Multipartite Gaussian entanglement of formation

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Entanglement of formation is a fundamental measure that quantifies the entanglement of bipartite quantum states. This measure has recently been extended into multipartite states, taking the name α-entanglement of formation. In this work we follow an analogous multipartite extension for the Gaussian version of entanglement of formation, and focusing on the finest partition of a multipartite Gaussian state, we show that this measure is fully additive and computable for three-mode Gaussian states.

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I. INTRODUCTION

Entanglement is a property of quantum mechanics that allows correlations beyond the classical limit. As such, it is considered a crucial resource that allows certain quantum protocols to be more efficient than their classical counterpart [1]. Several entanglement measures have been defined in the literature [1,2]; however, in general the quantification of their values is a challenging task.

Bipartite entanglement of formation (EoF) [3] is defined as the least expected amount of bipartite entropy of entanglement (EoE) required to create a state. In general, the quantification of bipartite EoF involves a minimization over infinite degrees of freedom, making it hard to compute [4]. Initial research focused on simple systems such as the two-qubit system [5,6], which led to analytical expressions for the measure.

An analogous measure, called Gaussian EoF (GEoF), focusing only on Gaussian states and operations, was defined by Wolf *et al.* [7]. A few years later, this measure was proven to be equal to EoF in the case of two-mode Gaussian states [8,9]. For these types of states, several efficient numerical methods and analytical expressions have been derived [7,10–12]. Recently, in Ref. [13], Szalay introduced a measure referred to as α-EoF, which is the multipartite extension of bipartite EoF. In this paper, we follow Wolf's approach and apply the notion of α-EoF onto the Gaussian regime. We show that α-GEoF is a computable multipartite entanglement measure. We utilize a special case of α-GEoF, which we refer to as the total GEoF (TGEoF), to quantify the *total entanglement* in a three-mode Gaussian system, in the sense that it is the sum of the entanglement of all internal partitions of the state.

Our paper is set out in the following way. In Sec. II, we introduce the conventions adopted in this paper. In Sec. III we review bipartite entanglement measures. In Sec. IV, we review α-entanglement measures [13] and introduce a special subset, referring to it as the TEoF. In Sec. V we apply α-entanglement measures to the Gaussian regime and prove TGEoF is fully additive. In Sec. VI, we consider the tripartite case and compute the TEoF for simple cases. We summarize and conclude our results in Sec. VII.

II. PRELIMINARIES

A. Modes, partitions, and subsystems

In the discrete variable case, the smallest subsystems are referred to as qudits (or qubits for two-level systems). In the continuous variable case, the smallest subsystems are referred to as modes. For simplicity, this paper will be utilizing the terminology mode, but in this context it can be used interchangeably with qudits if we are not considering the case of Gaussian states.

Let us consider an N-mode state $\hat{\rho}$. The state of the nth mode $\hat{\rho}_n$ can be found via the partial trace over all other modes:

$$\hat{\rho}_n \equiv \mathrm{Tr}_{\forall i\neq n}(\hat{\rho})\,. \tag{1}$$

$\hat{\rho}$ can be split into M partitions, via assigning each mode into one of the M partitions (where $N \geqslant M$). By doing this we introduce M subsystems, denoted $\{s_1, s_2, ..., s_M\}$. This defines the M partitioning, $\alpha = s_1|s_2|...|s_M$. Each subsystem s_j is defined as the reduced state, achieved through the partial trace over all other subsystems, i.e.,

$$\hat{\rho}_{s_n} \equiv \mathrm{Tr}_{\forall s_i\neq s_n}(\hat{\rho})\,. \tag{2}$$

B. von Neumann entropy

Before we get into the quantification of entanglement, we need to first define a function that a broad family of entanglement measures are based on, i.e., quantum entropy [14–16]. In particular, we focus on the von Neumann entropy, which for a state $\hat{\rho}$ is defined as

$$S(\hat{\rho}) \equiv -\mathrm{Tr}(\hat{\rho}\ln\hat{\rho})\,. \tag{3}$$

$S(\hat{\rho})$ is a basis-independent function, which vanishes for pure states. Also, note that it is fully additive for noncorrelated

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states (although subadditive in general), i.e.,

$$S(\hat{\rho}_{s_1} \otimes \hat{\rho}_{s_2}) = S(\hat{\rho}_{s_1}) + S(\hat{\rho}_{s_2}), \tag{4}$$

and concave

$$S\left(\sum_j p_j \hat{\rho}_j\right) \geqslant \sum_j p_j S(\hat{\rho}_j) . \tag{5}$$

C. Gaussian states

In the later part of this paper, we will be considering quantum systems comprised of bosonic Gaussian modes, $\hat{a}_n$ [17–20]. These bosonic annihilation operators satisfy the bosonic commutation relations $[\hat{a}_n, \hat{a}_m^\dagger] = \delta_m^n$, where δ is the Kronecker delta. For Gaussian states, the analysis of first and second moments [18] is sufficient to characterize the Wigner function of the state [21]. The first moment of an N-mode Gaussian state is fully characterized by its $2N$-dimensional displacement vector, $\vec{D}$. The second-order moment is described by its $2N \times 2N$ real symmetric covariance matrix [22], $\boldsymbol{\sigma}$. As a result, all Gaussian states can be written as $\hat{\rho}_{\sigma,\vec{D}}$.

