Twisted Holography: The Examples of 4d and 5d Chern-Simons Theories

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Statement of Contributions

This thesis is based on the following:

Chapter 2:

> *Topological Holography: The Example of The D2-D4 Brane System*. Nafiz Ishtiaque, Seyed Faroogh Moosavian, **Yehao Zhou**. SciPost Phys. 9, 017 (2020). arXiv:1809.00372.

Chapter 3:

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Chapter 4:

> *Feynman diagrams and Ω-deformed M-theory*. Jihwan Oh, **Yehao Zhou**. SciPost Phys. 10, 029 (2021). arXiv:2002.07343.
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Chapter 5 is based on my on-going and unpublished work.

Abstract

Twisted holography is a duality between a twisted supergravity, and a twisted supersymmetric gauge theory living on the D-branes in the supergravity. The main objectives of this duality is the comparison between the algebra of observables in the bulk twisted supergravity and the algebra of observables in the boundary twisted supersymmetric gauge theory.

In this thesis, two example of the twisted holography duality are explored. The bulk theory for the first example is the 4d topological-holomorphic Chern-Simons theory, which is expected to be dual to 2d BF theory with line defects. The algebra of observables in the 2d BF theory is computed by two methods: perturbation theory (Feynman diagrams), and phase space quantization. By holography duality this algebra is expected to be isomorphic to the algebra of bulk-boundary scattering process, and the latter is computed in this thesis using perturbative method.

The bulk theory for the second example is the 5d topological-holomorphic Chern-Simons theory, which is expected to be dual to the large-N limit of a family of 1d quantum mechanics built from the ADHM quivers. The generators and relations of the large-N limit algebra of observables in the 1d quantum mechanics are studied from algebraic point view. By holography duality, this algebra is expected to be the algebra of observables on the universal line defect coupled to the 5d Chern-Simons theory, and some nontrivial relations of the latter algebra are computed in this thesis using perturbative method. The surface defects and various fusion process between line and surface defects are also explored.

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Table of Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Holographic duality

Holography is one of the main active area of research in finding a theory of quantum gravity [154, 153]. The prime example of this concept is The AdS/CFT Correspondence [118]. In general, holography is a duality between two theories, referred to as a bulk theory and a boundary theory[118, 90, 158]. A familiar manifestation of the duality is an equality of the partition function of the two theories - the boundary partition function as a function of sources, and the bulk partition function as a function of boundary values of fields. This in turns implies that correlation functions of operators in the boundary theory can also be computed in the bulk theory by varying boundary values of its fields [90, 158]. This dictionary has been extended to include expectations values of non-local operators as well [117, 148, 162, 82]. This is a strong-weak duality, relating a strongly coupled boundary theory to a weakly coupled bulk theory. As is usual in strong-weak dualities, precise mathematical formulations and exact computations on both sides of the duality are hard in general.

Recently, a twisted version of The AdS/CFT Correspondence has been formulated by Costello and Li [28, 29], which makes the mathematical formulation of the duality possible. In their setup, the twisted holography relates the bulk twisted supergravity to the boundary twisted supersymmetric gauge theory. The bulk twisted supergravity, denoted by $\mathcal{T}_{\mathrm{bk}}$, is supergravity in a background where the bosonic ghost field takes some non-zero value [34, 35]. The boundary twisted supersymmetric gauge theory, denoted by $\mathcal{T}_{\mathrm{bd}}$, arises from D-branes in the twisted supergravity, considered as defects in the bulk theory. The main objectives considered in the twisted holography setting are

(1) The algebras of operators $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$ of the twisted supersymmetric gauge theory living on stack of N D-branes, after sending $N \to \infty$.

(2) The algebra of operators $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bk}})$ in perturbative twisted supergravity living on the location of the defect given by the stack of D-branes.

A particular nice feature of the twisted setup is that, the theories are usually drastically simplified, though still quite nontrivial. The theories can be furthermore simplified with the presence of Omega background [131, 132, 130, 161]. With the help of these simplifications, the exact computations becomes possible in some interesting examples [87, 139, 140, 84, 83, 28].

The main proposal of the twisted holography is the comparison between algebras of protected sub-sectors of observables of the bulk/boundary theories, which can be summarized as follows.

Conjecture 1.1.1 (Costello-Li, [29]). *$\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$ is Koszul dual to $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bk}})$.*

Before explaining the reason why Koszul duality is expected, we should mention that the Conjecture 1.1.1 is only expected to hold in the case that the brane sources no flux in the supergravity theory. In general, a deformation of the Koszul duality between $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$ and $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bk}})$ is expected, and this should be related to the curved Koszul duality in math literature [95], examples of this more general situation is explored in [31, 37, 38, 39]. The examples considered in this thesis (4d and 5d Chern-Simons theories) have the feature that branes source no flux in the supergravity background.

For the record of literature in this area, see [26] for an earlier example and also [98, 31, 70, 147, 111, 37, 137, 65, 112, 138, 136, 20, 67, 35, 36, 40] for follow-up and related works. For a recent and very readable review of Koszul duality aimed at physicists, we refer the reader to [142].

1.2 Why Koszul duality?

In this section we explain why in the first place we expect the Koszul duality between operator algebras of the bulk theory and the large N limit of the boundary theory. The discussion in this section is broadly general and somewhat impressionism, the aim of this section is to serve as a motivation behind the investigation into specific models in the body of this thesis.

Let us first briefly recall the notion of Koszul dual algebra in mathematical terminology [114]. Let A be a differential-graded algebra, i.e. an algebra

$$A = \bigoplus_{i \geq 0} A_i, \text{ such that } A_i \cdot A_j \subset A_{i+j}, \tag{1.1}$$

together with

$$d : A_i \to A_{i+1}, \text{ such that } d(a \cdot b) = d(a) \cdot b + (-1)^{\deg(a)} a \cdot d(b). \tag{1.2}$$

We furthermore assume that $A_0 = \mathbb{C}$, so there exists a differential-graded algebra homomorphism

$$\rho : A \to \mathbb{C}, \tag{1.3}$$

ρ is called the augmentation map. Define the Koszul dual algebra $A^!$ by

$$A^! = \mathrm{RHom}_A(\mathbb{C}, \mathbb{C}). \tag{1.4}$$

The algebra structure on $A^!$ is induced from the standard one on Hom. A presentation of $A^!$ in terms of bar-construction can be found in [114].

Let us go back to the holography, and lets us assume that the differential graded algebra A is the algebra of observables of some twisted supergravity theory $\mathcal{T}$ on $\mathbb{R}_t \times \mathbb{R}^d$ for $d \geq 2$ such that $\mathcal{T}$ is topological along $\mathbb{R}_t$. Consider a line defect on $\mathbb{R}_t \times \{0\}$, which comes from certain stack of N D-brane in the string-theory lift of $\mathcal{T}$ and taking $N \to \infty$. Then we remove this line defect and include the gravitational flux sourced by this defect, this will result in the back-reacted geometry with underlying topological space

$$\mathbb{R}_t \times (\mathbb{R}^d \backslash \{0\}) \cong \mathbb{R}_t \times \mathbb{R}_{r>0} \times S^{d-1} \cong \mathrm{AdS}_2 \times S^{d-1}. \tag{1.5}$$

In the physics holography language, the line defect is called a *black brane*, and the limit $r \to 0$ is called the *near-horizon limit*.

In the back-reacted geometry, we compactify the theory along S^{d-1} while keeping all the KK-modes, and assume that the compactified theory is topological on the remaining spacetime $\mathbb{R}_t \times \mathbb{R}_{r>0}$. Note that this assumption is satisfied for the examples that are considered in this thesis (4d and 5d Chern-Simons theories).

Next we put boundary conditions at $r = 0$ and $r = \infty$ respectively. In the 2d topological field theory [5], the boundary conditions at $r = 0$ form the category of right A-modules mod-A, and the boundary conditions at $r = \infty$ form the category of left A-modules A-mod.

To see this physically, we can choose a boundary condition and bring local operators in the bulk towards the boundary, this operation is associative, and the identity operator in the bulk becomes the identity on the boundary, therefore the boundary condition is a left or right A-module, depending on the direction of operator action.

After fixing a boundary condition M in A-mod or mod-A, the algebra of boundary observables is

$$\mathrm{RHom}_A(M, M), \tag{1.6}$$

where the multiplication is induced from Hom.

The specific boundary conditions that are considered in the holography correspondence are the following.

- At $r = 0$, we put Dirichlet boundary conditions, i.e. the right A-module A.
- At $r = \infty$, we put Neumann boundary conditions, i.e. the left A-module $\mathbb{C}$ (via the augmentation $A \to \mathbb{C}$).

Therefore the algebra of local operators at $r = 0$ is A itself, and the algebra of local operators at $r = \infty$ is the Koszul dual algebra $A^!$. The boundary theory at $r = \infty$ with the Neumann boundary condition is the theory $\mathcal{T}_{\mathrm{bd}}$ that is considered in the holography, hence we expect that

$$\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}}) \cong \mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bk}})^!, \tag{1.7}$$

given that various assumptions that we made along the way are satisfied.

1.2.1 Universal line defect

An important property of the Koszul dual algebra $A^!$ is the following fact [114]: there is an isomorphism between functors

$$\mathrm{Hom}_{\mathrm{dg-alg}}(A^!, -) \cong \mathrm{MC}(- \otimes A). \tag{1.8}$$

Here $\mathrm{MC}(B \otimes A)$ denotes the set of solutions to the Maurer-Cartan equation:

$$dx + \frac{1}{2}[x, x] = 0, \quad x \in (B \otimes A)_1 \tag{1.9}$$

in the differential-graded algebra $B \otimes A$. In other words, the Koszul dual algebra $A^!$ represents the functor $\mathrm{MC}(- \otimes A)$, this can be used as the definition of the Koszul dual algebra.

From the physics perspective, lets us still assume that the differential graded algebra A is the algebra of observables of the twisted supergravity $\mathcal{T}$ on $\mathbb{R}_t \times \mathbb{R}^d$, such that $\mathcal{T}$ is topological along $\mathbb{R}_t$. By the topological property, there exists a charge δ in the theory $\mathcal{T}$ of ghost number -1 such that

$$\{Q, \delta\} = \partial_t, \tag{1.10}$$

where Q is the differential (BRST charge) on A. Then we put a line defect on $\mathbb{R}_t \times \{0\}$, and look for one-dimensional topological quantum mechanics that can be coupled to the theory $\mathcal{T}$ along the line defect.

The general process of coupling a topological quantum mechanics to a bulk theory $\mathcal{T}$ along a line defect can be described as follows. Let B be the algebra of observables of the topological quantum mechanics, then a general coupling is constructed from topological descent

$$S_{\text{int}} = \mathrm{Pexp} \int_{\mathbb{R}_t} \delta x dt = \sum_{n \geq 1} \int_{t_1 \leq \cdots \leq t_n} \delta x(t_1) \cdots \delta x(t_n), \tag{1.11}$$

where x is an operator in $B \otimes A$ with ghost number one (so that the coupling is of ghost number zero). A physical coupling must be anomaly-free, i.e. $QS_{\text{int}} = 0$. We can compute the BRST variation

$$\begin{aligned} QS_{\text{int}} &= \sum_{n \geq 1} \sum_{i=1}^{n} \int_{t_1 \leq \cdots \leq t_n} \delta x(t_1) \cdots (\partial_t x - \delta Q x)(t_i) \cdots \delta x(t_n) \\ &= \sum_{n \geq 1} \sum_{i=1}^{n} \int_{t_1 \leq \cdots \leq t_n} \delta x(t_1) \cdots (-\delta Q x - \delta x \cdot x + x \cdot \delta x)(t_i) \cdots \delta x(t_n) \\ &= -\sum_{n \geq 1} \sum_{i=1}^{n} \int_{t_1 \leq \cdots \leq t_n} \delta x(t_1) \cdots \delta(Qx + \frac{1}{2}[x, x])(t_i) \cdots \delta x(t_n). \end{aligned} \tag{1.12}$$

The BRST variation of interaction term S_{int} vanishes if x satisfies the Maurer-Cartan equation:

$$Qx + \frac{1}{2}[x, x] = 0. \tag{1.13}$$

The above argument shows that $A^!$ is the algebra of observables of a *universal line defect* that can be coupled to $\mathcal{T}$.

Our speculation of Koszul duality rephrased in the twisted holography setup is that $\mathcal{T}_{\rm bd}$ is the *universal* theory that can be coupled to the bulk theory $\mathcal{T}_{\rm bk}$ along the defect.

Although the above discussion is only for the 1d defect, one should interpret that the universal defect is the "right" definition of Koszul duality and try to formulate the Koszul duality for higher dimensional defect. This is not the goal of this thesis. For discussion on Koszul dual chiral algebras, see [31, 37, 38, 39].

1.3 Relationship with Witten's prescription of holography

In the seminal works of holography correspondence [90, 158], two theories, $\mathcal{T}_{\rm bd}$ and $\mathcal{T}_{\rm bk}$ were considered on two manifolds M_1 and M_2 respectively, with the property that M_1 was conformally equivalent to the boundary of M_2. The theory $\mathcal{T}_{\rm bd}$ was considered with background sources, schematically represented by ϕ. The theory $\mathcal{T}_{\rm bk}$ was such that the values of its fields at the boundary ∂M_2 can be coupled to the fields of $\mathcal{T}_{\rm bd}$, then $\mathcal{T}_{\rm bk}$ was quantized with the fields ϕ as the fixed profile of its fields at the boundary ∂M_2. These two theories were considered to be holographic dual when their partition functions were equal:

$$Z_{\rm bd}(\phi) = Z_{\rm bk}(\phi)\,. \tag{1.14}$$

This is the main identity in Witten's prescription of holography.

This equality leads to an isomorphism of two algebras constructed from the two theories, as follows. Consider local operators O_i in $\mathcal{T}_{\rm bd}$ with corresponding sources ϕ^i. The partition function $Z_{\rm bd}(\phi)$ with these sources has the form:

$$Z_{\rm bd}(\phi) = \int \mathcal{D}X \exp\left(-\frac{1}{\hbar}S_{\rm bd} + \sum_i O_i\phi^i\right)\,, \tag{1.15}$$

where X schematically represents all the dynamical fields in $\mathcal{T}_{\rm bd}$. Correlation functions of the operators O_i can be computed from the partition function by taking derivatives with respect to the sources:

$$\langle O_1(p_1)\cdots O_n(p_n)\rangle = \frac{1}{Z_{\rm bd}(\phi)}\frac{\delta}{\delta\phi^1(p_1)}\cdots\frac{\delta}{\delta\phi^n(p_n)}Z_{\rm bd}(\phi)\Big|_{\phi=\phi_0}\,. \tag{1.16}$$

We can consider the algebra generated by the operators O_i using operator product expansion (OPE). However, this algebra is generally of singular nature, due to its dependence on the location of the operators and the possibility of bringing two operators too close to each other. In specific twisted supersymmetric gauge theories, we can consider protected sub-sectors of the operator spectrum that can generate algebras free from such contact singularity, so that a position independent algebra can be defined. This is the algebra $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$ that was introduced in the very beginning. Suppose the structure constants of the algebra $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$ are

$$O_i O_j = C_{ij}^k O_k \,. \tag{1.17}$$

In terms of the partition function and the sources the relation (1.17) becomes:

$$\frac{\delta}{\delta\phi^i}\frac{\delta}{\delta\phi^j} Z_{\mathrm{bd}}(\phi)\bigg|_{\phi=0} = C_{ij}^k\,\frac{\delta}{\delta\phi^k} Z_{\mathrm{bd}}(\phi)\bigg|_{\phi=\phi_0} . \tag{1.18}$$

The statement of duality (1.14) then tells us that the above equation must hold if we replace Z_{bd} by Z_{bk}:

$$\frac{\delta}{\delta\phi^i}\frac{\delta}{\delta\phi^j} Z_{\mathrm{bk}}(\phi)\bigg|_{\phi=0} = C_{ij}^k\,\frac{\delta}{\delta\phi^k} Z_{\mathrm{bk}}(\phi)\bigg|_{\phi=\phi_0} . \tag{1.19}$$

This gives us a realization of the operator algebra $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$ in the dual theory $\mathcal{T}_{\mathrm{bk}}$. This motivates us to define another algebra by taking functional derivatives of the partition function of $\mathcal{T}_{\mathrm{bk}}$ with respect to ϕ, as in (1.19). Let's call this algebra the *scattering algebra*, $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$.

From the above discussions, Witten's prescription of holography duality implies the following isomorphism:

$$\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}}) \cong \mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})\,. \tag{1.20}$$

Together with the previous Koszul duality isomorphism, the twisted holography gives two equivalent description of the algebra of boundary observables $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$ in terms of bulk theory $\mathcal{T}_{\mathrm{bk}}$:

(1) The algebra of observables on universally coupled defect, which can be computed by the anomaly-cancellation using Feynman diagrams.

(2) The algebra of bulk-boundary scatterings, which can be computed using Witten diagrams introduced in [158].

On the other hand, $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$ is defined as the large-N limit of algebra of observables of the world-volume theory on the stack of N D-branes supported at the defect. Therefore a comparison of $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$ between large-N limit presentation and one of the bulk presentation (i.e. $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ or $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bk}})^!$) would be a check of holography in the twisted setup.

1.4 Organization of the thesis

This thesis is dedicated to two examples of the twisted holography, whose bulk theories are 4d and 5d topological-holomorphic Chern-Simons theories. As we explained above, the main objectives are the comparison of the algebra of observable on the boundary $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bd}})$ in the large-N limit presentation versus the bulk presentation $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ or $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bk}})^!$. Along the way some additional features of the algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bd}})$ (e.g. coproduct) are also explored.

Chapters §2 and §2 are closely related. In chapter §2, we start from a brane setup involving N D2 branes and K D4 branes in a 6d topological string theory and describe the two theories that we claim to be holographic dual to each other. The world-volume theory on D4 branes is the 4d topological-holomorphic Chern-Simons theory with gauge group GL_K, and the world-volume theory on D2 branes is the 2d BF theory with gauge group GL_N. Next, in the section §2.3 we compute the local operator algebra of the world-volume theory on the stack of N D2 brane, using the Feynman diagram approach. This algebra is the Yangian $Y_\hbar(\mathfrak{gl}_K)$ in the limit $N \to \infty$. In the section §2.4 we show that the same algebra can be computed using Witten diagrams in the D4 brane theory. Some of technical computations in §2.3 and §2.4 are presented in appendix §A. In the section §2.5, we propose a string theory realization of the duality, and we show that the model of twisted holography that we have constructed in this chapter is a protected subsector of the more familiar model of holographic duality involving $\mathcal{N} = 4$ super Yang-Mills theory with defects. We identify the supersymmetric twists and Ω-deformation that reduce the $\mathcal{N} = 4$ duality setup to the topological setup presented in the earlier sections.

Chapter §3 views the 2d BF theory from a different perspective. Namely we investigate the geometry of the phase space of the 2d GL_N BF theory coupled to a quantum-mechanical system with GL_K flavour symmetry along a defect, and study the algebra of functions in this phase space. In §3.2 we show that the phase space $\mathcal{M}(N,K)$ can be embedded into the based loop group $L^-(\mathrm{GL}_K \times \mathrm{GL}_1)$ as a Poisson subvariety, and in §3.3 we show that the large-N limit of the $\mathcal{M}(N,K)$ is isomorphic to $L^-(\mathrm{GL}_K \times \mathrm{GL}_1)$. In §3.4 we compute Hilbert series of $\mathbb{C}[\mathcal{M}(N,K)]$ as well as Hilbert series of certain modules of it. The techniques of this computation is presented in appendix §B. In §3.5 we show that the quantization $\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ is a truncation of Yangian. The idea of the proof is to compare the defining ideal of $\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ in $Y_\hbar(\mathfrak{gl}_K \oplus \mathfrak{gl}_1)$ with the defining ideal of the Coulomb-branch presentation of truncated Yangian [18].

In chapter §4 we switch gears to the 5d topological-holomorphic Chern-Simons theory. We start with reviewing the 11 dimensional twisted supergravity background

$$(\mathbb{R}^3 \times \text{Taub-NUT})_{\text{topological}} \times (\mathbb{C}^2)_{\text{holomorphic}}$$

in section §4.1. In the limit of Taub-NUT circle shrink to zero radius, the theory reduces to 10 dimensional type IIA supergravity with D6 brane supported at the tip of Taub-NUT. After turning on the Ω-background, the effective theory on D6 brane becomes the 5d topological-holomorphic Chern-Simons theory. If we add M2 and M5 branes into the 11d supergravity, they become line and surface defects in 5d Chern-Simons theory after reduction to type IIA and turning on Ω-background. The algebra $\mathcal{A}$ of local observables on the M2 brane is the large-N limit of the quantized ring of functions on Nakajima quiver varieties associated to the ADHM quivers, and the algebra of local observables on the M5 brane is the mode algebra $\mathcal{W}_\infty$ of the W-infinity chiral algebra. Fusion of several M2 and M5 branes can be interpreted as coproduct between these algebras, see §4.1.10. We compute certain commutation relations of the algebra of universal defect coupled to 5d Chern-Simons along the M2 brane in §4.2, using Feynman diagrams, and show that they agree with the large-N limit presentation of $\mathcal{A}$. Some of the technical computation of integrals are presented in the appendix §D. We also compute the coproduct $\Delta_{\mathcal{A},\mathcal{A}}$, $\Delta_{\mathcal{W}_\infty,\mathcal{W}_\infty}$, and $\Delta_{\mathcal{A},\mathcal{W}_\infty}$ in §4.3, using Feynman diagrams, and show that they agree with the expected formulae derived using free-field realization in [73].

In chapter §5, we study the algebra $\mathcal{A}$ of observables on the M2 brane from a purely algebraic point of view. $\mathcal{A}$ is the large-N limit of the quantized ring of functions on Nakajima quiver varieties associated to the ADHM quivers, we review the quantization for general Nakajiam quiver varieties in the appendix §C. We write down the generators and relations of $\mathcal{A}$ in §5.1, partly using the Calogero representation (a free-field realization) of $\mathcal{A}$ worked out in §5.2. Note that they match with the presentation of the deformed double current algebra in the literature [89, 59, 60], and this answers a question of Costello [29, 2.1]. We also observe a simple relation between the Yangian of $\mathfrak{gl}_K$ and $\mathcal{A}$ in §5.5, and we use this observation to prove a conjecture of Costello [29, 2.3]. We mention the relation between cA and the Kac-Moody algebra of $\mathfrak{gl}_K$ in §5.6. And in the last section we write down the coproduct formula for $\mathcal{A}$ and show that it gives rise to a vertex coalgebra structure on $\mathcal{A}$.

The chapters of this thesis are technically independent of each other and they can be read independently.

Chapter 2

4d Chern-Simons Theory

In this chapter, we study a toy model for holographic duality. The model is constructed by embedding a stack of N D2-branes and K D4-branes (with one dimensional intersection) in a 6D topological string theory. The world-volume theory on the D2-branes (resp. D4-branes) is 2D BF theory (resp. 4D Chern-Simons theory) with GL_N (resp. GL_K) gauge group. We propose that in the large N limit the BF theory on $\mathbb{R}^2$ is dual to the closed string theory on $\mathbb{R}^2 \times \mathbb{R}_+ \times S^3$ with the Chern-Simons defect on $\mathbb{R} \times \mathbb{R}_+ \times S^2$. As a check for the duality we compute the operator algebra in the BF theory, along the D2-D4 intersection – the algebra is the Yangian of $\mathfrak{gl}_K$. We then compute the same algebra, in the guise of a scattering algebra, using Witten diagrams in the Chern-Simons theory. Our computations of the algebras are exact (valid at all loops).

2.1 Introduction and Summary

Holography is a duality between two theories, referred to as a bulk theory and a boundary theory, in two different space-time dimensions that differ by one [118, 90, 158]. A familiar manifestation of the duality is an equality of the partition function of the two theories - the boundary partition function as a function of sources, and the bulk partition function as a function of boundary values of fields. This in turns implies that correlation functions of operators in the boundary theory can also be computed in the bulk theory by varying boundary values of its fields [90, 158]. This dictionary has been extended to include expectations values of non-local operators as well [117, 148, 162, 82]. This is a strong-weak duality, relating a strongly coupled boundary theory to a weakly coupled bulk theory. As is usual in strong-weak dualities, exact computations on both sides of the duality are hard.

Topological theories have provided interesting examples of holographic dualities where exact computations are possible [87, 139, 140, 84, 83, 28].

Recently, it has shown that some instances of holography can be described as an algebraic relation, known as Koszul duality, between the operator algebras of the two dual theories [29]. It was previously known that the algebra of operators restricted to a line in the holomorphic twist of 4d $\mathcal{N} = 1$ gauge theory with the gauge group GL_K is the Koszul dual of the Yangian of $\mathfrak{gl}_K$ [26]. In light of the connection between Koszul duality and holography, this result suggests that if there is a theory whose local operator algebra is the Yangian of $\mathfrak{gl}_K$ then that theory could be a holographic dual to the twisted 4d theory. Since the inception of holography, brane constructions played a crucial role in finding dual theories. It turns out that the particular twisted 4d theory is the world-volume theory of K D4-branes[1] embedded in a particular 6d topological string theory [30]. Since the operators whose algebra is the Koszul dual of the Yangian lives on a line, it is a reasonable guess that we need to include some other branes that intersect this stack of D4-branes along a line. Beginning from such motivations we eventually find (and demonstrate in this chapter) that the correct choice is to embed a stack of N D2-branes in the 6d topological string theory so that they intersect the D4-branes along a line. The world-volume theory of the D2-branes is 2d BF theory with GL_N gauge group coupled to a fermionic quantum mechanics along the D2-D4 intersection. The algebra of gauge invariant local operators along this D2-D4 intersection is precisely the Yangian of $\mathfrak{gl}_K$.

This connected the D2 world-volume theory and the D4 world-volume theory via holography in the sense of Koszul duality. The connection between these two theories via holography in the sense of [90, 158] was still unclear. In this chapter we begin to establish this connection. We take the D2-brane world-volume theory to be our boundary theory. This implies that the closed string theory in some background, including the D4-brane theory should give us the dual bulk theory. In the boundary theory, we consider the OPE (operator product expansion) algebra of gauge invariant local operators, we argue that this algebra can be computed in the bulk theory by computing a certain algebra of scatterings from the asymptotic boundary in the limit $N \to \infty$. Our computation of the boundary local operator algebra using the bulk theory follows closely the computation of boundary correlation functions using Witten diagrams [158].

The Feynman diagrams and Witten diagrams we compute in this chapter have at most two loops, however, we would like to emphasize that the identification we make between the operator algebras and the Yangian is true at *all* loop orders. In the boundary theory

[1] We are following the convention of [3], according to which, by a topological Dp-brane we mean a brane with a p-dimensional world-volume.

(D2-brane theory) this will follow from the simple fact that, for the operator product that we shall compute, there will be no non-vanishing diagrams beyond two loops. In the bulk theory this follows from a certain classification of anomalies in the D4-brane theory [41] and independently from the very rigid nature of the deformation theory of the Yangian. We explain some of these mathematical aspects underlying our results in appendix §A.3.

A particular motivation for studying these topological/holomorphic theories and their duality is that these theories can be constructed from certain brane setup in string theory. We can identify these theories as certain supersymmetric subsectors of some theories on D-branes in type IIB string theory by applying supersymmetric twists and Ω-deformation.

The organization of the chapter is as follows. In §2.2 we start from a brane setup involving N D2-branes and K D4-branes in a 6d topological string theory and describe the two theories that we claim to be holographic dual to each other. In §2.3 we compute the local operator algebra in the D2-brane theory, this algebra will be the Yangian $Y(\mathfrak{gl}_K)$ in the limit $N \to \infty$. In §2.4 we show that the same algebra can be computed using Witten diagrams in the D4-brane theory. In the last section, §2.5, we propose a string theory realization of the duality.

2.2 The dual theories

2.2.1 Brane construction

The quickest way to introduce the theories we claim to be holographic dual to each other is to use branes to construct them. Our starting point is a 6d topological string theory, in particular, the product of the A-twisted string theory on $\mathbb{R}^4$ and the B-twisted string theory on $\mathbb{C}$ [30]. The brane setup is the following:

	$\mathbb{R}_v$	$\mathbb{R}_w$	$\mathbb{R}_x$	$\mathbb{R}_y$	$\mathbb{C}_z$	No. of branes
D2	0	$\times$	$\times$	0	0	N
D4	0	0	$\times$	$\times$	$\times$	K

(2.1)

The subscripts denote the coordinates we use to parametrize the corresponding directions, and it is implied that the complex direction is parametrized by the complex variable z, along with its conjugate variable $\bar{z}$.

Our first theory, denoted by $\mathcal{T}_{\text{bd}}$, is the theory of open strings on the stack of D2-branes. This is a 2d topological gauge theory with the complexified gauge group GL_N [30]. The intersection of the D2-branes with the D4-branes introduces a line operator in this theory. We describe this theory in §2.2.3.

Next, we consider the product of two theories, open string theory on the stack of D4-branes, and closed string theory on the 6d background sourced by the stack of D2-branes. The theory on the stack of D4-branes is a 4d analogue of Chern-Simons (CS) gauge theory with the complexified gauge group GL_K [30]. As it does in the theory on the D2-branes, the intersection between the D2 and the D4-branes introduces a line operator in this theory as well. This line sources a flux supported on the 3-sphere linking the line. Our bulk theory is the Kaluza-Klein compactification of the total 6d theory[2] on the 3-sphere. We describe the 4d CS theory in §2.2.5. Let us describe the closed sting theory in the next section.

2.2.2 The closed string theory

The closed string theory, denoted by $\mathcal{T}_{\text{cl}}$, is a product of Kodira-Spencer (also known as BCOV) theory [11, 34] on $\mathbb{C}$ and Kähler gravity [12] on $\mathbb{R}^4$, along with a 3-form flux sourced by the stack of D2-branes.[3] Fields[4] in this theory are given by:

$$\text{Set of fields,} \quad \mathcal{F} := \Omega^\bullet(\mathbb{R}^4) \otimes \Omega^{\bullet,\bullet}(\mathbb{C})\,, \tag{2.2}$$

i.e., the fields are differential forms on $\mathbb{R}^4$ and (p, q)-forms on $\mathbb{C}$.[5] The linearized BRST differential acting on these fields is a sum of the de Rham differential on $\mathbb{R}^4$ and the Dolbeault differential on $\mathbb{C}$, leading to the following equation of motion:

$$\left(\mathrm{d}_{\mathbb{R}^4} + \overline{\partial}_{\mathbb{C}}\right) \alpha = 0\,, \qquad \alpha \in \mathcal{F}\,. \tag{2.3}$$

The background field sourced by the D2-branes, let it be denoted by $F_3 \in \mathcal{F}$, measures the flux through a topological S^3 surrounding the D2-branes, it can be normalized as:

$$\int_{S^3} F_3 = N\,. \tag{2.4}$$

Note that the S^3 is only topological, i.e., continuous deformation of the S^3 should not affect the above equation. This is equivalent to saying that, the 3-form must be closed on

[2] 6d closed string theory coupled to 4d CS theory.

[3] This flux is analogous to the 5-form flux sourced by the stack of D4-branes in Maldacena's setup of AdS/CFT duality between $\mathcal{N} = 4$ super Yang-Mills and supergravity on $\mathrm{AdS}_5 \times S^5$ [118].

[4] In the BV formalism, including ghosts and anti-fields.

[5] We are not being careful about the degree (ghost number) of the fields since this will not be used in this chapter.

the complement of the support of the D2-branes:

$$\mathrm{d}_{\mathbb{R}^4\times\mathbb{C}} F_3(p) = 0\,, \quad p \notin \mathrm{D2}\,. \tag{2.5}$$

Here the differential is the de Rham differential for the entire space, i.e., $\mathrm{d}_{\mathbb{R}^4\times\mathbb{C}} = \mathrm{d}_{\mathbb{R}^4} + \overline{\partial}_{\mathbb{C}} + \partial_{\mathbb{C}}$. Moreover, as a dynamically determined background it is also constrained by the equation of motion (2.3). In addition to satisfying these equations, F_3 must also be translation invariant corresponding to the directions parallel to the D2-branes. The solution is:

$$F_3 = \frac{iN}{2\pi(v^2+y^2+z\overline{z})^2}(v\,\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}\overline{z} - y\,\mathrm{d}v\wedge\mathrm{d}z\wedge\mathrm{d}\overline{z} - 2\overline{z}\,\mathrm{d}v\wedge\mathrm{d}y\wedge\mathrm{d}z)\,. \tag{2.6}$$

In general, a closed string background like this might deform the theory on a brane, however, the pullback of the form (2.6) to the D4-branes vanishes:

$$\iota^* F_3 = 0\,, \tag{2.7}$$

where $\iota : \mathbb{R}^2_{x,y}\times\mathbb{C}_z \hookrightarrow \mathbb{R}^4_{v,w,x,y}\times\mathbb{C}_z$ is the embedding of the D4-branes into the entire space. So the closed string background leaves the D4-brane world-volume theory unaffected.[6]

The flux (2.6) signals a change in the topology of the closed string background:

$$\mathbb{R}^4_{v,w,x,y}\times\mathbb{C}_z \to \mathbb{R}^2_{w,x}\times\mathbb{R}_+\times S^3\,, \tag{2.8}$$

where the $\mathbb{R}_+$ is parametrized by $r := \sqrt{v^2+y^2+z\overline{z}}$. This change follows from requiring translation symmetry in the directions parallel to the D2-branes and the existence of an S^3 supporting the flux F_3. This S^3 is analogous to the S^5 in the D4-brane geometry supporting the 5-form flux sourced by the said D4-branes in Maldacena's AdS/CFT [118]. The coordinate r measures distance[7] from the location of the D2-branes. The $r\to 0$ region would be analogous to Maldacena's near horizon geometry. In our topological setting there is no distinction between near and distant, and we treat the entire $\mathbb{R}^2_{w,x}\times\mathbb{R}_+\times S^3$ as analogous to Maldacena's near horizon geometry. This makes $\mathbb{R}^2_{w,x}\times\mathbb{R}_+$ analogous to the AdS geometry. We recall that, in the AdS/CFT correspondence the location of the black

[6]The flux (2.6) is the only background turned on in the closed string theory. This can be argued as follows: The D2-branes introduce a 4-form source (the Poincaré dual to the support of the branes) in the closed string theory. This form can appear on the right hand side of the equation of motion (2.3) only for a 3-form field α, which can then have a non-trivial solution, as in (2.6). Furthermore, since the equation of motion (2.3) is free, the non-trivial solution for the 3-form field does not affect any other field.

[7]In the absence of a metric "distance" should be taken lightly. We really only distinguish between the two extreme cases, $r=0$ and $r=\infty$.

branes and the boundary of AdS correspond to two opposite limits of the non-compact coordinate transverse to the branes. In our case $r = 0$ corresponds to the location of the D2-branes, and we treat the plane at $r = \infty$, namely:

$$\mathbb{R}^2_{w,x} \times \{\infty\}\,, \tag{2.9}$$

as analogous to the asymptotic boundary of AdS.

The D4-branes in (2.1) appear as a defect in the closed string theory, they are analogous to the D5-branes that were considered in [85] or the D3-branes considered in [85, 86], in Maldacena's setup of AdS/CFT, where they were presented as holographic duals of Wilson loops in 4d $\mathcal{N} = 4$ super Yang-Mills. For the world-volume of these branes, the transformation (2.8) corresponds to:

$$\mathbb{R}^2_{x,y} \times \mathbb{C}_z \to \mathbb{R}_x \times \mathbb{R}_+ \times S^2\,, \tag{2.10}$$

where the $\mathbb{R}_+$ direction is parametrized by $r' := \sqrt{y^2 + z\bar{z}}$. The intersection of the boundary plane (2.9) and this world-volume is then the line:

$$\mathbb{R}_x \times \{\infty\}, \tag{2.11}$$

at infinity of r'. We draw a cartoon representing some aspects of the brane setup in figure 2.1.

We can now talk about two theories:

1. The 2d world-volume theory of the D2-branes. This is our analogue of the CFT (with a line operator) in AdS/CFT.
2. The effective[8] 3D theory on world-volume $\mathbb{R}^2_{w,x} \times \mathbb{R}_+$ with a defect supported on $\mathbb{R}_x \times \mathbb{R}_+$. This is our analogue of the gravitational theory in AdS background (with defect) in AdS/CFT.

To draw parallels once more with the traditional dictionary of AdS/CFT [118, 90, 158], we should establish a duality between the operators in the D2-brane world-volume theory and variations of boundary values of fields in the "gravitational" theory on $\mathbb{R}^2_{w,x} \times \mathbb{R}_+$ (the boundary is $\mathbb{R}_{w,x} \times \{\infty\}$). Both of these surfaces have a line operator/defect and this leads to two types of operators, ones that are restricted to the line, and others that can be placed

[8]Effective, in the sense that this is the Kaluza-Klein reduction of a 6d theory with three compact directions, though we don't want to loose any dynamics, i.e., we don't throw away massive modes.

anywhere. Local operators in a 2d surface are commuting, unless they are restricted to a line. Therefore, in both of our theories, we have non-commutative associative algebras whose centers consist of operators that can be placed anywhere in the 2d surface. For this chapter we are mostly concerned with the non-commuting operators:

1. Operators in the world-volume theory of the D2-branes that are restricted to the D2-D4 intersection.

2. Variations of boundary values of fields in the effective theory along the intersection (2.11) of the boundary $\mathbb{R}^2_{w,x} \times \{\infty\}$ and the defect on $\mathbb{R}_x \times \mathbb{R}_+$.

In physical string theory, the analogue of the D4-branes would be coupled to the closed string modes. In an appropriate low energy limit such gravitational couplings can be ignored, leading to the notion of *rigid holography* [1]. Since we are working with topological theory, we are assuming such a decoupling.

The computations in the "gravitational" side will be governed by the effective dynamics on the defect on $\mathbb{R}_x \times \mathbb{R}_+$. This is the Kaluza-Klein compactification of the world-volume theory of the D4-branes (with a line operator due to D2-D4 intersection). This 4d theory (which we describe in §2.2.5) is familiar from previous works such as [41]. Therefore we use the 4d dynamics, instead of the effective 2d one for our computations. In terms of Witten diagrams (which we compute in §2.4) this means that while we have a 1D boundary, the propagators are from the 4d theory and the bulk points are integrated over the 4d world-volume $\mathbb{R}^2 \times \mathbb{C}$. We take the boundary line to be at $y = \infty$ with some fixed coordinate z in the complex direction. In future we shall refer to this line as $\ell_\infty(z)$:

$$\ell_\infty(z) := \mathbb{R}_x \times \{y = \infty\} \times \{z\}\,. \tag{2.12}$$

A cartoon of our setup

Let us make a diagrammatic summary of our brane setup in Fig 2.1. In the figure we draw the non-compact part, namely $\mathbb{R}^2_{w,x} \times \mathbb{R}_+$, of the closed string background (the right hand side of (2.8)). We identify the location of the 2d black brane and the defect D4-branes, the asymptotic boundary $\mathbb{R}^2_{w,x} \times \{\infty\}$, and the intersection between the boundary and the defect. At the top of the picture, parallel to the asymptotic boundary, we also draw the D2-branes. We draw the D2-branes independently of the rest of the diagram because the D2-branes do not exist in the backreacted bulk, they become the black brane. However, traditionally, parallels are drawn between the asymptotic boundary and the brane sourcing

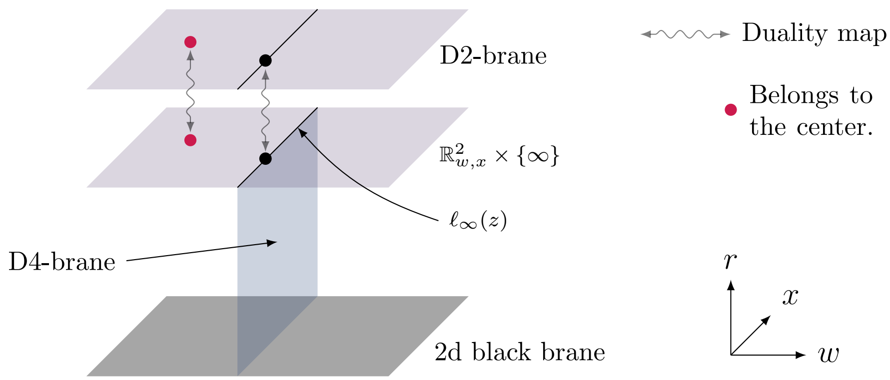


Figure 2.1: D2-brane, and the non-compact part of the backreacted bulk.

the bulk (the D2-brane in this case). The dots on the asymptotic boundary represent local variations of boundary values of fields in the bulk theory $\mathcal{T}_{\text{bk}}$. The corresponding dots on the D2-brane represent the local operators in the boundary theory $\mathcal{T}_{\text{bd}}$ that are dual to the aforementioned variations. By the duality map in the figure we schematically represent boundary excitations in the bulk theory corresponding to some local operators in the dual description of the same dynamics in terms of the boundary theory.

2.2.3 BF: The theory on D2-branes

This is a 2d topological gauge theory on the stack of N D2-branes (see (2.1)), supported on $\mathbb{R}^2_{w,x}$, with complexified gauge group GL_N. The field content of this theory is:

$$\begin{array}{cc} \text{Field} & \text{Valued in} \\ \hline \beta & \Omega^0(\mathbb{R}^2) \times \mathfrak{gl}_N \\ \alpha & \Omega^1(\mathbb{R}^2) \times \mathfrak{gl}_N \end{array} . \tag{2.13}$$

α is a Lie algebra valued connection and β is a Lie algebra valued scalar, both complex. The curvature of the connection is denoted as $\mathsf{F} = \mathrm{d}\alpha + \alpha \wedge \alpha$. The action is given by:

$$S_{\text{BF}} := \int_{\mathbb{R}^2_{w,x}} \mathrm{tr}_{\mathbf{N}}(\beta \mathsf{F}) \,, \tag{2.14}$$

where the trace is taken in the fundamental representation of $\mathfrak{gl}_N$.

We consider this theory in the presence of a line operator supported on $\mathbb{R}_x \times \{0\}$, caused by the intersection of the D2 and D4-branes. The line operator is defined by a fermionic quantum mechanical system living on it.[9] The fields in the quantum mechanics (QM) are K fundamental (of $\mathfrak{gl}_N$) fermions and their complex conjugates:

$$\begin{array}{cc} \text{Field} & \text{Valued in} \\ \hline \psi^i & \Omega^0(\mathbb{R}_x) \times \mathbf{N} \\ \overline{\psi}_i & \Omega^0(\mathbb{R}_x) \times \overline{\mathbf{N}} \end{array}\,, \qquad i \in \{1, \cdots, K\}\,, \tag{2.15}$$

where $\mathbf{N}$ refers to the fundamental representation of $\mathfrak{gl}_N$ and $\overline{\mathbf{N}}$ to the anti-fundamental. The fermionic system has a global symmetry $\mathrm{GL}_N \times \mathrm{GL}_K$. These fermions couple naturally to the $\mathfrak{gl}_N$ connection α of the BF theory. The action for the QM is given by:

$$S_{\mathrm{QM}} := \int_{\mathbb{R}_x} \left(\overline{\psi}_i \mathrm{d}\psi^i + \overline{\psi}_i \alpha \psi^i + \overline{\psi}_j A^j_i \psi^i \right)\,, \tag{2.16}$$

where we have introduced a *background* $\mathfrak{gl}_K$-valued gauge field $A \in \Omega^1(\mathbb{R}_x) \times \mathfrak{gl}_K$. Note that the terms in the above action are made $\mathfrak{gl}_N$ invariant by pairing up elements of $\mathbf{N}$ with elements of the dual space $\overline{\mathbf{N}}$.

Our first theory is this BF theory with the line operator, schematically:

$$\mathcal{T}_{\mathrm{bd}} := \mathrm{BF}_N \otimes_N \mathrm{QM}_{N\times K}\,, \tag{2.17}$$

where the subscripts on BF and QM refer to the symmetries (GL_N and $\mathrm{GL}_N \times \mathrm{GL}_K$ respectively) of the respective theories and the subscript on $\otimes$ implies that the GL_N is gauged. There are two types of gauge ($\mathfrak{gl}_N$) invariant operators in the theory:[10]

$$\text{for } n \in \mathbb{N}_{\geq 0}\,, \qquad \begin{array}{rl} \text{operators restricted to } \mathbb{R}_x\text{:} & O^i_j[n] := \frac{1}{\hbar}\overline{\psi}_j \beta^n \psi^i\,, \\ \text{operators } \textit{not} \text{ restricted to } \mathbb{R}_x\text{:} & O[n] := \frac{1}{\hbar}\mathrm{tr}_{\mathbf{N}} \beta^n\,. \end{array} \tag{2.18}$$

Unrestricted local operators in two topological dimensions can be moved around freely, implying that for any $n \geq 0$, the operator $O[n]$ commutes with all of the operators defined above.[11] The operator algebra of the 2d BF theory consists of all theses operators but for this chapter we focus on the non-commuting ones, in other words we, focus on the quotient

[9]This closely resembles the D3-D5 system in type IIB string theory considered in [85], there too a fermionic quantum mechanics lived on the intersection, giving rise to Wilson lines upon integrating out the fermions. Note that we could have considered bosons, instead of fermions, living on the line, without any significant change to our following computations. This would be similar to the D3-D3 system considered in [85, 86].

[10]The $\hbar^{-1}$ appears in these definitions because the action (2.16) will appear in path integrals as $\exp\left(-\hbar^{-1} S_{\mathrm{QM}}\right)$, which means functional derivatives with respect to A^i_j inserts operators that carry $\hbar^{-1}$.

[11]These operators are represented by the red dot on the D2-brane in figure 2.1.

of the full operator algebra of the boundary theory by its center.[12] We shall compute their Lie bracket in §2.3, which will establish an isomorphism with the Yangian. Had we included the commuting operators as well we would have found a *central extension* of the Yangian. In sum, the operator algebra we construct from the theory $\mathcal{T}_{\text{bd}}$ is:

$$\mathcal{A}^{\text{Op}}(\mathcal{T}_{\text{bd}}) := \left(O^i_j[n], O[n]\right)/(O[n])\,. \tag{2.19}$$

By the notation $(x, y, \cdots)$ we mean the algebra generated by the set of operators $\{x, y, \cdots\}$ over $\mathbb{C}$.

Remark 2.2.4 (A speculative link)*.* Note that it is possible to lift our D2 and D4 branes to type IIB string theory while maintaining a one dimensional intersection. This results in a D3-D5 setup (studied in particular in [85]) where on the D3 brane we find the $\mathcal{N} = 4$ Yang-Mills theory with a Wilson line.[13] In [57, 80, 78], the authors considered local operators in the $\mathcal{N} = 4$ Yang-Mills that are restricted to certain Wilson lines. With the proper choice of Wilson lines, Localization reduces this setup to 2d Yang-Mills theory with Wilson lines – local operator insertions along the Wilson lines in 4d reduce to local operator insertions along the Wilson lines in 2d [79]. 2d BF theory is the zero coupling limit of 2d Yang-Mills theory. We therefore expect the algebra constructed in this section to be related to the algebra constructed in the aforementioned references, at least in some limit.[14] The algebra in [78] would correspond to the $K = 1$ instance of our algebra, it may be an interesting check to compute the analogue of the algebra in [78] for higher K. △

2.2.5 4d Chern-Simons: The theory on D4-branes

This is a 4d gauge theory on the stack of K D4-branes, supported on $\mathbb{R}^2_{x,y} \times \mathbb{C}_z$ with the line $L := \mathbb{R}_x \times (0, 0, 0)$ removed and with the (complexified) gauge group GL_K. The notation of distinguishing directions by $\mathbb{R}$ and $\mathbb{C}$ is meant to highlight the fact that observables in this theory depend only on the topology of the real directions and depend holomorphically on the complex directions.[15] Due to the removed line, we can represent the topology of the support of this theory as (c.f. (2.10)):

$$M := \mathbb{R} \times \mathbb{R}_+ \times S^2\,. \tag{2.20}$$

[12] We shall similarly quotient out the center in the bulk theory as well.

[13] It is also interesting to note that the D5 brane in an Omega background reproduces the 4d CS theory [43].

[14] We thank Shota Komatsu for pointing out this interesting possibility.

[15] In particular, they are independent of the coordinates x and y that parametrize the $\mathbb{R}^2$, and depend holomorphically on z which parametrizes the $\mathbb{C}$.

The field of this theory is just a connection:

$$\begin{array}{cc} \text{Field} & \text{Valued in} \\ \hline A & \frac{\Omega^1(\mathbb{R}^2\times\mathbb{C}\backslash L)}{(\mathrm{d}z)}\otimes\mathfrak{gl}_K \end{array}\,. \tag{2.21}$$

The above notation simply means that A is a $\mathfrak{gl}_K$-valued 1-form without a $\mathrm{d}z$ component. The theory is defined by the action:

$$S_{\mathrm{CS}} := \frac{i}{2\pi}\int_M \mathrm{d}z\wedge \mathrm{CS}(A)\,, \tag{2.22}$$

where $\mathrm{CS}(A)$ refers to the standard Chern-Simons Lagrangian:

$$\mathrm{CS}(A) = \mathrm{tr}_{\mathbf{K}}\left(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right)\,, \tag{2.23}$$

where the trace is taken over the fundamental representation of $\mathfrak{gl}_K$. This theory is a 4d analogue of the, perhaps more familiar, 3D Chern-Simons theory. We shall therefore refer to it as the 4d Chern-Simons theory and sometimes denote it by CS^4_K or just CS.

The removal of the line L from $\mathbb{R}^2\times\mathbb{C}$ is caused by the D2-D4 brane intersection. Note that from the perspective of the CS theory, the D2-D4 intersection looks like a Wilson line. This means that we should be quantizing the CS theory on M with a background electric flux supported on the S^2 inside M. Alternatively, we can quantize the CS theory on $\mathbb{R}^4\times\mathbb{C}$ with a Wilson line inserted along L.[16] The choice of representation for the Wilson line is determined by the number, N, of D2-branes, let us denote this representation as $\varrho:\mathfrak{gl}_K\to V$. With this choice, the Wilson line is defined as the following operator:

$$W_\varrho(L) := P\exp\left(\int_L \varrho(A)\right)\,, \tag{2.24}$$

where $P\exp$ implies path ordered exponentiation, made necessary by the fact that the exponent is matrix valued. The above operator is valued in $\mathrm{End}(V)$. This in general means that the following expectation value:

$$\langle W_\varrho(L)\rangle = \frac{\int \mathcal{D}A\, W_\varrho(L)\exp\left(-\frac{1}{\hbar}S_{\mathrm{CS}}\right)}{\int\mathcal{D}A\,\exp\left(-\frac{1}{\hbar}S_{\mathrm{CS}}\right)}\,, \tag{2.25}$$

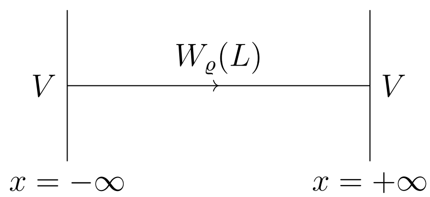
[16]Recall that in case of the BF theory the line operator at the D2-D4 intersection was described by a fermionic QM. We could do the same in this case. However, in this case it proves more convenient to integrate out the fermion, leaving a Wilson line in its place. The mechanism is the same that appeared for intersection of D3 and D5-branes in physical string theory [85].

is valued in $\mathrm{Hom}(\mathcal{H}_{-\infty} \otimes V, \mathcal{H}_{+\infty} \otimes V)$, where $\mathcal{H}_{\pm\infty}$ are the Hilbert spaces of the CS^4_K theory on the Cauchy surfaces perpendicular to L at $x = \pm\infty$, in the absence of the Wilson line. However, for the particular CS theory, these Hilbert spaces are trivial and we end up with a map that transports vectors in V from $x = -\infty$ to $x = +\infty$:

$$\langle W_\varrho(L) \rangle : V_{-\infty} \to V_{+\infty} \,. \tag{2.26}$$

In picture this operator may be represented as:

$\langle W_\varrho(L) \rangle$:



. (2.27)

The CS theory is quantized with some fixed boundary profile of the connection along the boundary $\mathbb{R}_x \times \{\infty\} \times S^2$.[17] To express the dependence of expectations values on this boundary value we put a subscript, such as $\langle W_\varrho(L) \rangle_A$. Since we are essentially interested in the Kaluza-Klein reduced theory on $\mathbb{R}_x \times \mathbb{R}_+$ we mostly care about the value of the connection along the boundary line (defined in (2.12)) $\ell_\infty(z) \subset \mathbb{R}_x \times \{\infty\} \times S^2$.

To define our second theory, we start with the product of the closed string theory and the CS theory, $\mathcal{T}_{\mathrm{cl}} \otimes \mathrm{CS}^4_K$, supported on $\mathbb{R}^2_{w,x} \times \mathbb{R}_+ \times S^3$ and compactify on S^3, our notation for this theory is the following:

$$\mathcal{T}_{\mathrm{bk}} := \pi^{S^3}_* \left(\mathcal{T}_{\mathrm{cl}} \otimes \mathrm{CS}^4_K \right) \,. \tag{2.28}$$

We can put the theory $\mathcal{T}_{\mathrm{bd}}$ (2.17) on the plane $\mathbb{R}_{w,x}$ "at infinity" of $\mathbb{R}^2_{w,x} \times \mathbb{R}_+$. This plane has a distinguished line $\mathbb{R}_x \times \{\infty\}$ (2.11) where the D4-brane world volume intersects.[18] Along this line we have the $\mathfrak{gl}_K$ gauge field which couples to the fermions of the QM in $\mathcal{T}_{\mathrm{bd}}$ (this coupling corresponds to the last term in (2.16)). Boundary excitations from arbitrary points on $\mathbb{R}_{w,x} \times \{\infty\}$ will correspond to operators in the BF theory that are commuting, since these local excitations on a plane are not ordered. The non-commutative algebra we are interested in in the BF theory is the algebra of operators restricted to a particular line. Similarly, in the "gravitational" side of the setup, we are interested in boundary excitations restricted to the line $\ell_\infty(z)$. Let us look a bit more closely at the coupling between the connection A and the fermions:

$$I_z := \frac{1}{\hbar} \int_{\ell_\infty(z)} \overline{\psi}^i A^j_i \psi_j \,, \qquad \ell_\infty(z) = \mathbb{R}_x \times \{y = \infty\} \times \{z\} \,. \tag{2.29}$$

[17] The boundary was chosen to respect the symmetry of the Wilson line along L.

[18] After aligning the v-coordinates of the plane and the D4-branes.

A small variation of z leads to coupling between the fermions and z-derivatives of the connection:

$$I_{z+\delta z} = \sum_{n=0}^{\infty} \frac{1}{\hbar} \int_{\ell_\infty(z)} \frac{(\delta z)^n}{n!} \overline{\psi}^i \partial_z^n A_i^j \psi_j \,. \tag{2.30}$$

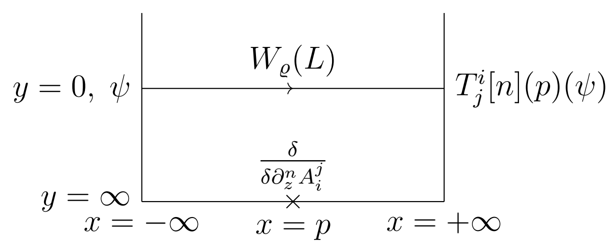
In the BF theory, the field β corresponds to the fluctuation of the D2-branes in the transverse $\mathbb{C}$ direction [30]. Therefore, we can interpret the above varied coupling term as saying that the operator in the boundary theory $\mathcal{T}_{\mathrm{bd}}$ that couples to the derivative $\partial_z^n A_i^j$ is precisely the operator $O_j^i[n] = \hbar^{-1} \overline{\psi}^i \beta^n \psi_j$ (c.f. (2.18), (2.19)). This motivates us to look at functional derivatives of $\langle W_\varrho(L) \rangle_A$ with respect to $\partial_z^n A_i^j$ at fixed points along $\ell_\infty(z)$, such as:

$$\frac{\delta}{\delta \partial_z^{n_1} A_{i_1}^{j_1}(p_1)} \cdots \frac{\delta}{\delta \partial_z^{n_m} A_{i_m}^{j_m}(p_m)} \langle W_\varrho(L) \rangle_A \,, \qquad p_1, \cdots, p_m \in \ell_\infty(z) \,. \tag{2.31}$$

Just as the expectation value $\langle W_\varrho(L) \rangle_A$ is $\mathrm{End}(V)$-valued, these functional derivatives are $\mathrm{End}(V)$-valued as well.[19] The action is given by applying the functional derivative on $\langle W_\varrho(L) \rangle_A (\psi)$ for any $\psi \in V$. Let us denote this operator as

$$\begin{gathered} T_j^i[n] : \ell_\infty(z) \times V \to V \,, \\ p \in \ell_\infty(z) \,, \qquad T_j^i[n](p) : \psi \mapsto \frac{\delta}{\delta \partial_z^n A_i^j(p)} \langle W_\varrho(L) \rangle_A (\psi) \,. \end{gathered} \tag{2.32}$$

which can be pictorially represented by slight modifications of (2.27):



(2.33)

Composition of these operators, such as $T_{j_1}^{i_1}(p_1) \cdots T_{j_m}^{i_m}(p_m)$, is defined by the expression (2.31). A more precise and computable characterization of these operators and their composition in terms of Witten diagrams [158] will be given in §2.4 (see (2.119)). Due to topological invariance along the x-direction, the operator $T_j^i[n](p)$ must be independent of the position p. However, since these operators are positioned along a line, their product

[19] After choosing a point along $\ell_\infty(z)$.

should be expected to depend on the ordering, leading to a non-commutative associative algebra. We can now define the second algebra to appear in our example of holography:

$$\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}}) := \left(T^i_j[n]\right) , \tag{2.34}$$

i.e., the complex algebra generated by the set $\{T^i_j[n]\}$.

Remark 2.2.6 (Center of the algebra). In the BF theory we mentioned gauge invariant operators that belong to the center of the algebra. Clearly, the holographic dual of those operators do not come from the CS theory, rather they come from the closed string theory. A 2-form field $\phi = \phi_{wx}\mathrm{d}w \wedge \mathrm{d}x + \cdots$ from the closed string theory deforms the BF theory as:

$$S_{\mathrm{BF}} \to S_{\mathrm{BF}} + \int_{\mathbb{R}^2_{w,x}} \mathrm{d}w \wedge \mathrm{d}x \left(\partial_z^n \phi_{wx}\right) \mathrm{tr}_{\mathbf{N}} \left(\beta^n\right) . \tag{2.35}$$

Functional derivatives with respect to the fields $\partial_z^n \phi_{w,x}$ placed at arbitrary locations on the asymptotic boundary $\mathbb{R}^2_{w,x} \times \{\infty\}$ correspond to inserting the operators $\mathrm{tr}_{\mathbf{N}}\beta^n$ in the BF theory.[20] As we did in the BF theory, we are going to ignore these operators now as well. △

After all this setup, we can present the main result of this chapter:

Theorem 2.2.7. *In the limit $N \to \infty$, both the algebra of local operators (2.19) along the line operator in the theory $\mathcal{T}_{\mathrm{bd}} = \mathrm{BF}_N \otimes_N \mathrm{QM}_{N\times K}$, and the algebra of scatterings from a line in the boundary (2.34) of the theory $\mathcal{T}_{\mathrm{bk}} = \pi_*^{S^3}\left(\mathcal{T}_{\mathrm{cl}} \otimes \mathrm{CS}^4_K\right)$ are isomorphic to the Yangian of $\mathfrak{gl}_K$, i.e.:*

$$\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}}) \overset{N\to\infty}{\cong} Y_\hbar(\mathfrak{gl}_K) \overset{N\to\infty}{\cong} \mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}}) . \tag{2.36}$$

The rest of the chapter is devoted to the explicit computations of these algebras.

2.3 $\mathcal{A}^{\mathrm{Op}}\left(\mathcal{T}_{\mathrm{bd}}\right)$ from $\mathrm{BF} \otimes \mathrm{QM}$ theory

In this section we prove the first half of our main result (Theorem 2.2.7):

Proposition 2.3.1. *The algebra $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$, defined in the context of 2d BF theory with the gauge group GL_N coupled to a 1D fermionic quantum mechanics with global symmetry $\mathrm{GL}_N \times \mathrm{GL}_K$, is isomorphic to the Yangian of $\mathfrak{gl}_K$ in the limit $N \to \infty$:*

$$\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}}) \overset{N\to\infty}{\cong} Y_\hbar(\mathfrak{gl}_K) . \tag{2.37}$$

[20]These functional derivatives are represented by the red dot on the asymptotic boundary in figure 2.1.

The BF theory coupled to a fermionic quantum mechanics was defined in §2.2.3, let us repeat the actions here:

$$S_{\mathcal{T}_{\text{bd}}} = S_{\text{BF}} + S_{\text{QM}}\,, \tag{2.38}$$

where:

$$S_{\text{BF}} = \int_{\mathbb{R}^2_{w,x}} \text{tr}_{\mathbf{N}}(\beta \text{d}\alpha + \beta[\alpha,\alpha]) \tag{2.39}$$

$$\text{and} \quad S_{\text{QM}} = \int_{\mathbb{R}_x} \left(\overline{\psi}_i \text{d}\psi^i + \overline{\psi}_i \alpha \psi^i\right)\,. \tag{2.40}$$

We no longer need the source term, i.e., the coupling to the background $\mathfrak{gl}_K$ connection (c.f. (2.16)). Let us determine the propagators now.

The BF propagator is defined as the 2-point correlation function:

$$\mathsf{P}^{\alpha\beta}(p,q) := \left\langle \beta^\alpha(p)\alpha^\beta(q)\right\rangle\,. \tag{2.41}$$

We choose a basis $\{\tau_\alpha\}$ of $\mathfrak{gl}_N$ which is orthonormal with respect to the trace $\text{tr}_{\mathbf{N}}$:

$$\text{tr}_{\mathbf{N}}(\tau_\alpha \tau_\beta) = \delta_{\alpha\beta}\,. \tag{2.42}$$

Then the two point correlation function becomes diagonal in the color indices:

$$\mathsf{P}^{\alpha\beta}(p,q) \equiv \delta^{\alpha\beta}\mathsf{P}(p,q)\,. \tag{2.43}$$

We shall often refer to just P as the propagator, it is determined by the following equation:[21]

$$\frac{1}{\hbar}\text{d}\mathsf{P}(0,p) = \delta^2(p)\text{d}w \wedge \text{d}x\,. \tag{2.44}$$

Once we impose the following gauge fixing condition:[22]

$$\text{d} \star \mathsf{P}(0,p) = 0\,, \tag{2.45}$$

the solution is (using translation invariance to replace the 0 with an arbitrary point):

$$\mathsf{P}(p,q) = \frac{\hbar}{2\pi}\,\text{d}\phi(p,q)\,, \tag{2.46}$$

[21] A minor technicality: $\mathsf{P}(p,q)$ is a 1-form on $\mathbb{R}^2_p \times \mathbb{R}^2_q$ and in (2.44), by $\mathsf{P}(0,p)$ we mean the pull-back of $\mathsf{P} \in \Omega^2(\mathbb{R}^4)$ by the diagonal embedding $\mathbb{R}^2 \hookrightarrow \mathbb{R}^2 \times \mathbb{R}^2$.

[22] This is the analogue of the Lorentz gauge.

where $\phi(p,q)$ is the angle (measured counter-clockwise) between the line joining p-q and any other reference line passing through p. In Feynman diagrams we shall represent this propagator as:

$$\mathsf{P}(p,q) = \; p \longrightarrow q \; . \tag{2.47}$$

Similarly, the propagator in the QM is defined by:

$$\frac{1}{\hbar}\partial_{x_2}\left\langle \overline{\psi}^a_i(x_1)\psi^j_b(x_2)\right\rangle = \delta^a_b\delta^j_i\delta^1(x_1-x_2)\,, \tag{2.48}$$

with the solution:

$$\left\langle \overline{\psi}^a_i(x_1)\psi^j_b(x_2)\right\rangle = \delta^a_b\delta^j_i\hbar\,\vartheta(x_2-x_1)\,, \tag{2.49}$$

where $\vartheta(x_2-x_1)$ is a unit step function. Anti-symmetry of the fermion fields dictates:

$$\left\langle \psi^j_b(x_1)\overline{\psi}^a_i(x_2)\right\rangle = -\left\langle \overline{\psi}^a_i(x_2)\psi^j_b(x_1)\right\rangle = -\delta^a_b\delta^j_i\hbar\,\vartheta(x_1-x_2)\,. \tag{2.50}$$

We take the step function to be:

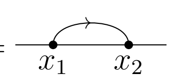
$$\vartheta(x) = \frac{1}{2}\mathrm{sgn}(x) = \begin{cases} 1/2 & \text{for } x>0 \\ 0 & \text{for } x=0 \\ -1/2 & \text{for } x<0 \end{cases}\;. \tag{2.51}$$

Then we can write:

$$\left\langle \overline{\psi}^a_i(x_1)\psi^j_b(x_2)\right\rangle = \left\langle \psi^j_b(x_1)\overline{\psi}^a_i(x_2)\right\rangle = \delta^a_b\delta^j_i\frac{\hbar}{2}\,\mathrm{sgn}(x_2-x_1)\,. \tag{2.52}$$

This propagator does not distinguish between ψ and $\overline{\psi}$ and it depends only on the order of the fields, not their specific positions. In Feynman diagrams we shall represent this propagator as:

$$\frac{\hbar}{2}\mathrm{sgn}(x_2-x_1) = \tag{2.53}$$



where the curved line refers to the propagator itself and the horizontal line refers to the support of the QM, i.e., the line $w=0$. We now move on to computing operator products that will give us the algebra $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$.

Remark 2.3.2 (Fermion vs. Boson - Propagator). We might as well have considered a bosonic QM instead of a fermionic QM. At present, this is an arbitrary choice, however, if one starts from some brane setup in physical string theory and reduce it to the topological setup we are considering by twists and Ω-deformations,[23] then depending on the starting setup one might end up with either statistics. Let us make a few comments about the bosonic case. In the first order formulation of bosonic QM the action looks exactly as in the fermionic action 2.40 except the fields would be commuting – let us denote the bosonic counterpart of $\overline{\psi}$ and ψ by $\overline{\phi}$ and ϕ respectively. Then, instead of the propagator (2.52), we would have the following propagator:[24]

$$- \left\langle \overline{\phi}_i^a(x_1)\phi_b^j(x_2) \right\rangle = \left\langle \phi_b^j(x_1)\overline{\phi}_i^a(x_2) \right\rangle = \delta_b^a \delta_i^j \frac{\hbar}{2} \mathrm{sgn}(x_2 - x_1)\,. \tag{2.54}$$

Note that the extra sign in the first term (compared to (2.52)) is consistent with the commutativity of the bosonic fields:

$$\left\langle \overline{\phi}_i^a(x_1)\phi_b^j(x_2) \right\rangle = \left\langle \phi_b^j(x_2)\overline{\phi}_i^a(x_1) \right\rangle\,. \tag{2.55}$$

The bosonic propagator (2.54) distinguishes between ϕ and $\overline{\phi}$, in that, the propagator is positive if $\phi(x_1)$ is placed before $\overline{\phi}(x_2)$, i.e., $x_1 < x_2$, and negative otherwise. △

2.3.3 Free theory limit, $\mathcal{O}(\hbar^0)$

Interaction in the quantum mechanics is generated via coupling to the $\mathfrak{gl}_N$ gague field (see (2.40)). Without this coupling, the quantum mechanics is free. In this section we compute the operator product between $O^i_j[m]$ and $O^k_l[n]$ in this free theory, which will give us the classical algebra.

Let us denote the operator product by $\star$, as in:

$$O^i_j[m] \star O^k_l[n]\,. \tag{2.56}$$

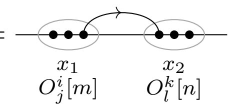
The classical limit of this product has an expansion in Feynman diagrams where we ignore all diagrams with BF propagators. Before evaluating this product let us illustrate the computations of the relevant diagrams by computing one exemplary diagram in detail.

[23] We describe one such specific procedure in §2.5.

[24] We have chosen the overall sign of the propagator to make comparision between Feynman diagrams involving bosonic operators and fermionic operators as simple as possible. However, the overall sign is not important for the determination of the algebra. The parameter $\hbar$ enters the algebra as the formal variable deforming the universal enveloping algebra $U(\mathfrak{gl}_K[z])$ to its Yangian, and the sign of $\hbar$ is irrelevant for this purpose.

Consider the following diagram:[25]

$$G^{ik}_{jl}[\triangle \cdot \blacktriangle](x_1, x_2) := \tag{2.57}$$



We are representing the operator $O^i_j[m] = \frac{1}{\hbar}\overline{\psi}^a_j(\beta^m)^b_a\psi^i_b$ by the symbol (•••) where the three dots represent the three fields $\overline{\psi}^a_j$, $(\beta^m)^b_a$, and ψ^i_b respectively. The coordinate below an operator in (2.57) represents the position of that operator and the lines connecting different dots are propagators. Depending on which dots are being connected a propagator is either the BF propagator (2.46) or the QM propagator (2.52). The value of the diagram is then given by:

$$\begin{aligned} G^{ik}_{jl}[\triangle \cdot \blacktriangle](x_1, x_2) &= \frac{1}{\hbar}\overline{\psi}^a_j(x_1)(\beta(x_1)^m)^b_a \frac{1}{2}\hbar\delta^c_b\delta^i_l \frac{1}{\hbar}(\beta(x_2)^n)^d_c\psi^k_d(x_2)\,, \\ &= \frac{1}{2\hbar}\delta^i_l\overline{\psi}_j(x_1)\beta(x_1)^m\beta(x_2)^n\psi^k(x_2)\,. \end{aligned} \tag{2.58}$$

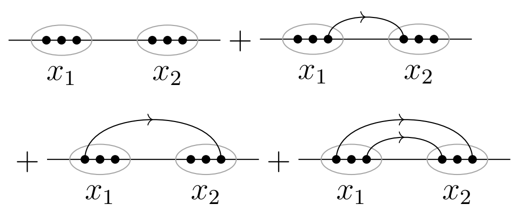
In the second line we have hidden away the contracted $\mathfrak{gl}_N$ indices. In computing the operator product (2.56) only the following limit of the diagram is relevant:

$$\lim_{x_2\to x_1} G^{ik}_{jl}[\triangle \cdot \blacktriangle](x_1, x_2) = \frac{1}{2\hbar}\delta^i_l\overline{\psi}_j\beta^{m+n}\psi^k = \frac{1}{2}\delta^i_l O^k_j[m+n]\,. \tag{2.59}$$

We have ignored the positions of the operators, because the algebra we are computing must be translation invariant. Reference to position only matters when we have different operators located at different positions.

We can now give a diagramatic expansion of the operator product (2.56) in the free theory:

$$O^i_j[m] \star O^k_l[n] \overset{x_2\to x1}{=} \tag{2.60}$$



We have omitted the labels for the operators in the diagrams. It is understood that the first operator is $O^i_j[m]$ and the second one is $O^k_l[n]$. Summing these four diagrams we find:

$$O^i_j[m] \star O^k_l[n] = O^i_j[m]O^k_l[n] + \frac{1}{2}\delta^i_l O^k_j[m+n] - \frac{1}{2}\delta^k_j O^i_l[m+n] + \frac{1}{4}\delta^i_l\delta^k_j \mathrm{tr}_{\mathbf{N}}\beta^{m+n}\,. \tag{2.61}$$

[25]The reader can ignore the elaborate symbols (triangles and as such) that we use to refer to a diagram. They are meant to systematically identify a diagram, but for practical purposes the entire expression can be thought of as an unfortunately long unique symbol assigned to a diagram, just to refer to it later on.

The product in the first term on the right hand side of the above equation is a c-number product, hence commuting. The sign of the third term comes from the first diagram in the second line in (2.60). In short, this comes about by commuting two fermions, as follows:

$$\lim_{x_2 \to x_1} G^{ik}_{jl}[\blacktriangle \cdot \triangle](x_1, x_2) = \frac{1}{2\hbar} \delta^k_j \psi^i \beta^{m+n} \overline{\psi}_l = -\frac{1}{2\hbar} \delta^k_j \overline{\psi}_l \beta^{m+n} \psi^i = -\frac{1}{2} \delta^k_j O^i_l[m+n] \,. \quad (2.62)$$

Using (2.61) we can compute the Lie bracket of the algebra $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$ in the classical limit:

$$\left[O^i_j[m], O^k_l[n]\right]_\star = \delta^i_l O^k_j[m+n] - \delta^k_j O^i_l[m+n] \,. \quad (2.63)$$

This is the Lie bracket in the loop algebra $\mathfrak{gl}_K[z]$.[26]

Remark 2.3.4 (Fermion vs. Boson - Classical Algebra). How would the bracket (2.63) be affected if we had a bosonic QM? It would not. The first and the fourth diagrams from (2.60) would still cancel with their counterparts when we take the commutator. The value of the second diagram, (2.59), remains unchanged. In computing the value of the third diagram (see (2.62)) we get an extra sign compared to the fermionic case because we don't pick up any sign by commuting bosonos, however, we pick up yet another sign from the propagator relative to the fermionic propagator (see Remark 2.3.2 – compare the bosonic (2.54) and fermionic (2.52) propagators).

