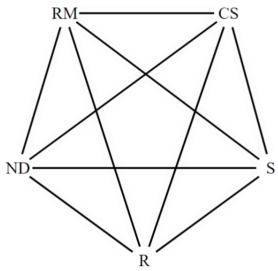

HOL Metatheory of Relevant Implication

Syntax and Semantics

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James T. Taylor

Monday 6th June, 2022

to my parents and sister

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Abstract

We mechanise two Hilbert systems, a Natural Deduction system, the Routley-Meyer semantics, and the Cover semantics for the Relevant Logic **R** in HOL4. We also show equivalence results between one of the Hilbert Systems and the other Hilbert system and the Natural Deduction system. We also show soundness and completeness results between the one of the Hilbert Systems and the two Semantic systems, thereby producing machine checked proofs of all of these results.

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Chapter 1

Introduction

This thesis uses HOL4 to mechanise two Hilbert-style proof systems and a Natural Deduction system for the logic of Relevant Implication (**R**) and show that these proof systems are equivalent. Furthermore it also mechanises the Routley-Meyer semantics and Goldblatt Cover semantics for **R** and uses HOL4 to produce machine checked proofs for soundness and completeness.

In this chapter, Section 1.1 provides a brief introduction to Relevant Implication, HOL4, and motivates the study of this topic. Section 1.2 outlines the motivating texts for this thesis while Section 1.3 describes the work done inside this thesis. Finally, Section 1.4 contains information on how to read this thesis.

1.1 Introduction

1.1.1 Relevant Implication

Relevant Implication arose in response to a number of philosophical concerns with the standard propositional logic of the time known as classical logic. The development of Relevant Implication is equally an attempt to ascribe a more human version of the meaning A *implies* B onto the $(\rightarrow)$.

Despite being the basis of reasoning in many fields, classical logic has some major flaws when we reflect the formalisation back onto human argument; first and formost amoung these flaws are the *paradoxes of material entailment* which we introduce by way of example:

The bottle on my desk has a squid in it and it also does not have a squid in it; therefore the earth orbits the moon

In our natural use of language, an argument of this form is invalid and would not be accepted as a valid argument as a reasonable person might question why a squid in a water bottle on anyone's desk could be used to determine how the cosmos operates. However in classical logic, when we represent *there is a squid in a bottle on my desk* as A, and *the earth orbits the moon* as B, and then the argument above becomes:

$$(A \wedge \neg A) \Rightarrow B$$

which is valid in classical logic. There are a number of these paradoxes that arise from classical logic generally, around how we read the implication symbol ($\rightarrow$).

One way to come to terms with these paradoxes is to claim that *formal logic* has little to do with natural argumentation and that drawing an analogy between them is not the point. A second view is to accept that the above argument is naturally valid and reject the premise that these are paradoxes in the first place. *Relevant Logics* (or *Relevance Logic* in the U.S.A.) were the result of taking a third way and saying that Logic should remain a model for how we naturally reason about the world; and if we are unable to prevent paradoxes like the *paradoxes of material entailment* then our current logic is inherently flawed and we need a better system.

Relevant Implication (also known as **R**) is one of these *Relevant Logic*s and was designed in the later half of the 20^{th} century. It is the oldest known system of *Relevant Logic* and according to Anderson and Belnap [1975, p.349] the implicational fragment was independantly forumlated by both Moh in 1950 and Church in 1951.

1.1.2 Metatheory

Metalogic is the study of different systems of logic and their properties by formally defining the system in terms of another logical system (called the *metasystem*). The *Metatheory* of a logic then is then just those properties that can be shown about, or between, systems; these are known as *metatheorems*. In the case of this thesis we will be looking a two different types of system using our *metasystem*; these are *syntactic* systems and *semantic systems*. Given some underlying language $\mathcal{L}$:

Syntactic systems are sets of rules that govern how we are able to manipulate expressions in $\mathcal{L}$. The set of all expressions that a syntactic system can produce is called the *theory* of that particular system.

Semantic systems on the other hand ascribe meaning (truth) to expressions. There exist many different types of semantic systems, but in the case of Relevant Implication, the semantics we will be looking at is with respect to model theory. Each model has a number of statements in the underlying language that are true, and others that are false.

The connection between these different types of systems are two *metatheorems* that are known as *soundness* and *completeness*. In the case of model theory the *soundness* result states that: All theorems p of the syntactic theory are true in every model in the semantic theory. The completeness result is the converse: If a proposition p is true in every model of the semantic theory, then it can be proved in the syntactic one.

1.1.3 HOL

HOL, standing for *Higher Order Logic*, is an interactive theorem prover based on Church's simple type theory [Slind and Norrish, 2008]. *Interactive* theorem proving (ITP), as distinct from automated theorem proving, usually requires some level of human interaction to prove results and is not expected to prove results on its own. Guiding the machine through the proof has a number of benefits, but there are two major benefits to this project that this control provides. The first is that a successful proof results in knowing that the theorem that was proved, was proved ultimately from the underlying axioms of the theorem prover, thus verifying the result (we say then that the proof has been mechanised, or mechanically checked). The second benefit is that it enables the programmer to follow already known proofs to the letter and note any deviations from preciously accepted proofs.

While we will introduce some HOL4 methods and functions throughout this thesis as we need them, it is imporant to talk about some of the basic syntax here:

- **Record types** are types that can have many subfields of diffferent types. For example, we may have a record type

$$\mathit{animal} = \langle \mathit{species} : \mathit{string}; \mathit{class} : \mathit{string}; \mathit{hungry} : \mathit{bool} \rangle$$

- **Inductive types** are types that are self-recursive and may be defined with bars between the different constructors.
- **Inductive definitions** are a special type of definition for predicates inside HOL that allows for induction over the constructor rather than just the argument to the defined predicate.
- **Inductive functions** are functions defined based off cases for inductive types, for example:

$$\begin{array}{lcl} \&[] & \stackrel{\text{def}}{=} & \tau \\ \&[p] & \stackrel{\text{def}}{=} & p \\ \&(p :: q :: lp) & \stackrel{\text{def}}{=} & p \ \& \ \&(q :: lp) \end{array}$$

1.1.4 But why mechanise R?

Relevant Logic is a well studied field from both mathematical and philosophical perspectives, yet the focus of mechanisation of proofs related to it are only with respect to the decidability and undecidability results of different logics [Larchey-Wendling, 2020, 2021; Dawson and Goré, 2017]. To date we have been unable to show mechanisations of any other metatheoretic results for Relevant Logics which is an oversight from the perspective of philosophical logic. We seek to provide an underlying syntactic and semantic basis from which further mechanisations of different extensions of **R** can be launched.

1.2 Motivating Texts

This thesis mechanises a number of different proofs from different texts, each of which are mentioned below:

Quantifiers, Propositions, and Identity by Goldblatt [2011][1] is the book where the Cover Semantics (Chapter 4) for Relevant Implication is defined and demonstrates soundness and completeness results. It also outlines the ideas behind "admissable semantics" and Cover Semantics, and shows that Quantified Relevant Implication is also sound and complete with resect to the natural extension of the semantic system.

A General Logic by Slaney [1990] outlines a class of Natural Deduction systems based around connecting propositions in two different ways, being set and multiset unions. Slaney also identifies the structural rules that give rise to Relevant Implication.

The Semantics of Entailment by Routley and Meyer [1973] is the text that defines a Kripke style Semantic System for Relevant Implication and demonstrates shows its soundness and completeness results.

Notes on Relevant Logic by Slaney [N.D.][2] provides a different axiomatisation for Relevant Implication which was used in Chapter 2.

1.3 Thesis Outline

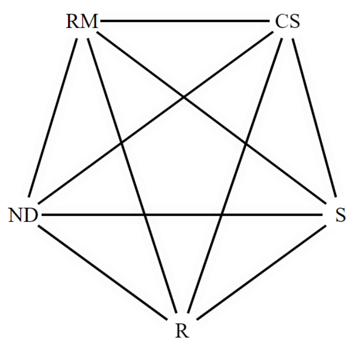


Figure 1.1: The Relevant Pentagram

[1] This is where the definitions for Goldblatt Proposition and Goldblatt Provable (which can be found at Definitions 2.1.1 and 2.1.2 in Chapter 2). While it should be noted that these are not the proper names for these constructs and they are not named as such in the text, they denote where these definitions come from.

[2] Same as above, but with Definitions 2.2.1 and 2.2.2.

Figure 1.1 illustrates the work that this thesis contains. R and S are two different Hilbert axiomatisations for Relevant Implication, ND is a Natural Deduction system for **R**, RM is the Routley-Meyer semantics, and CS is the Cover Semantics, each line representing a proof of syntactic or semantic equivalence.

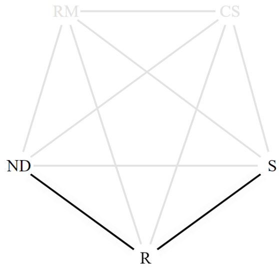
This thesis has a total of five chapters including this one. Chapter 2 introduces and mechanises the proof systems for **R** and establishes equivalence between them. Chapter 3 introduces and mechanises the Routley-Meyer semantics as well as shows that the Routley-Meyer semantics are both sound and complete with respect to **R**. Chapter 4 introduces and mechanises the Goldblatt Cover semantics as well as shows that these semantics are sound and complete with respect to **R**. Chapter 5 concludes this thesis by summarising the main results of each chapter and offers a perspective to the breadth and scope of immediate future work for the mechanisation of Relevant Logics.

1.4 How to read this thesis

This thesis attempts to outline exactly what structures and results that were needed to show the equivalence between the Hilbert and natural Dedcution proof systems. It also outlines what is needed to show the soundness and completeness results between the proof systems and the Routley-Meyer and the Goldblatt Cover semantic systems. In this sense, this thesis captures the entirety of what is required to show each target theorem.

While this thesis omits some proofs that are uninteresting or routine, care has been taken to ensure that any omission does not detract from the intention of this thesis. Furthermore all results used in this thesis, regardless of omission from this text, have been formally verified inside the HOL4 theorem prover.

The HOL source files used for this thesis can be on found on GitHub.



Chapter 2

Proof Systems

The first way that we may formalise a system of logic is to take a set of propositions that we hold to be true, called *axioms*; and a collection of rules that we can use to generate new propositions, called *rules of inference*. The *axioms* and *rules of inference* are the essence to a system of proving propositions called a *Hilbert system*. A *theorem* of a given Hilbert system is just a proposition that we can generate only using the axioms and the rules of inference. We tend to call the set of all theorems of a system the *theory* generated by the Hilbert System.

When talking about logic it can also be helpful to talk about a *sequent*, or what we can prove given some assumptions. We call what we prove the *conclusion* of the sequence and the extra propositions that we use, the *assumption* of the sequent. As sequents have more structure than propositions we need another way to interact with these structures, which is where *Natural Deduction systems* are particularly useful. A *Natural Deduction system* is normally characterised by a set of rules that interact with either the assumption, the conclusion, or sometimes both parts of the sequent.

It is important to note that proof systems are *syntactic* systems. This means that they only tell us *what* is true rather than providing any reason behind *why* it is true. While *syntax* only tells us one half of the story of any logic; there are still a number of reasons why we might decide to work with proof systems rather than grapple with a *semantic* system. For example, Natural Deduction systems make proving theorems significantly easier for logicians through the use of many proof rules and Hilbert Systems are very explicit as to what propositions should be taken to be true through the use of a large number of axioms.

This chapter focuses on the proof systems of relevant implication and the establishment of equivalency between these systems. Section 2.1 will introduce the axiomatic system that we use throughout this thesis and establish some results in **R**. In section 2.2 we look at a different axiomatisation of Relevant Implication and show equivalence between the two axiomatic systems. Finally Section 2.3 introduces a Natural Deduction system for **R** and then shows equivalence to the axiomatic systems.

2.1 A Relevant Hilbert System

In order to reason about relevant implication inside HOL, we need to create a new data type representing our propositions. In the characterisation of relevance implication in Goldblatt [2011], propositions are built up of: atomic symbols as g-VAR s with s a *string* type, an implication $\phi \rightarrow_g \psi$ between propositions, a conjunction $\phi \,\&_g\, \psi$ of propositions, the negation $\sim_g\phi$ of a proposition, or the propositional constant τ_g. Thus in HOL we introduce our propositions as inductive type called *g-prop*:

Definition 2.1.1 (Goldblatt Proposition)**.** A *Goldblatt Proposition* is formalised in HOL as an inductive type:

$$\textit{g-prop} \;=\; \textsf{g-VAR}\ \textit{string} \mid (\rightarrow_g)\ \textit{g-prop}\ \textit{g-prop} \mid (\&_g)\ \textit{g-prop}\ \textit{g-prop} \mid (\sim_g)\ \textit{g-prop} \mid \tau_g$$

We can also introduce disjunction, bi-conditional, and intensional conjunction[1] (also known as fusion) in the normal ways:

$$\begin{array}{lcl} A \vee_g B & \stackrel{\text{def}}{=} & \sim_g(\sim_g A \,\&_g \sim_g B) \\ A \leftrightarrow_g B & \stackrel{\text{def}}{=} & (A \rightarrow_g B) \,\&_g\, (B \rightarrow_g A) \\ A \circ_g B & \stackrel{\text{def}}{=} & \sim_g(A \rightarrow_g \sim_g B) \end{array}$$

We are now able to formalise the Hilbert system presented Goldblatt [2011] which we will use for the entirety of this thesis:

Definition 2.1.2 (Goldblatt provable)**.** A proposition is *Goldblatt Provable* (written

[1] An intuitive way to think about $(\circ_g)$ going ahead is that it behaves like classical conjunction does with classical implication, but when $\phi \circ_g \psi$ is true, it doesn't guarantee that either ϕ or ψ is also true.

$\vdash_{\boldsymbol{R}}$ [2]) if it is of the form of one of the following axioms

$\vdash_{\boldsymbol{R}} (A \rightarrow_g A)$	Identity
$\vdash_{\boldsymbol{R}} ((A \rightarrow_g B) \rightarrow_g (B \rightarrow_g C) \rightarrow_g A \rightarrow_g C)$	Suffixing
$\vdash_{\boldsymbol{R}} (A \rightarrow_g (A \rightarrow_g B) \rightarrow_g B)$	Assertion
$\vdash_{\boldsymbol{R}} ((A \rightarrow_g A \rightarrow_g B) \rightarrow_g A \rightarrow_g B)$	Contraction
$\vdash_{\boldsymbol{R}} (A \,\&_g\, B \rightarrow_g A)$	Conjunction Elimination L
$\vdash_{\boldsymbol{R}} (A \,\&_g\, B \rightarrow_g B)$	Conjunction Elimination R
$\vdash_{\boldsymbol{R}} ((A \rightarrow_g B) \,\&_g\, (A \rightarrow_g C) \rightarrow_g A \rightarrow_g B \,\&_g\, C)$	Conjunction Introduction
$\vdash_{\boldsymbol{R}} (A \rightarrow_g A \vee_g B)$	Disjunction Introduction L
$\vdash_{\boldsymbol{R}} (B \rightarrow_g A \vee_g B)$	Disjunction Introduction R
$\vdash_{\boldsymbol{R}} ((A \rightarrow_g C) \,\&_g\, (B \rightarrow_g C) \rightarrow_g A \vee_g B \rightarrow_g C)$	Disjunction Elimination
$\vdash_{\boldsymbol{R}} (A \,\&_g\, (B \vee_g C) \rightarrow_g A \,\&_g\, B \vee_g A \,\&_g\, C)$	Distribution
$\vdash_{\boldsymbol{R}} ((A \rightarrow_g \sim_g B) \rightarrow_g B \rightarrow_g \sim_g A)$	Contraposition
$\vdash_{\boldsymbol{R}} (\sim_g \sim_g A \rightarrow_g A)$	Double Negation Elimination

Or it is derivable from other *Goldblatt Provable* propositions using the following *rules of inference*:

$$\frac{\vdash_{\boldsymbol{R}} A \quad \vdash_{\boldsymbol{R}} B}{\vdash_{\boldsymbol{R}} (A \,\&_g\, B)} \text{ Adjunction} \qquad \frac{\vdash_{\boldsymbol{R}} A \quad \vdash_{\boldsymbol{R}} (A \rightarrow_g B)}{\vdash_{\boldsymbol{R}} B} \text{ Modus Ponens}$$

$$\frac{\vdash_{\boldsymbol{R}} A}{\vdash_{\boldsymbol{R}} (\tau_g \rightarrow_g A)} \;\tau_g \text{ Intro.} \qquad \frac{\vdash_{\boldsymbol{R}} (\tau_g \rightarrow_g A)}{\vdash_{\boldsymbol{R}} A} \;\tau_g \text{ Elim.}$$

We can now show a number of useful results about relevance implication inside this Hilbert system:

Proposition 2.1.3 (Some theorems of Relevance Logic)**.**

1.	$\vdash_{\boldsymbol{R}} ((A \rightarrow_g B \rightarrow_g C) \rightarrow_g B \rightarrow_g A \rightarrow_g C)$	Permutation
2.	$\vdash_{\boldsymbol{R}} (A \rightarrow_g \sim_g \sim_g A)$	Double Negation Introduction
3.	$\vdash_{\boldsymbol{R}} (A \leftrightarrow_g \sim_g \sim_g A)$	Double Negation Equivalence
4.	$\vdash_{\boldsymbol{R}} ((A \rightarrow_g B) \leftrightarrow_g (A \rightarrow_g \sim_g \sim_g B))$	
5.	$\vdash_{\boldsymbol{R}} ((A \rightarrow_g B) \leftrightarrow_g (\sim_g B \rightarrow_g \sim_g A))$	Contraposition alt.
6.	$\vdash_{\boldsymbol{R}} (A \circ_g B \rightarrow_g B \circ_g A)$	$(\circ_g)$ Commutative
7.	$\vdash_{\boldsymbol{R}} (A \circ_g (B \vee_g C) \leftrightarrow_g A \circ_g B \vee_g A \circ_g C)$	$(\circ_g)$ Distribution
8.	$\vdash_{\boldsymbol{R}} (A \,\&_g\, (A \rightarrow_g B) \rightarrow_g B)$	

Proof.

1. From application of suffixing, assertion, and modus ponens.

[2]The choice of $\vdash_{\boldsymbol{R}}$ rather than $\vdash_{\boldsymbol{G}}$ as our symbol for *Goldblatt Provable* is due to the fact that we will choose this system as a sort of 'source of truth' for what it means to be a theorem of relevance logic.

2. From the definition of Goldblatt provable and the definition of the bi-conditional connective.
3. Immediate from 2.
4. From the definition of Goldblatt provable, permutation, 3., and the definition of the bi-conditional connective.
5. It suffices to show that both directions of the Goldblatt bi-conditional are Goldblatt provable from the adjunction rule.

   ($\rightarrow_g$): Immediate from suffixing, left conjunction, modus ponens, contraposition, and 4.

   ($\leftarrow_g$): Immediate from suffixing, right conjunction, modus ponens, contraposition, and 4.

6.-8. These are involved, though uninteresting arguments.

□

We can also introduce a sort of replacement for sub-propositions:

Proposition 2.1.4 (Propositional Replacement)**.** If A, B are propositions and $A \leftrightarrow_g B$ then:

1. When A appears on either side of the ($\rightarrow_g$), ($\&_g$), or ($\leftrightarrow_g$) connectives, then that statement if Goldblatt provable if and only if the statement resulting from replacing A with B is Goldblatt provable.
2. Similarly, when A appears behind a ($\sim_g$) then $\sim_g A$ is Goldblatt provable if and only if $\sim_g B$ is.
3. Furthermore, A is Goldblatt provable if and only if B is.

In other words:

$$
\begin{aligned}
\vdash_{\boldsymbol{R}} (A \leftrightarrow_g B) \Rightarrow \\
&\vdash_{\boldsymbol{R}} ((A \rightarrow_g C) \leftrightarrow_g (B \rightarrow_g C)) \wedge \\
&\vdash_{\boldsymbol{R}} ((C \rightarrow_g A) \leftrightarrow_g (C \rightarrow_g B)) \wedge \\
&\vdash_{\boldsymbol{R}} (A \&_g C \leftrightarrow_g B \&_g C) \wedge \\
&\vdash_{\boldsymbol{R}} (C \&_g A \leftrightarrow_g C \&_g B) \wedge \\
&\vdash_{\boldsymbol{R}} (\sim_g A \leftrightarrow_g \sim_g B) \wedge \\
&\vdash_{\boldsymbol{R}} ((A \leftrightarrow_g C) \leftrightarrow_g (B \leftrightarrow_g C)) \wedge \\
&\vdash_{\boldsymbol{R}} ((C \leftrightarrow_g A) \leftrightarrow_g (C \leftrightarrow_g B)) \wedge \\
&(\vdash_{\boldsymbol{R}} A \Leftrightarrow \vdash_{\boldsymbol{R}} B)
\end{aligned}
$$

Proof. Immediate from the definitions of Goldblatt provable and the bi-conditional connective. □

It will also be useful to have a second understanding of intentional conjunction and how it behaves with implication:

Proposition 2.1.5 (Intensional Conjunction Rule)**.**

$$\vdash_{\boldsymbol{R}} (A \circ_g B \to_g C) \Leftrightarrow \vdash_{\boldsymbol{R}} (A \to_g B \to_g C)$$

Proof. Unwrapping the definition for $(\circ_g)$ we just need to show the bi-conditional:

$(\Rightarrow)$: By permutation and Propositions 2.1.3.5 and 2.1.4 it suffices to show that:

$$\vdash_{\boldsymbol{R}} (\sim_g C \to_g A \to_g \sim_g B)$$

which follows by Propositions 2.1.3.5 and 2.1.4 and the Goldblatt axioms.

$(\Leftarrow)$: The reverse of the previous direction.

□

2.2 A Second Hilbert System

In this section we briefly introduce an alternative Hilbert system and demonstrate an equivalence between these systems. We do this because there are multiple different ways that we could charaterise the same system and showing equivalence gives us assurance that these different charaterisations give us the same logic.

2.2.1 The Alternative System

Just like in the previous section, we need to define a new propositional type but there are some things that we will need to take into consideration with regards to the underlying language of this logic. In the Goldblatt presentation we defined the disjunction and intensional conjunction connectives from the other connectives when we could have had them as different symbols altogether. By defining them as such we may have added extra 'axioms' into the system without realising it. Therefore in this new system we will take disjunction and fusion to be their own symbols in the underlying language. This alternative system presented by Slaney [N.D.].

Definition 2.2.1 (Slaney Proposition)**.** A *Slaney Proposition* is formalised in HOL as an inductive type:

$$\begin{aligned} \textit{s-prop} \;=\; & \textsf{s-VAR}\ \textit{string} \mid (\to_s)\ \textit{s-prop s-prop} \mid (\&_s)\ \textit{s-prop s-prop} \\ & \mid (\vee_s)\ \textit{s-prop s-prop} \mid (\circ_s)\ \textit{s-prop s-prop} \mid \sim_s \textit{s-prop} \mid \tau_s \end{aligned}$$

As before, before we will also define the bi-conditional for this system:

$$A \leftrightarrow_s B \stackrel{\text{def}}{=} (A \to_s B) \,\&_s\, (B \to_s A)$$

Now we will introduce a somewhat similar Hilbert Axiomatisation with this new language:

Definition 2.2.2 (Slaney Provable)**.** A proposition is *Slaney provable* (written $\vdash_{\boldsymbol{S}}$) if it is of the form of the following axioms:

$\vdash_{\boldsymbol{S}} (A \rightarrow_s A)$	Identity
$\vdash_{\boldsymbol{S}} ((A \rightarrow_s B \rightarrow_s C) \rightarrow_s B \rightarrow_s A \rightarrow_s C)$	Permutation
$\vdash_{\boldsymbol{S}} ((A \rightarrow_s B) \rightarrow_s (B \rightarrow_s C) \rightarrow_s A \rightarrow_s C)$	Transitivity
$\vdash_{\boldsymbol{S}} ((A \rightarrow_s A \rightarrow_s B) \rightarrow_s A \rightarrow_s B)$	Contraction
$\vdash_{\boldsymbol{S}} (A \,\&_s B \rightarrow_s A)$	Conjunction Elimination L
$\vdash_{\boldsymbol{S}} (A \,\&_s B \rightarrow_s B)$	Conjunction Elimination R
$\vdash_{\boldsymbol{S}} ((A \rightarrow_s B) \,\&_s (A \rightarrow_s C) \rightarrow_s A \rightarrow_s B \,\&_s C)$	Conjunction Introduction
$\vdash_{\boldsymbol{S}} (A \rightarrow_s A \vee_s B)$	Disjunction Introduction L
$\vdash_{\boldsymbol{S}} (B \rightarrow_s A \vee_s B)$	Disjunction Introduction R
$\vdash_{\boldsymbol{S}} ((A \rightarrow_s C) \,\&_s (B \rightarrow_s C) \rightarrow_s A \vee_s B \rightarrow_s C)$	Disjunction Elimination
$\vdash_{\boldsymbol{S}} (A \,\&_s (B \vee_s C) \rightarrow_s A \,\&_s B \vee_s A \,\&_s C)$	Distribution
$\vdash_{\boldsymbol{S}} ((A \rightarrow_s \sim_s B) \rightarrow_s B \rightarrow_s \sim_s A)$	Contraposition
$\vdash_{\boldsymbol{S}} (\sim_s (\sim_s A) \rightarrow_s A)$	Double Negation Elimination

Or if it is derivable from other Slaney Provable propositions using the following rules:

$$\frac{\vdash_{\boldsymbol{S}} A \quad \vdash_{\boldsymbol{S}} B}{\vdash_{\boldsymbol{S}} (A \,\&_s B)}\ \text{Adjunction} \qquad \frac{\vdash_{\boldsymbol{S}} A \quad \vdash_{\boldsymbol{S}} (A \rightarrow_s B)}{\vdash_{\boldsymbol{S}} B}\ \text{Modus Ponens}$$

$$\frac{\vdash_{\boldsymbol{S}} A}{\vdash_{\boldsymbol{S}} (\tau_s \rightarrow_s A)}\ \tau_s\ \text{Intro.} \qquad \frac{\vdash_{\boldsymbol{S}} (\tau_s \rightarrow_s A)}{\vdash_{\boldsymbol{S}} A}\ \tau_s\ \text{Elim.}$$

$$\frac{\vdash_{\boldsymbol{S}} (A \circ_s B \rightarrow_s C)}{\vdash_{\boldsymbol{S}} (A \rightarrow_s B \rightarrow_s C)}\ (\circ_s)\ \text{Intro.} \qquad \frac{\vdash_{\boldsymbol{S}} (A \rightarrow_s B \rightarrow_s C)}{\vdash_{\boldsymbol{S}} (A \circ_s B \rightarrow_s C)}\ (\circ_s)\ \text{Elim.}$$

The differences to note of note between Hilbert system and Definition 2.1.2 are the replacement of assertion with permutation and the introduction of rules of inference for $(\circ_s)$

Also just like for the previous system, there are useful propositions that we can show immediately.