The ith element of the displacement vector is defined in the following way:

$$\vec{D}_i \equiv \mathrm{Tr}(\rho \hat{R}_i) , \tag{6}$$

where

$$\vec{R} \equiv (\hat{q}_1, ..., \hat{q}_N, \hat{p}_1, ..., \hat{p}_N)^T , \tag{7}$$

and we have defined $\hat{q}_n \equiv \hat{a}_n + \hat{a}_n^\dagger$ and $\hat{p}_n = (\hat{a}_n - \hat{a}_n^\dagger)/i$. The $\{i, i'\}$th element of the covariance matrix $\boldsymbol{\sigma}$ is defined in the following way:

$$\sigma_{ii'} \equiv \tfrac{1}{2}\big(\mathrm{Tr}\{[\hat{\rho}(\hat{R}_i\hat{R}_{i'} + \hat{R}_{i'}\hat{R}_i)]\} - 2\mathrm{Tr}(\hat{\rho}\hat{R}_i)\mathrm{Tr}(\hat{\rho}\hat{R}_{i'})\big) . \tag{8}$$

III. BIPARTITE ENTANGLEMENT MEASURES

A. Bipartite entropy of entanglement

EoE, $E_{s_1|s_2}$, is the typical way to quantify bipartite entanglement in pure states, $\hat{\psi} \equiv |\psi\rangle\langle\psi|$ [23]. This measure is given by the von Neumann entropy of the reduced state:

$$E_{s_1|s_2}(\hat{\psi}) \equiv S\big[\mathrm{Tr}_{s_2}(\hat{\psi})\big]. \tag{9}$$

As $\hat{\psi}$ is a pure state, EoE is invariant under exchange of subsystems, i.e., $E_{s_1|s_2}(\hat{\psi}) = E_{s_2|s_1}(\hat{\psi})$. This is a reliable bipartite entanglement measure, as it satisfies the following postulates [1,2]:

(1) $E_{s_1|s_2}$ is an indicator function for separability between the subsystem s_1 and s_2:

$$E_{s_1|s_2}(\hat{\psi}) = 0 \Leftrightarrow \hat{\psi} = \hat{\psi}_{s_1} \otimes \hat{\psi}_{s_2} . \tag{10}$$

(2) $E_{s_1|s_2}$ is nonincreasing on average under local operations and classical communications (LOCC), $\hat{\Lambda}_{s_1|s_2}$, where the locality is defined in terms of the subsystem s_1 and s_2 [2,3,24–27]:

$$E_{s_1|s_2}(\hat{\psi}) \geqslant \sum_j p_j E_{s_1|s_2}\big[\hat{\Lambda}_{j,s_1|s_2}(\hat{\psi})\big], \tag{11}$$

where

$$\hat{\Lambda}_{s_1|s_2}(\hat{\psi}) = \sum_j p_j \hat{\Lambda}_{j,s_1|s_2}(\hat{\psi}) \tag{12}$$

are pure LOCC suboperations [13,28].

B. Bipartite entanglement of formation

A natural way to extend an entanglement measure to mixed states is via the convex-roof extension [3,29–31]. EoF is defined as the convex-roof extension of EoE:

$$\mathcal{E}_{\mathrm{F},s_1|s_2}(\hat{\rho}) \equiv \inf_{\hat{\rho}=\sum_j p_j \hat{\psi}_j} \left\{\sum_j p_j E_{s_1|s_2}(\hat{\psi}_j)\right\} , \tag{13}$$

where "inf" becomes a "min" for discrete variable states, and the sum can be replaced with an integral when considering a continuum of pure states.

This is a reliable bipartite entanglement measure, as it satisfies the mixed state extension of the aforementioned postulates [13] and an extra one, i.e.,

(3) For pure states $\mathcal{E}_{\mathrm{F},s_1|s_2}$ reduces to the entropy of entanglement, i.e.,

$$\mathcal{E}_{\mathrm{F},s_1|s_2}(\hat{\psi}) = E_{s_1|s_2}(\hat{\psi}) . \tag{14}$$

As von Neumann entropy is concave, postulate 2 implies that bipartite EoF is also nonincreasing under LOCC [3,24]; $\mathcal{E}_{\mathrm{F},s_1|s_2}(\hat{\rho}) \geqslant \mathcal{E}_{\mathrm{F},s_1|s_2}[\hat{\Lambda}_{s_1|s_2}(\hat{\rho})]$.