2.3.5 Loop corrections from BF theory

Interaction in the BF theory comes from the following term in the BF action (2.39):

$$f_{\alpha\beta\gamma} \int_{\mathbb{R}^2} \beta^\alpha \alpha^\beta \wedge \alpha^\gamma \,, \quad (2.65)$$

where the structure constant $f_{\alpha\beta\gamma}$ comes from the trace in our orthonormal basis (2.42):

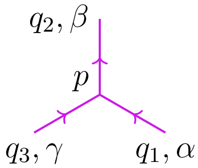
$$f_{\alpha\beta\gamma} = \mathrm{tr}_{\mathbf{N}}(\tau_\alpha[\tau_\beta, \tau_\gamma]) \,. \quad (2.66)$$

In Feynman diagrams this interaction will be represented by a trivalent vertex with exactly 1 outgoing and 2 incoming edges. Including the propagators for the edges, such a vertex

[26] The isomorphism is given by: $O^i_j[m] \mapsto z^m \mathrm{e}^j_i$, where e^j_i are the elementary matrices of dimension $K \times K$ satisfying the relation:

$$[\mathrm{e}^j_i, \mathrm{e}^l_k] = \delta^l_i \mathrm{e}^j_k - \delta^j_k \mathrm{e}^l_i \,. \quad (2.64)$$

will look like:

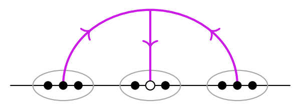


$$= \frac{\hbar^2}{(2\pi)^3} f^{\alpha\beta\gamma} \int_{p\in\mathbb{R}^2} \mathrm{d}_{q_1}\phi(p,q_1) \wedge \mathrm{d}_{q_2}\phi(p,q_2) \wedge \mathrm{d}_{q_3}\phi(p,q_3)\,, \qquad (2.67)$$

$$=: V^{\alpha\beta\gamma}(q_1,q_2,q_3)\,.$$

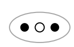
We have given the name $V^{\alpha\beta\gamma}$ to this vertex function.

Possibilities of Feynman diagrams are rather limited in the BF theory. In particular, there are no cycles.[27] This means that there is only one possible BF diagram that will appear in our computations, which is the following:



$$. \qquad (2.68)$$

The middle operator looks slightly different because this operator involves the connection α and an integration, as opposed to just the β field, to be specific,

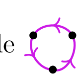


$$= \frac{1}{\hbar}\int_{\mathbb{R}} \overline{\psi}_i \alpha \psi^i\,. \qquad (2.69)$$

This term is the result of the insertion of the term coupling the fermions to the $\mathfrak{gl}_N$ connection in the QM action (2.40). In doing the above integrationover $\mathbb{R}$ we shall take $\overline{\psi}$ and ψ to be constant. In other words, we are taking derivatives of the fermions to be zero. The reason is that, the equations of motion for the fermions (derived from the action (2.40)), namely $\mathrm{d}\psi^i = -\mathsf{A}\psi^i$ and $\mathrm{d}\overline{\psi}_i = \mathsf{A}\overline{\psi}_i$, tell us that derivatives of the fermions are not gauge-invariant quantities – and we want to expand the operator product of gauge invariant operators in terms of other gauge invariant operators only.[28]

In the following we shall consider the diagram (2.68) with all possible fermionic propagators added to it.

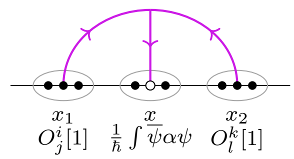
[27]By cycle we mean loop in the sense of graph theory. In this chapter when we write loop without any explanation, we mean the exponent of $\hbar$, as is customary in physics. This exponent is related but not always equal to the number of loops (graph theory). Therefore, we reserve the word loop for the exponent of $\hbar$, and the word cycle for what would be loop in graph theory.

Let us illustrate why there are no cycles in BF Feynman diagrams. Consider the cycle  . The three propagators in the cycle contribute the 3-form $\mathrm{d}\phi_1 \wedge \mathrm{d}\phi_2 \wedge \mathrm{d}\phi_3$ to a diagram containing the cycle, where the ϕ's are the angles between two successive vertices. However, due to the constraint $\phi_1+\phi_2+\phi_3 = 2\pi$, only two out of the three propagators are linearly independent. Therefore, their product vanishes.

[28]An alternative, and perhaps more streamlined, way to say this would be to formulate all the theories in the BV/BRST formalism, where operators are defined, a priori, to be in the cohomology of the BRST operator, which would exclude derivatives of the fermions to begin with.

0 **fermionic propagators**

We are mostly going to compute products of level 1 operators, i.e., $O^i_j[1]$, this is because together with the level 0 operators, they generate the entire algebra. Without any fermionic propagators, we just have the diagram (2.68):

$$G^{ik}_{jl}[\cdot\cdot](x_1, x_2) :=$$ 

. (2.70)

In future, we shall omit the labels below the operators to reduce clutter. In terms of the BF vertex function (2.67), the above diagram can be expressed as:

$$G^{ik}_{jl}[\cdot\cdot](x_1, x_2) = \frac{1}{\hbar^3}\overline{\psi}_j\tau_\alpha\psi^i\overline{\psi}\tau_\beta\psi\overline{\psi}_l\tau_\gamma\psi^k \int_{\mathbb{R}_x} V^{\alpha\beta\gamma}(x_1, x, x_2)\,. \tag{2.71}$$

We have used the expansions of $\beta = \beta^\alpha\tau_\alpha$ and $\alpha = \alpha^\beta\tau_\beta$ in the orthonormal $\mathfrak{gl}_N$ basis $\{\tau_\alpha\}$. As defined in (2.67), the vertex function $V^{\alpha\beta\gamma}$ is a 2d integral of a 3-form, therefore, the integration of the vertex function on a line gives us a number. It will be convenient to divide up the integral of the vertex function into three integrals depending on the location of the point x relative to x_1 and x_2:

$$\int_{\mathbb{R}_x} V^{\alpha\beta\gamma}(x_1, x, x_2) = \mathcal{V}^{\alpha\beta\gamma}_{\cdot||}(x_1, x_2) + \mathcal{V}^{\alpha\beta\gamma}_{|\cdot|}(x_1, x_2) + \mathcal{V}^{\alpha\beta\gamma}_{||\cdot}(x_1, x_2)\,, \tag{2.72}$$

where,

$$\mathcal{V}^{\alpha\beta\gamma}_{\cdot||}(x_1, x_2) := \int_{x<x_1} V^{\alpha\beta\gamma}(x_1, x, x_2) = \frac{\hbar^2}{24} f^{\alpha\beta\gamma}\,, \tag{2.73a}$$

$$\mathcal{V}^{\alpha\beta\gamma}_{|\cdot|}(x_1, x_2) := \int_{x_1<x<x_2} V^{\alpha\beta\gamma}(x_1, x, x_2) = \frac{\hbar^2}{24} f^{\alpha\beta\gamma}\,, \tag{2.73b}$$

$$\mathcal{V}^{\alpha\beta\gamma}_{||\cdot}(x_1, x_2) := \int_{x_2<x} V^{\alpha\beta\gamma}(x_1, x, x_2) = \frac{\hbar^2}{24} f^{\alpha\beta\gamma}\,. \tag{2.73c}$$

We evaluate these integrals in Appendix §A.1. Adding them up and substituting in (2.71) we get from the diagram (2.70):

$$G^{ik}_{jl}[\cdot\cdot](x_1, x_2) \overset{x_1\to x_2}{=} \frac{1}{8\hbar}\overline{\psi}_j\tau_\alpha\psi^i\overline{\psi}\tau_\beta\psi\overline{\psi}_l\tau_\gamma\psi^k f^{\alpha\beta\gamma}\,. \tag{2.74}$$

Since the $\mathfrak{gl}_N$ indices are all contracted, we can choose a particular basis to get an expression independent of any reference to $\mathfrak{gl}_N$. Choosing the elementary matrices as the basis we get the following expression:

$$G^{ik}_{jl}[\cdot\cdot] = \frac{\pi^2}{2\hbar}\overline{\psi}_j \mathrm{e}^a_b \psi^i \overline{\psi} \mathrm{e}^c_d \psi \overline{\psi}_l \mathrm{e}^e_f \psi^k f^{bdf}_{ace}\,. \tag{2.75}$$

Using the definition of the elementary matrices $(\mathrm{e}^a_b)^c_d = \delta^a_d \delta^c_b$ we get $\overline{\psi}_j \mathrm{e}^a_b \psi^i = \overline{\psi}^d_j\,(\mathrm{e}^a_b)^c_d\,\psi^i_c = \overline{\psi}^a_j \psi^i_b$ and in this basis the structure constant is:

$$f^{bdf}_{ace} = \delta^d_a \delta^f_c \delta^b_e - \delta^b_c \delta^d_e \delta^f_a\,. \tag{2.76}$$

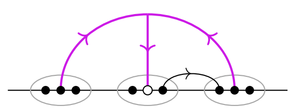
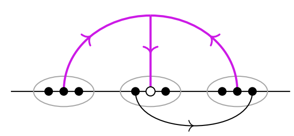
Using these expressions in (2.75) we get:

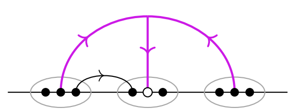
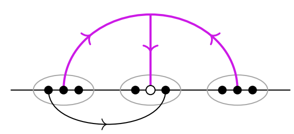
$$\begin{aligned} G^{ik}_{jl}[\cdot\cdot] &= \frac{1}{8\hbar}\left(\overline{\psi}_j \psi^m\,\overline{\psi}_m \psi^k\,\overline{\psi}_l \psi^i - \overline{\psi}_l \psi^m\,\overline{\psi}_m \psi^i\,\overline{\psi}_j \psi^k\right)\,, \\ &= \frac{1}{8}\hbar^2\left(O^m_j[0] O^k_m[0] O^i_l[0] - O^m_l[0] O^i_m[0] O^k_j[0]\right)\,. \end{aligned} \tag{2.77}$$

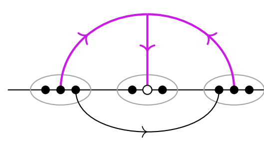
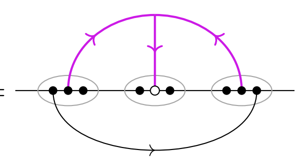
The above expression is anti-symmetric under the exchange $(i,j) \leftrightarrow (k,l)$, therefore, the contribution of this diagram to the Lie bracket (2.63) is twice the value of the diagram.

1 fermionic propagator

We have the following six diagrams:

$G^{ik}_{jl}[\cdot \triangle \cdot \blacktriangle] =$  , $G^{ik}_{jl}[\cdot \blacktriangle \cdot \triangle] =$  ,

$G^{ik}_{jl}[\triangle \cdot \blacktriangle \cdot] =$  , $G^{ik}_{jl}[\blacktriangle \cdot \triangle \cdot] =$  , (2.78)

$G^{ik}_{jl}[\triangle \cdot \cdot \blacktriangle] =$  , $G^{ik}_{jl}[\blacktriangle \cdot \cdot \triangle] =$  .

In all the above diagrams, the left and the right most operators are $O^i_j[1]$ and $O^k_l[1]$ respectively, and all the graphs are functions of x_1 and x_2, where these two operators are located. Let us explain the evaluation of the top left diagram in detail. Written explicitly, this diagram is:

$$G^{ik}_{jl}[\cdot\triangle\cdot\blacktriangle](x_1,x_2) = \frac{1}{\hbar^3}\int_{\mathbb{R}_x} \overline{\psi}_j(x_1)\tau_\alpha\psi^i(x_1)\overline{\psi}^a_m(x)\,(\tau_\beta)^b_a\left\langle \psi^m_b(x)\overline{\psi}^c_l(x_2)\right\rangle \\ \times(\tau_\gamma)^d_c\,\psi^k_d(x_2)\,V^{\alpha\beta\gamma}(x_1,x,x_2)\,, \tag{2.79}$$

where the two point correlation function is the QM propagator (2.52). The integrand above depends on the position only to the extend that they depend on the ordering of the positions, since we are only quantizing the constant modes of the fermions.[29] The propagator between the two fermions gives a propagator which depends on the sign of $x_2 - x$ (see (2.52), (2.53)), since we are integrating over x, this propagator will change sign depending on whether x is to the left or to the right of x_2.[30] Therefore, we can write this graph as:

$$\begin{aligned} G^{ik}_{jl}[\cdot\triangle\cdot\blacktriangle] &= \frac{1}{\hbar^2}\overline{\psi}_j\tau_\alpha\psi^i\overline{\psi}_l\tau_\beta\tau_\gamma\psi^k\left(\mathcal{V}^{\alpha\beta\gamma}_{\cdot||}+\mathcal{V}^{\alpha\beta\gamma}_{|\cdot|}-\mathcal{V}^{\alpha\beta\gamma}_{||\cdot}\right)\,,\\ &= \frac{1}{24}\overline{\psi}_j\tau_\alpha\psi^i\overline{\psi}_l\tau_\beta\tau_\gamma\psi^k\,f^{\alpha\beta\gamma} = \frac{1}{24}\overline{\psi}_j\tau_\alpha\psi^i\overline{\psi}_l\tau_\delta\psi^k\,f_{\beta\gamma}{}^{\delta}f^{\alpha\beta\gamma}\,. \end{aligned} \tag{2.80}$$

Due to the symmetry $f_{\beta\gamma}{}^{\delta}f^{\alpha\beta\gamma} = f_{\beta\gamma}{}^{\alpha}f^{\delta\beta\gamma}$, the above expression is symmetric under the exchange $(i,j)\leftrightarrow(k,l)$, therefore this diagram does not contribute to the Lie bracket (2.63). The diagrams $G^{ik}_{jl}[\cdot\blacktriangle\cdot\triangle]$, $G^{ik}_{jl}[\triangle\cdot\blacktriangle\cdot]$, and $G^{ik}_{jl}[\blacktriangle\cdot\triangle\cdot]$ do not contribute to the Lie bracket for exactly the same reason. The remaining two diagrams evaluate to the following expressions:

$$G^{ik}_{jl}[\triangle\cdot\cdot\blacktriangle] = \frac{1}{8\hbar}f^{\alpha\beta\gamma}\delta^i_l\overline{\psi}_j\tau_\alpha\tau_\gamma\psi^k\overline{\psi}\tau_\beta\psi\,, \tag{2.81a}$$

$$G^{ik}_{jl}[\blacktriangle\cdot\cdot\triangle] = -\frac{1}{8\hbar}f^{\alpha\beta\gamma}\delta^k_j\overline{\psi}_l\tau_\gamma\tau_\alpha\psi^i\overline{\psi}\tau_\beta\psi\,. \tag{2.81b}$$

Their sum is symmetric under the exchange $(i,j)\leftrightarrow(k,l)$,[31] and therefore these diagrams do not contribute to the Lie bracket either.

None of the diagrams with one fermionic propagator contributes to the Lie bracket.

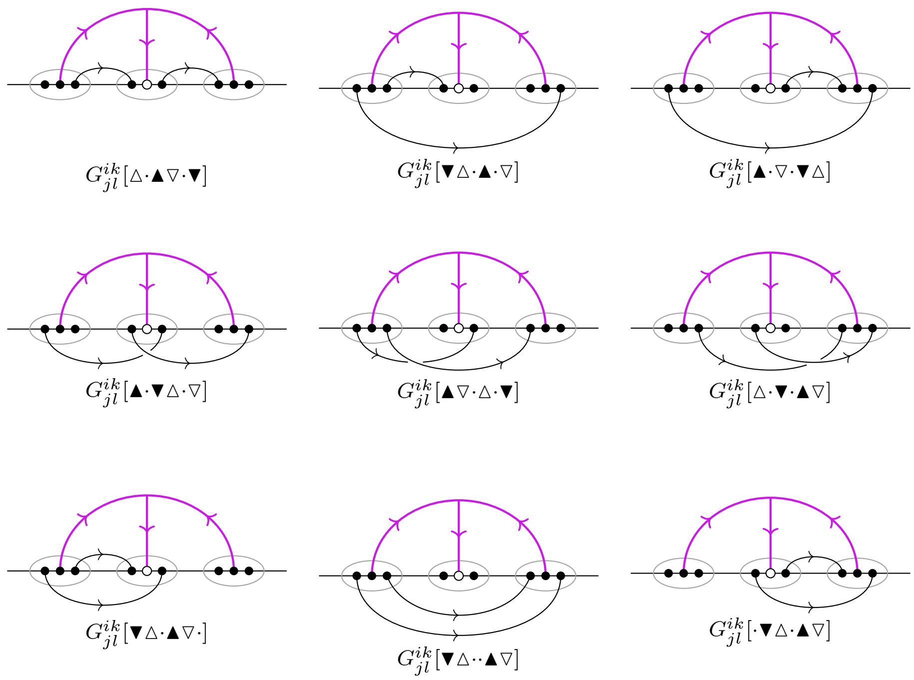
[29]Derivatives of the fermions are not gauge invariant.

[30]This is the reason why we computed the integrals (2.73) separately depending on the position of x.

[31]The opposite ordering of τ_α and τ_γ cancels the sign, using the anti-symmetry of the indices on the structure constant.

2 **fermionic propagators**

There are nine ways to join two pairs of fermions with propagators:



(2.82)

The left and the right most operators in all of the above diagrams are $O^i_j[1]$ and $O^k_l[1]$ respectively.

All three of the diagrams in the bottom line vanish. This is because joining all the fermions in two operators with propagators introduces a trace $\mathrm{tr}_{\mathbf{N}}(\tau_\alpha\tau_\beta)$ of $\mathfrak{gl}_N$ generators when the same color indices, α and β in this case, are contracted with the structure constant coming form the BF interaction vertex, as in $\mathrm{tr}_{\mathbf{N}}(\tau_\alpha\tau_\beta)f^{\alpha\beta\gamma}$. Since the trace is symmetric and the structure constant is anti-symmetric, these three diagrams vanish.

Computation also reveals the following relations:[32]

$$G^{ik}_{jl}[\blacktriangledown\,\triangle\cdot\blacktriangle\triangledown] = G^{ik}_{jl}[\blacktriangle\cdot\triangledown\cdot\blacktriangledown\,\triangle]\,,\quad G^{ik}_{jl}[\blacktriangle\triangledown\cdot\triangle\cdot\blacktriangledown] = G^{ik}_{jl}[\triangle\cdot\blacktriangledown\cdot\blacktriangle\triangledown]\,, \tag{2.83}$$

together with the fact that $G^{ik}_{jl}[\blacktriangledown\,\triangle\cdot\blacktriangle\cdot\triangledown]+G^{ik}_{jl}[\blacktriangle\triangledown\cdot\triangle\cdot\blacktriangledown]$ is symmetric under the exchange $(i,j)\leftrightarrow(k,l)$. The above relations and symmetry implies that when anti-symmetrized with respect to $(i,j)\leftrightarrow(k,l)$, the sum of the four diagrams appearing in the above relations

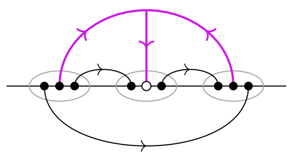
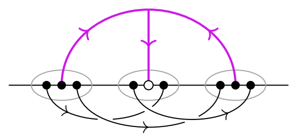
[32] Among the four diagrams at the top right 2×2 corner of (2.82).

vanish. In a similar vein, the sum $G^{ik}_{jl}[\vartriangle \cdot \blacktriangle \triangledown \cdot \blacktriangledown] + G^{ik}_{jl}[\blacktriangle \cdot \blacktriangledown\ \vartriangle \cdot \triangledown]$ also turns out to be symmetric under $(i,j) \leftrightarrow (k,l)$ and therefore these two diagrams do not contribute to the Lie bracket either.

None of the diagrams with two fermionic propagators contributes to the Lie bracket.

3 fermionic propagators

There are two ways to join all the fermions with propagators:

 ,  (2.84)

As before, the left and the right most operators are $O^i_j[1]$ and $O^k_l[1]$ respectively. Both of these diagrams are proportional to $\delta^i_l \delta^k_j$, in particular, they are symmetric under the exchange $(i,j) \leftrightarrow (k,l)$, and therefore do not contribute to the Lie bracket.

Lie bracket

Since only the diagram with zero fermionic propagator (2.77) survives the anti-symmetrization, the Lie bracket (2.63) up to $\mathcal{O}(\hbar^2)$ corrections becomes:

$$\begin{aligned}\left[O^i_j[1], O^k_l[1]\right]_\star &= \delta^i_l O^k_j[2] - \delta^k_j O^i_l[2] + G^{ik}_{jl}[\cdot\cdot] - G^{ki}_{lj}[\cdot\cdot]\,, \\ &= \delta^i_l O^k_j[2] - \delta^k_j O^i_l[2] + \frac{\hbar^2}{4}\left(O^m_j[0]O^k_m[0]O^i_l[0] - O^m_l[0]O^i_m[0]O^k_j[0]\right)\,. \end{aligned} \tag{2.85}$$

Though we have only computed up to 2-loops diagrams, this result is exact, because there are no more non-vanishing Feynman diagrams that can be drawn.

Since (2.85) is not among the standard relations of the Yangian that are readily available in the literature, we shall now make a change of basis to get to a standard relation. First note that, the product of operators in the right hand side of the above equation is not the operator product, this product is commutative (anti-commutative for fermions) and therefore we can write it in an explicitly symmetric form, such as:

$$O^m_j[0]O^k_m[0]O^i_l[0] = \left\{O^m_j[0], O^k_m[0], O^i_l[0]\right\}\,, \tag{2.86}$$

where the bracket means complete symmetriazation, i.e., for any three symbols O_1, O_2 and O_3 with a product we have:

$$\{O_1, O_2, O_3\} = \frac{1}{3!} \sum_{s \in S_3} O_{s(1)} O_{s(2)} O_{s(3)} \,, \tag{2.87}$$

where S_3 is the symmetric group of order 3!. With this symmetric bracket, let us now define:

$$Q^{ik}_{jl} := f^{iun}_{jvm} f^{vpq}_{uor} f^{rtk}_{qsl} \left\{ O^m_n[0], O^o_p[0], O^s_t[0] \right\} \,, \tag{2.88}$$

where f^{ijk}_{lmn} are the $\mathfrak{gl}_K$ structure constants in the basis of elementary matrices. Using the form of the $\mathfrak{gl}$ structure constant in the basis of elementary matrices (c.f. (2.76)) we can write:

$$Q^{ik}_{jl} = 3 \left\{ O^i_l, O^m_j, O^k_m \right\} - 3 \left\{ O^k_j, O^m_l, O^i_m \right\} + \delta^k_j \left\{ O^m_l, O^n_m, O^i_n \right\} - \delta^i_l \left\{ O^m_j, O^n_m, O^k_n \right\} \,. \tag{2.89}$$

We have ignored to write the [0] for each of the operators. Using the above expression we can re-write (2.85) as:

$$\left[O^i_j[1], O^k_l[1] \right]_\star = \delta^i_l \widetilde{O}^k_j[2] - \delta^k_j \widetilde{O}^i_l[2] + \frac{\hbar^2}{12} Q^{ik}_{jl} \,, \tag{2.90}$$

with the redefinition:

$$\widetilde{O}^k_j[2] := O^k_j[2] - \frac{\hbar^2}{12} \left\{ O^m_j, O^n_m, O^k_n \right\} \,. \tag{2.91}$$

Note that, $\left\{ O^m_j, O^n_m, O^k_n \right\}$ does indeed transform as an element of $\mathfrak{gl}_K$, since it only has a pair of fundamental-anti-fundamental $\mathfrak{gl}_K$ indices free. This makes the redefinition of $O^k_j[2]$ possible. The Lie bracket (2.90) is how the Yangian was presented in [41].

Remark 2.3.6 (Fermion vs. Boson - Quantum Algebra)*.* In Remark 2.3.4 we pointed out that the classical part of the algebra (2.90) remains unchanged if we replace the fermionic QM on the defect with a bosonic QM. This remains true at the quantum level – though a bit tedious, it can be readily verified by using the bosonic propagator (2.54) and keeping track of signs through the computations of this section without any other modifications. △

2.3.7 Large N limit: The Yangian

For finite N, there are some extra relations among the operators $O^i_j[n]$ that are not part of the Yangian algebra. These relations are simply a result of having finite dimensional matrices. We start by noting that the operators $O^i_j[m]$ act on the Hilbert space $\mathcal{H}^{\text{fer}}_{\text{QM}}$ of the quantum mechanics. This is a finite dimensional Hilbert space constructed by acting with the fermionic zero modes on the vacuum of the theory:

$$\mathcal{H}^{\text{fer}}_{\text{QM}} = \mathbb{C}|\Omega\rangle \oplus \bigoplus_{i,a} \mathbb{C}\psi^i_a|\Omega\rangle \oplus \bigoplus_{i,j,a,b} \mathbb{C}\psi^i_a\psi^j_b|\Omega\rangle + \cdots . \tag{2.92}$$

Considering the GL_N and GL_K indices on the fermions this Hilbert space can be decomposed into tensor products of representations of GL_K and GL_N as follows (see (A.9)):

$$\mathcal{H}^{\text{fer}}_{\text{QM}} = \bigoplus_{Y} \mathcal{H}^N_{Y^T} \otimes \overline{\mathcal{H}^K_Y}\,, \tag{2.93}$$

where Y is a Young tableaux, Y^T is the transpose of Y, $\mathcal{H}^N_{Y^T}$ (resp. $\mathcal{H}^K_Y$) is the GL_N (resp. GL_K) representation associated to the tableaux Y^T (resp. Y), and a bar over a representation denotes its dual. Any $d \times d$ matrix X satisfies a degree d polynomial equation:[33]

$$X^d = \sum_{i=0}^{d-1} c_i X^i\,. \tag{2.94}$$

Therefore, all the operators $O^i_j[m]$ satisfy some polynomial equation of degree $\dim \mathcal{H}^{\text{fer}}_{\text{QM}}$. Since the matrix B is an $N \times N$ matrix there are relations among its different powers, which can lead to relations among operators of the QM as well. In the limit $N \to \infty$ we do not need to worry about such truncations of the Yangian and we have the full Yangian. This positively concludes the first half of our main result (Theorem 2.2.7).

Remark 2.3.8 (Fermion vs. Boson – Hilbert Space). The Hilbert space as a representation of $\mathrm{GL}_N \times \mathrm{GL}_K$ differs between the fermionic description of the defect QM and the bosonic description. The fermionic Hilbert space (2.93) is finite dimensional because of the antisymmetry of the fermionic generators. There is no such exclusion principle for the bosons and the bosonic Hilbert space is infinite dimensional. The bosonic Hilbert space is (see (A.13)):

$$\mathcal{H}^{\text{bos}}_{\text{QM}} = \bigoplus_{Y} \mathcal{H}^N_{Y} \otimes \overline{\mathcal{H}^K_Y}\,, \tag{2.95}$$

where $\mathcal{H}^N_Y$ and $\mathcal{H}^K_Y$ are representations of GL_N and GL_K denoted by the same tableaux Y. △

[33] The relation between the coefficients appearing in (2.94) and X is the following [152]: if X has the characteristic polynomial $\sum_{i=0}^{d} a_i x^{d-i}$ with $a_0 = 1$ and u_i satisfy the recurrence relation $\sum_{i=0}^{d} a_i u_{d-i} = 0$, then $c_i = \sum_{j=i}^{d-1} a_{j-i} u_{d-j}$.

2.4 $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ from 4d Chern-Simons Theory

In this section we prove the second half of our main result (Theorem 2.2.7):

Proposition 2.4.1. *The algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$, defined in (2.34) in the context of 4d Chern-Simons theory, is isomorphic to the Yangian $Y_\hbar(\mathfrak{gl}_K)$:*

$$\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}}) \overset{N\to\infty}{\cong} Y_\hbar(\mathfrak{gl}_K)\,. \tag{2.96}$$

The 4d Chern-Simons theory with gauge group GL_K, also denoted by CS^4_K, is defined by the action (2.22), which we repeat here for convenience:

$$S_{\mathrm{CS}} := \frac{i}{2\pi}\int_{\mathbb{R}^2_{x,y}\times\mathbb{C}_z} \mathrm{d}z\wedge \mathrm{tr}_{\mathbf{K}}\left(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right)\,. \tag{2.97}$$

The trace in the fundamental representation defines a positive-definite metric on $\mathfrak{gl}_K$, moreover, we choose a basis of $\mathfrak{gl}_K$, denoted by $\{t_\mu\}$, in which the metric becomes diagonal:

$$\mathrm{tr}_{\mathbf{K}}(t_\mu t_\nu) \propto \delta_{\mu\nu}\,. \tag{2.98}$$

We consider this theory in the presence of a Wilson line in some representation $\varrho : \mathfrak{gl}_K \to \mathrm{End}(V)$, supported along the line L defined by $y = z = 0$:

$$W_\varrho(L) = P\exp\left(\int_L \varrho(A)\right)\,. \tag{2.99}$$

Consideration of fusion of Wilson lines to give rise to Wilson lines in tensor product representation shows that it is not only the connection A that couples to a Wilson line but also its derivatives $\partial_z^n A$ [41]. Furthermore, gauge invariance at the classical level requires that $\partial_z^n A$ couples to the Wilson line via a representation of the loop algebra $\mathfrak{gl}_K[z]$. So the line operator that we consider is the following:

$$P\exp\left(\sum_{n\geq 0}\varrho_{\mu,n}\int_L \partial_z^n A^\mu\right)\,, \tag{2.100}$$

where the matrices $\varrho_{\mu,n}\in \mathrm{End}(V)$ satisfy:

$$[\varrho_{\mu,m},\varrho_{\nu,n}] = f_{\mu\nu}{}^{\xi}\varrho_{\xi,m+n}\,. \tag{2.101}$$

The structure constant ${f_{\mu\nu}}^{\xi}$ is that of $\mathfrak{gl}_K$. In particular, we have $\varrho_{\mu,0} = \varrho(t_\mu)$.

In (2.21), A was defined to not have a $\mathrm{d}z$ component. The reason is that, due to the appearance of $\mathrm{d}z$ in the above action (2.97), the $\mathrm{d}z$ component of the connection A never appears in the action anyway.[34]

Though the theory is topological, in order to do concrete computations, such as imposing gauge fixing conditions, computing propagator, and evaluating Witten diagrams etc. we need to make a choice of metric on $\mathbb{R}^2_{x,y} \times \mathbb{C}_z$, we choose:[35]

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z\mathrm{d}\overline{z}\,. \tag{2.104}$$

For the GL_K gauge symmetry we use the following gauge fixing condition:

$$\partial_x A_x + \partial_y A_y + 4\partial_z A_{\overline{z}} = 0\,. \tag{2.105}$$

The propagator is defined as the two-point correlation function:

$$P^{\mu\nu}(v_1, v_2) := \langle A^\mu(v_1) A^\nu(v_2)\rangle\ . \tag{2.106}$$

Since in the basis of our choice the Lie algebra metric is diagonal (2.98), this propagator is proportional to a Kronecker delta in the Lie algebra indices:

$$P^{\mu\nu}(v_1, v_2) = \delta^{\mu\nu} P(v_1, v_2)\,, \tag{2.107}$$

where P is a 2-form on $\mathbb{R}^4_{v_1} \times \mathbb{R}^4_{v_2}$. We can fix one of the coordinates to be the origin, this amounts to taking the projection:

$$\varpi : \mathbb{R}^4_{v_1} \times \mathbb{R}^4_{v_1} \to \mathbb{R}^4_v\,, \qquad \varpi : (v_1, v_2) \mapsto v_1 - v_2 =: v\,. \tag{2.108}$$

Due to translation invariance, P can be written as a pullback of some 2-form on $\mathbb{R}^4$ by ϖ, i.e., $P = \varpi^* \overline{P}$ for some $\overline{P} \in \Omega^2(\mathbb{R}^4)$. The propagator P can be characterized as the

[34]Had we defined the space of connections to be $\Omega^1(\mathbb{R}^2_{x,y} \times \mathbb{C}_z) \otimes \mathfrak{gl}_K$, then, in addition to the usual GL_K gauge symmetry, we would have to consider the following additional gauge transformation:

$$A \to A + f\mathrm{d}z\,, \tag{2.102}$$

for arbitrary function $f \in \Omega^0(\mathbb{R}^2 \times \mathbb{C})$. We could fix this gauge by imposing:

$$A_z = 0\,. \tag{2.103}$$

This would get us back to the space $\big(\Omega^1(\mathbb{R}^2_{x,y} \times \mathbb{C}_z)/(\mathrm{d}z)\big) \otimes \mathfrak{gl}_K$.

[35]For this theory we follow the choices of [41] whenever possible.

Green's function for the differential operator $\frac{i}{2\pi\hbar}\mathrm{d}z \wedge \mathrm{d}$ that appears in the kinetic term of the action S_{CS}. For $\overline{P}$ this results in the following equation:

$$\frac{i}{2\pi\hbar}\mathrm{d}z \wedge \mathrm{d}\overline{P}(v) = \delta^4(v)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}\overline{z}\,, \tag{2.109}$$

The propagator P, and in turns $\overline{P}$, must also satisfy the gauge fixing condition (2.105):

$$\partial_x \overline{P}_x + \partial_y \overline{P}_y + 4\partial_z \overline{P}_{\overline{z}} = 0\,. \tag{2.110}$$

The solution to (2.109) and (2.110) is given by:

$$\overline{P}(x, y, z, \overline{z}) = \frac{\hbar}{2\pi}\frac{x\,\mathrm{d}y \wedge \mathrm{d}\overline{z} + y\,\mathrm{d}\overline{z} \wedge \mathrm{d}x + 2\overline{z}\,\mathrm{d}x \wedge \mathrm{d}y}{(x^2 + y^2 + z\overline{z})^2}\,. \tag{2.111}$$

The propagator $P(v_1, v_2)$ will be referred to as the *bulk-to-bulk* propagator, since the points v_1 and v_2 can be anywhere in the world-volume $\mathbb{R}^2_{x,y} \times \mathbb{C}_z$ of CS theory. To compute Witten diagrams we also need a *boundary-to-bulk* propagator. We will denote it as $\mathsf{K}_\mu(v, x) \equiv \mathsf{K}(v, x)t_\mu$, where $v \in \mathbb{R}^2_{x,y} \times \mathbb{C}_z$ and $x \in \ell_\infty(z)$ is restricted to the boundary line. The boundary-to-bulk propagator is a 1-form defined as a solution to the classical equation of motion:

$$\mathrm{d}z_v \wedge \mathrm{d}_v \mathsf{K}(v, x) = 0\,, \tag{2.112}$$

and by the condition that when pulled back to the boundary, in this case $\ell_\infty(z)$, it must become a delta function supported at x:

$$\varepsilon^* \mathsf{K}(x', x) = \delta^1(x' - x)\mathrm{d}x'\,, \qquad x' \in \ell_\infty(z) \tag{2.113}$$

where $\varepsilon : \ell_\infty(z) \hookrightarrow \mathbb{R}^2 \times \mathbb{C}$ is the embedding of the line in the larger 4d world-volume. As our boundary-to-bulk propagator we choose the following:

$$\mathsf{K}(v, x) = \mathrm{d}_v \theta(x_v - x) = \delta^1(x_v - x)\mathrm{d}x_v\,, \tag{2.114}$$

where x_v refers to the x-coordinate of the bulk point v. The function θ is the following step function:

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 1/2 & \text{for } x = 0 \\ 0 & \text{for } x < 0 \end{cases}\,. \tag{2.115}$$

Note that we have functional derivatives with respect to $\partial_z^n A$ for $n \in \mathbb{N}_{\geq 0}$. The propagator (2.114) corresponds to the functional derivative with $n = 0$. Let us denote the propagator corresponding to $\frac{\delta}{\delta \partial_z^n A}$, more generally, as K_n, and for $n \geq 0$, we modify the condition (2.113) by imposing:

$$\lim_{v \to x'} \varepsilon^* \partial_z^n \mathsf{K}(v, x) = \delta^1(x' - x) \mathrm{d}x' \,, \qquad x' \in \ell_\infty(z) \,. \tag{2.116}$$

This leads us to the following generalization of (2.114):

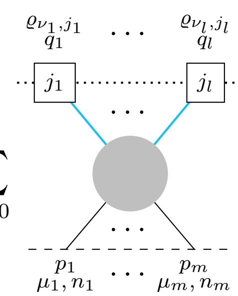
$$\mathsf{K}_n(v, x) = z_v^n \delta^1(x_v - x) \mathrm{d}x_v \,. \tag{2.117}$$

Apart from the two propagators, we shall need the coupling constant of the theory to compute Witten diagrams. The coupling constant of this theory can be read off from the interaction term in the action S_{CS}, it is:

$$\frac{i}{2\pi\hbar} f_{\mu\nu}{}^{\xi} \mathrm{d}z \,. \tag{2.118}$$

Now we can give a diagrammatic definition of the operators in the algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$, namely the ones defined in (2.32), and their products:

$$T_{\mu_1}[n_1](p_1) \cdots T_{\mu_m}[n_m](p_m) = \sum_{l=1}^{\infty} \sum_{j_i \geq 0} \tag{2.119}$$



Let us clarify some points about the picture. We have replaced the pair of fundamental-anti-fundamental indices on T with a single adjoint index. The bottom horizontal line represents the boundary line $\ell_\infty(z)$, and the top horizontal line represents the Wilson line in representation $\varrho : \mathfrak{gl}_K \to V$ at $y = 0$. The sum is over the number of propagators attached to the Wilson line and all possible derivative couplings. The orders of the derivatives are mentioned in the boxes. The points $q_1 \leq \cdots \leq q_l$ on the Wilson line are all integrated along the line without changing their order. The gray blob represents a sum over all possible graphs consistent with the external lines. We use different types of lines to represent different entities:

Bulk-to-bulk propagator, $P(v_1, v_2) =$ v_1 ——— v_2 ,

Boundary-to-bulk propagator, $\mathsf{K}(v, x) =$ v ——— x ,

The boundary line $\ell_\infty(z)$: - - - - - ,

Wilson line : ··········· . (2.120)

The labels μ_i, n_i below the points along the boundary line implies that the corresponding boundary-to-bulk propagator is $\mathsf{K}_{n_i} = z^{n_i}\mathsf{K}$ and that it carries a $\mathfrak{gl}_K$-index μ_i. Finally, the jth derivative of A^ν couples to the Wilson line via the matrix $\varrho_{\nu,j}$. Such a diagram with m boundary-to-bulk propagators and l bulk-to-bulk propagators attached to the Wilson lines will be evaluated to an element of $\mathrm{End}(V)$ which will schematically look like:

$$(\Gamma_{m\to l})^{\mu_1\cdots\mu_l}_{\nu_1\cdots\nu_m}\,\varrho_{\mu_1,j_1}\cdots\varrho_{\mu_l,j_l}\,, \tag{2.121}$$

where $(\Gamma_{m\to l})^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_l}$ is a number that will be found by evaluating the Witten diagram. Since the bulk-to-bulk propagator (2.111) is proportional to $\hbar$ and the interaction vertex (2.118) is proportional to $\hbar^{-1}$, each diagram will come with a factor of $\hbar$ that will be related to the Euler character of the graph.[36] In the following we start computing diagrams starting from $\mathcal{O}(\hbar^0)$ and up to $\mathcal{O}(\hbar^2)$, by the end of which we shall have proven the main result (Proposition 2.4.1) of this section.

Remark 2.4.2 (Diagrams as $m\to l$ maps, and deformation). Each $m\to l$ Witten diagram that appears in sums such as (2.119) can be interpreted as a map whose image is the value of the diagram:

$$\begin{aligned}\Gamma_{m\to l} &: \bigotimes_{i=1}^{m} z^{n_i}\mathfrak{gl}_K \to \bigotimes_{i=1}^{l} z^{j_i}\mathfrak{gl}_K \to \mathrm{End}(V)\,,\\ \Gamma_{m\to l} &: \bigotimes_{i=1}^{m} z^{n_i}t_{\mu_i} \mapsto (\Gamma_{m\to l})^{\mu_1\cdots\mu_l}_{\nu_1\cdots\nu_m}\,\varrho_{\mu_1,j_1}\cdots\varrho_{\mu_l,j_l}\,.\end{aligned} \tag{2.122}$$

As we shall see explicitly in our computations, diagrams in (2.119) without loops (diagrams of $\mathcal{O}(\hbar^0)$) define an associative product that leads to classical algebras such as $\mathrm{U}(\mathfrak{gl}_K[z])$. However, there are generally more diagrams in (2.119) involving loops (diagrams of $\mathcal{O}(\hbar)$ and higher order) that change the classical product to something else. Since loops in Witten or Feynman diagrams are the essence of the quantum interactions, classical algebras deformed by such loop diagrams are aptly called quantum groups (of course, why they are called groups is a different story entirely [24].) △

[36] In a Feynman diagram all propagators are proportional to $\hbar$ and the power of $\hbar$ of a diagram relates simply to the number of faces of the diagram, which is why $\hbar$ is called the loop counting parameter. In a Witten diagram the boundary-to-bulk propagators do not carry any $\hbar$ and therefore the power of $\hbar$ depends also on the number of boundary-to-bulk propagators. However, we are going to ignore this point and simply refer to the power of $\hbar$ in a diagram as the loop order of that diagram.

2.4.3 Relation to anomaly of Wilson line

As we shall compute relevant Witten diagrams of the 4d Chern-Simons theory in detail in later sections, we shall find that the computations are essentially similar to the computations of gauge anomaly of the Wilson line [41] in this theory. This of course is not a coincidence. To see this, let us consider the variation of the expectation value of the Wilson line, $\langle W_\varrho(L)\rangle_A$, as we vary the connection A along the boundary line $\ell_\infty(z)$:

$$\delta \langle W_\varrho(L)\rangle_A = \sum_{n=0}^{\infty} \int_{p\in\ell_\infty(z)} \frac{\delta}{\delta\partial_z^n A^\mu(p)} \langle W_\varrho(L)\rangle_A \, \delta\partial_z^n A^\mu(p)\,. \tag{2.123}$$

Let us make the following variation:

$$\delta\partial_z A^\mu(x) = \delta^1(x-p)\eta^\mu = \mathrm{d}_x\theta(x-p)\eta^\mu\,, \tag{2.124}$$

for some fixed Lie algebra element $\eta^\mu t_\mu \in \mathfrak{gl}_K$. Then we find:

$$\delta \langle W_\rho(L)\rangle_A = \frac{\delta}{\delta\partial_z A^\mu(p)} \langle W_\varrho(L)\rangle_A \, \eta^\mu\,. \tag{2.125}$$

An exact variation of the boundary value of the connection is like a gauge transformation that does not vanish at the boundary. In [41] it was proved that such a variation of the connection leads to a variation of the Wilson line which is a local functional supported on the line:

$$\delta \langle W_\varrho(L)\rangle_A = \left([\varrho_{\mu,1}, \varrho_{\nu,1}] + \Theta_{\mu,1,\nu,1}\right) \int_L \partial_z A^\mu \partial_z \mathsf{c}^\nu\,, \tag{2.126}$$

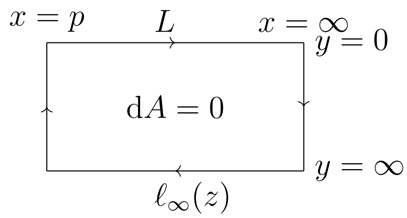
where c was the generator of the gauge transformation:

$$\partial_z \mathrm{d}\mathsf{c}^\mu = \delta\partial_z A^\mu\,, \tag{2.127}$$

$\rho_{\mu,1} \in \mathrm{End}(V)$ is part of the representation of $\mathfrak{gl}_K[z]$ that couples $\partial_z A^\mu$ to the Wilson line (see (2.100)), and $\Theta_{\mu,1,\nu,1}$, which is anti-symmetric in μ and ν, is a matrix that acts on V. Variations such as the above measure gauge anomaly associated to the line, though in our case it is not an anomaly since we are varying the connection at the boundary, and such "large gauge" transformations are not actually part of the gauge symmetry of the theory. The matrix $\Theta_{\mu,1,\nu,1}$ which signals the presence of anomaly is not an arbitrary matrix and in [41], all constraints on this matrix were worked out, we shall not need them at the moment. Comparing with (2.124) we see that for us $\partial_z c^\mu(x) = \theta(x-p)\eta^\mu$, which leads to:

$$\delta \langle W_\varrho(L)\rangle_A = \left(f_{\mu\nu}{}^{\xi}\varrho_{\xi,2} + \Theta_{\mu,1,\nu,1}\right) \int_{x>p} \partial_z A^\mu \eta^\nu\,, \tag{2.128}$$

where we have used the fact that the matrices $\varrho_{\mu,1}$ satisfy the loop algebra (2.101). The integral above is along L. The connection A above is a background connection satisfying the equation of motion, i.e., it is flat. Since the D4 world-volume, even in the presence of a Wilson line, has no non-contractible loop, all flat connections are exact. Symmetry of world-volume dictates in particular that the connection must also be translation invariant along the direction of the Wilson line L. By considering the integral of A along the following rectangle:



(2.129)

and using translation invariance in the x-direction along with Stoke's theorem, we can change the support of the integral in (2.128) from L to $\ell_\infty(z)$, to get:

$$\delta \langle W_\varrho(L)\rangle_A = \left(f_{\mu\nu}{}^{\xi}\varrho_{\xi,2} + \Theta_{\mu,1,\nu,1}\right) \int_{\ell_\infty(z)\ni x>p} \partial_z A^\mu \eta^\nu \,. \tag{2.130}$$

Comparing with (2.125) we find:

$$\frac{\delta}{\delta\partial_z A^\nu(p)} \langle W_\varrho(L)\rangle_A = \left(f_{\mu\nu}{}^{\xi}\varrho_{\xi,2} + \Theta_{\mu,1,\nu,1}\right) \int_{x>p} \partial_z A^\mu \,, \tag{2.131}$$

where the integral is now along the boundary line $\ell_\infty(z)$. This leads to the following relation between our algebra and anomaly:

$$\begin{aligned} &[T_\mu[1], T_\nu[1]] \\ &= \lim_{\iota\to 0}\left[\frac{\delta}{\delta\partial_z A^\mu(p+\iota)}\frac{\delta}{\delta\partial_z A^\nu(p)} - \frac{\delta}{\delta\partial_z A^\nu(p)}\frac{\delta}{\delta\partial_z A^\mu(p+\iota)}\right]\langle W_\varrho(L)\rangle_A \\ &= f_{\mu\nu}{}^{\xi}\varrho_{\xi,2} + \Theta_{\mu,1,\nu,1}\,. \end{aligned} \tag{2.132}$$

The first term with the structure constant gives us the loop algebra $\mathfrak{gl}_K[z]$, which is the classical result. The anomaly term is the result of 2-loop dynamics [41], i.e., it is proportional to $\hbar^2$. This term gives the quantum deformation of the classical loop algebra. This also explains why our two loop computation of the algebra is similar to the two loop computation of anomaly from [41].

At this point, we note that we can actually just use the result of [41] to find out what $\Theta_{\mu,1,\nu,1}$ is and we would find that the deformed algebra of the operators $T^\mu[n]$ is indeed the Yangian $Y_\hbar(\mathfrak{gl}_K)$. However, we think it is illustrative to derive this result from a direct computation of Witten diagrams.

2.4.4 Classical algebra, $\mathcal{O}(\hbar^0)$

Lie bracket

We denote a diagram by $\Gamma^d_{n\to m}$ when there are n boundary-to-bulk propagators, m propagators attached to the Wilson line, and the diagram is of order $\hbar^d$. If there are more than one such diagrams we denote them as $\Gamma^d_{n\to m,i}$ with $i = 1, \cdots$.

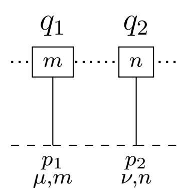
Our aim in this section is to compute the product $T_\mu[m](p_1)T_\nu[n](p_2)$ and eventually the commutator

$$[T_\mu[m], T_\nu[n]] := \lim_{p_2\to p_1} \left(T_\mu[m](p_1)T_\nu[n](p_2) - T_\nu[n](p_1)T_\mu[m](p_2)\right) , \tag{2.133}$$

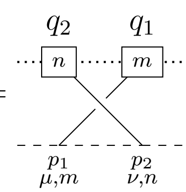
at 0-loop.[37]

We have the following two $2 \to 2$ diagrams:

$$\Gamma^0_{2\to 2,1}\left(\begin{smallmatrix} p_1 \\ \mu,m\end{smallmatrix};\begin{smallmatrix} p_2 \\ \nu,n\end{smallmatrix}\right) = $$



$$, \qquad \Gamma^0_{2\to 2,2}\left(\begin{smallmatrix} p_1 \\ \mu,m\end{smallmatrix};\begin{smallmatrix} p_2 \\ \nu,n\end{smallmatrix}\right) = $$



$$, \tag{2.134}$$

where a label m in a box on the Wilson line refers to the coupling between the Wilson line and the mth derivative of the connection. The first diagram evaluates to (note that $p_1 < p_2$ and $q_1 < q_2$):

$$\begin{aligned}\Gamma^0_{2\to 2,1}\left(\begin{smallmatrix} p_1 \\ \mu,m\end{smallmatrix};\begin{smallmatrix} p_2 \\ \nu,n\end{smallmatrix}\right) &= \int_{q_1<q_2} \mathrm{d}q_1\mathrm{d}q_2\, \delta^1(q_1 - p_1)\delta^1(q_2 - p_2)\varrho^\mu_m\varrho^\nu_n\,, \\ &= \varrho_{\mu,m}\varrho_{\nu,n}\,,\end{aligned} \tag{2.135}$$

and the second one (with $p_1 < p_2$ and $q_1 > q_2$):

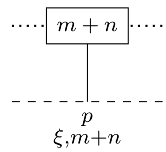
$$\begin{aligned}\Gamma^0_{2\to 2,2}\left(\begin{smallmatrix} p_1 \\ \mu,m\end{smallmatrix};\begin{smallmatrix} p_2 \\ \nu,n\end{smallmatrix}\right) &= \int_{q_1>q_2} \mathrm{d}q_2\mathrm{d}q_1\, \delta^1(q_1 - p_1)\delta^1(q_2 - p_2)\varrho_{\nu,n}\varrho_{\mu,m}\,, \\ &= 0\,.\end{aligned} \tag{2.136}$$

[37] $[T_\mu[m](p_1), T_\nu[n](p_1)]$ may be a more accurate notation but this algebra must be position invariant and therefore we shall ignore the position. Reference to the position only matters when different operators are positioned at different locations.

Therefore their contribution to the commutator is:

$$\begin{aligned} [T_\mu[m], T_\nu[n]] &= \lim_{p_2 \to p_1} \left(\Gamma^0_{2\to 2,1} \left(\begin{smallmatrix} p_1 \\ \mu,m \end{smallmatrix} ; \begin{smallmatrix} p_2 \\ \nu,n \end{smallmatrix} \right) - \Gamma^0_{2\to 2,1} \left(\begin{smallmatrix} p_1 \\ \nu,n \end{smallmatrix} ; \begin{smallmatrix} p_2 \\ \mu,m \end{smallmatrix} \right) \right) , \\ &= [\varrho_{\mu,m}, \varrho_{\nu,n}] = f_{\mu\nu}{}^{\xi} \varrho_{\xi,m+n} = f_{\mu\nu}{}^{\xi} T_\xi[m+n] , \end{aligned} \tag{2.137}$$

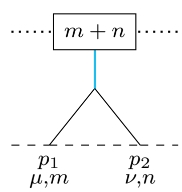
where the last equality is established by evaluating the diagram:



. (2.138)

The bracket (2.137) is precisely the Lie bracket in the loop algebra $\mathfrak{gl}_K[z]$. Note in passing that had we considered the same diagrams as the ones in (2.134) except with different derivative couplings at the Wilson line then the diagrams would have vanished, either because there would be more z-derivatives than z, or there would be less, in which case there would be z's floating around which vanish along the Wilson line located at $y = z = 0$.

There is one $2 \to 1$ diagram as well:



, (2.139)

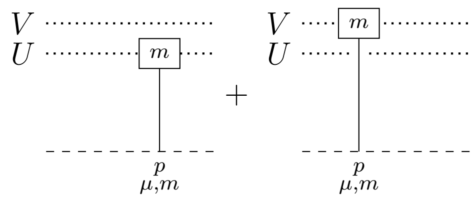
however, since the two boundary-to-bulk propagators are two parallel delta functions,[38] they never meet in the bulk and therefore the diagram vanishes. There are no more classical diagrams, so the Lie bracket in the classical algebra is just the bracket in (2.137).

Coproduct

Apart from the Lie algebra structure, the algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ also has a coproduct structure. This can be seen by considering the Wilson line in a tensor product representation, say $U \otimes V$. Such a Wilson line can be produced by considering two Wilson lines in representations U and V respectively and bringing them together, and asking how $T_\mu[n]$ acts on $U \otimes V$.

[38]i.e., their support are restricted to $x = p_1$ and $x = p_2$ respectively with $p_1 \neq p_2$, so they never intersect.

Since there are going to be multiple vector spaces in this section, let us distinguish the actions of $T_\mu[n]$ on them by a superscript, such as, $T_\mu^U[n]$, $T_\mu^V[n]$, etc. At the classical level the answer to the question we are asking is simply given by computing the following diagrams:



. (2.140)

Evaluation of these diagrams is very similar to that of the diagrams in (2.134) and the result is:

$$T_\mu^{U\otimes V}[m] = T_\mu^U[m] \otimes \mathrm{id}_V + \mathrm{id}_U \otimes T_\mu^V[m]\,. \tag{2.141}$$

This is the same coproduct structure as that of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_K[z])$.

Combining the results of this section and the previous one we find that, at the classical level we have an associative algebra with generators $T_\mu[n]$ with a Lie bracket and coproduct given by the Lie bracket of the loop algebra $\mathfrak{gl}_K[z]$ and the coproduct of its universla enveloping algebra. This identifies $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$, clasically, as the universal enveloping algebra itself:

Lemma 2.4.5. *The large N limit of the algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ at the classical level is the universal enveloping algebra $U(\mathfrak{gl}_K[z])$:*

$$\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})/\hbar \overset{N\to\infty}{\cong} U(\mathfrak{gl}_K[z]) \cong Y_\hbar(\mathfrak{gl}_K)/\hbar\,. \tag{2.142}$$

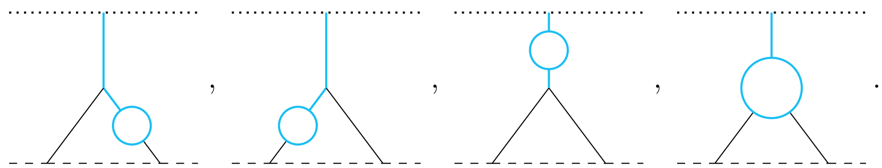
The reason why we need to take the large N limit is that, the operators $T_\mu[m]$ acts on a vector space which is finite dimensional for finite N. This leads to some extra relations in the algebra, which we can get rid of in the large N limit. A similar argument was presented for the operator algebra coming from the BF theory in §2.3.7 and the argument in the context of the CS theory will be explained in more detail in §2.4.7.

2.4.6 Loop corrections

1-loop, $\mathcal{O}(\hbar)$

Now we want to compute 1-loop deformation to both the Lie algebra structure and the coproduct structure of $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$.

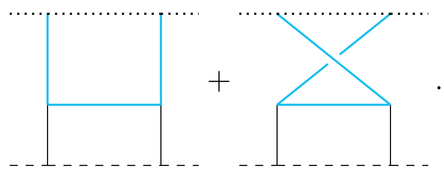
Lie bracket. The $2 \to 1$ diagrams at this loop order are:[39]



(2.143)

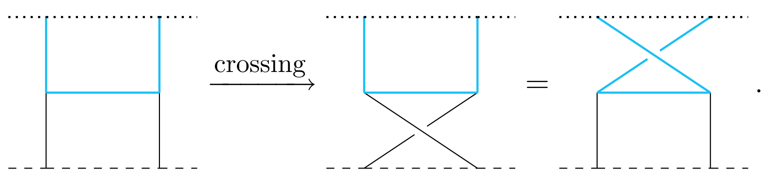
All of these vanish due to Lemma A.4.2 of §A.4.1.

The $2 \to 2$ diagrams at this loop order are:



(2.144)

Note that, since the bulk points are being integrated over, crossing the boundary-to-bulk propagators does not produce any new diagram, it just exchanges the two diagrams that we have drawn:



(2.145)

For this reason, in future we shall only draw diagrams up to crossing of the boundary-to-bulk propagators that are connected to bulk interaction vertices.

Now let us comment on the evaluation of the diagrams in (2.144). We start by doing integration by parts with respect the differential corresponding to either one of the two boundary-to-bulk propagators. As mentioned in §A.4.5, this gives two kinds of contributions, one coming from collapsing a bulk-to-bulk propagator, the other coming from boundary terms. Collapsing any of the bulk-to-bulk propagators leads to a configuration which will vanish due to Lemma A.4.3 (§A.4.1). Therefore, doing integration by parts will only result in a boundary term. Recall from the general discussion in §A.4.5 that only the boundary component of the integrals along the Wilson line can possibly contribute. Since there are two points on the Wilson line, let us call them p_1 and p_2, the domain of integration is:

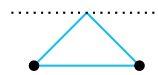
$$\Delta_2 = \{(p_1, p_2) \in \mathbb{R}^2 \,|\, p_1 < p_2\}\,. \tag{2.146}$$

[39]Sometimes we ignore to specify the derivative couplings at the Wilson line, when the diagrams we draw are vanishing regardless.

The boundary of this domain is:

$$\partial\Delta_2 = \{(p_1, p_2) \in \mathbb{R}^2 \,|\, p_1 = p_2\}\,. \tag{2.147}$$

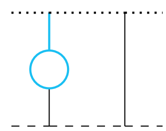
Once restricted to this boundary, both of the diagrams in (2.144) will involve a configuration such as the following:



, (2.148)

which vanishes due to Lemma A.4.2.[40] The diagrams (2.144) thus vanish.

There are four other $2 \to 2$ diagrams at 1-loop, they can be generated by starting with:



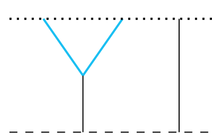
, (2.149)

and then

1. Permuting the two points on the Wilson line.
2. Permuting the two points on the boundary.
3. Simultaneously permuting the two points on the Wilson line and the two points on the boundary.

All of these diagrams vanish due to Lemma A.4.2.

There are also six $2 \to 3$ diagrams. All of these can be generated from the following:



, (2.150)

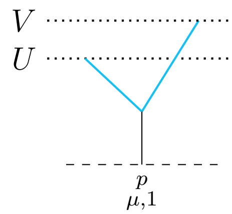
by permuting the points along the Wilson line and the boundary. However, due to Lemma A.4.3, all of these diagrams vanish.

[40]These diagrams actually require a UV regularization due to logarithmic divergence coming from the two points on the Wilson line being coincident. To regularize, the domain of integration needs to be restricted from Δ_2 to $\widetilde{\Delta}_2 := \{(p_1, p_2) \in \mathbb{R}^2 \,|\, p_1 \leq p_2 - \epsilon\}$ for some small positive number ϵ, which leads to the modified boundary equation $p_1 = p_2 - \epsilon$, however, this does not affect the arguments presented in the proof of Lemma A.4.2 (essentially because ϵ is a constant and $\mathrm{d}\epsilon = 0$, resulting in no new forms other than the ones considered in the proof), and therefore we are not going to describe the regularization of these diagrams in detail.

There are no more $2 \to m$ diagrams at 1-loop. Thus, we conclude that there is no 1-loop contribution to the Lie bracket in $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$.

Coproduct. We use the same superscript notation we used in §2.4.4 to distinguish between the actions of $T_\mu[m]$ on different vector spaces. The 1-loop diagram that deforms the classical coproduct is the following:

$$\Gamma^1_{1\to 2}\left(\begin{smallmatrix} p \\ \mu,1 \end{smallmatrix}\right) = \tag{2.151}$$



Happily for us, precisely this diagram was computed in eq. 5.6 of [41] to answer the question "how does a background connection couple to the product Wilson line?". The result of that paper involved an arbitrary background connection where we have our boundary-to-bulk propagator, so we just need to replace that with $\mathsf{K}_1(v,p) = z_v \delta^1(x_v - p)$ and we find:

$$\Gamma^1_{1\to 2}\left(\begin{smallmatrix} p \\ \mu,1 \end{smallmatrix}\right) = -\frac{\hbar}{2} f_\mu{}^{\nu\xi} T^U_\nu[0] \otimes T^V_\xi[0] \,. \tag{2.152}$$

This deforms the classical coproduct (2.141) as follows:

$$T^{U\otimes V}_\mu[1] = T^U_\mu[1] \otimes \mathrm{id}_V + \mathrm{id}_U \otimes T^V_\mu[1] - \frac{\hbar}{2} f_\mu{}^{\nu\xi} T^U_\nu[0] \otimes T^V_\xi[0] \,. \tag{2.153}$$

The exact same computation with K_0 instead of K_1 shows that $\Gamma^1_{1\to 2}\left(\begin{smallmatrix} p \\ \mu,0 \end{smallmatrix}\right) = 0$, i.e., the classical algebra of the 0th level operators remain entirely undeformed at this loop order.[41]

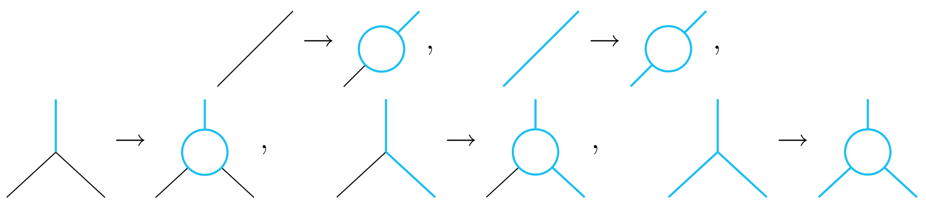
Thus we see that at 1-loop, the Lie algebra structure in $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ remains undeformed, but there is a non-trivial deformation of the coalgebra structure. At this point, there is a mathematical shortcut to proving that the algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$, including all loop corrections, is the Yangian. The proof relies on a uniqueness theorem (Theorem 12.1.1 of [24]) concerning the deformation of $\mathrm{U}(\mathfrak{gl}_K[z])$. Being able to use the theorem requires satisfying some technical conditions, we discuss this proof in Appendix A.3. This proof is independent of the rest of the chapter, where we compute two loop corrections to the commutator (2.137) which will directly show that the algebra is the Yangian.

[41]Note that the 0th level operators form a closed algebra which is nothing but the Lie algebra $\mathfrak{gl}_K$. Reductive Lie algebras belong to discrete isomorphism classes and therefore they are robust against continuous deformations. So the algebra of $T_\mu[0]$ will in fact remain undeformed at all loop orders. We will not make more than a few remarks about them in the future.

2-loops, $\mathcal{O}(\hbar^2)$

The number of 2-loop diagrams is too large to list them all, and most of them are zero. Instead of drawing all these diagrams let us mention how we can quickly identify a large portion of the diagrams that end up being zero.

Consider the following transformations that can be performed on a propagator or a vertex in any diagram:

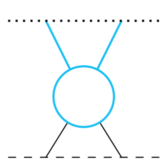
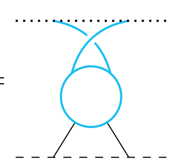


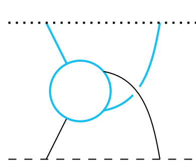
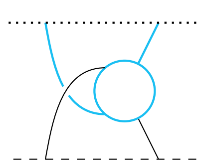
(2.154)

All these transformations increase the order of $\hbar$ by one, however, all the diagrams constructed using these modifications are zero due to Lemma A.4.2. We will therefore ignore such diagrams. Let us now identify potentially non-zero $2 \to m$ diagrams at 2-loops.

All 2-loop $2 \to 1$ diagrams are created from lower order diagrams using modifications such as (2.154). All of them vanish.

For $2 \to 2$ diagrams, ignoring those that are results of modifications such as (2.154) or that are product of lower order vanishing diagrams, we are left with the sum of the following diagrams:

$\Gamma^2_{2\to 2,1} =$  , $\Gamma^2_{2\to 2,2} =$  ,

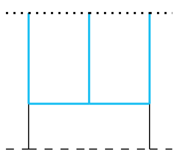
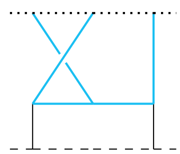
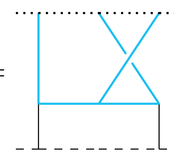
$\Gamma^2_{2\to 2,3} =$  , $\Gamma^2_{2\to 2,4} =$  .

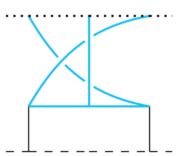
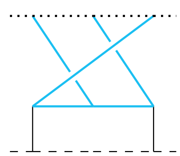
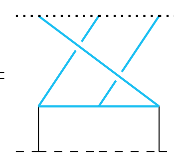
(2.155)

Let us first consider the first two diagrams $\Gamma^2_{2\to 2,1}$ and $\Gamma^2_{2\to 2,2}$. Collapsing any of the bulk-to-bulk propagators will result in a configuration where either Lemma A.4.2 or A.4.3 is applicable. Therefore, when we do integration by parts with respect to the differential in one of the two boundary-to-bulk propagators we only get a boundary term. The boundary corresponds to the boundary of Δ_2 (defined in (2.146)), and when restricted to this boundary, the integrand vanishes due to Lemma A.4.3, in the same way as for the diagrams in (2.144).[42]

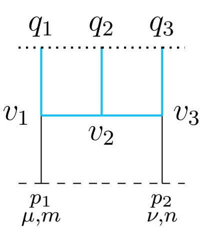
The diagrams $\Gamma^2_{2\to2,3}$ and $\Gamma^2_{2\to2,4}$ are symmetric under the exchange of the color labels associated to the boundary-to-bulk propagators, for a proof see the discussion following (A.40). So these diagrams don't contribute to the anti-symmetric commutator we are computing.

Now we come to the most involved part of our computations, $2 \to 3$ diagrams at 2-loops. We have the sum of the following diagrams:

$\Gamma^2_{2\to3,1} =$  , $\Gamma^2_{2\to3,2} =$  , $\Gamma^2_{2\to3,3} =$  ,

$\Gamma^2_{2\to3,4} =$  , $\Gamma^2_{2\to3,5} =$  , $\Gamma^2_{2\to3,6} =$  . (2.156)

All of these diagrams are in fact non-zero. We proceed with the evaluation of the diagram $\Gamma^2_{2\to3,1}$:

$\Gamma^2_{2\to3,1}\left(\begin{smallmatrix} p_1 \\ \mu,1\end{smallmatrix};\begin{smallmatrix} p_2 \\ \nu,1\end{smallmatrix}\right) =$ 

(2.157)

The $\mathfrak{gl}_K$ factor of the diagram is easily evaluated to be:

$$f_\mu{}^{\xi o} f_\xi{}^{\pi\rho} f_{\nu\pi}{}^{\sigma} \varrho(t_o)\varrho(t_\rho)\varrho(t_\sigma)\,. \tag{2.158}$$

The numerical factor takes a bit more care. Each of the bulk points $v_i = (x_i, y_i, z_i, \overline{z}_i)$ is integrated over $M = \mathbb{R}^2 \times \mathbb{C}$ and the points q_i on the Wilson line take value in the

[42]These diagrams are linearly divergent when the two points on the Wilson line are coincident and they require similar UV regularization as their 1-loop counterparts.

simplex $\Delta_3 = \{(q_1, q_2, q_3) \in \mathbb{R}^3 \,|\, q_1 < q_2 < q_3\}$. For the sake of integration we can partially compactify the bulk to $M = \mathbb{R} \times S^3$. So the domain of integration for this diagram is:

$$M^3 \times \Delta_3\,. \tag{2.159}$$

However, this domain needs regularization due to UV divergences coming from the points q_i all coming together. As in [41], we use a point splitting regulator, by restricting integration to the domain:

$$\widetilde{\Delta}_3 := \{(q_1, q_2, q_3) \in \Delta_3 \,|\, q_1 < p_3 - \epsilon\}\,, \tag{2.160}$$

for some small positive number ϵ. We are not going to discuss the regulator here, as it would be identical to the discussion in [41]. We shall now do integration by parts with respect to the differential in the propagator connecting p_1 and v_1. Note that collapsing any of the bulk-to-bulk propagators leads to a configuration where the vanishing Lemma A.4.3 applies. Therefore, contribution to the integral only comes from the boundary $M^3 \times \partial\widetilde{\Delta}_3$. The boundary of the simplex has three components, respectively defined by the constraints $q_1 = q_2$, $q_2 = q_3$, and $q_1 = q_3 - \epsilon$. However, when $q_1 = q_2$ or $q_2 = q_3$, we can use the vanishing Lemma A.4.2 and the integral vanishes. Therefore the contribution to the diagram comes only from integration over:

$$M^3 \times \{(q_1, q_2, q_3) \in \widetilde{\Delta}_3 \,|\, q_1 = q_3 - \epsilon\}\,. \tag{2.161}$$

Further simplification can be made using the fact that the propagator connecting p_2 and v_3 is $z\delta^1(x_3 - p_2)$. This restricts the integration over v_3 to $\{p_2\} \times S^3$. However, using translation symmetry in the x-direction we can fix the position of q_1 at $(0, 0, 0, 0)$ and allow the integration of v_3 over all of M. However, due to the presence of the delta function $\delta^1(x_3 - p_2)$ in the boundary-to-bulk propagator, x_3 and $p_1 = p_2 - \delta$ are rigidly tied to each other. This way, we end up with the following integration for the numerical factor:[43]

$$\begin{aligned}\frac{1}{2}\left(\frac{i}{2\pi\hbar}\right)^3 \int_{\substack{0<q_2<\epsilon \\ v_1, v_2, v_3}} &\mathrm{d}q_2 \mathrm{d}^4v_1 \mathrm{d}^4v_2 \mathrm{d}^4v_3 \theta(x_1 - x_3^-) z_1 z_3 P(v_2, v_1) \\ &\times P(v_3, v_2) P(q_1, v_1) P(q_2, v_2) P(q_3, v_3)\,,\end{aligned} \tag{2.162}$$

where $q_1 = (0, 0, 0, 0)$, $q_2 = (p_2, 0, 0, 0)$, $q_3 = (\epsilon, 0, 0, 0)$, and $x_3^- := x_3 - \delta$, and since all the forms that appear are even we have ignored the wedge product symbols to be economic.

Before evaluating the above integral, we note that the diagram $\Gamma^2_{2\to 3,4}$ evaluates to the

[43] The factor of 1/2 comes from diagram automorphisms.

same color factor and almost same numerical factor, except for a different step function:

$$\frac{1}{2}\left(\frac{i}{2\pi\hbar}\right)^3 \int_{\substack{0<q_2<\epsilon\\ v_1,v_2,v_3}} \mathrm{d}q_2\mathrm{d}^4v_1\mathrm{d}^4v_2\mathrm{d}^4v_3\theta(x_3-x_1^-)z_1z_3P(v_2,v_1) \times P(v_3,v_2)P(q_1,v_1)P(q_2,v_2)P(q_3,v_3)\,, \tag{2.163}$$

Since we have to sum over all the diagrams, we use the fact that:

$$\lim_{\delta\to 0}\big(\theta(x_1-x_3^-)+\theta(x_3-x_1^-)\big)=1\,, \tag{2.164}$$

to write:

$$\begin{aligned}&\lim_{p_2\to p_1}\left(\Gamma^2_{2\to 3,1}\left(\begin{smallmatrix}p_1\\ \mu,1\end{smallmatrix};\begin{smallmatrix}p_2\\ \nu,1\end{smallmatrix}\right)+\Gamma^2_{2\to 3,4}\left(\begin{smallmatrix}p_1\\ \mu,1\end{smallmatrix};\begin{smallmatrix}p_2\\ \nu,1\end{smallmatrix}\right)\right)\\ &=f_\mu{}^{\xi o}f_\xi{}^{\pi\rho}f_{\nu\pi}{}^{\sigma}\varrho(t_o)\varrho(t_\rho)\varrho(t_\sigma)\left(\frac{i}{2\pi\hbar}\right)^3\frac{1}{2}\int_{\substack{0<q_2<\epsilon\\ v_1,v_2,v_3}}\mathrm{d}q_2\mathrm{d}^4v_1\mathrm{d}^4v_2\mathrm{d}^4v_3\\ &\quad\times z_1z_3P(v_2,v_1)P(v_3,v_2)P(q_1,v_1)P(q_2,v_2)P(q_3,v_3)\,,\end{aligned} \tag{2.165}$$

Let us refer to the above integral by $\hbar^2 I_1$, so that we can write the right hand side of the above equation as:

$$\hbar^2 f_\mu{}^{\xi o}f_\xi{}^{\pi\rho}f_{\nu\pi}{}^{\sigma}\varrho(t_o)\varrho(t_\rho)\varrho(t_\sigma)\,I_1\,. \tag{2.166}$$

Similar considerations for the rest of the diagrams in (2.156) lead to similar expressions:

$$\lim_{p_2\to p_1}\left(\Gamma^2_{2\to 3,2}\left(\begin{smallmatrix}p_1\\ \mu,1\end{smallmatrix};\begin{smallmatrix}p_2\\ \nu,1\end{smallmatrix}\right)+\Gamma^2_{2\to 3,5}\left(\begin{smallmatrix}p_1\\ \mu,1\end{smallmatrix};\begin{smallmatrix}p_2\\ \nu,1\end{smallmatrix}\right)\right)=\hbar^2 f_\mu{}^{\xi o}f_\xi{}^{\pi\rho}f_{\nu\pi}{}^{\sigma}\varrho(t_\rho)\varrho(t_o)\varrho(t_\sigma)\,I_2\,, \tag{2.167a}$$

$$\lim_{p_2\to p_1}\left(\Gamma^2_{2\to 3,2}\left(\begin{smallmatrix}p_1\\ \mu,1\end{smallmatrix};\begin{smallmatrix}p_2\\ \nu,1\end{smallmatrix}\right)+\Gamma^2_{2\to 3,5}\left(\begin{smallmatrix}p_1\\ \mu,1\end{smallmatrix};\begin{smallmatrix}p_2\\ \nu,1\end{smallmatrix}\right)\right)=\hbar^2 f_\mu{}^{\xi o}f_\xi{}^{\pi\rho}f_{\nu\pi}{}^{\sigma}\varrho(t_o)\varrho(t_\sigma)\varrho(t_\rho)\,I_3\,, \tag{2.167b}$$

for two integrals I_2 and I_3 that are only slightly different from I_1.[44] To get the 2-loop contributions to the commutator $[T_\mu[1],T_\nu,[1]]$ we need only to anti-symmetrize the expressions (2.166), (2.167). Putting them together with the classical result (2.137) we get the Lie bracket up to 2-loops:

$$\begin{aligned}[T_\mu[1],T_\nu,[1]]=f_{\mu\nu}{}^{\xi}T_\xi[2]+2\hbar^2 f_{[\mu}{}^{\xi o}f_\xi{}^{\pi\rho}f_{\nu]\pi}{}^{\sigma}\big(&T_o[0]T_\rho[0]T_\sigma[0]\,I_1\\ &+T_\rho[0]T_o[0]T_\sigma[0]\,I_2+T_o[0]T_\sigma[0]T_\rho[0]\,I_3\big)\,,\end{aligned} \tag{2.168}$$

where we have replaced matrix products such as $\varrho(t_\rho)\varrho(t_o)\varrho(t_\sigma)$ with $T_\rho[0]T_o[0]T_\sigma[0]$ which is accurate up to the loop order shown. Thus we see that quantum corrections deform the classical Lie algebra of $\mathfrak{gl}_K[z]$.

