one of the pure Nash equilibria[5]. Conversely, from almost no starting points do the players converge to the mixed Nash equilibrium. Likewise, in Circular Mismatching Pennies almost no starting points converge to the mixed Nash equilibrium[6]. Instead, almost all starting points converge to a cycle which follows the length-six cycle in the response graph which defines the unique sink equilibrium. To summarise: in both games, from almost all starting points, the players *converge to the sinks of the response graph*, rather than the Nash equilibria. In a sense, the long-run outcomes of players in this evolutionary setting can best be modelled by, and solved using, preference games.

1.2.3 "If it ain't broke"

Irrespective of whether sink equilibria are a better predictor of the outcome of evolutionary games than Nash equilibria, there remains a seemingly reasonable objection to preference games: *they are unnecessary*, because the response graph is defined by every classical game, and so any insights we gain from it could still be gained without 'throwing away' payoff values. We briefly mention some counterarguments.

Firstly, in most cases, we do not throw away the payoffs; the payoffs never existed at all, and preference games simply do not require inventing them.

Secondly, if our goal is to use the response graph to analyse a game, we prefer, at least for pedagogical purposes, to use a model which isolates that which we are analysing. Conceptual clarity leads to insight; for instance, we show in Chapter 4 and Section 3.4.1 how the response graph allows us to *prove new results* on two-player zero-sum games clearly using combinatorial methods—despite these games having been studied for nearly eighty years (Von Neumann and Morgenstern, 1944).

Thirdly, preference games allow us to prove results over entire classes of games at once. As an example, in Chapter 4 we define a preference-zero-sum game as any game sharing a response graph with a zero-sum game. By proving results on this class, our results apply not only to zero-sum games but also a much larger collection of games.

Finally, even when payoffs *can* be plausibly modelled, we believe that preference games at least complement classical game theory by decoupling the influence of the graph structure and the values in a game, leading to a better-informed theory of games. In Section 3.3.2 we show how the Prisoner's Dilemma can be understood in terms of the independent influence of the graph structure and the payoff values. We show that in

[5]This reflects the 'coin-toss' analogy we used earlier.

[6]The interior starting points which converge to the Nash are only those on the diagonal line from (H, H, H) to (T, T, T). All other starting points converge to the sink equilibrium.

general the Nash equilibria are independent of Pareto efficient points, and that the Prisoner's Dilemma serves to highlight the general difficulty in understanding how the data in our game affect the solution concepts. This is the general topic of Section 3.4.

Ordinal games (Cruz and Simaan, 2000) are an existing model of game theory which also does not rely on payoffs. Preference games are more general than ordinal games (every ordinal game has a response graph, but each response graph may correspond to many ordinal games). We believe ordinal games are not as conceptually clear as preference games, lacking the graph-theoretic presentation. Further, we explain in Section 3.2 and Section 3.3 that preference games come from adopting an additional compelling axiom called **Relevance**. As an example, modelling Circular Mismatching Pennies as an ordinal game would require deciding whether Player 1 prefers (Heads, Heads, Heads) or (Tails, Tails, Tails), despite the fact that this does not affect their 'strategic' behaviour (see Section 3.2).

1.3 Structure

Here we provide an overview of the structure of the thesis. Chapters 3, 4 and 5 form the main contributions and are discussed below. Chapter 2 discusses the background and notation, and Chapter 6 gives a discussion of the related literature. Chapter 7 concludes and discusses future work and open problems. The appendices contain additional proofs and discussion that complement the main text.

Preference games are good models (Chapter 3)

In this chapter we explore the axiomatic foundations of preference game theory. We look at a variety of models for games, defined as equivalence classes of the standard utility formulation. We explore how game properties (such as equilibria, dominant and dominated strategies, and Pareto efficiency) relate to these equivalence classes. We discuss the Prisoner's Dilemma as a case for careful modelling of a game.

Preference games as a model follows naturally from adopting two axioms: **Ordinality**, which asserts that we only know an order over our choices, and **Relevance**, which asserts that we can only express a preference over profiles between which we can decide. These axioms are closely connected to relationship between the Von Neumann axioms (Von Neumann and Morgenstern, 1944) (which define games up to an affine transformation), Nash equilibria (which satisfy the **Relevance** axiom), and ordinal utility theory (which satisfies the **Ordinality** axiom). We show also that the labels of nodes in the response graph can be uniquely reconstructed up to renaming of strategies, showing that the game-theoretic structure is truly captured by the graph.

Finally, we give a thorough discussion of the desirable properties a game theory solution concepts should possess. In particular we examine the strengths and weaknesses of pure and mixed Nash equilibria, and discuss how the sink equilibrium satisfies these desirable properties.

Preference games give new insights into games (Chapter 4)

In this chapter we discuss some concepts from classical game theory, focusing on two-player zero-sum and potential games. We demonstrate that viewing these games from the perspective of preference game theory lends significant insight into their structure. Specifically, we identify the games preference-equivalent (having the same response graph) to potential games as the *acyclic games*, and those preference-equivalent to zero-sum games as those which are acyclic after reversing the order of preferences for one player.

From this we derive new results in classical game theory using the conceptual clarity and combinatorial methods of preference games. We show that the response graphs of the well-known 2x2 games Matching Pennies and Coordination play a key role in the structure of *all two-player games*: every non-iteratively-dominated strategy in every two-player game takes part in at least one 2x2 subgame with one of these response graphs. As a corollary we show that *every game preference-equivalent to both a zero-sum and potential game must be dominance-solvable*. We gain these insights into classical games through the methods of preference games.

Preference games give good dynamic predictions (Chapter 5)

In this chapter we explore how preference games relate to dynamic models of behaviour as explored in evolutionary game theory. Our main contribution of the first part is the **proof of an open problem** in evolutionary game theory, concerning the *existence of sink chain components in game dynamics*. This follows the recent work of Papadimitriou and Piliouras (2019) who suggest that the long-run outcomes of evolutionary games should be interpreted through the Fundamental Theorem of Dynamical Systems. This theorem shows that under a 'noisy' interpretation of a dynamical system, every point converges to collections of points called *chain components*. These chain components are ordered; point can move from near one chain component to another near another. In general, this order need not have minimal elements, but when it does these are known as *sink chain components*. These generalise pure Nash equilibria: a pure Nash is exactly the same as a *singleton sink chain component*. We prove that, for the replicator dynamic, *sink chain components always exist*.

Papadimitriou and Piliouras conjectured that when they exist, sink chain components always contain sink equilibria. We show that this is true under the replicator: the sink chain components, which always exist, contain sink equilibria. Thus, these complex dynamical objects (the sink chain components) are fundamentally governed by the preference structure of the game. We prove some additional properties of sink chain components and use our results from Chapter 4 to prove stronger results in some special cases.

Further, we show that for all discrete-time dynamics taking the form of a Markov chain on the response graph, the sink equilibria completely define the qualitative long-run properties. Specifically, the dimension and support of the limiting distributions and whether the chain converges to a unique one. In order words, all of these properties are defined by preference games.

1.4 A Comment on the Name 'Preference'

In defining this model of a game we encounter a problem. How should we name them? Many plausible choices, such as 'ordinal' or 'graphical' games, already have existing distinct meanings. Another option, 'response games', suggests a game with multiple turns, which preference games are not.

One criticism of the term 'preference games' is that all games involve preferences. However, the name reflects the fact that all games define, for fixed choices of other players, a *preference order* over each player's strategies, and this is usually vocalised as the 'order in which strategies are preferred'. The fact that all games involve preferences can then be recast as the fact that all games have an underlying preference game. This, along with the fact that the phrase 'preference game' is not already overloaded, is the motivator of our choice of name.

Chapter 2

Background and Notation

This thesis is largely mathematically self-contained, with most proofs given in full in the text or the appendix. The general assumption is that the reader is comfortable with mathematical abstraction and is familiar with basic concepts from game theory, linear algebra and graph theory. In Chapter 5 we will use a number of concepts from dynamical systems theory and evolutionary game theory. The text should be comprehensible to a reader unfamiliar with these topics, though one wishing to understand the proofs (particularly of Theorem 5.2.17) may require more background, for which Sandholm (2010) and Alongi and Nelson (2007) should provide sufficient preparation.

In this section we will introduce some standard definitions in each of these fields and the notation we will choose to use. In some areas (notably game theory) notation can differ wildly in style. We have chosen a style we believe is clear, for reasons we justify later, but which may look superficially different to what an experienced reader has seen before.

2.1 Graphs, Orders and Linear Algebra

We denote the set of integers $\{1, \ldots, n\}$ by $[n]$. A binary relation $\leq$ on a set X is a *preorder* if it is reflexive ($x \leq x$ for all x) and transitive ($x \leq y$, $y \leq z$ implies $x \leq z$). If it is also anti-symmetric ($x \leq y$, $y \leq x$ implies $y = x$), we call it a *partial order*. A *preference order* (also called a *preference relation* or a *weak order*) is a *total preorder*, that is a preorder where for any x and y either $x \leq y$ or $y \leq x$. A symmetric ($x \leq y$ implies $y \leq x$) preorder is an *equivalence relation*. Given an equivalence relation, a set of elements all equivalent to each other is known as an *equivalence class*, and these partition the set. An equivalence relation $\sim_1$ *refines* another $\sim_2$ if $x \sim_1 y$ implies $x \sim_2 y$. See, for instance, Grätzer (2002).

The following can be found in any text on graph theory, such as Diestel (2016). A *directed* *graph* or *digraph* is a pair $D = (N, A)$, where N is a finite set of *nodes* and $A \subseteq N \times N$ is a finite set of *arcs*. If $(x, y) \in A \implies (y, x) \in A$, that is, A is symmetric, then we say D is an *undirected graph* or simply a *graph*. Each directed graph has an associated undirected graph given by 'forgetting' the orientation of the arcs. We may denote an arc (x, y) by $x \longrightarrow y$. The nodes x and y are called the *endpoints* of the arc; x is called the *tail* and y the *head*. A digraph is *labelled* if we have a function $\ell : A \to L$, where L is some set of *labels*. Where an arc (x, y) is labelled, we may denote it by $x \xrightarrow{\ell} y$, where ℓ is the label. A *path* is a sequence $v_1, v_2, \ldots, v_n$ of distinct nodes where there is an arc $v_i \longrightarrow v_{i+1}$ for every i in $1, 2, \ldots, n-1$. If there is also an arc $v_n \longrightarrow v_1$, we call this a *cycle*. A digraph with no cycles is called *acyclic*. We denote a path from a node v to a node w by $v \rightsquigarrow w$, and we say w *is reachable from* v. Reachability defines a preorder on the nodes of a digraph. A set of nodes is *strongly connected* if every node in the set is reachable from every other, and the maximal strongly connected sets are called *strongly connected components*. Strongly connected components are the equivalence classes of the equivalence relation where two nodes are equivalent if there are between them in both directions. For any subset X of nodes, there is an associated digraph given by including exactly the arcs whose endpoints are in X. This is called the *subgraph induced by* X or simply an *induced subgraph*.

A map $\varphi : (N_1, A_2) \to (N_2, A_2)$ between two graphs is a *homomorphism* if $v \longrightarrow w \in A_1$ implies that $\varphi(v) \longrightarrow \varphi(w) \in A_2$. If a homomorphism is invertible and its inverse is also a homomorphism we call it an *isomorphism*. An isomorphism from a graph to itself is an *automorphism*. A graph (directed or not) is called *complete* if there are arcs between all possible pairs of nodes. The complete graphs are unique up to isomorphism for a given number n of nodes, and the undirected complete graph on n nodes is denoted K_n.

Given two undirected graphs G and H, the *Cartesian product* is the graph $G \square H$, whose node set is the set Cartesian product $N(G) \times N(H)$, where (v, w) and (v', w') are adjacent in $G \square H$ if $v = v'$ and w and w' are adjacent in H, or $w = w'$ and v and v' are adjacent in G (Hammack et al., 2011).

We call a non-negative vector $v \in \mathbb{R}^N$ a *distribution* if its entries sum to one. The set of distributions in $\mathbb{R}^N$ forms a $(N-1)$-simplex, and we denote this by Δ_N. Given two vectors $x, y \in \mathbb{R}^N$, we say x *Pareto dominates* y if $x[i] \geq y[i]$ for all $i \in [N]$. A point that is not Pareto dominated by another point is called *Pareto efficient*.

The *support* of a function is the elements of the domain which do not map to the zero in the codomain.

2.2 Game Theory

2.2.1 Games and profiles

Definition 2.2.1. A *game*[1] is a triple $([N], \{S_1, \dots, S_N\}, u : \prod_{i=1}^{N} S_i \to \mathbb{R}^N)$ where N is a natural number and each S_i is a finite set. We call the integers $[N]$ the *players*, the set S_i the *strategies* available to player i, and u the *utility function*.

We will always assume in this thesis that the sets of players and respective strategies are finite, and games are in *normal form*, so that they consist of a single 'round' of strategy selection[2].

As the players and strategies sets are implicitly defined by the utility function, we will often present a game by a utility function alone. An element of $\prod_{i=1}^{N} S_i$ (an assignment of strategies to players) we call a *profile*. For shorthand we denote the set $\prod_{i=1}^{N} S_i$ of profiles by Z. We call $u(p)_i$ the *payoff* of p to player i. We also write this as $u_i(p)$, where $u_i : \prod_{i=1}^{N} S_i \to \mathbb{R}$ denotes the payoff function for player i, which is the ith output of u. For a given player i, we call an assignment of strategies to all players other than i an *i-antiprofile*, or simply an *antiprofile* where i is clear. We generally denote antiprofiles by an overbar. If $p = (s_1, s_2, \dots, s_N)$ is a profile, then deleting the strategy s_i for player i gives an i-antiprofile, which we denote by $\bar{p}_{-i}$. Similarly, given a player i, i-antiprofile $\bar{p}_{-i}$, and a strategy $s \in S_i$ we obtain a profile p by assigning strategy s_i to player i. We can think of this as inserting s at index i in $\bar{p}_{-i}$, and we denote this operation by $s \curvearrowright \bar{p}_{-i}$. For a given player i, we denote the set of i-antiprofiles by $\bar{Z}_{-i}$.

Two profiles are *i-comparable* if they differ only in the strategy of player i; they are *comparable* if they are i-comparable for some player i. If two profiles are comparable, then there is exactly one i such that they are i-comparable.

We say a game is *strict* if the payoffs to player i in two i-comparable profiles are never equal. This can be used to simplify the discussion by avoiding the case where players are indifferent between strategies.

Definition 2.2.2. A *subgame* of a game $([N], \{S_1, \dots, S_n\}, u)$ is a game $([N], \{T_1, \dots, T_n\}, u')$ where for each i, $T_i \subseteq S_i$, and u' is u restricted to $\prod_{i=1}^{N} T_i$.

2.2.2 Mixed strategies, expected utility and Nash equilibria

The following is standard in game theory. For further reference, see Myerson (1997).

The *Nash equilibrium* is a foundational concept in game theory. We first define the restricted case, called a *pure Nash equilibrium*. Intuitively, this is a profile where no player can improve their payoff by a unilateral change of strategy.

[1] Sometimes called a *normal-form game*.

[2] Games with multiple 'turns' of play can still be represented this way (this is the justification for the name '*normal form*'). See Myerson (1997) or Von Neumann and Morgenstern (1944) for a full discussion.

Definition 2.2.3. Given a game u, a *pure Nash equilibrium* (PNE) is a profile p where for any player i and strategy $s \in S_i$, $u_i(p) \geq u_i(s \curvearrowright \bar{p}_{-i})$.

In game theory, we allow players to play a distribution over their strategies. We call such a distribution a *mixed strategy*. An assignment of mixed strategies to players we call a *mixed profile*, or simply a *point*. The set of mixed profiles on a game is given by $\prod_{i=1}^{N} \Delta_{|S_i|}$ where $\Delta_{|S_i|}$ are the distributions in $\mathbb{R}^{|S_i|}$. We generally denote $\prod_{i=1}^{N} \Delta_{|S_i|}$ simply by X. We call X the *strategy space* of the game. For a mixed profile $x \in X$, we write x^i for the distribution over player i's strategies, and x^i_s for the s-entry of player i's distribution, where $s \in S_i$. A mixed strategy whose support is a single strategy we call a *pure strategy*. There is a one-to-one correspondence between profiles and mixed profiles where all players use a pure strategy. We will sometimes call a profile a pure profile to distinguish it from a mixed profile. We can now generalise the notion of utility to mixed profiles using expectation.

Definition 2.2.4. Let u be a game, and x a mixed profile. The *expected utility function* of u is $\mathbb{U} : X \to \mathbb{R}^N$, where

$$\mathbb{U}(x) = \sum_{q \in Z} u(q) \prod_{i=1}^{N} x^i_{q_i}$$

Pure Nash equilibria can be generalised to mixed strategies using expected utility.

Definition 2.2.5. Given a game u, a *Nash equilibrium* (NE) is a mixed profile $x \in X$ where for any player i and mixed strategy s_i, $\mathbb{U}_i(x) \geq \mathbb{U}_i(s_i \curvearrowright \bar{x}_{-i})$.[3] That is, no player can improve their expected payoff by a unilateral change of mixed strategy.

The following is equivalent (see Myerson (1997) or the Appendix) to requiring that no player can improve their expected payoff by playing any pure strategy. It follows that PNEs are a special case of Nash equilibria where all the players play pure strategies. We will use the term *mixed Nash equilibrium* to mean any Nash equilibrium that is not pure—that is, at least one player plays a mixed strategy whose support is at least two distinct strategies.

A famous theorem of Nash establishes that Nash equilibria exist in all games.

Theorem 2.2.6 (Nash (1951))**.** *Every game has a Nash equilibrium.*

Definition 2.2.7. A mixed profile x is *Pareto efficient* if for every other mixed profile y, there is some player i such that $\mathbb{U}(x)_i \geq \mathbb{U}(y)_i$.

We say that a strategy $s \in S_i$ *dominates* a strategy $t \in S_i$ if $u_i(s \curvearrowright \bar{p}_{-i}) > u_i(t \curvearrowright \bar{p}_{-i})$ for every antiprofile $\bar{p}_{-i} \in \bar{Z}_{-i}$.[4] The strategy t is called *dominated*. If we delete some dominated strategy, other strategies can become dominated. This process is called *iterated elimination of dominated strategies*. Any strategy deleted during this process is

[3]Recall that we use the notation $s_i \curvearrowright \bar{x}_{-i}$ to mean inserting a (mixed) strategy s_i into the (mixed) antiprofile $\bar{x}_{-i}$.

[4]This is sometimes called *strict dominance*.

called *iteratively dominated*, and otherwise a strategy is said to *survive iterated dominance*. If a game has only one profile that survives iterated dominance, then that profile is the unique Nash equilibrium, and we call the game *dominance-solvable*. We call the subgame made up of strategies that survive iterated dominance the *residual subgame*.

A *correlated equilibrium* is a probability distribution $p : X \to \mathbb{R}$ over the mixed-strategy space of a game and x a point such that if all players play x then no player can improve their strategy by deviating, assuming the profiles are played according to the distribution p. Correlated equilibria will only appear tangentially in this thesis, and more details can be found in Aumann (1987); Myerson (1997).

2.2.3 The response graph

Definition 2.2.8. The *response graph* of a game $u : Z \to \mathbb{R}^N$ is the digraph $\mathcal{G}_u = (Z, A)$ where there is an arc $p \longrightarrow q \in A$ between profiles p and q if and only if they are i-comparable and $u(p)_i \leq u(q)_i$. That is, there is a player i who would be better off by deviating from strategy p to strategy q, holding the other players' strategies constant. If the arc $p \xrightarrow{x} q$ is labelled by the non-negative value $x := u(q)_i - u(p)_i$, then we call this the *labelled response graph* of u, written $\mathcal{LG}_u$. For a strict game, these labels are positive.

2.3 Dynamical Systems

In this thesis we consider both discrete- and continuous-time dynamical systems. A discrete-time dynamical system is simply a map $f : X \to X$ where X is usually assumed to be a compact metric space, and the associated *orbits* or *trajectories* of a point x are $f(x)$, $f^2(x) := f(f(x))$, $f^3(x) := f(f(f(x)))$, and so on. A continuous-time dynamical system we define by ordinary differential equations, that is equations of the form $\dot{x} = f(x)$ for $x \in X$, where X again is a compact metric space. In the cases we consider, the solutions define a *flow*. On games, X is usually taken to be the strategy space.

Definition 2.3.1. (Alongi and Nelson, 2007) Let X be a compact metric space. A *flow* is a continuous map $\varphi : X \times \mathbb{R} \to X$ where

1. $\varphi(\cdot, t) : X \to X$ is a homeomorphism for all $t \in \mathbb{R}$, and
2. $\varphi(\varphi(x, s), t) = \varphi(x, s + t)$ for all $x \in X$ and $s, t \in \mathbb{R}$.

The second argument is thought of as time. The flow defines the behaviour of the system from all starting points x. As is standard in dynamical systems, we will write flows in *power notation* as $\varphi^t(x) := \varphi(x, t)$. This reflects the fact that the flow axioms are similar to the power laws: $\varphi^0(x) = \mathrm{id}$ and $\varphi^t(\varphi^s(x)) = \varphi^{s+t}(x)$.

Definition 2.3.2. (Alongi and Nelson, 2007) If x is a point in X, and φ a flow, then $\mathcal{O}(x) = \{y \mid \varphi^t(x) = y,\ t \in \mathbb{R}\}$ is called the *orbit of* x. If we restrict t to the non-negative reals, we obtain $\mathcal{O}^+(x)$, the *forward orbit of* x. Likewise, for non-positive t we obtain the set $\mathcal{O}^-(x)$, the *backward orbit of* x.

A trajectory is some orbit. The existence and uniqueness theorem for ordinary differential equations establishes when an ordinary differential equation $\dot{x} = f(x)$ has a solution that is a differentiable flow, where the derivative of the flow is f. In the cases we consider in this paper, it is known that the differential equations define a unique well-behaved flow.

In this thesis, we shall consider only one dynamic in particular: the *replicator dynamic* (Hofbauer and Sigmund, 2003). This continuous-time dynamic is well-studied in evolutionary game theory. For a game u with N players and expected utility function $\mathbb{U}$, the replicator dynamic is defined by the following ordinary differential equation.

$$\dot{x}^p_s = x^p_s \left(\mathbb{U}_p(s \curvearrowright \bar{x}_{-p}) - \sum_{t \in S_p} x^p_t \mathbb{U}_p(t \curvearrowright \bar{x}_{-p}) \right)$$

where x is the mixed strategy profile, that is, a distribution over each player's strategy set for each player. The second term is the mean of the expected utility for player p given by playing any pure strategy t, with the average taken over all such strategies. The bracketed term is the *excess expected payoff* for strategy s, that is the difference between the expected payoff for always playing s and the mean expected payoff over all pure strategies.

Chapter 3

The Foundations of Preference Game Theory

3.1 What is a "game"?

The idea of a 'game' is to capture the notion of *strategic interaction*. With that in mind, how should we define the players' choices over their strategy sets, potentially dependent on the choices of other players?

In the introduction we discussed the standard game theory approach. This is to assign to each player a real-valued *payoff* according to each possible strategy profile. The notion of payoff is motivated by the obvious economic intuition. We then assume that players seek to maximise their payoff, and so will choose their strategy accordingly. This leads to the standard definition of a game in terms of a *utility function* $u : Z \to \mathbb{R}^N$ (Definition 2.2.1)[1]. This approach assumes tacitly that utilities *are* numbers; we can add, subtract and multiply them, we can take the expected value of distributions over them, we can compute the value of geometric series (as is done in the theory of repeated games (Myerson, 1978)), and we can define differential equations in our strategy space (as is done in evolutionary game theory (Sandholm, 2010)). For comparison with other definitions of a game we introduce later (Section 3.3.1), we call this model a **utility game**.

