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A Business Cycle Model of Speculation from a Viewpoint of Minsky and Shiller Ⅰ: Construction of Model and Its Local Analysis

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Abstract: We construct a 3-dimensional extension of the dynamic IS-LM model, in which the money demand function depends not only on income but also on a rate of change in expected income (RCEI). We demonstrate the occurrence of limit cycles in the extended IS-LM model. Our arguments are essentially derived from the remarkable viewpoint of H. P. Minsky and J. R. Shiller concerning financial markets. We assume that the money demand negatively correlates with RCEI. Such a negative correlation results from a speculative behavior. We demonstrate that the negative correlation is an important source of unstable equilibrium and therefore, business cycles. Firstly, we transform the extended IS-LM model into a 2-dimensional Liénard system and prove the occurrence of a stable limit cycle in the Liénard system. Secondly, by using a Hopf bifurcation theorem, we demonstrate the occurrence of a Hopf cycle in the extended 3-dimensionl IS-LM model. Our model possesses two types of self-fulfilling prophecy

Keywords: IS-LM Model; Money Demand; Speculation; Limit Cycle; Hopf Bifurcation

1. Introduction

The most well-known initial study concerning irrationality of the stock market was given by Shiller (1981). For an insightful consideration to this matter, see Rosser (2000). Shiller showed that stock prices move too much to be justified by the efficient markets hypothesis. It is naturally conceivable that such movements of stock prices result from a speculative motive of shareholding. That is, economic agents continue to purchase the stock with the expectation of an increase in a stock price. The Shiller's findings suggest that, through the movements of stock prices, speculation is closely related to business fluctuations. In this paper, based on the observation, we consider the relation between them. To do it, as a cause of the expectation of an increase in a stock price, we consider an increase in expected income in a business-cycle model that slightly extends the dynamic IS-LM model.[1]

We will here make one remark. The money market is excluded from the IS-MP model (i.e. the LM curve is excluded). See Romer (2000). However, following Minsky (1975, 1982) and Shiller (1981), we are interested in the money market.[2] Especially in this paper, we demonstrate that an important factor of short-run instability exists in the monetary market. It has been well known that the IS-LM model has been criticized. However, since the IS-LM model is very useful to explain clearly such an instability factor and no model of explaining the financial instability can be produced from the IS-MP model, this paper revives the IS-LM model.

As an example, we consider the case where economic agents expect future income to decrease. Since it is expected that the performances of many firms will decline, they expect many stock prices to decrease before future income decreases. Thus, the change in expectation about future stock prices will positively correlate with the change in expectation of the future income. On the other hand, if high substitutability between money and stock exists, then from the viewpoint of speculation, money demand will negatively correlate with the change in expectation about stock prices.

[1] There are many studies that focus the dynamic IS-LM model. Torre (1977) proved that a Hopf bifurcation occurs in the original dynamic IS-LM model. Gabisch and Lorenz (1987) considered a three-dimensional extension of the original dynamic IS-LM model and proved that a Hopf bifurcation occurs in that extension. For a similar model, see Boldrin (1984). On the other hand, Schinasi (1982) constructed an extension of the original dynamic IS-LM model, which has a government budget constraint. By using the Poincaré-Bendixson theorem, he proved that a stable limit cycle occurs in that extension. For a brief historical survey, see Owase (1991).

[2] From a different view point of ours, Mankiw (2006) also stresses the importance of considering the monetary economy.

Therefore, money demand is plausibly assumed to correlate negatively with the change in expectation regarding the future income.[3] Consequently, money demand is expected to be closely related to business cycles through changes in expectation.[4] However, there have been few theoretical arguments about such a relationship between money demand and business cycles. Following Taylor and O'Connell (1985), by assuming a high substitutability between money and stock[5], we demonstrate that the dependence of money demand on the change in expectation about the future income yields business cycles in the extended IS-LM model.

Here, we briefly explain our model. The extended IS-LM model includes the usual dynamic IS-LM model in the sense that the income dynamics of the model are caused by the difference between aggregate demand and aggregate supply and the interest dynamics are caused by the difference between money demand and money supply. However, unlike the usual IS-LM model, we also consider that agents form expectations about future income. Such expectations are formed from previous time series of income and are adaptively adjusted each time. Here, to consider the dependence of money demand on the change in expectation about the future income, we incorporate a variable on the rate of change in expected income into the money demand function.[6] We assume that as the rate of change in expected income decreases (resp. increases), money demand increases (resp. decreases). We investigate the influence of such a modification to the money demand function on dynamic behavior. To stress the instability caused by including the rate of change in expected income in the money demand function, we consider the case where the original (usual) IS-LM model is globally asymptotically stable.[7]

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[3] For the relationship between an expectation about the future income and money demand, see Matthews (1959, Section 8.2).

[4] Arthur C. Pigou is the first economist who stressed the importance of expectation in business cycles. See Pigou (1924). He stressed the relation between investment and the change in expected income. We, however, consider the relation between money demand and the change in expected income.

[5] Taylor and O'Connell (1985) demonstrated that such a high substitutability yields Hyman Minsky's financial crisis (Minsky (1975, 1982)). However, their model that is utilized to demonstrate such a crisis is different from our model and they do not consider the occurrence of business cycles. We demonstrate that the high substitutability yields business cycles.

[6] Friedman (1956) proposes a money demand function that depends on permanent income. Such a proposal is remarkable in the sense that a long-run viewpoint is incorporated into the money demand function. The permanent income defined by Friedman (1956) is closely related to expected income. We, however, assume that money demand depends not on expected income, but on the rate of change in expected income.

[7] It is possible that the IS-LM model is totally unstable. Actually, in order to prove the

2. The Model

In this section, we construct an extension of the IS-LM model, in which money demand depends on the rate of change in expected income (abbreviated as RCEI). Throughout this study, all functions are assumed to be continuously differentiable and all parameters are assumed to be positive. We denote income, expected income, interest rate, price level, money supply, investment, and consumption by Y, Y_e, R, P, M, I, and C, respectively.

We consider the following investment and consumption function:

$$I = I(Y, R), \text{ and } C = C(Y).$$

As usual, we assume

$$\partial I / \partial Y > 0, \quad \partial I / \partial R < 0, \quad dC / dY > 0.$$

These conditions are the same as those usually assumed in the original IS-LM model. For simplicity, it is convenient in the following argument to define the aggregate demand function:

$$H(Y, R) \equiv I(Y, R) + C(Y).$$

We next consider the money demand function. As discussed in Introduction, from the viewpoint of speculation which is based on the findings of Shiller (1981), it is plausible to assume that money demand correlates negatively with the change in expectation regarding the future income. Based on this consideration, we consider the following money demand function:

$$L = L(Y, R) - \Gamma(\dot{Y}_e).$$

The $L-$function denotes the usual money demand function. As usual, we assume

$$\partial L / \partial Y > 0^{8}, \quad \partial L / \partial R < 0.$$

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occurrence of a Hopf bifurcation, Torre (1977) considered the case where the IS-LM model is totally unstable.

