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Confused Entailment

Tore Fjetland Øgaard

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Abstract Priest argued in his paper *Fusion and Confusion* (Priest, 2015a) for a new concept of logical consequence over the relevant logic **B**, one where premises my be "confused" together. This paper develops Priest's idea. Whereas Priest uses a substructural proof calculus, this paper provides a Hilbert proof calculus for it. Using this it is shown that Priest's consequence relation is weaker than the standard Hilbert consequence relation for **B**, but strictly stronger than Anderson and Belnap's original relevant notion of consequence. Unlike the latter, however, Priest's consequence relation does not satisfy a variant of the variable sharing property. This paper shows that how it can be modified so as to do so. Priest's consequence relation turns out to be surprisingly weak in some respects. The prospects of strengthening it is raised and discussed in a broader philosophical context.

Keywords Consequence relations · Entailment · Relevant logic · Substructural logic · Validity

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1 Introduction

One of the key features of the standard account of logical consequence relation is that logical axioms, and logical truths more generally, follow from any set of assumptions. Anderson and Belnap initiated the relevant research program in the late fifties believing this to not be a true feature of logical consequence, or *entailment* as they preferred to call it: "the defect lies in the definition, which fails to take seriously the word 'from' in 'proof *from* hypothesis.' " (Anderson and Belnap, 1975, p. 18). As a replacement-concept, Anderson and Belnap put forth their own notion of entailment. Their suggestion for how to fix the "Official definition of a deduction" was to restrict the use of the rules in such a way as to make sure that at least some of the premises must be *used* in deriving the conclusion. They state in the first volume of *Entailment: The Logic of Relevance and Necessity* the so-called *Entailment theorem* as follows: "If there is a proof in **E** that $A_1, \ldots, A_n$ entail(s) B, then $(A_1 \& \ldots \& A_n) \rightarrow B$ is provable in **E**" (Anderson and Belnap, 1975, p. 278). The *entail(s)* on the left-hand side denote a use-restricted notion of Hilbert derivability, and so the theorem is a type of syntactic completeness theorem. The soundness-version of the theorem is quite easily shown to hold as well, and so their concept of entailment provides a notion of following from which can be expressed as certain logical truths. Since their notion of entailment interprets premises aggregation using extensional conjunction,[1] and so as to distinguish it from other notions of entailment, I'll refer Anderson and Belnap's notion as *C-entailment*, mnemonic for *conjunctive entailment*.

One of the key features of C-entailment is that it gives the so-called *variable sharing property* its true meaning: that property is simply that if $A \rightarrow B$ is a logical theorem in the Hilbertian sense of this, then A and B share a propositional variable. The property was set forth as a formal criteria to guarantee "the demand for relevance as between antecedent and consequent of an entailment" (Belnap, 1960, p. 144) and is as such a meaning-requirement upon entailment. Anderson and Belnap's preferred reading of the connective $\rightarrow$ of their favorite logic **E** was in fact as an *entailment conditional*: $A \rightarrow B$ is to be read *that A entails that B* (cf. Anderson and Belnap, 1975, A5). It is the belief that $\rightarrow$ is an object-language conditional which *expresses* entailment which mo-

Tore Fjetland Øgaard
Department of Philosophy, University of Bergen, PB 7805, 5020 Bergen, Norway E-mail: Tore.Ogaard@uib.no

[1] Anderson and Belnap used '&' for extensional conjunction; I'll use '$\wedge$'.

tivates the variable sharing property. The Entailment theorem, then, shows that one reasonable view of what $\rightarrow$ in fact expresses is that it expresses C-entailment.

Anderson and Belnap's specification of C-entailment only applies to logics with no more primitive rules than adjunction and modus ponens. The concept, along with the corresponding Entailment theorem, however, was generalized in Øgaard (2021a) so as to also apply to weak relevant logics such as **B**. **B** is a *contractionless* logic, and so $(A \land (A \rightarrow B)) \rightarrow B$ is not a logical theorem of it. Using C-entailment to explicate what logical consequence amounts to, then, has the definitive downside in the case of such logics since it fails to make *modus ponens* into a valid rule. For contractionless logics, then, it would seem that the choice is between giving up modus ponens and giving up relevance.

Priest has recently argued for a novel notion of logical consequence over the logic **B**. Priest's notion of consequence, however, is like the standard account in that logical theorems end up being consequences of any set of assumptions. Thus also Priest's consequence relation fails to be a properly relevant one. Despite this it is highly interesting—both in its own right, but also from a relevant perspective as it can easily be made into a relevant consequence relation. Priest allows premises not only to be conjunctively combined into antecedents for $\rightarrow$, but allows premises to be "confused" together. A *confusion* is intuitively any way of compiling premises using either extensional or intensional conjunction. The word *confusion* is a concatination of the three first letters of *conjunction* and the three first letters of *fusion*. *Fusion* is the name of the intensional conjunction connective $\circ$. However, confusions can also weaken in the Ackermann constant **t**. It, we shall see, is the source of Priest's notion of entailment—*P-entailment*—not standing up to Anderson and Belnap's requirement for relevance. It is shown, however, that if only confusions are purified so as not to allow **t** to be weakened in, then the resultant notion of P-entailment will satisfy Anderson and Belnap's meaning-requirement for relevance; both in the case of **B**, but also in general for any logic—with or without contraction—as long as it satisfies the standard variable sharing property. I argue that not only is the more stern notion of entailment motivated from relevance considerations, but also from Priest's other theoretical commitments. In particular, the most plausible account of how to formulate naïve set theory using Priest's proof system render the theory trivial also in the case of Priest's weak logic **B**. Doing away with **t**, it is noted, may be sufficient to thwart this.

Priest makes use of Restall (2000)'s substructural proof theory. To display its true colors, however, I will show that P-entailment is equivalent to *CF-entailment*, mnemonic for *confused entailment*, which is, like C-entailment, specified using a standard Hilbert calculus for **B**. Having the three consequences expressed as variants over the same calculus makes it easier to show that they are in fact different, but also to show how different in character they really are. It will emerge that in the case of **B**, CF-entailment is strictly stronger than C-entailment and strictly weaker than standard Hilbert derivability. One of the striking finds is that *Hypothetical Syllogism* fails to be a valid CF-entailment, and so this notion of logical consequence fails to make **B**'s $\rightarrow$ a transitive conditional. This, however, seems to be an unintended consequence as Priest nowhere else argues for the invalidity of this rule. Different ways of dealing with this will be discussed in the context of both Priest's philosophical interpretation of the Routley-Meyer semantics, but also his naïve theory of validity which is the true topic of Priest (2015a). Note, therefore, that Priest makes use of two different consequence relations: what I've called *P-entailment* is the one that pertains to "what it is to be provable from some information, such as that provided by the axioms of an axiom system" (Priest, 2015a, p. 59). This is not, however, the true relation of entailment or validity.[2] As we shall see, however, the two concepts of logical consequence are interestingly connected.

This paper focuses on developing Priest's idea. This is done by stating the theory in more familiar terms and by comparing it to other proposals, but also by way of pointing out some of its more surprising features. Priest, nor anyone else for that matter, has to my knowledge not discussed P-entailment in any great detail—neither technically nor philosophically. The current paper should not be read as arguing against Priest's proposal in any way, but rather as developing it and setting forth some challenges to be addressed in future work.

The plan for the paper is as follows: Section 2 defines the standard notion of Hilbert-derivability and Anderson and Belnap's notion of entailment. The axioms and rules of **B** are presented and the concept of variable sharing for consequence relations is defined. Section 3 explains enough of Restall's substructural proof theory without going into detail before section 4 digs into Priest's new notion of logical consequence, *P-entailment*. Section 5 compares Priest's notion of logical consequence to that of Anderson and Belnap's as well as to Hilbert-derivability and Section 6 discusses the fact that $\rightarrow$ fails to be a transitive conditional over P-entailment. Section 7 shows how to make P-entailment relevant before section 8 summarizes.

2 Hilbert proofs, logical theorems and C-entailments

In this section I will show how the standard Hilbert consequence relation for the relevant logic **B** is defined as well as how to define the concept of conjunctive entailment, *C-*

[2] These latter two terms are used interchangably in Priest (2015a).

entailment for short, for it. I will then state a version of Anderson and Belnap's "Entailment Theorem" for **B**.

Definition 1 (Parenthesis conventions) $\vee$, $\wedge$ and $\circ$ are to bind tighter than $\to$, and so I'll usually drop parenthesis enclosing conjunctions and disjunctions.

