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On the moments of ruin and recovery times☆

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Abstract

In this paper we consider the calculation of moments of the time to ruin and the duration of the first period of negative surplus. We present a recursive method by considering a discrete time compound Poisson process used by Dickson et al. [Astin Bull. 25 (2) (1995) 153]. With this method we will also be able to calculate approximations for the corresponding quantities in the classical model. Furthermore, for the classical compound Poisson model we consider some asymptotic formulae, as initial surplus tends to infinity, for the severity of ruin, which allow us to find explicit formulae for the moments of the time to recovery. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we consider recursive algorithms for the calculation of the moments of the random variables time to ruin and the duration of the first period of negative surplus (or time to recovery). As far as the time to ruin is concerned, much has been said in the actuarial literature about the study of the ruin probabilities, whether finite time or ultimate. The study of some existing moments like the expected value or the variance of the time to ruin can give another insight, although somehow limited. For practical purposes it may be useful for instance to have a quick look at how long it takes for ruin to occur, before going into more details like computing ruin probabilities. The proposed method is easy to implement and its outcome is easily interpreted. About the other random variable, the duration of negative surplus has been previously discussed by Egídio dos Reis (1993) and Dickson and Egídio dos Reis (1996). The former reference deals exclusively with the moments of this random variable only concerning the classical model, giving closed formulae which depend on the severity of ruin related quantities. In that paper it is also shown the relation between time to ruin and time to recovery when initial surplus is zero. This is important in our work too by providing a starting point in the proposed recursions, as we will see later in the text.

We use a discrete model as a discrete time compound Poisson process first introduced by Dickson and Waters (1991) and later retrieved by Dickson et al. (1995). Under this model we develop a recursion to compute moments

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of the above quantities, particularly the first two moments, provided they exist. This proposed discrete model not only has interest on his own but also provides numerical approximations for the corresponding quantities in the classical model. We will focus our attention to the discrete model in a way such that it provides an approximation to the classical continuous model.

As far as the time to ruin in the classical model is concerned, Lin and Willmot (2000) show formulae for the computation of the moments by means of renewal equations, particularly the first two moments. With their method, exact evaluation of the moments is straightforward when the claim size distribution is a combination of exponentials or a mixture of Erlang distributions.

Concerning the duration of negative surplus in the classical model, its moments depend on the same quantities of the severity of ruin as shown by Egídio dos Reis (1993). When initial surplus is zero the moments have a simple formula. For positive initial surplus (and also when it is zero) again, Lin and Willmot (2000) show expressions for the moments of the severity of ruin which will allow us to compute the moments of the time to recovery. Like in the case for the time to ruin they show that exact evaluations can be done easily for combinations of exponentials and mixture of Erlang claim size distributions. Furthermore, we show in this paper that we can also get a simple expression for the asymptotic moment generating function of the severity of ruin, as the initial surplus goes to infinity, which allows us to compute, at least numerically, moments for the severity of ruin as well as the related random variable time to recovery.

Approximations to the duration of negative surplus quantities in the classical model can be done via the severity of ruin using available methods, like the one presented by Dickson et al. (1995), who use the same discrete time model. The approach we present shows how we can do it dealing in full with the discrete model. Not only does it give exact results as far as the discrete model is concerned but it also provides the starting values (when the initial surplus is zero) for the recursions concerning the time to ruin by enhancing the relationship between time to ruin and time to recovery when the initial surplus is zero.

In the next section we introduce the basic continuous time surplus model as well as the discrete model that approximates the basic model, including definitions and notation. In Section 3 we present recursions for the moments of time to ruin for positive values of the initial surplus considering the discrete model. In Section 4 we consider for the discrete model the recursions for the duration of a first period of negative surplus, showing the relationship between time to recovery and severity of ruin. In Section 5 we discuss the starting values, with initial surplus equal to zero, for the recursions in the two previous sections, based on the fact that the time to ruin and the duration of negative surplus have the same distribution. In Section 6 we show some asymptotic formulae for the severity of ruin and, consequently, for the time to recovery considering the classic model. Finally, in the last section we consider a couple of examples showing the kind of approximations in the classical model we can expect to obtain from the recursions formerly discussed. Namely, an exponential, a combination of two exponentials and an Erlang(2) claim amount distributions.

2. Models and notation

We first introduce the classical compound Poisson model. Let $\{U(t)\}_{t\geq 0}$ be a classical continuous time surplus process, so that

$$U(t) = u + ct - S(t), \quad t \geq 0,$$

where u is the insurer's initial surplus, c the insurer's rate of premium income per unit time, $S(t) = \sum_{i=1}^{N(t)} X_i$ the aggregate claim amount up to time t, $N(t)$ the number of claims in the same time interval having a Poisson (λt) distribution, and $\{X_i\}_{i=1}^{\infty}$ a sequence of i.i.d. random variables representing the individual claim amounts. We denote by $P(x)$ and $p(x)$ the distribution and density function of X_i, respectively, with $P(0) = 0$, so that all claim amounts are positive. We also assume that the mean of X_i, which we denote by p_1 is finite. We will also assume throughout the paper the existence of higher moments than p_1, and we denote the kth moment about the origin of

X_i by p_k. We will further assume in some parts of the paper the existence of the moment generating function of X_i, which we denote by $m(t)$, and we state that clearly. We define a positive parameter θ to be such that

$$c = (1+\theta)\lambda p_1,$$

so that θ is the insurer's premium loading factor. Without loss of generality, we will make the following two assumptions: $c = p_1 = 1$. We will refer to this process as our *basic process*.

