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A Business Cycle Model of Speculation from a Viewpoint of Minsky and Shiller Ⅱ: Global Dynamic Analysis

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Abstract: We construct a 3-dimensional extension of the dynamic IS-LM model, in which the money demand function depends not only on income but also on a rate of change in expected income (RCEI). We demonstrate the occurrence of limit cycles in the extended IS-LM model. Our arguments are essentially derived from the remarkable viewpoint of H. P. Minsky and J. R. Shiller concerning financial markets. We assume that the money demand negatively correlates with RCEI. Such a negative correlation results from a speculative behavior. We demonstrate that the negative correlation is an important source of unstable equilibrium and therefore, business cycles. Firstly, we transform the extended IS-LM model into a 2-dimensional Liénard system and prove the occurrence of a stable limit cycle in the Liénard system. Secondly, by using a Hopf bifurcation theorem, we demonstrate the occurrence of a Hopf cycle in the extended 3-dimensionl IS-LM model. Our model possesses two types of self-fulfilling prophecy

Keywords: IS-LM Model; Money Demand; Speculation; Limit Cycle; Hopf Bifurcation

1. Introduction

Inspired by the important studies given by Minsky (1975, 1982) and Shiller (1981), Dohtani and Matsuyama (2022) constructed a simple three-dimensional (3D) business cycles model that gives an extension of the IS-LM model. The dynamic model demonstrates that speculation leads us to business fluctuations.

Using a Hopf bifurcation theorem, Dohtani and Matsuyama (2022) provided a local nonlinear analysis near the equilibrium point. In this paper, we provide a global analysis of the extended IS-LM model. Since the extended IS-LM model is 3D, it is difficult to perform the global analysis of the extended IS-LM model. So, by assuming an instant adjustment of interest rates (i.e., assuming that the money market instantly balances), we reduce the number of variables in the model to two and transform this two-dimensional (2D) model into the so-called Liénard system. By using the Liénard system, we perform the global analysis of the 2D model. Various results on Liénard system have been obtained so far. Especially among them, in this paper, we use two results on the existence of limit cycle and demonstrate the occurrence of business fluctuations in the extended IS-LM model.

2. Brief Explanation of the 3D Model

In this section, we briefly explain the extended IS-LM model, which given by Dohtani and Matsuyama (2022) (abbreviated by D-M). In the extended IS-LM model, money demand depends on the rate of change in expected income (abbreviated by RCEI). Throughout this study, all functions are assumed to be continuously differentiable and all parameters are assumed to be positive. We denote income, expected income, interest rate, price level, money supply, investment, and consumption by Y, Y_e, R, P, M, I, and C, respectively.

D-M considers the following investment and consumption function: $I = I(Y, R)$, and $C = C(Y)$. D-M assumes $\partial I / \partial Y > 0$, $\partial I / \partial R < 0$ and $dC / dY > 0$. For simplicity, D-M defines the aggregate demand function:

$$H(Y, R) \equiv I(Y, R) + C(Y).$$

D-M next considers the money demand function. Based on the findings of Shiller (1981), D-M assumes that money demand correlates negatively with the change in expectation regarding the future income and considers the following money demand function:

$$L = L(Y, R) - \Gamma(\overset{\bullet}{Y_e}).$$

The $L-$function denotes the usual money demand function. As usual, $\partial L / \partial Y > 0$ and $\partial L / \partial R < 0$ are assumed. On the other hand, to take the above-mentioned speculation into consideration, the $\Gamma-$function is incorporated. Moreover, D-M assumes the following:

Assumption 1: $\Gamma(0) = 0$, $\Gamma'(u) > 0$ for any $u \in R^1$, and $\sup\{\Gamma'(u) : u \in R^1\} < +\infty$.

For a detailed economic implication of Assumption 1, see D-M.

D-M assumes that expected income is adaptively adjusted: $\overset{\bullet}{Y_e} = \Psi(Y - Y_e)$. For the adaptive adjustment, see Section 3 of D-M. Under the assumption, the following extended IS-LM model is obtained:

$$\Omega \quad \begin{cases} \overset{\bullet}{Y} = \alpha\{H(Y, R) - Y\}, \\ \overset{\bullet}{R} = \beta\{L(Y, R) - \Gamma \circ \Psi(Y - Y_e) - M / P\}, \\ \overset{\bullet}{Y_e} = \Psi(Y - Y_e), \end{cases}$$

D-M provides System Ω in which the ideas of Minsky (1975, 1982) and Shiller (1981) are incorporated. The following usual IS-LM is obtained as the subsystem of System Ω:

$$\Omega_{IS-LM} \quad \begin{cases} \overset{\bullet}{Y} = \alpha\{H(Y, R) - Y\}, \\ \overset{\bullet}{r} = \beta\{L(Y, R) - M / P\}. \end{cases}$$

D-M calls System Ω_{IS-LM} the IS-LM subsystem. D-M works under the following assumptions:

Assumption 2: $1 > \partial H / \partial Y$ for any $(Y, R) \in R^2$.
Assumption 3: $\Psi(0) = 0$, $\Psi'(z) > 0$, and $\sup\{\Psi'(z) : z \in R^1\} < +\infty$.

Assumption 3 is clear. For the economic implications of Assumption 2, see D-M. The following result is obvious.

Lemma 1: System Ω has equilibrium points for income and the interest rate if and only if System Ω_{IS-LM} has them. Moreover, for two systems, the equilibrium points of income and the interest rate are the same.■

Proof: The proof is clear.■

To stress the instability that is caused by the dependence of money demand on RCEI, D-M considers the case where the IS-LM subsystem is globally stable. Assumption 2 guarantees the global stability of the IS-LM subsystem.