Definition 2 (H-entailment) A Hilbert proof of a formula A from a set of formulas Γ in the logic $\mathbf{L}$ is defined to be a finite list $A_1,\ldots,A_n$ such that $A_n = A$ and every $A_{i\leq n}$ is either a member of Γ, a logical axiom of $\mathbf{L}$, or there is a set $\Delta \subseteq \{A_j \mid j < i\}$ such that $\Delta \Vdash A_i$ is an instance of a rule of $\mathbf{L}$. The existential claim that there is such a proof is written $\Gamma \vdash^h_{\mathbf{L}} A$ and expressed as "Γ H-entails A in the logic $\mathbf{L}$" or "there exists a Hilbert-derivation of A from Γ in the logic $\mathbf{L}$". If $\varnothing \vdash^h_{\mathbf{L}} A$, then A is said to be a logical theorem of $\mathbf{L}$.

Definition 3 The following list of axioms and rules defines the logic **B**:

(A1)	$A \to A$
(A2)	$A \to A \vee B$ and $B \to A \vee B$
(A3)	$A \wedge B \to A$ and $A \wedge B \to B$
(A4)	$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
(A5)	$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
(A6)	$(A \to C) \wedge (B \to C) \to (A \vee B \to C)$
(A7)	$\sim\sim A \to A$
(A8)	$\bot \to A$
(A9)	$A \to (B \to A \circ B)$
(A10)	$\mathbf{t}$
(R1)	$\{A, A \to B\} \Vdash B$
(R2)	$\{A, B\} \Vdash A \wedge B$
(R3)	$\{A \to B\} \Vdash (C \to A) \to (C \to B)$
(R4)	$\{A \to B\} \Vdash (B \to C) \to (A \to C)$
(R5)	$\{A \to \sim B\} \Vdash B \to \sim A$
(R6)	$\{A \to (B \to C)\} \Vdash A \circ B \to C$
(R7)	$\{A\} \Vdash \mathbf{t} \to A$

Definition 4 $\mathbf{B}^\circ$ is identified as the $\mathbf{t}$-free version of **B**, that is, **B** without (A10) and (R7).

The following variant of Anderson and Belnap's definition of what *a proof that* $A_1,\ldots,A_n$ *entail(s)* B is to mean was first given in Øgaard (2021a):

Definition 5 (C-entailment) A proof that the set of formulas Γ C-entails A in the logic $\mathbf{L}$ is a Hilbert-proof $A_1,\ldots,A_n$ of A from Γ such that it is possible to mark the A_i's with # according to the following rules:

1. If $A_i \in \Gamma$, then A_i is marked.
2. R1 is not used on premises A_j and $A_j \to A_i$ where $A_j \to A_i$ is marked.
3. If A_i is obtained from A_j and $A_j \to A_i$ using R1, then A_i is marked if A_j is marked.
4. R2 is only used on premises which are either both marked or both unmarked.
5. If A_i is obtained from A_j and A_k using R2 and both of A_j and A_k are marked, then A_i is marked.
6. Any rule different from R1&R2 is only used on unmarked formulas.
7. No other formulas are marked.
8. As a consequence of (1–7), A_n is marked.

The existential claim that there is such a proof is written $\Gamma \vdash^c_{\mathbf{L}} A$ and expressed as "Γ C-entails A in the logic $\mathbf{L}$", or "there exists a proof that Γ C-entails A in the logic $\mathbf{L}$".

Theorem 1 *For* $\mathbf{L} \in \{\mathbf{B}, \mathbf{B}^\circ\}$, $\Gamma \vdash^c_{\mathbf{L}} B$ *if and only if for some conjunction* $A := A_1 \wedge \ldots \wedge A_n$, *where* $\{A_1,\ldots,A_n\} \subseteq \Gamma$, $\varnothing \vdash^h_{\mathbf{L}} A \to B$.

Proof *See Øgaard (2021a).*

In Øgaard (2019, sect. 5) it was shown that Anderson and Belnap's original C-entailment theorem extends to classical logic. This, then, is also the case for the concept of C-entailment as defined here. Even though the C-entailment theorem holds for a wide class of logics—relevant and irrelevant alike—one should not conclude that it is without interest from a relevant standpoint: The effect it has in combination with the variable sharing property is to transfer the latter property from being a property pertaining to *formulas*—a logic $\mathbf{L}$ has the variable sharing property just in case A and B share a propositional variable if $\varnothing \vdash^h_{\mathbf{L}} A \to B$—to one which pertains to the consequence relation itself. The following definition and corollary makes this precise:

Definition 6 A consequence relation $\vdash$ satisfies the consequential variable sharing property (CVSP) just in case for any antecedent aggregate of truth-constant-free formulas X and any truth-constant-free formula A, if $X \vdash A$, then A shares a propositional variable with a formula occurring in X.[3]

Corollary 1
- $\vdash^h_{\mathbf{L}}$ *does not satisfy (CVSP) for any logic* $\mathbf{L}$ *extending* $\mathbf{B}^\circ$.
- $\vdash^c_{\mathbf{B}}$ *and* $\vdash^c_{\mathbf{B}^\circ}$ *satisfy (CVSP).*

Proof *In the first case we have that* $\{A\} \vdash^h_{\mathbf{B}^\circ} B \to B$. *In the latter case, this follows from the fact that both* $\mathbf{B}^\circ$ *and* **B** *satisfy the variable sharing property (cf. Belnap, 1960) together with Thm. 1.*

One notable feature of C-entailment is that it yields no logical theorems: for any logic $\mathbf{L}$ we have that $\{A \mid \varnothing \vdash^c_{\mathbf{L}} A\} = \varnothing$. The simple reason for this is that the #-rules demands that the last formula in a $\vdash^c_{\mathbf{L}}$-proof must be #-marked. However, #-marks can only be introduced by formulas in the premise set, and so if there are no such formulas, there will simply be no #-marks to pass along to the conclusion.

[3] See Øgaard (2021c) for a discussion on how to extend the variable sharing property to formulas with truth-constants.

3 C-entailments in substructural proof theory

We have seen that Anderson and Belnap argued for a notion of logical consequence different from the standard Hilbertian one. Their notion of C-entailment is, however, not radically different from the Hilbertian notion. Relevant logics are sometimes claimed to be *substructural* in that the rules of premise combination lack some of the structural properties. Note, however, that C-entailment is a fully structural consequence relation. In more Tarski-familiar vocabulary we have that $\vdash^c_{\mathbf{B}}$ satisfies the following *closure conditions*:[4]

Theorem 2 *$\vdash^c_{\mathbf{B}}$ and $\vdash^c_{\mathbf{B}^\circ}$ are reflexive, transitive and monotone.*

Reflexivity: $\Gamma \vdash^c_{\mathbf{L}} A$ *for all* $A \in \Gamma$

Transitivity (cut): $\dfrac{\Gamma \vdash^c_{\mathbf{L}} A \qquad \Delta \vdash^c_{\mathbf{L}} B \text{ for all } B \in \Gamma}{\Delta \vdash^c_{\mathbf{L}} A}$

Monotonicity (weakening): $\dfrac{\Gamma \vdash^c_{\mathbf{L}} C}{\Gamma \cup \Delta \vdash^c_{\mathbf{L}} C}$

Proof *That $\vdash^c_{\mathbf{B}}$ is reflexive and monotone is obvious. That it is transitive is seen by noting that if $\Gamma \vdash^c_{\mathbf{B}} A$, then by Thm. 1 $\varnothing \vdash^h_{\mathbf{B}} \bigwedge \Gamma^\dagger \to A$ for some finite set $\Gamma^\dagger \subseteq \Gamma$. If furthermore $\Delta \vdash^c_{\mathbf{B}} B$ for all $B \in \Gamma$, we obtain using Thm. 1 together with the fact that if $\varnothing \vdash^h_{\mathbf{B}} C_1 \to D_1$ and $\varnothing \vdash^h_{\mathbf{B}} C_2 \to D_2$, then $\varnothing \vdash^h_{\mathbf{B}} (C_1 \wedge C_2) \to (D_1 \wedge D_2)$, that $\varnothing \vdash^h_{\mathbf{B}} \bigwedge \Delta^\dagger \to \bigwedge \Gamma^\dagger$ for some finite set $\Delta^\dagger \subseteq \Delta$. Using the rule (R4) it then follows that $\varnothing \vdash^h_{\mathbf{B}} \bigwedge \Delta^\dagger \to A$, and and so Thm. 1 finally yields that $\Delta \vdash^c_{\mathbf{B}} A$. The case for $\mathbf{B}^\circ$ is identical.*

