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Saturating systems and rank covering radius

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Abstract

We introduce the concept of a rank saturating system and outline its correspondence to a rank-metric code with a given covering radius. We consider the problem of finding the value of $s_{q^m/q}(k,\rho)$, which is the minimum $\mathbb{F}_q$-dimension of a q-system in $\mathbb{F}_{q^m}^k$ which is rank ρ-saturating. This is equivalent to the covering problem in the rank metric. We obtain upper and lower bounds on $s_{q^m/q}(k,\rho)$ and evaluate it for certain values of k and ρ. We give constructions of rank ρ-saturating systems suggested from geometry.

Keywords. Linear sets, projective systems, saturating systems, rank-metric codes, covering radius

MSC2020. 05B40, 11T71, 51E20, 52C17, 94B75

Contents

Introduction

The relationships between linear codes and sets of points in finite geometries have long been exploited by researchers [1, 12, 15, 16, 18, 22]. Indeed, the MDS conjecture was first posed by Segre as a problem on arcs in finite geometry [34]. A generator matrix or parity check matrix of a linear code can be constructed from a set of projective points. Supports of codewords correspond to complements of hyperplanes in a fixed projective set. This connection yields a 'dictionary' between these two fields, which allows one to apply methods from one domain to the other. This approach has been taken in constructing codes with bounded *covering radius*, related to *saturating sets* in projective space.

The geometry of rank-metric codes has recently been investigated [2,32]: rank-metric codes correspond to q-systems and linear sets. In this paper, we exploit this relationship further: we introduce the notion of a rank saturating system in correspondence with a rank-metric covering code.

The covering radius of a code is the least positive integer ρ such that the union of the spheres of radius ρ about each codeword equal the full ambient space. This fundamental coding theoretical parameter has been widely studied for codes in respect of the Hamming-metric [8,13,15–18,21], but very few papers on the subject have appeared in the literature on rank-metric codes [10,24]. The covering radius is an indicator of combinatorial properties of a code, such as *maximality* and is an invariant of code equivalence. It also gives a measure of its error-correcting capabilities, via a determination of the maximal weight of a correctable error. Several other communication problems can be expressed in terms of covering problems for the Hamming-metric [13]. For the rank-metric, the covering radius of a code has connections with min-rank problems, such as those that arise in *index-coding* [9].

A set $\mathcal{S}$ of points in the projective space $\mathrm{PG}(k-1,q)$ is call ρ-*saturating* if every point of $\mathrm{PG}(k-1,q)$ lies in a projective subspace spanned by $\rho+1$ points of $\mathcal{S}$ and ρ is the least integer with this property. Such sets are in bijection with equivalence classes of *covering codes*: if a ρ-saturating set $\mathcal{S}$ is identified with the columns of a parity check matrix of a code $\mathcal{C}$, then $\mathcal{C}$ has (Hamming) covering radius $\rho+1$. This yields an interesting connection between coding theory and finite geometry. A key question in this topic concerns the minimal cardinality of a saturating set for fixed q, k, and ρ. Translated to codes, this asks what the shortest length of an $\mathbb{F}_q$-code of redundancy k and covering radius $\rho+1$ is. A related and no less important problem is to obtain bounds on this number and to give constructions of codes or saturating sets that meet these bounds. Geometric methods to these problems have been considered in [15,17,18,21,37], wherein the two main approaches involve constructions using (1) *cutting* (or *strong*) blocking sets and (2) *mixed subgeometries*.

Rank-metric codes have been a source of intense research activity over the last number of years [5,10,11,20,23,24,30,33,35]. Such codes, as considered in this paper, are $\mathbb{F}_{q^m}$-linear subspaces for which the ambient space $\mathbb{F}_{q^m}^{1\times n}$ is endowed with the rank distance function. While there exists a more general description of rank-metric codes simply as linear spaces of matrices, the restriction to the $\mathbb{F}_{q^m}$-linear subspaces of $\mathbb{F}_{q^m}^{1\times n}$ has more immediate connections to finite geometry [2,5]. In this paper, we introduce the notion of an $[n,k]_{q^m/q}$ *rank* ρ-*saturating system*. In analogy with codes for the Hamming-metric, it turns out that a rank ρ-saturating system corresponds to a linear code of rank-metric covering radius ρ. Such codes have the property that every element of the ambient space is within rank-distance at most ρ of some codeword. In our analysis, we will use the notion of an $[n,k]_{q^m/q}$ system, which is simply an n-dimensional $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^m}^k$ whose $\mathbb{F}_{q^m}$-span is the full space and is a q-analogue of a projective system. Such q-*systems* have been used already in [2] and [32] to describe geometric aspects of rank-metric codes. Then an $[n,k]_{q^m/q}$ rank ρ-saturating system is one whose associated *linear set* is a $(\rho-1)$-saturating set in $\mathrm{PG}(k-1,q^m)$.

We write $s_{q^m/q}(k,\rho)$ to denote the minimum $\mathbb{F}_q$-dimension of a rank ρ-saturating system in $\mathbb{F}_{q^m}^k$. In this paper, we show that

$$\frac{m}{\rho}(k-\rho)+\rho \leq s_{q^m/q}(k,\rho) \leq m(k-\rho)+\rho. \tag{1}$$

While the lower bound of (1) arises from a combinatorial observation, the upper bound is constructive. Furthermore, using the notion of a *linear cutting blocking set* [1], and constructions from subgeometries we obtain sharper upper bounds for specific parameters by constructing rank saturating systems.

This paper is organised as follows. In Section 1 outline some background preliminaries. In Section 2 we introduce rank saturating systems give equivalent characterizations of them. In Section 3 we give upper and lower bounds on the minimum $\mathbb{F}_q$-dimension of a rank ρ-saturating system. In almost all cases, the bounds we establish turn out to be independent of q. In Section 4 we give constructions of of rank saturating systems using two approaches: one construction arises from linear cutting blocking sets and the other uses subgeometries. In the final section, we summarize the previous results and list some cases for which $s_{q^m/q}(k,\rho)$ is completely determined.

1 Background

Throughout this paper, q will denote a fixed prime power, while m, n, k will denote positive integers such that $n \leq km$ and $k \leq n$. We will write ρ to denote a positive integer in $\{1, \ldots, \min\{k, m\}\}$. Vectors will, as a rule, be column-vectors (unless specified otherwise). We write $[n]$ to denote the set $\{1, \ldots, n\}$.

The *projective geometry* $\mathrm{PG}(k-1, q)$ with underlying vector space $\mathbb{F}_q^k$ is defined as

$$\mathrm{PG}(k-1,q) := \left(\mathbb{F}_q^k \setminus \{0\}\right)/_\sim,$$

where $\sim$ denotes the equivalence relation on the non-zero elements of $\mathbb{F}_q^k$ defined by $u \sim v$ if and only if $u = \lambda v$ for some nonzero element $\lambda \in \mathbb{F}_q$.

For integers $0 \leq k \leq n$ and a prime power q, the *Gaussian binomial coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q$$

denotes the number of k-dimensional subspaces of an n-dimensional space over $\mathbb{F}_q$.

1.1 Linear codes

Let us start with some basic definitions of coding theory. Classically applied in noisy channel communication, code elements are often called words and therefore commonly represented as row-vectors. In this paper we will mainly consider the so-called rank-metric, but we will point out some relations with the more classical Hamming one.

Definition 1.1. We define the *Hamming distance* between $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in $\mathbb{F}_{q^m}^{1\times n}$ to be the number of coordinates in which they differ, that is,

$$d_H(u,v) := |\{i \in [n] : u_i \neq v_i\}|$$

and the *rank-distance* to be the $\mathbb{F}_q$-dimension of the vector space spanned by the difference of their coordinates, that is,

$$d_{\mathrm{rk}}(u,v) := \dim\langle u_i - v_i : i \in [n]\rangle_{\mathbb{F}_q}.$$

The *Hamming-weight* and the *rank-weight* of the vector u are respectively $\mathrm{wt}_H(u) := d_H(u, 0)$ and $\mathrm{wt}_{\mathrm{rk}}(u) := d_{\mathrm{rk}}(u, 0)$.

An $[n, k, d]_{q^m}$ (*Hamming-metric*) *code* $\mathcal{C}$ is a k-dimensional $\mathbb{F}_{q^m}$-subspace of $\mathbb{F}_{q^m}^{1\times n}$ such that

$$d = d_H(\mathcal{C}) := \min\{d_H(c, c') : c, c' \in \mathcal{C}, c \neq c'\}.$$

while an $[n,k,d]_{q^m/q}$ *rank-metric code* $\mathcal{C}$ is a k-dimensional $\mathbb{F}_{q^m}$-subspace of $\mathbb{F}_{q^m}^{1\times n}$ such that

$$d = d_{\mathrm{rk}}(\mathcal{C}) := \min\{d_{\mathrm{rk}}(c,c') : c,c' \in \mathcal{C}, c \neq c'\}.$$

Whenever the minimum distance is not known, we indicate them as $[n,k]_{q^m}$ and $[n,k]_{q^m/q}$ code respectively. Both are usually described with a *generator matrix* $G \in \mathbb{F}_{q^m}^{k\times n}$, which is a matrix whose rows generate $\mathcal{C}$.

The *dual code* of $\mathcal{C}$ (independent of the metric) is defined to be:

$$\mathcal{C}^\perp := \{v \in \mathbb{F}_{q^m}^{1\times n} : v \cdot c = 0 \;\forall\, c \in \mathcal{C}\},$$

where for all $x = (x_1,\dots,x_n), y = (y_1,\dots,y_n) \in \mathbb{F}_{q^m}^{1\times n}$ we have $x \cdot y := \sum_{j=1} x_j y_j$.

Let $\mathcal{C},\mathcal{C}'$ be two $[n,k]_{q^m}$ codes with generator matrix G,G' respectively. We say that $\mathcal{C}$ and $\mathcal{C}'$ are *equivalent*, and we write $\mathcal{C} \sim \mathcal{C}'$, if there exists a monomial matrix $M \in \mathbb{F}_{q^m}^{n\times n}$ such that $G' = GM$.

Let $\mathcal{C},\mathcal{C}'$ be two $[n,k]_{q^m/q}$ rank-metric codes with generator matrix G,G' respectively. We say that $\mathcal{C}$ and $\mathcal{C}'$ are *equivalent*, and we write again $\mathcal{C} \sim \mathcal{C}'$, if there exists an invertible matrix $A \in \mathbb{F}_q^{n\times n}$ such that $G' = GA$.

We will generally require that the codes we study are *non-degenerate* in the following sense.

Definition 1.2. An $[n,k]_{q^m}$ code $\mathcal{C}$ is (Hamming-)*nondegenerate* if for every $i \in [n]$ there exists $c \in \mathcal{C}$ such that $c_i \neq 0$.

An $[n,k]_{q^m/q}$ code $\mathcal{C}$ is (rank-)*nondegenerate* if the $\mathbb{F}_q$-span of the columns of any generator matrix of $\mathcal{C}$ has $\mathbb{F}_q$-dimension n.

Note that if a code is degenerate, then it can be embedded in an ambient space of smaller dimension.

It is proved in [Proposition 3.2 in [2]] that $\mathcal{C}$ is (rank-)nondegenerate if and only if for every $A \in \mathrm{GL}_n(q)$, the code $\mathcal{C}\cdot A$ is (Hamming-)nondegenerate. Note that, as already observed in [28, Corollary 6.5], nondegenerate rank-metric $[n,k]_{q^m/q}$ code may exist only if $n \leq mk$.

Definition 1.3. An $[n,k]_{q^m}$ code $\mathcal{C}$ is *projective* if $d_H(\mathcal{C}^\perp) \geq 3$.

We define a *projectivisation* of a code $\mathcal{C}$ to be a punctured code $\mathcal{C}^*$ (that is a code obtained by deleting some coordinates) such that $d_H((\mathcal{C}^*)^\perp) \geq 3$ of maximal length.