[8] Taylor and O'Connell (1985) constructs a system analogous to the IS-LM model in which income is replaced with a rate of profit. Moreover, under the high substitutability between money and stock, they demonstrate that their modified LM curve possesses a negative slope and the modified IS-LM model yields Hyman Minsky's financial crisis (i.e., instability of equilibrium). However, the present paper demonstrates that instability results from both the same substitutability and the dependence of money demand on the changing rate of expected income. Moreover, we demonstrate that the occurrence of business cycles results from such an

To take the above-mentioned speculation into consideration, the $\Gamma-$ function is incorporated. Throughout this study, we work under the following assumptions:

Assumption 1: $\Gamma(0)=0$, $\Gamma'(u)>0$ for any $u \in R^1$, and $\sup\{\Gamma'(u): u \in R^1\} < +\infty$.

We here present a detailed explanation of Assumption 1. The assumption $\Gamma'(u)>0$ implies the *high substitutability between money and stocks under a constant interest rate*. Unlike the usual IS-LM model, as stated in Introduction, our money demand function is assumed to depend on RCEI. Assumption 1 implies that there exists a negative correlation between money demand and RCEI. As a reason for such a correlation we have a speculative behavior. The money demand concerning a speculation will negatively correlate with the rate of change in stock prices. On the other hand, it is expected that stock prices will positively correlate with the future income. Therefore, from the viewpoint of a speculative motive, it is plausible to assume that the money demand negatively correlates with the rate of change in future income. Based such a consideration, we assume that the money demand concerning the speculative motive negatively correlates with RCEI. Thus, we see that Assumption 1 is economically plausible. As we will demonstrate later on, such a negative correlation given by Assumption 1 is an important source of business fluctuations.

Moreover, following Minsky (1975, 1982), we may consider expected bankruptcy probability to explain Assumption 1. Considering it, an increase (resp. decrease) in RCEI yields a decrease (resp. increase) in expected bankruptcy probability and a decrease (resp. increase) in bond demand. Consequently, an increase (resp. decrease) in RCEI yields a decrease (increase) in money demand. Therefore, from a viewpoint of not only stock demand but also bond demand, there are many possibilities that Assumption 1 is satisfied.

We assume that expected income is adaptively adjusted: $\dot{Y}_e = \Psi(Y - Y_e)$. Then, we obtain the following extended IS-LM model:

$$\Omega \quad \begin{cases} \dot{Y} = \alpha\{H(Y,R) - Y\}, \\ \dot{R} = \beta\{L(Y,R) - \Gamma \circ \Psi(Y - Y_e) - M/P\}, \\ \dot{Y}_e = \Psi(Y - Y_e), \end{cases}$$

instability.

The following usual IS-LM is obtained as the subsystem of System Ω :

$$\Omega_{IS-LM} \quad \begin{cases} \dot{Y} = \alpha\{H(Y,R) - Y\}, \\ \dot{r} = \beta\{L(Y,R) - M/P\}. \end{cases}$$

To make our argument clear, we call System Ω_{IS-LM} the IS-LM subsystem. Throughout this study, we work under the following assumptions:

Assumption 2: $1 > \partial H/\partial Y$ for any $(Y,R) \in R^2$.
Assumption 3: $\Psi(0) = 0$, $\Psi'(z) > 0$, and $\sup\{\Psi'(z) : z \in R^1\} < +\infty$.

Assumption 3 is a plausible assumption. Here, we briefly explain the economic implications of Assumption 2.

Assumption 2 implies that the propensity to invest and consume is not high. That is, $0 < \partial H/\partial Y = \partial I/\partial Y + \partial C/\partial Y < 1$ for any $(Y,R) \in R^2$. To prove the existence of a stable periodic path in such Keynesian business cycle models as Ω_{IS-LM} and the Kaldor model, it has been assumed that $\partial H/\partial Y = \partial I/\partial Y + \partial C/\partial Y > 1$ at the equilibrium. See, for example, Kaldor (1940), Chang and Smyth (1971), Torre (1977) and Schinasi (1982). Through these studies, it has been known that $\partial H/\partial Y > 1$ at the equilibrium is an important source of instability. For such an traditional instability condition, see also Gabisch and Lorenz (1987), Owase (1989) and Lorenz(1993). However, as stated above, we will demonstrate that the negative correlation between money demand and RCEI, which originates in speculative behaviors, is another important source of instability. Therefore, in Sections 4 and 5, without assuming the traditional instability condition, we demonstrate that the negative correlation makes the equilibrium unstable and a stable limit cycle emerges. However, in Section 6, by adding the traditional instability condition, we demonstrate that the extended IS-LM model yields chaotic fluctuations.

The following result is obvious.

Lemma 1: System Ω has equilibrium points for income and the interest rate if and only if System Ω_{IS-LM} has them. Moreover, for two systems, the equilibrium points of income and the interest rate are the same.■

Proof: The proof is clear.■

To stress the instability that is caused by the dependence of money demand on RCEI, we consider the case where the IS-LM subsystem is globally stable. Assumption 2 guarantees the global stability of the IS-LM subsystem.

Lemma 2: Suppose that the IS-LM subsystem Ω_{IS-LM} possesses an equilibrium point and that Assumption 2 is satisfied. Then System Ω_{IS-LM} is globally asymptotically stable.■

Proof: See the Appendix.■

3. Expected Income and Weighted Average of Income

Before starting our argument, we here briefly explain that the expected income, which is adjusted by linear $\Psi-$function, can be interpreted as some kind of weighted average of past income. We assume that the expected income is given by the weighted average of past income:.

$$Y_e(t) \equiv \int_{(-\infty,t]} W(\tau)Y(\tau)d\tau,$$

where $W(\tau)$ is the weight function satisfying the following natural conditions:

$$W'(\tau) < 0 \quad \text{かつ} \quad \int_{(-\infty,t]} W(\tau)d\tau = 1 \quad \text{for any } t \in R^1.$$

We here consider

$$W(\tau) = \zeta \exp(\zeta(\tau - t)), \quad \zeta > 0.$$

Then, we obtain the following result.

Lemma 3: $\int_{(-\infty,t]} W(\tau) = 1$ for any $t \in R^1$ and $\dot{Y}_e = \zeta(Y - Y_e)$.■

Proof: The proof of the first half follows from

$$\int_{(-\infty,t]} \zeta \exp(\zeta(\tau-t))d\tau = \zeta \exp(-\zeta t)\int_{(-\infty,t]} \exp(\zeta\tau)d\tau = \zeta \exp(-\zeta t)\left[\frac{\exp(\zeta\tau)}{\zeta}\right]_{-\infty}^{t} = 1.$$

Secondly, we prove the latter half. From the definition, we have

$$Y_e(t) = \exp(-\zeta t)\int_{(-\infty,t]} \zeta Y(\tau)\exp(\zeta\tau)d\tau.$$

Therefore, we obtain

$$\overset{\bullet}{Y}_e = -\zeta \exp(-\zeta t)\int_{(-\infty,t]} \zeta Y(\tau)\exp(\zeta\tau)d\tau + \exp(-\zeta t)\zeta \bullet Y(t)\exp(\zeta t)$$

$$= -\zeta \int_{(-\infty,t]} \zeta Y(\tau)\exp(\zeta(\tau-t)d\tau + \zeta Y(t) = \zeta\{Y(t) - Y_e(t)\}. \blacksquare$$

We see from Lemma 3 that the expected income, which is adjusted by linear $\Psi-$function, equals the above-mentioned weighted average of past income, whose weight function is given by $W(\tau) = \zeta \exp(\zeta(\tau - t))$.