Giving proofs using $\vdash^c_{\mathbf{L}}$ is a rather baroque matter. Note, then, that these consequence relations can quite easily be defined using a substructural proof calculus such as the sequent-style natural deduction calculus found in Restall (2000). It would take this paper too far afield to go further into the details of Restall's proof calculus. What I will comment on, and which will be of great use in the next section, however, is Restall's soundness and completeness result (Restall, 2000, thm. 4.3) which states in effect that the sequent $X \rhd A$ is provable in any of Restall's systems if and only if $\tau(X) \to A$ is a logical theorem of the corresponding Hilbert calculus.[5] '$\rhd$' is here the sequent symbol.[6] τ is a translation from antecedent structures X to formulas. Such structures—Priest calls them *bunches*—can be compiled from formulas using premise combinators, or *punctuation marks* as Restall calls them, which represents **t**, $\wedge$ and $\circ$: '**1**' is a null-ary premise combinator which stands for '**t**', and '$\oplus$' and '$\otimes$' are binary premise combinator which stand for, respectively, '$\wedge$' and '$\circ$'.[7] In particular, then, $\tau(A_1 \oplus \ldots \oplus A_n) = (A_1 \wedge \ldots \wedge A_n)$, and so Restall's theorem yields that the sequent $A_1 \oplus \ldots \oplus A_n \rhd B$ is provable in any of his systems if and only if $\bigwedge_{i \leq n} A_i \to B$ is a logical theorem of the corresponding Hilbert system. In the case of **B**, then, we get as a corollary that $\{A_1, \ldots, A_n\}$ C-entails B just in case the sequent $A_1 \oplus \ldots \oplus A_n \rhd B$ is provable in Restall's system corresponding to **B**.[8]

Without loss of generality we can simply demand that the bunch X always be a formula, and so instead of letting $A \oplus B$ be a bunch made up of two formulas combined using the extensional premise combinator '$\oplus$', and $A \otimes B$ be a bunch made up of two formulas combined using the intensional premise combinator '$\otimes$', we simply combine sentences using $\wedge$ and $\circ$, the extensional and intensional conjunctions. The null-ary '**1**' is simply to be replaced by '**t**'.[9]

4 Confused entailment and P-entailment

Graham Priest has made substantial contributions both the philosophy of, and the technical advancement of relevant logics over a number of years. His approach, however, has been to view relevant logics as sub-classical logics formulated in fully structural proof calculi in which every primitive rule of inference is treated on par with any other.[10]

[4]The same holds for any logic for which the C-entailment theorem applies to. This is a very large class; see Øgaard (2021a, § 3) for details.

[5]See Øgaard (2021b) for more on this result.

[6]Since this paper is primarily on Priest's new notion of logical consequence, I will follow him and write *sequent* where Restall uses *consecution*. Furthermore, Restall uses '$\vdash$' as the sequent symbol, whereas Priest uses '$\rhd$'.

[7]Again, I follow Priest's notation; Restall uses '0''to stand for **t**, the comma to stand for extensional conjunction and the semicolon to stand for intensional conjunction.

[8]Note that every structural rule is assumed for $\oplus$, and so we may ignore order in structure such as $A_1 \oplus \ldots \oplus A_n$. There is another detail, however, that needs to be commented upon a bit more carefully, namely the behaviour of the Church constant $\bot$. $\bot$ is intuitively the conjunction of every proposition. In Routley-Meyer semantics, $\bot$ is demanded to fail at every evaluation point. A consequence of this is that $C \to (\bot \to \bot)$ is valid in the semantics. This is a theorem of **DW**—**B** with (R5) replaced by its axiomatic version—but not of **B** as here defined. In other semantics, such as the algebraic semantics of Meyer and Routley (1972), however, $C \to (\bot \to \bot)$ can be made to fail. Since also Priest (2015a)'s rule for $\bot$ only yields (A8), I've chosen to stick to the more common way of adding $\bot$ in the case of **B**. Note, then, that Restall's rule for $\bot$, $\dfrac{X \rhd \bot}{Y(X) \rhd A}\,(\bot E)$, does yield $C \to (\bot \to \bot)$. However, by replacing the rule with $\dfrac{X \rhd \bot}{X \rhd A}\,(\bot E)$ —or equivalently as the axiom $\bot \rhd A$ which is Priest (2015a)'s rule for $\bot$—will allow Restall's soundness and completeness result to hold true with regards to how **B** is defined here. For more on this, see Øgaard (2021b)

Note that adding $C \to (\bot \to \bot)$ is not without its consequences: it was shown in Øgaard (2016, thm. 10) that if added to $\mathbf{BX}^d$—**B** as here defined where it includes both $\circ$ and **t**, but with excluded middle and the meta-rule of *reasoning by cases* added—then the naïve theory of truth is trivialized. Whether or not it trivializes without the added $\bot$ axiom is currently unknown.

[9]See Restall (2000, lem. 4.17) for a proof to the effect that there is no loss in generality involved in thus restricting our attention to formulas.

[10]See, for instance Priest (2006, ch. 18.3 & 19.8) and Priest (2008, ch. 10).

The exception, however, is found in the more recent article *Fusion and Confusion*. In it, Priest makes use of Restall's substructural proof calculus for the positive fragment of the logic **B**. His definition of logical consequence, when this is understood to be derivability from "information, such as that provided by the axioms of an axiom system" (Priest, 2015a, p. 59) is not C-entailment, however. The purpose of this section is to show just how to understand Priest's notion of logical consequence. To do so, we need the notion of a *confusion*:[11]

Definition 7 (Confusion) $C(\Sigma)$—the set of CONFUSIONS OVER A SET OF FORMULAS Σ—is inductively defined as follows:

- $\Sigma \subseteq C(\Sigma)$
- $\mathbf{t} \in C(\Sigma)$
- for any $D_1 \in C(\Sigma)$ and $D_2 \in C(\Sigma)$,
  - $D_1 \wedge D_2 \in C(\Sigma)$
  - $D_1 \circ D_2 \in C(\Sigma)$

We can now simplify Priest's definition that "A follows from Σ iff for some $X \in \mathfrak{B}(\Sigma)$[12], $X \rhd A$ is provable" (Priest, 2015a, p. 59) into the following definition:

Definition 8 (P-entailment) A set of formulas Σ P-ENTAILS A in the logic **B** just in case there is a confusion C of Σ such that $C \rhd A$ is a provable sequent in Restall's system corresponding to **B**.[13]

Since A is a logical theorem of **B** if and only if $\mathbf{t} \rhd A$ is a provable sequent in Restall's system for **B**, the familiar notion of a logical theorem remains intact: any logical theorem is P-entailed by *any* set of assumptions whatsoever. This is not the case for C-entailment, as we have seen, for which *nothing* follows from the empty set if premises. We have already seen that C-entailment satisfies the relevance property (CVSP). P-entailment does not:

Theorem 3 *P-entailment does not satisfy (CVSP).*

Proof $\{A\}$ *P-entails* $B \to B$ *for every formula* A *and* B *since* $\mathbf{t} \rhd B \to B$ *is a provable sequent in Restall's proof calculus for* **B** *and* $\mathbf{t}$ *is a confusion of* $\{A\}$.

Thus P-entailment is alike the Hilbertian consequence relation for **B** in that the consequential variable sharing property fails for it. For certain rather strong logics—logics like **RW**—it can be shown that P-entailment *reduces* to the corresponding Hilbert consequence relation.[14] As we shall see, however, this is not so in the case with **B**. To understand just how P-entailment for **B** differs from both Anderson and Belnap's C-entailment as well as Priest's previous standard Hilbertian understanding of logical consequence, I will provide a Hilbert-style proof calculus for the notion that I've called *CF-entailment*—short for *confused entailment*—which, as we shall see, turns out to be equivalent to P-entailment.

Definition 9 Γ CF-ENTAILS A in the logic **L** if and only if there is a Hilbert-proof of A from Γ in which any rule different from (R1) and (R2) are only used as *admissible* rules, i.e. just in case there is a Hilbert proof $A_1, \ldots, A_n$ of A from Γ which can be #-marked according to the following rules:

1. If $A_i \in \Gamma$, then A_i is marked.
2. If A_i is obtained using either (R1) or (R2), then A_i is marked if either of the premises are marked.
3. Rules different from (R1)&(R2) are only used on unmarked formulas.
4. No other formulas are marked.

The existential claim that there is such a proof is written $\Gamma \vdash^{cf}_{\mathbf{L}} A$ and expressed as "there exists a proof that Γ CF-entails A in the logic **L**".

Notice here that unlike C-entailment, a CF-proof of A does not require that A be marked as a consequence of the marking rules. CF-entailment, then, distinguishes between what Smiley (1963) calls "rules of proof" and "rules of inference", where modus ponens and adjunction are in the case at hand the only two rules of inference, whereas (R1)–(R7) are all rules of proof.[15]

The next task is to prove a "confused" entailment theorem for $\vdash^{cf}_{\mathbf{B}}$—Thm. 4 below. From it it will then follow that CF-entailment and P-entailment are extensionally identical. To prove the CF-entailment theorem, however, we first need to prove some lemmas.