We want to produce a discrete approximation to this basic process and we will consider Dickson et al. (1995) approach considering a discrete time Poisson process

$$U_{\mathrm{d}}(t) = u + t - S_{\mathrm{d}}(t)$$

for $t = 1, 2, \ldots$, with an initial reserve u ($u = 0, 1, 2, \ldots$) and $U_{\mathrm{d}}(0) = u$. Also, we have the following:

- $S_{\mathrm{d}}(t)$ is the aggregate claim amount up to time t, and we denote by $F(x,t)$ and $f(x,t)$ the distribution and density function of $S_{\mathrm{d}}(t)$.
- Individual claims are i.i.d. random variables on the non-negative integers with mean $\beta > 0$. Like in the classical model, we will require the existence of higher moments of individual claims, and we will denote the kth moment about the origin by b_k ($b_1 = \beta$).
- Premium income per unit time is 1.
- Expected number of claims per unit time is $\lambda_{\mathrm{d}} = \lambda/\beta$.

For simplicity we state $f_j = f_{\mathrm{d}}(j,1)$ and $F_j = F_{\mathrm{d}}(j,1)$ for $j = 0, 1, 2, \ldots$. Notice that $S_{\mathrm{d}}(t)$ is the sum of i.i.d. random variables, each with probability function $\{f_j\}_{j=0}^{\infty}$. If $X_{\mathrm{d},n}$ denotes the aggregate claim amount from time $n-1$ until time n, then $S_{\mathrm{d}}(t) = \sum_{n=1}^{t} X_{\mathrm{d},n}$ and f_j is the probability of $X_{\mathrm{d},n}$ taking value j. Values for f_j, $j = 0, 1, 2, \ldots$, can be obtained using Panjer's (1981) recursion.

Time to ruin in the basic process is denoted by T and defined as

$$T = \begin{cases} \inf\{t : U(t) < 0\}, \\ \infty \text{ if } U(t) \geq 0 \ \text{ for all } t > 0, \end{cases}$$

finite time ruin probability from some initial surplus $u \geq 0$ is defined as

$$\psi(u,t) = \Pr[T \leq t | U(0) = u],$$

and ultimate ruin probability $\psi(u) = \Pr[T < \infty | U(0) = u]$. Finite time survival probability is denoted by $\delta(u,t) = 1 - \psi(u,t)$ and ultimate survival probability by $\delta(u) = 1 - \psi(u) = \Pr[T = \infty | U(0) = u]$. It is well known that $\psi(0) = \lambda p_1/c = 1/(1+\theta)$.

For the discrete time model we will use two definitions of ruin, depending on whether or not a surplus of zero, other than at time zero, is regarded as ruin. Accordingly, we define time to ruin as

$$T_{\mathrm{d}} = \begin{cases} \min\{n : U_{\mathrm{d}}(n) < 0, \ n = 1, 2, 3, \ldots\}, \\ \infty \text{ if } U_{\mathrm{d}}(n) \geq 0 \ \text{ for } n = 1, 2, 3, \ldots, \end{cases}$$

$$T_{\mathrm{d}}^{*} = \begin{cases} \min\{n : U_{\mathrm{d}}(n) \leq 0, \ \ n = 1, 2, 3, \ldots\}, \\ \infty \text{ if } U_{\mathrm{d}}(n) > 0 \ \text{ for } n = 1, 2, 3, \ldots. \end{cases}$$

Discrete and finite time ruin probabilities for a given non-negative integer u are

$$\psi_{\mathrm{d}}(u,t) = \Pr[T_{\mathrm{d}} \leq t | U_{\mathrm{d}}(0) = u],$$

$$\psi_{\mathrm{d}}^{*}(u,t) = \Pr[T_{\mathrm{d}}^{*} \leq t | U_{\mathrm{d}}(0) = u].$$

Ultimate ruin probabilities are defined as $\psi_{\mathrm{d}}(u) = \lim_{t\to\infty} \psi_{\mathrm{d}}(u,t)$ and $\psi_{\mathrm{d}}^{*}(u) = \lim_{t\to\infty} \psi_{\mathrm{d}}^{*}(u,t)$, with $\delta_{\mathrm{d}}(u) = 1 - \psi_{\mathrm{d}}(u)$ and $\delta_{\mathrm{d}}^{*}(u) = 1 - \psi_{\mathrm{d}}^{*}(u)$, denoting the corresponding probabilities of ultimate survival. From Dickson and Waters (1991), we have $\psi_{\rm d}^*(0) = f_0\psi_{\rm d}(0) = \psi(0)$. We denote the probability functions of $T_{\rm d}$ and $T_{\rm d}^*$ as $\phi_{\rm d}(u,t)$ and $\phi_{\rm d}^*(u,t)$, respectively.

We also need to define the (defective) distributions of the severity of ruin, probability density and probability functions, for the basic process and discrete model, respectively. Accordingly, we have for $u \ge 0$ and $y > 0$,

$$G(u,y) = \Pr[T < \infty \text{ and } U(T) > -y | U(0) = u],$$

and for $u = 0, 1, 2, \ldots$ and $y = 1, 2, 3, \ldots$,

$$G_{\rm d}(u,y) = \Pr[T_{\rm d} < \infty \text{ and } U(T_{\rm d}) \ge -y | U_{\rm d}(0) = u],$$

$$G_{\rm d}^*(u,y) = \Pr[T_{\rm d}^* < \infty \text{ and } U(T_{\rm d}^*) > -y | U_{\rm d}(0) = u].$$

We denote the density and probability functions by $g(u,y)$, $g_{\rm d}(u,y)$, and $g_{\rm d}^*(u,y)$, respectively. Respective associated random variables are denoted as Y, $Y_{\rm d}$ and $Y_{\rm d}^*$.

We will let the surplus process continue if it falls below zero, i.e. if ruin occurs. Given that ruin occurs, the process is certain to recover to positive levels. Let us define this time as the recovery time or the duration of a negative surplus. Eventually, the process will drift to infinity and the number of occasions it falls below zero can be multiple. For more details see Dickson and Egídio dos Reis (1996).