IV. M-PARTITE ENTANGLEMENT MEASURES

A. α Separability

Entanglement can also exist among several partitions. There are several ways to divide an N-mode system into M partitions. To make a distinction between the partitioning, Szalay [13] introduced a hierarchy of separability classes. A pure state, $|\psi\rangle_\alpha$, is called "α separable" when

$$|\psi\rangle_\alpha \equiv \bigotimes_{s_i\in\alpha} |\psi_{s_i}\rangle, \tag{15}$$

where α denotes a specific partitioning of the N mode. For example, a pure five-mode state is 1|23|45-separable if and only if the state can be written in the following way:

$$|\psi\rangle_{1|23|45} = |\psi_1\rangle \otimes |\psi_{23}\rangle \otimes |\psi_{45}\rangle . \tag{16}$$

Then an α-separable mixed state can be written in the following way:

$$\hat{\rho}_\alpha = \sum_j p_j |\psi_j\rangle_\alpha \langle\psi_j|_\alpha . \tag{17}$$

We can then make a hierarchy for separability as follows: α precedes or equals β if all subsystem in α can be written as a subset or equal to a subsystem in β, i.e.,

$$\alpha \preceq \beta \Leftrightarrow \forall s_i \in \alpha,\ \exists s_{i'} \in \beta : s_i \subseteq s_{i'} . \tag{18}$$

If α has a finer partition than β (i.e. $\alpha \preceq \beta$), then a state which is α separable must also be β separable.

B. α-entropy of entanglement and α-entanglement of formation

1. α-von Neumann entropy

Let us define α-von Neumann entropy in the following way:

$$S_\alpha(\hat{\rho}) \equiv \frac{1}{2}\sum_{s_i\in\alpha} S(\hat{\rho}_{s_i}). \tag{19}$$

This is a measure that is well defined for all states $\hat{\rho}$. Due to the full additivity of S, S_α is also fully additive:

$$S_\alpha(\hat{\rho}_A \otimes \hat{\rho}_B) = S_{\alpha_A}(\hat{\rho}_A) + S_{\alpha_B}(\hat{\rho}_B)\,, \tag{20}$$

where α_C, $C \in \{A, B\}$, is the subset of α, which includes the part that overlaps with the system C.

2. α-entropy of entanglement and entanglement of formation

In the multipartite case, Szalay [13] defined the α-EoE of a pure state $\hat{\psi}$ to be

$$E_\alpha(\hat{\psi}) = S_\alpha(\hat{\psi})\,. \tag{21}$$

This measure can be interpreted as the sum of entanglement between the partitions.

α-EoF is defined as the convex-roof extension to α-EoE [13]:

$$\mathcal{E}_{\mathrm{F},\alpha}(\hat{\rho}) \equiv \inf_{\hat{\rho}=\sum_j p_j\hat{\psi}_j}\left\{\sum_j p_j E_\alpha(\hat{\psi}_j)\right\}. \tag{22}$$

α-EoE and EoF are reliable α-entanglement measures as they satisfy the same postulates as the bipartite case, except we must replace $s_1|s_2$ with α. α-entanglement measures also satisfy an extra postulate:

(4) E_α and $\mathcal{E}_{F,\alpha}$ must satisfy the multipartite monotonicity [13]:

$$E_\alpha(\hat{\rho}) \geqslant E_\beta(\hat{\rho}),\ \forall\alpha \preceq \beta\,, \tag{23}$$

$$\mathcal{E}_{\mathrm{F},\alpha}(\hat{\rho}) \geqslant \mathcal{E}_{\mathrm{F},\beta}(\hat{\rho}),\ \forall\alpha \preceq \beta\,. \tag{24}$$

This means that an entanglement measure of finer partition is sensitive to more entanglement within the system, hence giving a larger value.

C. Total entropy of entanglement and total entanglement of formation

1. Total entropy of entanglement

In this section we consider the finest partitioning of the α-entanglement measure and refer to it as the total entropy of entanglement (TEoE) and total entropy of formation (TEoF). This measure evaluates the entanglement between every mode that exists within the system. TEoE and TEoF satisfy the same postulates as α-entanglement measures with the finest partitioning.

For a pure N-mode state, $\hat{\psi}$, TEoE is defined in the following way:

$$\tilde{E}(\hat{\psi}) = \tilde{S}(\hat{\psi}) \equiv \frac{1}{2}\sum_{n=1}^{N} S[Tr_{\forall i\neq n}(\hat{\psi})]\,. \tag{25}$$

TEoE is the sum of all entanglement between each mode and the rest of the system. Due to multipartite monotonicity, this measure is also the largest pure entanglement measure out of the α-EoF. For this reason we refer to this quantity as the total entanglement within the system.

To highlight a feature of this measure, let us consider a two-mode entangled state, with a vacuum input in the third mode. In this case, this measure will reduce down to the bipartite entanglement between the two-mode entangled state, giving the total entanglement within the system. In comparison, a genuine tripartite entanglement measure [32,33] would be zero in this case, as there is only bipartite entanglement.

2. Total entanglement of formation

For an N-mode mixed state, $\hat{\rho}$, TEoF is defined in the following way:

$$\tilde{\mathcal{E}}_{\mathrm{F}}(\hat{\rho}) \equiv \inf_{\hat{\rho}=\sum_j p_j\hat{\psi}_j}\left\{\sum_j p_j \tilde{E}(\hat{\psi}_j)\right\}. \tag{26}$$

This measure quantifies the least-expected total entanglement that is required to create the mixed state. Even though this is a well-defined measure it is hard to compute, as there are infinite degrees of freedom for the set $\{p_j, \hat{\psi}_j\}$. In this paper we limit ourselves to a Gaussian convex-roof extension (i.e., the convex-roof extension is limited to an optimization over Gaussian states) to overcome this problem. This measure is an upper bound to EoF and only satisfies the Gaussian version of the aforementioned postulates [7].