4. Occurrence of Periodic Paths

In this section, we consider the extended IS-LM 3D model Ω. Since it is difficult to analyze the global dynamic behavior of the extended IS-LM system, we perform a local bifurcation analysis. To do so, we slightly specify the original extended IS-LM model. Throughout this section, we set

$$\Gamma(u) = mu\,.$$

The $\Gamma-$function satisfies Assumption 1. The parameter m displays the intensity of the dependence of money demand on RCEI. We can interpret that **the parameter m represents the intensity of speculation**. We have $\Gamma(\overset{\bullet}{Y}_e) = m\Psi(Y - Y_e)$. so that the extended IS-LM model Ω becomes

$$\overline{\Omega}\quad \begin{cases} \dot{Y} = \alpha\{H(Y,R) - Y\}, \\ \dot{R} = \beta\{L(Y,R) - m\Psi(Y - Y_e) - M/P\}, \\ \dot{Y_e} = \Psi(Y - Y_e). \end{cases}$$

For System $\overline{\Omega}$, we work under the following assumption.

Assumption 4: System $\overline{\Omega}$ possesses at least one equilibrium point $(Y^*, R^*, Y_e^*) \in R_+^3$, where $R_+^3 \equiv \{(x, y, z) \in R^3 : x > 0, y > 0, z > 0\}$.

There are many cases where Assumption 4 is satisfied.

Example 1: The interest in this section is in the occurrence of periodic paths in System $\overline{\Omega}$. As shown later, even if the IS-LM subsystem $\overline{\Omega}_{IS-LM}$ (i.e. System $\overline{\Omega}$ with $m = 0$) is linear, it is possible that the extended IS-LM model possesses cycles. In the case where the IS-LM subsystem is linear, it is easy to derive an existence condition of an equilibrium point in R_+^3. We here consider a more general case where the IS-LM subsystem is nonlinear. We set

$$H(Y,R) = h_1 Y - f(R) + B, \quad L(Y,R) = l_1 Y - g(R),$$

where $f(0) = 0$, $f'(R) > 0$, $g(0) = 0$, and $g'(R) > 0$. Then, the same conditions as those of the usual IS-LM model are satisfied. The equilibrium points of the IS-LM subsystem are given by the solutions to the equations:

$$M/p = L(Y,R) = l_1 Y - g(R) \quad \text{and} \quad Y = H(Y,R) = h_1 Y - f(R) + B.$$

Since Assumption 2 is assumed, we have $1 > h_1$. Therefore, we can easily prove that if

$$B/(1 - h_1) > M/(Pl_1),$$

The IS-LM subsystem $\overline{\Omega}_{IS-LM}$ possesses a unique equilibrium point $(Y^*, R^*) \in R_+^2$. See Figure 1. Moreover, since Assumption 2 is satisfied, we see from Lemma 2 that the IS-LM subsystem $\overline{\Omega}_{IS-LM}$ is globally asymptotically stable. Moreover, Lemma 1 yields that System $\overline{\Omega}$ also possesses a unique equilibrium point $(Y^*, R^*, Y_e^*) \in R_+^3$.■

Figure 1 about here.

Theorem 1: Under Assumptions 2 and 4, any path $(Y(t), R(t))$ of System $\overline{\Omega}$ with $m = 0$ converges to the equilibrium point (Y^*, R^*).

Proof: System $\overline{\Omega}$ with $m = 0$ contains the IS-LM model. We see directly from Lemma 2 that the IS-LM model is globally asymptotically stable. This completes the proof.■

Although Theorem 1 is clear, the economic implication of the result is important. So far, Keynesian business cycle models have demonstrated that the cause of equilibrium instability lies in the goods market.[9] However, Theorem 1 shows that **in System $\overline{\Omega}$, the cause of instability is merely in speculation in the money demand equation.**

We have

$$\partial L(Y^*, R^*)/\partial Y = l_1, \quad \partial H(Y^*, R^*)/\partial Y = h_1,$$

We here define

$$l_2 = -\partial L(Y^*, R^*)/\partial R, \quad h_2 = -\partial H(Y^*, R^*)/\partial R \quad \text{and} \quad Q = \Psi'(Y^* - Y_e^*) = \Psi'(0).$$

To demonstrate the possible occurrence of limit cycles in the IS-LM model, Torre (1977) considered the case where the equilibrium point of the IS-LM model is unstable. However, in the following, in the same way as before, to make clear how the dependence of money demand on RCEI (therefore, speculation) yields business cycles, we will consider the case where the IS-LM subsystem is globally asymptotically stable. As stated in Example 1, the IS-LM subsystem is globally asymptotically stable under Assumptions 2 and 4. In the following, we will prove that if the dependence of money demand on RCEI is incorporated into the globally stable IS-LM model, then the equilibrium point of the extended IS-LM model (i.e. System $\overline{\Omega}$ with $m \neq 0$) becomes unstable and it has periodic paths. Using a version of the Hopf bifurcation theorem, we demonstrate the occurrence of a periodic path. As a bifurcation parameter, we choose the parameter m. Since $\Gamma(u) = mu$, the parameter m displays the intensity of the

[9] See, for example, Kaldor (1940), Chang and Smyth (1971), Torre (1977), Schinasi (1982), Asada (1987), Gabisch and Lorenz (1987), Owase (1989, 1991), Lorenz (1993), Gandolfo (1996), Agliari and Dieci (2005), and Agliari, Dieci and Gardini (2007).

dependence of money demand on RCEI. We first prove the following lemma:

Lemma 4: Define

$$m^* = \frac{\alpha(1-h_1)\beta l_2 + \beta l_2 Q + \alpha(1-h_1)Q + \alpha h_2 \beta l_1 - \dfrac{\alpha(1-h_1)\beta l_2 Q + \alpha h_2 \beta l_1 Q}{\alpha(1-h_1) + \beta l_2 + Q}}{\alpha h_2 \beta Q}.$$

Then, under Assumption 2, we have $m^* > 0$.■

Proof: See Appendix.■

here are some versions of the Hopf bifurcation theorem. Among other things, since the equilibrium is independent of the parameter m, the following simple form is convenient for the derivation of our result.

Hopf Bifurcation Theorem[10]: Consider the system

$$\dot{x} = f(x;\eta), \quad x \in R^n, \quad \eta \in R^1.$$

We assume that the system has an equilibrium point x^* for any $\eta \in R^1$ and the following conditions are satisfied:

(C.15) The Jacobean matrix of the system, evaluated at x^*, has pure imaginary eigenvalues $\lambda(\eta_0)$ and $\overline{\lambda}(\eta_0)$. Moreover, the other eigenvalues have non-zero real parts.

(C.16) $\dfrac{d \operatorname{Re} \lambda(\eta_0)}{d\eta} \neq 0.$

Then, there exists a periodic path bifurcating from x^* at $\eta = \eta_0$.■

The Hopf bifurcation theorem yields the following result on the occurrence of business cycles in System $\overline{\Omega}$.

