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ABSTRACT

The dual risk model assumes that the surplus of a company decreases at a constant rate over time, and grows by means of upward jumps which occur at random times with random sizes. In the present work, we study the dual risk renewal model when the waiting times are phase-type distributed. Using the roots of the fundamental and the generalized Lundberg's equations, we get expressions for the ruin probability and the Laplace transform of the time of ruin for an arbitrary single gain distribution. Then, we address the calculation of expected discounted future dividends particularly when the individual common gains follow a phase-type distribution. We further show that the optimal dividend barrier does not depend on the initial reserve. As far as the roots of the Lundberg equations and the time of ruin are concerned, we address the existing formulae in the corresponding Sparre-Andersen insurance risk model for the first hitting time, and we generalize them to cover also the situations where we have multiple roots. We do that working a new approach and technique, approach we also use for working the dividends, unlike others, it can be also applied for every situation.

1. Introduction

We consider the dual risk model where the surplus or equity of the company is commonly described by the equation

$$U(t) = u - ct + S(t), \ t \geq 0, \ u \geq 0, \quad \text{with} \quad S(t) = \sum_{i=0}^{N(t)} X_i \quad \text{and} \quad X_0 \equiv 0.$$

Above, $u \geq 0$ is the initial surplus, c is the constant rate at which costs are paid, $\{X_i\}_{i=1}^{\infty}$ denote the sequence of random gains, $N(t)$ and $S(t)$ are the random number of gains and aggregate gains occurring up to time t, respectively. Corresponding random processes are denoted by $\{N(t), \ t \geq 0\}$ and $\{S(t), \ t \geq 0\}$. The model is called *dual* as opposed to the well-known Cramér–Lundberg insurance risk model, which consists of constant premiums instead of constant costs, and a sequence of claims rather than a sequence of gains. We will refer this one as the primal risk model. We denote by W_i the inter-arrival time between gains X_{i-1} and X_i, $i = 2, 3, \ldots$ (W_1 is the waiting time up gain X_1). We assume that $\{X_i\}_{i=1}^{\infty}$ and $\{W_i\}_{i=1}^{\infty}$ are sequences of i.i.d. random variables and independent from one another. Let $P(x)$ denote the common cumulative distribution function of the gains $\{X_i\}_{i=1}^{\infty}$, $p(x)$ and $\hat{p}(s)$ are the corresponding density and Laplace transform evaluated at s, respectively. We denote by $K(.), k(.)$ and $\hat{k}(.)$, respectively, the common cumulative distribution, density, and Laplace

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transform of $\{W_i\}_{i=1}^{\infty}$, which we will assume to be phase-type distributed. We assume the existence of $\mu_1 = E[X_1]$ and the *net profit condition*, i.e.

$$cE(W_1) < E(X_1) = \mu_1. \tag{1.1}$$

This condition means that on average gains are greater than expenses, per unit time. When compared to the primal model, the income condition is reversed.

The dual risk model has had an increasing interest in ruin theory in recent times. There are many possible interpretations for the model. We can look at the surplus as the amount of capital of a business engaged in research and development, where gains are random, at random instants, and costs are certain. More precisely, the company pays expenses which occur continuously along time for the research activity, gets occasional revenues according to a phase-type(n) distribution and of size driven by distribution $P(\cdot)$. Revenues can be interpreted as values of future gains from an invention or discovery, the decrease of surplus can represent costs of production, payments to employees, maintenance of equipment, etc.

Among pioneer works on the subject we can cite Cramér (1955), Takács (1967), Seal (1969), Bühlman (1970) and Gerber (1979). Recent works include those by Avanzi et al. (2007), Albrecher et al. (2008), Avanzi & Gerber (2008), Bayraktar & Egami (2008), Cheung & Drekic (2008), Gerber & Smith (2008), Song et al. (2008), Yang & Zhu (2008), Avanzi (2009), Ng (2009), Ng (2010), Cheung (2012), Afonso et al. (2013), Rodríguez-Martínez et al. (2015) and Sendova and Yang (2014). Many published works concerning the dual model deal with the compound Poisson model. We particularly reference to the work by Avanzi et al. (2007) that explains well where applications of the dual model are appropriate. On the same reason Bayraktar & Egami (2008) used it to model capital investments. Optimal strategies were analyzed by Avanzi et al. (2007), Avanzi & Gerber (2008) and in the review paper of Avanzi (2009), see also references therein. There are also works considering more general distributions. We can mention Rodríguez-Martínez et al. (2015), Wen & Yin (2012) and Sendova and Yang (2014), who studied ruin probabilities and dividend problems for a dual risk model with Erlang and generalized Erlang distributed inter-arrival times, respectively. We also underline the work by Afonso et al. (2013) who, among other problems, give a different view of the dividend problem calculation, by taking advantage of the relationship between the Cramér–Lundberg and the dual models. Most of the works on the dual model and on the discounted dividends problem assume that the inter-arrival times follow exponential, Erlang, or generalized Erlang distributions. We are going to extend results to the more general phase-type case, denoted as Ph(n).

In this paper, we study the dual risk model when the waiting times are phase-type distributed, generalizing the work of Rodríguez-Martínez et al. (2015) and extending the results presented in Bergel & Egídio dos Reis (2014) considering the Cramér–Lundberg insurance risk model. We will refer this as the Primal model. In this model we particularly refer to the works by Li (2008a) and Li (2008b) who works the first hitting time to an upper barrier and has correspondence to the time to ruin in the dual model. This correspondence has been addressed by Rodríguez-Martínez et al. (2015) and explains that it exists irrespective of the reverse income condition of the dual model. However, this correspondence is not full because in the Dual model both the standard and the generalized Lundberg equations have n roots with positive real parts, whereas in the primal model only the generalized Lundberg equation has n of those roots. The standard one has $n - 1$. Li (2008a) and Li (2008b) show results, in matrix form, for the Laplace transforms of the first hitting time. There is a correspondence to the Laplace transform of the time to ruin in the dual model. However, his formulation is valid when all of those roots are distinct only. We develop a new formulation that is also valid when we have multiplicity. For the Erlang(n) model there is no multiplicity, e.g. see Bergel & Egídio dos Reis (2015), for the generalized Erlang(n) we can have double roots, see Bergel & Egídio dos Reis (2016). In other Ph(n) models we can have higher multiplicity.

We now summarize our manuscript, as follows. In Section 2 we briefly introduce the phase-type distribution and the notation we use further in the paper. In Section 3 we study the fundamental and

the generalized Lundberg's equations and the role of its solutions. Although this had been studied, we want to introduce here equivalent equations and methods that will be subsequently used in the newer developments in the forthcoming sections. In Section 4 we get expressions for the ruin probability and the Laplace transform of the time to ruin for an arbitrary individual gain distribution. We also present some numerical analysis. Finally, in Section 5 we work on the problem of calculating the expected discounted dividends when the individual common gains follow a phase-type distribution. We could easily extend the methods and techniques for the computation of higher moments. We present some numerical illustrations.

2. The phase-type distribution

Phase-type distributions are the computational vehicle of much of modern applied probability. Typically, if a problem can be solved explicitly when the relevant distributions are exponentials, then the problem may admit an algorithmic solution involving a reasonable degree of computational effort, if one allows for the more general assumption of phase-type structure, and not in other cases. A proper knowledge of phase-type distributions seems therefore a must for anyone working in an applied probability area like risk theory.

We say that a distribution K on $(0, \infty)$ is phase-type(n) if K is the distribution of the lifetime of a terminating continuous time Markov process $\{J(t), t \geq 0\}$ with finitely many states and time homogeneous transition rates. More precisely, we define a terminating Markov process $\{J(t), t \geq 0\}$ with state space $E = \{1, 2, \ldots, n\}$ and intensity matrix $\mathbf{B}$ ($n \times n$) as the restriction to E of a Markov process $\{\bar{J}(t), 0 \leq t < \infty\}$ on $E_0 = E \cup \{0\}$ where 0 is some extra state which is absorbing, that is, $Pr(\bar{J}(t) = 0 | \bar{J}(0) = i) = 1$ for all $i \in E$ and where all states $i \in E$ are transient. This implies in particular that the intensity matrix for $\{\bar{J}(t)\}$ can be written in block-partitioned form as

$$\left(\begin{array}{c|c} \mathbf{B} & \mathbf{b}^{\mathsf{T}} \\ \hline \mathbf{0} & 0 \end{array} \right). \tag{2.1}$$

The $(1 \times n)$ vector $\mathbf{b} = (b_1, \ldots, b_n)$ is the exit rate vector, i. e., the ith component b_i gives the intensity in state i for leaving E and going to the absorbing state 0.

Note that since (2.1) is the intensity matrix of a non-terminating Markov process, the rows sums to zero which in matrix notation can be written as $\mathbf{b}^{\mathsf{T}} + \mathbf{B}\mathbf{1}^{\mathsf{T}} = \mathbf{0}$ where $\mathbf{1} = (1, 1, \ldots, 1)$ is the $(1 \times n)$ vector with all components equal to one. In particular we have

$$\mathbf{b}^{\mathsf{T}} = -\mathbf{B}\mathbf{1}^{\mathsf{T}}.$$

The intensity matrix $\mathbf{B}$ is denoted by $\mathbf{B} = (b_{ij})_{i,j=1}^{n}$. This matrix satisfies the conditions: $b_{ii} < 0$, $b_{ij} \geq 0$ for $i \neq j$, and $\sum_{j=1}^{n} b_{ij} \leq 0$ for $i = 1, \ldots, n$. The vector of entry probabilities is given by $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i \geq 0$ for $i = 1, \ldots, n$, and $\sum_{i=1}^{n} \alpha_i = 1$, so $Pr(\bar{J}(0) = i) = \alpha_i$.