Let us denote by $\tilde{T}$ and $\tilde{T}_{\rm d}$ the duration of the first period of negative surplus once ruin has occurred in the classical and discrete time models, respectively. These random variables depend on the initial surplus u. For the discrete time model, $\tilde{T}_{\rm d}$ stands for the recovery time up to non-ruin level zero, according to the first definition of ruin. Let $\alpha_{\rm d}(u,t)$ be the probability function of $\tilde{T}_{\rm d}$. We consider this function to be defective, i.e. $\alpha_{\rm d}(u,t)$ represents the probability that ruin occurs from initial surplus u and the surplus takes t periods to reach the level zero for the first time after $T_{\rm d}$. Hence, we have that $t = 1, 2, \ldots$ We denote by $\tilde{T}_{\rm d}^*$ the recovery time associated to the second definition of ruin, and $\alpha_{\rm d}^*(u,t)$ is its probability function. In Section 6 we use some conditional random variables, given that $T < \infty$. When this is the case we write the variables with a subscript c.

According to Dickson et al. (1995, Section 2), $\psi_{\rm d}(u\beta, \beta t)$ for some positive β is an approximation for $\psi(u,t)$. Furthermore, they explain that $\psi_{\rm d}^*(u\beta, \beta t)$ is a better approximation than $\psi_{\rm d}(u\beta, \beta t)$. As far as approximations to the basic process is concerned, we will consider the second definition of ruin in the discrete model for the different quantities we want to compute in this paper. We will compute approximate values for the conditional moments of time to ruin and time to recovery, given that $T < \infty$, from initial surplus u, denoted by $E[T^k|u]/\psi(u)$ and $E[\tilde{T}^k|u]/\psi(u)$, for $k = 1, 2, \ldots$ Respective approximations we consider will be $\beta^{-k}E[T_{\rm d}^{*k}|\beta u]/\psi_{\rm d}^*(\beta u)$ and $\beta^{-k}E[\tilde{T}_{\rm d}^{*k}|\beta u]/\psi_{\rm d}^*(\beta u)$.

3. On the time to ruin

For the time to ruin we can retrieve de Vylder and Goovaerts' (1988) formulae. Considering aggregate claims at the end of the first period in the discrete time model we have

$$\psi_{\rm d}(u,t) = \sum_{j=0}^{u+1} f_j \psi_{\rm d}(u+1-j, t-1) + (1 - F_{u+1}) \quad \text{for } t > 1, \tag{1}$$

with $\psi_{\rm d}(u,1) = \phi_{\rm d}(u,1) = 1 - F_{u+1}$. As far as the probability function is concerned, we get its respective version for $t > 1$,

$$\phi_{\rm d}(u,t) = \sum_{j=0}^{u+1} f_j \phi_{\rm d}(u+1-j, t-1). \tag{2}$$

We assume that the rth moment of $T_{\rm d}$, denoted as $E[{T_{\rm d}}^r|u]$, exists for a given initial surplus $u \geq 0$ ($r = 1, 2, \ldots$). As far as the classical model is concerned, Delbaen (1988) shows that the rth moment of the time to ruin depends on the existence of p_{r+1}. We can use formula (2) to find a recursion for $E[{T_{\rm d}}^r|u]$. The computation of $E[{T_{\rm d}}^r|u]$ can then be used for the approximation of the corresponding quantity in the classical model. In the last section we show examples with the kind of approximations we can make with this recursion. We use (Gerber's (1979) exact formulae when $P(x)$ is exponential. Also, we use Lin and Willmot (2000) formulae when $P(x)$ is a particular combination of exponentials and an Erlang(2). For $u = 0$ we can use formulae derived by Egídio dos Reis (1993), since in the classical model the conditional distribution of time to recovery and time to ruin, given ruin occurs, have the same distribution. We will consider the calculation of the expected value and variance only.

Using (2) we have, for a given value of $u \geq 0$,

$$
\begin{aligned}
E[T_{\rm d}^r|u] &= \sum_{i=1}^{\infty} i^r \phi_{\rm d}(u,i) = \phi_{\rm d}(u,1) + \sum_{i=2}^{\infty} i^r \sum_{j=0}^{u+1} f_j \phi_{\rm d}(u+1-j, i-1) \\
&= \phi_{\rm d}(u,1) + \sum_{j=0}^{u+1} f_j \sum_{i=2}^{\infty} i^r \phi_{\rm d}(u+1-j, i-1) = \phi_{\rm d}(u,1) + \sum_{j=0}^{u+1} f_j \sum_{i=1}^{\infty} (i+1)^r \phi_{\rm d}(u+1-j, i).
\end{aligned}
$$

Substituting

$$
(i+1)^r = \sum_{k=0}^{r} \binom{r}{k} i^k
$$

and interchanging summations we get

$$
\begin{aligned}
E[T_{\rm d}^r|u] &= \phi_{\rm d}(u,1) + \sum_{j=0}^{u+1} f_j \sum_{k=0}^{r} \binom{r}{k} \sum_{i=1}^{\infty} i^k \phi_{\rm d}(u+1-j, i) \\
&= \phi_{\rm d}(u,1) + \sum_{j=0}^{u+1} f_j \sum_{i=1}^{\infty} \phi_{\rm d}(u+1-j, i) + \sum_{j=0}^{u} f_j \sum_{k=1}^{r} \binom{r}{k} \sum_{i=1}^{\infty} i^k \phi_{\rm d}(u+1-j, i) \\
&= \phi_{\rm d}(u,1) + \sum_{j=0}^{u+1} f_j \psi_{\rm d}(u+1-j) + \sum_{j=0}^{u+1} f_j \sum_{k=1}^{r} \binom{r}{k} E[T_{\rm d}^k|u+1-j],
\end{aligned}
$$

since $T_{\rm d}$ is a defective random variable, i.e. $\sum_{i=1}^{\infty} \phi_{\rm d}(u,i) = \psi_{\rm d}(u)$,

$$
E[T_{\rm d}^r|u] = 1 - F_{u+1} + \sum_{j=0}^{u+1} f_j \psi_{\rm d}(u+1-j) + \sum_{k=1}^{r} \binom{r}{k} \sum_{j=0}^{u+1} f_j E[T_{\rm d}^k|u+1-j]. \tag{3}
$$