Lemma 1

$$\varnothing \vdash^{cf}_{\mathbf{B}} = \varnothing \vdash^{h}_{\mathbf{B}}$$

Proof (b) Any $\vdash^{cf}$*-proof is evidently also a* $\vdash^{h}$*-proof, so we need only show that if* $\varnothing \vdash^{h}_{\mathbf{B}} A$ *then* $\varnothing \vdash^{cf}_{\mathbf{B}} A$*. Let* $A_1, \ldots, A_n$ *be a* $\vdash^{h}$*-proof of* A *from* $\varnothing$*. Looking over the #-rules for CF-entailment it is evident that neither of the* A_i*'s can be marked. But then every application of every rule used in the* $\vdash^{h}$*-proof of* A *from* $\varnothing$ *is permissible by the standard set by* $\vdash^{cf}$*-proofs. Hence* $A_1, \ldots, A_n$ *is also a* $\vdash^{cf}$*-proof.*

Lemma 2 *If* $\Gamma \vdash^{cf}_{\mathbf{B}} B$ *and* $\Delta \vdash^{cf}_{\mathbf{B}} A$ *for every* $A \in \Gamma$*, then* $\Delta \vdash^{cf}_{\mathbf{B}} B$*.*

[11] The notion of a confusion was, to my knowledge, first introduced in Restall (2000) (def. 4.26). Restall allows $\top$ to be a member of $C(\Sigma)$, but since this is inconsequential and Priest does not, I've chosen to go with the stricter notion.

[12] $\mathfrak{B}(\Sigma)$ is the set of bunches corresponding to the set of confusions over the set of formulas Σ.

[13] Again I will have to refer the reader to Restall's book for details.

[14] This follows using Restall's deduction theorem for **RW** (Restall, 2000, p. 87).

[15] See Humberstone (2010) for a nice discussion of Smiley's distinction.

Proof *I'll give a general description of why the lemma holds; details are left for the reader. In the case of Hilbert proofs, this is a simple matter of cutting and pasting proofs. The general recipe is to take a proof $B_1, \dots B_n$ of B from Γ and replace every B_i occuring in Γ with a proof of B_i from Δ.*

In the case of CF-proofs, however, we have to make sure that the #-rules are obeyed. Note, then, that since any $B_i \in \Gamma$ must be #-marked, any such B_i can only figure as a premise of either modus ponens (R1) or adjunction (R2). It can happen that B_i in a proof $C_1, \dots, C_{i_m}$ of B_i from Δ is not #-marked, however.[16] *Since B_i is only needed as a premise for (R1) or (R2) in the proof of B, however, a B_i unmarked by # will do just as nicely. The R1-conclusion or R2-conclusion of CF-proofs is marked if either the of the premises are marked. Thus replacing the marked B_i by a proof of B_i where, then, B_i need not be marked, may result in further formulas not being marked. Neither this, however, has any effect as the #-rules only restrict certain rules to only apply to unmarked premises.*[17]

Lemma 3 *$\Sigma \vdash^{cf}_{\mathbf{B}} C$ for every $C \in \mathcal{C}(\Sigma)$.*

Proof *Use (A9), (A10), (R1) and (R2).*

Lemma 4 *If for some confusion C of A, $\Gamma \vdash^{cf}_{\mathbf{B}} C \to B$, then $\Gamma \cup \{A\} \vdash^{cf}_{\mathbf{B}} B$.*

Proof *If $\Gamma \vdash^{cf}_{\mathbf{L}} C \to B$, then since by Lem. 3 $\{A\} \vdash^{cf}_{\mathbf{B}} C$ for every $C \in \mathcal{C}(\{A\})$, it follows that $\Gamma \cup \{A\} \vdash^{cf}_{\mathbf{B}} B$.*

Lemma 5 *If $\Gamma \cup \{A\} \vdash^{cf}_{\mathbf{B}} B$, then for some confusion C of A, $\Gamma \vdash^{h}_{\mathbf{B}} C \to B$.*

Proof *Assume that $A_1, \dots, A_n$ is a cf-proof of B from $\Gamma \cup \{A\}$. The proof is an inductive proof to the effect that $\Gamma \vdash^{h}_{\mathbf{L}} C_i \to A_i$ for every $i \le n$, where the C_i's are all confusions of $\{A\}$. If A_i is A, let C_i be A. If A_i is either* **t**, *a member of Γ, or an axiom, or obtained from any A_j for $j < i$ using one of the restricted rules, let C_i be* **t**. *In all of these cases it is clear that $\Gamma \vdash^{h}_{\mathbf{L}} C_i \to A_i$.*

Now for the inductive part. Assume for inductive hypothesis (IH) that $\Gamma \vdash^{h}_{\mathbf{L}} C_i \to A_i$ and $\Gamma \vdash^{h}_{\mathbf{L}} C_j \to A_j$. There are two cases to consider.

(1) Assume that A_j is $A_i \to A_k$ and A_k is obtained from A_i and A_j using (R1). Then from $\Gamma \vdash^{h}_{\mathbf{L}} C_i \to A_i$ we can derive $\Gamma \vdash^{h}_{\mathbf{L}} (A_i \to A_k) \to (C_i \to A_k)$ using (R4). Since we have assumed that $\Gamma \vdash^{h}_{\mathbf{L}} C_j \to (A_i \to A_k)$, it follows by transitivity that $\Gamma \vdash^{h}_{\mathbf{L}} C_j \to (C_i \to A_k)$, and so that $\Gamma \vdash^{h}_{\mathbf{L}} C_j \circ C_i \to A_k$ using ($R\circ_1$).

(2) If A_k is obtained from A_i and A_j using (R2), then A_j is the formula $A_i \wedge A_j$. Using (A3) and (A5) one easily obtains $\Gamma \vdash^{h}_{\mathbf{L}} C_i \wedge C_j \to A_i \wedge A_j$ from (IH).

[16] For instance, let B_i be a logical axiom, $\Gamma = \{B_i\}$, and $\Delta = \varnothing$.

[17] I would like to thank the reviewer for insisting that a proof beyond a general remark on cutting and pasting proofs was needed and for in essense suggesting the proof given here.

Theorem 4 **(CF-entailment theorem)** *$\Gamma \vdash^{cf}_{\mathbf{B}} B$ if and only if $\varnothing \vdash^{h}_{\mathbf{B}} C \to B$ for some $C \in \mathcal{C}(\Gamma)$.*

Proof *($\Leftarrow$) If $\varnothing \vdash^{h}_{\mathbf{B}} C \to B$ for some $C \in \mathcal{C}(\Gamma)$, then by Lem. 1 we get that $\varnothing \vdash^{cf}_{\mathbf{B}} C \to B$. Lem. 4 then yields that $\Gamma \vdash^{cf}_{\mathbf{B}} B$.*

($\Rightarrow$) Assume that $\Gamma \vdash^{cf}_{\mathbf{B}} B$. Since proofs are finite we get that $\Gamma' \vdash^{cf}_{\mathbf{L}} B$ for some finite set $\Gamma' \subseteq \Gamma$. Let $A =_{df} \bigwedge \Gamma'$. Then $\Gamma' \vdash^{cf}_{\mathbf{L}} A$, and since $\{A\} \vdash^{cf}_{\mathbf{L}} D$ for every $D \in \Gamma'$, it follows by Lem. 2 that $\{A\} \vdash^{cf}_{\mathbf{L}} B$. By Lem. 5 $\varnothing \vdash^{h}_{\mathbf{L}} C \to B$ for some confusion C of $\{A\}$. The result now follows since C is also a confusion of Γ.