Below we list expressions of most of the quantities of interest related to K, density, distribution, Laplace transform, mean and the jth derivative of $k(t)$ at 0:

$$\begin{aligned} k(t) &= \boldsymbol{\alpha} e^{\mathbf{B}t} \mathbf{b}^{\mathsf{T}}, \quad t \geq 0, \\ K(t) &= 1 - \boldsymbol{\alpha} e^{\mathbf{B}t} \mathbf{1}^{\mathsf{T}}, \quad t \geq 0, \\ \hat{k}(s) &= \boldsymbol{\alpha} (s\mathbf{I} - \mathbf{B})^{-1} \mathbf{b}^{\mathsf{T}}, \\ E[W_1] &= -\boldsymbol{\alpha} \mathbf{B}^{-1} \mathbf{1}^{\mathsf{T}}, \\ k^{(j)}(0) &= \boldsymbol{\alpha} \mathbf{B}^{j} \mathbf{b}^{\mathsf{T}}, \quad j \geq 0, \end{aligned} \tag{2.2}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix.

From this point on we consider $K(t)$ to be a phase-type(n) distribution, shortly Ph(n), of the inter-arrival times, and we call our model the phase-type(n) dual risk model. It is important to notice that we can write the corresponding *net profit condition* (1.1) as

$$-c\boldsymbol{\alpha}\mathbf{B}^{-1}\mathbf{1}^{\mathsf{T}} < \mu_1. \tag{2.3}$$

3. Lundberg's equations

In this section we study the fundamental and the generalized Lundberg's equations

$$E\left[e^{-s(X_1 - cW_1)}\right] = 1, \qquad E\left[e^{-\delta W_1}e^{-s(X_1 - cW_1)}\right] = 1, \ s \in \mathbb{C}, \ \delta > 0, \tag{3.1}$$

see e.g. Landriault & Willmot (2008) or Rodríguez-Martínez et al. (2015). As we can see from the works of Gerber & Shiu (2005) and Ren (2007), these equations can be expressed in the form, respectively,

$$\hat{k}(-cs)\hat{p}(s) = 1, \qquad \hat{k}(\delta - cs)\hat{p}(s) = 1. \tag{3.2}$$

Remark 3.1: A very important result we will use in the rest of our paper is the observation that for a phase-type(n) dual risk model the Lundberg's equations have exactly n roots with positive real parts, see Albrecher & Boxma (2005). Denote them by $\rho_1, \cdots, \rho_n$.

Remark 3.2: Unlike the generalized Lundberg's equation, in the phase-type(n) primal risk model, the standard one only has $n - 1$ roots with positive real parts. When comparing to de dual model this is due to the reversed income condition. So that the random variable first hitting time in the primal model has a proper distribution and the time to ruin in the dual is a defective random variable.

The roots of the Lundberg's equations play an important role in the calculation of many quantities that are fundamental in risk and ruin theory. Namely, the ultimate and finite time ruin probabilities, the Laplace transform of the ruin time, the expected discounted future dividends. All those calculations depend on the nature of the roots of the Lundberg's equation, particularly those roots with positive real parts. A study on the multiplicity of the roots can be found in Bergel & Egídio dos Reis (2014, 2016).

Note that in order to solve Equation (3.2) numerically we need to determine a rational expression for the Laplace transform $\hat{k}(\delta - cs)$. Since

$$\hat{k}(\delta - cs) = \boldsymbol{\alpha}((\delta - cs)\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}^{\mathsf{T}}, \tag{3.3}$$

the main difficulty is to compute the inverse matrix $((\delta - cs)\mathbf{I} - \mathbf{B})^{-1}$. Before we go further we give some definitions from linear algebra.

Definition 3.1: Let $\mathbf{A} = (a_{i,j})_{i,j=1}^{n}$ be a $n \times n$ matrix. Define, for the given subindices $1 \le i_1 < i_2 < \ldots < i_k \le n$,

$$\mathbf{M}_{i_1,i_2\ldots i_k}(\mathbf{A}) = \det \begin{pmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \cdots & a_{i_1,i_k} \\ a_{i_2,i_1} & a_{i_2,i_2} & \cdots & a_{i_2,i_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_k,i_1} & a_{i_k,i_2} & \cdots & a_{i_k,i_k} \end{pmatrix}, \ 1 \le k \le n.$$

These are the minors $k \times k$ of the matrix $\mathbf{A}$ obtained by deleting the row and the column that meet in a_{ii} for $i \notin \{i_1, i_2, \ldots, i_k\}$. Then

$$tr_k(\mathbf{A}) = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \mathbf{M}_{i_1,i_2\ldots i_k}(\mathbf{A}). \tag{3.4}$$

We call $tr_k(\mathbf{A})$ the k-generalized trace of the matrix A. In particular, $tr_1(\mathbf{A}) = tr(\mathbf{A}) = trace(\mathbf{A})$, and $tr_n(\mathbf{A}) = det(\mathbf{A})$.

Note that the sum in (3.4) above has $\binom{n}{k}$ summands. Using this definition enables us to express the characteristic polynomial of the matrix $\mathbf{B}$ as

$$det(s\mathbf{I} - \mathbf{B}) = \sum_{i=0}^{n} (-1)^{n-i} tr_{n-i}(\mathbf{B})s^i.$$

Moreover, the inverse matrix $(s\mathbf{I} - \mathbf{B})^{-1}$ can be obtained as follows:

Theorem 3.1: *The inverse matrix $(s\mathbf{I} - \mathbf{B})^{-1}$ has the expression*

$$(s\mathbf{I} - \mathbf{B})^{-1} = \frac{N(s, \mathbf{B})}{det(s\mathbf{I} - \mathbf{B})},$$

where the matrix $N(s, \mathbf{B})$ takes the form

$$N(s, \mathbf{B}) = \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1-i} (-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-1-i-j} \right) s^i.$$

Proof: We prove that $(s\mathbf{I} - \mathbf{B})^{-1}(s\mathbf{I} - \mathbf{B}) = \mathbf{I}$ or, equivalently, that

$$(s\mathbf{I} - \mathbf{B})N(s, \mathbf{B}) = det(s\mathbf{I} - \mathbf{B})\mathbf{I}.$$

If we denote by $\mathbf{a}_i$ the $n \times n$ matrix given by

$$\mathbf{a_i} = \sum_{j=0}^{n-1-i} (-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-1-i-j},$$

then

$$(s\mathbf{I} - \mathbf{B})N(s, \mathbf{B}) = (s\mathbf{I} - \mathbf{B}) \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1-i} (-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-1-i-j} \right) s^i$$

$$= (s\mathbf{I} - \mathbf{B}) \sum_{i=0}^{n-1} \mathbf{a}_i s^i$$

$$= \mathbf{a}_{n-1}s^n + \sum_{i=1}^{n-1} (\mathbf{a}_{i-1} - \mathbf{a}_i\mathbf{B})s^i - \mathbf{a}_0\mathbf{B}.$$

Now we can easily verify that $\mathbf{a}_{n-1} = \mathbf{I}$. Since

$$det(\mathbf{BI} - \mathbf{B}) = \sum_{j=0}^{n} (-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-j} = 0,$$

we get $-\mathbf{a}_0\mathbf{B} = (-1)^n det(\mathbf{B})\mathbf{I}$, and

$$\mathbf{a}_{i-1} - \mathbf{a}_i\mathbf{B} = \sum_{j=0}^{n-i} (-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-i-j} - \left(\sum_{j=0}^{n-1-i} (-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-1-i-j} \right) \mathbf{B}$$

$$= (-1)^{n-i} tr_{n-i}(\mathbf{B})\mathbf{I}.$$

Therefore,

$$(s\mathbf{I} - \mathbf{B})N(s, \mathbf{B}) = \mathbf{I}s^n + \sum_{i=1}^{n-1} ((-1)^{n-i} tr_{n-i}(\mathbf{B})\mathbf{I})s^i + (-1)^n det(\mathbf{B})\mathbf{I}$$

$$= \sum_{i=0}^{n} ((-1)^{n-i} tr_{n-i}(\mathbf{B})\mathbf{I})s^i = det(s\mathbf{I} - \mathbf{B})\mathbf{I}.$$

This completes the proof. □

From Theorem 3.1 and (3.3) we get the rational expression for the Lundberg's Equation (3.2). The generalized Lundberg's equation for the phase-type(n) dual risk model becomes

$$\frac{det((\delta - cs)\mathbf{I} - \mathbf{B})}{\alpha N(\delta - cs, \mathbf{B})\mathbf{b}^\mathsf{T}} = \hat{p}(s), \tag{3.5}$$

and we obtain the corresponding fundamental Lundberg's equation by setting $\delta = 0$ in Equation (3.5)

$$\frac{det((-cs)\mathbf{I} - \mathbf{B})}{\alpha N(-cs, \mathbf{B})\mathbf{b}^\mathsf{T}} = \hat{p}(s). \tag{3.6}$$

Although the new expressions for the Lundberg's equations found in (3.5) and (3.6) are already in rational form, they are not adequate for our purposes. What we need are expressions that show a natural connection with other parts of this manuscript. The reason for this will be clear in Section 4 when we will calculate quantities like the Laplace transform of the time of ruin and the ruin probability using integro-differential equations. It turns out that these integro-differential equations can be expressed using polynomial forms, denoted as $B_\delta(\cdot)$ and $q_\delta(\cdot)$, and these polynomial forms can be used instead to rewrite the generalized and fundamental Lundberg's Equations (3.5) and (3.6). This is shown next.