[44] These integrals can be performed and their values are $I_2=I_3=\frac{1}{72}\left(2-\frac{3}{\pi^2}\right)$, $I_1=\frac{1}{36}\left(1+\frac{3}{\pi^2}\right)$ though we postpone computing them until we no longer need to compute them.

2.4.7 Large N limit: The Yangian

The deformed Lie bracket (2.168) may not look quite like the standard relations of the Yangian found in the literature, but we can choose a different basis to get to the standard relations, which we shall do momentarily.[45] However, for finite N, our algebra has more relations. Recall that the generators $T_\mu[1]$ act on the space V where classically V is a representation space, $\varrho : \mathfrak{gl}_K[z] \to \mathrm{End}(V)$, of the loop algebra $\mathfrak{gl}_K[z]$ and the representation ρ was determined by the number N. The representation ϱ depends on N because ρ is the representation that couples the $\mathfrak{gl}_K$ connection A to the Wilson line generated by integrating out $N \times K$ fermions. The representation is essentially the Hilbert space (2.93) of the fermionic QM that lives on the line. The important point for us is that, for finite N, ϱ is finite dimensional. This implies that, as we discussed at the end of §(2.3.5), the generators $T_\mu[1]$ satisfy degree d polynomial equations where $d = \dim(V)$. In the limit $N \to \infty$ these relations disappear and we have our isomorphism with the Yangian $Y(\mathfrak{gl}_K)$.

The Yangian in a more standard basis

To get to a standard defining bracket for the Yangian, we change basis as follows. There is an ambiguity in $T_\xi[2]$. In (2.137) it was equal to $\varrho_{\xi,2}$ at the classical level, but it can be shifted at 2-loops (i.e., by a term proportional to $\hbar^2$) by the image $\vartheta(t_\xi)$ for an arbitrary $\mathfrak{gl}_K$-equivariant map $\vartheta : \mathfrak{gl}_K \to \mathrm{End}(V)$. This shift simply corresponds to a different choice for the counterterm that couples $\partial_z^2 A^\xi$ to the Wilson line. Using this freedom we want to replace products such as $\varrho(t_o)\varrho(t_\rho)\varrho(t_\sigma)$ with the totally symmetric product $\{\varrho(t_o), \varrho(t_\rho), \varrho(t_\sigma)\}$ (defined in (2.87)). To this end, Consider the difference:

$$\Delta_{\mu\nu} := 2\hbar^2 f_{[\mu}{}^{\xi o} f_{\xi}{}^{\pi\rho} f_{\nu]\pi}{}^{\sigma} \left(\varrho(t_o)\varrho(t_\rho)\varrho(t_\sigma) - \{\varrho(t_o), \varrho(t_\rho), \varrho(t_\sigma)\}\right) . \tag{2.169}$$

The square brackets around μ and ν in the above equation implies anti-symmetrization with respect to μ and ν. The difference $\Delta_{\mu\nu}$ can be viewed as the image of the following $\mathfrak{gl}_K$-equivariant map:

$$\Delta : \wedge^2 \mathfrak{gl}_K \to \mathrm{End}(V) , \qquad \Delta : t_\mu \wedge t_\nu \mapsto \Delta_{\mu\nu} . \tag{2.170}$$

We now propose the following lemma:

Lemma 2.4.8. *The map Δ factors through $\mathfrak{gl}_K$, i.e., $\Delta : \wedge^2\mathfrak{gl}_K \to \mathfrak{gl}_K \to \mathrm{End}(V)$.*

[45] We can also appeal to the uniqueness theorem 12.1.1 of [24], in conjunction with the result of Appendix A.3, to conclude that the deformed algebra must be the Yangian $Y_\hbar(\mathfrak{gl}_K)$.

The proof of this lemma involves some algebraic technicalities which we relegate to the Appendix §A.5. The utility of this lemma is that, it establishes the difference (2.169) as the image of an element of $\mathfrak{gl}_K$ which, according to our previous argument, can be absorbed into a redefinition of $\varrho_{\xi,2}$ (equivalently $T_\xi[2]$). Therefore, with a new $T_\xi^{\mathrm{new}}[2]$ we can rewrite (2.168) as:

$$[T_\mu[1], T_\nu, [1]] = f_{\mu\nu}{}^\xi T_\xi^{\mathrm{new}}[2] + \hbar^2(I_1 + I_2 + I_3)Q_{\mu\nu}\,, \tag{2.171}$$

where we have also defined:

$$Q_{\mu\nu} := 2f_{[\mu}{}^{\xi o} f_\xi{}^{\pi\rho} f_{\nu]\pi}{}^\sigma \left\{T_o[0], T_\rho[0], T_\sigma[0]\right\}\,. \tag{2.172}$$

The reason why we have postponed presenting the evaluations of the individual integrals I_1, I_2, and I_3 is that we don't need their individual values, only the sum, and precisely this sum was evaluated in eq. (E.23) of [41] with the result:

$$I_1 + I_2 + I_3 = \frac{1}{12}\,. \tag{2.173}$$

We can therefore write (ignoring the "new" label on $T_\xi[2]$):

$$[T_\mu[1], T_\nu[1]] = f_{\mu\nu}{}^\xi T_\xi[2] + \frac{\hbar^2}{12}Q_{\mu\nu}\,. \tag{2.174}$$

This is the relation for the Yangian that was presented in §8.6 of [41] and how to relate this to other standard relations of the Yangian was also discussed there. This is also the exact relation we found in the boundary theory (c.f. (2.90)). Note furthermore that, if we had used the relation between our algebra and anomaly (2.132) to derive the algebra Lie bracket, we would have arrived at precisely the same conclusion, as the second term in (2.174) is indeed the anomaly of a Wilson line (c.f. eq. 8.35 of [34]).

Thus we see that the algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$, defined in (2.34), at 2-loops, is the Yangian $Y_\hbar(\mathfrak{gl}_K)$:

$$\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})/\hbar^3 \overset{N\to\infty}{\cong} Y_\hbar(\mathfrak{gl}_K)/\hbar^3\,. \tag{2.175}$$

The two loop result in the BF theory was exact. The above two loop result is exact as well. Though we do not prove this by computing Witten diagrams, we can argue using the form of the algebra in terms of anomaly (2.132). In [41] it was shown that there are no anomalies beyond two loops. This concludes our second proof of Proposition 2.4.1.[46]

[46]The first one, which is significantly more abstract, being in Appendix A.3.

2.5 String Theory Construction of The Duality

The topological theories we have considered so far can be constructed from a certain brane setup in type IIB string theory and then applying a twist and an omega deformation. This brane construction will show that the algebras we have constructed are infact certain supersymmetric subsectors of the well studied $\mathcal{N} = 4$ SYM theory with defect and its holographic dual. We dscribe this construction below.

2.5.1 Brane Configuration

Our starting brane configuration involves a stack of N D3 branes and K D5 branes in type IIB string theory on a 10d target space of the form $\mathbb{R}^8 \times C$ where C is a complex curve which we take to be just the complex plane $\mathbb{C}$. The D5 branes wrap $\mathbb{R}^4 \times C$ and the D3 branes wrap an $\mathbb{R}^4$ which has a 3d intersection with the D5 branes. Let us express the brane configuraiton by the following table:

	0	1	2	3	4	5	6	7	8	9
	$\mathbb{R}^4$				C		$\mathbb{R}^4$			
$D5$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$				
$D3$	$\times$		$\times$	$\times$				$\times$		

(2.176)

The world-volume theory on the D5 branes is the 6d $\mathcal{N} = (1, 1)$ SYM theory coupled to a 3d defect preserving half of the supersymmetry. Similarly, the world-volume theory on the D3 branes is the 4d SYM theory coupled to a 3d defect preserving half of the supersymmetry. To this setup we apply a particular twist, i.e., we choose a nilpotent supercharge and consider its cohomology.

2.5.2 Twisting Supercharge

From the 6d Perspective

We use Γ_i with $i \in \{0, \cdots, 9\}$ for 10d Euclidean gamma matrices. We also use the notation:

$$\Gamma_{i_1 \cdots i_n} := \Gamma_{i_1} \cdots \Gamma_{i_n} \,. \tag{2.177}$$

Type IIB has 32 supercharges, arranged into two Weyl spinors of the same 10 dimensional chirality – let us denote them as Q_l and Q_r. A general linear combination of them is written as $\epsilon_L Q_l + \epsilon_R Q_r$ where ϵ_L and ϵ_R are chiral spinors parametrizing the supercharge. The chirality constraints on them are:

$$i\Gamma_{0\cdots 9}\epsilon_L = \epsilon_L\,, \qquad i\Gamma_{0\cdots 9}\epsilon_R = \epsilon_R\,. \tag{2.178}$$

We shall discuss constraints on the supercharge by describing them as constraints on the parametrizing spinors.

The supercharges preserved by the D5 branes are constrained by:

$$\epsilon_R = i\Gamma_{012345}\epsilon_L\,. \tag{2.179}$$

This reduces the number of supercharges to 16. The D3 branes imposes the further constraint:

$$\epsilon_R = i\Gamma_{0237}\epsilon_L\,. \tag{2.180}$$

This reduces the number of supercharges by half once more. Therefore the defect preserves just 8 supercharges. Since ϵ_R is completely determined given ϵ_L, in what follows we refer to our choice of supercharge simply by referring to ϵ_L.

We want to perform a twist that makes the D5 world-volume theory topological along $\mathbb{R}^4$ and holomorphic along C. This twist was described in [43]. Let us give names to the two factors of $\mathbb{R}^4$ in the 10d space-time:

$$M := \mathbb{R}^4_{0123}\,, \qquad M' := \mathbb{R}^4_{6789}\,. \tag{2.181}$$

The spinors in the 6d theory transform as representations of Spin(6) under space-time rotations. $\mathcal{N} = (1,1)$ algebra has two left handed spinors and two right handed spinors transforming as $\mathbf{4}_l$ and $\mathbf{4}_r$ respectively.[47] The subgroup of Spin(6) preserving the product structure $\mathbb{R}^4 \times C$ is $\mathrm{Spin}(4)_M \times \mathrm{U}(1)$. Under this subgroup $\mathbf{4}_l$ and $\mathbf{4}_r$ transform as $(\mathbf{2},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{2})_{+1}$ and $(\mathbf{2},\mathbf{1})_{+1} \oplus (\mathbf{1},\mathbf{2})_{-1}$ respectively, where the subscripts denote the U(1) charges. Rotations along M' act as R-symmetry on the spinors – the spinors transform as representations of $\mathrm{Spin}(4)_{M'}$ such that $\mathbf{4}_+$ transforms as $(\mathbf{2},\mathbf{1})$ and $\mathbf{4}_-$ transforms as $(\mathbf{1},\mathbf{2})$. In total, under the symmetry group $\mathrm{Spin}(4)_M \times \mathrm{U}(1) \times \mathrm{Spin}(4)_{M'})$ the 16 supercharges of the 6d theory transform as:

$$((\mathbf{2},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{2})_{+1}) \otimes (\mathbf{2},\mathbf{1}) \oplus ((\mathbf{2},\mathbf{1})_{+1} \oplus (\mathbf{1},\mathbf{2})_{-1}) \otimes (\mathbf{1},\mathbf{2})\,. \tag{2.182}$$

[47] There are two of each chirality because the R-symmetry is $\mathrm{Sp}(1) \times \mathrm{Sp}(1) = \mathrm{Spin}(4)_{M'}$ such that the two left handed spinors transform as a doublet of one Sp(1) and the two right handed spinors transform as a doublet of the other Sp(1).

The twist we seek is performed by redefining the the space-time isometry:

$$\mathrm{Spin}(4)_M \rightsquigarrow \mathrm{Spin}(4)_M^{\mathrm{new}} \subseteq \mathrm{Spin}(4)_M \times \mathrm{Spin}(4)_{M'}\,, \tag{2.183}$$

where the subgroup $\mathrm{Spin}(4)_M^{\mathrm{new}}$ of $\mathrm{Spin}(4)_M \times \mathrm{Spin}(4)_{M'}$ consists of elements $(x, \theta(x))$ which is defined by the isomorphism $\theta : \mathrm{Spin}(4)_M \xrightarrow{\sim} \mathrm{Spin}(4)_{M'}$. More, explicitly, the isomorphism acts as:

$$\theta(\Gamma_{\mu\nu}) = \Gamma_{\mu+6,\nu+6}\,, \qquad \mu,\nu \in \{0,1,2,3\}\,. \tag{2.184}$$

The generators of the new $\mathrm{Spin}(4)_M^{\mathrm{new}}$ are then:

$$\Gamma_{\mu\nu} + \Gamma_{\mu+6,\nu+6}\,. \tag{2.185}$$

After this redefinition, the symmetry $\mathrm{Spin}(4)_M \times \mathrm{U}(1) \times \mathrm{Spin}(4)_{M'}$ of the 6d theory reduces to $\mathrm{Spin}(4)_M^{\mathrm{new}} \times \mathrm{U}(1)$ and under this group the representation (2.182) of the supercharges becomes:

$$2(\mathbf{1},\mathbf{1})_{-1} \oplus (\mathbf{3},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{3})_{-1} \oplus 2(\mathbf{2},\mathbf{2})_{+1}\,. \tag{2.186}$$

We thus have two supercharges that are scalars along M, both of them have charge -1 under the U(1) rotation along C. We take the generator of this rotation to be $-i\Gamma_{45}$, then if ϵ is one of the scalar (on M) supercharges that means:

$$i\Gamma_{45}\epsilon = \epsilon\,. \tag{2.187}$$

We identify the supercharge ϵ by imposing invariance under the new rotation generators on M, namely (2.185):

$$(\Gamma_{\mu\nu} + \Gamma_{\mu+6,\nu+6})\epsilon = 0\,. \tag{2.188}$$

The constraints (2.179) and (2.180) put by the D-branes and the U(1)-charge on C (2.187) together are equivalent to the following four independent constraints:

$$i\Gamma_{\mu,\mu+6}\epsilon = \epsilon\,, \qquad \mu\{0,1,2,3\}\,. \tag{2.189}$$

Together with the chirality constraint (2.178) in 10d we therefore have 5 equations, each reducing the number degrees of freedom by half. Since a Dirac spinor in 10d has 32 degrees of freedom, we are left with $32 \times 2^{-5} = 1$ degree of freedom, i.e., we have a unique supercharge,[48] which we call Q. It was shown in [43] that the supercharge Q is nilpotent:

$$Q^2 = 0\,, \tag{2.190}$$

[48]Note that without using the constraint put by the D3 branes we would get *two* supercharges that are scalar on M, i.e., there are two superhcarges in the 6d theory (by itself) that are scalar on M.

and the 6d theory twisted by this Q is topological along M – which is simply a consequence of (2.188) – and it is holomorphic along C. The latter claim follows from the fact that there is another supercharge in the 2d space of scalar (on M) supercharges in the 6d theory, let's call it Q', which has the following commutator with Q:

$$\{Q, Q'\} = \partial_{\overline{z}}\,, \tag{2.191}$$

where $z = \frac{1}{2}(x^4 - ix^5)$ is the holomorphic coordinate on C. This shows that $\overline{z}$-dependence is trivial (Q-exact) in the Q-cohomology.

From the 4d Perspective

What is new in our setup compared to the setup considered in [43] is the stack of D3 branes. We can figure out what happens to the world-volume theory of the D3 branes – we get the *Kapustin-Witten (KW)* twist [107], as we now show. The equations (2.189) can be used to to get the following six (three of which are independent) equations:

$$\begin{aligned} &(\Gamma_{02} + \Gamma_{68})\epsilon = 0\,, \qquad (\Gamma_{03} + \Gamma_{69})\epsilon = 0\,, \qquad (\Gamma_{23} + \Gamma_{89})\epsilon = 0\,, \\ &(\Gamma_{07} + \Gamma_{16})\epsilon = 0\,, \qquad (\Gamma_{27} + \Gamma_{18})\epsilon = 0\,, \qquad (\Gamma_{37} + \Gamma_{19})\epsilon = 0\,. \end{aligned} \tag{2.192}$$

These are in fact the equations that defines a scalar supercharge in the KW twist of $\mathcal{N} = 4$ theory on $\mathbb{R}^4_{0237}$ for a particular homomorphism from space-time ismoetry to the R-symmetry.[49] Space-time isometry of the theory on $\mathbb{R}^4_{0237}$ acts on the spinors as $\text{Spin}(4)_{\text{iso}}$, generated by the six generators:

$$\Gamma_{\mu\nu}\,, \qquad \mu, \nu \in \{0, 2, 3, 7\} \text{ and } \mu \neq \nu\,. \tag{2.193}$$

Rotations along the transverse directions act as R-symmetry, which is Spin(6), though the subgroup of the R-symmetry preserving the product structure $C \times \mathbb{R}^4_{1689}$ is $\text{U}(1) \times \text{Spin}(4)_{\text{R}}$. The KW twist is constructed by redefining space-time isometry to be a Spin(4) subgroup of $\text{Spin}(4)_{\text{iso}} \times \text{Spin}(4)_{\text{R}}$ consisting of elements $(x, \vartheta(x))$ where $\vartheta : \text{Spin}(4)_{\text{iso}} \xrightarrow{\sim} \text{Spin}(4)_{\text{R}}$ is an isomorphism. The particular isomorphism that leads to the equations (2.192) is:

$$\begin{aligned} &\Gamma_{02} \mapsto \Gamma_{68}\,, \qquad \Gamma_{03} \mapsto \Gamma_{69}\,, \qquad \Gamma_{23} \mapsto \Gamma_{89}\,, \\ &\Gamma_{07} \mapsto \Gamma_{16}\,, \qquad \Gamma_{27} \mapsto \Gamma_{18}\,, \qquad \Gamma_{37} \mapsto \Gamma_{19}\,. \end{aligned} \tag{2.194}$$

[49] Note that we ar using subscripts simply to refer to particular directions.

Remark 2.5.3 (A member of a $\mathbb{CP}^1$ family of twists). In [107] it was shown that there is a family of KW twists parametrized by $\mathbb{CP}^1$. The unique twist (by the supercharge Q) we have found is a specific member of this family. Let us identify which member that is.

The $\mathbb{CP}^1$ family comes from the fact that there is a 2d space of scalar (on M) supercharges (in (2.186)) in the twisted theory.[50] Also note from the original representation of the spinors (2.182) that the two scalar supercharges come from spinors transforming as $(\mathbf{1}, 2)$ and $(\mathbf{2}, 1)$ under the original isometry $\text{Spin}(4)^{\text{old}}$.[51] Let us choose two $\text{Spin}(4)^{\text{new}}$ scalar spinors with opposite $\text{Spin}(4)^{\text{old}}$ chiralities and call them ϵ_l and ϵ_r. The $\text{Spin}(4)^{\text{old}}$ chirality operator is $\Gamma^{\text{old}} := \Gamma_{0237}$. Let us choose ϵ_l and ϵ_r in such a way that they are related by the following equation:

$$\epsilon_r = N\epsilon_l \quad \text{where} \quad N = \frac{1}{4}(\Gamma_{06} + \Gamma_{28} + \Gamma_{39} + \Gamma_{17})\,. \tag{2.195}$$

This relation is consistent with the spinors being $\text{Spin}(4)^{\text{new}}$ invariant because N anticommutes with $\text{Spin}(4)^{\text{new}}$ (thus invariant spinors are still invariant after being operated with N), as well as with Γ^{old} (chaning $\text{Spin}(4)^{\text{old}}$ chirality). An arbitrary scalar supercharge in the twisted theory is a complex linear combination of ϵ_l and ϵ_r, such as $\alpha\epsilon_l + \beta\epsilon_r$, however, since the overall normalization of the spinor is irrelevant, the true parameter identifying a spinor is the ratio $t := \beta/\alpha \in \mathbb{CP}^1$. Furthermore, due to the equations (2.192), N^2 acts as -1 on any $\text{Spin}(4)^{\text{new}}$ scalar, leading to:

$$\epsilon_l = -N\epsilon_r\,. \tag{2.196}$$

To see the value of the twisting parameter t for the supercharge identified by the equations (2.189) (in addition to the 10d chirality (2.178)), we first pick a linear combination $\epsilon := \epsilon_l + t\epsilon_r$ with $t \in \mathbb{CP}^1$. Then using (2.196) and (2.189) we get:

$$-i\epsilon = N\epsilon = \epsilon_r - t\epsilon_l\,, \tag{2.197}$$

where the first equality follows from (2.189) and the second from (2.196). Equating the two sides we find the twisting parameter:

$$t = i\,. \tag{2.198}$$

△

[50] Though we began the discussion with a view to identifying topological-holomorphic twist of 6d $\mathcal{N} = (1,1)$ theory, what we found in the process in particular are supercharges that are scalar on M. If we forget that we had a 6d theory on $M \times C$ and just consider a theory on M with rotations on C being part of the R-symmetry then, first of all, we find a $\mathcal{N} = 4$ SYM theory on M and the twist we described is precisely the KW twist.

[51] We are writing $\text{Spin}(4)^{\text{old}}$ instead of $\text{Spin}(4)_M$ since the support of the 4d theory is not $M \equiv \mathbb{R}^4_{0123}$ but $\mathbb{R}^4_{0237}$.

From the 3d Perspective

Finally, at the 3 dimensional D3-D5 intersection lives a 3d $\mathcal{N} = 4$ theory consisting of bifundamental hypermultiplets coupled to background gauge fields which are restrictions of the gauge fields from the D3 and the D5 branes [81]. Considering Q-cohomology for the 3d theory reduces it to a topological theory as well. To identify the topological 3d theory we note that for the twisting parameter $t = i$, the 4d theory is an analogue of a 2d B-model[52] [107] and this can be coupled to a 3d analogue of the 2d B-model[53] – a B-type topological twist of 3d $\mathcal{N} = 4$ is called a *Rozanski-Witten (RW)* twist [149]. The flavor symmetry of the theory is $\mathrm{U}(N) \times \mathrm{U}(K)$ which acts on the hypers and is gauged by the background connections.

We can reach the same conclusion by analyzing the constraints on the twisting supercharge viewed from the 3d point of view. The bosonic symmetry of the 3d theory includes $\mathrm{SU}(2)_{\text{iso}} \times \mathrm{SU}(2)_H \times \mathrm{SU}(2)_C$ where $\mathrm{SU}(2)_{\text{iso}}$ is the isometry of the space-time $\mathbb{R}^3_{023}$, $\mathrm{SU}(2)_C$ are rotations in $\mathbb{R}^3_{689}$, and $\mathrm{SU}(2)_H$ are rotations in $\mathbb{R}^3_{145}$. The hypers in the 3d theory come from strings with one end attached to the D5 branes and another end attached to the D3 branes. Rotations in $\mathbb{R}^3_{145}$ – the R-symmetry $\mathrm{SU}(2)_H$ – therefore act on the hypers. This means that $\mathrm{SU}(2)_H$ acts on the Higgs branch of the 3d theory. This leaves the other R-symmetry group $\mathrm{SU}(2)_C$ which would act on the coulomb branch of the theory if the theory had some dynamical 3d vector multiplets. We now note that the topological twist, from the 3d perspective, involves twisting the isometry $\mathrm{SU}(2)_{\text{iso}}$ with the R-symmetry group $\mathrm{SU}(2)_C$, as evidenced explicitly by the three equations in the first line of (2.192). This particular topological twist (as opposed to the topological twist using the other R-symmetry $\mathrm{SU}(2)_H$) of 3d $\mathcal{N} = 4$ is indeed the RW twist [32].

To summerize, taking cohomology with respect to the supercharge Q leaves us with the KW twist (twisting parameter $t = i$) of $\mathcal{N} = 4$ SYM theory on $\mathbb{R}^4$ with gauge group $\mathrm{U}(N)$ and a topological-holomorphic twist of $\mathcal{N} = (1, 1)$ theory on $\mathbb{R}^4 \times C$ with gauge group $\mathrm{U}(K)$, and these two theories are coupled via a 3d RW theory of bifundamental hypers with flavor symmetry $\mathrm{U}(N) \times \mathrm{U}(K)$ gauged by background connections.[54]

[52]In particular, the 4d Theory on $\mathbb{R}^2 \times T^2$ can be compactified on the two-torus T^2 to get a B-model on $\mathbb{R}^2$.

[53]We want to be able to take the 3d theory on $\mathbb{R}^2 \times S^1$ and compactify it on S^1 to get a B-model on $\mathbb{R}^2$. If we have a 4d theory on $\mathbb{R}^2 \times T^2$ coupled to a 3d theory on $\mathbb{R}^2 \times S^1$, compactifying on T^2 should not make the two systems incompatible.

[54]Though it is customary to decouple the central $\mathrm{U}(1)$ subgroup from the gauge groups as it doesn't interact with the non-abelian part, our computations look somewhat simpler if we keep the $\mathrm{U}(1)$.

2.5.4 Omega Deformation

We start by noting that the dimensional reduction of the topological-holomorphic 6d theory from $\mathbb{R}^4 \times C$ to $\mathbb{R}^4$ reduces it to the KW twist of $\mathcal{N} = 4$ SYM on the $\mathbb{R}^4$.[55] This observation allows us to readily tailor the results obtained in [43] about omega deformation of the 6d theory to the case of omega deformation of 4d KW theory.

The fundamental bosonic field in the 10d $\mathcal{N} = 1$ SYM theory is the connection A_I where $I \in \{0, \cdots, 9\}$. When dimensionally reduced to 6d, this becomes a 6d connection A_M with $M \in \{0, \cdots, 5\}$ and four scalar fields ϕ_0, ϕ_1, ϕ_2, and ϕ_3 which are just the remaining four components of the 10d connection. The $\mathrm{Spin}(4)_M$ space-time isometry acts on the first four components of the connection, namely A_0, A_1, A_2, and A_3 via the vector representation. The four scalars – ϕ_0, ϕ_1, ϕ_2, and ϕ_3 – transform under the vector representation of the R-symmetry $\mathrm{Spin}(4)_{M'}$. Once twisted according to (2.183), only the diagonal subgroup $\mathrm{Spin}(4)_M^{\mathrm{new}}$ of $\mathrm{Spin}(4)_M \times \mathrm{Spin}(4)_{M'}$ acts on the fields, under which the first four components of the connection and the four scalars transform in the same way[56] and therefore we can package them together into one complex valued gauge field:

$$\mathcal{A}_\mu := A_\mu + i\phi_\mu \,, \qquad \mu \in \{0, 1, 2, 3\} \,. \tag{2.199}$$

We also write the remaining components of the connection in complex coordinates on C:

$$A_z := A_4 + iA_5 \quad \text{and} \quad A_{\overline{z}} := A_4 - iA_5 \,. \tag{2.200}$$

It was shown in [43] that this topological-holomorphic 6d theory can be viewed as a 2d gauged B-model on $\mathbb{R}^2_{23}$ where the fields are valued in maps $\mathrm{Map}(\mathbb{R}^2_{01} \times \mathbb{C}, \mathfrak{gl}_K)$. This is a vector space which plays the role of the Lie algebra of the 2d gauge theory. From the 2d point of view $\mathcal{A}_2$ and $\mathcal{A}_3$ are part of a connection on $\mathbb{R}^2_{23}$ and there are four chiral multiplets with the bottom components $\mathcal{A}_0, \mathcal{A}_1, A_z$, and $A_{\overline{z}}$. The 2d theory consists of a superpotential which is a holomorphic function of these chiral multiplets – the superpotential can be written conveniently in terms of a one form $\widetilde{\mathcal{A}} := \mathcal{A}_0 \mathrm{d}x^0 + \mathcal{A}_1 \mathrm{d}x^1 + A_z \mathrm{d}z + A_{\overline{z}} \mathrm{d}\overline{z}$ on $\mathbb{R}^2_{01} \times C$ consisting of these chiral fields:[57]

$$W(\mathcal{A}_0, \mathcal{A}_1, A_z, A_{\overline{z}}) = \int_{\mathbb{R}^2_{01} \times C} \mathrm{d}z \wedge \mathrm{tr}\left(\widetilde{\mathcal{A}} \wedge \mathrm{d}\widetilde{\mathcal{A}} + \frac{2}{3} \widetilde{\mathcal{A}} \wedge \widetilde{\mathcal{A}} \wedge \widetilde{\mathcal{A}} \right) \,. \tag{2.201}$$

[55] Both the 6d $\mathcal{N} = (1, 1)$ SYM and the 4d $\mathcal{N} = 4$ SYM are dimensional reductions of the 10d $\mathcal{N} = 1$ SYM and dimensional reduction commutes with the twisting procedure.

[56] Apart from the inhomogeneous transformation of the connection.

[57] Up to some overall numerical factors.

The superpotential is the action functional of a 4d CS theory on $\mathbb{R}^2_{01} \times C$ for the connection $\widetilde{\mathcal{A}}$.

One of the results of [43] is the following: Ω-deformation applied to this topological-holomorphic 6d theory with respect to rotation on $\mathbb{R}^2_{23}$ reduces the the theory to a 4d CS theory on $\mathbb{R}^2_{01} \times C$ with *complexified* gauge group GL_K .

The twisted 4d theory (the D3 world-volume theory) wraps the plane $\mathbb{R}^2_{23}$ as well and therefore is affected by the Ω-deformation. By noting that the 4d theory is a dimensional redcution of the 6d theory from $\mathbb{R}^4 \times C$ to $\mathbb{R}^4$ and assuming that Ω-deformation commutes with dimensional reduction,[58] we can deduce what the Ω-deformed version of the twisted 4d theory is. This will be a 2d gauge theory with complexified gauge group GL_N and the action will be the dimensional reduction of the 4d CS action (2.201) from $\mathbb{R}^2 \times C$ to $\mathbb{R}^2$ – this is the 2d BF theory where $A_{\overline{z}}$ plays the role of the B field:

$$\begin{aligned}\int_{\mathbb{R}^2\times C} \mathrm{d}z \wedge \mathrm{CS}(A_{\mathbb{R}^2\times C}) \xrightarrow{\text{Reduce on } C} & \int_{\mathbb{R}^2} \mathrm{tr} A_{\overline{z}} \left(\mathrm{d}A_{\mathbb{R}^2} + \frac{1}{2} A_{\mathbb{R}^2} \wedge A_{\mathbb{R}^2} \right) \\ = & \int_{\mathbb{R}^2} \mathrm{tr} A_{\overline{z}} F(A_{\mathbb{R}^2}) \,,\end{aligned} \tag{2.202}$$

where, as before, $\overline{z}$ is the anti-holomorphic coordinate on C.

Finally, it was shown in [161] that the RW twist of a 3d $\mathcal{N}=4$ theory on $\mathbb{R}^2_\Omega \times \mathbb{R}$ with only hypers reduces, upon Ω-deformation with respect to rotation in the plane $\mathbb{R}^2_\Omega$, to a free quantum mechanics on $\mathbb{R}$. A slight modification of this result, involving background connections gauging the flavor symmetry of the hypers leads to the result that the omega deformed theory is a gauged quantum mechanics, the kind of theory we have considered on the defect in the 2d BF theory.[59]

2.5.5 Takeaway from the Brane Construction

Via supersymmetric twists and Ω-deformation, we have made contact with precisely the setup we have considered in this chapter. We have a 4d CS theory with gauge group GL_K and a 2d BF theory with gauge group GL_N and they intersect along a topological line supporting a gauged quantum mechanics with $\mathrm{GL}_K \times \mathrm{GL}_N$ symmetry. We thus claim that the topological holographic duality that we have established in this chapter is indeed a topological subsector of the standard holographic duality involving defect $\mathcal{N}=4$ SYM.

[58] Alternatively, one can redo the localization computations of [43] for the 4d case, confirming that Ω-deformation does indeed commute with dimensional reduction.

[59] The bosonic version, which leads to the same Yangian with minor modifications to the computations as remarked in 2.3.2, 2.3.4, and 2.3.6.

2.6 Concluding Remarks and Future Works

In the previous sections we have been able to match a subsector of the operator algebra in the 2d BF theory with a line defect, with a subsector of the scattering algebra in a 3D closed string theory with a surface defect. The subsectors of operators we focused on are restricted to the defects on both sides of the duality. While this matching provides a non-trivial check of the proposed holographic duality, several immediate questions and new directions arise that were not addressed in the chapter. Let us comment on a few such issues that we think are interesting topics to pursue for future research.

Central extensions on two sides of the duality: To ease computation we restricted our attention to the quotients of the full operator algebra and scattering algebra by their centers. The inclusion of the central operators will change the associative structure of the algebras. A stronger statement of duality will be to compare the centrally extended Yangians coming from the boundary and the bulk theory.

Brane probes: Using branes in the bulk to probe local operators in the boundary theory has been a useful tool [120, 4]. In our setup, a brane must be *Lagrangian* in the A-twisted $\mathbb{R}^4$ directions. Looking at the brane setup (2.1) (which we reproduce in (2.204) for convenience) we see that the real directions of the D2 and D4 branes are Lagrangian with respect to the following symplectic form:

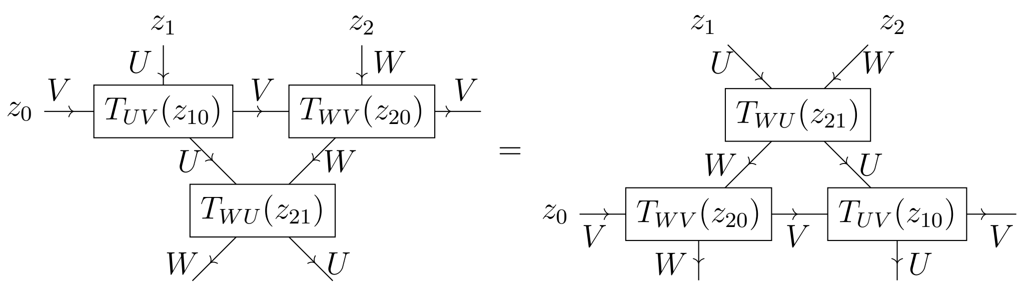
$$\mathrm{d}v \wedge \mathrm{d}x + \mathrm{d}w \wedge \mathrm{d}y\,. \tag{2.203}$$

This leaves the possibility of two more different embeddings for D2-branes:

	$\mathbb{R}_v$	$\mathbb{R}_w$	$\mathbb{R}_x$	$\mathbb{R}_y$	$\mathbb{C}_z$
D2	0	$\times$	$\times$	0	0
D4	0	0	$\times$	$\times$	$\times$
D2′	$\times$	0	0	$\times$	z
D2″	$\times$	$\times$	0	0	z

(2.204)

The D2′-branes are Wilson lines in the CS theory on the D4-branes perpendicular to the original Wilson line at thte D2-D4 intersection. Such crossing Wilson lines were studied in [41, 42] with the result that this corssing (of two Wilson lines carrying representations U and V of $\mathfrak{gl}_K$ respectively) inserts an operator $T_{VU}(z) : U \otimes V \to V \otimes U$ in the CS theory which solves the Yang-Baxter equation, which is described more easily with diagrams:



, (2.205)

where z_1, z_2, and z_3 are the spectral parameters (location in the complex plane) of the lines carrying representations V, U, and W respectively, and $z_{21} := z_2 - z_1$ and so on. Solutions of the above equation are closely tied to Quantum Groups. The operators $T_{UV}(z)$, which are commonly referred to as R-matrices, can be explicitly constructed using Feynman diagrams [41]. When the complex directions of the theory are parametrized by $\mathbb{C}$ (as in our case), these R-matrices are rational functions of z. If we choose U and W to be the fundamental representation of $\mathfrak{gl}_K$, then by providing an incoming and an outgoing fundamental state, we can view $\langle j|T_{\mathbf{K}V}(z)|i\rangle$ as a map $\mathsf{T}^i_j(z) : V \to V$ which has an expansion is z^{-1}:

$$\mathsf{T}^i_j(z) = \mathrm{id}_V\, \delta^i_j - \hbar \sum_{n\geq 0} \left(-z^{-1}\right)^{n+1} T^i_j[n]\,, \qquad (2.206)$$

where the $T^i_j[n]$ are precisely the operators that generate the scattering algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ (see (2.32) and (2.34)). This suggests that in the dual picture we should be able to interpret the D2$'$ branes as a generating function for the operators $O^i_j[n]$.

The interpretations of the D2$''$ branes are missing on both sides of the duality.

Finite N duality: Most of our computations were insensitive to the size of N. After computing the relevant algebras in both sides of the duality we considered their limits when $N \to \infty$ in §2.3.7 and §2.4.7. It was only in this limit that the algebras become the Yangian. However, it would be a stronger check if we could match the algebras at finite N, when they are quotients of the Yangian by some extra relations. In the CS theory we mentioned that these relations came from the fact that our operators were acting on some finite dimensional vector space, in the BF theory the dual operators were acting on a vector space of the same dimension. So the relations that resulted this way are the same. However, there were some extra relations in the BF side, where we argued that $O^i_j[N]$ was a linear combination of $O^i_j[n]$'s with $n < N$. To have a duality these relations should have an explanation in the CS side as well.

Duality for other quantum groups: In [41, 42] it was shown that by replacing our complex direction $\mathbb{C}$ with the punctured plane $\mathbb{C}^\times$ or an elliptic curve, we can get, instead of the Yangian, the trigonometric or elliptic solutions to the Yang-Baxter equation (2.205). It will be interesting to have an analogous analysis of holographic duality for the corresponding quantum groups as well.

Chapter 3

Phase Space of 2d BF Theory

In the previous chapter, it was shown that the algebra of local operators in $2d$ BF theory with gauge group GL_N coupled to a $1d$ fermionic[1] quantum mechanics with global symmetry GL_K (the boundary side) and the algebra of scattering states computed using Witten digrams of $4d$ Chern-Simons theory with gauge group GL_K (the bulk side) match and in the large-N limit approach the Yangian (see Theorem 1 on page 12 of [98]). Concretely, let B be the the B-field of BF theory and $(\overline{\psi}, \psi)$ are the fields of quantum mechanical system living on the line. It was shown in [98] that the subalgebra of a subset of local gauge-invariant operators of this system, which are given by

$$\overline{\psi}_a B^n \psi^b, \qquad a, b = 1, \cdots, K, \qquad n \geq 0, \tag{3.1}$$

in the large-N limit is the Yangian of $\mathfrak{gl}_K$.

In this chapter, we study the same problem from the perspective of the geometry of the phase space of $2d$-$1d$ coupled system. On of the reasons we are studying the problem from the phase-space perspective is that it allows us to make statements about some aspects of this example of twisted holography at finite N. Our strategy is to fist characterize the (classical or quantized) algebra in the large-N limit and then find the finite-N algebra as the quotient of the large-N limit algebra by an ideal. Let the phase space of the coupled $2d$-$1d$ system be denoted as $\mathcal{M}(N, K)$, which is parameterized by $(B, \psi, \overline{\psi})$.

[1]There is nothing special about quantum mechanics to be fermionic. We pointed out it since it was used in [98] for explicit computations. We could consider bosonic quantum mechanics and the resulting operator algebra would be the same. However, the brane configuration which leads to the fermionic vs bosonic quantum mechanics would be different [85, 86].

Summary of the results

As we have explained so far, we study the phase space of the $2d$ BF theory coupled to a $1d$ quantum-mechanical system. The basic logic of the chapter is to first study the classical phase space $\mathcal{M}(K,N)$ and its ring of functions $\mathbb{C}(K,N)$, and finally their large-N limit. We then study modules for these algebras. Then, we considered the quantization of the classical phase space and the deformation quantization $\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ of its ring of functions, which leads to the algebra. We study its structure, especially its coproduct and its identification with the Coulomb-branch algebra of $3d$ $\mathcal{N}=4$ theories.

Section 3.1 is devoted to the review of physical holographic setup. The main results of this work can be summarized as follows.

In Section 3.2, we investigate the geometry of the phase space of BF theory coupled to our quantum-mechanical system and study the algebra of functions in this phase space. The main results of this section are the following

1. The first result is concerned with the structure of the phase space; we show that $\mathcal{M}(N,K)$ is a normal affine variety of dimension $2NK$. This is shown in Corollary 3.2.8.

2. $\mathbb{C}[\mathcal{M}(N,K)]$, the algebra of functions on $\mathcal{M}(N,K)$ is generated by the set $\{\overline{\psi}B^n\psi; \mathrm{Tr}(B^n)\}$. Note that the operators $\mathrm{Tr}(B^n)$ are dual to the gravitons while determinant $(\det(B^n))$ and subdeterminant operators are dual to giant gravitons in the bulk. We then find that $\mathcal{M}(N,1)\simeq\mathbb{A}^{2N}$. Furthermore, by defining the morphism
$$\begin{aligned}\eta_{ab}:\mathcal{M}(N,K)&\to\mathcal{M}(1,K)\\(B,\psi,\overline{\psi})&\to(B,\psi_b,\overline{\psi}_a);\end{aligned}\tag{3.2}$$
we show that the products of η_{ab} for various a and b is a closed embedding
$$\prod_{1\le a,b\le K}\eta_{ab}:\mathcal{M}(N,K)\longrightarrow\mathcal{M}(N,1)\times_{\mathbb{A}^{(N)}}\mathcal{M}(N,1)\times_{\mathbb{A}^{(N)}}\cdots\times_{\mathbb{A}^{(N)}}\mathcal{M}(N,1),\tag{3.3}$$
where the right hand side has K^2 copies of $\mathcal{M}(N,1)$. This is achieved in Proposition 3.2.12.

3. Next, we define the following Poisson structure on $\mathcal{M}(N,K)$ by
$$\begin{aligned}\{\psi_{ia},\overline{\psi}_{bj}\}&=\delta_{ab}\delta_{ij},\\\{B_{mn},B_{pq}\}&=\delta_{pn}B_{mq}-\delta_{mq}B_{pn},\\\{B_{mn},\overline{\psi}_{bj}\}&=\{B_{mn},\psi_{ia}\}=0,\end{aligned}\tag{3.4}$$

Defining $T^{(n)}_{ab} \equiv \overline{\psi}_a B^n \psi_b$, with the convention $T^{(-1)}_{ab} = \delta_{ab}$, we show that

$$\{T^{(p)}_{ab}, T^{(q)}_{cd}\} = \sum_{i=-1}^{\min(p,q)-1} \left(T^{(p+q-1-i)}_{cb} T^{(i)}_{ad} - T^{(i)}_{cb} T^{(p+q-1-i)}_{ad} \right). \tag{3.5}$$

In Section 3.3, we consider the large-N limit of $\mathcal{M}(N,K)$ and its ring of functions $\mathbb{C}[\mathcal{M}(N,K)]$. The main results of this section as as follows

1. We show that $\bigcup_N \mathcal{M}(N,K)$ is Zariski-dense in $L^-\mathrm{GL}_K \times L^-\mathrm{GL}_1$, where $L^-\mathrm{GL}_K$ is the loop group defined in (3.28). This result is the content of Theorem 3.3.4.
2. Using this result, we then show that $\mathcal{M}(\infty,K) \cong L^-\mathrm{GL}_K \times L^-\mathrm{GL}_1$. This in turn would imply that

$$\mathbb{C}[\mathcal{M}(\infty,K)] \cong \mathbb{C}[L^-\mathrm{GL}_K] \otimes \mathbb{C}[L^-\mathrm{GL}_1]. \tag{3.6}$$

Section 3.4 is devoted to study of modules for $\mathbb{C}[\mathcal{M}(N,K)]$. The main result of this section is the computation of Hilbert series for $\mathbb{C}[\mathcal{M}(N,K)]$ in Theorem 3.4.4 and its large-N limit $\mathbb{C}[\mathcal{M}(\infty,K)]$ in Proposition 3.4.15.

In Section 3.5, we move to the quantization $\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ of the ring of functions $\mathbb{C}[\mathcal{M}(N,K)]$ on the phase space. Quantization amounts to replace the Poisson brackets (3.3) with commutators and studying the resulting algebras. The main results of this section are

1. We first prove the commutator of $T^{(n)}_{ab}$

$$[T^{(p)}_{ab}, T^{(q)}_{cd}] = \hbar \sum_{i=-1}^{\min(p,q)-1} \left(T^{(i)}_{cb} T^{(p+q-1-i)}_{ad} - T^{(p+q-1-i)}_{cb} T^{(i)}_{ad} \right). \tag{3.7}$$

   This is equivalent to the RTT relation if one defines the generating functions $T_{ab}(z)$ (the RTT generators) of $T^{(n)}_{ab}$ by the following power-series expansion at $z \to \infty$

$$T_{ab}(z) \equiv \sum_{n \geq -1} T^{(n)}_{ab} z^{-n-1} = \delta_{ab} + \overline{\psi}_a \frac{1}{z-B} \psi_b.$$

2. We next present one of the main results of this work, i.e. we show that the surjective map

$$Y_\hbar(\mathfrak{gl}_K \oplus \mathfrak{gl}_1) := Y_\hbar(\mathfrak{gl}_K) \otimes Y_\hbar(\mathfrak{gl}_1) \twoheadrightarrow \mathbb{C}_\hbar[\mathcal{M}(N,K)] \tag{3.8}$$

exists for any N. We present two arguments for the existence of this map in Theorem 3.5.5 and in Section 3.5.6. We also prove a particular observation of [51] according to which the quantum determinant of Yangian, whose coefficient determine the center, determine is given in terms of Capelli's determinant, defined in (3.55).

3. The coproduct of the quantized algebra is constructed in Section 3.5.15.

4. Finally, we explain the identification between the quantized ring of functions on the phase space $\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ and the Coulomb-branch algebra of certain $3d$ $\mathcal{N}=4$ quiver gauge theories.

Some details are relegated to the appendices. The Hall-Littlewood polynomial has been reviewed in Appendix B.1. Geometrization of the Jing operators, which are used in giving a vertex-algebra definition of the Hall-Littlewood polynomials is explained in Appendix B.2.

3.1 The Holographic Setup

In this section, we briefly review the twisted holography setup of [98].

The starting point is $6d$ topological string theory on $\mathbb{R}^4 \times \mathbb{C}$, where the theory is A-twisted along $\mathbb{R}^4$ and B-twisted along $\mathbb{C}$. These theories coming from a configuration of branes[2], which is summarized in Table 3.1.

	$\mathbb{R}_x$	$\mathbb{R}_y$	$\mathbb{R}_v$	$\mathbb{R}_w$	$\mathbb{C}_z$	# of branes
D2	$\times$			$\times$		N
D4	$\times$		$\times$		$\times$	K

Table 3.1: The brane configuration that realizes our twisted holography setup. The subscripts on $\mathbb{R}_x$ et al denote the coordinates along that direction. We have used the same conventions as [98]. The last column denotes the number of branes.

There are four main ingredients at play [29]: 1) the theory of open strings on the stack of

[2]Similar to [98], we are following the convention used in [3] where Dp-brane in topological string theory have p-dimensional world-volume in spacetime.

D2 branes, which is the $2d$ BF theory with gauge group GL_N with the following action

$$S_{\mathrm{BF}} = \int_{\mathbb{R}^2_{x,w}} \mathrm{Tr}_{\mathbf{N}}(BF_A), \tag{3.9}$$

where $B \in \Omega^0(\mathbb{R}^2_{x,w}, \mathfrak{gl}_N)$ is an adjoint-valued scalar, $A \in \Omega^1(\mathbb{R}^2_{x,w}, \mathfrak{gl}_N)$ is the gauge field for the gauge group GL_N with curvature $F_A = \mathsf{d}_A A = \mathsf{d}A + A \wedge A$, and the trace is taken over the fundamental representation of $\mathfrak{gl}_N$, which we have denoted as $\mathbf{N}$. This plays the role of the boundary side of the correspondence; 2) the theory on the stack of D4-branes, which is $4d$ Chern-Simons theory with gauge group GL_K [26, 27]. Since we do not need this theory in this chapter, we are not describing its details and refer the reader to [98]; 3) the $1d$ intersection of the two sets of branes, which introduces a line operator in both theories: the line operator in the BF-theory side is described by a quantum mechanics with fermionic degrees of freedom: $\psi^a \in \Omega^0(\mathbb{R}_x, \mathbf{N})$ with $a = 1, \cdots, K$[3], and the conjugate field $\overline{\psi}_a \in \Omega^0(\mathbb{R}_x, \overline{\mathbf{N}})$, where bar denotes the antifundamental representation. The action of this theory is given by

$$S_{\mathrm{QM}} = \int_L \sum_{a=1}^{K} \overline{\psi}_a (\mathsf{d} + A)\psi^a, \tag{3.10}$$

where we have denoted $\mathbb{R}_x$ as L. A carton of the setup is shown in Figure 3.1. In the $4d$ Chern-Simons theory side, we get a Wilson line taking values in some representation of $\mathfrak{gl}_K[\![z]\!] = \prod_{n\geq 0} \mathfrak{gl}_K \otimes z^n$ (at least classically). 4) The bulk closed-string theory sector, which is a mixture of the Kähler gravity along $\mathbb{R}^4_{x,y,v,w}$ and BCOV theory along $\mathbb{C}_z$ [11, 12, 45, 34]. Furthermore, we turn on a background 3-form flux field, which is sourced by D2-branes. This could deform the topology of the closed-string background and also the theory living on the stack of D4-branes. It turns out that the topology of the closed-string background is deformed to $\mathbb{R}^2_{x,w} \times \mathbb{R}_{+,r} \times S^3$, where $r \equiv (y^2 + v^2 + z\bar{z})^{\frac{1}{2}}$ parameterizes $\mathbb{R}_+$, and the background 3-form field measures the flux through S^3. The value of this flux is nothing but the number of D2-branes, which is N. On the other hand, it turns out that the pullback of the 3-form to D4-branes vanishes, and hence the theory of D4 branes is not deformed in the presence of this flux. Note that the theory on D4-branes could in principle be coupled to the modes in the closed-string theory living in the bulk. In the setup considered in [98], it is assumed that in the large-N limit, there is no such coupling. This has been called rigid holography in the physics literature [1]. Taking this point into account, the theory that effectively plays the role of bulk side in our twisted holography setup is $4d$ Chern-Simons

[3]Note that there is a change of notation compared to [98]. Here, we have used $a = 1, \cdots, K$ for the global symmetry indices while $i = 1, \cdots, K$ has been used for the global symmetry indices in [98].

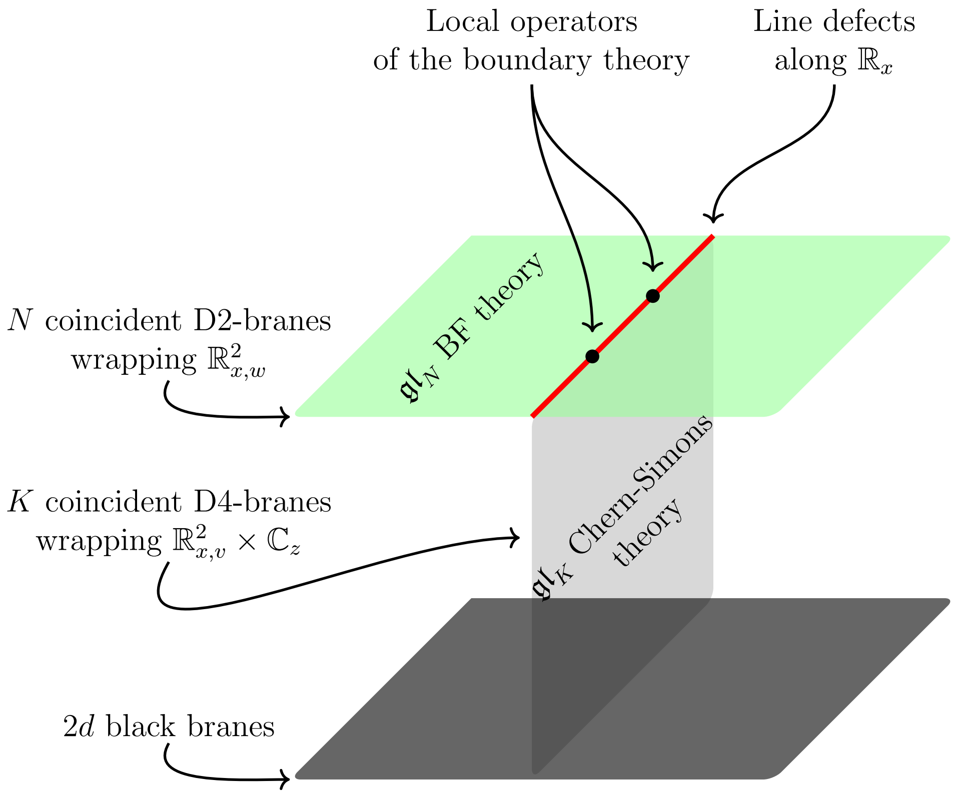


Figure 3.1: The holographic setup. N coincident D2-branes are hosting a $\mathfrak{gl}_N$ BF theory. These branes should be thought of as the imaging of D2-branes deep in the bulk which are sourcing the bulk fields. At the bottom of the figure, we have shown $2d$ black branes which are the D2-branes in the backreacted geometry of the bulk. A $4d$ $\mathfrak{gl}_K$ Chern-Simons theory lives on the of K coincident D4-branes. The intersection of the two stack of branes is a line defect on which a fermionic quantum-mechanical system lives.

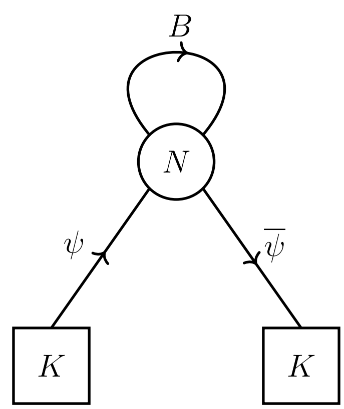


Figure 3.2: The quiver description of the phase space.

theory and we need to consider Witten diagrams of this theory in the computations of scattering through the bulk.

We would like to analyze the phase space of $2d$ BF theory coupled to a $1d$ quantum mechanics and its geometry. We choose the gauge $A = 0$. The equations of motion are

$$\begin{aligned} \mathsf{d}\psi^a &= \mathsf{d}\overline{\psi}_a = 0, \\ \mathsf{d}B - \sum_{a=1}^{K} \psi^a \overline{\psi}_a \delta_{w=0} &= 0. \end{aligned} \tag{3.11}$$

The solution is that ψ^a and $\overline{\psi}_a$ are constant along the line defect, B is constant on the regions $w < 0$ and $w > 0$, and

$$B_{w>0} - B_{w<0} = \sum_{a=1}^{K} \psi^a \overline{\psi}_a.$$

So the phase space is parametrized by $B_{w>0}$, ψ^a and $\overline{\psi}_a$, modulo the GL_N action. This is the quiver variety (categorical quotient) associated to the framed quiver in Figure 3.2. Let us denote this quiver variety by $\mathcal{M}(N, K) = \mathrm{Rep}(N, K)/\mathrm{GL}_N$, where $\mathrm{Rep}(N, K)$ is the linear space of representations of quiver in Figure 3.2. We study this space in Section 3.2.

3.2 Geometry of the Phase Space $\mathcal{M}(N, K)$

In this section, we analyze the phase space of the coupled theory $S_{\mathrm{BF}} + S_{\mathrm{QM}}$, where S_{BF} and S_{QM} are given by (3.9) and (3.10), respectively.

3.2.1 Singularities and resolution

When $K = 1$, $\mathcal{M}(N, 1)$ is the Zastava space $\mathcal{Z}^N_{\mathfrak{sl}_2}$ studied in [61]. Recall that the degree $\mathcal{Z}^N_{\mathfrak{sl}_2}$ is defined by the degree N based quasi-map space of the flag variety of SL_2. Explicitly, this is the space of polynomials $Q(z), P(z)$ such that $\deg Q < N$ and P is a monic polynomial of degree N. In particular, $\mathcal{Z}^N_{\mathfrak{sl}_2}$ is isomorphic to the affine space $\mathbb{A}^{2N}$. The isomorphism between $\mathcal{M}(N, 1)$ and $\mathcal{Z}^N_{\mathfrak{sl}_2}$ is given by the map

$$(B, \psi, \overline{\psi}) \mapsto (P(z) = \det(z - B), Q(z) = \overline{\psi}\,\mathrm{adj}(z - B)\psi). \tag{3.12}$$

Here $\mathrm{adj}(z - B)$ is the adjugate matrix of $z - B$.

For general K, then same argument in [61, Section 2] shows that the Laumon resolution $\mathcal{P}^N \to \mathcal{Z}^N$ factors through $\mathcal{M}(N, K)$. Here $\mathcal{P}^N$ is the parabolic Laumon space, i.e. the moduli space of degree N rank K subsheaves $\mathcal{F}$ of rank $2K$ trivial vector bundle on $\mathbb{P}^1$ such that $\mathcal{F}|_\infty$ is a sub-bundle and is a fixed rank K flag of $\mathbb{C}^{2K}$, and $\mathcal{Z}^N$ is the parabolic Zastava space associated to SL_{2K} and the parabolic subgroup $P \subset \mathrm{SL}_{2K}$ which stabilizes a fixed rank K flag in $\mathbb{C}^{2K}$ [17]. In fact $\mathcal{P}^N$ is isomorphic to the moduli space of stable representations of the quiver in Figure 3.2, denoted by $\mathcal{M}^s(N, K)$, where the stability condition is that if $V \subset \mathbb{C}^N$, $B(V) \subset V$ and $\mathrm{im}\,(\psi) \subset V$ then $V = \mathbb{C}^N$. The stability condition implies that the action of GL_N on the stable representations is free, thus $\mathcal{M}^s(N, K)$ is smooth. Since the parabolic Zastava space $\mathcal{Z}^N$ is affine, the Laumon resolution $\mathcal{P}^N \to \mathcal{Z}^N$ factors through the affinization of $\mathcal{P}^N = \mathcal{M}^s(N, K)$, which is $\mathcal{M}(N, K)$, and $\mathcal{M}(N, K) \to \mathcal{Z}^N$ is finite since Laumon resolution is proper. Moreover, $\mathcal{P}^N \to \mathcal{Z}^N$ is isomorphism on the locus where the subsheaf $\mathcal{F}$ is a sub-bundle, this corresponds to a map (instead of just a quasi-map) from $\mathbb{P}^1$ to Grassmannian $\mathrm{Gr}(K, 2K)$ which sends ∞ to identity. We call this locus the "regular" locus, and it has a quiver description as well: it parametrizes quiver representations that are stable and also co-stable, i.e. if $V \subset \mathbb{C}^N$, $B(V) \subset V$ and $V \subset \ker(\overline{\psi})$ then $V = 0$. The semi-simplification map $\mathcal{M}^s(N, K) \to \mathcal{M}(N, K)$ is also isomorphism on the regular locus, since GL_N acts on a stable and co-stable representation freely with closed orbit. This implies that the morphism $\mathcal{M}(N, K) \to \mathcal{Z}^N$ is birational. Since $\mathcal{M}(N, K)$ is affine quotient of a smooth variety normal $\mathcal{M}(N, K)$ is normal, thus

$\mathcal{M}(N,K) \to \mathcal{Z}^N$ is the normalization. It turns out that $\mathcal{M}(N,K) \to \mathcal{Z}^N$ is isomorphism, which will be proven in a more general context elsewhere.

Summarize the above discussions, we have morphisms of varieties:

$$\mathcal{P}^N \cong \mathcal{M}^s(N,K) \longrightarrow \mathcal{M}(N,K) \cong \mathcal{Z}^N,$$

such that $\mathcal{M}^s(N,K) \longrightarrow \mathcal{M}(N,K)$ is a resolution of singularities, and it is isomorphism when restricted on $\mathcal{M}(N,K)^{\mathrm{reg}}$.

Lemma 3.2.2. *For the resolution of singularities $f : \mathcal{M}^s(N,K) \longrightarrow \mathcal{M}(N,K)$, we have $\mathcal{O}_{\mathcal{M}(N,K)} \cong Rf_*\mathcal{O}_{\mathcal{M}^s(N,K)}$, i.e.*

(1) $R^i f_* \mathcal{O}_{\mathcal{M}^s(N,K)} = 0$ for $i > 0$,

(2) the natural homomorphism $\mathcal{O}_{\mathcal{M}(N,K)} \to f_*\mathcal{O}_{\mathcal{M}^s(N,K)}$ is isomorphism.

The key to the proof of Lemma 3.2.2 is Grauert-Riemenschneider vanishing theorem, we recall it here:

Theorem 3.2.3 (Grauert-Riemenschneider Vanishing)**.** *Let $h : X \longrightarrow Y$ be a resolution of singularities in characteristic zero, then $R^i h_*(\mathcal{K}_X) = 0$ for $i > 0$. Moreover let $\mathcal{L}$ be an ample line bundle on X, then $R^i h_*(\mathcal{K}_X \otimes \mathcal{L}) = 0$ for $i > 0$. Here $\mathcal{K}_X$ is the canonical line bundle of X.*

For a proof (of a more general version of this theorem), see [109, Corollary 2.68]. We would like to apply this theorem to $f : \mathcal{M}^s(N,K) \longrightarrow \mathcal{M}(N,K)$, but the sheaf in the theorem is the canonical sheaf, not the structure sheaf. This is not an issue, because:

Lemma 3.2.4. *The canonical line bundle on $\mathcal{M}^s(N,K)$ is trivial.*

Proof. Denote by $\mathcal{V}$ the tautological sheaf on $\mathcal{M}^s(N,K)$, which is the descent of $\mathbb{C}^N$ along the quotient $\mathrm{Rep}^s(N,K) \to \mathcal{M}^s(N,K)$, and denote by W the framing vector space, then there is a short exact sequence

$$0 \longrightarrow \mathrm{End}(\mathcal{V}) \longrightarrow \mathrm{End}(\mathcal{V}) \oplus W \otimes \mathcal{V}^* \oplus W^* \otimes \mathcal{V} \longrightarrow T_{\mathcal{M}^s(N,K)} \longrightarrow 0. \tag{3.13}$$

Here $T_{\mathcal{M}^s(N,K)}$ is the tangent sheaf of $\mathcal{M}^s(N,K)$. From this short exact sequence we get

$$\mathcal{K}_{\mathcal{M}^s(N,K)} = \det T^*_{\mathcal{M}^s(N,K)} \cong \det(W \otimes \mathcal{V}^*) \otimes \det(W^* \otimes \mathcal{V}) \cong \mathcal{O}_{\mathcal{M}^s(N,K)}.$$

□

Proof of Lemma 3.2.2. Since the canonical line bundle on $\mathcal{M}^s(N,K)$ is trivial, we have

$$R^i f_* \mathcal{O}_{\mathcal{M}^s(N,K)} \cong R^i f_* \mathcal{K}_{\mathcal{M}^s(N,K)} = 0,$$

for $i > 0$, by Grauert-Riemenschneider vanishing theorem. Since $\mathcal{M}(N,K)$ is normal and f is birational, we also have $\mathcal{O}_{\mathcal{M}(N,K)} \cong f_* \mathcal{O}_{\mathcal{M}^s(N,K)}$. □

Corollary 3.2.5. *$\mathcal{M}(N,K)$ has rational Gorenstein singularities.*

Proof. $\mathcal{M}(N,K)$ has rational singularities by Lemma 3.2.2. Then the dualizing sheaf $\omega_{\mathcal{M}(N,K)}$ is

$$\omega_{\mathcal{M}(N,K)} \cong Rf_* \mathcal{K}_{\mathcal{M}^s(N,K)} \cong Rf_* \mathcal{O}_{\mathcal{M}^s(N,K)} \cong \mathcal{O}_{\mathcal{M}(N,K)},$$

which is a line bundle, thus $\mathcal{M}(N,K)$ has Gorenstein singularities. □

3.2.6 Factorization

There is an obvious morphism:

$$\mathfrak{f}_{N_1,N_2} : \mathcal{M}(N_1,K) \times \mathcal{M}(N_2,K) \longrightarrow \mathcal{M}(N_1+N_2,K), \tag{3.14}$$

$$(B^{(1)}, \psi^{(1)}, \overline{\psi}^{(1)}) \times (B^{(2)}, \psi^{(2)}, \overline{\psi}^{(2)}) \mapsto \left(\begin{bmatrix} B^{(1)} & 0 \\ 0 & B^{(2)} \end{bmatrix}, \begin{bmatrix} \psi^{(1)} \\ \psi^{(2)} \end{bmatrix}, \begin{bmatrix} \overline{\psi}^{(1)} & \overline{\psi}^{(2)} \end{bmatrix}. \right) \tag{3.15}$$

Consider the natural projection

$$\Phi_N : \mathcal{M}(N,K) \longrightarrow \mathbb{A}^{(N)}. \tag{3.16}$$

Here Φ_N maps a triple $(B, \psi, \overline{\psi})$ to the coefficients of the characteristic polynomial of B, and $\mathbb{A}^{(N)}$ is the N'th symmetric product of affine line $\mathbb{A}^1$, which parametrizes coefficients of the characteristic polynomial of B. Denote by $\left(\mathbb{A}^{(N_1)} \times \mathbb{A}^{(N_2)}\right)_{\text{disj}}$ the open subset of $\mathbb{A}^{(N_1)} \times \mathbb{A}^{(N_2)}$ such that eigenvalues of $B^{(1)}$ is disjoint from eigenvalues of $B^{(2)}$. Analogous to the $K = 1$ case discussed in [61], we have the following factorization isomorphism

Proposition 3.2.7. *The restriction of $\mathfrak{f}_{N_1,N_2}$ on $\left(\mathbb{A}^{(N_1)} \times \mathbb{A}^{(N_2)}\right)_{\text{disj}}$ is isomorphism:*

$$\mathfrak{f}_{N_1,N_2} : (\mathcal{M}(N_1,K) \times \mathcal{M}(N_2,K))_{\text{disj}} \cong \mathcal{M}(N_1+N_2,K) \times_{\mathbb{A}^{(N_1+N_2)}} \left(\mathbb{A}^{(N_1)} \times \mathbb{A}^{(N_2)}\right)_{\text{disj}}.$$

Here $(\mathcal{M}(N_1,K) \times \mathcal{M}(N_2,K))_{\text{disj}}$ is the restriction of $\mathcal{M}(N_1,K) \times \mathcal{M}(N_2,K)$ on $\left(\mathbb{A}^{(N_1)} \times \mathbb{A}^{(N_2)}\right)_{\text{disj}}$.

Corollary 3.2.8. *$\mathcal{M}(N,K)$ is a normal affine variety of dimension $2NK$.*

Proof. $\mathcal{M}(N, K)$ is normal and affine since is the quotient of an affine space by GL_N, we only need to show that its dimension is $2NK$. By the factorization isomorphism, it suffices to show that $\dim \mathcal{M}(1, K) = 2K$. Note that $\mathcal{M}(1, K)$ is isomorphic to the $\mathbb{A}^1$ times the space of $K \times K$ matrices of rank ≤ 1 and the latter has dimension $2K - 1$. □

Using the normalization map $\mathcal{M}(N, K) \to \mathcal{Z}^N$ we have the following result:

Proposition 3.2.9. *The morphism $\Phi_N : \mathcal{M}(N, K) \to \mathbb{A}^{(N)}$ is equidimensional.*

Proof. It suffices to show that the projection $\Phi'_N : \mathcal{Z}^N \to \mathbb{A}^{(N)}$ is equidimensional, since $\mathcal{M}(N, K) \to \mathcal{Z}^N$ is finite. Here we prove that the central fiber $\Phi'^{-1}_N(0)$ has dimension $(2K - 1)N$, and the dimensions for other fibers follow from factorization isomorphism.

To compute $\dim \Phi'^{-1}_N(0)$, we use the description of the central fiber for parabolic Zastava in [17, 3.5], and obtain that

$$\Phi'^{-1}_N(0) \cong \overline{\mathrm{Gr}}_P^{N\theta} \cap \mathrm{Gr}_{U(P^-)} \subset \mathrm{Gr}_{\mathrm{SL}_{2K}}. \tag{3.17}$$

Here $P \subset \mathrm{SL}_{2K}$ is the parabolic subgroup which stabilizes a fixed rank K flag in $\mathbb{C}^{2K}$, $U(P^-)$ is the unipotent radical of the opposite of P, and $\theta = \mathrm{diag}(1, 0, \cdots, 0, -1)$ is the longest coroot of $\mathfrak{sl}_{2K}$. Then $\overline{\mathrm{Gr}}_P^{N\theta} \cap \mathrm{Gr}_{U(P^-)} \subset \overline{\mathrm{Gr}}^{N\theta} \cap \mathrm{Gr}_{U(B^-)}$ and the latter has dimension $\langle N\theta, \check{\rho} \rangle = (2K - 1)N$, thus $\dim \Phi'^{-1}_N(0) \leq (2K - 1)N$. Since the generic fiber of Φ_N has dimension $(2K - 1)N$, we also have the other direction of inequality $\dim \Phi'^{-1}_N(0) \geq (2K - 1)N$. Hence $\dim \Phi'^{-1}_N(0) = (2K - 1)N$. □

Corollary 3.2.10. *The morphism $\Phi_N : \mathcal{M}(N, K) \to \mathbb{A}^{(N)}$ is flat.*

Proof. This follows from Proposition 3.2.9, Corollary 3.2.5, and the miracle flatness theorem [151, Tag 00R4] □

3.2.11 Generators of $\mathbb{C}[\mathcal{M}(N, K)]$

By invariant theory, the algebra of functions on $\mathcal{M}(N, K)$, denoted by $\mathbb{C}[\mathcal{M}(N, K)]$, is generated by

$$\mathrm{Tr}(B^n),\ \overline{\psi}_a B^m \psi_b. \tag{3.18}$$

Here $1 \leq n \leq N$, $0 \leq m \leq N-1$ and $1 \leq a, b \leq K$. When $K = 1$, it turns out that there is no relations between these generators, i.e. $\mathrm{Tr}(B), \cdots, \mathrm{Tr}(B^N), \overline{\psi}\psi, \overline{\psi}B\psi, \cdots, \overline{\psi}B^{N-1}\psi$ give rise to an isomorphism $\mathcal{M}(N,1)\cong\mathbb{A}^{2N}$. In fact, since we know that $\dim\mathcal{M}(N,1)=2N$ and the map $\mathcal{M}(N,1)\to\mathbb{A}^{2N}$ is closed embedding, the map must be an isomorphism by dimensional reason.

For general K, let us fix a pair of integers a,b, then the functions $\mathrm{Tr}(B^n),\overline{\psi}_a B^m\psi_b$ give rise to a morphism $\eta_{ab}:\mathcal{M}(N,K)\to\mathcal{M}(1,K)$ sending a triple $(B,\psi,\overline{\psi})$ to $(B,\psi_b,\overline{\psi}_a)$. From the above discussions, we have

Proposition 3.2.12. *The product of η_{ab} is a closed embedding*

$$\prod_{1\le a,b\le K}\eta_{ab}:\mathcal{M}(N,K)\longrightarrow\mathcal{M}(N,1)\times_{\mathbb{A}^{(N)}}\cdots\times_{\mathbb{A}^{(N)}}\mathcal{M}(N,1),\tag{3.19}$$

where the right hand side has K^2 copies of $\mathcal{M}(N,1)$. Moreover, $\prod_{1\le a,b\le K}\eta_{ab}$ is compatible with factorization isomorphism $\mathfrak{f}_{N_1,N_2}$.

3.2.13 Poisson structure

Let us introduce a Poisson structure on the space of $(B,\psi,\overline{\psi})$ as following

$$\{\psi_{ia},\overline{\psi}_{bj}\}=\delta_{ab}\delta_{ij},\ \{B_{mn},B_{pq}\}=\delta_{pn}B_{mq}-\delta_{mq}B_{pn},\ \{B_{mn},\overline{\psi}_{bj}\}=\{B_{mn},\psi_{ia}\}=0.\tag{3.20}$$

Here we treat $\psi,\overline{\psi}$ as usual bosonic variables, i.e. commute instead of anti-commute with each other. This Poisson structure comes from the classical limit of $U_\hbar(\mathfrak{gl}_N)\otimes\mathrm{Weyl}_{NK}$, where Weyl_{NK} is the Weyl algebra generated by $\psi,\overline{\psi}$. It is easy to see that the Poisson structure is equivariant under the GL_N action, so it descends to $\mathcal{M}(N,K)$.

Remark 3.2.14. This is **not** the Poisson structure for the Zastava space. In fact, when $K=1$, this Poisson structure on $\mathbb{C}[\mathcal{M}(N,1)]$ is trivial, see the Theorem 3.2.15 below.

