Decoupling cross-quadrature correlations using passive operations

Syed M. Assad [1,2,*] Mile Gu,[1,3,5,†] Xiaoying Li,[4] and Ping Koy Lam [2,1]
[1] *Nanyang Quantum Hub, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore 639673, Republic of Singapore*
[2] *Centre for Quantum Computation and Communication Technology, Department of Quantum Science, Research School of Physics and Engineering, Australian National University, Canberra ACT 2601, Australia*
[3] *Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore, Republic of Singapore*
[4] *College of Precision Instrument and Opto-Electronics Engineering, Key Laboratory of Opto-Electronics Information Technology, Ministry of Education, Tianjin University, Tianjin 300072, China*
[5] *Complexity Institute, Nanyang Technological University, Singapore 639673, Republic of Singapore*

(Received 9 April 2020; accepted 23 July 2020; published 19 August 2020)

Quadrature correlations between subsystems of a Gaussian quantum state are fully characterized by its covariance matrix. For example, the covariance matrix determines the amount of entanglement or decoherence of the state. Here, we establish when it is possible to remove correlations between conjugate quadratures using only passive operations. Such correlations are usually undesired and arise due to experimental cross-quadrature contamination. Using the Autonne-Takagi factorization, we present necessary and sufficient conditions to determine when such removal is possible. Our proof is constructive, and whenever it is possible we obtain an explicit expression for the required passive operation.

DOI: 10.1103/PhysRevA.102.022615

I. INTRODUCTION

The decomposition of Gaussian quantum systems has proven to be a fruitful subject of research. For instance, the textbook examples of Williamson (see [1,2]) and Braunstein [3] tell us that any Gaussian state can be decomposed through beamsplitters, phase shifters, and single-mode squeezers into uncorrelated thermal states. This is useful for designing quantum gates [4]. More generally, instead of demanding the complete diagonalization of the state, it can also be transformed into another that has specific kinds of correlations. Early examples of this are the Simon and Duan *et al.* standard forms (see [5,6], respectively): using local squeezing and phase shifts to bring an entangled state into some standard form of correlations. This turned out to be important in advancing our understanding of Gaussian entanglement.

All the transformations above require the use of active operations and bring the state to a form that does not have any cross-quadrature correlations. Active operations are those that require an external source of energy, for example, squeezing, while passive operations are those that do not [7]. Active operations are usually more difficult to implement in a real device compared to passive operations which can be implemented almost free of errors using beamsplitters and phase shifts [8]. When restricted to only passive operations, a generic Gaussian state cannot be diagonalized; it can only be brought to standard forms that remain correlated. There exist conditions with which one can check whether a Gaussian state can be diagonalized by a passive operation [2,9]. These conditions are always satisfied when the Gaussian states are pure [3,9].

Here, instead of requiring the state to be fully diagonalized, we report a necessary and sufficient condition under which the correlations between conjugate quadrature variables can be entirely removed using passive operations only. This is stated in the following theorem.

Theorem 1. Let $\mathbf{a} = [a_1, \ldots, a_n, a_1^\dagger, \ldots, a_n^\dagger]$ be a vector collecting the annihilation and creation operators of n modes. Let

$$S_{jk} = \frac{1}{2}\mathrm{Tr}[\rho(\mathbf{a}_j\mathbf{a}_k^\dagger + \mathbf{a}_k^\dagger\mathbf{a}_j)] = \begin{bmatrix} \mathsf{X} & \mathsf{Y} \\ \mathsf{Y}^* & \mathsf{X}^* \end{bmatrix}_{jk}$$

be the complex covariances of an n-mode Gaussian state ρ having zero mean $\mathrm{Tr}[\rho\,\mathbf{a}] = 0$. Then S can be brought into a cross-quadrature decorrelated form using passive operations if and only if there exist an Autonne-Takagi factorization of Y: $\mathsf{Y} = \mathsf{Z}^\dagger\mathsf{Y}_0\mathsf{Z}^*$ and a diagonal matrix R with entries in $\{1, i\}$ such that $\mathsf{R}^\dagger\mathsf{Z}\mathsf{X}\mathsf{Z}^\dagger\mathsf{R}$ is real. Furthermore, the required passive operation is given by Z up to swapping of quadratures determined by R.

The crux of the theorem is the diagonalization of Y, which is given to us by the Autonne-Takagi factorization [10,11].