At the other extreme, we assume that if told of the choices of strategy of all other players, each player is capable of expressing a *preference order* over their own strategy set. This is what we defined as a *preference game*. In general, this order will depend on what the other players have chosen to play. Formally, supposing s and t are strategies for player i in such a game, and $\bar{q}_{-i}$ is an antiprofile[2], then player i can determine which of s or

[1] Recall from Chapter 2 that Z is the set of strategy profiles: $Z := \prod_{i=1}^{N} S_i$

[2] A choice of strategies for all players other than i, see Chapter 2.

t they prefer given $\bar{q}_{-i}$, and nothing more. In particular they cannot quantify *by how much* that strategy is preferred[3]. Because each ordering depends on the antiprofile $\bar{q}_{-i}$, given a *mixed antiprofile* it might not be possible for player i to deduce which of s and t they prefer[4]. As before, we assume then that players choose strategies that they prefer, given the choices of the other players. Concretely:

Definition 3.1.1. Given players $[N]$ with strategy sets $S_1, \ldots, S_N$, a *preference game* is a map p_i for each player i, with $p_i : \prod_{j \neq i}^{N} S_j \to \mathcal{T}(S_i)$, where $\mathcal{T}(S_i)$ is the set of preference orders on S_i.

Preference games are appealing in the sense that they require only basic assumptions on individual preferences, but they are limited in that we cannot apply the same suite of mathematical tools. In particular, Nash's existence theorem does not hold, because mixed Nash equilibria may not be describable in an preference game.

Every utility game has an associated preference game, because the payoffs to the players define an order on a player's strategies given strategies for all other players[5]. The converse is not true; a given preference game is generally consistent with many different utility games. By working with preference games, we are allowing that the modeller *cannot necessarily distinguish between all the utility games which give rise to this preference game.*

Our characterisation of utility games was somewhat generous. Often, in accordance with Von Neumann and Morgenstern (1944), utilities are only assumed to be real numbers *up to a positive affine transformation*. Such games we refer to as **affine games**, and we define these and other game classes in Section 3.3. The theoretical motivation for utility in affine games comes from the axiomatic derivation in Von Neumann and Morgenstern (1944).

Utility as an economic concept has been the subject of much debate in the past [cite all]. Consequently, weakenings of its assumptions are well-known in the literature. One of the best-studied is *ordinal utility theory*, initiated in Pareto (1919). In this theory, we drop the assumptions that individuals can provide a utility value to their choices, and instead assume only the existence of a rank order over their choices. This is the motivation behind concepts such as *Pareto efficiency*. The assumption of ordinality in games has

[3]By asserting that preferences form a weak order, we are assuming that preference is transitive for a fixed $\bar{q}_{-i}$. In some fields, such as social choice theory (Sen, 2017), one considers the effects of weakening the assumption of transitivity of preference. We will not delve into this in this thesis, except to say to following: weakening preference to be non-total (a partial order) or non-transitive (not an order) is essentially impossible in classical game theory, because preferences are defined by real numbers, which are always totally ordered. By contrast, preference in a preference game is defined by a directed graph, and a directed graph can easily represent non-transitive or non-total relations, with very little modification to the theory.

[4]That is not to say that we can say nothing at all about responses to mixed antiprofiles in preference games. For instance, if given $\bar{q}_{-i}$ player i prefers s to t to r, then they prefer some mixture of s and t to q. However we cannot say whether a mixture of s and r is preferred to t.

[5]If we define a game by the induced order over all *mixed* strategy profiles, then we obtain an affine game; this follows from the Von Neumann axioms (3.2).

been studied for some time, with the resultant games known as ordinal games (Cruz and Simaan, 2000; Durieu et al., 2008). While ordinal games are quite similar to preference games, preference games are more general. That is, every ordinal game has an associated preference game, but a preference game may be compatible with many ordinal games.

We show in this section that preference games are motivated by a combination of ordinal utility theory and the axioms of the 'best-response' or 'strategic' equivalence. That is, preference games follow from two axioms[6].

1. **Preferences are ordinal**: we can only distinguish between the ordering of payoffs, not their magnitude.
2. **Preferences are relevant**: we can only distinguish between preferences over options we can choose between.

The former axiom is what distinguishes *ordinal games* (Definition 3.3.5) from *affine games* (Definition 3.3.2); the latter is what distinguishes *strategic games* (Definition 3.3.3) from affine games. The **Relevance** axiom is motivated by the fact that preferences over options we cannot choose do not affect our behaviour, and so should not influence the outcome of a game. This intuition is reasonable because Nash equilibria (and other natural game theory concepts, see Figure 3.4) are 'invariants' (Definition 3.3.1) of strategic games, and so obey the relevance axiom. The **Ordinality** axiom is motivated by the difficulty of assigning utility values in application areas. Pareto efficient points are an example of an invariant of ordinal games.

3.2 Axioms for Preference Games

In this section we give an axiomatic treatment of preference games. The treatment is conceptual, so we will informally assert results whose proofs we will provide in later sections, particularly summarised in Figure 3.4.

The concept of utility has a long and complex history in economics. Even as Von Neumann and Morgenstern introduced the first concepts of game theory in the 1940s, utility had been thoroughly questioned, with the authors pointing out that:

> Many economists will feel that we are assuming far too much ... and that our standpoint is a retrogression from the more cautious modern technique of 'indifference curves.'

Consequently, Von Neumann and Morgenstern did not simply embrace utility, but instead presented a careful axiomatic treatment, demonstrating that if individuals possessed a total order over their strategies, and individuals could compare 'convex combinations' of strategies, and these comparisons were compatible in some reasonable ways, then preferences could be faithfully represented by numbers which preserved the order

[6] Of course, these axioms are actually weakenings of our existing axioms. However, by making these axioms we are asserting that our analysis should not depend on information we cannot reliably model.

and were compatible with convex combinations. Further, this representation was unique up to a positive affine transformation.

While these games can describe almost all[7] solution concepts of a game generally studied, it turns out that many (Nash, correlated equilibria, dominance, Pareto efficiency) can be defined using weaker axioms.

In this section we explore two weakenings of these axioms, and show how these important game concepts can be constructed in these weaker axiomatic settings. In the following section we will use these axioms to inspire concrete definitions. The notion of preference games follows naturally from the adoption of *both* of these axioms.

3.2.1 Relevance: we shouldn't care about choices we cannot make

Consider the standard Rock-Paper-Scissors game, where Rock defeats Scissors, Scissors defeats Paper, and Paper defeats Rock, with a tie occurring when two players play the same strategy. We assume that players receive some positive payoff for winning and a negative payoff for losing. These payoffs may be different in different winning and losing profiles.

As a player, if my opponent's strategy is known (say, Rock), then I can express the order in which I prefer my own strategies (in this case, (1) Paper, (2) Rock, (3) Scissors). However, having fixed some utilities, I can express much more than this. For instance, I can also express my preference over the strategy profiles $(Rock, Rock)$ and $(Scissors, Paper)$. The question is, what influence does this preference have on the game, from a strategic perspective?

The answer is: very little. We can modify the game to make this preference arbitrarily positive or negative without altering the Nash equilibria, correlated equilibria, dominant and dominated strategies. This seems surprising, but there is a good axiomatic reason: neither player can ever choose between these profiles. Players can only modify their own strategy, and so can only 'choose' between comparable profiles. All of these solution concepts depend only on the the relative payoff between comparable profiles. If the goal is to analyse these properties, we should ignore data which does not affect them. Motivated by this, we adopt the following axiom.

Axiom 1 (Relevance)**.** We should only model the relationship between *comparable* profiles.

We find in (Myerson, 1997, p. 52) the statement that two games are equivalent from the perspective of decision theory if and only if they are equivalent up to an affine transformation, as in the Von Neumann–Morgenstern definition. However, this statement implicitly assumes that players make decisions over all profiles, not only those they can

[7]Concepts like *social welfare*, defined as the sum of players' utilities, cannot be described by affine games. This is because the positive coefficients may differ for each player, and these affect such a sum.

choose between. In fact, Myerson (1997) does briefly discuss this the relevance axiom[8] implicitly under the name of *best-response equivalence*. He ends this discussion with the point:

> The distinction between these two equivalence concepts depends on whether we admit that one player's preference over the possible strategies of another player may be meaningful and relevant to the analysis of our game.

The equivalence induced by this axiom is closely related to *strategic equivalence*, which we discuss this in the next section. The importance of this equivalence class has only recently begun to be appreciated in the literature, and it has led to significant insight, such as the decomposition of games in Candogan et al. (2011) into potential, harmonic and non-strategic components.

Some game properties do not satisfy the **Relevance** axiom. The most important example is *Pareto efficiency*. However, this does satisfy a different weakening of the Von Neumann–Morgenstern axioms.

3.2.2 Ordinality

> "Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk." – Leopold Kronecker
>
> (God made the integers, all else is the work of man.)

Ordinality is the idea that individuals only possess an *order*, and not a valuation, over their preferences. Some of the earliest work on this idea can be found in Pareto (Pareto, 1919). It is an appealing concept, both because of its simplicity and the modelling difficulty inherent in assigning cardinal values to preferences. In many application areas, such as computer science and biology, there is no clear choice of utility function, and instead we can model only an ordering in which the options are preferred. Consequently:

Axiom 2 (Ordinality)**.** We should only model the ordinal relationship between profiles.

Recalling the Rock-Paper-Scissors game, ordinality asserts that it is the order that each player prefers the profiles that matters. This agrees with our intuition about the game. We think about the outcomes as 'Win', 'Draw' or 'Lose', and we prefer them in that order; the associated values are never mentioned in the English-language description of the game.

As was the case for the **Relevance** axiom above, several important solution concepts depend only on the ordering of payoffs (**Ordinality**). These include pure Nash equilibria, dominance and Pareto efficiency. If our goal is to analyse these concepts, we should assume ordinal payoffs.

[8] Myerson also separately uses a different axiom titled 'Relevance'—this is not the same as what we discuss here.

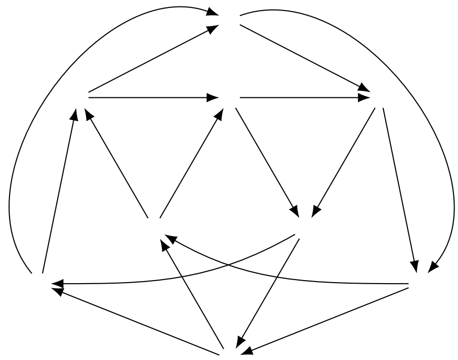


Figure 3.1: Rock-Paper-Scissors as a preference game

3.2.3 Putting them together: Preference

Let us again consider the Rock-Paper-Scissors game. As was the case with the Circular Matching and Mismatching Pennies games in the Introduction, the English-language description, describing which strategies defeat which others, is intuitively *everything we need to know about the game*. Certainly, it is enough for us to play the game successfully. When we adopt both the **Relevance** and **Ordinality** axioms this is precisely the information we are left with: an ordering over each player's strategies, for every fixed combination of the other players' strategies. More precisely:

Axiom 3 (Preference)**.** We should only model the ordinal relationship between comparable profiles.

This is exactly what preference games define. Also, see Figure 3.1[9], which depicts the response graph of Rock-Paper-Scissors[10].

It follows that the solution concepts which are defined in preference games (and so satisfy this axiom) also satisfy both the **Ordinality** and **Relevance** axioms. Pure Nash equilibria[11], dominance, and iterated dominance are examples of such. These are each compelling analysis tools for a game (where they exist), and preference games are the natural domain in which to study them.

[9] Those from a graph theory background may notice that this graph is non-planar. Indeed, following Wagner's Theorem (Wagner, 1937) it contains the complete graph K_5 as a minor—we encourage the interested reader to verify this. It can, however, be embedded on a Möbius strip in a natural way.

[10] We can present this response graph without node labels, because this can be uniquely reconstructed up to renaming of strategies; see Section 3.3.3.

[11] The ordinal games literature is often motivated, just as we do here, by the fact that PNEs are an invariant of ordinal games. This is true, but the stronger statement that PNEs are an invariant of preference games is also true.

3.3 Equivalence Relations and Classes of Games

Already in this thesis we have seen utility, affine and preference games discussed and compared as models of strategic behaviour. In this section we identify these games with *equivalence relations* on the set of utility games. An equivalence relation on a game can be thought of as how much 'data' is available to the modeller. For instance, given a profile in a utility game the modeller has access to the 'data' of the payoffs to all players. By contrast, in a preference game a modeller given a player and an antiprofile only possesses the 'data' of an order over that player's strategies.

Figure 3.2 depicts the relationship between the equivalence relations we discuss, ordered by *refinement*. Those classes requiring 'the most data' appear on the left-hand side, those requiring the least are to the right. As discussed in 3.1, this relation defines that every utility game has a unique associated affine game, which has a unique associated ordinal game, and so on. Conversely, each of these containments is *proper*; every ordinal game is induced by many different affine games, et cetera.

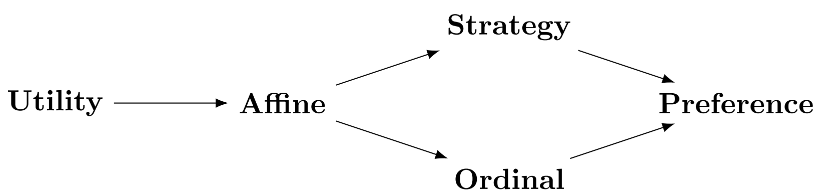


Figure 3.2: The Hasse diagram (Grätzer, 2002) of the refinement order on classes of games.

To understand which concepts can be defined in each class of game, we use the concept of an invariant.

Definition 3.3.1. Given an equivalence relation, an *invariant* is a property that is constant on all members of each equivalence class.

As an example, whether a profile is a Nash equilibrium is not changed by a positive affine transformation, and so it is an invariant of affine games. We use invariants to formalise which solution concepts 'can be defined' in each equivalence relation. We will use the terms *affine-invariant*, *ordinal-invariant*, *strategy-invariant* and *preference-invariant* to describe invariants of each of these equivalence relations.

In Section 3.3.2 we discuss a famous game: the *Prisoner's Dilemma* (Myerson, 1997). This result, originally developed by Flood and Dresher and presented in this form by Tucker (Rapoport and Chammah, 1970), demonstrates the counter-intuitive result that self-interested players, even with full knowledge of the game, may choose to play strategies which lead to an outcome that is bad for all of them. Formally, the equilibrium solution of the game may be one which is not *Pareto efficient*. We show in this section that this result can be reinterpreted simply as a statement about equivalence relations on games: the 'data' that determines the Pareto efficient points is *independent* of the 'data' that determines the Nash equilibrium. We call this result the *Independence Theorem*.

We can think of this theorem in an adversarial context. If you provide me with an affine game and *any* strategy profile p, I can construct an affine game with *identical Nash equilibria* where p is a Pareto efficient point (or not, as desired).

This example illustrates the importance of careful modelling. If we only have enough 'data' in our game to calculate the Nash equilibria[12], then we *cannot say anything* about Pareto efficiency, and we might mislead ourselves if we attempt to do so. If we wish to discuss these concepts, we must ensure that we can truly model the 'extra information' required. This example motivates our thinking about preference games in the same way: if we only know a preference relation over individual strategies, we should model our game as a preference game, rather than construct a utility function to fit it—otherwise our results may be artifacts of our utility function and not of our underlying model.

3.3.1 The classes we will consider

In this section we give definitions of the various classes of games we will consider. Throughout, a *game*, when unqualified, refers to a *utility game*, and follows Definition 2.2.1 from Chapter 2. We also assume each game has N players with strategy sets $S_1, \ldots, S_N$, with set of profiles $Z := \prod_{i=1}^{N} S_i$. In all cases, two games can only be equivalent under any of these relations if their player and strategy sets are the same. We discuss in Section 3.3.3 the equivalences induced by symmetries among players and strategies.

Affine

We begin with the standard equivalence relation defined by Von Neumman utility theory.

Definition 3.3.2. Two games u and v are *affine-equivalent* if there exists a positive vector $a \in \mathbb{R}^N_{++}$ and a vector $b \in \mathbb{R}^N$ such that for any profile $p \in \Delta$ and player i, $u(p)_i = a_i v(p)_i + b_i$. The set of equivalence classes of games up to affine equivalence we denote by **Affine**, and a given equivalence class we can an *affine game*.

Strategy

The next equivalence allows the utility to differ by an affine transformation where the constant depends both on the player *and the given antiprofile*.

Definition 3.3.3. Two games u and v are *strategically-equivalent* (Candogan et al., 2011) if there exists a positive vector $a \in \mathbb{R}^N$ and, for every player i and antiprofile $\bar{p}$ there exists a constant $b_{i,\bar{p}} \in \mathbb{R}$ such that for any strategy $s \in S_i$, $u_i(s \curvearrowright \bar{p}) = a_i v_i(s \curvearrowright \bar{p} + b_{i,\bar{p}}$. The set of equivalence classes of games up to strategic equivalence we denote by **Strategy**, and we call an individual equivalence class a *strategy game*.

[12] that is, we only know our game up to *strategic equivalence*,

Note that we allow the constant term $b_{i,\bar{p}}$ to vary depending on the player i and antiprofile $\bar{p}$. By contrast, in **Affine**, b_i is constant for a given i. This terminology follows Candogan et al. (2011), and reflects the fact that games with the same equilibrium sets are called strategically-equivalent, and as we show later, Nash equilibria are invariants of strategically-equivalent games. Recalling the definition of the *labelled* response graph introduced in Chapter 2, we can rephrase strategic equivalence another way.

Theorem 3.3.4. *Two games u and v are strategically equivalent if and only if their labelled response graphs are equal up to a rescaling by a positive constant for each player.*

This result was presented in Candogan et al. (2011), where it formed the basis of their decomposition theory for games up to strategic equivalence based on the notion of graph flows. Additionally, note the relationship between strategic equivalence and the **Relevance** axiom. We show in the Appendix that strategy games depend only on the *relative utility* $u(s \curvearrowright \bar{p})_i - u(t \curvearrowright \bar{p})_i$ between two i-comparable profiles $s \curvearrowright \bar{p}$ and $t \curvearrowright \bar{p}$, and so satisfy the **Relevance** axiom.

Ordinal

Ordinal-equivalence is similar to preference-equivalence, in that it considers only the order induced by each player's preferences. However, unlike in preference games, in ordinal games one has available a ranking over all profiles in the game, while in preference games we have only a ranking for each fixed antiprofile. Ordinal games satisfy the **Ordinality** axiom, because they depend only on the order over profiles for each player.

Definition 3.3.5. Two games u and v are *ordinal-equivalent* if for each i the total orders on $u_i(Z)$ and $v_i(Z)$ are the same. The set of equivalence classes of games up to ordinal-equivalence we denote by **Ordinal**.

Preference

We now define preference-equivalence.

Definition 3.3.6. A u and v be games. If for any player i, strategies $s, t \in S_i$, and antiprofile $\bar{p}_{-i}$ we have $u(s \curvearrowright^i \bar{p}_{-i})_i \geq u(t \curvearrowright^i \bar{p}_{-i})_i \Leftrightarrow v(s \curvearrowright^i \bar{p}_{-i})_i \geq v(t \curvearrowright^i \bar{p}_{-i})_i$, we say u and v are *preference-equivalent*. The set of equivalence classes of games up to preference-equivalence we denote by **Preference**.

The resultant equivalence classes are exactly the same as our direct definition of preference games in Definition 3.1.1.

Proposition 3.3.7. Two games u and v are preference-equivalent if and only if their underlying preference games are equal.

Thus we can alternatively define preference games as a natural equivalence relation on utility games. This relation refines both ordinal- and strategic-equivalence, and so satisfies both the **Relevance** and **Ordinality** axioms, as we asserted previously. Preference-equivalence is defined by the order over comparable profiles, and so preference games satisfy the **Preference** axiom.

As for strategic-equivalence, and as we discussed in the Introduction, preference-equivalence is the same as having an identical response graph. For this reason, we will generally not distinguish between a preference game and its associated response graph. This allows us to prove theorems and state properties of preference games from a purely graph-theoretic perspective.

Theorem 3.3.8. *Two game u and v are preference-equivalent if and only if their response graphs are equal*[13].

3.3.2 Overmodelling: a new look at the Prisoner's Dilemma

In this section we look at a concrete example of how our modelling decisions are reflected in our analysis of a game. We will use what is arguably the most famous example of a game: the Prisoner's Dilemma (Rapoport and Chammah, 1970).

Table 3.1: The Prisoner's Dilemma game.

	B stays silent	B betrays
A stays silent	$-1, -1$	$-3, 0$
A betrays	$0, -3$	$-2, -2$

In the original presentation, the game models the following scenario. Two prisoners, call them A and B, are arrested on a minor charge. Both are suspected of a more serious crime, and are approached, separately, by the prosecutor with the same offer. If A gives evidence against B for the serious charge, and B says nothing, then A walks free while B will serve the maximum sentence of three years, and vice versa. If both A and B choose to stay silent, and not betray the other, then each will serve a sentence of one year on the minor charge. However, if both A and B choose to betray each other then they will both serve two years.

This is a 2-player, 2-strategy game, whose payoff table is given in Table 3.1. In this table the first element of each pair is the payoff to player A and the second is the payoff to player B. The key property of this game is that the game-theoretic outcome—the unique pure Nash equilibrium, where both players choose to betray each other—is not Pareto efficient, because both players staying silent would make them both strictly better off. The nature of the paradox stems from our intuition suggesting that individual incentives should lead to good social outcomes. The purpose of the Prisoner's Dilemma is to dispel this misconception.

[13] Note the subtlety: we say *equal* rather than *isomorphic* because some response graphs have non-trivial symmetries. However we show in Section 3.3.3 that all such symmetries are caused by renaming of individual players and strategies. There we prove a stronger version of this theorem: two games are preference-equivalent (up to renaming of players or strategies) if and only if their game graphs are isomorphic.

The paradox is strengthened by the fact that this prediction is very compelling: for both players, the 'Betray' strategy is dominant, so each player prefers to play this strategy *regardless* of what the other chooses. Games with a dominant strategy for all players always have a unique pure Nash equilibrium (and so requires no assumptions about players being willing to play mixed strategies). From a strategic perspective, it is hard to doubt that this is the outcome of the game.

However, we now recast the Prisoner's Dilemma as an example of a more general theorem, showing that Pareto efficiency, as a property of a game, is independent of Nash equilibria.

Theorem 3.3.9 (Independence Theorem)**.** *Pareto efficiency and Nash equilibria are independent. That is, given any game u and Pareto inefficient profile p, there is a game u with the same Nash equilibria as u where p is Pareto efficient; similarly, if p is Pareto efficient in u, there is a game w with the same equilibria as u where p is Pareto inefficient.*

Proof. Let p be a profile that is Pareto efficient, and let r be a profile that differs from p in every entry. We will construct a strategically-equivalent game u' where r Pareto dominates p. Let $\bar{r}_{-i}$ be the antiprofile given by removing the ith entry of r. Then for each i define $b_{i,\bar{r}_{-i}} := 1 + \max_{s \in S_i} |u_i(s \curvearrowright \bar{r}_{-i})|$. Finally, define

$$u'_i(q) = \begin{cases} u_i(q) + b_{i,\bar{r}_{-i}} + u_i(p) & \text{if } \bar{q}_{-i} = \bar{r}_{-i} \\ u_i(q) & \text{otherwise} \end{cases}$$

As u' is formed by adding a constant to u that depends only on the player i and antiprofile $\bar{q}_{-i}$, u' is strategically equivalent to u. However, $u'_i(r) = u_i(r) + b_{i,\bar{r}_{-i}} + u_i(p) > u_i(p) = u'_i(p)$, so r Pareto dominates p.