Theorem 2: Suppose that Assumptions 1 to 3 are satisfied. Then, System $\overline{\Omega}$

[10] For this form of the Hopf bifurcation theorem, see Guckenheimer and Holmes (1983, Theorem 3.4.2). For applications to economics, see Lorenz (1993) and Gandolfo(1996).

undergoes a Hopf bifurcation at $m = m^*$. ■

Proof: See Appendix.■

Theorem 2 demonstrates that System $\overline{\Omega}$ undergoes a Hopf bifurcation at $m = m^*$ and, therefore, a periodic path (called a Hopf cycle). In the following, we numerically observe that as the bifurcation parameter m becomes large, the Hopf cycle becomes large. Thus, as the dependence of money demand on RCEI becomes large, the intensity of the instability becomes large and therefore, the Hopf cycle also becomes large.

Example 2: We consider the case where the IS-LM subsystem is linear. We set

$$\alpha = 1, \quad \beta = 14, \quad B = 400, \quad h_1 = 0.8, \quad h_2 = 300, \quad l_1 = 0.3, \quad l_2 = 200,$$
$$M / p = 80, \quad \Psi(z) = 16\text{Arctan}(z/20) + 0.3z.$$

Then, we have

$$\Psi'(z) = \frac{4}{5(z/20)^2 + 5} + 0.6 > 0 \quad \text{and} \quad \lim_{z \to \pm\infty} \Psi'(z) = 0.6$$

For the graph of the Ψ – function, see Figure 2. Clearly, Assumptions 1 to 3 are satisfied. The extended IS-LM model is given by

$$\overline{\overline{\Omega}} \quad \begin{cases} \dot{Y} = 0.8Y - 300R + 400 - Y, \\ \dot{R} = 14[0.3Y - 200R - 20m\{\text{Arc}\tan((Y - Y_e)/20) + 0.6(Y - Y_e)\} - 80], \\ \dot{Y}_e = 20\text{Arc}\tan((Y - Y_e)/20) + 0.6(Y - Y_e). \end{cases}$$

On the other hand, the IS-LM subsystem in System $\overline{\overline{\Omega}}$ is given by

$$\overline{\overline{\Omega}}_{IS-LM} \quad \begin{cases} \dot{Y} = 0.8Y - 300R + 400 - Y, \\ \dot{R} = 14(0.3Y - 200R - 80), \end{cases}$$

From Lemma 2, we see that System $\overline{\overline{\Omega}}_{IS-LM}$ is globally asymptotically stable. First, we briefly observe that the dynamic behavior of System $\overline{\overline{\Omega}}_{IS-LM}$ is non-rotatory. See Figure 3. The blue curve of Part 1 of Figure 3 is a typical path of System $\overline{\overline{\Omega}}_{IS-LM}$, which tends straight toward the equilibrium. The time series of the path of Part (1) is described by the blue curve of Part 2 of Figure 3. Figure 3 shows that System $\overline{\overline{\Omega}}_{IS-LM}$

is strongly stable. The typical path approaches the LM curve very quickly and converges toward the equilibrium, clinging to the LM curve. This implies that the money market equilibrates very quickly. Next, in the case where the IS-LM subsystem has such strong stability, we consider the dynamic behavior of System $\overline{\overline{\Omega}}$. We continue to consider the same parameters as above. Figure 4 describes the dynamic behavior of System $\overline{\overline{\Omega}}$. The black curve in Part 1 in Figure 4 describes a path of System $\overline{\overline{\Omega}}$ with $m = 0$ that corresponds to System $\overline{\overline{\Omega}}_{IS-LM}$. The path quickly tends to the equilibrium point of System $\overline{\overline{\Omega}}$. The blue dot of Part 1 describes the equilibrium point. Part 1 describes that System $\overline{\overline{\Omega}}$ with $m = 0$ is strongly stable. Next, we consider the case of $m \neq 0$. It follows from Theorem 2 that System $\overline{\overline{\Omega}}$ undergoes a Hopf bifurcation at $m = m^* \fallingdotseq 0.97608342$. The dynamic behaviors of System $\overline{\overline{\Omega}}$ are described in Parts 2 to 4 in Figure 4. The black curves of Parts 2 to 4 in Figure 4 describes the paths of System $\overline{\overline{\Omega}}$ with $m = 0.976 < m^*$, $m = 0.9768 > m^*$, and $m = 0.979 > m^*$, respectively. The blue dot of Part 2 also describes the equilibrium point. The path of Part 2 slowly tends to the equilibrium point of System $\overline{\overline{\Omega}}$. The blue closed paths of Parts 3 and 4 describe the periodic paths of System $\overline{\overline{\Omega}}$ with $m = 0.9768 > m^*$ and $m = 0.979 > m^*$, respectively. Each black path converges to each blue periodic path. Theorem 2 suggests that the periodic paths obtained in this example are Hopf cycles. Moreover, Parts 3 and 4 suggest that the periodic paths obtained in this example are stable. Moreover, we see from Figure 4 that as the intensity m of the dependence of money demand on RCEI becomes large, the emerging cycle becomes large.■

Figures 2, 3 and 4 about here.

Thus, we demonstrate that the extended IS-LM 3D model $\overline{\Omega}$ possesses a stable periodic path. From Figure 4, we obtain the following first observation:

Observation 1: Our model shows that if the intensity m of the dependence of money demand on RCEI is zero (Part 1), the economy is strongly stable and moreover, that as the intensity m is large to some extent (Parts 2 to 4), the economy yields fluctuation. That is, as economic agents become speculative, the economic yields fluctuation.■

Moreover, from Part 2 of Figure 1 and Figure 4, we also obtain the following second observation:

Observation 2: Our model shows that as the intensity m of the sensitivity of money demand to RCEI becomes large, the emerging cycle becomes large. This implies that as economic agents become more speculative, economic fluctuations are larger.■

These observations may be related to the important Shiller's findings (Shiller (1984)) concerning movements of stock prices. Thus, our model gives a theoretical explanation of such findings.

6. Conclusion and Final Remarks

Our base model is the dynamic IS-LM model: the rate of change in income depends on the difference between aggregate demand and aggregate supply and the rate of change in interest depends on the difference between money demand and money supply. The usual money demand function of the IS-LM model depends on income and the interest rate. Moreover, based on Shiller (1981), we stressed the speculative motive of money demand and considered the money demand function depending on not only income and the interest rate but also the RCEI. By incorporating such a money demand function, we extended the dynamic IS-LM model. In the extended IS-LM model, the expected income is adaptively adjusted. Thus, the extended IS-LM model is 3D and includes the usual IS-LM model. The equilibrium income and equilibrium interest rate of the extended IS-LM model are the same as those of the IS-LM model. We demonstrated that the negative correlation between money demand and RCEI is an important source of instability and therefore, of business cycles. In demonstrating it, we assumed that the money demand depends on the RCEI.

Our most important assumptions are as follows. First one is the assumption of speculative behavior that is closely related to the well-known Shiller's findings concerning the stock market. Second one is the assumption of high substitutability between money and stock. Such a substitutability was stressed by Taylor and O'Connell (1985) to demonstrate Minsky's financial crisis, though the substitutability is closely related to the speculative behavior. Thus, our results were derived from the important viewpoints of Minsky and Shiller.