The generalized Lundberg's equation can be written as

$$B_\delta(-s) = q_\delta(-s)\hat{p}(s), \qquad s \in \mathbb{C}, \tag{3.7}$$

where B_δ and q_δ are polynomials in s given by

$$B_\delta(s) = \frac{det(\mathbf{B} - \delta\mathbf{I} - cs\mathbf{I})}{det(\mathbf{B})} = \sum_{i=0}^{n} B_i \left(s + \frac{\delta}{c} \right)^i$$

and

$$q_\delta(s) = \sum_{j=0}^{n-1} \tilde{B}_j \left(s + \frac{\delta}{c} \right)^j.$$

The equivalent fundamental Lundberg's equation (for $\delta = 0$) is

$$B(-s) = q(-s)\hat{p}(s), \qquad s \in \mathbb{C}. \tag{3.8}$$

The coefficients B_i and $\tilde{B}_j$ of the polynomials B and q, respectively, are given by the following expressions

$$B_i = (-c)^i \frac{tr_{n-i}(\mathbf{B})}{det(\mathbf{B})}, \qquad \tilde{B}_j = \sum_{i=j+1}^{n} B_i \left(\frac{1}{c}\right)^{i-j} k^{(i-1-j)}(0).$$

Theorem 3.2: *Expressions (3.5) and (3.7) are equivalent forms of the generalized Lundberg's equation. Corresponding expressions (3.6) and (3.8) represent the fundamental Lundberg's equation.*

Proof: The proof is simple and follows by rearranging and comparing the coefficients of the above-mentioned versions of the Lundberg's equations. Namely, we have to prove that

$$\frac{det((\delta - cs)\mathbf{I} - \mathbf{B})}{\alpha N(\delta - cs, \mathbf{B})\mathbf{b}^{\mathsf{T}}} = \frac{B_\delta(-s)}{q_\delta(-s)}. \tag{3.9}$$

From the left-hand side we have

$$B_\delta(-s) = \frac{det(\mathbf{B} - \delta\mathbf{I} + cs\mathbf{I})}{det(\mathbf{B})} = \frac{(-1)^n}{det(\mathbf{B})} det((\delta - cs)\mathbf{I} - \mathbf{B}),$$

and from the right-hand side:

$$\begin{aligned}
q_\delta(-s) &= \sum_{j=0}^{n-1} \tilde{B}_j \left(\frac{\delta}{c} - s\right)^j = \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} B_i \left(\frac{1}{c}\right)^{i-j} k^{(i-1-j)}(0) \left(\frac{\delta}{c} - s\right)^j \\
&= \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} (-1)^i c^i \frac{tr_{n-i}(\mathbf{B})}{det(\mathbf{B})} \left(\frac{1}{c}\right)^i \alpha \mathbf{B}^{i-1-j} \mathbf{b}^{\mathsf{T}} c^j \left(\frac{\delta}{c} - s\right)^j \\
&= \alpha \sum_{j=0}^{n-1} \sum_{i=0}^{n-j-1} (-1)^{n-i} \frac{tr_i(\mathbf{B})}{det(\mathbf{B})} \mathbf{B}^{n-1-i-j} (\delta - cs)^j \mathbf{b}^{\mathsf{T}} \\
&= \frac{(-1)^n}{det(\mathbf{B})} \alpha N(\delta - cs, \mathbf{B}) \mathbf{b}^{\mathsf{T}}.
\end{aligned}$$

This proves (3.9). □

Remark 3.3: Alternatively, we can write

$$B_\delta(s) = \frac{det(\mathbf{B} - \delta\mathbf{I} - cs\mathbf{I})}{det(\mathbf{B})} = \sum_{i=0}^{n} B_{i,\delta} \, s^i, \tag{3.10}$$

$$q_\delta(s) = \sum_{j=0}^{n-1} \tilde{B}_{j,\delta} \, s^j, \tag{3.11}$$

where the coefficients $B_{i,\delta}$ and $\tilde{B}_{j,\delta}$ are given by

$$B_{i,\delta} = (-c)^i \frac{tr_{n-i}(\mathbf{B} - \delta\mathbf{I})}{det(\mathbf{B})},$$

$$\tilde{B}_{j,\delta} = \sum_{i=0}^{n-1-j} B_{1+i+j,\delta} \left(\sum_{l=0}^{i} \binom{i}{l}(-\delta)^l k^{(i-l)}(0)\right) \left(\frac{1}{c}\right)^{1+i}.$$

These forms are going to be used in the following section.

4. The time to ruin and its Laplace transform

In this section we study the ruin probability and the Laplace transform of the time to ruin in the phase-type(n) dual risk model. Let

$$T_u = \begin{cases} \min\{t > 0 : U(t) = 0 \mid U(0) = u\} \\ \infty \ \text{if} \ U(t) \geq 0 \quad \forall t \geq 0 \end{cases}$$

be the time to ruin, $\psi(u) = P(T_u < \infty)$ be the ultimate ruin probability and

$$\psi(u, \delta) = E[e^{-\delta T_u} I(T_u < \infty) \mid U(0) = u]$$

be the Laplace transform of the time to ruin, where $\delta > 0$ and $I(.)$ is the indicator function. This Laplace transform can be interpreted as the expected value of one monetary unit received at the time of ruin discounted at the constant force of interest δ.

In particular, we can obtain the ultimate ruin probability $\psi(u)$ as a limiting case of the Laplace transform of the time to ruin, since

$$\lim_{\delta \to 0} \psi(u, \delta) = \psi(u). \tag{4.1}$$

Conditioning on the time and the amount of the first gain, we find that the Laplace transform of the time to ruin for the phase-type(n) dual risk model satisfies the renewal equation

$$\psi(u, \delta) = \left(1 - K\left(\frac{u}{c}\right)\right) e^{-\delta\left(\frac{u}{c}\right)} + \int_0^{\frac{u}{c}} k(t) e^{-\delta t} \int_0^\infty p(x)\psi(u - ct + x, \delta)\mathrm{d}x \ \mathrm{d}t.$$

Note that the above equation is valid for any renewal model with density k and distribution K. Changing variables $s = u - ct$, we get

$$\psi(u, \delta) = \left(1 - K\left(\frac{u}{c}\right)\right) e^{-\delta\left(\frac{u}{c}\right)} + \frac{1}{c} \int_0^u k\left(\frac{u-s}{c}\right) e^{-\delta\left(\frac{u-s}{c}\right)} W_\psi(s, \delta) \ \mathrm{d}s, \tag{4.2}$$

where $W_\psi(s, \delta) = \int_0^\infty p(x)\psi(s + x, \delta)\mathrm{d}x$.

Before we continue further, we state the following lemma, which will be useful in a subsequent theorem.

Lemma 4.1:   *Let B_δ, q_δ be the polynomials described in (3.7), Section 3 for the generalized Lundberg's equation, and consider the following differential operators*

$$B_\delta(\mathcal{D}) = \sum_{i=0}^n B_i \left(\mathcal{D} + \frac{\delta}{c}\right)^i = \sum_{i=0}^n B_{i,\delta} \ \mathcal{D}^i, \quad q_\delta(\mathcal{D}) = \sum_{j=0}^{n-1} \tilde{B}_j \left(\mathcal{D} + \frac{\delta}{c}\right)^j = \sum_{j=0}^{n-1} \tilde{B}_{j,\delta} \ \mathcal{D}^j. \tag{4.3}$$

for $\mathcal{D} = \frac{\mathrm{d}}{\mathrm{d}u}$. Then the following properties hold

$$B_\delta(\mathcal{D})\left[k\left(\frac{u-s}{c}\right) e^{-\delta\left(\frac{u-s}{c}\right)}\right] = 0,$$

$$B_\delta(\mathcal{D})\left[\left(1 - K\left(\frac{u}{c}\right)\right) e^{-\delta\left(\frac{u}{c}\right)}\right] = 0.$$

Proof:

$$B_\delta(\mathcal{D})\left[k\left(\frac{u-s}{c}\right)e^{-\delta\left(\frac{u-s}{c}\right)}\right] = \sum_{i=0}^{n} B_{i,\delta}\mathcal{D}^i\left[\boldsymbol{\alpha}e^{\mathbf{B}\left(\frac{u-s}{c}\right)}\mathbf{b}^\mathsf{T}e^{-\delta\left(\frac{u-s}{c}\right)\mathbf{I}}\right]$$

$$= \boldsymbol{\alpha}\left[\sum_{i=0}^{n} B_{i,\delta}\mathcal{D}^i(e^{(\mathbf{B}-\delta\mathbf{I})\left(\frac{u-s}{c}\right)})\right]\mathbf{b}^\mathsf{T}$$

$$= \boldsymbol{\alpha}\left[\sum_{i=0}^{n} B_{i,\delta}\left(\frac{1}{c}\right)^i(\mathbf{B}-\delta\mathbf{I})^i e^{(\mathbf{B}-\delta\mathbf{I})\left(\frac{u-s}{c}\right)}\right]\mathbf{b}^\mathsf{T}$$

$$= \boldsymbol{\alpha}\left[B_\delta\left(\frac{1}{c}(\mathbf{B}-\delta\mathbf{I})\right)\right]e^{(\mathbf{B}-\delta\mathbf{I})\left(\frac{u-s}{c}\right)}\mathbf{b}^\mathsf{T}$$

$$= \boldsymbol{\alpha}\left[\frac{det(\mathbf{B}-\delta\mathbf{I}-c\mathbf{I}\left(\frac{1}{c}(\mathbf{B}-\delta\mathbf{I})\right))}{det(\mathbf{B})}\right]e^{(\mathbf{B}-\delta\mathbf{I})\left(\frac{u-s}{c}\right)}\mathbf{b}^\mathsf{T} = 0,$$

Analogously, we can see $B_\delta(\mathcal{D})\left[\left(1-K\left(\frac{u}{c}\right)\right)e^{-\delta\left(\frac{u}{c}\right)}\right] = 0$. □

We can obtain a formula for the Laplace transform of the time of ruin $\psi(u,\delta)$ solving the following integro-differential equation with boundary conditions:

Theorem 4.1: *The Laplace transform of the time of ruin $\psi(u,\delta)$ satisfies the integro-differential equation*

$$B_\delta(\mathcal{D})\psi(u,\delta) = q_\delta(\mathcal{D})W_\psi(u,\delta), \tag{4.4}$$