Lemma 2: Suppose that the IS-LM subsystem Ω_{IS-LM} possesses an equilibrium point and that Assumption 2 is satisfied. Then System Ω_{IS-LM} is globally asymptotically stable.■

Proof: See Appendix of D-M.■

4. Global Dynamics of the Reduced 2D Model

Using a Hopf bifurcation theorem, D-M demonstrated the occurrence of Hopf cycle in the extended IS-LM model that is 3D. Such a result is local in the sense that works merely in a neighborhood of the equilibrium point. In this paper, we try to perform a global analysis of the extended IS-LM model. To do so, D-M considers two simplifications: dimension reduction and linear approximations of some functions. Firstly, by assuming that the money market reaches equilibrium instantly, D-M reduces the number of variables in the model Ω . D-M assumes that the IS-LM subsystem is linear. In this case, it is shown soon after that the main nonlinearity of the IS-LM subsystem is included in the dependence of money demand on RCEI. We set

$$L = L(Y,R) - \Gamma(\dot{Y}_e) \equiv l_1 Y - l_2 R - \Gamma \circ \Psi(Y - Y_e), \quad H(Y,R) \equiv h_1 Y - h_2 R + B$$

The solution of $L = M / P$ for R is given by

$$R = \frac{l_1}{l_2} Y - \frac{\Gamma \circ \Psi(Y - Y_e)}{l_2} - \frac{M}{l_2 P},$$

where we define

$$l \equiv l_1 / l_2, \quad A \equiv M / l_2 P \quad \text{and} \quad \Lambda(u) \equiv \Gamma(u) / l_2.$$

Then, we have

$$H(Y, R) \equiv h_1 Y - h_2 R + B = (h_1 - h_2 l) Y + h_2 \Lambda \circ \Psi(Y - Y_e) + h_2 A + B.$$

Assumption 2 yields $h_1 \in (0,1)$. Then, the extended IS-LM model becomes

$$\Omega^{\#} \quad \begin{cases} \dot{Y} = \alpha\{(h_1 - h_2 l - 1) Y + h_2 \Lambda \circ \Psi(Y - Y_e) + h_2 A + B\}, \\ \dot{Y}_e = \Psi(Y - Y_e). \end{cases}$$

The equilibrium point of System $\Omega^{\#}$ is given by

$$(Y^*, Y_e^*) = (\frac{h_2 A + B}{1 + h_2 l - h_1}, \frac{h_2 A + B}{1 + h_2 l - h_1}).$$

It should be noted here that Assumption 2 yields $1 + h_2 l - h_1 > 1 - h_1 = 1 - \partial H / \partial Y > 0.$ In this case, the goods market system $\Omega^{\#}$ is given as $\dot{Y} = (h_1 - h_2 l - 1) Y + h_2 A + B.$ It should be also noted here that since $h_1 - h_2 l - 1 < 0$, we obtain that System $\Omega^{\#}$ in this case is globally asymptotically stable and therefore, as we demonstrate in the following, the source of instability is included in the $\Lambda-$function that is closely related to the speculative behavior. To transform to a Liénard system, we define the affine transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/\alpha s & -1/\alpha s \\ 0 & -1 \end{bmatrix} \begin{bmatrix} Y - K/s \\ Y_e - K/s \end{bmatrix}, \quad s \equiv 1 + h_2 l - h_1, \quad K \equiv h_2 A + B,$$

Then, we have $x = (Y - Y_e)/\alpha s$ and $y = -Y_e + K/s.$ This yields

$$Y = \alpha s x + K/s - y.$$

Therefore, we have

$$\dot{x} = \frac{\dot{Y} - \dot{Y}_e}{\alpha s} = \frac{\alpha[-sY + h_2 \Lambda \circ \Psi(Y - Y_e) + K] - \Psi(Y - Y_e)}{\alpha s}$$

$$= -Y + \frac{K}{s} + \frac{h_2 \Lambda \circ \Psi(\alpha s x) - \Psi(\alpha s x)/\alpha}{s} = y - \{\alpha s x - \frac{h_2 \Lambda \circ \Psi(\alpha s x) - \Psi(\alpha s x)/\alpha}{s}\}.$$

$$\dot{y} = -\dot{Y}_e = -\Psi(\alpha s x).$$

We now define

$$\Phi(x) \equiv \alpha s x - \frac{h_2 \Lambda \circ \Psi(\alpha s x) - \Psi(\alpha s x)/\alpha}{s}.$$

Then, we have the Liénard system:

$$\Omega_{Li} \quad \begin{cases} \dot{x} = y - \Phi(x), \\ \dot{y} = -\Psi(\alpha s x). \end{cases}$$

In the following, we analyze the Liénard system Ω_{Li}. It should be noted here that the qualitative properties of the dynamics of System Ω_{Li} are the same as those of System $\Omega^{\#}$. For example, the existence, uniqueness and stability of the equilibria and limit cycles in System Ω_{Li} are inherited from $\Omega^{\#}$. We have the following lemma.