Theorem 2: Autonne-Takagi factorization. Let Y be a complex symmetric matrix. Then there exists a unitary matrix Z such that $\mathsf{Y} = \mathsf{Z}^\dagger\mathsf{Y}_0\mathsf{Z}^*$, with Y_0 real, non-negative, and diagonal.

The diagonal entries of Y_0 are the singular values of Y in any desired order. The uniqueness property of Z is stated in the Appendix. Essentially, the physical situation of interest is a correlated state with unwanted correlations between some of the conjugate quadratures and we are concerned with the conditions under which these unwanted correlations can be removed using only passive operations. We mean "conjugate quadratures" in a more general sense—*any* quadrature pairs, q_j and p_k with j not necessarily equal to k and where

*cqtsma@gmail.com
†gumile@ntu.edu.sg

$[q_j, p_k] = i\delta_{jk}$. In other words, theorem 1 identifies those states that are composed of q correlations and p correlations plus passive operations. As a corollary, it also identifies states which cannot be constructed by passive operations on initially uncorrelated, squeezed or otherwise, single modes. The proof of the theorem is constructive in that the required passive operation is obtained whenever it exists. It turns out to be, up to local rotations, just Z given by the Autonne-Takagi's factorization, which is very convenient.

We note that Autonne-Takagi's factorization makes its appearance in multimode quantum optics [12,13] that resembles the approach we have taken here, but there is one important difference—we consider the factorization of quantum states rather than the decomposition of unitaries for determining supermodes as is the case in multimodal theories.

II. PROOF OF THEOREM 1

In what follows, we prove Theorem 1. We work with the complex covariance matrix which can be obtained from the quadrature covariance matrix by the change of variables [14]

$$a_j = \frac{q_j + ip_j}{\sqrt{2}} \quad \text{and} \quad a_j^\dagger = \frac{q_j - ip_j}{\sqrt{2}}. \tag{1}$$

The reason for working in such a basis is twofold. First, the conjugate quadratures have vanishing correlations if and only if both matrices X and Y are real. Second, passive operations take the simple form

$$\begin{bmatrix} \mathsf{E} & 0 \\ 0 & \mathsf{E}^* \end{bmatrix}$$

with E unitary due to the symplectic conditions. A direct calculation shows that the covariance matrix transforms as $\mathsf{E} : (\mathsf{X}, \mathsf{Y}) \mapsto (\mathsf{E}\mathsf{X}\mathsf{E}^\dagger, \mathsf{E}\mathsf{Y}\mathsf{E}^\intercal)$ under passive operations, whence it follows that the problem of decoupling conjugate variables is reduced to finding a unitary matrix E such that $\mathsf{E}\mathsf{X}\mathsf{E}^\dagger$ and $\mathsf{E}\mathsf{Y}\mathsf{E}^\intercal$ are simultaneously real. We can now proceed to prove the main result.

Proof: Forward direction. Suppose S is the covariance matrix of a state ρ the cross-quadrature correlations of which can be removed by a passive operation Q. In other words, after applying Q, the cross-quadrature correlations $\{q_j, p_k\} = 0$, where to simplify notations we use $\{q_j, p_k\}$ to mean $\frac{1}{2}\mathrm{Tr}[\rho(q_j p_k + p_k q_j)]$. In the complex representation, denoting the transformed matrix as $\mathsf{X}_1 = \mathsf{Q}\mathsf{X}\mathsf{Q}^\dagger$ and $\mathsf{Y}_2 = \mathsf{Q}\mathsf{Y}\mathsf{Q}^\intercal$, the transformed covariance matrix has entries

$$[\mathsf{X}_1]_{jk} = \{a_j, a_k^\dagger\} = \frac{\{q_j, q_k\}}{2} + \frac{\{p_j, p_k\}}{2},$$
$$[\mathsf{Y}_2]_{jk} = \{a_j, a_k\} = \frac{\{q_j, q_k\}}{2} - \frac{\{p_j, p_k\}}{2},$$