Proposition 2.2.3 (Some theorems of the Slaney System)**.**

1. $\vdash_{\boldsymbol{S}} (A \rightarrow_s (A \rightarrow_s B) \rightarrow_s B)$ Assertion
2. $\vdash_{\boldsymbol{S}} (A \leftrightarrow_s \sim_s (\sim_s A))$ Double Negative Equivalence
3. $\vdash_{\boldsymbol{S}} ((A \rightarrow_s B) \leftrightarrow_s (\sim_s B \rightarrow_s \sim_s A))$ Contrapositive Alt

We can also show propositional replacement for Slaney propositions:

Proposition 2.2.4 (Propositional Replacement (Slaney))**.** If A, B are propositions and $A \leftrightarrow_s B$ then:

1. When A appears on either side of the $(\rightarrow_s)$, $(\&_s)$, $(\vee_s)$, $(\circ_s)$, or $(\leftrightarrow_s)$ connectives, then that statement is Slaney Provable if and only if the statement resulting from replacing A with B is Slaney Provable.
2. Similarly, when A appears behind a $\sim_s$ then $\sim_s A$ is Slaney Provable if and only if $\sim_s B$ is.
3. Furthermore, A is Slaney Provable if and only if B is.

$$\begin{aligned}
\vdash_{\boldsymbol{S}} (A \leftrightarrow_s B) \Rightarrow & \\
& \vdash_{\boldsymbol{S}} ((A \rightarrow_s C) \leftrightarrow_s (B \rightarrow_s C)) \wedge \\
& \vdash_{\boldsymbol{S}} ((C \rightarrow_s A) \leftrightarrow_s (C \rightarrow_s B)) \wedge \\
& \vdash_{\boldsymbol{S}} (\sim_s A \leftrightarrow_s \sim_s B) \wedge \\
& \vdash_{\boldsymbol{S}} (A \&_s C \leftrightarrow_s B \&_s C) \wedge \\
& \vdash_{\boldsymbol{S}} (C \&_s A \leftrightarrow_s C \&_s A) \wedge \\
& \vdash_{\boldsymbol{S}} (A \vee_s C \leftrightarrow_s B \vee_s C) \wedge \\
& \vdash_{\boldsymbol{S}} (C \vee_s A \leftrightarrow_s C \vee_s A) \wedge \\
& \vdash_{\boldsymbol{S}} (A \circ_s C \leftrightarrow_s B \circ_s C) \wedge \\
& \vdash_{\boldsymbol{S}} (C \circ_s A \leftrightarrow_s C \circ_s A) \wedge \\
& \vdash_{\boldsymbol{S}} ((A \leftrightarrow_s C) \leftrightarrow_s (B \leftrightarrow_s C)) \wedge \\
& \vdash_{\boldsymbol{S}} ((C \leftrightarrow_s A) \leftrightarrow_s (C \leftrightarrow_s A)) \wedge \\
& (\vdash_{\boldsymbol{S}} A \Leftrightarrow \vdash_{\boldsymbol{S}} B)
\end{aligned}$$

Corollary 2.2.5.

$$\begin{aligned}
\vdash_{\boldsymbol{S}} (A \leftrightarrow_s B) \wedge \vdash_{\boldsymbol{S}} (C \leftrightarrow_s D) \Rightarrow & \\
& \vdash_{\boldsymbol{S}} ((A \rightarrow_s C) \leftrightarrow_s (B \rightarrow_s D)) \wedge \\
& \vdash_{\boldsymbol{S}} (A \&_s C \leftrightarrow_s B \&_s D) \wedge \\
& \vdash_{\boldsymbol{S}} (A \vee_s C \leftrightarrow_s B \vee_s D) \wedge \\
& \vdash_{\boldsymbol{S}} (A \circ_s C \leftrightarrow_s B \circ_s D)
\end{aligned}$$

We can now show that there holds an equivalence between the $(\vee_s)$ and $(\circ_s)$ connectives and their defined counterparts.

Proposition 2.2.6 ($(\vee_s)$ Equivalence)**.**

$$\vdash_{\boldsymbol{S}} (A \vee_s B \leftrightarrow_s \sim_s (\sim_s A \&_s \sim_s B))$$

Proof.

$(\rightarrow_s)$: This follows directly from the Slaney Provable axioms.

$(\leftarrow_s)$: Because of equivalence of double negation, replacement, and contrapositive we only need to show that:

$$\vdash_{\boldsymbol{S}} (\sim_s (A \vee_s B) \rightarrow_s \sim_s A \&_s \sim_s B)$$

This follows by first applying conjunction introduction and then it is immedate using the Slaney axioms, the equivalence of double negative, and replacement.

□

Proposition 2.2.7 (($\circ_s$) Equivalence)**.**

$$\vdash_{\boldsymbol{S}} (A \ \circ_s \ B \ \leftrightarrow_s \ \sim_s (A \ \rightarrow_s \ \sim_s B))$$

Proof.

($\rightarrow_s$): This follows Intensional conjunction rule, contraposition, and axioms about implication.

($\leftarrow_s$): It follows from the axioms that:

$$\vdash_{\boldsymbol{S}} (A \ \rightarrow_s \ B \ \rightarrow_s \ A \ \circ_s \ B)$$

This direction then follows from the properties of implication and contraposition.

□

2.2.2 Equivalence to Goldblatt

Given that we have shown with Propositions 2.2.6 and 2.2.7 that ($\vee_s$) and ($\circ_s$) could have been reasonably defined in terms of ($\&_s$), ($\rightarrow_s$), and ($\sim_s$) we and that these Hilbert systems share a number of simularities; it seems reasonable that there should be a sort of equivalence between theorems in one system and theorems in the other; or rather, a result that looks something like:

$$\vdash_R A \iff \vdash_S A$$

But it can not be exactly this. Why? Every term, predicate, and function in HOL has an explicit type, so A, $\vdash_{\boldsymbol{R}}$, and $\vdash_{\boldsymbol{S}}$ all have their own types in the above equivalency. Referring back to Definitions 2.1.2 and 2.2.2 we note that $\vdash_{\boldsymbol{R}}$ is of type *g-prop* $\rightarrow$ *bool* and similarly $\vdash_{\boldsymbol{S}}$ is of type *s-prop* $\rightarrow$ *bool*, raising the question: what is the type of A? The answer is, of course, that the type for A is undefined as it cannot be both a *g-prop* and a *s-prop* at the same time. This means that we will need to establish a different, but similar, form of equivalence between these systems. The idea is that, if we can create two translation functions between the propositional types, and these functions also preserve validity, then the systems prove the same theorems under equivalence. That is we are attempting to show the following theorem:

Theorem 2.2.8. The Goldblatt and Slaney proof systems are equivalent: given a Goldblatt proposition A and a Slaney proposition B there exist translation functions s : *g-prop* $\rightarrow$ *s-prop* and g : *s-prop* $\rightarrow$ *g-prop* such that the following conditions hold:

1. $\vdash_{\boldsymbol{R}} \ A \ \Rightarrow \ \vdash_{\boldsymbol{S}} (s \ A)$
2. $\vdash_{\boldsymbol{S}} B \ \Rightarrow \vdash_{\boldsymbol{R}} \ (g \ B)$
3. $\vdash_{\boldsymbol{R}} \ A \ \Leftrightarrow \vdash_{\boldsymbol{R}} \ (g \ (s \ A))$
4. $\vdash_{\boldsymbol{S}} B \ \Leftrightarrow \ \vdash_{\boldsymbol{S}} (s \ (g \ B))$

That is:

$$\begin{array}{l} \vdash \exists\, s\ g.\ \forall\, A\ B. \\ \quad (\vdash_{\boldsymbol{R}}\ A\ \Rightarrow\ \vdash_{\boldsymbol{S}}\ (s\ A))\ \wedge\ (\vdash_{\boldsymbol{S}}\ B\ \Rightarrow \vdash_{\boldsymbol{R}}\ (g\ B))\ \wedge \\ \quad (\vdash_{\boldsymbol{R}}\ (g\ (s\ A))\ \Leftrightarrow \vdash_{\boldsymbol{R}}\ A)\ \wedge\ (\vdash_{\boldsymbol{S}}\ (s\ (g\ B))\ \Leftrightarrow\ \vdash_{\boldsymbol{S}}\ B) \end{array}$$

We will first establish the translation functions θ_s : *g-prop* $\rightarrow$ *s-prop* and θ_g : *s-prop* $\rightarrow$ *g-prop* that we will be using for the above theorem:

Definition 2.2.9. θ_s is a function of type *g-prop* $\rightarrow$ *s-prop* as defined below:

$$\begin{array}{lcl} \theta_s\ (\text{g-VAR}\ A) & \stackrel{\text{def}}{=} & \text{s-VAR}\ A \\ \theta_s\ (A\ \rightarrow_g\ B) & \stackrel{\text{def}}{=} & \theta_s\ A\ \rightarrow_s\ \theta_s\ B \\ \theta_s\ (A\ \&_g\ B) & \stackrel{\text{def}}{=} & \theta_s\ A\ \&_s\ \theta_s\ B \\ \theta_s\ (\sim_g A) & \stackrel{\text{def}}{=} & \sim_s\ (\theta_s\ A) \\ \theta_s\ \tau_g & \stackrel{\text{def}}{=} & \tau_s \end{array}$$

Definition 2.2.10. θ_g is a function of type *s-prop* $\rightarrow$ *g-prop* as defined below:

$$\begin{array}{lcl} \theta_g\ (\text{s-VAR}\ A) & \stackrel{\text{def}}{=} & \text{g-VAR}\ A \\ \theta_g\ (A\ \rightarrow_s\ B) & \stackrel{\text{def}}{=} & \theta_g\ A\ \rightarrow_g\ \theta_g\ B \\ \theta_g\ (A\ \&_s\ B) & \stackrel{\text{def}}{=} & \theta_g\ A\ \&_g\ \theta_g\ B \\ \theta_g\ (A\ \vee_s\ B) & \stackrel{\text{def}}{=} & \theta_g\ A\ \vee_g\ \theta_g\ B \\ \theta_g\ (A\ \circ_s\ B) & \stackrel{\text{def}}{=} & (\theta_g\ A)\ \circ_g\ (\theta_g\ B) \\ \theta_g\ (\sim_s\ A) & \stackrel{\text{def}}{=} & \sim_g \theta_g\ A \\ \theta_g\ \tau_s & \stackrel{\text{def}}{=} & \tau_g \end{array}$$

Lemma 2.2.11. A Slaney proposition A is equivalent to its translation $\theta_s\ (\theta_g\ A)$. That is:

$$\vdash_{\boldsymbol{S}}\ (A\ \leftrightarrow_s\ \theta_s\ (\theta_g\ A))$$

Proof. After inducting on the structure of A, the base and inductive cases (excepting ($\vee_s$) and ($\circ_s$)) follow by simplifying against the definitions of the translation functions and immediately follow from the properties of ($\leftrightarrow_s$) and the Slaney axioms. In the case of ($\vee_s$) and ($\circ_s$):

($\vee_s$): Simplifying with the definitions for the translation functions and the definition ($\vee_g$) we then need to show that

$$\vdash_{\boldsymbol{S}}\ (A\ \vee_s\ B\ \leftrightarrow_s\ \sim_s\ (\sim_s\ (\theta_s\ (\theta_g\ A))\ \&_s \sim_s\ (\theta_s\ (\theta_g\ B))))$$

Using Proposition 2.2.6, we note that it suffices to show that:

$$\vdash_{\boldsymbol{S}}\ (A\ \vee_s\ B\ \leftrightarrow_s\ \theta_s\ (\theta_g\ A)\ \vee_s\ \theta_s\ (\theta_g\ B))$$

but then this follows immediately from Corollary 2.2.5.

($\circ_s$): Same argument as above but instead of ($\vee_g$) and Proposition 2.2.6, we will use ($\circ_s$) and proposition 2.2.7.

□

We now have all that we need for our proof of equivalence.

Proof of Theorem 2.2.8.

1. Induct on Goldblatt Provable. It then follows immediately from the Slaney Provable rules, Assertion (Slaney), and that ($\vee_s$) and ($\circ_s$) have an equivalent representation.
2. Induct on Slaney Provable. It then follows immediately from Goldblatt Provable rules, Permutation (Goldblatt), and the intentional implication rule.
3. Inducting on the structure of A, we get that $\theta_g\ (\theta_s\ A)\ =\ A$.
4. Follows from Lemma 2.2.11 and replacement.

□

2.3 A Natural Deduction System

While Hilbert system presents a theory of logic as a collection of axioms and ways to transform propositions using rules of inference, it is not the only way that we could present a logic. A *Natural Deduction* system has very few 'axioms' (if any) and instead encodes the structure of the logic inside many different rules that govern how connectives behave under certain conditions. It should be noted that Natural Deduction operates using *sequents*, or *assumption-conclusion* pairs, where the conclusion is a result of the assumption, rather than propositions by themselves (like seen in Hilbert systems). An *assumption* is generally a structured collection of propositions while a conclusion is just a proposition by itself.

Just as we demonstrated an equivalent Hilbert system in the previous section, we shall consider a Natural Deduction system for Relevant Implication and show that it is equivalent to the previous Hilbert systems. From here on, we will be using the Goldblatt Axiomatization and language and as such, will be dropping all of the subscripts for the language, that is ($\rightarrow_g$) becomes ($\rightarrow$) and so on.

2.3.1 Natural Deduction for Relevant Logic

For sequents in classical logic, assumptions are put together by way of a single connective ($,$) which is representative of the set union of these assumptions. We are also allowed to move assumptions into the conclusion by way of implication introduction. For example:

$$\frac{a_1, ..., a_{n-1}, a_n \Vdash c}{a_1, ..., a_{n-1} \Vdash a_n \rightarrow c}$$

But the comma $(,)$ could also be read as the conjunction $(\wedge)$ between assumptions and still preserve meaning. This enables us to weaken the assumptions by adding new assumptions to the sequent that are irrelevant to the proof and then move them into the conclusion as the antecedent to an implication like so:

$$\cfrac{\cfrac{\Gamma \Vdash c}{\Gamma, a \Vdash c}}{\Gamma \Vdash a \to c}$$

This poses a problem for the creation of a Natural Deduction system for relevant implication where we are still able to weaken assumptions and shift assumptions into the conclusion while also maintaining that we avoid paradoxes of material implication.

Luckily Slaney [1990] gives us a solution by structuring our assumption with not one, but two different connectives: The first $(,)$ is coupled with conjunction, while the second $(;)$ is associated with implication. Of course, in HOL this means that we will need a new data-type for these connectives and, taking inspiration from Slaney [1990], we will also call these bunches:

Definition 2.3.1 (Bunches)**.** A *Bunch* is formalized in HOL as an inductive type:

$$\mathit{Bunch} \;=\; \mathsf{PROP}\ \mathit{g\text{-}prop} \mid (,)\ \mathit{Bunch}\ \mathit{Bunch} \mid (;)\ \mathit{Bunch}\ \mathit{Bunch}$$

We drop PROP for clarity whenever *g-prop* appears explicitly in a bunch.

Slaney [1990] also introduces a way for us to access a privileged position inside a bunch, which can be manipulated while keeping the rest of the structure as it is. In our formalisation, we will take this to be a sort of context with a hole and a replacement function that takes both a bunch and a context and produces a new bunch by filling the hole in the context with the bunch:

Definition 2.3.2 (B Contexts)**.** A *B-Context* is formalised in HOL as an inductive type:

$$\begin{aligned}\mathit{B\text{-}Context} \;=\; &\mathsf{HOLE} \mid (,)\ \mathit{B\text{-}Context}\ \mathit{Bunch} \mid (,)\ \mathit{Bunch}\ \mathit{B\text{-}Context}\\ &\mid (;)\ \mathit{B\text{-}Context}\ \mathit{Bunch} \mid (;)\ \mathit{Bunch}\ \mathit{B\text{-}Context}\end{aligned}$$

Definition 2.3.3 (Replacement)**.** Let Γ be a *B-Context* and B and X be of type *Bunch* then REPLACE (written as $\Gamma(X)$) is a function of type *B-Context* $\to$ *Bunch* $\to$ *Bunch* and is defined as:

$$\begin{aligned}\mathsf{HOLE}(X) &\stackrel{\mathsf{def}}{=} X\\ (\Gamma\ ,\ B)(X) &\stackrel{\mathsf{def}}{=} \Gamma(X)\ ,\ B\\ (B\ ,\ \Gamma)(X) &\stackrel{\mathsf{def}}{=} B\ ,\ \Gamma(X)\\ (\Gamma\ ;\ B)(X) &\stackrel{\mathsf{def}}{=} \Gamma(X)\ ;\ B\\ (B\ ;\ \Gamma)(X) &\stackrel{\mathsf{def}}{=} B\ ;\ \Gamma(X)\end{aligned}$$

While $(,)$ and $(;)$ in Definition 2.3.2 are the same symbols as seen in Definition 2.3.1, they are not the same object. We only use these symbols in this datatype because we take a *B-Context* together with Definition 2.3.3 these symbols effectively have the same meaning as their *Bunch* counterparts.

With these definitions in place we can now consider the rules of the Natural Deduction system.

Definition 2.3.4 (Natural Deduction)**.** If X is a *Bunch*, and A is a *g-prop*, then $X \Vdash_{\boldsymbol{R}} A$ is a *Relevant Sequent* if it is of the form:

$$\frac{}{A \Vdash_{\boldsymbol{R}} A}\ \boldsymbol{A}$$

Or can be derived from other *Relevant Sequents* using the following deduction rules:

$$\frac{X \Vdash_{\boldsymbol{R}} A \,\&\, B}{X \Vdash_{\boldsymbol{R}} A} \quad \frac{X \Vdash_{\boldsymbol{R}} A \,\&\, B}{X \Vdash_{\boldsymbol{R}} B}\ (\&)\boldsymbol{E} \qquad \frac{X \Vdash_{\boldsymbol{R}} A}{X \Vdash_{\boldsymbol{R}} {\sim\sim}A}\ ({\sim\sim})\boldsymbol{I}$$

$$\frac{X \Vdash_{\boldsymbol{R}} A}{X \Vdash_{\boldsymbol{R}} A \vee B} \quad \frac{X \Vdash_{\boldsymbol{R}} B}{X \Vdash_{\boldsymbol{R}} A \vee B}\ (\vee)\boldsymbol{I} \qquad \frac{X \Vdash_{\boldsymbol{R}} {\sim\sim}A}{X \Vdash_{\boldsymbol{R}} A}\ ({\sim\sim})\boldsymbol{E}$$

$$\frac{X \Vdash_{\boldsymbol{R}} A \quad Y \Vdash_{\boldsymbol{R}} B}{X \,,\, Y \Vdash_{\boldsymbol{R}} A \,\&\, B}\ (\&)\boldsymbol{I}$$

$$\frac{X \,;\, A \Vdash_{\boldsymbol{R}} B}{X \Vdash_{\boldsymbol{R}} A \rightarrow B}\ (\rightarrow)\boldsymbol{I} \qquad \frac{X \Vdash_{\boldsymbol{R}} A \rightarrow B \quad Y \Vdash_{\boldsymbol{R}} A}{X \,;\, Y \Vdash_{\boldsymbol{R}} B}\ (\rightarrow)\boldsymbol{E}$$

$$\frac{X \,;\, A \Vdash_{\boldsymbol{R}} B \quad Y \Vdash_{\boldsymbol{R}} {\sim}B}{X \,;\, Y \Vdash_{\boldsymbol{R}} {\sim}A}\ \boldsymbol{RAA}$$

$$\frac{X \Vdash_{\boldsymbol{R}} A \vee B \quad \Gamma(A) \Vdash_{\boldsymbol{R}} C \quad \Gamma(B) \Vdash_{\boldsymbol{R}} C}{\Gamma(X) \Vdash_{\boldsymbol{R}} C}\ (\vee)\boldsymbol{E}$$

Or can be derived from other *Relevant Sequents* using the following structural rules:

$$\frac{\Gamma(X \,,\, Y) \Vdash_{\boldsymbol{R}} A}{\Gamma(Y \,,\, X) \Vdash_{\boldsymbol{R}} A}\ (,)\ \textsf{Comm.} \qquad \frac{\Gamma(X) \Vdash_{\boldsymbol{R}} A}{\Gamma(X \,,\, Y) \Vdash_{\boldsymbol{R}} A}\ \textsf{Weakening}$$

$$\frac{\Gamma((X \,,\, Y) \,,\, Z) \Vdash_{\boldsymbol{R}} A}{\Gamma(X \,,\, Y \,,\, Z) \Vdash_{\boldsymbol{R}} A}\ (,)\ \textsf{Assoc. LR} \qquad \frac{\Gamma(X \,,\, Y \,,\, Z) \Vdash_{\boldsymbol{R}} A}{\Gamma((X \,,\, Y) \,,\, Z) \Vdash_{\boldsymbol{R}} A}\ (,)\ \textsf{Assoc. RL}$$

$$\frac{\Gamma(X \,,\, X) \Vdash_{\boldsymbol{R}} A}{\Gamma(X) \Vdash_{\boldsymbol{R}} A} \; (\,,\,) \text{ Idem. LR} \qquad \frac{\Gamma(X) \Vdash_{\boldsymbol{R}} A}{\Gamma(X \,,\, X) \Vdash_{\boldsymbol{R}} A} \; (\,,\,) \text{ Idem. RL}$$

$$\frac{\Gamma(\tau \,;\, X) \Vdash_{\boldsymbol{R}} A}{\Gamma(X) \Vdash_{\boldsymbol{R}} A} \; (\,;\,) \text{ Identity LR} \qquad \frac{\Gamma(X) \Vdash_{\boldsymbol{R}} A}{\Gamma(\tau \,;\, X) \Vdash_{\boldsymbol{R}} A} \; (\,;\,) \text{ Identity RL}$$

$$\frac{\Gamma(X \,;\, Y) \Vdash_{\boldsymbol{R}} A}{\Gamma(Y \,;\, X) \Vdash_{\boldsymbol{R}} A} \; (\,;\,) \text{ Comm.} \qquad \frac{\Gamma((X \,;\, Y) \,;\, Z) \Vdash_{\boldsymbol{R}} A}{\Gamma(X \,;\, Y \,;\, Z) \Vdash_{\boldsymbol{R}} A} \; (\,;\,) \text{ Assoc. LR}$$

$$\frac{\Gamma(X \,;\, X) \Vdash_{\boldsymbol{R}} A}{\Gamma(X) \Vdash_{\boldsymbol{R}} A} \; (\,;\,) \text{ Idem. LR}$$

Clarification should be made here regarding the system presented in Slaney [1990] and Definition 2.3.4. Slaney [1990] uses what he calls the 0^{th} assumption and assigns an identity-like behaviour with $(\,;\,)$. Slaney [1990] states that this 0^{th} assumption is the set of all logical truths. We, on the other hand, have defined our Bunches in terms of propositions so we can't use this set in our Bunches. We instead give the properties of the 0^{th} to τ as we hope that we can use τ to derive any logical truth that we would need in a given proof (we show this in the next subsection 2.3.2).

We can now work our way through a couple of propositions that we can produce within this natural deduction system.