V. α-GAUSSIAN ENTANGLEMENT OF FORMATION

A. von Neumann entropy and α-EoE for Gaussian states

For Gaussian states, the von Neumann entropy of a state, $\hat{\rho}_{\sigma,\vec{D}}$, is fully characterized by its covariance matrix. The von Neumann entropy of an N-mode Gaussian state with covariance matrix $\boldsymbol{\sigma}$ can be calculated as follows [34]:

$$S(\boldsymbol{\sigma}) = \frac{1}{2}\sum_{n=1}^{N} h(\nu_n)\,, \tag{27}$$

where ν_n is the nth symplectic eigenvalue of $\boldsymbol{\sigma}$, and

$$h(x) \equiv \frac{x_+}{2}\log_2\left(\frac{x_+}{2}\right) - \frac{x_-}{2}\log_2\left(\frac{x_-}{2}\right), \tag{28}$$

with $x_\pm \equiv x \pm 1$ an auxiliary function.

As the von Neumann entropy is fully characterized by its covariance matrix, α-EoE of a pure state, $\hat{\psi}_{\pi,\vec{D}}$, is also fully characterized by its covariance matrix. The α-EoE of a pure state with covariance matrix $\boldsymbol{\pi}$ is calculated as follows:

$$E_\alpha(\boldsymbol{\pi}) = \frac{1}{2}\sum_{s_i\in\alpha} S(\boldsymbol{\pi}^{(s_i)})\,. \tag{29}$$

A covariance matrix is pure if and only if $\det(\boldsymbol{\pi}) = 1$. The superscript (s_i) denotes the removal of all rows and columns except for the subsystem s_i in the covariance matrix. In the density matrix representation, this is equivalent to a trace of all subsystems except s_i; hence $\det(\boldsymbol{\pi}^{(s_i)}) \neq 1$ in general.

B. α-Gaussian entanglement of formation

A mixed Gaussian state $\rho_{\sigma,\vec{D}}$ can be decomposed into a mixture of pure Gaussian states in the following way:

$$\hat{\rho}_{\sigma,\vec{D}} = \int d\pi d\vec{D}' \; \mu(\pi, \vec{D}')\hat{\psi}_{\pi,\vec{D}'} \,, \tag{30}$$

where μ is the probability density of $\hat{\rho}_{\pi,\vec{D}'}$. In Ref. [7] the authors defined the bipartite Gaussian entanglement of formation (GEoF), and analogously, we define the α-GEoF as follows:

$$\mathcal{E}_{G,\alpha}(\hat{\rho}_{\sigma,\vec{D}}) \equiv \inf_{\mu}\left\{ \int d\pi d\vec{D}' \; \mu(\pi, \vec{D}')E_\alpha(\pi) \right.$$
$$\left. |\hat{\rho}_{\sigma,\vec{D}} = \int d\pi d\vec{D}' \; \mu(\pi, \vec{D}')\hat{\psi}_{\pi,\vec{D}'} \right\}. \tag{31}$$

This definition involves a minimization over infinite degrees of freedom; however, by following the analysis of Ref. [7], we find that Eq. (31) reduces to the following expression:

$$\mathcal{E}_{G,\alpha}(\sigma) = \inf_{\pi}\{E_\alpha(\pi)|\sigma = \pi + \varphi\} \,, \tag{32}$$

where φ is a positive semidefinite matrix. This equation has finite free parameters and therefore is a computable entanglement measure. In Appendix A we utilize Eq. (32) to prove the additivity of TGEoF.

VI. TOTAL GAUSSIAN ENTANGLEMENT OF FORMATION FOR THREE-MODE STATES

A. Mixed three-mode Gaussian states

Gaussian local unitary operations (GLUO, refer to Appendix B) are a class of Gaussian unitary operations comprised of single-mode squeezers and phase shifters. For mixed three-mode states, we can utilize GLUO to reduce the state into the standard form [32,35]:

$$\sigma_{sf} = \begin{bmatrix} a_1 & e_1 & e_3 & 0 & 0 & e_4 \\ e_1 & a_2 & e_6 & 0 & 0 & e_7 \\ e_3 & e_6 & a_3 & 0 & e_8 & 0 \\ 0 & 0 & 0 & a_1 & e_2 & e_5 \\ 0 & 0 & e_8 & e_2 & a_2 & e_9 \\ e_4 & e_7 & 0 & e_5 & e_9 & a_3 \end{bmatrix}. \tag{33}$$

As GLUO do not affect the entanglement, we can reduce Eq. (32) to the following:

$$\mathcal{E}_{G,\alpha}(\sigma) = \inf_{\pi}\{E(\pi)|\sigma_{sf} - \pi \geqslant 0\} \,. \tag{34}$$

In the next section, we fully parametrize π.