Lemma 3: Suppose that Assumptions 1 and 3 are satisfied. Then, the origin is the unique equilibrium point of the Liénard System Ω_{Li}.■

Proof: See Appendix.■

Through System Ω_{Li}, we consider the global dynamic behavior of the extended IS-LM in which the money demand depends on RCEI. We start from a local stability analysis of the equilibrium. The Jacobean matrix of Ω_{Li} that is estimated at the equilibrium is given by

$$J = \begin{bmatrix} -\Phi'(0) & 1 \\ -\alpha\Psi'(0) & 0 \end{bmatrix}.$$

Therefore, as stated above, the equilibrium point of Ω_{Li} (therefore, $\Omega^{\#}$) is completely unstable in the sense that all real parts of the eigenvalues of J are positive:

Lemma 4: Suppose Assumptions 1 to 3 and the following condition are satisfied. Then, the equilibrium point of the extended IS-LM model is completely unstable.

(C.1) $\Lambda'(0) > \dfrac{s + \Psi'(0)/\alpha}{h_2 \Psi'(0)} \equiv \Lambda_H > 0.$ ■

Proof: Assumptions 1 to 3 and (C.1) yield that $\det J = \alpha \Psi'(0) > 0$ and $\mathrm{tr} J = -\Phi'(0) = -s + h_2 \Lambda'(0) \bullet \Psi'(0) - \Psi'(0)/\alpha > 0.$ This completes the proof.■

Condition (C.1) is satisfied, provided that $\Lambda'(0)$ (the sensitivity of money demand to RCEI) is sufficiently large. Lemmas 2 and 4 demonstrate that the extended IS-LM model, in which the money demand depends on RCEI, possesses a new instability source that does not appear in the original IS-LM model. See for example Torre (1977), Schinasi (1982), Gabisch and Lorenz (1987) and Lorenz (1993). It should be noted that the instability source in this paper is purely monetary and related to speculation.

By using a version of the Hopf bifurcation theorem, Torre (1977) proved the existence of a limit cycle in the IS-LM subsystem in Ω . It should be note that, unlike Torre (1977), the IS-LM subsystem of our model is globally stable. Although, like Torre (1977), we later provide a result about the existence of a Hopf bifurcation, it is a local result. In the following, we will provide more important global results.

The chief reason that by slightly discarding generality (i.e. the above assumption of the money market clearing) we take the trouble to derive the Liénard system is to utilize two useful mathematical results about the existence and the unique existence of a stable limit cycle, which are given as follows.

Dragilëv Theorem[1]: Consider the Liénard system:

$$\Sigma \quad \begin{cases} \dot{w} = z - h(w), \\ \dot{z} = -g(w). \end{cases}$$

System Σ possesses at least one stable limit cycle under the following conditions:

(C.2) $wg(w) > 0$ for any $w \neq 0$ and $\lim_{w \to \pm\infty} G(w) = +\infty,$ where

$$G(w) = \int_0^w g(u)du;$$

[1] The Dragilëv Theorem was proved by Dragilëv (1952). A simpler proof of this theorem is provided by Yanqian et al. (1986, Theorem 5.1); see also Nemytskii and Stepanov (1989, pp.140-146).

(C.3) $wh(w) < 0$ when $w \neq 0$ and $|w|$ is sufficiently small;

(C.4) There are constants $M > 0$ and $K > K'$ such that $h(w) \geq K$ for any $w > M$ and $h(w) \leq K'$ for any $w < -M$. ■

Proof: See Yanqian (1986, Theorem 5.1).■

A zero x^* of a differentiable function $f: U \subseteq R^1 \to R^1$ is said to be transversal if $f'(x^*) \neq 0$.

Carletti–Villari Theorem: System Σ of Class C^1 possesses a unique stable limit cycle under the following conditions:

(C.5) The $h-$function has only three real transversal zeros, located at $w_0 = 0$ and $w_2 < 0 < w_3$. Moreover, the $h-$function is monotone increasing outside the interval $[w_2, w_3]$;

(C.6) $h'(0) < 0$;

(C.7) There exists a $\delta > 0$ such that $h'(w) > 0$ for $|w| > \delta$;

(C.8) $wg(w) > 0$ for any $w \neq 0$;

(C.9) $\limsup_{w \to +\infty} |G(w) + h(w)| = +\infty$ and $\limsup_{w \to -\infty} |G(w) - h(w)| = +\infty$;

(C.10) $G(w) = G(-w)$. ■

Proof: See Carletti and Villari (2005).■

Utilizing these two results, we prove the existence and the unique existence of a stable limit cycle in System Ω_{Li}.

Theorem 1: System Ω_{Li} (therefore, $\Omega^{\#}$) possesses at least one stable limit cycle under Assumptions 1 to 3, (C.1), and one of the following assumptions:

(C.11.1) $\lim_{z \to \pm\infty} \Psi(z) = \pm\infty$ and $\lim_{u \to \pm\infty} \Gamma'(u) = 0$,

(C.11.2) $\lim_{z \to \pm\infty} \Psi'(z) = 0$. ■

Proof: See Appendix.■

We make one remark about Condition (C.11). As a typical Ψ – function that satisfies the first half of Condition (C.11.1), we have $\Psi(z) = \gamma z \quad (\gamma > 0).$ In this sense, the first half of Condition (C.11.1) is natural. Since the explanation of Condition (C.11.2) is almost the same as that of the latter half of Condition (C.11.1), we only explain the latter half of Condition (C.11.1). Since we assume $\Gamma'(u) > 0$ in Assumption 1, a typical Γ – function that satisfies Condition (C.11.1) incorporates a sigmoid nonlinearity into System Ω_{Li}.