which are real. Since Y_2 is a real symmetric matrix, it has a spectral decomposition $\mathsf{Y}_2 = \mathsf{R}_1^\intercal \mathsf{Y}_1 \mathsf{R}_1$ [15], where R_1 is a real orthogonal matrix and Y_1 is a real (but not necessarily positive) diagonal matrix the entries of which are the eigenvalues of Y_2. To obtain the Autonne-Takagi decomposition, consider a passive unitary (but not necessarily real) transformation $\mathsf{R} : (a_j, a_j^\dagger) \mapsto (ia_j, -ia_j^\dagger)$ on Y_1 for every $j \in J$ where J is the set containing all indices j for which $[\mathsf{Y}_1]_{jj}$ is negative. This corresponds to a rotation of the quadratures $\mathsf{R} : (q_j, p_j) \mapsto (p_j, -q_j)$ for $j \in J$. In matrix form, R is diagonal with entries

$$[\mathsf{R}]_{jk} = \begin{cases} 1 & \text{for } j = k \notin J \\ i & \text{for } j = k \in J \\ 0 & \text{for } j \neq k \end{cases}.$$

Applying this to Y_1 brings it to a non-negative diagonal matrix $\mathsf{Y}_0 = \mathsf{R}\mathsf{Y}_1\mathsf{R}^\intercal$ since

$$\mathsf{R} : \{a_j, a_j\} \mapsto \begin{cases} -\{a_j, a_j\} & \text{for } \{a_j, a_j\} < 0 \\ \{a_j, a_j\} & \text{for } \{a_j, a_j\} \geqslant 0 \end{cases}.$$

Putting everything together, we arrive at the Autonne-Takagi decomposition of Y as

$$\mathsf{Y} = \underbrace{\mathsf{Q}^\dagger \mathsf{R}_1^\intercal \mathsf{R}^\dagger}_{\mathsf{Z}^\dagger} \mathsf{Y}_0 \underbrace{\mathsf{R}^* \mathsf{R}_1 \mathsf{Q}^*}_{\mathsf{Z}^*}.$$

Then X transforms as

$$\mathsf{Z}\mathsf{X}\mathsf{Z}^\dagger = \mathsf{R}\mathsf{R}_1^*\mathsf{Q}\mathsf{X}\mathsf{Q}^\dagger\mathsf{R}_1^\intercal\mathsf{R}^\dagger = \mathsf{R}\underbrace{\mathsf{R}_1^*\mathsf{X}_1\mathsf{R}_1^\intercal}_{\mathsf{X}_0}\mathsf{R}^\dagger,$$

where X_0 is a real (symmetric) matrix since both X_1 and R_1 are real. This implies $\mathsf{R}^\dagger\mathsf{Z}\mathsf{X}\mathsf{Z}^\dagger\mathsf{R}$ is real, which completes the proof. ■

Proof: Reverse direction. Let Z be the unitary matrix in the Autonne-Takagi factorization of Y: $\mathsf{Y} = \mathsf{Z}^\dagger\mathsf{Y}_0\mathsf{Z}^*$ and R be a diagonal matrix with entries in $\{1, i\}$ such that $\mathsf{R}^\dagger\mathsf{Z}\mathsf{X}\mathsf{Z}^\dagger\mathsf{R}$ is real. The passive transformation $\mathsf{R}^\dagger\mathsf{Z}$ results in $\mathsf{R}^\dagger\mathsf{Z} : (\mathsf{X}, \mathsf{Y}) \mapsto (\mathsf{R}^\dagger\mathsf{Z}\mathsf{X}\mathsf{Z}^\dagger\mathsf{R}, \mathsf{R}^\dagger\mathsf{Z}\mathsf{Y}\mathsf{Z}^\intercal\mathsf{R}^*)$. The first term is real by assumption. The second term

$$\mathsf{R}^\dagger\mathsf{Z}\mathsf{Y}\mathsf{Z}^\intercal\mathsf{R}^* = \mathsf{R}^\dagger\mathsf{Z}\mathsf{Z}^\dagger\mathsf{Y}_0\mathsf{Z}^*\mathsf{Z}^\intercal\mathsf{R}^* = \mathsf{R}^\dagger\mathsf{Y}_0\mathsf{R}^*$$

is also real since Y_0 is a real diagonal matrix. When X and Y are simultaneously real, it follows from direct substitution that the quadrature covariance matrix has no cross-quadrature correlations. ■