Now suppose that p is not Pareto efficient. We assume without loss of generality that all utilities are positive, as there is always an affine-equivalent game with positive utilities. Let $M = \max_{q \in Z} \sum_i u_i(q)$ be the maximum *social welfare* (sum of utilities) over profiles in u. Define

$$u'_i(q) = \begin{cases} u_i(q) + M + 1 & i = 1 \text{ or } 2, \bar{q}_{-i} = \bar{p}_{-i} \\ u_i(q) & \text{otherwise} \end{cases}$$

Again this is a strategically equivalent game. We show that p has maximal social welfare. The social welfare of p is: $\sum_i u'_i(p) = 2M + 2 + \sum_i u_i(p) \geq 2M + 2$. Any other profile q differs from p in at least one place, and so $\sum_i u'_i(q) \leq M + 1 + \sum_i u_i(q) \leq 2M + 1$, and so p has maximal social welfare in u', and thus must be Pareto efficient. □

In light of this theorem, the example of the Prisoner's Dilemma seems more to be the rule rather than the exception. In fact, it follows that we can construct games with identical equilibrium structure where any of the four profiles in the Prisoner's Dilemma are Pareto efficient, or not. Seen from the perspective of equivalence classes, and using the facts from Figure 3.4 that Nash are strategic-invariants and Pareto-efficient points are ordinal-invariants, we are really saying that these two equivalence classes (**Ordinal** and **Strategy**) are quite different, and that fixing the strategic structure leaves us free to modify the ordinal structure. In the Prisoner's Dilemma specifically, the Nash equilibrium is pure, and these are preference-invariants—so we also show that **Preference** and **Ordinal** are quite distinct classes.

This example highlights the importance of the **Relevance** axiom introduced in Section 3.2.1. In this proof we modify the payoffs differences between non-comparable profiles to move the Pareto efficient points while holding the Nash equilibria constant. We might say that all our changes were 'irrelevant' in the sense that the games should be considered equivalent by the **Relevance** axiom. Thus, the Pareto-efficient points of a game are *determined entirely by our preferences over options we can never choose between.*

The paradox assumes that Pareto efficiency and Nash equilibria are both equally intrinsic properties of a game. Instead, the theorem shows that the influence of real-valued payoffs is complex, and Pareto efficiency and Nash equilibria stem from distinct aspects of the payoff structure (the underlying ordinal and strategy games). As a preference game, there is no paradox, because Pareto efficiency is not defined.

3.3.3 Symmetries

There is another natural form of equivalence of games, quite distinct from those above. This equivalence stems from the following observation: the analysis of a game should be invariant under *renaming* of strategies and *reordering* of players. This equivalence relation is generally implicit in game theory, and is enforced by choosing arbitrary names for players and strategies, and is assumed to be generally innocuous[14].

However, modelling 'arbitrary information' is generally a clunky way of handling equivalence. As we established in the Introduction and in the previous section, this approach can lead to highly misleading results. In preference games there is a completely natural solution: graph isomorphism.

As an example, consider the two well-known 2x2 games: Coordination (see Section 3.4.1) and Chicken (see Myerson (1997)). In Coordination the two players prefer to choose the same strategy; in Chicken the two players prefer to choose different strategies. Their response graphs are shown in Figure 3.3.

These response graphs are not *equal*, but they are *isomorphic*—they differ only in the labels. If the response graph truly defines the preference game, our analysis should not differ between two isomorphic games. These games are equivalent up to renaming of strategies: switching the labels of the strategy for one player makes these games identical.

[14] An example of how one can explicitly represent this theorem in equivalence classes of isomorphisms is given in Mertens (2004). As we see here, in preference games this symmetry is already captured implicitly.

(a) Coordination (b) Chicken

Figure 3.3: The preference structure of Chicken and Coordination

For the theory of preference games, this raises an important question. Is it well-defined to present a preference game as a digraph with unlabelled nodes? If we are given a response graph without the information of which profiles correspond to which nodes, a priori we do not even know how to determine if it *is a response graph*. Luckily, it turns out that preference games and response graphs are well-behaved—we can uniquely reconstruct the associated preference game from an unlabelled response graph, up to renaming of the players and strategies, and we can do so in linear time. We prove this in the appendix, but we state here the main results, which include a characterisation of response graphs.

Proposition 3.3.10. A directed graph is the response graph of a game if and only if its undirected structure is the Cartesian product $\square_{i=1}^{N} K_{|S_i|}$ of N complete graphs of sizes $|S_i|$, which is directed such that each copy of each complete graph is a preference order.

Theorem 3.3.11 (Reconstruction Theorem)**.** *Given a directed graph, we can determine in linear time if it is isomorphic to the response graph of some game, and if so we can construct the associated preference game in linear time, and it is unique up to renaming of strategies and players.*

This is good news for preference game theory, because it says that this set of graphs really does capture preference games, and we can present them up to isomorphism (that is, without explicit node labels) in the natural graph-theoretic way without losing any game-theoretic information. As an example, consider the Rock-Paper-Scissors response graph in Figure 3.1. This still contains all the information of the Rock-Paper-Scissors game. In short, *preference games really are graphs.*

3.4 Solution Concepts for Games

In this section we discuss solution concepts for games, such as dominance, pure and mixed Nash equilibria, and Pareto efficiency. We first look at how these concepts relate to the equivalence classes we introduced in Section 3.3.1. Our findings are summarised in Figure 3.4, with the formal statements and proofs in Section A.2.2. This diagram shows which game properties are invariants of which game classes. As each equivalence relation is a refinement of those above it, each invariant of a class is an invariant of all classes above that one. For instance, pure NEs are an invariant of **Preference**, and so must also be invariants of **Strategy**, **Ordinal** and **Affine**. Further, this diagram is as

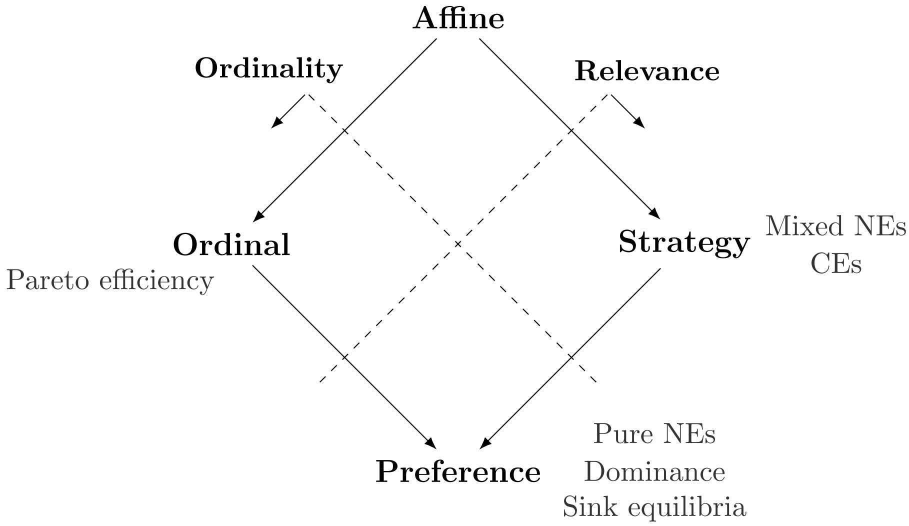


Figure 3.4: Equivalence classes and their invariant properties. Each class is labelled by the invariants which are not invariants of the classes beneath. Moving from affine to ordinal games or strategic to preference games involves adopting the **Ordinality** axiom; likewise, moving from affine to strategy games or ordinal to preference games involves adopting the **Relevance** axiom.

strong as possible—each property is *not* an invariant of any below it, so, for instance, mixed Nash equilibria are not invariant under preference-equivalence.

To begin, we introduce the four 2x2 preference games; these will serve as motivating examples throughout this thesis. After that we discuss more generally what we would like from a 'good' solution concept, focusing on three compelling desirata:

1. (**Stability**) a solution should be 'stable' and nearby strategies should converge to it, under some assumptions on how players update their strategy, and

2. (**Existence**) a solution should exist in all games, and

3. (**Computability**) a solution should be efficiently computable.

We discuss the ways that existing solutions do and do not satisfy these criteria, with a particular emphasis on the Nash equilibrium. We find that mixed Nash are not generally stable or efficiently computable, and that conversely the solutions which are stable and computable tend to be *preference-invariants*, notably dominance, iterated dominance and pure NEs. These do not satisfy existence. We show that sink equilibria exist and are computable, and we lay out the argument that they are stable, which we examine formally using evolutionary game theory in Chapter 5.

3.4.1 Prototypical examples: 2x2 games

To motivate the study of solution concepts, we will introduce four well-known two-player, two-strategy games (2x2 games), which will reappear often in this thesis. We call these games Matching Pennies[15], Coordination, Single-dominance, and Double-dominance. While these four games are of course not sufficient to demonstrate the full spectrum of games in game theory, they are clear enough to establish much of the intuition used throughout the subject, and so are ubiquitous in introductory and advanced textbooks (under various names). We will contract their names to MP, CO, SD, and DD respectively. Their payoff tables are presented in Table 3.2.

Table 3.2: The standard 2x2 games. Player A is the row player and B is the column player.

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

(a) Matching Pennies

	H	T
H	$1, 1$	$0, 0$
T	$0, 0$	$2, 2$

(b) Coordination

	H	T
H	$0, 0$	$0, 1$
T	$1, 1$	$1, 0$

(c) Single-dominance

	H	T
H	$0, 0$	$-1, 1$
T	$1, -1$	$2, 2$

(d) Double-dominance

These games work as follows. In Coordination, players each choose either Heads (H) or Tails (T), and both players prefer to make the same choice. In Matching Pennies, players again choose H or T, but now only player A wishes for their choices to match, while player B prefers if the choices *do not* match. In Single-dominance, as the name suggests, player A always prefers to play T, while player B prefers to play whatever player A plays. In Double-dominance, both players prefer to play T regardless of the other player's choice.

Each of these games serves a pedagogical purpose for introducing solution concepts. Double-dominance explains the concept of *dominant* and *dominated* strategies, and is used to construct the Prisoner's Dilemma, which we discussed in Section 3.3.2. Single-dominance shows how we extend this idea to *iterated dominance*. Coordination shows that multiple Nash equilibria can exist in a game, while Matching Pennies shows that games may have no pure Nash equilibria. We discuss this in the next section.

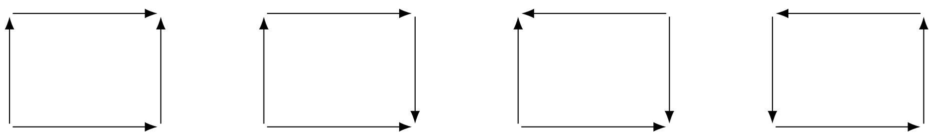
As 2x2 games are the simplest non-trivial games, they are a natural choice for demonstration of ideas. However, this is even simpler from the perspective of preference games: these four games are *the only four possible 2x2 preference games*[16], up to symmetry in

[15] Not to be confused with the three-player Circular Matching Pennies game discussed in the introduction. In fact, by historical naming convention, this game is closer to Circular *Mis*matching Pennies.

[16] Not including non-strict games.

the players and strategies. That is, these are the only non-isomorphic response graphs on four nodes, so every strict 2x2 utility game has the response graph of one of these. We show the response graphs in Figure 3.5. This is distinct from the ordinal games case, where there are many more distinct 2x2 games; see, for instance, Cruz and Simaan (2000).

Again, the English-language descriptions of each game said nothing about payoffs. In fact, the key pedagogical properties that we learn from each game are invariant under preference-equivalence[17]: every game that is preference-equivalent to CO has two pure NEs and one mixed NE, every game preference-equivalent to MP has a unique NE that is fully mixed, every game preference-equivalent to Double-dominance has two dominant strategies and every game preference-equivalent to Single-dominance has an iterated dominance solution (Proposition A.2.15). Our choice of payoffs in Table 3.2 was arbitrary, and from hereon in we shall treat MP, CO, SD and DD as preference games, and so consider only their structure in Figure 3.5. From a pedagogical perspective, we also believe this presentation is the clearest way to teach these ideas.



(a) DD (b) SD (c) CO (d) MP

Figure 3.5: The four non-isomorphic response graphs of 2x2 games.

3.4.2 What do we want from a solution concept?

Suppose now that we are analysing a game, and determining as we proceed what tools may be of use. First, we set out what we would like to achieve: **we want to understand how the players will behave (possibly in the long-run) when presented with the strategic interaction under consideration**. We set out three criteria, which we will justify as we proceed.

1. (**Stability**) a solution should be ‘stable’ and nearby strategies should converge to it, under some assumptions on how players update their strategy, and
2. (**Existence**) a solution should exist in all games, and
3. (**Computability**) a solution should be efficiently computable.

We begin by thinking about the solution concept of *dominance*. When people’s preferences over their strategies are *independent* of the other players, the problem becomes

[17] This includes up to symmetry. Recall that this is a stronger statement than being invariant under ordinal equivalence. Note also that this is distinct from the Prisoner’s Dilemma case, because there we discussed the additional question of Pareto efficiency of points, which is not preference-invariant. As a 2x2 preference game, the Prisoner’s Dilemma has the structure of Double-dominance, and so as expected has a unique pure NE.

trivial: players play dominant strategies, if they have them, and they do not play dominated ones. By finite induction, we can plausibly extend our prediction to say that players will not play iteratively dominated strategies. We can easily compute (Conitzer and Sandholm, 2005) those strategies which survive iterated dominance, and so reduce our analysis of the game to this subgame (**Computability**). Further, small deviations of the strategies of players, or of the payoffs, will not modify our prediction in the long run (**Stability**). In dominance-solvable games, this gives us a unique outcome which is quite compelling. The DD and SD games are dominance-solvable, so we now analyse these games.

Iterated dominance feels plausible as a prediction, and yet suffers from an obvious flaw: it only applies to the special case of games with dominated strategies. Dominance-solvable games are a particularly special case. That is, this solution concept doesn't satisfy **Existence**. Clearly, we would like to say more in general.

Now consider PNEs. As we established in the introduction, (strict) PNEs are stable under small modifications of the game, and they are robust against small perturbations of players' strategies (**Stability**). PNEs also generalise iterated dominance, allowing us to make use of that solution concept in the special cases where it applies. Similar to dominance, PNEs can be efficiently computed (linear in the size of the game) (**Computability**). The generalisation of iterated dominance to PNEs comes at the cost of uniqueness, as the CO game shows, and this leads to the general problem of equilibrium selection (Harsanyi et al., 1988), which we do not address in this thesis. Still, assuming we end up 'close' to a PNE, players will fall into it, and continue to play this equilibrium (see Section 5.2). We can use PNEs to analyse CO: players end up playing one of the two pure equilibria.

Note that these two seemingly compelling (where they exist) solution concepts are both preference-invariants (Figure 3.4).

Nash equilibria, and the importance of existence

The analysis tools we have developed thus far cannot say anything about the MP game, because it has neither dominated strategies nor PNEs.

Nash's existence theorem provides us with one solution: mixed Nash equilibria, which exist in all games. The power of this result is rooted in its generality. Its influence was such that game theory exploded in popularity, and became a central part of economics. Nash equilibria are the standard solution concept in game theory (Myerson, 1997), and the criteria by which all other solution concepts are judged.

Using Nash equilibria we can always state something about a game (its equilibria), and the concept generalises our compelling notions of PNEs and dominance. But are mixed Nash equilibria a solution to our problem, in the sense of being how players will behave in the long run?

In evolutionary game theory, where long-run notions of strategic update are studied,

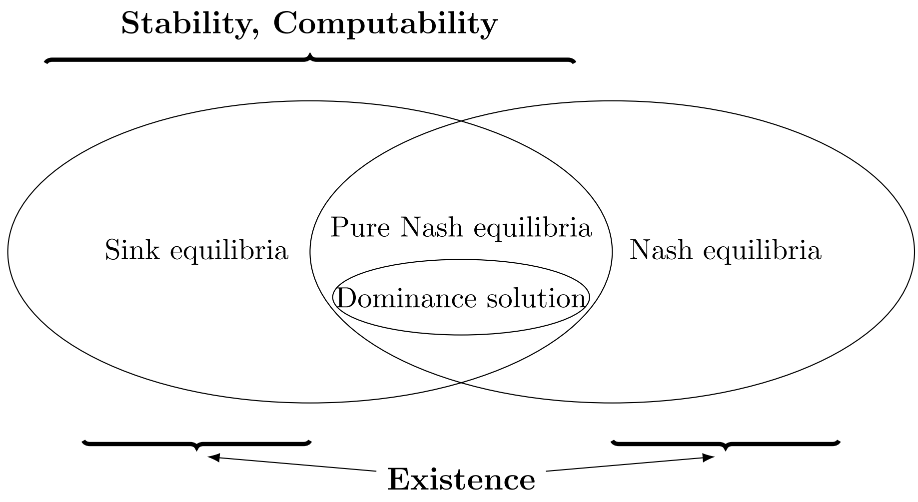


Figure 3.6: The relationship between solution concepts and our desired properties.

the answer is no. Dynamics (the abstraction of strategic update rules) cannot generically converge to Nash equilibria [cite all], as we saw in the Introduction. Mixed Nash equilibria can be thought of as similar to saddle points of a function (this is precise for *potential games* (Definition 4.1.1) like CO—there, the unique mixed Nash equilibrium is truly a saddle point of the potential function), with the update of players working like gradient descent; in this framework, we expect to almost always end up at local minima, and almost never stop at saddle points.

Another blow for mixed Nash equilibria comes from algorithmic game theory: they are not tractable to compute (Daskalakis et al., 2009). As analysts, we can't make a prediction we can't compute; equally concerningly, the players in the game can't be expected to play strategies that they themselves cannot compute.

Figure 3.6 summarises the situation. Generalising pure Nash equilibria to Nash equilibria gave us **Existence**, but came at the cost of the **Stability** and **Computability** possessed by PNEs. What's more, as PNEs are preference-invariant, the generalisation to mixed Nash equilibria also requires the addition of the *entire concept of utilities*. By contrast, sink equilibria generalise PNEs in a different direction, retaining **Computability**, because the sink connected component can be found in linear time. They also satisfy **Existence**. Finally, they seem to be **Stable**: for dynamics defined by Markov chains and the replicator dynamic, the sink equilibria define the long-run outcomes. Specifically, for the replicator dynamic, sink equilibria are always contained in *sink chain components* (Definition 5.1.8), which are the long-run outcomes of the dynamic in a precise mathematical sense (see Section 5.1). As we show in Figure 5.1, the sink chain components precisely correspond to the sink equilibria in the eight games we study in this thesis, including MP, DD, SD, CO, Circular Matching and Mismatching Pennies. This is the topic of Chapter 5.

Sink equilibria satisfy our criteria for a solution, while satisfying our **Ordinality** and **Relevance** axioms, and so making minimal requirements of the modeller. As a well-motivated model with a compelling solution concept, we conclude that we should study preference games.

Chapter 4

Preference in Classical Game Theory

In this chapter we discuss how preference game theory can add to classical (utility) game theory. Specifically, we focus on two-player games. We demonstrate that preference games

1. allow us to prove existing game theory results easily and clearly, and
2. this gain in clarity allows us to prove new results in classical game theory.

4.1 Zero-sum and Potential games

In this section we discuss two famous classes of games: zero-sum games (Von Neumann and Morgenstern, 1944) and potential games (Monderer and Shapley, 1996). A priori, neither of these classes seem to be closely related to preference games—in fact, neither is even an affine invariant. However, we show that thinking in terms of equivalence classes gives us insight into these games. First, we prove a relationship between zero-sum and potential games when considered up to strategic equivalence; we call this result the *zero-sum–potential duality theorem*[1] (Theorem 4.2.3). This relationship was noted in Hwang and Rey-Bellet (2020b,a), but we extend the ideas to preference games. In fact this relationship suggests a close connection to physics, and to dynamics (which we discuss in Chapter 5). That is, strategically potential games are connected to potential dynamics, and strategic zero-sum games to Hamiltonian dynamics, a connection that is discussed in Balduzzi et al. (2018) but which is generalised through preference games.

This result allows us to find simple and intuitive characterisations of *preference-potential* and *preference-zero-sum* games, that is, those which are preference-equivalent to a zero-sum or potential game respectively. These classes have never before been identified. This is a much broader class of games than zero-sum and potential games, because

[1] Indeed, there is a duality between these classes of games.

these include any game whose response graph is isomorphic to the response graph of any zero-sum (respectively potential) game. We then give a graph-theoretic proof of a new theorem in classical game theory, which asserts that in every two-player game, each rationalisable strategy must be contained in 2x2 subgame with other rationalisable strategies that has the response graph of Matching Pennies or Coordination (Theorem 4.2.9). This shows that these preference games play an important role in all two-player games. Combining this with our characterisations of preference-zero-sum and preference-potential games, we prove that every two-player game that is both preference-zero-sum and preference-potential is dominance-solvable, tying together three important classes from game theory. We call this result the Zero-Sum–Potential–Dominance Theorem (4.2.12).

4.1.1 Potential games

We begin with the definition of a potential game, due to Monderer and Shapley.

Definition 4.1.1. A game u is called a *potential game* if there is a function $\phi : Z \to \mathbb{R}$ such that for every pair of i-comparable profiles p and q, $\phi(p) - \phi(q) = u(p)_i - u(q)_i$.

That is, the relative payoffs to each player can be defined by a single real-valued function, named the *potential function*, by analogy with physics. There, a dynamic f is called *potential* if $f = \nabla\varphi$, where φ is a real-valued function. This means that f is a gradient vector field. As we know from vector calculus (Hubbard and Hubbard, 2015), such vector fields are exactly those that are *conservative*. Additionally, the fundamental theorem of calculus holds, and so f is *path-independent*—that is, the path integral of f is always the difference between the values of φ at the endpoints.

In game theory, potential games are notable because they guarantee the existence of a pure Nash equilibrium[2] (Monderer and Shapley, 1996).

Intuitively, the existence of a potential function prevents cycles of preference. This idea is well-captured by preference games—in fact, as we will show, a game is preference-potential if and only if its response graph is acyclic. In fact, the relationship goes further, with potential games also satisfying a form of 'path-independence', once considered up to strategic equivalence (in the sense of Definition 3.3.3).

Definition 4.1.2. A game u is strategically-potential if it is strategically equivalent to some potential game.

Definition 4.1.3. Given a labelled response graph of a game u and an undirected path $p = v_1, v_2, \ldots, v_n$ between them, the *path-sum* of p is the (signed) sum of arc labels along p, that is

$$\mathrm{pathsum}(p) = \sum_{i=1}^{n-1} (u(v_i)_{p_i} - u(v_{i+1})_{p_i})$$

[2] Because pure NEs are preference-invariant, this result generalises easily to all preference-potential games.

where p_i is the unique player such that v_i and v_{i+1} are p-comparable. A game is *path-independent* if for any nodes v and w, all paths from v to w have the same path-sum.

Lemma 4.1.4. A game u is strategically potential if and only if there is a constant $a_i > 0$ for each player i such that the labelled response graph of $v = (a_1 u_1, \ldots, a_n u_n)$ is path-independent.

That is strategically potential games are exactly path-independent labelled response graphs, possibly with a rescaling for each player. Using this result, it is easy to see that a (strict) potential game must have an acyclic response graph, as otherwise we could not have path-independence.[3]

Recall the analogy with path-independence and conservative vector fields in calculus. Here the path integral is replaced with the sum over weights on a path in the response graph. Again we find that the value of this 'path integral' is equal to the difference in potential between the two endpoints of the path.

We want to know which games are preference-equivalent to a potential game. The answer comes easily out of Lemma 4.1.4:

Corollary 4.1.5. A game is preference-potential if and only if its response graph is acyclic.

Proof. It is obvious from Lemma 4.1.4 that this is a necessary condition. For sufficiency, observe that if the response graph is acyclic, we can construct a preference-equivalent game with a potential function by traversing the response graph in reverse topological order. □

It is obvious now why preference-potential games always have a pure Nash equilibrium: as acyclic graphs, every strongly connected component is a singleton, and so all sink equilibria are singletons, and singleton sink equilibria are PNEs. It also is immediate that no preference-potential game ever has a subgame isomorphic to MP, because this is a cycle.

4.1.2 Zero-sum games

In a similar vein to our discussion of potential games, we will consider the class of *zero-sum games*. While zero-sum games can exist in games with any number of players, we will focus on the two player case. These games are both famous and important[4] in game theory, and were the subject of Von Neumann and Morgenstern's original book (Von Neumann and Morgenstern, 1944). There are a number of powerful theorems and properties that hold only for zero-sum games (for instance the *minimax theorem* and the notion of

[3] A non-strict game could have a non-strict cycle, where the potential difference between all nodes on this cycle is zero. In the graph this would be a cycle where each pair of adjacent nodes has an arc in both directions.