Lemma 2 and Theorem 1 prove that the dependence of money demand on RCEI makes the system unstable and yields business cycles. Moreover, by adding some considerably restrictive conditions to the conditions in Theorem 1, we have a result showing the unique existence of a stable limit cycle:

Theorem 2: System Ω_{Li} (therefore, $\Omega^{\#}$) possesses a unique stable limit cycle under Assumptions 1 to 3, (C.1), one of (C.11), and the following conditions:

(C.12) $\quad \Psi(\alpha s x) = \gamma x \quad (\gamma > 0),$

(C.13) $\quad \Gamma''(u)u < 0$ for any $u \neq 0$.■

Proof: See Appendix.■

Example 1: We set

$$\alpha = 1, \quad h_1 = 0.8, \quad h_2 = 100, \quad l = 0.0032, \quad \Psi(\alpha s x) = 0.2x,$$
$$\text{and} \quad \Lambda(u) = m\text{Arctan}(0.1u),$$

where the parameter m displays the intensity of the dependence of money demand on RCEI. This represents **the intensity of speculation**. See D-M. Assumptions 1 to 3 are satisfied. Under these settings, we have

$$s = 0.52 \quad \text{and} \quad \Gamma(u) = l_2 m\text{Arctan}(0.1u).$$

If $m > 0.36,$ then we obtain

$$\Lambda'(0) = 0.1m > \Lambda_H = (0.52 + 0.2)/(0.2 \times 100) = 0.036.$$

Then, all the conditions of Theorem 1 are satisfied. Therefore, we see that System Ω_{Li}

with these parameters possesses a stable limit cycle. Moreover,

$$\Gamma''(u)u = -\frac{0.0002 l_2 m u^2}{(0.01u^2+1)^2} < 0 \quad \text{for any} \quad u \neq 0.$$

The conditions of Theorem 2 are also satisfied. Therefore, we also see that the stable limit cycle is uniquely yielded. The dynamic behavior of System Ω_{Li} is described in Figure 1. The paths in Figure 1 rotate clockwise. The blue closed curves of Part 1 and the inside blue closed curve of Part 2 in Figure 1 are the same and describe a stable limit cycle of System Ω_{Li} with $m = 1$. On the other hand, the black curves of Part 1 in Figure 1 are paths starting at the interior and exterior points of the limit cycle. Moreover, the outside blue closed curve of Part 2 in Figure 1 describes the limit cycle of System Ω_{Li} with $m = 1.2$. Part 2 demonstrates that as the intensity of speculation becomes large, the amplitude of the emerging limit cycle becomes large.■

Figure 1 about here.

Next, we consider the globally asymptotic stability of System Ω_{Li}. By using the Olech Theorem (see Appendix 7), we obtain the following result.

Theorem 3: System Ω_{Li} (therefore, $\Omega^{\#}$) is globally asymptotically stable under Assumptions 2 and 3, and the following condition:

(C.14) $$\frac{s + \inf\{\Psi'(\alpha s x)/\alpha : x \in R^1\}}{h_2 \bullet \sup\{\Psi'(\alpha s x) : x \in R^1\}} > \Lambda'(u) \quad \text{for any} \quad u \in R^1. ■$$

Proof: See Appendix.■

Lemma 5 and Theorem 3 show that as $\Lambda'(u)$ is small, the market economy becomes globally asymptotically stable. The largeness of $\Lambda'(u)$ represents the intensity of speculation, for example, as shown by the parameter m in Example 1. Therefore, we see that as the intensity of speculation is small, the market economy becomes globally asymptotically stable. In the following, we provide such a numerical example.

Example 2: We set $\Lambda(u) = \text{Arctan}(0.03u)$. Then, Assumptions 2 and 3 are satisfied. Moreover, the $\Psi-$function and the other parameters are the same as those of Example

1. Then, we have

$$\Lambda'(u) = 0.03 < \Lambda_H = \frac{0.52 + 0.2}{0.2 \times 100} = 0.036 \quad \text{for any} \quad u \in R^1 .$$

Therefore, all the conditions of Theorem 3 are satisfied. Thus, we see that System Ω_{Li} is globally asymptotically stable. The dynamic behavior of System Ω_{Li} is described in Figure 2. The black path in Figure 2 rotates clockwise and tends to the equilibrium.■

Figure 2 about here.

6. Conclusion and Final Remarks

In this paper, by assuming that the money market instantly balances, we reduce the number of variables in the model to two and transform this two-dimensional model into the so-called Liénard system. By using the Liénard system, we demonstrate the occurrence of limit cycles (i.e. business fluctuations) in the extended IS-LM model. Using the extended IS-LM model, D-M analytically demonstrated that *speculation* (Shiller (1981)) and *high substitutability between money and stocks* (Minsky (1975, 1982)) and Shiller (1981) are "purely" monetary causes of business cycles. Although the result of D-M is local, our result is global. Thus, our result reinforces the local result of D-M.

We expect that the extended IS-LM 3D model with a liquidity trap generates strange attractors. Researches in such a direction must remain in the future.

7. Appendix

In this appendix, we prove Lemma 3 and Theorems 1 to 3.