For instance, if we want to compute the mean of $T_{\rm d}$ we get, solving for $E[T_{\rm d}|u+1]$ from (3) with $r = 1$, and $u = 0, 1, 2, \ldots$,

$$
E[T_{\rm d}|u+1] = f_0^{-1} \left(E[T_{\rm d}|u] - \left(1 - F_{u+1} + \sum_{j=0}^{u+1} f_j \psi_{\rm d}(u+1-j) + \sum_{j=1}^{u+1} f_j E[T_{\rm d}|u+1-j] \right) \right). \tag{4}
$$

If we want to compute the variance of the (defective) random variable T_{d}, we will need to compute $E[T_{\mathrm{d}}^2|u]$. From (3), we have

$$E[T_{\mathrm{d}}^2|u] = 1 - F_{u+1} + \sum_{j=0}^{u+1} f_j \psi_{\mathrm{d}}(u+1-j) + 2\sum_{j=0}^{u+1} f_j E[T_{\mathrm{d}}|u+1-j] + \sum_{j=0}^{u+1} f_j E[T_{\mathrm{d}}^2|u+1-j], \tag{5}$$

from which we get, solving for $E[T_{\mathrm{d}}^2|u+1]$, for $u = 0, 1, 2, \ldots$,

$$E[T_{\mathrm{d}}^2|u+1] = f_0^{-1}\left(E[T_{\mathrm{d}}^2|u] - \left(1 - F_{u+1} + \sum_{j=0}^{u+1} f_j \psi_{\mathrm{d}}(u+1-j) + 2\sum_{j=0}^{u+1} f_j E[T_{\mathrm{d}}|u+1-j] + \sum_{j=1}^{u+1} f_j E[T_{\mathrm{d}}^2|u+1-j]\right)\right). \tag{6}$$

Taking a look at the recursion for $E[T_{\mathrm{d}}|u+1]$ in (4) we see that we need all the ultimate ruin probabilities $\psi_{\mathrm{d}}(j)$ for $j = 0, 1, \ldots, u+1$, as well as all the mean values $E[T_{\mathrm{d}}|j]$ for $j = 0, 1, \ldots, u$. The recursion for the second moment, (6), will also need the same ruin probabilities as well as all the mean values up to $E[T_{\mathrm{d}}|u+1]$ and all the previous second moment values, from 0 up to u. For the computation of the ruin probabilities we can use the recursions by Dickson et al. (1995, formulae (3.3) and (3.8)).

We need to find a formula for the starting value, when $u = 0$. We will come to this later. Let us first deal with the computation of the moments for the duration of the first period of negative surplus, or time to recovery to positive values after ruin has occurred. The reason for this approach is that we will pay attention to the relationship between T_{d} and $\tilde{T}_{\mathrm{d}}$ when $u = 0$.

To compute the moments of T_{d}^*, we note that Dickson and Waters (1991) explain that for $u = 1, 2, \ldots$, $\phi_{\mathrm{d}}^*(u,t) = \phi_{\mathrm{d}}(u-1,t)$, hence $E[T_{\mathrm{d}}^{*r}|u] = E[T_{\mathrm{d}}^r|u-1]$.

4. On the time to recovery

We consider the discrete time model. We will let the surplus process continue if it falls below zero, i.e. if ruin occurs. After ruin has occurred, the process is certain to recover to positive levels at some point. Eventually, the process will drift to infinity and the number of occasions on which it falls below zero can be multiple. For more details see Dickson and Egídio dos Reis (1996) who consider this discrete model. We note that the recovery time $\tilde{T}_{\mathrm{d}}$ stands for the time that the surplus after having fallen below zero recovers to level zero for the first time. Its probability function has been defined as $\alpha(u,t)$ for $t = 1, 2, \ldots$.

Let us define $\tilde{T}_{\mathrm{d}}(x)$ as the time that the surplus process $U_{\mathrm{d}}(t)$ starting from zero takes to reach a fixed positive level x ($x = 1, 2, \ldots$) for the first time. Its probability function is given by

$$\frac{x}{t} f_{\mathrm{d}}(t-x, t), \tag{7}$$

with $t \geq x$ (see Gerber (1979, p. 21)). That is, we consider

$$U_{\mathrm{d}}(t) = t - S_{\mathrm{d}}(t),$$

and

$$\tilde{T}_{\mathrm{d}}(x) = \min\{t : U_{\mathrm{d}}(t) = x | U_{\mathrm{d}}(0) = 0\} \quad \text{for } x = 1, 2, 3, \ldots,$$

where $S_{\mathrm{d}}(t)$ is defined as in Section 2. $S_{\mathrm{d}}(t)$ is the aggregate claim amount up to time t, so its distribution is a discrete compound Poisson with Poisson parameter λ_{d}. We further assume that the moment generating function of individual claims in this discrete model exists and we denote it by $m_{\mathrm{d}}(r)$. Hence, the moment generating function of $S_{\mathrm{d}}(t)$ is

$$M_{S_{\mathrm{d}}(t)}(r) = [M_{X_{\mathrm{d}}}(r)]^t = \exp\{\lambda_{\mathrm{d}} t(m_{\mathrm{d}}(r) - 1)\},$$

since $M_{X_{\mathrm{d}}}(r) = \exp\{\lambda_{\mathrm{d}}(m_{\mathrm{d}}(r) - 1)\}$. We can find the moment generating function of $\tilde{T}_{\mathrm{d}}(x)$ by means of the martingale method used by Gerber (1990) for the corresponding compound Poisson continuous time model. In this case we have a discrete time martingale