Proposition 2.3.5 ($(\,;\,)$ Assoc. RL)**.**

$$\frac{\Gamma(X \,;\, Y \,;\, Z) \Vdash_{\boldsymbol{R}} A}{\Gamma((X \,;\, Y) \,;\, Z) \Vdash_{\boldsymbol{R}} A}$$

Proof. Immediate from $(\,;\,)$ Assoc. LR and $(\,;\,)$ Comm. □

There are also a number of useful results that we derive from the definition of $(\Vdash_{\boldsymbol{R}})$

Proposition 2.3.6 (Some Relevant Sequents)**.**

1. $A \to B \Vdash_{\boldsymbol{R}} {\sim}B \to {\sim}A$
2. $A \,\&\, B \Vdash_{\boldsymbol{R}} A$
3. $A \,\&\, B \Vdash_{\boldsymbol{R}} B$
4. $A \Vdash_{\boldsymbol{R}} A \vee B$
5. $B \Vdash_{\boldsymbol{R}} A \vee B$
6. $A \Vdash_{\boldsymbol{R}} {\sim}{\sim}A$
7. ${\sim}{\sim}A \Vdash_{\boldsymbol{R}} A$

2.3.2 From the Hilbert System to Natural Deduction and Back Again

In the last chapter we claimed that the reason that we give the identity behaviour, with respect to $(\,;\,)$, to τ was because we hoped that we would be able to derive any logical truth from it using the Natural Deduction rules. We shall show that now:

Theorem 2.3.7. If A is Goldblatt Provable, then $\tau \Vdash_{\boldsymbol{R}} A$ is a valid sequent of the Natural Deduction system, that is:

$$(\vdash_{\boldsymbol{R}} A) \;\Rightarrow\; \tau \Vdash_{\boldsymbol{R}} A$$

Proof. Induct on $\vdash_{\boldsymbol{R}}$. We skip the cases that follow immediately from already shown results or the natural deduction rules:

Suffixing: It suffices to show that $((A \to B) \mathbin{;} B \to C) \mathbin{;} A \Vdash_{\boldsymbol{R}} C$ by using $(\mathbin{;})$ Identity RL, and Definition 2.3.3. We can then get $(B \to C) \mathbin{;} (A \to B) \mathbin{;} A \Vdash_{\boldsymbol{R}} C$ through repeated applications of **A** and $(\to)$**E**. The result then follows from structural rules. The whole Natural Deduction proof is shown here for clarity, though we will omit it in the other parts of this proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \to C \Vdash_{\boldsymbol{R}} B \to C \qquad \dfrac{A \to B \Vdash_{\boldsymbol{R}} A \to B \qquad A \Vdash_{\boldsymbol{R}} A}{(A \to B) \mathbin{;} A \Vdash_{\boldsymbol{R}} B}(\to)\boldsymbol{E}}{(B \to C) \mathbin{;} (A \to B) \mathbin{;} A \Vdash_{\boldsymbol{R}} C}(\to)\boldsymbol{E}}{((B \to C) \mathbin{;} A \to B) \mathbin{;} A \Vdash_{\boldsymbol{R}} C}(\mathbin{;})\text{Assoc. RL}}{((A \to B) \mathbin{;} B \to C) \mathbin{;} A \Vdash_{\boldsymbol{R}} C}(\mathbin{;})\text{Comm.}}{(A \to B) \mathbin{;} B \to C \Vdash_{\boldsymbol{R}} A \to C}(\to)\boldsymbol{I}}{A \to B \Vdash_{\boldsymbol{R}} (B \to C) \to A \to C}(\to)\boldsymbol{I}}{\tau \mathbin{;} A \to B \Vdash_{\boldsymbol{R}} (B \to C) \to A \to C}(\mathbin{;})\text{Identity. RL}}{\tau \Vdash_{\boldsymbol{R}} (A \to B) \to (B \to C) \to A \to C}(\to)\boldsymbol{I}$$

Assertion: It suffices to show that $A \mathbin{;} A \to B \Vdash_{\boldsymbol{R}} B$. This result then follows from $(\to)$**E** and $(\mathbin{;})$ Comm.

Contraction: This follows largely from $(\mathbin{;})$ Idem. LR and $(\mathbin{;})$ Assoc. LR

Conj Introduction: It suffices to show that $(A \to B) \,\&\, (A \to C) \mathbin{;} A \Vdash_{\boldsymbol{R}} B \,\&\, C$ but we can get $(A \to B) \,\&\, (A \to C) \mathbin{;} A \Vdash_{\boldsymbol{R}} B$ and $(A \to B) \,\&\, (A \to C) \mathbin{;} A \Vdash_{\boldsymbol{R}} C$ by using $(\to)$**E** together with $(\&)$**E** L and $(\&)$**E** R respectively. The result then follows from $(\mathbin{,})$ Idem. LR.

Disj Elimination: It suffices to show $(A \to C) \,\&\, (B \to C) \mathbin{;} A \vee B \Vdash_{\boldsymbol{R}} C$ and we can show both $(A \to C) \,\&\, (B \to C) \mathbin{;} A \Vdash_{\boldsymbol{R}} C$ and $(A \to C) \,\&\, (B \to C) \mathbin{;} B \Vdash_{\boldsymbol{R}} C$ like in the previous case. This result then follows from $(\vee)$**E**.

Distribution: It suffices to show $A \,\&\, (B \vee C) \Vdash_{\boldsymbol{R}} A \,\&\, B \vee A \,\&\, C$. We can show that $A \,\&\, (B \vee C) \Vdash_{\boldsymbol{R}} A$ and $A \,\&\, (B \vee C) \Vdash_{\boldsymbol{R}} B \vee C$ from $(\&)$**E** L and $(\&)$**E** R. From this we can get both:

$$A \,\&\, (B \vee C) \mathbin{,} B \Vdash_{\boldsymbol{R}} A \,\&\, B \vee A \,\&\, C$$
$$A \,\&\, (B \vee C) \mathbin{,} C \Vdash_{\boldsymbol{R}} A \,\&\, B \vee A \,\&\, C$$

By using $(\&)$**I** and $(\vee)$**I**. The result then follows from $(\vee)$**E** and $(\mathbin{,})$ Idem. LR.

Contraposition: It suffices to show $(A \rightarrow \sim B) \;;\; B \Vdash_{\boldsymbol{R}} \sim A$. Then we then get the result from **RAA** and $(\rightarrow)$**E**.

□

After this result we may wonder if every valid sequent can be taken to a proposition and furthermore, if the original sequent can be reconstructed from that proposition. To show this we need a function that takes bunches to propositions.

Definition 2.3.8. β is a function of type *Bunch* $\rightarrow$ *g-prop* given by:

$$\begin{array}{lcl} \beta(A) & \stackrel{\text{def}}{=} & A \\ \beta(X \,,\, Y) & \stackrel{\text{def}}{=} & \beta(X) \;\&\; \beta(Y) \\ \beta(X \,;\, Y) & \stackrel{\text{def}}{=} & \beta(X) \circ \beta(Y) \end{array}$$

The first of two theorems we will want to show to establish that every valid sequent of the Natural Deduction system has an associated Goldblatt Provable proposition:

Theorem 2.3.9. If $X \Vdash_{\boldsymbol{R}} A$ is a valid sequent of the Natural Deduction system, then $\beta(X) \rightarrow A$ is Goldblatt Provable, that is:

$$X \Vdash_{\boldsymbol{R}} A \Rightarrow \vdash_{\boldsymbol{R}} \beta(X) \rightarrow A$$

In order to show this we need the following lemmata:

Lemma 2.3.10.

$$\vdash_{\boldsymbol{R}} \beta(\Gamma(A \vee B)) \rightarrow \beta(\Gamma(A)) \vee \beta(\Gamma(B))$$

Sketch. We induct on Γ. Noting that Γ is a *B-Context* there are 2 $(\,,\,)$ cases and two $(\,;\,)$ cases. The $(\,,\,)$ cases hinge on the commutativity of $(\&)$ and the distribution axiom and can then be proven directly from the Goldblatt Axioms. The $(\,;\,)$ cases rely on the commutativity of $(\circ)$ and the fact that it is also distributive over $(\vee)$ (Propositions 2.1.3.6 and 2.1.3.7). □

Lemma 2.3.11.

$$(\vdash_{\boldsymbol{R}} \beta(X) \rightarrow \beta(Y)) \Rightarrow \forall \Gamma.\ \vdash_{\boldsymbol{R}} \beta(\Gamma(X)) \rightarrow \beta(\Gamma(Y))$$

Proof. Inducting on the structure of Γ we get that the HOLE and $(\,,\,)$ cases are immediate from the Goldblatt Axioms. When then *Bunch* sits on the right of the $(\,;\,)$ the result follows from the definitions of $(\circ)$ and $(\leftrightarrow)$ and both the Goldblatt Axioms and Proposition 2.1.3.5.

When the *Bunch* sits on the left of the $(\,;\,)$ we first show that, by the commutativity of $(\circ)$ and Proposition 2.1.4, that the required result follows from showing $\vdash_{\boldsymbol{R}} \beta(\Gamma(X)) \circ \beta(Z) \rightarrow \beta(\Gamma(Y)) \circ \beta(Z)$ which follows by the above argument. □

Proof (Theorem 2.3.9). First we induct on ($\Vdash_{\boldsymbol{R}}$). Most of the 23 cases are immediate from Goldblatt axioms or follow easily from the Natural Deduction structural rules as well as Lemma 2.3.11. Thus we will explore the only complicated case which is for $(\vee)$**E**:

We note that $\vdash_{\boldsymbol{R}} \beta(\Gamma(A)) \vee \beta(\Gamma(B)) \rightarrow C$ holds from the inductive assumptions and the Goldblatt provable rule. Then, by the Goldblatt Axioms, it suffices to show $\vdash_{\boldsymbol{R}} \beta(\Gamma(A)) \vee \beta(\Gamma(B)) \rightarrow C$. By specializing Lemma 2.3.11 appropriately we get that this follows from Lemma 2.3.10, Suffixing and Modus Ponens. □

Theorem 2.3.9 shows that we can take sequents to Goldblatt provable proposition, but not that the original sequent is recoverable from the Goldblatt proposition, though this is definitely the case:

Theorem 2.3.12. If $\beta(X) \rightarrow A$ is Goldblatt Provable, then $X \Vdash_{\boldsymbol{R}} A$ is a valid sequent inside the Natural Deduction system.

$$(\vdash_{\boldsymbol{R}} \beta(X) \rightarrow A) \Rightarrow X \Vdash_{\boldsymbol{R}} A$$

We will need the following lemma to show this:

Lemma 2.3.13.

$$X \Vdash_{\boldsymbol{R}} \beta(X)$$

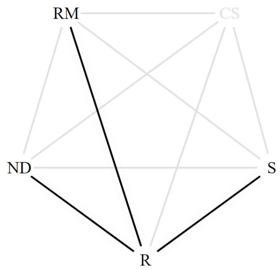
Proof. Induct on the structure of X. The base and (**,**) cases follow immediately from the Natural Deduction rules. For the (**;**) case we use the definition of ($\circ$) and then apply **RAA**. Then we only need to show that $X \,;\, \beta(X) \rightarrow {\sim}\beta(Y) \Vdash_{\boldsymbol{R}} {\sim}\beta(Y)$ and $Y \Vdash_{\boldsymbol{R}} {\sim}{\sim}\beta(Y)$. But these follow by using $({\sim}{\sim})\boldsymbol{I}$ for $Y \Vdash_{\boldsymbol{R}} {\sim}{\sim}\beta(Y)$ and $(\rightarrow)$**E** and (**;**) Comm. along with **A** and Replacement for $X \,;\, \beta(X) \rightarrow {\sim}\beta(Y) \Vdash_{\boldsymbol{R}} {\sim}\beta(Y)$. □

Proof (Theorem 2.3.12). This suffices to show that given $\vdash_{\boldsymbol{R}} \beta(X) \rightarrow A$ we can get $\tau \,;\, X \Vdash_{\boldsymbol{R}} A$. Then it follows immediately from $(\rightarrow)$**E**, Lemma 2.3.13, and Theorem 2.3.7. □

2.3.3 Summary

This chapter introduced, formalised, and showed the equivalence of two different Hilbert systems and a Natural Deduction system for relevance logic. Inside the HOL4 development: the introduction of the Goldblatt System (in `GoldblattRLScript.sml`) consists of 146 lines of source code; the introduction of the Slaney system (in `SlaneyRLScript.sml`) consists of 244 lines of source code; showing the equivalence between the Goldblatt and Slaney systems requires a further 102 lines of source code in `GoldblattSlaneyEquivScript.sml`; and the introduction and equivalence results of the Natural Deduction system consists of 434 lines of HOL4 in

`NaturalDeductionScript.sml`; for a total of 926 lines of HOL4. There was also further work done in support of this chapter in the `RLRulesScript.sml` notably Propositions 2.1.3.6 and 2.1.3.7. Now that we have thoroughly explored our *syntactic* systems, it follows that we should now consider our first *semantic* one.



Chapter 3

The Routley-Meyer Semantics

While *syntactic* systems are primarily concerned with transforming expressions to determine what is a theorem of the system, *semantic* systems gives us an alternative way of looking at logic by providing *meaning* (in our case, truth) to statements.

One method of ascribing meaning to propositions is through the consideration of *possible worlds*, which is the idea that something must be true only if we can show it is necessarily true in all alternative realities that are consistent with what we already know. By way of example: Suppose you have an opaque bottle on your desk and picked it up only to find that it was heavier than it should be. You would have to conclude that the bottle contained something as you cannot conceive of a world whereby the bottle is both empty and weighs more than it ought. However you would not know what was in the bottle as you could conceive both a reality where the bottle is full of cursed treasure and another where the bottle contains a live squid.

The *possible worlds* semantics saw massive success throughout the 1960s with its application to the modal logics due to the introduction of a binary relation over a set of worlds [Blackburn et al., 2002]. Each world in this case is considered to be a place where propositions hold. In modal logic this relationship (R) denotes *accessibility*, this means that when we have $R\,x\,y$ for worlds x and y then we consider y to be accessible from x; or rather if we find ourselves in the world x (though we wouldn't know we were in x) then we would consider y to be a *possible world* [Blackburn et al., 2002].

In 1973 Richard Routley (later Sylvan) and Robert Meyer introduced a possible world semantic that extended some of the advances made in modal logic by using a ternary relation R over the set of worlds as well as a dual operator over worlds, $*$ [Routley and Meyer, 1973]. Routley and Meyer [1973] propose a reading of $R\ w\ x\ y$ that states: the world y is compatible relative to w and x together. It is natural then that the *rules* of w are just propositions of the form $A \rightarrow B$ that w asserts to be true. Then if x asserts

A, then B is just a result of applying the rules of w to x, and as such y asserts B. For example, if we had a rule that holds at world such as w that says: If your bottle is heavier than normal, then it contains a squid. And in the world x we had that the bottle was heavier than it should be, then the world y is compatible with the information that your bottle contains a squid. x^* on the other hand acts as a dual world to x. In this, we mean that if there exists a *possible world* that does not explicitly reject a set of statements then there must exist a *possible world* that asserts that all of these statements are true, ergo if there is a world that does not explicately state that there isn't a squid in the bottle, then there is a world that asserts that there is a squid in your bottle. We also choose a world that we read as the *logical world* and it is where we evaluate truth for the whole model.

An important note to make is that quite unlike semantics for modal logic, we also do not require that worlds be *consistent*. In some cases it is necessary that some worlds be inconsistent otherwise we could show that $A \;\&\; {\sim}A \;\rightarrow\; B$ is a rule of all possible worlds, including the *logical world* for any choice of A and B that this should not be derivable.

The connection between semantic and syntactic systems are two metatheorems called *soundness* and *completeness* results. A *soundness* result between a syntactic theory and a semantic theory states that if proposition p is a theorem of the syntactic theory, then p holds in every model of the semantic theory. We may also say that the syntactic theory is *sound* with respect to the semantic theory. Conversely a *completeness* result states that if a proposition holds in every model of the semantic theory, then it is a theorem of the syntactic theory. Ergo, the syntactic theory is *complete* with respect to the semantic theory.

This chapter introduces the Routley-Meyer Semantics and establishes soundness and completeness results between the Goldblatt axioms and this system. In Section 3.1 we present the Routley-Meyer Semantic theory and show that our Hilbert System is sound with respect to these semantics. In Section 3.2 we will formalise the full completeness proof for relevant implication.

3.1 The Routley-Meyer System

As with the formalisations given in Chapter 2, we will first begin with the types of structures that we will be working with in the Routley-Meyer semantics:

Definition 3.1.1. An α *FRAME* is a record type with fields W, 0, R, and $*$. W is a set of worlds of type α. 0 is of type α, R is a ternary relation over α, $*$ is an operator over α.

$$\alpha\ \mathit{FRAME} = \langle\ \mathsf{W}\ :\ \alpha\ \rightarrow\ \mathit{bool};\ \mathsf{0}\ :\ \alpha;\ \mathsf{R}\ :\ \alpha\ \rightarrow\ \alpha\ \rightarrow\ \alpha\ \rightarrow\ \mathit{bool};\ *\ :\ \alpha\ \rightarrow\ \alpha\ \rangle$$

When $\mathcal{F}$ is an α *FRAME* and we have $R_{\mathcal{F}}\,0\,x\,y$ we write $x\ \leq_{\mathcal{F}}\ y$ as shorthand.

Accompanying the α *FRAME* datatype we shall also consider a valuation function which takes our propositional letters and assigns them sets of worlds. An α *MODEL* is a record type with fields $\mathcal{F}$ being a α *FRAME*, and V begin a map of type: *string* $\rightarrow \alpha \rightarrow$ *bool*:

$$\alpha \textit{ MODEL} = \langle\ \mathcal{F}\ :\ \alpha \textit{ FRAME};\ V\ :\ \textit{string} \rightarrow \alpha \rightarrow \textit{bool}\ \rangle$$

Now that we have these basic dataypes we introduce a set of conditions on members of the datatypes to generate *frames* and *models* for relevant implication:

Definition 3.1.2 (R-Frame). If $\mathcal{F}$ is of type α *FRAME* and then it is an R-Frame if it has the following closure properties:

1. $0 \in \mathcal{F}^W$

2. If $x \in \mathcal{F}^W$ then so is x^*

$\mathcal{F}$ must also satisfy the following postulates that make $\mathcal{F}$ a member of the class of logics that contain Relevance Logic [Routley et al., 1982, pp.298-299].

3. When we have $x \in \mathcal{F}^W$ we also have $x \leq_{\mathcal{F}} x$

4. When we have x, x', y, y', z, $z' \in \mathcal{F}^W$ such that $x' \leq_{\mathcal{F}} x$, $y' \leq_{\mathcal{F}} y$, and $z \leq_{\mathcal{F}} z'$ then $R_{\mathcal{F}}\ x\ y\ z \Rightarrow R_{\mathcal{F}}\ x'\ y'\ z'$

5. When $x \in \mathcal{F}^W$ we have $x^{**} = x$

6. When we have w, x, $y \in \mathcal{F}^W$ then we have $R_{\mathcal{F}}\ w\ x\ y \Rightarrow R_{\mathcal{F}}\ w\ y^*\ x^*$

Fo Relevant Implication we also require that $\mathcal{F}$ satisfy the following three propositions [Routley, 1980]:

7. When we have $x \in \mathcal{F}^W$ we also have $R_{\mathcal{F}}\ x\ x\ x$

8. When we have x, y, $z \in \mathcal{F}^W$ then we have $R_{\mathcal{F}}\ x\ y\ z \Rightarrow R_{\mathcal{F}}\ y\ x\ z$

9. If we have w, x, y, z, $a \in \mathcal{F}^W$ where we have both $R_{\mathcal{F}}\ w\ x\ a$ and $R_{\mathcal{F}}\ a\ y\ z$ then there exists a world $b \in \mathcal{F}^W$ such that both $R_{\mathcal{F}}\ x\ y\ b$ and $R_{\mathcal{F}}\ w\ b\ z$

In other words:

$$
\begin{aligned}
&\vdash \textsf{R-Frame}\ \mathcal{F} \iff \\
&\quad 0 \in \mathcal{F}^W \land (\forall x.\ x \in \mathcal{F}^W \Rightarrow x^* \in \mathcal{F}^W) \land \\
&\quad (\forall x.\ x \in \mathcal{F}^W \Rightarrow x \leq_{\mathcal{F}} x) \land \\
&\quad (\forall x\ x'\ y\ y'\ z\ z'. \\
&\qquad x \in \mathcal{F}^W \land y \in \mathcal{F}^W \land z \in \mathcal{F}^W \land x' \in \mathcal{F}^W \land y' \in \mathcal{F}^W \land \\
&\qquad z' \in \mathcal{F}^W \land x' \leq_{\mathcal{F}} x \land y' \leq_{\mathcal{F}} y \land z \leq_{\mathcal{F}} z' \land R_{\mathcal{F}}\ x\ y\ z \Rightarrow \\
&\qquad\quad R_{\mathcal{F}}\ x'\ y'\ z') \land (\forall x.\ x \in \mathcal{F}^W \Rightarrow x^{**} = x) \land \\
&\quad (\forall w\ x\ y. \\
&\qquad R_{\mathcal{F}}\ w\ x\ y \land x \in \mathcal{F}^W \land y \in \mathcal{F}^W \land w \in \mathcal{F}^W \Rightarrow R_{\mathcal{F}}\ w\ y^*\ x^*) \land \\
&\quad (\forall x.\ x \in \mathcal{F}^W \Rightarrow R_{\mathcal{F}}\ x\ x\ x) \land \\
&\quad (\forall x\ y\ z.\ x \in \mathcal{F}^W \land y \in \mathcal{F}^W \land z \in \mathcal{F}^W \land R_{\mathcal{F}}\ x\ y\ z \Rightarrow R_{\mathcal{F}}\ y\ x\ z) \land \\
&\quad \forall w\ x\ y\ z\ a. \\
&\qquad x \in \mathcal{F}^W \land y \in \mathcal{F}^W \land z \in \mathcal{F}^W \land w \in \mathcal{F}^W \land a \in \mathcal{F}^W \land R_{\mathcal{F}}\ w\ x\ a \land \\
&\qquad R_{\mathcal{F}}\ a\ y\ z \Rightarrow \\
&\qquad\quad \exists b.\ R_{\mathcal{F}}\ x\ y\ b \land R_{\mathcal{F}}\ w\ b\ z \land b \in \mathcal{F}^W
\end{aligned}
$$

We will then take our R-Frame and assign a valuation function taking propositional variable to worlds and creating an α ***MODEL***. But not all valuation functions are created equal.

As we are taking 0 to be the *logical world* and that we read $R_{\mathcal{M}}\ w\ x\ y$ as y is compatible with the result of applying the rules of w to x; then the natural reading of $x \leq_{\mathcal{M}} y$ is naturally y is compatible with the result of applying the rules of *logic* to x. A desirable condition on our *logical world* is: by applying the rules of the logical world, we cannot change the valuation of the propositional variables; as such we will need a condition on the valuation function to reflect this.

What we need is: when x and y are worlds such that $x \leq_{\mathcal{F}} y$ then when s is a propositional variable and $x \in \mathcal{M}^V\ s$ then $y \in \mathcal{M}^V\ s$. We call this the *hereditary property* as y inherits it membership of $\mathcal{M}^V\ s$ from its relationship to x.

Definition 3.1.3 (Hereditary property)**.** An α ***MODEL***, $\mathcal{M}$ has the *hereditary property* if and only if whenever we both have $x \leq_{\mathcal{M}} y$ and $x \in \mathcal{M}^V\ s$ then we also have $y \in \mathcal{M}^V\ s$:

$$\vdash \textsf{Hereditary}\ \mathcal{M} \iff \forall x\ y\ s.\ x \leq_{\mathcal{M}} y \land x \in \mathcal{M}^V\ s \Rightarrow y \in \mathcal{M}^V\ s$$

If an α ***MODEL*** $\mathcal{M}$ has the hereditary property, then we say that $\mathcal{M}$ is *hereditary*.

Definitions 3.1.2 and 3.1.3 then lead us to the formal definition of an R-Model:

Definition 3.1.4 (R-Model)**.** If $\mathcal{M}$ is of type α ***MODEL*** then it is an R-Model if $\mathcal{M}^{\mathcal{F}}$ is an R-Frame and $\mathcal{M}$ is hereditary.

$$\vdash \textsf{R-Model}\ \mathcal{M} \iff \textsf{R-Frame}\ \mathcal{M}^{\mathcal{F}} \land \textsf{Hereditary}\ \mathcal{M}$$

As we consider each of the worlds in $\mathcal{M}^W$ to be a place where propositions can be true, we define a relationship $\models$ that is defined as follows:

Definition 3.1.5 (Holds)**.** Given a R-Model $\mathcal{M}$ and a world $w \in \mathcal{M}^W$ then when a *g-prop* p holds at w in the model $\mathcal{M}$ then we write $\mathcal{M}, w \models p$. When $p = \textsf{g-VAR}\ s$ for some *string* s then p holds at w if and only if $w \in \mathcal{M}^V\ s$. When $p = A \mathbin{\&} B$ then p holds at w if and only if $\mathcal{M}, w \models A$ and $\mathcal{M}, w \models B$. When $p = {\sim}A$ then p holds at w if and only if $\neg\mathcal{M}, w^* \models A$. When $p = A \to B$ p holds at w holds if and only if whenever $R_{\mathcal{M}}\ w\ x\ y$ for $x, y \in \mathcal{M}^W$ and $\mathcal{M}, x \models A$ then we have that $\mathcal{M}, y \models B$. When $p = \tau$ then p holds at w if and only if $0 \leq_{\mathcal{M}} w$.

$$\begin{aligned}
&\mathcal{M}, w \models (\textsf{g-VAR}\ s) \overset{\text{def}}{=} w \in \mathcal{M}^W \land w \in \mathcal{M}^V\ s \\
&\mathcal{M}, w \models (A \mathbin{\&} B) \overset{\text{def}}{=} w \in \mathcal{M}^W \land \mathcal{M}, w \models A \land \mathcal{M}, w \models B \\
&\mathcal{M}, w \models ({\sim}A) \overset{\text{def}}{=} w \in \mathcal{M}^W \land \neg\mathcal{M}, w^* \models A \\
&\mathcal{M}, w \models (A \to B) \overset{\text{def}}{=} \\
&\quad w \in \mathcal{M}^W \land \\
&\quad \forall x\ y.\ x \in \mathcal{M}^W \land y \in \mathcal{M}^W \land R_{\mathcal{M}}\ w\ x\ y \land \mathcal{M}, x \models A \Rightarrow \mathcal{M}, y \models B \\
&\mathcal{M}, w \models \tau \overset{\text{def}}{=} w \in \mathcal{M}^W \land 0 \leq_{\mathcal{M}} w
\end{aligned}$$

We also say that a proposition p holds at a model if and only if it holds at 0 and we write $\mathcal{M} \models p$.