Proof of Lemma 3: Suppose that System Ω_{IS-LM} possesses more than one equilibrium. Then, there is an $e \neq 0$ such that $0 = \Psi(0) = \Psi(e)$. Without loss of generality, we here assume $e > 0$. Then, from the mean value theorem there is an $v \in [0, e]$ such that $\Psi'(v) = 0$. This contradicts Assumption 3. The contradiction proves that the solution of $\Psi(x) = 0$ is only zero. Then, we have

$$y = \Phi(0) \equiv -\{h_2 \Lambda \circ \Psi(0) - \Psi(0)\} / s = 0.$$

This completes the proof.■

Proof of Theorem 1: Defining $h(x) \equiv \Phi(x)$ and $g(x) \equiv \Psi(\alpha s x)$, we prove that all the conditions of the Dragilëv Theorem are satisfied. The first part of (C.2) of the Dragilëv Theorem follows from Assumption 3. Moreover, from Assumption 3, there exists an $\bar{x} \in R^1_+ = \{x \in R^1 : x > 0\}$ and a $E > 0$ such that $g(x) \geq E$ for any $x > \bar{x}$ and $g(x) \leq -E$ for any $x < -\bar{x}$. Then, we have

$$\int_{\bar{x}}^{x} g(u)du \geq E(x - \bar{x}) = E(|x| - \bar{x}) \quad \text{for any} \quad x > \bar{x},$$

$$\int_{-\bar{x}}^{x} g(u)du = -\int_{x}^{-\bar{x}} g(u)du \geq E(-\bar{x} - x) = E(|x| - \bar{x}) \quad \text{for any} \quad x < -\bar{x}$$

Therefore, defining $G(x) \equiv \int_0^x g(u)du$, we have

(A.1.1) $$G(x) = \int_0^{\bar{x}} g(u)du + \int_{\bar{x}}^{x} g(u)du \geq \int_0^{\bar{x}} g(u)du + E(|x| - \bar{x}) \qquad \text{for any} \quad x > \bar{x},$$

(A.1.2) $$G(x) = \int_0^{-\bar{x}} g(u)du + \int_{-\bar{x}}^{x} g(u)du \geq \int_0^{-\bar{x}} g(u)du + E(|x| - \bar{x}) \qquad \text{for any} \quad x < -\bar{x},$$

The last part of (C.2) of the Dragilëv Theorem follows directly from (A.1). We now prove (C.3) and (C.4) of the Dragilëv Theorem. We have

$$h'(0) = \alpha s - \alpha h_2 \Lambda'(0) \Psi'(0) + \Psi'(0).$$

Condition (C.1) yields

(A.2) $$h'(0) < 0.$$

Then, we have

$$h'(x) < 0 \quad \text{when} \quad |x| \quad \text{is sufficiently small.}$$

From Assumption 3, we have

(A.3) $$h(0) = \Phi(0) \equiv -\frac{h_2 \Lambda \circ \Psi(0) - \Psi(0)/\alpha}{s} = 0.$$

Therefore, we obtain

$$xh(x) < 0 \text{ when } x \neq 0 \text{ and } |x| \text{ is sufficiently small.}$$

This proves (C.3). We now prove (C.4). We first consider the case where (C.11.1) is satisfied. We see from (C.11.1) that

$$\lim_{u \to \pm\infty} \Lambda'(u) = \lim_{u \to \pm\infty} \Gamma'(u)/l_2 = 0.$$

(C.11.1) yield $\lim_{x \to \pm\infty} \Psi(\alpha sx) = \pm\infty$ and we have

(A.4) $$\lim_{x \to \pm\infty} \Lambda'(\Psi(\alpha sx)) = 0.$$

Moreover, since $g(x) = \Psi(\alpha sx)$, Assumption 3 yields

(A.5) $$\sup\{g'(x) : x \in R^1\} = \sup\{\alpha s \Psi'(\alpha sx) : x \in R^1\} < +\infty.$$

From Assumption 3, we see $g'(x) > 0$ for any $x \in R^1$. Therefore, it follows from (A.4) and (A.5) that there exists $U > 0$ and $W > 0$ such that for any $|x| > W$

(A.6) $$\begin{aligned} h'(x) &= \alpha s - \alpha h_2 \Lambda'(\Psi(\alpha sx))\Psi'(\alpha sx) + \Psi'(\alpha sx) \\ &\geq \alpha s - \alpha h_2 \Lambda'(\Psi(\alpha sx))\Psi'(\alpha sx) > U. \end{aligned}$$

This implies that there are constants $M > 0$ and $K > 0 > K'$ such that

(A.7) $$h(x) > K \text{ for any } x > M \text{ and } h(x) < K' \text{ for any } x < -M.$$

This proves (C.4). We next consider the case where (C.11.2) is satisfied. It follows directly from Assumption 1 that $\sup\{\Lambda'(\Psi(\alpha sx)) : x \in R^1\} < +\infty$. Therefore, from Condition (C.11.2), we obtain $\lim_{x \to \pm\infty} h_2 \Lambda'(\Psi(\alpha sx))\Psi'(\alpha sx) = 0$. Thus, we obtain (A.6). The same argument as above proves (C.4). Thus, we complete the proof.■

Proof of Theorem 2: We define $h(x) \equiv \Phi(x)$, $g(x) \equiv \Psi(\alpha sx) = \gamma x$. We prove that all the conditions of the Carletti–Villari Theorem are satisfied. Since the conditions of Theorem 1 are satisfied under those of Theorem 2, we can use the proof of Theorem 1.