[4] These are separate things!

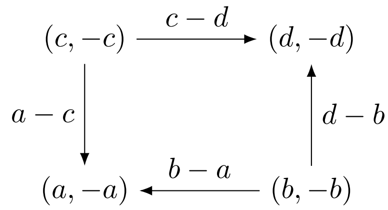


Figure 4.1: The structure of Coordination in a zero-sum game.

the *value* of a game). As these games are such useful tools, we wish to understand the influence of zero-sum-ness on the structure of a game.

Definition 4.1.6. A two-player game u is zero-sum if $u(p)_1 + u(p)_2 = 0$ for any profile p.

Intuitively, zero-sum games capture the notion that one player's gain is always another player's loss. This model closely captures the recreational games from which game theory takes its name; there, if one player wins, the other must lose. From a preference perspective, this suggests that the interests of the players in a zero-sum game are never aligned. Hearing this, one might suspect that games like Coordination never occur in zero-sum games. This guess does turn out to be correct, and the insight gained leads to a characterisation of these games.

What are preference-zero-sum games?

Our goal is to understand the following class of games:

Definition 4.1.7. A two-player game is *preference-zero-sum* if there is a preference-equivalent game that is zero-sum.

It seems, a priori, that the definition of a zero-sum game depends pivotally on utilities—it is not even invariant under addition of an arbitrary constant. We might wonder whether anything non-trivial can be said about the preference-zero-sum games, or whether every game is preference-zero-sum. This turns out not to be the case, as the following example shows.

Example 4.1.8 (Coordination is not preference-zero-sum)**.** Consider any 2x2 zero-sum game, with payoffs a, b, c and d for player 1 in the differing profiles, and their negations for player 2. The setup is shown in Figure 4.1. Also shown in the figure are the direction of preferences needed to make such a game have the structure of Coordination. This occurs if all of the relative payoffs $c-d$, $b-a$, $a-c$ and $d-b$ are positive. However this implies that $c > d$, $b > a$, $a > c$ and $d > b$, giving the impossible cycle $d > b > a > c > d$. Hence no zero-sum game can have this response graph.

This example used only the response graph of Coordination and so demonstrates that preference-zero-sum games are a non-trivial class of games. Usefully, the structure of the proof also suggests a way of proceeding. The critical fact was the existence of a *preference cycle* in the underlying utilities. In some sense then, zero-sum games are *acyclic*, after the application of some transformation. From the results in the previous section, this should suggest a connection to preference-potential games, which we characterised as those with acyclic game graphs.

4.2 Zero-Sum–Potential Duality

The connection between zero-sum and potential games in two players is best stated up to strategic equivalence.

Definition 4.2.1. A game is *strategically zero-sum* if it is strategically equivalent to some zero-sum game.

The relationship between potential and zero-sum games can be described by a *reflection* operation on the payoffs of one player.

Definition 4.2.2. Let $u = (u_1, u_2)$ be a two-player game. The *reflected game* $\hat{u}$ is $(u_1, -u_2)$.[5]

There are in fact two reflected games, $(-u_1, u_2)$ and $(u_1, -u_2)$. These are 'strategic reversals' of each other as $(u_1, -u_2) = -(-u_1, u_2)$. Since the order of players is generally not important (see Section 3.3.3), we can choose either, and all our theorems here follow with either choice.

Then, finally, we get the following theorem:

Theorem 4.2.3 (Zero-sum–Potential Duality)**.** *Let $u = (u_1, u_2)$ be a two-player game. Then u is strategically zero-sum if and only if $(u_1, -u_2)$ is strategically potential.*[6]

Proof. ($\Rightarrow$) Suppose u is zero-sum. Then the payoff in any profile (s_i, s_j) is $(x_{i,j}, -x_{i,j})$ for some real $x_{i,j}$. In the reflected game $\hat{u}$ the payoff is $(x_{i,j}, x_{i,j})$. This game is a potential game, with potential function $f : Z \to \mathbb{R}$, $f(s_i, s_j) = x_{i,j}$.

($\Leftarrow$) Suppose now that u is potential, with potential function $f : Z \to \mathbb{R}$. Then the game v given by $v(s_i, s_j) = (f(s_i, s_j), -f(s_i, s_j))$ is clearly a zero-sum game, and its reflection $\hat{v}$ is a potential game with potential f by the above. But then $\hat{v}$ is strategically-equivalent to u, since they share the same potential function (Proposition A.3.2). By Lemma A.3.3, $\hat{u}$ is strategically-equivalent to v if and only if u is strategically equivalent to $\hat{v}$. □

That is, a game is strategically zero-sum if and only if its reflection is strategic potential. Thus, in two-players, the notion of zero-sum-ness is intimately connected to potential and thus to acyclicity.

[5] Where $-u_2 : Z \to \mathbb{R}$ has the obvious meaning of $(-u_2)(x) = -u_2(x)$.
[6] This is also true for the reflection in player 1.

One direction of this result is obvious and well-known. That is that the reflection of a zero-sum game is a potential game. In fact, the reflection of a zero-sum game is an *identical-interest game* (both players receive the same payoff in all profiles). It is the opposite direction that is surprising and gives this theorem its power: up to strategic equivalence, all two-player potential games are reflected zero-sum games. This point can be seen as a consequence of the decomposition in Hwang and Rey-Bellet (2020b).

Making use of Lemma 4.1.4 from the previous section we can state more concretely that:

Corollary 4.2.4. A two-player game is strategically zero-sum if and only if it is affine-equivalent to a game whose reflection is path-independent.

Nash equilibria in two-player zero-sum games have many special properties. Most notably, the problem of finding a Nash equilibrium of a two-player zero-sum game is equivalent to linear programming. Since Nash are preserved under the strategic equivalence, this characterisation of the set of strategically zero-sum games is naturally of interest.

It is also interesting to note the relationship between the reflection operation, potential games, and Hamilonians in physics and dynamical systems. In dynamics, a vector field f is *Hamiltonian* if its structure is $f = J\nabla\varphi$ for some real-valued function φ, where J is a matrix with block structure $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, where I is the identity (Alongi and Nelson, 2007). That is, f is a gradient field that has been transformed by reversing half of its variables—but this is essentially the definition of a zero-sum game as a potential game transformed by reversing the preference of one player.

These ideas have received recent interest, particularly in the machine learning communities. In Balduzzi et al. (2018) the authors identify zero-sum games as a special case of so-called *Hamiltonian games*, which are related to potential games through an analogous transformation. That work was more specific than what we present here, without using strategic equivalence or identifying this duality in two players. It was also shown in Papadimitriou and Piliouras (2018) that zero-sum games under the replicator dynamic (Chapter 5) possess a Hamiltonian function, which can be used to demonstrate cyclic behaviour.

4.2.1 Characterising preference-zero-sum games

Using the duality theorem we can now easily derive a characterisation of preference-zero-sum games. Given that a game is preference-equivalent to a potential game if and only if it is acyclic, we can deduce that a game is preference-zero-sum if and only if its reflection is acyclic!

Theorem 4.2.5 (Preference zero-sum games)**.** *A two-player game is preference-zero-sum if and only if its reflection is acyclic.*

Note that if a game's reflection in one player is acyclic, then its reflection in the other player is simply the game with all preferences reversed, which must also be acyclic.

Figure 4.2: The structure of Coordination (left) and its reflection in player 2, which has the structure of Matching Pennies (right). Note that reflecting the game in the preferences of player 1 also swaps between these two graph structures.

Hence the choice of player in this theorem does not matter.

With this theorem in mind, we can return to Example 4.1.8. The reflection of Coordination is Matching Pennies—that is, a cycle—and so we conclude immediately that Coordination is not preference-zero-sum. This is shown in Figure 4.2.

In fact, given that the reflection of a zero-sum game cannot have any cycles, no subgame of a zero-sum game can have the structure of CO either.

Corollary 4.2.6. Every two-player preference-zero-sum game has no subgame whose response graph is isomorphic to CO.

Proof. If u is preference-zero-sum and has a subgame isomorphic to CO, then that subgame is isomorphic to MP in the reflection $\hat{u}$. But this is acyclic by Theorem 4.2.5, so no preference-zero-sum has such a subgame.two-player □

This allows us to immediately prove an otherwise non-trivial result about zero-sum games, which we can also now generalise to preference-zero-sum games.

Theorem 4.2.7. *A strict preference-zero-sum game can have at most one pure NE.*

Proof. Suppose for contradiction that a preference-zero-sum game u had two distinct pure NEs, $p = (p_1, p_2)$ and $q = (q_1, q_2)$. Being a strict game, these cannot share a strategy for either player, so $p_1 \neq q_1$ and $p_2 \neq q_2$. But then the four profiles (p_1, p_2), (p_1, q_2), (q_1, p_2) and (q_1, q_2) form a subgame isomorphic to CO, which is impossible in a preference-zero-sum game (Corollary 4.2.6). Hence there can be at most one pure NE. □

In fact, a stronger version of this theorem exists. If a zero-sum game has a PNE, then it is unique among all (pure and mixed) equilibria. Surprisingly, this uniqueness extends to sink equilibria, which unlike PNEs do exist in all games.

Lemma 4.2.8 (Zero-sum games have a unique sink equilibrium)**.** If u is two-player preference-zero-sum, then u has one sink connected component, and one source connected component (these may be the same).

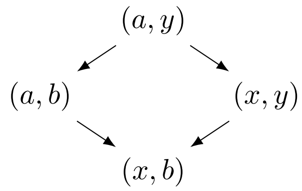


Figure 4.3: In a preference-zero-sum game, two profiles with no paths between them must have the structure of DD.

Proof. Suppose for contradiction that (a, b) and (x, y) are profiles in two distinct sink connected components, so there are no paths from one to the other. Then consider (a, y) and (x, b). Without loss of generality, we have the picture in Figure 4.3. Because there are no paths between these, both (a, y) and (x, b) have either only inward or outward arcs from (a, b) and (x, y). Since the structure of CO cannot occur (Corollary 4.2.6, we conclude that this subgame has the structure of DD, but then there is a node (x, b) succeeding both (x, y) and (a, b), contradicting the fact that they are sink connected components. The case for source connected components is identical. □

While generic[7] zero-sum games have unique Nash equilibria, this novel result shows that this uniqueness extends to sink equilibria in preference-zero-sum games, a much more general class. In some sense, the uniqueness is caused by the preference structure.

4.2.2 The importance of Matching Pennies and Coordination

The 2x2 preference games Matching Pennies and Coordination have now appeared several times in our discussion, both to motivate solution concepts (Section 3.4.1) and in the previous section, to motivate and explain the reflection operation. While these games are useful examples, one does not expect that these games encompass all the possibilities of game theory, and indeed this is not true. However, it turns out that in two-player games MP and CO are the *source of all strategic complexity*, in a sense we make precise in the following.

We say a game contains an MP or CO subgame if there is a 2x2 subgame whose response graph is isomorphic to Matching Pennies or Coordination respectively. This property is a preference-invariant. We prove the following.

Theorem 4.2.9. *In any two-player strict game that is not dominance-solvable, every strategy that survives iterated dominance is contained in a MP or CO subgame inside the residual subgame*[8].

Proof. We assume that all iteratively dominated strategies have been removed. As the game is not dominance-solvable, there are at least two strategies remaining for both

[7]In a precise sense, see Quint and Shubik (1997).
[8]The subgame that survives iterated dominance (Chapter 2).

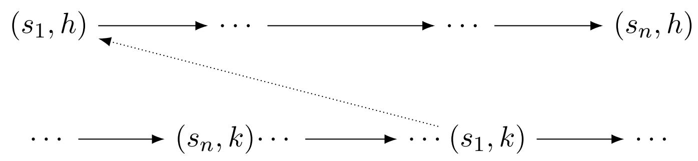


Figure 4.4: A diagram showing the subgraph of the game graph for the profiles where player 2 plays h or k. The dotted arc $(s_1, k) \longrightarrow (s_1, h)$ is by assumption of case (1); this arc is reversed in case (2).

players and no strategy dominates another. Now we assume for contradiction that there are no MP or CO subgames in the game.

Let h be some strategy for player 2, and $s_1, \ldots, s_n$ the strategies for player 1, ordered by payoff for player 1 when player 2 plays h, that is, $u_1(s_1, h) < u_1(s_2, h) < \cdots < u_1(s_n, h)$. Since no strategy dominates any other, s_n cannot dominate s_1, so there is some strategy k for player 2 where $u_1(s_n, k) < u_1(s_1, k)$. Let $t_1, \ldots, t_n$ denote the preference order on strategies for player 1 when player 2 plays k.

There are two cases, for which the argument is symmetric while only reversing the role of MP and CO. In case (1) player 2 prefers k when player 1 plays s_1, so $u_2(s_1, k) > u_2(s_i, h)$, and case (2) is the opposite, where $u_2(s_1, k) < u(s_i, h)$. Assuming case (1), we obtain the picture shown in Figure 4.4.

Now let s_i be any strategy with $u_1(s_i, k) < u_1(s_1, k)$. Since s_1 is least preferred by player 1 when player 2 plays h, we also have $u_1(s_1, h) < u_1(s_i, h)$. By case (1) we also have $u_2(s_1, k) < u(s_i, h)$. Thus we cannot also have $u_2(s_i, h) < u_2(s_i, k)$, as this would give an MP subgame. Thus for any such s_i, $u_2(s_i, h) > u_2(s_i, k)$. In particular, this implies that $u_2(s_n, h) > u_2(s_n, k)$.

Now let s_j be any strategy with $u_1(s_j, k) > u_1(s_n, k)$. Since s_n is most preferred by player 1 when player 2 plays h, we also have $u_1(s_j, h) < u_1(s_n, h)$. By the above, we also have $u_2(s_n, h) > u_2(s_n, k)$. Thus we cannot also have $u_2(s_j, h) < u_2(s_j, k)$, as this would give a CO subgame. Thus for any such s_i, $u_2(s_i, h) > u_2(s_i, k)$. These two cases are summarised graphically in Figure 4.5.

However, all strategies $s \in S_1$ have now been handled, and so we find that strategy h dominates strategy k for player 2, contradicting our original assumption. Case (2) follows identical reasoning, swapping CO and MP and concluding with k dominating h. □

Note firstly that while this theorem is entirely a result of classical game theory (in that it applies to all two-player games) its statement and proof exploited the graph structure of MP and CO and made no reference to utility. This result is reminiscent of characterisation of graphs by their forbidden subgraphs (Diestel, 2016). This is an

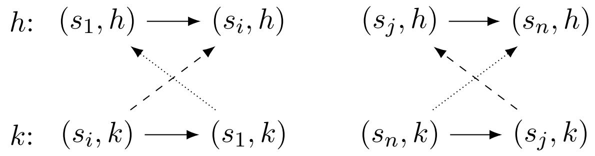


Figure 4.5: An illustration of the proof, with horizontal arrows representing preference for player 1 and vertical arrows preference for player 2. The dotted arc is given, and the figure shows how avoiding MP and CO fixes the choice of direction for the dashed arc.

example of the preference approach to games clarifying our understanding of classical game theory topics.

Clearly, in dominance-solvable games there cannot be a MP or CO subgame in the residual subgame, because it contains only one profile, and at least four profiles are needed! Phrased another way, this theorem proves that any two-player game containing no MP or CO subgames *must be dominance-solvable* and so must have have a unique pure Nash equilibrium. Two-player games without MP or CO are trivial in some sense, and so MP and CO are responsible for bringing strategic complexity to a game.

Note that a strategy that survives iterated dominance may take part in many MP or CO subgames within the residual subgame, but always at least one. In some cases it is only one, even when many 2x2 subgames exist. As an example, consider Figure 4.6. This is a preference-zero-sum game with no dominated strategies, but it has a PNE. The PNE profile cannot participate in a MP subgame, and the strategies played in this PNE participate separately in precisely one MP subgame each.

Interestingly, this game and its reversal (the game given by reversing all arcs) are special in a certain sense. This is the *unique* 3x3 two-player preference-zero-sum game that has no dominated strategies and a PNE. As no such games exist with fewer than 3x3 strategies, this is the *minimal example of such a game.* Its reversal (which is isomorphic, except the PNE becomes a source) is the minimal preference-zero-sum game where the sink equilibrium is not a subgame. We omit the proofs in the interest of space. Because these games are interesting we give them names: we call Figure 4.6 the *Inner Diamond game* (for its shape) and its reversal the *Outer Diamond game.* We will return to them in Chapter 5.

4.2.3 The Zero-Sum–Potential–Dominance Theorem

To wrap up this chapter we discuss results combining our findings on preference-zero-sum and preference-potential games in two players. First, we showed in Corollary 4.2.6 and Corollary 4.1.5 that preference-zero-sum games have no CO subgames and preference-potential games have no MP subgames. Using Theorem 4.2.9, we can now immediately prove a partial converse: preference-zero-sum games always have an MP, and preference-

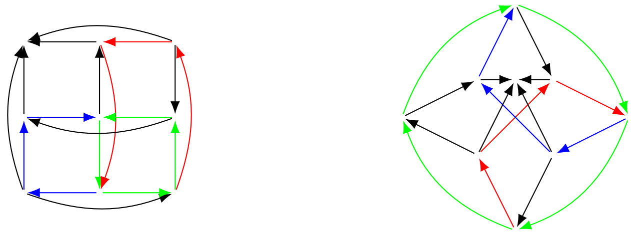


Figure 4.6: (The *Inner Diamond* game) A preference-zero-sum game with a PNE. This is the minimal such game. There are three MP subgames, which are coloured red, blue and green.

potential games always have a CO, unless they are dominance-solvable.

Theorem 4.2.10. *Every non-dominance-solvable preference-zero-sum game contains an MP subgame in its residual subgame. Similarly, every non-dominance-solvable preference-potential game contains a CO subgame in its residual subgame.*

Proof. In a non-dominance-solvable two-player game, there is a CO or MP subgame in the residual subgame (Theorem 4.2.9). If it is preference-zero-sum, it cannot have a CO (Corollary 4.2.6), and so it has an MP subgame. Likewise, preference-potential games must have a CO, as they have no MP subgames. □

Finally, we consider games that are both preference-zero-sum and preference-potential. Intuitively, such games must be trivial in some sense. Even before we had proved Theorem 4.2.9, we could easily establish that such games have *exactly one PNE*.

Lemma 4.2.11. Any game that is preference-zero-sum and preference-potential has exactly one PNE.

Proof. By Theorem 4.2.7, preference-zero-sum games have at most one PNE. However, as acyclic graphs (Corollary 4.1.5), preference-potential games have at least one PNE, and so the result is proved. □

This result was presented as an application of the methods in Hwang and Rey-Bellet (2020b), though only stated up to strategic equivalence. Here we have proved it both more easily and more generally. However, using Theorem 4.2.9 we can prove a much stronger theorem, which ties together several of the results from this section.

Theorem 4.2.12 (Zero-Sum–Potential–Dominance Theorem)**.** *Let u be a game that is preference-zero-sum and preference-potential. Then u is dominance-solvable.*

Proof. By Theorem 4.2.5 and Corollary 4.1.5 we know that both u and its reflection must be acyclic. Since MP is a cycle, and CO is a reflected cycle, no 2x2 subgame in u can be isomorphic to MP or CO. Finally, by Theorem 4.2.9, u must be dominance-solvable. □

This theorem ties together our characterisations of preference-potential games, preference-zero-sum games and iterated dominance (a property which is already preference-invariant). As concrete examples, observe that both DD and SD are preference-zero-sum and preference-potential, and thus are dominance-solvable.

To the best of our knowledge, nothing close to this theorem has been stated in the past. There are even cases, such as Li et al. (2020), of the explicit study of the more-restricted case of zero-sum and potential games, without noting that these games are dominance-solvable.

This form of the theorem, in terms of preference games, implies the result up to all other forms of equivalence discussed in Section 3.3. As a corollary, any game which is zero-sum and potential is dominance-solvable. However, this Theorem can only be given in its full generality for preference games.

Chapter 5

Preference in Evolutionary Game Theory

Evolutionary game theory is a form of game theory which focuses on the dynamic processes that players use to adjust their strategy. Its original motivation comes from biology, where it has modelled the strategy of evolving populations of competing species (Smith and Price, 1973).

Evolutionary game theory embraces the approach that all strategies used in practice are heuristics. The intuitive idea is that decision-makers do not have custom algorithms for deciding the optimal decision in any scenario, but rather have a generic algorithm which is improved over time *with experience*. That is, players have a learning mechanism to alter their strategies; evolutionary game theory places this learning mechanism on centre stage. The power of this idea has gained importance since the discovery that computing Nash equilibria is intractable (Daskalakis et al., 2009), even in two-player games (Chen et al., 2009)—thus, even 'completely rational players' cannot play perfectly. With the growth in the understanding of learning in computer science, the relationship between games and learning has been a fruitful and important research topic (Kleinberg et al., 2011; Fabrikant and Papadimitriou, 2008; Omidshafiei et al., 2019; Mertikopoulos et al., 2018).

As an example, consider a game such as chess or Go. Here, computing the best-response to a given board position is generally infeasible. Instead, both human and AI players use heuristics, which they update by learning based on past performance. The importance of this problem in AI has motivated significant work in algorithmic and evolutionary game theory.

Evolutionary game theory defines the *learning* or *strategy update* procedure as a differential or difference equation on the mixed strategy space of the game, which defines a *dynamical system*. The problem of analysing the game becomes the problem of understanding the long-run behaviour of this dynamical system.

In this chapter we explore how evolutionary game theory relates to preference games. Our motivation for preference games (Chapter 3) still apply here; we cannot, for instance, precisely model the utility of a population of species choosing a given strategy—our predictions, therefore, ought not to depend too closely on these values[1]. Hence, we seek a **preference-invariant concept which captures the long-run behaviour of dynamics on games.**

A priori, it seems that our quest is not promising; dynamics are notoriously complex, and proving even simple properties can be prohibitive. We would expect then that preference games have little to say. This turns out to not be true: preference games, and the solution concept of sink equilibria, turn out to be deeply connected to the long-run outcome of dynamics on games, in a way that Nash equilibria are not.

We explore two specific cases: the continuous-time replicator dynamic (Sandholm, 2010), and the discrete-time dynamics defined by a Markov chain on the response graph. The former is the best-studied evolutionary game dynamic (Sandholm, 2010; Hofbauer and Sigmund, 2003), and is motivated both by simple models of biological imitation and as the continuous-time limit of the *Multiplicative Weights Update* algorithm, an important online learning algorithm (Arora et al., 2012). Under the replicator dynamic, we prove an open problem of Papadimitriou and Piliouras (2019) establishing that *sink chain components*—a compelling dynamic notion of long-run outcome motivated by the Fundamental Theorem of Dynamical Systems (Conley, 1978)—always exist, and *always contain a sink equilibrium*. We conjecture that the relationship between sink chain components and sink equilibria is one-to-one. We further conjecture that this containment is tight, and in fact the sink chain components are exactly the union of subgames contained entirely within the *content* of the sink equilibria (see Definition 5.2.20). This says that the long-run outcomes of the replicator are entirely *preference-invariant properties that are fundamentally defined by the sink equilibrium.*We use our results from Chapter 4 to prove a one-to-one correspondence in the special case of preference-potential and preference-zero-sum two-player games, and prove some new general results relating sink equilibria and sink chain components. As an application, we analyse the behaviour of the replicator on six games (MP, CO, SD, DD, Circular Matching and Mismatching Pennies, and the Inner and Outer Diamond games), and summarise our results in Figure 5.1, demonstrating a one-to-one correspondence between sink chain components and sink equilibria in each case.

Secondly, we show that under Markov dynamics we can prove a very strong result: **the sink equilibria define the entire long-run outcome of the game**. Specifically, (1) from any initial distribution over profiles, the game converges to a distribution over sinks; (2) each sink has a unique *stationary distribution*, and a distribution is stationary if and only if it is a convex combination of these; (3) the dynamics converges to a stationary distribution if and only if every sink is *aperiodic*, a property that is preference-invariant.

[1] This is similar to the concept of *structural stability* in dynamical systems.