By using a version of he Hopf bifurcation theorem, we proved the occurrence of a periodic path (i.e. a Hopf cycle) and numerically demonstrated that the Hopf cycle is stable. Moreover, we numerically demonstrated that as the intensity of the dependence of expected income on RCEI becomes large, the amplitude of the emerging cycle becomes large. This implies that as economic agents become more speculative, economic fluctuations are larger. This fact may be related to the Shiller's findings (Shiller (1981)).

Finally, we should be made one important remark. Our model possesses two types of self-fulfilling prophecy. The first one is as follows. In our model, economic agents expect that the economy fluctuates. This expectation leads economic agents to speculative behaviors. On the other hand, the speculative behaviors yield business fluctuation. Thus, **the expectation of economic fluctuations comes true through the speculative behaviors**. We next consider the second one. From Observation 2, we see the following. If economic agents expect that the economy fluctuates more largely, economic agents become more speculative. This expectation leads economic agents to more intense speculative behaviors. This also implies a self-fulfilling prophecy. Thus, **the expectation of the enlargement of economic fluctuation also comes true through the speculative behaviors**.

7. Appendix

In this appendix, we prove Lemmas 2 and 4 and Theorems 1 and 2. We start from the statement of the Olech Theorem on global asymptotic stability:

Olech Theorem: Suppose that the following system possesses an equilibrium point.

$$\Sigma \quad \begin{cases} \dot{w} = F(w,z), \\ \dot{z} = G(w,z). \end{cases}$$

The equilibrium point is globally asymptotically stable (i.e., the equilibrium point is stable and any path converges to the equilibrium point), under the following conditions:

(C.17) $\partial F/\partial w + \partial G/\partial z < 0$ for any $(w,z) \in R^2$;
(C.18) $\partial F/\partial w \bullet \partial G/\partial z - \partial F/\partial z \bullet \partial G/\partial w > 0$ for any $(w,z) \in R^2$;
(C.19) $\partial F/\partial w \bullet \partial G/\partial z \neq 0$ or $\partial F/\partial z \bullet \partial G/\partial w \neq 0$ for any $(w,z) \in R^2$.■

Proof: See Olech (1963).■

We now start from the proof of Lemma 2.

Proof of Lemma 2: Define

$$F(Y,R) \equiv \alpha\{H(Y,R) - Y\} \quad \text{and} \quad G(Y,R) \equiv \beta\{L(Y,R) - M/P\}.$$

From Assumption 2 and the assumption of the original IS-LM model, we see that

$$\partial F/\partial Y = \alpha(\partial H/\partial Y - 1) < 0, \quad \partial G/\partial R = \beta\partial L/\partial R < 0,$$
$$\partial F/\partial R = \alpha\partial H/\partial R < 0, \quad \partial G/\partial Y = \beta\partial L/\partial Y > 0$$

for any $(Y,R) \in R^2$. Thus, (C.17) to (C.19) of the Olech Theorem are satisfied for System Ω_{IS-LM}. Thus, the proof follows directly from the Olech Theorem.■

Proof of Lemma 4: Assumption 2 yields that $0 < 1 - \partial H/\partial Y = 1 - h_1$ for any $(Y,R) \in R^2$. Therefore, we have

$$m^* = \frac{(1-h_1)\beta l_2 + \beta l_2 Q + (1-h_1)Q + h_2\beta l_1 - \dfrac{(1-h_1)\beta l_2 Q + h_2\beta l_1 Q}{1-h_1+\beta l_2+Q}}{h_2\beta Q}$$

$$= \frac{(1-h_1)\beta l_2 + (1-h_1)Q}{h_2\beta Q} + \frac{\beta l_2 Q + h_2\beta l_1 - \dfrac{(1-h_1)\beta l_2 Q + h_2\beta l_1 Q}{1-h_1+\beta l_2+Q}}{h_2\beta Q}$$

$$= \frac{(1-h_1)\beta l_2 + (1-h_1)Q}{h_2\beta Q} + \frac{\dfrac{h_2\beta l_1(1-h_1) + \beta^2 l_2^2 Q + \beta l_2 Q^2 + h_2\beta^2 l_1 l_2}{1-h_1+\beta l_2+Q}}{h_2\beta Q}.$$

Therefore, since $1 > h_1$ and the other parameters are positive, we obtain $m^* > 0$. ■

Before proving Theorem 2 concerning the occurrence of a Hopf bifurcation, we prepare the following two results:

Lemma 5: Consider the cubic equation: $x^3 + ax^2 + b(\eta)x + c = 0$. Suppose that the following conditions are satisfied.

(C.18) $\quad a > 0, \quad b(\eta^*) > 0,$

(C.19) $\quad ab(\eta^*) - c = 0.$

Then, the cubic equation with $\eta = \eta^*$ possesses complex conjugate pure imaginary solutions and a non-zero real solution.■

Proof: See Gandolfo (1996, pp.220-221).■

Lemma 6: Suppose that all the conditions of Lemma 5 are satisfied. We denote by $\lambda_1(\eta)$, $\lambda_2(\eta)$, and $\lambda_3(\eta)$ the solutions to the cubic equation of Lemma 5. Then, $\lambda_1(\eta)$, $\lambda_2(\eta)$, and $\lambda_3(\eta)$ are continuously differentiable at $\eta = \eta^*$. From Lemma 5, without loss of generality, we assume that $\lambda_2(\eta^*)$ and $\lambda_3(\eta^*)$ are pure imaginary and $\lambda_1(\eta^*)$ is a non-zero real. Suppose that the following condition is satisfied:

(C.20) $\quad db(\eta^*)/d\eta \neq 0.$

Then, $d\,\mathrm{Re}\,\lambda_i(\eta^*)/d\eta \neq 0 \quad (i = 2,3)$. ■

Proof: For the proof of the first half, see Guckenheimer and Holmes (1983, Theorem 3.4.2). We now prove the latter part. From (C.18) and the relationship between the solutions and the coefficients, we obtain

(A.13.1) $\quad -a = \lambda_1(\eta) + \lambda_2(\eta) + \lambda_3(\eta),$
(A.13.2) $\quad b(\eta) = \lambda_1(\eta)\lambda_2(\eta) + \lambda_2(\eta)\lambda_3(\eta) + \lambda_3(\eta)\lambda_1(\eta) > 0,$
(A.13.3) $\quad -c = \lambda_1(\eta)\lambda_2(\eta)\lambda_3(\eta).$

Since $\lambda_1(\eta^*)$ is non-zero and $\lambda_1(\eta)$ is continuous at $\eta = \eta^*$, we see that there is a neighborhood of $\eta = \eta^*$, in which $\lambda_1(\eta)$ is non-zero and $\lambda_i(\eta)$ $(i = 2,3)$ is complex conjugate. In the neighborhood, we see from (A.13) that

$$\begin{aligned} b(\eta) &= \{\lambda_2(\eta) + \lambda_3(\eta)\}\lambda_1(\eta) + \lambda_2(\eta)\lambda_3(\eta) \\ &= -a\lambda_1(\eta) - \{\lambda_1(\eta)\}^2 - c/\lambda_1(\eta). \end{aligned}$$

Therefore, it follows from (C.20) that

(A.14) $$0 \neq \frac{db(\eta^*)}{d\eta} = [-a - 2\lambda_1(\eta^*) + c/\{\lambda_1(\eta^*)\}^2]\frac{d\lambda_1(\eta^*)}{d\eta}.$$

Therefore, we have

(A.15) $$\frac{d\lambda_1(\eta^*)}{d\eta} \neq 0$$

Defining $\mathrm{Re}\{\lambda_2(\eta)\} = \mathrm{Re}\{\lambda_3(\eta)\} \equiv \lambda_{real}(\eta)$, (A.13.1) yields $-a = \lambda_1(\eta) + 2\lambda_{real}(\eta)$.