A code is projective if and only if in any generator matrix no two columns are linearly dependent. In a projectivisation one erases the minimum number of columns to obtain a projective code. There are different ways to do that, all leading to equivalent codes. For this reason, it makes sense to talk about 'the' projectivisation.

Definition 1.4. The (Hamming) *covering radius* of a code $\mathcal{C} \leq \mathbb{F}_{q^m}^{1\times n}$ is the integer

$$\rho_H(\mathcal{C}) := \max\{\min\{d_H(x,c) : c \in \mathcal{C}\} : x \in \mathbb{F}_{q^m}^{1\times n}\}.$$

The *rank covering radius* of a code $\mathcal{C} \leq \mathbb{F}_{q^m}^{1\times n}$ is the integer

$$\rho_{\mathrm{rk}}(\mathcal{C}) := \max\{\min\{d_{\mathrm{rk}}(x,c) : c \in \mathcal{C}\} : x \in \mathbb{F}_{q^m}^{1\times n}\}.$$

Equivalently, the (rank) covering radius of a code $\mathcal{C} \leq \mathbb{F}_{q^m}^{1\times n}$ is the minimum value r such that the union of the spheres of (rank) radius r about each codeword is equal to the full ambient space $\mathbb{F}_{q^m}^{1\times n}$. Covering radii are invariant under equivalence.

We summarize some well-known results on the covering radius (c.f. [10, 13]).

Lemma 1.5. Let $\mathcal{C}, \mathcal{D} \leq \mathbb{F}_{q^m}^{1\times n}$ be a pair of rank-metric codes. The following hold.

(a) If $\mathcal{C} \subseteq \mathcal{D}$, then $\rho_{\mathrm{rk}}(\mathcal{C}) \geq \rho_{\mathrm{rk}}(\mathcal{D})$.

(b) If $\mathcal{C} \subsetneq \mathcal{D}$, then $\rho_{\mathrm{rk}}(\mathcal{C}) \geq d_{\mathrm{rk}}(\mathcal{D})$.

(c) If $|\mathcal{C}| \geq 2$ and $\mathcal{C} \subsetneq \mathbb{F}_{q^m}^{1\times k}$, then $d_{\mathrm{rk}}(\mathcal{C}) - 1 < 2\rho_{\mathrm{rk}}(\mathcal{C})$.

An $[n, k, d]_{q^m/q}$ code is called *maximal* if it is not contained in any (possibly non-linear) code $\mathcal{D} \subseteq \mathbb{F}_{q^m}^{1\times n}$ such that $d_{rk}(\mathcal{D}) = d$. Clearly a cardinality-optimal code is also maximal.

Lemma 1.6 (The Supercode Lemma, [13])**.** Let $\mathcal{C}$ be an $[n, k, d]_{q^m/q}$ code with $|\mathcal{C}| \geq 2$. Then $\mathcal{C}$ is maximal if and only if $\rho_{\mathrm{rk}}(\mathcal{C}) \leq d - 1$.

Example 1.7. Let $\alpha = (\alpha_i : i \in [n]) \in \mathbb{F}_{q^m}^{1\times n}$ have rank-weight n over $\mathbb{F}_q$. An $[n, k, n-k+1]_{q^m/q}$ code with generator matrix

$$G_{n,k,i} = \left(\alpha_j^{q^{i(t-1)}}\right)_{t\in[k],j\in[n]}$$

is called as a *generalised Gabidulin code*. We denote it by $\mathcal{G}_{n,k,i}(\alpha)$. Its dual code is the generalised Gabidulin code $\mathcal{G}_{n,n-k,i}(\alpha)$. Such codes meet the rank-metric Singleton bound and are hence maximal, being optimal. Therefore, from the Supercode Lemma, we have $\rho_{\mathrm{rk}}(\mathcal{G}_{n,k,i}(\alpha)) \leq n - k$. On the other hand, $\mathcal{G}_{n,k,i}(\alpha) \lneq \mathcal{G}_{n,k+1,i}(\alpha)$ and so from Lemma 1.5, we have $\rho_{\mathrm{rk}}(\mathcal{G}_{n,k,i}(\alpha)) \geq d_{\mathrm{rk}}(\mathcal{G}_{n,k+1,i}(\alpha)) = n - k$. It follows that $\rho_{\mathrm{rk}}(\mathcal{G}_{n,k,i}(\alpha)) = n - k$.

We recall the Dual Distance and External Distance Bounds for rank-metric codes [10, 19], which we state for $\mathbb{F}_{q^m}$-linear rank-metric codes.

Theorem 1.8 (External Distance Bound, [10])**.** Let $\mathcal{C}$ be a $[n, k]_{q^m/q}$ rank-metric code and let

$$s(\mathcal{C}) := |\{\mathrm{wt}_{\mathrm{rk}}(c) : c \in \mathcal{C},\, c \neq 0\}|.$$

Then $\rho_{\mathrm{rk}}(\mathcal{C}^{\perp}) \leq s(\mathcal{C})$.

Theorem 1.9 (Dual Distance Bound, [10])**.** Let $\mathcal{C}$ be a $[n, k]_{q^m/q}$ rank-metric code,

$$\rho_{\mathrm{rk}}(\mathcal{C}^{\perp}) \leq m - d_{rk}(\mathcal{C}) + 1.$$

1.2 q-Systems and linear sets

There is a classical way to associate a set of points in $\mathcal{P} \subseteq \mathrm{PG}(k-1, q^m)$ to a projective code in the Hamming metric. The idea is simply to take representatives in $\mathbb{F}_{q^m}^k$ of the points of $\mathcal{P}$ and put them as columns of a $k \times |\mathcal{P}|$ generator matrix G over $\mathbb{F}_{q^m}$ of a code. As in the rank-metric case, such codes depend on the ordering of the points and on their chosen vector representatives, but different choices yield equivalent codes. We will call any code in this equivalence class a projective code *associated* to $\mathcal{P}$ and we will denote it by $\mathcal{C}_{\mathcal{P}}$. The same can be done for multisets of points, in which case we arrive at non-projective codes, but we will not consider these in this work. This geometric vision of codes leads to many interesting connections between objects in finite geometry and properties of linear codes. In particular, the Hamming metric can be read from this set of points: for any $u \in \mathbb{F}_{q^m}^{1\times k}$,

$$\mathrm{wt}_H(uG) = n - |\mathcal{P} \cap \langle u\rangle^{\perp}| \tag{2}$$

In the rank-metric, there is analogous vision, which associates q-systems to codes. We will now introduce these objects.

Definition 1.10. An $[n,k]_{q^m/q}$ *system* is an n-dimensional $\mathbb{F}_q$-space $\mathcal{U} \subseteq \mathbb{F}_{q^m}^k$ such that $\langle \mathcal{U} \rangle_{\mathbb{F}_{q^m}} = \mathbb{F}_{q^m}^k$. A *generator matrix* for $\mathcal{U}$ is a $k \times n$ matrix over $\mathbb{F}_{q^m}$ whose columns form an $\mathbb{F}_q$-basis for $\mathcal{U}$. Two $[n,k]_{q^m/q}$ systems $\mathcal{U}$ and $\mathcal{V}$ are called *equivalent* if there exists ϕ in $\mathrm{GL}_k(\mathbb{F}_{q^m})$ such that $\phi(\mathcal{U}) = \mathcal{V}$.

A standard way to obtain $[n,k]_{q^m/q}$ systems is to associate them to nondegenerate rank-metric codes. So, given a nondegenerate rank-metric code $\mathcal{C}$, we may associate to it a system $\mathcal{U}$ by taking a generator matrix of $\mathcal{C}$ and defining $\mathcal{U}$ to be the $\mathbb{F}_q$-span of its columns. This clearly depends on the choice of the matrix, but if we change the generator matrix we obtain an equivalent system. We will call any system $\mathcal{U}$ in this equivalence class a system *associated* to $\mathcal{C}$. For a more detailed description of this correspondence, which involves also the rank-metric, the reader is referred to [2, §3]. We just point out one important result which is the q-analogue of (2): for any $u \in \mathbb{F}_{q^m}^{1\times k}$,

$$\mathrm{wt}_{\mathrm{rk}}(uG) = n - \dim_{\mathbb{F}_q}(\mathcal{U} \cap \langle u \rangle^{\perp})$$

In this paper we will show new connections between rank-metric codes, viewed as covering codes, and q-systems.

In order to do this, we will need the definition of linear sets. Such objects were introduced by Lunardon in [29] in order to construct blocking sets and they are subject of intense research over the last years. An in-depth treatment of linear sets can be found in [31].

Definition 1.11. Let $\mathcal{U}$ be an $[n,k]_{q^m/q}$ system. The $\mathbb{F}_q$-*linear set* in $\mathrm{PG}(k-1,q^m)$ of rank n associated to $\mathcal{U}$ is the set

$$L_{\mathcal{U}} := \{\langle u \rangle_{\mathbb{F}_{q^m}} : u \in \mathcal{U} \setminus \{0\}\},$$

where $\langle u \rangle_{\mathbb{F}_{q^m}}$ denotes the projective point corresponding to u.

Remark 1.12. The original definition of linear sets does not assume that $\langle \mathcal{U} \rangle_{\mathbb{F}_{q^m}} = \mathbb{F}_{q^m}^k$. However, if $\dim_{\mathbb{F}_{q^m}}(\langle \mathcal{U} \rangle_{\mathbb{F}_{q^m}}) = h < k$, then, up to equivalence, we may assume $\mathcal{U} \subseteq \mathbb{F}_{q^m}^h$ with $\langle \mathcal{U} \rangle_{\mathbb{F}_{q^m}} = \mathbb{F}_{q^m}^h$, and then study $L_{\mathcal{U}}$ in $\mathrm{PG}(h-1,q^m)$.

Let $0 \neq v \in \mathbb{F}_{q^m}^k$ and $P \in \mathrm{PG}(k-1,\mathbb{F}_{q^m})$ be the projective point associated to v. We define the *weight* of P in $L_{\mathcal{U}}$ as the integer

$$\mathrm{wt}_{\mathcal{U}}(P) := \dim_{\mathbb{F}_q}(\mathcal{U} \cap \langle v \rangle_{\mathbb{F}_{q^m}}).$$

Definition 1.13. A linear set $L_{\mathcal{U}}$ is *scattered* if $\mathrm{wt}_{\mathcal{U}}(P) = 1$ for each $P \in L_{\mathcal{U}}$.

Any linear set $L_{\mathcal{U}}$ of rank n satisfies

$$|L_{\mathcal{U}}| \leq \frac{q^n-1}{q-1}. \tag{3}$$

Clearly a linear set $L_{\mathcal{U}}$ is scattered if and only if equality holds in (3).

As above, a linear set $L_{\mathcal{U}}$ being a set of set of points in $\mathrm{PG}(k-1,q^m)$, it can be associated to an $[|L_{\mathcal{U}}|,k]_{q^m}$ code $\mathcal{C}_{L_{\mathcal{U}}}$. We obtain the diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}_{L_{\mathcal{U}}} \\ \downarrow & & \uparrow \\ \mathcal{U} & \longrightarrow & L_{\mathcal{U}} \end{array} \tag{4}$$

which allows one to associate a (projective) Hamming code to a nondegenerate rank-metric one.

Definition 1.14. Let $\mathcal{C}$ be a $[n,k]_{q^m/q}$ rank-metric code. We call the code $\mathcal{C}_{L_{\mathcal{U}}}$ obtained as in (4) the *projective Hamming code associated to* $\mathcal{C}$.

Remark 1.15. We highlight the fact that the projective Hamming code associated to a rank-metric code defined above is not, in general, the associated Hamming-metric code described in [2, §4.2]. The two definitions coincide if and only if the underlying linear set is *scattered* (see for example [2, §4.1]). If the linear set is scattered, the Hamming code associated to an $[n,k]_{q^m/q}$ code has length $(q^n-1)/(q-1)$. Otherwise, it is shorter (it is its projectivisation). Let us remark that Hamming codes associated to scattered linear sets have been already considered in [6] and [38].