What does this mean? It means that we have a way of testing if the correlations between conjugate variables can be removed—diagonalize Y to obtain the matrix Z using the Autonne-Takagi factorization and subsequently compute $\mathsf{Z}\mathsf{X}\mathsf{Z}^\dagger$. If Y is a full-rank matrix with nondegenerate eigenvalues and $\mathsf{Z}\mathsf{X}\mathsf{Z}^\dagger$ cannot be transformed to a real matrix by a diagonal matrix R, then the correlations cannot be decoupled. This is certainly the case if $\mathsf{Z}\mathsf{X}\mathsf{Z}^\dagger$ has any entries that are neither real nor purely imaginary. On the other hand, if all the entries of $\mathsf{Z}\mathsf{X}\mathsf{Z}^\dagger$ are real, then Z is the passive operation that we are after. If some entries are purely imaginary then in addition to Z additional local rotations R are required. If Y is singular or has degenerate eigenvalues, then we have some freedom in choosing Z to make the entries of $\mathsf{R}^\dagger\mathsf{Z}\mathsf{X}\mathsf{Z}^\dagger\mathsf{R}$ real.

When S corresponds to a *pure state*, the matrix Z gives the passive operation required to create it from a product of independent squeezed states. However, if S is mixed, our result implies that it is sometimes impossible to create by passive operations on *any* independent states, or even on states possessing only q correlations and p correlations. One

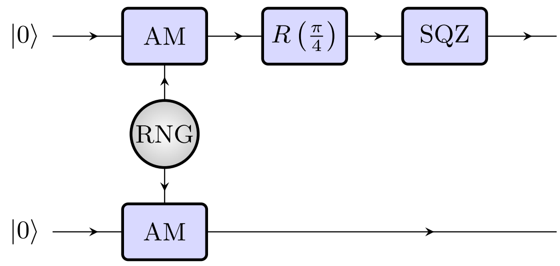

FIG. 1. The output state with quadrature covariance matrix given by (2) has cross-quadrature correlations that cannot be removed by passive operations. AM: Amplitude modulator. RNG: Gaussian random number generator with variance 1/2. $R(\frac{\pi}{4})$: $\pi/4$ phase shifter. SQZ: 3-dB squeezer.

example is the state with quadrature covariance matrix

$$\mathcal{S} = \frac{1}{2}\begin{bmatrix} 3 & 0.5 & 1 & 0 \\ 0.5 & 0.75 & 0.5 & 0 \\ 1 & 0.5 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

which can be created by the scheme in Fig. 1. The squeezing operation "locks in" the cross-quadrature correlations and makes it impossible to be removed using passive operations only.

III. TWO-MODE EXAMPLE

We illustrate our result by working through an example. Consider a two-mode Gaussian state having the following quadrature covariance matrix:

$$\mathcal{S} = \begin{bmatrix} m & 0 & c & 0 \\ 0 & m & 0 & -c \\ c & 0 & n & s \\ 0 & -c & s & n \end{bmatrix}$$

with all m, n, c, and s positive. We want to determine if this state can be brought into a cross-quadrature decorrelated form. The basis transformation (1) represented by the unitary matrix

$$\mathcal{L} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & i \\ 1 & -i & 0 & 0 \\ 0 & 0 & 1 & -i \end{bmatrix}$$

transforms the quadrature covariance matrix into the complex covariance matrix

$$\mathsf{S} = \mathcal{L}\mathcal{S}\mathcal{L}^\dagger = \begin{bmatrix} m & 0 & 0 & c \\ 0 & n & c & is \\ 0 & c & m & 0 \\ c & -is & 0 & n \end{bmatrix},$$

which identifies X and Y as

$$\mathsf{X} = \begin{bmatrix} m & 0 \\ 0 & n \end{bmatrix} \text{ and } \mathsf{Y} = \begin{bmatrix} 0 & c \\ c & is \end{bmatrix}.$$

The Autonne-Takagi factorization of $\mathsf{Y} = \mathsf{Z}^\dagger \mathsf{Y}_0 \mathsf{Z}^*$ is given by

$$\mathsf{Z} = e^{i\pi/4}\begin{bmatrix} -i\sqrt{t} & \sqrt{1-t} \\ \sqrt{1-t} & -i\sqrt{t} \end{bmatrix}$$

and

$$\mathsf{Y}_0 = \frac{1}{2}\begin{bmatrix} \sqrt{4c^2+s^2} - s & 0 \\ 0 & \sqrt{4c^2+s^2} + s \end{bmatrix}$$