Therefore, we have

$$0 = \frac{d\lambda_1(\eta)}{d\eta} + 2\frac{d\lambda_{real}(\eta)}{d\eta}.$$

This equation and (A.15) imply

$$0 \neq \frac{d\lambda_1(\eta^*)}{d\eta} = -2\frac{d\lambda_{real}(\eta^*)}{d\eta} = -2\frac{d\,\mathrm{Re}\{\lambda_i(\eta^*)\}}{d\eta}, \quad i = 2,3.$$

This completes the proof of the second half.■

By using Lemmas 5 and 6, we prove Theorem 2.

Proof of Theorem 2: The Jacobean matrix of $\overline{\Omega}$ is given by

$$J = \begin{bmatrix} -\alpha(1-h_1) & -\alpha h_2 & 0 \\ \beta l_1 - \beta m Q & -\beta l_2 & \beta Q m \\ Q & 0 & -Q \end{bmatrix}.$$

Then, the characteristic equation of J is given by

$$\begin{aligned} 0 &= \lambda^3 + \{\alpha(1-h_1) + \beta l_2 + Q\}\lambda^2 \\ &\quad + \{\alpha(1-h_1)\beta l_2 + \beta l_2 Q + Q\alpha(1-h_1) + \alpha h_2 \beta l_1 - \alpha h_2 \beta Q m\}\lambda \\ &\quad + \alpha(1-h_1)\beta l_2 Q + \alpha h_2 \beta l_1 Q \\ &= \lambda^3 + g_1\lambda^2 + (E - mD)\lambda + g_3, \end{aligned}$$

where

$$g_1 \equiv \alpha(1-h_1) + \beta l_2 + Q, \quad E \equiv \alpha(1-h_1)\beta l_2 + \beta l_2 Q + Q\alpha(1-h_1) + \alpha h_2 \beta l_1,$$
$$D \equiv \alpha h_2 \beta Q, \text{ and } g_3 \equiv \alpha(1-h_1)\beta l_2 Q + \alpha h_2 \beta l_1 Q.$$

We now prove that all conditions of Lemma 5 are satisfied at $m = m^*$. From the definition of m^*, we have

$$\text{(A.16)} \qquad m^* = \frac{\alpha(1-h_1)\beta l_2 + \beta l_2 Q + \alpha(1-h_1)Q + \alpha h_2 \beta l_1 - \dfrac{\alpha(1-h_1)\beta l_2 Q + \alpha h_2 \beta l_1 Q}{\alpha(1-h_1) + \beta l_2 + Q}}{\alpha h_2 \beta Q}$$

$$= \frac{E - g_3 / g_1}{D}.$$

We define

$$g_2(m^*) \equiv E - m^* D.$$

Then, it follows from (A.16) that

(A.17) $g_1 g_2(m^*) - g_3 = 0$

On the other hand, from Assumption 2 and the parameter definitions, we have

$$1 - h_1 > 0, \quad l_1 = \partial L(Y^*, R^*)/\partial Y > 0, \quad l_2 = -\partial L(Y^*, R^*)/\partial R > 0,$$
$$h_2 = -\partial H(Y^*, R^*)/\partial R > 0 \quad \text{and} \quad \Psi'(0) = Q > 0.$$

Therefore, we obtain

(A.18) $g_1 > 0$ and $g_3 > 0$.

From (A.17) and (A.18), we obtain

(A.19) $0 < g_3 / g_1 = g_2(m^*)$.

Thus, from (A.17) to (A.19), we can use Lemma 7 to determine that the characteristic equation possesses complex conjugate pure imaginary solutions and a non-zero real solution. Since $g_2(m) = E - mD$ and $D = \alpha h_2 \beta Q > 0$, we have

(A.20) $0 > -D = \dfrac{dg_2(m^*)}{dm} (\neq 0)$.

Then, from Lemma 6 and (A.20), denoting one of the complex conjugate solutions by $\lambda(m)$, we obtain

$$\frac{d \operatorname{Re} \lambda(m^*)}{dm} \neq 0.$$

Thus, the Hopf Bifurcation Theorem shows that System $\overline{\Omega}$ undergoes a Hopf bifurcation at $m = m^*$. This completes the proof.■

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Figure Captions

Figure 1. Existence and uniqueness of the equilibrium point.

Figure 2. Graph of the Ψ – function.

Figure 3. Strong stability of the reduced IS-LM model: (1) phase plane; (2) time series.

Figure 4. Emergence and growth of Hopf cycle.