2 Rank saturating systems

In this section we will introduce the main object of the paper. We will study its properties and relations with covering codes in the rank-metric.

Let us start with the notion of a saturating set.

Definition 2.1. Let $\mathcal{S} \subseteq \mathrm{PG}(k-1,q^m)$.

(a) A point $Q \in \mathrm{PG}(k-1,q^m)$ is said to be *ρ-saturated* by $\mathcal{S}$ if there exist $\rho+1$ points $P_1,\ldots,P_{\rho+1} \in \mathcal{S}$ such that $Q \in \langle P_1,\ldots,P_{\rho+1}\rangle_{\mathbb{F}_{q^m}}$. We also say that $\mathcal{S}$ *ρ-saturates* Q.

(b) The set $\mathcal{S}$ is called *ρ-saturating* set of $\mathrm{PG}(k-1,q^m)$ if every point $Q \in \mathrm{PG}(k-1,q^m)$ is ρ-saturated by $\mathcal{S}$ and ρ is the smallest value with this property.

It is well-known (see, for example [27, Theorem 11.1.2]) that an $[n,n-k]_{q^m}$ code has (Hamming) covering radius ρ if every element of $\mathbb{F}_{q^m}^k$ is a linear combination of ρ columns of a generator matrix of the dual code and ρ is the smallest value with such a property. The correspondence between projective systems and linear codes leads to a correspondence between $(\rho-1)$-saturating sets of size n in $\mathrm{PG}(k-1,q^m)$ and the duals of $[n,n-k]_{q^m}$ codes of covering radius ρ. In defining the q-analogue of such saturating sets, we arrive at a q-analogue of the correspondence. We give this as follows.

Definition 2.2. An $[n,k]_{q^m/q}$ system $\mathcal{U}$ is *rank ρ-saturating* if $L_{\mathcal{U}}$ is a $(\rho-1)$-saturating set in $\mathrm{PG}(k-1,q^m)$. We call such a linear set a *linear* $(\rho-1)$*-saturating set*.

Such definition is clearly invariant under equivalence of q-systems. The following result offers a characterization of rank saturating systems which we will use extensively in the remainder of this paper.

Theorem 2.3. Let $\mathcal{U}$ be an $[n,k]_{q^m/q}$ system and $\{u_1,\ldots,u_n\}$ an $\mathbb{F}_q$-basis of it. The following are equivalent:

(a) $\mathcal{U}$ is rank ρ-saturating;

(b) for each vector $v \in \mathbb{F}_{q^m}^k$ there exists $\lambda \in \mathbb{F}_{q^m}^{1\times n}$ with $\mathrm{wt}_{rk}(\lambda) \leq \rho$ such that

$$v = \lambda_1 u_1 + \ldots + \lambda_n u_n,$$

and ρ is the smallest value with this property;

(c)

$$\mathbb{F}_{q^m}^k = \bigcup_{\mathcal{S}\colon \mathcal{S} \leq_{\mathbb{F}_q} \mathcal{U}\colon \dim_{\mathbb{F}_q} \mathcal{S} \leq \rho} \langle \mathcal{S}\rangle_{\mathbb{F}_{q^m}}$$

and ρ is the smallest integer with this property.

Proof. (a) $\Rightarrow$ (b) Let $0 \neq v \in \mathbb{F}_{q^m}^k$ and $Q = \langle v \rangle \in \mathrm{PG}(k-1, q^m)$. Since $\mathcal{U}$ is rank ρ-saturating, there exists ρ points $P_1 = \langle w_1 \rangle, \dots, P_\rho = \langle w_\rho \rangle$ such that

$$v = \gamma_1 w_1 + \dots \gamma_\rho w_\rho$$

with $\gamma_i \in \mathbb{F}_{q^m}$. Now $w_1, \dots, w_\rho$ are in $L_\mathcal{U}$, so that, if $u_1, \dots, u_n$ is an $\mathbb{F}_q$-basis of $\mathcal{U}$, we have

$$v = \gamma_1(\mu_{1,1}u_1 + \dots + \mu_{1,n}u_n) + \dots + \gamma_\rho(\mu_{\rho,1}u_1 + \dots + \mu_{\rho,n}u_n)$$

with $\mu_{i,j} \in \mathbb{F}_q$. We can reorder the terms so that

$$v = \underbrace{(\gamma_1\mu_{1,1} + \dots + \gamma_\rho\mu_{\rho,1})}_{\lambda_1} u_1 + \dots + \underbrace{(\gamma_1\mu_{1,n} + \dots + \gamma_\rho\mu_{\rho,n})}_{\lambda_n} u_n.$$

Now, call $\gamma = (\gamma_1, \dots, \gamma_\rho) \in \mathbb{F}_{q^m}^{1\times\rho}$, $M = (\mu_{i,j}) \in \mathbb{F}_q^{\rho\times n}$, and $\lambda = \mathbb{F}_{q^m}^{1\times n}$. We have

$$\lambda = \gamma M,$$

so that $\mathrm{wt}_{\mathrm{rk}}(\lambda) \leq \rho$.
(b) $\Rightarrow$ (c) Take $v \in \mathbb{F}_{q^m}^k$. This is a linear combination

$$v = \lambda_1 u_1 + \dots + \lambda_n u_n \tag{5}$$

with $\dim_{\mathbb{F}_q} \langle \lambda_1, \dots, \lambda_n \rangle_{\mathbb{F}_q} \leq \rho$. Let $\mathcal{S} = \langle \lambda_1, \dots, \lambda_n \rangle_{\mathbb{F}_q}$. By (5), $v \in \langle \mathcal{S} \rangle_{\mathbb{F}_{q^m}}$.
(c) $\Rightarrow$ (a) Take $Q = \langle v \rangle \in \mathrm{PG}(k-1, q^m)$. There exists $\mathcal{S}$, an $\mathbb{F}_q$-subspace of $\mathcal{U}$ with $\dim_{\mathbb{F}_q} \mathcal{S} \leq \rho$, such that $v \in \langle \mathcal{S} \rangle_{\mathbb{F}_{q^m}}$. Let $\{w_1, \dots, w_\rho\}$ be a set containing a basis of $\mathcal{S}$ over $\mathbb{F}_q$ and let $P_1, \dots, P_\rho$ be their corresponding projective points, so that $\langle w_i \rangle_{\mathbb{F}_{q^m}}$. These clearly belong to $L_\mathcal{U}$. Since $v \in \langle \mathcal{S} \rangle_{\mathbb{F}_{q^m}}$, $Q \in \langle P_1, \dots, P_\rho \rangle_{\mathbb{F}_{q^m}}$. □

Remark 2.4. Note that (b) does not depend on the choice of the $\mathbb{F}_q$-basis of $\mathcal{U}$. Indeed, consider two $\mathbb{F}_q$-bases $\mathcal{B} = \{u_1, \dots, u_n\}$ and $\mathcal{B}' = \{u_1', \dots, u_n'\}$ of $\mathcal{U}$. Then, for each $i \in \{1, \dots, n\}$ we have that $u_i = \sum_{j=1}^n a_j u_j'$ with $a_j \in \mathbb{F}_q$ for $j \in \{1, \dots, n\}$. For this reason, we have that

$$\sum_{i=1}^n \lambda_i u_i = \sum_{i=1}^n \lambda_i \left(\sum_{j=1}^n a_j u_j' \right) = \sum_{j=1}^n \left(\sum_{i=1}^n \lambda_i a_j \right) u_j' = \sum_{i=1}^n \lambda_i' u_i'.$$

that implies $\mathrm{wt}_{\mathrm{rk}}(\lambda) = \mathrm{wt}_{\mathrm{rk}}(\lambda')$.

The following theorem shows that also in the rank-metric case there is a correspondence between saturating systems and (rank) covering codes. In this sense, it is a further confirmation of the benefit of the geometric point of view on rank-metric codes.

Theorem 2.5. Let $\mathcal{U}$ be an $[n,k]_{q^m/q}$ system associated to a code $\mathcal{C}$. The following are equivalent:

(a) $\mathcal{U}$ is rank ρ-saturating;

(b) $\rho_{\mathrm{rk}}(\mathcal{C}^\perp) = \rho$.

Proof. (a) $\Rightarrow$ (b) Let $w \in \mathbb{F}_{q^m}^{1\times n}$, G be a generator matrix for $\mathcal{C}$ and $v = Gw^T \in \mathbb{F}_{q^m}^k$. Since $\mathcal{U}$ is rank ρ-saturating, by condition (b) of Theorem 2.3, there exists $\lambda \in \mathbb{F}_{q^m}^{1\times n}$ with $\mathrm{wt}_{\mathrm{rk}}(\lambda) \leq \rho$ such that $v = G\lambda^T$. Then $G(w^T - \lambda^T)$, so that $w - \lambda \in \mathcal{C}^\perp$. Since ρ is the least integer with this property, we may conclude that $\rho_{\mathrm{rk}}(\mathcal{C}^\perp) = \rho$.
(b) $\Rightarrow$ (a) Let $v \in \mathbb{F}_{q^m}^k$ and G be a generator matrix for $\mathcal{C}$. Let z any vector in $\mathbb{F}_{q^m}^{1\times n}$ such that $v = Gz^T$. By the definition of rank covering radius, there exists $w \in \mathcal{C}^\perp$ (i.e. such that $Gw^T = 0$) such that $\mathrm{wt}_{\mathrm{rk}}(z - w) \leq \rho$. Call $\lambda = z - w$. We have $v = Gz^T = G(z^T - w^T) = G\lambda^T$. Since ρ is the least integer with this property, we may conclude that $\mathcal{U}$ is rank ρ-saturating. □

Corollary 2.6. Let $\mathcal{C}$ be an $[n,k]_{q^m/q}$ rank-metric code and $\mathcal{U}$ the $[n,k]_{q^m/q}$ system associated to it. Then

$$\rho_{\mathrm{rk}}(\mathcal{C}^\perp) = \rho_H((\mathcal{C}_{L_\mathcal{U}})^\perp),$$

where $\mathcal{C}_{L_\mathcal{U}}$ is the projective Hamming code associated to $\mathcal{C}$.

Proof. This follows immediately by Theorem 2.5 and by the definition of rank ρ-saturating system. □

Corollary 2.7. Let $\mathcal{U}$ be a rank ρ-saturating $[n,k]_{q^m/q}$ system associated to a code $\mathcal{C}$. Then

$$\rho \leq s(\mathcal{C}) \quad \text{and} \quad \rho \leq m - d_{rk}(\mathcal{C}) + 1,$$

where $s(\mathcal{C}) := |\{\mathrm{wt}_{\mathrm{rk}}(c) : c \in \mathcal{C},\, c \neq 0\}|$.

Proof. These are direct consequences of the External Distance Bound (Theorem 1.8) and of the Dual Distance Bound (Theorem 1.9). □

Corollary 2.8. Let $\mathcal{C}$ be an $[n,k]_{q^m/q}$ generalized Gabidulin code and let $\mathcal{U}$ be the $[n,k]_{q^m/q}$ system associated to $\mathcal{C}$. Then $\mathcal{U}$ is rank k-saturating.

Proof. The statement follows immediately from the fact that $\rho_{\mathrm{rk}}(\mathcal{C}^\perp) = k$. □

3 Bounds on the dimension of rank saturating systems

The classical covering problem, as presented for example in [13], is as follows: given n and ρ, what is the smallest number of spheres (with respect to the metric considered) of radius ρ that can be placed in such a way that every vector in a vector space of dimension n is contained in their union, i.e. such that the union of the spheres of radius ρ *covers* this n-dimensional space? In the framework of linear codes, this is equivalent to asking how large the rate of a code (that is the ratio between the dimension of the code and n) must be in order to obtain a covering of the ambient space by spheres centred at codewords. In terms of $[n,k]_{q^m/q}$ rank ρ-saturating systems, by Theorem 2.5 this translates into asking how small can n be with respect to k, ρ being fixed.