We are now in a position to state the main theorem:

Theorem 5 *P-entailment for $\boldsymbol{B}$ is extensionally identical to CF-entailment.*

Proof *Using Restall's soundness and completeness theorem get that Γ P-entails A if and only if $\varnothing \vdash^{h}_{\mathbf{B}} C \to A$ for some confusion $C \in \mathcal{C}(\Gamma)$. Using Thm. 4 it then follows that Γ P-entails A if and only if Γ CF-entails A.*

The above theorem easily extends to stronger logics than **B** . It does not, however, extend to logics without the Ackermann constant as the next theorem shows:

Theorem 6 *P-entailment and CF-entailment are not extensionally identical in the case of $\boldsymbol{B}^\circ$.*

Proof *Using Restall's soundness and completeness theorem get that $\{A\}$ P-entails B in $\boldsymbol{B}^\circ$ if and only if $\varnothing \vdash^{h}_{\mathbf{B}^\circ} C \to B$ for some* **t***-free confusion $C \in \mathcal{C}\{A\}$.*

Let A be the propositional variable p and B be $q \to q$ so that A and B share no propositional variable. Since $\boldsymbol{B}^\circ$ satisfies the variable sharing property it follows that $\varnothing \nvdash^{h}_{\mathbf{B}^\circ} C \to B$ for every **t***-free confusion $C \in \mathcal{C}\{A\}$ and thus that $\{A\}$ does not P-entail B. However, $p^\#, q \to q$ is a CF-proof of B from A and thus B is CF-entailed by $\{A\}$ in $\boldsymbol{B}^\circ$.*

Corollary 2 *CF-entailment for $\boldsymbol{B}$ does not satisfy (CFSP).*

Proof *This follows from Thm. 3 together with Thm. 5.*

Since C-entailment for **B** does satisfy the consequential variable sharing property and CF-entailment does not, the latter notion of logical consequence does not hold up to the standard argued to be correct by Anderson and Belnap. Just how different, then, is CF-entailment from the standard Hilbertian notion of entailment, and just how different is it really from C-entailment? This is the question investigated in the next section.

5 Comparing C-, CF-, and H-entailment for B

We have now seen how to define three different consequence relations using the same set of axioms and rules, namely H-, CF- and C-entailment. In this section I will show that they are interestingly related. I will only focus on the case of **B**. In this case it turns out that CF-entailment is strictly stronger than C-entailment, but strictly weaker than H-entailment. CF-entailment turns out to be a rather weak consequence relation. The next section will discuss the prospects of strengthening it within the confinement of some of Priest's other theoretical commitments.

Notice that CF-entailment allows both the rules (R1) and (R2) to be applied unrestrictedly in any deductive situation. This is good as Priest is keen to emphasize that especially modus ponens is "one of the rules of inference" (Priest, 2015a, p. 59). As shown in Thm. 7 below, this is not the case with C-entailment in the case of **B**. We have already seen that CF-entailment differs from C-entailment in that the latter does, but the former does not satisfy the consequential variable sharing property. To realize just how different CF-entailment and C-entailment are, however, note that these consequence relations differ radically in terms of whether they allow modus ponens and adjunction as rules applicable to any deductive situation:

Theorem 7

(1) $\{(A \to A) \to B\} \vdash^{cf}_{\mathbf{B}} B$	$\{(A \to A) \to B\} \nvdash^{c}_{\mathbf{B}} B$
(2) $\{A, A \to B\} \vdash^{cf}_{\mathbf{B}} B$	$\{A, A \to B\} \nvdash^{c}_{\mathbf{B}} B$
(3) $\{A\} \vdash^{cf}_{\mathbf{B}} A \wedge (B \to B)$	$\{A\} \nvdash^{c}_{\mathbf{B}} A \wedge (B \to B)$

Proof *(1) The **B**-logically true $\to$ sentence corresponding to the CF-entailment is*

$$\mathbf{t} \wedge ((A \to A) \to B) \to (\mathbf{t} \wedge ((A \to A) \to B) \to B).$$

*An easy CF-proof is simply $((A \to A) \to B)^{\#}, A \to A, B^{\#}$. With regards to C-entailment: It is well known that $((A \to A) \to B) \to B$ is not a theorem of **B**, and so Thm. 1 yields that $\{(A \to A) \to B\} \nvdash^{c}_{\mathbf{B}} B$.*

*(2) Since $\varnothing \vdash^{h}_{\mathbf{B}} A \wedge (A \to B) \to (A \wedge (A \to B) \to B)$ it is easy to obtain a proof that $\{A, A \to B\} \vdash^{cf}_{\mathbf{B}} B$. It is, again, well known that $A \wedge (A \to B) \to B$ is not a theorem of **B**, and so Thm. 1 yields that $\{A, A \to B\} \nvdash^{c}_{\mathbf{B}} B$.*

*(3) $A^{\#}, B \to B, (A \wedge (B \to B))^{\#}$ is a CF-proof to the effect that $\{A\} \vdash^{cf}_{\mathbf{B}} A \wedge (B \to B)$ and $A \wedge \mathbf{t} \to A \wedge (B \to B)$ is a **B**-logically true $\to$ sentence corresponding to it. It follows from Thm. 1, however, that $\{A\} \nvdash^{c}_{\mathbf{B}} A \wedge (B \to B)$ since $\varnothing \nvdash^{h}_{\mathbf{B}} A \to A \wedge (B \to B)$.*

CF-entailment, unlike C-entailment, is alike Hilbert derivability in treating adjunction and modus ponens as applicable to any deductive situation. Beyond this, however, it is quite different: The following theorem shows that all of the

$$\begin{array}{c} 5 \\ \uparrow \\ 4 \\ \uparrow \\ 3 \\ \uparrow \\ 2 \\ \uparrow \\ 1 \\ \uparrow \\ 0 \end{array} \qquad \begin{array}{l} [\![\mathbf{t}]\!] = 3 \\ [\![A]\!] = 4 \\ [\![B]\!] = 1 \\ [\![C]\!] = 2 \\ [\![D]\!] = 3 \end{array}$$

$$\begin{array}{c|cccccc|c} \to & 0 & 1 & 2 & 3 & 4 & 5 & \sim \\ \hline 0 & 3 & 3 & 3 & 3 & 5 & 5 & 5 \\ 1 & 0 & 3 & 3 & 3 & 5 & 5 & 4 \\ 2 & 0 & 1 & 3 & 3 & 4 & 5 & 3 \\ 3 & 0 & 0 & 2 & 3 & 4 & 5 & 2 \\ 4 & 0 & 0 & 1 & 2 & 4 & 5 & 1 \\ 5 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \end{array} \qquad \begin{array}{c|cccccc} \circ & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 2 & 2 & 5 \\ 2 & 0 & 1 & 2 & 2 & 3 & 5 \\ 3 & 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 4 & 4 & 4 & 4 & 4 & 5 \\ 5 & 4 & 4 & 5 & 5 & 5 & 5 \end{array}$$

Fig. 1 A model for **B**

rules (R3)–(R7) fail as CF-entailments, and so these rules are *not* applicable in every deductive situation. In addition, also modus tollens fails, as does the *Substitution of Coimplicants* rule—to infer D with every A_1 replaced by A_2 from premises $A_1 \leftrightarrow A_2$ together with D. Lastly, and even more shockingly, CF-entailment fails to treat $\to$ as a transitive conditional.

Theorem 8

(1) $\{C \to B\} \nvdash^{cf}_{\mathbf{B}} (A \to C) \to (A \to B)$
(2) $\{C \to B\} \nvdash^{cf}_{\mathbf{B}} (B \to A) \to (C \to A)$
(3) $\{A \to {\sim}C\} \nvdash^{cf}_{\mathbf{B}} C \to {\sim}A$
(4) $\{A \to (B \to C)\} \nvdash^{cf}_{\mathbf{B}} A \circ B \to C$
(5) $\{B\} \nvdash^{cf}_{\mathbf{B}} \mathbf{t} \to B$

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(6) $\{A \to {\sim}C, C\} \nvdash^{cf}_{\mathbf{B}} {\sim}A$
(7) $\{A_1 \leftrightarrow A_2, D\} \nvdash^{cf}_{\mathbf{B}} D({}^{A_1}/{}_{A_2})$
(8) $\{A \to D, D \to C\} \nvdash^{cf}_{\mathbf{B}} A \to C$

Proof *By using Thm. 4 this will follow if we can show the following:*

(1) $\forall E \in C(\{C \to B\}) : \varnothing \nvdash^{h}_{\mathbf{B}} E \to ((A \to C) \to (A \to B))$
(2) $\forall E \in C(\{C \to B\}) : \varnothing \nvdash^{h}_{\mathbf{B}} E \to ((B \to A) \to (C \to A))$
(3) $\forall E \in C(\{A \to {\sim}C\}) : \varnothing \nvdash^{h}_{\mathbf{B}} E \to (C \to {\sim}A)$
(4) $\forall E \in C(\{A \to (B \to C)\}) : \varnothing \nvdash^{h}_{\mathbf{B}} E \to (A \circ B \to C)$
(5) $\forall E \in C(\{B\}) : \varnothing \nvdash^{h}_{\mathbf{B}} E \to (\mathbf{t} \to B)$

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(6) $\forall E \in C(\{A \to {\sim}C, C\}) : \varnothing \nvdash^{h}_{\mathbf{B}} E \to {\sim}A$
(7) $\forall E \in C(\{D \leftrightarrow A, D \to C\}) : \varnothing \nvdash^{h}_{\mathbf{B}} E \to (D \to C)({}^{D}/{}_{A})$
(8) $\forall E \in C(\{A \to D, D \to C\}) : \varnothing \nvdash^{h}_{\mathbf{B}} E \to (A \to C)$