Condition (C.5) is proved at the end. Conditions (C.6) and (C.7) of the Carletti–Villari Theorem follow directly from (A.2) and (A.6), respectively. Moreover, it follows from (C.12) that $G(x)$ is an even function. Therefore, (C.10) of he Carletti–Villari Theorem is satisfied. On the other hand, (C.8) follows directly from (C.12). We now prove (C.9). It follows from (A.3) and (A.6) that

$$h(x)=\int_0^x h'(u)du > \int_0^W h'(u)du + U(x-W) \quad \text{for any} \quad x>W,$$

$$h(x)=\int_0^x h'(u)du < \int_0^{-W} h'(u)du + U(x+W) \quad \text{for any} \quad x<-W.$$

This yields $\lim_{x\to\pm\infty} h(x)=\pm\infty.$ Therefore, it follows from (C.2) that

$$\limsup_{x\to\pm\infty}\left|\int_0^x g(u)du \pm h(x)\right| = +\infty.$$

This proves (C.9). Finally, we prove (C.5). First, we consider the case of $x>0$. Then, (C.13) implies $\Lambda''(x)=\Gamma''(x)/l_2<0,$ so that noting $\Psi(\alpha sx)=\gamma x$ we have

(A.8) $$h''(x)=-h_2\gamma^2\Lambda''(\gamma x)/s>0.$$

Thus, $h'(x)$ is strictly monotonically increasing. Therefore, (A.2) and (A.6) show that $h'(x)$ has a unique zero in $x>0$. We denote the zero by $x_1>0$. Then, it follows from (A.2) and (A.8) that

(A.9) $h'(x_1)=0$, $h'(x)<0$ for any $x\in(0,x_1)$ and $h'(x)>0$ for any $x\in(x_1,+\infty)$.

From (A.3) and (A.9), we see that $h(x_1)<0$. On the other hand, (A.6) shows that there is a $x_2\in(x_1,+\infty)$ such that $h(x_2)>0$. Therefore, there is an $x^*\in(x_1,x_2)$ such that $h(x^*)=0$. Moreover, it follows from (A.9),

(A.10) $$\#\{x\in(0,+\infty):h(x)=0\}=\{x^*\}.$$

See Figure 3. Next, we consider the case of $x<0$. (C.13) implies $\Lambda''(x)=\Gamma''(x)/l_2>0,$ so that noting $\Psi(\alpha sx)=\gamma x$ we have

$$h''(x) = -h_2\gamma^2 \Lambda''(\gamma x)/s < 0.$$

Thus, $h'(x)$ is strictly monotonically decreasing. Therefore, (A.2) and (A.6) show that $h'(x)$ has a unique zero in $x < 0$. We denote the zero by $x_3 < 0$. In the similar way as above, we obtain that

(A.11) $h'(x_3) = 0$, $h'(x) < 0$ for any $x \in (x_3, 0)$ and $h'(x) > 0$ for any $x \in (-\infty, x_3)$.

From (A.3) and (A.11), we see that $h(x_3) > 0$. On the other hand, (A.6) shows that there is a $x_4 \in (-\infty, x_3)$ such that $h(x_4) < 0$. Therefore, there is an $x^{**} \in (x_4, x_3)$ such that $h(x^{**}) = 0$. Moreover, it follows from (A.11),

(A.12) $\#\{x \in (-\infty, 0) : h(x) = 0\} = \{x^{**}\}$.

Thus, from (A.3), (A.10) and (A.13) we see that $h(x)$ has exactly three real transversal zeros:

$$\#\{x \in R^1 : h(x) = 0\} = \{x^{**}, 0, x^*\}.$$

This proves (C.5). Thus, we complete the proof.■

Figure 3 about here.

In proving Theorem 3, we use the following theorem:

Olech Theorem: Suppose that the following system possesses an equilibrium point.

$$\Sigma \quad \begin{cases} \dot{w} = F(w, z), \\ \dot{z} = G(w, z). \end{cases}$$

The equilibrium point is globally asymptotically stable (i.e., the equilibrium point is stable and any path converges to the equilibrium point), under the following conditions:

(C.17) $\partial F / \partial w + \partial G / \partial z < 0$ for any $(w, z) \in R^2$;

(C.18) $\partial F/\partial w \bullet \partial G/\partial z - \partial F/\partial z \bullet \partial G/\partial w > 0$ for any $(w,z) \in R^2$;
(C.19) $\partial F/\partial w \bullet \partial G/\partial z \neq 0$ or $\partial F/\partial z \bullet \partial G/\partial w \neq 0$ for any $(w,z) \in R^2$.■

Proof: See Olech (1963).■

Proof of Theorem 3: Define

$$F(x,y) \equiv y - \Phi(x), \quad G(x,y) \equiv -\Psi(sx).$$

From Condition (C.14), we see that

$$\partial F/\partial x = -\Phi'(x), \quad \partial G/\partial x = -s\Psi'(sx) < 0, \quad \partial F/\partial y = 1, \quad \partial G/\partial y = 0.$$

Condition (C.14) yields

$$\Lambda'(\Psi(\alpha sx)) < \frac{s + \inf\{\Psi'(\alpha sx)/\alpha : x \in R^1\}}{h_2 \bullet \sup\{\Psi'(\alpha sx) : x \in R^1\}} < \frac{s + \Psi'(\alpha sx)/\alpha}{h_2 \Psi'(\alpha sx)}.$$

This yields that

$$s > h_2 \Lambda'(\Psi(\alpha sx))\Psi'(\alpha sx) - \Psi'(\alpha sx)/\alpha. \quad \text{T}$$

Therefore, we obtain

$$\begin{aligned}\Phi'(x) &= \alpha s - \frac{\alpha s h_2 \Lambda'(\Psi(\alpha sx))\Psi'(\alpha sx) - \alpha s \Psi'(\alpha sx)/\alpha}{s} \\ &= \alpha s - \{\alpha h_2 \Lambda'(\Psi(\alpha sx))\Psi'(\alpha sx) - \Psi'(\alpha sx)\} \\ &= \alpha[s - \{h_2 \Lambda'(\Psi(\alpha sx))\Psi'(\alpha sx) - \Psi'(\alpha sx)/\alpha\}] > 0\end{aligned}$$