Definition 3.1. Let $s_{q^m/q}(k,\rho)$ denote the minimal $\mathbb{F}_q$-dimension of a rank ρ-saturating system in $\mathbb{F}_{q^m}^k$. Whenever we prove a result on $s_{q^m/q}(k,\rho)$ that does not depend on q (which is everywhere except in Remark 4.12) we will use the simplified notation $s_m(k,\rho)$.

The rest of this paper is devoted to obtaining bounds on this quantity: we will first give a lower bound and then provide upper bounds arising from explicit constructions of rank ρ-saturating systems.

We will use the following result, shown in [24, Corollary 2.3].

Lemma 3.2 ([24])**.** Let a, b be positive integers, with $b \leq a$. Then

$$\begin{bmatrix} a \\ b \end{bmatrix}_q < \frac{q^{b(a-b)}}{(1/q)_\infty},$$

where $(1/q)_\infty := \prod_{i=1}^{\infty}(1 - q^{-i})$.

The following result has been obtained with a slightly different approach in [25, Proposition 14][1].

[1] We note a small typographical error in [25, Proposition 14].

Theorem 3.3. Let $\mathcal{U}$ be a rank ρ-saturating $[n,k]_{q^m/q}$ system. Then

$$\begin{bmatrix} n \\ \rho \end{bmatrix}_q \geq q^{m(k-\rho)}.$$

In particular

$$n \geq \frac{m}{\rho}(k-\rho)+\rho.$$

Proof. Let us consider the set Π_ρ of all $\mathbb{F}_{q^m}$-subspaces spanned by ρ $\mathbb{F}_q$-linearly independent elements of $\mathcal{U}$; since the $\mathbb{F}_q$-dimension of these subspaces is ρ, the rank of their coefficients is at most ρ. As $\mathcal{U}$ saturates $\mathbb{F}_{q^m}^k$, from Theorem 2.5, we know that Π_ρ must cover the latter, i.e. that $\mathbb{F}_{q^m}^k = \bigcup_{V\in\Pi_\rho} V$. Therefore,

$$\begin{bmatrix} n \\ \rho \end{bmatrix}_q \cdot q^{m\rho} \geq q^{mk}.$$

From Lemma 3.2

$$\begin{bmatrix} a \\ b \end{bmatrix}_q < (1/q)_\infty^{-1} \cdot q^{b(a-b)}, \text{for } a,b \in \mathbb{N}.$$

So

$$(1/q)_\infty^{-1} \cdot q^{\rho(n-\rho)} \cdot q^{m\rho} > q^{mk}$$

Since $(1/q)_\infty^{-1} \leq q$ for all $q > 2$, we obtain the result in this case. Suppose now that $q = 2$. In the case $\rho > 1$, the result follows since $f(2) \leq 2^2$, while the case $\rho = 1$ is trivial. □

By Theorem 3.3, we obtain an immediate lower bound:

$$s_m(k,\rho) \geq \frac{m}{\rho}(k-\rho)+\rho. \tag{6}$$

Note that in the case $\rho = 1$, the bound of (6) is attained:

$$s_m(k,1) = m(k-1)+1.$$

To see this, take a vector $v \in \mathbb{F}_{q^m}^k$, $v \neq 0$ and $v' \notin \langle v\rangle_{\mathbb{F}_{q^m}}^\perp$. Consider

$$\mathcal{U} = \langle v'\rangle_{\mathbb{F}_q} + \langle v\rangle_{\mathbb{F}_{q^m}}^\perp.$$

This is an $[m(k-1)+1,k]_{q^m/q}$ system and it is clearly a rank 1-saturating system. In $\mathrm{PG}(k-1,q^m)$, up to equivalence, it corresponds to an $\mathbb{F}_q$-cone whose base is an $\mathbb{F}_{q^m}$-hyperplane and whose vertex is a point external to the hyperplane. Let $\mathcal{C}$ be the code whose generator matrix has the elements of an $\mathbb{F}_q$-basis of the system $\mathcal{U}$ as its columns. The dual code $C^\perp$ is an $[m(k-1)+1, m(k-1)+1-k]_{q^m/q}$ with rank covering radius 1 and it is the shortest code with this property for this dimension and m.

We now will obtain upper bounds on $s_m(k,\rho)$. To start with, we give a generalization of the previous construction. In $\mathrm{PG}(k-1,q^m)$, up to equivalence, this corresponds to all $\mathbb{F}_q$-lines connecting the points of an $\mathbb{F}_q$-subspace of dimension $\rho-1$ and of an $\mathbb{F}_{q^m}$-subspace of dimension $k-\rho-1$, non-intersecting.

Theorem 3.4. Let $\mathbb{F}_{q^m} = \mathbb{F}_q[\alpha]$. The $[m(k-\rho)+\rho,k]_{q^m/m}$ system $\mathcal{U}$ whose generator matrix is

$$G := \left[\begin{array}{c|c|c|c|c} I_\rho & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline \mathbf{0} & I_{k-\rho} & \alpha I_{k-\rho} & \cdots & \alpha^{m-1} I_{k-\rho} \end{array}\right]$$

is rank ρ-saturating. In particular, we have that

$$s_m(k,\rho) \leq m(k-\rho)+\rho.$$

Proof. The system $\mathcal{U}$ is defined as follows:

$$\mathcal{U} = \left\{ \left(\frac{u}{\omega} \right) : u \in \mathbb{F}_q^{\rho}, \omega \in \mathbb{F}_{q^m}^{k-\rho} \right\}.$$

Let $v \in \mathbb{F}_{q^m}^k$ and suppose that $v_{i_1}, \ldots, v_{i_r} \neq 0$, for some $i_j \in [\rho]$. Then v can be expressed as:

$$v = v_{i_1} \begin{pmatrix} e_{i_1} \\ \hline \frac{\omega_1}{v_{i_1}} \\ \vdots \\ \frac{\omega_{k-\rho}}{v_{i_1}} \end{pmatrix} + v_{i_2} \begin{pmatrix} e_{i_2} \\ \hline \frac{\omega_1}{v_{i_2}} \\ \vdots \\ \frac{\omega_{k-\rho}}{v_{i_2}} \end{pmatrix} + \ldots + v_{i_r} \begin{pmatrix} e_{i_r} \\ \hline \frac{\omega_1}{v_{i_r}} \\ \vdots \\ \frac{\omega_{k-\rho}}{v_{i_r}} \end{pmatrix},$$

where $e_1, \ldots, e_\rho$ is the standard basis of $\mathbb{F}_q^\rho$, and each of these $r \leq \rho$ vectors belongs to $\mathcal{U}$.

Any vector whose first ρ coordinates are non-zero requires exactly ρ vectors in $\mathcal{U}$ and hence the system is rank ρ-saturating. □

Clearly, we have equality between the lower and the upper bound for $\rho = 1$ and for $\rho = k$. So the bound of (6) is attained. We now study some properties of the function $s_m(k, \rho)$.

Lemma 3.5. Let $\mathcal{U}$ be a rank ρ-saturating $[n, k]_{q^m/q}$ system. Suppose that $\mathcal{U}$ has an $\mathbb{F}_q$-basis $\{u_1, \ldots, u_n\} \subseteq \mathbb{F}_{q^m}^k$ with the property that

$$u_n = \lambda \sum_{j=1}^{n-1} l_{\rho+1,j} u_j$$

for some $l_{\rho+1,j} \in \mathbb{F}_q, 1 \leq j \leq n-1$ and $\lambda \in \mathbb{F}_{q^m}$. Then $\mathcal{U}' = \langle u_1, \ldots, u_{n-1} \rangle_{\mathbb{F}_q}$ is a rank ρ'-saturating $[n-1, k]_{q^m/q}$ system satisfying $\rho' \leq \rho + 1$.

Proof. For any vector $v \in \mathbb{F}_{q^m}^k$,

$$v = \sum_{i=1}^{\rho} \lambda_i \sum_{j=1}^{n} l_{i,j} u_j$$

for some $\lambda_i \in \mathbb{F}_{q^m}$ and $l_{i,j} \in \mathbb{F}_q$. Therefore,

$$\begin{aligned} v &= \sum_{i=1}^{\rho} \lambda_i \sum_{j=1}^{n-1} l_{i,j} u_j + \sum_{i=1}^{\rho} \lambda_i l_{i,n} u_n \\ &= \sum_{i=1}^{\rho} \lambda_i \sum_{j=1}^{n-1} l_{i,j} u_j + \sum_{i=1}^{\rho} \lambda_i l_{i,n} \lambda \sum_{j=1}^{n-1} l_{\rho+1,j} u_j \\ &= \sum_{i=1}^{\rho+1} \lambda_i \sum_{j=1}^{n-1} l_{i,j} u_j, \end{aligned}$$

where $\lambda_{\rho+1} = \sum_{i=1}^{\rho} \lambda_i l_{i,n} \lambda \in \mathbb{F}_{q^m}$. □

Remark 3.6. The hypothesis of Lemma is verified if and only if $L_{\mathcal{U}}$ is not scattered.

Using similar arguments as in the classical Hamming case (see [27, §11.5]), we have the following results.

Theorem 3.7 (Monotonicity)**.** The following hold:

(a) If $\rho < \min\{k, m\}$, then $s_m(k, \rho + 1) \leq s_m(k, \rho)$.

(b) $s_m(k,\rho) \leq s_m(k+1,\rho) - 1$.

(c) If $\rho < m$, then $s_m(k+1,\rho+1) \leq s_m(k,\rho) + 1$.

Proof. (a) Let $n > k$ and let $n = s_m(k,\rho)$. Let $G \in \mathbb{F}_{q^m}^{k\times n}$ be a generator matrix associated to a rank ρ-saturating $[n,k]_{q^m/q}$ system $\mathcal{U}$. We may assume that $G = [I_k|u_{k+1}\ldots,u_{n-1},y]$ for some $y, u_i \in \mathcal{U}$. Assume further, that over all such choices of $\mathcal{U}$ and G, that y has minimal rank weight.

If $\mathrm{wt}_{\mathrm{rk}}(y^T) = 1$ then $\mathcal{U}$ satisfies the hypothesis of Lemma 3.6 and so there exists a $(\rho+1)$-rank saturating system of length $n-1$.

We thus assume that $\mathrm{wt}_{\mathrm{rk}}(y^T) \geq 2$. Choose some coordinate value $\theta \in \mathbb{F}_{q^m}\backslash\mathbb{F}_q$ of y that is not contained in the $\mathbb{F}_q$-span of $\{y_j : j \in [n], y_j \neq \theta\}$, let $J = \{j \in [n] : y_j = \theta\}$ and define $y' := y - \theta\sum_{j\in J} e_j$. Consider the matrix $G' = [I_k|u_{k+1}\ldots,u_{n-1},y']$ and the rank ρ'-saturating system $\mathcal{U}'$ spanned by its columns (they are not necessarily linearly independent over $\mathbb{F}_q$). Let $w \in \mathbb{F}_{q^m}^k$. There exists $z \in \mathbb{F}_{q^m}^n$ of rank at most ρ such that $w = Gz$. Therefore,

$$\begin{aligned}
w &= Gz \\
&= \sum_{i=1}^{k} z_i e_i + \sum_{i=k+1}^{n-1} z_i u_i + z_n(y' + \theta\sum_{j\in J} e_j) \\
&= \sum_{i\in[k]\backslash J} z_i e_i + \sum_{j\in J}(z_j + \theta z_n)e_j + \sum_{i=k+1}^{n-1} z_i u_i + z_n y' \\
&= G'z',
\end{aligned}$$

where $z' = z + \theta z_n \sum_{j\in J} e_j$. Clearly, $\mathrm{wt}_{\mathrm{rk}}((z')^T) \leq \mathrm{wt}_{\mathrm{rk}}(z^T) + 1 \leq \rho + 1$ and so $\rho' \leq \rho + 1$. If $\rho' = \rho + 1$, then we have $s_m(k,\rho+1) \leq n = s_m(k,\rho)$ and hence the statement of the theorem will follow.