$$\{\exp\{-f(s)U_{\mathrm{d}}(t) + st\},$$

where $f(s)$ is some function of $s \leq 0$ such that

$$s = f(s) - \lambda_{\mathrm{d}}[m_{\mathrm{d}}(f(s)) - 1].$$

Hence, we get that

$$E[\mathrm{e}^{s\tilde{T}(x)}] = \mathrm{e}^{f(s)x}.$$

The expression above leads to, for instance using the cumulant generating function (see Gerber, 1990)),

$$E[\tilde{T}(x)] = \frac{x}{(1 - \lambda_{\mathrm{d}}\beta)}, \qquad V[\tilde{T}(x)] = \frac{x\lambda_{\mathrm{d}} b_2}{(1 - \lambda_{\mathrm{d}}\beta)^3} = \frac{xV[X_i]}{(1 - \lambda_{\mathrm{d}}\beta)^3}. \tag{8}$$

We see that, like in the continuous model, the mean of $\tilde{T}(x)$ equals x divided by the expected profit per unit time (c is equal to 1). Getting back to our main problem with the random variable $\tilde{T}_{\mathrm{d}}$, we have that if the deficit at the time of ruin is j, $j = 1, 2, \ldots$, then the probability that the surplus returns to zero at time $T_{\mathrm{d}} + t$ ($t \geq j$) is given by (7). Hence,

$$\alpha_{\mathrm{d}}(u, t) = \sum_{j=1}^{t} g_{\mathrm{d}}(u, j)\frac{j}{t} f_{\mathrm{d}}(t - j, t) \quad \text{for } t = 1, 2, 3, \ldots. \tag{9}$$

If we want to compute the rth moment of $\tilde{T}_{\mathrm{d}}$ for a given u, we have that

$$E[\tilde{T}_{\mathrm{d}}{}^r|u] = \sum_{t=1}^{\infty} t^r \alpha_{\mathrm{d}}(u, t) = \sum_{t=1}^{\infty} t^r \sum_{j=1}^{t} g_{\mathrm{d}}(u, j)\frac{j}{t} f_{\mathrm{d}}(t - j, t) = \sum_{j=1}^{\infty} g_{\mathrm{d}}(u, j)\sum_{t=j}^{\infty} jt^{r-1} f_{\mathrm{d}}(t - j, t)$$

after interchanging the order of the summations. In another way,

$$E[\tilde{T}_{\mathrm{d}}{}^r|u] = \sum_{j=1}^{\infty} g_{\mathrm{d}}(u, j)E[\tilde{T}(j)^r] = E[E[\tilde{T}(Y_{\mathrm{d}})^r|Y_{\mathrm{d}}]|u],$$

where Y_{d} denotes the severity of ruin (defective) with probability function $g_{\mathrm{d}}(u, y)$. From here we get, for instance,

$$E[\tilde{T}_{\mathrm{d}}|u] = \frac{E[Y_{\mathrm{d}}|u]}{(1 - \lambda_{\mathrm{d}}\beta)}, \quad E[\tilde{T}_{\mathrm{d}}{}^2|u] = \frac{E[Y_{\mathrm{d}}|u]\lambda_{\mathrm{d}} b_2}{(1 - \lambda_{\mathrm{d}}\beta)^3} + \frac{E[Y_{\mathrm{d}}{}^2|u]}{(1 - \lambda_{\mathrm{d}}\beta)^2}.$$

We can use the recursions in Dickson et al. (1995, Section 4) for the moments of Y_{d}. Like its continuous analogue, once we get the moments of the severity of ruin we will be able to compute the moments of the duration of a negative surplus.

We will approximate the first two moments of $\tilde{T}$ by using the moments

$$E[\tilde{T}_{\rm d}^{*}|u]=\frac{E[Y_{\rm d}^{*}|u]}{(1-\lambda_{\rm d}\beta)},\qquad E[\tilde{T}_{\rm d}^{*2}|u]=\frac{E[Y_{\rm d}^{*}|u]\lambda_{\rm d}b_2}{(1-\lambda_{\rm d}\beta)^3}+\frac{E[Y_{\rm d}^{*2}|u]}{(1-\lambda_{\rm d}\beta)^2}$$

calculating for $u=1,2,\ldots,$

$$E[Y_{\rm d}^{*}|u]=E[Y_{\rm d}|u-1]-\psi_{\rm d}(u-1),\qquad E[Y_{\rm d}^{*2}|u]=E[Y_{\rm d}^{2}|u-1]-2E[Y_{\rm d}|u-1]+\psi_{\rm d}(u-1).$$

5. Formulae for $u=0$

For $u=0$ we have from (9) that

$$\alpha_{\rm d}(0,t)=\sum_{j=1}^{t}g_{\rm d}(0,j)\frac{j}{t}f_{\rm d}(t-j,t)\quad \text{for } t=1,2,3,\ldots,$$

and Dickson and Egídio dos Reis (1996) have defined using the second definition of ruin in Section 2 that for $t=1,2,\ldots,$

$$\alpha_{\rm d}^{*}(0,t)=\sum_{j=1}^{t}g_{\rm d}^{*}(0,j)\frac{j}{t}f_{\rm d}(t-j,t),$$

where $g_{\rm d}^{*}(0,y)=1-F_y=f_0g_{\rm d}(0,y)$ (see Dickson et al., 1995)), giving for $t=1,2,\ldots,\alpha_{\rm d}^{*}(0,t)=f_0\alpha_{\rm d}(0,t)$. Also, Dickson and Egídio dos Reis (1996) have showed that $\phi_{\rm d}^{*}(0,t+1)=\alpha_{\rm d}^{*}(0,t)$. It is easy to show that $\phi_{\rm d}^{*}(u,t+1)=f_0\phi_{\rm d}(0,t)$, hence $\alpha_{\rm d}(0,t)=\phi_{\rm d}(0,t)$. This is the discrete counterpart of the relationship between time to ruin and the duration of a negative surplus in the classical model explained by Egídio dos Reis (1993).