It follows from immediately from this definition that:

Corollary 3.1.6 (($\vee$) Holds)**.** When $\mathcal{M}$ is an R-Model and $w \in \mathcal{M}^W$ we have that $\mathcal{M}, w \models (A \vee B)$ if and only if at least one of $\mathcal{M}, w \models A$ or $\mathcal{M}, w \models B$

$$\vdash \textsf{R-Model}\ \mathcal{M} \land w \in \mathcal{M}^W \Rightarrow \mathcal{M}, w \models (A \vee B) \Leftrightarrow \mathcal{M}, w \models A \vee \mathcal{M}, w \models B$$

It is also the case that the hereditary property extends to all formulae:

Lemma 3.1.7 (Hereditary Lemma)**.** When $\mathcal{M}$ is an R-Model and x and y are worlds of $\mathcal{M}$ such that $x \leq_{\mathcal{M}} y$, then when a proposition p holds at x then it also holds at y:

$$\vdash \textsf{R-Model}\ \mathcal{M} \land x \in \mathcal{M}^W \land y \in \mathcal{M}^W \land \mathcal{M}, x \models p \land x \leq_{\mathcal{M}} y \Rightarrow \mathcal{M}, y \models p$$

Proof. Induct on the structure of p. The g-VAR and (&) cases are immediate using Definitions 3.1.3, 3.1.4, and 3.1.5.

($\to$)**:** We suppose that $\mathcal{M}, x \models (A \to B)$ and $R_{\mathcal{M}}\ y\ a\ b$ with $a, b \in \mathcal{M}^W$ where $\mathcal{M}, a \models A$. It suffices to show that $R_{\mathcal{M}}\ x\ a\ b$ by Definition 3.1.5. From Definition 3.1.2.4 all that is require is to show that $a \leq_{\mathcal{M}} a$ and $b \leq_{\mathcal{M}} b$, both of which follow from Definition 3.1.2.3.

(${\sim}$)**:** We suppose that $\mathcal{M}, x \models ({\sim}A)$ which means that $\neg\mathcal{M}, x^* \models A$ and we are required to show that $\neg\mathcal{M}, y^* \models A$. But we know that $y^* \leq_{\mathcal{M}} x^*$ by Definition 3.1.2.6. Then, due to the inductive hypothesis, this case follows from Definition 3.1.2.2.

(τ): We suppose that $\mathcal{M}, x \models \tau$ which just means that $0 \leq_{\mathcal{M}} x$ and we need to show that $0 \leq_{\mathcal{M}} y$. By definition 3.1.2.4 we get that all we need to show is $0 \leq_{\mathcal{M}} 0$ and $y \leq_{\mathcal{M}} y$, these follow from Definitions 3.1.2.7 and 3.1.2.3 respectively.

□

3.1.1 Soundness

In Sections 2.2 and 2.3 we showed that the new proof systems that we were introducing proved exactly the same theorems that the Goldblatt Axiomatisation could prove. In this subsection we continue with this trajectory by showing one direction of a sort of equivalence between the semantic and syntactic theories. *Soundness* for **R** and the class of R-Models states: if p is a theorem and $\mathcal{M}$ is an R-Model, then $\mathcal{M} \models p$.

We will first state a useful lemma before showing the soundness result:

Lemma 3.1.8 (Contraction Lemma)**.** If $\mathcal{F}$ is an R-Frame with worlds w, x, and y that are related such that $R_{\mathcal{F}}\ w\ x\ y$ then we can always construct an intermediate world b such that $R_{\mathcal{F}}\ w\ x\ b$ and $R_{\mathcal{F}}\ b\ x\ y$:

$$\vdash \text{R-Frame } \mathcal{F} \wedge R_{\mathcal{F}}\ w\ x\ y \wedge w \in \mathcal{F}^W \wedge x \in \mathcal{F}^W \wedge y \in \mathcal{F}^W \Rightarrow \exists x'.\ x' \in \mathcal{F}^W \wedge R_{\mathcal{F}}\ w\ x\ x' \wedge R_{\mathcal{F}}\ x'\ x\ y$$

We call Lemma 3.1.8 the Contraction Lemma as the direct result is that if in some model $\mathcal{M}$ based off $\mathcal{F}$ has a world w such that $\mathcal{M}, w \models (A \to A \to B)$ then it is the case that $\mathcal{M}, w \models (A \to B)$.

Theorem 3.1.9 (Soundness)**.** The Goldblatt Axioms are *sound* with respect to the Routley-Meyer Semantics

$$\vdash\ \vdash_{\boldsymbol{R}} p \wedge \text{R-Model } \mathcal{M} \Rightarrow \mathcal{M} \models p$$

Proof. First we induct on $\vdash_{\boldsymbol{R}}$. Many of the resulting cases are uninteresting, tedious, and/or follow as a result of the hereditary lemma and application of the R-Frame rules. There are however, a few exceptions. After simplifying with Definitions 3.1.2 and 3.1.5:

Contraction: Let aab be a world where $\mathcal{M}, aab \models (A \to A \to B)$ such that $aab \leq_{\mathcal{M}} ab$. We then need $\mathcal{M}, ab \models (A \to B)$, so suppose that we have a world a such that $\mathcal{M}, a \models A$ and $R_{\mathcal{M}}\ ab\ a\ b$ for some world b.

By the fact that we already have $\mathcal{M}, aab \models (A \to A \to B)$, we need only show the existance of worlds x, x', and y such that they are all in $\mathcal{M}^W$, $\mathcal{M}, x \models A$, $\mathcal{M}, x' \models A$, and y relates to these worlds like so: $R_{\mathcal{M}}\ aab\ x'\ y \wedge R_{\mathcal{M}}\ y\ x\ b$. This works as $R_{\mathcal{M}}\ aab\ x'\ y$ guarantees that $\mathcal{M}, y \models (A \to B)$ and then $R_{\mathcal{M}}\ y\ x\ b$ give us $\mathcal{M}, b \models B$.

There is of course only one world where we know A holds, so x and x' are assigned to a leaving us the problem of finding our y world. With help from the aptly named

Contraction Lemma (3.1.8) it suffices to show only that $R_{\mathcal{M}}\ aab\ a\ b$. We then use Definitions 3.1.2.4 followed by 3.1.2.3 and 3.1.2.8 to get the required result.

Contradiction: We suppose that there is a world x where $\mathcal{M}, x \models (A \to {\sim}B)$ which relates to another world y by $x \leq_{\mathcal{M}} y$ with the goal in mind of $\mathcal{M}, y \models (B \to {\sim}A)$. We then suppose there exists worlds a and b such that $R_{\mathcal{M}}\ y\ a\ b$ such that $\mathcal{M}, a^* \models A$ and $\mathcal{M}, b \models B$ with the idea to derive a contradiction.

First we show that $R_{\mathcal{M}}\ x\ a^*\ b^*$ by first applying Definition 3.1.2.4 and then 3.1.2.2, 3.1.2.3, and 3.1.2.6.

But because of this we now know that $\neg\mathcal{M}, b^{**} \models B$ because we know that $\mathcal{M}, x \models (A \to {\sim}B)$ and $\mathcal{M}, a^* \models A$. Because of Definition 3.1.2.5 we derive a contradiction.

□

3.2 Completeness

As in the previous section, a *completeness* result states that if a proposition holds in every model of the semantic theory, then it is a theorem of the syntactic theory. For Relevance Logic, this looks like this:

$$(\forall\, \mathcal{M}.\ \textsf{R-Model}\ \mathcal{M} \Rightarrow \mathcal{M} \models p) \Rightarrow \vdash_{\boldsymbol{R}} p$$

We will hold off on proving this statement until the end of this section but the idea is: if p is a non-theorem of the Goldblatt Axioms, then we should be able to construct an R-Model $\mathcal{M}$ where p does not hold, or rather $\mathcal{M} \not\models p$. We shall first briefly introduce some ideas that will be used extensively throughout this section.

We utilise the HOL list type, given by type α *list*, extensively throughout this section and we will briefly state a few facts that will be helpful to understanding the next definitions:

- Lists can be expressed in explicately where every element of the list is defined and seperated by a semicolon, for example $[a;\ b;\ c]$.
- **Append** is a function that takes an object a and a list l and returns another list with a as the first value and all others preceeding it, for example $a :: l$.
- **Concatenation** is given by ($+\!\!+$) and represents joining one list to another head to tail.
- **Filter** is a function that a function f from elements of the list to *bool* and a list l. It returns another list l' which is precisely the list containing elements of l that satisfy f. We only use the case where $f = (\lambda\, x.\ x \neq a)$ for some predefined a and as such we use the statement $l - a$ to mean: remove all elements of a from l.

- (set) is a function that takes a list l to the set of all of its elements. We omit explicitly showing this function, prefering to use statements such as $l \subseteq L$ to mean that all the elements of l are in L

Definition 3.2.1 (List Conjunction)**.** Where [] is the empty list of type *g-prop list*; p,q are of type *g-prop*; and γ is of type *g-prop list* we define the conjunction over a list ($\mathcal{E}$) as below:

$$\begin{array}{lll} \mathcal{E}[] & \stackrel{\text{def}}{=} & \tau \\ \mathcal{E}[p] & \stackrel{\text{def}}{=} & p \\ \mathcal{E}(p :: q :: \gamma) & \stackrel{\text{def}}{=} & p \;\&\; \mathcal{E}(q :: \gamma) \end{array}$$

By design $\mathcal{E}$ behaves very similarly to (&) and we will briefly outline some of its properties now.

$\mathcal{E}$ naturally extends the adjunction rule per the Goldblatt Axiomatisation:

Proposition 3.2.2 ($\mathcal{E}$ Adjunction)**.** If we have a list of Goldblatt propositions γ then $\mathcal{E}\gamma$ is also a theorem:

$$\vdash \gamma \subseteq \boldsymbol{R} \Rightarrow \vdash_{\boldsymbol{R}} (\mathcal{E}\gamma)$$

It also naturally extends conjunction elimination and introduction:

Proposition 3.2.3 ($\mathcal{E}$ Elimination)**.** If δ and γ are lists then it follows that:

$$\begin{array}{ll} 1. & \vdash \delta \neq [] \Rightarrow \vdash_{\boldsymbol{R}} (\mathcal{E}(\delta +\!\!+ \gamma) \rightarrow \mathcal{E}\delta) \\ 2. & \vdash \gamma \neq [] \Rightarrow \vdash_{\boldsymbol{R}} (\mathcal{E}(\delta +\!\!+ \gamma) \rightarrow \mathcal{E}\gamma) \end{array}$$

Further if p is a proposition that appears in γ then:

$$3. \quad \vdash \mathsf{MEM}\ p\ \gamma \Rightarrow \vdash_{\boldsymbol{R}} (\mathcal{E}\gamma \rightarrow p)$$

Proposition 3.2.4 ($\mathcal{E}$ Introduction)**.** If a proposition q implies every member of a non-empty list of propositions γ then it implies $\mathcal{E}\gamma$.

$$\vdash \gamma \neq [] \wedge (\forall p.\ \mathsf{MEM}\ p\ \gamma \Rightarrow \vdash_{\boldsymbol{R}} (q \rightarrow p)) \Rightarrow \vdash_{\boldsymbol{R}} (q \rightarrow \mathcal{E}\gamma)$$

We can also split and conjoin lists as we wish:

Proposition 3.2.5. If γ and δ are both non-empty lists of propositions then:

$$\begin{array}{l} \vdash \alpha \neq [] \wedge \beta \neq [] \Rightarrow \\ \quad \vdash_{\boldsymbol{R}} (\mathcal{E}\alpha \;\&\; \mathcal{E}\beta \rightarrow \mathcal{E}(\alpha +\!\!+ \beta)) \wedge \\ \quad \vdash_{\boldsymbol{R}} (\mathcal{E}(\alpha +\!\!+ \beta) \rightarrow \mathcal{E}\alpha \;\&\; \mathcal{E}\beta) \end{array}$$

Finally we can remove all instances of a proposition A from a list γ and conjunct the resulting list $(\gamma - A)$ with A to imply the original list.

Proposition 3.2.6.

$$\vdash \gamma \neq [] \wedge \mathsf{MEM}\ A\ \gamma \wedge (\gamma - A) \neq [] \Rightarrow \\ \vdash_{\boldsymbol{R}} (\mathcal{E}(\gamma - A)\ \&\ A \to \mathcal{E}\gamma)$$

With the definition of $\mathcal{E}$ we will find it useful to introduce a new relation:

Definition 3.2.7 (Conjunctive Consequence)**.** Where Γ is a set of *g-prop*s and p is a *g-prop* we say that p is a conjunctive consequence of Γ (Written $\Gamma \vdash^{\&}_{\boldsymbol{R}} p$) when there exists a list γ of *g-prop*s the elements of which are in Γ such that $\vdash_{\boldsymbol{R}} (\mathcal{E}\gamma \to p)$. That is:

$$\vdash \Gamma \vdash^{\&}_{\boldsymbol{R}} p \Leftrightarrow \exists \gamma.\ \gamma \neq [] \wedge \gamma \subseteq \Gamma \wedge \vdash_{\boldsymbol{R}} (\mathcal{E}\gamma \to p)$$

We naturally extend this idea further by replacing $\boldsymbol{R}$ with any theory Θ and writing $\Gamma \vdash^{\&}_{\Theta} p$. That is:

$$\vdash \Gamma \vdash^{\&}_{\Theta} p \Leftrightarrow \exists \gamma.\ \gamma \neq [] \wedge \gamma \subseteq \Gamma \wedge \mathcal{E}\gamma \to p \in \Theta$$

We differentiate between these two symbols by calling $(\vdash^{\&}_{\Theta})$ conjunctive consequence with respect to Θ.

Which naturally leads us to ask about sets that are closed under $(\vdash^{\&}_{\boldsymbol{R}})$ and $(\vdash^{\&}_{\Theta})$:

Definition 3.2.8 (R-Theory)**.** We call a set of propositions Γ an R-Theory if it is closed under conjunctive consequence:

$$\vdash \mathsf{R\text{-}Theory}\ \Gamma \Leftrightarrow \forall p.\ \Gamma \vdash^{\&}_{\boldsymbol{R}} p \Rightarrow p \in \Gamma$$

We can qualify R-Theories further:

Definition 3.2.9. Given an R-Theory Γ we say that Γ is Regular if all theorems of $\boldsymbol{R}$ are in Γ, that is:

$$\vdash \mathsf{Regular}\ \Gamma \Leftrightarrow \mathsf{R\text{-}Theory}\ \Gamma \wedge \forall p.\ \vdash_{\boldsymbol{R}} p \Rightarrow p \in \Gamma$$

We can also say that Γ is Prime if whenever a disjunct $A \vee B \in \Gamma$ then one of the disjuncts is also in Γ:

$$\vdash \mathsf{Prime}\ \Gamma \Leftrightarrow \\ \mathsf{R\text{-}Theory}\ \Gamma \wedge \forall A\ B.\ A \vee B \in \Gamma \Rightarrow A \in \Gamma \vee B \in \Gamma$$

Finally, if an R-Theory Γ is both Regular and Prime, then we say that it is Ordinary:

$$\vdash \mathsf{Ordinary}\ \Gamma \Leftrightarrow \mathsf{Prime}\ \Gamma \wedge \mathsf{Regular}\ \Gamma$$

It follows immediately from this definition that:

Corollary 3.2.10. When Γ is an R-Theory, then $\&\gamma$ appears in Γ if and only if each conjunct also appears in Γ:

$$\vdash \textsf{R-Theory}\ \Gamma \ \wedge\ \gamma \neq [] \ \Rightarrow\ \&\gamma \ \in\ \Gamma \ \Leftrightarrow\ \gamma \subseteq \Gamma$$

Corollary 3.2.11. When Γ is an R-Theory, and γ is a non-empty list such that; if, for each element B of γ we have $A \rightarrow B \in \Gamma$, then we have that $A \rightarrow \&\gamma \in \Gamma$:

$$\vdash \gamma \neq [] \ \wedge\ \textsf{R-Theory}\ \Gamma \ \wedge\ (\forall B.\ \textsf{MEM}\ B\ \gamma \ \Rightarrow\ A \rightarrow B \in \Gamma) \ \Rightarrow \\ A \rightarrow \&\gamma \in \Gamma$$

Corollary 3.2.12. When Γ is Prime, then when $\sim\&\gamma$ appears in Γ it is also the case there is a member of γ, x such that $\sim x \in \Gamma$:

$$\vdash \textsf{Prime}\ A \ \wedge\ \sim\&\gamma \in A \ \wedge\ \gamma \neq [] \ \Rightarrow\ \exists x.\ \textsf{MEM}\ x\ \gamma \ \wedge\ \sim x \in A$$

We can also extend the idea behind R-Theory to be based on any Ordinary set of propositions:

Definition 3.2.13 (Θ-Theory)**.** Similarly to Definition 3.2.8 we call Γ a Θ-Theory if Θ is Ordinary and Γ is closed under conjunctive consequence with respect to Θ:

$$\vdash \Theta\textsf{-Theory}\ \Gamma \ \Leftrightarrow\ \textsf{Ordinary}\ \Theta \ \wedge\ \forall p.\ \Gamma \vdash^{\&}_{\Theta} p \ \Rightarrow\ p \in \Gamma$$

We shall now look at a final interesting property of ($\vdash^{\&}_{\boldsymbol{R}}$) in depth. If we can have $\Gamma \cup \{p\} \vdash^{\&}_{\boldsymbol{R}} q$ with $\Gamma \vdash^{\&}_{\boldsymbol{R}} p$, then we can eliminate p and get $\Gamma \vdash^{\&}_{\boldsymbol{R}} q$. Even though we only use this result in a minor way in the next section, it is still an important result by itself as it shows that we actually cannot get anywhere by creating a chain of consequences ($\vdash^{\&}_{\boldsymbol{R}}$) starting at Γ that we wound not be able to get directly from Γ itself.

Theorem 3.2.14.

$$\vdash \Gamma \vdash^{\&}_{\boldsymbol{R}} p \ \wedge\ \Gamma \cup \{p\} \vdash^{\&}_{\boldsymbol{R}} q \ \Rightarrow\ \Gamma \vdash^{\&}_{\boldsymbol{R}} q$$

Proof. The case where $p \in \Gamma$ holds trivially so we will assume that $p \notin \Gamma$, similarly we will excuse the case where $\Gamma \vdash^{\&}_{\boldsymbol{R}} q$ as it is also trivial. This leaves us just with the case where we have $\gamma \subseteq \Gamma$, $\vdash_{\boldsymbol{R}} (\&\gamma \rightarrow p)$, and $\delta \subseteq \Gamma \cup \{p\}$, $\vdash_{\boldsymbol{R}} (\&\delta \rightarrow q)$, and MEM $p\ \delta$ with the goal in mind to find a non-empty list, all members of which are in Γ that implies q.

The list that we want is of course $(\delta - p) +\!\!+ \gamma$ as it then suffices to show that $\vdash_{\boldsymbol{R}} (\&((\delta - p) +\!\!+ \gamma) \rightarrow \&\delta)$ because of suffixing and that $((\delta - p) +\!\!+ \gamma) \subseteq \Gamma$.

But showing $\vdash_{\boldsymbol{R}} (\&((\delta - p) +\!\!+ \gamma) \rightarrow \&\delta)$ turns out to be relatively easy. First we use Proposition 3.2.4 to show that we only need to show for every member r of δ that $\vdash_{\boldsymbol{R}} (\&((\delta - p) +\!\!+ \gamma) \rightarrow r)$. If $r = p$ then this result holds because of Proposition 3.2.3.1, that and that γ implies p. When $r \neq p$ then we know that MEM $r\ (\delta - p)$ and this result follows straightforwardly from Proposition 3.2.3.3. □

3.2.1 Building a logical world

In order to show completeness through contraposition, then for any given non-theorem p we need an R-Model where p does not appear inside the set that we will use to represent the privileged world.

In order to do this we will first need a world that rejects p:

Definition 3.2.15. Where p is a proposition, $\boldsymbol{R}$ is the set of all theorems under the Goldblatt axiomatisation, and $\{A_n\}_{n \in \mathbb{N}}$ is an enumeration over $\mathcal{U}(: \textit{g-prop})$;[1] we define $\theta_n(p)$ inductively:

$$\begin{array}{lcl} \theta_0(p) & \stackrel{\mathsf{def}}{=} & \boldsymbol{R} \\ \theta_{n\,+\,1}(p) & \stackrel{\mathsf{def}}{=} & \texttt{if}\ \theta_n(p)\ \cup\ \{A_n\}\ \vdash^{\&}_{\boldsymbol{R}}\ p\ \texttt{then}\ \theta_n(p)\ \texttt{else}\ \theta_n(p)\ \cup\ \{A_n\} \end{array}$$

We also define $\Theta(p)$ as the union of all $\theta_n(p)$

$$\Theta(p) \stackrel{\mathsf{def}}{=} \bigcup \{ \theta_n(p) \mid n \in \mathcal{U}(: \textit{num}) \}$$

We shoud note here that when $\vdash_{\boldsymbol{R}} p$ then p is still in $\Theta(p)$, so it is not the case that $\Theta(p)$ rejects p in all cases. However, as we will only be looking at cases where $\nvdash_{\boldsymbol{R}} p$, we can nominally talk about $\Theta(p)$ as the set that rejects p.

Corollary 3.2.16. $\Theta(p)$ contains all Goldblatt provable propositions:

$$\vdash \boldsymbol{R} \subseteq \Theta(p)$$

Corollary 3.2.17. When we have $n \leq m$ then we have that $\theta_n(p) \subseteq \theta_m(p)$ (which we characterise as):

$$\vdash q \in \theta_n(p) \land n \leq m \Rightarrow q \in \theta_m(p)$$

Given how we have constructed $\Theta(p)$ it should feel right that if we have a finite subset of $\Theta(p)$ then there should be a $\theta_n(p)$ which contains all members of the subset. This turns out to be an important result for many proofs regarding $\Theta(p)$ so we will show it now:

Lemma 3.2.18. If s is a finite set of *g-prop* then it is a subset of $\Theta(p)$ if and only if there exists an n such that $s \subseteq \theta_n(A)$:

$$\vdash \mathsf{FINITE}\ s \Rightarrow s \subseteq \Theta(A) \Leftrightarrow \exists n.\ s \subseteq \theta_n(A)$$

Proof. First we induct on FINITE and use Definition 3.2.15 to get that $s \subseteq \theta_n(A)$ and $p \in \theta_m(A)$ with the idea to find an r such that $p \in \theta_r(A) \land s \subseteq \theta_r(A)$.

If $m \leq n$ then we can just use Corollary 3.2.17 to choose $r = n$. When $n < m$ we instead choose $r = m$ and use Corollary 3.2.17 again along with the definition of subset. ☐

[1] It is important to note that this $\mathcal{U}(: \textit{g-prop})$ is countable and this was also formalised though we will omit it from this thesis.

It is important to note that when p is not a theorem then it is the case that p is not a conjunctive consequence of $\Theta(p)$ and futhermore we cannot add any further propositions to $\Theta(p)$ without having p as a conjunctive consequence. We refer to this property informally throughout the rest of this chapter as *maximal exclusion*:[2]

Lemma 3.2.19.

$$\vdash\ \nvdash_{\boldsymbol{R}}\ p\ \Rightarrow\ (\Theta(p) \nvdash^{\&}_{\boldsymbol{R}} p)\ \wedge\ \forall q.\ q\ \notin\ \Theta(p)\ \Rightarrow\ \Theta(p)\ \cup\ \{q\}\ \vdash^{\&}_{\boldsymbol{R}}\ p$$

Proof. The first conjunct is relatively straightforward as we use the definition of $(\vdash^{\&}_{\boldsymbol{R}})$ and then Lemma 3.2.18 to show that $\Theta(p)\ \vdash^{\&}_{\boldsymbol{R}}\ p$ if and only if $p\ \in\ \boldsymbol{R}$ leading to a contradiction.

We will show the second conjunct by contraposition, that is given $\nvdash_{\boldsymbol{R}}\ p$ and $\Theta(p)\ \cup\ \{q\} \nvdash^{\&}_{\boldsymbol{R}} p$ we will show that $q\ \in\ \Theta(p)$. Of course this just means that we need to show that there is an n such that $q\ \in\ \theta_n(p)$.

Where we use $\eta(q)$ to denote the position of q inside our enumeration of formulae $\{A_n\}_{n\in\mathbb{N}}$ we choose $n\ =\ \eta(q)\ +\ 1$ and then by Definition 3.2.15 we only need to show that:

$$\theta_{\eta(q)}(p)\ \cup\ \{q\} \nvdash^{\&}_{\boldsymbol{R}} p$$

Using the definition of $(\vdash^{\&}_{\boldsymbol{R}})$ this is just the same as showing $\neg(\ \gamma\ \subseteq\ \theta_{\eta(q)}(p) \cup \{q\})$ from the assumption that $\neg(\ \gamma\ \subseteq\ \Theta(p) \cup \{q\})$ which is the case because of Lemma 3.2.18. ☐

It immediately follows from this and Theorem 3.2.14 that:

Corollary 3.2.20. When p is not a theorem of the Goldblatt Axioms, then we have that $\Theta(p)$ is an R-Theory:

$$\vdash\ \nvdash_{\boldsymbol{R}}\ p\ \Rightarrow\ \textsf{R-Theory}\ \Theta(p)$$

One of the consequences of Lemma 3.2.19 is that for any $q\ \notin\ \Theta(p)$ there is always a list γ such that each of its elements are either in $\Theta(p)$ or are equal to q itself ($\gamma\ \subseteq\ \Theta(p)\ \cup\ \{q\}$) such that $\vdash_{\boldsymbol{R}}\ (\mathcal{\&}\gamma\ \rightarrow\ p)$. This may seem like an obvious restatement of Lemma 3.2.19 but it leaves us with the beginnings of an important property that our type of *maximal exclusion* has:

Lemma 3.2.21. Let $\nvdash_{\boldsymbol{R}}\ p$ and $q\ \notin\ \Theta(p)$, then there exists a proposition $c\ \in\ \Theta(p)$ such that $\vdash_{\boldsymbol{R}}\ (c\ \&\ q\ \rightarrow\ p)$:

$$\vdash\ \nvdash_{\boldsymbol{R}}\ p\ \wedge\ q\ \notin\ \Theta(p)\ \Rightarrow\ \exists c.\ \vdash_{\boldsymbol{R}}\ (c\ \&\ q\ \rightarrow\ p)\ \wedge\ c\ \in\ \Theta(p)$$

We omit the full proof due to it being case heavy:

[2] We deliberately did not define this notion as a HOL predicate because we use very similar notions in the following subsections and there is no neat way of abstracting this concept across multiple types of conjunctive consequences and different structures.