B. Pure three-mode Gaussian states

By utilizing GLUO, $\mathbf{L}$, we can reduce any π to the standard form [36,37]. For the three-mode pure state, the standard form is [38]

$$\pi_{sf} = \begin{bmatrix} a_1 & e_{12}^+ & e_{13}^+ & 0 & 0 & 0 \\ e_{12}^+ & a_2 & e_{23}^+ & 0 & 0 & 0 \\ e_{13}^+ & e_{23}^+ & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & e_{12}^- & e_{13}^- \\ 0 & 0 & 0 & e_{12}^- & a_2 & e_{23}^- \\ 0 & 0 & 0 & e_{13}^- & e_{23}^- & a_3 \end{bmatrix}, \tag{35}$$

where $e_{ij}^{\pm}$ are a function of a_1, a_2, and a_3. For π_{sf} to be a physical covariance matrix the inequality $|a_i - a_j| \leqslant a_k - 1$ must be satisfied [32]. All pure states can then be decomposed in the following way:

$$\pi = L\pi_{sf}(a_1, a_2, a_3)L^T \,. \tag{36}$$

In general, L has nine free parameters, and hence the minimization of Eq. (34) can be conducted over 12 free parameters. A numerical code which scans over all possible π with a finite-size step for these 12 free parameters can be created. The condition $(\sigma_{sf}^{(n)} - \pi^{(n)}) \geqslant 0$ gives a finite range for all local squeezing operations, a_1, a_2, and a_3. The phase parameters are limited to $0 \geqslant \phi \geqslant 2\pi$.

C. q-p states

In this section we consider a special class of states where we can reduce the number of free parameters to 6. In special cases, the standard form of the mixed state reduces to the following form:

$$\sigma_{qp} = \begin{bmatrix} a_1 & e_1 & e_3 & 0 & 0 & 0 \\ e_1 & a_2 & e_6 & 0 & 0 & 0 \\ e_3 & e_6 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & e_2 & e_5 \\ 0 & 0 & 0 & e_2 & a_2 & e_9 \\ 0 & 0 & 0 & e_5 & e_9 & a_3 \end{bmatrix}. \tag{37}$$

We will refer to these states as q-p states. q-p states have the property that the $\hat{q}$ quadrature is completely uncorrelated to the $\hat{p}$ quadrature. This means that we can write the following:

$$\sigma_{qp} = \sigma_{\hat{q}} \oplus \sigma_{\hat{p}} \,. \tag{38}$$

Following the analysis in Ref. [7], we prove that the optimum pure state to create such a state must also be a q-p state:

$$\pi_{qp} = \pi_{\hat{q}} \oplus \pi_{\hat{p}} \,. \tag{39}$$

These states only have six free parameters, which greatly reduces the complexity of the problem.

Proposition 1. Consider a q-p state σ_{qp}. For every pure state, $\pi \leqslant \sigma$, there exists a q-p pure state $\pi'_{qp} \leqslant \sigma$ which satisfies the following:

$$\tilde{E}(\pi) \geqslant \tilde{E}(\pi'_{qp}) \,. \tag{40}$$

Proof. Any Gaussian pure state π can be written in the following way [7,39]:

$$\pi(X, Y) = \begin{bmatrix} X & XY \\ YX & YXY + X^{-1} \end{bmatrix}, \tag{41}$$

where $X > 0$ and Y are a real symmetric $N \times N$ matrix with $X > 0$. For q-p states, $Y = 0$. For every $\sigma_{qp} \geqslant \pi(X, Y)$, we have the following [7]:

$$\sigma_{qp} \geqslant \pi(X, Y) \Rightarrow \sigma_{qp} \geqslant \pi(X, 0) \,. \tag{42}$$

We also have that the determinant of the single mode $\pi^{(n)}(X, Y)$ is always larger than $\pi^{(n)}(X, \mathbf{0})$:

$$\det[\pi^{(n)}(X, \mathbf{0})] \leqslant \det[\pi^{(n)}(X, Y)] \,. \tag{43}$$

The entropy of a single-mode state is computed to be;

$$S(\sigma_n) = h(\sqrt{\det(\sigma^{(n)})}). \tag{44}$$

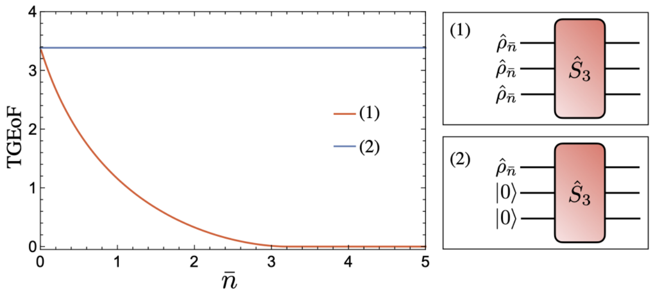

FIG. 1. A plot demonstrating how TGEoF changes with input noise. The red line represents the TGEoF for a thermal input, $\hat{\rho}_{\bar{n}}$, in all three modes. The blue line represents the TGEoF for thermal input in one mode, with all other modes being a vacuum input.

As this is true for every mode, combining Eqs. (44) and (43) gives the following:

$$\tilde{E}[\boldsymbol{\pi}^{(n)}(\boldsymbol{X},\boldsymbol{0})] \leqslant \tilde{E}[\boldsymbol{\pi}^{(n)}(\boldsymbol{X},\boldsymbol{Y})]. \tag{45}$$

Equations (42) and (45) complete the proof. ■

D. Numerical results

Consider a two-mode Gaussian state where one of the modes is thermal while the others are vacuum. When a two-mode squeezer is applied to such a state, the bipartite GEoF is constant regardless of the number of photons in the thermal mode [38,40,41].