The boundary conditions of (4.4) *are given by*

$$\psi(0,\delta) = 1,$$

$$\frac{d^i}{du^i}\psi(u,\delta)\Big|_{u=0} = \left(-\frac{\delta}{c}\right)^i - \sum_{j=0}^{i-1}\frac{1}{c^i}\binom{i}{j}(-\delta)^j k^{(i-1-j)}(0)$$

$$+ \sum_{j=0}^{i-1}\left(\sum_{l=0}^{i-1-j}\frac{1}{c^{i-j}}\binom{i-1-j}{l}(-\delta)^l k^{(i-1-j-l)}(0)\right)W_\psi^{(j)}(0,\delta), \tag{4.5}$$

$$i = 1,\ldots,n-1.$$

Proof: We proceed taking successive derivatives of $\psi(u,\delta)$ using the renewal Equation (4.2). We want to prove the equation $B_\delta(\mathcal{D})\psi(u,\delta) = q_\delta(\mathcal{D})W_\psi(u,\delta)$. The jth derivative of $\psi(u,\delta)$ with respect to u is given by

$$\frac{d^j}{du^j}\psi(u,\delta) = \left[\left(-\frac{\delta}{c}\right)^j\left(1-K\left(\frac{u}{c}\right)\right) - \sum_{i=0}^{j-1}\frac{1}{c^j}\binom{j}{i}(-\delta)^i k^{(j-1-i)}\left(\frac{u}{c}\right)\right]e^{-\delta\left(\frac{u}{c}\right)}$$

$$+ \sum_{i=0}^{j-1}\left(\sum_{l=0}^{j-1-i}\left(\frac{1}{c}\right)^{j-i}\binom{j-1-i}{l}(-\delta)^l k^{(j-1-i-l)}(0)\right)W_\psi^{(i)}(u,\delta)$$

$$+ \frac{1}{c}\int_0^u\left[\sum_{i=0}^{j}\frac{1}{c^j}\binom{j}{i}(-\delta)^i k^{(j-i)}\left(\frac{u-s}{c}\right)\right]e^{-\delta\left(\frac{u-s}{c}\right)}W_\psi(s,\delta)ds,$$

for $j = 1, \ldots, n-1$. Hence, we obtain

$$
\frac{d^j}{du^j} \psi(u, \delta) \bigg|_{u=0} = \left(-\frac{\delta}{c}\right)^j - \sum_{i=0}^{j-1} \frac{1}{c^j} \binom{j}{i} (-\delta)^i k^{(j-1-i)}(0)
$$

$$
+ \sum_{i=0}^{j-1} \left(\sum_{l=0}^{j-1-i} \left(\frac{1}{c}\right)^{j-i} \binom{j-1-i}{l} (-\delta)^l k^{(j-1-i-l)}(0) \right) W_\psi^{(i)}(0, \delta),
$$

for $j = 1, \ldots, n-1$.

Now we apply the differential operator $B_\delta(\mathcal{D})$ to $\psi(u, \delta)$

$$
B_\delta(\mathcal{D}) \psi(u, \delta) = \underbrace{B_\delta(\mathcal{D}) \left[\left(1 - K\left(\frac{u}{c}\right)\right) e^{-\delta\left(\frac{u}{c}\right)} \right]}_{=0} + B_\delta(\mathcal{D}) \left(\frac{1}{c} \int_0^u k\left(\frac{u-s}{c}\right) e^{-\delta\left(\frac{u-s}{c}\right)} W_\psi(s, \delta) \, ds \right)
$$

$$
= \sum_{j=0}^{n} B_{j,\delta} \mathcal{D}^j \left(\frac{1}{c} \int_0^u k\left(\frac{u-s}{c}\right) e^{-\delta\left(\frac{u-s}{c}\right)} W_\psi(s, \delta) ds \right)
$$

$$
= \sum_{j=0}^{n} B_{j,\delta} \left[\sum_{i=0}^{j-1} \left(\sum_{l=0}^{j-1-i} \left(\frac{1}{c}\right)^{j-i} \binom{j-1-i}{l} (-\delta)^l k^{(j-1-i-l)}(0) \right) W_\psi^{(i)}(u, \delta) \right.
$$

$$
\left. + \frac{1}{c} \int_0^u \left(\sum_{i=0}^{j} \frac{1}{c^j} \binom{j}{i} (-\delta)^i k^{(j-i)}\left(\frac{u-s}{c}\right) \right) e^{-\delta\left(\frac{u-s}{c}\right)} W_\psi(s, \delta) ds \right]
$$

$$
= \sum_{j=1}^{n} B_{j,\delta} \left[\sum_{i=0}^{j-1} \left(\sum_{l=0}^{i} \left(\frac{1}{c}\right)^{i+1} \binom{i}{l} (-\delta)^l k^{(i-l)}(0) \right) W_\psi^{(j-1-i)}(u, \delta) \right]
$$

$$
+ \frac{1}{c} \int_0^u \underbrace{B_\delta(\mathcal{D}) \left[k\left(\frac{u-s}{c}\right) e^{-\delta\left(\frac{u-s}{c}\right)} \right]}_{=0} W_\psi(s, \delta) ds
$$

$$
= \sum_{j=0}^{n-1} \sum_{i=0}^{n-1-j} B_{1+i+j,\delta} \underbrace{\left(\sum_{l=0}^{i} \left(\frac{1}{c}\right)^{i+1} \binom{i}{l} (-\delta)^l k^{(i-l)}(0) \right)}_{=\tilde{B}_{j,\delta}} W_\psi^{(j)}(u, \delta)
$$

$$
= \sum_{j=0}^{n-1} \tilde{B}_{j,\delta} W_\psi^{(j)}(u, \delta) = q_\delta(\mathcal{D}) W_\psi(u, \delta).
$$

This completes the proof.   □

For the phase-type(n) dual risk model, we have found that the Laplace transform of the time of ruin can be written as follows

Theorem 4.2:

$$
\psi(u, \delta) = \sum_{i=1}^{L} \sum_{j=1}^{\beta_i} a_{ij,\delta} u^{j-1} e^{-\rho_i u}, \tag{4.6}
$$

where $a_{ij,\delta}$ are some constants, $\rho_1, \ldots, \rho_L$ are the only roots of the generalized Lundberg's equation which have positive real parts, and ρ_i has multiplicity β_i, with $\sum_{i=1}^{L} \beta_i = n$.

Proof: It is very simple to verify that if ρ is a single root of the generalized Lundberg's equation $B_\delta(-s) = q_\delta(-s)\hat{p}(s)$ then the function $f(u) = e^{-\rho u}$ satisfies the integro-differential equation $B_\delta(\mathcal{D})f(u) = q_\delta(\mathcal{D})W_f(u)$, where $W_f(u) = \int_0^\infty p(x)f(u+x)\mathrm{d}x$.

Moreover, we can show that if ρ is a root of the generalized Lundberg's equation with multiplicity $\beta \geq 1$ then the functions $f(u) = u^{j-1}e^{-\rho u}$, $j = 1, \ldots, \beta$ are all solutions of the same integro-differential equation. We will prove that $B_\delta(\mathcal{D})f(u) = q_\delta(\mathcal{D})W_f(u)$. We have

$$f^{(k)}(u) = \sum_{l=0}^{j-1}\left(\prod_{m=0}^{l-1}(k-m)\right)(-\rho)^{k-l}\binom{j-1}{l}u^{j-1-l}e^{-\rho u}.$$

Then from the left-hand side

$$\begin{aligned}
B_\delta(\mathcal{D})f(u) &= \sum_{k=0}^{n}B_{k,\delta}f^{(k)}(u) \\
&= \sum_{k=0}^{n}B_{k,\delta}\sum_{l=0}^{j-1}\left(\prod_{m=0}^{l-1}(k-m)\right)(-\rho)^{k-l}\binom{j-1}{l}u^{j-1-l}e^{-\rho u} \\
&= \sum_{l=0}^{j-1}\left(\sum_{k=l}^{n}B_{k,\delta}\left(\prod_{m=0}^{l-1}(k-m)\right)(-\rho)^{k-l}\right)\binom{j-1}{l}u^{j-1-l}e^{-\rho u} \\
&= \sum_{l=0}^{j-1}B_\delta^{(l)}(-\rho)\binom{j-1}{l}u^{j-1-l}e^{-\rho u}.
\end{aligned} \tag{4.7}$$

From the right-hand side, we have

$$\begin{aligned}
W_f(u) &= \int_0^\infty f(u+x)p(x)\mathrm{d}x = \int_0^\infty (u+x)^{j-1}e^{-\rho(u+x)}p(x)\mathrm{d}x \\
&= \int_0^\infty \sum_{i=0}^{j-1}\binom{j-1}{i}u^{j-1-i}x^i e^{-\rho x}e^{-\rho u}p(x)\mathrm{d}x \\
&= \sum_{i=0}^{j-1}\binom{j-1}{i}u^{j-1-i}e^{-\rho u}\int_0^\infty x^i e^{-\rho x}p(x)\mathrm{d}x \\
&= \sum_{i=0}^{j-1}\binom{j-1}{i}u^{j-1-i}e^{-\rho u}(-1)^i\hat{p}^{(i)}(\rho).
\end{aligned}$$

Therefore,

$$\begin{aligned}
q_\delta(\mathcal{D})W_f(u) &= \sum_{k=0}^{n-1}\tilde{B}_{k,\delta}W_f^{(k)}(u) \\
&= \sum_{i=0}^{j-1}\binom{j-1}{i}(-1)^i\hat{p}^{(i)}(\rho)\sum_{k=0}^{n-1}\tilde{B}_{k,\delta}\frac{d}{d^k}(u^{j-1-i}e^{-\rho u}) \\
&= \sum_{i=0}^{j-1}\binom{j-1}{i}(-1)^i\hat{p}^{(i)}(\rho)\sum_{l=0}^{j-1-i}q_\delta^{(l)}(-\rho)\binom{j-1-i}{l}u^{j-1-i-l}e^{-\rho u}
\end{aligned}$$

$$= \sum_{i=0}^{j-1} \binom{j-1}{i}(-1)^i \hat{p}^{(i)}(\rho) \sum_{l=i}^{j-1} q_\delta^{(l-i)}(-\rho) \binom{j-1-i}{l-i} u^{j-1-l} e^{-\rho u}$$

$$= \sum_{l=0}^{j-1} \left[\sum_{i=0}^{l} (-1)^i \binom{l}{i} q_\delta^{(l-i)}(-\rho) \hat{p}^{(i)}(\rho) \right] \binom{j-1}{l} u^{j-1-l} e^{-\rho u}. \tag{4.8}$$

Since the root ρ has multiplicity $\beta \geq 1$, it satisfies the equations

$$B_\delta^{(l)}(-\rho) = \sum_{i=0}^{l} (-1)^i \binom{l}{i} q_\delta^{(l-i)}(-\rho) \hat{p}^{(i)}(\rho), \; l = 0, 1, \ldots \beta - 1, \tag{4.9}$$

which implies that expressions (4.7) and (4.8) are identical, thus proving our statement.