Sketch. If $\vdash_{\boldsymbol{R}} (\mathcal{E}\gamma \rightarrow p)$ where $\gamma \subseteq \Theta(p) \cup \{q\}$, then the c we want is $\mathcal{E}(\gamma - q)$. □

This lemma gives us the last tool we need to show that $\Theta(p)$ is Prime:

Theorem 3.2.22 ($\Theta(p)$ is Prime)**.**

$$\vdash \nvdash_{\boldsymbol{R}} p \Rightarrow \textsf{Prime}\ \Theta(p)$$

Proof. By contraposition it suffices to show that given $\nvdash_{\boldsymbol{R}} p$ and $a \notin \Theta(p) \wedge b \notin \Theta(p)$ that we show $a \vee b \notin \Theta(p)$.

Of course because of Lemma 3.2.21 we know that there are c and d in $\Theta(p)$ such that both $\vdash_{\boldsymbol{R}} (c \,\&\, a \rightarrow p)$ and $\vdash_{\boldsymbol{R}} (d \,\&\, b \rightarrow p)$. Through the use of the Goldblatt axiomatisation we get that $\vdash_{\boldsymbol{R}} (c \,\&\, d \,\&\, (a \vee b) \rightarrow p)$ and from Definition 3.2.8 we get that $c \,\&\, d \in \Theta(A)$.

Assuming $a \vee b \in \Theta(p)$ we derive a contradiction as we get that $\Theta(p) \vdash^{\&}_{\boldsymbol{R}} p$ which fails because of Lemma 3.2.19. □

Corollary 3.2.23.

$$\vdash \nvdash_{\boldsymbol{R}} p \Rightarrow \textsf{Ordinary}\ \Theta(p)$$

And as a result of Corollary 3.2.23 we get that:

Corollary 3.2.24.

$$\vdash \nvdash_{\boldsymbol{R}} p \Rightarrow \Theta(p)\text{-}\textsf{Theory}\ \Gamma \Leftrightarrow \forall q.\ \Gamma \vdash^{\&}_{\Theta(p)} q \Rightarrow q \in \Gamma$$

3.2.2 Constructing Canonical Models

In this subsection we construct the α *MODEL* that we will use for completeness proof. But first we have one last definition to look at:

Definition 3.2.25 (APPLYING)**.** Let w and x be sets of Goldblatt Propositions then w *applied to* x (written $w(x)$) is the set of propositions B where $\mathcal{E}\gamma \rightarrow B \in w$ and $\gamma \subseteq x$:

$$\vdash w(x) = \{\, p \mid \exists \gamma.\ \gamma \neq [] \wedge \mathcal{E}\gamma \rightarrow p \in w \wedge \gamma \subseteq x \,\}$$

We also get the following results:

Proposition 3.2.26. That is, if $\gamma \subseteq w(x)$ when θ-Theory w for some θ. Then $\mathcal{E}\gamma \in w(x)$:

$$\vdash \theta\text{-}\textsf{Theory}\ w \wedge \gamma \subseteq w(x) \wedge \gamma \neq [] \Rightarrow \mathcal{E}\gamma \in w(x)$$

Proposition 3.2.27. When $\nvdash_{\boldsymbol{R}} p$ and w is a $\Theta(p)$-Theory then we get that $w(x)$ is also a $\Theta(p)$-Theory:

$$\vdash \theta\text{-}\textsf{Theory}\ w \Rightarrow \theta\text{-}\textsf{Theory}\ w(x)$$

These show that w imposes its closure properties onto $w(x)$. Now we have everything we need to define our frame:

Definition 3.2.28 (Canonical-Frame)**.** The *canonical frame* rejecting a proposition p is defined by the set of worlds $\mathfrak{F}(p)^W$ being given by sets of Goldblatt propositions that are both Prime and are $\Theta(p)$-Theory. The privileged world where truth is evaluated is given by $\Theta(p)$. Our Ternary relation $R_{\mathfrak{F}(p)}$ is given by inclusion of applied worlds, that is $R_{\mathfrak{F}(p)}\ x\ y\ z \ \Leftrightarrow\ x(y)\ \subseteq\ z$.[3] And finally, our star operator is given by taking the set of all propositions that are not negated in the world x, that is $p\ \in\ x^* \ \Leftrightarrow\ {\sim}p\ \notin\ x$. In HOL this looks like:

$$
\begin{aligned}
&\mathfrak{F}(p) \stackrel{\mathsf{def}}{=} \\
&\quad \langle \mathsf{W}\ :=\ \{\, w \mid \mathsf{Prime}\ w\ \wedge\ \Theta(p)\text{-}\mathsf{Theory}\ w\,\};\ \mathsf{0}\ :=\ \Theta(p); \\
&\quad\ \ \mathsf{R}\ := \\
&\quad\quad (\lambda\, x\ y\ z. \\
&\quad\quad\quad x(y)\ \subseteq\ z\ \wedge\ x\ \in\ \{\, w \mid \mathsf{Prime}\ w\ \wedge\ \Theta(p)\text{-}\mathsf{Theory}\ w\,\}\ \wedge \\
&\quad\quad\quad y\ \in\ \{\, w \mid \mathsf{Prime}\ w\ \wedge\ \Theta(p)\text{-}\mathsf{Theory}\ w\,\}\ \wedge \\
&\quad\quad\quad z\ \in\ \{\, w \mid \mathsf{Prime}\ w\ \wedge\ \Theta(p)\text{-}\mathsf{Theory}\ w\,\}); \\
&\quad\ \ *\ :=\ (\lambda\, x.\ \{\, p \mid {\sim}p\ \notin\ x\,\})\rangle
\end{aligned}
$$

It should be noted that the canonical frame can only reject the proposition p if p is not Goldblatt Provable. However, for our purposes we will not come across a situation where our chosen p is a theorem of the Goldblatt axiomatisation, so we are nominally fine in saying that this is the canonical frame *rejecting* p.

Similarly we define our canonical model:

Definition 3.2.29 (Canonical-Model)**.** The *canonical model* rejecting a proposition is given by the frame $\mathfrak{F}(p)$ and the valuation over the propositional variables is just set inclusion. So if s is a string and w a world in the frame then $\mathfrak{M}(p),\ w \models\ (\mathsf{g\text{-}VAR}\ s)$ if and only if $\mathsf{g\text{-}VAR}\ s\ \in\ w$

$$\mathfrak{M}(p)\ \stackrel{\mathsf{def}}{=}\ \langle \mathcal{F}\ :=\ \mathfrak{F}(p);\ V\ :=\ (\lambda\, s.\ \{\, w \mid \mathsf{g\text{-}VAR}\ s\ \in\ w\,\})\rangle$$

As the goal behind creating $\mathfrak{M}(p)$ is to construct an R-Frame and valuation function such that p does not hold at $\mathfrak{M}(p)^{\mathcal{F}}.0$ then the first thing to show is that the frame that $\mathfrak{M}(p)$ is based off is actually an R-Frame.

Theorem 3.2.30 ($\mathfrak{F}(p)$ is an R-frame)**.** Where $\nvdash_{\boldsymbol{R}}\ p$, then the Canonical frame rejecting p is an R-Frame.

$$\vdash\ \nvdash_{\boldsymbol{R}}\ p\ \Rightarrow\ \mathsf{R\text{-}Frame}\ \mathfrak{F}(p)$$

[3] Note the extra conditions on $R_{\mathfrak{F}(p)}$ in the HOL definition, this is because we want a nessarsary condition of the ternary relation to be that all arguments to $R_{\mathfrak{F}(p)}$ must be members of HOLtm(Canonical-Frame p).W. This is so that we only need to worry about the case where all worlds in question are in the world set.

Showing the first eight postulates of R-Frame are relatively straightforward. Though we will first need to show the following lemmata:

Lemma 3.2.31. Where we have that $\nvdash_{\boldsymbol{R}}\ p$ then it is the case that $\Theta(p)$ is closed under conjunctive consequence with respect to itself.

$$\vdash\ \nvdash_{\boldsymbol{R}}\ p\ \Rightarrow\ \Theta(p)\text{-Theory}\ \Theta(p)$$

Proof. Given that $\gamma\ \subseteq\ \Theta(p)$ and $\&\gamma\ \rightarrow\ q\ \in\ \Theta(p)$ we are required to show that $q\ \in\ \Theta(p)$. We get that because $\Theta(p)$ is an R-Theory that $\&\gamma\ \in\ \Theta(p)$. From here we then use R-Theory along with Proposition 2.1.3.8 to show the result. □

Together with Theorem 3.2.22 gives us that $\Theta(p)$ is in fact inside $\mathfrak{F}(p)^W$. We also need that $*$ is its own inverse (or equivalently is of order 2), note that we do not need p to be a non-theorem for this case:

Lemma 3.2.32.

$$\vdash x\ \in\ \mathfrak{F}(p)^W\ \Rightarrow\ x^{**}\ =\ x$$

Proof. This is immediate from the fact that we can get Ordinary $\Theta(p)$ from $\Theta(A)$-Theory x. The result then follows from the regularity of $\Theta(p)$ and that $\vdash_{\boldsymbol{R}}\ (a\ \rightarrow\ \sim\sim a)$ and $\vdash_{\boldsymbol{R}}\ (\sim\sim a\ \rightarrow\ a)$. □

Finally we also need to that $\mathfrak{F}(p)^W$ is closed under $*$:

Lemma 3.2.33. Where we have that $\nvdash_{\boldsymbol{R}}\ p$ and $x\ \in\ \mathfrak{F}(p)^W$ then we also know that $x^*\ \in\ \mathfrak{F}(p)^W$

$$\vdash x\ \in\ \mathfrak{F}(p)^W\ \wedge\ \nvdash_{\boldsymbol{R}}\ p\ \Rightarrow\ x^*\ \in\ \mathfrak{F}(p)^W$$

Though we omit the proof due to its length.

Now we could go on to show postulates 1-8 from Definition 3.1.2 right now, but we will hold off on the full proof of Theorem 3.2.30 as it turns out that showing that the final postulate 9 holds for $\mathfrak{F}(p)$ when p is a non-theorem is a bit more involved.

First and foremost we should consider what would be the natural choice of world for the choice of b. We would need that everything in $x(y)$ to be known to be in b and that $w(b)$ be a subset of z so, if $q\ \notin\ z$ and $p\ \rightarrow\ q\ \in\ w$ then $p\ \notin\ b$. Finally we would also need to show that b is $\Theta(p)$-Theory and Prime. To do this we are going to use a trick we have already seen before and construct b in the exact same way we constructed $\Theta(p)$:

Definition 3.2.34. Where Θ, i, and r are sets of *g-prop*, and $\{A_n\}_{n\in\mathbb{N}}$ is an enumeration over $\mathcal{U}$(: *g-prop*) then we define $b_n(\Theta,\ i,\ r)$ inductively as:

$$b_0(\Theta,\ i,\ r)\ \stackrel{\text{def}}{=}\ i$$

$$b_{n\,+\,1}(\Theta,\ i,\ r)\ \stackrel{\text{def}}{=}$$

$\quad$ `if` $\exists A.\ A\ \in\ r\ \wedge\ (b_n(\Theta,\ i,\ r)\ \cup\ \{A_n\})\vdash^{\&}_{\Theta}\ A$ `then` $b_n(\Theta,\ i,\ r)$

$\quad$ `else` $b_n(\Theta,\ i,\ r)\ \cup\ \{A_n\}$

We then take the union of $b_n(\Theta, i, r)$ over n to get:

$$B(\Theta, i, r) \stackrel{\mathsf{def}}{=} \bigcup \{ b_n(\Theta, i, r) \mid n \in \mathcal{U}(:\mathit{num}) \}$$

We can then think about $B(\Theta, i, r)$ as the world that contains all of i and rejects everything in r under Θ.

Due to the constraints HOL4 we have to be explicit about the initial set i, rejection set r, and which Θ we will be defining our rejection with respect to otherwise we would not be able to show this holds for any choice of z.

Given that Definition 3.2.34 by design shares has remarkable similarities to Definition 3.2.15, and due to the fact that it shares a similar argument to Lemma 3.2.18, we will state without proof that:

Proposition 3.2.35. If s is a finite set of *g-prop* then it is a subset of $B(\Theta, i, r)$ if and only if there exists an n such that $s \subseteq b_n(\Theta, i, r)$:

$$\vdash \mathsf{FINITE}\ s \Rightarrow s \subseteq B(\Theta, i, r) \Leftrightarrow \exists n.\ s \subseteq b_n(\Theta, i, r)$$

As we reject everything in r and assume everything in i; the natural choice for b when showing Definition 3.1.2.9 is:

$$B(\Theta(p), x(y), \{ p \mid \exists q.\ p \rightarrow q \in w \wedge q \notin z \})$$

We shall now show that this set satisfies the conditions for membership to $\mathfrak{F}(p)^W$ when several preconditions hold with the idea in our heads that we will show that these preconditions hold in the proof of Theorem 3.2.30.

Lemma 3.2.36. $B(\Theta(p), x(y), \{ p \mid \exists q.\ p \rightarrow q \in w \wedge q \notin z \})$ is an $\Theta(p)$-Theory when the following three preconditions hold:

1. $\nvdash_{\boldsymbol{R}}\ p$
2. $\Theta(p)$-Theory w
3. $w(B(\Theta(p), x(y), \{ p \mid \exists q.\ p \rightarrow q \in w \wedge q \notin z \})) \subseteq z$

Or rather:

$$\begin{aligned} \vdash\ & \nvdash_{\boldsymbol{R}}\ p \wedge \Theta(p)\text{-}\mathsf{Theory}\ w\ \wedge \\ & w(B(\Theta(p), x(y), \{ p \mid \exists q.\ p \rightarrow q \in w \wedge q \notin z \})) \subseteq z \Rightarrow \\ & \quad \Theta(p)\text{-}\mathsf{Theory}\ B(\Theta(p), x(y), \{ p \mid \exists q.\ p \rightarrow q \in w \wedge q \notin z \}) \end{aligned}$$

Proof. We are required to show that given

1. $\&\gamma \rightarrow C \in \Theta(p)$
2. $\gamma \subseteq B(\Theta(p), x(y), \{ p \mid \exists q.\ p \rightarrow q \in w \wedge q \notin z \})$

that

$$C \in B(\Theta(p), x(y), \{ p \mid \exists q.\ p \rightarrow q \in w \wedge q \notin z \})$$

But this is just the same as showing that there exists an n for which this holds and we will choose $n = \eta(C) + 1$ because this value of n for which we add C into

$$b_n(\Theta(p), x(y), \{ p \mid \exists q.\ p \to q \in w \wedge q \notin z \}).$$

It then suffices derive a contradiction from assuming that there is exists a D and q such that $D \to q \in w$, $q \notin z$, and

$$(b_{\eta(p)}(\Theta(A), x(y), \{ p \mid \exists q.\ p \to q \in w \wedge q \notin z \}) \cup \{C\}) \vdash^{\&}_{\Theta(p)} D.$$

Where δ is the list that satisfies the above $(\vdash^{\&}_{\Theta(p)})$ relation; we know that because we have $\mathcal{E}\delta \to B \in \Theta(A)$ then we also have $(B \to q) \to \mathcal{E}\delta \to q \in \Theta(A)$. We also note that we can derive a contradiction by showing that:

$$q \in w(B(\Theta(A), x(y), \{ p \mid \exists q.\ p \to q \in w \wedge q \notin z \}))$$

Naturally the choice of subset that implies that q is the one that implies all of δ, so we choose $(\delta - p) +\!\!\!+\ \gamma$. This choice of $(\delta - p) +\!\!\!+\ \gamma$ results in three subgoals, two of which are related to the membership of $(\delta - p) +\!\!\!+\ \gamma$ which is obviously in

$$B(\Theta(p), x(y), \{ p \mid \exists q.\ p \to q \in w \wedge q \notin z \})$$

and are omitted. The other is whether $\mathcal{E}((\delta - p) +\!\!\!+\ \gamma) \to q \in w$ but this should be obvious if tedious to show after stating that it follows immediately from showing $\mathcal{E}\delta \to q \in w$ so is also omitted. □

Looking at how $B(\theta, i, r)$ has been constructed to exclude everything that could imply a proposition outside of r and it contains all other propositions, we can informally say that $B(\theta, i, r)$ *maximally excludes* r. In this sense we will propose a variant of Lemma 3.2.21 for our specific $B(\Theta(p), x(y), \{ p \mid \exists q.\ p \to q \in w \wedge q \notin z \})$.

Proposition 3.2.37. Let p be a non-theorem, x be any set of Goldblatt propositions, z be Prime, and w a $\Theta(p)$-Theory. Assume further that $B(\Theta(p), x, \{ p \mid \exists q.\ p \to q \in w \wedge q \notin z \})$ is non-empty.

Whenever C is not in $B(\Theta(p), x, \{ p \mid \exists q.\ p \to q \in w \wedge q \notin z \})$ then we know that there is a $D \notin z$ such that $d \in B(\Theta(p), x, \{ p \mid \exists q.\ p \to q \in w \wedge q \notin z \})$ and $d \,\&\, C \to D \in w$:

$$\begin{aligned}
\vdash\ & \nvdash_{\boldsymbol{R}}\ p \wedge \mathsf{Prime}\ z \wedge \Theta(p)\text{-}\mathsf{Theory}\ w \wedge \\
& \Theta(p)\text{-}\mathsf{Theory}\ B(\Theta(p), x, \{ p \mid \exists q.\ p \to q \in w \wedge q \notin z \}) \wedge \\
& C \notin B(\Theta(p), x, \{ p \mid \exists q.\ p \to q \in w \wedge q \notin z \}) \wedge \\
& B(\Theta(p), x, \{ p \mid \exists q.\ p \to q \in w \wedge q \notin z \}) \neq \emptyset \Rightarrow \\
& \quad \exists D\ d. \\
& \qquad D \notin z \wedge d \in B(\Theta(p), x, \{ p \mid \exists q.\ p \to q \in w \wedge q \notin z \}) \wedge \\
& \qquad d \,\&\, C \to D \in w
\end{aligned}$$

Naturally, like in the proof of the primality of $\Theta(p)$ we use the preceding proposition to show that $B(\Theta(p),\, x(y),\, \{\, p \mid \exists q.\ p \;\rightarrow\; q \;\in\; w \;\wedge\; q \;\notin\; z \,\})$ is prime. We will omit the proof as the argument follows similarly to that of Theorem 3.2.22, except that we use Proposition 3.2.37 the regularity of $\Theta(p)$ to use the rules of the Goldblatt axioms, and that z is Prime to derive a contradiction.

Proposition 3.2.38. $B(\Theta(p),\, x(y),\, \{\, p \mid \exists q.\ p \;\rightarrow\; q \;\in\; w \;\wedge\; q \;\notin\; z \,\})$ is Prime when the following four preconditions hold:

1. $\nvdash_{\boldsymbol{R}}\ p$
2. $\Theta(p)$-Theory w
3. Prime z
4. $w(B(\Theta(p),\, x(y),\, \{\, p \mid \exists q.\ p \;\rightarrow\; q \;\in\; w \;\wedge\; q \;\notin\; z \,\})) \;\subseteq\; z$

Or rather:

$$\begin{aligned}&\vdash\ \nvdash_{\boldsymbol{R}}\ p \;\wedge\; \text{Prime } z \;\wedge\; \Theta(p)\text{-Theory } w \;\wedge\\ &\quad w(B(\Theta(p),\, x(y),\, \{\, p \mid \exists q.\ p \;\rightarrow\; q \;\in\; w \;\wedge\; q \;\notin\; z \,\})) \;\subseteq\; z \;\Rightarrow\\ &\qquad \text{Prime } B(\Theta(p),\, x(y),\, \{\, p \mid \exists q.\ p \;\rightarrow\; q \;\in\; w \;\wedge\; q \;\notin\; z \,\})\end{aligned}$$

Now we are able to show that $\mathfrak{F}(p)$ is an R-Frame in full:

Proof (Theorem 3.2.30). We show that given a non-theorem p each of the nine conditions for R-Frames hold:

1. See Lemma 3.2.31.
2. See Lemma 3.2.33.
3. This straightforwardly follows from the fact that every world in $\mathfrak{F}(p)^W$ is a $\Theta(p)$-Theory.
4. Suppose we have worlds x, x', y, y', z, and z' such that $x' \;\leq_{\mathfrak{F}(p)}\; x$, $y' \;\leq_{\mathfrak{F}(p)}\; y$, $z \;\leq_{\mathfrak{F}(p)}\; z'$, and $R_{\mathfrak{F}(p)}\ x\ y\ z$ with the goal to show $R_{\mathfrak{F}(p)}\ x'\ y'\ z'$.

   By the definition of $\mathfrak{F}(p)$ this is just the same as having $\Theta(p)(x') \;\subseteq\; x$, $\Theta(p)(y') \;\subseteq\; y$, $\Theta(p)(z) \;\subseteq\; z'$, and $x(y) \;\subseteq\; z$ with the goal of $x'(y') \;\subseteq\; z'$. As $\vdash_{\boldsymbol{R}} (A \rightarrow A)$ is in $\Theta(p)$ we can show that $x' \;\subseteq\; x$, $y' \;\subseteq\; y$, and $z \;\subseteq\; z'$. This means that it suffices to show that $x'(y') \;\subseteq\; x(y)$ which holds by Definition 3.2.25.
5. See Lemma 3.2.32.
6. Suppose we have worlds w, x, and y such that $R_{\mathfrak{F}(p)}\ w\ x\ y$ with the goal of showing $R_{\mathfrak{F}(p)}\ w\ y^*\ x^*$. From Lemma 3.2.33 we have that these x^* and y^* are in $\mathfrak{F}(p)^W$. So by the definition of $\mathfrak{F}(p)$ this means that we only need to show:

   $$w(\{\, A \mid {\sim}A \;\notin\; y \,\}) \;\subseteq\; \{\, A \mid {\sim}A \;\notin\; x \,\}.$$

   Or rather, given $\&\gamma \;\rightarrow\; a \;\in\; w$ and ${\sim}a \;\in\; x$ show that there is an element of γ whose negation is in y.

It is the case that $\sim a \rightarrow \sim \mathcal{E}\gamma \in w$ due to contraposition being a theorem, and thus in $\Theta(p)$. It is then obvious that $\sim \mathcal{E}\gamma \in y$, so the result follows from Proposition 3.2.12.

7. We need to show that $x(x) \subseteq x$, so given $\gamma \subseteq x$ and $\mathcal{E}\gamma \rightarrow a \in x$ show $a \in x$. But because $\Theta(p)$-Theory x and Proposition 2.1.3.8 the result follows from the regularity of $\Theta(p)$.

8. Follows similarly to 7. but follows from Assertion instead of Proposition 2.1.3.8.

9. Last but not least we are required to show that for any choice of worlds w, x, y, z, and a where $w(x) \subseteq a$ and $a(y) \subseteq z$. Then there exists a world b such that $x(y) \subseteq b$ and $w(b) \subseteq z$. We choose

$$b = B(\Theta(A), x(y), \{ p \mid \exists q.\, p \rightarrow q \in w \wedge q \notin z \})$$

$x(y) \subseteq b$ holds trivially. To show that $w(b) \subseteq z$ we let $\gamma \subseteq b$ and $\mathcal{E}\gamma \rightarrow B \in w$ with the the goal to show that $B \in z$. Using Proposition 3.2.35 and the finiteness of γ this is just the same as assuming that there exists an n such that

$$\gamma \subseteq b_n(\Theta(A), x(y), \{ p \mid \exists q.\, p \rightarrow q \in w \wedge q \notin z \})$$

We then induct on n:

The base case requires that we show that $\gamma \subseteq x(y) \Rightarrow \beta \in z$. By Proposition 3.2.26 this amounts to showing that it holds given that $\mathcal{E}\gamma \in x(y)$. By $a(y) \subseteq z$ we have that we only need to show that there exists a $\delta \subseteq y$ such that $\mathcal{E}\delta \rightarrow B \in a$ but we know that there exists a $\delta \subseteq y$ such that $\mathcal{E}\delta \rightarrow \mathcal{E}\gamma \in x$.

If we assume that these δ are the same and then all we need is that $\mathcal{E}\delta \rightarrow B \in a$. But this is true if we show that $(\mathcal{E}\delta \rightarrow \mathcal{E}\gamma) \rightarrow \mathcal{E}\delta \rightarrow B \in w$ as $w(x) \subseteq a$, which of course holds because of permutation and that w is a $\Theta(p)$-Theory.

The inductive step follows immediately.

So to take stock, we have now that $x(y) \subseteq b$ and $w(b) \subseteq z$ but we do not have that $b \in \mathfrak{F}(p)^W$. But this is as simple as showing that b is a $\Theta(p)$-Theory and that is it Prime. $\Theta(p)$-Theory holds immediately from Lemma 3.2.36 and Prime follows immediately from this fact and Proposition 3.2.38.