Define $T^{(n)}_{ab}=\overline{\psi}_a B^n\psi_b$, and we use the convention $T^{(-1)}_{ab}=\delta_{ab}$, then denote by $T_{ab}(z)$ the power series expanded at $z\to\infty$:

$$T_{ab}(z)=\sum_{n\ge -1}T^{(n)}_{ab}z^{-n-1}=\delta_{ab}+\overline{\psi}_a\frac{1}{z-B}\psi_b.$$

Proposition 3.2.15. *The Poisson brackets between $T^{(k)}_{ab}$ are:*

$$\{T^{(p)}_{ab},T^{(q)}_{cd}\}=\sum_{i=-1}^{\min(p,q)-1}\left(T^{(p+q-1-i)}_{cb}T^{(i)}_{ad}-T^{(i)}_{cb}T^{(p+q-1-i)}_{ad}\right).\tag{3.21}$$

And for all $n\ge 1$, $\mathrm{Tr}(B^n)$ is Poisson central.

Proof. This is the classical limit of (3.49). □

Remark 3.2.16. In [98], another presentation of Poisson structure is obtained:

$$\{\psi_{ia}, \overline{\psi}_{bj}\} = \delta_{ab}\delta_{ij},\ \{B_{mn}, B_{pq}\} = \delta_{np}\sum_a \overline{\psi}_{aq}\psi_{ma} - \delta_{mq}\sum_a \overline{\psi}_{an}\psi_{pa},$$
$$\{B_{mn}, \overline{\psi}_{bj}\} = \{B_{mn}, \psi_{ia}\} = 0, \tag{3.22}$$

3.2.17 Multiplication morphism

Apart from the obvious factorization map (3.14), there is another map

$$\mathfrak{m}_{N_1,N_2} : \mathcal{M}(N_1, K) \times \mathcal{M}(N_2, K) \longrightarrow \mathcal{M}(N_1 + N_2, K), \tag{3.23}$$
$$(B^{(1)}, \psi^{(1)}, \overline{\psi}^{(1)}) \times (B^{(2)}, \psi^{(2)}, \overline{\psi}^{(2)}) \mapsto \left(\begin{bmatrix} B^{(1)} & \psi^{(1)}\overline{\psi}^{(2)} \\ 0 & B^{(2)} \end{bmatrix}, \begin{bmatrix} \psi^{(1)} \\ \psi^{(2)} \end{bmatrix}, \begin{bmatrix} \overline{\psi}^{(1)} & \overline{\psi}^{(2)} \end{bmatrix} \right). \tag{3.24}$$

We have the following elementary property of the multiplication morphism.

Proposition 3.2.18. *The multiplication morphism $\mathfrak{m}_{N_1,N_2}$ is dominant.*

Proof. It suffices to prove that the composition $\mathfrak{f}^{-1}_{N_1,N_2} \circ \mathfrak{m}_{N_1,N_2}$ is dominant when restricted on $\left(\mathbb{A}^{(N_1)} \times \mathbb{A}^{(N_2)}\right)_{\mathrm{disj}}$. First of all, we construct a $\mathrm{GL}_{N_1} \times \mathrm{GL}_{N_2}$ equivariant map

$$\widetilde{\mathfrak{m}}_{N_1,N_2} : (\mathrm{Rep}(N_1, K) \times \mathrm{Rep}(N_2, K))_{\mathrm{disj}} \longrightarrow (\mathrm{Rep}(N_1, K) \times \mathrm{Rep}(N_2, K))_{\mathrm{disj}},$$

such that $\widetilde{\mathfrak{m}}_{N_1,N_2}$ descends to $\mathfrak{f}^{-1}_{N_1,N_2} \circ \mathfrak{m}_{N_1,N_2}$ after taking the quotient by $\mathrm{GL}_{N_1} \times \mathrm{GL}_{N_2}$. The construction is as follows. If the spectra of B_1 and B_2 are disjoint from each other, then linear map $\mathrm{Mat}(N_1, N_2) \to \mathrm{Mat}(N_1, N_2), X \mapsto B_1X - XB_2$ is an isomorphism. Let A be the unique $N_1 \times N_2$ matrix such that

$$B^{(1)}A - AB^{(2)} = \psi^{(1)}\overline{\psi}^{(2)} \text{ holds.}$$

Then we can use the matrix

$$\begin{bmatrix} 1 & A \\ 0 & 1 \end{bmatrix} \text{ to diagonalize } \begin{bmatrix} B^{(1)} & \psi^{(1)}\overline{\psi}^{(2)} \\ 0 & B^{(2)} \end{bmatrix}$$

and it accordingly maps $\left[\overline{\psi}^{(1)}, \overline{\psi}^{(2)}\right]$ to $\left[\overline{\psi}^{(1)}, \overline{\psi}^{(2)} - \overline{\psi}^{(1)} A\right]$ and $\left[\psi^{(1)}, \psi^{(2)}\right]^{\mathrm{t}}$ to $\left[\psi^{(1)} + A\psi^{(2)}, \psi^{(2)}\right]^{\mathrm{t}}$. Hence we define $\widetilde{\mathfrak{m}}_{N_1,N_2}$ as

$$(B^{(1)}, \psi^{(1)}, \overline{\psi}^{(1)}) \times (B^{(2)}, \psi^{(2)}, \overline{\psi}^{(2)}) \mapsto (B^{(1)}, \psi^{(1)} + A\psi^{(2)}, \overline{\psi}^{(1)}) \times (B^{(2)}, \psi^{(2)}, \overline{\psi}^{(2)} - \overline{\psi}^{(1)} A). \tag{3.25}$$

Notice that the tangent map $d\widetilde{\mathfrak{m}}_{N_1,N_2}$ is an isomorphism at any point $(B^{(1)}, 0, \overline{\psi}^{(1)}) \times (B^{(2)}, \psi^{(2)}, 0)$, so $\widetilde{\mathfrak{m}}_{N_1,N_2}$ is generically étale thus it is dominant. Then it follows that $\mathfrak{m}_{N_1,N_2}$ is dominant. □

Proposition 3.2.19. *The multiplication morphism $\mathfrak{m}_{N_1,N_2}$ has following properties*

(1) $\mathfrak{m}_{N_1,N_2}$ *is Poisson,*

(2) $\mathfrak{m}_{N_1+N_2,N_3} \circ (\mathfrak{m}_{N_1,N_2} \times \mathrm{Id}) = \mathfrak{m}_{N_1,N_2+N_3} \circ (\mathrm{Id} \times \mathfrak{m}_{N_2,N_3})$*, i.e. multiplication is associative.*

The proposition will be evident once we make connection to the multiplication map on the loop group in the next section. Note that the factorization map $\mathfrak{f}_{N_1,N_2}$ is not Poisson in general.

3.2.20 Embedding $\mathcal{M}(N, K) \hookrightarrow \mathcal{M}(N', K)$

Suppose that $N < N'$, then we have a morphism

$$\iota_{N,N'} : \mathcal{M}(N, K) \longrightarrow \mathcal{M}(N', K), \tag{3.26}$$

$$(B, \psi, \overline{\psi}) \mapsto \left(\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \psi \\ 0 \end{bmatrix}, \begin{bmatrix} \overline{\psi} & 0 \end{bmatrix} \right). \tag{3.27}$$

Note that $\iota^*_{N,N'}(\mathrm{Tr}(B^n)) = \mathrm{Tr}(B^n), \iota^*_{N,N'}(T^{(m)}_{ab}) = T^{(m)}_{ab}$, so $\iota^*_{N,N'}$ is surjective, thus $\iota_{N,N'}$ is a closed embedding.

Proposition 3.2.21. *The embedding $\iota_{N,N'}$ has following properties*

(1) $\iota_{N',N''} \circ \iota_{N,N'} = \iota_{N,N''}$*,*

(2) $\iota_{N,N'}$ *is Poisson,*

(3) $\mathfrak{m}_{N'_1,N'_2} \circ \left(\iota_{N_1,N'_1} \times \iota_{N_2,N'_2}\right) = \iota_{N_1+N_2,N'_1+N'_2} \circ \mathfrak{m}_{N_1,N_2}$.

Proof. Property (1) is obvious from definition of $\iota_{N,N'}$, (2) is a corollary of Proposition 3.2.15, only (3) needs explanation. Using property (1), the proof of (3) reduces to the cases of either $N_1' = N_1, N_2' = N_2 + 1$ or $N_1' = N_1 + 1, N_2' = N_2$. The first case is obvious from the definition of embedding and multiplication morphism, so we only need to consider the case when $N_1' = N_1 + 1, N_2' = N_2$. It amounts to showing that

$$\left(\begin{bmatrix} B^{(1)} & 0 & \psi^{(1)}\overline{\psi}^{(2)} \\ 0 & 0 & 0 \\ 0 & 0 & B^{(2)} \end{bmatrix}, \begin{bmatrix} \psi^{(1)} \\ 0 \\ \psi^{(2)} \end{bmatrix}, \begin{bmatrix} \overline{\psi}^{(1)} & 0 & \overline{\psi}^{(2)} \end{bmatrix}\right)$$

is equivalent to

$$\left(\begin{bmatrix} B^{(1)} & \psi^{(1)}\overline{\psi}^{(2)} & 0 \\ 0 & B^{(2)} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \psi^{(1)} \\ \psi^{(2)} \\ 0 \end{bmatrix}, \begin{bmatrix} \overline{\psi}^{(1)} & \overline{\psi}^{(2)} & 0 \end{bmatrix}\right)$$

under the action of some matrix $W \in \mathrm{GL}_{N_1+N_2+1}$. It is elementary to check that

$$W = \begin{bmatrix} \mathrm{Id}_{N_1} & 0 \\ 0 & w_{N_2}w_{N_2-1}\cdots w_1 \end{bmatrix}$$

does the job, where $w_i \in \mathrm{GL}_{N_2+1}$ switches row i and row $i+1$. □

3.3 Large-N Limit

In this section we use the embeddings $\iota_{N,N'} : \mathcal{M}(N,K) \hookrightarrow \mathcal{M}(N',K)$ constructed in the previous section to define the large-N limit of the family $\mathcal{M}(N,K)$ as the spectrum of $\mathbb{C}^\times$-finite elements in the inverse limit of algebras $\mathbb{C}[\mathcal{M}(N,K)]$, and show that the large-N limit is isomorphic to the Poisson group $L^-(\mathrm{GL}_K \times \mathrm{GL}_1)$, defined below. It is known that $L^-(\mathrm{GL}_K \times \mathrm{GL}_1)$ quantizes to the Yangian $Y_\hbar(\mathfrak{gl}_K \oplus \mathfrak{gl}_1)$, and we will explore the quantized version of the larg-N limit in the next section.

Definition 3.3.1. Define $\mathbb{C}[\mathcal{M}(\infty,K)]$ to be the subalgebra of $\varprojlim_N \mathbb{C}[\mathcal{M}(N,K)]$ generated by $T^{(n)}_{ab}$ and $\mathrm{Tr}(B^m)$, for all $n,m \in \mathbb{Z}_{\geq 0}$ and $1 \leq a,b \leq K$. And then define $\mathcal{M}(\infty,K) = \mathrm{Spec}\mathbb{C}[\mathcal{M}(\infty,K)]$.

Denote by $L^-\mathrm{GL}_K$ the group of power series

$$1 + \sum_{i=1}^{\infty} g_i z^{-i},\ g_i \in \mathfrak{gl}_K, \tag{3.28}$$

here the group structure is the multiplication of power series in matrices. Consider the morphism

$$i_N = (\pi_N, \varphi_N) : \mathcal{M}(N, K) \to L^- \mathrm{GL}_K \times L^- \mathrm{GL}_1, \tag{3.29}$$

$$(B, \psi, \overline{\psi}) \mapsto \left(1 + \overline{\psi} \frac{1}{z - B} \psi,\ \frac{1}{z^N} \det(z - B)\right), \tag{3.30}$$

which is a closed embedding because $T^{(n)}_{ab}$ and $\mathrm{Tr}(B^m)$ generate $\mathbb{C}[\mathcal{M}(N, K)]$. Here $(z - B)^{-1}$ is expanded as a power series of matrices in z^{-1}. It is known that $L^- \mathrm{GL}_K$ is a Poisson-Lie group scheme whose Poisson structure comes from the Manin triple

$$(\mathfrak{gl}_K(\!(z^{-1})\!), z^{-1}\mathfrak{gl}_K[\![z^{-1}]\!], \mathfrak{gl}_K[z]).$$

Explicitly, let $T^{(n)}_{ab}, n \geq -1$ be the function on $L^- \mathrm{GL}_K$ that takes the value of ab component of g_{n+1} and we use the convention that $T^{(-1)}_{ab} = \delta_{ab}$, then the Poisson structure on $L^- \mathrm{GL}_K$ is determined by the equation

$$(u - v)\{T_{ab}(u), T_{cd}(v)\} = T_{ad}(v)T_{cb}(u) - T_{ad}(u)T_{cb}(v), \text{ where } T_{ab}(u) = \sum_{i=-1}^{\infty} T^{(i)}_{ab} u^{-i-1}. \tag{3.31}$$

Compare equation (3.31) with equation (3.21), and we have

Proposition 3.3.2. *The morphism $i_N : \mathcal{M}(N, K) \to L^- \mathrm{GL}_K \times L^- \mathrm{GL}_1$ is Poisson.*

Proposition 3.3.3. *i_N is compatible with embedding $\iota_{N,N'}$ and multiplication $\mathfrak{m}_{N_1,N_2}$, i.e.*

(1) $i_{N'} \circ \iota_{N,N'} = i_N$,

(2) $i_{N_1+N_2} \circ \mathfrak{m}_{N_1,N_2} = \mathfrak{m} \circ (i_{N_1} \times i_{N_2})$.

Here $\mathfrak{m} : L^-(\mathrm{GL}_K \times \mathrm{GL}_1) \times L^-(\mathrm{GL}_K \times \mathrm{GL}_1) \to L^-(\mathrm{GL}_K \times \mathrm{GL}_1)$ is the multiplication map of the group $L^-(\mathrm{GL}_K \times \mathrm{GL}_1)$.

Proof. (1) is obvious from definition. (2) can be shown by direct computation. If $(B^{(1)}, \psi^{(1)}, \overline{\psi}^{(1)})$ is a point in $\mathcal{M}(N_1, K)$ and $(B^{(2)}, \psi^{(2)}, \overline{\psi}^{(2)})$ is a point in $\mathcal{M}(N_2, K)$, then $\pi_{N_1+N_2} \circ \mathfrak{m}_{N_1,N_2}$

maps this pair of representations to

$$
\begin{aligned}
&1+\begin{bmatrix}\overline{\psi}^{(1)} & \overline{\psi}^{(2)}\end{bmatrix}\left(z-\begin{bmatrix}B^{(1)} & \psi^{(1)}\overline{\psi}^{(2)}\\ 0 & B^{(2)}\end{bmatrix}\right)^{-1}\begin{bmatrix}\psi^{(1)}\\ \psi^{(2)}\end{bmatrix}\\
&=1+\begin{bmatrix}\overline{\psi}^{(1)} & \overline{\psi}^{(2)}\end{bmatrix}\left(z-\begin{bmatrix}B^{(1)} & \psi^{(1)}\overline{\psi}^{(2)}\\ 0 & B^{(2)}\end{bmatrix}\right)^{-1}\begin{bmatrix}\psi^{(1)}\\ \psi^{(2)}\end{bmatrix}\\
&=1+\overline{\psi}^{(1)}\frac{1}{z-B^{(1)}}\psi^{(1)}+\overline{\psi}^{(2)}\frac{1}{z-B^{(2)}}\psi^{(2)}\\
&\ +\sum_{i,j=0}^{\infty}\overline{\psi}^{(1)}\left(B^{(1)}\right)^{i}\psi^{(1)}\overline{\psi}^{(2)}\left(B^{(2)}\right)^{j}\psi^{(2)}z^{-i-j-2}\\
&=\left(1+\overline{\psi}^{(1)}\frac{1}{z-B^{(1)}}\psi^{(1)}\right)\left(1+\overline{\psi}^{(2)}\frac{1}{z-B^{(2)}}\psi^{(2)}\right).
\end{aligned}
$$

And we also have $\varphi_{N_1+N_2}\circ\mathfrak{m}_{N_1,N_2}=\mathfrak{m}\circ(\varphi_{N_1}\times\varphi_{N_2})$ by the multiplicativity of determinants of block diagonal matrices. □

Proof of Proposition 3.2.19. (1) follows from Proposition 3.3.2 and the fact that the Poisson structure on $L^-(\mathrm{GL}_K\times\mathrm{GL}_1)$ makes it a Poisson-Lie group, i.e. $\mathfrak{m}$ is Poisson. (2) is a direct consequence of Proposition 3.3.3. □

Since i_N is compatible with $\iota_{N,N'}$, it makes sense to take the ind-scheme $\bigcup_N\mathcal{M}(N,K)$ inside $L^-\mathrm{GL}_K\times L^-\mathrm{GL}_1$.

Theorem 3.3.4. *$\bigcup_N\mathcal{M}(N,K)$ is Zariski-dense in $L^-\mathrm{GL}_K\times L^-\mathrm{GL}_1$.*

Proof. It suffices to show that for every N, there exists N' such that $L^-_N\mathrm{GL}_K\times L^-_N\mathrm{GL}_1$ is a closed subscheme of $\mathcal{M}(N',K)$. Denote by $\mathfrak{m}=\mathfrak{m}_{L^-\mathrm{GL}_K}\times\mathfrak{m}_{L^-_N\mathrm{GL}_1}$ the multiplication map on $L^-\mathrm{GL}_K\times L^-\mathrm{GL}_1$. We make two observations

(1) $L^-_1\mathrm{GL}_K\times\{1\}\subset\mathcal{M}(K,K)$. This is because $S\times\{1\}\subset\mathcal{M}(K,K)$, where S is the subvariety of $L^-_1\mathrm{GL}_K$:

$$1+\frac{g}{z},\ g\in\mathfrak{gl}_K\text{ such that }\mathrm{rank}(g)\le 1.$$

then we can apply the multiplication $\mathfrak{m}$ K times to obtain $L^-_1\mathrm{GL}_K\times\{1\}$, more precisely, we have following linear algebra fact:

– Every matrix $M \in \mathfrak{gl}_K$ can be written as a linear combination $M = X_1 + \cdots + X_K$ such that $\operatorname{rank}(X_i) \leq 1$ and $X_i X_j = 0$ if $i < j$.

This can be interpreted as

$$1 + \frac{M}{z} = \left(1 + \frac{X_1}{z}\right) \cdots \left(1 + \frac{X_K}{z}\right),$$

which is exactly what we want to show. To show this fact, we notice that the statement is true for M if and only if it is true for AMA^{-1} for some $A \in \mathrm{GL}_K$, so without loss of generality, we assume that M is a Jordan block J_λ, and then take $X_i = a_i^{\mathrm{t}} b_i$, where

$$\begin{aligned} a_i &= (0, \cdots, 0, \lambda, 1, 0, \cdots, 0),\ i < K \text{ and } i\text{'th component is } \lambda, \\ a_K &= (0, \cdots, 0, \lambda), \\ b_i &= (0, \cdots, 0, 1, 0, \cdots, 0),\ i\text{'th component is } 1. \end{aligned}$$

If M is a direct sum of Jordan blocks, then we take X_i associated to each individual block.

(2) $\{1\} \times L_1^- \mathrm{GL}_1 \subset \mathcal{M}(1, K)$, this is because $\{1\} \times L_1^-$ is the the image of points $(b, 0, 0) \subset \mathcal{M}(1, K)$.

(3) The multiplication map $\mathfrak{m}_{L^-\mathrm{GL}_K} : L_1^- \mathrm{GL}_K \times L_N^- \mathrm{GL}_K \to L_{N+1}^- \mathrm{GL}_K$ is dominant. In effect, the tangent map $d\mathfrak{m}_{L^-\mathrm{GL}_K}$ at the point $\left(1, 1 + 1/z + \cdots + 1/z^N\right)$ is

$$\left(\frac{X}{z}, \frac{Y_1}{z}, \cdots, \frac{Y_i}{z^i}, \cdots, \frac{Y_N}{z^N}\right) \mapsto \frac{X + Y_1}{z}, \cdots, \frac{X + Y_N}{z^N}, \frac{X}{z^{N+1}} \tag{3.32}$$

where left hand side is a tangent vector at $\left(1, 1 + 1/z + \cdots + 1/z^N\right)$, and right hand side is a tangent vector at $1 + 1/z + \cdots + 1/z^N \in L_{N+1}^- \mathrm{GL}_K$. Since $X, Y_1, \cdots, Y_N$ take value in all matrices in $\mathfrak{gl}_K$, the linear map (3.32) is surjective and thus is an isomorphism by dimension counting. It follows that $\mathfrak{m}_{L^-\mathrm{GL}_K} : L_1^- \mathrm{GL}_K \times L_N^- \mathrm{GL}_K \to L_{N+1}^- \mathrm{GL}_K$ is étale at the point $\left(1, 1 + 1/z + \cdots + 1/z^N\right)$, thus it is generically étale, and dominant.

Combine (1) and (2) and use the multiplication $\mathfrak{m}$ (which is compatible with the multiplications of $\mathcal{M}(N, K)$), then we have an inclusion $L_1^- \mathrm{GL}_K \times L_1^- \mathrm{GL}_1 \subset \mathcal{M}(K+1, K)$. (3) implies that $\mathfrak{m} : \left(L_1^- \mathrm{GL}_K \times L_1^- \mathrm{GL}_1\right) \times \left(L_N^- \mathrm{GL}_K \times L_N^- \mathrm{GL}_1\right) \to L_{N+1}^- \mathrm{GL}_K \times L_{N+1}^- \mathrm{GL}_1$ is dominant. By induction on N, we have inclusions $L_N^- \mathrm{GL}_K \times L_N^- \mathrm{GL}_1 \subset \mathcal{M}((K+1)N, K)$. This concludes the proof. □

Corollary 3.3.5. *$\mathcal{M}(\infty, K) \cong L^{-}\mathrm{GL}_K \times L^{-}\mathrm{GL}_1$, i.e.*

$$\mathbb{C}[\mathcal{M}(\infty, K)] \cong \mathbb{C}[L^{-}\mathrm{GL}_K] \otimes \mathbb{C}[L^{-}\mathrm{GL}_1]. \tag{3.33}$$

3.4 Modules of $\mathbb{C}[\mathcal{M}(N, K)]$ and their Hilbert series

Recall that we have a resolution of singularities $f : \mathcal{M}^s(N, K) \longrightarrow \mathcal{M}(N, K)$, where $\mathcal{M}^s(N, K)$ is the moduli space of stable representations of the quiver in the Figure 3.2. The action of gauge group GL_N on the space of stable representations $\mathrm{Rep}^s(N, K)$ is free, so the quotient map $\mathrm{Rep}^s(N, K) \to \mathcal{M}^s(N, K)$ is a principal GL_N-bundle. The gauge node vector space $\mathbb{C}^N$ is a trivial bundle on $\mathrm{Rep}(N, K)$ but it is endowed with a non-trivial equivariant structure under the action of GL_N, then it descend to a locally free sheaf $\mathcal{V}$ on $\mathcal{M}^s(N, K)$ since the GL_N action on the stable locus is free. We call this locally free sheaf $\mathcal{V}$ the *tautological sheaf*, and call its determinant line bundle the *tautological line bundle*, denoted by Det.

Lemma 3.4.1. *The tautological line bundle* Det *is ample on $\mathcal{M}^s(N, K)$.*

This lemma will be proven in the next subsection. Apply the Grauert-Riemenschneider vansihing theorem 3.2.3 to the tautological line bundle Det, we have

$$H^i(\mathcal{M}^s(N, K), \mathsf{Det}^{\otimes n}) = 0, \text{ for all } \ i > 0 \text{ and } \ n \geq 0. \tag{3.34}$$

Definition 3.4.2. The $\mathbb{C}[\mathcal{M}(N, K)]$ module of level n, denoted by $\Gamma(N, K, n)$, is defined by the global section of n'th power of tautological line bundle, i.e.

$$\Gamma(N, K, n) = \Gamma(\mathcal{M}^s(N, K), \mathsf{Det}^{\otimes n}). \tag{3.35}$$

In this section, we compute the Hilbert series of $\mathbb{C}[\mathcal{M}(N, K)]$ and $\Gamma(N, K, n)$. Before starting, let us introduce some notations and explain what we are going to compute.

The quiver in Figure 3.2 admits an action of $\mathrm{GL}_K \times \mathbb{C}^\times_q \times \mathbb{C}^\times_t$, where GL_K is the flavour symmetry which acts on the framing vector space, $\mathbb{C}^\times_q$ scales B by $B \mapsto q^{-1}B$, and $\mathbb{C}^\times_t$ scales $\overline{\psi}$ by $\overline{\psi} \mapsto t^{-1}\overline{\psi}$. The convention of the inverse q^{-1} and t^{-1} is such that the functions $\mathrm{Tr}(B^n)$ and $\overline{\psi}_a B^m \psi_b$ scales by q^n and $q^m t$ respectively (since functions are dual to the space). Although $\mathbb{C}[\mathcal{M}(N, K)]$ is infinite dimensional, every $\mathbb{C}^\times_q \times \mathbb{C}^\times_t$-weight space of $\mathbb{C}[\mathcal{M}(N, K)]$ is finite dimensional (we will see it later), thus it makes sense to regard $\mathbb{C}[\mathcal{M}(N, K)]$ as an element in $K_{\mathrm{GL}_K}(\mathrm{pt})[\![q, t]\!]$. Similarly, the same properties hold for $\Gamma(N, K, n)$. The goal of this section is to compute the these elements.

Definition 3.4.3. Let $K_{\mathrm{GL}_K}(\mathrm{pt}) = \mathbb{Q}[x_1^{\pm}, \cdots, x_K^{\pm}]^{S_K}$, where S_K is the permutation group acting on $x_1, \cdots x_K$. We use shorthand notation $f(x)$ for a function of $x_1, \cdots, x_K$, and $f(x^{-1}) = f(x_1^{-1}, \cdots, x_K^{-1})$. Denote by $Z_{N,K}(x; q, t)$ the element of $\mathbb{C}[\mathcal{M}(N, K)]$ in $K_{\mathrm{GL}_K}(\mathrm{pt})[\![q, t]\!]$, and denote by $Z^{(n)}_{N,K}(x; q, t)$ the element of $\Gamma(N, K, n)$ in $K_{\mathrm{GL}_K}(\mathrm{pt})[\![q, t]\!]$.

By Lemma 3.4.1, we have $Z^{(n)}_{N,K}(x; q, t) = \chi(\mathcal{M}^s(N, K), \mathsf{Det}^{\otimes n})$. The case $K = 1$ is trivial: The functions $\mathrm{Tr}(B), \cdots, \mathrm{Tr}(B^N), \overline{\psi}\psi, \overline{\psi}B\psi, \cdots, \overline{\psi}B^{N-1}\psi$ give rise to an isomorphism $\mathcal{M}(N, 1) \cong \mathbb{A}^{2N}$. The Lemma 3.4.6 below, together with the fact that the Hilbert-Chow map for Hilbert scheme of points on smooth curve is isomorphism, implies that $\mathcal{M}^s(N, K) \cong \mathcal{M}(N, K)$. In fact, Det in this case is a trivial bundle, with $\mathbb{C}^\times_q \times \mathbb{C}^\times_t$-weight $(1, 0)$, thus

$$Z^{(n)}_{N,1}(x; q, t) = q^n Z_{N,1}(x; q, t) = \frac{q^n}{(q; q)_N (t; q)_N}. \tag{3.36}$$

Here we use the q-Pochhammer symbol $(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$. The case when $K > 1$ is trickier. In principal, one can use the localization technique to get a formula of $\chi(\mathcal{M}^s(N, K), \mathsf{Det}^{\otimes n})$ in terms of summation over fixed points, but it involves complicated denominators that make it hard to extract the power series in q and t explicitly. What we will actually do, is to reduce the computation to Euler character of vector bundles on Quot scheme, which is related to the affine Grassmannian of GL_K, and finally apply the known results on the geometry of the affine Grassmannian of GL_K to finish the calculation. We present the final result here and explain the calculation in steps afterwards.

Theorem 3.4.4. *The Hilbert series of* $\Gamma(N, K, n)$ *is*

$$Z^{(n)}_{N,K}(x; q, t) = \frac{1}{(q; q)_N} \sum_{\underline{\mu}} t^{|\underline{\mu}|} H_{\underline{\mu}+(n^N)}(x; q) s_{(\mu_1)}(x^{-1}) \cdots s_{(\mu_N)}(x^{-1}). \tag{3.37}$$

Here the summation is over arrays $\underline{\mu} = (\mu_1, \cdots, \mu_N) \in \mathbb{Z}^N_{\geq 0}$, (n^N) *is the array consisting of* N *copies of* n, *i.e.* $(n, n, \cdots, n)$, $|\underline{\mu}| = \sum_{i=1}^N \mu_i$, $s_{(\mu_i)}(x)$ *is the Schur polynomial of the partition* (μ_i), *and* $H_{\underline{\lambda}}(x; q)$ *is the generalized transformed Hall-Littlewood polynomial of the array* $\underline{\lambda}$, *defined in* (B.6).

3.4.5 Reduction steps

Recall that the stability condition in the definition of $\mathcal{M}^s(N, K)$ is that if $V \subset \mathbb{C}^N$, $B(V) \subset V$ and $\mathrm{im}\,(\psi) \subset V$ then $V = \mathbb{C}^N$, in particular the sub-quiver consisting of arrows (B, ψ) is stable under the same stability condition, so we have:

Lemma 3.4.6. *The moduli of stable representations $\mathcal{M}^s(N,K)$ is a vector bundle over the Quot scheme of $\mathbb{A}^1$ which parametrizes length N quotients of $\mathcal{O}_{\mathbb{A}^1}^{\oplus K}$, denoted by $\mathrm{Quot}^N(\mathbb{A}^1,\mathcal{O}_{\mathbb{A}^1}^{\oplus K})$:*

$$\begin{array}{c}\mathcal{M}^s(N,K)=\mathbb{V}(\mathcal{V}\otimes W^*)\\ \Big\downarrow p\\ \mathrm{Quot}^N(\mathbb{A}^1,\mathcal{O}_{\mathbb{A}^1}^{\oplus K})\end{array}$$

Here $\mathcal{V}$ is the tautological sheaf on $\mathrm{Quot}^N(\mathbb{A}^1,\mathcal{O}_{\mathbb{A}^1}^{\oplus K})$, and W is the framing vector space.

Proof. Consider a point $(B,\psi,\overline{\psi})\in\mathcal{M}^s(N,K)$, the action B on $\mathbb{C}^N$ makes it into a $\mathbb{C}[z]$-module such that z acts as B. The stability on (B,ψ) is equivalent to that $\mathbb{C}^N$ is a quotient module of a free module of rank K. This gives rise to a morphism $p:\mathcal{M}^s(N,K)\to\mathrm{Quot}^N(\mathbb{A}^1,\mathcal{O}_{\mathbb{A}^1}^{\oplus K})$, and the extra information in $\mathcal{M}^s(N,K)$ compared to the Quot scheme is a homomorphism from the universal quotient $\mathcal{V}$ to the framing vector space W, so $\mathcal{M}^s(N,K)$ is represented by $\mathbb{V}(\mathcal{V}\otimes W^*)$. □

Lemma 3.4.6 implies that

$$\chi(\mathcal{M}^s(N,K),\mathsf{Det}^{\otimes n})=\sum_{m=0}^{\infty}t^m\chi(\mathrm{Quot}^N(\mathbb{A}^1,\mathcal{O}_{\mathbb{A}^1}^{\oplus K}),\mathrm{Sym}^m(\mathcal{V}\otimes W^*)\otimes\mathsf{Det}^{\otimes n}).\qquad(3.38)$$

Here in each summand, $\chi(\mathrm{Quot}^N(\mathbb{A}^1,\mathcal{O}_{\mathbb{A}^1}^{\oplus K}),\mathrm{Sym}^m(\mathcal{V}\otimes W^*))$ is in $K_{\mathrm{GL}_K}(\mathrm{pt})[\![q]\!]$. So the computation of $\mathbb{C}[\mathcal{M}(N,K)]$ boils down to the computation of equivariant Euler characters of sheaves on the Quot scheme.

The Quot scheme has a nice structure: there is morphism $h:\mathrm{Quot}^N(\mathbb{A}^1,\mathcal{O}_{\mathbb{A}^1}^{\oplus K})\longrightarrow\mathbb{A}^{(N)}$ where $\mathbb{A}^{(N)}$ is the N'th symmetric product of $\mathbb{A}^1$, which is identified with the Hilbert scheme of N points on $\mathbb{A}^1$ and h is the Hilbert-Chow morphism for the Quot scheme. In the language of quivers, h maps (B,ψ) to the spectrum of B, regarded as a divisor of degree N on $\mathbb{A}^1$.

Lemma 3.4.7. *The central fiber $h^{-1}(0)$ of the Hilbert-Chow morphism $h:\mathrm{Quot}^N(\mathbb{A}^1,\mathcal{O}_{\mathbb{A}^1}^{\oplus K})\longrightarrow\mathbb{A}^{(N)}$, endowed with reduced scheme structure, is isomorphic to $\overline{\mathrm{Gr}}^{N\omega_1}_{\mathrm{GL}_K}$ in the affine Grassmannian $\mathrm{Gr}_{\mathrm{GL}_K}$. Here $\omega_1=(1,0,\cdots,0)$ is the first fundamental coweight of GL_K.*

Proof. The central fiber $h^{-1}(0)$ represents submodules of $\mathbb{C}[z]^{\oplus K}$ whose cokernels are finite of length N and are supported at 0, so by formal gluing theorem [151, Tag 0BP2], $h^{-1}(0)$ represents submodules of $\mathbb{C}[\![z]\!]^{\oplus K}$ whose cokernels are finite of length N, this is $\overline{\mathrm{Gr}}^{N\omega_1}_{\mathrm{GL}_K}$. □

Proposition 3.4.8. *The Hilbert-Chow morphism* $h : \mathrm{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K}) \longrightarrow \mathbb{A}^{(N)}$ *is flat.*

Proof. By the deformation theory, $\mathrm{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K})$ is smooth of dimension NK. $h^{-1}(0)_{\mathrm{red}} \cong \overline{\mathrm{Gr}}^{N\omega_1}_{\mathrm{GL}_K}$ has dimension $(K-1)N$, which equals to $\dim \mathrm{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K}) - \dim \mathbb{A}^{(N)}$, thus h is flat along $h^{-1}(0)$ by miracle flatness theorem [151, Tag 00R4]. Since flatness is an open condition, h is flat in an open neighborhood of $h^{-1}(0)$. Since Hilbert-Chow morphism h is proper, there is an open neighborhood U of $0 \in \mathbb{A}^{(N)}$ such that $h|_{h^{-1}(U)}$ is flat. Finally h is equivariant under the $\mathbb{C}^\times$ action on $\mathbb{A}^1$ which contracts $\mathbb{A}^{(N)}$ to 0, so the flatness is transported from U to the whole $\mathbb{A}^{(N)}$. □

Proposition 3.4.8 provides a tool that reduces the computation of Euler character to the central fiber. In effect, to compute $\chi(\mathrm{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K}), \mathcal{F})$ for a locally free sheaf $\mathcal{F}$, we can apply $\mathbb{C}^\times_q$-localization to its derived pushforward $Rh_*(\mathcal{F})$:

$$\chi(\mathrm{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K}), \mathcal{F}) = \chi(\mathbb{A}^{(N)}, Rh_*(\mathcal{F})) = \frac{\chi(h^{-1}(0), \mathcal{F}|_{h^{-1}(0)})}{\prod_{i=1}^N (1-q^i)}, \tag{3.39}$$

where in the last equation we use the proper base change (since $\mathcal{F}$ is flat over $\mathbb{A}^{(N)}$ by Proposition 3.4.8), and the denominator comes from the tangent space of $\mathbb{A}^{(N)}$ at 0 which has $\mathbb{C}^\times_q$-weights $-1, \cdots, -N$.

Proposition 3.4.9. *The central fiber* $h^{-1}(0)$ *is isomorphic to* $\overline{\mathrm{Gr}}^{N\omega_1}_{\mathrm{GL}_K}$ *as a scheme.*

Proof. In view of Lemma 3.4.7, the proposition is equivalent to that $h^{-1}(0)$ is reduced. Since h is flat with domain and codomain being smooth, $h^{-1}(0)$ is a Cohen-Macaulay scheme, therefore it is enough to show that $h^{-1}(0)$ is generically reduced. We claim that h is smooth at the point $z^{N\omega_1}$. Assume that the claim is true, then h is smooth in an open neighborhood of $z^{N\omega_1}$, thus $h^{-1}(0)$ is generically reduced.

The claim follows from the deformation theory of $\mathrm{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K})$. Namely, if $e_1, \cdots, e_K$ is the basis of $\mathcal{O}_{\mathbb{A}^1}^{\oplus K}$, then $z^{N\omega_1}$ corresponds to short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{A}^1}^{\oplus K} \longrightarrow Q \longrightarrow 0$$

such that $\mathcal{E}$ is the subsheaf of $\mathcal{O}_{\mathbb{A}^1}^{\oplus K}$ generated by $z^N e_1, e_2, \cdots, e_K$. Then the tangent space of $\mathrm{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K})$ at $z^{N\omega_1}$ is

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{A}^1}}(\mathcal{E}, Q).$$

In particular, the tangent space contains $\mathrm{Hom}_{\mathcal{O}_{\mathbb{A}^1}}(z^N\mathcal{O}_{\mathbb{A}^1}, \mathcal{O}_{\mathbb{A}^1}/z^N\mathcal{O}_{\mathbb{A}^1})$ as a subspace, and the latter projects isomorphically onto the tangent space of $\mathbb{A}^{(N)}$ at 0. In particular, the tangent map at $z^{N\omega_1}$ is surjective, thus h is smooth at $z^{N\omega_1}$. □

Note that the restriction of the tautological line bundle Det to the central fiber $h^{-1}(0)$ is exactly the determinant line bundle $\mathcal{O}(1)$ on the affine Grassmannian. This enables us to prove the aforementioned Lemma 3.4.1.

Proof of Lemma 3.4.1. It is well-known that the determinant line bundle $\mathcal{O}(1)$ on the affine Grassmannian is ample [164], thus the restriction of the tautological line bundle Det to the central fiber $h^{-1}(0)$ is ample. Since Hilbert-Chow morphism h is proper, there is an open neighborhood U of $0 \in \mathbb{A}^{(N)}$ such that $\mathsf{Det}|_{h^{-1}(U)}$ is ample relative to U. And h is equivariant under the $\mathbb{C}^\times$ action on $\mathbb{A}^1$ which contracts $\mathbb{A}^{(N)}$ to 0, so the relative ampleness is transported from U to the whole $\mathbb{A}^{(N)}$, i.e. Det is relatively ample over $\mathbb{A}^{(N)}$. Since $\mathbb{A}^{(N)}$ is affine, Det on the Quot scheme $\mathrm{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K})$ is ample. Since the projection $\mathcal{M}^s(N,K) \to \mathrm{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K})$ is a vector bundle, the pullback of Det, which is the tautological line bundle on $\mathcal{M}^s(N,K)$, is ample. □

By the Lemma 3.4.1 and localization formula (3.39), we reduce the calculation to

$$\begin{aligned} &Z^{(n)}_{N,K}(x;q,t) = \chi(\mathcal{M}^s(N,K), \mathsf{Det}^{\otimes n}) \\ =&\frac{1}{(q;q)_N}\sum_{m=0}^{\infty} t^m \chi(\overline{\mathrm{Gr}}^{N\omega_1}_{\mathrm{GL}_K}, \mathrm{Sym}^m(\mathcal{V}\otimes W^*)\otimes \mathcal{O}(n)), \end{aligned} \tag{3.40}$$

where $\mathcal{V}$ is the restriction of the universal quotient sheaf to $\overline{\mathrm{Gr}}^{N\omega_1}_{\mathrm{GL}_K}$.

Remark 3.4.10. One can show that $\mathrm{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K})$ is isomorphic to $\overline{\mathrm{Gr}}^{\omega_1,\cdots,\omega_1}_{\mathrm{GL}_K,\mathbb{A}^{(N)}}$, defined as the closure of $\mathrm{Sym}^N\left(\mathrm{Gr}^{\omega_1}\times\mathbb{A}^1\right)|_{\mathbb{A}^{(N)}\setminus\mathbb{A}^{(N-1)}}$ in the symmetrized Beilinson-Drinfeld Grassmannian of GL_K on $\mathbb{A}^{(N)}$, here $\mathbb{A}^{(N-1)} \hookrightarrow \mathbb{A}^{(N)}$ embeds diagonally. Moreover the isomorphism is $\mathrm{GL}_K \times \mathbb{C}_q^\times$-equivariant and commutes with projections to $\mathbb{A}^{(N)}$:

$$\begin{array}{ccc} \mathrm{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K}) & \longrightarrow & \overline{\mathrm{Gr}}^{\omega_1,\cdots,\omega_1}_{\mathrm{GL}_K,\mathbb{A}^{(N)}} \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & \mathbb{A}^{(N)} & \end{array}$$

Here π is the structure map of symmetrized Beilinson-Drinfeld Grassmannian.

Furthermore, one can show that the Picard groups of $\mathcal{M}^s(N,K)$ and $\mathrm{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K})$ are generated by the tautological line bundle, i.e.

$$\mathrm{Pic}(\mathcal{M}^s(N,K)) = \mathrm{Pic}(\mathrm{Quot}^N(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^{\oplus K})) = \mathbb{Z}\cdot\mathsf{Det}.$$

3.4.11 Calculation on affine Grassmannian

It remains to do the calculation on affine Grassmannian for

$$\sum_{m=0}^{\infty} t^n \chi(\overline{\mathrm{Gr}}^{N\omega_1}_{\mathrm{GL}_K}, \mathrm{Sym}^m(\mathcal{V}\otimes W^*)\otimes \mathcal{O}(n)) = \chi(\overline{\mathrm{Gr}}^{N\omega_1}_{\mathrm{GL}_K}, S^\bullet_t(\mathcal{V}\otimes W^*)\otimes \mathcal{O}(n)).$$

Here we use the notation $S^\bullet_t(\mathcal{V}\otimes W^*) = \bigoplus_{m\geq 0} t^n \,\mathrm{Sym}^m(\mathcal{V}\otimes W^*)$. To start with, note that there is a convolution map on $\mathrm{Gr}_{\mathrm{GL}_K}$:

$$m : \mathrm{Gr}^{\omega_1}_{\mathrm{GL}_K} \widetilde{\times} \overline{\mathrm{Gr}}^{(N-1)\omega_1}_{\mathrm{GL}_K} \longrightarrow \overline{\mathrm{Gr}}^{N\omega_1}_{\mathrm{GL}_K},$$

see appendix (B.12) for definition of the convolution product. The key property of the convolution product is that

$$\mathcal{O} \cong Rm_*\mathcal{O}. \tag{3.41}$$

See the proof of appendix B.2.5 for an explanation of this isomorphism. Here $\mathcal{O}$ is the structure sheaves, we omit the subscripts labelling the domain and codomain, since the meaning of the homomorphism is clear. In view of (3.41), we have

$$\chi(\overline{\mathrm{Gr}}^{N\omega_1}_{\mathrm{GL}_K}, S^\bullet_t(\mathcal{V}\otimes W^*)\otimes \mathcal{O}(n)) = \chi\left(\mathrm{Gr}^{\omega_1}_{\mathrm{GL}_K} \widetilde{\times} \overline{\mathrm{Gr}}^{(N-1)\omega_1}_{\mathrm{GL}_K}, S^\bullet_t(m^*\mathcal{V}\otimes W^*)\otimes \mathcal{O}(n)\right).$$

Let us write $\mathcal{V}_N$ for $\mathcal{V}$ to indicate the rank of the gauge group.

Lemma 3.4.12. *$m^*\mathcal{V}_N$ is an extension of $p^*\mathcal{V}_1$ by the twist of $\mathcal{V}_{N-1}$, denoted by $\widetilde{\mathcal{V}}_{N-1}$, i.e. there is a short exact sequence*

$$0 \longrightarrow \widetilde{\mathcal{V}}_{N-1} \longrightarrow m^*\mathcal{V}_N \longrightarrow p^*\mathcal{V}_1 \longrightarrow 0.$$

Here $p : \mathrm{Gr}^{\omega_1}_{\mathrm{GL}_K} \widetilde{\times} \overline{\mathrm{Gr}}^{(N-1)\omega_1}_{\mathrm{GL}_K} \to \overline{\mathrm{Gr}}^{N\omega_1}_{\mathrm{GL}_K}$ is the projection to the first component map, and $\widetilde{\mathcal{V}}_{N-1}$ is the sheaf $\mathrm{GL}_K(\mathcal{K}) \overset{\mathrm{GL}_n(\mathcal{O})}{\times} \mathcal{V}_{N-1}$.

Proof. $\mathcal{V}_N$ is the universal quotient of $\mathbb{C}[\![z]\!]^{\oplus K}$. Denote the kernel by L_N. Then the pullback of $\mathcal{V}_N$ to the twisted product $\mathrm{Gr}^{\omega_1}_{\mathrm{GL}_K} \widetilde{\times} \overline{\mathrm{Gr}}^{(N-1)\omega_1}_{\mathrm{GL}_K}$ is by definition the extension of $\mathcal{V}_1$ by $\mathcal{V}_{N-1}$, except that in the definition of $\mathcal{V}_{N-1}$ the free module $\mathbb{C}[\![z]\!]^{\oplus K}$ is replaced by L_1 (this is the meaning of twist). □

Note that $\mathcal{V}_1$ is of rank one, so it is by definition the determinant line bundle $\mathcal{O}(1)$ on the affine Grassmannian $\mathrm{Gr}_{\mathrm{GL}_K}$ restricted on $\mathrm{Gr}^{\omega_1}_{\mathrm{GL}_K}$. The convolution map easily generalizes to multiple copies of $\mathrm{Gr}_{\mathrm{GL}_K}$:

$$m : \mathrm{Gr}^{\omega_1}_{\mathrm{GL}_K}\widetilde{\times}\cdots\widetilde{\times}\mathrm{Gr}^{\omega_1}_{\mathrm{GL}_K} \longrightarrow \overline{\mathrm{Gr}}^{\omega_1}_{\mathrm{GL}_K},$$

and we can apply Lemma 3.4.12 recursively and see that $m^*\mathcal{V}_N$ is a consecutive extension of (twisted) $\mathcal{O}(1)$. Since we only care about the Euler character, we can forget about the extension structure and focus on the K-theory class, in other words, we have:

$$\begin{aligned}
&\chi(\overline{\mathrm{Gr}}^{N\omega_1}_{\mathrm{GL}_K}, S^\bullet_t(\mathcal{V}\otimes W^*)\otimes\mathcal{O}(n))\\
=&\chi(\mathrm{Gr}^{\omega_1}_{\mathrm{GL}_K}\widetilde{\times}\cdots\widetilde{\times}\mathrm{Gr}^{\omega_1}_{\mathrm{GL}_K}, S^\bullet_t((\mathcal{O}(1)+\widetilde{\mathcal{O}(1)}+\cdots+\widetilde{\mathcal{O}(1)})\otimes W^*)\otimes(\mathcal{O}(n)\widetilde{\boxtimes}\cdots\widetilde{\boxtimes}\mathcal{O}(n)))\\
=&\sum_{\underline{\mu}} T^{|\underline{\mu}|}\chi(\mathrm{Gr}^{\omega_1}_{\mathrm{GL}_K}\widetilde{\times}\cdots\widetilde{\times}\mathrm{Gr}^{\omega_1}_{\mathrm{GL}_K}, \mathcal{O}(\mu_1+n)\widetilde{\boxtimes}\cdots\widetilde{\boxtimes}\mathcal{O}(\mu_N+n))\chi(S^{\mu_1}(W^*))\cdots\chi(S^{\mu_N}(W^*)).
\end{aligned} \tag{3.42}$$

Here the summation is over arrays $\underline{\mu}=(\mu_1,\cdots,\mu_N)\in\mathbb{Z}^N_{\geq 0}$, $|\underline{\mu}|=\sum_{i=1}^N\mu_i$, and $\chi(S^k(W^*))$ is the GL_K-equivariant K-theory class of the k'th symmetric tensor product of W^*, where W is the fundamental representation of GL_K. It is well-known that $\chi(S^k(W^*))=s_{(k)}(x^{-1})$, where $s_{(k)}(x)$ is the Schur polynomial of the partition (k). Finally, the remaining part of the computation, which is the character of $\mathcal{O}(\mu_1+n)\widetilde{\boxtimes}\cdots\widetilde{\boxtimes}\mathcal{O}(\mu_N+n)$, is related to a well-understood family of symmetric functions, the transformed Hall-Littlewood polynomial. In fact we have

$$\chi(\mathrm{Gr}^{\omega_1}_{\mathrm{GL}_K}\widetilde{\times}\cdots\widetilde{\times}\mathrm{Gr}^{\omega_1}_{\mathrm{GL}_K}, \mathcal{O}(\mu_1+n)\widetilde{\boxtimes}\cdots\widetilde{\boxtimes}\mathcal{O}(\mu_N+n)) = H_{\underline{\mu}+(n^N)}(x;q). \tag{3.43}$$

where $H_{\underline{\mu}}(x;q)$ is the generalized transformed Hall-Littlewood polynomial of the array $\underline{\mu}+(n^N)$ (see (B.6)). For the derivation of this formula, see Corollary B.2.4 in the appendix.

3.4.13 $N\to\infty$ limit

Recall that $\mathbb{C}[\mathcal{M}(\infty,K)]$ is the subalgebra of $\varprojlim_N \mathbb{C}[\mathcal{M}(N,K)]$ generated by $T^{(n)}_{ab}$ and $\mathrm{Tr}(B^m)$, for all $n,m\in\mathbb{Z}_{\geq 0}$ and $1\leq a,b\leq K$ (Definition 3.3.1).

Lemma 3.4.14. *$\mathbb{C}[\mathcal{M}(\infty,K)]$ contains all $T\times\mathbb{C}^\times_q\times\mathbb{C}^\times_t$ eigenvectors in $\varprojlim_N \mathbb{C}[\mathcal{M}(N,K)]$, where $T\subset\mathrm{GL}_K$ is the maximal torus.*

Proof. We claim that for fixed $n \in \mathbb{Z}_{\geq 0}$, the dimension of $\mathbb{C}_q^\times$-weight n space of $\mathbb{C}[\mathcal{M}(N,K)]$ stabilizes when $N \gg 0$, more precisely there exists N such that for all $N' > N$ the kernel of $\mathbb{C}[\mathcal{M}(N',K)] \twoheadrightarrow \mathbb{C}[\mathcal{M}(N,K)]$ has $\mathbb{C}_q^\times$-weights $> n$. To see why this is true, we take N such that $L_n^-\mathrm{GL}_K \times L_n^-\mathrm{GL}_1 \subset \mathcal{M}(N,K)$ (N can be $(n+1)K$ according to the proof of Theorem 3.3.4), then $\ker(\mathbb{C}[\mathcal{M}(N',K)] \twoheadrightarrow \mathbb{C}[\mathcal{M}(N,K)])$ is a subquotient of $\ker(\mathbb{C}[L^-\mathrm{GL}_K \times L^-\mathrm{GL}_1] \twoheadrightarrow \mathbb{C}[L_n^-\mathrm{GL}_K \times L_n^-\mathrm{GL}_1])$, and the latter is an ideal generated by elements of $\mathbb{C}_q^\times$-weights greater than n.

Now assume that $a \in \varprojlim_N \mathbb{C}[\mathcal{M}(N,K)]$ is a $T \times \mathbb{C}_q^\times \times \mathbb{C}_t^\times$ eigenvector, and let its $\mathbb{C}_q^\times$ be n. Then there exists N such that for all $N' > N$ the kernel of $\mathbb{C}[\mathcal{M}(N',K)] \twoheadrightarrow \mathbb{C}[\mathcal{M}(N,K)]$ has $\mathbb{C}_q^\times$-weights greater than n. Consider the image of a in $\mathbb{C}[\mathcal{M}(N,K)]$, denoted by $\overline{a}$, and take a $T \times \mathbb{C}_q^\times \times \mathbb{C}_t^\times$-equivariant lift of $\overline{a}$ along the projection $\mathbb{C}[L^-\mathrm{GL}_K \times L^-\mathrm{GL}_1] \twoheadrightarrow \mathbb{C}[\mathcal{M}(N,K)]$, and we denote the lift by a', then $a - a'$ has $\mathbb{C}_q^\times$-weight n and is zero in $\mathbb{C}[\mathcal{M}(N,K)]$, thus $a - a'$ is in the kernel of $\mathbb{C}[\mathcal{M}(N',K)] \twoheadrightarrow \mathbb{C}[\mathcal{M}(N,K)]$ for all $N' > N$, which forces $a = a'$ in $\mathbb{C}[\mathcal{M}(N',K)]$ because of weight consideration, therefore $a = a'$ in $\varprojlim_N \mathbb{C}[\mathcal{M}(N,K)]$. □

Proposition 3.4.15. *The Hilbert series of $\mathbb{C}[\mathcal{M}(\infty,K)]$ equals to the $N \to \infty$ limit of Hilbert series of $\mathbb{C}[\mathcal{M}(N,K)]$, i.e.*

$$\mathbb{C}[\mathcal{M}(\infty,K)] = \frac{1}{(q;q)_\infty} \lim_{N\to\infty} \sum_{\underline{\mu}} t^{|\underline{\mu}|} H_{\underline{\mu}}(x;q) s_{(\mu_1)}(x^{-1}) \cdots s_{(\mu_N)}(x^{-1}) \tag{3.44}$$

Proof. The $N \to \infty$ limit of Hilbert series of $\mathbb{C}[\mathcal{M}(N,K)]$ enumerates $T \times \mathbb{C}_q^\times \times \mathbb{C}_t^\times$ eigenvectors in $\varprojlim_N \mathbb{C}[\mathcal{M}(N,K)]$, which is the same as $T \times \mathbb{C}_q^\times \times \mathbb{C}_t^\times$ eigenvectors in $\mathbb{C}[\mathcal{M}(\infty,K)]$, by Lemma 3.4.14. □

On the other hand, $\mathbb{C}[\mathcal{M}(\infty,K)]$ is freely generated by $\overline{\psi}_a B^n \psi_b, \mathrm{Tr}(B^m)$, which makes its Hilbert series easily computed by

$$\mathrm{PE}\left((t + tq + tq^2 + \cdots)\chi(\mathfrak{gl}_K)\right) \mathrm{PE}(q + q^2 + \cdots). \tag{3.45}$$

Here $\chi(\mathfrak{gl}_K)$ is the character of the adjoint representation of GL_K, and PE is the plethestic exponential. Note that $\chi(\mathfrak{gl}_K)$ can be written as a symmetric function $1 + \frac{s_{\underline{\lambda}_{\mathrm{ad}}}(x)}{h_K(x)}$, where $\underline{\lambda}_{\mathrm{ad}}$ is the Young tableaux corresponding to the adjoint representation of SL_K, and $h_K(x) = x_1 x_2 \cdots x_K$. Moreover,

$$\mathrm{PE}(q + q^2 + \cdots) = \prod_{i=1}^{\infty} \frac{1}{1 - q^i} = \frac{1}{(q;q)_\infty}.$$

Compare equation 3.44 with 3.45, we get the following interesting equation, which we do not know other way to prove.

Corollary 3.4.16.

$$\lim_{N\to\infty}\sum_{\underline{\mu}} t^{|\underline{\mu}|}H_{\underline{\mu}}(x;q)s_{(\mu_1)}(x^{-1})\cdots s_{(\mu_N)}(x^{-1}) = \mathrm{PE}\left(\frac{t}{1-q}\left(1+\frac{s_{\underline{\lambda}_{\mathrm{ad}}}(x)}{h_K(x)}\right)\right). \tag{3.46}$$

Here $\underline{\lambda}_{\mathrm{ad}}$ *is the Young tableaux corresponding to the adjoint representation of* SL_K*,* $h_K(x) = x_1x_2\cdots x_K$*, and* PE *is the plethestic exponential.*

3.5 Quantization of $\mathcal{M}(N,K)$

In this section we study the quantization of $\mathcal{M}(N,K)$, namely we quantizes the Poisson structure (3.20) to the commutation relation:

$$[\psi_{ia},\overline{\psi}_{bj}] = \hbar\delta_{ab}\delta_{ij},\ [B_{mn},B_{pq}] = \hbar(\delta_{pn}B_{mq}-\delta_{mq}B_{pn}),\ [B_{mn},\overline{\psi}_{bj}] = [B_{mn},\psi_{ia}] = 0. \tag{3.47}$$

This is the algebra $U_\hbar(\mathfrak{gl}_N)\otimes \mathrm{Weyl}_{NK}$, and we define the quantized ring of functions on the phase space $\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ by the invariant part $(U_\hbar(\mathfrak{gl}_N)\otimes \mathrm{Weyl}_{NK})^{\mathrm{GL}_N}$. Since GL_N is reductive, we have $\mathbb{C}_\hbar[\mathcal{M}(N,K)]/(\hbar) = \mathbb{C}[\mathcal{M}(N,K)]$. Note that $\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ possesses a natural grading by setting

$$\deg(\psi) = 0, \deg(\overline{\psi}) = 1, \deg(B) = 1, \deg(\hbar) = 1. \tag{3.48}$$

Lemma 3.5.1. $\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ *is flat over* $\mathbb{C}[\hbar]$*.*

Proof. Since $U_\hbar(\mathfrak{gl}_N)\otimes \mathrm{Weyl}_{NK}$ is flat over $\mathbb{C}[\hbar]$, the subalgebra $\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ is $\hbar$-torsion free, thus it is also flat over $\mathbb{C}[\hbar]$. □

Remark 3.5.2. On the stable moduli $\mathcal{M}^s(N,K)$ there is a notion of quantized structure sheaf, namely, consider the completion of $U_\hbar(\mathfrak{gl}_N)\otimes \mathrm{Weyl}_{NK}$ in the $\hbar$-adic topology, this allows us to localize it in the Zariski topology of the affine space $\mathrm{Rep}(N,K)$, and by taking GL_N-invariant on the open locus of stable representations $\mathrm{Rep}^s(N,K)$, we get a sheaf of flat $\mathbb{C}[\![\hbar]\!]$-algebras on $\mathcal{M}^s(N,K)$, denoted by $\widehat{\mathcal{O}}_{\mathcal{M}^s(N,K)}$. By construction we have $\widehat{\mathcal{O}}_{\mathcal{M}^s(N,K)}/(\hbar) = \mathcal{O}_{\mathcal{M}^s(N,K)}$. This sheaf is related to $\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ as follows. By construction we have a natural homomorphism of algebras $\mathbb{C}_\hbar[\mathcal{M}(N,K)] \to \Gamma(\mathcal{M}^s(N,K),\widehat{\mathcal{O}}_{\mathcal{M}^s(N,K)})$, which preserves the grading (3.48). On the other hand, by Lemma 3.2.2 below, we have

- $H^i(\mathcal{M}^s(N,K), \widehat{\mathcal{O}}_{\mathcal{M}^s(N,K)}) = 0$, for $i > 0$.
- $\Gamma(\mathcal{M}^s(N,K), \widehat{\mathcal{O}}_{\mathcal{M}^s(N,K)})$ is a flat $\mathbb{C}[\![\hbar]\!]$-algebra, which quantizes $\mathbb{C}[\mathcal{M}(N,K)]$.

Since $\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ is generated by positive degree elements, we conclude that $\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ is naturally identified with the subalgebra of homogeneous elements in $\Gamma(\mathcal{M}^s(N,K), \widehat{\mathcal{O}}_{\mathcal{M}^s(N,K)})$.

$T^{(n)}_{ab} = \overline{\psi}_a B^n \psi_b$ and $\mathrm{Tr}(B^k)$ generate $\mathbb{C}[\mathcal{M}(N,K)]$, so they generate $\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ as well, and it is easy to see that $\mathrm{Tr}(B^k)$ commutes with all elements in $U_\hbar(\mathfrak{gl}_N) \otimes \mathrm{Weyl}_{NK}$, therefore $\mathrm{Tr}(B^k)$ is central. We denote by $T_{ab}(z)$ the power series expanded at $z \to \infty$:

$$T_{ab}(z) = \sum_{n \ge -1} T^{(n)}_{ab} z^{-n-1} = \delta_{ab} + \overline{\psi}_a \frac{1}{z-B} \psi_b.$$

Proposition 3.5.3. *The commutators between $T^{(k)}_{ab}$ are:*

$$[T^{(p)}_{ab}, T^{(q)}_{cd}] = \hbar \sum_{i=-1}^{\min(p,q)-1} \left(T^{(i)}_{cb} T^{(p+q-1-i)}_{ad} - T^{(p+q-1-i)}_{cb} T^{(i)}_{ad} \right). \tag{3.49}$$

Proof. It is easy to see that (3.49) is equivalent to

$$[T^{(p+1)}_{ab}, T^{(q)}_{cd}] - [T^{(p)}_{ab}, T^{(q+1)}_{cd}] = \hbar \left(T^{(p)}_{cb} T^{(q)}_{ad} - T^{(q)}_{cb} T^{(p)}_{ad} \right).$$

We compute the left hand side of the above equation:

$$\begin{aligned}
&[T^{(p+1)}_{ab}, T^{(q)}_{cd}] - [T^{(p)}_{ab}, T^{(q+1)}_{cd}] = \overline{\psi}_{am} \overline{\psi}_{cr} \left([(B^{p+1})_{mn}, (B^q)_{rs}] - [(B^p)_{mn}, (B^{q+1})_{rs}] \right) \psi_{bn} \psi_{ds} \\
&= \hbar \overline{\psi}_{am} \overline{\psi}_{cr} \left(\sum_{i=1}^{q} (B^{i-1})_{rn} (B^{p+1+q-i})_{ms} - (B^{i+p})_{rn} (B^{q-i})_{ms} \right) \psi_{bn} \psi_{ds} \\
&- \hbar \overline{\psi}_{am} \overline{\psi}_{cr} \left(\sum_{i=1}^{q+1} (B^{i-1})_{rn} (B^{p+1+q-i})_{ms} - (B^{i+p-1})_{rn} (B^{q+1-i})_{ms} \right) \psi_{bn} \psi_{ds} \\
&= \hbar \overline{\psi}_{am} \overline{\psi}_{cr} \left((B^p)_{rn} (B^q)_{ms} - (B^q)_{rn} (B^p)_{ms} \right) \psi_{bn} \psi_{ds},
\end{aligned}$$

which is exactly the right hand side. ∎

Remark 3.5.4. The commutators (3.49) is equivalent to the RTT equation

$$[T_{ab}(u), T_{cd}(v)] = \frac{\hbar}{u-v} \left(T_{cb}(u) T_{ad}(v) - T_{cb}(v) T_{ad}(u) \right). \tag{3.50}$$

The classical embedding $\mathcal{M}(L,K) \hookrightarrow \mathcal{M}(N,K)$ for $L<N$ can be quantized as follows. Consider the left ideal of $U_\hbar(\mathfrak{gl}_N)\otimes \mathrm{Weyl}_{NK}$ generated by B_{ij} and ψ_{ia} for all $L<i,j\le N$ and $1\le a\le K$, denote it by $I^0_{L,N}$, then $I_{L,N} := (U_\hbar(\mathfrak{gl}_N)\otimes \mathrm{Weyl}_{NK})^{\mathrm{GL}_{N-L}} \cap I^0_{L,N}$ is a two-sided ideal in $(U_\hbar(\mathfrak{gl}_N)\otimes \mathrm{Weyl}_{NK})^{\mathrm{GL}_{N-L}}$, where GL_{N-L} acts on indices $L<i,j\le N$. It is easy to see that

$$(U_\hbar(\mathfrak{gl}_N)\otimes \mathrm{Weyl}_{NK})^{\mathrm{GL}_{N-L}} = (U_\hbar(\mathfrak{gl}_L)\otimes \mathrm{Weyl}_{LK}) \oplus I_{L,N}$$

as vector spaces, thus $U_\hbar(\mathfrak{gl}_L)\otimes \mathrm{Weyl}_{LK} = (U_\hbar(\mathfrak{gl}_N)\otimes \mathrm{Weyl}_{NK})^{\mathrm{GL}_{N-L}}/I_{L,N}$. Restricting to $\mathbb{C}_\hbar[\mathcal{M}(N,K)] = (U_\hbar(\mathfrak{gl}_N)\otimes \mathrm{Weyl}_{NK})^{\mathrm{GL}_N}$, we get a map $\mathbb{C}_\hbar[\mathcal{M}(N,K)] \to \mathbb{C}_\hbar[\mathcal{M}(L,K)]$ between graded algebras, and this quantizes the embedding $\mathcal{M}(L,K) \hookrightarrow \mathcal{M}(N,K)$. This map is surjective because it is surjective modulo $\hbar$.

Theorem 3.5.5. *For every N there is a surjective map of algebras*

$$Y_\hbar(\mathfrak{gl}_K\oplus\mathfrak{gl}_1) := Y_\hbar(\mathfrak{gl}_K)\otimes Y_\hbar(\mathfrak{gl}_1) \twoheadrightarrow \mathbb{C}_\hbar[\mathcal{M}(N,K)].$$

Here we define $Y_\hbar(\mathfrak{gl}_1)$ as the algebra $\mathbb{C}[L^-\mathrm{GL}_1][\hbar]$. These maps are compatible in the sense that for $N>L$ the diagram

$$\begin{array}{ccc} Y_\hbar(\mathfrak{gl}_K\oplus\mathfrak{gl}_1) & \longrightarrow & \mathbb{C}_\hbar[\mathcal{M}(N,K)] \\ & \searrow & \downarrow \\ & & \mathbb{C}_\hbar[\mathcal{M}(L,K)] \end{array}$$

commutes. Moreover, the intersection of ideals of these maps is zero.

Proof. After quantization, we need to be careful about taking determinant. Instead of taking coefficients in the characteristic polynomial of B, we use the natural generators $\mathrm{Tr}(B^k)$. More precisely, write $\mathbb{C}[L^-\mathrm{GL}_1] = \mathbb{C}[m_1,m_2,\cdots]$, where m_i is the function that takes the value of a_i in the power series $1+\sum_{i\ge 1} a_i z^{-i} \in L^-\mathrm{GL}_1$, and define the "power sum" generators $p_1,p_2,\cdots$ by

$$\sum_{n\ge 1}\frac{p_n}{nz^n} = -\log\left(1+\sum_{n\ge 1}\frac{m_n}{z^n}\right).$$

We define the map $Y_\hbar(\mathfrak{gl}_1)\to \mathbb{C}_\hbar[\mathcal{M}(N,K)]$ by $p_n\mapsto \mathrm{Tr}(B^n)$. Let the RTT generators of $Y_\hbar(\mathfrak{gl}_K)$ be $\mathsf{T}(u)$, and we define $Y_\hbar(\mathfrak{gl}_K)\to \mathbb{C}_\hbar[\mathcal{M}(N,K)]$ by $\mathsf{T}(u)\mapsto T(u)$. Then $Y_\hbar(\mathfrak{gl}_K)\otimes Y_\hbar(\mathfrak{gl}_1)\to \mathbb{C}_\hbar[\mathcal{M}(N,K)]$ is surjective since it is surjective modulo $\hbar$. The compatibility is clear from construction. The intersection of kernels is zero because $\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ is flat over $\mathbb{C}[\hbar]$ and the intersection of kernels modulo $\hbar$ is zero. $\square$

3.5.6 Another map from $Y_\hbar(\mathfrak{gl}_K \oplus \mathfrak{gl}_1)$ to $\mathbb{C}_\hbar[\mathcal{M}(N,K)]$

Recall that the phase space $\mathcal{M}(N,K)$ is actually a Hamiltonian reduction of $(B_+, B_-, \psi, \overline{\psi})$ by the moment map equation $B_+ - B_- = \psi\overline{\psi}$. In the previous discussions we use the convention $B = B_+$, so there is another set of generators $\overline{\psi}B_-^k\psi$ and $\mathrm{Tr}(B_-^k)$ of $\mathbb{C}[\mathcal{M}(N,K)]$. It is easy to see that the subalgebra in $\mathbb{C}[\mathcal{M}(N,K)]$ generated by $\overline{\psi}B_+^k\psi$ is the same as the subalgebra generated by $\overline{\psi}B_-^k\psi$. However, the subalgebra generated by $\mathrm{Tr}(B_+^k)$ is not the same as the subalgebra generated by $\mathrm{Tr}(B_-^k)$. This means that we have two distinct maps from $\mathbb{C}[L^-\mathrm{GL}_1]$ to $\mathbb{C}[\mathcal{M}(N,K)]$.

After quantization, the commutation relation between B_- are

$$[B_{-,ij}, B_{-,kl}] = \hbar(\delta_{il}B_{-,kj} - \delta_{kj}B_{-,il}).$$

Definition 3.5.7. The quantum moment map $\mu : \mathfrak{gl}_N \to U_\hbar(\mathfrak{gl}_N) \otimes U_\hbar(\mathfrak{gl}_N) \otimes \mathrm{Weyl}_{NK}$ is

$$\mu(E_{ij}) = B_{+,ij} - B_{-,ij} - \psi_i\overline{\psi}_j + \hbar N\delta_{ij}. \tag{3.51}$$

And the quantum Hamiltonian reduction $(U_\hbar(\mathfrak{gl}_N) \otimes U_\hbar(\mathfrak{gl}_N) \otimes \mathrm{Weyl}_{NK}) /\!\!/_\mu \mathrm{GL}_N$ is defined as the GL_N invariant of $U_\hbar(\mathfrak{gl}_N) \otimes U_\hbar(\mathfrak{gl}_N) \otimes \mathrm{Weyl}_{NK}$ quotient by the left ideal generated by $\mu(\mathfrak{gl}_N)$. Denote the quantum Hamiltonian reduction by $\mathcal{A}_{N,K}$.

Obviously there are two isomorphisms between $\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ and $\mathcal{A}_{N,K}$, corresponding to two set of generators which are packaged in the generating functions

$$\begin{aligned} T_{ab}(u) &= \delta_{ab} + \overline{\psi}_a \frac{1}{u - B_+}\psi_b, Z(u) = 1 - \hbar\mathrm{Tr}\left(\frac{1}{u - B_+}\right), \\ \overline{T}_{ab}(u) &= \delta_{ab} + \overline{\psi}_a \frac{1}{u + B_-}\psi_b, \overline{Z}(u) = 1 - \hbar\mathrm{Tr}\left(\frac{1}{u + B_-}\right). \end{aligned}$$

The relations between two sets of generators are summarized in the next lemma.

Lemma 3.5.8.

$$T_{ab}(u)\overline{T}_{bc}(-u) = \delta_{ac}, \tag{3.52}$$

$$T_{ab}(u)\overline{T}_{ba}(-u + K\hbar) = KZ(u)\overline{Z}(-u + K\hbar) \tag{3.53}$$

Proof. First of all, we compute

$$
\begin{aligned}
T_{ab}(u)\overline{T}_{bc}(w) &= \delta_{ac} + \overline{\psi}_a \frac{1}{u-B_+}\psi_c + \overline{\psi}_a \frac{1}{w+B_-}\psi_c + \overline{\psi}_a \frac{1}{u-B_+}\psi_b\overline{\psi}_b \frac{1}{w+B_-}\psi_c \\
&= \delta_{ac} + \overline{\psi}_a \frac{1}{u-B_+}\psi_c + \overline{\psi}_a \frac{1}{w+B_-}\psi_c + \overline{\psi}_a \frac{1}{u-B_+}(B_+-B_-)\frac{1}{w+B_-}\psi_c \\
&= \delta_{ac} + (u+w)\overline{\psi}_a \frac{1}{u-B_+}\frac{1}{w+B_-}\psi_c.
\end{aligned}
$$

Taking $w=-u$, we get $T_{ab}(u)\overline{T}_{bc}(-u)=\delta_{ac}$. Contracting with δ_{ac}, we get

$$
\begin{aligned}
&T_{ab}(u)\overline{T}_{ba}(w) = K + (u+w)\mathrm{Tr}\left(\frac{1}{w+B_-}\psi\overline{\psi}\frac{1}{u-B_+}\right) - K\hbar(u+w)\mathrm{Tr}\left(\frac{1}{w+B_-}\frac{1}{u-B_+}\right) \\
&= K + (u+w)\mathrm{Tr}\left(\frac{1}{w+B_-}(B_+-B_-)\frac{1}{u-B_+}\right) + \hbar(u+w)\mathrm{Tr}\left(\frac{1}{w+B_-}\right)\mathrm{Tr}\left(\frac{1}{u-B_+}\right) \\
&- K\hbar(u+w)\mathrm{Tr}\left(\frac{1}{w+B_-}\frac{1}{u-B_+}\right).
\end{aligned}
$$

Here the second equality follows from moment map condition. Taking $w=-u+K\hbar$, we get

$$
T_{ab}(u)\overline{T}_{ba}(-u+K\hbar) = K\left(1-\hbar\mathrm{Tr}\left(\frac{1}{u-B_+}\right)\right)\left(1-\hbar\mathrm{Tr}\left(\frac{1}{-u+K\hbar+B_-}\right)\right).
$$

□

Recall that the quantum determinant of $T(u)$ is defined as

$$
\mathrm{qdet}T(u) = \sum_{\sigma\in\mathfrak{S}_K}\mathrm{sgn}(\sigma)T_{\sigma(1),1}(u+\frac{K-1}{2}\hbar)\cdots T_{\sigma(K),K}(u-\frac{K-1}{2}\hbar). \tag{3.54}
$$

It is proposed in [51] that quantum determinant of $T(u)$ should be related to Capelli's determinant of $B_\pm$, we prove it in the next proposition.