Thus the support of the long-run outcome, the support of all fixed points, and the conditions under which the chain converges *are all preference-invariants.*

5.1 Beyond Fixed Points

In this section we discuss the 'long-run' or 'asymptotic' behaviour of a dynamical system, and use this to understand how we should think about evolutionary games. In particular we contrast 'stability', a long-run notion, with 'stationarity' and show how mixed Nash equilibria, being generally stationary but not stable, are not good predictors of the long-run behaviour of evolutionary games. We discuss the notion of *chains*, a 'noisy' interpretation of the evolution of a dynamical system which is used in the Fundamental Theorem of Dynamical Systems, and which reflects the computational limitations of real-world systems. Following Papadimitriou and Piliouras (2018), we show the natural 'stable' notion in a dynamical system is the *sink chain components*, a dynamic concept which in games generalises the pure Nash equilibria. This perspective on games has lately been gaining momentum and interest, particularly in the algorithmic game theory and machine learning communities (Balduzzi et al., 2018; Omidshafiei et al., 2019; Vlatakis-Gkaragkounis et al., 2020). In the next section we will relate these ideas to preference games and the replicator dynamic.

Our discussion is inspired by Papadimitriou and Piliouras (2019, 2018), and we invite the interested reader to explore these texts.

5.1.1 Stability and stationarity

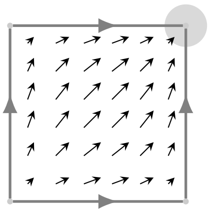
We begin with two similar but subtly different statements.

1. "Players end up playing a Nash equilibrium."
2. "If players are playing a Nash equilibrium, then they will continue to do so."

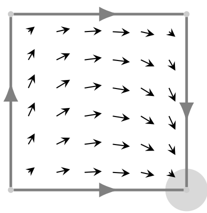
The former asserts that a Nash equilibrium is a *long-run* outcome of the game, while the latter states that a Nash is stationary in the sense that *if* currently played it will continue to be played. Both statements are 'evolutionary' because they implicitly refer to some strategic adjustment procedure defining how the players choose and update their strategies.

We can formalise both concepts using dynamical systems. Regions where players 'end up in the long run' (if they start nearby) are called *asymptotically stable.* A 'stationary' point is a *fixed point* of the dynamics. These two things are generally not the same. A fixed point may be unstable, having few or no nearby points which converge to it. Likewise, an asymptotically stable set may contain many points, of which none may be fixed; an example is a limit cycle (Strogatz, 2018).

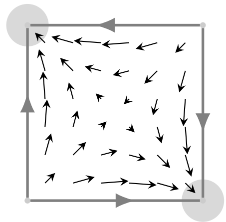
Under many natural choices of dynamics (Sandholm, 2010), the second statement is true: Nash equilibria are fixed points. In particular this is true for the replicator.



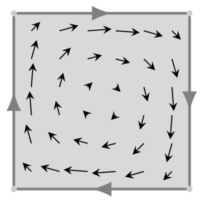
(a) Double-Dominance



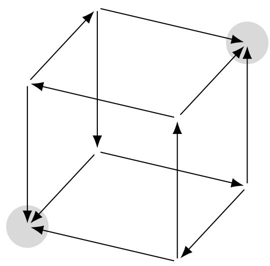
(b) Single-Dominance



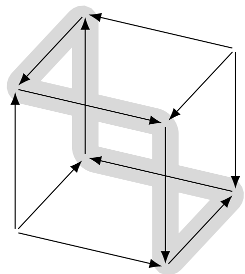
(c) Coordination



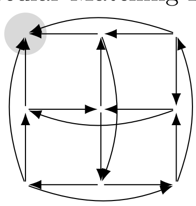
(d) Matching Pennies



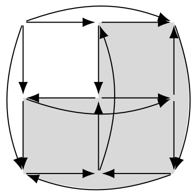
(e) Circular Matching Pennies



(f) Circular Mismatching Pennies



(g) Inner Diamond



(h) Outer Diamond

Figure 5.1: The sink chain components (identified in grey) of the replicator dynamic (Section 2.3) on the games we have discussed in this thesis. For the 2x2 games, we also display the vector field of the replicator. In each case there is a one-to-one correspondence between sink equilibria and sink chain components. For 2x2 games, these sets are preference-invariant, and we conjecture this is generically true. We present these results in Section 5.2.4, using the theory we develop in this chapter.

Lemma 5.1.1. (Sandholm, 2010) Every Nash equilibrium is a fixed point of the replicator dynamic.

However, there are **no dynamics** where all points converge to the Nash equilibria in all games (Benaïm et al., 2012; Milionis et al., 2022; Hart and Mas-Colell, 2003). In evolutionary game theory, Nash equilibria are generically *not* the long-run outcome, despite popular intuition. Some intuition for this result can be gained from the Instability Theorem (Theorem 1.1.1). That theorem states that the utility function is unstable around mixed equilibria, and so if the dynamics closely follow the utility function they cannot be stable in this region.

In fact the problem is exactly caused by the *mixed* Nash equilibria, because pure NEs do satisfy both of these properties.

Proposition 5.1.2 (PNEs are stable and stationary)**.** Under the replicator (and many other dynamics), every pure Nash equilibrium is an asymptotically stable fixed point.

For a proof, see Vlatakis-Gkaragkounis et al. (2020). This also follows as an immediate Corollary of Lemma 5.2.23, which we prove later. Once again we find that pure Nash satisfy desirable properties that do not carry over to mixed Nash. From the dynamics, we can see why: *the property we care about is stability, not stationarity.* Stability is a predictive tool, while stationarity is not. Instead of generalising asymptotically stable fixed points (pure Nash) to fixed points (mixed Nash) we should generalise them to (minimal) asymptotically stable sets.

The minimality criterion is important because asymptotically stable sets may have complex internal structure; as an example, they may contain other asymptotically stable sets. Naturally, one wishes to break the system into a few pieces that are as simple as possible, joined by connecting orbits. To formalise this idea, we need the dynamic concept of *recurrence*.

5.1.2 The Fundamental Theorem of Dynamical Systems

In one-dimensional dynamical systems on a compact space, every point is either fixed or converges to a fixed point. In two dimensions, the famous Poincaré-Bendixson theorem (Coddington and Levinson, 1955) establishes that all points are either fixed, on a cycle, or converge to one of these. Thus our simple 1-d picture is recovered once we relax the criteria of fixed points to *periodic points* (points on a cyclic orbit). Poincaré then introduced the notion of recurrence, with the goal of further generalising this behaviour.

Definition 5.1.3. Let φ be the flow of a dynamical system. A point x is *periodic* if there is some time $T > 0$ such that $\varphi(x, T) = x$. It is *Poincaré recurrent* if the orbit of x returns arbitrarily close to x infinitely often.

It is easy to see that Poincaré recurrent points generalise periodic ones which themselves generalise fixed points. These inclusions are proper as well. Unfortunately, our simple picture where all points converge to cycles or fixed points breaks down in three or more dimensions. Generalising this picture to arbitrarily dynamical systems was a central goal of the field for many years, which culminated in the notion of *chain recurrence* and the Fundamental Theorem of Dynamical Systems.

Definition 5.1.4. Let ϕ be the flow of a dynamical system on a compact metric space X. Let x and y be points in X. There is an ϵ-*chain* from x to y if there is a finite sequence of points $x_1, x_2, \ldots, x_n$ with $x = x_1$ and $y = x_n$, and times $t_1, \ldots, t_n \in [1, \infty)$ such that $\mathbf{d}(\phi^{t_i}(x_i), x_{i+1}) < \epsilon$.[2]

Definition 5.1.5. If there is an ϵ-chain from x to y for *all* $\epsilon > 0$ we say there is a *chain orbit* from x to y, and write $x \rightsquigarrow y$. We say x and y are *chain equivalent* if $x \rightsquigarrow y$ and $y \rightsquigarrow x$ and write $x \sim y$. If $x \sim x$ we say x is *chain recurrent*.

We call the relation $x \rightsquigarrow y$ the *chain order* on points, and say x *chain precedes* y (respectively, y *chain succeeds* x). This relation is transitive, though not reflexive, as in general not all points are chain recurrent. However, if we restrict our attention to those points which are chain recurrent then we do obtain a preorder. Every preorder has an associated partial order given by grouping points into equivalence classes. Each equivalence class under chain equivalence is called a *chain component*, and the chain components are thus naturally partially ordered. Given a chain recurrent point x, we denote its chain component by $[x]$. We denote the set of all chain recurrent points of ϕ by $\mathcal{R}(\phi)$.

Note that chain recurrence generalises Poincaré recurrence. The notion of an ϵ-chain also has a compelling computational interpretation. Suppose that Alice is computing the orbit of a point x under a flow φ. An adversary Bob wishes to convince her that a point y is in the orbit of x. If $x \rightsquigarrow y$, then Bob can always convince Alice that y is in the orbit of x only by manipulating the round-off error of her computation, regardless of how much precision Alice has available[3]. Thus, as limited computational beings we cannot distinguish between a 'true' orbit of the dynamical system and a chain orbit. Intuitively, the chain orbits are the orbits where infinitesimally small 'noise' is allowed.

While the notion of the chain orbit is compelling in its own right, its importance comes from its role in the Fundamental Theorem of Dynamical Systems (also called Conley's Theorem). For the statement, we follow the presentations of Alongi and Nelson (2007); Sandholm (2010).

Definition 5.1.6 (Conley (1978); Alongi and Nelson (2007))**.** Let X be a complete metric space, ϕ a flow on X and let $\mathcal{R}(\phi)$ be the set of chain recurrent points of X. A *complete Lyapunov function* for ϕ is a continuous function $L : X \to \mathbb{R}$ such that

1. $L(\phi^t(x))$ is a decreasing function of t for all $x \in X \setminus \mathcal{R}(\phi)$,

[2] Note that we require in the definition of an ϵ-chain that each time t_i is bounded below by 1. Without this bound all points could reach all others by ϵ-chains, giving a trivial relation. However the value of 1 is arbitrary; replacing this with any fixed constant with give the same relation $x \rightsquigarrow y$ (Alongi and Nelson, 2007)

[3] Assuming it is finite.

2. for $x, y \in \mathcal{R}(\phi)$, $x \sim y$ if and only if $L(x) = L(y)$, and
3. $L(\mathcal{R}(\phi))$ is nowhere dense.

Theorem 5.1.7 (Fundamental Theorem of Dynamical Systems (Conley, 1978))**.** *Let X be a compact metric space and ϕ a flow on X. Then ϕ has a complete Lyapunov function $L : X \to \mathbb{R}$.*

Informally, Conley's theorem states that every dynamical system splits into two parts: the chain components, which are the generalisation of fixed points, and the remaining points (sometimes called *chain transitive*), which are the connecting orbits between. All chain transitive points converge to a chain component.

5.1.3 Sink chain components: a dynamic extension of pure Nash

However, from the perspective of prediction we are not interested in *all* chain components. Recall that we desire *stability* rather than *stationarity*, and chain components are generalisations of fixed points, so some chain components are not stable. The ones which represent the 'long-run outcomes' of the dynamical system are those that are *minimal* in the chain order. As an example, a repelling fixed point is always a chain component, but it is not a reasonable 'long-run outcome' of the game.

Definition 5.1.8. A chain component is called a *sink chain component* if it is minimal in the partial order on chain components.

When minimal asymptotically stable sets exist, they are sink chain components. However this generalisation is the appropriate one from a computational and mathematical perspective. See Alongi and Nelson (2007); Papadimitriou and Piliouras (2018) for more discussion. By extension, *every PNE is a sink chain component*, and so they are the correct dynamic generalisation of PNEs.

5.1.4 A historical note

The field of dynamical systems was initially motivated by problems such as the three-body problem, and significant early work was done by Henri Poincaré (Poincaré, 1881). In working towards understanding the complex behaviour of dynamics, he developed the methods that would later become algebraic topology, by considering discrete invariants of the space (much as we do in this chapter). In this work he also proved a theorem which later turned out to be equivalent to Brouwer's fixed-point theorem.

Brouwer proved the full version of his famous fixed-point theorem in 1910, using the methods developed by Poincaré. Forty years later, his result would be used by Nash to prove the existence of Nash equilibria.

As pointed out by Papadimitriou and Piliouras (2019), while Brouwer's theorem was one of the most powerful topological theorems of Nash's era, we have available new results to which Nash did not have access (the Fundamental Theorem of Dynamical Systems). It is wise then to use this knowledge to develop a better notion of solution concept. More than that though, Brouwer's theorem *was itself* originally motivated by problems from dynamical systems—it therefore makes sense to use the century of additional dynamical systems research that has refined our understanding of concepts beyond fixed points.

5.2 Preference in the Replicator Dynamic

Recalling the criteria we laid out for a solution concept back in Chapter 3, we see that sink chain components, through their connection with the Fundamental Theorem of Dynamical Systems, are a solution concept that is **stable**. This is in contrast to the (mixed) Nash equilibrium, as we have established. On the other hand, they do not generally meet the other two criteria of **Existence** and **Computability**. First, chain components are complex topological objects which can be very hard to identify, and so there is no obvious tractable way to compute them. Second, there may in general be uncountably many chain components, and so *sink* chain components may not even exist.

Our goal in this chapter is to construct preference-invariant properties that capture the long-run outcome of dynamical systems. In this section we show that the sink equilibrium satisfies this criteria under the replicator dynamic. First, we prove that under the replicator, *sink chain components always exist*, a problem that was conjectured in Papadimitriou and Piliouras (2019). This is important, because we know a solution concept will not be seen as compelling without a global proof of existence. Second, we prove that every sink chain component contains a sink equilibrium (though they generically do not contain Nash equilibria), and so sink equilibria are a preference-invariant underlying sink chain components, and so meet our three desired criteria of **Stability**, **Existence** and **Computability**.

In addition, we prove several useful new theorems describing preference-invariant properties of the replicator flow, and apply these to the games in Figure 5.1. Finally, we conjecture some stronger theorems relating chain components and the response graph, and prove these in some special cases of preference-potential and preference-zero-sum games.

5.2.1 Dynamical systems

Before we can prove our results, we establish some notions from dynamical systems, in particular the concept of an *attractor*. These results can be found in numerous works on the subject; we will mostly use the presentation from Sandholm (2010). We have omitted proofs of well-known results (and provided references) but included proofs for those results which we could not easily find in the literature. As usual, if a result has neither a citation nor a proof in the text then the proof can be found in the Appendix.

In general, we will assume our dynamic defines a flow (Definition 2.3.1), as this is the case for the replicator. Such dynamics are *time-reversible*, so if ϕ is the flow then ϕ^{-1} is also a flow. This has a nice interpretation for the replicator on a game; if ϕ is the flow of the replicator on a game u, then ϕ^{-1} is the flow of the replicator on the reversed game $-u$.

Proposition 5.2.1 (Reversal)**.** Let u be a game, and ϕ the replicator flow on u. Then ϕ^{-1} is the flow on $-u$.

We also assume the space, denoted X, is a compact metric space. When this is a game dynamic this is the space of mixed profiles.

Attractors and repellors

Definition 5.2.2 (Sandholm (2010))**.** Let A be a compact, non-empty invariant set under a flow ϕ on a compact space X. If there is a neighbourhood U of A such that

$$\lim_{t\to\infty} \sup_{x\in U} \mathbf{d}(\phi^t(x), A) = 0$$

then we call A an *attractor*. An attractor of the time-reversed flow ϕ^{-t} we call a *repellor*.

There are many different, equivalent ways of defining an attractor.

Proposition 5.2.3 (Sandholm (2010))**.** The following are equivalent:

- A is an attractor;
- A is non-empty, compact, asymptotically stable and invariant under ϕ;
- There is a neighbourhood U of A where

$$A = \bigcap_{t\geq 0} \mathrm{cl}\left(\bigcup_{s\geq t} \phi^s(U)\right)$$

- There is an open set B with $\phi^T(\mathrm{cl}(B)) \subset B$ for some time $T > 0$, and $A = \bigcap_{t\geq 0} \phi^t(B)$.
- As above, but with the additional restriction that B be forward-invariant under ϕ. Such a B is called a *trapping region*.
- As above, but where $\phi^t(\mathrm{cl}(B)) \subset B$ *for all* $t \geq 0$.
- As above, but with a closed set K where $\phi^t(K) \subset \mathrm{int}\, K$.

Lemma 5.2.4. Any finite non-empty intersection of attractors is an attractor.

Definition 5.2.5. We write $\mathcal{A}_Y$ for the set of attractors of ϕ wholly containing a set Y. If $Y = y$ is a singleton we simplify notation by writing $\mathcal{A}_y$ rather than $\mathcal{A}_{\{y\}}$.

Each attractor has a *dual repellor*, defined in a natural way.

Lemma 5.2.6 (Sandholm (2010))**.** Let A be an attractor, with B a trapping region for A. Then $A^* := \bigcap_{t\leq 0} \phi^t(X \setminus B)$ is a repellor, which we call the *dual repellor of A*.

Attractors and repellors are dual in the sense that A^* is an attractor of the time-reversed flow ϕ^{-1}, and in this flow A is its dual repellor. Attractors are closely connected to concept of ϵ-chains that we introduced in the previous section.

Lemma 5.2.7 (Sandholm (2010))**.** If A is an attractor, and $x \in A$, then $x \rightsquigarrow y$ implies $y \in A$.

In English, this states that chain orbits never leave (but possibly enter) attractors. It follows that if $x \rightsquigarrow y$ then $\mathcal{A}_x \subseteq \mathcal{A}_y$. Dually, a chain orbit never enters repellors, but may leave them. Attractor-repellor pairs can characterise the entire chain recurrent set.

Theorem 5.2.8 (Sandholm (2010); Alongi and Nelson (2007); Conley (1978))**.**

$$\mathcal{R}(\phi) = \bigcap (A \cup A^*)$$

We can now prove a partial converse to Lemma 5.2.7.

Lemma 5.2.9. If x is chain recurrent, and $\mathcal{A}_x \subseteq \mathcal{A}_y$, then $x \rightsquigarrow y$.

It follows immediately that

Corollary 5.2.10. If x and y are chain recurrent points then $x \sim y$ if and only if $\mathcal{A}_x = \mathcal{A}_y$.

Dually, $x \sim y$ if they are contained in all the same repellors. This Corollary is an important concrete test for when two points are chain equivalent, and we make significant use of it.

5.2.2 The replicator dynamic

Aside from being the most well-studied game dynamic, the replicator on a game has some specific properties as a dynamical system which we will use in our results. This is in contrast with the results in the previous section which apply to general dynamical systems. The first important property is that it is subgame-invariant—that is, the support of a point is invariant along its orbit, so mixed profiles which begin in a subgame always remain in that subgame. Further, the orbit of a point in a subgame is *independent of the rest of the game*, so the replicator flow on a subgame is identical to the replicator flow on a game which has the structure of that subgame. Formally,

Theorem 5.2.11 (Subgame-independence of the replicator)**.** *Let X be the mixed-strategy space of a game u, and Y be the mixed-strategy space of a subgame u' of u. The flow $\phi_u|_Y$ of the replicator on u restricted to Y is identical to $\phi_{u'}$, the replicator flow on u'.*

This theorem makes it easy to analyse the flow of the replicator on a subgame of a game using induction on subgames. In particular, an arc of the response graph can be considered as a subgame of a game where one player has two strategies and all other players have one strategy. In this case the replicator flow simplifies to the 1-dimensional logistic equation, with the flow in the direction of higher payoff. Using this result, we can state the first connection between the response graph and ϵ-chains.

Lemma 5.2.12 (Omidshafiei et al. (2019))**.** If v and w are pure profiles, and there is a path in the response graph from v to w, then $v \rightsquigarrow w$ under the replicator.

This is easy to prove, because we know how the replicator flows towards higher payoff along a single arc. Upon nearing the pure profile at the end of the arc, an ϵ-jump will take us to the subsequent arc of the path. It follows that:

Corollary 5.2.13 (Omidshafiei et al. (2019))**.** If H is a set of pure profiles which make up a connected component of the response graph, then all points in H are contained in exactly one chain component. It is therefore well-defined to write $[H]$ for this component.

All points on 1-dimensional subspaces (arcs) between nodes in H are also contained in $[H]$. In fact, if all the pure profiles in any subgame Y are contained in the same connected component H, then all mixed strategies with support Y are in $[H]$—we prove this new result in Theorem 5.2.21, but to do so we need to understand attractors in the replicator.

Volume preservation and interior attractors

The replicator dynamic has another important property: after a differentiable change of variables, it preserves the volume of all sets on the interior of a game (Sandholm, 2010; Vlatakis-Gkaragkounis et al., 2020). This means that no asymptotically stable set can exist entirely in the interior of the state space.

Theorem 5.2.14 (Vlatakis-Gkaragkounis et al. (2020))**.** *Let G be a game, with X the mixed strategy space and ϕ the flow of the replicator dynamic. Then there are no attractors contained in the interior of X.*

This does not mean that there are no attractors at all—only that each must contain at least some points on the boundary of X. Inductive repetition of this theorem combined with the flow along edges gives us:

Lemma 5.2.15. Every attractor in ϕ contains a lower set of nodes in the response graph.

Proof. Consider an attractor A of ϕ. By Theorem 5.2.14, A must contain some points which are on the boundary of X. Every point on the boundary is contained in some proper subgame Y of X. Now consider any such subgame Y, and let A' be the intersection of A with Y. Since subgames are invariant under the replicator (Theorem 5.2.11), A' must be an invariant set. If U is the trapping region for A, and U' is the intersection of U with Y, then again using subgame-invariance we find that A' is an attractor in Y with U' its trapping region.

By Theorem 5.2.11, $\phi|_Y$ is identical to the flow on the game u' given by Y. However, Theorem 5.2.14 also applies to this game, and so u' has no interior attractors, implying A' intersects the boundary of Y, which again is a yet smaller subgame. By induction, we find that every attractor contains a pure profile.

Finally by Lemma 5.2.12 and Lemma 5.2.7, all graph nodes reachable from this pure profile in the response graph are also in this attractor. The attractor therefore contains a set of nodes in the response graph that has no outgoing arcs. □

This result appears as Theorem 3 in Vlatakis-Gkaragkounis et al. (2020). It follows that since every lower set of nodes in the response graph contains a sink equilibrium, *every attractor of the replicator contains a sink equilibrium.*

5.2.3 The existence theorem for sink chain components

In this section we prove the following conjecture.

Conjecture 5.2.16 (Papadimitriou and Piliouras (2019))**.** There is always at least one sink chain component for games under the replicator dynamic.

In fact, we prove the following stronger statement which shows the strong relationship between sink chain components in the replicator and the response graph. Not only do sink chain components exist, but all chain recurrent points in the game can reach one via a chain orbit. Combining this with the Fundamental Theorem of Dynamical Systems, which states that all points converge to chain recurrent ones, we find that all points under the replicator 'end up at' a sink chain component.

Theorem 5.2.17 (Existence of sink chain components)**.** *Let ϕ be the flow of the replicator dynamic on a game, and let $x \in X$ be a chain recurrent point. Then there exists a pure profile y with $x \rightsquigarrow y$ where y is in a sink equilibrium H contained in a sink chain component $[H]$. Dually, there exists a pure profile z with $z \rightsquigarrow x$ where z is in a source connected component K contained in a source chain component $[K]$.*

Proof. First we define a preorder on the set of sink equilibria. If K and H are graph sinks, we say $K \preceq H$ if and only if $\mathcal{A}_K \supseteq \mathcal{A}_H$. This is clearly a preorder because it inherits reflexivity and transitivity. By finiteness, this order has at least one minimal element, and we denote the minimal elements by $M_1, \ldots, M_m$. What we will show is that the sink chain components are exactly the chain components $[M_1], \ldots, [M_m]$ (which may not all be distinct).

(*Claim:* Every chain recurrent point has a chain orbit to some M_i.)