□

Theorem 3.2.39.

$$\vdash\ \nvdash_{\boldsymbol{R}}\ p \Rightarrow \text{R-Model}\ \mathfrak{M}(p)$$

Proof. From Definition 3.1.4 and Theorem 3.2.30 we only need to show that the hereditary condition holds. It suffices from Definition 3.2.28 that if g-VAR $s \in x$ for any given propositional constant s, then g-VAR $s \in \Theta(p)(x)$. But we know from Corollary 3.2.16 that $\Theta(p)$ contains all axioms, in particular it contains $\vdash_{\boldsymbol{R}}$ (g-VAR $s \rightarrow$ g-VAR s) for every given s which gives us the result we want. □

3.2.3 Completing Completeness

In our canonical model construction we have taken a collection of sets and defined our relations between them based on the contents of these sets. What this does not do, however, is tell us at which worlds our propositions are holding. What we really want is some connection between the propositions inside these worlds and where propositions hold. We tend to call a result of this type a *truth lemma* and we continue this tradition here:

Lemma 3.2.40 (Truth Lemma)**.**

$$\nvdash_{\boldsymbol{R}} p \Rightarrow \forall q\, w.\, w \in \mathfrak{M}(p)^W \Rightarrow \mathfrak{M}(p), w \models q \Leftrightarrow q \in w$$

The complex part about this proof, is dealing with the case where q is of the form $A \rightarrow B$. The way that we will show this the contrapositive statement by letting $A \rightarrow B \notin w$ and $R_{\mathfrak{F}(p)}\ w\ x\ y$, and then constructing worlds x and y such that $A \in x$ and $B \notin y$.

We already have a method of constructing worlds with the canonical frame: choose a set of propositions that we want the world to definitely have and another set of propositions that we want the world to reject and then iteratively add all the propositions we can to the starting set with the condition that we do not end up implying the rejection set:

Definition 3.2.41. Where Θ, i, r, and w are sets of *g-prop*, and $\{A_n\}_{n\in\mathbb{N}}$ is an enumeration over $\mathcal{U}(:\textit{g-prop})$ we define $x_n(\Theta, i, r, w)$ inductively as:

$$x_0(\Theta, i, r, w) \stackrel{\text{def}}{=} i$$
$$x_{n+1}(\Theta, i, r, w) \stackrel{\text{def}}{=}$$
$$\texttt{if}\ \exists p.\ p \in r \wedge w(x_n(\Theta, i, r, w) \cup \{A_n\}) \vdash^{\&}_{\Theta} p\ \texttt{then}$$
$$x_n(\Theta, i, r, w)$$
$$\texttt{else}\ x_n(\Theta, i, r, w) \cup \{A_n\}$$

We then take the union of $x_n(\Theta, i, r, w)$ over n to get:

$$X(\Theta, i, r, w) \stackrel{\text{def}}{=} \bigcup \{ x_n(\Theta, i, r, w) \mid n \in \mathcal{U}(:\textit{num}) \}$$

Similarly we define:

Definition 3.2.42. Where Θ, i, and r are sets of *g-prop*, and $\{A_n\}_{n\in\mathbb{N}}$ is an enumeration over $\mathcal{U}(:\textit{g-prop})$ we define $y_n(\Theta, i, r)$ inductively as:

$$y_0(\Theta, i, r) \stackrel{\text{def}}{=} i$$
$$y_{n+1}(\Theta, i, r) \stackrel{\text{def}}{=}$$
$$\texttt{if}\ \exists p.\ p \in r \wedge (y_n(\Theta, i, r) \cup \{A_n\}) \vdash^{\&}_{\Theta} p\ \texttt{then}\ y_n(\Theta, i, r)$$
$$\texttt{else}\ y_n(\Theta, i, r) \cup \{A_n\}$$

We then take the union of $y_n(\Theta, i, r)$ over n to get:

$$Y(\Theta, i, r) \stackrel{\text{def}}{=} \bigcup \{ y_n(\Theta, i, r) \mid n \in \mathcal{U}(: num) \}$$

Naturally we want $x = X(\Theta(p), \{A\}, \{B\}, w)$ and $y = Y(\Theta(p), w(X(\Theta(p), \{A\}, \{B\}, w)), \{B\})$ for any given proposition A and B as A is contained in x while B is explicitly rejected by $w(x)$ and y starts with $w(x)$ so it will be related to x under w but also maintains its rejection of B.

Just like with our treatment of $B(\Theta, i, r)$ in Definition 3.2.34 we show that both these worlds are $\Theta(p)$-Theory when few conditions are met:

Proposition 3.2.43. When p is a non-theorem, w is a $\Theta(p)$-Theory, and $A \rightarrow B \notin w$ we have that $X(\Theta(p), \{A\}, \{B\}, w)$ is a $\Theta(p)$-Theory:

$$\vdash \not\vdash_{\boldsymbol{R}}\ p \wedge \Theta(p)\text{-Theory } w \wedge A \rightarrow B \notin w \Rightarrow \Theta(p)\text{-Theory } X(\Theta(p), \{A\}, \{B\}, w)$$

Proposition 3.2.44. When p is a non-theorem, x is a $\Theta(p)$-Theory, and $B \notin x$ we have that $Y(\Theta(p), x, \{B\})$ is a $\Theta(p)$-Theory:

$$\vdash \not\vdash_{\boldsymbol{R}}\ p \wedge B \notin x \wedge \Theta(p)\text{-Theory } x \Rightarrow \Theta(p)\text{-Theory } Y(\Theta(p), x, \{B\})$$

And similarly they are both Prime:[4]

Proposition 3.2.45. When p is a non-theorem, w is a Prime $\Theta(p)$-Theory, and $A \rightarrow B \notin w$ we have that $X(\Theta(p), \{A\}, \{B\}, w)$ is Prime:

$$\vdash \not\vdash_{\boldsymbol{R}}\ p \wedge \Theta(p)\text{-Theory } w \wedge \text{Prime } w \wedge A \rightarrow B \notin w \Rightarrow \text{Prime } X(\Theta(p), \{A\}, \{B\}, w)$$

Proposition 3.2.46. When p is a non-theorem, x is a $\Theta(p)$-Theory, and $B \notin x$ we have that $Y(\Theta(p), x, \{B\})$ is a Prime:

$$\vdash \not\vdash_{\boldsymbol{R}}\ p \wedge \Theta(p)\text{-Theory } x \wedge B \notin x \Rightarrow \text{Prime } Y(\Theta(p), x, \{B\})$$

Now we can go ahead and proof the Truth Lemma:

Proof (Lemma 3.2.40). Setting p we then induct on the structure of A. Taking w to be the world in question:

g-VAR s: Holds from the definition of $\mathfrak{M}(p)^V$.

[4] We formalised primality here by first showing similar Proposition to 3.2.21 and 3.2.37 for each construction.

$A \rightarrow B$: The backwards direction here is rather trivial so we will only show the forwards direction i.e. show $A \rightarrow B \in w$ from

$$\forall x\ y.\ x \in \mathfrak{F}(p)^W \wedge y \in \mathfrak{F}(p)^W \wedge R_{\mathfrak{F}(p)}\ w\ x\ y \wedge \mathfrak{M}(p), x \models A \Rightarrow B \in y$$

Through contraposition we assume that $A \rightarrow B \notin w$ and then seek to show that there exists worlds x and y such that $A \in x$ and $B \notin y$ when $R_{\mathfrak{F}(p)}\ w\ x\ y$ i.e.

$$\exists x\ y.\ (x \in \mathfrak{F}(p)^W \wedge y \in \mathfrak{F}(p)^W \wedge R_{\mathfrak{F}(p)}\ w\ x\ y \wedge \mathfrak{M}(p), x \models A) \wedge B \notin y$$

We will take:

1. $x = X(\Theta(p), \{A\}, \{B\}, w)$
2. $y = Y(\Theta(p), w(X(\Theta(p), \{A\}, \{B\}, w)), \{B\})$

And the first thing to show is that these are both worlds of $\mathfrak{F}(p)$:

$X(\Theta(p), \{A\}, \{B\}, w)$ is in the canonical frame immediately from Propositions 3.2.43 and 3.2.45.

$Y(\Theta(p), w(X(\Theta(p), \{A\}, \{B\}, w)), \{B\})$ requires that we show that:

$$B \notin w(X(\Theta(p), \{A\}, \{B\}, w))$$
$$\Theta(p)\text{-Theory } w(X(\Theta(p), \{A\}, \{B\}, w))$$

and then follows immediately from Propositions 3.2.44 and 3.2.46.

$B \notin w(X(\Theta(p), \{A\}, \{B\}, w))$ is a natural result given Definition 3.2.41 but is tedious to show. It follows however from the following condition on $X(\Theta(p), \{A\}, \{B\}, w)$.

$$\vdash \forall_{\boldsymbol{R}}\ p \wedge B \in w(X(\Theta(p), \{A\}, \{B\}, w)) \Rightarrow \exists \gamma.\ \&\gamma \rightarrow B \in w \wedge \gamma \subseteq \{A\} \wedge \gamma \neq []$$

$\Theta(p)$-Theory $w(X(\Theta(p), \{A\}, \{B\}, w))$ is immediate from Proposition 3.2.27.

Now that we have that both our choice of $x = X(\Theta(p), \{A\}, \{B\}, w)$ and $y = Y(\Theta(p), w(X(\Theta(p), \{A\}, \{B\}, w)), \{B\})$ are worlds in the canonical frame, we only need to show that:

1. $w(X(\Theta(p), \{A\}, \{B\}, w)) \subseteq Y(\Theta(p), w(X(\Theta(p), \{A\}, \{B\}, w)), \{B\})$
2. $A \in X(\Theta(p), \{A\}, \{B\}, w)$
3. $B \notin Y(\Theta(p), w(X(\Theta(p), \{A\}, \{B\}, w)), \{B\})$

1. and 2. are immediately obvious from Definitions 3.2.41 and 3.2.42, while 3. is a natural result of the definition of Y-WORLD though it is tedious to show.

$A \& B$: We need to show that $A \in w$ and $B \in w$ if and only if $A \& B \in w$. As w is a $\Theta(p)$-Theory and $\Theta(p)$ is Regular, the showing the result is straightforward.

$\sim A$: From Theorem 3.2.30 we know that $w^* \in \mathfrak{F}(p)^W$ from the closure property of $*$. The result then follows from the definition of $\mathfrak{M}(p)$.

τ: We are required to show $\Theta(p) \leq_{\mathfrak{F}(p)} w \Leftrightarrow \tau \in w$:

($\Rightarrow$): Both τ and $\tau \rightarrow \tau$ are inside $\Theta(p)$ as $\Theta(p)$ is Regular. Given the definition of $R_{\mathfrak{F}(p)}$ it follows that $\tau \in w$.

($\Leftarrow$): Let $\tau \in w$. We have that $w(\Theta(p)) \subseteq w$ from the fact that $\mathfrak{F}(p)$ is an R-Frame and Definitions 3.1.2.3 and 3.1.2.8. We then just need to show then that $\Theta(p)(\Theta(p)) \subseteq w$.

This is the equivalent to showing that if there is a $\gamma \subseteq \Theta(p)$ such that $\gamma \rightarrow x \in \Theta(p)$ then $x \in w$. As we already know that $w(\Theta(p)) \subseteq w$, all we need is to show is that $x \in w(\Theta(p))$, or rather that there exists a δ such that $\delta \subseteq \Theta(p)$ and $\mathcal{E}\delta \rightarrow x \in w$.

Let $\delta = [x]$. x is in $\Theta(p)$ as a result of Lemma 3.2.31 and $x \rightarrow x \in w$ is a result of $\vdash_{\boldsymbol{R}} (\tau \rightarrow x \rightarrow x)$ and that w is a $\Theta(p)$-Theory completing the proof.

□

Armed with the truth (lemma) we can now complete our completeness result:

Theorem 3.2.47 (Completeness)**.** The Goldblatt Axioms are *complete* with respect to the Routley-Meyer Semantics

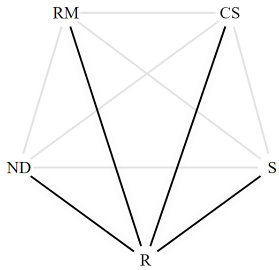
$$\vdash (\forall \mathcal{M}.\ \text{R-Model}\ \mathcal{M} \Rightarrow \mathcal{M} \models p) \Rightarrow \vdash_{\boldsymbol{R}} p$$

Proof. By contraposition we suppose that $\nvdash_{\boldsymbol{R}} p$ and all that is required is to show that there exists an R-Model $\mathcal{M}$ where $\mathcal{M} \not\models p$. Of course the R-Model that we want is $\mathfrak{M}(p)$ which we know is an R-Model because of Theorem 3.2.39. By the Truth Lemma (3.2.40) it suffices to show that $p \notin \mathfrak{F}(p).0$. But recall that $\Theta(p)$ maximally excludes p (Meaning that $\Theta(p) \nvdash^{\&}_{\boldsymbol{R}} p$ and adding anything further means that $\Theta(p) \cup \{q\} \vdash^{\&}_{\boldsymbol{R}} p$) when p is not a theorem of $\boldsymbol{R}$ (see Lemma 3.2.19). We then assume that $p \in \Theta(p)$ and derive a contradiction using $\vdash_{\boldsymbol{R}} (p \rightarrow p)$. □

3.3 Summary

This chapter introduced the Routley-Meyer semantic system for Relevant Implication and formalised both semantic soundness and completeness between the Goldblatt axiomatisation and the Routley-Meyer system. Inside the HOL development, this required 2212 lines of source code contained entirely inside `RMSemanticsScript.sml`.

We shall now take a look at an alternative semantic system for $\boldsymbol{R}$.



Chapter 4

Cover Semantics

In contrast to the Routley-Meyer's semantics being based off a worlds relationship to other worlds determining truth, the Cover System Semantics described by Goldblatt [2011] is a semantic system based off the idea that properties of *worlds* are *locally determined* rather than *relationally determined*. This notion is largely inspired from the field of topology in that we are able to take a point and construct a covering of that point using all open subsets about a point an use results on those open subsets to determine properties about that point. In topology we say that a property is *locally true* of a point p if it is true of all open subsets about that point; we also say that a property is *locally determined* of a point p if the property holding at all open subsets about p implies that the property holds at p [Goldblatt, 2011].

Inside the Cover Semantics there are two particularly important relationships that we should mention. The first is a binary relationship between worlds ($\leqslant$) and the second is a relationship between sets of worlds and a world ($\triangleright$). The natural reading of $x \leqslant y$, as provided by Goldblatt [2011], is that if x contains some information, then y also contains that infomation; so we can think of ($\leqslant$) as a sort of refinement of information. $Z \triangleright x$ on the otherhand operates differently its natural reading is that x consists of precisely the information that is shared between the members of Z, that is if p is a propostion that is true of all members in Z then it is true at x; that is, it is a *locally determined* property.

We should mention the Routley-Meyer system has one major perk over the Cover-Semantics: It is similar enough to the Kripke semantics for modal logic that adopters of Relevant Logics would be able to understand it without too much hassle. This raises the question, why bother with Cover Semantics at all? The answer is that Fine [1989] showed that the expected extension of the Routley-Meyer Semantics to Quantified Relevant Logic was incomplete. This meant that while a semantical system based off the Routley-Meyer semantics was eventually found, it gives an unintuitive meaning to the quantifiers. Cover

Semantics on the other has already been shown to be sound and complete with respect to the Quantified extention of **R**.

This chapter introduces the Cover Semantics for unquantified relevant implication and establishes soundness and completeness results between the Goldblatt axioms and this system. In Section 4.1 we present Cover Semantics as described by Goldblatt [2011] and we will also establish *semantic soundness* between the Hilbert System and this alternate semantic system. In Section 4.2 we will formalise *semantic completeness* for relevant implication with respect to the cover semantics.

4.1 The Goldblatt System

Before we consider defining the Cover Semantics formally, we should first define a notion of Preorder. A preorder is defined mathematically as a binary relation over a set X such that it is both reflexive and transitive. In HOL there is already a defined predicate for preorder as well as reflexivity and transitivity, but these are defined for all of $\mathcal{U}(:\alpha)$ i.e. a function is only reflexive if it is reflexive over all of $\mathcal{U}(:\alpha)$.

For our purposes in this chapter what we need however is a predicate that captures preorder over a defined set with elements of type α rather than the universe of $\alpha, \mathcal{U}(:\alpha)$.

Definition 4.1.1 (Preorder)**.** A binary operator R is a Preorder over a set X if

1. For all x in X we have that $R\ x\ x$
2. For all x, y, z in X we have that if $R\ x\ y$ and $R\ y\ z$, then $R\ x\ z$.

$$\begin{array}{l} \mathsf{Preorder}\ R\ X \stackrel{\mathsf{def}}{=} \\ (\forall x.\ x \in X \Rightarrow R\ x\ x) \land \\ \forall x\ y\ z.\ x \in X \land y \in X \land z \in X \land R\ x\ y \land R\ y\ z \Rightarrow R\ x\ z \end{array}$$

4.1.1 Beginning with Cover Systems

Let us first define what structures we will be working with:

Definition 4.1.2. An α *COVER-SYSTEM* is a record type with fields W, $\leqslant$, $\triangleright$. W is a set of worlds of type α, $\leqslant$ is a binary relation over α, and $\triangleright$ is a relation between sets containing elements of α and elements of α.

$$\begin{array}{l} \alpha\ \textit{COVER-SYSTEM} = \langle \\ \quad \mathsf{W} : \alpha \to \textit{bool}; \\ \quad \leqslant : \alpha \to \alpha \to \textit{bool}; \\ \quad \triangleright : (\alpha \to \textit{bool}) \to \alpha \to \textit{bool} \\ \rangle \end{array}$$

We should keep in mind that the only members of this datatype that are interesting for our purposes are the ones where $\leqslant$ is a Preorder over W. With this in mind we will define an *upwardly-closed set* or *upset* with respect to a α *COVER-SYSTEM*.

Definition 4.1.3 (Upset)**.** A set X is an Upset with respect to a α *COVER-SYSTEM* $\mathcal{C}$ if

1. $X \subseteq \mathcal{C}^W$
2. $\forall d\, e.\ d \in X \wedge e \in \mathcal{C}^W \wedge \mathcal{C}.\leqslant d\ e \Rightarrow e \in X$

or rather:

$$\vdash \mathsf{Upset}\ \mathcal{C}\ X \Leftrightarrow$$
$$X \subseteq \mathcal{C}.\mathsf{W} \wedge \forall d\, e.\ d \in X \wedge e \in \mathcal{C}.\mathsf{W} \wedge \mathcal{C}.\leqslant d\ e \Rightarrow e \in X$$

We can also define the *upward closure* of a set X:

Definition 4.1.4. If X is a set, then the *upward closure* with respect to an α *COVER-SYSTEM* $\mathcal{C}$ is the set of all elements in $\mathcal{C}^W$ such that there exists an $x \in X$ and $\mathcal{C}.\leqslant x\ u$:

$$\uparrow_{\mathcal{C}} X \stackrel{\mathsf{def}}{=} \{\, u \mid u \in \mathcal{C}.\mathsf{W} \wedge \exists x.\ x \in X \wedge \mathcal{C}.\leqslant x\ u \,\}$$

When X is a singleton $\{\, x \,\}$ we drop the set notation and write $\uparrow_{\mathcal{C}} x$ for clarity.

We are now able to define a Cover System:

Definition 4.1.5. A α *COVER-SYSTEM* $\mathcal{C}$ is a Cover-System if the following are true:

1. $\mathcal{C}.\leqslant$ is a Preorder over $\mathcal{C}^W$
2. If $\mathcal{C}.\leqslant x\ y$ then both x and y are in $\mathcal{C}^W$
3. If $x \in \mathcal{C}^W$ then there exists a Z such that $\mathcal{C}.\triangleright Z\ x$
4. If $x \in \mathcal{C}^W$ and $\mathcal{C}.\triangleright Z\ x$ then $Z \subseteq \uparrow_{\mathcal{C}} x$

In other words:

$$\begin{aligned}
&\vdash \mathsf{Cover\text{-}System}\ \mathcal{C} \Leftrightarrow \\
&\quad \mathsf{Preorder}\ \mathcal{C}.\leqslant \mathcal{C}.\mathsf{W} \wedge \\
&\quad (\forall x\, y.\ \mathcal{C}.\leqslant x\ y \Rightarrow x \in \mathcal{C}.\mathsf{W} \wedge y \in \mathcal{C}.\mathsf{W}) \wedge \\
&\quad (\forall x.\ x \in \mathcal{C}.\mathsf{W} \Rightarrow \exists Z.\ \mathcal{C}.\triangleright Z\ x) \wedge \\
&\quad \forall x\, Z.\ x \in \mathcal{C}.\mathsf{W} \wedge \mathcal{C}.\triangleright Z\ x \Rightarrow Z \subseteq \uparrow_{\mathcal{C}} x
\end{aligned}$$

As we now have the Preorder condition on $\mathcal{C}$ we can claim that a set X is an *upset* if it is invariant under it upward closure *upward closure*.

Proposition 4.1.6. A set X is an Upset with respect to a Cover-System CS if and only if it is invarient under *upward closure*. That is:

$$\vdash \mathsf{Cover\text{-}System}\ \mathcal{C} \Rightarrow \mathsf{Upset}\ \mathcal{C}\ X \Leftrightarrow \uparrow_{\mathcal{C}} X = X$$

We will also define a function j $\mathcal{C}$ over subsets of $\mathcal{C}^W$:

Definition 4.1.7.

$$\mathsf{j}\ \mathcal{C}\ X \stackrel{\mathsf{def}}{=} \{\, w \mid w \in \mathcal{C}.\mathsf{W} \wedge \exists Z.\ \mathcal{C}.\triangleright\ Z\ w \wedge Z \subseteq X \,\}$$

We should ask what $\mathsf{j}\ \mathcal{C}$ actually means. If $w \in \mathsf{j}\ \mathcal{C}\ X$ then w has a covering that is constrained inside of X, or with the natural reading from the introduction to this Section, all members of Z are members of X so being a member of X is *locally holds* of w. In this sense $\mathsf{j}\ \mathcal{C}\ X$ can be considered the set of *local members* of X.

Definition 4.1.8 (Localized)**.** We say that a set X is Localized in a Cover System CS when all local members of X are in X

$$\mathsf{Localized}\ \mathcal{C}\ X \Leftrightarrow \mathsf{j}\ \mathcal{C}\ X \subseteq X$$

Definition 4.1.9 (Is-Prop)**.** We say that a set is a *Proposition* with respect to a Relevant-CS $\mathcal{S}$ if it is both Localized and an Upset with respect to $\mathcal{S}$.

$$\mathsf{Is\text{-}Prop}\ \mathcal{S}\ X \Leftrightarrow \mathsf{Localized}\ X \wedge \mathsf{Upset}\ X$$

We also can show that:

Lemma 4.1.10. If $\mathcal{C}$ is a Cover System, and X is an Upset in $\mathcal{C}$ then all members of X are also *local members* of X. Or rather:

$$\vdash \mathsf{Cover\text{-}System}\ \mathcal{C} \wedge \mathsf{Upset}\ \mathcal{C}\ X \Rightarrow X \subseteq \mathsf{j}\ \mathcal{C}\ X$$

4.1.2 Making Cover Systems Relevant

In order to make this semantic appropriate for Relevant Implication, we need to introduce a number of extra relationships to our cover systems:

Definition 4.1.11. An α *R-COVER-SYSTEM* is a record type with fields W, $\leqslant$, $\triangleright$, $\cdot$, ε, and $\perp$. Like in Definition 4.1.5, W is a set of worlds of type α, $(\leqslant)$ is a binary relation over α, and $(\triangleright)$ is a relation between sets containing elements of α and and elements of α. $(\cdot)$ is a binary operator over elements of α, ε is a specific element of type α, and $(\perp)$ is a binary relation over elements of α.

$$\begin{array}{l}\alpha\ \textit{R-COVER-SYSTEM} = \langle \\ \quad \mathsf{W} : \alpha \to \mathit{bool}; \\ \quad \leqslant\ : \alpha \to \alpha \to \mathit{bool}; \\ \quad \triangleright : (\alpha \to \mathit{bool}) \to \alpha \to \mathit{bool}; \\ \quad \cdot : \alpha \to \alpha \to \alpha; \\ \quad \varepsilon : \alpha; \\ \quad \perp : \alpha \to \alpha \to \mathit{bool} \\ \rangle \end{array}$$

With the new operator $(\cdot)$ we write can use the shorthand $x \cdot Y = \{ x \cdot y \mid y \in Y \}$ and similarly $X \cdot y = \{ x \cdot y \mid x \in X \}$ for fusion over sets. Similarly, for $(\perp)$ we can use the shorthand $x \perp Y \Leftrightarrow \forall y.\, y \in Y \Rightarrow x \perp y$ and $X \perp y \Leftrightarrow \forall x.\, x \in X \Rightarrow x \perp y$.

As an α *R-COVER-SYSTEM* is also just an extension of an α *COVER-SYSTEM* we note that $\lambda X.\, \mathsf{j}\ X$, $\lambda X.$ Upset X, and Localized naturally extend to α *R-COVER-SYSTEM*.