Proposition 3.5.9. *Let $C_+(u)$ be the Capelli's determinant of B_+*

$$
C_+(u) = \sum_{\sigma\in\mathfrak{S}_N}\mathrm{sgn}(\sigma)(u-(N-1)\hbar-B_+)_{\sigma(1),1}\cdots(u-B_+)_{\sigma(N),N}, \tag{3.55}
$$

and similarly let $C_-(u)$ be the the Capelli's determinant of $-B_-$, then

$$
\mathrm{qdet}T(u) = (-1)^N\frac{C_-(-u+\frac{K-1}{2}\hbar)}{C_+(u+\frac{K-1}{2}\hbar)}. \tag{3.56}
$$

Proof. Let $f(u) = \mathrm{qdet}T(u) \cdot C_+(u + \frac{K-1}{2}\hbar)/C_-(-u + \frac{K-1}{2}\hbar)$, then compare the quantum Liouville formula [124]:

$$T_{ab}(u)\overline{T}_{ba}(-u + K\hbar) = K\frac{\mathrm{qdet}T(u - \frac{K-1}{2}\hbar)}{\mathrm{qdet}T(u - \frac{K+1}{2}\hbar)}, \tag{3.57}$$

with Lemma 3.5.8, we get $f(u)/f(u-\hbar) = 1$, so $f(u)$ does not depend on u, thus $f(u) = \lim_{u\to\infty} f(u) = (-1)^N$, i.e. $\mathrm{qdet}T(u) = (-1)^N C_-(-u + \frac{K-1}{2}\hbar)/C_+(u + \frac{K-1}{2}\hbar)$. □

Now we have RTT generator $T(u)$ and its inverse $\overline{T}(-u)$, then the J-generators of the Yangian for $\mathfrak{sl}_K$ can be obtained from them, in fact one define $B_{\mathrm{avr}} = \frac{1}{2}(B_+ + B_-)$, and

$$J^{(n)}_{ab} = \overline{\psi}_a B^n_{\mathrm{avr}} \psi_b, \tag{3.58}$$

then $J^{(0)}_{ab}$ are generators of $\mathfrak{gl}_K$ and they act on $J^{(1)}_{ab}$ as adjoint representation, and

$$[J^{(1)}_{ab}, J^{(1)}_{cd}] = \hbar(\delta_{bc}J^{(2)}_{ad} - \delta_{ad}J^{(2)}_{cb}) + \frac{\hbar}{4}(J^{(0)}_{ed}J^{(0)}_{ae}J^{(0)}_{cb} - J^{(0)}_{eb}J^{(0)}_{ce}J^{(0)}_{ad}). \tag{3.59}$$

The above commutation relation shows that $\tilde{J}^{(0)}_{ab} = J^{(0)}_{ab} - \frac{1}{K}\delta_{ab}J^{(0)}_{cc}$ and $\tilde{J}^{(1)}_{ab} = J^{(1)}_{ab} - \frac{1}{K}\delta_{ab}J^{(1)}_{cc}$ generate the image of the subalgebra $Y_\hbar(\mathfrak{sl}_K) \subset Y_\hbar(\mathfrak{gl}_K \oplus \mathfrak{gl}_1)$.

3.5.10 Defining ideal of $\mathbb{C}_\hbar[\mathcal{M}(N, K)]$

In this subsection we present some observations about the ideal of the quotient map $Y_\hbar(\mathfrak{gl}_K \oplus \mathfrak{gl}_1) \twoheadrightarrow \mathbb{C}_\hbar[\mathcal{M}(N, K)]$.

Definition 3.5.11. Fix N, define a power series $\mathsf{C}(u) = z^N + \sum_{n>0} \mathsf{C}_n u^{N-n}$ with coefficients $\mathsf{C}_n \in Y_\hbar(\mathfrak{gl}_1)$ by

$$1 - \frac{N\hbar}{u} - \hbar\sum_{n>0}\frac{p_n}{u^{n+1}} = \frac{\mathsf{C}(u-\hbar)}{\mathsf{C}(u)}. \tag{3.60}$$

Here p_n are the power sum generators of $Y_\hbar(\mathfrak{gl}_1)$. Let RTT generator of $Y_\hbar(\mathfrak{gl}_K)$ be $\mathsf{T}(u) = 1 + \sum_{n\geq 0}\mathsf{T}^{(n)}u^{-n-1}$, and write the quantum minor of $\mathsf{T}(u)$ for row indices $\underline{a} = (a_1 < \cdots < a_i)$ and column indices $\underline{b} = (b_1 < \cdots < b_i)$ as

$$\mathsf{T}_{\underline{a},\underline{b}}(u) = \sum_{\sigma\in\mathfrak{S}_i}\mathrm{sgn}(\sigma)\mathsf{T}_{\sigma(a_1),b_1}(u + \frac{i-1}{2}\hbar)\cdots\mathsf{T}_{\sigma(a_i),b_i}(u - \frac{i-1}{2}\hbar). \tag{3.61}$$

Remark 3.5.12. Let $C(u)$ be the Capelli's determinant of B, then by the quantum Newton's formula [124], we have

$$1-\hbar\mathrm{Tr}\left(\frac{1}{u-B}\right)=\frac{C(u-\hbar)}{C(u)},\tag{3.62}$$

therefore the image of $\mathsf{C}(u)$ in $\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ is $C(u)$. In the classical limit $\hbar\to 0$, $C(u)$ is the $\det(u-B)$, and $C_n\equiv(-1)^n m_n \mod \hbar$, where m_n are the generators of $\mathbb{C}[L^-\mathrm{GL}_1]$ that take the value of a_n in the power series $1+\sum_{n\ge 1}a_n z^{-n}\in L^-\mathrm{GL}_1$.

Theorem 3.5.13. *The kernel of* $Y_\hbar(\mathfrak{gl}_K\oplus\mathfrak{gl}_1)\twoheadrightarrow\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ *is generated by all coefficients for negative powers in* u *in the power series*

$$\mathsf{C}(u),\ \mathsf{C}(u+\frac{i-1}{2}\hbar)\mathsf{T}_{\underline{a},\underline{b}}(u),\tag{3.63}$$

for all $\underline{a}=(a_1<\cdots<a_i),\underline{b}=(b_1<\cdots<b_i)$ *and all* $1\le i\le K$.

Proof. First of all, we show that (3.63) are mapped to polynomials. For $\mathsf{C}(u)$, its image is the Capelli's determinant $C(u)$ of B, which is a polynomial. Note that $C(u)$ is known to be noncommutative version of characteristic polynomial in the sense that $C(B)=0$ [124], thus we have recursion relations: $T^{(m)}_{ab}+\sum_{n=1}^N C_n T^{(m-n)}_{ab}=0$ for all $m\ge N$, which is equivalent to that $C(u)T_{ab}(u)$ is a polynomial. It follows from (3.56) that $C(u+\frac{K-1}{2}\hbar)\mathrm{qdet}T(u)$ is a polynomial. Next we consider the embedding $\mathbb{C}_\hbar[\mathcal{M}(N,i)]\hookrightarrow\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ by $B\mapsto B$ and $\psi_{is}\mapsto\psi_{ia_s}$ and $\overline{\psi}_{si}\mapsto\overline{\psi}_{a_si}$. This implies that $C(u+\frac{i-1}{2}\hbar)T_{\underline{a},\underline{a}}(u)$ are polynomials for all $\underline{a}=(a_1<\cdots<a_i)$ and all $1\le i\le K$. After taking commutators with $T^{(0)}_{ab}$ for various indices a and b, we see that all coefficients for negative powers in u in the power series $C(u+\frac{i-1}{2}\hbar)T_{\underline{a},\underline{b}}(u)$ are $\hbar$-torsion, and by the flatness (Lemma 3.5.1) they must be zero. Thus we see that (3.63) are mapped to polynomials.

Next we show that the kernel of $Y_\hbar(\mathfrak{gl}_K\oplus\mathfrak{gl}_1)\twoheadrightarrow\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ is generated by all coefficients for negative powers in u in the power series (3.63). By the flatness over $\mathbb{C}[\hbar]$ (Lemma 3.5.1), it suffices to show that they generate the ideal modulo $\hbar$. In fact, we claim that the scheme defined by vanishing of those coefficients modulo $\hbar$ is reduced and irreducible of dimension $2NK$, this implies the result. To prove the claim, we write down the image of (3.63) in the Drinfeld's generators of $Y_\hbar(\mathfrak{gl}_K\oplus\mathfrak{gl}_1)$:

$$H_i(u)=\mathsf{C}(u)^{\delta_{i,1}}\frac{A_{i-1}(u+\frac{\hbar}{2})A_{i+1}(u+\frac{\hbar}{2})}{A_i(u)A_i(u+\hbar)}\tag{3.64}$$

$$E_i(u)=\mathsf{C}(u+\frac{i-1}{2}\hbar)\mathsf{T}_{\underline{i},\underline{i}^+}(u)A_i(u)^{-1},\ F_i(u)=\mathsf{C}(u+\frac{i-1}{2}\hbar)A_i(u)^{-1}\mathsf{T}_{\underline{i}^+,\underline{i}}(u),\tag{3.65}$$

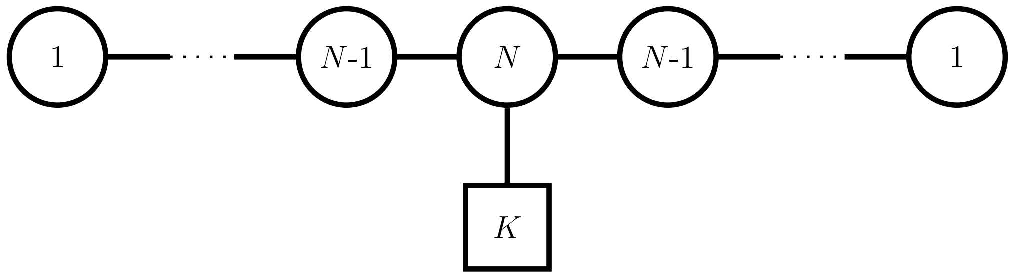


Figure 3.3: The quiver diagram for the Higgs-branch description

where $1 \le i \le K-1$, $A_0(u) = 1$, $A_i(u) = \mathsf{C}(u + \frac{i-1}{2}\hbar)\mathsf{T}_{\underline{i},\underline{i}}(u)$, $\underline{i} = (1 < \cdots < i)$ and $\underline{i}^+ = (1 < \cdots < i-1 < i+1)$. Compare this formula with [18, Corollary B.17] we conclude that the quotient of $Y_\hbar(\mathfrak{gl}_K \oplus \mathfrak{gl}_1)$ by the ideal generated by all coefficients for negative powers in u in the power series (3.63) is the truncated Yangian $Y_0^{N\lambda}[m_1^L, \cdots, m_N^L, m_1^R, \cdots, m_N^R]_\hbar$ for $\mathfrak{sl}_K$, where $\lambda = \omega_1 + \omega_{K-1}$ and ω_i is the i'th fundamental coweight of $\mathfrak{sl}_K$, and the mass parameters $W_L(u) = \prod_{i=1}^N (u - m_i^L), W_R(u) = \prod_{i=1}^N (u - m_i^R)$ are identified through

$$W_L(u) = \mathsf{C}(u - \frac{1}{2}\hbar),\ W_R(u) = (-1)^N \mathsf{C}_-(-u + \frac{K}{2}\hbar). \tag{3.66}$$

Since $Y_0^{N\lambda}[m_1^L, \cdots, m_N^L, m_1^R, \cdots, m_N^R]_\hbar/(\hbar) = \mathbb{C}[\overline{\mathcal{W}}^{N\lambda}_{0,\mathrm{SL}_K}]$ and $\overline{\mathcal{W}}^{N\lambda}_{0,\mathrm{SL}_K}$ is a reduced and irreducible scheme of dimension $2NK$, the theorem follows. □

Remark 3.5.14. We actually find an explicit S-duality isomorphism between quantized Higgs branch of the 3*d* $\mathcal{N} = 4$ gauge theory associated to the quiver in Figure 3.3 and the quantized Coulomb branch of the 3*d* $\mathcal{N} = 4$ gauge theory associated to the quiver in Figure 3.4. The generator $A_i(u)$ for $1 \le i \le K-1$ are mapped to $\prod_{r=1}^N (u - \hbar - w_{i,r})$, where $w_{i,r}$ is the r'th equivariant parameter of the i'th gauge node. The subalgebra of $\mathbb{C}_\hbar[\mathcal{M}(N, K)]$ generated by $A_i(u)$ for $1 \le i \le K-1$ is known as the *Gelfand-Zeitlin* subalgebra.

3.5.15 Quantized coproduct

It is well-known that truncated Yangian has coproduct

$$\Delta : Y_0^{(N_1+N_2)\lambda}[m_1^L, \cdots, m_{N_1+N_2}^L, m_1^R, \cdots, m_{N_1+N_2}^R]_\hbar \longrightarrow \tag{3.67}$$
$$Y_0^{N_1\lambda}[m_1^L, \cdots, m_{N_1}^L, m_1^R, \cdots, m_{N_1}^R]_\hbar \otimes_{\mathbb{C}[\hbar]} Y_0^{N_2\lambda}[m_{N_1+1}^L, \cdots, m_{N_1+N_2}^L, m_{N_1+1}^R, \cdots, m_{N_1+N_2}^R]_\hbar$$

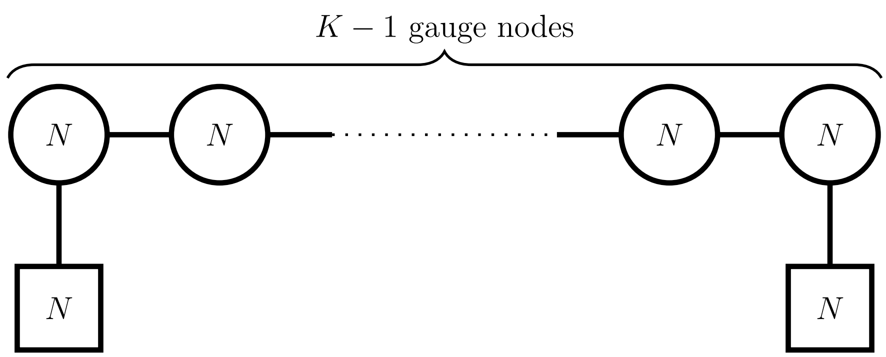


Figure 3.4: The quiver diagram for the Coulomb-branch description. The corresponding gauge theory is mirror-dual to the one described by the quiver in Figure 3.3.

which is compatible with the coproduct on $Y_\hbar(\mathfrak{sl}_K)$. In the RTT generators, we can write the coproduct explicitly as

$$\Delta(T_{ab}(u)) = T_{ac}(u) \otimes T_{cb}(u),\ \Delta(C(u)) = C(u) \otimes C(u). \tag{3.68}$$

or equivalently

$$\Delta(\overline{T}_{ab}(u)) = \overline{T}_{cb}(u) \otimes \overline{T}_{ac}(u),\ \Delta(C_-(u)) = C_-(u) \otimes C_-(u). \tag{3.69}$$

Note that this coproduct is a map of $Y_\hbar(\mathfrak{gl}_K \oplus \mathfrak{gl}_1)$-bimodules.

An interesting feature of this coproduct is that it does not comes from a truncation of coproduct for $Y_\hbar(\mathfrak{gl}_K \oplus \mathfrak{gl}_1)$. In fact (3.62) together with (3.68) implies that $1 - \hbar \mathrm{Tr}\left(\frac{1}{u-B}\right)$ is group-like, and we compute that $\Delta(\mathrm{Tr}(B)) = \mathrm{Tr}(B) \otimes 1 + 1 \otimes \mathrm{Tr}(B) - \hbar N_1 N_2$. The rank of the truncation explicitly enters the coproduct formula, this means that we need to upgrade the rank N into a variable in the large N limit. Namely we define the $\mathbb{C}[\hbar]$-bialgebra $Y_{\hbar,\delta}(\mathfrak{gl}_K \oplus \mathfrak{gl}_1)$ as the Yangian extended by central element δ, i.e. $Y_\hbar(\mathfrak{gl}_K \oplus \mathfrak{gl}_1)[\delta]$, and the coproduct

$$\Delta(\mathsf{T}_{ab}(u)) = \mathsf{T}_{ac}(u) \otimes \mathsf{T}_{cb}(u),\ \Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta, \tag{3.70}$$
$$\Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n - \hbar \sum_{i=0}^{n-1} p_i \otimes p_{n-1-i},$$

where $p_n, n > 0$ are the power sum generators of $Y_\hbar(\mathfrak{gl}_1)$ and $p_0 := \delta$, and the counit

$$\epsilon(\mathsf{T}_{ab}(u)) = \delta_{ab},\ \epsilon(p_n) = \epsilon(\delta) = 0. \tag{3.71}$$

Under the natural quotient map $Y_{\hbar,\delta}(\mathfrak{gl}_K \oplus \mathfrak{gl}_1) \twoheadrightarrow \mathbb{C}_\hbar[\mathcal{M}(N,K)]$ defined as $\mathsf{T}(u) \mapsto T(u), p_n \mapsto \mathrm{Tr}(B^n), \delta \mapsto N$, the coproduct (3.70) truncates to (3.68).

Motivated by (3.60), we define power series $\mathsf{A}(u) = 1 + \sum_{n>0} \mathsf{A}_n u^{-n}, \mathsf{A}_n \in Y_{\hbar,\delta}(\mathfrak{gl}_1)$ by

$$\left(1 - \frac{\hbar}{u}\right)^{-\delta} \left(1 - \hbar \sum_{n\geq 0} \frac{p_n}{u^{n+1}}\right) = \frac{\mathsf{A}(u-\hbar)}{\mathsf{A}(u)}. \tag{3.72}$$

Then the second line of (3.70) can be written in a compact form

$$\Delta(\mathsf{A}(u)) = \mathsf{A}(u) \otimes \mathsf{A}(u). \tag{3.73}$$

In fact $Y_{\hbar,\delta}(\mathfrak{gl}_K \oplus \mathfrak{gl}_1)$ is a $\mathbb{C}[\hbar]$-Hopf algebra with antipode S

$$S(\mathsf{T}(u)) = \mathsf{T}^{-1}(u),\ S(\delta) = -\delta,\ S(\mathsf{A}(u)) = \mathsf{A}^{-1}(u). \tag{3.74}$$

3.5.16 Quantized phase space and Coulomb branch algebra

In this subsection we give a conceptual understanding of the identification between the quantized phase space $\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ and Coulomb branch algebra associated to the quiver in Figure 3.4.

Given a quiver Q, we denote by $\mathcal{A}^\hbar_C(Q)$ the quantum Coulomb branch algebra associated to the quiver Q with all mass deformation turned on [18], i.e.

$$\mathcal{A}^\hbar_C(Q) := \mathrm{H}_*^{(\mathrm{GL}(V)_\mathcal{O}\times \mathrm{GL}(W)_\mathcal{O})\rtimes \mathbb{C}^\times}(\mathcal{R}),$$

see [18] for more details.

Example 3.5.17. It is known that the quantum Coulomb branch algebra of the quiver of Figure 3.5 is the Weyl algebra $\mathrm{Weyl}_{N(K+N)}$. Its classical limit is the generalized transverse slice $\overline{\mathcal{W}}^{\lambda_N^*}_{w_0(\lambda_N^*)}$, where λ_N is the N'th fundamental coweight of $\mathrm{GL}_{N(K+N)}$ and w_0 is the longest element of the Weyl group of $\mathrm{GL}_{N(K+N)}$ and $\lambda_N^* = -w_0(\lambda_N)$. The projection $\mathrm{GL}_{N(K+N)}(\!(z)\!) \to \mathrm{Gr}_{\mathrm{GL}_{N(K+N)}}$ identifies $\overline{\mathcal{W}}^{\lambda_N^*}_{w_0(\lambda_N^*)}$ with the cotangent bundle of the orbit $U_{\lambda_N} \cdot z^{-\lambda_N}$, where U_{λ_N} is the unipotent group whose Lie algebra is the -1 eigenspace of λ_N.

Example 3.5.18. The 3*d* $\mathcal{N} = 4$ gauge theory associated to the following quiver

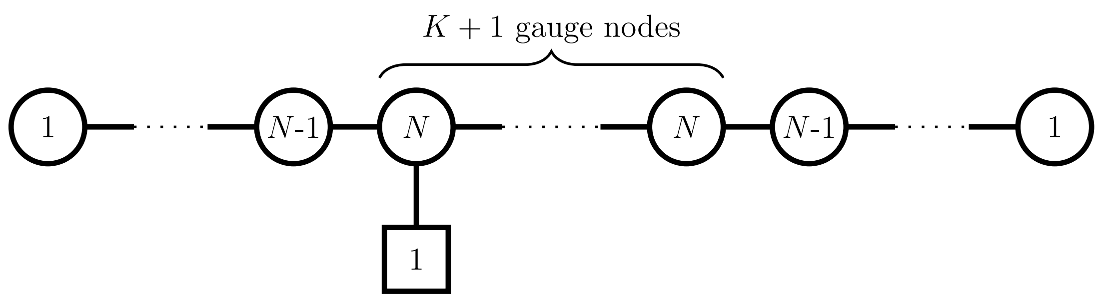


Figure 3.5: The quiver for the Weyl algebra $\mathrm{Weyl}_{N(K+N)}$.

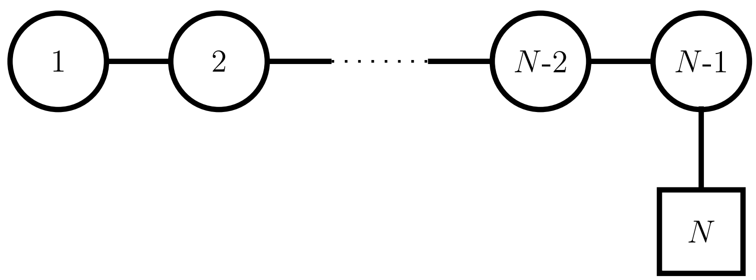


Figure 3.6: The quiver diagram of $\mathcal{T}[\mathrm{SU}(N)]$ theory.

is known as $\mathcal{T}[\mathrm{SU}(N)]$, its Coulomb branch algebra is isomorphic to $U_\hbar(\mathfrak{gl}_N)$. An explicit way to see this isomorphism is by looking at the evaluation representation of $Y_\hbar(\mathfrak{gl}_N)$: $\mathsf{T}_{ij}(u) \mapsto \delta_{ij} + \frac{E_{ij}}{u}$, where E_{ij} are the generators of $U_\hbar(\mathfrak{gl}_N)$ satisfying relations $[E_{ij}, E_{kl}] = \hbar(\delta_{jk}E_{il} - \delta_{il}E_{kj})$. Define

$$A_n(u) = u^{[n]}\mathsf{T}_{\underline{n},\underline{n}}(u),\ u^{[n]} := (u + \frac{n-1}{2}\hbar)(u + \frac{n-3}{2}\hbar)\cdots(u - \frac{n-1}{2}\hbar),$$

where $\mathsf{T}_{\underline{n},\underline{n}}(u)$ is the quantum determinant of the submatrix of $\mathsf{T}(u)$ consisting of first n rows and first n columns. Write $A_n(u) = u^n + \sum_{i>0} A_n^{(i)} u^{n-i}$, then the kernel of $Y_\hbar(\mathfrak{gl}_N) \twoheadrightarrow U_\hbar(\mathfrak{gl}_N)$ contains $A_n^{(p)}$ for all $p > n$ and for all $1 \le n \le N$. In the Drinfeld generators, we have

$$H_n(u) = \frac{A_{n-1}(u+\frac{\hbar}{2})A_{n+1}(u+\frac{\hbar}{2})}{A_n(u)A_n(u+\hbar)},\ 1 \le n \le N-1,\ A_0(u) = 1.$$

Compare with [18, Corollary B.17] we conclude that the quotient of $Y_\hbar(\mathfrak{gl}_N)$ by the ideal generated by $A_n^{(p)}$ for all $p > n$ and for all $1 \le n \le N$ (and invert $\hbar$ if possible) is the truncated Yangian $Y_0^{N\omega_{N-1}}[m_1, \cdots, m_N]_\hbar$ for $\mathfrak{sl}_N$, where ω_{N-1} is the $(N-1)$'st fundamental coweight of $\mathfrak{sl}_N$, and the mass parameters $W(u) = \prod_{i=1}^N (u - m_i)$ are identified as $W(u) = A_N(u + \hbar)$. The classical limit of $Y_0^{N\omega_{N-1}}[m_1, \cdots, m_N]_\hbar$ is the function ring of $\overline{\mathcal{W}}^{N\omega_{N-1}}_{0,\mathrm{SL}_N}$, which is reduced and irreducible of dimension N^2. On the other hand, the classical limit of $U_\hbar(\mathfrak{gl}_N)$ is the function ring of $\mathfrak{gl}_N^*$, which has dimension N^2 and embeds into $\overline{\mathcal{W}}^{N\omega_{N-1}}_{0,\mathrm{SL}_N}$ as a closed subscheme, thus $U_\hbar(\mathfrak{gl}_N)$ is isomorphic to the Coulomb branch algebra of quiver in the Figure 3.6, argued in the same way as Theorem 3.5.13.

Recall that balanced subquiver $Q^{\mathrm{bal}} \subset Q$ is formed by those edge-loop-free nodes $i \in Q_0$ such that $2\dim V_i = \dim W_i + \sum_j a_{ij} \dim V_j$ where a_{ij} is the number of edges between i and j. It is well-known that Q^{bal} is a union of finite ADE quivers, unless Q^{bal} is a union of connected components of Q of affine type with zero framing on them. It is shown in [18] that if it is not the latter case then the corresponding ADE group, denoted by $\widetilde{L}^{\mathrm{bal}}$, acts on the Coulomb branch algebra $\mathcal{A}^\hbar_C(Q)$, such that the infinitesimal action is generated by $\frac{1}{\hbar}[H_i^{(1)}, \bullet], \frac{1}{\hbar}[E_i^{(1)}, \bullet], \frac{1}{\hbar}[F_i^{(1)}, \bullet]$ for those $i \in Q_0^{\mathrm{bal}}$.

Example 3.5.19. In the case that Q is of ADE type with gauge dimension vector $\mathbf{v}$ and flavour dimension vector $\mathbf{w}$, the classical Coulomb branch $\mathcal{M}_C(Q)$ is the Poisson variety $\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}$, where $\lambda = \sum_{i \in Q_0} \mathbf{w}_i \lambda_i, \mu = \lambda - \sum_{i \in Q_0} \mathbf{v}_i \alpha_i$, $\lambda^* = -w_0(\lambda)$, λ_i are fundamental coweights and α_i are fundamental coroots and w_0 is the longest element in the Weyl group of G. It is shown in [19, Example A.5] that L^{bal} action can be identified with the natural action of $\mathrm{Stab}_G(\mu^*)$ on $\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}$ when μ is dominant. This holds for general μ. In fact we can take a dominant ν such that $\langle \nu, \check{\alpha}_i \rangle = 0, \forall i \in Q_0^{\mathrm{bal}}$ and $\mu + \nu$ is dominant, then the shift map $i_{0,\nu^*} : \mathbb{C}[\overline{\mathcal{W}}^{\lambda^*+\nu^*}_{\mu^*+\nu^*}] \to \mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}]$ commutes with the action of $\mathrm{Stab}_G(\mu^*) \subset \mathrm{Stab}_G(\mu^* + \nu^*)$. Since i_{0,ν^*} is Poisson and preserves $E_i^{(1)}, F_i^{(1)}, H_i^{(1)}$ for $i \in Q_0^{\mathrm{bal}}$, it follows that the action of L^{bal} constructed in [19, Proposition A.3] commutes with the shift map. Since the action of L^{bal} agrees with the natural one for $\mathrm{Stab}_G(\mu^*)$ on $\overline{\mathcal{W}}^{\lambda^*+\nu^*}_{\mu^*+\nu^*}$, and the shift map is birational and equivariant for both of actions, these two actions agree on $\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}$ as well.

Remark 3.5.20. Suppose that there is another action of $\widetilde{L}^{\mathrm{bal}}$ on $\mathcal{A}^\hbar_C(Q)$ which acts trivially on $\hbar$, not necessarily the one constructed in [19, Appendix A], such that these two actions agree after modulo $\hbar$ and mass parameters (generators of $\mathrm{H}^*_{\mathrm{GL}(W)}(\mathrm{pt})$), then these two actions must agree on $\mathcal{A}^\hbar_C(Q)$. In fact $\mathcal{A}^\hbar_C(Q)$ is a flat deformation of $\mathcal{A}^\hbar_C(Q)/(\hbar, \mathrm{mass})$ and the deformation spaces of modules for reductive group are trivial.

Consider a quiver Q containing following part

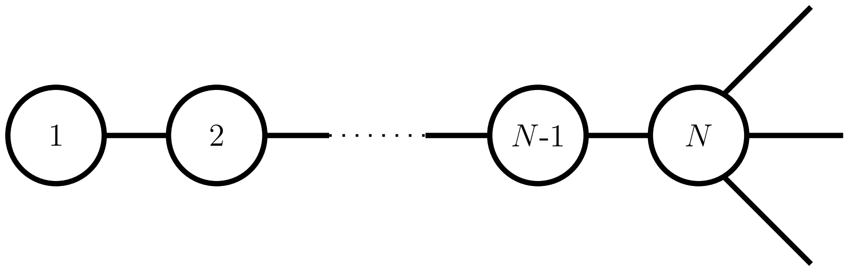


Figure 3.7: The quiver Q.

Then $\mathcal{A}^{\hbar}_C(Q)$ admits an action of SL_N, and also a grading ($\mathbb{C}^\times$ action) coming from $\pi_0(\mathrm{Gr}_{\mathrm{GL}_N})$ which commutes with the SL_N action, thus $\mathcal{A}^{\hbar}_C(Q)$ admits an action of GL_N. Denote the following quiver by Q'

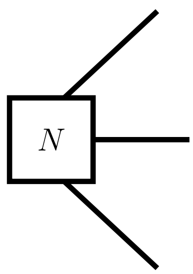


Figure 3.8: The quiver Q'.

then we have

Lemma 3.5.21. $\mathcal{A}^{\hbar}_C(Q') \cong \mathcal{A}^{\hbar}_C(Q)^{\mathrm{GL}_N}$.

Proof. Consider the affine Grassmannian $\mathrm{Gr}_{\mathrm{GL}_N}$ and denote by $\mathcal{A}_Q$ (resp. $\mathcal{A}_{Q'}$) the ring object in $D_{\mathrm{GL}_N(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{\mathrm{GL}_N})$ coming from pushing forward of the dualing complex on the BFN space of triples corresponding to quiver gauge theory Q (resp. Q'), see [19]. Then we have $\mathcal{A}_Q \cong \mathcal{A}_R \overset{!}{\otimes} \mathcal{A}_{Q'}$ [19], where $\mathcal{A}_R$ is the regular ring object with a natural GL_N action (which is called the right action in [19]). Therefore we have

$$\begin{aligned}\mathcal{A}^{\hbar}_C(Q)^{\mathrm{GL}_N} &= \mathrm{H}^*_{\mathrm{GL}_N(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{\mathrm{GL}_N}, \mathcal{A}_R \overset{!}{\otimes} \mathcal{A}_{Q'})^{\mathrm{GL}_N} = \mathrm{H}^*_{\mathrm{GL}_N(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{\mathrm{GL}_N}, \mathrm{IC}_0 \overset{!}{\otimes} \mathcal{A}_{Q'}) \\ &= \mathrm{Ext}^*_{\mathrm{GL}_N(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{IC}_0, \mathcal{A}_{Q'}) = \mathcal{A}^{\hbar}_C(Q').\end{aligned}$$

□

Example 3.5.22. In the example of $\mathrm{Weyl}_{N(K+N)}$, $\mathrm{Stab}_{\mathrm{SL}_{N(K+N)}}(w_0(\lambda_N^*)) = \mathrm{SL}_N \times \mathrm{SL}_{N+K}$ acts on $\mathrm{Weyl}_{N(K+N)} = \mathrm{Diff}_\hbar(\mathbb{C}^N \otimes \mathbb{C}^{K+N})$ naturally via regarding $\mathbb{C}^N$ as fundamental representation of SL_N and $\mathbb{C}^{K+N}$ as antifundamental representation of SL_{K+N}, and modulo $\hbar$ the action becomes the natural one on the cotangent bundle of the orbit $U_{\lambda_N} \cdot z^{-\lambda_N}$, thus by the above remark we see that the action of $L^{\mathrm{bal}} = \mathrm{SL}_N \times \mathrm{SL}_{N+K}$ on $\mathrm{Weyl}_{N(K+N)}$ is the natural one described above. Moreover the grading coming from $\pi_0(\mathrm{Gr}_{\mathrm{GL}_N})$ is that $\deg E_N^{(p)} = -1, \deg F_N^{(p)} = 1$, and this grading enlarges the action of $\mathrm{SL}_N \times \mathrm{SL}_{N+K}$ to the action of $\mathrm{GL}_N \times \mathrm{GL}_{N+K}$ on $\mathrm{Weyl}_{N(K+N)} = \mathrm{Diff}_\hbar(\mathbb{C}^N \otimes \mathbb{C}^{K+N})$ via regarding $\mathbb{C}^N$ as fundamental representation of GL_N and $\mathbb{C}^{K+N}$ as antifundamental representation of GL_{K+N}.

Applying Lemma 3.5.21 to the quiver in the Figure 3.5 with $K = 0$, then we see that $U_\hbar(\mathfrak{gl}_N) \cong \mathrm{Weyl}_{N^2}^{\mathrm{GL}_N}$, which is nothing but the free field realization of $U_\hbar(\mathfrak{gl}_N)$. Then it follows that $\mathbb{C}_\hbar[\mathcal{M}(N,K)] \cong \mathrm{Weyl}_{N(K+N)}^{\mathrm{GL}_N \times \mathrm{GL}_N}$ where the action comes from the restriction of $\mathrm{GL}_N \times \mathrm{GL}_{N+K}$ to $\mathrm{GL}_N \times \mathrm{GL}_N$. Apply Lemma 3.5.21 again, followed by removing the edge between flavours as it has no effect on Coulomb branch, we see that $\mathbb{C}_\hbar[\mathcal{M}(N,K)]$ is the Coulomb branch algebra of the quiver in Figure 3.4.

Remark 3.5.23. Apply Lemma 3.5.21 to the quiver in the Figure 3.6 with N replaced by $N + K$, and we see that

$$U_\hbar(\mathfrak{gl}_{N+K})^{\mathrm{GL}_N} \cong Y_0^\lambda[m_1^L, \cdots, m_N^L, m_1^R, \cdots, m_{N+K}^R]_\hbar,$$

where the right-hand-side is a truncated Yangian for $\mathfrak{sl}_K$ and $\lambda = N\omega_1 + (N+K)\omega_{K-1}$. This is known as the centralizer construction of Yangian in the literature [124].

Chapter 4

5d Chern-Simons Theory with Line and Surface Defects

In [35], Costello and Li developed a beautiful formalism, which prescribes a way to topologically twist supergravity. Combining with the classical notion of topological twist of supersymmetric quantum field theory [156, 157], we are now able to explore a topological sector for both sides of AdS/CFT correspondence. It was further suggested in [28] a systematic method of turning an Ω-background, which plays an important roles [131, 2, 132, 130, 161, 129] in studying supersymmetric field theories, in the twisted supergravity.

Topological twist along with Ω-deformation enables us to study a particular protected sub-sector of a given supersymmetric field theory [134, 100, 5, 135], which is localized not only in the field configuration space but also in the spacetime. Interesting dynamics usually disappear along the way, but as a payoff, we can make a more rigorous statement on the operator algebra.

The topological holography is an exact isomorphism between the operator algebras of gravity and field theory. In this chapter, we will focus on a particular example of topological holography: the correspondence of the operator algebra of M-theory on a certain background parametrized by ϵ_1, ϵ_2, which localizes to 5d non-commutative GL_K Chern-Simons theory with non-commutativity parameter ϵ_2 [1], and the operator algebra of the worldvolume theory of M2-brane, which localizes to 1d topological quantum mechanics(TQM). In

[1]The 5d CS theoriy that appears in this thesis is always meant to be a certain variant of the usual 5d CS theory with a topological-holomorphic twist and with non-commutativity turned on in the holomorphic directions.

particular, [29] proved that two operator algebras are Koszul dual [29] to each other.

The important first step of the proof was to impose a BRST-invariance of the 5d GL_K CS theory coupled with the 1d TQM. 5d CS theory is a renormalizable, and self-consistent theory [34]. However, in the presence of the topological defect that couples 1d TQM and 5d CS theory, certain Feynman diagrams turn out to have non-zero BRST variations. For the combined, interacting theory to be quantum mechanically consistent, the BRST variations of the Feynman diagrams should combine to give zero. This procedure magically reproduces the algebra commutation relations that define 1d TQM operator algebra, $\mathcal{A}_{\epsilon_1,\epsilon_2}$. Intriguingly, one can extract non-perturbative information in the protected operator algebra from the perturbative calculation.

In fact, both the algebra of local operators in 5d CS theory and the 1d TQM operator algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ are deformations of the universal enveloping algebra of the Lie algebra $\mathrm{Diff}_{\epsilon_2}(\mathbb{C})\otimes\mathfrak{gl}_K$ over the ring $\mathbb{C}[\![\epsilon_1]\!]$. Deformation theory tells us that the space of deformations of $U(\mathrm{Diff}_{\epsilon_2}(\mathbb{C})\otimes\mathfrak{gl}_K)$ is the second Hochschild cohomology $\mathrm{HH}^2(U(\mathrm{Diff}_{\epsilon_2}(\mathbb{C})\otimes\mathfrak{gl}_K))$. Although this Hochschild cohomology is known to be hard to compute, there is still a clever way of comparing these two deformations [29]: notice that both of the algebras are defined compatibly for super groups $\mathrm{GL}_{K+R|R}$, and their deformations are compatible with transition maps $\mathrm{GL}_{K+R|R}\hookrightarrow\mathrm{GL}_{K+R+1|R+1}$, so there are induced transition maps between Hochschild cohomologies $\mathrm{HH}^2(U(\mathrm{Diff}_{\epsilon_2}(\mathbb{C})\otimes\mathfrak{gl}_{K+R+1|R+1}))\to\mathrm{HH}^2(U(\mathrm{Diff}_{\epsilon_2}(\mathbb{C})\otimes\mathfrak{gl}_{K+R|R}))$, hence the equivalence class of deformations are actually elements in the limit

$$\lim_{R\to\infty}\mathrm{HH}^2(U(\mathrm{Diff}_{\epsilon_2}(\mathbb{C})\otimes\mathfrak{gl}_{K+R|R})) \tag{4.1}$$

and the limit is well-understood [2], it turns out that the space of all deformations is essentially one-dimensional: a free module over $\mathbb{C}[\kappa]$ where κ is the central element $1\otimes\mathrm{Id}_K$. Hence the algebra of local operators in 5d CS theory and the 1d TQM operator algebra are isomorphic up to a κ-dependent reparametrization

$$\hbar\mapsto\sum_{i=1}^{\infty}f_i(\kappa)\hbar^i \tag{4.2}$$

where $f_i(\kappa)$ are polynomials in κ.

Later, in [70] the same algebra with $K=1$ was defined using the gauge theory approach, and a combined system of M2-branes and M5-branes were studied. In this case the algebra of observables $\mathcal{A}$ on M2 brane is isomorphic to the 1-shifted affine Yangian of $\mathfrak{gl}_1$ [155, 108,

[2] The actual computation in [29] is more subtle, and will not be used in this work.

89, 59, 60], and the algebra of observables $\mathcal{W}_\infty$ on M5 brane is isomorphic to the affine Yangian of $\mathfrak{gl}_1$ [143, 64, 144]. Importantly, the algebras have three parameters ϵ_1, ϵ_2, ϵ_3, which are the parameters of Omega deformations turned on three complex planes as a part of the eleven-dimension supergravity background. Depending on the orientations of the M2 branes(extending over one of the three complex planes) and the M5 branes(extending over two of the three complex planes) on the three Omega deformed planes, the description of the theories on the membrane worldvolume changes; however, both of $\mathcal{A}$ and $\mathcal{W}_\infty$ have triality [70, 72, 63] under the cyclic permutation of the deformation parameters.

Crucially, [70] noticed GL_1 CS should be treated separately from GL_K CS theory with $K > 1$, since the algebras differ drastically and the ingredients of the Feynman diagram are different in GL_1 CS, due to the non-commutativity. As a result, operator algebra isomorphism should be re-assessed.

Our work was motivated by the observation, and we will solve the following problem in a part of next three chapters.

- The simplest algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ commutator, which has ϵ_1 correction.

The problem will be solved by two complementary methods:

(1) Calculation by algebraic method, which is done in chapter §5.

(2) Using Feynman diagrams whose non-trivial BRST variation lead to the commutator, which is done in section §4.2.

At this point, one may wonder about the algebraic structure of a network of M2 and M5 branes extending over different complex planes. In [70], the authors conjectured a fusion of $\mathcal{A}$'s and interpreted an end of M2 branes on M5 branes as a degenerate module of a truncated version of $\mathcal{W}_\infty$ [72]. Moreover, recently the authors of [73] discovered a full algebraic structure governing intersecting M2-M5 branes. Key algebraic relations used to assemble the elements of the brane system are $\Delta_{\mathcal{A},\mathcal{A}}$, $\Delta_{\mathcal{A},\mathcal{W}_\infty}$, $\Delta_{\mathcal{W}_\infty,\mathcal{W}_\infty}$, coproducts of $\mathcal{A}$ and $\mathcal{W}_\infty$. They are induced by properly defined fusions of M2 and M5 branes. The strategy of [73] was to use a free field realization of both $\mathcal{A}$ and $\mathcal{W}_\infty$ algebras. This is the boundary field theory derivation in the context of the twisted M-theory.

One of the objectives of the section §4.3 is to reproduce the coproducts of the M2-M5 brane system by a perturbative computation in the gravity side of the twisted M-theory. By the gravity side of the twisted M-theory, we mean the 5d topological holomorphic Chern-Simons theory, which is obtained as a result of localization of the Omega deformed twisted

M-theory. The philosophy of our approach is simple to state. By probing the entire theory enriched with defects using the perturbative method[3], we will decode the non-perturbative algebraic structure of the defects.

We interpret the coproduct $\Delta_{\mathcal{A},\mathcal{A}} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ as a fusion of two Wilson lines and the coproduct $\Delta_{\mathcal{W}_\infty,\mathcal{W}_\infty} : \mathcal{W}_\infty \to \mathcal{W}_\infty \otimes \mathcal{W}_\infty$ as a fusion of two surface defects, and compute the OPE of both defects in the 5d Chern-Simons theory background. Importantly, the quantum corrections in the coproduct relations are captured by 1-loop Feynman diagrams in the perturbation theory of the 5d Chern-Simons theory coupled with the defects.

Consistent with the logic under [73], which was used to explain the mixed coproduct $\Delta_{\mathcal{A},\mathcal{W}_\infty} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{W}_\infty$, we will impose gauge invariance of the intersecting M2, M5 brane configuration coupled to the 5d Chern-Simons theory, and reproduce the mixed coproduct. Again, the quantum corrections in the $\mathcal{A} \to \mathcal{A} \otimes \mathcal{W}_\infty$ coproduct are captured by 1-loop Feynman diagrams in the 5d Chern-Simons theory that have vertices on both kinds of defects.

4.1 Twisted holography via Koszul duality

Twisted holography is the duality between the protected sub-sectors of full supersymmetric AdS/CFT [118, 90, 158], obtained by a topological twist and Ω-background both turned on in the field theory side and supergravity side. The most glaring aspect of twisted holography[4] is an correspondence between operator algebra in both sides, which is manifested by a rigorous Koszul duality. Moreover, the information of physical observables such as Witten diagrams in the bulk side that match with correlation functions in the boundary side is fully captured by OPE algebra in the twisted sector [71].

This section is prepared for a quick review of twisted holography for non-experts. The idea was introduced in [35] and studied in various examples [28, 29, 98, 31, 70, 37] with or without Ω-deformation. The reader who is familiar with [28] can skip most of this section, except for §4.1.2, §4.1.3, and §4.1.9, where we set up the necessary conventions for the rest of this paper. These sections can be skipped as well, if the reader is familiar with [70]. Also, see a complementary review of the formalism in the section 2 of [70].

[3] A similar set-up but using a non-perturbative method to find the algebraic data of a coupled system can be found in the bootstrap program for a BCFT, for instance [113].

[4] A similar line of development was made in [15, 122], using twisted $\mathbb{Q}$-cohomology, where $\mathbb{Q}$ is a particular combination of a supercharge Q and a conformal supercharge S [6]. In the sense of [134], $\mathbb{Q}$-cohomology is equivalent to Q_V-cohomology, where Q_V is the modified scalar supercharge in Ω-deformed theories.

After defining the notion of twisted supergravity in §4.1.1, we will focus on a particular (twisted and Ω-deformed) M-theory background on $\mathbb{R}_t \times \mathbb{C}^2_{\mathrm{NC}} \times \mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2} \times \mathbb{C}_{\epsilon_3}$, where NC means non-commutative, and ϵ_i stands for Ω-background related to $U(1)$ isometry with a deformation parameter ϵ_i in §4.1.2. N $M2$ branes extending $\mathbb{R}_t \times \mathbb{C}_{\epsilon_1}$ leads to the field theory side. As we will explain in §4.1.3, a bare operator algebra isomorphism between twisted supergravity and twisted M2-brane worldvolume theory is given by an interaction Lagrangian between two systems. Due to this interaction, a perturbative gauge anomaly appears in various Feynman diagrams, and a careful cancellation of the anomaly will give a consistent quantum mechanical coupling between two systems. Strikingly, the anomaly cancellation condition itself leads to a complete operator algebra isomorphism, by fixing algebra commutators. This will be described in section §4.2. To discuss holography, it is necessary to include the effect of taking a large N limit and the subsequent deformation in the spacetime geometry. We will illustrate the concepts in §4.1.8. In §4.1.9, we will explain how to introduce M5-brane in the system and describe the role of M5-brane in the gravity and field theory side. In short, the degree of freedom on M5-brane will form a module of the operator algebra of M2-brane. Similar to the M2-brane case, the anomaly cancellation condition for M5-brane uniquely fixes the structure of the module.

4.1.1 Twisted supergravity

Before discussing the topological twist of supergravity, it would be instructive to recall the same idea in the context of supersymmetric field theory and make an analog from the field theory example.

Given a supersymmetric field theory, we can make it topological by redefining the generator of the rotation symmetry M using the generator of the R-symmetry R.

$$M' = M + R \tag{4.3}$$

As a part of Lorentz symmetry is redefined, supercharges, which were previously spinor(s), split into a scalar Q, which is nilpotent

$$Q^2 = 0, \tag{4.4}$$

and a 1-form Q_μ. Because of the nilpotency of Q, one can define the notion of Q-cohomology.

Following anti-commutator explains the topological nature of the operators in Q-cohomology: a translation is Q-exact.

$$\{Q, Q_\mu\} = P_\mu \tag{4.5}$$

To go to the particular Q-cohomology, one needs to turn off all the infinitesimal super-translation ϵ_Q except for the one that parametrizes the particular transformation δ_Q generated by Q.

More precisely, if we were to start with a gauge theory, which is quantized with BRST formalism, the physical observables are defined as BRST cohomology, with respect to some Q_{BRST}. The topological twist modifies Q_{BRST}, and the physical observables in the resulting theory are given by Q'_{BRST}-cohomology.

$$Q'_{\text{BRST}} = Q_{\text{BRST}} + Q \tag{4.6}$$

As an example, consider $3d\,\mathcal{N}=4$ supersymmetric field theory. The Lorentz symmetry is $SU(2)$ and R-symmetry is $SU(2)_H \times SU(2)_C$, where H stands for Higgs and C stands for Coulomb. There are two ways to re-define the Lorentz symmetry algebra, and we choose to twist with $SU(2)_C$, as this will be used in the later discussion. In other words, one redefines

$$M' = M + R_C \tag{4.7}$$

The resulting scalar supercharge is obtained by identifying two spinor indices, one of Lorentz symmetry α and one of $SU(2)_C$ R-symmetry a

$$Q^{\alpha}_{a\dot{a}} \mapsto Q^{a}_{a\dot{a}} \tag{4.8}$$

and taking a linear combination.

$$Q = Q^{+}_{1\bar{1}} + Q^{-}_{1\bar{2}} \tag{4.9}$$

This twist is called Rozansky-Witten twist [149] and will be used in twisting our M2-brane theory.

One way to start thinking about the topological twist of supergravity is to consider a brane in the background of the "twisted" supergravity. If one places a brane in a twisted supergravity background, it is natural to guess that the worldvolume theory of the brane should also be topologically twisted coherently with the prescribed twisted supergravity background.

Given the intuition, let us define twisted supergravity, following [35]. In supergravity, the supersymmetry is a local (gauge) symmetry, a fermionic part of super-diffeomorphism. As usual in gauge theories, one needs to take a quotient by the gauge symmetry, and this is done by introducing a ghost field. As supersymmetry is a fermionic symmetry, the corresponding ghost field is a bosonic spinor, q. Twisted supergravity is defined as supergravity in a background where the bosonic ghost q takes a non-zero value.

It is helpful to recall how we twist a field theory to have a better picture for presumably unfamiliar non-zero bosonic ghost. One can think the infinitesimal super-translation parameter ϵ that appears in the global supersymmetry transformation as a rigid limit of the bosonic ghost q. For instance, in 4d $\mathcal{N} = 1$ holomorphically twisted field theory [133, 102, 25, 150], with Q paired with ϵ_+, the supersymmetry transformation of the bottom component ϕ of anti-chiral superfield $\bar{\Psi} = (\bar{\phi}, \bar{\psi}, \bar{F})$ transforms as

$$\delta\phi = \bar{\epsilon}\bar{\psi}, \quad \delta\bar{\psi} = i\epsilon_+\bar{\partial}\bar{\phi} + i\epsilon_-\partial\bar{\phi} + \bar{\epsilon}\bar{F} \tag{4.10}$$

As we focus on Q-cohomology, we set $\epsilon_+ = 1$, $\epsilon_- = \bar{\epsilon} = 0$, then the equations reduce into

$$\delta\bar{\phi} = 0, \quad \delta\bar{\psi} = i\bar{\partial}\bar{\phi} \tag{4.11}$$

In the similar spirit, in the twisted supergravity, we control the twist by giving non-zero VEV to components of the bosonic ghost q.

Indeed, [35] proved that by turning on non-zero bosonic spinor vacuum expectation value $\lambda q \rightarrow\neq 0$ with $q_\alpha \Gamma^{\alpha\beta}_\mu q_\beta = 0$ for a vector gamma matrix, one can obtain the effect of topological twisting. We can now compare with the field theory case above (4.4): $Q^2 = 0$ with $Q \neq 0$. One can think of ϵ_Q as a rigid limit of q.

The operator algebra of twisted type IIB supergravity is isomorphic to that of Kodaira-Spencer theory [11]. The following diagram gives a pictorial definition of the two theories, which turned out to be isomorphic to each other.

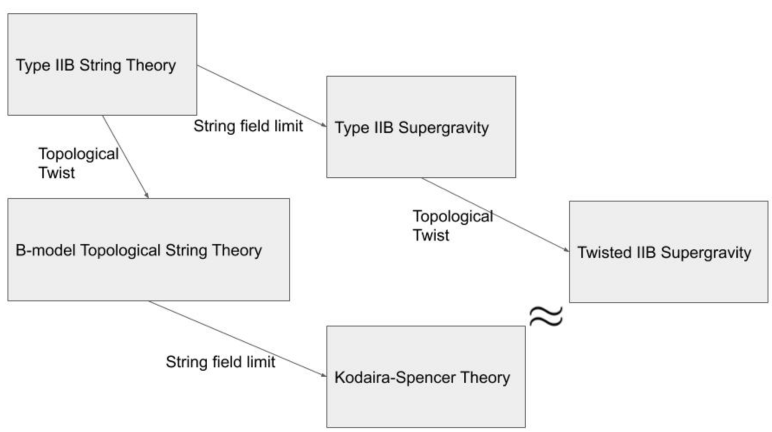


Figure 4.1: Starting from type IIB string theory, one can obtain the same theory by taking two operations 1. String field limit, 2. Topological twist, in any order.

Notice that the topological twist in the first column of the picture is the twist applied on the worldsheet string theory[5], whereas in the second column is the twist on the target space theory.

Lastly, there are two types of twists available: a topological twist and a holomorphic twist, and it is possible to turn on the two different types of twists in the two different directions of the spacetime. The mixed type of twists is called a topological-holomorphic twist, for example, [106]. Different from a topological twist, a holomorphic twist makes only the anti-holomorphic translation to be Q-exact; after the twist we have Q and Q_z such that

$$\{Q, Q_z\} = \partial_{\bar{z}} \tag{4.12}$$

Hence, the holomorphic translation is physical(not Q-exact), and there exist non-trivial dynamics arising from this. [35, 28] showed that it is possible to discuss a holomorphic twist in the supergravity. It is important to have a holomorphic direction to keep the non-trivial dynamics, as we will later see.

4.1.2 Ω-deformed M-theory

Similar to the previous section, we will start reviewing the notion of Ω-deformation of topologically twisted field theory. To define Ω-background, one first needs an isometry, typically $U(1)$, generated by some vector field V on a plane where one wants to turn on the Ω-background. Ω-deformation is a deformation of topologically twisted field theory. Physical observables are in the modified Q_V cohomology, which satisfies

$$Q_V^2 = \mathcal{L}_V, \quad \text{where } Q_V = Q + i_{V^\mu} Q_\mu \tag{4.13}$$

where $\mathcal{L}_V$ is a conserved charge associated with V, and i_{V^μ} is a contraction with the vector field V^μ, reducing the form degree by 1.

As the RHS of (4.13) is non-trivial, Q_V cohomology only consists of operators, which are fixed by the action of L_V such that $L_V\mathcal{O} = 0$. Hence, effectively, the theory is defined in two fewer dimensions, if the isometry group is $U(1)$. More generally, one can turn on Ω-background in the n planes, and the dynamics of the original theory defined on D-dimensions localizes on $D - 2n$ dimensions.

[5]We thank Kevin Costello, who pointed out that the arrow from Type IIB string theory to B-model topological string theory is still mysterious in the following sense. In Ramond-Ramond formalism, as the super-ghost is in the Ramond sector and it is hard to give it a VEV. In the Green-Schwarz picture surely it should work better, but there are still problems there, as the world-sheet is necessarily embedded in space-time whereas in the B model that is not allowed.

[28] proposed a prescription for turning Ω-background in twisted 11d supergravity; we need a 3-form field ϵC, along with $U(1)$ isometry generated by a vector field ϵV, where ϵ is a constant, measuring the deformation. Similar to the field theory description, in this background($\lambda q \to, C \neq 0$), the bosonic ghost q squares into the vector field, ϵV to satisfy the 11d supergravity equation of motion.

$$q^2 = q_\alpha (\Gamma^{\alpha\beta})_\mu q_\beta = \epsilon V_\mu \tag{4.14}$$

The Ω-background localizes the supergravity field configuration into the fixed point of the $U(1)$ isometry. From now on, we will call Ω-background with parametrized by ϵ_i as Ω_{ϵ_i} background. More generally, one can turn on multiple Ω_{ϵ_i}-backgrounds in the separate 2-planes, which we will denote as $\mathbb{C}_{\epsilon_i}$.

The twisted and Ω-deformed 11d background that we will focus in this paper is

$$\text{11d SUGRA: } (\mathbb{R}_t \times \mathbb{C}_{\epsilon_1} \times TN_{K;\epsilon_2,\epsilon_3})_{\text{topological}} \times (\mathbb{C}_z \times \mathbb{C}_w)_{\text{holomorphic}} \tag{4.15}$$

where $TN_{K;\epsilon_2,\epsilon_3}$ is Taub-NUT space with A_K-singularity at origin, which can be thought of as $S^1_{\epsilon_2}$-fibration over $\mathbb{R} \times \mathbb{C}_{\epsilon_3}$. In the background we are interested in, we have, out of the 7 topological directions, 6 directions equipped with an Omega background $\Omega_{\epsilon_1} \times \Omega_{\epsilon_2} \times \Omega_{\epsilon_3}$ with a Calabi-Yau condition $K\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$. The twist is implemented with the bosonic ghost chosen such that holomorphic twist in $\mathbb{C}_z \times \mathbb{C}_w$ directions, and topological twist in $\mathbb{R}_t \times \mathbb{C}_{\epsilon_1} \times TN_{K;\epsilon_2,\epsilon_3}$ directions[6]. The 3-form is

$$C = V^d \wedge d\bar{z} \wedge d\bar{w} \tag{4.16}$$

where V^d is 1-form, which is a Poincare dual of the vector field V on $\mathbb{C}_{\epsilon_2}$ plane, and z, w are holomorphic coordinates on $\mathbb{C}_z \times \mathbb{C}_w$.

The twisted holography is the duality between the protected subsector of M2-brane and the localized supergravity, due to the Ω-background. We first want to introduce $M2$ branes and establish the explicit isomorphism at the level of operator algebras. Place N-stacks of M2-branes on

$$\text{M2-brane: } \mathbb{R}_t \times \{\cdot\} \times \mathbb{C}_{\epsilon_1} \times \{\cdot\} \tag{4.17}$$

For the concrete computation, it is convenient to go to type IIA frame by reducing along an M-theory circle. We pick the M-theory circle as $S^1_{\epsilon_2}$, which is in the direction of the vector field V.[7]

[6] As remarked, if one introduces branes, the worldvolume theory inherits the particular twist that is turned on in the particular direction that the branes extend.

[7] For a different purpose, to make contact with Y-algebra system, type IIB frame is better, but we will not pursue this direction in this thesis.

After reducing on $S^1_{\epsilon_2}$, the Taub-NUT geometry maps into K-stacks of D6-brane and N-stacks of M2-branes map to N-stacks of D2-branes.

$$\begin{aligned} \text{Type IIA SUGRA} &: \mathbb{R}_t \times \mathbb{C}_z \times \mathbb{C}_w \times \mathbb{C}_{\epsilon_1} \times \mathbb{R} \times \mathbb{C}_{\epsilon_3} \\ \text{D6-branes} &: \mathbb{R}_t \times \mathbb{C}_z \times \mathbb{C}_w \times \mathbb{C}_{\epsilon_1} \\ \text{D2-branes} &: \mathbb{R}_t \times \qquad\quad \times \mathbb{C}_{\epsilon_1} \end{aligned} \tag{4.18}$$

and 3-form C-field reduces into a B-field, which induces a non-commutativity $[z, w] = \epsilon_2$ on $\mathbb{C}_z \times \mathbb{C}_w$.

$$B = \epsilon_2 d\bar{z} \wedge d\bar{w} \tag{4.19}$$

There are two types of contributions to gravity side: (1) closed strings in type IIA string theory, (2) open strings on the D6-brane. It was shown in [28] that we can completely forget about the closed strings. The reason is in the presence of the non-commutativity, the holomorphically twisted supergravity background (B-model) is the same as the topologically twisted background (A-model) equipped with a B-field. As we are working in the supergravity limit, where there is no instanton effect, we can also ignore the effect from a B-field. Hence, for closed string, the background becomes topological A-model without instanton effect, which is trivial. Therefore, the open strings from the D6-brane entirely capture gravity side.

D6-brane worldvolume theory is 7d SYM, and it localizes on 5d non-commutative GL_K Chern-Simons on $\mathbb{R}_t \times \mathbb{C}_z \times \mathbb{C}_w$ due to Ω_{ϵ_1}-background on $\mathbb{C}_{\epsilon_1}$ [43]. The 5d Chern-Simons theory is not the typical Chern-Simons theory, as it inherits a topological twist in $\mathbb{R}_t$ direction and a holomorphic twist in $\mathbb{C}_z \times \mathbb{C}_w$ direction, in addition to the non-commutativity. As a result, a gauge field only has 3 components

$$A = A_t dt + A_{\bar{z}} d\bar{z} + A_{\bar{w}} d\bar{w} \tag{4.20}$$

and the action takes the following form.

$$S_{5d \text{ CS}} = \frac{1}{\epsilon_1} \int_{\mathbb{R}_t \times \mathbb{C}_z \times \mathbb{C}_w} dz dw \left(A \star_{\epsilon_2} dA + \frac{2}{3} A \star_{\epsilon_2} A \star_{\epsilon_2} A \right) \tag{4.21}$$

The star product $\star_{\epsilon_2}$ is the standard Moyal product induced from the non-commutativity of $\mathbb{C}_z \times \mathbb{C}_w$: $[z, w] = \epsilon_2$. The Moyal product between two holomorphic functions[8] f and g is defined as

$$f \star_\theta g = m \circ e^{\frac{\theta}{2} \partial_z \wedge \partial_w} (f \otimes g) = fg + \frac{\theta}{2} \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial w} - \frac{\partial g}{\partial z} \frac{\partial f}{\partial w} \right) + \mathcal{O}(\theta^2) \tag{4.22}$$

[8]The Moyal product is extended to a product on the Dolbeault complex $\Omega^{0,*}(\mathbb{C}_z \times \mathbb{C}_w)$ by the same formula, except that the product between two functions becomes a wedge product between two forms.

The gauge transformation $\Lambda \in \Omega^0(\mathbb{R} \times \mathbb{C}_z \times \mathbb{C}_w) \otimes \mathfrak{gl}_K$ acting on the gauge field A is

$$A \mapsto A + d\Lambda + [\Lambda, A], \text{ where } [\Lambda, A] = \Lambda \star_{\epsilon_2} A - A \star_{\epsilon_2} \Lambda \tag{4.23}$$

The field theory side is defined on N D2-branes, which extend on $\mathbb{R}_t \times \mathbb{C}_{\epsilon_1}$. This is the 3d $\mathcal{N} = 4$ gauge theory with 1 fundamental hypermultiplet and 1 adjoint hypermultiplet. Since the D2-branes are placed on a topologically twisted supergravity background, the theory inherits the topological twist, which is the Rozansky-Witten twist. We will work on $\mathcal{N} = 2$ notation, then each of $\mathcal{N} = 4$ hypermultiplet splits into a chiral and an anti-chiral $\mathcal{N} = 2$ multiplet. We denote the scalar bottom component of the fundamental chiral and anti-chiral multiplet as I_i^a and J_a^i, and that of adjoint multiplets as X_j^i and Y_j^i, where a, b are GL_K flavour indices and i, j are GL_N gauge indices. Those scalars parametrize the hyper-Kahler target manifold $\mathcal{M}$, which has a non-degenerate holomorphic symplectic structure. This structure turns the ring of holomorphic functions on $\mathcal{M}$ into a Poisson algebra with the following basic Poisson brackets:

$$\{J_a^i, I_j^b\} = \delta_a^b \delta_j^i, \quad \{X_j^i, Y_l^k\} = \delta_l^i \delta_j^k. \tag{4.24}$$

It is known that the gauge-invariant combinations of Q-cohomology of Rozansky-Witten twisted $\mathcal{N} = 4$ theory is equivalent to the Higgs branch chiral ring. The elements of Higgs branch chiral ring are gauge invariant polynomials of I, J, X, and Y:

$$I^a \mathrm{Sym}(X^m Y^n) J_b, \quad \mathrm{Tr}\mathrm{Sym}(X^m Y^n) \tag{4.25}$$

where $\mathrm{Sym}(\bullet)$ means fully symmetrized polynomial of the monomial $\bullet$.

Upon imposing the F-term relation[9]

$$[X, Y]_j^i - I_j^a J_a^i + \epsilon_2 \delta_j^i = 0, \tag{4.26}$$

one can show two words in (4.25) are related by following relation:

$$I^a \mathrm{Sym}(X^m Y^n) J_a = \epsilon_2 \mathrm{Tr}\, \mathrm{Sym}(X^m Y^n). \tag{4.27}$$

Note that the ϵ_2 factor, which was previously introduced as a measure for the non-commutativity in the 5d CS theory, acts as an FI parameter in the 3d $\mathcal{N} = 4$ gauge theory. In the Ω_{ϵ_1}-background, the Higgs branch chiral ring is quantized to an algebra and

[9] Physically, one needs to impose the F-term relation, as it is a part of defining condition for the supersymmetric vacua, as a critical locus of our specific 3d $\mathcal{N} = 4$ superpotential. Algebraically, the F-term relation forms an ideal of the ring of holomorphic functions on $\mathcal{M}$.

the support of the operator algebra in 3d $\mathcal{N} = 4$ theory also localizes to the fixed point of the Ω_{ϵ_1}-background. Therefore, the theory effectively becomes 1d TQM (Topological Quantum Mechanics) [53, 7, 23]

$$S_{1d\ \mathrm{TQM}} = \frac{1}{\epsilon_1}\int_{\mathbb{R}_t} \mathrm{Tr}(\epsilon_2 A_t + XD_tY + ID_tJ)dt. \tag{4.28}$$

In summary, two sides of twisted holography are 5d non-commutative Chern-Simons theory and 1d TQM. Until now, we have not quite taken a large N limit and resulting back-reaction that will deform the geometry. The large N limit will be crucial for the operator algebra isomorphism to work and we will illustrate this point in section §4.1.8.

4.1.3 Comparing elements of operator algebra

As 5d Chern-Simons theory has a trivial equation of motion: the curvature $F(A) = 0$, all the operators have positive ghost numbers. Also, since $\mathbb{R}_t$ direction is topological, the fields do not depend on t. As a result, operator algebra consist of ghosts $c(z, w)$ with holomorphic dependence on coordinates of $\mathbb{C}_z \times \mathbb{C}_w$. The elements are then Fourier modes of the ghosts:

$$c^a_b[m, n] = \partial^m_z \partial^n_w c^a_b(0, 0), \tag{4.29}$$

where a, b are $\mathfrak{gl}_K$ indices. Note that ghost fields $c^a_b[m, n]$ are understood as the linear dual to the elements in the Lie algebra of gauge transformations $\mathbb{C}[z, w]_{\epsilon_2} \otimes \mathfrak{gl}_K$, where $\mathbb{C}[z, w]_{\epsilon_2}$ is the space of holomorphic functions on $\mathbb{C}^2$ with commutators

$$[f, g] = f \star_{\epsilon_2} g - g \star_{\epsilon_2} f. \tag{4.30}$$

An equivalent way to write $\mathbb{C}[z, w]_{\epsilon_2}$ is the ring $\mathrm{Diff}_{\epsilon_2}(\mathbb{C})$ of ϵ_2-differential operators on $\mathbb{C}$, where w is interpreted as the coordinate and z is the differential operator $\epsilon_2\partial_w$. The algebra of classical observables $\mathrm{Obs}^{\mathrm{cl}}_{\epsilon_2}$ of 5d CS theory is generated by ghost fields $c^a_b[m, n]$ with anti-commutativity relations, i.e. $\wedge^*((\mathrm{Diff}_{\epsilon_2}\mathbb{C} \otimes \mathfrak{gl}_K)^\vee)$, and the BRST differential is the dual of the LIe bracket, which is the Chevalley-Eilenberg differential, thus $\mathrm{Obs}^{\mathrm{cl}}_{\epsilon_2}$ is the Chevalley-Eilenberg algebra of cochains on the Lie algebra of gauge transformations $\mathrm{Diff}_{\epsilon_2}\mathbb{C} \otimes \mathfrak{gl}_K$, denote by $C^*(\mathrm{Diff}_{\epsilon_2}\mathbb{C} \otimes \mathfrak{gl}_K)$. At the quantum level, the operator algebra $\mathrm{Obs}^{\mathrm{cl}}_{\epsilon_2}$ receives deformations, and we denoted it by $\mathrm{Obs}_{\epsilon_1,\epsilon_2}$.

For the 1d topological quantum mechanics, the defining commutation relations come from the quantization of the Poisson brackets deformed by Ω_{ϵ_1}-background:

$$[J^i_a, I^b_j] = \epsilon_1 \delta^b_a \delta^i_j, \quad [X^i_j, Y^k_l] = \epsilon_1 \delta^i_l \delta^k_j \tag{4.31}$$

We will write the F-term relation with explicit gauge indices as follows.

$$X^i_k Y^k_j - X^k_j Y^i_k - I^a_j J^i_a + \epsilon_2 \delta^i_j = 0 \tag{4.32}$$

Let us define operators as

$$e^a_b[m,n] = \frac{1}{\epsilon_1} I^a \mathrm{Sym}(X^m Y^n) J_b, \quad t[m,n] = \frac{1}{\epsilon_1} \mathrm{TrSym}(X^m Y^n), \tag{4.33}$$

and they are related by $e^a_a[m,n] = \epsilon_2 t[m,n]$. As we will see later in the chapter §5, the commutation relations among those generators $e^a_b[m,n], t[m,n]$ are independent of N, which allows us to *define* the large N limit algebra to be generated by $e^a_b[m,n], t[m,n]$ with corresponding relations (see Lemma 5.1.3 and Proposition 5.1.4 for detail). We call such algebra the ADHM algebra, denote by $\mathcal{A}^{(K)}_{\epsilon_1,\epsilon_2}$.

Assumption 4.1.4. For the rest of this chapter, we assume that $\epsilon_2 \neq 0$, so $\mathcal{A}^{(K)}_{\epsilon_1,\epsilon_2}$ is generated by $e^a_b[m,n]$.

Just from counting of degrees of freedom, there is a one-to-one correspondence between $c^a_b[m,n]$ and $e^a_b[m,n]$. More precisely, they are dual generators in the sense of Koszul duality. The main result of [29] is that

Theorem 4.1.5 ([29, 16.0.1])**.** *There is an isomorphism*

$$\mathrm{Obs}^!_{\epsilon_1,\epsilon_2} \cong \mathcal{A}^{(K)}_{\epsilon_1,\epsilon_2} \tag{4.34}$$

between the Koszul dual of algebra of local observables $\mathrm{Obs}_{\epsilon_1,\epsilon_2}$ *in 5d* GL_K *Chern-Simons theory, and the ADHM algebra* $\mathcal{A}^{(K)}_{\epsilon_1,\epsilon_2}$*, when* $K > 1$.

Here $A^!$ is the Koszul dual of an algebra A. For example, it is known that the Koszul dual for Chevalley-Eilenberg cochain complex $C^*(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$, is the universal enveloping algebra $U(\mathfrak{g})$ [26], in particular

$$\mathrm{Obs}^!_{\epsilon_1=0,\epsilon_2} \cong U(\mathrm{Diff}_{\epsilon_2}(\mathbb{C}) \otimes \mathfrak{gl}_K). \tag{4.35}$$

The algebra $U(\mathrm{Diff}_{\epsilon_2}(\mathbb{C}) \otimes \mathfrak{gl}_K)$ is called the double current algebra in the literature [59]. The proof to the Theorem 4.1.5 consists of three parts. First, one shows that $\mathrm{Obs}^!_{\epsilon_1=0,\epsilon_2}$ is isomorphic to $\mathcal{A}^{(K)}_{\epsilon_1=0,\epsilon_2}$, i.e. the latter is isomorphic to the double current algebra. Next, one checks two algebras' commutation relations match in the $\mathcal{O}(\epsilon_1)$ order, where one-loop Feynman diagram is used to compute the first order correction to $\mathrm{Obs}^!_{\epsilon_1=0,\epsilon_2}$. Finally, one proves the uniqueness of the deformation of $U(\mathrm{Diff}_{\epsilon_2}(\mathbb{C})\otimes\mathfrak{gl}_K)$ that ensures all order matching.

One of our goals is to extend the $\mathcal{O}(\epsilon_1)$ order matching to $K = 1$. It may seem trivial compared to higher K, but it turns out that the perturbation computation is more complicated. We will give the detail for the computation of Feynman diagrams in section §4.2.

Notation: In the later discussions, we write $\mathcal{A}_{\epsilon_1,\epsilon_2}$ for $\mathcal{A}^{(1)}_{\epsilon_1,\epsilon_2}$.

4.1.6 ADHM algebra for $K = 1$

In chapter §5, we will derive a set of commutation relations between generators $t[m,n] \in \mathcal{A}_{\epsilon_1,\epsilon_2}$ that determine all other relations (see §5.3.6 for details), of which the simplest ones are:

$$\begin{aligned}
[t[3,0],t[0,3]] &= 9t[2,2] + \frac{3}{2}\big(\sigma_2 t[0,0] - \sigma_3 t[0,0]t[0,0]\big)\\
[t[2,1],t[1,2]] &= 3t[2,2] - \frac{1}{2}\big(\sigma_2 t[0,0] - \sigma_3 t[0,0]t[0,0]\big)
\end{aligned} \tag{4.36}$$

where

$$\sigma_2 = \epsilon_1^2+\epsilon_2^2+\epsilon_1\epsilon_2, \quad \sigma_3 = -\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2). \tag{4.37}$$

For the convenience of later discussions, we also introduce the notation:

$$T[m,n] = \frac{\epsilon_2}{\epsilon_1}\mathrm{Tr}\ \mathrm{Sym}(X^mY^n) = \frac{1}{\epsilon_1} I\mathrm{Sym}(X^mY^n)J \in \mathcal{A}_{\epsilon_1,\epsilon_2} \tag{4.38}$$

Our final goal is to reproduce the $\mathcal{A}_{\epsilon_1,\epsilon_2}$ algebra from the anomaly cancellation of 1-loop Feynman diagrams in 5d Chern-Simons theory. So, it is important to have commutation relations that yield $\mathcal{O}(\epsilon_1)$ term in the right hand side, where ϵ_1 is a loop counting parameter in 5d CS theory.

To compare the commutation relation to that from 5d Chern-Simons calculation, we need to make sure if the parameters of ADHM algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ are the same as those in 5d CS theory. From [29], the correct parameter dictionary[10] is

$$(\epsilon_1)_{\mathrm{ADHM}} = (\epsilon_1)_{\mathrm{CS}}, \quad \left(\epsilon_2 + \frac{1}{2}\epsilon_1\right)_{\mathrm{ADHM}} = (\epsilon_2)_{\mathrm{CS}}. \tag{4.39}$$

[10]We thank Davide Gaiotto, who pointed out this subtlety.