Therefore, we have $\partial F/\partial x < 0$ for any $x \in R^1$. It is now easy to see that System Ω_{Li} satisfies all conditions of the Olech Theorem. Theorem 3 follows from the Olech Theorem.■

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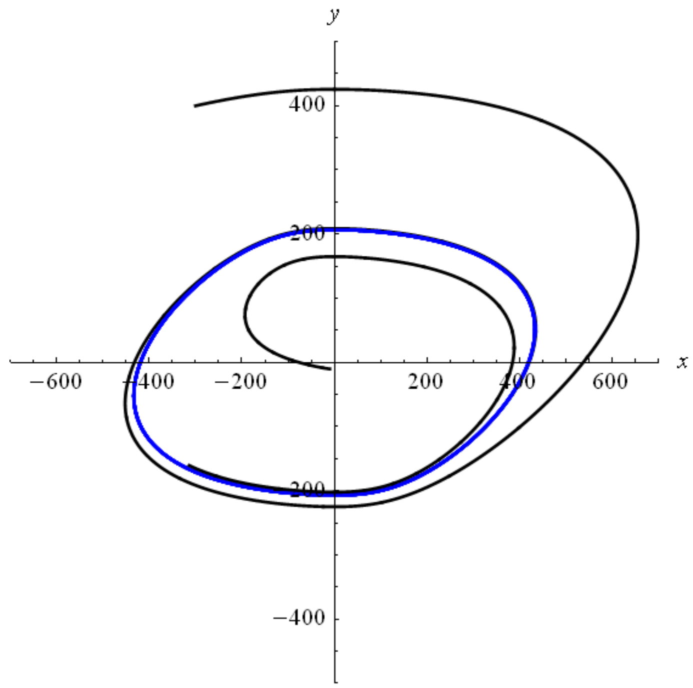
Figure Captions

Figure 1. Stable limit cycles of System Ω_{Li}: (1) a typical stable limit cycle; (2) the amplitude of limit cycle that becomes larger parallel to the intensity of speculation *m*.

Figure 2. Global stability.

Figure 3. The *h*-function



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−600
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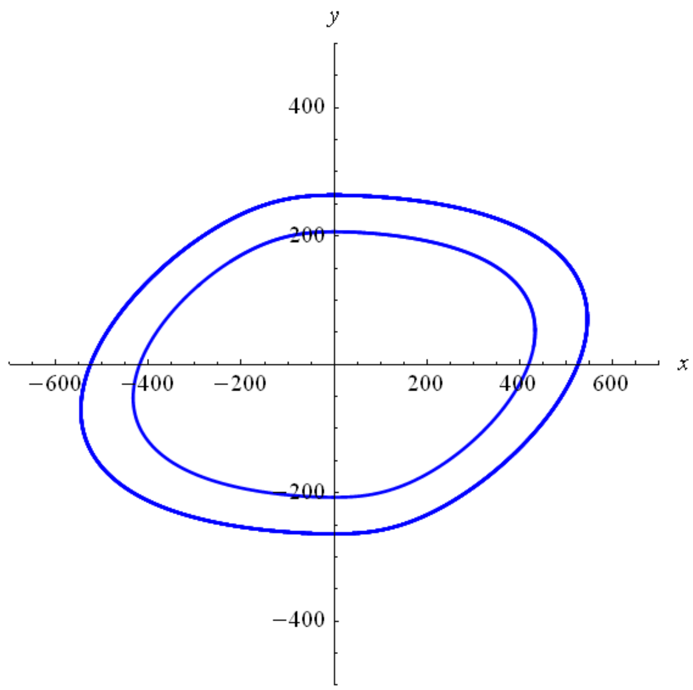
−200
200
400
600
x
−200
−400

(1)



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x
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(2)