The model in Fig. 1 consists of a displayed partial ordering over which conjunction and disjunction are to be interpreted as, respectively, infimum and supremum. The matrices show how $\to$, $\sim$ and $\circ$ are to be interpreted. $\mathbf{t}$ is interpreted as the least designated element, and so every element above it in the displayed ordering are designated elements. A formula is validated by a model if it is evaluated to one of the designated elements, and a rule is said to hold in the *model just in case any instance of the rule is designatedness-preserving.*[18]

1. $[\![(A \to C) \to (A \to B)]\!] = 0$, *but* $[\![E]\!] \in \{1,2,3\}$ *for every confusion* $E \in C(\{C \to B\})$ *and so every such implication* $E \to ((A \to C) \to (A \to B))$ *is evaluated to* 0.
2. *Same as (1)*
3. $[\![C \to {\sim}A]\!] = 1$, *but* $[\![E]\!] \in \{2,3\}$ *for every* $E \in C(\{A \to {\sim}C\})$ *and so every such implication* $E \to (C \to {\sim}A)$ *is evaluated to* 1.
4. $[\![A \circ B \to C]\!] = 1$, *but* $[\![E]\!] \in \{2,3\}$ *for every* $E \in C(\{A \to (B \to C)\})$ *and so every such implication* $E \to (A \circ B \to C)$ *is evaluated to either* 1 *or* 0.
5. $[\![\mathbf{t} \to B]\!] = 0$, *but* $[\![E]\!] \in \{1,2,3\}$ *for every* $E \in C(\{C\})$ *and so every such implication* $E \to (\mathbf{t} \to B)$ *is evaluated to* 0.
6. *Similar as (3).*
7. $(D \to C)(^{D}/_{A})$ *is simply the formula* $A \to C$, *and* $[\![A \to C]\!] = 1$. *Since* $[\![D \leftrightarrow A]\!] = 2$ *and* $[\![D \to C]\!] = 2$, $[\![E]\!] \in \{2,3\}$ *for every* $E \in C(\{D \leftrightarrow A, D \to C\})$. *Thus every such implication* $E \to (D \to C)(^{D}/_{A})$ *is evaluated to either* 1 *or* 0.
8. $[\![A \to C]\!] = 1$, *but* $[\![E]\!] \in \{2,3\}$ *for every* $E \in C(\{A \to D, D \to C\})$ *and so every such implication* $E \to (A \to C)$ *is evaluated to either* 1 *or* 0.

All of (1)–(8) in Thm. 8 hold in the case of $\vdash^h_{\mathbf{B}}$, and so CF-entailment is a much weaker concept of logical consequence than H-entailment. It seems, then, that if CF-entailment is the concept that explicates what is derivable from "information [...] provided by the axioms of an axiom system", then very little seems indeed to follow that involves the intensional $\{\mathbf{t}, {\sim}, \circ, \to\}$-part of the language. The following two corollaries specifies just how CF-entailment compares to C-entailment and standard Hilbert consequence.

Corollary 3

$$\vdash^c_{\mathbf{B}} \subsetneq \vdash^{cf}_{\mathbf{B}} \subsetneq \vdash^h_{\mathbf{B}}$$

Proof *That the subset-relation holds follows simply by noting that any* $\vdash^c$*-proof is a* $\vdash^{cf}$*-proof which in turn is a* $\vdash^h$*-proof. That the subset-relations are* proper *follows from Thm. 7 and Thm. 8.*

C-entailment is a weaker notion of logical consequence than CF-entailment. There is, however, a precise condition under which the distinction between the two notions collapses, namely if we can unrestrictedly use logical axioms, which in the presence of **t**, simplifies to having **t** as a premise.

Lemma 6 *If* $\{\mathbf{t}\} \subseteq \Sigma$, *then* $\Sigma \vdash^c_{\mathbf{B}} C$ *for every* $C \in C(\Sigma)$.

Proof *Use (A9), (R1) and (R2); details left for the reader.*

Corollary 4

$$\Gamma \cup \{\mathbf{t}\} \vdash^c_{\mathbf{B}} A \Longleftrightarrow \Gamma \vdash^{cf}_{\mathbf{B}} A$$

Proof *Assume first that* $\Gamma \cup \{\mathbf{t}\} \vdash^c_{\mathbf{B}} A$. *Since proofs are finite it follows that* $\Gamma^\dagger \cup \{\mathbf{t}\} \vdash^c_{\mathbf{B}} A$ *for some finite set* $\Gamma^\dagger \subseteq \Gamma$. *From Thm. 1 (and some fiddling) it then follows that* $\varnothing \vdash^h_{\mathbf{B}} \bigwedge \Gamma^\dagger \wedge \mathbf{t} \to A$, *and then from Lem. 1 that* $\varnothing \vdash^{cf}_{\mathbf{B}} \bigwedge \Gamma^\dagger \wedge \mathbf{t} \to A$. *Since* $\Gamma^\dagger \wedge \mathbf{t}$ *is a confusion of* Γ *it follows from Thm. 4 that* $\Gamma \vdash^{cf}_{\mathbf{B}} A$.

Assume that $\Gamma \vdash^{cf}_{\mathbf{B}} A$. *From Thm. 4 it follows that* $\varnothing \vdash^h_{\mathbf{B}} C \to A$ *for some confusion* $C \in C(\Gamma)$. *From Thm. 1 it follows that* $\{C\} \vdash^c_{\mathbf{B}} A$, *and from Lem. 6 that* $\Gamma \cup \{\mathbf{t}\} \vdash^c_{\mathbf{B}} C$. *Since* $\vdash^c_{\mathbf{B}}$ *is transitive (Thm. 2) it then follows that* $\Gamma \cup \{\mathbf{t}\} \vdash^c_{\mathbf{B}} A$.

6 The prospects of a stronger confused entailment

We have seen that CF-entailment is a very weak notion of logical consequence. Now Priest is no stranger to weak relevant logics. However, in most of Priest's writings wherein an intensional implication operator such as $\to$ is utilized, Priest utilizes consequence relations at least as strong as $\vdash^h_{\mathbf{B}}$. The notable exception is Priest (2016) where Priest makes use of the relevant logic $\mathbf{N}_4$ with a standard Hilbertian consequence relation defined over it.[19] Neither of the pre- and suffixing rules (R3) and (R4), nor contraposition or modus tollens hold for $\vdash^h_{\mathbf{N}_4}$. Neither does Substitution of Coimplicants hold for $\vdash^h_{\mathbf{N}_4}$.[20] However, transitivity of $\to$ is a valid rule of inference for both $\vdash^h_{\mathbf{N}_4}$ and $\vdash^h_{\mathbf{B}}$, but as we have seen, not for CF-entailment.

Faced with this, Priest could either strengthen CF-entailment somehow, to supply an interpretation under which the failure of the transitivity of $\to$ under CF-entailment is acceptable, or simply to abandon CF-entailment altogether. Let's briefly look into these options in turn. The most obvious way of strengthening CF-entailment so as to validate transitivity for $\to$ is by way of adding axioms to **B**. One way to do so would be to add the axiom called *Conjunctive Syllogism*:

(ConSyll) $(A \to B) \wedge (B \to C) \to (A \to C)$.

ConSyll, however, suffices for trivializing naïve truth theory in the presence of $\circ$, and so would not suite Priest's other ambitions.[21] Another natural suggestion would be to

[18] The model was found with the help of `MaGIC`—an acronym for *Matrix Generator for Implication Connectives*—which is an open source computer program created by John K. Slaney (Slaney (1995)).

[19] See Priest (2008, ch. 9) for a tableaux system and semantics for $\mathbf{N}_4$.

[20] To see why, note that both $A \wedge B \leftrightarrow B \wedge A$ and $(A \wedge B \to A) \to (A \wedge B \to A)$ are logical theorems of $\mathbf{N}_4$, but that $(A \wedge B \to A) \to (B \wedge A \to A)$ is not since $\to$ formulas can be given arbitrary truth-values at "non-normal worlds".