If $y' \in \mathbb{F}_q^k$, then $\mathcal{U}'$ is an $[n-1,k]_{q^m/q}$ system and so $\rho' = \rho + 1$.

Suppose then that $y' \notin \mathbb{F}_q^k$. Since $\mathrm{wt}_{\mathrm{rk}}((y')^T) < \mathrm{wt}_{\mathrm{rk}}(y^T)$, by our choice of $\mathcal{U}$ and G, it must be the case that $\rho' \neq \rho$. Suppose now that $\rho' \leq \rho - 1$. If $\mathrm{wt}_{\mathrm{rk}}((y')^T) = 1$ then $\mathcal{U}'$ satisfies the hypothesis of Lemma 3.6 and so there exists a rank ρ''-saturating $[n-1,k]_{q^m/q}$ system $\mathcal{U}''$ with $\rho'' \leq \rho$, yielding a contradiction to the fact that $n = s_m(k,\rho)$. We hence assume that $\mathrm{wt}_{\mathrm{rk}}((y')^T) \geq 2$. Apply a similar argument as before to produce a matrix $G' = [I_k|u_{k+1}\ldots,u_{n-1},y'']$ with associated rank ρ''-saturating system $\mathcal{U}''$ satisfying $\rho'' \leq \rho' + 1 \leq \rho$ and $\mathrm{wt}_{\mathrm{rk}}((y'')^T) < \mathrm{wt}_{\mathrm{rk}}(y^T)$. Again, by our choice of G and $\mathcal{U}$, it must be the case that $\rho'' \leq \rho - 1$. Continue, iterating the same argument to produce a sequence of generator matrices $G^{(i)} = [I_k|u_{k+1},\ldots,u_{n-1},y^{(i)}]$ and associated $[n-1,k]_{q^m/q}$ rank $\rho^{(i)}$-saturating systems $\mathcal{U}^{(i)}$ with $\mathrm{wt}_{\mathrm{rk}}((y^{(i)})^T) < \mathrm{wt}_{\mathrm{rk}}((y^{(i-1)})^T)$ at each step. This sequence will terminate at some r for which $\mathrm{wt}_{\mathrm{rk}}((y^{(r)})^T) = 1$, in which case we may apply Lemma 3.6 to arrive at a contradiction. We deduce that $\rho' = \rho + 1$ and so the result follows.

(b) Let $n > k$ and let $n = s_m(k,\rho)$. Let $G = [\,I_{k+1}|\,A\,] \in \mathbb{F}_{q^m}^{(k+1)\times n}$ be a generator matrix of a rank ρ-saturating $[n,k+1]_{q^m/q}$ system $\mathcal{U}$. Consider the matrix $G' = [\,I_k|\,A'\,] \in \mathbb{F}_{q^m}^{k\times(n-1)}$ found by deleting the first column and row of G. Let $w' \in \mathbb{F}_{q^m}^k$ and let $w = (0,w')^T \in \mathbb{F}_{q^m}^{k+1}$. Since $\mathcal{U}$ is rank ρ-saturating, there exists $z \in \mathbb{F}_{q^m}^n$ of rank at most ρ such that $w = Gz$ and so $w' = G'z'$, where $z' = (z_2,\ldots,z_n)^T$. Since $\mathrm{wt}_{\mathrm{rk}}((z')^T) \leq \mathrm{wt}_{\mathrm{rk}}(z^T) \leq \rho$, then G' generates an $[n-1,k]_{q^m/q}$ rank ρ'-saturating system $\mathcal{U}'$ with $\rho' \leq \rho$. Therefore, by (a),

$$s_m(k,\rho) \leq s_m(k,\rho') \leq n - 1 = s_m(k+1,\rho) - 1.$$

(c) Let $n = s_m(k,\rho)$. Let $G \in \mathbb{F}_{q^m}^{k\times n}$ be a generator matrix of a rank ρ-saturating $[n,k]_{q^m/q}$ system $\mathcal{U}$. Consider the matrix

$$G' = \begin{bmatrix} G & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{F}_{q^m}^{(k+1)\times(n+1)},$$

which generates a rank ρ'-saturating $[n+1,k+1]_{q^m/q}$ system $\mathcal{U}'$. It is straightforward to check that for any $w \in \mathbb{F}_{q^m}^{k+1}$, there exists $z \in \mathbb{F}_{q^m}^{n+1}$ of rank at most $\rho+1$ such that $w = G'z$. Again by (a), we have

$$s_m(k+1,\rho+1) \le s_m(k+1,\rho') \le n+1 \le s_m(k,\rho)+1.$$

□

In the following, we define the direct sum of systems to obtain recursive bounds, in analogy with [18, 37].

Definition 3.8. Let $\mathcal{U}$ be an $[n,k]_{q^m/q}$ system and $\mathcal{U}'$ be $[n',k']_{q^m/q}$ system. Suppose that $\mathcal{U}$ and $\mathcal{U}'$ come from two codes $\mathcal{C}$ and $\mathcal{C}'$, whose generator matrices are G and G' respectively. We call the *direct sum* of $\mathcal{U}$ and $\mathcal{U}'$, which we denote by $\mathcal{U}\oplus\mathcal{U}'$, the $[n+n',k+k']_{q^m/q}$ system associated to the direct sum of $\mathcal{C}$ and $\mathcal{C}'$, i.e. the code whose generator matrix is

$$G \oplus G' := \begin{bmatrix} G & 0 \\ 0 & G' \end{bmatrix}.$$

Theorem 3.9 (Direct Sum)**.** Let $\mathcal{U}$ be an $[n,k]_{q^m/q}$ system that is rank ρ-saturating and $\mathcal{U}'$ be an $[n',k']_{q^m/q}$ system that is rank ρ'-saturating. Then $\mathcal{U}\oplus\mathcal{U}'$ is an $[n+n',k+k']_{q^m/q}$ system that is rank ρ''-saturating, where

$$\rho'' \le \rho + \rho'. \tag{7}$$

In particular, if $\rho+\rho' \le \min\{k+k',m\}$

$$s_m(k+k',\rho+\rho') \le s_m(k,\rho) + s_m(k',\rho')$$

Proof. Let $v \in \mathbb{F}_{q^m}^{k+k'}$. By the definition of rank saturating systems, there exist ρ vectors in $\mathcal{U}$, say $u_1,\ldots,u_\rho$ and ρ' vectors in $\mathcal{U}'$, say $u'_1,\ldots,u'_\rho$, such that $(v_1,\ldots,v_k)^T$ is an $\mathbb{F}_{q^m}$-linear combination of $u_1,\ldots,u_\rho$ and $(v_{k+1},\ldots,v_{k+k'})^T$ is an $\mathbb{F}_{q^m}$-linear combination of $u'_1,\ldots,u'_\rho$.

Adding k' and k zeros to these vectors in a natural way, we obtain $\rho+\rho'$ vectors in $\mathcal{U}\oplus\mathcal{U}'$ that span ν. Hence, (7) holds.

Let $\mathcal{U}$ and $\mathcal{U}'$ have $\mathbb{F}_q$-dimension $s_m(k,\rho)$ and $s_m(k',\rho')$, respectively. Then $\mathcal{U}\oplus\mathcal{U}'$ has $\mathbb{F}_q$-dimension $s_m(k,\rho)+s_m(k',\rho')$. Now, $\mathcal{U}\oplus\mathcal{U}'$ is rank ρ''-saturating with $\rho''\le\rho+\rho'$, so that, by Theorem 3.7,

$$s_m(k+k',\rho+\rho') \le s_m(k+k',\rho'') \le s_m(k,\rho)+s_m(k',\rho'),$$

if $\rho+\rho' \le \min\{k+k',m\}$. □

Remark 3.10. The inequality (7) may be strict, as the following example shows. Let $\mathbb{F}_{16} = \mathbb{F}_2[\alpha]$ with $\alpha^4 = \alpha+1$. Let $\mathcal{U}$ the $[2,2]_{16/2}$ system and $\mathcal{U}'$ the $[3,1]_{16/2}$ system defined, respectively, by

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad G' = \begin{bmatrix} 1 & \alpha & \alpha^5 \end{bmatrix}.$$

The system $\mathcal{U}$ is rank 2-saturating and the system $\mathcal{U}'$ is rank 1-saturating, while $\mathcal{U}\oplus\mathcal{U}'$ is rank 2-saturating (one can check it directly with Magma).

Corollary 3.11.

$$s_m(tsh, ts) \leq t \cdot s_m(sh, s).$$

Proof. We proceed by induction on t (for $t = 1$ it is clear). By Theorem 3.9 and by induction hypothesis, we get

$$s_m(tsh, ts) \leq s_m((t-1)sh, (t-1)s) + s_m(sh, s) \leq (t-1)s_m(sh, s) + s_m(sh, s) = t \cdot s_m(sh, s).$$

□

Proposition 3.12. Let $\mathcal{C}_i$ be an $[n_i, k_i]_{q^m/q}$ code with rank covering radius ρ_i for $i \in \{1, 2\}$. Let $f : \mathbb{F}_{q^m}^{1\times n_1} \longrightarrow \mathbb{F}_{q^m}^{1\times n_2}$ be a map. Then the code

$$\mathcal{C} := \{(u, f(u) + v) : u \in \mathcal{C}_1, v \in \mathcal{C}_2\}$$

is an $[n_1 + n_2, k_1 + k_2]_{q^m/q}$ code such that $\rho_{\mathrm{rk}}(\mathcal{C}^\perp) \leq \rho_1 + \rho_2$.

Proof. $\mathcal{C}$ has a generator matrix of the form

$$G = \begin{bmatrix} G_1 & G' \\ 0 & G_2 \end{bmatrix},$$

where G_i is an generator matrix for $\mathcal{C}_i$ for each i and $G' \in \mathbb{F}_{q^m}^{k_1\times n_2}$. Let $\mathcal{U}_i$ be the system generated by G_i and let $\mathcal{U}'$ be the system generated by G'. Each $\mathcal{U}_i$ is rank ρ_i-saturating and $\mathcal{U}_1 + \mathcal{U}'$ is ρ'-saturating for some $\rho' \leq \rho_1$. Then, with the same argument of Theorem 3.9, we have that the $[n_1 + n_2, k_1 + k_2]_{q^m/q}$ system generated by G is rank ρ''-saturating, with $\rho'' \leq \rho_1 + \rho_2$. The conclusion follows from Theorem 2.5. □

4 Constructions

In this section, we present some geometric constructions of rank saturating systems of small $\mathbb{F}_q$-dimension, following the lines of [15,18,21,37], wherein, as we have already mentioned, the two main approaches involve constructions using cutting blocking sets and mixed subgeometries.

4.1 Construction with linear cutting blocking sets

Let us first introduce the notion of a cutting blocking set.

Definition 4.1. A subset $\mathcal{M} \subseteq \mathrm{PG}(k-1, q)$ is a *cutting blocking set* (or *strong blocking set*) if for every hyperplane $\mathcal{H}$ of $\mathrm{PG}(k-1, q)$, we have:

$$\langle \mathcal{M} \cap \mathcal{H} \rangle = \mathcal{H}.$$

Such sets were introduced in [16], with the original name of *strong blocking sets*, in connection to ρ-saturating sets. More explicitly, we have the following result.

Theorem 4.2 (Theorem 3.2. of [16])**.** Any cutting blocking set in a subgeometry $\mathrm{PG}(k-1, q)$ of $\mathrm{PG}(k-1, q^{k-1})$ is a $(k-2)$-saturating set in $\mathrm{PG}(k-1, q^{k-1})$.

In [7], they were reintroduced, with the name of *cutting blocking sets*, in order to construct a particular family of minimal codes.