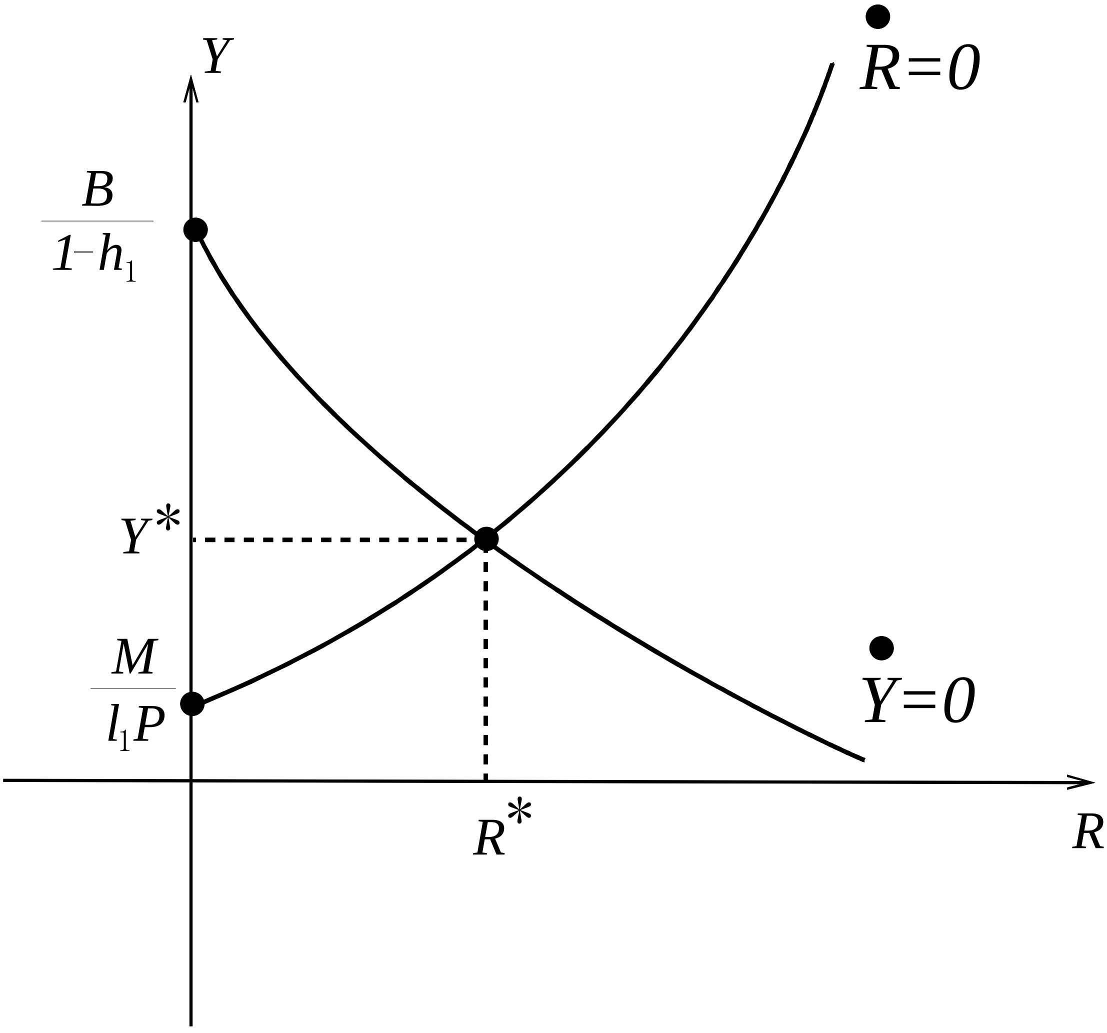


Figure 1

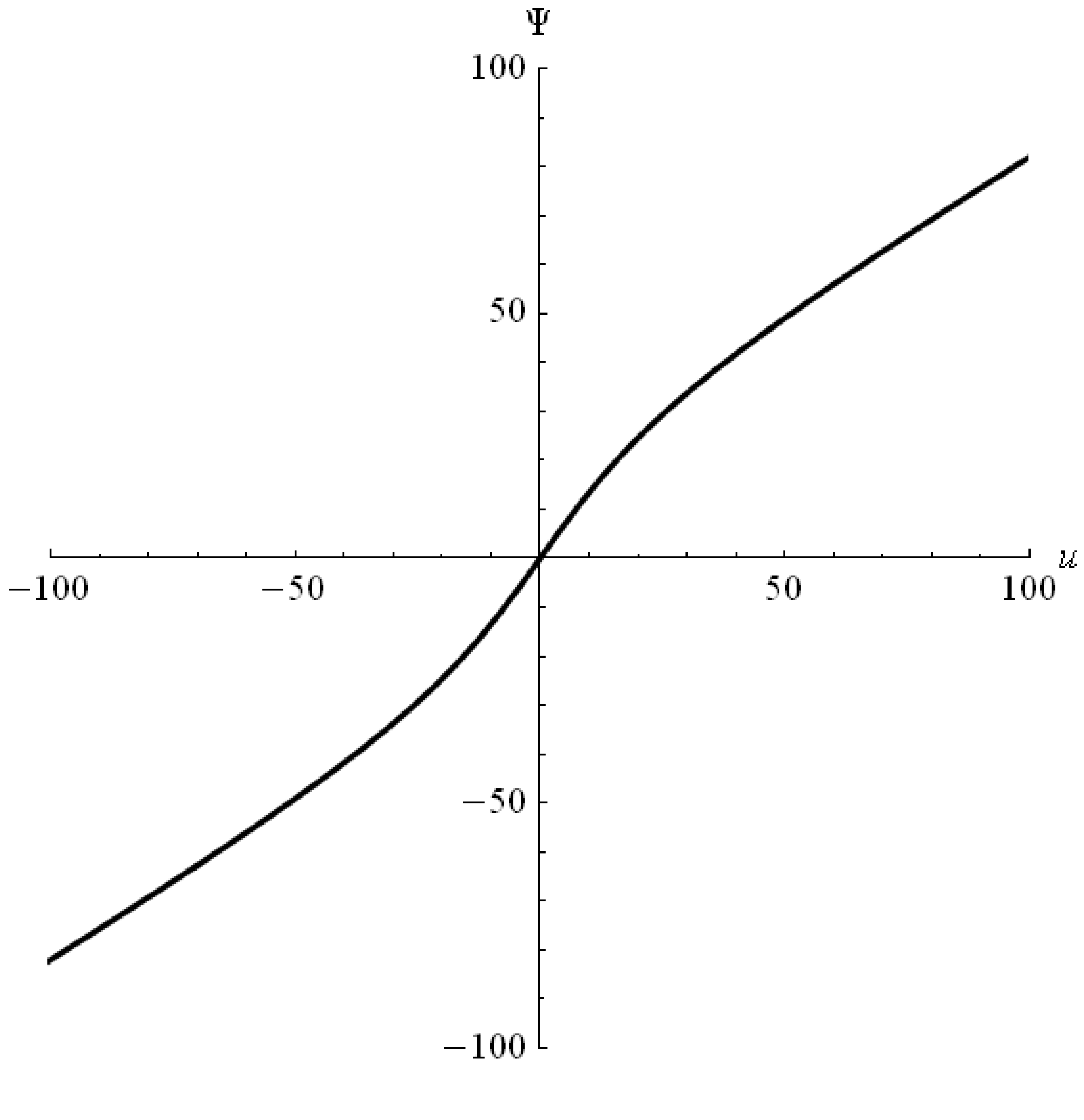
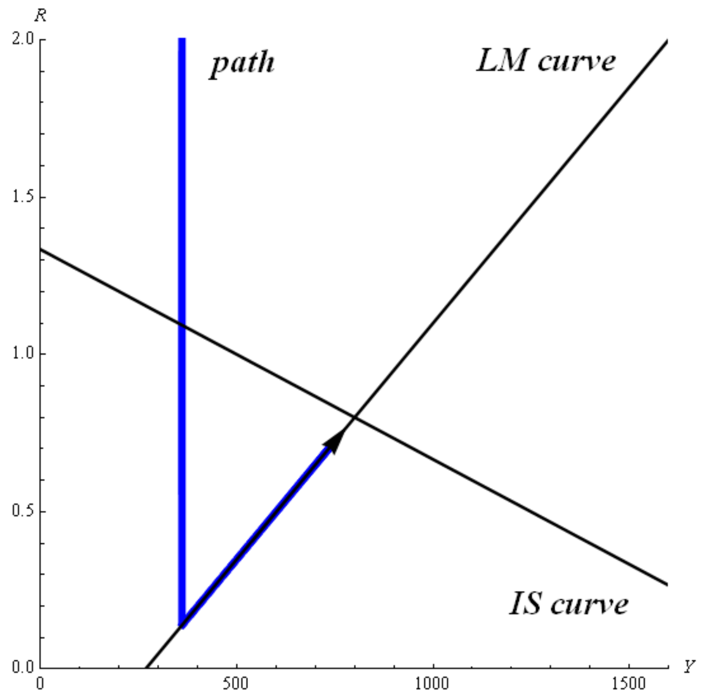


Figure 2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1500
Y
path
LM curve
IS curve