With this structure, we define Relevant-CS:

Definition 4.1.12. A Relevant-CS $\mathcal{S}$ is an α *R-COVER-SYSTEM* that satisfies the following closure properties:

1. $x \leqslant y \Rightarrow x \in \mathcal{S}^W \wedge y \in \mathcal{S}^W$
2. $Z \rhd x \Rightarrow x \in \mathcal{S}^W \wedge Z \subseteq \mathcal{S}^W$
3. $x \in \mathcal{S}^W \wedge y \in \mathcal{S}^W \Rightarrow x \cdot y \in \mathcal{S}^W$
4. $x \perp y \Rightarrow x \in \mathcal{S}^W \wedge y \in \mathcal{S}^W$

And, assuming that all variables are already in $\mathcal{S}^W$, the following statements hold:

6. $\mathcal{S}^W$, $\mathcal{S}.\leqslant$, and $\mathcal{S}.\rhd$ together are a Cover-System
7. $x \cdot \varepsilon = x$ and $\varepsilon \cdot x = x$
8. $(\cdot)$ is *commutative* and *associative* over $\mathcal{S}^W$
9. If $x \leqslant x'$ and $y \leqslant y'$ then it is the case that $(x \cdot y) \leqslant (x' \cdot y')$
10. $(x \cdot x) \leqslant x$ for all x
11. If $Z \rhd x$ and $y \in \mathcal{S}^W$ then $(Z \cdot y) \rhd (x \cdot y)$
12. If $x \leqslant x'$, $y \leqslant y'$ and $x \perp y$ then $x' \perp y'$
13. If $Z \rhd x$ and $Z \perp \varepsilon$ then it is the case that $x \perp \varepsilon$
14. If $(x \cdot y) \perp z$ then $(x \cdot z) \perp y$

In other words:

$$
\begin{array}{l}
\vdash \textsf{Relevant-CS}\ \mathcal{S} \Leftrightarrow \\
\quad \varepsilon \in \mathcal{S}^W \wedge (\forall x\, y.\ x \leqslant y \Rightarrow x \in \mathcal{S}^W \wedge y \in \mathcal{S}^W) \wedge \\
\quad (\forall x\, Z.\ Z \rhd x \Rightarrow x \in \mathcal{S}^W \wedge Z \subseteq \mathcal{S}^W) \wedge \\
\quad (\forall x\, y.\ x \in \mathcal{S}^W \wedge y \in \mathcal{S}^W \Rightarrow x \cdot y \in \mathcal{S}^W) \wedge \\
\quad (\forall x\, y.\ x \perp y \Rightarrow x \in \mathcal{S}^W \wedge y \in \mathcal{S}^W) \wedge \textsf{Cover-System}\ \mathfrak{C} \wedge \\
\quad (\forall x.\ x \in \mathcal{S}^W \Rightarrow x \cdot \varepsilon = x) \wedge (\forall x.\ x \in \mathcal{S}^W \Rightarrow \varepsilon \cdot x = x) \wedge \\
\quad (\forall x\, y.\ x \in \mathcal{S}^W \wedge y \in \mathcal{S}^W \Rightarrow x \cdot y = y \cdot x) \wedge \\
\quad (\forall x\, y\, z. \\
\qquad x \in \mathcal{S}^W \wedge y \in \mathcal{S}^W \wedge z \in \mathcal{S}^W \Rightarrow x \cdot (y \cdot z) = (x \cdot y) \cdot z) \wedge \\
\quad (\forall x\, x'\, y\, y'.\ x \leqslant x' \wedge y \leqslant y' \Rightarrow (x \cdot y) \leqslant (x' \cdot y')) \wedge \\
\quad (\forall x.\ x \in \mathcal{S}^W \Rightarrow (x \cdot x) \leqslant x) \wedge \\
\quad (\forall x\, y\, Z.\ Z \rhd x \wedge y \in \mathcal{S}^W \Rightarrow (Z \cdot y) \rhd (x \cdot y)) \wedge \\
\quad (\forall x\, x'\, y\, y'.\ x \leqslant x' \wedge y \leqslant y' \wedge x \perp y \Rightarrow x' \perp y') \wedge \\
\quad (\forall x\, Z.\ Z \rhd x \wedge Z \perp \varepsilon \Rightarrow x \perp \varepsilon) \wedge \\
\quad \forall x\, y\, z. \\
\qquad x \in \mathcal{S}^W \wedge y \in \mathcal{S}^W \wedge z \in \mathcal{S}^W \wedge (x \cdot y) \perp z \Rightarrow (x \cdot z) \perp y
\end{array}
$$

The important mechanism behind Relevant-CSs is postulate 13. which makes orthogonality to ε a *locally determined property.* As we read $Z \rhd x$ to mean that x is precisely what statements the collection of possible worlds Z agree on, and each member of Z disagrees with ε in some way, then there is some information that all members of Z agree on that is in conflict with ε.

Now we will define some other useful predicates over the set of worlds $\mathcal{S}^W$.

Definition 4.1.13 (Orthocomplement, Orthojoin, and Implication)**.** Inside a Relevant-CS $\mathcal{S}$, the *Orthocomplement* of a set X (Written $X^\perp$) is the set of of worlds that are orthogonal to all worlds in x:

$$X^\perp \stackrel{\textsf{def}}{=} \{\, y \mid y \in \mathcal{S}^W \wedge \forall x.\ x \in X \Rightarrow y \perp x \,\}$$

The *orthojoin* of two sets X and Y (Written $(\mathcal{S} + X)\ Y$) is then the double application of orthogonality to, or *orthoclosure* of, the union of the two sets:

$$X + Y \stackrel{\textsf{def}}{=} (X \cup Y)^{\perp\perp}$$

We also define the implication under $\mathcal{S}$ from X to Y as the set of worlds w such that w fused to all members of X is a subset of Y:

$$X \hookrightarrow Y \stackrel{\textsf{def}}{=} \{\, w \mid w \in \mathcal{S}^W \wedge \{\, w \cdot x \mid x \in X \,\} \subseteq Y \,\}$$

We can show the following properties of any Relevant-CS:

Lemma 4.1.14.

1. $w \perp u$ is true if and only if $(w \cdot u) \perp \varepsilon$.
2. $X^{\perp}$ is $\mathsf{Localized}$ in, and an Upset of, $\mathcal{S}$. Thus it is a proposition of $\mathcal{S}$
3. If Y is an Upset of $\mathcal{S}$ then so is $X \hookrightarrow Y$.
4. If Y is $\mathsf{Localized}$ in $\mathcal{S}$ then so is $X \hookrightarrow Y$.
5. If $w \perp X$ then $a \perp \mathsf{j}\ X$. Or, $X^{\perp} \subseteq (\mathsf{j}\ X)^{\perp}$
6. $\mathsf{j}\ X \subseteq (\mathsf{j}\ X)^{\perp\perp}$ and $(\mathsf{j}\ X)^{\perp\perp} \subseteq X^{\perp\perp}$
7. If X is an Upset of $\mathcal{S}$ then, $w \perp u$ if and only if $w \perp \mathsf{j}\ X$. Or, $X^{\perp} = (\mathsf{j}\ X)^{\perp}$
8. If X is an Upset of $\mathcal{S}$ then: $X \subseteq \mathsf{j}\ X$, $\mathsf{j}\ X \subseteq \mathsf{j}\ X^{\perp\perp}$, and $\mathsf{j}\ X^{\perp\perp} = X^{\perp\perp}$

$\vdash \mathsf{Relevant\text{-}CS}\ \mathcal{S} \wedge x \in \mathcal{S}^W \wedge y \in \mathcal{S}^W \Rightarrow x \perp y \Leftrightarrow (x \cdot y) \perp \varepsilon$
$\vdash \mathsf{Relevant\text{-}CS}\ \mathcal{S} \wedge X \subseteq \mathcal{S}^W \Rightarrow \mathsf{Is\text{-}Prop}\ \mathcal{S}\ X^{\perp}$
$\vdash \mathsf{Relevant\text{-}CS}\ \mathcal{S} \wedge X \subseteq \mathcal{S}^W \wedge Y \subseteq \mathcal{S}^W \wedge \mathsf{Upset}\ Y \Rightarrow \mathsf{Upset}\ (X \hookrightarrow Y)$
$\vdash \mathsf{Relevant\text{-}CS}\ \mathcal{S} \wedge X \subseteq \mathcal{S}^W \wedge Y \subseteq \mathcal{S}^W \wedge \mathsf{Localized}\ Y \Rightarrow$
$\quad \mathsf{Localized}\ (X \hookrightarrow Y)$
$\vdash \mathsf{Relevant\text{-}CS}\ \mathcal{S} \wedge x \in \mathcal{S}^W \wedge X \subseteq \mathcal{S}^W \wedge x \perp X \Rightarrow x \perp \mathsf{j}\ X$
$\vdash \mathsf{Relevant\text{-}CS}\ \mathcal{S} \wedge X \subseteq \mathcal{S}^W \Rightarrow \mathsf{j}\ X \subseteq (\mathsf{j}\ X)^{\perp\perp} \wedge (\mathsf{j}\ X)^{\perp\perp} \subseteq X^{\perp\perp}$
$\vdash \mathsf{Relevant\text{-}CS}\ \mathcal{S} \wedge \mathsf{Upset}\ X \wedge X \subseteq \mathcal{S}^W \wedge x \in \mathcal{S}^W \Rightarrow$
$\quad x \perp X \Leftrightarrow x \perp \mathsf{j}\ X$
$\vdash \mathsf{Relevant\text{-}CS}\ \mathcal{S} \wedge X \subseteq \mathcal{S}^W \wedge \mathsf{Upset}\ X \Rightarrow$
$\quad X \subseteq \mathsf{j}\ X \wedge \mathsf{j}\ X \subseteq (\mathsf{j}\ X)^{\perp\perp} \wedge ((\mathsf{j}\ X)^{\perp\perp} = X^{\perp\perp})$

Proof. The only non-routine case of this Lemma is 2.[1] which we will show now:

2. After applying Definitions 4.1.9, 4.1.8, and 4.1.13 we show:

   ($\Leftarrow$): Given $X \subseteq \mathcal{S}^W$ show that $\{\, y \mid y \in \mathcal{S}^W \wedge \forall x.\ x \in X \Rightarrow y \perp x \,\}$ is an Upset. This direction follows straightforwardly from Definition 4.1.12.12.

   ($\Rightarrow$): We use 1. to show that we are required to show that $(x \cdot y) \perp \varepsilon$ given $Z \rhd x$ and $\forall z.\ z \in Z \Rightarrow (z \cdot y) \perp \varepsilon$ Which follows immedately from Definitions 4.1.12.11 and 4.1.12.13.

□

It follows immediately from Lemma 4.1.14 that:

Corollary 4.1.15. If $\mathcal{S}$ is a relevant cover system and X is an Upset with respect to $\mathcal{S}$ then when $X = X^{\perp\perp}$ we have that X is a *Proposition.*

$$\vdash \mathsf{Relevant\text{-}CS}\ \mathcal{S} \wedge X \subseteq \mathcal{S}^W \wedge \mathsf{Upset}\ X \wedge (X = X^{\perp\perp}) \Rightarrow \mathsf{Is\text{-}Prop}\ \mathcal{S}\ X$$

[1] We should note however, that these results do build off each other and are proved in ascending order

4.1.3 Model Systems, Models, and Soundness

Now that we have thoroughly defined the Cover systems that we will be working with we should now define the systems that we use to establish validity. In the Cover semantics, we augment the Cover-Semantics with a specific set of *Propositions* and closed under $(\perp)$, $(\cap)$, $(\hookrightarrow)$, and $(+)$.

Definition 4.1.16. A Model-System is a Relevant-CS together with a collection of subsets of $\mathcal{S}^W$ *Prop*, called the *admissable propositions* of the such that for all X and Y in *Prop* we have that:

1. $\uparrow_{\mathcal{S}} \varepsilon \in \mathit{Prop}$.
2. X is an Upset.
3. $X = X^{\perp\perp}$
4. $X^{\perp} \in \mathit{Prop}$
5. $X \cap Y \in \mathit{Prop}$
6. $X \hookrightarrow Y \in \mathit{Prop}$
7. $X + Y = \mathsf{j}\ (X \cup Y)$

$$\begin{array}{l} \vdash \textsf{Model-System}\ \mathcal{S}\ \mathit{Prop} \Leftrightarrow \\ \quad \textsf{Relevant-CS}\ \mathcal{S} \wedge \uparrow_{\mathcal{S}} \varepsilon \in \mathit{Prop} \wedge (\forall X.\ X \in \mathit{Prop} \Rightarrow \textsf{Upset}\ X) \wedge \\ \quad (\forall X.\ X \in \mathit{Prop} \Rightarrow X = X^{\perp\perp}) \wedge (\forall X.\ X \in \mathit{Prop} \Rightarrow X^{\perp} \in \mathit{Prop}) \wedge \\ \quad (\forall X\ Y.\ X \in \mathit{Prop} \wedge Y \in \mathit{Prop} \Rightarrow X \cap Y \in \mathit{Prop}) \wedge \\ \quad (\forall X\ Y.\ X \in \mathit{Prop} \wedge Y \in \mathit{Prop} \Rightarrow X \hookrightarrow Y \in \mathit{Prop}) \wedge \\ \quad \forall X\ Y.\ X \in \mathit{Prop} \wedge Y \in \mathit{Prop} \Rightarrow X + Y = \mathsf{j}\ (X \cup Y) \end{array}$$

We then assign a function M that takes propositions from the language of $\mathbf{R}$ and assigns them to *Propositions* inside *Prop*.

Definition 4.1.17. A *Model Function* M is a function that is inductively defined from g-prop into *Prop* such that each propositional variable is assigned a set inside *Prop*, τ is assigned $\uparrow_{\mathcal{S}} \varepsilon$. Then $M\ (A\ \&\ B)$ goes to $M\ A \cap M\ B$, $M\ (A \rightarrow B)$ goes to $M\ A \hookrightarrow M\ B$, and $M\ ({\sim}A)$ goes to $(M\ A)^{\perp}$:

$$\begin{array}{l} \vdash \textsf{Model-Function}\ \mathcal{S}\ \mathit{Prop}\ M \Leftrightarrow \\ \quad (\forall a\ A.\ (M\ (\textsf{g-VAR}\ a) = A) \Rightarrow A \in \mathit{Prop}) \wedge (M\ \tau = \uparrow_{\mathcal{S}} \mathcal{S}.\varepsilon) \wedge \\ \quad (\forall A\ B.\ M\ (A\ \&\ B) = M\ A \cap M\ B) \wedge \\ \quad (\forall A\ B.\ M\ (A \rightarrow B) = M\ A \hookrightarrow M\ B) \wedge \forall A.\ M\ ({\sim}A) = (M\ A)^{\perp} \end{array}$$

Corollary 4.1.18. It is also the case that $M\ (A \vee B)$ goes to $M\ A + M\ B$.

$$\vdash \textsf{Model-Function}\ \mathcal{S}\ \mathit{Prop}\ M \Rightarrow \forall A\ B.\ M\ (A \vee B) = M\ A + M\ B$$

A Model-System and Model-Function together form a *Model* of the Cover semantics. With this we are able to determine the validitity of a formula with respect to the Model-System and Model-Function pair.

Definition 4.1.19. Given a Model-System, $\mathcal{S}$, $Prop$, and a Model-Function M defined over $\mathcal{S}$ and $Prop$, as well as a world w inside $\mathcal{S}^W$. Proposition A Holds at w when $w \in M\ A$:

$$\vdash (\mathcal{S},\ Prop,\ M)\, w \models A \iff \\ w \in M\ A \wedge \text{Model-Function}\ \mathcal{S}\ Prop\ M \wedge \text{Model-System}\ \mathcal{S}\ Prop$$

When this is the case we write $(\mathcal{S},\ Prop,\ M)\, w \models A$. We also say that a proposition holds at a model, if and only if it holds at ε and we write $(\mathcal{S},\ Prop,\ M) \models A$.

Corollary 4.1.20. We can establish the following conditions on Holds:

1. $(\mathcal{S},\ Prop,\ M)\, w \models (\text{g-VAR}\ s)$ if and only if $w \in M\ (\text{g-VAR}\ s)$
2. $(\mathcal{S},\ Prop,\ M)\, w \models \tau$ if and only if $\mathcal{S}.\varepsilon \leqslant w$
3. $(\mathcal{S},\ Prop,\ M)\, w \models \sim\! A$ if and only if $\forall u.\ (\mathcal{S},\ Prop,\ M)\, u \models A \Rightarrow w \perp u$
4. $(\mathcal{S},\ Prop,\ M)\, w \models (A\ \&\ B)$ if and only if $(\mathcal{S},\ Prop,\ M)\, w \models A \wedge (\mathcal{S},\ Prop,\ M)\, w \models B$
5. $(\mathcal{S},\ Prop,\ M)\, w \models (A \rightarrow B)$ if and only if $\forall u.\ (\mathcal{S},\ Prop,\ M)\, u \models A \Rightarrow (\mathcal{S},\ Prop,\ M)\, w \cdot u \models B$

In other words:

$$\begin{array}{l} \vdash \text{Model-System}\ \mathcal{S}\ Prop \wedge \text{Model-Function}\ \mathcal{S}\ Prop\ M \wedge w \in \mathcal{S}.\mathsf{W} \Rightarrow \\ \quad (\forall s.\ (\mathcal{S},\ Prop,\ M)\, w \models (\text{g-VAR}\ s) \iff w \in M\ (\text{g-VAR}\ s)) \wedge \\ \quad ((\mathcal{S},\ Prop,\ M)\, w \models \tau \iff \mathcal{S}.\varepsilon \leqslant w) \wedge \\ \quad (\forall A.\ (\mathcal{S},\ Prop,\ M)\, w \models \sim\! A \iff \forall u.\ (\mathcal{S},\ Prop,\ M)\, u \models A \Rightarrow w \perp u) \wedge \\ \quad (\forall A\ B. \\ \quad\quad (\mathcal{S},\ Prop,\ M)\, w \models (A\ \&\ B) \iff \\ \quad\quad\quad (\mathcal{S},\ Prop,\ M)\, w \models A \wedge (\mathcal{S},\ Prop,\ M)\, w \models B) \wedge \\ \quad \forall A\ B. \\ \quad\quad (\mathcal{S},\ Prop,\ M)\, w \models (A \rightarrow B) \iff \\ \quad\quad\quad \forall u.\ (\mathcal{S},\ Prop,\ M)\, u \models A \Rightarrow (\mathcal{S},\ Prop,\ M)\, w \cdot u \models B \end{array}$$

With these conditions we are now able to the soundness result for the Cover Semantics.

Theorem 4.1.21 (Soundness)**.** The Goldblatt Axioms are *sound* with respect to the Cover Semantics:

$$\vdash\ \vdash_{\boldsymbol{R}} p \wedge \text{Model-System}\ \mathcal{S}\ Prop \wedge \text{Model-Function}\ \mathcal{S}\ Prop\ M \Rightarrow \\ (\mathcal{S},\ Prop,\ M) \models p$$

Proof. First we will induct on $\vdash_{\boldsymbol{R}}$. Most of these cases we will omit for brevity as they are largely routine, but we will look at a couple of the more interesting cases:

Suffixing: After simplifying with Definitions 4.1.17 and 4.1.13 we are required to show that $\mathcal{S}.\varepsilon \cdot x \in \mathcal{S}.\mathsf{W}$ and $(\mathcal{S}.\varepsilon \cdot x) \cdot y \in \mathcal{S}.\mathsf{W}$ for x and y in $\mathcal{S}^W$ all of which hold because of the properties of Relevant-CS. Then we have to show that $((\mathcal{S}.\varepsilon \cdot x) \cdot y) \cdot z \in M\ C$ for $z \in M\ A$ while assuming both:

1. $\forall x'. (\exists x. (x' = y \cdot x) \wedge x \in M\ B) \Rightarrow x' \in M\ C$
2. $\forall x'. (\exists x''. (x' = x \cdot x'') \wedge x'' \in M\ A) \Rightarrow x' \in M\ B$

By 1. we only need to produce a x' that satisfies $(((\mathcal{S}.\varepsilon \cdot x) \cdot y) \cdot z = y \cdot x') \wedge x' \in M\ B$ which is of course $x' = x \cdot z$. We get $((\mathcal{S}.\varepsilon \cdot x) \cdot y) \cdot z = y \cdot (x \cdot z)$ by the conditions on Fusion and we get $x \cdot z \in M\ B$ immediately.

Contraction: After simplifying with Definitions 4.1.17 and 4.1.13, and resolving the trivial cases, we are required to show that $x \cdot y \in M\ B$ given $x \in \mathcal{S}^W$ and $y \in M\ A$.

We note that Upset $(M\ B)$ which means that it suffices to show that $x \cdot y \in (\text{to-CS } \mathcal{S}).\mathsf{W}$ and that there exists a d such that $d \in M\ B \wedge d \leqslant (x \cdot y)$. The first condition is immediate and setting $d = (x \cdot y) \cdot y$ makes $((x \cdot y) \cdot y) \leqslant (x \cdot y)$ obvious due to the conditions of how Fusion interacts with $(\leqslant)$.

Contradiction: After simplifying with Definitions 4.1.17 and 4.1.13, and resolving the trivial cases, we get that we need to show that $(x \cdot y) \perp z$ given that $x \in \mathcal{S}^W$ and $y \in M\ B$ and $z \in M\ A$ While also assuming:

1. $\forall x''. x'' \in M\ A \Rightarrow x \cdot x'' \in \mathcal{S}.\mathsf{W} \wedge \forall x'. x' \in M\ B \Rightarrow (x \cdot x'') \perp x'$

We then specialise 1. with z and the result then follow from the conditions on $(\perp)$.

□

4.2 Completeness

For the cover semantics, our completeness result will take the form of:

$$\begin{aligned}&(\forall \mathcal{S}\ \mathit{Prop}\ M.\\ &\quad \text{Model-System } \mathcal{S}\ \mathit{Prop} \wedge \text{Model-Function } \mathcal{S}\ \mathit{Prop}\ M \Rightarrow\\ &\qquad (\mathcal{S}, \mathit{Prop}, M) \models p) \Rightarrow\\ &\ \vdash_{\boldsymbol{R}}\ p\end{aligned}$$

As we did for the Routley-Meyer Semantics, we will construct a canonical Model-System to show completeness with respect to the cover semantics. However we will be using it in a competely different way. The idea is to show that if p is valid in all $\mathcal{S}$ then it must be valid in the canonical Model-System, and then we will show that if p is valid in the canonical Model-System then it is a theorem of the Goldblatt Hilbert System.

But first we will look at some useful constructs:

Definition 4.2.1 (*Consequence*)**.** We will define a relationship over *g-prop* which we will call the *consequence* relation (Though it could also be read as deducibility).

$$A \vdash_{\boldsymbol{R}} B \overset{\text{def}}{=} \vdash_{\boldsymbol{R}} (A \rightarrow B)$$

We should then read $A \vdash_{\boldsymbol{R}} B$ as B is a *consequence* of A.[2]

And naturally we will observe the following results of Definition 4.2.1

Proposition 4.2.2 (Some results of entails)**.**

1. $\text{Preorder}\ (\vdash_{\boldsymbol{R}})\ \mathcal{U}(:\textit{g-prop})$
2. $A \vdash_{\boldsymbol{R}} A$
3. $A \vdash_{\boldsymbol{R}} B \wedge B \vdash_{\boldsymbol{R}} C \Rightarrow A \vdash_{\boldsymbol{R}} C$
4. $A \vdash_{\boldsymbol{R}} B \,\&\, C \Leftrightarrow A \vdash_{\boldsymbol{R}} B \wedge A \vdash_{\boldsymbol{R}} C$
5. $A \vdash_{\boldsymbol{R}} C \wedge B \vdash_{\boldsymbol{R}} C \Rightarrow A \vee B \vdash_{\boldsymbol{R}} C$
6. $A \vdash_{\boldsymbol{R}} {\sim}B \Rightarrow B \vdash_{\boldsymbol{R}} {\sim}A$
7. $A \vdash_{\boldsymbol{R}} B \Rightarrow {\sim}B \vdash_{\boldsymbol{R}} {\sim}A$
8. $A \vdash_{\boldsymbol{R}} {\sim}{\sim}A \wedge {\sim}{\sim}A \vdash_{\boldsymbol{R}} A$
9. $A \circ B \vdash_{\boldsymbol{R}} B \circ A$
10. $A \vdash_{\boldsymbol{R}} B \Rightarrow A \circ C \vdash_{\boldsymbol{R}} B \circ C \wedge C \circ A \vdash_{\boldsymbol{R}} C \circ B$
11. $A \vdash_{\boldsymbol{R}} B \wedge C \vdash_{\boldsymbol{R}} D \Rightarrow A \circ C \vdash_{\boldsymbol{R}} B \circ D \wedge C \circ A \vdash_{\boldsymbol{R}} D \circ B$
12. $A \circ B \vdash_{\boldsymbol{R}} C \Leftrightarrow A \vdash_{\boldsymbol{R}} B \rightarrow C$
13. $A \circ B \circ C \vdash_{\boldsymbol{R}} A \circ (B \circ C) \wedge A \circ (B \circ C) \vdash_{\boldsymbol{R}} A \circ B \circ C$
14. $\tau \circ A \vdash_{\boldsymbol{R}} A \wedge A \vdash_{\boldsymbol{R}} \tau \circ A$
15. $A \vdash_{\boldsymbol{R}} A \circ A$

We naturally call the set of all propositions that are the consequence of A the *prinicpal theory* of A and define:

Definition 4.2.3 (Principal Theory)**.** The *principal theory* of A (Or just theTheory of A) is the set of all propositions are *consequents* of A under $(\vdash_{\boldsymbol{R}})$. We write $A^{\vdash}$ to denote this set.

$$A^{\vdash} \overset{\text{def}}{=} \{\, B \mid A \vdash_{\boldsymbol{R}} B \,\}$$

We also can derive the following closure and containment properties of *principal theories* from Proposition 4.2.2

Proposition 4.2.4 (properties of $A^{\vdash}$)**.**

1. $B \in A^{\vdash} \wedge B \vdash_{\boldsymbol{R}} C \Rightarrow C \in A^{\vdash}$
2. $B \in A^{\vdash} \wedge C \in A^{\vdash} \Rightarrow B \,\&\, C \in A^{\vdash}$
3. $A \vdash_{\boldsymbol{R}} B \Leftrightarrow B^{\vdash} \subseteq A^{\vdash}$
4. $A \vdash_{\boldsymbol{R}} B \wedge B \vdash_{\boldsymbol{R}} A \Leftrightarrow A^{\vdash} = B^{\vdash}$

[2]Note that this is just a special case of Definition 3.2.7.