Definition 4.3. An $[n, k]_{q^m}$ code $\mathcal{C}$ is *minimal* if for every $c, c' \in \mathcal{C}$, $\{i : c'_i \neq 0\} \subseteq \{i : c_i \neq 0\}$ implies $c' = \lambda c$ for some $\lambda \in \mathbb{F}_{q^m}$.

Such codes have been the subject of intense research over the last twenty years. In [1,36] it is shown that they are the geometrical counterparts of minimal codes, via the correspondence introduced in Subsection 1.2. Similarly to saturating sets, one of the main problem in the theory of minimal codes is the construction of short families, that is equivalent to construct small strong blocking sets. Some recent results can be found in [3,4,26].

The q-analogue of a cutting blocking set is defined as follows.

Definition 4.4. A $[n,k]_{q^m/q}$ system $\mathcal{U}$ is called a *linear cutting blocking set* if for every $\mathbb{F}_{q^m}$-hyperplane $\mathcal{H}$ we have $\langle \mathcal{H} \cap \mathcal{U} \rangle_{\mathbb{F}_{q^m}} = \mathcal{H}$.

Linear cutting blocking sets were introduced recently in [2], in connection with minimal codes in the rank-metric. In order to define these last, we need to introduce the notion of rank-support: for a word $c \in \mathbb{F}_{q^m}^{1\times n}$ and an ordered basis $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ of $\mathbb{F}_{q^m}/\mathbb{F}_q$, let $\Gamma(c) \in \mathbb{F}_q^{n\times m}$ be the matrix such that

$$c_i = \sum_{j=1}^{m} \Gamma(c)_{ij}\gamma_j.$$

The Γ-*support* of a vector $c \in \mathbb{F}_{q^m}^n$ is the column space of $\Gamma(c)$. It is easy to prove that it is independent of the chosen basis. We then call it simply the *rank-support* and we denote it by $\sigma^{rk}(c)$.

Definition 4.5. An $[n,k]_{q^m/q}$ code $\mathcal{C}$ is *minimal* if for every $c, c' \in \mathcal{C}$, $\sigma^{rk}(c') \subseteq \sigma^{rk}(c)$ implies $c' = \lambda c$ for some $\lambda \in \mathbb{F}_{q^m}$.

As shown in [2], a system is a linear cutting blocking set if and only if the associated rank-metric code is minimal.

We will show that, as in the classical setting, linear cutting blocking sets give rise to rank saturating systems.

Theorem 4.6. Let $\mathcal{U}$ be an $[n,k]_{q^m/q}$ system that is a linear cutting blocking set. Then $\mathcal{U}$ is a rank $(k-1)$-saturating $[n,k]_{q^{m(k-1)}/q}$ system.

Proof. The system $\mathcal{U}$ is a linear cutting blocking set in $\mathbb{F}_{q^m}^k$, so that the $[n,k]_{q^m/q}$ code $\mathcal{C}$ associated to $\mathcal{U}$ is a minimal code in the rank metric by [2, Corollary 5.7.]. Then the projective Hamming code $\mathcal{C}_{L_\mathcal{U}}$ associated to $\mathcal{C}$ is a minimal code in the Hamming metric by [2, Theorem 5.13] (indeed, $\mathcal{C}_{L_\mathcal{U}}$ is the projectivisation of the code $\mathcal{C}^H$ in that reference, and we are using also the trivial fact that a code is minimal if and only if its projectivisation is minimal). Hence $L_\mathcal{U}$ is a cutting blocking set in $\mathrm{PG}(k-1,q^m)$. Then $L_\mathcal{U}$ is a $(k-2)$-saturating set in $\mathrm{PG}(k-1,q^{m(k-1)})$ by Theorem 4.2. By definition, this means that $\mathcal{U}$ is a rank $(k-1)$-saturating $[n,k]_{q^{m(k-1)}/q}$ system. □

Corollary 4.7. For every $m,k \geq 2$,

$$k+m-1 \leq s_{m(k-1)}(k,k-1) \leq 2k+m-2.$$

Proof. By [2, Corollary 6.11.], for every $m,k \geq 2$, there exists a $[2k+m-2,k]_{q^m/q}$ linear cutting blocking set. So the upper bound is a direct consequence of Theorem 4.6. The lower bound is the one by Theorem 3.3. □

Remark 4.8. Quite remarkably, the lower bound coincides with the one for linear cutting blocking set given in [2, Corollary 5.10]. Note however that in [5] it is proved that the bound is not sharp for linear cutting blocking sets when $m < (k-1)^2$. It would be interesting to know if a similar result holds also for saturating systems.

We reformulate here a result in [2].

Theorem 4.9 (Theorem 6.3 [2])**.** Let $n \geq m+2$ and $\mathcal{U}$ a $[n,3]_{q^m/q}$ system. If $L_{\mathcal{U}}$ is scattered, then $\mathcal{U}$ is a linear cutting blocking set.

Corollary 4.10. Suppose that $m \not\equiv 3,5 \mod 6$ and $m \geq 4$. Then

$$s_{2m}(3,2) = m+2.$$

Proof. As in [2, Theorem 6.7], by Theorem 4.9 and known results on scattered linear sets (see [6, 14]), under these hypothesis $[m+2,3]_{q^m/q}$ linear cutting blocking sets exist. So by Theorem 4.6, rank 2-saturating $[m+2,3]_{q^{2m}/q}$ exist. The equality comes from the fact the in this case the upper bound meets the lower bound. □

Example 4.11. Let λ in $\mathbb{F}_{16}$ such that $\lambda^4 = \lambda+1$. The $[6,3]_{16/2}$ system with generator matrix

$$G = \begin{pmatrix} \lambda^4 & \lambda^{10} & \lambda^8 & \lambda^3 & \lambda^9 & \lambda^7 \\ \lambda^{14} & \lambda^8 & \lambda & \lambda^8 & 0 & \lambda^8 \\ \lambda^{10} & 0 & \lambda^6 & \lambda^5 & \lambda^{11} & \lambda^3 \end{pmatrix}.$$

is a linear cutting blocking set, as shown in [2, Example 6.9]. So the $[6,3]_{256/2}$ system $\mathcal{U}$ with the same generator matrix is a rank 2-saturating system. It has the smallest $\mathbb{F}_2$-dimension. The linear set $L_{\mathcal{U}}$ is scattered.

Remark 4.12. In [5] it is shown that, for all q, there exists an $[8,4]_{q^3/q}$ linear cutting blocking set. Therefore, by Theorem 4.6 we get the existence of an $[8,4]_{q^9/q}$ system that is rank 3-saturating. Hence

$$6 \leq s_9(4,3) \leq 8.$$

In this case, their construction is independent of q. On the other hand, for $q = 2^h$ with h odd, they show that the $[8,4]_{q^4/q}$ system

$$\mathcal{U} = \left\{ \begin{pmatrix} x \\ y \\ x^q + y^{q^2} \\ x^{q^2} + y^q + y^{q^2} \end{pmatrix} : x, y \in \mathbb{F}_{q^4} \right\}.$$

is a linear cutting blocking set (but for h even the result is not true anymore) and, by Theorem 4.6 a $[8,4]_{q^{12}/q}$ rank 3-saturating system. So

$$7 \leq s_{q^{12}/q}(4,3) \leq 8$$

for $q = 2^h$ with h odd. Note that, for h even $\mathcal{U}$ may eventually be still a rank 3-saturating system, but Theorem 4.6 is not sufficient anymore. It would be interesting to know whether such an example of dependence on q exists also for saturating systems.

Finally, in [5] they prove the following: if $[t,k]q^m/q$ is a linear cutting blocking set, then one can construct a $[t+m,k+1]_{q^m/q}$ linear cutting blocking set. In our terms, by Theorem 4.6 we get that if a $[t,k]q^m/q$ linear cutting blocking set exists, then

$$s_{q^{m(k-1)}/q}(k,k-1) \leq t \quad \text{and} \quad s_{q^{mk}/q}(k+1,k) \leq t+m.$$

4.2 A construction from subgeometries

In this subsection, we outline a construction that exploits the the properties of particular subgeometries of $\mathrm{PG}(k-1,q^m)$, i.e. those arising from subfields of $\mathbb{F}_{q^m}$.

For the purposes of exposition, we start with a special case, which will serve as an example of a more general construction. In $\mathrm{PG}(k-1,q^4)$, up to equivalence, this corresponds to all $\mathbb{F}_q$-lines connecting the points of an $\mathbb{F}_q$-plane and of an $\mathbb{F}_{q^2}$-subspace of dimension $k-4$, non-intersecting.

Proposition 4.13. Let $\mathbb{F}_{q^2}=\mathbb{F}_q[\alpha]$. For $k\geq 3$, the $[2k-3,k]_{q^4/q}$ system $\mathcal{U}$ defined by

$$\mathcal{U}=\left\{\begin{pmatrix} u \\ \hline w\end{pmatrix} : u\in\mathbb{F}_q^3, w\in\mathbb{F}_{q^2}^{k-3}\right\},$$

which has an associated generator matrix given by:

$$G=\left[\begin{array}{c|c|c} I_3 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & I_{k-3} & \alpha I_{k-3}\end{array}\right],$$

is rank 3-saturating. In particular, we have:

$$s_4(k,3)\leq 2k-3.$$

Proof. Fix $\beta_1,\beta_2\in\mathbb{F}_{q^4}$ such that $\mathbb{F}_{q^4}=\mathbb{F}_{q^2}+\langle\beta_1,\beta_2\rangle_{\mathbb{F}_q}$, i.e. $\mathbb{F}_{q^4}=\mathbb{F}_q[\alpha,\beta_1,\beta_2]$. For any $w\in\mathbb{F}_{q^4}$, write $w=\pi_{\beta_1}(w)\beta_1+\pi_{\beta_2}(w)\beta_2+\pi_{\mathbb{F}_{q^2}}(w)$ for $\pi_{\beta_1}(w),\pi_{\beta_2}(w)\in\mathbb{F}_q$ and $\pi_{\mathbb{F}_{q^2}}(w)\in\mathbb{F}_{q^2}$.

Consider a vector $v=(v_1,\ldots,v_k)^T\in\mathbb{F}_{q^4}^k$; we will show that $v=\lambda^{(1)}u^{(1)}+\lambda^{(2)}u^{(2)}+\lambda^{(3)}u^{(3)}$ for some $\lambda^{(1)},\lambda^{(2)},\lambda^{(3)}\in\mathbb{F}_{q^4}$ and $u^{(1)},u^{(2)},u^{(3)}\in\mathcal{U}$.

We first define the functions

$$\begin{aligned}\varphi_1 :\ & \mathbb{F}_{q^4}\times\mathbb{F}_{q^4}\longrightarrow\mathbb{F}_q \\ & (x_1,x_2)\longmapsto \pi_{\beta_1}(x_1)^{-1}\pi_{\beta_1}(x_2)\end{aligned}$$

and

$$\begin{aligned}\varphi_2 :\ & \mathbb{F}_{q^4}\times\mathbb{F}_{q^4}\times\mathbb{F}_{q^4}\longrightarrow\mathbb{F}_{q^2} \\ (x_1,x_2,x_3)\longmapsto\ & \frac{\pi_{\mathbb{F}_{q^2}}(x_2)-\pi_{\mathbb{F}_{q^2}}(x_1)\varphi_1(x_1,x_2)}{\pi_{\beta_2}(x_2)-\pi_{\beta_2}(x_1)\varphi_1(x_1,x_2)}(\pi_{\beta_2}(x_3)-\pi_{\beta_2}(x_1)\varphi_1(x_1,x_3)).\end{aligned}$$

Suppose first that

(I) $\pi_{\beta_1}(v_1)\neq 0$,

(II) $\pi_{\beta_2}(v_2)\neq\pi_{\beta_2}(v_1)\varphi_1(v_1,v_2)$,

(III) $\pi_{\mathbb{F}_{q^2}}(v_3)\neq\pi_{\mathbb{F}_{q^2}}(v_1)\varphi_1(v_1,v_3)+\varphi_2(v_1,v_2,v_3)$.