Hence, the commutation relation that we are supposed to match from the 5d computation is

$$[t[2,1],t[1,2]] = 3t[2,2] - \frac{1}{2}\Big(\big(\epsilon_2^2 + \frac{3}{4}\epsilon_1^2\big)t[0,0] + \big(\epsilon_1\epsilon_2^2 - \frac{\epsilon_1^3}{4}\big)t[0,0]t[0,0] \Big) \tag{4.40}$$

There is one term in the RHS of (4.40) that is in $\mathcal{O}(\epsilon_1)$ order:

$$[t[2,1],t[1,2]] = \mathcal{O}(\epsilon_1^0) - \frac{1}{2}\epsilon_1\epsilon_2^2 t[0,0]t[0,0] + \mathcal{O}(\epsilon_1^2) \tag{4.41}$$

We will try to recover the $\mathcal{O}(\epsilon_1)$ term from 5d Feynman diagram calculation[11] in section §4.2.

4.1.7 Koszul duality

Let us explain why in the first place we can expect the Koszul duality between operator algebras of 5d Chern-Simons and the large N limit of 1d topological quantum mechanics. Further details on Koszul duality can be found in [116, 69, 68, 70, 37].

The 5d theory is defined on $\mathbb{R}_t \times \mathbb{C}^2_{\text{NC}}$, where $\mathbb{R}_t$ is topological and $\mathbb{C}^2_{\text{NC}}$, and 1d TQM couples to the 5d theory along $\mathbb{R}_t$. As explained in (4.5), there is a scalar supercharge Q and 1-form supercharge δ that anti-commute to give a translation operator P_t. We can build a topological line defect action using topological descent

$$\text{Pexp} \int_{-\infty}^{\infty} [\delta, x(t)]\,, \text{ where } x(t) = \sum_{m,n} c^a_b[m,n]e^b_a[m,n]. \tag{4.42}$$

The BRST variation of (4.42) vanishes if $x(t)$ satisfies the Maurer-Cartan equation:

$$Qx + \frac{1}{2}\{x,x\} = 0. \tag{4.43}$$

Now recall that one of the equivalent characterization of Koszul dual algebra of a differential-graded algebra A[12] is the following universal property: for any other differential-graded algebra B, there is an isomorphism

$$\text{Hom}_{\text{dg-alg}}(A^!, B) \cong \text{MC}(B \otimes A), \tag{4.44}$$

[11] The basis used in the Feynman diagram computation is $T[m,n]$, not $t[m,n]$. However, the change of basis does not affect any computation because the $\mathcal{O}(\epsilon_1)$ term in (4.41) is quadratic in t.

[12] One additional technical assumption is that A has an augmentation $\rho : A \to \mathbb{C}$.

which is functorial in B, where the left-hand-side is the set of differential-graded algebra morphisms, and the right-hand-side is the set of Maurer-Cartan elements in $B \otimes A$. In other word $A^!$ is the universal differential-graded algebra which solves the Maurer-Cartan equation in the tensor product with A. In the physics language, if A is the algebra of observables of some topological-holomorphic field theory, then $A^!$ is the algebra of observables of the universal 1d topological defect that can be coupled to the bulk field theory.

Back to the 5d Chern-Simons situation, the equation (4.42) is the universal way to couple a line defect to the 5d Chern-Simons theory, hence it is natural to expect the Koszul duality between $\mathrm{Obs}_{\epsilon_1,\epsilon_2}$ and $\mathcal{A}^{(K)}_{\epsilon_1,\epsilon_2}$. So the coupling between the 5d ghosts and gauge invariant polynomials of 1d TQM is given by

$$S_{\text{int}} = \int_{\mathbb{R}_t} \mathrm{Tr}(e[m,n]\partial_z^m \partial_w^n A)dt. \tag{4.45}$$

Now that we have three types of Lagrangians:

$$S_{5d\ \mathrm{CS}} + S_{1d\ \mathrm{TQM}} + S_{\text{int}} \tag{4.46}$$

Quantum mechanically, for the 5d Chern-Simons theory to be compatible with the M2 brane line defect, all correlation functions or Feynman diagrams that involve vertices on both the defect and the bulk should be invariant under the BRST transformation

$$Q_{\mathrm{BRST}}A = \mathrm{d}c + [A,c], \quad Q_{\mathrm{BRST}}c = -\frac{1}{2}[c,c] \tag{4.47}$$

where c is a scalar ghost. The bracket does not vanish in general even that we are considering GL_1 gauge theory, since $\mathbb{C}_z \times \mathbb{C}_w$ is non-commutative.

4.1.8 Large-N-limit and a back-reaction of N M2-branes

Although we have not discussed explicitly about taking large N limit, but it was being used implicitly in the construction of the algebra $\mathcal{A}^{(K)}_{\epsilon_1,\epsilon_2}$ which makes it a crucial step towards the holography. However, it is important to notice that large N is not necessary for Koszul duality, but it is important for holography.[13]

The general philosophy of AdS/CFT [118] teaches us that the back-reaction of N M2-branes will deform the spacetime geometry. In our case, since the closed strings completely decouple from the analysis, the back-reaction is only encoded in the interaction related

[13] We thank an anonymous referee who made this point.

to the open strings. More precisely, the back-reaction is already encoded in the 5d-1d interaction Lagrangian (4.45), a part of which we reproduce below.

$$S_{\text{back}} = \frac{1}{K}\int_{\mathbb{R}_t} e^a_a[0,0]c^b_b[0,0]dt. \tag{4.48}$$

Here, we can explicitly see N in $t[0,0] = \frac{1}{\epsilon_2}e^a_a[0,0]$, as

$$t[0,0] = \frac{1}{\epsilon_1}\text{Tr}(1) = \frac{N}{\epsilon_1}. \tag{4.49}$$

After taking large N limit, N becomes an element of the algebra $\mathcal{A}^{(K)}_{\epsilon_1,\epsilon_2}$, which is coupled to the trace of the zeroth Fourier mode of the 5d ghost, $c^a_a[0,0]$.

4.1.9 M5-brane in Ω-deformed M-theory

In the Ω-deformed M-theory background, in addition to N-stacks of $M2$ branes, we may introduce N'-stacks of M5 branes. M2 and M5 branes extend in 1 and 2 real directions, respectively in the 5d Chern-Simons theory, and can be considered as line and surface defects with their degrees of freedom interacting with the 5d Chern-Simons theory. For simplicity we only discuss the case $K = 1$.

	0	1	2	3	4	5	6	7	8	9	10
Geometry	$\mathbb{R}_t$	$\mathbb{C}_{\epsilon_1}$		$\mathbb{C}_z$		$\mathbb{C}_w$		$\mathbb{C}_{\epsilon_2}$		$\mathbb{C}_{\epsilon_3}$	
$M2$	×	×	×								
$M5$						×	×	×	×	×	×
5d CS	×			×	×	×	×				

Table 4.1: M2, M5-brane and 5d Chern-Simons theory. In general, $M2$ branes may extend over $\mathbb{R}_t \times \mathbb{C}_{\epsilon_i}$ and $M5$ branes may extend over $\mathbb{C}_{\text{z or w}} \times \mathbb{C}_{\epsilon_i} \times \mathbb{C}_{\epsilon_j}$, where $i, j \in \{1, 2, 3\}$.

Let us fix the orientation of the N' M5 branes so that they extend over $\mathbb{C}_w \times \mathbb{C}_{\epsilon_2} \times \mathbb{C}_{\epsilon_3}$. We are interested in the M5 brane theory on $\mathbb{C}_w$, as the M5 branes intersect with the 5d Chern-Simons theory along $\mathbb{C}_w$. For this, it is rather convenient to go to the IIA frame(by compactifying the M-theory circle $S^1 \in \mathbb{C}_{\epsilon_2}$). In the type IIA frame, the theory on $\mathbb{C}_w$ consists of D4-D6 strings, with 8 ND directions; this gives rise to a pair of chiral fermions ψ, ψ' with a Lagrangian

$$\int_{\mathbb{C}_w} dz\text{Tr}\psi(\bar{\partial} + A)\psi' \tag{4.50}$$

Also, the resulting algebra consists of modes of various currents labeled by its conformal dimension n: $W^{(n)} = \psi\partial_w^{n-1}\psi'$, where n runs from 1 to N'. [28] proposed a mathematically rigorous way to take the large N' limit and showed that the M5 brane algebra is $\mathcal{W}_\infty$.

Note that another intuitive way to understand the M5 brane algebra is via AGT set-up [2]. N' M5 brane worldvolume theory is the 6d $(2,0)$ theory of $A_{N'-1}$ type on 1 holomorphic direction $\mathbb{C}_w$ and 4 topological directions $\mathbb{C}_{\epsilon_2} \times \mathbb{C}_{\epsilon_3}$ with an Omega background turned on both of topological planes [159]. Localizing on the locus of the Omega background, we get a $\mathcal{W}_\infty$ algebra on the holomorphic plane [160, 159, 8, 14].

The coupling between the currents in the theory of the M5 branes and the gauge field of the 5d Chern-Simons theory is given by [14]

$$\int_{\mathbb{C}_w} dw W^{(m)} \partial_w^{m-1} A. \tag{4.51}$$

To see an explicit coupling between the m'th mode of $W_m^{(n)}$ and 5d gauge field, let us expand $W^{(n)}$ in w:

$$\sum_{m\in\mathbb{Z}} W_n^{(m)} \int_{\mathbb{C}} w^{-m-n} \partial_w^{m-1} A dw. \tag{4.52}$$

Therefore, the n'th mode of $W^{(m)}$ current $W_n^{(m)}$ couples to $w^{-m-n}(\partial_w^{m-1}A)dw$.

Quantum mechanically, for the 5d Chern-Simons theory to be compatible with the surface defect from the M5 branes, all correlation functions or Feynman diagrams that involve vertices both to the defect and the bulk should be invariant under the BRST transformation $A \mapsto dc + [A, c]$, where c is a scalar ghost.

4.1.10 Coproducts of M2, M5 brane algebra

Recently, [73] proposed a recipe to fuse $\mathcal{A}$ and $\mathcal{W}_\infty$. There are two types of fusion, which we will respectively call homogeneous fusion and heterotic fusion.

The homogeneous fusion

The homogeneous fusion is between the same type of defects. As there are two types of defects, we have two homogeneous fusions: a fusion of line defects with each other and

[14] A similar example of the surface operator was discussed in [44] in the context of 4d Chern-Simons theory.

a fusion of surface defects with each other. The operation of the two homegenous fusions is given as follows

- Place two M2 branes at separate points in one of the holomorphic directions $\mathbb{C}_w$ and bring them together.
- Place two M5 branes at separate points in the topological direction $\mathbb{R}_t$ and bring them together.

We may consider this operation as an OPE of two defects $\mathcal{D}_1$, $\mathcal{D}_2$ that leads to a single defect $\mathcal{D}$. Therefore, we may ponder about the relation among the operator algebras $\mathcal{A}(\mathcal{D}_1)$, $\mathcal{A}(\mathcal{D}_2)$, $\mathcal{A}(\mathcal{D})$, associated to $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}$. The fusion process is a 2-to-1 operation from the bulk algebra point of view, and Koszul-dually [15] it induces an 1-to-2 operation, which will be called coproducts $\Delta_{\mathcal{A},\mathcal{A}}$, $\Delta_{\mathcal{W}_\infty,\mathcal{W}_\infty}$ on each $\mathcal{A}$ and $\mathcal{W}_\infty$.

$$\begin{aligned} \Delta_{\mathcal{A},\mathcal{A}} &: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \\ \Delta_{\mathcal{W}_\infty,\mathcal{W}_\infty} &: \mathcal{W}_\infty \to \mathcal{W}_{\infty,1} \otimes \mathcal{W}_{\infty,2}. \end{aligned} \tag{4.53}$$

Physically, we may see the existence of the coproducts in the bulk side through Feynman diagrams with a bulk 3-point vertex, which has two internal legs connecting to 2 defects participating in the fusion and 1 external leg.

We visualized the process so far in Figure 4.2.

[15] The Koszul dual algebra $A^!$ of an algebra A has the functorial property that $\mathrm{Hom}_{\mathrm{algebra}}(A^!, B) \cong$ Maurer-Cartan$(B \otimes A)$, where the Maurer-Cartan elements in $B \otimes A$ is interpreted as the coupling between two systems with algebra of local observables B and A. Fusion of two line operators with operator algebra $A^!$ gives rise to a Maurer-Cartan element in $A^! \otimes A^! \otimes A$ and this induces a map $A^! \to A^! \otimes A^!$.

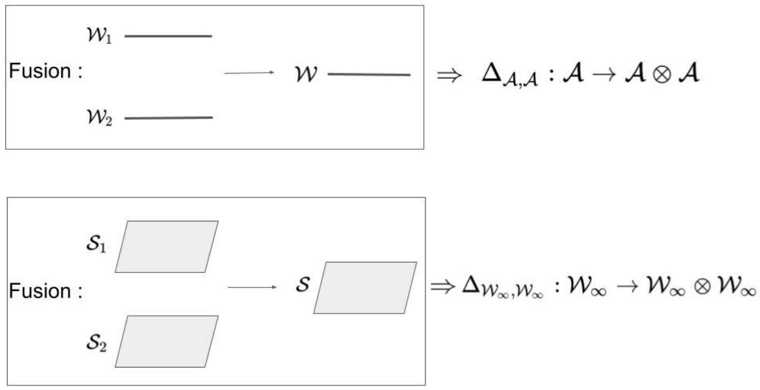


Figure 4.2: The top figure schematically describes that the Wilson line fusion induces the coproduct in $\mathcal{A}$. The bottom figure shows the surface operator fusion induces the coproduct in $\mathcal{W}_\infty$.

Now, let us write down the representative example of the coproduct $\Delta_{\mathcal{A},\mathcal{A}}$ [73] that we will try to reproduce in the next section:

$$t_{2,0} \to t'_{2,0} + \tilde{t}_{2,0} + 2\sigma_3 \sum_{m,n\geq 0} \mathrm{d}_{m,n} t'_{0,m} \tilde{t}_{0,n} \tilde{w}^{-m-n-2}. \tag{4.54}$$

$t_{2,0}$, $t'_{2,0}$, $\tilde{t}_{2,0}$ are elements of $\mathcal{A}$, $\mathcal{A}_1$, $\mathcal{A}_2$. $d_{m,n}$ is a combinatorial factor that depends on m and n. $\tilde{w}$ is a separation of two line defects in the $\mathbb{C}_w$-plane.

The coproduct $\Delta_{\mathcal{A},\mathcal{A}}$ comes from the fusion of two Wilson lines. If we bring three Wilson lines together in the $\mathbb{C}_w$ plane, then they fuse without ambiguity, which means that the fusion is associative. Koszul-dually, this means that the coproduct $\Delta_{\mathcal{A},\mathcal{A}}$ should satisfy coassociativity in some sense, and this is mathematically captured by the notion of vertex coalgebra [96]. In the next chapter §5, we make our observation rigorous by proving that $\mathcal{A}$ equipped with $\Delta_{\mathcal{A},\mathcal{A}}$ satisfies the axioms of the vertex coalgebra, see Proposition 5.7.2 for detail.

The heterotic fusion

The heterotic fusion is between different types of defects: a line and a surface, or M2 branes intersecting with M5 branes. Different from the case of homogeneous fusions, which have a simple interpretation as an OPE of defects, the heterotic fusion is subtle. The coproduct for the heterotic fusion is induced by imposing a gauge-invariant condition on the M2-M5 junction configuration.

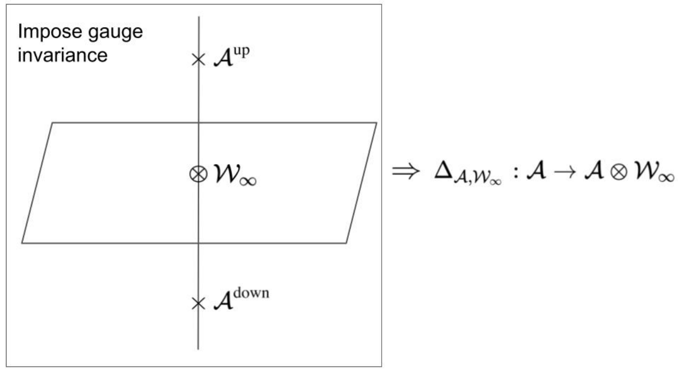


Figure 4.3: Imposing the gauge-invariance of the coupled system of the line defect and the surface defect induces the coproduct $\Delta_{\mathcal{A},\mathcal{W}_\infty}$.

Imposing gauge-invariance of the entire coupled system leads to the following schematic relation between various operators in the system:

$$t^{\text{up}}_{n,m} \cdot O + W^{(n+1)}_{m-n} \cdot O + (\dots) \cdot O - O \cdot t^{\text{down}}_{n,m} = 0. \tag{4.55}$$

where $\mathcal{O}$ represents the junction between the line and the surface, and $(\dots)$ is a sum of polynomials of elements of $\mathcal{A}$ and $\mathcal{W}_\infty$ that can be seen as quantum corrections. By arranging the terms in (4.55) as

$$O \cdot (t^{\text{down}}_{n,m}) = (t^{\text{up}}_{n,m} + W^{(n+1)}_{m-n} + (\dots)) \cdot O, \tag{4.56}$$

and comparing the LHS and the RHS, we can notice that the gauge invariance induces a map between $\mathcal{A}$ and $\mathcal{A} \otimes \mathcal{W}_\infty$:

$$\Delta_{\mathcal{A},\mathcal{W}_\infty} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{W}_\infty. \tag{4.57}$$

The representative example [73] of the coproduct $\Delta_{\mathcal{A},\mathcal{W}_\infty}$ is

$$t_{2,0} \to t_{2,0} + V_{-2} + \sigma_3 \sum_{n=1}^{\infty} n W^{(1)}_{-n-1} W^{(1)}_{n-1} + \sigma_3 \sum_{n=1}^{\infty} n W^{(1)}_{-n-1} t_{0,n-1}. \tag{4.58}$$

In the RHS, $t_{2,0}$ and V_{-2} are implicitly $t_{2,0} \otimes 1$ and $1 \otimes V_{-2}$, so both are elements of $\mathcal{A} \otimes \mathcal{W}_\infty$.

4.2 Perturbative calculations in 5d GL_1 CS theory coupled to 1d ADHM quantum mechanics

In this section, we will provide a derivation of the $G = \mathrm{GL}_N$, $\hat{G} = \mathrm{GL}_1$ ADHM algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ using the perturbative calculation in 5d GL_1 CS. We will see the result from the perturbative calculation matches with the expectation (4.41). The strategy, which we will spell out in this section, is to compute the $\mathcal{O}({\epsilon_1}^1)$ order gauge anomaly of various Feynman diagrams in the presence of the line defect from $M2$ brane($\mathbb{R}^1 \times \{0\} \subset \mathbb{R}^1 \times \mathbb{C}^2_{\mathrm{NC}}$). Imposing a cancellation of the anomaly for the 5d CS theory uniquely fixes the algebra commutation relations.

Purely working in the weakly coupled 5d CS theory, we will derive the representative commutation relations of the ADHM algebra (4.41):

- Algebra commutation relation

$$[t[2,1], t[1,2]] = \ldots + \epsilon_1 \epsilon_2^2 t[0,0]t[0,0] + \ldots \tag{4.59}$$

  where $t[n,m]$ is a basis element of $\mathcal{A}_{\epsilon_1,\epsilon_2}$.

As we commented in §4.1.6, the algebra basis used in the Feynman diagram computation is $T[m,n]$, which is related to $t[m,n]$ by rescaling with ϵ_2. The effect of the change of basis is trivial in (4.59), so we will interchangeably use $t[m,n]$ and $T[m,n]$ without loss of generality.

4.2.1 Ingredients of Feynman diagrams

To set-up the Feynman diagram computations, we recall the 5d GL_1 Chern-Simons theory action on $\mathbb{R}_t \times \mathbb{C}^2_{\mathrm{NC}}$.

$$S = \frac{1}{\epsilon_1} \int_{\mathbb{R}_t \times \mathbb{C}^2_{\mathrm{NC}}} dz dw \left(A \star_{\epsilon_2} dA + \frac{2}{3} A \star_{\epsilon_2} A \star_{\epsilon_2} A \right) \tag{4.60}$$

with $|\epsilon_1| \ll |\epsilon_2| \ll 1$. In components, the 5d gauge field A can be written as

$$A = A_t dt + A_{\bar{z}} d\bar{z} + A_{\bar{w}} d\bar{w} \tag{4.61}$$

with all the components are smooth holomorphic functions on $\mathbb{R}^1 \times \mathbb{C}^2_{\mathrm{NC}}$.

Now, we want to collect all the ingredients of the Feynman diagram computation. It is convenient to rewrite (4.60) as

$$S = \frac{1}{\epsilon_1}\int_{\mathbb{R}^1\times\mathbb{C}^2_{\mathrm{NC}}} dzdw\left(AdA + \frac{2}{3}A(A\star_{\epsilon_2}A)\right) \tag{4.62}$$

(4.62) is equivalent to (4.60) up to a total derivative. From the kinetic term of the Lagrangian, we can read off the following information:

- 5d gauge field propagator P is a solution of

$$dz\wedge dw\wedge dP = \delta_{t=z=w=0}. \tag{4.63}$$

  That is,

$$P(v_1,v_2) = \langle A(v_1)A(v_2)\rangle = \frac{\bar{z}_{12}d\bar{w}_{12}dt_{12} - \bar{w}_{12}d\bar{z}_{12}dt_{12} + t_{12}d\bar{z}_{12}d\bar{w}_{12}}{d_{12}^5} \tag{4.64}$$

  where

$$v_i = (t_i,z_i,w_i),\quad d_{ij} = \sqrt{t_{ij}^2+|z_{ij}|^2+|w_{ij}|^2},\quad t_{ij}=t_i-t_j \tag{4.65}$$

From the three-point coupling in the Lagrangian, we can extract 3-point vertex. This is not immediate, as the theory is defined on non-commutative background. Different from GL_N CS, where the leading contribution of the 3-point vertex was AAA, the leading contribution of the 3-point coupling of the GL_1 gauge bosons starts from $\mathcal{O}(\epsilon_2)A\partial_z A\partial_w A$. The reason is following:

$$\begin{aligned}
&\int dz\wedge dw\wedge A\wedge(A\star_{\epsilon_2}A)\\
&=\int A\wedge\left((A_t dt + A_{\bar{z}}d\bar{z} + A_{\bar{w}}d\bar{w})\star(A_t dt + A_{\bar{z}}d\bar{z} + A_{\bar{w}}d\bar{w})\right)\\
&=\int dz\wedge dw\wedge A\wedge\left[dt\wedge d\bar{z}\,(A_t\star A_{\bar{z}} - A_{\bar{z}}\star A_t) + \ldots\right]\\
&=\int dz\wedge dw\wedge A\wedge\left[dt\wedge d\bar{z}\,(0 + 2\epsilon_2\,(\partial_z A_t\partial_w A_{\bar{z}} - \partial_w A_t\partial_z A_{\bar{z}})) + \ldots\right]\\
&=2\epsilon_2\int dz\wedge dw\wedge A\wedge\left[dt\wedge d\bar{z}(\partial_z A_t\partial_w A_{\bar{z}} - \partial_w A_t\partial_z A_{\bar{z}})\right] + \mathcal{O}(\epsilon_2^2)
\end{aligned} \tag{4.66}$$

Note that for GL_N case, SL_N Lie algebra factors attached to each A prevents the $\mathcal{O}(\epsilon_2^0)$ term to vanish. Still, $\mathrm{GL}_1\subset\mathrm{GL}_N$ part of A contributes as $\mathcal{O}(\epsilon_2)$, but it can be ignored, since we take $\epsilon_2\ll 1$.

Hence, in GL_1 CS, the 3-point $A\partial_z A\partial_{z_2}A$ coupling contributes as

- Three-point vertex $\mathcal{I}_{3pt}$:

$$\mathcal{I}_{3pt} = \epsilon_2 dz \wedge dw \tag{4.67}$$

Now, we are ready to introduce the line defect into the theory and study how it couples to 5d gauge fields. Classically, $t[n_1, n_2]$ couples to the mode of 5d gauge field by

$$\int_{\mathbb{R}} t[n_1, n_2] \partial_z^{n_1} \partial_w^{n_2} A dt \tag{4.68}$$

The last ingredient of the bulk Feynman diagram computation comes from the interaction (4.68).

- One-point vertex $\mathcal{I}_{1pt}^A$:

$$\mathcal{I}_{1pt}^A = \begin{cases} t[n_1, n_2]\delta_{t,z,w} & \text{if } \partial_z^{n_1}\partial_w^{n_2} A \text{ is a part of an internal propagator} \\ t[n_1, n_2]\partial_z^{n_1}\partial_w^{n_2} A & \text{if } \partial_z^{n_1}\partial_w^{n_2} A \text{ is an external leg} \end{cases} \tag{4.69}$$

Lastly, the loop counting parameter is ϵ_1. Each of the propagator is proportional to ϵ_1 and the internal vertex is proportional to ϵ_1^{-1}. Hence, 0-loop order($\mathcal{O}({\epsilon_1}^0)$) Feynman diagrams may contain the same number of internal propagators and internal vertices and 1-loop order($\mathcal{O}(\epsilon_1)$) diagrams may contain one more internal propagators than internal vertices.

Until now, we have collected all the components of the 5d perturbative computation (4.64), (4.67), (4.68), and (4.69). With these, let us see what Feynman diagrams have non-zero BRST variations and how the cancelation of BRST variations of different diagrams leads to the ADHM algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$.

4.2.2 Feynman diagram

The goal of this section is derive the $\mathcal{O}(\epsilon_1)$-term of $[t[2,1], t[1,2]]$ by Feynman diagrams. We interpret the commutator $[t[2,1], t[1,2]]$ as the following difference between two tree level diagrams

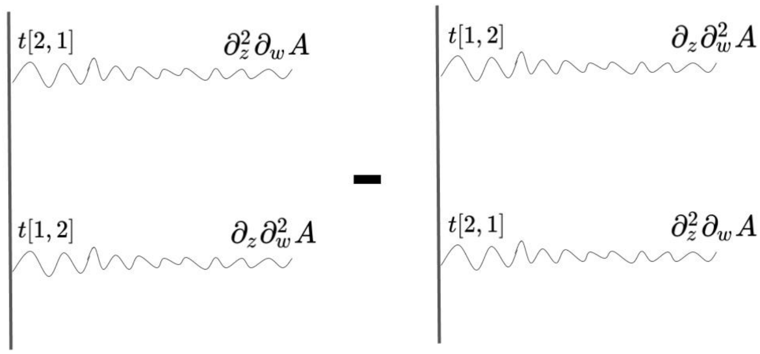


Figure 4.4: There is no internal propagators, but just external ghosts for 5d gauge fields, which directly interact with 1d QM. The minus sign in the middle literally means that we take a difference between two amplitudes. In the left diagram $t[1,2]$ vertex is located at $t=0$ and $t[2,1]$ is at $t=\epsilon$. In the right diagram, $t[1,2]$ is at $t=-\epsilon$ and $t[2,1]$ at $t=0$.

The amplitude of the diagram is

$$[t[2,1],t[1,2]]\,\partial_z^2\partial_w A_1\partial_z\partial_w^2 A_2 \tag{4.70}$$

so the BRST variation of the amplitude is proportional to

$$[t[2,1],t[1,2]]\,\partial_z^2\partial_w A_1\partial_z\partial_w^2 c_2+[t[2,1],t[1,2]]\,\partial_z^2\partial_w c_1\partial_z\partial_w^2 A_2 \tag{4.71}$$

Note that the BRST variation on A fields is $Q_{\mathrm{BRST}}A=\partial c$. At $\mathcal{O}(\epsilon_1)$ level, this diagram will cancel all anomalies coming from one-loop diagrams with two external legs coupled to $\partial_z^2\partial_w A$ and $\partial_z\partial_w^2 A$ respectively. Let's enumerate those diagrams, there are two types of diagrams:

(1) See figure 4.5.

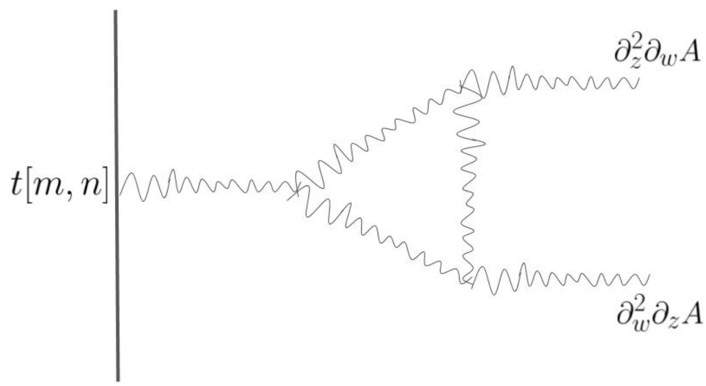


Figure 4.5: A diagram, which has a vanishing amplitude.

(2) See figure 4.6.

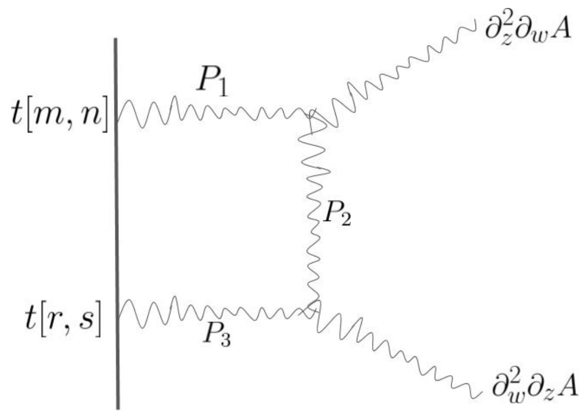


Figure 4.6: The vertical solid line represents the time axis, where 1d topological defect is supported. Internal wiggly lines stand for 5d gauge field propagators P_i, and the external wiggly lines stand for 5d gauge field A.

For the first diagram, we claim that the amplitude is always zero. This can be seen as follows. Let $\mathbb{C}^\times$ act on z and w by rotation with weight 1, then propagators has weight -2. For the interaction vertex, it contains the integration measure $dz \wedge dw$ together with ∂_z

and ∂_w in the interaction term, so the total weight of the interaction vertex is zero. Each external leg is of weight 3. Hence, the total weight of the amplitude is $-2-m-n<0$, i.e. it's not invariant under the $\mathbb{C}^\times$-rotation symmetry, so the amplitude must be zero.

For the second diagram, we will follow the approach shown in [41] and show that the diagram has a nonvanishing amplitude if and only if $m=n=r=s=0$. And in the case that it is nonzero, it has a nonvanishing gauge anomaly consequently, under the BRST variation $Q_{\mathrm{BRST}}A = \partial c$.

Let's do the same analysis on the second diagram as the first one, i.e., let $\mathbb{C}^\times$ act on z and w by rotation with weight 1, then the total weight of the amplitude is $-n-m-r-s$. Hence, the diagram is nonzero only if $m=n=r=s=0$. In the following discussion, we will focus on he case $m=n=r=s=0$.

We first integrate over the first vertex $(P_1\ \partial_z^2\partial_w A\ P_2)$ and then integrate over the second vertex$(P_2\ \partial_z\partial_w^2 A\ P_3)$.

First vertex$(P_1\ \partial_z^2\partial_w A\ P_2)$

First, we focus on computing the integral over the first vertex:

$$\epsilon_1\epsilon_2^2\int_{v_1} dw_1\wedge dz_1\wedge\partial_{z_1}P_1(v_0,v_1)\wedge\partial_{z_2}\partial_{w_1}P_2(v_1,v_2)(z_1^2w_1\partial_{z_1}^2\partial_{w_1}A) \tag{4.72}$$

Note that ∂_{z_1} and ∂_{w_1} comes from the three-point coupling at v_1:

$$\epsilon_2 A\wedge\partial_{z_1}A\wedge\partial_{w_1}A \tag{4.73}$$

And ∂_{z_2} comes from the 3-pt coupling at v_2:

$$\epsilon_2 A\wedge\partial_{z_2}A\wedge\partial_{w_2}A \tag{4.74}$$

We will consider ∂_{w_2} later when we treat the second vertex.

The factor $z_1^2w_1\partial_{z_1}^2\partial_{w_1}A$ is for the external leg attached to v_1, which is $c[2,1]$. In short, this is an ansatz, and we can start without fixing m,n in $c[m,n]$. However, we will see that the integral converges to a finite value only with this particular choice of (m,n). For a simple presentation, we will drop $\partial_{z_1}^2\partial_{w_1}A$, and recover it later.

After some manipulation, which we refer to **Lemma 1** in Appendix D, (4.72) becomes

$$-\int_{v_1} dt_1dz_1d\bar{z}_1dw_1d\bar{w}_1\frac{|z_1|^2|w_1|^2\bar{z}_2(\bar{w}_{12}dt_2-t_{12}d\bar{w}_2)}{d_{01}^5d_{12}^9} \tag{4.75}$$

The integral 4.75 can be further simplified by using the typical Feynman integral technique, which can be found in **Lemma 2** in Appendix D. We are left with

$$\bar{z}_2(\bar{w}_2 dt_2 - t_2 d\bar{w}_2)\left(\frac{c_1}{d_{02}^5} + \frac{c_2 w_2^2}{d_{02}^7} + \frac{c_3 z_2^2}{d_{02}^7} + \frac{c_4 z_2^2 w_2^2}{d_{02}^9}\right) \tag{4.76}$$

with c_i being a constant. Note that all terms in the parenthesis have a same order of divergence. Therefore, it suffices to focus on the first term to check the convergence of the full integral(we still need to do v_2 integral.)

We will explicitly show the calculation for the first term, and just present the result for the second, third, and fourth term in (D.9). They are all non-zero and finite. We will denote the first term as $\mathcal{P}$, which is 1-form.

Second vertex($\mathcal{P}\ \partial_z \partial_w^2 A\ P_3$)

Now, let us do the integral over the second vertex(v_2). The remaining things are organized into

$$\int_{v_2} \mathcal{P} \wedge \partial_{w_2} P_3(v_2, v_3) \wedge dz_2 \wedge dw_2 (z_2 w_2^2 \partial_{z_2} \partial_{w_2}^2 A) \tag{4.77}$$

where we dropped forms related to v_3, as we do not integrate over it. ∂_{w_2} comes from the 3-pt coupling at v_2:

$$\epsilon_2 A \wedge \partial_{z_2} A \wedge \partial_{w_2} A \tag{4.78}$$

The factor $z_2 w_2^2 \partial_{z_2} \partial_{w_2}^2 A$ is for the external leg attached to v_2, which corresponds to $c[1,2]$. Again, this is an ansatz. We will see that only this integral converges and does not vanish. We will drop $\partial_{z_2} \partial_{w_2}^2 A$ and recover it later.

The integral (4.77) is simplified to

$$\int_{v_2} -\frac{|z_2|^2 |w_2|^4}{d_{02}^5 d_{23}^7} dt_2 d\bar{z}_2 d\bar{w}_2 dw_2 dz_2 \tag{4.79}$$

The intermediate steps can be found in **Lemma 3** in Appendix D.

Now, it remains to evaluate the delta function at the third vertex and use Feynman technique to evaluate the integral. By **Lemma 4** in Appendix D, we are left with

$$(\text{const})\epsilon_1 \epsilon_2^2 t[0,0] t[0,0] \partial_{z_1}^2 \partial_{z_2} A_1 \partial_{z_1}^1 \partial_{z_2}^2 A_2 \tag{4.80}$$

The BRST variation of the amplitude is

$$(\text{const})\epsilon_1 \epsilon_2^2 t[0,0] t[0,0] \partial_{z_1}^2 \partial_{z_2} A_1 \partial_{z_1}^1 \partial_{z_2}^2 c_2 \tag{4.81}$$

This indicates that the theory is quantum mechanically inconsistent, as it has a Feynman diagram that has nonzero BRST variation. However, as long as there is another diagram whose BRST variation is proportional to the same factors We can cancel the anomaly.

Hence, imposing BRST invariance of the sum of Feynman diagrams, we bootstrap the possible 1d TQM that can couple to 5d GL_1 CS.

An obvious choice is the tree-level diagram where $(\partial_{z_1}A)(\partial_{z_2}A)$ appears explicitly: By equating (4.81) and (4.71), we get

$$[t[2,1], t[1,2]] = \epsilon_1\epsilon_2^2 t[0,0]t[0,0] + \dots \tag{4.82}$$

Therefore, we have reproduced the $\mathcal{O}(\epsilon_1)$ part of the ADHM algebra $\mathcal{A}_{\epsilon_1,\epsilon_2}$ commutation relation from the Feynman diagram computation:

$$[t[2,1], t[1,2]]_{\epsilon_1} = \epsilon_1\epsilon_2^2 t[0,0]t[0,0] \tag{4.83}$$

where $[-,-]_{\epsilon_1}$ is the $\mathcal{O}(\epsilon_1)$-part of the commutator.

4.3 Perturbative calculations of the defect fusions

In this section, we will give a twisted holographic derivation of the various coproducts, which we reviewed in the previous section.

The original derivation [73] of the coproducts $\Delta_{\mathcal{A},\mathcal{A}} : \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ and $\Delta_{\mathcal{W}_\infty,\mathcal{W}_\infty} : \mathcal{W}_\infty \to \mathcal{W}_\infty \otimes \mathcal{W}_\infty$, which are induced by the homogeneous fusion, was purely algebraic, appealing to the free field realization of $\mathcal{A}$ and $\mathcal{W}_\infty$ [146, 145]. We will explain how to take an OPE of two identical type defects and produce a single defect by computing 1-loop Feynman diagrams. The RHS of the coproducts $\Delta_{\mathcal{A},\mathcal{A}}$, $\Delta_{\mathcal{W}_\infty,\mathcal{W}_\infty}$ naturally emerges as a fusion coefficient of the resulting single defect. We will first state the result in §4.3.1 with a diagrammatic explanation. Using the various ingredients of the Feynman diagram collected in §4.3.3, we give an explicit Feynman diagram computation in §4.3.4, §4.3.6.

The philosophy of the argument that leads to the coproducts $\mathcal{A} \to \mathcal{A}\otimes\mathcal{W}_\infty$ was to impose the gauge-invariance of the intersecting M2-M5 configuration. [73] derived the coproduct by utilizing purely algebraic properties of $\mathcal{A}$ and $\mathcal{W}_\infty$. As the system couples to the bulk 5d Chern-Simons theory, imposing the gauge invariance implicitly assumes the gauge-invariance of the entire system. We will explain how to compute the possible gauge anomaly of a collection of Feynman diagrams, where defects interact with the bulk. By imposing the vanishing anomaly condition, we reproduce the coproduct $\mathcal{A} \to \mathcal{A}\otimes\mathcal{W}_\infty$. We will first state the result in §4.3.2 with a diagrammatic explanation and give an explicit Feynman diagram computation in §4.3.5.

In §4.3.7, we propose a conjecture about the fusion between two transverse surface defects. Different from the fusion between two parallel surface defects, we will see a line operator as one of the byproducts.

Note that the coproducts that we are dealing with are all truncated in the first order of σ_3. We prove the dual statement in the 5d Chern-Simons side in §4.3.8.

Our calculation is based on the integral technique developed in [41] in the context of 4d Chern-Simons theory. The authors discussed an OPE between two Wilson lines and show that it gives a composite Wilson line. We will sometimes rely on our previous paper [137], as well.

4.3.1 Holographic interpretation of the homogeneous fusion

Given two parallel Wilson lines, placed on the $\mathbb{C}_w$ plane at $w = 0$, $w = \tilde{w}$, when they approach each other, $\tilde{w} \to 0$, we obtain a single Wilson line. We will directly compute the OPE of two Wilson lines in the 5d Chern-Simons background using Feynman diagrams.

At the tree level, the OPE of two Wilson lines associated with $t'_{2,0}$, $\tilde{t}_{2,0}$[16] is trivial and the OPE is simply given by a single Wilson line associated with $t'_{2,0} \otimes 1 + 1 \otimes \tilde{t}_{2,0}$. Hence, the tree level OPE gives

$$(t'_{2,0} \otimes 1 + 1 \otimes \tilde{t}_{2,0}) \int \partial_z^2 A. \tag{4.84}$$

On the other hand, the OPE becomes nontrivial at the 1-loop level, as there is an obvious correction coming from the 3-point vertex of the 5d Chern-Simons theory that couples two Wilson lines, as shown in the figure below.

[16] We distinguish two algebra elements in different Wilson lines by prime and tilde.

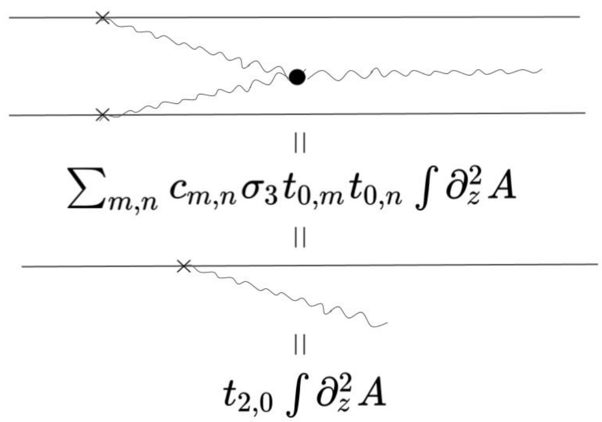


Figure 4.7: The top figure shows the quantum correction on the Wilson line OPEs from the interaction with the 5d Chern-Simons theory. The formula($\sim \sigma_3 t_{0,m} t_{0,n}$) for the fused Wilson line can be obtained by computing the Feynman diagram. As the representation associated with $\partial_z^2 A$ is $t_{2,0}$, the OPE directly gives the coproduct formula $\Delta_{\mathcal{A},\mathcal{A}} : t_{2,0} \to \ldots \sigma_3 t_{0,m} t_{0,n}$.

Combining the tree level and the 1-loop level computation, we obtain a single fused Wilson line

$$\left(t'_{2,0} + \tilde{t}_{2,0} + 2\sigma_3 \sum_{m,n \geq 0} \mathrm{d}_{m,n} t'_{0,m} \tilde{t}_{0,n} \tilde{w}^{-m-n-2} \right) \int \partial^2 z A. \tag{4.85}$$

Since $\int \partial_z^2 A$ couples to $t_{2,0}$ according to (4.45), the fusion induces an embedding map

$$\Delta_{\mathcal{A},\mathcal{A}} : t_{2,0} \to t'_{2,0} + \tilde{t}_{2,0} + 2\sigma_3 \sum_{m,n \geq 0} \mathrm{d}_{m,n} t'_{0,m} \tilde{t}_{0,n} \tilde{w}^{-m-n-2}. \tag{4.86}$$

This is exactly (4.54). As the tree level is trivial, we will only give an explicit derivation of the 1-loop term in §4.3.4.

We can similarly analyze the surface defect fusion. Given two parallel surface defects, placed on $\mathbb{R}_t$ direction at $t = 0$, $t = -\epsilon$, when we approach them together by taking $\epsilon \to 0$, we obtain a single surface defect. We will directly compute the OPE of two surface defects in the 5d Chern-Simons background using Feynman diagrams.

We will present the nontrivial part of the OPE, which is at 1-loop order, as shown in the figure below.

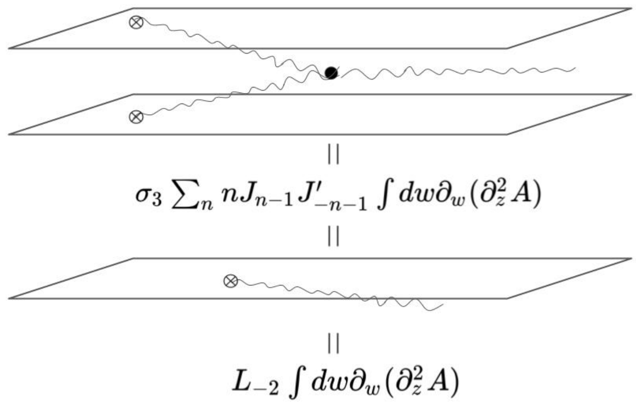


Figure 4.8: The top figure shows the quantum correction on the surface defect OPEs from the interaction of the two surface defects with the 5d Chern-Simons theory. The formula($\sim \sigma_3 J_{n-1}J'_{-n-1}$) for the fused surface defect can be obtained by computing the Feynman diagram. As the representation associated with $\partial_w A$ is L_{-2}, the OPE directly gives the coproduct formula $\Delta_{\mathcal{W}_\infty,\mathcal{W}_\infty} : L_{-2} \to \ldots \sigma_3 J_{n-1}J'_{-n-1}$.

From the 1-loop computation, we obtain a single fused surface defect

$$\ldots + \sigma_3 \sum_{n=-\infty}^{\infty} nJ_{n-1}J'_{-n-1} \int dw\partial_w(\partial_z^2 A). \tag{4.87}$$

Since $\int \partial_w A$ couples to L_{-2} according to (4.52), the fusion induces an embedding map $\Delta_{\mathcal{W}_\infty,\mathcal{W}_\infty}$.

$$L_{-2} \to \ldots + \sigma_3 \sum_{n=-\infty}^{\infty} nJ_{n-1}J'_{-n-1}. \tag{4.88}$$

The basic coproduct $\Delta_{\mathcal{W}_\infty,\mathcal{W}_\infty}$ was not explicitly presented in [73], but it was hiding in a composed coproduct $\mathcal{A} \to \mathcal{A} \otimes \mathcal{W}_\infty \otimes \mathcal{W}_\infty$. On the other hand, from [72] we expect the basic coproduct $T \to J \otimes J$, where T is spin-2 current and J is a spin-1 current. (4.88) is essentially the relevant $\mathcal{O}(\sigma_3)$ order term hiding in the RHS of (2.41) of [73]. We will give a check in §4.3.6.

4.3.2 Holographic interpretation of the heterotic fusion

We will derive the coproduct $\Delta_{\mathcal{A},\mathcal{W}_\infty}$, based on the gauge invariance of the M2-M5 brane junction configuration. One way to discuss the gauge-invariance of the coupled system is by computing the amplitude of a collection of Feynman diagrams that involve vertices on the defects.

To figure out the collection of Feynman diagrams, one needs to consider the LHS(an element of $\mathcal{A}$) of the boundary coproduct relation (4.58) and determine the 5d gauge mode that would couple to it. The next step is to write down all Feynman diagrams whose amplitude is proportional to the 5d gauge mode.

The LHS of the second line of (4.58) is $t_{2,0}$ and it couples to $\partial_z^2 A$. The following diagram represents the one associated with the LHS.

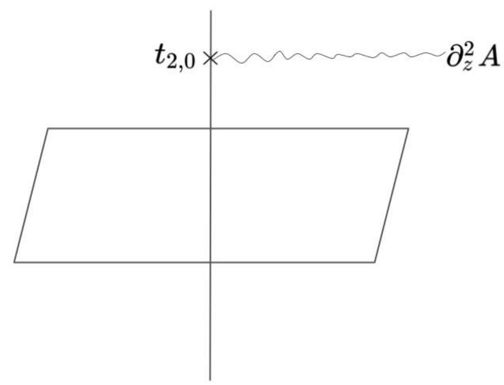


Figure 4.9: The Feynman diagram associated with the LHS of (4.58): $t_{2,0}$.

The amplitude of the Feynman diagram is trivially

$$t_{2,0}\partial_z^2 A. \tag{4.89}$$

Its BRST variation (4.47) is

$$t_{2,0}\partial_z^2(Q_{\text{BRST}}A). \tag{4.90}$$

On the other hand, up to $\mathcal{O}(\sigma_3)$ order, there are three more diagrams, whose amplitudes are proportional to $\partial_z^2 A$. They are

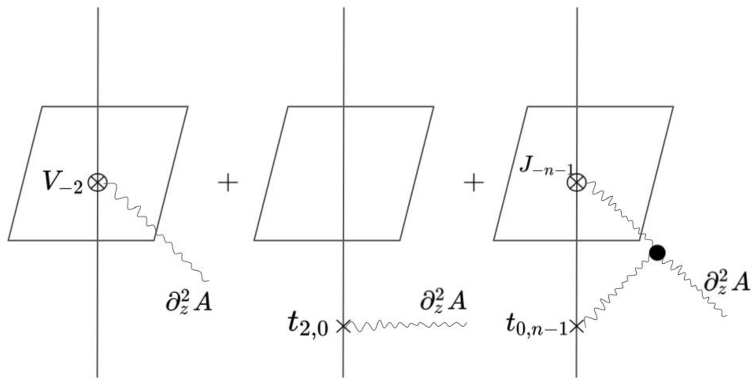


Figure 4.10: The Feynman diagram associated with the RHS of (4.58): $t_{2,0} + V_{-2} + \sum_n nJ_{n-1}t_{0,n-1}$.

The sum of the amplitudes of the Feynman diagram is

$$\left(t_{2,0} + V_{-2} + \sum_{n=1}^{\infty} nJ_{-n-1}t_{0,n-1}\right)\partial_z^2 A. \tag{4.91}$$

Its BRST variation (4.47) is

$$\left(t_{2,0} + V_{-2} + \sum_{n=1}^{\infty} nJ_{-n-1}t_{0,n-1}\right)\partial_z^2 (Q_{\text{BRST}}A). \tag{4.92}$$

For the defect-enriched 5d Chern-Simons theory to be anomaly free, there must be a cancellation between (4.90) and (4.92), which leads to the coproduct relation that we have already seen in the second line of (4.58)[17]:

$$t_{2,0} \to t_{2,0} + V_{-2} + \sum_{n=1}^{\infty} nJ_{-n-1}t_{0,n-1} \tag{4.93}$$

As the tree level $\mathcal{O}(\sigma_3)$ computation is trivial, we will only provide an explicit 1-loop $\mathcal{O}(\sigma_3)$ computation in §4.3.5.

[17]We thank Miroslav Rapčák, who pointed out the previous typos in the following formula.

4.3.3 Ingredients of Feynman diagrams

To prepare for the computation of the Feynman diagrams shown in the previous subsections, we recall the5d $U(1)$ Chern-Simons theory [28, 43](also, see the nice description of the related 4d Chern-Simons theory [26] in [29]) with a leading order action given as [18]

$$\frac{1}{\sigma_3}\int_{\mathbb{R}_t\times\mathbb{C}_z\times\mathbb{C}_w}(AdA+A\{A,A\})dzdw, \tag{4.95}$$

where $\sigma_3^{-1}=(\epsilon_1\epsilon_2\epsilon_3)^{-1}$ is the equivariant volume of $\mathbb{C}_{\epsilon_1}\times\mathbb{C}_{\epsilon_2}\times\mathbb{C}_{\epsilon_3}$ and

$$A=A_t dt+A_{\bar{z}}d\bar{z}+A_{\bar{w}}d\bar{w}, \tag{4.96}$$

and $\{A,A\}$ is the holomorphic Poisson bracket defined as

$$\{A,A\}=\frac{\partial A}{\partial z}\frac{\partial A}{\partial w}-\frac{\partial A}{\partial w}\frac{\partial A}{\partial z}. \tag{4.97}$$

This is nonzero, since A is a 1-form, not a function.

From the Lagrangian, we write down the ingredients of the Feynman diagrams that involve the 5d Chern-Simons theory and two types of defects.

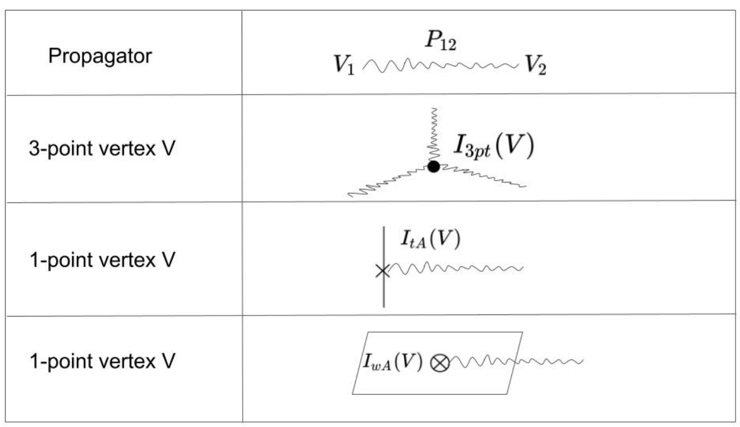


Figure 4.11: A table of ingredients of the Feynman diagrams in the 5d Chern-Simons theory coupled with the line and the surface defects.

[18] In the original paper of Costello [28], the action took a different form as

$$\frac{1}{\epsilon_1}\int_{\mathbb{R}_t\times\mathbb{C}_z\times\mathbb{C}_w}(A*_{\epsilon_2}dA+A*_{\epsilon_2}A*_{\epsilon_2}A)dzdw, \tag{4.94}$$

where $*$ is a Moyal product combined with the wedge product. The equivalent action (4.95), which makes the triality among ϵ_i's manifest, was suggested in [70], and it will be more convenient in our computation.

From the kinetic term of the 5d Chern-Simons Lagrangian

$$\frac{1}{\sigma_3} dz \wedge dw \wedge A \wedge dA, \tag{4.98}$$

we can read off the gauge field propagator:

- 5d gauge field propagator P is a solution of

  $$dz \wedge dw \wedge dP = \delta_{t=z=w=0}. \tag{4.99}$$

  That is,

  $$P_{12} = P(v_1, v_2) = \langle A(v_1) A(v_2) \rangle = \frac{\bar{z}_{12} d\bar{w}_{12} dt_{12} - \bar{w}_{12} d\bar{z}_{12} dt_{12} + t_{12} d\bar{z}_{12} d\bar{w}_{12}}{d_{12}^5} \tag{4.100}$$

  where

  $$\begin{aligned} v_i = (t_i, z_i, w_i), \quad d_{ij} = \sqrt{t_{ij}^2 + |z_{ij}|^2 + |w_{ij}|^2}, \\ t_{ij} = t_i - t_j, \quad z_{ij} = z_i - z_j, \quad w_{ij} = w_i - w_j. \end{aligned} \tag{4.101}$$

From the 3-point coupling

$$\frac{1}{\sigma_3} dz \wedge dw \wedge A \wedge \left(\frac{\partial A}{\partial z} \wedge \frac{\partial A}{\partial w} - \frac{\partial A}{\partial w} \wedge \frac{\partial A}{\partial z} \right), \tag{4.102}$$

we read off

- Three-point vertex $\mathcal{I}_{3pt}$:

  $$\mathcal{I}_{3pt} = \frac{1}{\sigma_3} dz \wedge dw (\partial_z \partial_w). \tag{4.103}$$

Each of the partial derivatives acts on one of three legs that attaches to the vertex.

From (4.98), (4.102), we can see that the loop counting parameter is σ_3: each of the propagator is proportional to σ_3 and the internal vertex is proportional to σ_3^{-1}. Therefore, a given Feynman diagram with v 3-point vertices and e internal propagators is proportional to σ_3^{e-v}.

Next, consider the line defect coupled to the 5d Chern-Simons theory. Classically, $t_{m,n}$ couples to the mode of the 5d gauge field by

$$\int_{\mathbb{R}} t_{m,n} \partial_z^m \partial_w^n A \tag{4.104}$$

From (4.104), we read off

- One-point vertex $\mathcal{I}_{tA}$:

$$\mathcal{I}_{tA} = \delta^{(5)}_{t,z,w} t_{m,n} \partial_z^m \partial_w^n A \tag{4.105}$$

Lastly, consider the surface defect coupled to the 5d Chern-Simons theory. Classically, $W_n^{(m)}$ couples to the mode of the 5d gauge field by

$$\int_{\mathbb{C}_w} W_n^{(m)} \cdot w^{-m-n} \partial_w^{m-1} A dw \tag{4.106}$$

From (4.106), we read off

- One-point vertex $\mathcal{I}_{wA}$:

$$\mathcal{I}_{wA} = \delta^{(3)}_{t,z} W_n^{(m)} w^{-m-n} \partial_w^{m-1} A dw \tag{4.107}$$

As usual in the Feynman diagram computation, we will use (4.108) in the following sub-sections, when we evaluate the final integrals.

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{(xA+(1-x)B)^{\alpha+\beta}}. \tag{4.108}$$

Along with it, we used *Mathematica* to compute various integrals; we submitted an ancillary notebook that collects the integral computations.

4.3.4 $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ coproduct

We will derive the meromorphic coproducts of the M2 brane algebra using the perturbative Feynman diagram computation in 5d Chern-Simons theory. The target relation that we want to derive from the 5d Chern-Simons side is

$$t_{2,0} \to \ldots + \sigma_3 \sum_{m,n\geq 0} (\text{const})_{m,n} t_{0,m} t_{0,n} \tilde{z}^{-m-n-2}. \tag{4.109}$$

We will use the technique developed in [41], where the authors computed the OPE of two Wilson lines using the relevant Feynman diagram in the 4d Chern-Simons theory[19].

Using the ingredients given in §4.3.3, we can decorate the 1-loop Feynman diagram shown in §4.3.1 as follows.

[19]The Yangian coproduct was more explicitly discussed in [42] in the context of the 4d Chern-Simons theory.

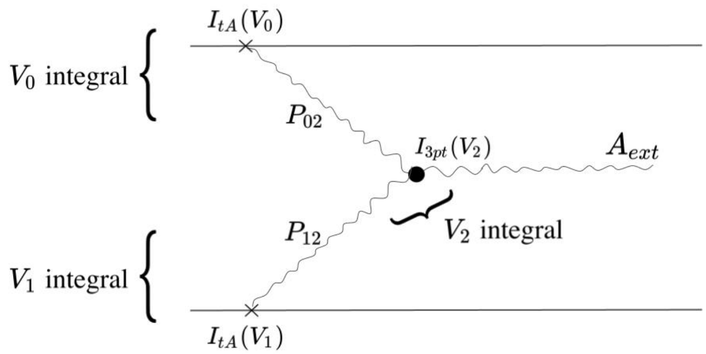


Figure 4.12: The 1-loop Feynman diagram associated with the $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ coproduct. All the ingredients are explicitly displayed.

The amplitude is

$$\begin{aligned}\sigma_3 t_{0,m} t_{0,n} \int_{V_2} dz_2 dw_2 A_{\text{ext}} \int_{V_0} \delta^{(2)}(z_0)\delta^{(2)}(w_0)\partial_{w_0}^m \partial_{z_2} P_{02} \\ \times \int_{V_1} \delta^{(2)}(z_1)\delta^{(2)}(w_1 - \tilde{w})\partial_{w_1}^n \partial_{w_2} P_{12}.\end{aligned} \tag{4.110}$$

where we used (4.103), (4.105) for $I_{3pt}(V_2)$, $I_{tA}(V_0)$ and $I_{tA}(V_1)$, respectively.

There are three floating vertices, so there are three integrals to do. Let us first do V_0, V_1 integrals and use them in the final V_2 integral.

$$\int_{V_0} \delta^{(2)}(z_0)\delta^{(2)}(w_0)\partial_{w_0}^m \partial_{z_2} P_{02}. \tag{4.111}$$

Since $\delta^{(2)}(z_0)\delta^{(2)}(w_0) \sim dz_0 d\bar{z}_0 dw_0 d\bar{w}_0$, we can project most of the terms in P_{02}, and get

$$(-1)^m \frac{7}{2} \cdot \frac{9}{2} \cdots \frac{7+2m-2}{2} \int_{V_0} \delta^{(2)}(z_0)\delta^{(2)}(w_0) \bar{w}_2^m \bar{z}_2 \frac{(\bar{z}_2 d\bar{w}_2 - \bar{w}_2 d\bar{z}_2) dt_0}{\sqrt{t_{02}^2 + |w_{02}|^2 + |z_{02}|^2}^{7+2m}} \tag{4.112}$$

After shifting $t_0 \to t_0 + t_2$, and evaluating two delta functions, we do the t_0 integral. The result is

$$(-1)^m \frac{8\Gamma(3+m)}{15} \frac{\bar{w}_2^m \bar{z}_2 (\bar{z}_2 d\bar{w}_2 - \bar{w}_2 d\bar{z}_2)}{(|w_2|^2 + |z_2|^2)^{m+3}}. \tag{4.113}$$

Now, let us do V_1 integral.

$$\int_{V_1} \delta^{(2)}(z_1)\delta^{(2)}(w_1 - \tilde{w})\partial_{w_1}^n \partial_{w_2} P_{12}, \tag{4.114}$$

Taking into account of the z_1, w_1 delta function, we simplify it into

$$(-1)^n \frac{7}{2} \cdot \frac{9}{2} \cdots \frac{7+2n-2}{2} \int_{-\infty}^{\infty} dt_1 \frac{(\bar{\tilde{w}} - \bar{w}_2)^{n+1}(\bar{z}_2 d\bar{w}_2 + (\bar{\tilde{w}} - \bar{w}_2)d\bar{z}_2)}{\sqrt{t_1^2 + |\tilde{w} - w_2|^2 + |z_2|^2}^{7+2n}}. \tag{4.115}$$

Doing the t_1 integral we get

$$(-1)^m \frac{8\Gamma(3+n)}{15} \frac{(\bar{\tilde{w}} - \bar{w}_2)^{n+1}(\bar{z}_2 d\bar{w}_2 + (\bar{\tilde{w}} - \bar{w}_2)d\bar{z}_2)}{(|\bar{\tilde{w}} - w_2|^2 + |z_2|^2)^{n+3}}. \tag{4.116}$$

We can then combine (4.113), (4.116), and the 3-point interaction vertex $\sigma_3 dz_2 dw_2$, and set up the V_2 integral. To be concise, let us omit the constant factors and reintroduce them at the end.

$$\int_{V_2} (\sigma_3 dz_2 dw_2) \frac{\bar{w}_2^m \bar{z}_2(\bar{z}_2 d\bar{w}_2 - \bar{w}_2 d\bar{z}_2)}{(|w_2|^2 + |z_2|^2)^{m+3}} \frac{(\bar{\tilde{w}} - \bar{w}_2)^{n+1}(\bar{z}_2 d\bar{w}_2 + (\bar{\tilde{w}} - \bar{w}_2)d\bar{z}_2)}{(|\bar{\tilde{w}} - w_2|^2 + |z_2|^2)^{n+3}} A_{\text{ext}}. \tag{4.117}$$

We then expand[20] $A_{\text{ext}}(z_2)$ in z_2 and notice that the only nonvanishing part of the integral comes from one of the modes of A_{ext}.

$$A_{\text{ext}} = \ldots + z_2^2 \partial_{z_2}^2 A_{\text{ext}} \tag{4.118}$$

Simplifying the numerator, we get

$$\sigma_3 \int_{V_2} \frac{\bar{w}_2^m (\bar{\tilde{w}} - \bar{w}_2)^{n+1} \bar{z}_2^2 \bar{\tilde{w}} (z_2^2 \partial_{z_2}^2 A)}{(|w_2|^2 + |z_2|^2)^{m+3} (|\tilde{w} - w_2|^2 + |z_2|^2)^{n+3}} |dw_2|^2 |dz_2|^2. \tag{4.119}$$

We can apply Feynman integral (4.108) here and get

$$\int_0^1 x^{m+2}(1-x)^{n+2} \int_{V_2} \frac{\bar{w}_2^m (\bar{\tilde{w}} - \bar{w}_2)^{n+1} \bar{z}_2^2 \bar{\tilde{w}} (z_2^2 \partial_{z_2}^2 A) |dw_2|^2 |dz_2|^2 dx}{((1-x)(|w_2|^2 + |z_2|^2) + x(|\tilde{w} - w_2|^2 + |z_2|^2))^{m+n+6}}. \tag{4.120}$$

We can rewrite the denominator into $(|w_2 - x\tilde{w}|^2 + |z_2|^2 + x(1-x)|\tilde{w}|^2)^{m+n+6}$, and shift $w_2 \to w_2 + x\tilde{w}$. Then, the above becomes

$$\int_0^1 x^{m+2}(1-x)^{n+2} \int_{V_2} \partial_{z_2}^2 A \frac{(\bar{w}_2 + x\bar{\tilde{w}})^m ((1-x)\bar{\tilde{w}} - \bar{w}_2)^{n+1} |z_2|^4 \bar{\tilde{w}}}{(|w_2|^2 + |z_2|^2 + x(1-x)|\tilde{w}|^2)^{m+n+6}} |dw_2|^2 |dz_2|^2 dx. \tag{4.121}$$

When we work in the radial coordinates (r_z, θ_z), (r_w, θ_w) on each $\mathbb{C}_z$, $\mathbb{C}_w$ planes, it becomes manifest that all the terms with non-zero powers of $\bar{w}_2$ in the numerator become zero under the θ_w integral.

[20] See the discussion around equation (3.20) of [41].

Hence, only one term in the expanded numerator survives:

$$\bar{\tilde{w}}^{m+n+2}\int_0^1 x^{2m+2}(1-x)^{2n+3}\int_{V_2}\partial_{z_2}^2 A\frac{|z_2|^4}{(|w_2|^2+|z_2|^2+x(1-x)|\tilde{w}|^2)^{m+n+6}}|dw_2|^2|dz_2|^2dx. \tag{4.122}$$

Note that if the external leg were $\partial_{z_2}^n A$ with $n\neq 2$, the amplitude vanishes, and the only non-vanishing condition under θ_z integral is $n=2$.

In the radial coordinates, we can evaluate the integral explicitly:

$$\begin{aligned}&\bar{\tilde{w}}^{m+n+2}\int_0^1 x^{2m+2}(1-x)^{2n+3}\int_{\mathbb{R}_t}\partial_{z_2}^2 A\int_0^\infty\int_0^\infty\frac{4\pi^2 r_z^5 r_w}{(r_z^2+r_w^2+x(1-x)|\tilde{w}|^2)^{m+n+6}}dr_zdr_w\\ =&\ \frac{2\pi^2}{\prod_{i=2}^5(i+m+n)}\tilde{w}^{-m-n-2}\int_{\mathbb{R}_t}\partial_{z_2}^2 A\int_0^1 x^{m-n}(1-x)^{n-m+1}dx\\ =&\ \frac{\pi^2}{\prod_{i=2}^5(i+m+n)}\tilde{w}^{-m-n-2}\int_{\mathbb{R}_t}\partial_{z_2}^2 A.\end{aligned} \tag{4.123}$$

The integration in the second line converges if and only if $m=n$ or $m=n+1$. Reintroducing the numerical factors that were omitted, we arrive at

$$\sigma_3\sum_{0\le m-n\le 1}c_{m,n}t_{0,m}t_{0,n}\tilde{w}^{-m-n-2}\int_{\mathbb{R}_t}\partial_{z_2}^2 A, \tag{4.124}$$

where

$$c_{m,n}=(-1)^{m+n}\left(\frac{8\pi}{15}\right)^2(m+n+1)!. \tag{4.125}$$

We have obtained a single composite Wilson line associated with the tensor product representation $t_{0,m}\otimes t_{0,n}\in\mathcal{A}\otimes\mathcal{A}$. Due to the coupling (4.45), the tensor product representation can be equally understood as $t_{2,0}\in\mathcal{A}$.

Therefore, we have derived the 1-loop part of the seed coproduct relation of $\Delta_{\mathcal{A},\mathcal{A}}:\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$.

$$t_{2,0}\to\ldots+\sigma_3\sum_{0\le m-n\le 1}c_{m,n}t_{0,m}\otimes t_{0,n}w^{-m-n-2}. \tag{4.126}$$

We should emphasize that although we have spent most of the space to compute the integral, it is only for checking and showing that the integral converges to a finite quantity for a particular component of the expansion of A_{ext} (4.118). More emphasis should be placed on the selection rule that determines which structure constants to vanish or not. In the present case, the selection rule restricts the RHS of the coproduct to have only $t'_{0,m}\tilde{t}_{0,n}$.

One can still compare the structure constant $c_{m,n}$ in (4.126) and its Koszul dual structure constant $d_{m,n}$ in (4.54). In general, we do not expect a precise equality between two; there can be overall numerical factor. For instance, let us recall [41], where the author compared the OPE in $C^*(\mathbb{C}_{\epsilon_2}[z_1,z_2]\otimes\mathfrak{gl}_1)$

$$\{\partial_{z_1}^p\partial_{z_2}^q X, \partial_{z_1}^k\partial_{z_2}^l X\} = \sum \epsilon_1\epsilon_2^{(r+s+m+n-p-q-k-l)/2-1} A^{p,q,k,l}_{r,s,m,n}(\partial_{z_1}^r\partial_{z_2}^s X)(\partial_{z_1}^m\partial_{z_2}^n X), \quad (4.127)$$

where $\partial_{z_1}^p\partial_{z_2}^q X \in C^*(\mathbb{C}_{\epsilon_2}[z_1,z_2]\otimes\mathfrak{gl}_1)$, and the Koszul dual OPE in $U(\mathbb{C}_{\epsilon_2}[z_1,z_2]\otimes\mathfrak{gl}_1)$

$$[z_1^r z_2^s, z_1^m z_2^n] = \sum \epsilon_1\epsilon_2^{(r+s+m+n-p-q-k-l)/2-1} A^{p,q,k,l}_{r,s,m,n}\frac{m!n!r!s!}{p!q!k!l!}(z_1^p z_2^q)(z_1^k z_2^l), \quad (4.128)$$

where $z_1^m z_2^n \in U(\mathbb{C}_{\epsilon_2}[z_1,z_2]\otimes\mathfrak{gl}_1)$. The analogue of $d_{m,n}$ in (4.54) is the structure constant $A^{p,q,k,l}_{r,s,m,n}$ that appears in (4.127) and the analogue of $c_{m,n}$ in (4.126) is the structure constant $A^{p,q,k,l}_{r,s,m,n}$ multiplied by the numerical factor that follows. In this case, the structure constants of Koszul dual pair algebra are related the numerical constant.

4.3.5 $\mathcal{A}\to\mathcal{A}\otimes\mathcal{W}_\infty$ coproduct

We will derive the coproducts of M2 brane algebra and M5 brane algebra using the perturbative Feynman diagram computation in 5d Chern-Simons theory. The target relation that we want to derive from the 5d Chern-Simons side is

$$t_{2,0} \to \ldots + \sigma_3\sum_{n=1}^{\infty} nW^{(1)}_{-n-1}W^{(1)}_{n-1} + \sigma_3\sum_{n=1}^{\infty} nW^{(1)}_{-n-1}t_{0,n-1}. \quad (4.129)$$

This situation is similar to the intersecting M2-M5 brane configuration studied in [70, 137]. To derive the coproduct relation holographically, we will follow [137], where we computed the Feynman diagrams involving a line and a surface defect.

Let us first write the RHS of (4.129) in the manifest form of $\mathcal{A}\otimes\mathcal{W}_\infty$, by recalling the embedding $\rho(J_{n-1}) = t_{0,n}$.

$$\sigma_3\sum_{n=1}^{\infty} nJ_{-n-1}J_{n-1} + \sigma_3\sum_{n=1}^{\infty} nJ_{-n-1}t_{0,n-1} = \sigma_3\sum_{n=1}^{\infty} nJ_{-n-1}t_{0,n-1}. \quad (4.130)$$

Using the ingredients given in §4.3.3, we can decorate the 1-loop Feynman diagram shown in §4.3.2 as follows.

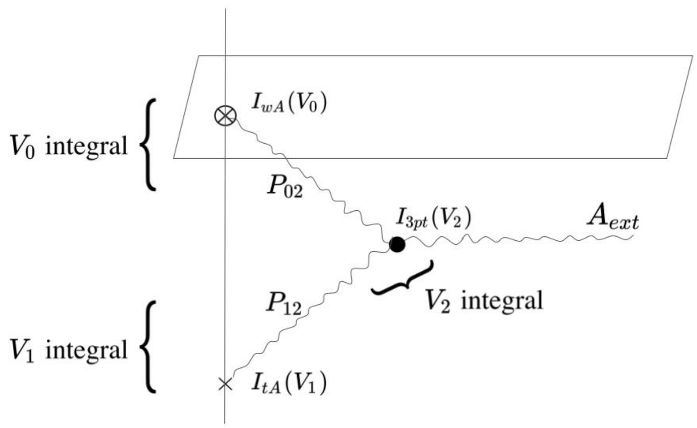


Figure 4.13: The 1-loop Feynman diagram associated with the $\mathcal{A} \to \mathcal{A} \otimes \mathcal{W}_\infty$ coproduct. All the ingredients are explicitly displayed.

The amplitude is

$$\begin{aligned}\sigma_3 J_{-n-1} t_{0,n-1} \int_{V_2} A_{\text{ext}} dz_2 dw_2 \int_{V_0} \delta(t_0)\delta^{(2)}(z_0)\partial_{z_2}(w_0^n P_{02}) dw_0 \\ \times \int_{V_1} \delta(t_1 - \epsilon)\delta^{(2)}(z_1)\delta^{(2)}(w_1)\partial_{w_2}\partial_{w_1}^{n-1} P_{12},\end{aligned} \tag{4.131}$$

where we used (4.103), (4.106), (4.105) for $I_{3pt}(V_2)$, $I_{wA}(V_0)$ and $I_{tA}(V_1)$, respectively.