Since the functions $u^{j-1} e^{-\rho_i u}$, $i = 1, \ldots, L$; $j = 1, \ldots, \beta_i$ are linearly independent, any solution of $B_\delta(\mathcal{D})f(u) = q_\delta(\mathcal{D})W_f(u)$ can be expressed in the following way

$$f(u) = \sum_{i=1}^{L} \sum_{j=1}^{\beta_i} b_{ij} u^{j-1} e^{-\rho_i u}, \tag{4.10}$$

for some constants b_{ij}.

Since the Laplace transform of the time of ruin satisfies an integro-differential equation of the same form, $B_\delta(\mathcal{D})\psi(u,\delta) = q_\delta(\mathcal{D})W_\psi(u,\delta)$, it can be written as

$$\psi(u,\delta) = \sum_{i=1}^{L} \sum_{j=1}^{\beta_i} a_{ij,\delta} u^{j-1} e^{-\rho_i u}.$$

Using the boundary conditions (4.5) we can determine the constants $a_{ij,\delta}$ that correspond to $\psi(u,\delta)$ in the following way

$$\psi(0,\delta) = \sum_{i=1}^{L} a_{i1,\delta} = 1,$$

$$\frac{d^m}{du^m}\psi(u,\delta)\bigg|_{u=0} = \frac{d^m}{du^m} \sum_{i=1}^{L} \sum_{j=1}^{\beta_i} a_{ij,\delta} u^{j-1} e^{-\rho_i u}\bigg|_{u=0}$$

$$\left(-\frac{\delta}{c}\right)^m - \sum_{j=0}^{m-1} \frac{1}{c^m} \binom{m}{j}(-\delta)^j k^{(m-1-j)}(0)$$

$$+ \sum_{j=0}^{m-1} \left(\sum_{l=0}^{m-1-j} \frac{1}{c^{m-j}} \binom{m-1-j}{l}(-\delta)^l k^{(m-1-j-l)}(0) \right) W_\psi^{(j)}(0,\delta),$$

$$m = 1, \ldots, n-1.$$

where $W_\psi(u,\delta) = \int_0^\infty p(x) \left[\sum_{i=1}^{L} \sum_{j=1}^{\beta_i} a_{ij,\delta}(u+x)^{j-1} e^{-\rho_i(u+x)} \right] dx$.

Regardless of multiplicities, this gives a system of n equations on the n unknowns constants $a_{ij,\delta}, i = 1, \ldots, L$; $j = 1, \ldots, \beta_i$, that can be solved using standard linear algebra methods. □

Remark 4.1: If all the roots with positive real parts of the generalized Lundberg's equation are single (multiplicity 1), then we write the Laplace transform of the time of ruin in the following way

$$\psi(u,\delta) = \sum_{i=1}^{n} a_{i,\delta} e^{-\rho_i u},$$

and the constants $a_{i,\delta}$ can be found using the boundary conditions (4.5), which is equivalent to solving the following system of n equations on the n unknowns $a_{i,\delta}$:

$$\sum_{i=1}^{n} a_{i,\delta} = 1,$$

and

$$\sum_{i=1}^{n} a_{i,\delta} \left[(-\rho_i)^j - \hat{p}(\rho_i) \sum_{m=0}^{j-1} \left(\sum_{l=0}^{j-1-m} \frac{1}{c^{j-m}} \binom{j-1-m}{l} (-\delta)^l k^{(j-1-m-l)}(0) \right) (-\rho_i)^m \right]$$

$$= \left(-\frac{\delta}{c} \right)^j - \sum_{m=0}^{j-1} \frac{1}{c^j} \binom{j}{m} (-\delta)^m k^{(j-1-m)}(0), \quad j = 1,\ldots,n.$$

Example 4.1: For $n=2$, the Laplace transform of the time of ruin in the phase-type(2) model has the expression

$$\psi(u,\delta) = \frac{\rho_2 - \frac{\delta}{c} + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^\mathsf{T}(\hat{p}(\rho_2) - 1)}{\rho_2 - \rho_1 + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^\mathsf{T}(\hat{p}(\rho_2) - \hat{p}(\rho_1))} e^{-\rho_1 u}$$

$$- \frac{\rho_1 - \frac{\delta}{c} + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^\mathsf{T}(\hat{p}(\rho_1) - 1)}{\rho_2 - \rho_1 + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^\mathsf{T}(\hat{p}(\rho_2) - \hat{p}(\rho_1))} e^{-\rho_2 u},$$

where $\rho_1, \rho_2 > 0$ are real and solutions of $B_\delta(-s) = q_\delta(-s)\hat{p}(s)$.

4.1. The ruin probability

The ultimate ruin probability as set in (4.1) can be obtained from Equation (4.2) calculating the limit as $\delta \to 0$, giving

$$\psi(u) = 1 - K\left(\frac{u}{c}\right) + \int_0^{\frac{u}{c}} k(t) \int_0^\infty p(x)\psi(u - ct + x)dx \, dt. \qquad (4.11)$$

In the exponential case, $n = 1$, we can see Afonso et al. (2013). Also, Gerber (1979) found that $\psi(u) = e^{-\rho u}$, where ρ is the unique positive root of the fundamental Lundberg's equation.

As $\delta \to 0$, the corresponding integro-differential equation and ruin probability are given by the following corollaries

Corollary 4.1: *The ruin probability $\psi(u)$ satisfies the following integro-differential equation*

$$B(\mathcal{D})\psi(u) = q(\mathcal{D})W(u), \qquad (4.12)$$

where $W(u) = \int_0^\infty p(x)\psi(u+x)dx$ and B, q are the same polynomials described before for the fundamental Lundberg's Equation (3.8). The operator $\mathcal{D}$ is the differentiation with respect to u, as before.

The boundary conditions of (4.12) *are given by*

$$\psi(0) = 1,$$

$$\frac{\mathrm{d}^j}{\mathrm{d}u^j}\psi(u)\bigg|_{u=0} = -\frac{1}{c^j}k^{(j-1)}(0) + \sum_{i=0}^{j-1}\frac{1}{c^{i+1}}k^{(i)}(0)W^{(j-1-i)}(0),$$

$$j = 1,\ldots,n-1.$$

(4.13)

Corollary 4.2: *The ultimate ruin probability $\psi(u)$ can be written in the general form*

$$\psi(u) = \sum_{i=1}^{L}\sum_{j=1}^{\beta_i} a_{ij,0}u^{j-1}e^{-\rho_i u},$$

where $\rho_1,\ldots,\rho_L$ are the only roots of the Fundamental Lundberg's equation which have positive real parts, and ρ_i has multiplicity β_i, with $\sum_{i=1}^{L}\beta_i = n$.

Example 4.2: *For $n = 2$, the ruin probability in the phase-type(2) model has the expression*

$$\psi(u) = \frac{\rho_2 + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^{\mathsf{T}}(\hat{p}(\rho_2) - 1)}{\rho_2 - \rho_1 + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^{\mathsf{T}}(\hat{p}(\rho_2) - \hat{p}(\rho_1))}e^{-\rho_1 u}$$

$$-\frac{\rho_1 + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^{\mathsf{T}}(\hat{p}(\rho_1) - 1)}{\rho_2 - \rho_1 + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^{\mathsf{T}}(\hat{p}(\rho_2) - \hat{p}(\rho_1))}e^{-\rho_2 u},$$

where $\rho_1, \rho_2 > 0$ are real and solutions of $B(-s) = q(-s)\hat{p}(s)$.

5. Expected discounted dividends

In this section, we consider a barrier strategy for dividend calculation in terms of a dividend barrier b. Although we just consider results for the expected discounted future dividends we could extend the presented methods to higher moments. Any time the regulated surplus upcrosses b the excess is paid as a dividend. From that payment instant the process restarts from level b and that repeats whenever it occurs in the future until ruin.

Let $\{D_i\}_{i=1}^{\infty}$ be the sequence of the dividend payments and let $D(u, b)$ be the aggregate discounted dividends, at force of interest δ. Let τ_i be the arrival time of D_i, then

$$D(u, b) = \sum_i e^{-\delta\tau_i}D_i.$$

We denote by $V(u, b) = E[D(u, b)]$, the expected value of $D(u, b)$.

Note that

$$V(u, b) = E[u - b + D(b, b)] = u - b + V(b, b), \quad u \geq b.$$

(5.1)

The expected discounted dividends $V(u, b)$ satisfy the following renewal equation:

$$V(u, b) = \int_0^{\frac{u}{c}} k(t)e^{-\delta t}\left[\int_0^{b-u+ct} V(u - ct + y, b)p(y)\mathrm{d}y\right.$$

$$\left. + \int_{b-u+ct}^{\infty} \tilde{V}(u - ct + y, b)p(y)\mathrm{d}y\right]\mathrm{d}t, \quad \text{for } u < b,$$

with

$$\tilde{V}(x, b) = E[D(x, b)] = E[x - b + D(b, b)] = x - b + V(b, b), \quad x \geq b.$$

Differentiating the renewal equation with respect to u produces an integro-differential equation for $V(u, b)$.