Let

$$\lambda^{(1)}:=\frac{v_1}{\pi_{\beta_1}(v_1)},$$

$$u^{(1)}:=\begin{pmatrix}\pi_{\beta_1}(v_1)\\ \pi_{\beta_1}(v_2)\\ \pi_{\beta_1}(v_3)\\ \hline \pi_{\beta_1}(v_4)\\ \vdots \\ \pi_{\beta_1}(v_k)\end{pmatrix},$$

$$\lambda^{(2)} := \beta_2 + \frac{\pi_{\mathbb{F}_{q^2}}(v_2) - \pi_{\mathbb{F}_{q^2}}(v_1)\varphi_1(v_1, v_2)}{\pi_{\beta_2}(v_2) - \pi_{\beta_2}(v_1)\varphi(v_1, v_2)} = \frac{v_2 - \lambda^{(1)}u_2^{(1)} - \pi_{\beta_1}(v_2 - \lambda^{(1)}u_2^{(1)})\beta_1}{\pi_{\beta_2}(v_2 - \lambda^{(1)}u_2^{(1)})},$$

$$u^{(2)} := \left(\begin{array}{c} 0 \\ \pi_{\beta_2}(v_2) - \pi_{\beta_2}(v_1)\varphi_1(v_1, v_2) \\ \pi_{\beta_2}(v_3) - \pi_{\beta_2}(v_1)\varphi_1(v_1, v_3) \\ \hline \pi_{\beta_2}(v_4) - \pi_{\beta_2}(v_1)\varphi_1(v_1, v_4) \\ \vdots \\ \pi_{\beta_2}(v_k) - \pi_{\beta_2}(v_1)\varphi_1(v_1, v_4) \end{array}\right) = \left(\begin{array}{c} 0 \\ \pi_{\beta_2}(v_2) \\ \pi_{\beta_2}(v_3) \\ \hline \pi_{\beta_2}(v_4) \\ \vdots \\ \pi_{\beta_2}(v_k) \end{array}\right) - \left(\begin{array}{c} 0 \\ \pi_{\beta_2}(\lambda^{(1)}u_2^{(1)}) \\ \pi_{\beta_2}(\lambda^{(1)}u_3^{(1)}) \\ \hline \pi_{\beta_2}(\lambda^{(1)}u_4^{(1)}) \\ \vdots \\ \pi_{\beta_2}(\lambda^{(1)}u_k^{(1)}) \end{array}\right),$$

$$\begin{aligned} \lambda^{(3)} &:= \pi_{\mathbb{F}_{q^2}}(v_3) - \pi_{\mathbb{F}_{q^2}}(v_1)\varphi_1(v_1, v_3) - \varphi_2(v_1, v_2, v_3) \\ &= v_3 - \lambda^{(1)}u_3^{(1)} - \lambda^{(2)}u_3^{(2)} - \pi_{\beta_1}(v_3 - \lambda^{(1)}u_3^{(1)} - \lambda^{(2)}u_3^{(2)})\beta_1 - \pi_{\beta_2}(v_3 - \lambda^{(1)}u_3^{(1)} - \lambda^{(2)}u_3^{(2)})\beta_2 \end{aligned}$$

$$\begin{aligned} u^{(3)} &:= \left(\begin{array}{c} 0 \\ 0 \\ 1 \\ \hline 0 \\ \vdots \\ 0 \end{array}\right) + \frac{1}{\lambda^{(3)}} \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline \pi_{\mathbb{F}_{q^2}}(v_4) - \pi_{\mathbb{F}_{q^2}}(v_1)\varphi_1(v_1, v_4) - \varphi_2(v_1, v_2, v_4) \\ \vdots \\ \pi_{\mathbb{F}_{q^2}}(v_k) - \pi_{\mathbb{F}_{q^2}}(v_1)\varphi_1(v_1, v_k) - \varphi_2(v_1, v_2, v_k) \end{array}\right) \\ &= \left(\begin{array}{c} 0 \\ 0 \\ 1 \\ \hline 0 \\ \vdots \\ 0 \end{array}\right) + \frac{1}{\lambda^{(3)}} \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline \pi_{\mathbb{F}_{q^2}}(v_4) \\ \vdots \\ \pi_{\mathbb{F}_{q^2}}(v_k) \end{array}\right) - \frac{1}{\lambda^{(3)}} \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline \pi_{\mathbb{F}_{q^2}}(\lambda^{(1)}u_4^{(1)}) \\ \vdots \\ \pi_{\mathbb{F}_{q^2}}(\lambda^{(1)}u_k^{(1)}) \end{array}\right) - \frac{1}{\lambda^{(3)}} \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline \pi_{\mathbb{F}_{q^2}}(\lambda^{(2)}u_4^{(2)}) \\ \vdots \\ \pi_{\mathbb{F}_{q^2}}(\lambda^{(2)}u_k^{(2)}) \end{array}\right). \end{aligned}$$

Direct computations show that $v = \lambda^{(1)}u^{(1)} + \lambda^{(2)}u^{(2)} + \lambda^{(3)}u^{(3)}$. Since $u^{(1)}, u^{(2)} \in \mathbb{F}_q^k \subseteq \mathcal{U}$ and $u^{(3)} \in \mathbb{F}_q^3 \times \mathbb{F}_{q^2}^{k-3} \subseteq \mathcal{U}$, we have that $\rho_{\mathrm{rk}}(\mathcal{U}) \leq 3$.

We now consider the possibility that one or more of the assumptions (I)-(III) do not hold. We will show that the argument holds with some minor modifications.

(I) Suppose that $\pi_{\beta_1}(v_1) = 0$.

  (a) If there exists an index $i \in \{2, \dots, k\}$ such that $\pi_{\beta_1}(v_i) \neq 0$, repeat the passages written above replacing v_1 with v_i.

  (b) Otherwise, if there does not exist any $i \in \{2, \dots, k\}$ such that $\pi_{\beta_1}(v_i) \neq 0$, set $\lambda^{(1)} = v_1$, $u^{(1)} = (1, 0, \dots, 0)^T$, and replace $\pi_{\beta_1}(v_1)^{-1}$ with the value zero in the formula for φ_1.

(II) Suppose that $\pi_{\beta_2}(v_2) = \pi_{\beta_2}(v_1)\varphi_1(v_1, v_2)$.

  (a) If there exists an index $i \in \{3, \dots, k\}$ such that

$$\pi_{\beta_2}(v_i) \neq \pi_{\beta_2}(v_1)\varphi_1(v_1, v_i),$$

  repeat the passages written above replacing v_2 with v_i.

(b) Otherwise, if there does not exist any $i \in \{3, \dots, k\}$ such that

$$\pi_{\beta_2}(v_i) \neq \pi_{\beta_2}(v_1)\varphi_1(v_1, v_i),$$

then set $\lambda^{(2)} = v_2 - \lambda^{(1)} u_2^{(1)}$, $u^{(2)} = (0, 1, \dots, 0)^T$, replace

$$(\pi_{\beta_2}(v_2) - \pi_{\beta_2}(v_1)\varphi_1(v_1, v_3))^{-1}$$

with the value zero in the determination of $\lambda^{(3)}, u^{(3)}$.

(III) Suppose that that $\pi_{\mathbb{F}_{q^2}}(v_3) \neq \pi_{\mathbb{F}_{q^2}}(v_1)\varphi_1(v_1, v_3) + \varphi_2(v_1, v_2, v_3)$.

(a) If there exists an index $i \in \{4, \dots, k\}$ such that

$$\pi_{\mathbb{F}_{q^2}}(v_i) \neq \pi_{\mathbb{F}_{q^2}}(v_1)\varphi_1(v_1, v_i) + \varphi_2(v_1, v_2, v_i),$$

then replace v_3 with v_i in the determination of $\lambda^{(3)}$, $u^{(3)}$.

(b) Otherwise, if there does not exist any $i \in \{4, \dots, k\}$ such that

$$\pi_{\mathbb{F}_{q^2}}(v_i) \neq \pi_{\mathbb{F}_{q^2}}(v_1)\varphi_1(v_1, v_i) + \varphi_2(v_1, v_2, v_i),$$

then the process has already terminated in the Step (II) and it is enough to set $\lambda^{(3)} = 0$ and $u^{(3)} = (0, \dots, 0)^T$.

To conclude the proof, we show that $\mathcal{U}$ is exactly 3-saturating. Let $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{F}_{q^4}$ be linearly independent over $\mathbb{F}_q$, and let $\overline{v} = (\gamma_1, \gamma_2, \gamma_3, 0, \dots, 0)^T \in \mathcal{U}$. Due to the linear independence of the γ_i over $\mathbb{F}_q$, it is not possible to saturate $\overline{v}$ with fewer than 3 elements of $\mathcal{U}$. □

The idea of the previous proof above, which has much to share with Gram-Schmidt algorithm, allows us to obtain a construction which generalises Proposition 4.13. In PG$(h + (r-1)t, q^{rt})$, up to equivalence, this corresponds to all $\mathbb{F}_q$-lines connecting the points of an $\mathbb{F}_q$-subspace of dimension $(r-1)t$ and of an $\mathbb{F}_{q^t}$-subspace of dimension $h-1$, non-intersecting.

Theorem 4.14. Let $r, t \geq 2$ and $\mathbb{F}_{q^t} = \mathbb{F}_q[\alpha]$. For $h \geq 0$, the $[th+t(r-1)+1, h+t(r-1)+1]_{q^{rt}/q}$ system $\mathcal{U}$ defined by:

$$\mathcal{U} = \left\{ \left(\frac{u}{w} \right) : u \in \mathbb{F}_q^{(r-1)t+1}, w \in \mathbb{F}_{q^t}^h \right\},$$

which has an associated generator matrix given by:

$$G = \left[\begin{array}{c|c|c|c|c} I_{(r-1)t+1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & I_h & \alpha I_h & \dots & \alpha^{t-1} I_h \end{array} \right],$$

is rank $((r-1)t+1)$-saturating. In particular,

$$s_{rt}(h + (r-1)t + 1, (r-1)t + 1) \leq th + (r-1)t + 1.$$

Proof. Let $\{1, \beta_1, \dots, \beta_{(r-1)t+1}\}$ be a basis of $\mathbb{F}_{q^{rt}}$ over $\mathbb{F}_{q^t}$. We start by supposing that

$$\pi_{\beta_\ell}(v_\ell - \sum_{i=1}^{\ell-1} \lambda^{(i)} u_\ell^{(i)}) \neq 0, \text{ for } \ell \in \{1, \dots, (r-1)t\},$$

and that

$$\pi_{\mathbb{F}_{q^t}}(v_{(r-1)t+1}) \neq \sum_{i=1}^{(s-1)t} \pi_{\mathbb{F}_{q^t}}(\lambda^{(i)} u_{(r-1)t+1}^{(i)}).$$

If this is not the case, we may proceed as in the proof of Proposition 4.13.