Figure 1

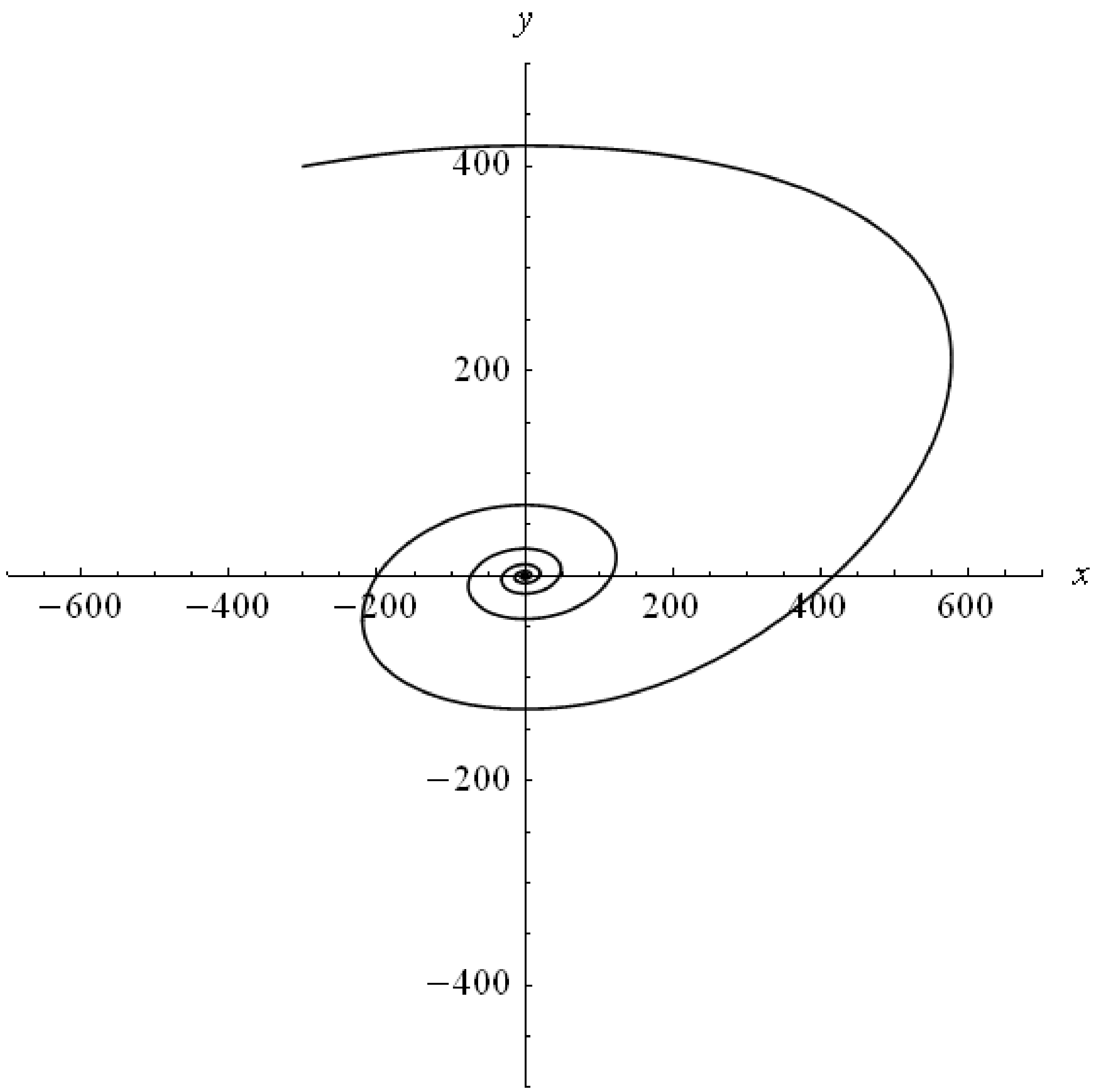

Figure 2

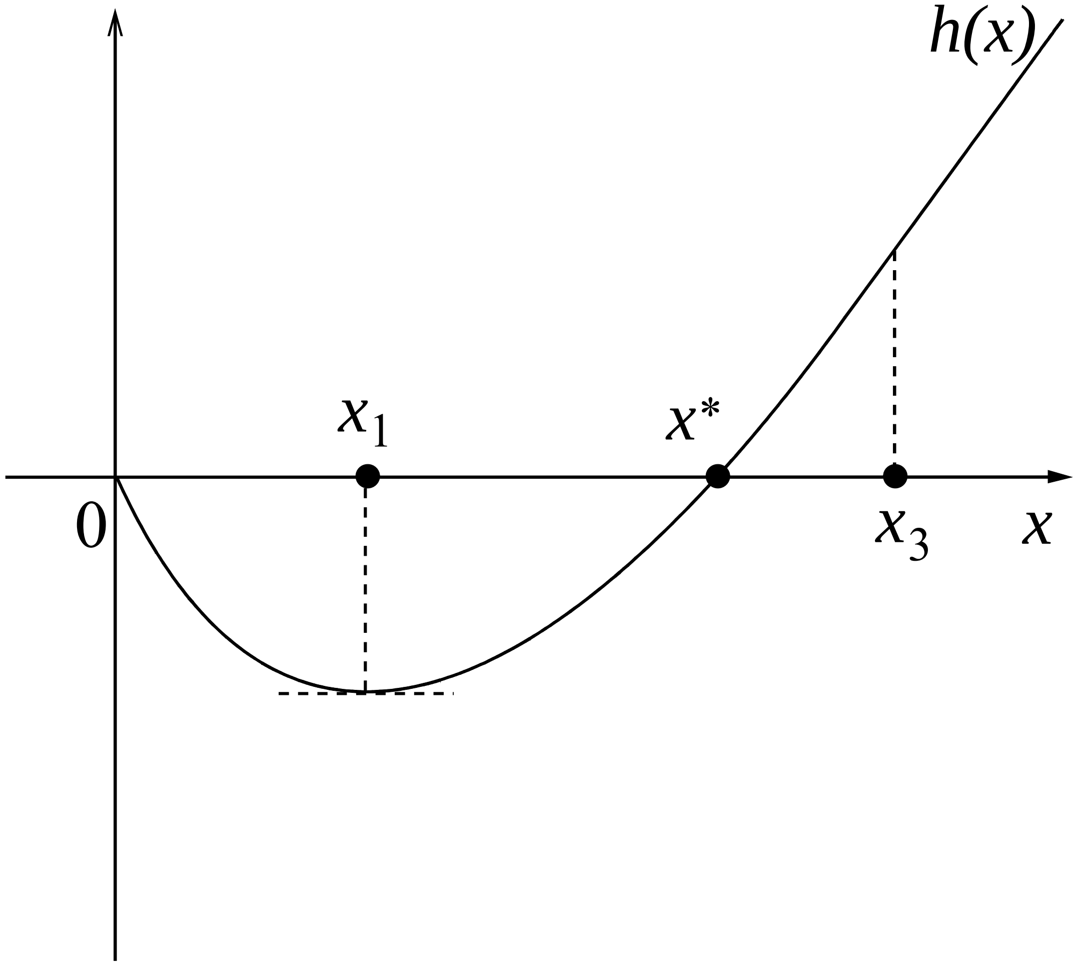

Figure 3