Hence, we can establish the starting values for recursions in Section 3, having for the first two moments

$$E[T_{\rm d}|0]=\frac{E[Y_{\rm d}|0]}{(1-\lambda_{\rm d}\beta)},\qquad E[T_{\rm d}^{2}|0]=\frac{E[Y_{\rm d}|0]\lambda_{\rm d}b_2}{(1-\lambda_{\rm d}\beta)^3}+\frac{E[Y_{\rm d}^{2}|0]}{(1-\lambda_{\rm d}\beta)^2},$$

and from Dickson et al. (1995),

$$E[Y_{\rm d}^{*}|0]=\tfrac{1}{2}(E[S_{\rm d}(1)^2]-E[S_{\rm d}(1)]),\qquad E[Y_{\rm d}^{*2}|0]=\tfrac{1}{3}E[S_{\rm d}(1)^3]-\tfrac{1}{2}E[S_{\rm d}(1)^2]+\tfrac{1}{6}E[S_{\rm d}(1)],$$

with $E[Y_{\rm d}^{k}|0]=f_0^{-1}E[Y_{\rm d}^{*k}|0]$ for $k=1,2,\ldots$.

6. Some further comments on the classical model

In this section we will assume that the moment generating function of $P(x)$ exists. Like in the discrete model, the moments of the duration of a negative surplus rely on the existence of the corresponding moments of the severity of ruin in the classical model. Hence we focus on the severity of ruin quantities. For $u=0$, the moments of the severity of ruin have a simple expression. We will show that we can also have a simple expression for the asymptotic moments, as $u\to\infty$, which will allow us to compute easily, at least numerically, the corresponding moments of the random variable time to recovery. For practical use the asymptotic formulae can be quite useful when the numerical calculation reveals to be unstable, like the method by Dickson et al. (1995), giving inaccurate values for very high values of u.

It is easy to show that $E[Y^k|u=0]=\lambda p_{k+1}/c(k+1)$, $k=1,2,\ldots$, whenever p_{k+1} exists. If $m(t)$ exists, we can also express the moment generating function of Y as

$$M_Y(0,t)=\frac{\lambda}{ct}[m(t)-1]$$

(see for instance Egídio dos Reis (1993)). Let us work now the asymptotic moment generating function of Y as $u\to\infty$.

From Gerber (1974) we have that the conditional density of the severity of ruin, given that ruin occurs, denoted as $\tilde{g}(\infty,y)$, is given by

$$\tilde{g}(\infty,y)=\frac{\lambda R}{c\delta(0)}\int_0^\infty \mathrm{e}^{Rx}[1-P(x+y)]\,\mathrm{d}x, \tag{10}$$

where R is the adjustment coefficient satisfying the following equation:

$$\frac{\lambda}{c}\int_0^\infty \mathrm{e}^{Rx}[1-P(x)]\,\mathrm{d}x=1. \tag{11}$$

Eq. (10) can be expressed as

$$\tilde{g}(\infty,y)=\frac{\lambda R}{c\delta(0)}\mathrm{e}^{-Ry}\int_y^\infty \mathrm{e}^{Rx}[1-P(x)]\,\mathrm{d}x. \tag{12}$$

Hence, the moment generating function corresponding to the above density, denoted as $M_{Y_c}(\infty,t)$ becomes

$$M_{Y_c}(\infty,t)=\frac{\lambda R}{c\delta(0)}\int_0^\infty \mathrm{e}^{(t-R)y}\int_y^\infty \mathrm{e}^{Rx}[1-P(x)]\,\mathrm{d}x,$$

giving for $t\neq R$,

$$M_{Y_c}(\infty,t)=\frac{\lambda R}{c\delta(0)(t-R)}\left(\int_0^\infty \mathrm{e}^{tx}[1-P(x)]\,\mathrm{d}x-1\right),$$

interchanging the order of the integration and introducing (11). Expressing in another way we have

$$M_{Y_c}(\infty,t)=\frac{R}{\delta(0)(t-R)}(M_Y(0,t)-1), \tag{13}$$

since $g(0,x)=\lambda[1-P(x)]/c$. For $t=R$ we get Gerber's (1974) formula, i.e

$$M_{Y_c}(\infty,R)=\delta(0)^{-1}\left(\frac{\lambda}{c}m'(R)-1\right),$$

where $m'(R)$ denotes the derivative of $m(t)$ evaluated at R.

From expression (13), we get

$$E[Y_c|\infty]=\frac{1}{R}-\frac{E[Y|u=0]}{\delta(0)},\qquad E[{Y_c}^2|\infty]=\frac{2}{R^2}-\frac{2E[Y|u=0]}{\delta(0)}-\frac{E[Y^2|u=0]}{\delta(0)}.$$

If we take the expression for $E[Y_c|\infty]$, we obtain (Gerber's (1979), p. 128) upper bound for R, since $E[Y_c|\infty]>0$. If we express (12) as

$$\tilde{g}(\infty,y)=\frac{R}{\delta(0)}\mathrm{e}^{-Ry}\left(1-\int_0^y \frac{\lambda}{c}\mathrm{e}^{Rx}[1-P(x)]\,\mathrm{d}x\right),$$

we see that this density can be viewed as a combination of an exponential (with mean $1/R$) and some other density, with weights $1/\delta(0)$ and $-\psi(0)/\delta(0)$, respectively.