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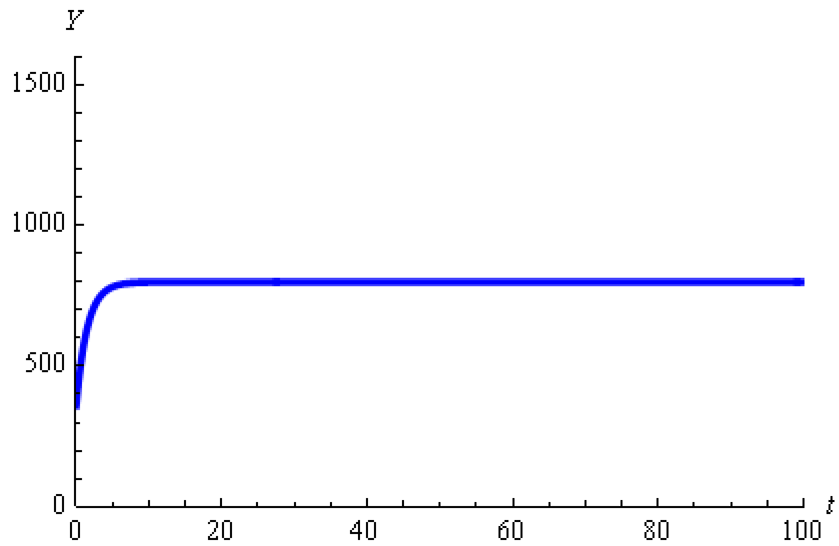
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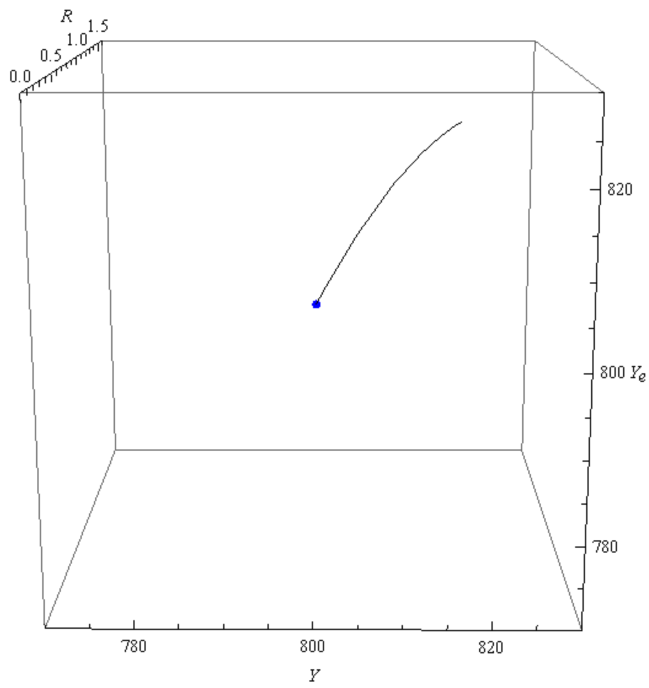
20
40
60
80
100
t

(2)

Figure 3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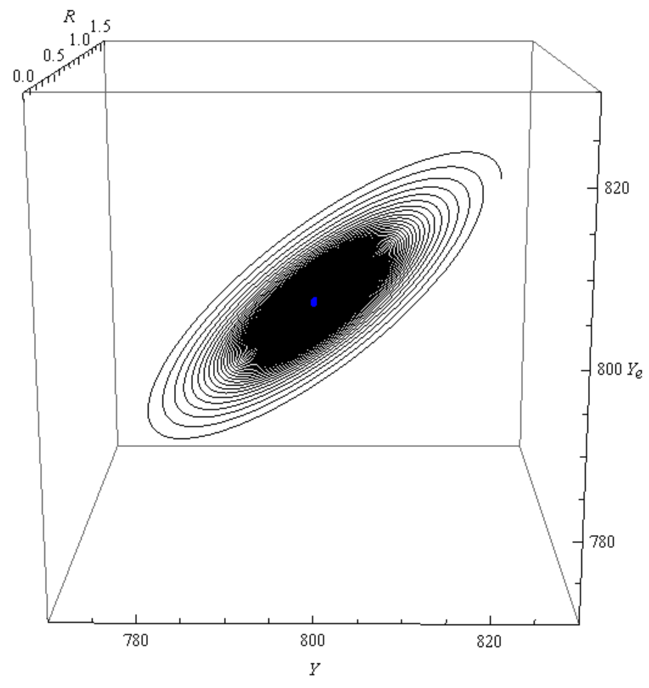
780
800
820
Y

(1)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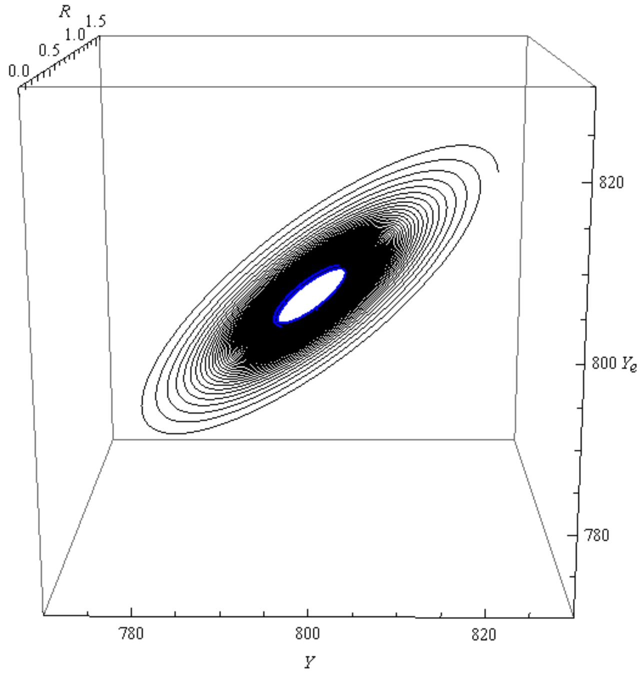
Y

(2)

Figure 4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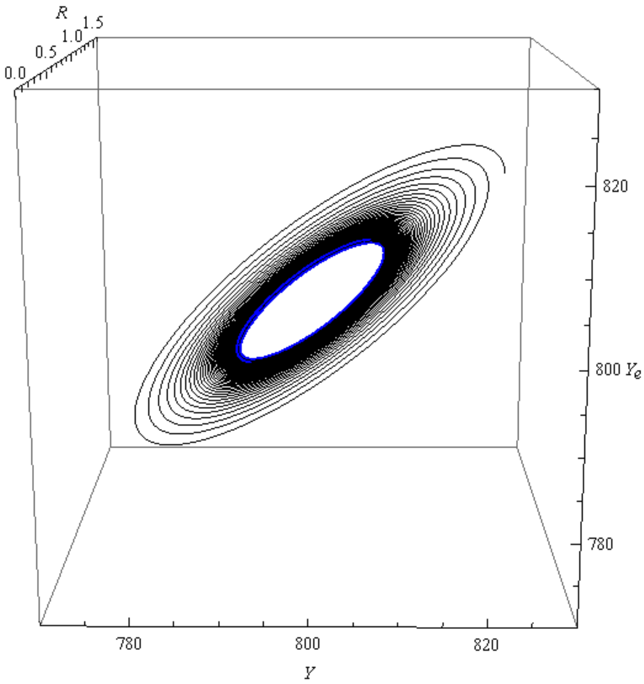
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Y

(4)

Figure 4