Theorem 5.1: *The expected discounted dividends $V(u, b)$ satisfy the integro-differential equation*

$$B_\delta(\mathcal{D})V(u, b) = q_\delta(\mathcal{D})W(u, b), \quad u < b, \tag{5.2}$$

where

$$W(u, b) = \int_u^b V(x, b)p(x - u)\mathrm{d}x + \int_b^\infty \widetilde{V}(x, b)p(x - u)\mathrm{d}x.$$

and $B_\delta(\mathcal{D}), q_\delta(\mathcal{D})$ are defined as in (4.3). *The boundary conditions of* (5.2) *are given by*

$$V(0, b) = 0,$$

$$\frac{\mathrm{d}^i}{\mathrm{d}u^i}V(u, b)\bigg|_{u=0} = \sum_{j=0}^{i-1}\left(\sum_{l=0}^{i-1-j}\frac{1}{c^{i-j}}\binom{i-1-j}{l}(-\delta)^l k^{(i-1-j-l)}(0)\right)W^{(j)}(0, b),$$

$$i = 1, \ldots, n - 1. \tag{5.3}$$

Proof: The proof follows the same methodology as that of Theorem 4.1. $\qquad\square$

Because of the additional information of a barrier level b in $V(u, b)$, we cannot solve the equation

$$B_\delta(\mathcal{D})V(u, b) = q_\delta(\mathcal{D})W(u, b), \tag{5.4}$$

to find an expression for $V(u, b)$ as we did for the Laplace transform of the time to ruin $\psi(u, \delta)$. There, we did not need to specify a particular density function $p(x)$ for the gain amounts, here we do and we show this in the following remark:

Remark 5.1: Consider the conditions that must be met by ρ when we insert $f(u) = e^{-\rho u}$ in (5.4). On the left-hand side we have

$$B_\delta(\mathcal{D})f(u) = B_\delta(-\rho)e^{-\rho u}. \tag{5.5}$$

On the right-hand side we get, denoting $W_f(u, b)$,

$$\begin{aligned}
W_f(u, b) &= \int_0^{b-u} f(x + u)p(x)\mathrm{d}x + \int_{b-u}^\infty (x + u - b + f(b))p(x)\mathrm{d}x \\
&= \int_0^{b-u} e^{-\rho(x+u)}p(x)\mathrm{d}x + \int_{b-u}^\infty (x + u - b + e^{-\rho b})p(x)\mathrm{d}x \\
&= e^{-\rho u}\hat{p}(\rho) + \int_b^\infty (x - b + e^{-\rho b} - e^{-\rho x})p(x - u)\mathrm{d}x,
\end{aligned}$$

and

$$q_\delta(\mathcal{D})W_f(u, b) = q_\delta(-\rho)\hat{p}(\rho)e^{-\rho u} + \int_b^\infty (x - b + e^{-\rho b} - e^{-\rho x})q_\delta(\mathcal{D})p(x - u)\mathrm{d}x. \tag{5.6}$$

Comparing Equations (5.5) and (5.6) we obtain

$$(B_\delta(-\rho) - q_\delta(-\rho)\hat{p}(\rho))e^{-\rho u} = \int_b^\infty (x - b + e^{-\rho b} - e^{-\rho x})q_\delta(\mathcal{D})p(x - u)\mathrm{d}x, \quad \forall u \geq 0. \tag{5.7}$$

If ρ was a root of the generalized Lundberg's equation $B_\delta(-s) = q_\delta(-s)\hat{p}(s)$, the left-hand side of (5.7) would be zero. On the other side, the right-hand side is not necessarily zero since $q_\delta(\mathcal{D})p(x - u)$ may not be zero.

Indeed, we have to assume a particular distribution for the gain amounts. For the rest of this manuscript, we assume that the gain amounts follow a phase-type(m) distribution and we use the annihilator method to find $V(u, b)$. See similar approach in Rodríguez-Martínez et al. (2015).

Following the notation in Section 2, consider the case when the gains X_i follow a phase-type(m) distribution $P(x)$ with representation ($\boldsymbol{\alpha}', \mathbf{B}', \mathbf{b}'$). Let $\rho_1, \ldots, \rho_n$ be the roots of the generalized Lundberg's equation $B_\delta(-s) = q_\delta(-s)\hat{p}(s)$ with positive real parts, and $\rho_n, \ldots, \rho_{n+m}$ be the roots with negative real parts. For simplicity, assume that all those roots are distinct (although this is not the case in general, see Bergel & Egídio dos Reis (2014) or Bergel & Egídio dos Reis (2016)).

Because of condition (5.1), we cannot write the solutions of (5.2) as a linear combination of n exponential functions as we did before in the cases of the ruin probability and the Laplace transform of the time of ruin. We will need more than n exponential functions, the exact required number will depend on the nature of the distribution of the single gains $P(x)$. However, we can apply the annihilator approach known from the theory of ordinary differential equations to find the appropriate solutions.

We can rewrite $W(u, b)$ as

$$W(u, b) = \int_u^b V(x, b)p(x - u)\mathrm{d}x + \int_b^\infty (x - b + V(b, b))p(x - u)\mathrm{d}x \tag{5.8}$$
$$= \int_u^b V(x, b)p(x - u)\mathrm{d}x + \int_b^\infty \widetilde{V}(x, b)p(x - u)\mathrm{d}x,$$

with $\widetilde{V}(x, b) = x - b + V(b, b)$. The idea is to find a linear differential operator that will annihilate $p(x - u)$ (where the variable is u), so that when we apply this operator to the integro-differential Equation (5.2) we obtain a linear homogeneous differential equation of a higher degree. We apply the annihilator operator, denoted as $A(\mathcal{D}) = \mathrm{Det}(\mathbf{I_m}\mathcal{D} + \mathbf{B}')$, at both sides of the integro-differential equation

$$B_\delta(\mathcal{D})V(u, b) = q_\delta(\mathcal{D})W(u, b),$$

where $\mathbf{I}_m$ is the identity $m \times m$ matrix, and we obtain an homogeneous integro-differential equation of degree $m + n$.

Theorem 5.2: *When $P(x)$ is phase-type(m) the solution of $V(u, b)$ is of the form*

$$V(u, b) = \sum_{l=1}^{n+m} a_l(b)e^{-\rho_l u}, \quad u < b, \tag{5.9}$$

where $\rho_l, l = 1, \ldots, n, n + 1, \ldots, n + m$ are the roots of the generalized Lundberg's equation, n with positive real parts and m with negative real parts, and the coefficients $a_l(b)$, depending on b, are found using the boundary n conditions (5.3), and the identity

$$\boldsymbol{\alpha}' \left[\sum_{l=1}^{n+m} a_l(b)e^{-\rho_l b} \left((\rho_l \mathbf{I_m} - \mathbf{B}')^{-1}\mathbf{B}' + \mathbf{I_m} \right) - \mathbf{B}'^{-1} \right] = \mathbf{0}, \tag{5.10}$$

which gives another m conditions. We obtain a system of $m + n$ equations on the $m + n$ unknowns $a_l(b)$.

Proof: Let $p(x - u) = \boldsymbol{\alpha}'e^{\mathbf{B}'}(x - u)\mathbf{b}'^{\mathsf{T}}$. The annihilator operator $A(\mathcal{D})$ can be expanded as

$$A(\mathcal{D}) = \mathrm{Det}(\mathbf{I_m}\mathcal{D} + \mathbf{B}') = \sum_{i=0}^m tr_{m-i}(\mathbf{B}')\mathcal{D}^i.$$

This operator annihilates $p(x - u)$

$$A(\mathcal{D})p(x - u) = \sum_{i=0}^{m} tr_{m-i}(\mathbf{B}')\mathcal{D}^i \left(\boldsymbol{\alpha}' e^{\mathbf{B}'(x-u)} \mathbf{b}'^{\mathsf{T}} \right)$$

$$= \boldsymbol{\alpha}' \left[\sum_{i=0}^{m} tr_{m-i}(\mathbf{B}')\mathcal{D}^i e^{\mathbf{B}'(x-u)} \right] \mathbf{b}'^{\mathsf{T}}$$

$$= \boldsymbol{\alpha}' \left[\underbrace{\sum_{i=0}^{m} tr_{m-i}(\mathbf{B}')(- \mathbf{B}')^i \, e^{\mathbf{B}'(x-u)}}_{\det(\mathbf{B}' - \mathbf{I_m} \mathbf{B}') = A(-\mathbf{B}') = 0} \mathbf{b}'^{\mathsf{T}} \right] = 0.$$

Since

$$V(u, b) = \sum_{l=1}^{n+m} a_l(b)e^{-\rho_l u},$$

we will prove that $V(u, b)$ satisfies the homogeneous integro-differential equation

$$A(\mathcal{D})[B_\delta(\mathcal{D})V(u, b)] = A(\mathcal{D})[q_\delta(\mathcal{D})W(u, b)],$$

or equivalently,

$$B_\delta(\mathcal{D})[A(\mathcal{D})V(u, b)] = q_\delta(\mathcal{D})[A(\mathcal{D})W(u, b)]. \tag{5.11}$$

We have

$$W(u, b) = \int_u^b V(x, b)p(x - u)\mathrm{d}x + \int_b^\infty (x - b + V(b, b))p(x - u)\mathrm{d}x$$

$$= \sum_{l=1}^{n+m} a_l(b)e^{-\rho_l u}\hat{p}(\rho_l) + \sum_{l=1}^{n+m} a_l(b)e^{-\rho_l b}\boldsymbol{\alpha}'(\rho_l \mathbf{I_m} - \mathbf{B}')^{-1}\mathbf{B}' e^{\mathbf{B}'(x-u)} \mathbf{1}^{\mathsf{T}}$$

$$- \boldsymbol{\alpha}'(\mathbf{B}')^{-1}e^{\mathbf{B}'(x-u)}\mathbf{1}^{\mathsf{T}} + \boldsymbol{\alpha}' e^{\mathbf{B}'(x-u)}\mathbf{1}^{\mathsf{T}} \sum_{l=1}^{n+m} a_l(b)e^{-\rho_l b},$$

so, in the right-hand side of (5.11), we have

$$A(\mathcal{D})W(u, b) = \sum_{l=1}^{n+m} a_l(b)A(- \rho_l)\hat{p}(\rho_l)e^{-\rho_l u}.$$

On the left-hand side, we have

$$A(\mathcal{D})V(u, b) = A(\mathcal{D}) \sum_{l=1}^{n+m} a_l(b)e^{-\rho_l u} = \sum_{l=1}^{n+m} a_l(b)A(- \rho_l)e^{-\rho_l u}.$$

Then,

$$B_\delta(\mathcal{D})[A(\mathcal{D})V(u,b)] = \sum_{l=1}^{n+m} a_l(b)A(-\rho_l)B_\delta(-\rho_l)e^{-\rho_l u},$$

$$q_\delta(\mathcal{D})[A(\mathcal{D})W(u,b)] = \sum_{l=1}^{n+m} a_l(b)A(-\rho_l)q_\delta(-\rho_l)\hat{p}(\rho_l)e^{-\rho_l u}.$$

This proves that $V(u,b)$ satisfies (5.11), because $B_\delta(-\rho_l) = q_\delta(-\rho_l)\hat{p}(\rho_l)$ for the values $l = 1,\ldots n+m$.