The next few definitions are quite important as they allow us to create well-defined functions for fusion and orthogonality inside our Canonical System.

Definition 4.2.5 (Equivalence Class)**.** The *equivalence class*[3] of a Goldblatt proposition A is the set of propositions where both $A \vdash_{\boldsymbol{R}} B$ and $B \vdash_{\boldsymbol{R}} A$.

$$[A] \stackrel{\mathsf{def}}{=} \{ B \mid B \vdash_{\boldsymbol{R}} A \wedge A \vdash_{\boldsymbol{R}} B \}$$

We also say that a set X is an equivalence class if there exists a proposition A such that $X = [A]$ or $X = \emptyset$.

$$\textsf{is-EQUIV}\ X \Leftrightarrow \exists A.\ (X = [A]) \vee (X = \emptyset)$$

If we know that X can be expressed as $A^{\vdash}$ for some A then we say that A generates X. X could also reasonably have more than one generator so it makes sense to talk about the set of generators of X:

Definition 4.2.6 (Generators)**.** A *generator* of a set X is a propostition A such that $A^{\vdash} = X$. The *generators* of$X$ is the set all such As:

$$\textsf{gens}\ X \stackrel{\mathsf{def}}{=} \{ A \mid A^{\vdash} = X \}$$

It is also the case that any two generators of the same set are in the same equivalence class from Proposition 4.2.4.4, so:

Corollary 4.2.7. It is the case that the set of *generators* of a set X form an *equivilance class*:

$$\vdash \textsf{is-EQUIV}\ (\textsf{gens}\ X)$$

Finally we can consider taking the union of every $A^{\vdash}$ when A appears inside another set X.

Definition 4.2.8 (Union of Theories)**.** Given a set X we say that the $\textsf{Theory-}\bigcup$ of X is just the union of all $A^{\vdash}$ where A appears in X.

$$\textsf{Theory-}\bigcup X \stackrel{\mathsf{def}}{=} \bigcup \{ A^{\vdash} \mid A \in X \}$$

When it is the case that our X is a non-empty equivalence class of a Goldblatt proposition, then the union of all the theories generated by members of X is precisely $A^{\vdash}$

Corollary 4.2.9. The union of all theories generated by $[A]$ is precicely $A^{\vdash}$.

$$\vdash \textsf{Theory-}\bigcup [A] = A^{\vdash}$$

Now we are able to define the relations we will use for Fusion and Orthogonality:

[3] We are borrowing terminology from the formal notion of equivalence class, but we don't actually show that this is indeed an actual equivalence class

Definition 4.2.10. For sets of Goldblatt propostions x and y we define the operator($\cdot_{\mathfrak{C}}$) to be:

$$X \cdot_{\mathfrak{C}} Y \stackrel{\mathsf{def}}{=} \mathsf{Theory}\text{-}\bigcup \{ x \circ y \mid x \in \mathsf{gens}\ X \wedge y \in \mathsf{gens}\ Y \}$$

Definition 4.2.11. For sets of Goldblatt propostions x and y we define the relation ($\perp_{\mathfrak{C}}$) to be:

$$X \perp_{\mathfrak{C}} Y \stackrel{\mathsf{def}}{=} \exists A\ B.\ A \in \mathsf{gens}\ X \wedge B \in \mathsf{gens}\ Y \wedge A \vdash_{\boldsymbol{R}} {\sim} B$$

When Goldblatt [2011] defined fusion and orthogonality for the canonical system, it was only defined it over the canonical frame's set of worlds. This meant that Goldblatt only needed to define the function across elements of the form $A^{\vdash}$ leading to a simplified . In HOL we are not able to do this, so we propose ($\cdot_{\mathfrak{C}}$) and ($\perp_{\mathfrak{C}}$) as above and an operator and relation over all sets of propositions. We can show that these agree with Goldblatt's definitions below:

Theorem 4.2.12. Where A and B are Goldblatt propositions then:

$$A^{\vdash} \cdot_{\mathfrak{C}} B^{\vdash} = A \circ B^{\vdash}$$

Proof. We simplify with Definition 4.2.10 And then prove equality using the anti-symmetry of ($\subseteq$)

- $\supseteq$: We rewrite with Definition 4.2.8 then it suffices to show that there exists $y \in \mathsf{gens}\ A^{\vdash}$ and $z \in \mathsf{gens}\ B^{\vdash}$ such that $x \in y \circ z^{\vdash}$ given that we already have $x \in A \circ B^{\vdash}$. The result follows immedately.
- $\subseteq$: This requires that given $y \in \mathsf{gens}\ A^{\vdash}$, $z \in \mathsf{gens}\ B^{\vdash}$, and $x \in y \circ z^{\vdash}$, show that $x \in A \circ B^{\vdash}$. This then follows from Propositions 4.2.2.3, 4.2.2.11, and 4.2.4.4.

□

Theorem 4.2.13. Where A and B are Goldblatt propositions then:

$$A^{\vdash} \perp_{\mathfrak{C}} B^{\vdash} \Leftrightarrow A \vdash_{\boldsymbol{R}} {\sim} B$$

Proof. This follows immediately from Propositions 4.2.2.3 and 4.2.2.7. □

4.2.1 The Canonical System and Completeness

Now that we have all the underlying stuctures defined we will now move onto constructing our canonical system:

Definition 4.2.14. The Canonical System $\mathfrak{C}$ is a set of worlds $\mathfrak{C}^W$ given by the set of $A^{\vdash}$ where A is a Goldblatt Proposition. A refinement operation ($\leqslant_{\mathfrak{C}}$) given by containment, that is for worlds x and y, $x \leqslant_{\mathfrak{C}} y \Leftrightarrow x \subseteq y$. A Covering relation given by $Z \rhd_{\mathfrak{C}} x \Leftrightarrow$

$\bigcap Z = x$ where Z is a subset of $\mathfrak{C}^W$ and x is in $\mathfrak{C}^W$. Our $\varepsilon = \tau^{\vdash}$. Fusion and Orthogonality are given by Defintions 4.2.10 and 4.2.11:

$$
\begin{array}{l}
\mathfrak{C} \stackrel{\mathsf{def}}{=} \\
\quad \langle \mathsf{W} := \{ A^{\vdash} \mid A \in \mathcal{U}(:\textit{g-prop}) \}; \\
\quad\; \leqslant := \\
\quad\;\; (\lambda\, x\; y. \\
\quad\quad\quad x \subseteq y \wedge x \in \{ A^{\vdash} \mid A \in \mathcal{U}(:\textit{g-prop}) \} \wedge \\
\quad\quad\quad y \in \{ A^{\vdash} \mid A \in \mathcal{U}(:\textit{g-prop}) \}); \\
\quad\; \rhd := \\
\quad\;\; (\lambda\, Z\; x. \\
\quad\quad\quad (\bigcap Z = x) \wedge x \in \{ A^{\vdash} \mid A \in \mathcal{U}(:\textit{g-prop}) \} \wedge \\
\quad\quad\quad Z \subseteq \{ A^{\vdash} \mid A \in \mathcal{U}(:\textit{g-prop}) \}); \; \varepsilon := \tau^{\vdash}; \; \cdot := \cdot_{\mathfrak{C}}; \\
\quad\; \bot := \bot_{\mathfrak{C}} \rangle
\end{array}
$$

Now that we have our choice of Canonical System we should make sure that it is actually a Relevant Cover System:

Theorem 4.2.15. $\mathfrak{C}$ is a *Relevant Cover System*:

$$\vdash \textsf{Relevant-CS}\ \mathfrak{C}$$

Proof. Most of the 19 cases needed to show this result are immediate given the definition of $\mathfrak{C}$ and Proposition 4.2.2 the two exceptions are as follows:

The first exception is: given $Z \rhd_{\mathfrak{C}} x$ and $y \in \mathfrak{C}^W$ then show $(Z \cdot_{\mathfrak{C}} y) \rhd_{\mathfrak{C}} (x \cdot_{\mathfrak{C}} y)$. We need to show first that $A^{\vdash} \cdot_{\mathfrak{C}} B^{\vdash}$ and $Z \cdot_{\mathfrak{C}} B^{\vdash}$ are well formed, both of which are a result of Theorem 4.2.12. Then we have to show that $\bigcap (Z \cdot_{\mathfrak{C}} B^{\vdash}) = A^{\vdash} \cdot_{\mathfrak{C}} B^{\vdash}$ which we will do by using the anti-symmetry of $(\subseteq)$

- $\supseteq$: Let $x \in (A^{\vdash} \cdot_{\mathfrak{C}} B^{\vdash})$ and $\bigcap Z = A^{\vdash}$, then given $C^{\vdash} \in Z$ show that $x \in (C^{\vdash} \cdot_{\mathfrak{C}} B^{\vdash})$. Using Theorem 4.2.12 we have that this is the same as saying that given $x \in A \circ B^{\vdash}$ show $x \in C \circ B^{\vdash}$. It then suffices to show that $C \vdash_{\boldsymbol{R}} A$ which follows from $\bigcap Z = A^{\vdash}$.
- $\subseteq$: Let $\bigcap Z = A^{\vdash}$ and assume:

  $$\forall\, x.\ x \in Z \Rightarrow D \in (x \cdot_{\mathfrak{C}} B^{\vdash})$$

  This simplifies down to showing $D \in A \circ B^{\vdash}$. We are able to show that for all propositions C in Z then $B \rightarrow D \in C^{\vdash}$ and that it suffices to show that $B \rightarrow D \in A^{\vdash}$ by using Proposition 4.2.2. The result then follows straightforwardly.

The second exception is: given $Z \rhd_{\mathfrak{C}} x$ and $Z \bot_{\mathfrak{C}} \varepsilon$ then show $x \bot_{\mathfrak{C}} \varepsilon$. From this we have that $\forall\, C.\ C^{\vdash} \in Z \Rightarrow C^{\vdash} \bot_{\mathfrak{C}} \tau^{\vdash}$ and that $\bigcap Z = A^{\vdash}$ and we need to show that $A^{\vdash} \bot_{\mathfrak{C}} \tau^{\vdash}$. We note that it suffices to show that $\sim\!\tau \in \bigcap Z$ because of Theorem 4.2.13 but we note again that we have $\forall\, C.\ C^{\vdash} \in Z \Rightarrow C \vdash_{\boldsymbol{R}} \sim\!\tau$ due to Theorem 4.2.13. The result then follows immediately. ☐

We will also simultaniously define the set of *Propositions* that we will use as the basis for an Model-System:

Definition 4.2.16. Where $|A|_{\boldsymbol{R}}$ denotes the set of worlds in $\mathfrak{C}^W$ which contain A

$$|A|_{\boldsymbol{R}} \stackrel{\mathsf{def}}{=} \{\, w \mid A \in w \wedge w \in \mathfrak{C}^W \,\}$$

We define the *canonical admissable propoistions* ($Prop_{\mathfrak{C}}$) as:

$$Prop_{\mathfrak{C}} \stackrel{\mathsf{def}}{=} \{\, |A|_{\boldsymbol{R}} \mid A \in \mathcal{U}(:\textit{g-prop}) \,\}$$

There is also an equivalence between the Goldblatt Propositions and the structure of our set of admissable propoistions $Prop_{\mathfrak{C}}$.

Lemma 4.2.17 (Properties of $|\cdot|_{\boldsymbol{R}}$)**.**

1. $\uparrow_{\mathfrak{C}} (A^{\vdash}) = |A|_{\boldsymbol{R}}$
2. $|A|_{\boldsymbol{R}} \cap |B|_{\boldsymbol{R}} = |A \,\&\, B|_{\boldsymbol{R}}$
3. $|A|_{\boldsymbol{R}}^{\perp} = |{\sim}A|_{\boldsymbol{R}}$
4. $|A|_{\boldsymbol{R}} \hookrightarrow_{\mathfrak{C}} |B|_{\boldsymbol{R}} = |A \rightarrow B|_{\boldsymbol{R}}$
5. $|A|_{\boldsymbol{R}} + |B|_{\boldsymbol{R}} = |A \vee B|_{\boldsymbol{R}}$

Proof. We ommit 1. and 2. as as they are routine proofs.

3. We show equality though the anti-symmetry of ($\subseteq$):

   $\supseteq$: We are required to show that given $x \in \mathfrak{C}^W$ and

   $$\forall y.\, A \in y \wedge y \in \mathfrak{C}^W \Rightarrow x \perp_{\mathfrak{C}} y$$

   That we can show ${\sim}A \in x$. We first specialise the above statement with $A^{\vdash}$ and then simpilfying against Theorem 4.2.13. The result then follows straighforwardly from the definition of $\mathfrak{C}$.

   $\subseteq$: Follows straightforwardly using Theorem 4.2.13 and Propositions 4.2.2.3 and 4.2.2.6.

4. We show equality though the anti-symmetry of ($\subseteq$):

   $\subseteq$: This direction requires that we show $A \rightarrow B \in C^{\vdash}$ given

   $$\{\, C^{\vdash} \cdot_{\mathfrak{C}} x' \mid A \in x' \wedge \exists A.\, x' = A^{\vdash} \,\} \subseteq \{\, w \mid B \in w \wedge \exists A.\, w = A^{\vdash} \,\}$$

   We can then show that this is equivalent to showing $A \rightarrow B \in C^{\vdash}$ from

   $$A \in A^{\vdash} \Rightarrow \exists D.\, B \in C \circ A^{\vdash} \wedge (C \circ A^{\vdash} = D^{\vdash})$$

   The result then follows immediately from Proposition 4.2.2.2 and 4.2.2.12.

$\supseteq$: Immediate from Definition 4.1.13.

5. Immediate given 2. and 3.

$\square$

Theorem 4.2.18. $\mathfrak{C}$ together with $Prop_{\mathfrak{C}}$ constitutes a Model-System

$$\vdash \textsf{Model-System}\ \mathfrak{C}\ Prop_{\mathfrak{C}}$$

Proof. Most of the cases are immediate from the definitions of $\mathfrak{C}$ and $Prop_{\mathfrak{C}}$ given Lemma 4.2.17. The only case that isn't immediate is showing that $X + Y = \mathsf{j}\ (X \cup Y)$.

As X and Y are in $Prop_{\mathfrak{C}}$ then they are of the form $|A|_{\boldsymbol{R}}$ and $|B|_{\boldsymbol{R}}$ for some propositions A and B. We will show the required equality though the anti-symmetry of $(\subseteq)$.

$\supseteq$: By Definition 4.1.13 we have that we need to show $\mathsf{j}\ (|A|_{\boldsymbol{R}} \cup |B|_{\boldsymbol{R}}) \subseteq (|A|_{\boldsymbol{R}} \cup |B|_{\boldsymbol{R}})^{\perp\perp}$ and this follows straightforwardly from Theorem 4.2.15 and Lemma 4.1.14.6.

$\subseteq$: It suffices to show that there exists a C such that when $C^{\vdash} \in |A \vee B|_{\boldsymbol{R}}$ then $C^{\vdash} \in \mathsf{j}\ (|A|_{\boldsymbol{R}} \cup |B|_{\boldsymbol{R}})$. It is also the case that:

$$\uparrow_{\mathfrak{C}} (C^{\vdash}) = |C|_{\boldsymbol{R}}$$

We then need to show that there exists a cover $Z \rhd_{\mathfrak{C}} C^{\vdash}$ such that $Z \subseteq |A|_{\boldsymbol{R}} \cup |B|_{\boldsymbol{R}}$ and we naturally pick.

$$Z = |C|_{\boldsymbol{R}} \cap |A|_{\boldsymbol{R}} \cup |C|_{\boldsymbol{R}} \cap |B|_{\boldsymbol{R}}$$

This means that we need to show that $\bigcap Z = C^{\vdash}$ which we will again show by anti-symmetry.

The backwards direction relies on $Z \subseteq |C|_{\boldsymbol{R}}$, and Propostion 4.2.2.3.

The forwards direction relies Lemma 4.2.17.2 and on the fact that if x is a proposition and $x \in (C\ \&\ A)^{\vdash}$ as well as $x \in (C\ \&\ B)^{\vdash}$ then it must be in $C^{\vdash}$.

$\square$

Now that we actually have that $\mathfrak{C}$ and $Prop_{\mathfrak{C}}$ together set up a Model-System we have everything we need to show completeness, which we will show by way of a chain of implications.

Theorem 4.2.19 (Model System Characterisation of $\boldsymbol{R}$)**.** For any formula p the following are equivalent:

1. $\vdash_{\boldsymbol{R}}\ p$
2. $\forall\, \mathcal{S}\ Prop\ M.$
   Model-System $\mathcal{S}\ Prop\ \wedge$ Model-Function $\mathcal{S}\ Prop\ M \Rightarrow$
   $(\mathcal{S}, Prop, M) \models p$

3. $\forall M.$ Model-Function $\mathfrak{C}$ $Prop_{\mathfrak{C}}$ $M \Rightarrow (\mathfrak{C}, Prop_{\mathfrak{C}}, M) \models p$

Proof.

$1 \Rightarrow 2$: See Theorem 4.1.21.

$2 \Rightarrow 3$: Follows as an immediate consequence of Theorem 4.2.18.

$3 \Rightarrow 1$: We choose our $M = (\lambda x.\ |x|_{\boldsymbol{R}})$. Then we are required to show that this is a Model-Function with respect to the $\mathfrak{C}$ and $Prop_{\mathfrak{C}}$. The g-VAR case follows from Definition 4.2.16. The (&), ($\rightarrow$), and ($\sim$) cases follow from Lemma 4.2.17.2, .4, and .3 respectively. And the τ case follows from the definition of ε and Lemma 4.2.17.4.

Now that we have our model function, we only need to show that when

$$(\mathfrak{C},\ Prop_{\mathfrak{C}},\ (\lambda x.\ |x|_{\boldsymbol{R}})) \models p$$

then p which, when unpacking the definitions for $\mathfrak{C}$, $Prop_{\mathfrak{C}}$, EQUIV-W, and ($\models$) gives us that we only need to show that $\vdash_{\boldsymbol{R}}\ p$ given $\tau\ \vdash_{\boldsymbol{R}}\ p$ which follows immediately.

□

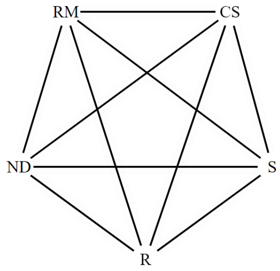
Which yields the completeness result:

Corollary 4.2.20 (Completeness)**.** The Goldblatt Axioms are *complete* with respect to the Cover Semantics:

$$\begin{array}{l} \vdash (\forall \mathcal{S}\ Prop\ M. \\ \quad \text{Model-System } \mathcal{S}\ Prop \land \text{Model-Function } \mathcal{S}\ Prop\ M \Rightarrow \\ \quad\quad (\mathcal{S},\ Prop,\ M) \models p) \Rightarrow \\ \ \vdash_{\boldsymbol{R}}\ p \end{array}$$

4.3 Summary

This chapter introduced the Goldblatt Cover semantics for Relevant Implication and formalised both semantic soundness and completeness between the Goldblatt axiomatisation and the Goldblatt Cover semantics. Inside the HOL development, this required 1286 lines of source code contained entirely inside `CoverSemanticsScript.sml` and completes our investigation of Relevant Implication.



Chapter 5

Conclusion

This thesis mechanised the Relevant Logic **R** as two different Hilbert Systems, a Natural Deduction system, the Routley-Meyer Semantics for **R**, and the Goldblatt Cover Semantics for **R**. It also provided equivalence results between the Hilbert systems as well as between the Natural Deduction system and Goldblatt axiomatisation. Finally we also showed both Soundness and Completeness results between the Goldblatt axiomatisation and both of the semantic systems that were mechanised. With these results we are able to show all equivalence results between our different characterisations of **R** and thus complete the Relevant Pentagram with red denoting mechanised proofs and black denoting equivalence results that are immediate from these proofs (see Figure 5.1).

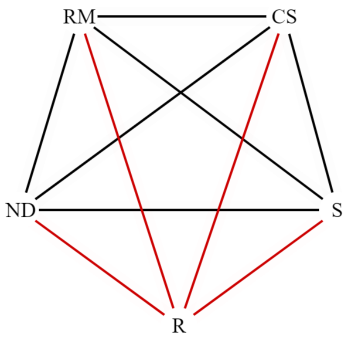


Figure 5.1: Mechanised results (red) from left to right:
Equivalence of ND and R: Theorems 2.3.7, 2.3.9 and 2.3.12; Soundness and Completeness of RM and R: Theorems 3.1.9 and 3.2.47; Soundness and Completeness of CS and R: Theorem 4.1.21 and Corollary 4.2.20; Equivalence of S and R: Theorem 2.2.8

On reflection of this mechanisation, while it was successful beyond doubt in its goals, there are a number of design decisions in definitions that meant that in order to achieve the desired results there would have had to be a number of reproved results for similar constructs. For example, the definitions for $(\vdash^{\&}_{\boldsymbol{R}})$ and $(\vdash^{\&}_{\Theta})$ are incredibly similar, and could be merged into a single HOL definition. As an immediate result, R-Theory and Θ-Theory would become a single definition, and the definition of Θ-Theory would have to be weakened so that Θ was not required to be Ordinary as **R** is not nessisarily Prime. This would not matter in the grand scheme of the results we were trying to show as we were only in situations where Θ that are either Ordinary or equal to **R**, but it would simplify many of the proofs themselves and reduce cases where we were proving theorems twice.

5.1 Future Work

As a final note, there are several different routes that could build off the work done in this thesis. Here we will provide a non-exhaustive list of directions that future research could take and outlines the first steps in their potential mechanisation in HOL.

Alternative Relevant Logics: Routley et al. [1982, pp. 287-290] express the the basic Relevant Logic **B** along with a number of different systems that are build off **B** (including **R**) by adding new axioms schemes to the Hilbert system. Routley et al. [1982, pp. 298-301] also state what conditions on the ternary relation R added to the *B model structures* correspond to which axiom schema. One could extend the work here by building the axiom system for **B**, showing soundness and completeness between **B** and the class of all *B model structures* and then showing that the resulting classes acheived by adding further restrictions on R correspond with the expected axioms.

Fine's incompleteness proof for RQ: **RQ** is the first order extension of **R** and was shown to be incomplete with respect to the *consistant domain semantics* which was considered to be the natural quanitified extension of the Routley-Meyer system [Routley, 1980]. A mechanisation of incompleteness result by Fine [1989] between **RQ** and the *consistant domain semantics* will entail extending the mechanisation of **R** to **RQ** and the Routley-Meyer semantics to the consistant domain semantics.

Goldblatt cover completeness for RQ: Goldblatt [2011, pp. 231-239] demonstrates that there is an extension of the cover semantics that is sound and complete with respect to **RQ**. This would involve similar work to the incompleteness result above, that is, extending **R** to **RQ** and the cover semantics to its quantified extension before then showing the soundness and completeness results.

Decidability of Relevant Fragments: Many fragments of **R** have been shown to be decidable such as $\mathbf{R}_{\to}$, $\mathbf{R}_{\to,\&}$, $\mathbf{R}_{\to,\sim}$, and **LR** [Dunn and Restall, 2002]. $\mathbf{R}_{\to}$ at least has been already been mechanised by Larchey-Wendling [2020] while Dawson and Goré [2017] have verified that most of the decidability proof of $\mathbf{T}_{\to}$ by Bimbó and Dunn [2013] but have not yet shown the proof in its entirety. To mechanise this fully in HOL4 this would require that the axiomatisation for whatever logic fragment to be shown to be decidable, be implemented entirely in lambda calculus in order to establish formal decidability, though there is flexibility in this.

Undecidability of R: Urquhart [1984] shows that **E**, **R**, and **T** are undecidable through showing that it is equivalent to the *word problem* by defining lattice-like on subsets of propositions. In order to mechanise the undecidability of any of these logics in HOL one would have to first show that the proofs that Urquhart [1984] relies on are true, in particular the undecidability of the *word problem* which has not yet been done in HOL, though it has been shown to be undecidable in Coq [Larchey-Wendling, 2021]. Then, just like in showing decidability, one would have to implement the axiomatisation entirely in lambda calculus before showing the proof.

Modal and Modal-Like Relevant Logics: These are logics that extend any of the Relevant Logics, such as **B** or **R** with one or many modal or modal-like operators $\Box$ as well as axiomatic schema that govern them. Introducing modalities extends Relevant Logics to allow for them to reason about knowledge (epistemic relevant logic) or time (temporal relevant logic), among others. Fuhrmann [1990] showed that a number of these modal extensions were complete with respect to the unreduced Routley-Meyer semantics. The next steps for this extension would be to extend the Routley-Meyer semantics to the unreduced case, show that **R** is sound and complete with respect to the unreduced case, and then extend the unreduced system and **R** to its modal extension before showing completeness.

It should be noted that extending the work of this thesis in these directions will likely only result in gains inside the study of Relevant Logic due to there being limited use of Relevant Logics outside of the field of philosophical logic. There may be a few exceptions to this, however, given that there is some minor research being done by Cheng [2004] where a Temporal extension of Relevant Logic is used as the basis of a class of anticipatory systems; or the use of an further extension of Relevant Logic as a basis for Air Traffic Control by Han et al. [2016]. At this stage however, it is still too early to say whether this research will yield use cases for Relevant Logic outside of philosophy.

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