[21] The result is due to Dunn and Slaney and can be found in Routley et al. (1982, pp. 366f).

strengthen the pre- and suffixing rules of **B** to their axiomatic version, namely

$$(A \to B) \to ((C \to A) \to (C \to B))$$
$$(A \to B) \to ((B \to C) \to (A \to C)).$$

However, these axioms are interderivable with the axioms

$$(A \circ B) \circ C \to (A \circ (B \circ C))$$
$$(A \circ B) \circ C \to (B \circ (A \circ C))$$

and Priest explicitly states that fusion is not associative given his functional interpretation of the Routley-Meyer semantics (Priest, 2015a, p. 60).[22] There are, however, alternatives. One such is to weaken the mentioned transitivity axioms to their ***t**-enthymematic* versions. In the case of ConSyll this yields $(A \to B) \wedge (B \to C) \wedge \mathbf{t} \to (A \to C)$ which like ConSyll also trivializes naïve truth theory in the presence of $\circ$.[23] The **t**-enthymematic versions of the pre- and suffixng axioms, that is

$$(A \to B) \wedge \mathbf{t} \to ((C \to A) \to (C \to B))$$
$$(A \to B) \wedge \mathbf{t} \to ((B \to C) \to (A \to C)),$$

however, seem more promising. These can be added to **B** without making fusion associative. It can also be added to the $\circ$-free fragment of **BX**—**B** with excluded middle added—without trivializing the naïve theory of truth and even the naïve theory of sets.[24] The same goes for modus tollens and contraposition more generally which can be obtained as true CF-entailments by simply opting for the axiomatic version of the contraposition rule (R5).[25] Even the **t** rule (R7) can be strengthened to the axiom $A \wedge \mathbf{t} \to (\mathbf{t} \to A)$ and would if thus strengthened yield a corresponding true CF-entailment without trivializing the two mentioned naïve theories.[26] Thus all the primitive rules of **B** can strengthened so as to yield corresponding CF-entailments. However, whether or not naïve truth theory is non-trivial over **BX** *with* $\circ$ is currently unknown.[27] Naïve set theory, it turns out, is trivial even over **B**.[28]

Even though strengthening CF-entailment seem possible without thereby trivializing the naïve theory truth, it remains to be investigated whether or not these strengthenings are acceptable by the light of Priest's functional interpretation of the Routley-Meyer semantics. This, however, is an issue that is beyond the scope of the current paper.

The other option was to find an interpretation under which the failure of the transitivity of $\to$ over CF-entailment is acceptable. Even though I will not provide such an interpretation here, it seems pertinent to note at least how Priest intends $\to$ to be read. Priest does point out that

> one of the motivating thoughts of early relevant logic was exactly to have a connective in the language which expressed entailment. Thus, validity is expressed by the logical truth of the conditional, as a matter of definition. And an inference from A to B is valid iff A suffices logically for B [...] (Priest, 2015a, p. 60)

Validity, then, is not an account of sound inference, according to Priest; modus ponens, for instance, is not a valid rule of inference seeing as $A \wedge (A \to B) \to B$ is not logically true. It is, however, "one of the rules of inference" (Priest, 2015a, p. 59), meaning that the antecedent CF-entails the conclusion. Let's call such a rule *acceptable*. Thus some rules are acceptable even though they are not valid, although all valid rules are acceptable. CF-entailment takes premises to be *sets* of formulas and is a fully structural consequence relation, whereas validity takes premises to be structured into bunches and is a *substructural* consequence relation. There is an easy way of translating away this substructural feature, however, seeing as any inference with X as its premise bunch and A as its conclusion is valid if and only if $X \rhd A$ is provable (Priest, 2015a, p. 59). Using Restall's soundness and completeness result, then, it follows that the inference from X to A is valid if and only if $\emptyset \vdash^h_{\mathbf{B}} \tau(X) \to A$ if and only if (Thm. 1) A is C-entailed by $\tau(X)$. The notion of validity, then, is at heart Anderson and Belnap's notion of C-entailment. The twist is simply that the notion of premises are different: Anderson and Belnap adhered to the standard set-based notion of premises, whereas Priest accepts a more fine-grained one.

Note, then, that even if an interpretation is given under which it is acceptable that $\to$ is not transitive over CF-entailment, it is still a fact that *validity* is transitive: if both A suffices logically for B and B for C, then A suffices logically

[22] See Beall et al. (2012) and Priest (2015b) for more on this interpretation of the semantics.

[23] The proof is almost identical to the mentioned proof by Dunn and Slaney, but uses the Curry-sentence $\lambda \leftrightarrow (T\langle\lambda\rangle \circ (T\langle\lambda\rangle \wedge \mathbf{t}) \to \bot)$ instead of $\lambda \leftrightarrow (T\langle\lambda\rangle \circ T\langle\lambda\rangle \to \bot)$.

[24] I should emphasize that (Priest, 2015a) only considers the positive fragment of **B**, and so the status of excluded middle is not touched in the paper currently under consideration.

[25] **B** with (R5) replaced by $(A \to {\sim}B) \to (B \to {\sim}A)$ is called **DW**.

[26] Brady showed in Brady (1989) that naïve set theory, and therefore also naïve truth theory, is non-trivial in a certain logic extending the $\circ$- and **t**-free fragment of **DW** augmented by both ConSyll and excluded middle. That the construction also allows for the Ackermann constant was to my knowledge first noted in Beall (2009, pp. 121ff). That it can be strengthened to the mentioned axiomatic version is first noted here. So is the fact that Brady's construction validates the **t**-enthymematic versions of the pre- and suffixing axiom. The proofs are rather straightforward, and so I leave them to the interested reader. Brady's construction in Brady (2006, § 6.3) on the other hand, validates even the pre- and suffixing axioms, although does not allow for the Ackermann constant, not even in rule form.

[27] **B** is a sublogic of infinitely-valued Łukasiewics logic and so the naïve theory of truth is non-trivial over **B**. See Hájek et al. (2000) for details on this. Note, however, that the theory has no ω-model over this logic. Whether or not this extends to **B** is currently unknown.

[28] I'll come back to this fact in the next section.

for C. Since, however, there is no confusion $E \in C(\{A \to D, D \to C\})$ such that the fact that A suffices logically for C is validly inferable from E, there is no way of expressing this within the object language. Thus any such interpretation will also have to admit that there are fact about validity that are not inexpressible even within the theory of naïve validity which is the context in which Priest's proposal is situated.

The last mentioned option was that of abandoning CF-entailment. This might seem radical, but it need not be, although it does have a definitive downside. Note that CF-entailment is appealed to so as to explain why certain rules may be used, even though they fail to ensure valid inference. Instead of using CF-entailment for this, one could use the standard notion of H-entailment and thus allow *every* rule to be on par with regards to logical consequence for axiomatic theories. The question, then, is whether doing so would, to paraphrase Priest (2015a, p. 60), replace a piece of the jigsaw with one not as nicely fitting.

Note first of all that CF-entailment simply *is* H-entailment for stronger relevant logics—logics like **RW** and **R** which only have modus ponens and adjunction as primitive rules. CF-entailment and H-entailment come apart, however, not only for **B** , but for a range of weaker relevant logics which have more primitive rules than only these two. We have seen that Priest distinguishes between valid rules of inference and acceptable rules—rules such as modus ponens which aren't unrestrictedly valid. Priest's notion of what I've called an acceptable rule, however, is definable using the concept of validity together with that of a confusion. Opting for H-entailment as an account of acceptable rules, however, would for logics such as **B** amount to having rules acceptable for reasons unexplainable using the notion of validity. As such it would be to replace a nicely fitting piece with a piece from another jigsaw.

Going forward, then, I would suggest focusing on a standard Hilbert calculus instead of Restall's consecution calculus in investigating the notion of CF-entailment, and specifically then to focus on Smiley's distinction between rules of inference and rules of proof. If modus ponens and adjunction are the only rules of inference, then some explanation as to why axiomatic theories are not closed under rules such as (R3) and (R4)—the pre- and suffixing rules—are in order. If, on the other hand, rules such as (R3) and (R4) are instead to be viewed as rules of inference, then an account explaining their relation to validity would be welcomed: are axiomatic theories closeable under such rules by mere brute fact, or is there always a confusion of the premises which validly entails the conclusion. If the latter: is there a unified account such that for any instance of the rule, a specific confusion of the premises validly entails the conclusion in the same manner as modus ponens is valid in that the inference from $(A \wedge (A \to B)) \circ (A \wedge (A \to B))$ to B is always valid (cf. Thm. 7(2)).

Even though, in the case of **B**, CF-entailment improves upon C-entailment in validating modus ponens, the drawback from a relevant perspective is that C-entailment satisfies the consequential variable sharing property, whereas CF-entailment does not. This, as we shall see, may be remedied by replacing **B** by its **t**-free sibling $\mathbf{B}^{\circ}$. The next section proves this and argues that doing so is also motivated given some of Priest's other theoretical commitments.

7 Relevant Confused Entailment

We have seen that Priest's concept of logical consequence is quite different from that of Anderson and Belnap. Even though the variable sharing property was formulated so as to apply to logically true $\to$-statements, it is the relation of entailment itself that Anderson and Belnap sought to free from paradoxes. Thus it is important to note that their concept of conjunctive entailment—*C-entailment*—itself satisfies variable sharing. Priest's P-entailment is too much alike Hilbert-entailment to do so. As we have seen, however, Anderson and Belnap's notion of entailment is closely related to that of Priest. It turns out that it is quite easy to make P-entailment relevant—for it to satisfy the consequential variable sharing property, that is—while still allowing modus ponens to be an unrestricted rule of inference, something we have seen is not the case for Anderson and Belnap's C-entailment in the case of **B**. The trick is simply to prune away **t**.