Let us omit all constant terms and reintroduce them at the end. There are three floating vertices, so there are three integrals to do. Let us first do V_0, V_1 integrals and use them in the final V_2 integral.

$$\int_{V_0} \delta(t_0)\delta^{(2)}(z_0) w_0^n \partial_{z_2} P_{02}. \tag{4.132}$$

Since $\delta(t_0)\delta^{(2)}(w_0) \sim dt dw_0 d\bar{w}_0$, we can project most of the terms in P_{02}. After performing t_0, z_0 integral, we get

$$-\int_{\mathbb{C}_{w_0}} \frac{w_0^n \bar{z}_2(\bar{z}_2 dt_2 + t_2 d\bar{z}_2)}{\sqrt{t_2^2 + |z_2|^2 + |w_{02}|^2}^7} |dw_0|^2. \tag{4.133}$$

After shifting $w_0 \to w_0 + w_2$, we get

$$-\int_{\mathbb{C}_{w_0}} \frac{\bar{z}_2(w_0 + w_2)^n (t_2 d\bar{z}_2 + \bar{z}_2 dt_2)}{\sqrt{t_2^2 + |z_2|^2 + |w_0|^2}^7} |dw_0|^2. \tag{4.134}$$

Working in the radial coordinate of $\mathbb{C}_{w_0}$ plane, only one term in the expanded $(w_0+w_2)^n$ survives. Performing the w_0 integral in the radial coordinate, we get

$$-\frac{2\pi}{3}\frac{w_2^n\bar{z}_2}{\sqrt{t_2^2+|z_2|^2}^5}(t_2 d\bar{z}_2+\bar{z}_2 dt_2). \tag{4.135}$$

Now, let us do V_1 integral.

$$\int_{V_1}\delta(t_2+\epsilon)\delta^{(2)}(z_1)\delta^{(2)}(w_1)\partial_{w_1}^{n-1}\partial_{w_2}P_{12}. \tag{4.136}$$

As there are 3 delta functions, we can easily get

$$\left((-1)^n\frac{7}{2}\cdot\frac{9}{2}\cdots\frac{7+2n-2}{2}\right)\frac{-\bar{w}_2^n(-\bar{z}_2 d\bar{w}_2 dt_2+\bar{w}_2 d\bar{z}_2 dt_2-(\epsilon+t_2)d\bar{z}_2 d\bar{w}_2)}{\sqrt{(\epsilon+t_2)^2+|z_2|^2+|w_2|^2}^{5+2n}}. \tag{4.137}$$

The numerical factor in front can be written in terms of Γ function and will be incorporated later in the final formula.

We can then combine (4.135), (4.137), and the 3-point interaction vertex $\sigma_3 dz_2 dw_2$, and set up the V_2 integral.

$$\sigma_3\int_{V_2}dz_2 dw_2\frac{|w_2|^{2n}\bar{z}_2(\bar{z}_2 dt_2+t_2 d\bar{z}_2)(-\bar{z}_2 d\bar{w}_2 dt_2+\bar{w}_2 d\bar{z}_2 dt_2-(\epsilon+t_2)d\bar{z}_2 d\bar{w}_2)}{\sqrt{t_2^2+|z_2|^2}^5\sqrt{|\epsilon+t_2|^2+|w_2|^2+|z_2|^2}^{5+2n}}A_{\text{ext}}. \tag{4.138}$$

We then expand[21] $A_{\text{ext}}(z_2)$ in z_2 and notice that the only nonvanishing part of the integral comes from one of the modes of A_{ext}.

$$A_{\text{ext}}=\ldots+z_2^2\partial_{z_2}^2 A_{\text{ext}}. \tag{4.139}$$

Substituting it in and simplifying the numerator, we get

$$\sigma_3\int_{V_2}\frac{|w_2|^{2n}|z_2|^4(\epsilon+2t_2)}{\sqrt{t_2^2+|z_2|^2}^5\sqrt{(\epsilon+t_2)^2+|w_2|^2+|z_2|^2}^{5+2n}}dt_2|dw_2|^2|dz_2|^2. \tag{4.140}$$

Note that if the external leg were not $z_2^2\partial_{z_2}^2 A$, but $z_2^n\partial_{z_2}^n A$ with $n\neq 2$, the z_2-integral would vanish.

[21] See the discussion around equation (3.20) of [41].

We can now apply Feynman integral (4.108) on (4.140). Omitting Γ functions for now, and setting $\epsilon = 1$, we get

$$\sigma_3 \int_0^1 dx \int_{V_2} \frac{\sqrt{x^{2n+3}(1-x)^3}|w_2|^{2n}|z_2|^4(2t_2+1)}{((1-x)(t_2^2+|z_2|^2)+x(|w_2|^2+|z_2|^2+(1+t_2)^2))^{n+5}}|dw_2|^2|dz_2|^2 dt_2. \quad (4.141)$$

We can rewrite the denominator into $(|z_2|^2 + x|w_2|^2 + (t_2+x)^2 + x(1-x))^{m+n+6}$, and work in radial coordinates (r_z, θ_z), (r_w, θ_w) for both $\mathbb{C}_z$, $\mathbb{C}_w$ planes. Then, the above becomes

$$4\pi^2\sigma_3 \int_0^1 dx \sqrt{x^{2n+3}(1-x)^3} \int \frac{r_w^{2n+1} r_z^5 (2t_2+1)}{(r_z^2 + x r_w^2 + (t_2+x)^2 + x(1-x))^{n+5}}. \quad (4.142)$$

Then, shift $t_2 \to t_2 - x$, and rescale $r_w \to r_w/\sqrt{x}$. Using the fact that the integral domain for t_2 is $(-\infty, \infty)$, a term with an odd power of t_2 vanishes.

$$4\pi^2\sigma_3 \int_0^1 dx \sqrt{x^3(1-x)^3}(1-2x) \int_0^\infty dr_z \int_0^\infty dr_w \int_{-\infty}^\infty dt_2 \frac{r_w^{2n+1} r_z^5}{(r_w^2 + r_z^2 + t_2^2 + x(1-x))^{n+5}}. \quad (4.143)$$

The final integral is straightforward to evaluate and it gives

$$\sigma_3 \frac{\pi^4}{256} \frac{\Gamma(1+n)}{\Gamma(5+n)}. \quad (4.144)$$

Re-introducing all omitted numerical factors, we arrive at

$$\sum_n c_n \sigma_3 J_{-n-1} t_{0,n-1} \partial_{z_2}^2 A, \quad (4.145)$$

where

$$c_n = \frac{\pi^4}{144} \frac{n!}{2n+5}. \quad (4.146)$$

As it has the external leg $\partial_{z_2}^2 A$, this Feynman diagram mixes with Figure 4.9 and the first two of Figure 4.10. The BRST variation (4.47) of these Feynman diagrams should sum to zero for anomaly-free coupled systems. Hence, we recover the desired coproduct relation.

$$t_{2,0} \to t_{2,0} + \sigma_3 \sum_{n=1}^{\infty} (\text{const}) J_{-n-1} t_{0,n-1} + \dots . \quad (4.147)$$

We should emphasize that although we have spent most of the space to compute the integral, it is only for checking and showing that the integral converges to a finite quantity for a particular component of the expansion of A_{ext} (4.139). More emphasis should be placed on the selection rule that determines which structure constants to vanish or not. In the present case, the selection rule restricts the RHS of the coproduct to have only $J_{-n-1}t_{0,n-1}$. Similar remark that we made at the end of §4.3.4 applies for the numerical part of the structure constant.

4.3.6 $\mathcal{W}_\infty \to \mathcal{W}_\infty \otimes \mathcal{W}_\infty$ coproduct

We will derive the coproducts of M5 brane algebra using the perturbative Feynman diagram computation in 5d Chern-Simons theory. The target relation that we want to derive from the 5d Chern-Simons side is

$$L_{-2} \to \ldots + \sigma_3 \sum_{n=-\infty}^{\infty} n J_{n-1} J'_{-n-1}. \tag{4.148}$$

We will use the technique developed in [41], where the authors computed the OPE of two Wilson lines by computing the relevant Feynman diagram in the 4d Chern-Simons theory.

Using the ingredients given in §4.3.3, we can decorate the 1-loop Feynman diagram shown in §4.3.1 as follows.

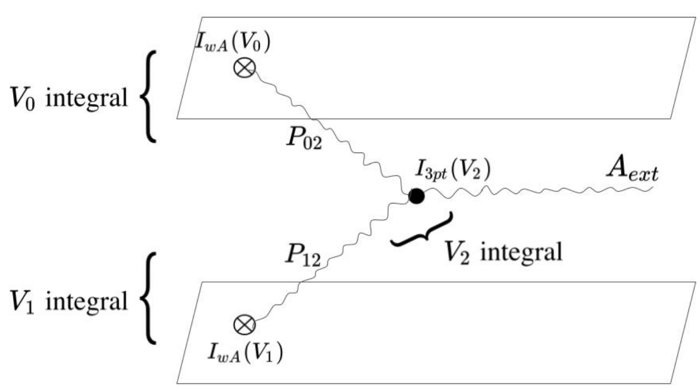


Figure 4.14: The 1-loop Feynman diagram associated with the $\mathcal{W}_\infty \to \mathcal{W}_\infty \otimes \mathcal{W}_\infty$ coproduct. All the ingredients are explicitly displayed.

The amplitude is

$$\begin{aligned} \sigma_3 J_{n-1} J'_{-n-1} \int_{V_2} dz_2 dw_2 (w_2 \partial_{w_2} A_{\text{ext}}) \int_{V_0} \delta(t_0) \delta^{(2)}(z_0) dw_0 \partial_{z_2} (w_0^n P_{02}) \\ \times \int_{V_1} \delta(t_1 - (-\epsilon)) \delta^{(2)}(z_1) dw_1 \partial_{w_2} (w_1^{-n} P_{12}), \end{aligned} \tag{4.149}$$

where we used (4.103), (4.106) for $I_{3pt}(V_2)$, $I_{wA}(V_0)$ and $I_{wA}(V_1)$, respectively. Here, we started by inserting the explicit term of the expansion $A_{\text{ext}} = \ldots + w_2\partial_{w_2}A_{\text{ext}} + \ldots$ in (4.149), as the LHS of (4.148) tells us that the external leg should be proportional to $\partial_{w_2}A_{\text{ext}}$. Hence, the computation in this subsection should be thought of as a check, not a derivation.

There are three floating vertices, so there are three integrals to do. Let us first do V_0, V_1 integrals and use them in the final V_2 integral.

$$\int_{V_0} \delta(t_0)\delta^{(2)}(z_0)w_0^n\partial_{z_2}P_{02}. \tag{4.150}$$

Since $\delta(t_0)\delta^{(2)}(z_0) \sim dt_0dz_0\bar{z}_0$, we can project most of the terms in P_{02}, and get

$$-\frac{5}{2}\int_{\mathbb{C}_{w_0}} \frac{w_0^n\bar{z}_2(\bar{z}_2dt_2 + t_2d\bar{z}_2)}{\sqrt{t_2^2+|z_2|^2+|w_{02}|^2}^7}|dw_0|^2. \tag{4.151}$$

After shifting $w_0 \to w_0 + w_2$, expanding $(w_0+w_2)^n$ in the numerator, and working in the radial coordinates (r_w, θ_w) of $\mathbb{C}_{w_0}$, we can project everything but one term:

$$-\frac{5}{2}\int_0^\infty \frac{w_2^n\bar{z}_2(t_2d\bar{z}_2 + \bar{z}_2dt_2)}{\sqrt{t_2^2+|z_2|^2+r_w^2}^7}(2\pi r_wdr_w). \tag{4.152}$$

Doing r_w integral, we have

$$-\pi\frac{w_2^n\bar{z}_2(t_2d\bar{z}_2 + \bar{z}_2dt_2)}{\sqrt{t_2^2+|z_2|^2}^5}. \tag{4.153}$$

Now, let us do V_1 integral.

$$\int_{V_1} \delta(t_1+\epsilon)\delta^{(2)}(z_1)(w_1)^{-n}\partial_{w_2}P_{12}, \tag{4.154}$$

Taking into account of the t_1, w_1 delta function, we simplify it into

$$\frac{5}{2}\int \frac{w_1^{-n}(\bar{z}_2dt_2 + (t_2+\epsilon)d\bar{z}_2)\bar{w}_{12}}{\sqrt{(t_2+\epsilon)^2+|z_2|^2+|w_{12}|^2}^7}|dw_1|^2. \tag{4.155}$$

Shifting $w_1 \to w_1 + w_2$, and expanding $(w_1+w_2)^{-n}$ in w_1/w_2 and in w_2/w_1 respectively in the region of convergence, we have

$$\begin{aligned}\frac{5}{2}\int_{0\le|w_1|\le|w_2|} &\frac{\bar{w}_1w_2^{-n}\left(1 - n\frac{w_1}{w_2} + \ldots\right)(\bar{z}_2dt_2 + (t_2+\epsilon)d\bar{z}_2)}{\sqrt{(t_2+\epsilon)^2+|z_2|^2+|w_1|^2}^7}|dw_1|^2 \\ &+ \frac{5}{2}\int_{|w_2|\le|w_1|<\infty} \frac{\bar{w}_1w_1^{-n}\left(1 - n\frac{w_2}{w_1} + \ldots\right)(\bar{z}_2dt_2 + (t_2+\epsilon)d\bar{z}_2)}{\sqrt{(t_2+\epsilon)^2+|z_2|^2+|w_1|^2}^7}|dw_1|^2.\end{aligned} \tag{4.156}$$

In the radial coordinates (r_w, θ_w) of $\mathbb{C}_{w_1}$, it is clear that only one term in the expansion in the first parenthesis survives, and reduces to

$$\frac{5}{2}\int_0^{|w_2|} \frac{-nw_2^{-n-1} r_w^2(\bar{z}_2 dt_2 + (t_2+\epsilon) d\bar{z}_2)}{\sqrt{(t_2+\epsilon)^2 + |z_2|^2 + r_w^2}^7}(2\pi r_w) dr_w. \tag{4.157}$$

Doing the r_w integral, we get

$$-\frac{\pi n w_2^{-n-1}(\bar{z}_2 dt_2 + (t_2+\epsilon) d\bar{z}_2)}{3}\left(\frac{1}{\sqrt{(t_2+\epsilon)^2+|z_2|^2}^3} + \frac{2((t_2+\epsilon)^2+|z_2|^2)+5|w_2|^2}{\sqrt{(t_2+\epsilon)^2+|z_2|^2+|w_2|^2}^5}\right). \tag{4.158}$$

We can then combine (4.153), (4.158), the 3-point interaction vertex $\sigma_3 dz_2 dw_2$, and the external leg A. This sets up the V_2 integral. To be concise, let us omit the constant factors and reintroduce them at the end.

$$\sigma_3 \int dw_2 dt_2 |dz_2|^2 \frac{\bar{z}_2^2(2t_2+\epsilon)\partial_{w_2} A}{\sqrt{t_2^2+|z_2|^2}^5}\left(\frac{1}{\sqrt{(t_2+\epsilon)^2+|z_2|^2}^3} + \frac{2((t_2+\epsilon)^2+|z_2|^2)+5|w_2|^2}{\sqrt{(t_2+\epsilon)^2+|z_2|^2+|w_2|^2}^5}\right). \tag{4.159}$$

We may further expand[22] $\partial_{w_2} A(z_2)$ and notice the only nonvanishing piece comes from

$$\partial_{w_2} A = \ldots + z_2^2 \partial_{z_2}^2(\partial_{w_2} A). \tag{4.160}$$

Substituting it in and simplifying the integral, we have

$$\begin{aligned}\sigma_3 \int dw_2(\partial_{w_2}(\partial_{z_2}^2 A)) \int dt_2 |dz_2|^2 \frac{|z_2|^4(2t_2+\epsilon)}{\sqrt{t_2^2+|z_2|^2}^5}\Bigg(&\frac{1}{\sqrt{(t_2+\epsilon)^2+|z_2|^2}^3} \\ &+ \frac{2((t_2+\epsilon)^2+|z_2|^2)+5|w_2|^2}{\sqrt{(t_2+\epsilon)^2+|z_2|^2+|w_2|^2}^5}\Bigg).\end{aligned} \tag{4.161}$$

Let us apply Feynman integral (4.108) to each of two terms, omitting Γ functions for now, and setting $\epsilon = 1$. For the first term, we get

$$\sigma_3 \int dw_2(\partial_{w_2}(\partial_{z_2}^2 A)) \int_0^1 dx \sqrt{(1-x)^3 x} \int \frac{|z_2|^4(2t_2+1) dt_2 |dz_2|^2}{((1-x)(t_2^2+|z_2|^2) + x((t_2+1)^2+|z_2|^2))^4}. \tag{4.162}$$

[22] See the discussion around equation (3.20) of [41].

We can rewrite the denominator into $((t_2+x)^2+|z_2|^2+x(1-x))^4$, and shift $t_2 \to t_2 - x$. Since the t_2-integral domain is $(-\infty,\infty)$, the t_2-linear term vanishes. Then, the above becomes

$$\sigma_3 \int dw_2(\partial_{w_2}(\partial_{z_2}^2 A)) \int_0^1 dx \sqrt{(1-x)^3 x} \int dt_2 |dz_2|^2 \frac{|z_2|^4(1-2x)}{(t_2^2+|z_2|^2+x(1-x))^4}. \tag{4.163}$$

Working in radial coordinates (r_z,θ_z) on $\mathbb{C}_{z_2}$ plane, we can perform the integral straightforwardly as

$$\frac{\pi}{36}\sigma_3 \int dw_2 \partial_{w_2}(\partial_{z_2}^2 A). \tag{4.164}$$

Similarly, for the second term of (4.161), we apply Feynman integral (4.108).

$$\begin{aligned}\sigma_3 \int dw_2(\partial_{w_2}(\partial_{z_2}^2 A)) \int_0^1 dx \sqrt{(1-x)^5 x^3} \\ \times \int \frac{|z_2|^4(2t_2+1)(2((t_2+1)^2+|z_2|^2)+5|w_2|^2)dt_2|dz_2|^2}{((1-x)(t_2^2+|z_2|^2)+x((t_2+1)^2+|z_2|^2+|w_2|^2))^5}.\end{aligned} \tag{4.165}$$

We can rewrite the denominator into $((t_2+x)^2+|z_2|^2+x|w_2|^2+x(1-x))^4$, shift $t_2 \to t_2 - x$, and re-scale $w_2 \to w_2/\sqrt{x}$. Since the t_2-integral domain is $(-\infty,\infty)$, the t_2-linear term vanishes. Then, the above becomes

$$\begin{aligned}\sigma_3 \int dw_2(\partial_{w_2}(\partial_{z_2}^2 A)) \int_0^1 dx \sqrt{(1-x)^5} x^2 \\ \times \int dt_2 |dz_2|^2 \frac{|z_2|^4(1-2x)(2(t_2^2+(1-x)^2+|z_2|^2)+5|w_2|^2/x)}{(t_2^2+|z_2|^2+|w_2|^2+x(1-x))^4}.\end{aligned} \tag{4.166}$$

Working in the radial coordinates (r_z,θ_z) of $\mathbb{C}_{z_2}$ plane, we can check all the terms in the integrand nicely converge under the r_z, t_2, x integrals and (4.166) evaluate to

$$\sigma_3 \int dw_2 \partial_{w_2}(\partial_{z_2}^2 A)\left(\frac{\pi}{256}+\frac{\pi}{48}\left(\frac{3092}{3465}+4|w_2|^2\right)+\frac{\pi^2}{12288}\right). \tag{4.167}$$

As we are working in the holomorphic supergravity background in the $\mathbb{C}_w$ direction, the term $|w_2|^2 = w_2\bar{w}_2$ with an extra anti-holomorphic dependence on $\bar{w}_2$ must be Q-exact, and we may safely drop it.

Combining (4.164) and (4.167), and re-introducing all the omitted constant factors, we arrive at[23]

$$\sigma_3 \; nJ_{n-1}J'_{-n-1}(\text{const}) \int dw_2 \partial_{w_2}(\partial_{z_2}^2 A), \tag{4.168}$$

[23]It is unclear how to intrerpret $\partial_{z_2}^2$ acting on A, as the coupling does not give any information on the z coordinate, but just modes in $\mathbb{C}_w$ plane.

where

$$(\text{const}) = \frac{\pi^2}{3}\frac{\Gamma(4)}{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}\frac{\Gamma(5)}{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right)}\left(\frac{73\pi}{2304}+\frac{\pi}{48}\left(\frac{3092}{3465}\right)+\frac{\pi^2}{12288}\right). \tag{4.169}$$

We have obtained a single composite surface operator associated to the tensor product representation $J_{n-1} \otimes J'_{-n-1} \in \mathcal{W}_\infty \otimes \mathcal{W}_\infty$. Let us look at the external leg $\partial_{w_2}\partial_{z_2}^2 A$, and recall the coupling (4.52). Since it only tells us about the coupling between the w_2 modes of the 5d gauge field and $\mathcal{W}_\infty$ modes, we do not understand what $\partial_{z_2}^2$ means in terms of Koszul duality. Focusing on $\partial_{w_2} A$, as it couples to L_{-2}, the tensor product representation $J_{n-1} \otimes J'_{-n-1}$ can be equally understood as $L_{-2} \in \mathcal{W}_\infty$; it induces the coproduct.

Therefore, we have derived the 1-loop part of the basic coproduct relation of $\Delta_{\mathcal{W}_\infty,\mathcal{W}_\infty} : \mathcal{W}_\infty \to \mathcal{W}_\infty \otimes \mathcal{W}_\infty$

$$L_{-2} \to \ldots + \sigma_3 \sum_{n\geq 1} (const) n J_{n-1} J'_{-n-1}. \tag{4.170}$$

This is the expected coproduct formula for $\mathcal{W}_{2,0,0} \to \mathcal{W}_{1,0,0} \otimes \mathcal{W}_{1,0,0}$. $\mathcal{W}_{2,0,0}$ is a direct sum of a Virasoro algebra, which provides the mode L_{-2}, and an affine Kac-Moody algebra $\hat{\mathfrak{u}}(1)$. $\mathcal{W}_{1,0,0}$ is an affine Kac-Moody algebra $\hat{\mathfrak{u}}(1)$, according to [72].

4.3.7 A comment on the fusion of transverse surface defects

In this section we consider a pair of transverse holomorphic surface defects. Since there is a $\mathrm{SL}_2(\mathbb{C})$ symmetry, we can assume that this pair of surface defects are supported on $\mathbb{C}_z$ and $\mathbb{C}_w$ respectively.

We conjecture that a fusion of two transverse surface defects will give a line operator as a quantum correction in 1-loop order, along with the transverse surface operators. Since we do not have a candidate field theory result for the transverse surface defect fusion, we will not specify a particular mode of $\mathcal{W}_\infty$ algebra that would appear in the coproduct in this subsection. We already have all the ingredients of this calculation. We will frequently draw them from the previous subsections.

We would like to compute the 1-loop correction to the OPE between two transverse surface defects. Diagramatically, it is given by the following figure.

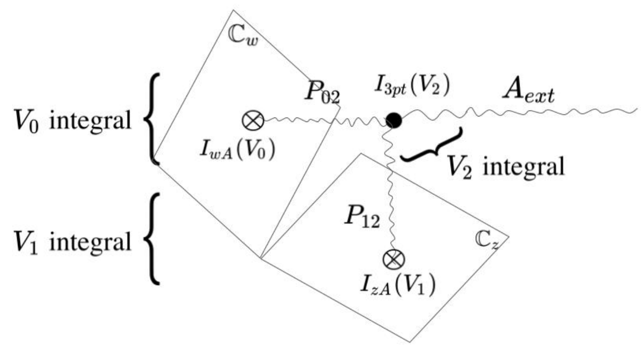


Figure 4.15: The 1-loop Feynman diagram for the OPE between two transverse surface defects on $\mathbb{C}_z$ and $\mathbb{C}_w$ planes. All the ingredients are explicitly displayed.

The amplitude is schematically

$$\begin{aligned}\sigma_3(\dots)\int_{V_2} dz_2 dw_2 A_{\text{ext}} \int_{V_0} \delta(t_0)\delta^{(2)}(z_0) dw_0 \partial_{z_2}(\dots P_{02}) \\ \times \int_{V_1} \delta(t_1)\delta^{(2)}(z_1) dw_1 \partial_{w_2}(\dots P_{12}),\end{aligned} \tag{4.171}$$

where ...'s depend on the detail of the modes of the $\mathcal{W}_\infty$ on each of the vertices $I_{wa}(V_0)$ and $I_{zA}(V_1)$. Since the omitted parts do not affect the structural result that we claim, we will not specify those throughout this subsection.

As we have learned how to do the integral along the surface defect in the previous subsection, for each V_0, V_1 integral, we will draw the result from there:

$$\begin{aligned}\int_{V_0} \delta(t_0)\delta^{(2)}(z_0)(\dots)\partial_{z_2} P_{02} = (\text{const})\frac{(\dots)\bar{z}_2(t_2 d\bar{z}_2 + \bar{z}_2 dt_2)}{\sqrt{t_2^2 + |z_2|^2}^5} \\ \int_{V_1} \delta(t_1)\delta^{(2)}(z_1)(\dots)\partial_{w_2} P_{12} = (\text{const})\frac{(\dots)\bar{w}_2(t_2 d\bar{w}_2 + \bar{w}_2 dt_2)}{\sqrt{t_2^2 + |w_2|^2}^5}.\end{aligned} \tag{4.172}$$

We can then combine (4.172) and the 3-point interaction vertex $\sigma_3 dz_2 dw_2$, and the external leg A_{ext}. This sets up the V_2 integral.

$$\sigma_3 \int_{V_2} dz_2 dw_2 \frac{(\dots)\bar{z}_2(t_2 d\bar{z}_2 + \bar{z}_2 dt_2)}{\sqrt{t_2^2 + |z_2|^2}^5} \frac{(\dots)\bar{w}_2(t_2 d\bar{w}_2 + \bar{w}_2 dt_2)}{\sqrt{t_2^2 + |w_2|^2}^5} A_{\text{ext}}. \tag{4.173}$$

Expanding the numerator, we can observe three objects with a σ_3 factor omitted.

$$\int (\dots) A_{\text{ext}} dw_2 \int \frac{(\dots) dz_2 d\bar{z}_2 dt_2}{\sqrt{(t_2^2+|z_2|^2)(t_2^2+|w_2|^2)}^5} + \int (\dots) A_{\text{ext}} dz_2 \int \frac{(\dots) dw_2 d\bar{w}_2 dt_2}{\sqrt{(t_2^2+|z_2|^2)(t_2^2+|w_2|^2)}^5} + \int (\dots) A_{\text{ext}} \int \frac{(\dots) dz_2 d\bar{z}_2 dw_2 d\bar{w}_2}{\sqrt{(t_2^2+|z_2|^2)(t_2^2+|w_2|^2)}^5}. \tag{4.174}$$

Depending on $(\dots)$ in the numerators, combined with a proper term from the expansion of A_{ext} in z or w, each integral may or may not produce non-zero answers. As our primary purpose is to see the structure, let us now assume that each integral gives a nonzero answer.

The second integrals in each term evaluate to finite constants, which we again denote by the uniform format $(\dots)$.

$$\sigma_3(\text{const}) \int_{\mathbb{C}_w} (\dots) A_{\text{ext}} dw_2 + \sigma_3(\text{const}) \int_{\mathbb{C}_z} (\dots) A_{\text{ext}} dz_2 + \sigma_3(\text{const}) \int_{\mathbb{R}_t} (\dots) A_{\text{ext}}. \tag{4.175}$$

Therefore, in the most general case, we would obtain either a surface operator on $\mathbb{C}_w$, surface operator on $\mathbb{C}_z$, or a line operator on $\mathbb{R}_t$ as a result of the fusion of two transverse surface defects, especially from the 1-loop quantum correction.

4.3.8 1-loop exactness of the Feynman diagrams

All basic coproducts $\Delta_{\mathcal{A},\mathcal{A}}$, $\Delta_{\mathcal{A},\mathcal{W}_\infty}$, $\Delta_{\mathcal{W}_\infty,\mathcal{W}_\infty}$ that we have tried to reproduce so far truncate at $\mathcal{O}(\sigma_3)$ order [73]. However, in principle, all the diagrams that we have discussed may have higher loop corrections on the internal 3-point vertex and one of the propagators. To match with the algebraic result of [73], we need to argue that such higher corrections vanish. Note that [42] showed the 1-loop exactness of Yangian coproduct using the 4d Chern-Simons Feynman diagrams.

Let us start with the potential higher loop corrections to the internal 3-point vertex.

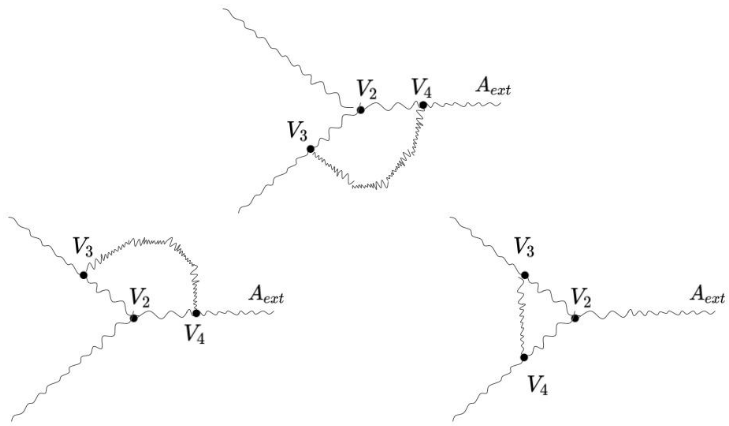


Figure 4.16: Three possible higher loop corrections. In addition to the already existed vertices V_0, V_1, V_2 that were used in the computation in the previous subsections, it contains two extra vertices V_3, V_4, and three extra propagators P_{32}, P_{34}, P_{24}. We distinguish the internal propagators and the external leg, by showing A_{ext} explicitly on the external leg.

Two new internal vertices V_3, V_4 and three new internal propagators P_{32}, P_{34}, P_{24} will introduce an extra factor of $\sigma_3 = \sigma_3^{3-2}$. Hence, with these further corrections, the diagrams, presented in §4.3, are proportional to $\mathcal{O}(\sigma_3^2)$.

We will argue the vanishing of the higher loop corrections without introducing complicated integrals again since we have learned the rule of the game from the 1-loop computations in the previous subsections. For simplicity, we will focus on the left corner diagram, but the other diagrams are equivalent, as it will turn out soon.

By (4.103), each of the new 3-point vertices V_3, V_4 will introduce $\mathcal{I}_{3pt}(V_i)$: a factor of σ_3, a vertex integral $\int_{V_i} dz \wedge dw$, and partial derivatives ∂_z, ∂_w associated with the vertex coordinate that we integrate over.

The partial derivatives act on the two of the propagators emitting from the vertex V_3 to other vertices V_2, V_4, and effectively produce a multiplicative factor

$$\frac{\bar{z}_{32}\bar{w}_{34}}{d_{32}^2 d_{34}^2}. \tag{4.176}$$

Similarly, the partial derivatives that act on the propagator emitting from the vertex V_4 to a vertex V_2 and the external leg, and effectively produce a multiplicative factor

$$\frac{\bar{z}_{24}}{d_{24}^2}\partial_{w_4}. \tag{4.177}$$

where ∂_{w_4} is assumed to act on A_{ext}.

Considering the three new propagators $P_{32}P_{24}P_{43}$, the multiplicative factor introduced by the addition of the new bridge is

$$\sigma_3 \frac{\bar{z}_{32}\bar{w}_{34}\bar{z}_{24}}{d_{32}^2 d_{34}^2 d_{24}^2} P_{32} \wedge P_{24} \wedge P_{43} \partial_w. \tag{4.178}$$

The numerator of P_{ij} is an anti-holomorphic 2-form on the 2-point configuration space of V_i and V_j.

When we multiply the three propagators, recalling the definition (4.100), we see it is precisely zero.[24]

$$P_{32} \wedge P_{24} \wedge P_{43} = 0. \tag{4.179}$$

Since this vanishing property only depends on the three encircling propagators, the argument remains the same for the other two diagrams in Figure 4.16 that we have not discussed. Therefore, there is no higher loop correction on the internal vertex V_2.

Next, we consider the potential higher loop corrections on the external leg and the propagators.

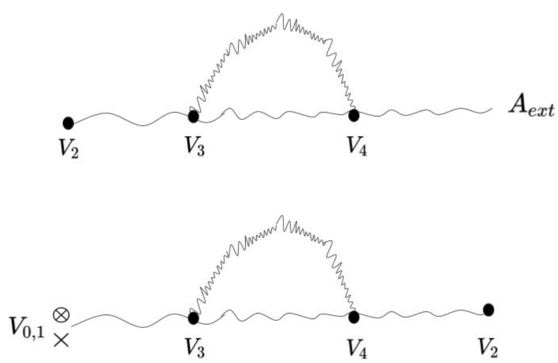


Figure 4.17: The first diagram is a loop correction on the external leg($\sim A_{\text{ext}}$) and the second diagram is a loop correction on one of the internal propagators P_{02}, P_{12} that we have worked with in the previous subsections.

A similar analysis that was applied on Figure 4.16 goes through, and we see the multiplicative factor introduced by the new bridge is proportional to

$$P_{34} \wedge P_{43} \tag{4.180}$$

[24]We used a *Mathematica* package "grassmann" developed by Matthew Headrick in this computation.

for both diagrams in Figure 4.17. Again, (4.180) is precisely zero. Therefore, there is no higher loop correction to the internal propagators and the external leg.

We conjecture that the vanishing phenomena[25] are generally the case for the combinations of anti-holomorphic 2-form of the 5d Chern-Simons propagators with their subscripts in the form of "a trace of a product of matrices ":

$$P_{i_1 i_2} \wedge P_{i_2 i_3} \wedge \ldots \wedge P_{i_n i_1} = 0. \tag{4.181}$$

[25] The similar vanishing phenomena were observed in 4d Chern-Simons theory [98], see §A.4.1.

Chapter 5

M2 Brane Algebras: Algebraic Point of View

In this chapter, we study the M2 brane algebra from a purely algebraic point of view. Namely the M2 brane algebra is the uniform-in-N algebra of the quantized ring of functions on the ADHM moduli space (see §4.1.3). We will spell out its generators and relations in §5.1, using direct computation together with the (faithful) Calogero representation that is explicitly written down in §5.2, then we define the uniform-in-N algebra using the generators and relations in §5.1 and we will prove a flatness theorem of the M2 brane algebra (Theorem 5.3.3), next we observe a simple relation between the Yangian of $\mathfrak{gl}_K$ and the M2 brane algebra in §5.5 and prove a conjecture of Costello (Theorem 5.5.1), we also explain a relation between a certain degeneration limit of M2 brane algebra to the Kac-Moody algebra of $\mathfrak{gl}_K$ in §5.6, and finally we write down the meromorphic coproduct structure of the M2 brane algebra in §5.7 and show that it gives rise to a vertex coalgebra structure on the M2 brane algebra (Proposition 5.7.2).

5.1 Generators and relations

The ADHM quiver Q consists of one edge loop and one framing. The Cartan matrix C_Q is zero in this case, therefore it is easy to verify that (C.19) is satisfied for all $\mathbf{v}, \mathbf{w}$, i.e. $(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}})$ is always good for the Jordan quiver. Introduce the convention $\mathbf{v} = N$ and $\mathbf{w} = K$, $\hbar = \epsilon_1$, $\mathbf{t} = \epsilon_2$, and denote the deformed quantum Nakajima quiver variety by $\mathsf{B}_N^{(K)} = \mathbb{C}_{\epsilon_1}[\mathcal{M}_{\epsilon_2}(N, K)]$, it is also called the *quantized Gieseker variety*. In this case Proposition C.1.18 and Theorem C.1.19 can be summarized as follows.

Lemma 5.1.1. $\mathsf{B}_N^{(K)}$ *is a free* $\mathbb{C}[\epsilon_1,\epsilon_2]$*-module, and*

$$\mathsf{B}_N^{(K)}/(\epsilon_1) \cong \mathbb{C}[\mathcal{M}_{\epsilon_2}(N,K)]. \tag{5.1}$$

Moreover the scheme $\mathcal{M}_c(N,K)$ *is reduced* $\forall c \in \mathbb{C}$.

The affine schemes $\mathcal{M}_c(N,K)$, known as Gieseker varieties, have been extensively studied. In the case of $K=1$, $\mathcal{M}_0(N,1)=S^N(\mathbb{C}^2)$, i.e. the N'th symmetric product of $\mathbb{C}^2$.

By invariant theory, $\mathsf{B}_N^{(K)}$ is generated by GL_N-invariant monomials in $\{X,Y,I,J\}$ with relations

$$\begin{aligned}[X^i_j, Y^k_l] =&\epsilon_1\delta^i_l\delta^k_j,\ [J^j_a, I^b_i]=\epsilon_1\delta^j_i\delta^b_a,\\ g(X,Y,I,J)&\left(:[X,Y]^i_j: -I_jJ^i+\epsilon_2\delta^i_j\right)=0,\end{aligned} \tag{5.2}$$

and other commutations between symbols X,Y,I,J are zero. Here $g(X,Y,I,J)$ means arbitrary polynomials in X,Y,I,J, and normal ordering convention is such that Y is put at the left of X.

Definition 5.1.2. In order to present the relations in a nice form, we slightly enlarge $\mathsf{B}_N^{(K)}$ by inverting ϵ_1, and define

$$e^a_{b;n,m}=\frac{1}{\epsilon_1}I^a\mathrm{Sym}(X^nY^m)J_b,\ t_{n,m}=\frac{1}{\epsilon_1}\mathrm{Tr}(\mathrm{Sym}(X^nY^m)).$$

They are generators of $\mathsf{B}_N^{(K)}[\epsilon_1^{-1}]$. It is clear from the moment map equation that $e^a_{a;n,m}=\epsilon_2 t_{n,m}$ and $t_{0,0}=\frac{N}{\epsilon_1}$. Moreover, the following relations are easily derived from definition.

Lemma 5.1.3. $t_{0,0}$ *is central, and* $e^a_{b;0,0}$ *act on* $\mathsf{B}_N^{(K)}[\epsilon_1^{-1}]$ *as generators of* $\mathfrak{gl}_K$,

$$[e^a_{b;0,0}, e^c_{d;n,m}]=\delta^c_b e^a_{d;n,m}-\delta^a_d e^c_{b;n,m}. \tag{5.3}$$

The linear span of $t_{2,0},t_{1,1},t_{0,2}$ *acts on* $\mathsf{B}_N^{(K)}[\epsilon_1^{-1}]$ *as* $\mathfrak{sl}_2$:

$$[t_{2,0},e^a_{b;n,m}]=2me^a_{b;n+1,m-1},\ [t_{1,1},e^a_{b;n,m}]=(m-n)e^a_{b;n,m},\ [t_{0,2},e^a_{b;n,m}]=-2ne^a_{b;n-1,m+1} \tag{5.4}$$

And moreover

$$[t_{1,0},e^a_{b;n,m}]=me^a_{b;n,m-1},\ [t_{0,1},e^a_{b;n,m}]=ne^a_{b;n-1,m}. \tag{5.5}$$

The above lemma together with three other commutation relations presented in the next proposition completely determine all other commutation relations.

Proposition 5.1.4. *Let $\epsilon_3 = -K\epsilon_1 - \epsilon_2$, then*

$$[e^a_{b;1,0}, t_{3,0}] = 0. \tag{5.6}$$

$$\begin{aligned}[e^a_{b;1,0}, e^c_{d;0,n}] = &\delta^c_b e^a_{d;1,n} - \delta^a_d e^c_{b;1,n} - \frac{\epsilon_3 n}{2}\left(\delta^c_b e^a_{d;0,n-1} + \delta^a_d e^c_{b;0,n-1}\right) - n\epsilon_1 \delta^c_d e^a_{b;0,n-1} \\ &- \epsilon_1 \sum_{m=0}^{n-1} \frac{m+1}{n+1} \delta^a_d e^c_{f;0,m} e^f_{b;0,n-1-m} - \epsilon_1 \sum_{m=0}^{n-1} \frac{n-m}{n+1} \delta^c_b e^a_{f;0,m} e^f_{d;0,n-1-m} \\ &+ \epsilon_1 \sum_{m=0}^{n-1} e^a_{d;0,m} e^c_{b;0,n-1-m}\end{aligned} \tag{5.7}$$

$$\begin{aligned}[t_{3,0}, t_{0,n}] =&3nt_{2,n-1} + \frac{n(n-1)(n-2)}{4}(\epsilon_1^2 - \epsilon_2\epsilon_3)t_{0,n-3} \\ &- \frac{3\epsilon_1}{2}\sum_{m=0}^{n-3}(m+1)(n-2-m)(e^a_{c;0,m}e^c_{a;0,n-3-m} + \epsilon_1\epsilon_2 t_{0,m}t_{0,n-3-m}).\end{aligned} \tag{5.8}$$

Equations (5.6), (5.7), (5.8) together with Lemma 5.1.3 determine all the other commutation relations. For example (5.7) implies that $[e^a_{b;1,0}, t_{0,n}] = ne^a_{b;0,n-1}$, together with (5.6) this in turn implies that

$$\begin{aligned}[t_{3,0}, e^a_{b;0,n}] &= \frac{1}{n+1}[t_{3,0}, [e^a_{b;1,0}, t_{0,n+1}]] = \frac{1}{n+1}[e^a_{b;1,0}, [t_{3,0}, t_{0,n+1}]] \\ &= 3ne^a_{b;2,n-1} + \frac{n(n-1)(n-2)}{4}(\epsilon_1^2 - \epsilon_2\epsilon_3)e^a_{b;0,n-3} + \text{quadratic+cubic}.\end{aligned} \tag{5.9}$$

The equation (5.6) is obvious from definition, but the other two equations in Proposition 5.1.4 are less obvious, of which the proof will be deferred until the next section. It turns out that (5.7) is more or less transparent in the Calogero representation, but the computation of (5.8) in the Calogero representation or in the original definition seems to be hard. In order to proceed, we notice that (5.8) is equivalent to the next two equations:

$$[t_{2,1}, t_{0,n}] = 2nt_{1,n}, \tag{5.10}$$

$$\begin{aligned}[t_{2,1}, t_{1,n}] =&(2n-1)t_{2,n} + \frac{n(n-1)}{4}(\epsilon_2\epsilon_3 - \epsilon_1^2)t_{0,n-2} + \\ &+ \frac{3\epsilon_1}{2}\sum_{m=0}^{n-2}\frac{(m+1)(n-1-m)}{n+1}(e^a_{c;0,m}e^c_{a;0,n-2-m} + \epsilon_1\epsilon_2 t_{0,m}t_{0,n-2-m}).\end{aligned} \tag{5.11}$$

This comes from $[t_{3,0}, t_{0,n}] = \frac{1}{2}[[t_{2,0}, t_{2,1}], t_{0,n}] = \frac{1}{2}[t_{2,0}, [t_{2,1}, t_{0,n}]] - n[t_{2,1}, t_{1,n-1}]$.

5.2 Calogero representation

We choose $\theta = -1$ then $\mathcal{N}^\theta(Q, \mathbf{v}, \mathbf{w})$ is isomorphic to the Quot scheme Quot_N^K parametrizing length-N quotients of $\mathcal{O}_{\mathbb{C}}^{\oplus K}$. The Hilbert-Chow map $\mathrm{Quot}_N^K \to \mathrm{Sym}^N(\mathbb{C})$ sends a quiver data (I, Y) to the spectrum of Y. Restricted to the open locus where spectra of Y are distinct, Quot_N^K is isomorphic to product of N copies of $\mathbb{P}^{K-1}$ fibered over the base $\mathrm{Sym}^N(\mathbb{C})_{\mathrm{disj}}$.

Let E^a_b be the image of the generator $e^a_b \in U(\mathfrak{gl}_K)$ under the Beilinson–Bernstein map $U(\mathfrak{gl}_K) \to D^{\epsilon_2/\epsilon_1}(\mathbb{P}^{K-1})$, where $D^{\epsilon_2/\epsilon_1}(\mathbb{P}^{K-1})$ is the ring of $\mathcal{O}(1)^{\otimes \frac{\epsilon_2}{\epsilon_1}}$-twisted differential operators on $\mathbb{P}^{K-1}$. In particular we have

$$E^a_a = \frac{\epsilon_2}{\epsilon_1}, \quad E^a_c E^c_b = -\frac{\epsilon_1 + \epsilon_3}{\epsilon_1} E^a_b, \quad E^a_b E^b_a = -\frac{(\epsilon_1 + \epsilon_3)\epsilon_2}{\epsilon_1^2}. \tag{5.12}$$

It is not hard to see the following:

Lemma 5.2.1. *Composing the Calogero representation* $\mathsf{B}_N^{(K)}[\epsilon_1^{-1}] \hookrightarrow D^{\epsilon_2}_{\epsilon_1}(\mathrm{Quot}_N^K)[\epsilon_1^{-1}]$ *with the restriction map* $D^{\epsilon_2}_{\epsilon_1}(\mathrm{Quot}_N^K) \hookrightarrow D^{\epsilon_2}_{\epsilon_1}(\mathbb{P}^{K-1} \times \cdots \times P^{K-1} \times \mathbb{C}^N_{\mathrm{disj}})$*, then* $t_{2,0}$ *and* $e^a_{b;0,n}$ *are mapped to*

$$t_{2,0} \mapsto \epsilon_1 \sum_{i=1}^N \partial^2_{y_i} - 2\sum_{i<j}^N \frac{\epsilon_1 \Omega_{ij} + \epsilon_2}{(y_i - y_j)^2}, \qquad e^a_{b;0,n} \mapsto \sum_{i=1}^N E^a_{b,i} y_i^n, \tag{5.13}$$

where $(y_1, \cdots, y_N)$ *is the coordinate on* $\mathbb{C}^N$*,* $E^a_{b,i}$ *is the* E^a_b *for the i'th* $\mathbb{P}^{K-1}$*, and* $\Omega_{ij} = E^a_{b,i} E^b_{a,j}$ *is the quadratic Casimir between* i *and* j*.*

Proof. We diagonalize $Y = H\mathrm{diag}(y_1, \cdots, y_N)H^{-1}$, and define

$$u^a_i = (IH)^a_i, \quad v^i_a = (H^{-1}J)^i_a, \quad E^a_{b,i} = \frac{1}{\epsilon_1} u^a_i v^i_b. \tag{5.14}$$

The commutators between u and v are

$$[v^i_a, u^b_j] = \epsilon_1 \delta^b_a \delta^i_j. \tag{5.15}$$

Therefore u^a_i are the projective coordinates on the i'th $\mathbb{P}^{K-1}$ and v^i_a are the differential operators on it, and $E^a_{b,i}$ satisfy the $\mathfrak{gl}_K$ commutation relations, namely $e^a_b \mapsto E^a_{b,i}$ is a free-field realization of $\mathfrak{gl}_K$.

From the diagonalization $Y = HDH^{-1}$ where D is the diagonal matrix, we read out the tangent map $dY = [dH \cdot H^{-1}, Y] + HdDH^{-1}$, and in the dual basis the above equation becomes

$$\partial_{H^i_j} = \frac{1}{\epsilon_1}(H^{-1} : [Y, X] :)^j_i, \quad \partial_{y_i} = \frac{1}{\epsilon_1} : (H^{-1}XH)^i_i :, \tag{5.16}$$

here we use the identification $X^i_j = \epsilon_1 \partial_{Y^j_i}$, and the normal ordering such that X is always at the right-hand-side of H and Y. Let $\overline{X}^i_j =: (H^{-1}XH)^i_j :$, then

$$\overline{X}^i_j = \begin{cases} \frac{\epsilon_1}{y_i - y_j} : (\partial_H \cdot H)^i_j :, & i \neq j \\ \epsilon_1 \partial_{y_i}, & i = j, \end{cases} \tag{5.17}$$

and the quantum moment map equation becomes

$$u^a_i v^j_a = \begin{cases} -\epsilon_1 : (\partial_H \cdot H)^i_j :, & i \neq j \\ \epsilon_2, & i = j. \end{cases} \tag{5.18}$$

Thus the image of $e^a_{b;0,n} = \frac{1}{\epsilon_1} I^a Y^n J_b$ is

$$\frac{1}{\epsilon_1} \sum_{i=1}^{N} u^a_i y^n_i v^i_b = \sum_{i=1}^{N} E^a_{b,i} y^n_i,$$

and the image of $t_{2,0} = \frac{1}{\epsilon_1} \overline{X}^i_j \overline{X}^j_i$ is

$$\epsilon_1 \sum_{i=1}^{N} \partial^2_{y_i} - \frac{2}{\epsilon_1} \sum_{i<j}^{N} \frac{u^a_i v^j_a u^b_j v^i_b}{(y_i - y_j)^2},$$

and it is easy to see that $\frac{1}{\epsilon_1} u^a_i v^j_a u^b_j v^i_b = \epsilon_1 \Omega_{ij} + \epsilon_2$, this proves our claim. □

From the above lemma, we can derive the formula for more generators.

$$t_{0,n} \mapsto \frac{1}{\epsilon_1} \sum_{i=1}^{N} y^n_i, \qquad t_{1,n} \mapsto \sum_{i=1}^{N} \left(\frac{n}{2} y^{n-1}_i + y^n_i \partial_{y_i} \right), \tag{5.19}$$

$$e^a_{b;1,n} \mapsto \epsilon_1 \sum_{i=1}^{N} E^a_{b,i} \left(\frac{n}{2} y^{n-1}_i + y^n_i \partial_{y_i} \right) + \epsilon_1 \sum_{i<j}^{N} \frac{y^{n+1}_i - y^{n+1}_j}{n+1} \frac{E^a_{c,i} E^c_{b,j} - E^a_{c,j} E^c_{b,i}}{(y_i - y_j)^2} \tag{5.20}$$

$$t_{2,n} \mapsto \epsilon_1 \sum_{i=1}^{N} \left(\frac{n(n-1)}{4} y_i^{n-2} + n y_i^{n-1} \partial_{y_i} + y_i^n \partial_{y_i}^2 \right) - \frac{2}{n+1} \sum_{i<j}^{N} \frac{y_i^{n+1} - y_j^{n+1}}{(y_i - y_j)^3} (\epsilon_1 \Omega_{ij} + \epsilon_2). \tag{5.21}$$

We can compute more relations in the Calogero representation.

$$\begin{aligned} e^a_{c;0,m} e^c_{b;0,n} &= -\frac{\epsilon_1 + \epsilon_3}{\epsilon_1} e^a_{b;0,m+n} + \sum_{i<j}^{N} y_i^m y_j^n E^a_{c,i} E^c_{b,j} + y_i^n y_j^m E^c_{b,i} E^a_{c,j}, \\ e^a_{b;0,m} e^b_{a;0,n} &= -\frac{(\epsilon_1 + \epsilon_3)\epsilon_2}{\epsilon_1} t_{0,m+n} + \sum_{i<j}^{N} (y_i^m y_j^n + y_i^n y_j^m) \Omega_{ij} \\ &= -\frac{\epsilon_2 \epsilon_3}{\epsilon_1} t_{0,m+n} - \epsilon_1 \epsilon_2 t_{0,m} t_{0,n} + \frac{1}{\epsilon_1} \sum_{i<j}^{N} (y_i^m y_j^n + y_i^n y_j^m)(\epsilon_1 \Omega_{ij} + \epsilon_2). \end{aligned} \tag{5.22}$$

Proof of Equation (5.7). The left hand side of (5.7) can be written as

$$\begin{aligned} [e^a_{b;1,0}, e^c_{d;0,n}] &= \epsilon_1 \sum_{i=1}^{N} [E^a_{b,i} \partial_{y_i}, E^c_{d,i} y_i^{n-1}] + \epsilon_1 \sum_{i<j}^{N} [\frac{E^a_{f,i} E^f_{b,j} - E^a_{f,j} E^f_{b,i}}{y_i - y_j}, E^c_{d,i} y_i^n + E^c_{d,j} y_j^n] \\ &= \epsilon_1 \sum_{i=1}^{N} \left([E^a_{b,i}, E^c_{d,i}](n y_i^{n-1} + y_i^n \partial_{y_i}) + n E^c_{d,i} E^a_{b,i} y_i^n \right) + \epsilon_1 \sum_{i<j}^{N} \frac{y_i^n - y_j^n}{y_i - y_j} (E^a_{d,i} E^c_{b,j} + E^a_{d,j} E^c_{b,i}) \\ &\quad - \epsilon_1 \delta^a_d \sum_{i<j}^{N} \frac{E^c_{f,i} E^f_{b,j} y_i^n - E^c_{f,j} E^f_{b,i} y_j^n}{y_i - y_j} - \epsilon_1 \delta^c_b \sum_{i<j}^{N} \frac{E^f_{d,i} E^a_{f,j} y_i^n - E^f_{d,j} E^a_{f,i} y_j^n}{y_i - y_j} \\ &= \delta^c_b e^a_{d;1,n} - \delta^a_d e^c_{b;1,n} + \frac{\epsilon_3 n}{2} \left(\delta^c_b e^a_{d;0,n-1} + \delta^a_d e^c_{b;0,n-1} \right) - \epsilon_1 n \delta^a_d e^c_{b;0,n-1} \\ &\quad - \epsilon_1 \delta^a_d \sum_{m=0}^{n-1} \frac{m+1}{n+1} e^c_{f;0,m} e^f_{b;0,n-1-m} - \epsilon_1 \sum_{m=0}^{n-1} \frac{n-m}{n+1} \delta^c_b e^a_{f;0,m} e^f_{d;0,n-1-m} \\ &\quad + \epsilon_1 n \sum_{i=1}^{N} (E^c_{d,i} E^a_{b,i} - E^a_{d,i} E^c_{b,i}) y_i^{n-1} + \epsilon_1 \sum_{m=0}^{n-1} e^a_{d;0,m} e^c_{b;0,n-1-m}. \end{aligned}$$

Use the identity $E^c_{d,i} E^a_{b,i} - E^a_{d,i} E^c_{b,i} = \delta^a_d E^c_{b,i} - \delta^c_d E^a_{b,i}$, we get the right hand side of (5.7). □

Proof of Equation (5.10). In the Calogero representation we have

$$[t_{2,1}, t_{0,n}] = \sum_{i=1}^{N} [\partial_{y_i} + y_i \partial_{y_i}^2, y_i^n] = \sum_{i=1}^{N} \left(n^2 y_i^{n-1} + 2n y_i^n \partial_{y_i} \right) = 2n t_{1,n}.$$

□

Proof of Equation (5.11). The left hand side of (5.11) can be written as

$$\begin{aligned}
[t_{2,1}, t_{1,n}] =& \epsilon_1 \sum_{i=1}^{N} [\partial_{y_i} + y_i \partial_{y_i}^2, \frac{n}{2} y_i^{n-1} + y_i^n \partial_{y_i}] - \sum_{i<j}^{N} (\epsilon_1 \Omega_{ij} + \epsilon_2) [\frac{y_i + y_j}{(y_i - y_j)^2}, y_i^n \partial_{y_i} + y_j^n \partial_{y_j}] \\
=& \epsilon_1 \sum_{i=1}^{N} \left(\frac{n(n-1)^2}{2} y_i^{n-2} + n(2n-1) y_i^{n-1} \partial_{y_i} + (2n-1) y_i^n \partial_{y_i}^2 \right) \\
&+ \sum_{i<j}^{N} (\epsilon_1 \Omega_{ij} + \epsilon_2) \frac{3(y_i y_j^n - y_i^n y_j) - (y_i^{n+1} - y_j^{n+1})}{(y_i - y_j)^3},
\end{aligned}$$

And the relevant summations that we encounter in the right hand side of (5.11) can be written as

$$\begin{aligned}
&\frac{\epsilon_1}{2} \sum_{m=0}^{n-2} (m+1)(n-1-m)(e^a_{c;0,m} e^c_{a;0,n-2-m} + \epsilon_1 \epsilon_2 t_{0,m} t_{0,n-2-m}) \\
&= -\frac{(n+1)n(n-1)}{12} \epsilon_2 \epsilon_3 t_{0,n-2} \\
&+ \sum_{i<j}^{N} \frac{(n-1)(y_i^{n+1} - y_j^{n+1}) + (n+1)(y_i y_j^n - y_i^n y_j)}{(y_i - y_j)^3} (\epsilon_1 \Omega_{ij} + \epsilon_2).
\end{aligned} \tag{5.23}$$

Now we can see that two sides of (5.11) agree by direct computation using (5.23). □

5.3 Uniform-in-N algebra

Definition 5.3.1. Let the $\mathbb{C}[\epsilon_1, \epsilon_2]$-algebra $\mathcal{A}^{(K)}$ be generated by $t_{n,m}, e^a_{b;n,m}$ with relations $t_{n,m} = \epsilon_2 e^a_{a;n,m}$, and those in Lemma 5.1.3 and Proposition 5.1.4, and $[e^a_{b;1,0}, t_{0,n}] = n e^a_{b;0,n-1}$. $\mathcal{A}^{(K)}$ is called the ADHM algebra in chapter §4. Let $\mathsf{B}^{(K)}$ be the $\mathbb{C}[\epsilon_1, \epsilon_2]$-algebra generated by $t'_{n,m} = \epsilon_1 t_{n,m}, e'^a_{b;n,m} = \epsilon_1 e^a_{b;n,m}$ with relations $t'_{n,m} = e'^a_{a;n,m}$ and those obtained from scaling of Lemma 5.1.3 and Proposition 5.1.4 by ϵ_1^2, and finally $[e'^a_{b;1,0}, t'_{0,n}] = n\epsilon_1 e'^a_{b;0,n-1}$.

Obviously, there is an algebra homomorphism $\mathsf{B}^{(K)} \to \mathcal{A}^{(K)}$, which is an isomorphism when localized to $\mathbb{C}[\epsilon_1^{\pm 1}, \epsilon_2]$. In fact $\mathsf{B}^{(K)} \to \mathcal{A}^{(K)}$ is injective, this is due to the flatness of the algebras, see Theorem 5.3.3. Moreover there are surjective algebra homomorphisms

$$p_N : \mathsf{B}^{(K)} \twoheadrightarrow \mathsf{B}_N^{(K)},$$
$$e'^{a}_{b;n,m} \mapsto I^a \mathrm{Sym}(X^n Y^m) J_b,\ t'_{n,m} \mapsto \mathrm{Tr}\mathrm{Sym}(X^n Y^m).$$

Definition 5.3.2. The image of $\prod_N p_N : \mathsf{B}^{(K)} \to \prod_N \mathsf{B}_N^{(K)}$ is called the uniform-in-N algebra $\mathbb{C}_{\epsilon_1}[\mathcal{M}_{\epsilon_2}(\bullet, K)]$.

Theorem 5.3.3. *The canonical map $\mathsf{B}^{(K)} \to \mathbb{C}_{\epsilon_1}[\mathcal{M}_{\epsilon_2}(\bullet, K)]$ is an isomorphism. Moreover $\mathcal{A}^{(K)}$ and $\mathsf{B}^{(K)}$ are free $\mathbb{C}[\epsilon_1, \epsilon_2]$-modules.*

Proof. The situation here is similar to that of [29], and we borrow the idea from there. First of all, note that $\mathsf{B}^{(K)}/(\epsilon_1)$ is the commutative algebra *freely* generated by $e'^{a}_{b;n,m}, t'_{n,m}$ for $(n,m) \in \mathbb{N}^2$ and $(a,b) \neq (K,K)$, and their images in $\mathbb{C}[\mathcal{M}_{\epsilon_2}(\bullet, K)]$ are algebraically independent for generic ϵ_2 by [29, Proposition 15.0.2], thus $\mathsf{B}^{(K)}/(\epsilon_1) \to \mathbb{C}[\mathcal{M}_{\epsilon_2}(\bullet, K)]$ is injective therefore it is isomorphic. In other word the kernel of $\mathsf{B}^{(K)} \to \mathbb{C}_{\epsilon_1}[\mathcal{M}_{\epsilon_2}(\bullet, K)]$ is contained in the ideal (ϵ_1). By the flatness of $\mathbb{C}_{\epsilon_1}[\mathcal{M}_{\epsilon_2}(\bullet, K)]$, if $\epsilon_1 f$ is in the kernel, then f is in the kernel too. This implies that the kernel is contained in $\cap_n (\epsilon_1^n) = 0$, thus $\mathsf{B}^{(K)} \to \mathbb{C}_{\epsilon_1}[\mathcal{M}_{\epsilon_2}(\bullet, K)]$ is an isomorphism.

Fix an order for the generators $e'^{a}_{b;n,m}, t'_{n,m}$ and we can form a putative basis of $\mathsf{B}^{(K)}$ given by the normal-ordered monomials in $e'^{a}_{b;n,m}, t'_{n,m}$. Modulo ϵ_1, they form a basis of $\mathsf{B}^{(K)}/(\epsilon_1)$ over $\mathbb{C}[\epsilon_2]$, therefore they generated $\mathsf{B}^{(K)}$ as $\mathbb{C}[\epsilon_1, \epsilon_2]$-module by the graded Nakayama lemma. And moreover they do not have nontrivial linear relations, in fact any such relations must be divisible by ϵ_1, and by the flatness of $\mathsf{B}^{(K)}$, ϵ_1 can be subtracted from linear relation and we can get a relation not divisible by ϵ_1, which must be trivial. This shows the freeness of $\mathsf{B}^{(K)}$.

For the freeness of $\mathcal{A}^{(K)}$, fix an order for the generators $e^{a}_{b;n,m}, t_{n,m}$ and we can form a putative basis of $\mathcal{A}^{(K)}$ given by the normal-ordered monomials in $e^{a}_{b;n,m}, t_{n,m}$. Note that $\mathcal{A}^{(K)}/(\epsilon_1)$ is the universal envelope of a Lie algebra generated by $e^{a}_{b;n,m}, t_{n,m}$, so the set of the normal-ordered monomials forms a basis for $\mathcal{A}^{(K)}/(\epsilon_1)$ by PBW theorem. In particular they generate $\mathcal{A}^{(K)}$ as $\mathbb{C}[\epsilon_1, \epsilon_2]$-module by the graded Nakayama lemma. If there is a nontrivial relation among those normal-ordered monomials, then we can multiple ϵ_1^n so that it lies in the image of $\mathsf{B}^{(K)}$, but we have shown that normal-ordered monomials in $e'^{a}_{b;n,m}, t'_{n,m}$ form a basis of $\mathsf{B}^{(K)}$, therefore the relation must be trivial. This shows the freeness of $\mathcal{A}^{(K)}$. $\square$

Proposition 5.3.4. *On the locus $\epsilon_2 \neq 0$, the algebra $\mathcal{A}^{(K)}/(\epsilon_1)$ is isomorphic to the universal enveloping algebra of $D_{\epsilon_2}(\mathbb{C}) \otimes \mathfrak{gl}_K$, where $D_{\epsilon_2}(\mathbb{C})$ is the algebra of ϵ_2-differential operators on $\mathbb{C}$.*

Proof. It is obvious from the commutation relations that $\mathcal{A}^{(K)}/(\epsilon_1)$ is the universal enveloping algebra of the Lie algebra spanned by $e^a_{b;n,m}$. Let the coordinate on $\mathbb{C}$ be y and its differential x such that $[x, y] = \epsilon_2$, then one consider the map

$$e^a_{b;n,m} \mapsto \mathrm{Sym}(x^n y^m) \otimes e^a_b, \tag{5.24}$$

where e^a_b are the corresponding generators of $\mathfrak{gl}_K$, this gives a one-two-one correspondence between generators $e^a_{b;n,m}$ and a basis of $D_{\epsilon_2}(\mathbb{C}) \otimes \mathfrak{gl}_K$. It is easy to see that the above map preserves the commutators in Lemma 5.1.3 and Proposition 5.1.4 modulo ϵ_1, i.e. it is a Lie algebra morphism. Thus $\mathcal{A}^{(K)}/(\epsilon_1) \cong U(D_{\epsilon_2}(\mathbb{C}) \otimes \mathfrak{gl}_K)$. ☐

Remark 5.3.5. The algebra $U(D_{\epsilon_2}(\mathbb{C}) \otimes \mathfrak{gl}_K)$ is known as the double current algebra in the literature [89, 59, 60]. Compare the generators and relations of $\mathcal{A}^{(K)}$ with [59, Proposition 4.2.13], we see that $\mathcal{A}^{(K)}$ is isomorphic to the deformed double current algebra (DDCA). This provides an affirmative answer to a question of Costello [29, 2.1].

5.3.6 Commutation relations in the case $K = 1$

In this subsection we write down the commutation relations in the case $K = 1$ which is relevant to the subsection §4.1.6. Again, $t_{0,0}$ is central, and the linear span of $t_{2,0}, t_{1,1}, t_{0,2}$ acts on $\mathcal{A}^{(1)}$ as $\mathfrak{sl}_2$:

$$[t_{2,0}, t_{n,m}] = 2m t_{n+1,m-1},\ [t_{1,1}, t_{n,m}] = (m-n) t_{n,m},\ [t_{0,2}, t_{n,m}] = -2n t_{n-1,m+1}. \tag{5.25}$$

And

$$[t_{1,0}, t_{n,m}] = m t_{n,m-1},\ [t_{0,1}, t_{n,m}] = n t_{n-1,m}. \tag{5.26}$$

Moreover

$$[t_{3,0}, t_{0,n}] = 3n t_{2,n-1} + \frac{n(n-1)(n-2)}{4}\sigma_2 t_{0,n-3} + \frac{3\sigma_3}{2}\sum_{m=0}^{n-3}(m+1)(n-2-m) t_{0,m} t_{0,n-3-m}. \tag{5.27}$$

where

$$\sigma_2 = \epsilon_1^2 + \epsilon_2^2 + \epsilon_1\epsilon_2, \quad \sigma_3 = -\epsilon_1\epsilon_2(\epsilon_1 + \epsilon_2). \tag{5.28}$$

These relations determine all other commutation relations.

5.4 Other choices of generators

In the definition of $\mathsf{B}^{(K)}$, we use the symmetrization $e'^a_{b;n,m} = I^a \mathrm{Sym}(X^n Y^m) J_b, t'_{n,m} = \mathrm{Tr}\mathrm{Sym}(X^n Y^m)$ to define the universal-in-N generators, it turns out that non-symmetrized operators also exists in universal-in-N algebra $\mathsf{B}^{(K)}$.

Let us introduce some notation first. For an array $\mathbf{r}$ of binaries of length l, i.e. components of $\mathbf{r}$ are 0 or 1, define $\mathbf{r}(X,Y)$ to be the length-l letter such that at the i'th place is X if $\mathbf{r}_i = 0$ or Y if $\mathbf{r}_i = 1$. For example if $\mathbf{r} = (0,1,0)$ then $\mathbf{r}(X,Y) = XYX$. We write $|\mathbf{r}| = (n,m)$ if there are n zeroes and m ones in $\mathbf{r}$.

Lemma 5.4.1. *There exist a set of polynomials $f^a_{b;\mathbf{r}}$ and $g_{\mathbf{r}}$ of variables $e'^c_{d;p,q}, t'_{r,s} \in \mathsf{B}^{(K)}$ for $p,r < n$ and $q,s < m$ and $1 \le c,d \le K$, such that the image of $f^a_{b;\mathbf{r}}$ in $\mathsf{B}^{(K)}_N$ equals to $I^a \mathbf{r}(X,Y) J_b - I^a \mathrm{Sym}(X^n Y^m) J_b$, and the image of $g_{\mathbf{r}}$ in $\mathsf{B}^{(K)}_N$ equals to* $\mathrm{Tr}(\mathbf{r}(X,Y)) - \mathrm{Tr}\mathrm{Sym}(X^n Y^m)$. *Here $(n,m) = |\mathbf{r}|$.*

Proof. The statement obviously holds for $n = 0$. Now assume that the statement holds for all (n',m') such that $n' < n$. One can write

$$I^a \mathbf{r}(X,Y) J_b - I^a \mathrm{Sym}(X^n Y^m) J_b = \frac{1}{(m+n)!} \sum_{g \in \mathfrak{S}_{n+m}} I^a (\mathbf{r}(X,Y) - g \cdot \mathbf{r}(X,Y)) J_b,$$

as a sum over permutations $g \in \mathfrak{S}_{n+m}$, and $g \cdot \mathbf{r}(X,Y)$ means permuting the letter $\mathbf{r}(X,Y)$ using g. By decomposing g into product of permutations of neighboring letters, we only need to show that for any pair of arrays of binaries $\mathbf{r}_1, \mathbf{r}_2$ such that $|\mathbf{r}_1| + |\mathbf{r}_2| = (n-1, m-1)$, there exists polynomial $f^a_{b;\mathbf{r}_1,\mathbf{r}_2}$ of variables $e'^c_{d;p,q}, t'_{r,s} \in \mathsf{B}^{(K)}$ for $p,r < n$ and $q,s < m$ and $1 \le c,d \le K$, such that the image of $f^a_{b;\mathbf{r}_1,\mathbf{r}_2}$ in $\mathsf{B}^{(K)}_N$ equals to $I^a \mathbf{r}_1(X,Y)[X,Y]\mathbf{r}_2(X,Y)) J_b$. Using quantum moment map equation and commutation relations, we can rewrite it as

$$I^a \mathbf{r}_1(X,Y)[X,Y]\mathbf{r}_2(X,Y)) J_b = I^a \mathbf{r}_1(X,Y) J_c I^c \mathbf{r}_2(X,Y)) J_b - \epsilon_3 I^a \mathbf{r}_1(X,Y) \mathbf{r}_2(X,Y)) J_b,$$

thus the statement for $\mathbf{r}$ follows from induction. □

Definition 5.4.2. For an array $\mathbf{r}$ of binaries such that $|\mathbf{r}| = (n,m)$, define

$$e'^a_{b;\mathbf{r}} = e'^a_{b;n,m} + f^a_{b;\mathbf{r}}, \quad t'_{\mathbf{r}} = t'_{n,m} + g_{\mathbf{r}}.$$

Note that $e'^a_{b;\mathbf{r}}$ does not depend on the choice of $f^a_{b;\mathbf{r}}$, since the image of $e'^a_{b;\mathbf{r}}$ in $\mathsf{B}^{(K)}_N$ is $I^a \mathbf{r}(X,Y) J_b$ for all N, and the intersection of kernels of projections $\mathsf{B}^{(K)} \to \mathsf{B}^{(K)}_N$ is zero. Similarly $t'_{\mathbf{r}}$ does not depend on the choice of $g_{\mathbf{r}}$.

Lemma 5.4.1 shows that the coordinate transform between $e'^{a}_{b;\mathbf{r}}$ and $e'^{a}_{b;n,m}$ is triangular. For every $(n,m)\in\mathbb{N}^2$, choose an array of binaries $\mathbf{r}_{n,m}$ with $|\mathbf{r}_{n,m}|=(n,m)$, then $e'^{a}_{b;\mathbf{r}_{n,m}}$ and $t'_{\mathbf{r}_{n,m}}$ is a set of generators of $\mathsf{B}^{(K)}$, in fact fix an order for $e'^{a}_{b;\mathbf{r}_{n,m}}$, $(a,b)\neq(K,K)$ and $t'_{\mathbf{r}_{n,m}}$, then normal-ordered monomials in $e'^{a}_{b;\mathbf{r}_{n,m}}$ and $t'_{\mathbf{r}_{n,m}}$ is a basis of the free $\mathbb{C}[\epsilon_1,\epsilon_2]$-module $\mathsf{B}^{(K)}$.

5.5 $\mathcal{B}$-algebra and Yangian

Recall that if $A=\oplus_{i\in\mathbb{Z}}A^i$ is a $\mathbb{Z}$-graded algebra with homogeneous components A^i, then one can define a new algebra $\mathcal{B}(A)$, called the $\mathcal{B}$-algebra

$$\mathcal{B}(A)=A^0/\left(\sum_{i>0}A^i\cdot A^{-i}\right). \tag{5.29}$$

Note that if A is commutative, then $\mathrm{Spec}\mathcal{B}(A)=(\mathrm{Spec}A)^{\mathbb{C}^\times}$, where $\mathbb{C}^\times$-action on $\mathrm{Spec}A$ is induced from grading.

There is a natural grading on $\mathsf{B}^{(K)}_N$ by letting degree of X,I to be 1 and degree of Y,J to be -1. This grading is uniform in N, and we obtain a grading on $\mathsf{B}^{(K)}$ such that $\deg e'^{a}_{b;n,m}=\deg t'_{n,m}=n-m, \deg\epsilon_1=\deg\epsilon_2=0$. The following is conjectured by Costello [29, 2.3].

Theorem 5.5.1. *On the locus $\epsilon_2\neq 0$, there is an algebra isomorphism*

$$\mathcal{B}(\mathsf{B}^{(K)})\cong Y_{\epsilon_1}(\mathfrak{gl}_K)[\epsilon_2] \tag{5.30}$$

between $\mathcal{B}$-algebra of $\mathsf{B}^{(K)}$ and the Yangian of $\mathfrak{gl}_K$ freely joint with ϵ_2.

We prove the theorem as follows. First we construct a homomorphism from $Y_{\epsilon_1}(\mathfrak{gl}_K)$ to the degree zero piece of $\mathsf{B}^{(K)}_N$, such that the image of generators are uniform in N, so we obtain a homomorphism $Y_{\epsilon_1}(\mathfrak{gl}_K)\to\mathsf{B}^{(K)}$ with image in the degree zero piece. It will be transparent from construction that the composition $Y_{\epsilon_1}(\mathfrak{gl}_K)[\epsilon_2]\to\mathcal{B}(\mathsf{B}^{(K)})$ is bijective on the locus $\epsilon_2\neq 0$.