Consider some chain recurrent point x. Now define

$$\mathcal{M}_x := \bigcap_{A \in \mathcal{A}_x} \{\text{sink equilibria contained in } A\}$$

Recall that by Lemma 5.2.15 each such A contains a sink equilibrium. Now suppose for contradiction that $\mathcal{M}_x$ is empty. Then for each sink equilibrium $H_1, \ldots, H_n$ in our game there exists attractors $B_1, \ldots, B_n$, each containing x, such that B_i does not contain H_i. But then by Lemma 5.2.4 $\bigcap_{i=1}^{n} B_i$ is an attractor, which by Lemma 5.2.15 contains at least one sink equilibrium—but this is a contradiction of our construction. Hence $\mathcal{M}_x$ is not empty. Let K be some sink in $\mathcal{M}_x$. By definition of $\mathcal{M}_x$, $\mathcal{A}_x \subseteq \mathcal{A}_K$.

There must exist some minimal sink M_i with $M_i \preceq K$ and so $\mathcal{A}_x \subseteq \mathcal{A}_K \subseteq \mathcal{A}_{M_i}$, and by Lemma 5.2.9 there is a chain orbit $x \rightsquigarrow y$ from x to every node y in M_i.

(*Claim:* The $[M_i]$s are sink chain components.)

If y is in M_i and there is a chain orbit $y \rightsquigarrow z$, then by the above argument there is some M_j with $z \rightsquigarrow M_j$. But then by transitivity there is a flow $M_i \rightsquigarrow M_j$, which by Lemma 5.2.7 implies that $\mathcal{A}_{M_i} \subseteq \mathcal{A}_{M_j}$, but since each M_i is minimal in this order we must have $\mathcal{A}_{M_i} = \mathcal{A}_{M_j}$. It follows from Corollary 5.2.10 that $[M_i] = [M_j]$. Hence $z \in [M_i]$ and so $[M_i]$ is a sink chain component. □

It follows immediately that sink chain components must always exist, and that they must always contain sink equilibria, and likewise source chain components exist and contain source connected components of the response graph. Despite this proof involving a complex topological object, our proof again made use of the combinatorial structure of preference games, and made no references to utility. In the next section we will explore the power of this result.

5.2.4 Sink chain components and sink equilibria

In this section we use our results to deepen our understanding of the relationship between chain components of the replicator flow and connected components of the response graph. In particular, we give the formal reasoning behind Figure 5.1. We know already from the previous section that sink chain components contain sink equilibria, and since distinct chain components are disjoint, each sink equilibrium is in at most one. These are powerful results. The replicator dynamic has access to a great deal of additional information beyond simply the response graph, because it depends on the relative utilities. Thus our result describes the sink and source chain components of the replicator over *the entire* preference-equivalence class of games.

We naturally wish to know how tight these results are, and whether the sink chain component is ever much larger than a sink connected component which it contains. In particular, when is there a one-to-one correspondence between sink chain components and sink equilibria? We prove some theorems establishing stronger relationships for some special cases. First, using our proof we make concrete the criteria under which this is true.

Proposition 5.2.18. Let H be a sink equilibrium. Then $[H]$ is a sink chain component if and only if there is no sink equilibrium K with $\mathcal{A}_K \supset \mathcal{A}_H$.

Proposition 5.2.19. Let S be a sink chain component containing a sink equilibrium H. Then H is the unique sink equilibrium in S if and only if there is no sink equilibrium K with $\mathcal{A}_K = \mathcal{A}_H$.

Together, we find that there is a one-to-one correspondence if and only if the attractor sets of distinct sink equilibria are always incomparable. We conjecture that there is always such a one-to-one correspondence (see Section 5.2.5). First we show a stronger lower bound on the points contained in a sink chain component.

Definition 5.2.20. Let W be a collection of pure profiles in a game u. The *content* of W is the set of all mixed strategy profiles x where all pure profiles in the support of x are in W.

The content is always a topological subspace of the mixed strategy space. If W is a connected component of the response graph, then the content of W is topologically connected. We now show that if W is connected in the graph, the content of W is always a subset of $[W]$.

Theorem 5.2.21. *Let H be a connected component of the response graph. Then under the replicator, the content of H is contained in $[H]$.*

As a corollary, whenever the response graph is strongly connected the *entire mixed-strategy space is chain recurrent*. An example is the Matching Pennies game (Figure 5.1d). Now we consider when an upper bound on the sink chain component is possible.

Definition 5.2.22. A subgraph of the response graph is *attracting* if there are no paths out of it. Dually it is *repelling* if there are no paths into it.

With this definition, sink equilibria are minimal attracting subgraphs. It is hard to determine in general if these are also attracting in a dynamic sense, but when they are *subgames* this is always the case.

Lemma 5.2.23. If Y is an attracting subgame, then it is an attractor under the replicator. Dually, repelling subgames are repellors.

Pure NEs are attracting singleton subgames—by this lemma, they are always attractors under the replicator. Combining this result with Theorem 5.2.21 and Proposition 5.2.18 gives us:

Corollary 5.2.24. If a sink equilibrium H is a subgame Y, then $[H]$ is exactly the content of H.

Proof. Sink equilibria are attracting, so H is an attracting subgame, and thus Y is an attractor under the replicator (Lemma 5.2.23). By Theorem 5.2.21, all points in the content of Y are in $[H]$, and so are chain equivalent. However by Proposition 5.2.18 this containment is only strict if there is another sink equilibrium K contained in all attractors containing H—this is impossible because Y is itself an attractor (in fact a minimal attractor). □

Preference-zero-sum and preference-potential games

It is useful to consider some special cases of games. The natural place to begin are two-player zero-sum and potential games. In fact, because we are interested in the structure of the response graph, we consider preference-zero-sum and preference-potential games. Preference-potential games can be particularly easily handled using our results.

Theorem 5.2.25 (Sink chain components in preference-potential games)**.** *In a preference-potential game under the replicator, the sink chain components are exactly* $\{p_1\}, \ldots, \{p_m\}$ *where* $p_1, \ldots, p_m$ *are the pure Nash equilibria.*

Proof. Each PNE is an attracting subgame, so by Corollary 5.2.24 the associated sink chain component is precisely that profile. Since every sink chain component contains a sink equilibrium, and thus a PNE, these are precisely the sink chain components. □

Recalling the Coordination game, preference-potential games can have mixed Nash equilibrium, but these are *never in the sink chain components*, while sink equilibria (the PNEs) are always the sink chain components. While such results are known for potential games, we point out that this result applies to all preference-potential games, and indeed any game where every sink equilibrium is a subgame (this includes, for instance, the Inner Diamond game, and any *weakly acyclic games* (Young, 1993)).

Preference-zero-sum games are trickier to analyse, but we can make use of Lemma 4.2.8.

Lemma 5.2.26. Every preference-zero-sum game has precisely one sink chain component.

Proof. By Lemma 4.2.8, every preference-zero-sum game has one sink equilibrium and thus one sink chain component (Theorem 5.2.17). □

Recall again that this class of games is much larger than zero-sum games. It follows immediately that there is a one-to-one correspondence between sink chain components and sink equilibria in both preference-zero-sum and preference-potential games.

Applications

Understanding convergence of specific games under the replicator is difficult. It is often the topic of entire papers, such as Papadimitriou and Piliouras (2018), which characterises the chain recurrent set for zero-sum and potential games.

As another example, Kleinberg et al. (2011) focuses on establishing convergence to the best-response cycle in Circular Mismatching Pennies, as we discussed in the introduction, in the special case when a player receives a payoff of 0 for matching the player to their left, of payoff of 1 for playing H when the adjacent player chose T, and a payoff of M for playing T when the adjacent player chose H. We extend these results by describing the structure of the sink and source chain components in all games with the response graph of Circular Mismatching Pennies.

In this section we analyse the replicator on the games we have discussed in this thesis, and summarise the results in Figure 5.1. Some of these games have been analysed before[4], but all our results will use only the preference structure, and so apply to the *entire preference-equivalence class* of these games.

- **SD and DD and CO**: As preference-potential games, Theorem 5.2.25 tells us the sink chain components are precisely the pure Nash equilibria, for any game with this preference structure.

- **Matching Pennies**: As a strongly connected graph, the entire mixed-strategy space is always a single chain component (Theorem 5.2.21).

- **Circular Matching and Mismatching Pennies**: All sink equilibria in Circular Matching Pennies are PNEs, so these are precisely the sink chain components (Theorem 5.2.25). In Circular Mismatching Pennies, both sources are repellors (Lemma 5.2.23), and so are exactly the two source chain components. By Lemma 5.2.26 there is a unique sink chain component containing the nodes and arcs on the cycle (this is the content of the sink equilibrium).

  In the standard utility form (Figure 1.1b), the sink is an attractor, so is a minimal attractor and so the sink chain component is precisely the content of the sink equilibrium. In Kleinberg et al. (2011), the authors give a range of other payoff values for which the sink is also an attractor, and we conjecture that the sink is an attractor for every game preference-equivalent to Circular Mismatching Pennies.

- **Inner and Outer Diamond (Figure 5.1)**: The Inner Diamond has a pure Nash equilibria which is the unique sink chain component. The Outer Diamond is preference zero-sum, and so has a unique sink chain component (Lemma 5.2.26) containing all profiles other than the source (which is a repellor, by Lemma 5.2.23), and the content of these profiles (Theorem 5.2.21).

  No game with the preference structure of the Outer Diamond ever has an interior Nash equilibrium[5], and so there is never an interior fixed point of the replicator. For true zero-sum games with the structure of the Outer Diamond, we can use this fact and a result of Papadimitriou and Piliouras (2018) to prove that all interior trajectories converge to the content of the sink equilibrium, establishing in that case that the sink chain component is exactly this set.

[4] As an example, the result that every point in the Matching Pennies game is chain recurrent under the replicator is presented initially in Papadimitriou and Piliouras (2016, 2018), and reproved using different methods in Balduzzi et al. (2018). Our proof uses only the response graph structure.

[5] This can be verified by considering the kernel of the matrix of arc labels.

5.2.5 Conjectures

These results have allowed for significant insight, but many questions remain. Here we list some of the most important problems.

In all the cases above we found that not only does each sink chain component contain a sink equilibrium, but in fact exactly one, and each sink equilibrium is contained in a sink component. We conjecture that both are true.

Conjecture 5.2.27. Every sink equilibrium is contained in a sink chain component.

Conjecture 5.2.28. Every sink chain component contains exactly one sink equilibrium.

Propositions 5.2.18 and 5.2.19 express these statements in terms of the set of attractors containing each sink equilibrium. If these statements are true, it would imply there is a one-to-one correspondence between sink chain components and sink equilibria, which would make sink equilibria a true stand-in for the sink chain components, and would be a very interesting result. Of course, even if such a one-to-one correspondence existed, it wouldn't necessarily specify precisely which points are in sink chain components. The following stronger conjecture does do this:

Conjecture 5.2.29. The sink chain components of a game are exactly the content of the sink equilibria.

This would also imply that the sink chain components of the replicator are completely preference-invariant, which would be an important result. Another strong conjecture, generalising the one-to-one correspondence, is the following:

Conjecture 5.2.30. The reachability order embeds in the chain order.

By embeds we mean that there is an order-isomorphism between the reachability order and a suborder of the chain order. This would mean that the reachability order would define the key parts of the chain order, including its minimal and maximal elements (Theorem 5.2.17), thus providing a complete combinatorial representation of the chain order on a game. This is a strong conjecture, and may be false, but we do not currently know of a counterexample. These are important questions for future work.

5.3 Markov Dynamics

In this section we examine a simple but important class of discrete-time dynamics: Markov chains over the response graph of the game. This dynamic has recently been used to rank strategies in machine learning (Omidshafiei et al., 2019). This model of strategic update also has a simple conceptual motivation. Suppose that we have a single profile currently being played, but at a random time any player (currently playing some strategy s) may randomly deviate to any strategy t which is preferred to s given the current antiprofile, and the probably of each deviation is fixed given s, t and the antiprofile. If the probability of any player deviating at any given time is small (this is known as the 'small mutation-rate assumption') then we obtain a Markov chain M on the set of profiles where $M_{p,q}$ is positive if and only if either $p = q$ or there is an arc $p \longrightarrow q$ in the response graph. We do not assume any further criteria on M. Any such dynamic we call a *Markov dynamic*.

The 'evolution' of this dynamic is given by iterating the matrix M. If π_0 is our initial distribution over the profiles (which are the states of the Markov chain), then the distributions at each time step are $\pi_1 := \pi_0 M$, $\pi_2 := \pi_1 M = \pi_0 M^2$...[6] Given some initial distribution π_0 we wish to understand the long-term behaviour of the players in our game. As an example, in some cases a Markov chain converges to a *stationary distribution* π (which is one where $\pi M = \pi$), where π does not depend on the initial distribution π_0, and so represents the probability of each profile occurring in the long run.

In this section we show that the long-run behaviour of Markov dynamics on any game is a preference-invariant.

Theorem 5.3.1 (Markov dynamics are preference-invariant in the long run)**.** *Let M be a Markov dynamic on a game u, and let $S_1, \dots, S_m$ be the sinks of the response graph of u. Then*

1. *the weight on all non-sink nodes converges to zero;*
2. *for each sink S_i there is a unique stationary distribution σ_i whose support is S_i;*
3. *a distribution π is stationary if and only if it is a convex combination of $\sigma_1, \dots, \sigma_m$;*
4. *the dynamic converges to a stationary distribution for any initial distribution π_0 if every sink (as a subgraph) is aperiodic (the gcd of the length of all cycles is one).*

This says that all the key long-run properties of a Markov dynamics are preference-invariant. This includes whether the chain will converge, the dimension of the simplex of stationary distributions, and the support of a point in the limit as the iterations go to infinity. This is regardless of how we actually define our chain, and so in particular does not depend on the values of the utility function u. The specific choice of transition probabilities will determine the vectors σ_i, and these combined with our prior distribution will determine which specific distribution we converge to, assuming that the chain does converge.

We give a self-contained[7] proof of Theorem 5.3.1. Fundamentally, the proof is an extension of the Perron-Frobenius theorem, a famous theorem in linear algebra with very significant application to the theory of Markov chains.

While much of content of this theorem has been proved before (for instance, parts (2) and (3) are stated in (Sandholm, 2010, Chapter 11), and part (1) is stated in Candogan et al. (2013)), the connection with the graph structure is rarely emphasised, and in most discussion of Markov chains the graph is assumed to be strongly connected (that is, the

[6] Markov chains are row-stochastic by convention, so we multiply from the right.

[7] With the exception of the Perron-Frobenius theorem, which we do not prove. Proofs of this theorem can be found in many places, such as (Brualdi and Cvetkovic, 2008).

chain is *irreducible*). For the theory of preference games this result shows the importance of the underlying combinatorial structure.

Chapter 6

Related Work

Generally, references to the literature can be found throughout. In this chapter we give a more thorough summary of related ideas and methods.

Although we know of no prior work which has proposed the response graph as a model of a game, it has been used to varying degrees to analyse games, particularly in the recent algorithmic game theory literature (Roughgarden, 2010; Omidshafiei et al., 2020). Random walks on the response graph are sometimes called *Nash dynamics* (Fabrikant and Papadimitriou, 2008; Mirrokni and Skopalik, 2009), and they are defined as Markov chains on the response graph, which we studied in Section 5.3. Omidshafiei et al. (2019); Candogan et al. (2013) both make use of Nash dynamics, with Candogan et al. (2013) showing convergence to sinks. The labelled game graph forms a key part of the decompositions in Candogan et al. (2011); Hwang and Rey-Bellet (2020b). The sink equilibrium first appeared in Goemans et al. (2005), though this was define in terms of the best-response graph. It has received significant interest since in algorithmic game theory (Fabrikant and Papadimitriou, 2008; Roughgarden, 2010).

The most similar work to ours is Papadimitriou and Piliouras (2019), which advocates for a reinterpretation of games on the basis of chain components of the dynamics, and suggests that sink equilibria could present an analogue for sink chain components. Although the use of chains to analyse games extends as far back as Akin and Losert (1984) and more recently Benaïm et al. (2012), the explicit argument for the chain components approach to game theory begins in Papadimitriou and Piliouras (2018). This argument has been developed by subsequent papers, including Papadimitriou and Piliouras (2019); Omidshafiei et al. (2019, 2020); Milionis et al. (2022). A nearly identical set of three criteria to ours (Section 3.4) are laid out for a predictive solution concept in Fabrikant and Papadimitriou (2008).

General arguments against the Nash equilibria have a longer history (Harsanyi et al., 1988; Hofbauer and Sigmund, 2003). These have strengthened since the discovery that

Nash equilibria cannot be efficiently computed (Daskalakis et al., 2009). The increasing importance of game theory in machine learning and computer science has accelerated the move away from Nash to solutions which can better make predictions (Kleinberg et al., 2011; Milionis et al., 2022). Hart and Mas-Colell (2003); Benaïm et al. (2012); Milionis et al. (2022) all present forms of impossibility theorems for convergence of dynamics to Nash in general. Kleinberg et al. (2011) examines the Circular Mismatching Pennies game (under the name *Asymmetric Cyclic Matching Pennies*) and showed that the sink equilibrium is both the dynamic outcome under the replicator and can have significantly higher social welfare than the Nash equilibrium.

6.1 Foundations

Within game theory, the axiomatic study of utility in games begins with Von Neumann and Morgenstern (1944). Extensive discussion of the relation between the axioms of decision theory and game theory can be found in Myerson (1997).

The original solution concept for a two-player game is called a *minimax strategy*, and was introduced in Von Neumann and Morgenstern. This is a strategy that minimises the opponents maximum gain; the authors showed that these strategies form an equilibrium. This notion was generalised by Nash, who introduced what is now known as the *Nash equilibrium*. An extensive literature exists on special cases and extensions of the Nash equilibrium. See, for instance Myerson (1978), or Myerson (1997); Fudenberg and Tirole (1991) for general references.

The two most important equivalence relations on games in this paper are strategic and ordinal equivalence. Both are discussed in different terminology in Mertens (2004), which also gives an example of their relevant impacts on a game which parallels our own discussion of the Prisoner's Dilemma, though without the Independence Theorem.

Strategic equivalence appears as best-response equivalence in Myerson (1997). A thorough investigation is given in Morris and Ui (2004), with a focus on applications to potential games. Work on ordinal games has included studying the location of Pareto sets (Barany et al., 1992) and identifying appropriate generalisations of Nash to this setting (Cruz and Simaan, 2000; Ben Amor et al., 2017). Ben Amor et al. (2017) highlight the particular importance of ordinality in algorithmic game theory. Durieu et al. (2008) present a proof, similar to our own, that a game is *best-response potential* if and only if it is acyclic.

6.2 Zero-sum and Potential games

The study of the relationship between zero-sum and potential games begins seemingly with Candogan et al. (2011), which makes use of strategic equivalence, represented on the labelled response graph, to decompose a game into potential, non-strategic and harmonic components. This was followed by Hwang and Rey-Bellet (2020b), where a decomposition precisely in terms of strategically-zero-sum and strategically-potential games, noting in the two-player case the connection between these. This is further discussed in Hwang and Rey-Bellet (2020a), which discusses methods for identifying such games. However, being defined up to strategic equivalence, these papers do not identify the associated preference structures.

The recent paper Balduzzi et al. (2018) gives a discussion of the Hamiltonian–potential duality, and uses this for a alternate proof that all points in Matching Pennies are chain recurrent. Again, the graph structure is not discussed.

6.3 Dynamics

There are a number of similar analyses of the behaviour of the replicator and related dynamics (such as *Follow-the-Regularised Leader* (FTRL)) in games. The study of Poincaré recurrent points under the replicator begins with Piliouras and Shamma (2014), who analysed zero-sum games and showed the interior is Poincaré recurrent when there is an interior Nash equilibrium. Building on this, Papadimitriou and Piliouras (2016, 2018); Mertikopoulos et al. (2018) consider the chain recurrent sets of games under the replicator, with a particular focus on zero-sum and potential games. These papers describe the chain recurrent sets of the Matching Pennies and Coordination games. Boone and Piliouras (2019) generalise these results to single-population games under certain conditions. Similarly, Balduzzi et al. (2018) discusses *Hamiltonian* (which are generalisations of zero-sum) and potential games, and provides a dynamic intended to converge to Nash in both cases.

Papadimitriou and Piliouras (2019, 2018) outline the importance of understanding the chain recurrent sets of the replicator, which motivates our own findings. Vlatakis-Gkaragkounis et al. (2020) demonstrated that pure NEs are stable under the replicator, while mixed NEs are not. This paper also proves that the replicator has no interior attractors, and argues that developments of these results are an important direction for future work. A similar theorem is also proved in Omidshafiei et al. (2019), which uses both the replicator and Markov dynamics to analyse the evolutionary strength of strategies in complex games.

In an analogous way to our work, Czechowski and Piliouras (2021) analyses the replicator in succinct games purely in terms of the combinatorial structure defining that game. For certain special structures, the game can be shown to satisfy dynamic properties similar to those guaranteed in two dimensions by the Poincaré-Bendixson theorem.

Chapter 7

Concluding Remarks

7.1 Conclusion

In this thesis we presented the case for preference games as a model of game theory. We explored the relationhsip with preference games and sink equilibria and utility games and and Nash equilibria. We presented an axiomatic derivation of preference games and an intuitive argument for why they are good models. We then proved a number of important results using preference games, including the Independence Theorem (3.3.9) and Reconstruction Theorem (3.3.11). In Chaopter 4 we showed how preference games provided new insight into zero-sum games and potential games, and then proved Theorem 4.2.9, which demonstrated the role of MP and CO in two-player games and the Zero-sum–Potential–Dominance Theorem (4.2.12). In Chapter 5 we proved the existence of sink chain components under the replicator dynamic (Theorem 5.2.17), and gave a number of results describing how they relate to sink equilibria. We used these results to analyse the long-run behaviour of the replicator dynamic on a number of example games.

7.2 Future Work

Preference games have shown themselves to be a surprisingly rich and well-motivated strategic model. As this monograph is currently the only work on the subject, further study seems clearly warranted. Game theory is a large and diverse field, while in this thesis we considered only finite normal-form games with perfect information. Extending the ideas of preference to a greater classes of games is a significant direction for future research. Further, extending both the structural results of Chapter 4 and strengthening the evolutionary results of Chapter 5 would further motivate the preference games approach. In addition to these generic goals, there are some more specific questions:

- (Extensive-form games): How do we extend the preference games approach to extensive-form games (games with multiple turns)? This seems conceptually reasonable but has not been explored specifically.
- (Asymmetric information): A key part of modern game theory is the study of games with incomplete or asymmetric information. How should these concepts be represented in a preference game?
- ("Moves by Nature"): Some games are inherently probabilistic, a concept which in the game theory is sometimes modelled as strategies played by "Nature". How should this be incorporated into a preference game?
- (Better- or best-response?): Our model of a preference game was defined by the response graph. There is another graph associated to a game, which is known as the *best-response graph*. This is a subgraph of the response graph with the same node set, where only arcs that are 'best-responses' for that player are retained (Goemans et al., 2005). Pure Nash equilibria are also the singleton sinks of the best-response graph, but we do not know which other of our results also hold in this more restricted setting, and we believe that many no longer hold. Exploring this alternative model would be an interesting direction for study.
- (Other than the replicator): Many of our dynamical systems results were specific to the replicator, but some can possibly be generalised. For instance, for what class of dynamics do sink chain components exist and contain sink equilibria in all games?
- (Dynamics conjectures): We conjectured in Section 5.2.5 some stronger versions of our results relating sink chain components and sink equilibria. In particular, we conjectured there is a one-to-one correspondence between these classes. A proof of this result would be a significant advance.

We believe that these lines of inquiry will lead to a new understanding of strategic interactions.

Appendix A

Appendix: Proofs

A.1 Chapter 1

For several of the following results, an alternative characterisation of strategic equivalence will be useful, along with the notion of the *relative utility*.

Definition A.1.1. If u is a game, then the *relative utility function* for player i is the function $\Delta u_i : S_i^2 \times \bar{Z}_{-i} \to \mathbb{R}$ defined by $\Delta u_i(s,t;\bar{q}) := u(s \curvearrowleft \bar{q})_i - u(t \curvearrowleft \bar{q})_i$. Similarly, given a mixed antiprofile $\bar{x}$ we define the *relative expected utility* by

$$\Delta \mathbb{U}_i(s,t;\bar{x}) = \sum_{\bar{q}\in\bar{Z}_{-i}} \bar{x}^1_{\bar{q}_1} \dots \bar{x}^{N-1}_{\bar{q}_{N-1}} \Delta u(s,t;\bar{q})_i = \mathbb{U}(e_s \curvearrowleft \bar{x})_i - \mathbb{U}(e_t \curvearrowleft \bar{x})_i$$

The last equality can be justified by expanding the definition.