Now define the following:

$$\lambda^{(1)} := \frac{v_1}{\pi_{\beta_1}(v_1)}, \qquad u^{(1)} := \left(\begin{array}{c} \pi_{\beta_1}(v_1) \\ \pi_{\beta_1}(v_2) \\ \vdots \\ \pi_{\beta_1}(v_{(r-1)t+1}) \\ \hline \pi_{\beta_1}(v_{(r-1)t+2}) \\ \vdots \\ \pi_{\beta_1}(v_{th+t(s-1)+1}) \end{array}\right),$$

and, for $\ell \in \{2, \dots, (r-1)t\}$, we set

$$\lambda^{(\ell)} := \frac{v_\ell - \sum_{i=1}^{\ell-1}(\lambda^{(i)}u_\ell^{(i)}) - \sum_{j=1}^{\ell-1}\pi_{\beta_j}(v_\ell - \sum_{i=1}^{j}\lambda^{(i)}u_\ell^{(i)})\beta_j}{\pi_{\beta_\ell}(v_\ell - \sum_{i=1}^{\ell-1}\lambda^{(i)}u_\ell^{(i)})},$$

$$u^{(\ell)} := \left(\begin{array}{c} \mathbf{0}_{(\ell-1)\times 1} \\ \pi_{\beta_\ell}(v_\ell) \\ \pi_{\beta_\ell}(v_{\ell+1}) \\ \vdots \\ \pi_{\beta_\ell}(v_{(r-1)t+1}) \\ \hline \pi_{\beta_\ell}(v_{(r-1)t+2}) \\ \vdots \\ \pi_{\beta_\ell}(v_{th+t(s-1)+1}) \end{array}\right) - \sum_{i=1}^{\ell-1}\left(\begin{array}{c} \mathbf{0}_{(\ell-1)\times 1} \\ \pi_{\beta_\ell}(\lambda^{(i)}u_\ell^{(i)}) \\ \pi_{\beta_\ell}(\lambda^{(i)}u_{\ell+1}^{(i)}) \\ \vdots \\ \pi_{\beta_\ell}(\lambda^{(i)}u_{(r-1)t+1}^{(i)}) \\ \hline \pi_{\beta_\ell}(\lambda^{(i)}u_{(r-1)t+2}^{(i)}) \\ \vdots \\ \pi_{\beta_\ell}(\lambda^{(i)}u_{th+t(s-1)+1}^{(i)}) \end{array}\right).$$

Finally, we define

$$\lambda^{((r-1)t+1)} := \pi_{\mathbb{F}_{q^t}}(v_{(r-1)t+1}) - \sum_{i=1}^{(s-1)t}\pi_{\mathbb{F}_{q^t}}(\lambda^{(i)}u_{(r-1)t+1}^{(i)}),$$

and

$$u^{(r-1)t+1} := \left(\begin{array}{c} \mathbf{0}_{((r-1)t)\times 1} \\ 1 \\ \hline \frac{\pi_{\mathbb{F}_{q^t}}(v_{(r-1)t+2})}{\lambda^{((r-1)t+1)}} \\ \vdots \\ \frac{\pi_{\mathbb{F}_{q^t}}(v_{th+t(s-1)+1})}{\lambda^{((r-1)t+1)}} \end{array}\right) - \frac{1}{\lambda^{((r-1)t+1)}}\sum_{i=1}^{(r-1)t}\left(\begin{array}{c} \mathbf{0}_{((r-1)t+1)\times 1} \\ \hline \pi_{\mathbb{F}_{q^t}}(\lambda^{(i)}u_{(r-1)t+2}^{(i)}) \\ \vdots \\ \pi_{\mathbb{F}_{q^t}}(\lambda^{(i)}u_{th+t(s-1)+1}^{(i)}) \end{array}\right).$$

In order to prove that $\mathcal{U}$ is $((r-1)t+1)$-saturating, we show that the following hold:

(i) $v_\ell = \sum_{i=1}^{\ell}\lambda^{(i)}u_\ell^{(i)}$ for $\ell \in \{1, \dots, (r-1)t\}$,

(ii) $\pi_{\beta_\ell}(v_k) = \pi_{\beta_\ell}\left(\sum_{i=1}^{\ell}\lambda^{(i)}u_k^{(i)}\right) = \pi_{\beta_\ell}\left(\sum_{i=1}^{j}\lambda^{(i)}u_k^{(i)}\right)$ for $\ell \in \{1, \dots, (r-1)t\}$, $k \in \{\ell + 1, \dots, th + t(s-1) + 1\}$, and $j \in \{\ell, \dots, (r-1)t + 1\}$,

(iii) $\sum_{i=1}^{(r-1)t+1}\lambda^{(i)}u^{(i)} = v$.

Direct computations show that

$$\lambda^{(1)}u_1^{(1)} = \frac{v_1}{\pi_{\beta_1}(v_1)}\pi_{\beta_1}(v_1) = v_1,$$

and that, furthermore:

$$\begin{aligned}\sum_{i=1}^{\ell}\lambda^{(i)}u_\ell^{(i)} =&\lambda^{(\ell)}u_\ell^{(\ell)} + \sum_{i=1}^{\ell-1}(\lambda^{(i)}u_\ell^{(i)})\\ =&\frac{v_\ell - \sum_{i=1}^{\ell-1}(\lambda^{(i)}u_\ell^{(i)}) - \sum_{j=1}^{\ell-1}\pi_{\beta_j}(v_\ell - \sum_{i=1}^{j}\lambda^{(i)}u_\ell^{(i)})\beta_j}{\pi_{\beta_\ell}(v_\ell - \sum_{i=1}^{\ell-1}\lambda^{(i)}u_\ell^{(i)})}\pi_{\beta_\ell}\left(v_\ell - \sum_{i=1}^{\ell-1}\lambda^{(i)}u_\ell^{(i)}\right)\\ &+\sum_{i=1}^{\ell-1}(\lambda^{(i)}u_\ell^{(i)})\\ =&v_\ell - \sum_{i=1}^{\ell-1}(\lambda^{(i)}u_\ell^{(i)}) + \sum_{i=1}^{\ell-1}(\lambda^{(i)}u_\ell^{(i)})\\ =&v_\ell.\end{aligned}$$

Hence (i) holds.

We now prove (ii), noting that by construction, $\pi_{\beta_i}(\lambda^{(i)}) = 1$, for $i \leq (r-1)t$. Moreover, for $\ell \in \{1, \dots, (r-1)t\}$, it is straightforward to show that the second equality in (ii) holds, as $\lambda^{(i)} \in \langle \beta_i, \beta_{i+1}, \dots, \beta_{(r-1)t}\rangle_{\mathbb{F}_q} + \mathbb{F}_{q^2}$ and $u^{(i)} \in \mathbb{F}_q^{th+t(s-1)+1}$ for $i \leq (r-1)t$. Firstly, we have that

$$\pi_{\beta_1}(\lambda^{(1)}u_k^{(1)}) = \frac{\pi_{\beta_1}(v_1)}{\pi_{\beta_1}(v_1)}u_1^{(1)} = \pi_{\beta_1}(v_1).$$

Consider now $\ell \in \{2, \dots, (r-1)t\}$. By construction, we have that:

$$\begin{aligned}\pi_{\beta_\ell}\left(\sum_{i=1}^{\ell}\lambda^{(i)}u_k^{(i)}\right) =&\pi_{\beta_\ell}(\lambda^{(\ell)}u_k^{(\ell)}) + \pi_{\beta_\ell}\left(\sum_{i=1}^{\ell-1}(\lambda^{(i)}u^{(i)})\right)\\ =&\pi_{\beta_\ell}(v_\ell) - \sum_{i=1}^{\ell-1}\pi_{\beta_\ell}(\lambda^{(i)}u_k^{(i)}) + \pi_{\beta_\ell}\left(\sum_{i=1}^{\ell-1}(\lambda^{(i)}u_k^{(i)})\right)\\ =&\pi_{\beta_\ell}(v_\ell) - \sum_{i=1}^{\ell-1}\pi_{\beta_\ell}(\lambda^{(i)}u^{(i)})_k + \sum_{i=1}^{\ell-1}\pi_{\beta_\ell}(\lambda^{(i)}u^{(i)})\\ =&\pi_{\beta_\ell}(v_\ell),\end{aligned}$$

which implies (ii).

To prove (iii) it remains to show that $\pi_{\mathbb{F}_{q^t}}(v_k) = \pi_{\mathbb{F}_{q^t}}\left(\sum_{i=1}^{(r-1)t+1}\lambda^{(i)}u_k^{(i)}\right)$, for $k \in \{(r-1)t+1, \dots, th+t(s-1)+1\}$.

By construction, for $k \in \{(r-1)t+1, \dots, th+t(s-1)+1\}$ we have:

$$\begin{aligned}\pi_{\mathbb{F}_{q^t}}\left(\sum_{i=1}^{(r-1)t+1}\lambda^{(i)}u_k^{(i)}\right) =&\lambda^{((r-1)t+1)}u_k^{((r-1)t+1)} + \sum_{i=1}^{(r-1)t}\lambda^{(i)}u_k^{(i)}\\ =&\pi_{\mathbb{F}_{q^t}}(v_{(r-1)t+1}) - \sum_{i=1}^{(s-1)t}\pi_{\mathbb{F}_{q^t}}(u_k^{(i)}) + \sum_{i=1}^{(r-1)t}\lambda^{(i)}u_k^{(i)}\\ =&\pi_{\mathbb{F}_{q^t}}(v_k).\end{aligned}$$

Since $u^{(\ell)} \in \mathbb{F}_q^{(r-1)t+h+1} \subseteq \mathcal{U}$, for $\ell \in \{1, \ldots, (r-1)t\}$, and $u^{(r-1)t+1} \in \mathbb{F}_q^{(r-1)t} \times \mathbb{F}_{q^t}^h \subseteq \mathcal{U}$ we have that $\rho_{\mathrm{rk}}(\mathcal{U}) \leq (r-1)t+1$. Moreover, using the same argument as in the proof of Proposition 4.13, $\mathcal{U}$ is exactly $((r-1)t+1)$-saturating.

□

Remark 4.15. Note that for $h \geq 0$,

$$\frac{rth + (t(r-1)+1)^2}{t(r-1)+1} \leq s_{rt}((r-1)t+1+h, (r-1)t+1) \leq th + (r-1)t + 1$$

and the difference between the upper and the lower bound is

$$\frac{(r-1)t(t-1)}{(r-1)t+1} \cdot h \ \sim \ (t-1)h \qquad \text{for } r \to \infty.$$

Note that, for $t = 2$ and $h = 1$, for all $r \geq 2$ the difference is strictly less than 1, so that

$$s_{2r}(2r, 2r-1) = 2r+1.$$

5 Conclusion

For the convenience of the reader, we summarize the main results on $s_m(k,\rho)$ proved in this paper. First, by Theorem 3.3 and Theorem 3.4,

$$\frac{m}{\rho}(k-\rho) + \rho \leq s_m(k,\rho) \leq m(k-\rho) + \rho.$$

The Monotonicity Theorem (Theorem 3.7) and the Direct Sum Theorem (Theorem 3.9) state that, for all positive integers m, k, k', $\rho \in \{1, \ldots, \min\{k,m\}$, $\rho' \in \{1, \ldots, \min\{k', m\}$,

(a) If $\rho < \min\{k,m\}$, then $s_m(k, \rho+1) \leq s_m(k,\rho)$.

(b) $s_m(k,\rho) < s_m(k+1,\rho)$.

(c) If $\rho < m$, then $s_m(k+1, \rho+1) \leq s_m(k,\rho) + 1$.

(d) If $\rho + \rho' \leq \min\{k+k', m\}$, $s_m(k+k', \rho+\rho') \leq s_m(k,\rho) + s_m(k',\rho')$.

The upper bound is sharpened for particular cases: for every $r, k \geq 2$, thanks to the construction using linear cutting blocking sets (Corollary 4.7) we have

$$s_{r(k-1)}(k, k-1) \leq 2k + r - 2,$$

and, using subgeometries (Theorem 4.14), we have, for $t, s \geq 2$, $h \geq 0$

$$s_{tr}(t(r-1)+1+h, t(r-1)+1) \leq th + t(r-1) + 1.$$

Finally, we list some cases for which $s_m(k,\rho)$ is determined, namely:

$$\begin{aligned} s_m(k,1) &= m(k-1)+1, && \text{for all } m,k \geq 2,\\ s_m(k,k) &= k, && \text{for all } m,k \geq 2,\\ s_{2r}(3,2) &= r+2, && \text{for } r \neq 3,5 \bmod 6 \text{ and } r \geq 4,\\ s_{2r}(2r,2r-1) &= 2r+1, && \text{for all } r \geq 2. \end{aligned}$$

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