We aim to replicate an analogous result in the tripartite case, utilizing TGEoF. We consider a case where a three-mode squeezer, $\hat{S}_3$ (details of this operation can be found in Appendix C), is applied to an input with with all three modes which are thermal with an average of $\bar{n}$ particles. Since the output state is a q-p state, we conduct an numerical optimization over the q-p state to obtain Fig. 1. We repeat this process in the case where $\hat{S}_3$ is applied to an input with one mode which is thermal and the rest being a vacuum. TGEoF is constant when there is only one thermal input, which is an analogous result to the two-mode case.

VII. CONCLUSION

In this paper, we utilized the analysis of Ref. [13] on multipartite entanglement measures and applied it to the Gaussian regime. We successfully demonstrated that the degrees of freedom for this measure reduces down to a finite one for all Gaussian states. In particular, we were interested in a special case of α-GEoF, TGEoF, which quantifies the least expected total entanglement that is required to create the state. We proved that this measure is fully additive. In the last section we quantified its value for simple three-mode Gaussian states and demonstrated that this measure displayed analogous features to the two-mode case.

An interesting future research direction would be to compare TGEoF and TEoF. For the two-mode case, it has been proven that TGEoF and TEoF coincide with each other for Gaussian states [8,9]. It would be beneficial to prove that this can be extended to the N-mode case. Combined with the result that TGEoF is additive, as proven by this paper, the additivity of TEoF would then be proven for Gaussian states in general.

In this paper we were particularly interested in TEoF; however, there are other interesting α-EoF measures. In particular, there is an α-EoF which quantifies the genuine tripartite entanglement within a three-mode system [13,33]. We refer to this measure as a genuine tripartite entanglement measure, as it vanishes for all states which are not genuinely tripartite entangled states. A recent paper [33] looked into finding an upper bound to this measure for the DV case. It would be interesting to apply this to the Gaussian regime and investigate how useful the measure is.

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APPENDIX A: ADDITIVITY OF TOTAL GAUSSIAN ENTANGLEMENT OF FORMATION

Proposition 2. TGEoF for Gaussian states $\boldsymbol{\sigma} = \boldsymbol{\sigma}_A \oplus \boldsymbol{\sigma}_B$ is fully additive, i.e.,

$$\tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\sigma}_A \oplus \boldsymbol{\sigma}_B) = \tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\sigma}_A) + \tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\sigma}_B)\,. \tag{A1}$$

where $\boldsymbol{\sigma}_A$ and $\boldsymbol{\sigma}_B$ is an N-mode and N'-mode Gaussian state, respectively.

Proof. TGEoF is by construction subadditive, i.e.,

$$\tilde{\mathcal{E}}_{G}(\boldsymbol{\sigma}_A \oplus \boldsymbol{\sigma}_B) \leqslant \tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\sigma}_A) + \tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\sigma}_B)\,, \tag{A2}$$

and thus its additivity can be shown by proving that TGEoF is superadditive too, i.e.,

$$\tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\sigma}_A \oplus \boldsymbol{\sigma}_B) \geqslant \tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\sigma}_A) + \tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\sigma}_B)\,. \tag{A3}$$

The Gaussian state $\boldsymbol{\sigma} = \boldsymbol{\sigma}_A \oplus \boldsymbol{\sigma}_B$ can decomposed as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_A \oplus \boldsymbol{\sigma}_B = \boldsymbol{\pi} + \boldsymbol{\varphi}\,, \tag{A4}$$

$$\boldsymbol{\sigma}_A = \boldsymbol{\pi}^{(A)} + \boldsymbol{\varphi}^{(A)},\;\; \boldsymbol{\sigma}_B = \boldsymbol{\pi}^{(B)} + \boldsymbol{\varphi}^{(B)}\,, \tag{A5}$$

where $\boldsymbol{\pi}$ is a pure Gaussian state and $\boldsymbol{\varphi}$ is a positive semidefinite matrix. For any $\boldsymbol{\varphi} \geqslant 0$, the TGEoF for the states $\boldsymbol{\sigma}_A$ and $\boldsymbol{\sigma}_B$ satisfies

$$\tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\pi}^{(A)}) \geqslant \tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\pi}^{(A)} + \boldsymbol{\varphi}^{(A)}) = \tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\sigma}_A)\,, \tag{A6a}$$

$$\tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\pi}^{(B)}) \geqslant \tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\pi}^{(B)} + \boldsymbol{\varphi}^{(B)}) = \tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\sigma}_B)\,, \tag{A6b}$$

so we have

$$\tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\pi}^{(A)}) + \tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\pi}^{(B)}) \geqslant \tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\sigma}_A) + \tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\sigma}_B)\,. \tag{A7}$$