Theorem A.1.2 (Instability)**.** *Let p be a mixed Nash equilibrium in a strict game. For any player i, in* every neighbourhood *of p, there is a point p' where player i's unique best response is a pure strategy.*

Proof of Theorem 1.1.1. Let Y be the subgame that is the support of p. Choose any antiprofile $\bar{q}$ in $\bar{Y}$, and let $s_1, \dots, s_n$ be the order in which player i prefers (strictly) their strategies in this antiprofile. At p we have $\Delta\mathbb{U}(s,t;\bar{p})_i = 0$ for all pairs of strategies s and t in Y_i. Now consider the path $\alpha\bar{p} + (1-\alpha)\bar{q}$ in $\bar{Y}$, parameterised by $\alpha \in [0,1]$. For a fixed s and t in Y_i:

$$\Delta\mathbb{U}(s,t;\alpha\bar{p} + (1-\alpha)\bar{q})_i = \sum_{\bar{w}\in\bar{Y}} \Delta u(s,t;\bar{w})_i \prod_{j\neq i} (\alpha\bar{p}_{j,\bar{w}} + (1-\alpha)\bar{q}_{j\bar{w}})$$

which is a polynomial of degree $N-1$ in the parameter α. We know that this polynomial has a zero at $\bar{p}$. As a polynomial, either all roots are isolated or it is a constant, but it cannot be a constant because when $\alpha = 1$ we obtain $\Delta u(s, t; \bar{q})_i$, which is non-zero by strictness. Thus for any infinitesimal positive α we must have the expected utilities of all strategies be unequal (since the above is true for all pairs of strategies s and t), so there is a best-response which is a pure strategy. □

A.2 Chapter 3

In this section we give proofs of our claimed results from Chapter 3. In the following proofs we will denote ordinal-, strategic- and preference-equivalence between two games u and v by $u \sim_O v$, $u \sim_S v$ and $u \sim_P v$ respectively.

Strategic equivalence can be expressed in terms of the relative utility function.

Lemma A.2.1. $u \sim_S v$ if and only if $\Delta u_i = a_i \Delta v_i$, for some $a_i > 0$ for all players i.

Proof. In one direction, $u \sim_S v \implies u(s \curvearrowright \bar{q})_i = a_i v(s \curvearrowright \bar{q})_i + b_{i,\bar{q}}$ for all players i, $s \in S_i$ and $\bar{q} \in \bar{Z}_{-i}$. This respectively implies, for any $t \in S_i$, $u(s \curvearrowright \bar{q})_i - u(t \curvearrowright \bar{q}) = a_i v(s \curvearrowright \bar{q})_i + b_{i,\bar{q}} - a_i v(t \curvearrowright \bar{q})_i + b_{i,\bar{q}}$ giving $\Delta u_i(s, t; \bar{q}) = a_i \Delta v_i(s, t; \bar{q})$.

For the converse, suppose $\Delta u_i(s, t; \bar{q}) = a_i \Delta v_i(s, t; \bar{q})$ for all i, $s, t \in S_i$ and $\bar{q} \in \bar{Z}_{-i}$. Then

$$
\begin{aligned}
& u(s \curvearrowright \bar{q})_i - u(t \curvearrowright \bar{q}) = a_i v(s \curvearrowright \bar{q})_i - a_i v(t \curvearrowright \bar{q})_i \\
\Leftrightarrow\ & u(s \curvearrowright \bar{q})_i - a_i v(s \curvearrowright \bar{q})_i = u(t \curvearrowright \bar{q}) - a_i v(t \curvearrowright \bar{q})_i
\end{aligned}
$$

However, for fixed i and $\bar{q}$, this equality is true for any s and t in S_i. Consequently, for any $s \in S_i$, $u(s \curvearrowright \bar{q})_i - a_i v(s \curvearrowright \bar{q})_i = b_{i,\bar{q}}$ for some constant $b_{i,\bar{q}}$ which does not depend on s. Thus $u(s \curvearrowright \bar{q})_i = a_i v(s \curvearrowright \bar{q})_i + b_{i,\bar{q}}$. □

Theorem A.2.2 (Theorem 3.3.4)**.** *Two games u and v are strategically equivalent if and only if their labelled response graphs are equal up to rescaling by a positive constant for each player.*

Proof. By the above lemma, $u \sim_S v$ if and only if $\Delta u_i = a_i \Delta v_i$. The direction of an arc $s \curvearrowright \bar{q} \xrightarrow{x} t \curvearrowright \bar{q}$ is given by the sign of the relative utility function, and its label x is given by the absolute value, and so the labelled response graphs of u and v differ only by rescaling by the constants a_i. □

Proposition A.2.3 (Proposition 3.3.7)**.** Two games u and v are preference-equivalent if and only if their underlying preference games are equal.

Proof. Let p_i^u and p_i^v be the underlying preference functions of u and v for a given player i. Then

$$\begin{aligned}
u \sim_P v &\Leftrightarrow \forall i,\ \ s,t \in S_i, \bar{q} \in \bar{Z}_{-i}\,(u(s \curvearrowright \bar{q})_i \geq u(t \curvearrowright \bar{q})_i \Leftrightarrow v(s \curvearrowright \bar{q})_i \geq v(t \curvearrowright \bar{q})_i)\\
&\Leftrightarrow \forall i,\ \ s,t \in S_i, \bar{q} \in \bar{Z}_{-i}\,(\Delta u_i(s,t;\bar{q}) \geq 0 \Leftrightarrow \Delta v_i(s,t;\bar{q}) \geq 0)\\
&\Leftrightarrow \forall i, \bar{q} \in \bar{Z}_{-i}\,(\forall s,t \in S_i(s \leq t \text{ in } p_i^u(\bar{q}) \Leftrightarrow s \leq t \text{ in } p_i^v(\bar{q}))\\
&\Leftrightarrow \forall i, \bar{q}\,(p_i^u(\bar{q}) = p_i^v(\bar{q}))\\
&\Leftrightarrow \forall i, p_i^u = p_i^v
\end{aligned}$$

□

Theorem A.2.4 (Theorem 3.3.8)**.** *Two game u and v are preference-equivalent if and only if their response graphs are equal.*

Proof. Assuming u and v have the same player and strategy sets, the response graphs G_u and G_v can only differ in the orientation of their arcs. If q and r are profiles, then $q \longrightarrow r \ \in G_u$ iff $q = s \curvearrowright^i \bar{q}$, $r = t \curvearrowright^i \bar{q}$ for some player i, $s,t \in S_i$ and $\bar{q} \in \bar{Z}_{-i}$, and $\Delta u_i(s,t;\bar{q}) \geq 0$. Since $u \sim_P v$, $\Delta u_i(s,t;\bar{q}) \geq 0 \Leftrightarrow \Delta v_i(s,t;\bar{q}) \geq 0$ we have $q \longrightarrow r \ \in G_u$ iff $q \longrightarrow r \ \in G_v$. Thus the response graphs are equal. □

A.2.1 Symmetry

Proposition A.2.5 (Proposition 3.3.10)**.** A directed graph is the response graph of a game if and only if its undirected structure is the Cartesian product $\square_{i=1}^N K_{|S_i|}$ of N complete graphs of varying sizes S_i, which is directed such that each copy of each complete graph is a preference order.

Proof. The node set is the profile set $Z := \prod_{i=1}^N S_i$. Initially we focus on the undirected structure, where two profiles are adjacent if and only if they are comparable for some player. The nodes of $\square_{i=1}^N K_{|S_i|}$ are in a one-to-one correspondence with Z, given by choosing a strategy $s \in S_i$ for each node in each $K_{|S_i|}$. Now by the definition of the Cartesian product (Hammack et al., 2011), there is an arc between two distinct nodes $(\alpha_1, \dots, \alpha_N)$ and $(\beta_1, \dots, \beta_N)$ in a Cartesian product of N graphs if and only if they differ in one entry j, and α_j is adjacent to β_j in the kth graph of the product. In this case, each graph is complete, and so we find that $(\alpha_1, \dots, \alpha_N)$ is adjacent to $(\beta_1, \dots, \beta_N)$ if and only if they differ in one place, but this is the definition of being comparable as strategy profiles.

For the direction, observe that fixing an i-antiprofile $\bar{q}$ defines an induced subgraph that is isomorphic to $K_{|S_i|}$, whose nodes are $s \curvearrowright \bar{q}$ for each $s \in S_i$. The arcs on this complete subgraph must be in order of the preference order on i's strategies given $\bar{q}$, giving the result. □

Theorem A.2.6 (Theorem 3.3.11)**.** *Given a directed graph, we can determine in linear time if it is isomorphic to the response graph of some game, and if so we can construct the associated preference game in linear time, and it is unique up to renaming of strategies.*

Proof. Here we describe the method for reconstructing a preference game from a response graph. First, note that identifying a tuple label for each node requires determining if the undirected structure is a product of complete graphs. Once these tuples (the strategy profiles) have been constructed, it is straightforward to determine if they are oriented to be a total order for each antiprofile. We thus must show the desired undirected structure. We let G be the undirected form of the given graph.

We show that, assuming that G is a product of complete graphs, this method finds a valid labelling. Since this method checks all arcs, if G is not a product of complete graphs then there will be no valid choice of labelling. For now, assume G is indeed a product of complete graphs.

We must first choose our strategy names. Choose an arbitrary node v. Since G is a product graph, its neighbours split into N sets $M_1, \dots, M_N$, where any pair of nodes in the same M_i are adjacent, and no nodes in different M_is are adjacent. This partition is clearly unique, and M_i is the comparable profiles for each player i, with N the number of players. The strategy sets have sizes $\kappa_1, \dots, \kappa_N$, where $\kappa_i = |M_i| + 1$. Let $Z = \prod_{i=1}^{N} [\kappa_i]$. Label v by $(1, \dots, 1) \in Z$. Iterate through each M_i, and label the jth element of M_i by $(1, \dots, j+1, \dots, 1)$, where $j+1$ is in the ith place. We have now given each strategy an arbitrary name, and the order of iteration of M_is fixed an order on the players. So far, all labellings were possible, and now each strategy has appeared at least once. We show that the labels of all remaining profiles are uniquely determined by our given choice of strategy names.

We create a set Visited. Place v in Visited. We will denote the label of a node x by $\ell(x)$. During the main phase of the algorithm we will visit all nodes in a breadth-first-search from v, at each point labelling all of their neighbours. Each node x is labelled on an iteration of one of its neighbours y; we refer to y as the *parent* of x. In the above, v is the parent of all of its neighbours. We will maintain the following loop invariants: for every node in Visited, all of its neighbours are labelled, and the labels of any adjacent labelled nodes differ in only one place. So far, this is true.

Let x be the next node in the breadth-first-search, which is labelled because it is adjacent to some node in Visited. For each neighbour y of x, either y is in Labelled or it is unlabelled. Let w be x's parent (so w and x are adjacent). By the invariant, all neighbours of w are labelled. As y is not labelled, it is not adjacent to w. $\ell(x)$ differs from $\ell(w)$ in one index i, and $\ell(y)$ must differ from $\ell(x)$ in exactly one index $j \neq i$. Without loss of generality, $\ell(w) = (\dots, a, \dots, b, \dots)$, $\ell(x) = (\dots, \alpha, \dots, b, \dots)$ and $\ell(y)$ must be $(\dots, \alpha, \dots, \beta, \dots)$. It remains to determine what is the index j and strategy $\beta \in [\kappa_j]$.

However, as a product graph, w is adjacent to some node z whose label must be $\ell(z) =$

$(\ldots, a, \ldots, \beta, \ldots)$, and this node is labelled because w is in Visited. This node is also adjacent to y (their labels differ in one place), and so from its two labelled neighbours x and z we can concretely determine the unique valid labelling of y. Repeat this process to label all (not-yet-labelled) neighbours of x. Both loop invariants remain true after this iteration, so by induction this algorithm gives a unique labelling, after initially selecting the strategy names and player order.

To show this algorithm takes a linear number of steps in the number of arcs, observe that each arc is examined a bounded number of times. Let $a \longrightarrow b$ be an arc, and assume that a is labelled first. On the iteration that a is labelled, we examine a's neighbours each once (to find labelled neighbours), and so examine this arc once. On the iteration where b is labelled (assuming this not the iteration we visit a), we again examine this arc once. When we visit a or b we examine this arc at most twice, and so overall we examine the arc at most six times, giving an overall complexity that is linear in the number of arcs in the graph, which in a strict game is equal to $\left(\prod_{i=1}^{N}[\kappa_i]\right)\left(\sum_{i=1}^{N}[\kappa_i]-1\right)/2$. □

A.2.2 Invariants

In this section we justify the assertions of Figure 3.4 concerning invariants of various classes of game.

Proposition A.2.7. A profile in a game is a pure Nash equilibrium if and only if it is a singleton sink of the response graph.

Proposition A.2.8. A strategy s for player i is dominated by a strategy t if and only if for every antiprofile $\bar{p}$, there is an arc $s \curvearrowright^i \bar{p} \longrightarrow t \curvearrowright^i \bar{p}$.

These two facts are straightforward restatements of the definition, and we leave them as an exercise for the reader. Since we have now expressed PNEs and dominance in terms of the arcs of the response graph, it follows that these are preference-invariants, using the fact that a preference game is defined by its response graph.

Recall the more surprising fact that this remains true up to isomorphism of the response graphs. That is, if two games have isomorphic response graphs, then PNEs and dominated strategies still exist, they are simply shuffled around by the isomorphism.

Pareto points are ordinal invariants

Lemma A.2.9. u and v are ordinal-equivalent if and only if $u_i = f_i \circ v_i$, where f_i is some monotone transformation, for all players i.

Proof. If $u_i = f_i \circ v_i$, then $u_i(Z) = f_i(v_i(Z))$. $v_i(Z)$ is finite preference relation, and a monotone transformation does not alter the order of elements, and so the preference relation on $u_i(Z) = f_i(v_i(Z))$ and $v_i(Z)$ must be the same, so $u \sim_O v$.

If $u \sim_O v$, then the orders of $u_i(Z)$ and $v_i(Z)$ are equal. This set is finite so we can define a map $f_i : v_i(Z) \to u_i(Z)$ where $v_i(p) \mapsto u_i(p)$. Because the orders are the same, this map is monotone, and $u_i(p) = f_i(v_i(p))$. □

Proposition A.2.10. Pareto efficient points are ordinal-invariant.

Proof. Let $f = (f_1, \ldots, f_N)$ be a transformation that is monotone in each f_i. Let $x, y \in \mathbb{R}^N$ be vectors. Since a monotone transformation preserves order, $x[i] \geq y[i]$ if and only if $f_i(x[i]) \geq f_i(y[i])$, and so x Pareto dominates y if and only if $f(x)$ Pareto dominates $f(y)$. Pareto efficient points are unchanged under a monotone transformation. If $u \sim_O v$, then $u = f \circ v$ (Lemma A.2.9), where $f = (f_1, \ldots, f_N)$ is monotone in each component. Thus the Pareto efficient points are identical in ordinal-equivalent games. □

The fact that Pareto efficient points are *not* invariants of preference games follows from our discussion in Section 3.3.2.

Mixed Nash equilibria are strategic-invariants

In order to prove that Nash equilibria are strategic invariants, we will need some further results. Recall that our original definition of Nash required that no player can improve their expected utility by unilaterally changing mixed strategy. We show first that this can be weakened to requiring only that no unilateral change to any *pure* strategy can improve any player's payoff.

Theorem A.2.11 (Myerson (1997))**.** *Let x be a mixed profile in a game u. Then x is Nash iff $\mathbb{U}(x)_i \geq \mathbb{U}(t \curvearrowright \bar{x}_{-i})_i$ for any player i and strategy $t \in S_i$.*

This simplifies the problem, but we can do better still. We wish to show that Nash equilibria depend only on the relative payoffs, as these define strategic equivalence up to a positive scaling (Lemma A.2.1). Further, one's willingness to deviate from a point depends only on the other players' choices of mixed strategy. Neither of these facts are obvious in this characterisation of the Nash.

Lemma A.2.12 (Myerson (1997))**.** At a Nash equilibrium, all pure strategies in the support of the Nash receive the same expected payoff.

Theorem A.2.13. *Let x be a mixed profile in a game u. Then x is Nash iff for any player i and strategy $s \in S_i$ in the support of x^i, for any strategy $t \in S_i$, $\nabla\mathbb{U}(s, t; \bar{x}_{-i}) \geq 0$.*

Proof. $\mathbb{U}(x)_i \geq \mathbb{U}(t \curvearrowright \bar{x}_{-i})$ if and only if $\mathbb{U}(s \curvearrowright \bar{x}_{-i}) \geq \mathbb{U}(t \curvearrowright \bar{x}_{-i}) \Leftrightarrow \Delta\mathbb{U}_i(s, t; \bar{x}) \geq 0$, for any strategy $s \in S_i$ in the support of x^i, and so this definition is equivalent to that in Theorem A.2.11. □

Note that if both s and t are in the support of P for player i, and x is Nash, then it must be that $\Delta\mathbb{U}(s,t;\bar{x}_{-i}) = \Delta\mathbb{U}(t,s;\bar{x}_{-i}) = 0$. Similarly, if $\Delta\mathbb{U}(s,t;\bar{x}_{-i}) > 0$ at a Nash equilibrium x, then t cannot be in the support of x^i.

This theorem makes it obvious that Nash equilibria depend only on relative payoffs, and also that the property of whether an individual player is willing to deviate depends only on the choices of other players. Now we can prove that Nash are strategy invariants.

Corollary A.2.14. Let u and v be strategically-equivalent games. Then they have the same set of Nash equilibria.

Proof. Let x be any Nash of u. We show x is a Nash equilibrium of v, and the opposite direction follows by symmetry. Let i be a player, s a strategy in the support of x^i, and t any strategy for player i. By Theorem A.2.13, $\Delta\mathbb{U}(s,t;\bar{x}_{-i})_i \geq 0$. By strategic equivalence, for any antiprofile $\bar{q}$ in the support of $\bar{x}_{-i}$ we have $u(s \curvearrowright \bar{q})_i = a_i v(s \curvearrowright \bar{q})_i + b_{i,\bar{q}}$. Now let t be any other strategy for player i, and let $\Delta\mathbb{V}$ be the relative expected utility in v. Then

$$\begin{aligned}
\Delta\mathbb{U}(s,t;\bar{x})_i &= \sum_{\bar{q}\in\bar{Z}_{-i}} \bar{x}^1_{\bar{q}_1}\dots\bar{x}^{N-1}_{\bar{q}_{N-1}} \left(u(s\curvearrowright\bar{q})_i - u(t\curvearrowright\bar{q})_i\right) \\
&= \sum_{\bar{q}\in\bar{Z}_{-i}} \bar{x}^1_{\bar{q}_1}\dots\bar{x}^{N-1}_{\bar{q}_{N-1}} \left(a_i v(s\curvearrowright\bar{q})_i + b_{i,\bar{q}} - a_i v(t\curvearrowright\bar{q})_i - b_{i,\bar{q}}\right) \\
&= a_i \sum_{\bar{q}\in\bar{Z}_{-i}} \bar{x}^1_{\bar{q}_1}\dots\bar{x}^{N-1}_{\bar{q}_{N-1}} \left(v(s\curvearrowright\bar{q})_i - v(t\curvearrowright\bar{q})_i\right) \\
&= a_i \Delta\mathbb{V}(s,t;\bar{x})_i \\
&\geq 0
\end{aligned}$$

and so x is also a Nash of v. □

A.2.3 2x2 games

Proposition A.2.15. The following is true.

- Every game that is preference-equivalent to CO has two pure NEs and one mixed NE,
- every game preference-equivalent to MP has a unique NE that is fully mixed,
- every game preference-equivalent to DD has two dominant strategies and
- every game preference-equivalent to SD has an iterated dominance solution.

Proof. Dominace, iterated dominance and PNEs are all preference-invariants. The non-trivial part of the theorem is showing every game preference-equivalent to CO or MP has a unique fully-mixed NE. Assume the strategies are A and B for each player, and let α and β denote the probabilities of players 1 and 2 respectively playing A. There is a mixed NE where player 2 is indifferent between A and B given player 1 plays α. That is $\alpha\Delta u_2(A,B;A) + (1-\alpha)\Delta u_2(A,B;B) = 0$. In either MP or CO, these have opposite sign, so we get $\alpha x - (1-\alpha)y = 0$, where $x = |\Delta u_2(A,B;A)|$ and $y = |\Delta u_2(A,B;B)|$. This gives a unique solution $\alpha = x/(x+y) \in (0,1)$, for any positive values of x and y. The same reasoning applies to player 2, and so in both CO and MP there is a unique fully-mixed Nash equilibrium. □

A.3 Chapter 4

Lemma A.3.1 (Lemma 4.1.4)**.** A game u is strategically potential if and only if there is a constant $a_i > 0$ for each player i such that the labelled response graph of $v = (a_1 u_1, \ldots, a_n u_n)$ is path-independent.

Proof. First we show that a game u is potential if and only if it is path-independent.

Suppose u is potential with potential function f, and let $p_1 = v, x_1, \ldots, x_n, w$ and $p_2 = v, y_1, \ldots, y_m, w$ be two paths between v and w. The path-sum is

$$\begin{aligned}
\text{pathsum}(p_1) &= u(v)_p - u(x_1)_p + \sum_{i=2}^{n-1}(u(x_i)_{p_i} - u(x_{i+1})_{p_i}) + u(x_n)_{p'} - u(w)_{p'} \\
\text{pathsum}(p_1) &= f(v) - f(x_1) + \sum_{i=2}^{n-1}(f(x_i) - f(x_{i+1})) + f(x_n) - f(w) \\
\text{pathsum}(p_1) &= f(v) - f(x_1) + f(x_1) - f(x_n) + f(x_n) - f(w) \\
\text{pathsum}(p_1) &= f(v) - f(w)
\end{aligned}$$

where we have used the fact that the sum is telescoping. By identical reasoning, $\text{pathsum}(p_2) = f(v) - f(w) = \text{pathsum}(p_1)$.

For the converse, suppose u is path-independent. Define an order $v \preceq w$ if the path-sum of any path from v to w is non-negative. This is well-defined because all such paths have the same path-sum, and since the graph is connected all elements are comparable under this relation. This is reflexive, and we can see that it is transitive by the following. If $v \preceq w \preceq t$, then the path-sum from v to t is the sum of the path-sums of paths from v to w and w to t respectively, and these are each non-negative, so the path-sum from v to t is also non-negative. This order must have minimal elements as it is finite. Choose one, call it z. Define a potential function f as follows. Set $f(z) = 0$, and for any other node x, define $f(x) = \text{pathsum}(p_{x\to z})$, where $p_{x\to z}$ is any undirected path from x to z.

Now we show this is indeed a potential function. Let v and w be i-comparable profiles. Choose paths $p_{v\to z}$ and $p_{z\to w}$. Since u is path-independent, the path-sum along the one-step path $p = u, v$, which is $u(v)_i - u(w)_i$, must be equal to the path-sum of the concatenated path $p_{u\to z}p_{z\to v}$, and this is equal to $f(u) + (-f(v))$, so f is a potential function.

The theorem is completed by recalling Theorem 3.3.4, which asserts that two games are strategically equivalent if and only if their labelled response graphs differ by a positive rescaling in each player. □

Proposition A.3.2. If u and v are potential games with potential functions with the same potential function f, then u and v are strategically-equivalent.

Proof. For any i, $s, t \in S_i$ and $\bar{q} \in \bar{Z}_{-i}$, $\Delta u_i(s,t;\bar{q}) = f(s \frown \bar{q}) - f(t \frown \bar{q}) = \Delta v_i(s,t;\bar{q})$. Hence $\Delta u = \Delta v$ and so $u \sim_S v$ (Lemma A.2.1). □

Lemma A.3.3. For two-player games u and v, $u \sim_S v$ if and only if $\hat{u} \sim_S \hat{v}$.