with $t = (1 + s/\sqrt{4c^2+s^2})/2$. This results in

$$\mathsf{Z}\mathsf{X}\mathsf{Z}^\dagger = \begin{bmatrix} n(1-t) + mt & -i\sqrt{t(1-t)}(m-n) \\ i\sqrt{t(1-t)}(m-n) & nt + m(1-t) \end{bmatrix}$$

which has entries that are all real or purely imaginary, and is transformed to a real matrix by

$$\mathsf{R} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

so that finally we have

$$\mathsf{R}^\dagger\mathsf{Z}\mathsf{X}\mathsf{Z}^\dagger\mathsf{R} = \begin{bmatrix} n(1-t) + mt & \sqrt{t(1-t)}(m-n) \\ \sqrt{t(1-t)}(m-n) & nt + m(1-t) \end{bmatrix}.$$

This means that the state S can be brought to a cross-quadrature decorrelated form and the passive operation that does this is $\mathsf{R}^\dagger\mathsf{Z}$. This can be factorized as

$$\mathsf{R}^\dagger\mathsf{Z} = \begin{bmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i3\pi/4} \end{bmatrix}\begin{bmatrix} \sqrt{t} & \sqrt{1-t} \\ -\sqrt{1-t} & \sqrt{t} \end{bmatrix}\begin{bmatrix} e^{-i\pi/2} & 0 \\ 0 & 1 \end{bmatrix}$$

which is realized by a beamsplitter of transmissivity t and three phase shifts: $\pi/4$ and $-3\pi/4$ at the outputs and $-\pi/2$ at the input port.

The expert reader might have recognized that the state S can in fact be cross-quadrature decorrelated through the simpler transformation

$$\mathsf{R}^\dagger\mathsf{Z} = \begin{bmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{bmatrix}$$

requiring just two phase shifts. This shows that when it is possible to decorrelate the conjugate quadratures the procedure we presented is not the only way to do so. The condition that Y be diagonalized can be relaxed—all we need to decouple q and p is for Y to be transformed into a real matrix after applications of the passive operation—this real matrix need not be diagonal or non-negative. In terms of implementations, this would mean that the required operation might be simpler; for instance, we can do away with the beamsplitter in the example considered.

IV. DISCUSSIONS

An immediate application of theorem 1 is to the calculation of the "squeezing of formation" [16]. This quantity measures how much squeezing is required to create a given state and indicates the degree of nonclassicality of the state. Squeezing of formation is invariant under passive operations because these transformations do not require any squeezing. This means that the result of this paper can be used to simplify complicated states to a form in which the squeezing of formation can be directly calculated. For example, a brute force computation of the squeezing of formation for a two-mode Gaussian state involves an optimization over six free parameters. However, by first transforming the state to a quadrature-decorrelated form, if it is possible, this computation reduces to a simple one parameter optimization problem [17].

There is also an interesting connection with the generation of cluster states. A cluster state has multiple quantum modes with correlations between each mode [18–20]. Many of these can be shown to possess correlations only between the q's and between the p's, such as the two-dimensional square cluster. However, in real devices for generating cluster states there are imperfections which give rise to correlations between q and p. This implies that our result might be useful for identifying if an ideal cluster state can be recovered using only passive operations.

What can be said about a state with cross-quadrature correlations which cannot be removed by passive operations? While most theoretical work on Gaussian quantum information considers cross-quadrature decorrelated states, almost every state realized experimentally would have some cross-quadrature correlations that cannot be decoupled using only passive operations. However, if we are also allowed to add correlated noise in the form of random Gaussian quadrature displacements, then any state can be cross-quadrature decorrelated. One obvious question is then the following: what is the least amount of noise required to achieve such decorrelation?

ACKNOWLEDGMENTS

We acknowledge H. Jeng for preparing an earlier version of the paper. We thank B. Shajilal, T. Michel, and S. Tserkis for useful discussions. This work is supported by the Australian Research Council under the Centre of Excellence for Quantum Computation and Communication Technology (Grants No. CE110001027, No. CE170100012, and No. FL150100019), the National Research Foundation (NRF). Singapore, under its NRFF Fellow programme (Award No. NRF-NRFF2016-02), the Singapore Ministry of Education Tier 1 Grant No. MOE2017-T1-002-043, Grant No. FQXi-RFP-1809 from the Foundational Questions Institute and Fetzer Franklin Fund (a donor-advised fund of Silicon Valley Community Foundation). Any opinions, findings and conclusions or recommendations expressed in this material are those of the author(s) and do not reflect the views of National Research Foundation, Singapore.