Hence, we can find the asymptotic moment generating function of the duration of a negative surplus, conditional on $T < \infty$,

$$M_{\tilde{T}_c}(\infty, s) = M_{Y_c}(\infty, f(s)),$$

where $f(s)$ is some function of s such that $s = f(s)c - \lambda[m(f(s)) - 1]$ and $s, f(s) \leq 0$ (see Egídio dos Reis, 1993)). Using $M_Y(0, t)$ above we can express $M_{\tilde{T}_c}(\infty, s)$ as

$$M_{\tilde{T}_c}(\infty, s) = \frac{Rs}{c\delta(0) f(s)(R - f(s))}.$$

7. Examples

In this section we present three examples for which we considered the calculation of the first two moments of both ruin and the recovery times.

As far as time to ruin is considered we first need to compute the first two moments of the severity of ruin when $u = 0$. For the time to recovery, we need the corresponding moments of the severity of ruin for the corresponding initial surplus. For these situations, as shown by Dickson et al. (1995), we will need the first three moments of the individual claim amount distribution. Panjer and Lutek (1993) describe a method which provides the discretization of the individual claim amount distribution that preserves the original moments which we have adopted. Dickson et al. (1995) refer to some problems with this discretization method. We have used the software Mathematica for the calculations in the discretization procedure.

In the examples we show values for $E[T_c|u]$ and $E[{T_c}^2|u]$ together with the respective approximating values $\beta^{-1}E[T_{\mathrm{d}}^*|\beta u]/\psi_{\mathrm{d}}^*(\beta u)$ and $\beta^{-2}E[T_{\mathrm{d}}^{*2}|\beta u]/\psi_{\mathrm{d}}^*(\beta u)$. We do not show values for the same quantities of $\tilde{T}_c$ as it is obvious that they depend solely on the approximations for the respective moments of the severity of ruin, and Dickson et al. (1995) already showed examples for the severity of ruin random variable using the same discrete model. We have set $c = 1$, so that $\lambda = 1/(1+\theta)$ and $\theta = 0.1$. In all the computations below, we have used a $\beta = 100$.

Example 1 (Exponential claim amounts). We considered exponentially distributed claim amounts, i.e. $P(x) = 1 - \mathrm{e}^{-\alpha x}$, $x \geq 0$, and we have set $\alpha = 1$, so that it has a mean 1. We can find easily the required moments for both the time to ruin and the time to recovery. Considering the time to ruin, we get from Gerber (1979) an expression for the moment generating function of T_c, which is given by

$$E[\mathrm{e}^{sT}|T < \infty] = \frac{c}{\lambda}(\alpha - f(s))\,\mathrm{e}^{-(f(s)-R)u},$$

where $f(s)$ is some function of s such that

$$s = f(s)c - \lambda[m(f(s)) - 1]$$

for $R \leq f(s) < \alpha$ and $R = \alpha - \lambda/c$ is the adjustment coefficient. We note that $f(s)$ is uniquely defined for $s \leq 0$ from $R \leq f(s) < \alpha$, and that $f(0) = R$ since $\lambda + cR = \lambda m(R)$. A graph of the above function is shown by Egídio dos Reis (1993). If we take the cumulant generating function, it is easy to show that

$$E[T_c|u] = \frac{1 + \lambda u/c}{\alpha c - \lambda} = E[T_c|0]\left(1 + \frac{\lambda}{c}u\right)$$

and

$$V[T_c|u] = \frac{\alpha c + \lambda + 2\alpha\lambda u}{(\alpha c - \lambda)^3} = V[T_c|0] + \frac{2\alpha\lambda}{(\alpha c - \lambda)^3}u,$$

Table 1
First and second moments of T_c for exponential claim size

u	(1)	(2)	(2)/(1)	(4)	(5)	(5)/(4)
0	11	10.99500	0.99955	2662	2661.88985	0.99996
1	21	21.00500	1.00024	5402	5402.20968	1.00004
2	31	31.00499	1.00016	8342	8342.30952	1.00004
3	41	41.00499	1.00012	11482	11482.40935	1.00004
4	51	51.00499	1.00010	14822	14822.50918	1.00003
5	61	61.00499	1.00008	18362	18362.60902	1.00003
6	71	71.00499	1.00007	22102	22102.70885	1.00003
7	81	81.00499	1.00006	26042	26042.80868	1.00003
8	91	91.00499	1.00005	30182	30182.90852	1.00003
9	101	101.00499	1.00005	34522	34523.00835	1.00003
10	111	111.00499	1.00004	39062	39063.10818	1.00003
15	161	161.00499	1.00003	64762	64763.60735	1.00002
20	211	211.00499	1.00002	95462	95464.10651	1.00002
30	311	311.00499	1.00002	171862	171865.10481	1.00002
40	411	411.00499	1.00001	268262	268266.10298	1.00002
50	511	511.00499	1.00001	384662	384667.10113	1.00001
100	1011	1011.00633	1.00001	1266662	1266673.63702	1.00001

and we can compare the approximations for these moments given by using the appropriate discrete model described earlier.

In Table 1 we show values for $E[T_c|u]$ and $E[{T_c}^2|u]$ together with the respective approximating values $\beta^{-1}E[T_\mathrm{d}^*|\beta u]/\psi_\mathrm{d}^*(\beta u)$ and $\beta^{-2}E[T_\mathrm{d}^{*2}|\beta u]/\psi_\mathrm{d}^*(\beta u)$ for different values of initial surplus u. The key for the table is the following: (1) and (4) show the true values of $E[T_c|u]$ and $E[{T_c}^2|u]$, (2) and (5) show the approximations for these quantities, respectively; columns 4 and 7 of the table show the ratios (2)/(1) and (5)/(4), respectively.

Example 2 (Combination of exponentials claim amounts). In this example we took a combination of two exponentials presented by Gerber et al. (1987, Section 5) which we rescaled to have mean 1. That is $p(x) = 7\,\mathrm{e}^{-(7/4)x} - 7\,\mathrm{e}^{-(7/3)x}$, $x \geq 0$.