Now, we want $V(u,b)$ to be a solution of our integro-differential Equation (5.2), as in Theorem 5.1. Since solutions of (5.11) include those of $B_\delta(\mathcal{D})V(u,b) = q_\delta(\mathcal{D})W(u,b)$, we want to know which are the extra conditions that must be satisfied by the coefficients $a_l(b)$ of $V(u,b)$ for this purpose. Replacing $V(u,b)$ in $B_\delta(\mathcal{D})V(u,b) = q_\delta(\mathcal{D})W(u,b)$ we obtain

$$0 = \sum_{l=1}^{n+m} a_l(b)\underbrace{[B_\delta(-\rho_l) = q_\delta(-\rho_l)\hat{p}(\rho_l)]}_{=0}e^{-\rho_l u}$$

$$= \boldsymbol{\alpha}'\left[\sum_{l=1}^{n+m} a_l(b)e^{-\rho_l b}\left((\rho_l\mathbf{I_m} - \mathbf{B}')^{-1}\mathbf{B}' + \mathbf{I_m}\right) - \mathbf{B}'^{-1}\right]q_\delta(-\mathbf{B}')e^{\mathbf{B}'(x-u)}\mathbf{1}^{\mathsf{T}}, \; \forall u \geq 0.$$

This proves that the identity holds

$$\boldsymbol{\alpha}'\left[\sum_{l=1}^{n+m} a_l(b)e^{-\rho_l b}\left((\rho_l\mathbf{I_m} - \mathbf{B}')^{-1}\mathbf{B}' + \mathbf{I_m}\right) - \mathbf{B}'^{-1}\right] = \mathbf{0}.$$

Using this identity and the boundary conditions (5.3) we obtain a system of $m + n$ equations that allow us to find the $m + n$ coefficients $a_l(b)$ in $V(u,b)$. $\qquad\square$

Example 5.1: Assume that $K(t)$ is Ph(2) distributed ($n = 2$) and $P(x)$ is Ph(2) distributed ($m = 2$), with representations $(\boldsymbol{\alpha}, \mathbf{B}, \mathbf{b})$ and $(\boldsymbol{\alpha}', \mathbf{B}', \mathbf{b}')$, respectively.

The net profit condition is $-c\boldsymbol{\alpha}\mathbf{B}\mathbf{1}^{\mathsf{T}} < -\boldsymbol{\alpha}'\mathbf{B}'\mathbf{1}^{\mathsf{T}}$ and the generalized Lundberg's equation becomes

$$B_\delta(-s)\bar{B}(s) = q_\delta(-s)\bar{q}(s), \tag{5.12}$$

where

$$B_\delta(-s) = 1 - c\frac{\mathrm{tr}(\mathbf{B})}{\det(\mathbf{B})}\left(\frac{\delta}{c} - s\right) + \frac{c^2}{\det(\mathbf{B})}\left(\frac{\delta}{c} - s\right)^2,$$

$$q_\delta(-s) = 1 + \frac{c}{\det(\mathbf{B})}\boldsymbol{\alpha}\mathbf{b}^{\mathsf{T}}\left(\frac{\delta}{c} - s\right),$$

$$\bar{B}(s) = 1 - \frac{\mathrm{tr}(\mathbf{B}')}{\det(\mathbf{B}')}s + \frac{1}{\det(\mathbf{B}')}s^2,$$

$$\bar{q}(s) = 1 + \frac{1}{\det(\mathbf{B}')}\boldsymbol{\alpha}'\mathbf{b}'^{\mathsf{T}}s.$$

Let

$$V(u,b) = \sum_{l=1}^{4} a_l(b)e^{-\rho_l u}.$$

Table 1. Values of $V(u, b)$.

				b				
u	3	5	6	7	8	10	15	20
2	14.0478	18.5161	18.6277	18.2447	17.6847	16.4611	13.656	11.3231
3	15.9118	20.9727	21.0991	20.6653	20.031	18.6451	15.4678	12.8254
5	17.9118	23.5692	23.7112	23.2237	22.5109	20.9533	17.3828	14.4132
10	22.9118	28.5692	28.7183	28.1731	27.3464	25.4359	21.1015	17.4967
15	27.9118	33.5692	33.7183	33.1731	32.3464	30.4359	25.4503	21.1026
20	32.9118	38.5692	38.7183	38.1731	37.3464	35.4359	30.4503	25.4504

The exponents ρ_l's are the four roots of (5.12). Assume that ρ_1, ρ_2 have positive real parts and ρ_3, ρ_4 have negative real parts. The coefficients $a_l(b)$'s are obtained using the corresponding boundary conditions (5.3)

$$V(0, b) = \sum_{l=1}^{4} a_l(b) = 0,$$

$$\frac{d}{du} V(u, b) \bigg|_{u=0} = \frac{1}{c} k(0) W(0, b) = -\sum_{l=1}^{4} \rho_l a_l(b), \quad \text{or}$$

$$0 = \sum_{l=1}^{4} a_l(b) \left(\frac{k(0)}{c} \hat{p}(\rho_l) + \rho_l \right),$$

and the additional constrains (5.10), giving

$$\sum_{l=1}^{4} a_l(b) e^{-\rho_l b} \rho_l \boldsymbol{\alpha}'(\rho_l \mathbf{I_2} - \mathbf{B}')^{-1} = \boldsymbol{\alpha}' \mathbf{B}'^{-1}, \quad \text{with} \quad \boldsymbol{\alpha}' = (\alpha_1', \alpha_2'), \quad \mathbf{B}' = \begin{pmatrix} b_{11}' & b_{12}' \\ b_{21}' & b_{22}' \end{pmatrix},$$

or

$$\sum_{l=1}^{4} a_l(b) \left(\frac{e^{-\rho_l b} \rho_l (\alpha_1'(\rho_l - b_{22}') + \alpha_2' b_{21}')}{\det(\rho_l \mathbf{I_2} - \mathbf{B}')} \right) = \frac{\alpha_1' b_{22}' - \alpha_2' b_{21}'}{\det(\mathbf{B}')},$$

$$\sum_{l=1}^{4} a_l(b) \left(\frac{e^{-\rho_l b} \rho_l (\alpha_1' b_{12}' + \alpha_2'(\rho_l - b_{11}'))}{\det(\rho_l \mathbf{I_2} - \mathbf{B}')} \right) = \frac{-\alpha_1' b_{12}' + \alpha_2' b_{11}'}{\det(\mathbf{B}')}.$$

If we set the values for the parameters $c = 1$, $\delta = 0.05$, and

$$\boldsymbol{\alpha} = (0.2, 0.8), \quad \mathbf{B} = \begin{pmatrix} -3 & 2 \\ 4 & -7 \end{pmatrix}, \quad \boldsymbol{\alpha}' = (0.7, 0.3), \quad \mathbf{B}' = \begin{pmatrix} -2 & 1 \\ 5 & -5 \end{pmatrix},$$

then $\rho_1 = 8.41055$, $\rho_2 = 0.949785$, $\rho_3 = -0.0374676$ and $\rho_4 = -6.22287$.

In Table 1 we show numerical values for $V(u, b)$ for some choices of (u, b). We can observe that for a fixed u we have a maximal $V(u, b)$ for a value of b between 5 and 7. In the following, we devote our study to the optimal barrier level b and show that it is independent of u.

5.1. Optimal dividends

For a given initial capital u, let b^* denote the optimal value of the barrier b that maximizes the expected discounted dividends $V(u, b)$. Avanzi et al. (2007) show that for a dual model with exponentially

distributed inter-arrival times the value of b^* is independent of u. The same situation occurs for a dual model with phase-type(n) distributed inter-gain times and phase-type(m) distributed gain amounts. Also, the optimal level is independent of the initial surplus.

Theorem 5.3: b^* *is independent of the initial surplus u.*

Proof: For a given initial surplus $u_0 \geq 0$ let b_0^* be the optimal barrier level that maximizes the expected discounted dividends, $V(u_0, b)$ is maximal at $b = b_0^*$ and

$$\frac{\partial}{\partial b} V(u_0, b)\Big|_{b=b_0^*} = 0, \quad \text{for} \quad u = u_0.$$

The idea of this proof is to show that

$$\frac{\partial}{\partial b} V(u, b)\Big|_{b=b_0^*} = 0, \quad \forall u \geq 0.$$

From (5.1), we have $\forall u \geq b_0^*$ that

$$\frac{\partial}{\partial b} V(u, b)\Big|_{b=b_0^*} = 0 = -1 + \frac{d}{d b} V(b, b)\Big|_{b=b_0^*} \quad \Rightarrow \quad \frac{d}{d b} V(b, b)\Big|_{b=b_0^*} = 1.$$

Since we have $V(0, b) \equiv 0$ then clearly

$$\frac{\partial}{\partial b} V(0, b)\Big|_{b=b_0^*} = 0, \quad \text{for} \quad u = 0.$$

It only remains to show that

$$\frac{\partial}{\partial b} V(u, b)\Big|_{b=b_0^*} = 0, \quad 0 < u < b_0^*.$$

Previously in Theorem 5.1 we have found that in the phase-type(n) dual risk model the expected discounted dividends $V(u, b)$ satisfy the integro-differential equation

$$B_\delta(\mathcal{D}) V(u, b) = q_\delta(\mathcal{D}) W(u, b),$$

where

$$W(u, b) = \int_u^b V(y, b) p(y - u) dy + \int_b^\infty (y - b + V(b, b)) p(y - u) dy.$$