APPENDIX: UNIQUENESS OF AUTONNE-TAKAGI DECOMPOSITION

For completeness, this Appendix recalls the uniqueness properties of the Autonne-Takagi decomposition. See, for example, the textbook by Horn and Johnson [15] for proofs.

Let Y be an $n \times n$ complex symmetric matrix of rank r. Let $\lambda_1, \ldots, \lambda_d$ be the distinct positive singular values of Y, in any given order with respective multiplicities $n_1, \ldots, n_d$. Let $\mathsf{Y}_0 = \lambda_1 \mathbb{1}_{n_1} \oplus \ldots \oplus \lambda_d \mathbb{1}_{n_d} \oplus \mathbb{0}_{n-r}$; the zero block is missing if Y is nonsingular. Let U and V be unitary. Then the Autonne-Takagi decomposition of Y: $\mathsf{Y} = \mathsf{U}\mathsf{Y}_0\mathsf{U}^\mathsf{T} = \mathsf{V}\mathsf{Y}_0\mathsf{V}^\mathsf{T}$ if and only if $\mathsf{V} = \mathsf{U}\mathsf{Q}$, with $\mathsf{Q} = \mathsf{Q}_1 \oplus \ldots \oplus \mathsf{Q}_d \oplus \mathsf{W}$ where each Q_j is an $n_j \times n_j$ real orthogonal matrix and W is an $(n-r) \times (n-r)$ unitary matrix. If the singular values of Y are distinct (that is, if $d \geqslant n-1$), then $\mathsf{V} = \mathsf{U}\mathsf{D}$, in which $\mathsf{D} = \mathrm{diag}(d_1, \ldots, d_n)$ with $d_j = \pm 1$ for each $j = 1, \ldots, n-1$. The last entry $d_n = e^{i\theta}$ if Y is singular ($d = n-1$), otherwise $d_n = \pm 1$ if Y is nonsingular ($d = n$).

[1] J. Williamson, Amer. J. Math. **58**, 141 (1936).

[2] R. Simon, N. Mukunda, and B. Dutta, Phys. Rev. A **49**, 1567 (1994).

[3] S. L. Braunstein, Phys. Rev. A **71**, 055801 (2005).

[4] Y. Shiozawa, J.-i. Yoshikawa, S. Yokoyama, T. Kaji, K. Makino, T. Serikawa, R. Nakamura, S. Suzuki, S. Yamazaki, W. Asavanant, S. Takeda, P. van Loock, and A. Furusawa, Phys. Rev. A **98**, 052311 (2018).

[5] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).

[6] L. M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000).

[7] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Rev. Mod. Phys. **84**, 621 (2012).

[8] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Phys. Rev. Lett. **73**, 58 (1994).

[9] Arvind, B. Dutta, N. Mukunda, and R. Simon, Pramana **45**, 471 (1995).

[10] L. Autonne, *Sur les matrices hypohermitiennes et sur les matrices unitaires*, Ann. Univ. Lyon Nouvelle Srie I (Rey, 1915), Vol. 38, pp. 1–77.

[11] T. Takagi, Japan. J. Math. **1**, 83 (1925).

[12] G. Cariolaro and G. Pierobon, Phys. Rev. A **94**, 062109 (2016).

[13] F. Arzani, C. Fabre, and N. Treps, Phys. Rev. A **97**, 033808 (2018).

[14] B. L. Schumaker and C. M. Caves, Phys. Rev. A **31**, 3093 (1985).

[15] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University, Cambridge, England, 2012).

[16] M. Idel, D. Lercher, and M. M. Wolf, J. Phys. A **49**, 445304 (2016).

[17] S. Tserkis, J. Thompson, A. P. Lund, T. C. Ralph, P. K. Lam, M. Gu, and S. M. Assad, arXiv:2004.13948 (2020).

[18] N. C. Menicucci, P. van Loock, M. Gu, C. Weedbrook, T. C. Ralph, and M. A. Nielsen, Phys. Rev. Lett. **97**, 110501 (2006).

[19] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).

[20] R. Raussendorf, D. E. Browne, and H. J. Briegel, Phys. Rev. A **68**, 022312 (2003).