Proof. $\Delta(-u_2)(s,t;\bar{q}) = (-u_2)(s \frown \bar{q}) - (-u_2)(t \frown \bar{q}) = -(u_2(s \frown \bar{q}) - u_2(t \frown \bar{q})) = -\Delta u_2(s,t;\bar{q})$. But then $u \sim_S v$ iff $(\Delta u_1, \Delta u_2) = (a_1 \Delta v_1, a_2 \Delta v_2)$ iff $(\Delta u_1, -\Delta u_2) = (a_1 \Delta v_1, -a_2 \Delta v_2)$ iff $\hat{u} \sim_S \hat{v}$. □

Lemma A.3.4. For two-player games u and v, $u \sim_P v$ if and only if $\hat{u} \sim_P \hat{v}$.

Proof. For some $s, t \in S_2$ and $r \in S_1$, $\Delta(-u_2)(s,t;r) \geq 0$ iff $\Delta u_2(s,t;r) \leq 0$ iff $\Delta v_2(s,t;r) \leq 0$ iff $\Delta(-v_2)(s,t;r) \geq 0$, and so $u \sim_P v$ iff $\hat{u} \sim_P \hat{v}$. □

Theorem A.3.5 (Theorem 4.2.5)**.** *A two-player game is preference zero-sum if and only if its reflection is acyclic.*

Proof. By Corollary 4.1.5, a game has an acyclic game graph if and only if it is preference potential.

Suppose u is preference zero-sum, so $u \sim_P z$ where z is zero-sum. By Theorem 4.2.3, $\hat{z}$ is potential, and by Lemma A.3.4, $\hat{u} \sim_P z$. For the converse, suppose $\hat{u}$ is preference potential, so $\hat{u} \sim_P w$, where w is a potential game. Then by Theorem 4.2.3 $\hat{w} \sim_S z$, where z is a zero-sum game. By Lemma A.3.4, $u \sim_P \hat{w} \sim_S z$, so $u \sim_P z$, and u is preference zero-sum. □

A.4 Chapter 5

Proposition A.4.1 (Proposition 5.2.1)**.** Let u be a game, and ϕ the replicator flow on u. Then ϕ^{-1} is the flow on $-u$.

Proof. Recall that ϕ is defined by this ordinary differential equation.

$$\dot{x}^p_s = x^p_s \left(\sum_{\bar{q} \in \bar{Z}_{-p}} u(s \frown \bar{q})_p \prod_{\substack{j=1 \\ j \neq p}}^{N} x^j_{\bar{q}_j} - \sum_{t \in S_p} x^p_t \sum_{\bar{q} \in \bar{Z}_{-p}} u(t \frown \bar{q})_p \prod_{\substack{j=1 \\ j \neq p}}^{N} x^j_{\bar{q}} \right)$$

But then in $-u$, we have the replicator flow defined by:

$$\dot{x}_s^p = x_s^p \left(\sum_{\bar{q} \in \bar{Z}_{-p}} (-u)(s \curvearrowright \bar{q})_p \prod_{\substack{j=1 \\ j \neq p}}^{N} x_{\bar{q}_j}^j - \sum_{t \in S_p} x_t^p \sum_{\bar{q} \in \bar{Z}_{-p}} (-u)(t \curvearrowright \bar{q})_p \prod_{\substack{j=1 \\ j \neq p}}^{N} x_{\bar{q}}^j \right)$$
$$= -x_s^p \left(\sum_{\bar{q} \in \bar{Z}_{-p}} u(s \curvearrowright \bar{q})_p \prod_{\substack{j=1 \\ j \neq p}}^{N} x_{\bar{q}_j}^j - \sum_{t \in S_p} x_t^p \sum_{\bar{q} \in \bar{Z}_{-p}} u(t \curvearrowright \bar{q})_p \prod_{\substack{j=1 \\ j \neq p}}^{N} x_{\bar{q}}^j \right)$$

which is precisely the negation of the original ODE, so the flow is the reversed flow. □

Lemma A.4.2 (Lemma 5.2.4)**.** Any finite non-empty intersection of attractors is an attractor.

Proof. Let A and B be attractors with $A \cap B$ non-empty, and let K_A and K_B be closed sets with $\phi^T(K_A) \subseteq \operatorname{int} K_A$ and $\phi^T(K_B) \subseteq \operatorname{int} K_B$ and $A = \bigcap_{t \geq 0} \phi^t(K_A)$ and $B = \bigcap_{t \geq 0} \phi^t(K_B)$. We can choose the same arbitrary $T > 0$, by Proposition 5.2.3. Let $\delta_A := \mathbf{d}(\operatorname{bd} K_A, A)$ and $\delta_B := \mathbf{d}(\operatorname{bd} K_B, B)$. Since $A \subset \phi^T(K_A)$ and $B \subset \phi^T(K_B)$, these compact sets are disjoint and so δ_A and δ_B are positive.

Now $A \cap B$ is also non-empty (by assumption) and compact. Observe that $\operatorname{bd}(K_A \cap K_B) \subseteq \operatorname{bd} K_A \cup \operatorname{bd} K_B$. Let x be in $\operatorname{bd}(K_A \cap K_B)$. If $x \in \operatorname{bd} K_A$ then $\mathbf{d}(x, A \cap B) \geq \delta_A$, and likewise if $x \in \operatorname{bd} K_B$ then $\mathbf{d}(x, A \cap B) \geq \delta_B$. Hence $\inf_{x \in \operatorname{bd}(K_A \cap K_B)} \mathbf{d}(x, A \cap B) \geq \min\{\delta_A, \delta_B\} > 0$. Hence $A \cap B$ does not intersect $\operatorname{bd}(K_A \cap K_B)$. However $A \cap B \subseteq K_A \cap K_B$, and so $A \cap B \subset \operatorname{int}(K_A \cap K_B)$. Since A and B are invariant sets, $A \cap B$ is invariant, and so $A \cap B \subseteq \phi^t(K_A \cap K_B)$ for any $t \geq 0$, so $A \cap B \subseteq \cap_{t \geq 0} \phi^t(K_A \cap K_B)$. But $\cap_{t \geq 0} \phi^t(K_A \cap K_B) \subseteq \bigcap_{t \geq 0} \left(\phi^t(K_A) \cap \phi^t(K_B)\right) = \bigcap_{t \geq 0} \phi^t(K_A) \cap \bigcap_{t \geq 0} \phi^t(K_B) = A \cap B$, by invariance, and so $A \cap B = \bigcap_{t \geq 0} \phi^t(K_A \cap K_B)$ and so $A \cap B$ is an attractor. By induction, any non-empty finite intersection of attractors is also an attractor. □

Lemma A.4.3 (Lemma 5.2.9)**.** If x is chain recurrent, and $\mathcal{A}_x \subseteq \mathcal{A}_y$, then $x \rightsquigarrow y$.

Proof. Fix some $\epsilon > 0$, and let $C_\epsilon(x)$ be the set of points reachable by ϵ-chains from x. Note that $C_\epsilon(x)$ is open. If $C_\epsilon(x)$ is all of X, then $y \in C_\epsilon(x)$. Otherwise, consider $V = N_{\epsilon/2}(C_\epsilon(x))$, that is the set of points within $\epsilon/2$ distance of $C_\epsilon(x)$. This is open, and further $\phi^1(V) \subseteq C_\epsilon(x) \subset \operatorname{int}(V)$, because otherwise there would be an ϵ-chain from x to some point outside $C_\epsilon(x)$, contradicting its definition. Hence V is a trapping region for some attractor $A \subseteq C_\epsilon(x) \subseteq V$. By Theorem 5.2.8, $x \in A \cup A^*$, and since $x \in V$ and $A^* \subseteq X \setminus V$ we have $x \in A$. But then $y \in A$, since y is contained in all attractors containing x, and so $y \in A \subseteq C_\epsilon(x)$. Since ϵ was arbitrary, $x \rightsquigarrow y$. □

Theorem A.4.4 (Theorem 5.2.11). *Let X be the mixed-strategy space of a game u, and Y be the mixed-strategy space of a subgame u' of u. The flow $\phi_u|_Y$ of the replicator on u restricted to Y is identical to $\phi_{u'}$, the replicator flow on u'.*

Proof. Let x be a point in the subspace Y. Observe from the definition of the replicator that for any player p with strategy t outside of Y, $x_t^p = 0$ and so $\dot{x}_t^p = 0$. Now we show that the derivative of strategies in Y only depends on the values within Y.

Let s be a strategy for player p inside Y. Then First, let $S_i = T_i \cup R_i$, where T_i is the strategies for player i in subgame Y, and R_i is the remaining strategies for player i.

$$\begin{aligned}\dot{x}_s^p &= x_s^p \left(\mathbb{U}(s \curvearrowright \bar{x})_p - \sum_{t \in S_p} x_t^p \mathbb{U}(t \curvearrowright \bar{x})_p \right) \\ &= x_s^p \left(\sum_{\bar{q} \in \bar{Z}_{-p}} u(s \curvearrowright \bar{q})_p \prod_{\substack{j=1 \\ j \neq p}}^{N} x_{\bar{q}_j}^j - \sum_{t \in S_p} x_t^p \sum_{\bar{q} \in \bar{Z}_{-p}} u(t \curvearrowright \bar{q})_p \prod_{\substack{j=1 \\ j \neq p}}^{N} x_{\bar{q}}^j \right)\end{aligned}$$

But for any $\bar{q}$ outside of $\bar{Y}_{-p}$, $\prod_{\substack{j=1 \\ j \neq p}}^{N} x_{\bar{q}_j}^j = 0$, because at least one of $x_{\bar{q}_j}^j$ is zero. Similarly, for $t \in S_p$, $x_t^p = 0$ if t is outside of Y for p. Thus we find

$$\dot{x}_s^p = x_s^p \left(\sum_{\bar{q} \in \bar{Y}_{-p}} u(s \curvearrowright \bar{q})_p \prod_{\substack{j=1 \\ j \neq p}}^{N} x_{\bar{q}_j}^j - \sum_{t \in Y_p} x_t^p \sum_{\bar{q} \in \bar{Y}_{-p}} u(t \curvearrowright \bar{q})_p \prod_{\substack{j=1 \\ j \neq p}}^{N} x_{\bar{q}}^j \right)$$

where $\bar{Y}_{-p}$ are the antiprofiles in Y, and $Y_p \subseteq S_p$ are the strategies in S_p used in Y. This is precisely the definition of the replicator flow in the game u' defined by restricting u to Y. □

Proposition A.4.5 (Proposition 5.2.18). Let H be a sink equilibrium. Then $[H]$ is a sink chain component if and only if there is no sink equilibrium K with $\mathcal{A}_K \supset \mathcal{A}_H$.

Proposition A.4.6 (Proposition 5.2.19). Let S be a sink chain component containing a sink equilibrium H. Then H is the unique sink equilibrium in S if and only if there is no sink equilibrium K with $\mathcal{A}_K = \mathcal{A}_H$.

Theorem A.4.7 (Theorem 5.2.21). *Let H be a connected component of the response graph. Then under the replicator, the content of H is contained in $[H]$.*

Proof. First, observe that the base case where Y is a pure profile is trivial, as its content is itself, so is in $[H]$.

Now suppose for induction that X is the mixed strategy space of a game contained in the connected component H. By Theorem 5.2.11, we can assume X is a game, because it does not matter if X is a subgame of a larger game. The boundary of X are all smaller subgames, so by the inductive hypothesis, the boundary of X is entirely contained in $[H]$. Now let x be a point in the interior of X. By Corollary 5.2.10, x is in $[H]$ if and only if it is contained in all the same attractors and repellors as the boundary of X. Suppose not. Then there is an attractor or repellor A containing the boundary but not x, but its dual repellor/attractor must be therefore be in the interior of X, but this contradicts Theorem 5.2.14. Thus the whole subgame X is contained in $[H]$. □

Lemma A.4.8 (Lemma 5.2.23)**.** If Y is an attracting subgame, then it is an attractor. Dually, repelling subgames are repellors.

Proof. Fix a player p, and let s and t be strategies for p, with $s \in Y_p$ and $t \notin Y_p$. We show there is a positive-measure region around Y where $\Delta\mathbb{U}_p(s,t;\bar{x})$ is always positive, and so all trajectories converge to having weight on s and none on t. Let $\alpha = \min_{\bar{q}\in\bar{Y}} \Delta u_p(s,t;\bar{q})$ and $\beta = \min_{\bar{q}\in\bar{X}\setminus\bar{Y}} \Delta u_p(s,t;\bar{q})$. As Y is attracting, α must be positive. If β is also positive then s dominates t making the result trivial, so we assume that β is negative. Then

$$\begin{aligned}
\Delta\mathbb{U}_p(s,t;\bar{x}) &= \sum_{\bar{q}} \Delta u_p(s,t;\bar{q}) \prod_{\substack{j=1\\ j\neq p}}^{N} x^j_{\bar{q}} \\
&= \sum_{\bar{q}\in\bar{Y}} \Delta u_p(s,t;\bar{q}) \prod_{\substack{j=1\\ j\neq p}}^{N} x^j_{\bar{q}} + \sum_{\bar{q}\in\bar{X}\setminus\bar{Y}} \Delta u_p(s,t;\bar{q}) \prod_{\substack{j=1\\ j\neq p}}^{N} x^j_{\bar{q}} \\
&\geq \sum_{\bar{q}\in\bar{Y}} \alpha \prod_{\substack{j=1\\ j\neq p}}^{N} x^j_{\bar{q}} + \sum_{\bar{q}\in\bar{X}\setminus\bar{Y}} \beta \prod_{\substack{j=1\\ j\neq p}}^{N} x^j_{\bar{q}} \\
&= W^{s,t}_p \alpha + (1-W)\beta
\end{aligned}$$

where $W_p := \sum_{\bar{q}\in\bar{Y}} \prod_{\substack{j=1\\ j\neq p}}^{N} x^j_{\bar{q}}$ is the total weight on profiles in Y, which is one minus the weight outside Y because x defines a distribution. This is positive if $W_p \geq \beta/(\alpha+\beta)$, which is always a positive number strictly less than one. Taking the maximum W_p over all s, t and p gives a positive-measure region in which all points converge to Y. This is also true for points in subgames bordering Y, by Theorem 5.2.11. □

A proof of Theorem 5.3.1

First we introduce the graph of a matrix.

Definition A.4.9. (Brualdi and Cvetkovic, 2008) Let M be an $N \times N$ matrix. The *graph of* M is the directed graph G_M with N nodes where there is an arc from node i

to node j if $M_{ij} \neq 0$.[1]

Definition A.4.10. (Brualdi and Cvetkovic, 2008) A non-negative matrix M is *irreducible* if for every i, j, there exists a $k \in \mathbb{N}_0$ such that $M^k_{i,j} > 0$. It is *reducible* otherwise.

This can be rephrased entirely in terms of the graph.

Lemma A.4.11. (Brualdi and Cvetkovic, 2008) A non-negative matrix M is irreducible if and only if G_M is strongly connected.

We will also need the related concept of the period of a matrix.

Definition A.4.12. Let M be a non-negative matrix. The *period* of M is the greatest common divisor of the length of cycles in G_M. A matrix is *aperiodic* if it has period one.

The Perron-Frobenius theorem, proved in a restricted form by Oskar Perron (Perron, 1907) and later extended by Georg Frobenius (Frobenius, 1912), showed that when a non-negative matrix is irreducible and aperiodic it has a unique largest eigenvalue, and converges to a unique positive eigenvector in that eigenspace. For stochastic matrices, this eigenvalue is always one, and so we have:

Theorem A.4.13 (Perron-Frobenius Theorem)**.** *Let M be a non-negative irreducible stochastic matrix. Then there exists a unique stochastic vector π such that $\pi M = \pi$, and* $\lim_{k\to\infty} M^k = \pi\,\mathbf{1}$. *We call this the* Perron-Frobenius vector*.*

It is worth noting that both irreducibility and aperiodicity are defined in terms of the graph of the matrix only. Thus, when considering Markov chains on games, these key properties are *preference-invariants*. Each of these properties contributes independently to the theorem, with aperiodicity ensuring convergence and irreducibility ensuring uniqueness. We will now proceed with our proof. In the following, Z is the state space of the chain M. We denote by $\cdot|_S$ the restriction of a matrix or vector to some subset S of its entries.

Lemma A.4.14. Let i be a non-sink node, M a stochastic matrix and x_0 an arbitrary stochastic vector. Then $\lim_{k\to\infty}(x_0 M^k)_i = 0$.

Proof. Let T be the nodes which are not in sinks, and let S be the nodes that are in sinks. We show that the sum of weights over T decreases monotonically as M is iterated, and so $\lim_{k\to\infty} \sum_{i\in T}(x_0 M^k)_i = 0$. S has no outgoing arcs, and so

$$1 - \sum_{i\in T}(x_0 M) = \sum_{i\in S}(x_0 M) = \sum_{i\in S}\sum_{j\in X} M_{i,j} x_j = \sum_{j\in X} x_j \sum_{i\in S} M_{i,j} = \sum_{j\in X} x_j \geq \sum_{j\in S} x_j$$

where the sum $\sum_{i\in S} M_{i,j} = 1$ by the fact that S has no outgoing arcs and M is row-stochastic. This inequality holds with equality only if there is no weight on nodes in T, and so we find that $\sum_{i\in S}(x_0 M^k)_i$ increases monotonically in k, and so $\sum_{in\in T}(x_0 M^k)_i$

[1] It is interesting how this definition mirrors that of the response graph. For Markov dynamics, this is precisely the response graph.

decreases monotonically, and so the weight on any individual node in T converges to zero. □

Proof of Theorem 5.3.1. Lemma A.4.14 establishes the first claim of the theorem.

Lemma A.4.15. Let S be a subset of states of the Markov chain M with no outgoing arcs, and let π be a vector whose support is S. Then πM is $\pi M|_S$ padded with zeros for the entries not in S.

Proof. Then $(\pi M)_j = \sum_{i \in X} M_{i,j}\pi_j = \sum_{i \in S} M_{i,j}\pi_j$. Since $M_{i,j} = 0$ if $j \notin S$, πM is exactly $\pi|_S M|_S$ padded with zeros for the entries not in S. □

Now, we know by Perron-Frobenius that every irreducible stochastic matrix has a unique positive stochastic eigenvector. If S is some fixed sink, let τ_S be the unique stationary distribution of $M|_S$, which is irreducible as S is a sink, and let σ_i be τ_i padded with zeros. Let π be some stationary dsitribution whose support is S. By Lemma A.4.15, πM is $\pi M|_S$ padded with zeros, and so $\pi|_S M|_S = \pi|_S$. Thus $\pi|_S = \tau_S$ by uniqueness, and so $\pi = \sigma_i$. This proves the second claim.

Lemma A.4.16. A distribution π is stationary for M if it is a convex combination of $\sigma_1, \ldots, \sigma_m$.

Proof. One direction is straightforward, by linearity. If $\pi = \sum_i \alpha_i \sigma_i$, then $\pi M = \sum_i \alpha_i \sigma_i M = \sum_i \alpha_i \sigma_i = \pi$, so any convex combination of stationary distributions is itself stationary.

Now suppose that π is an arbitrary stationary distribution. By Lemma A.4.14, the support of π can only contain nodes which are in sinks. Let $S_1, \ldots, S_m$ be the sinks, and let $\pi_1, \ldots, \pi_m$ be the restriction of π to each sink, padded with zeros. Then by linearity $\pi = \pi M = \sum_{i=1}^m \pi_i M$. By our above argument, for every i, $\pi_i M$ is exactly $\pi_i M|_{S_i}$ padded with zeros outside S. But then $\pi_i = \pi_i M|_{S_i}$, and all such vectors are positive multiples of σ_i, so $\pi_i = \alpha_i \sigma_i$ for some $\alpha_i > 0$. Thus $\pi = \sum_{i=1}^m \alpha_i \sigma_i$, and these α_is must sum to one as π is a distribution. □

Finally, we show convergence. Let $\epsilon > 0$ be given. Let S be the set of sink nodes, and T the set of non-sink nodes, and we write $x_T^{(n)}$ and $x_S^{(n)}$ for the vectors with support T and S respectively with $x^{(n)} = x_0 M^n = x_S^{(n)} + x_T^{(n)}$. We will show that the sequence $x^{(n)}$ is Cauchy and thus converges.

$$\begin{aligned} \|x_0 M^n - x_0 M^{n-1}\| &= \|(x_S^{(n)} - x_S^{(n-1)}) + (x_T^{(n)} - x_T^{(n-1)})\| \\ &\le \|x_S^{(n)} - x_S^{(n-1)}\| + \|x_T^{(n)} - x_T^{(n-1)}\| \end{aligned}$$

Since we know that $\sum_{t \in T} x_t^{(n)} \to 0$ as $n \to \infty$ (Lemma A.4.14), there exists a number K where $\sum_{t \in T} x_t^{(n)} < \epsilon/3$ for all $n \ge K$. Now we assume $n > K$. We can write $x_S^{(n)} =$

$x_S^{(n-1)} M|_S + x_{T\to S}^{(n)}$, where $x_{T\to S}^{(n)} = \sum_{i\in T} M_{i,j} x_i^{(n-1)}$, for those $j \in S$, and 0 otherwise. By iterating this $n-K$ times, we get $x_S^{(n)} = x_S^{(K)}(M|_S)^{n-K} + \sum_{j=0}^{n-N-1} x_{T\to S}^{(n-j)}(M|_S)^j$. The sum term defines the component of $x_S^{(n)}$ which depends on T since iteration K. The magnitude of this vector is bounded above by $\sum_{t\in T} x_t^{(K)}$ (the total weight on nodes in T at step K) which is itself less than $\epsilon/3$ by assumption. Thus

$$\begin{aligned}
\|x_0 M^n - x_0 M^{n-1}\| &\leq \|x_S^{(K)}(M|_S)^{n-K} - x_S^{(K)}(M|_S)^{n-K-1} \\
&+ \sum_{j=0}^{n-N-1} x_{T\to S}^{(n-j)}(M|_S)^j - \sum_{j=0}^{n-N-2} x_{T\to S}^{(n-j-1)}(M|_S)^j\| + \|x_T^{(n)} - x_T^{(n-1)}\| \\
&\leq \|x_S^{(K)}(M|_S)^{n-K} - x_S^{(K)}(M|_S)^{n-K-1}\| \\
&+ \|\sum_{j=0}^{n-N-1} x_{T\to S}^{(n-j)}(M|_S)^j - \sum_{j=0}^{n-N-2} x_{T\to S}^{(n-j-1)}(M|_S)^j\| + \|x_T^{(n)} - x_T^{(n-1)}\| \\
&\leq \left\| x_S^{(K)}(M|_S)^{n-K} - x_S^{(K)}(M|_S)^{n-K-1} \right\| + \epsilon/3 + \|x_T^{(n)} - x_T^{(n-1)}\|
\end{aligned}$$

Finally, as $n \to \infty$ $x_T^{(n)} \to 0$, by Lemma A.4.14, so there exists an n_1 where the third term is less than $\epsilon/3$ for all $n \geq n_1$. Also, by the Perron-Frobenius theorem, $x_S^{(K)}(M|_S)^n$ converges to a stationary vector as $n \to \infty$ because $x_S^{(K)}$ has support S, and so $\|x_S^{(K)}(M|_S)^{n-K} - x_S^{(K)}(M|_S)^{n-K-1}\|$ is a Cauchy sequence, and so there exists n_2 where $\|x_S^{(K)}(M|_S)^{n-K} - x_S^{(K)}(M|_S)^{n-K-1}\| < \epsilon/3$ for all $n \geq n_2$. Thus for all $n > \max\{n_1, n_2\}$ we have $\|x^{(n)} - x^{(n-1)}\|M < \epsilon$, so this sequence is Cauchy, and thus converges as the space is Euclidean. The limit is a distribution (because the simplex is closed) and is stationary because it is invariant under right-multiplication by M by construction. □

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