Although Priest does subscribe to variable sharing as a correct principle, it is quite possible that he would rest content with the property pertaining to the relation of validity itself, and not also to that of CF-entailment. As we shall see, however, variable sharing extends to CF-entailment given one of Priest's other theoretical commitments, viz. upholding the naïve theory of sets as a non-trivial theory. The stricter notion of CF-entailment, then, seems to be a better fit for Priest.

There are two sensible ways of pruning away the Ackermann constant. The first is to retain **B**, but to update the definition of P-entailment:

Definition 10 (Pure Confusion) $\mathcal{P}_{\mathcal{C}}(\Sigma)$—the set of PURE confusions over a set of formulas Σ—is inductively defined as follows:

- $\Sigma \subseteq \mathcal{P}_{\mathcal{C}}(\Sigma)$
- for any $D_1 \in \mathcal{P}_{\mathcal{C}}(\Sigma)$ and $D_2 \in \mathcal{P}_{\mathcal{C}}(\Sigma)$,
  - $D_1 \wedge D_2 \in \mathcal{P}_{\mathcal{C}}(\Sigma)$
  - $D_1 \circ D_2 \in \mathcal{P}_{\mathcal{C}}(\Sigma)$

Definition 11 ($\mathbf{P}_{\mathcal{P}}$-entailment) A set of formulas Σ $\mathrm{P}_{\mathcal{P}}$-ENTAILS A in the logic **B** just in case there is a pure confusion C of Σ such that $C \rhd A$ is a provable sequent in Restall's system corresponding to **B**.

Notice first of all that $\mathrm{P}_{\mathcal{P}}$-entailment is alike C-entailment in not having any logical theorems—there simply is no **t**-free confusion of the empty set of formulas. $\mathrm{P}_{\mathcal{P}}$-entailment is also alike C-entailment with regards to the consequential variable sharing property:

Theorem 9 *$P_{\mathcal{P}}$-entailment for **B** satisfies (CVSP).*

Proof *Let $\Sigma \cup \{A\}$ be* **t**-*free. Using Restall's soundness and completeness result described in section. 3, one can then prove that Σ $P_{\mathcal{P}}$-entails A if and only if $\varnothing \vdash^{h}_{\mathbf{B}} C \to A$ for some* **t**-*free confusion C of Σ. Thus it follows from the fact that **B** satisfies the variable sharing property that $P_{\mathcal{P}}$-entailment for **B** satisfies the consequential variable sharing property.*

We saw in Thm. 7 that modus ponens holds for CF-entailment, and therefore, given Thm. 5 that it also holds for P-entailment. Note, then, that modus ponens also holds for $\mathrm{P}_{\mathcal{P}}$-entailment since $(A \wedge (A \to B)) \circ (A \wedge (A \to B) \rhd B$ is a provable sequent in Restall's system for **B**. $\mathrm{P}_{\mathcal{P}}$-entailment, however, is unlike P-entailment (but alike C-entailment) in that $\{A\}$ does not $\mathrm{P}_{\mathcal{P}}$-entail $A \wedge (B \to B)$ and neither does $\{(A \to A) \to B\}$ $\mathrm{P}_{\mathcal{P}}$-entail B.[29] Like P-entailment, however, $\mathrm{P}_{\mathcal{P}}$-entailment is a very weak consequence relations: neither it suffices for making $\to$ a transitive conditional in the sense that $\{A \to B, B \to C\}$ does not $\mathrm{P}_{\mathcal{P}}$-entail $A \to C$ (cf. Thm. 8). Whether or not this is in the end acceptable, or whether it can be improved upon somehow, remains to be investigated.

t is often introduced to increase expressivity: Ackermann (1956), for instance, introduced what amounts to $\sim$**t** in order to define modal operators, and Anderson and Belnap used propositional quantifiers to in effect define a constant akin to **t** in order to define intuitionistic implication within **E**.[30] Priest appeals in for instance Priest (2006, sect. 18.3) to **t** in order to define restricted quantification and subset-hood in naïve set theory. One can, however, desire too much of a good thing. As we have seen, **t** is the culprit which makes P-entailment irrelevant. It is also a key component in the triviality proof for naïve set theory shown forth in Øgaard (2016, appendix A) using Restall's proof system for **B**. The proof relies heavily on the presence of both $\circ$ and **t**. As already mentioned, however: naïve set theory is non-trivial in **B** without $\circ$. The other option, then, an option which seems to be a better fit with some of Priest's other theoretical commitments, is to keep the definition of $\mathrm{P}_{\mathcal{P}}$-entailment, but to abandon **t** completely and thus to replace **B** by its **t**-free version $\mathbf{B}^{\circ}$.[31]

8 Conclusion

The standard concept of logical consequence has it that logical truths follow from any set of assumptions regardless of whether the premises assumed are relevant to the consequence or not. Anderson and Belnap sought to replace this notion with one for which this is not the case. They believed that for a set of premises to entail something, that something has to be meaning-related to the premises. This meaning-relatedness was formalized as the variable sharing property. Even though the property was from its conception stated as one concerning logically true $\to$-statements, its true intent is to sort *consequence relations* into those upholding the strictures of relevance and those, like the standard Hilbertian one, that do not.

Anderson and Belnap's own concept of entailment was formalized as *C-entailment*, that $\{A_1, \ldots, A_n\}$ entails B just in case in case $A_1 \wedge \ldots \wedge A_n \to B$ is a logical truth. This is a decent notion of entailment for some logics, but for contractionless logics for which $(A \wedge (A \to B)) \to B$ is not a logical truth, it yields that modus ponens fails to be a valid rule of inference. Priest accepted this consequence in *Fusion and Confusion*, but defined a new notion of logical consequence which is to explain why certain such rules may be relied upon regardless. Whereas C-entailment takes premises to be merely conjuncted together, the new notion allows premises to be "confused" together—joined together using both extensional and intensional conjunction. Since, however, also the Ackermann constant can be weakened in, Priest's notion of logical consequence turns out to be an irrelevant one. This, it was shown, can be easily remedied by simply pruning away the Ackermann constant. Priest's notion of entailment is a highly novel one and this paper is the first in depth investigation into it. Priest's starting point is the positive fragment of the weak relevant logic **B**. It has been shown that Priest's notion of consequence—*P-entailment*—is strictly stronger than Anderson and Belnap's C-entailment, but strictly weaker than Hilbert-derivability. It was shown that although it does validate modus ponens, it fails to validate a large number of other logical laws; most disturbingly the rule called *Hypothetical Syllogism*, that $A \to C$ follows

[29] That this is so is easily verified using Slaney's MaGIC—use **B** augmented with the axiom $A \leftrightarrow A \circ A$. I leave the details as an exercise for the reader.

[30] See Anderson and Belnap (1961, § IV) as well as Anderson et al. (1992, §§ 35–36).

[31] I should emphasize, however, that whether or not naïve set theory is trivial in $\mathbf{B}^{\circ}$ is as of yet unknown. Note that Priest (2015a) is primarily a discussion of the theory of *naïve validity* which Priest discusses in the context of the naïve theory of truth. Note, then, that these theories are regarded as extending the *logic*, not as axiomatic theories. For instance, Priest notion of naïve truth theory is that it *augments* the proof system by in effect adding as logically true, every sequent $A \rhd T\langle A\rangle$ and $T\langle A\rangle \rhd A$. In terms of CF-entailment, this translates to allowing $A \leftrightarrow T\langle A\rangle$ as well as any instance of the self-reference schema as a *logical* axiom, and so every rule may be applied to it unrestrictedly. Note, then, that the mentioned triviality proof for naïve set theory over **B** relies on an unrestricted rule of extensionality—that is a primitive rule $\forall x(x \in a \leftrightarrow x \in b) \Vdash \forall y(a \in y \leftrightarrow b \in y)$ on par with, say modus ponens—as well as the abstraction schema on par with the logical axioms. The requirements of the triviality proof, then, are on line with Priest's view of the naïve theories of truth and validity. See Øgaard (2021b) for more on ways of augmenting Restall's consecution calculus.

from $A \to B$ together with $B \to C$. Opting for a stronger logic would solve this, although whether or not this can be done within the confinement of Priest's other theoretical commitments is then where the challenge lies. Some of the problems facing Priest's account have been raised in this paper, although they remain to be properly investigated.

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