The N-mode state $\boldsymbol{\pi}^{(A)}$ and N'-mode state $\boldsymbol{\pi}^{(B)}$ in the above inequality can also be decomposed as follows:

$$\boldsymbol{\pi}^{(A)} = \boldsymbol{\pi}_A + \boldsymbol{\varphi}_A\,, \tag{A8a}$$

$$\boldsymbol{\pi}^{(B)} = \boldsymbol{\pi}_B + \boldsymbol{\varphi}_B\,, \tag{A8b}$$

and again for arbitrary $\boldsymbol{\varphi}_A \geqslant 0$ and $\boldsymbol{\varphi}_B \geqslant 0$ we have

$$\tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\pi}_A) \geqslant \tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\pi}_A + \boldsymbol{\varphi}_A) = \tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\pi}^{(A)})\,, \tag{A9a}$$

$$\tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\pi}_B) \geqslant \tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\pi}_B + \boldsymbol{\varphi}_B) = \tilde{\mathcal{E}}_{\rm G}(\boldsymbol{\pi}^{(B)})\,, \tag{A9b}$$

which implies

$$\tilde{\mathcal{E}}_{\mathrm{G}}(\boldsymbol{\pi}_A)+\tilde{\mathcal{E}}_{\mathrm{G}}(\boldsymbol{\pi}_B)\geqslant\tilde{\mathcal{E}}_{\mathrm{G}}(\boldsymbol{\pi}^{(A)})+\tilde{\mathcal{E}}_{\mathrm{G}}(\boldsymbol{\pi}^{(B)})\,. \tag{A10}$$

Since $\boldsymbol{\pi}_A$ and $\boldsymbol{\pi}_B$ are pure states, their TGEoF is equivalent to their entropy of entanglement, i.e.,

$$\tilde{\mathcal{E}}_{\mathrm{G}}(\boldsymbol{\pi}_A)=\tilde{E}(\boldsymbol{\pi}_A)=\tilde{S}(\boldsymbol{\pi}_A)\,, \tag{A11a}$$

$$\tilde{\mathcal{E}}_{\mathrm{G}}(\boldsymbol{\pi}_B)=\tilde{E}(\boldsymbol{\pi}_B)=\tilde{S}(\boldsymbol{\pi}_B)\,, \tag{A11b}$$

and for arbitrary $\boldsymbol{\varphi}_A\geqslant 0$ and $\boldsymbol{\varphi}_B\geqslant 0$ we get

$$\tilde{S}(\boldsymbol{\pi}^{(A)})=\tilde{S}(\boldsymbol{\pi}_A+\boldsymbol{\varphi}_A)\geqslant\tilde{S}(\boldsymbol{\pi}_A)\,, \tag{A12a}$$

$$\tilde{S}(\boldsymbol{\pi}^{(B)})=\tilde{S}(\boldsymbol{\pi}_B+\boldsymbol{\varphi}_B)\geqslant\tilde{S}(\boldsymbol{\pi}_B), \tag{A12b}$$

which combined with the inequality (A7) and (A10) turns into

$$\tilde{S}(\boldsymbol{\pi}^{(A)})+\tilde{S}(\boldsymbol{\pi}^{(B)})\geqslant\tilde{\mathcal{E}}_{\mathrm{G}}(\boldsymbol{\sigma}_A)+\tilde{\mathcal{E}}_{\mathrm{G}}(\boldsymbol{\sigma}_B)\,. \tag{A13}$$

We now notice that for any $(N+N')$-mode state $\boldsymbol{\sigma}$ we have

$$\tilde{S}(\boldsymbol{\sigma}_{s_1s_2})=\tilde{S}\big(\boldsymbol{\sigma}^{(s_1)}_{s_1s_2}\big)+\tilde{S}\big(\boldsymbol{\sigma}^{(s_2)}_{s_1s_2}\big), \tag{A14}$$

and thus the left-hand side of the inequality (A13) becomes

$$\tilde{S}(\boldsymbol{\pi}^{(A)})+\tilde{S}(\boldsymbol{\pi}^{(B)})=\tilde{S}(\boldsymbol{\pi})=\tilde{E}(\boldsymbol{\pi})\,. \tag{A15}$$

Given that the above equality is true for every $\boldsymbol{\pi}$ satisfying Eq. (A4), it should be also true for the "optimal" $\boldsymbol{\pi}_o$ that gives the TGEoF of the global state $\boldsymbol{\sigma}=\boldsymbol{\sigma}_A\oplus\boldsymbol{\sigma}_B$ in Eq. (A4), i.e.,

$$\tilde{\mathcal{E}}_{\mathrm{G}}(\boldsymbol{\sigma})=\tilde{\mathcal{E}}_{\mathrm{G}}(\boldsymbol{\sigma}_A\oplus\boldsymbol{\sigma}_B)=\tilde{E}(\boldsymbol{\pi}_o)\,. \tag{A16}$$

Combining the above equations (A15) and (A16) with the inequality (A13), we get

$$\tilde{\mathcal{E}}_{\mathrm{G}}(\boldsymbol{\sigma}_A\oplus\boldsymbol{\sigma}_B)\geqslant\tilde{\mathcal{E}}_{\mathrm{G}}(\boldsymbol{\sigma}_A)+\tilde{\mathcal{E}}_{\mathrm{G}}(\boldsymbol{\sigma}_B)\,, \tag{A17}$$

which completes the proof. ■

APPENDIX B: GAUSSIAN LOCAL UNITARY OPERATIONS

In this section we introduce a useful class of operations, Gaussian local unitary operations. GLUO are operations which do not increase or decrease the amount of entanglement. By definition, these operations are a subset of LOCC (here, locality is defined with respect to each mode), which means that they cannot increase the entanglement. As these operations are locally reversible (i.e., unitary in terms of the Heisenberg picture), they cannot decrease the entanglement.