Moreover, assuming that the gain amounts follow another phase-type(m) distribution, with density function $p(x) = \boldsymbol{\alpha}' e^{\mathbf{B}' x} \mathbf{b}'^{\mathsf{T}}$, we were able to write an expression of $V(u, b)$ of the form (5.9)

$$V(u, b) = \sum_{l=1}^{n+m} a_l(b) e^{-\rho_l u}.$$

Since

$$\frac{\partial}{\partial b} W(u,b)\Big|_{b=b_0^*} = \int_u^{b_0^*} \frac{\partial}{\partial b} V(y,b)\Big|_{b=b_0^*} p(y-u)\mathrm{d}y$$

$$+ \underbrace{\left(-1 + \frac{\mathrm{d}}{\mathrm{d}b} V(b,b)\Big|_{b=b_0^*}\right)}_{=0} \int_{b_0^*}^{\infty} p(y-u)\mathrm{d}y$$

$$= \int_u^{b_0^*} \frac{\partial}{\partial b} V(y,b)\Big|_{b=b_0^*} p(y-u)\mathrm{d}y,$$

then for $0 < u < b_0^*$ we have that

$$B_\delta(\mathcal{D}) \frac{\partial}{\partial b} V(u,b)\Big|_{b=b_0^*} = q_\delta(\mathcal{D}) \frac{\partial}{\partial b} W(u,b)\Big|_{b=b_0^*},$$

or equivalently

$$B_\delta(\mathcal{D}) \frac{\partial}{\partial b} V(u,b)\Big|_{b=b_0^*} = q_\delta(\mathcal{D}) \left[\int_u^{b_0^*} \frac{\partial}{\partial b} V(y,b)\Big|_{b=b_0^*} p(y-u)\mathrm{d}y \right], \quad 0 < u < b_0^*. \tag{5.13}$$

When we replace

$$\frac{\partial}{\partial b} V(u,b)\Big|_{b=b_0^*} = \sum_{l=1}^{n+m} a_l'(b_0^*)e^{-\rho_l u}$$

in (5.13) we get an identity of exponential functions in terms of the coefficients $a_l'(b_0^*)$ which is valid for all u in $(0, b_0^*)$, as follows.

Let's define the function

$$F(u) = \frac{\partial}{\partial b} V(u,b)\Big|_{b=b_0^*} = \sum_{l=1}^{n+m} a_l'(b_0^*)e^{-\rho_l u}.$$

Then (5.13) becomes

$$B_\delta(\mathcal{D})F(u) = q_\delta(\mathcal{D}) \left[\int_u^{b_0^*} F(y)p(y-u)\mathrm{d}y \right], \quad 0 < u < b_0^*. \tag{5.14}$$

On the left-hand side of (5.14) we calculate $B_\delta(\mathcal{D})F(u)$,

$$B_\delta(\mathcal{D})F(u) = \sum_{l=1}^{n+m} a_l'(b_0^*)B_\delta(\mathcal{D})e^{-\rho_l u} = \sum_{l=1}^{n+m} a_l'(b_0^*)B_\delta(-\rho_l)e^{-\rho_l u}. \tag{5.15}$$

On the right-hand side of (5.14) we compute $q_\delta(\mathcal{D}) \left[\int_u^{b_0^*} F(y)p(y-u)\mathrm{d}y \right]$.

Recall that $p(y-u) = \boldsymbol{\alpha}' e^{\mathbf{B}'(y-u)} \mathbf{b}'^{\mathsf{T}}$, therefore

$$\int_u^{b_0^*} e^{-\rho_l y} p(y-u)\mathrm{d}y = e^{-\rho_l u}\hat{p}(\rho_l) - e^{-\rho_l u} \int_{b_0^*-u}^{\infty} e^{-\rho_l y} p(y)\mathrm{d}y$$

$$= e^{-\rho_l u}\hat{p}(\rho_l) - e^{-\rho_l u}\int_{b_0^*-u}^{\infty} e^{-\rho_l y}\boldsymbol{\alpha}' e^{\mathbf{B}'(y)}\mathbf{b}'^{\mathsf{T}}dy$$

$$= e^{-\rho_l u}\left[\hat{p}(\rho_l) - \int_{b_0^*-u}^{\infty}\boldsymbol{\alpha}' e^{(\mathbf{B}'-\rho_l\mathbf{I})y}\mathbf{b}'^{\mathsf{T}}dy\right]$$

$$= e^{-\rho_l u}\left[\hat{p}(\rho_l) - \boldsymbol{\alpha}'\int_{b_0^*-u}^{\infty} e^{(\mathbf{B}'-\rho_l\mathbf{I})y}dy\ \mathbf{b}'^{\mathsf{T}}\right]$$

$$= e^{-\rho_l u}\left[\hat{p}(\rho_l) + \boldsymbol{\alpha}'(\mathbf{B}'-\rho_l\mathbf{I})^{-1}e^{(\mathbf{B}'-\rho_l\mathbf{I})(b_0^*-u)}\ \mathbf{b}'^{\mathsf{T}}\right]$$

$$= e^{-\rho_l u}\left[\hat{p}(\rho_l) + \boldsymbol{\alpha}'(\mathbf{B}'-\rho_l\mathbf{I})^{-1}e^{(\mathbf{B}'-\rho_l\mathbf{I})b_0^*}e^{-\mathbf{B}'u}\ \mathbf{b}'^{\mathsf{T}}\right].$$

Hence,

$$q_\delta(\mathcal{D})\int_u^{b_0^*} e^{-\rho_l y}p(y-u)dy = q_\delta(-\rho_l)e^{-\rho_l u}\hat{p}(\rho_l)$$
$$+ \boldsymbol{\alpha}'(\mathbf{B}'-\rho_l\mathbf{I})^{-1}e^{(\mathbf{B}'-\rho_l\mathbf{I})b_0^*}q_\delta(-\mathbf{B}')e^{-\mathbf{B}'u}\ \mathbf{b}'^{\mathsf{T}},$$

and,

$$q_\delta(\mathcal{D})\int_u^{b_0^*} F(y)p(y-u)dy = \sum_{l=1}^{n+m} a_l'(b_0^*)q_\delta(\mathcal{D})\int_u^{b_0^*} e^{-\rho_l y}p(y-u)dy$$
$$= \sum_{l=1}^{n+m} a_l'(b_0^*)q_\delta(-\rho_l)e^{-\rho_l u}\hat{p}(\rho_l)$$
$$+ \sum_{l=1}^{n+m} a_l'(b_0^*)\boldsymbol{\alpha}'(\mathbf{B}'-\rho_l\mathbf{I})^{-1}e^{(\mathbf{B}'-\rho_l\mathbf{I})b_0^*}q_\delta(-\mathbf{B}')e^{-\mathbf{B}'u}\mathbf{b}'^{\mathsf{T}}. \quad (5.16)$$

Expressions in (5.15) and (5.16) are equal,

$$\sum_{l=1}^{n+m} a_l'(b_0^*)B_\delta(-\rho_l)e^{-\rho_l u} = \sum_{l=1}^{n+m} a_l'(b_0^*)q_\delta(-\rho_l)e^{-\rho_l u}\hat{p}(\rho_l)$$
$$+ \sum_{l=1}^{n+m} a_l'(b_0^*)\boldsymbol{\alpha}'(\mathbf{B}'-\rho_l\mathbf{I})^{-1}e^{(\mathbf{B}'-\rho_l\mathbf{I})b_0^*}q_\delta(-\mathbf{B}')e^{-\mathbf{B}'u}\ \mathbf{b}'^{\mathsf{T}}.$$

So,

$$\sum_{l=1}^{n+m} a_l'(b_0^*)[B_\delta(-\rho_l) - q_\delta(-\rho_l)\hat{p}(\rho_l)]e^{-\rho_l u} = \sum_{l=1}^{n+m} a_l'(b_0^*)\boldsymbol{\alpha}'(\mathbf{B}'-\rho_l\mathbf{I})^{-1}e^{(\mathbf{B}'-\rho_l\mathbf{I})b_0^*}q_\delta(-\mathbf{B}')e^{-\mathbf{B}'u}\ \mathbf{b}'^{\mathsf{T}}.$$

Since $\rho_1, \ldots, \rho_{m+n}$ are the roots of the generalized Lundberg's equation then $B_\delta(-\rho_l) = q_\delta(-\rho_l)\hat{p}(\rho_l)$. Thus,

$$0 = \sum_{l=1}^{n+m} a_l'(b_0^*)\boldsymbol{\alpha}'(\mathbf{B}'-\rho_l\mathbf{I})^{-1}e^{(\mathbf{B}'-\rho_l\mathbf{I})b_0^*}q_\delta(-\mathbf{B}')e^{-\mathbf{B}'u}\ \mathbf{b}'^{\mathsf{T}}$$

$$= \left[\underbrace{\sum_{l=1}^{n+m} a_l'(b_0^*)\boldsymbol{\alpha}'(\mathbf{B}' - \rho_l\mathbf{I})^{-1}e^{(\mathbf{B}'-\rho_l\mathbf{I})b_0^*}}_{=0} \right] q_\delta(-\mathbf{B}')e^{-\mathbf{B}'u}\,\mathbf{b}'^{\mathsf{T}}, \ \forall \ u \in (0, b_0^*),$$

since the above identity is valid for all u in the interval $(0, b_0^*)$. For simplicity, we have assumed that the roots $\rho_1, \ldots, \rho_{m+n}$ are all distinct, then the vectors $\boldsymbol{\alpha}'(\mathbf{B}' - \rho_l\mathbf{I})^{-1}e^{(\mathbf{B}'-\rho_l\mathbf{I})b_0^*}$ are linearly independent and we obtain

$$a_l'(b_0^*) = 0, \ \forall \ l = 1, \ldots, m + n.$$

This proves that

$$\frac{\partial}{\partial b}V(u,b)\bigg|_{b=b_0^*} = 0, \quad 0 < u < b_0^*.$$

Therefore, we have proven that the optimal barrier level is independent of u.

□

Remark 5.2: The result holds if we assume multiplicities higher than 1 in the roots ρ_l's.

Example 5.2: In Example 5.1 the optimal value of the barrier level is $b^* = 5.61986$, with $V(b^*, b^*) = 24.3976$.

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