Following the method by Lin and Willmot (2000), and using a similar notation we get $\psi(u) = -0.00952025\,\mathrm{e}^{-3.05264u} + 0.918611\,\mathrm{e}^{-0.121604u}$, $E[T_c|u] = \psi_1(u)/\psi(u)$ and $E[{T_c}^2|u] = \psi_2(u)/\psi(u)$, where

$$\begin{aligned}\psi_1(u) &= \mathrm{e}^{-3.17424u}[\mathrm{e}^{0.121604u}(0.8221 + 0.000996986u) + \mathrm{e}^{3.05264u}(6.72892 + 9.28231u)],\\ \psi_2(u) &= \mathrm{e}^{-18.8022u}[93.7952\,\mathrm{e}^{18.6806u}(0.699241 + u)(18.6476 + u)\\ &\qquad -0.000104407\,\mathrm{e}^{15.7496u}(-609.786 + u)(2255.98 + u)].\end{aligned}$$

Table 2 shows values for $E[T_c|u]$ and $E[{T_c}^2|u]$ together with the respective approximating values $\beta^{-1}E[T_\mathrm{d}^*|\beta u]/\psi_\mathrm{d}^*(\beta u)$ and $\beta^{-2}E[T_\mathrm{d}^{*2}|\beta u]/\psi_\mathrm{d}^*(\beta u)$ for different values of initial surplus u. The key for this table is the same as in Example 1.

If we look at the approximating values and compare with the previous examples we see that they show a similar pattern, although not as good as before. It is readable that the algorithm show signs of instability for very high values of the initial surplus.

Example 3 (Erlang(2) claim amounts). In this example we consider a Gamma(2,2) claim amount distribution with p.d.f $p(x) = 4x\,\mathrm{e}^{-2x}$, $x \geq 0$.

From Egídio dos Reis (1993) we know that $\psi(u) = -0.010092\,\mathrm{e}^{-2.96841u} + 0.919183\,\mathrm{e}^{-0.122502u}$. We computed the values for $E[T_c|u]$ and $E[{T_c}^2|u]$ given by Lin and Willmot (2000) formulae with the help of Mathe-

Table 2
First and second moments of T_c for a combination of two exponentials claim size

u	(1)	(2)	(2)/(1)	(4)	(5)	(5)/(4)
0	8.30612	8.30112	0.99940	1503.30279	1503.21944	0.99994
1	17.48729	17.49228	1.00029	3419.11003	3419.28422	1.00005
2	27.53791	27.54290	1.00018	5691.23339	5691.50765	1.00005
3	37.63947	37.64445	1.00013	8176.60121	8176.97602	1.00005
4	47.74400	47.74899	1.00010	10866.72068	10867.19605	1.00004
5	57.84872	57.85370	1.00009	13761.08329	13761.65918	1.00004
6	67.95344	67.95842	1.00007	16859.65866	16860.33506	1.00004
7	78.05816	78.06314	1.00006	20162.44498	20163.22187	1.00004
8	88.16288	88.16786	1.00006	23669.44216	23670.31950	1.00004
9	98.26761	98.27259	1.00005	27380.65017	27381.62796	1.00004
10	108.37233	108.37731	1.00005	31296.06903	31297.14723	1.00003
15	158.89594	158.90091	1.00003	53936.32596	53937.90605	1.00003
20	209.41956	209.42450	1.00002	81681.85394	81683.93635	1.00003
30	310.46678	310.47160	1.00002	152488.72308	152491.82578	1.00002
40	411.51401	411.51840	1.00001	243716.67644	243720.91937	1.00002
50	512.56124	512.56420	1.00001	355365.71403	355371.72337	1.00002
100	1017.79737	1016.91321	0.99913	1219927.16536	1221303.49553	1.00113

Table 3
First and second moments of T_c for a Gamma(2,2) claim size

u	(1)	(2)	(2)/(1)	(4)	(5)	(5)/(4)
0	8.25000	8.24500	0.99939	1482.25000	1482.16735	0.99994
1	17.39325	17.39824	1.00029	3377.48192	3377.65549	1.00005
2	27.44252	27.44751	1.00018	5634.27252	5634.54637	1.00005
3	37.54960	37.55459	1.00013	8105.68849	8106.06317	1.00005
4	47.66036	47.66535	1.00010	10782.23238	10782.70795	1.00004
5	57.77136	57.77635	1.00009	13663.28456	13663.86101	1.00004
6	67.88237	67.88736	1.00007	16748.80468	16749.48201	1.00004
7	77.99338	77.99837	1.00006	20038.79012	20039.56834	1.00004
8	88.10439	88.10938	1.00006	23533.24073	23534.11983	1.00004
9	98.21541	98.22039	1.00005	27232.15649	27233.13647	1.00004
10	108.32642	108.33141	1.00005	31135.53739	31136.61826	1.00003
15	158.88148	158.88647	1.00003	53719.41914	53721.00443	1.00003
20	209.43654	209.44153	1.00002	81414.92960	81417.01931	1.00003
30	310.54667	310.55166	1.00002	152140.83665	152143.93529	1.00002
40	411.65679	411.66179	1.00001	243313.25854	243317.36726	1.00002
50	512.76692	512.77193	1.00001	354932.19526	354937.32262	1.00001
100	1018.31754	1018.38733	1.00007	1219724.60140	1219786.85688	1.00005

matica, as their expressions lead to infinite series. Table 3 shows values for $E[T_c|u]$ and $E[{T_c}^2|u]$ together with the respective approximating values $\beta^{-1}E[T_{\mathrm{d}}^*|\beta u]/\psi_{\mathrm{d}}^*(\beta u)$ and $\beta^{-2}E[T_{\mathrm{d}}^{*2}|\beta u]/\psi_{\mathrm{d}}^*(\beta u)$, produced by the proposed recursions, for different values of initial surplus u. The key for this table is the same as in the previous examples.

The approximating figures in this case show a similar accuracy when compared with the previous examples.

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