We introduce the GLUO of an N-mode state as follows:

$$\boldsymbol{L}\equiv\bigoplus_{n=1}^{N}\boldsymbol{L}_n\,, \tag{B1}$$

where $\boldsymbol{L}_n$ is the GLUO in each mode. Each GLUO can be decomposed through the Bloch Messiah decomposition [42,43] as

$$\boldsymbol{L}_n=\boldsymbol{L}(\phi_n')\boldsymbol{L}(r_n)\boldsymbol{L}(\phi_n)\,, \tag{B2}$$

where

$$\boldsymbol{L}(\phi)\equiv\begin{bmatrix}\cos(\phi) & \sin(\phi)\\ -\sin(\phi) & \cos(\phi)\end{bmatrix} \tag{B3}$$

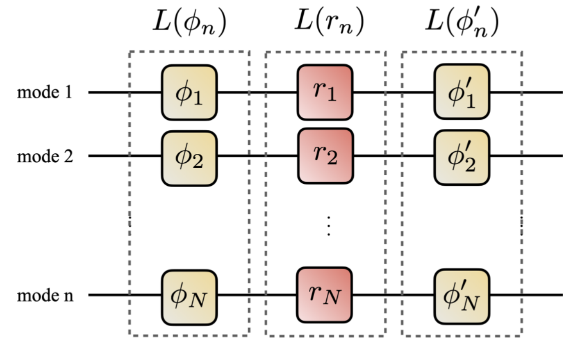

FIG. 2. A schematic decomposition of all GLUO operations.

corresponds to phase rotations, and

$$\boldsymbol{L}(r)\equiv\begin{bmatrix}e^r & 0\\ 0 & e^{-r}\end{bmatrix} \tag{B4}$$

corresponds squeezing operations. A schematic diagram of this decomposition for GLUO is shown in Fig. 2.

APPENDIX C: SYMMETRIC THREE-MODE SQUEEZING OPERATION

The Heisenberg evolution of a three-mode squeezing operation is as follows [44]:

$$\hat{S}_3^\dagger\hat{a}_i\hat{S}_3=\cosh(r)\hat{a}_i+\sinh(r)\Big(-\frac{1}{3}\hat{a}_i^\dagger+\frac{2}{3}(\hat{a}_j^\dagger+\hat{a}_k^\dagger)\Big)\,. \tag{C1}$$

The symplectic matrix representation of the three-mode operation is given by

$$\boldsymbol{S}_3(r_3)=\begin{bmatrix}\alpha_+ & \beta_+ & \beta_+ & 0 & 0 & 0\\ \beta_+ & \alpha_+ & \beta_+ & 0 & 0 & 0\\ \beta_+ & \beta_+ & \alpha_+ & 0 & 0 & 0\\ 0 & 0 & 0 & \alpha_- & \beta_- & \beta_-\\ 0 & 0 & 0 & \beta_- & \alpha_- & \beta_-\\ 0 & 0 & 0 & \beta_- & \beta_- & \alpha_-\end{bmatrix}, \tag{C2}$$

where we have defined the following:

$$\alpha_\pm\equiv\cosh(r_3)\mp\frac{\sinh(r_3)}{3}\,,\qquad \beta_\pm\equiv\pm\frac{2\sinh(r_3)}{3}\,. \tag{C3}$$

We obtain the GhZ/W state [32] when we apply this operator onto the vacuum state. In the standard form [38], this state can be written in the following way:

$$\begin{aligned}\boldsymbol{\pi}_{GhZ/W,\mathrm{sf}}(r_3)&\equiv\big(\boldsymbol{S}_3\boldsymbol{S}_3^T\big)_{\mathrm{sf}}\\ &=\begin{bmatrix}\alpha' & \beta'_+ & \beta'_+ & 0 & 0 & 0\\ \beta'_+ & \alpha' & \beta'_+ & 0 & 0 & 0\\ \beta'_+ & \beta'_+ & \alpha' & 0 & 0 & 0\\ 0 & 0 & 0 & \alpha' & \beta'_- & \beta'_-\\ 0 & 0 & 0 & \beta'_- & \alpha' & \beta'_-\\ 0 & 0 & 0 & \beta'_- & \beta'_- & \alpha'\end{bmatrix},\end{aligned} \tag{C4}$$

where

$$\begin{aligned}\alpha'&\equiv\frac{1}{3}\sqrt{9\cosh(2r_3)^2-\sinh(2r_3)^2}\,,\\ \beta_\pm&\equiv\pm\frac{|2\sinh(2r_3)|}{3}\sqrt{\frac{3\cosh(2r_3)\pm|\sinh(2r_3)|}{3\cosh(2r_3)\mp|\sinh(2r_3)|}}\,.\end{aligned} \tag{C5}$$

The Bloch-Messiah decomposition [42,43] of this operator can be found in a straightforward fashion by setting the local squeezers to be equal with $2\pi/3$ phase differences.

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