Let us construct the homomorphism from $Y_{\epsilon_1}(\mathfrak{gl}_K)$ to $\mathsf{B}^{(K)}_N$. Denote by $T^a_{b;n}$ the RTT generators of $Y_{\epsilon_1}(\mathfrak{gl}_K)$ satisfying commutation relations

$$[T^a_b(u),T^c_d(v)]=\frac{\epsilon_1}{u-v}\left(T^c_b(u)T^a_d(v)-T^c_b(v)T^a_d(u)\right), \tag{5.31}$$

where $T^a_b(u) = \delta^a_b + \sum_{n\geq 0} T^a_{b;n} u^{-n-1}$. The computation in [126] shows that the map

$$T^a_{b;n} \mapsto I^a (YX)^n J_b \tag{5.32}$$

extends to a $\mathbb{C}[\epsilon_1]$-algebra homomorphism from $Y_{\epsilon_1}(\mathfrak{gl}_K)$ to $\mathsf{B}^{(K)}_N$. Discussion in the last section shows that these maps are uniform in N and gives rise to the map

$$T^a_{b;n} \mapsto e'^a_{b;\mathbf{r}_n}, \tag{5.33}$$

where $\mathbf{r}_n$ is the length-$2n$ array $(1, 0, \cdots, 1, 0)$. This is a homomorphism from $Y_{\epsilon_1}(\mathfrak{gl}_K)$ to $\mathsf{B}^{(K)}$ with image in the degree zero piece.

Proof of Theorem 5.5.1. After inverting ϵ_2, we have $t'_{n,m} = e'^a_{a;n,m}/\epsilon_2$, so $\mathsf{B}^{(K)}[\epsilon_2^{-1}]$ is generated by $e'^a_{b;n,m}$. From the last section, $\mathsf{B}^{(K)}[\epsilon_2^{-1}]$ is generated by $e'^a_{b;n,m}$ $(n \neq m)$ and $e'^a_{b;\mathbf{r}_n}$. Consider the partial order on $\mathbb{N}^2$ such that $(n, m) \preceq (n', m')$ if $n - m \leq n' - m'$, and then refine it to a total order on the set of generators $e'^a_{b;n,m}$ and $e'^a_{b;\mathbf{r}_n}$. Then $\mathsf{B}^{(K)}[\epsilon_2^{-1}]$ has a basis consisting of decreasing-order monomials in $e'^a_{b;n,m}$ and $e'^a_{b;\mathbf{r}_n}$, in particular the degree zero component of $\mathsf{B}^{(K)}$ has a direct sum decomposition

$$\left(\bigoplus \mathbb{C}[\epsilon_1, \epsilon_2^{\pm 1}] \cdot \text{normal-ordered monomials in } e'^a_{b;\mathbf{r}_n}\right) \oplus \left(\bigoplus_{n>m} \mathbb{C}[\epsilon_1, \epsilon_2^{\pm 1}] \cdot e'^a_{b;n,m} \cdot (\cdots)\right).$$

Thus we see that $\mathcal{B}(\mathsf{B}^{(K)}[\epsilon_2^{-1}])$ is a free $\mathbb{C}[\epsilon_1, \epsilon_2^{\pm 1}]$-module with a basis consisting of normal-ordered monomials in $e'^a_{b;\mathbf{r}_n}$. Since the homomorphism $Y_{\epsilon_1}(\mathfrak{gl}_K)[\epsilon_2] \to \mathcal{B}(\mathsf{B}^{(K)})$ maps $T^a_{b;n}$ to $e'^a_{b;\mathbf{r}_n}$, and it is well-known that normal-ordered monomials in $T^a_{b;n}$ is a basis of Yangian, the theorem is proven. □

5.6 Relation to Poisson current algebra and Kac-Moody algebra

Define the traceless version of e generators

$$\bar{e}^a_{b;n,m} = e^a_{b;n,m} - \frac{\epsilon_2}{K} \delta^a_b t_{n,m},$$

then commutation relations in Lemma 5.1.3 and equations (5.6) remain the same form with e replaced by $\bar{e}$. Equations (5.7) and (5.8) are modified:

$$\begin{aligned}
[\bar{e}^a_{b;1,0}, \bar{e}^c_{d;0,n}] = {} & \delta^c_b \bar{e}^a_{d;1,n} - \delta^a_d \bar{e}^c_{b;1,n} + \frac{\epsilon_2\epsilon_3 n}{K} t_{0,n-1} \left(\frac{\delta^a_b \delta^c_d}{K} - \delta^a_d \delta^c_b \right) \\
& + \frac{n}{K}(\epsilon_3 \delta^c_d \bar{e}^a_{b;0,n-1} - \epsilon_2 \delta^a_b \bar{e}^c_{d;0,n-1}) - \frac{\epsilon_3 n}{2} \left(\delta^c_b \bar{e}^a_{d;0,n-1} + \delta^a_d \bar{e}^c_{b;0,n-1} \right) \\
& - \epsilon_1 \sum_{m=0}^{n-1} \frac{m+1}{n+1} \delta^a_d \bar{e}^c_{f;0,m} \bar{e}^f_{b;0,n-1-m} - \epsilon_1 \sum_{m=0}^{n-1} \frac{n-m}{n+1} \delta^c_b \bar{e}^a_{f;0,m} \bar{e}^f_{d;0,n-1-m} \\
& + \epsilon_1 \sum_{m=0}^{n-1} \bar{e}^a_{d;0,m} \bar{e}^c_{b;0,n-1-m}.
\end{aligned} \tag{5.34}$$

$$\begin{aligned}
[t_{3,0}, t_{0,n}] = {} & 3n t_{2,n-1} + \frac{n(n-1)(n-2)}{4} (\epsilon_1^2 - \epsilon_2\epsilon_3) t_{0,n-3} \\
& - \frac{3\epsilon_1}{2} \sum_{m=0}^{n-3} (m+1)(n-2-m) \left(\bar{e}^a_{c;0,m} \bar{e}^c_{a;0,n-3-m} + \frac{(2\epsilon_2 - \epsilon_3)\epsilon_2}{K} t_{0,m} t_{0,n-3-m} \right).
\end{aligned} \tag{5.35}$$

Define a filtration degree function on generators $\bar{e}, t$ by

$$\deg(\bar{e}^a_{b;n,m}) = n+m, \quad \deg(t_{n,m}) = n+m+2, \tag{5.36}$$

and define the degree of a monomial in $\bar{e}, t$ by the sum of degrees of its components. This induces a filtration on $\mathcal{A}^{(K)}$. For the convenience of later discussions, we write the $\bar{e}$ generators as $J^a_{n,m}$, where J^a are the $\mathfrak{sl}_K$ generators with structure constant f^{ab}_c and Killing form κ^{ab}.

Proposition 5.6.1. *Commutators in $\mathcal{A}^{(K)}$ can be schematically written as*

$$[J^a_{n,m}, J^b_{p,q}] = f^{ab}_c J^c_{n+p,m+q} - \kappa^{ab}(nq - mp) \frac{\epsilon_2\epsilon_3}{K} t_{n+p-1,m+q-1} + \textit{lower degree terms}, \tag{5.37}$$

$$[t_{n,m}, J^a_{p,q}] = (nq - mp) J^a_{n+p-1,m+q-1} + \textit{lower degree terms}. \tag{5.38}$$

$$[t_{n,m}, t_{p,q}] = (nq - mp) t_{n+p-1,m+q-1} + \textit{lower degree terms}. \tag{5.39}$$

Proof. The defining relations in Lemma 5.1.3 and equations (5.6), (5.34), and (5.35) are obviously of the above form. In particular (5.37) is satisfied for $(n, m) = (1, 0), p = 0$ and all q, (5.38) and (5.39) are satisfied for $m = p = 0$, $n \leq 3$ and all q. Using the adjoint action of $t_{2,0}$, we see that (5.37) is satisfied for $(n, m) = (1, 0)$ and all (p, q), and (5.38) and (5.39) are satisfied for $m = 0, n \leq 3$ and all (p, q). Then using the adjoint action of $t_{0,2}$, we see that (5.37) is satisfied for $n + m \leq 1$ and all (p, q), and (5.38) and (5.39) are satisfied for $m + n \leq 3$ and all (p, q). Next we proceed by induction. Assume that (5.37), (5.38) and (5.39) are satisfied for $(n + m) \leq s$ and all (p, q), then

$$
\begin{aligned}
[J^a_{s+1,0}, J^b_{p,q}] &= -\frac{1}{s}[[t_{2,1}, J^a_{s,0}], J^b_{p,q}] = -\frac{1}{s}([t_{2,1}, [J^a_{s,0}, J^b_{p,q}]] - [J^a_{s,0}, [t_{2,1}, J^b_{p,q}]]) \\
&= -\frac{1}{s}[t_{2,1}, f^{ab}_c J^c_{s+p,q} - \kappa^{ab} sq \frac{\epsilon_2\epsilon_3}{K} t_{s+p-1,q-1} + \text{lower degree terms}] \\
&\quad + \frac{1}{s}[J^a_{s,0}, (2q-p) J^b_{p+1,q} + \text{lower degree terms}] \\
&= f^{ab}_c J^c_{s+1+p,q} - \kappa^{ab}(s+1) q \frac{\epsilon_2\epsilon_3}{K} t_{s+p,q-1} + \text{lower degree terms},
\end{aligned}
$$

so (5.37) is satisfied for $(n, m) = (s+1, 0)$ and all (p, q). Using the adjoint action of $t_{0,2}$, we see that (5.37) is satisfied for $n + m \leq s + 1$ and all (p, q). By induction on s, we see that (5.37) is satisfied for m, n, p, q. The other two equations is proven using similar induction argument. □

From the above description of commutation relations, we immediately see that the associated graded algebra is related to the Poisson current algebra, defined as the universal envelope of $\mathfrak{gl}_K \otimes \mathcal{O}(\mathbb{C}^2)$, where $\mathbb{C}^2$ is endowed with a Poisson bracket.

Corollary 5.6.2. *The associated graded algebra* $\operatorname{gr} \mathcal{A}^{(K)}$ *with respect to the filtration* (5.36) *is the universal enveloping algebra of the Lie algebra* $\mathfrak{gl}_K \otimes \mathcal{O}(\mathbb{C}^2)$ *with Lie brackets*

$$
[A \otimes f, B \otimes g] = [A, B] \otimes fg + (A, B)_{-\frac{\epsilon_2\epsilon_3}{K},1}\{f, g\}, \tag{5.40}
$$

where $A, B \in \mathfrak{gl}_K$*, and* $(A, B)_{-\frac{\epsilon_2\epsilon_3}{K},1}$ *is the symmetric form that equals to* $-\frac{\epsilon_2\epsilon_3}{K}\kappa$ *on* $\mathfrak{sl}_K$ *and* 1 *on* $\mathfrak{gl}_1$*, and* $f, g \in \mathcal{O}(\mathbb{C}^2)$ *with Poisson bracket* $\{z, w\} = 1$*.* $J^a_{n,m}$ *is mapped to* $J^a \otimes z^n w^m$ *and* $t_{n,m}$ *is mapped to* $1_K \otimes z^n w^m$*.*

Since $t_{0,0}$ is central in $\mathcal{A}^{(K)}$, we can regard $\mathcal{A}^{(K)}$ as a $\mathbb{C}[\epsilon_1, \epsilon_2, t_{0,0}]$-algebra, and it is free as a $\mathbb{C}[\epsilon_1, \epsilon_2, t_{0,0}]$-module. Let us add the inverse square root $t_{0,0}^{-\frac{1}{2}}$ to the algebra $\mathcal{A}^{(K)}$, and consider the $\mathbb{C}[\epsilon_1, \epsilon_2, t_{0,0}^{-\frac{1}{2}}]$-subalgebra $\widetilde{\mathcal{A}}^{(K)} \subset \mathcal{A}^{(K)}[t_{0,0}^{-\frac{1}{2}}]$ generated by

$$
\tilde{J}^a_{n,m} = t_{0,0}^{-\frac{n+m}{2}} J^a_{n,m}, \quad \tilde{t}_{n,m} = t_{0,0}^{-\frac{n+m}{2} - \delta_{n,m}} t_{n,m},
$$

then Proposition 5.6.1 implies that

Corollary 5.6.3. *$\widetilde{\mathcal{A}}^{(K)}$ is a free $\mathbb{C}[\epsilon_1, \epsilon_2, t_{0,0}^{-\frac{1}{2}}]$-module, and $\widetilde{\mathcal{A}}^{(K)}/(t_{0,0}^{-\frac{1}{2}})$ is the universal enveloping algebra of the Lie algebra $\mathcal{O}(\mathbb{C}^2) \otimes \mathfrak{gl}_K/\mathbb{C} \cdot 1_K$ with Lie bracket*

$$[A \otimes f, B \otimes g] = [A, B] \otimes fg + (A, B)_{-\frac{\epsilon_2\epsilon_3}{K},1}\pi^*\left(\oint_{|z|=1} \{f, g\}\frac{dz}{2\pi iz}\right), \tag{5.41}$$

where $\pi : \mathbb{C}^2 \to \mathbb{C}, (z, w) \mapsto zw$, and $\oint_{|z|=1}$ maps an one-form on $\mathbb{C}^2$ to a function on $\mathbb{C}$. $\tilde{J}^a_{n,m}$ is mapped to $J^a \otimes z^n w^m$ and $\tilde{t}_{n,m}$ is mapped to $1_K \otimes z^n w^m$.

Moreover, there is a surjective Lie algebra map from $\mathcal{O}(\mathbb{C}^2) \otimes \mathfrak{gl}_K/\mathbb{C} \cdot 1_K$ to the affine Lie algebra $\widehat{\mathfrak{gl}}(K)_{-\frac{\epsilon_2\epsilon_3}{K},1}$, where the latter has generators J^a_n, α_m $(m \neq 0)$, and $\mathbf{c}$, and Lie brackets $[J^a_n, J^b_m] = f^{ab}_c J^c_{n+m} - n\delta_{n,-m}\frac{\epsilon_2\epsilon_3}{K}\kappa^{ab}\mathbf{c}$, $[\alpha_n, \alpha_m] = n\delta_{n,-m}\mathbf{c}$, and other brackets are zero. The map is given by

$$\tilde{J}^a_{n,m} \mapsto J^a_{n-m}, \quad \tilde{t}_{n,m} \mapsto \alpha_{n-m}\ (n \neq m), \quad \tilde{t}_{n,n} \mapsto \frac{1}{n+1}\mathbf{c}.$$

Proof. Proposition 5.6.1 implies that the commutators are schematically of the form

$$\begin{aligned}[A_{n,m}, B_{p,q}] =& [A, B]_{n+p,m+q} + \delta_{n+p,m+q}(A, B)_{-\frac{\epsilon_2\epsilon_3}{K},1}(nq - mp)\tilde{t}_{n+p-1,m+q-1} \\ &+ t_{0,0}^{-\frac{1}{2}} \cdot (\text{Polynomial in } t_{0,0}^{-\frac{1}{2}}).\end{aligned}$$

where A, B are $\tilde{J}$ or $\tilde{t}$ and we regard $\tilde{t}$ as the diagonal $\mathfrak{gl}_1$ part of $\mathfrak{gl}_K$. Thus $\widetilde{\mathcal{A}}^{(K)}/(t_{0,0}^{-\frac{1}{2}})$ is the universal enveloping algebra of $\mathcal{O}(\mathbb{C}^2) \otimes \mathfrak{gl}_K/\mathbb{C} \cdot 1_K$ with the Lie brackets (5.41).

For the freeness of $\widetilde{\mathcal{A}}^{(K)}$, fix an order for the generators $A_{n,m}$ and we can form a putative basis of $\widetilde{\mathcal{A}}^{(K)}$ given by the normal-ordered monomials in $A_{n,m}$. Note that normal-ordered monomials form a basis $\widetilde{\mathcal{A}}^{(K)}/(t_{0,0}^{-\frac{1}{2}})$ by PBW theorem. In particular they generate $\mathcal{A}^{(K)}$ as $\mathbb{C}[\epsilon_1, \epsilon_2, t_{0,0}^{-\frac{1}{2}}]$-module by the graded Nakayama lemma. They also form a basis when localized to $\mathbb{C}[t_{0,0}^{\pm\frac{1}{2}}]$, because after localization $\widetilde{\mathcal{A}}^{(K)}$ is the same as $\mathcal{A}^{(K)}$ of which the set of normal-ordered monomials form a basis. In particular normal-ordered monomials are linear independent over $\mathbb{C}[\epsilon_1, \epsilon_2, t_{0,0}^{-\frac{1}{2}}]$. This shows the freeness of $\mathcal{A}^{(K)}$.

The surjective map to $\widehat{\mathfrak{gl}}(K)_{-\frac{\epsilon_2\epsilon_3}{K},1}$ follows directly by computation. □

5.7 Meromorphic coproduct

Consider the rational map $\mathbb{C}^{N_1}_{\text{disj}} \times \mathbb{C}^{N_2}_{\text{disj}} \dashrightarrow \mathbb{C}^{N_1+N_2}_{\text{disj}}$ sending $(y^{(1)}_1, \cdots, y^{(1)}_{N_1}) \times (y^{(2)}_1, \cdots, y^{(2)}_{N_2})$ to $(y^{(1)}_1, \cdots, y^{(1)}_{N_1}, y^{(2)}_1, \cdots, y^{(2)}_{N_2})$. This is not a globally-defined map since $y^{(1)}_i$ might collide with $y^{(2)}_j$. Alternatively, one can consider the parametrized version of the above rational map $m : \mathbb{C}^{N_1}_{\text{disj}} \times \mathbb{C}^{N_2}_{\text{disj}} \times \mathbb{P}^1 \dashrightarrow \mathbb{C}^{N_1+N_2}_{\text{disj}}$ sending $(y^{(1)}_1, \cdots, y^{(1)}_{N_1}) \times (y^{(2)}_1, \cdots, y^{(2)}_{N_2}) \times (w)$ to $(y^{(1)}_1 + w, \cdots, y^{(1)}_{N_1} + w, y^{(2)}_1, \cdots, y^{(2)}_{N_2})$, where w is the coordinate on $\mathbb{P}^1$. Then the non-defined loci for m on $\mathbb{C}^{N_1}_{\text{disj}} \times \mathbb{C}^{N_2}_{\text{disj}} \times \mathbb{P}^1$ is union of hyperplanes $y^{(1)}_i + w = y^{(2)}_j$ and the infinity divisor $w = \infty$. Since the hyperplanes do not intersect with the infinity divisor, we can take the formal neighborhood of $w = \infty$ and localize to get a genuine map

$$m : \mathbb{C}^{N_1}_{\text{disj}} \times \mathbb{C}^{N_2}_{\text{disj}} \times \mathrm{Spec}\mathbb{C}(\!(w^{-1})\!) \to \mathbb{C}^{N_1+N_2}_{\text{disj}}.$$

It maps function ring $\mathbb{C}[y^{(1)}_i, y^{(2)}_j, (y^{(1)}_{i_1} - y^{(1)}_{i_2})^{-1}, (y^{(2)}_{j_1} - y^{(2)}_{j_2})^{-1}, (y^{(1)}_i - y^{(2)}_j)^{-1}]$ to $\mathbb{C}[y^{(1)}_i, y^{(2)}_j, (y^{(1)}_{i_1} - y^{(1)}_{i_2})^{-1}, (y^{(2)}_{j_1} - y^{(2)}_{j_2})^{-1}](\!(w^{-1})\!)$ by

$$\begin{gathered} y^{(1)}_i \mapsto y^{(1)}_i + w, \quad y^{(2)}_j \mapsto y^{(2)}_j, \\ \frac{1}{y^{(1)}_{i_1} - y^{(1)}_{i_2}} \mapsto \frac{1}{y^{(1)}_{i_1} - y^{(1)}_{i_2}}, \quad \frac{1}{y^{(2)}_{j_1} - y^{(2)}_{j_2}} \mapsto \frac{1}{y^{(2)}_{j_1} - y^{(2)}_{j_2}} \\ \frac{1}{y^{(1)}_i - y^{(2)}_j} \mapsto \sum_{n=0}^{\infty} w^{-n-1} (y^{(2)}_j - y^{(1)}_i)^n. \end{gathered} \tag{5.42}$$

We call such map a *meromorphic coproduct*, denoted by $\Delta(w)_{N_1,N_2}$. It is coassociative in the obvious sense, in fact it satisfies a more basic property:

Lemma 5.7.1. *Meromorphic coproducts are **local** in the sense that, if we decompose $N = N_1 + N_2 + N_3$ into three clusters, then for any $f \in \mathcal{O}(\mathbb{C}^N_{\text{disj}})$, two elements*

$$(\mathrm{Id} \otimes \Delta(w)_{N_2,N_3})\Delta(z)_{N_1,N_2+N_3} f, \quad (P \otimes \mathrm{Id})(\mathrm{Id} \otimes \Delta(z)_{N_1,N_3})\Delta(w)_{N_2,N_1+N_3} f,$$

are expansions of the same element in $\mathcal{O}(\mathbb{C}^{N_1}_{\text{disj}} \times \mathbb{C}^{N_2}_{\text{disj}} \times \mathbb{C}^{N_3}_{\text{disj}})[\![z^{-1}, w^{-1}, (z-w)^{-1}]\!][z, w]$, where $P : \mathcal{O}(\mathbb{C}^{N_2}_{\text{disj}}) \otimes \mathcal{O}(\mathbb{C}^{N_1}_{\text{disj}}) \to \mathcal{O}(\mathbb{C}^{N_1}_{\text{disj}}) \otimes \mathcal{O}(\mathbb{C}^{N_2}_{\text{disj}})$ is the permutation operator.

Proof. After taking two-step meromorphic coproduct, $y^{(1)}_i \mapsto y^{(1)}_i + z, y^{(2)}_j \mapsto y^{(2)}_j + w, y^{(3)}_k \mapsto y^{(3)}_k$, and those $(y_i - y_j)^{-1}$ are mapped accordingly and then expanded in power series. Thus we immediately see that $\Delta(w)_{N_2,N_3}\Delta(z)_{N_1,N_2+N_3} f$ and $\Delta(z)_{N_1,N_3}\Delta(w)_{N_2,N_1+N_3} f$ are expansions of the same rational function. □

The meromorphic coproduct can be defined for (twisted) differential operators as well, i.e. there exists

$$\Delta(w)_{N_1,N_2} : D^{\epsilon_2}_{\epsilon_1}((\mathbb{P}^{K-1}\times\mathbb{C})^{N_1+N_2}_{\text{disj}}) \to D^{\epsilon_2}_{\epsilon_1}((\mathbb{P}^{K-1}\times\mathbb{C})^{N_1}_{\text{disj}}) \otimes D^{\epsilon_2}_{\epsilon_1}((\mathbb{P}^{K-1}\times\mathbb{C})^{N_2}_{\text{disj}})(\!(w^{-1})\!),$$

which also satisfies the locality in the Lemma 5.7.1. Restricted to the image of $\mathsf{B}^{(K)}_N$ in $D^{\epsilon_2}_{\epsilon_1}((\mathbb{P}^{K-1}\times\mathbb{C})^{N}_{\text{disj}})$, we obtain a formula for the meromorphic coproduct for the generators of $\mathsf{B}^{(K)}_N$:

$$\Delta(w)(e^a_{b;0,n}) = 1\otimes e^a_{b;0,n} + \sum_{m=0}^{n}\binom{n}{m} w^{n-m} e^a_{b;0,m}\otimes 1,$$

$$\Delta(w)(t_{2,0}) = 1\otimes t_{2,0} + t_{2,0}\otimes 1 - 2\epsilon_1 \sum_{m,n\geq 0}\frac{(m+n+1)!}{m!n!w^{n+m+2}}(-1)^n(e^a_{b;0,n}\otimes e^b_{a;0,m} + \epsilon_1\epsilon_2 t_{0,n}\otimes t_{0,m}). \tag{5.43}$$

From the above formulae we see $\Delta(w)_{N_1,N_2}$ maps $\mathsf{B}^{(K)}_{N_1+N_2}$ to $\mathsf{B}^{(K)}_{N_1}\otimes\mathsf{B}^{(K)}_{N_2}(\!(w^{-1})\!)$, and the coproduct formulae are independent of N_1, N_2, thus the family $\Delta(w)_{N_1,N_2}$ induces a uniform-in-N meromorphic coproduct $\Delta(w) : \mathsf{B}^{(K)} \to \mathsf{B}^{(K)}\otimes\mathsf{B}^{(K)}(\!(w^{-1})\!)$ and similarly a meromorphic coproduct for $\mathcal{A}^{(K)}$. It turns out the uniform-in-N locality for the meromorphic coproduct can be put into more general framework called the *vertex coalgebra*.

Recall that a vertex coalgebra [96] is a vector space V together with linear maps

- Coproduct $\Delta(w) : V \to V\otimes V(\!(w^{-1})\!)$, and write $\Delta(w)v = \sum_{n\in\mathbb{Z}}\Delta_n(v)w^{-n-1}$,
- Covacuum $\mathfrak{C} : V \to \mathbb{C}$,

satisfying the following axioms:

(1) Left counit: $\forall v\in V$,

$$(\mathfrak{C}\otimes \mathrm{Id})\Delta(w)v = v.$$

(2) Cocreation: $\forall v\in V$,

$$(\mathrm{Id}\otimes\mathfrak{C})\Delta(w)v \in V[w], \text{ and } \lim_{w\to 0}(\mathrm{Id}\otimes\mathfrak{C})\Delta(w)v = 0.$$

(3) Translation: let $T = (\mathrm{Id}\otimes\mathfrak{C})\Delta_{-2}$, then

$$\frac{\mathrm{d}}{\mathrm{d}w}\Delta(w) = \Delta(w)T - (\mathrm{Id}\otimes T)\Delta(w).$$

(3) Locality: $\forall v \in V$, two elements

$$(\mathrm{Id} \otimes \Delta(w))\Delta(z)v, \quad (P \otimes \mathrm{Id})(\mathrm{Id} \otimes \Delta(z))\Delta(w)v,$$

are expansions of the same element in $(V \otimes V \otimes V)[\![z^{-1}, w^{-1}, (z-w)^{-1}]\!][z, w]$.

We can similarly define vertex coalgebra over some base ring R.

Proposition 5.7.2. *The meromorphic coproduct induce vertex coalgebra structures on $\mathcal{A}^{(K)}$ and $\mathsf{B}^{(K)}$ over the base ring $\mathbb{C}[\epsilon_1, \epsilon_2]$.*

Proof. We prove the theorem for $\mathcal{A}^{(K)}$, and the proof for $\mathsf{B}^{(K)}$ is analogous. Let us define the covacuum $\mathfrak{C} : \mathcal{A}^{(K)} \to \mathbb{C}[\epsilon_1, \epsilon_2]$ by mapping on generators $\mathfrak{C}(t_{n,m}) = \mathfrak{C}(e^a_{b;n,m}) = 0$ and extending it to an algebra map. Then the left counit and cocreation axioms are easily checked for (5.43), thus these two axioms are satisfied for all elements in $\mathcal{A}^{(K)}$ since $\mathfrak{C} \otimes \mathrm{Id}$ and $\mathrm{Id} \otimes \mathfrak{C}$ are algebra homomorphisms. It remains to check the translation axiom.

Note that the operator $T = (\mathrm{Id} \otimes \mathfrak{C})\Delta_{-2} : \mathcal{A}^{(K)} \to \mathcal{A}^{(K)}$ is a derivation, since we can write

$$T = \lim_{w \to 0} \frac{\mathrm{d}}{\mathrm{d}w}(\mathrm{Id} \otimes \mathfrak{C})\Delta(w),$$

and the derivative operator is a derivation. Since $T(t_{2,0}) = 0$ and $T(e^a_{b;0,n}) = ne^a_{b;0,n-1}$, we conclude that T is the same as the adjoint action of $t_{1,0}$. For finite N, the operator T exists, in fact $t_{1,0}$ is mapped to $\sum_{i=1}^N \partial_{y_i}$ in $D^{\epsilon_2}_{\epsilon_1}((\mathbb{P}^{K-1} \times \mathbb{C})^N_{\mathrm{disj}})$, thus

$$\begin{aligned}
&\Delta(w)_{N_1,N_2} T(f(y^{(1)}_i, y^{(2)}_j)) - (\mathrm{Id} \otimes T)\Delta(w)_{N_1,N_2} f(y^{(1)}_i, y^{(2)}_j) \\
&= \left(\sum_{k=1}^{N_1} \frac{\partial f}{\partial y^{(1)}_k}\right)(y^{(1)}_i + w, y^{(2)}_j) + \left(\sum_{k=1}^{N_2} \frac{\partial f}{\partial y^{(2)}_k}\right)(y^{(1)}_i + w, y^{(2)}_j) - \sum_{k=1}^{N_2} \partial_{y^{(2)}_k}\left(f(y^{(1)}_i + w, y^{(2)}_j)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}w} f(y^{(1)}_i + w, y^{(2)}_j),
\end{aligned}$$

for all functions f on $\mathbb{C}^{N_1+N_2}_{\mathrm{disj}}$, and this equation extends to hold for differential operators on $(\mathbb{P}^{K-1} \times \mathbb{C})^N_{\mathrm{disj}}$ by linearity. In particular, the translation axiom is satisfied for all $\Delta(w)_{N_1,N_2}$, and it is therefore satisfied for the uniform-in-N coproduct $\Delta(w)$. □

Chapter 6

Conclusion

Throughout this thesis, we concretely demonstrated in examples of 4d and 5d Chern-Simons theories, the computation of the algebra of local observables in the boundary, and the algebra of bulk-boundary universal coupling or scattering process. These examples serve as toy models for the twisted holography duality.

The example of 4d Chern-Simons theory is a (quasi) topological subsector of the physical AdS/CFT correspondence between 4d $\mathcal{N} = 4$ super Yang-Mills theory with a domain wall and Type IIB supergravity on $\mathrm{AdS}_5 \times S^5$ with a probing D5-brane. The relation is given by a certain topological-holomorphic twist and Ω-deformation (§2.5.2).

The example of 5d Chern-Simons theory is a (quasi) topological subsector of the physical AdS/CFT correspondence between M2-brane SCFT and M-theory on $\mathrm{AdS}_4 \times S^7$ background. The $\mathbb{C}_{\epsilon_1} \times \mathrm{TN}_K^{\epsilon_2,\epsilon_3}$ on which the Ω-background is turned on can be regarded as the transverse direction of an $\mathrm{AdS}_2 \times S^3$ inside $\mathrm{AdS}_4 \times S^7$, and the $\mathrm{AdS}_2 \times S^3$ is the back-reacted geometry of the line defect in $\mathbb{R} \times \mathbb{C}^2_{\mathrm{NC}}$. Note that the space of protected sphere correlation functions in the M2 brane SCFT is identified with the space of twisted traces of $\mathcal{A}^{(K)}$ [65], it would be interesting to look into the space of twisted traces of $\mathcal{A}^{(K)}$, which might help to understand better the structure of protected sphere correlation functions.

In the algebraic studies of M2 brane algebra $\mathcal{A}^{(K)}$ (Chapter §5), some questions and conjectures of Costello are answered and proved. One interesting fact we would like to point out is that the algebra $\mathcal{A}^{(K)}$ has a degeneration limit which is isomorphic to the $\mathfrak{gl}_K$ Kac-Moody algebra (Corollary 5.6.3), this completes the derivation of the emergence of Kac-Moody algebra in the matrix model studied by Dorey *et al* [56].

An important lesson that we learn from these studies is

- The perturbative method (Feynman diagram or Witten diagram) is powerful, one could in principle compute the algebra of local observables in arbitrary order in $\hbar$. However, as we have seen in our computation, the perturbative method has its limit that as the loop order grows, the complication of integrals increases drastically. In practice, one should not expect to compute the whole algebra unless there are small-loop exactness result, like the uniqueness of deformation ([24, Theorem 12.1.1] and [29, Theorem 16.0.1]).

The large-N presentation of the algebra of boundary observables, on the other hand, can be studied by algebraic or algbro-geometric method, at least in the examples that we studied in this thesis. One could argue that this is partly because our examples are well-engineered such that the models manifest connections to mathematics. In fact, the connection to mathematics can be traced back to the topological/holomorphic nature of these models, and one could imagine this feature to present in other examples of twisted holography. Here is one examples in this "wishing-list":

- The bulk theory is 6d holomorphic Chern-Simons theory coupled to the BCOV theory, one could put a holomorphic line (2d) defect and ask for the universal chiral algebra that can be coupled to the bulk theory. This universal chiral algebra is interpreted as the Koszul dual to the bulk algebra of observables. The mode algebra of this chiral algebra is some kind of three-parameter quantum group, generalizing the dictionary that in 4d Chern-Simons we get Yangian which is an one-parameter quantum group, and in 5d Chern-Simons we get affine Yangian which is a two-parameter quantum group.

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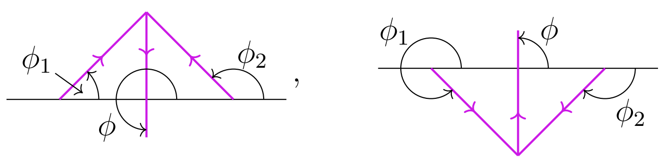
APPENDICES

Appendix A

Techniques in 4d Chern-Simons and 2d BF Perturbative Computations

A.1 Integrating the BF interaction vertex

In this appendix we evaluate the integrals in (2.73).



(A.1)

We split up each integral into two, based on whether the bulk point is above or below the line operator. We use angular coordinates defined as in the above diagrams. One subtlety is that, from the definition of the propagators in the Cartesian coordinate we can see that the integrand[1] is even under reflection with respect to the line. So, we just have to make sure that when we divide up the integral in the aforementioned way, even when written in angular coordinates, the integrand does not change sign under reflection. With this in

[1]including the measure

mind, the integrals we have to evaluate are:

$$\mathcal{V}^{\alpha\beta\gamma}_{\cdot||}(x_1,x_2) = \frac{\hbar^2}{(2\pi)^3} f^{\alpha\beta\gamma} \int_0^{2\pi} \mathrm{d}\phi_1 \int_{\phi_1}^{\pi} \mathrm{d}\phi_2 \left(\int_{\pi}^{\phi_1+\pi} \mathrm{d}\phi + \int_{\pi}^{\phi_1-\pi} \mathrm{d}\phi \right) ,$$
$$\mathcal{V}^{\alpha\beta\gamma}_{|\cdot|}(x_1,x_2) = \frac{\hbar^2}{(2\pi)^3} f^{\alpha\beta\gamma} \int_0^{2\pi} \mathrm{d}\phi_1 \int_{\phi_1}^{\pi} \mathrm{d}\phi_2 \left(\int_{\phi_1+\pi}^{\phi_2+\pi} \mathrm{d}\phi + \int_{\phi_1-\pi}^{\phi_2-\pi} \mathrm{d}\phi \right) ,$$
$$\mathcal{V}^{\alpha\beta\gamma}_{||\cdot}(x_1,x_2) = \frac{\hbar^2}{(2\pi)^3} f^{\alpha\beta\gamma} \int_0^{2\pi} \mathrm{d}\phi_1 \int_{\phi_1}^{\pi} \mathrm{d}\phi_2 \left(\int_{\phi_2+\pi}^{2\pi} \mathrm{d}\phi + \int_{\phi_2-\pi}^{0} \mathrm{d}\phi \right) .$$

All three terms are equal to $\frac{\hbar^2}{24} f^{\alpha\beta\gamma}$.

A.2 Quantum Mechanical Hilbert Spaces

A.2.1 Fermionic

The quantum mechanical action (2.16) is written in terms of fermions $\overline{\psi}$ and ψ that transform under $\mathrm{GL}_N \times \mathrm{GL}_K$ according to the representations $V := \overline{\mathbf{N}} \otimes \mathbf{K}$ and $\overline{V} := \mathbf{N} \otimes \overline{\mathbf{K}}$ respectively. The kinetic term in the action is first order in derivative, which establishes $\overline{\psi}$ and ψ as canonically conjugate variables, in other words, the phase space of the QM is:

$$V \oplus \overline{V} = T^* V \,. \tag{A.2}$$

The Hilbert space of this theory can now be written as the space of functions on V – since V is a *fermionic* vector space, functions on this space can be written as *anti-symmetric* polynomials in the dual vectors:

$$\mathcal{H}^{\text{fer}} = \mathscr{O}(V) = \wedge^{\bullet}(\overline{V}) \,. \tag{A.3}$$

Let us look at the anti-symmetric polynomials of degree n, which can be defined as the subspace of $\overline{V}^{\otimes n}$ where S_n acts by sign – S_n being the permutation group of n objects:

$$\wedge^n \left(\overline{V}^{\otimes n} \right) = \mathrm{Hom}_{S_n} \left(\varepsilon, \overline{V}^{\otimes n} \right) \cong \mathrm{Hom}_{S_n} \left(\varepsilon, \mathbf{N}^{\otimes n} \otimes \overline{\mathbf{K}}^{\otimes n} \right) \,. \tag{A.4}$$

Here ε is the one dimensional sign representation of the symmetric group S_n. Using *Schur-Weyl duality* we can decompose spaces such as $\mathbf{N}^{\otimes n}$ into irreducible representations of $S_n \times \mathrm{GL}_N$:

$$\wedge^n \left(\overline{V}^{\otimes n} \right) = \bigoplus_{|Y|=|Y'|=n} \mathrm{Hom}_{S_n} \left(\varepsilon, \pi_Y \otimes \mathcal{H}^N_Y \otimes \pi_{Y'} \otimes \overline{\mathcal{H}^K_{Y'}} \right) , \tag{A.5}$$

where Y and Y' are Young tableau, the sum is over tableau containing n boxes, π_Y is the irreducible representation of S_n parametrized by the tableaux Y, $\mathcal{H}^K_{Y'}$ is the irreducible representation of GL_K parametrized by the tableaux Y' and $\overline{\mathcal{H}^K_{Y'}}$ is its dual. Since we are computing S_n equivariant Hom, we can focus on the S_n representations:

$$\mathrm{Hom}_{S_n}(\varepsilon, \pi_Y \otimes \pi_{Y'}) \cong \mathrm{Hom}_{S_n}(\varepsilon \otimes \overline{\pi_Y}, \pi_{Y'}) = \mathrm{Hom}_{S_n}(\varepsilon \otimes \pi_Y, \pi_{Y'}) \,, \tag{A.6}$$

where we have used the fact that representations of S_n are self-dual. Now, tensoring with the sign representation exchanges the role of rows and columns in a Yaoung tableau parametrizing a representation of S_n, and by Schur's lemma, there is exactly one (up to scalar multiples) map of representations between two irreducible representations if they are isomorphic and no such map if they are not. These two facts tell us that:

$$\mathrm{Hom}_{S_n}(\varepsilon \otimes \pi_Y, \pi_{Y'}) = \delta_{Y^T, Y'}\mathbb{C} \,, \tag{A.7}$$

where Y^T denotes the transpose of the tableaux Y. This leaves just one sum in (A.5):

$$\wedge^n \left(\overline{V}^{\otimes n}\right) = \bigoplus_{|Y|=n} \mathcal{H}^N_{Y^T} \otimes \overline{\mathcal{H}^K_Y} \,. \tag{A.8}$$

The full fermionic Hilbert space (A.3) is then the following sum:

$$\mathcal{H}^{\mathrm{fer}} = \bigoplus_{Y} \mathcal{H}^N_{Y^T} \otimes \overline{\mathcal{H}^K_Y} \,. \tag{A.9}$$

Note that, this is a finite sum, since the tableau Y can have at most K rows and at most N columns – this is of course a consequence of exclusion principle for fermions.

A.2.2 Bosonic

Let us replace the fermions in the action (2.16) with bosons and change nothing else. Representations of the bosons are the same as their fermionic counterpart and therefore we still have the phase space $T^*(V)$ where $V = \overline{\mathbf{N}} \oplus \mathbf{K}$. The difference, compared to the fermionic case, is that the Hilbert space now consists of *symmetric* polynomials in $\overline{V}$ (c.f. (A.3)):

$$\mathcal{H}^{\mathrm{bos}} = \mathrm{Sym}^\bullet(\overline{V}) \,. \tag{A.10}$$

Then, instead of (A.4) we have:

$$\mathrm{Sym}^n \left(\overline{V}^{\otimes n}\right) = \mathrm{Hom}_{S_n} \left(\mathbb{C}, \mathbf{N}^{\otimes n} \otimes \overline{\mathbf{K}}^{\otimes n}\right) \,, \tag{A.11}$$

where $\mathbb{C}$ is the trivial representation of S_n. Following a similar computation as we did for the fermionic case we now end up with the following Hom between representations of S_n (c.f. (A.7)):

$$\mathrm{Hom}_{S_n}(\pi_Y, \pi_{Y'}) = \delta_{Y,Y'}\mathbb{C}\,, \tag{A.12}$$

which leads to the following description of the bosonic Hilbert space:

$$\mathcal{H}^{\mathrm{bos}} = \bigoplus_Y \mathcal{H}_Y^N \otimes \overline{\mathcal{H}_Y^K}\,. \tag{A.13}$$

Note that, as opposed to the fermionic case, we now have no restriction on the number of columns of Y (number of rows is restricted to be at most $\min(N, K)$) and therefore the Hilbert space is infinite dimensional, as expected given the lack of any exclusion principle for bosons.

A.3 Yangian from 1-loop Computations

At the end of §2.4.6, by computing 1-loop diagrams, we concluded that quantum corrections deform the coalgebra structure of the classical Hopf algebra $U(\mathfrak{gl}_K[z])$. Since $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ is an algebra to begin with, we conclude that at one loop, we have a deformation of the classical algebra as a Hopf algebra. We are using the term "deformation" (alternatively, "quantization") in the sense of Definition 6.1.1 of [24], which essentially means that:

- $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bd}})$ becomes the classical algebra $U(\mathfrak{gl}_K[z])$ in the classical limit $\hbar \to 0$.
- $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ is isomorphic to $U(\mathfrak{gl}_K[z])[\![\hbar]\!]$ as a $\mathbb{C}[\![\hbar]\!]$-module.
- $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ is a *topological* Hopf algebra (with respect to $\hbar$-adic topology).

The reason that we adhere to these conditions is that, there is a well known uniqueness theorem (Theorem 12.1.1 of [24]) which says that the Yangian is the unique deformation of $U(\mathfrak{gl}_K[z])$ in the above sense. Therefore, if we can show that our algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ satisfies all these conditions and it is a nontrivial deformation of $U(\mathfrak{gl}_K)$ then we can conclude that it is the Yangian. From 1-loop computations we already know that it is a non-trivial deformation. That the first condition in the list above is satisfied is the content of Lemma 2.4.5. The second condition is satisfied because $\hbar$ acts on the generators of our algebra by simply multiplying the external propagators by $\hbar$ in the relevant Witten diagrams, this action does not distinguish between classical diagrams and higher loop diagrams. Satisfying the last condition is less trivial. While it seems known to people working in the field, we were unable to find a reference to cite, therefore, for the sake of completion, we provide a proof in this appendix, that the algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ is indeed an ($\hbar$-adic)*topological* Hopf algebra.

We shall prove this by reconstructing the algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ from its representations. As mentioned in §2.2.5, representations of this algebra are carried by Wilson lines, which form an abelian monoidal category. A morphism between two representations V and U in this category is constructed by computing the expectation value of two Wilson lines in representations U and $V^\vee$ and providing a state at one end of each of the lines. For example, if ϱ and ϱ' are two homomorphisms from $\mathfrak{gl}_K$ to U and $V^\vee$ respectively, then for two lines L and L' in the topological plane of the CS theory and any $\psi \otimes \chi^\vee \in U \otimes V^\vee$, the expectation value $\langle W_\varrho(L) W_{\varrho'}(L')\rangle\,(\psi \otimes \chi^\vee)$ is a morphism $V \to U$.

Classically, these same Wilson lines carry representations of the classical algebra $U(\mathfrak{gl}_K[z])$. When viewed as representations of the deformed (alternatively, quantized) algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$, we shall call the category of Wilson lines as the *quantized category* and viewed as representations of $U(\mathfrak{gl}_K[z])$ we shall refer to the category as the *classical category*. Given any two Wilson lines U and V, any non-trivial morphism between them in the quantized category is a quantization of a non-trivial morphism in the classical category.[2] In fact, there is a one-to-one correspondence between morphisms between two lines in the classical category and the morphisms between the same lines in the quantized category.

For the sake of proof, let us abstract the information we have. We start with a $\mathbb{C}$-linear rigid abelian monoidal category $\mathcal{C} = \mathrm{Rep}_{\mathbb{C}}(H)$ which is the representation category of a Hopf algebra H. We then find a $\mathbb{C}[\![\hbar]\!]$-linear abelian monoidal category $\mathcal{C}_\hbar$, whose objects are representations of some, yet unknown, Hopf algebra $H_\hbar$, with the following properties:

- $\mathrm{ob}(\mathcal{C}_\hbar) = \mathrm{ob}(\mathcal{C})\,,$
- $\mathrm{Hom}_{\mathcal{C}_\hbar}(U,V) \cong \mathrm{Hom}_{\mathcal{C}}(U,V)[\![\hbar]\!]$ as $\mathbb{C}[\![\hbar]\!]$-modules.

Given this information we shall now prove that $H_\hbar$ is unique and that it is topological with respect to $\hbar$-adic topology.

[2] Recall that a morphism between two Wilson lines is the expectation value of the lines when provided with a state at one end. A classical morphism is computed with classical diagrams and its quantization amounts to adding loop diagrams. A zero morphism is constructed by providing zero states, this is independent of quantization, i.e., a quantized morphism is zero, if the provided states are zero, but then so is the original classical morphism.

A.3.1 Tannakian formalism

The aim of this formalism is to realize certain abelian rigid monoidal categories as the representation (or corepresentation) categories of Hopf algebras (possibly with extra structures). To avoid running into some subtlety in the beginning (we shall explain the subtlety later in this section), we first consider the reconstruction from the category of corepresentations.

Reconstruction from corepresentation. Let k be a field, $\mathcal{C}$ an abelian (resp. abelian monoidal and $\mathrm{End}(1) = k$) category such that morphisms are k-bilinear, and let R be a commutative algebra over k – if there is an exact faithful (resp. monoidal) functor ω from $\mathcal{C}$ to $\mathrm{Mod}_f(R)$[3] such that the image of ω is inside the full subcategory $\mathrm{Proj}_f(R)$[4], then we shall say that $\mathcal{C}$ has a *fiber functor* ω to $\mathrm{Mod}_f(R)$.

Theorem A.3.2 (Tannakian Reconstruction for Coalgebra and Bialgebra)**.** *With the notation above, if moreover R is a local ring or a PID*[5]*, then there exists a unique flat R-coalgebra (resp. R-bialgebra) A, up to unique isomorphism, such that A represents the endomorphism of ω in the sense that $\forall M \in IndProj_f(R)$* [6]

$$Hom_R(A, M) \cong Nat(\omega, \omega \otimes M)\,.$$

Moreover, there is a functor $\phi : \mathcal{C} \to Corep_R(A)$ which makes the following diagram commutative

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ \phi\ } & Corep_R(A) \\ & \searrow^{\omega} & \downarrow{\scriptstyle forget} \\ & & Mod_f(R) \end{array}$$

and ϕ is an equivalence if $R = k$.

Our strategy in proving this theorem basically follows [54]. First of all, we need the following

Lemma A.3.3. *$\mathcal{C}$ is both Noetherian and Artinian.*

[3] finitely generated modules of R

[4] finitely generated projective modules of R

[5] PID=Principal Ideal Domain

[6] $\mathrm{IndProj}_f(R)$ means category of inductive limit of finite projective R-modules, which is equivalent to category of flat R-modules.

Proof. Take $X \in \mathrm{ob}(\mathcal{C})$, and an ascending chain X_i of subobjects of X, apply the functor ω to this chain, so that $\omega(X_i)$ is an ascending chain of finitely generated projective submodules of finitely generated projective module $\omega(X)$, thus there is an index j such that $\mathrm{rank}(\omega(X_j)) = \mathrm{rank}(\omega(X))$. Now the quotient of $\omega(X)$ by $\omega(X_j)$ is $\omega(X/X_j)$, which is again finitely generated projective, so it has zero rank, hence trivial. Faithfulness of ω implies that X/X_j is zero, i.e. $X = X_j$, so $\mathcal{C}$ is Noetherian. It follows similarly that $\mathcal{C}$ is Artinian as well. □

Next, we define a functor

$$\otimes : \mathrm{Proj}_f(R) \times \mathcal{C} \to \mathcal{C}$$

by sending (R^n, X) to X^n, recall that every finitely generated projective module over a local ring or a PID is free, thus isomorphic to R^n for some n. Define $\underline{\mathrm{Hom}}(M, X)$ to be $M^\vee \otimes X$. For $V \subset M$ and $Y \subset X$, we define the transporter of V to Y to be

$$(Y : V) := \mathrm{Ker}(\underline{\mathrm{Hom}}(M, X) \to \underline{\mathrm{Hom}}(V, X/Y))$$

We now have the following:

Lemma A.3.4. *Take the full abelian subcategory $\mathcal{C}_X$ of $\mathcal{C}$ generated by subquotients of X^n, consider the largest subobject P_X of $\underline{\mathrm{Hom}}(\omega(X), X)$ whose image in $\underline{\mathrm{Hom}}(\omega(X)^n, X^n)$ under diagonal embedding is contained in $(Y : \omega(Y))$ for all subobjects Y of X^n and all n. Then the Theorem (A.3.2) is true for $\mathcal{C}_X$ with coalgebra defined by $A_X := \omega(P_X)^\vee$.*

Proof. P_X exists because $\mathcal{C}$ is Artinian. Notice that ω takes $\underline{\mathrm{Hom}}(M, X)$ to $\mathrm{Hom}_R(M, X)$ and $(Y : V)$ to $(\omega(Y) : V)$, so it takes P_X, which is defined by

$$\bigcap (\underline{\mathrm{Hom}}(\omega(X), X) \cap (Y : \omega(Y)))$$

to

$$\bigcap (\mathrm{End}_R(\omega(X)) \cap (\omega(Y) : \omega(Y))) \ .$$

Hence $\omega(P_X)$ is the largest subring of $\mathrm{End}_R(\omega(X))$ stabilizing $\omega(Y)$ for all $Y \subset X^n$ and all n. It's a finitely generated projective R module by construction, and so is A_X. Note that only finitely many intersection occurs because $\underline{\mathrm{Hom}}(\omega(X), X)$ is Artinian.

Next, take any flat R module M,[7] since $\mathcal{C}_X$ is generated by subquotients of X, an element $\lambda \in \mathrm{Nat}(\omega, \omega \otimes M)$ is completely determined by it is value on X, so $\lambda \in \mathrm{End}_R(\omega(X)) \otimes M$. Since $- \otimes_R M$ is an exact functor, we have:

$$\begin{aligned} &\bigcap (\mathrm{Hom}_R(\omega(X), \omega(X) \otimes_R M) \cap (\omega(Y) \otimes_R M : \omega(Y))) \\ &= \Big(\bigcap (\mathrm{End}_R(\omega(X)) \cap (\omega(Y) : \omega(Y)))\Big) \otimes_R M \,. \end{aligned}$$

This follows because there are only finitely many intersections and finite limit commutes with tensoring with flat module. Therefore,

$$\lambda \in \omega(P_X) \underset{R}{\otimes} M \,.$$

Conversely, every element in $\omega(P_X) \otimes_R M$ gives rise to a natural transform in the way described above. Hence we establish the isomorphism

$$\mathrm{Nat}(\omega, \omega \otimes M) \cong \omega(P_X) \otimes_R M \cong \mathrm{Hom}_R(A_X, M) \,.$$

A_X is unique up to unique isomorphism (as a flat R module) because it represents the functor $M \mapsto \mathrm{Nat}(\omega, \omega \otimes M)$.

Next, we shall define a co-action of A_X on ω, a counit and a coproduct on A_X which makes A_X an R-coalgebra and ω a corepresentation:

$$\rho \in \mathrm{Nat}(\omega, \omega \otimes A_X) \cong \mathrm{End}_R(A_X)$$

corresponds to the identity map of A_X, and

$$\epsilon \in \mathrm{Hom}_R(A_X, R) \cong \mathrm{Nat}(\omega, \omega)$$

corresponds to Id_ω. The co-action ρ tensored with Id_{A_X} gives a natural transform between $\omega \otimes A_X$ and $\omega \otimes A_X \otimes A_X$, whose composition with ρ gives the following commutative diagram:

$$\begin{array}{ccc} \omega & \xrightarrow{\rho} & \omega \otimes A_X \\ & \searrow^{\psi} & \downarrow{\scriptstyle \rho \otimes \mathrm{Id}_{A_X}} \\ & & \omega \otimes A_X \otimes A_X \end{array} \,.$$

[7] Recall that a R module is flat if and only if it is a filtered colimit of finitely generated projective modules.

Take Δ to be the image of ψ in $\mathrm{Hom}_R(A_X, A_X \otimes_R A_X)$. It follows from definition that A_X is counital and $\rho : \omega \to \omega \otimes A_X$ is a corepresentation. It remains to check that Δ is coassociative.

Observe that the essential image of $\omega \otimes A_X$ is a subcategory of the essential image of ω, hence every functor that shows up here can be restricted to $\omega \otimes A_X$, in particular, ρ, whose restriction to $\omega \otimes A_X$ is obviously $\rho \otimes \mathrm{Id}_{A_X}$. It follows from the definition that

$$(\rho \otimes \mathrm{Id}_{A_X}) \circ \rho = (\mathrm{Id}_\omega \otimes \Delta) \circ \rho \in \mathrm{Nat}(\omega, \omega \otimes A_X \otimes A_X)\,.$$

Restrict this equation to $\omega \otimes A_X$ and we get

$$(\rho \otimes \mathrm{Id}_{A_X} \otimes \mathrm{Id}_{A_X}) \circ (\rho \otimes \mathrm{Id}_{A_X}) = (\mathrm{Id}_\omega \otimes \mathrm{Id}_{A_X} \otimes \Delta) \circ (\rho \otimes \mathrm{Id}_{A_X})\,.$$

Composing with ρ, the LHS corresponds to $(\Delta \otimes \mathrm{Id}_{A_X}) \circ \Delta$ and the RHS corresponds to $(\mathrm{Id}_{A_X} \otimes \Delta) \circ \Delta$ whose equality is exactly the coassociativity of A_X.

It follows that $\forall Z \in \mathcal{C}_X$,

$$\rho(Z) : \omega(Z) \to \omega(Z) \otimes_R A_X$$

gives $\omega(Z)$ a A_X corepresentation structure and this is functorial in Z, thus ω factors through a $\phi : \mathcal{C}_X \to \mathrm{Corep}_R(A_X)$.

Back to the uniqueness of A_X. It has been shown that it is unique up to unique isomorphism as a flat R module. Additionally, if $\phi : A_X \to A'_X$ is an isomorphism such that it induces identity transformation on the functor $M \mapsto \mathrm{Nat}(\omega, \omega \otimes M)$ then, ϕ automatically maps the triple (Δ, ϵ, ρ) to $(\Delta', \epsilon', \rho')$, so ϕ is a coalgebra isomorphism.

Finally, it remains to show that when $R = k$, ϕ is essentially surjective[8] and full:

- Essentially Surjective: If $M \in \mathrm{Corep}_k(A_X)$, then define

  $$\widetilde{M} := \mathrm{Coker}(M \otimes \omega(P_X) \otimes P_X \rightrightarrows M \otimes P_X)\,,$$

  where two arrows are $\omega(P_X)$ representation structure of M and P_X respectively, then

  $$\omega(\widetilde{M}) = M \underset{\omega(P_X)}{\otimes} \omega(P_X) = M\,.$$

[8] In fact, ϕ is essentially surjective even without the assumption that $R = k$.

- Full: If $f : M \to N$ is a A_X-corepresentation morphism, then by the k-linearlity of $\mathcal{C}_X$, f lifts to morphisms

$$f \otimes \mathrm{Id}_{P_X} : M \otimes P_X \to N \otimes P_X \,,$$

and

$$f \otimes \mathrm{Id}_{\omega(P_X)} \otimes \mathrm{Id}_{P_X} : M \otimes \omega(P_X) \otimes P_X \to N \otimes \omega(P_X) \otimes P_X \,.$$

Thus, passing to cokernel gives rise to $\widetilde{f} : \widetilde{M} \to \widetilde{N}$ which is mapped to f by ω.

□

Next we move on to recover the category $\mathcal{C}$ by its subcategories $\mathcal{C}_X$. Define an index category I such that its objects are isomorphism classes of objects in $\mathcal{C}$, denoted by X_i for each index i, and a unique arrow from i to j if X_i is a subobject of X_j. I is directed because for any two objects Z and W, they are subobjects of $Z \oplus W$. Observe that if X is a subobject of Y, then $\mathcal{C}_X$ is a full subcategory of $\mathcal{C}_Y$, so a functorial restriction

$$\mathrm{Hom}_R(A_Y, M) \cong \mathrm{Nat}(\omega_Y, \omega_Y \otimes M) \to \mathrm{Nat}(\omega_X, \omega_X \otimes M) \cong \mathrm{Hom}_R(A_Y, M)\,,$$

gives rise to a coalgebra homomorphism $A_X \to A_Y$. Futhermore, this homomorphism is injective because $\omega(P_Y) \to \omega(P_X)$ is surjective, otherwise $\mathrm{Coker}(\omega(P_Y) \to \omega(P_X))$ will be mapped to the zero object in $\mathrm{Corep}_R(A_Y)$, which contradicts with ω being faithful.

Lemma A.3.5. *Define the coalgbra*

$$A := \varinjlim_{i \in I} A_{X_i}\,,$$

then it is the desired coalgebra in Theorem A.3.2.

Proof. A is flat because it is an inductive limit of flat R modules. Moreover

$$\mathrm{Hom}_R(A, M) = \varprojlim_{i \in I} \mathrm{Hom}_R(A_{X_i}, M) \cong \varprojlim_{i \in I} \mathrm{Nat}(\omega_{X_i}, \omega_{X_i} \otimes M) = \mathrm{Nat}(\omega, \omega \otimes M)\,,$$

which gives the desired functorial property and this implies that A is unique up to unique isomorphism. Finally, when $R = k$, the functor ϕ is defined and it is fully faithful because it is fully faithful on each subcategory $\mathcal{C}_{X_i}$. It's also essentially surjective because every corepresentation V of A comes from a corepresentation of a finite dimensional sub-coalgebra of A,[9] and A is a filtered union of sub-coalgebras A_{X_i}, so V comes from a corepresentation of some A_{X_i}. □

[9] Take a basis $\{e_i\}$ for V, the co-action ρ takes e_i to $\sum_j e_j \otimes a_{ji}$, then it is easy to see that $\mathrm{span}\{a_{ji}\}$ is a finite dimensional sub-coalgebra of A.

Proof of Theorem A.3.2. It remains to prove the theorem when $\mathcal{C}$ is monoidal. This amounts to including $m : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$ and $e : \mathbb{1} \to \mathcal{C}$ with associativity and unitarity constrains, where $\mathbb{1}$ is the trivial tensor category with objects $\{0, 1\}$ and only nontrivial morphisms are $\mathrm{End}(1) = k$. Using the isomorphism:

$$\mathrm{Hom}_R(A \otimes_R A, A \otimes_R A) \cong \mathrm{Nat}(\omega \boxtimes \omega, \omega \boxtimes \omega \otimes A \otimes_R A)\,,$$

we get a homomorphism

$$\tau : \mathrm{Hom}_R(A \otimes_R A, M) \to \mathrm{Nat}(\omega \boxtimes \omega, \omega \boxtimes \omega \otimes M)\,.$$

It is an isomorphism because for each pair of subcategories $(\mathcal{C}_X, \mathcal{C}_Y)$

$$\begin{aligned}\mathrm{Hom}_R(A_X \otimes_R A_Y, M) &\cong \mathrm{Hom}_R(A_X, R) \otimes_R \mathrm{Hom}_R(A_Y, M)\\ &\cong \mathrm{Nat}(\omega_X, \omega_X) \otimes_R \mathrm{Nat}(\omega_Y, \omega_Y \otimes M)\\ &\cong \mathrm{Nat}(\omega_X \boxtimes \omega_Y, \omega_X \boxtimes \omega_Y \otimes M)\end{aligned}$$

and it is compatible with the homomorphism given above, so after taking limit, τ is an isomorphism. We also have a homomorphism:

$$\mathrm{Nat}(\omega, \omega \otimes M) \to \mathrm{Nat}(\omega \boxtimes \omega, \omega \boxtimes \omega \otimes M)\,,$$

by taking any $\alpha \in \mathrm{Nat}(\omega, \omega \otimes M)$, and composing with the isomorphism $\omega \boxtimes \omega(X \boxtimes Y) \cong \omega(X \otimes Y)$. This homomorphism in turn becomes a homomorphism

$$\mu : A \otimes_R A \to A\,.$$

And the obvious isomorphism

$$\mathrm{Hom}_R(R, M) = M \to \mathrm{Nat}(\omega_{\mathbb{1}}, \omega_{\mathbb{1}} \otimes M)\,,$$

together with the unit functor $e : \mathbb{1} \to \mathcal{C}$ give a homomorphism

$$\iota : R \to A\,.$$

All of the homomorphisms are functorial with respect to M so μ and ι are homomorphisms between coalgebras. Now the associativity and unitarity of monoidal category $\mathcal{C}$ translates into associativity and unitarity of μ and ι, which are exactly conditions for A to be a bialgebra. This concludes the proof of Theorem A.3.2. □

Remark A.3.6. In the statement of Theorem A.3.2, it is assumed that R is a local ring or a PID, for the following technical reason: we want to introduce the functor

$$\otimes : \mathrm{Proj}_f(R) \times \mathcal{C} \to \mathcal{C}$$

which is defined by sending (R^n, X) to X^n. This is feasible only if every finite projective module is free, which is not always true for an arbitary ring. Nevertheless, this is true when R is local or a PID. It is tempting to eliminate this assumption when $\mathcal{C}$ is rigid, since we only use the $\underline{\mathrm{Hom}}(\omega(X), X)$ to define the crucial object P_X, and there is no need to define a $\underline{\mathrm{Hom}}$ when the category is rigid. In fact, there is no loss of information if we define P_X by

$$\bigcap \left(\underline{\mathrm{Hom}}(X, X) \cap (Y : Y)\right) ,$$

then the fiber functor ω takes P_X to

$$\bigcap \left(\mathrm{End}_R(\omega(X)) \cap (\omega(Y) : \omega(Y))\right) ,$$

since ω is monoidal by definition and a monoidal functor between rigid monoidal categories preserves duality and thus preserves inner Hom. △

Following the above remark, we drop the assumption on ring R and state the following version of Tannakian reconstruction for Hopf algebras:

Theorem A.3.7 (Tannakian Reconstruction for Hopf Algebra)**.** *Let R be a commutative k-algebra, $\mathcal{C}$ a k-linear abelian rigid monoidal category (resp. abelian rigid braided monoidal) with a fiber functor ω to $Mod_f(R)$, then there exists a unique flat R-Hopf algebra A (resp. R-coquasitriangular Hopf algebra), up to unique isomorphism, such that A represents the endomorphism of ω in the sense that $\forall M \in IndProj_f(R)$*

$$Hom_R(A, M) \cong Nat(\omega, \omega \otimes M) .$$

Moreover, there is a functor $\phi : \mathcal{C} \to Corep_R(A)$ which makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\phi} & Corep_R(A) \\ & \underset{\omega}{\searrow} & \downarrow{\scriptstyle forget} \\ & & Mod_f(R) \end{array}$$

and ϕ is an equivalence if $R = k$.

Sketch of proof. The idea of proof basically follows [103]. Accoring to Remark A.3.6 and Theorem A.3.2, there exists a bialgebra A which satisfies all conditions in the theorem, so it remains to prove that there are compatible structures on A when $\mathcal{C}$ has extra structures.

(a) **$\mathcal{C}$ is rigid.** This means that there is an equivalence between k-linear abelian monoidal categories

$$\sigma : \mathcal{C} \to \mathcal{C}^{op} \,,$$

by taking the right dual of each object, so it turns into an isomophism between R modules

$$\sigma : \mathrm{Nat}(\omega, \omega \otimes M) \to \mathrm{Nat}(\omega^{op}, \omega^{op} \otimes M) \,.$$

According to the functoriality of the construction of the bialgebra A, there is a bialgebra isomorphism:

$$\mathcal{S} : A \to A^{op} \,,$$

put it in another way, a bialgebra anti-automorphism of A. To prove that it satisfies the required compatibility:

$$\mu \circ (\mathcal{S} \otimes \mathrm{Id}) \circ \Delta = \iota \circ \epsilon = \mu \circ (\mathrm{Id} \otimes \mathcal{S}) \circ \Delta \,,$$

we observe that $\iota \circ \epsilon$ gives the natural transformation

$$\mathrm{Id} \otimes \rho_{\omega(1)} : \omega(X) = \omega(X) \otimes \omega(1) \mapsto \omega(X) \otimes \rho(\omega(1)) \,,$$

but 1 is the trivial corepresentation of A, so $\rho(\omega(1))$ is canonically identified with $\omega(1)$, so $\iota \circ \epsilon$ is just the identity morphism on $\omega(X)$. On the other hand, $\mu \circ (\mathcal{S} \otimes \mathrm{Id}) \circ \Delta$ corresponds to the homomorphism

$$\omega(X) \to \omega(X) \otimes \omega(X)^{\vee} \otimes \omega(X) \to \omega(X) \otimes \omega(X^{\vee} \otimes X) \to \omega(X) \otimes \omega(1) = \omega(X)$$

which is identity by the rigidity of $\mathcal{C}$, hence $\mu \circ (\mathcal{S} \otimes \mathrm{Id}) \circ \Delta = \iota \circ \epsilon$. The other equation is similiar.

(b) $\mathcal{C}$ **is rigid braided.** This means that there is a natural transformation:

$$r : \omega \boxtimes \omega \to \omega \boxtimes \omega\,,$$

which gives the braiding. This corresponds to a homomorphism of R-modules

$$\mathcal{R} : A \otimes A \to R\,,$$

let's define it to be the universal R-matrix. The fact that r is a natural transformation is equivalent to the diagram below being commutative

$$\begin{array}{ccccc} \omega(U)\otimes\omega(V) & \xrightarrow{\rho\otimes\rho} & \omega(U)\otimes\omega(V)\otimes A\otimes A & \xrightarrow{\mathrm{Id}\otimes\mathrm{Id}\otimes\mu} & \omega(U)\otimes\omega(V)\otimes A \\ \downarrow{\scriptstyle r} & & & & \downarrow{\scriptstyle r\otimes\mathrm{Id}} \\ \omega(V)\otimes\omega(U) & \xrightarrow{\rho\otimes\rho} & \omega(V)\otimes\omega(U)\otimes A\otimes A & \xrightarrow{\mathrm{Id}\otimes\mathrm{Id}\otimes\mu} & \omega(V)\otimes\omega(U)\otimes A \end{array}$$

which in turn translates to the following equation of $\mathcal{R}$:

$$\mathcal{R}_{12}\circ\mu_{24}\circ(\Delta\otimes\Delta) = \mathcal{R}_{23}\circ\mu_{13}\circ\tau_{13}\circ(\Delta\otimes\Delta)\,,$$

where $\tau : A\otimes A \to A\otimes A$ sends $x\otimes y$ to $y\otimes x$. The compactibility of r with the identity

$$\begin{array}{ccc} \omega(X) & \longrightarrow & \omega(X)\otimes\omega(1) \\ \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle r} \\ \omega(X) & \longleftarrow & \omega(1)\otimes\omega(X) \end{array}\,,$$

translates to $\mathcal{R}\circ(\mathrm{Id}_A\otimes 1) = \epsilon$. And symmetrically $\mathcal{R}\circ(1\otimes\mathrm{Id}_A) = \epsilon$.

Finally, the hexagon axiom of braiding:

$$\begin{array}{ccc} & (\omega(X)\otimes\omega(Y))\otimes\omega(Z) & \\ \swarrow{\scriptstyle r\otimes 1} & & \searrow \\ (\omega(Y)\otimes\omega(X))\otimes\omega(Z) & & \omega(X)\otimes(\omega(Y)\otimes\omega(Z)) \\ \downarrow & & \downarrow{\scriptstyle r} \\ \omega(Y)\otimes(\omega(X)\otimes\omega(Z)) & & (\omega(Y)\otimes\omega(Z))\otimes\omega(X) \\ \searrow{\scriptstyle 1\otimes r} & & \swarrow \\ & \omega(Y)\otimes(\omega(Z)\otimes\omega(X)) & \end{array}\,,$$

translates to the commutativity of the diagram

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{\mathrm{Id}\otimes\mathrm{Id}\otimes\Delta} & A\otimes A\otimes A\otimes A \\ \downarrow{\scriptstyle \mu\otimes\mathrm{Id}} & & \downarrow{\scriptstyle \mathcal{R}_{13}\cdot\mathcal{R}_{24}} \\ A\otimes A & \xrightarrow{\qquad\mathcal{R}\qquad} & R \end{array} ,$$

and the same hexagon but with r^{-1} instead of r gives another one:

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{\Delta\otimes\mathrm{Id}\otimes\mathrm{Id}} & A\otimes A\otimes A\otimes A \\ \downarrow{\scriptstyle \mathrm{Id}\otimes\mu} & & \downarrow{\scriptstyle \mathcal{R}_{14}\cdot\mathcal{R}_{23}} \\ A\otimes A & \xrightarrow{\qquad\mathcal{R}\qquad} & R \end{array} .$$

So we end up confirming all the properties that universal R-matrix should satisfy, and we conclude that A is indeed a coquasitriangular Hopf algebra.

□

Reconstruction from representation It is tempting to dualize everything above to formalize the Tannakian reconstruction for the category of representations. In other words, we can take the dual of A instead of A itself, and a corepresentation becomes the representaion, and when the category has extra structures, those structures will be dualized, for example, when $\mathcal{C}$ is a k-linear abelian rigid braided monoidal category, it should come from the representation category of a flat R-quasitriangular Hopf algebra, since the dual of those diagrams involved in the proof of Theorem A.3.7 are exactly properties of universal R-matrix of a quasitriangular Hopf algebra.

This is naive because the statement:

$$\mathrm{Hom}_R(U, V\otimes A) \cong \mathrm{Hom}_R(U\otimes A^*, V)\,,$$

is not true in general, since A can be infinite dimensional, thus the naive dualizing procedure is not feasible. To resolve this subtlety, we observe that A is constructed from a filtered colimit of finite projective R-modules, each is an R-coalgebra, and any finitely generated corepresentation of A comes from a corepresentation of a finite coalgebra, so it is natural to define the action of A^* on those modules by factoring through some finite quotient A^*_X

for some $X \in \mathrm{ob}(\mathcal{C})$. Similiarly, the multiplication structure on A^* can be defined by first projecting down to some finite quotient and taking multiplication

$$A^* \otimes A^* = \varprojlim_{i \in I} A_{X_i} \otimes \varprojlim_{i \in I} A_{X_i} \to A_{X_i} \otimes A_{X_i} \to A_{X_i}$$

which is compatible with transition map $A_{X_j} \to A_{X_i}$ then taking the inverse limit gives the multiplication of A^*. For antipode $\mathcal{S}$, its dual is a map $A^* \to A^*$.

On the other hand, the comultiplication on A^*, is still subtle. If we dualize the multiplication of A, cut-off at some finite submodule

$$A_{X_i} \otimes A_{X_j} \to A\,,$$

we only get an inverse system of morphisms from A^* to $A^*_{X_i} \otimes A^*_{X_j}$ and the latter's inverse limit is $A^* \widehat{\otimes} A^*$, instead of $A^* \otimes A^*$. So we actually get a *topological Hopf algebra* with topological basis

$$N_i := \ker(A^* \to A^*_{X_i})\,,$$

so that the comultiplication is continuous. Similiarly the counit, multiplication, and anipode are continuous as well. Finally when $\mathcal{C}$ is braided, there exists an invertible element $\mathcal{R} \in A^* \widehat{\otimes} A^*$, and the dual of the structure homomorphism in A is exactly the condition that $\mathcal{R}$ is the universal R-matrix of a topological quasitriangular Hopf algebra.

So we can restate Theorem A.3.7 in terms of representations of topological Hopf algebras:

Theorem A.3.8. *Let R be a commutative k-algebra, $\mathcal{C}$ a k-linear abelian rigid monoidal category (resp. abelian rigid braided monoidal) with a fiber functor ω to $Mod_f(R)$, then there exists a unique topological R-Hopf algebra H (resp. R-quasitriangular Hopf algebra) which is an inverse limit of finite projective R-modules endowed with discrete topology, up to unique isomorphism, such that H represents the endomorphism of ω in the sense that*

$$H \cong Nat(\omega, \omega)\,.$$

Moreover, there is a functor $\phi : \mathcal{C} \to Rep_R(H)$ which sends an object in $\mathcal{C}$ to a continuous representation of H and makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ \phi\ } & Rep_R(H) \\ & \searrow^{\omega} & \Big\downarrow{\scriptstyle forget} \\ & & Mod_f(R) \end{array} ,$$

and ϕ is an equivalence if $R = k$.

Application to Quantization We now consider the case that we have a category $\mathcal{C}_\hbar$, which is a *quantization* of the category of representations of some Hopf algebra H over $\mathbb{C}$. The quantization, namely $\mathcal{C}_\hbar$, of $\mathrm{Rep}_{\mathbb{C}}(H)$ is a $\mathbb{C}$-linear abelian monoidal category which has the same set of generators as $\mathrm{Rep}_{\mathbb{C}}(H)$, together with a fiber functor $\omega_\hbar : \mathcal{C}_\hbar \to \mathrm{Mod}_f(\mathbb{C}[\![\hbar]\!])$ which acts on generators of $\mathrm{Rep}_{\mathbb{C}}(H)$ by tensoring with $\mathbb{C}[\![\hbar]\!]$, and

$$\mathrm{Hom}_{\mathcal{C}_\hbar}(X, Y) \cong \mathrm{Hom}_{\mathcal{C}_\hbar}(X, Y)/\hbar = \mathrm{Hom}_{\mathrm{Rep}_{\mathbb{C}}(H)}(X, Y)$$

for any pair of generators X and Y. For example, the classical algebra of local observables in 4D Chern-Simons theory is $U(g[z])$, the universal enveloping algebra of Lie algebra $g[z]$, which has the category of representations generated by classical Wilson lines. Quantized Wilson lines naturally generated a $\mathbb{C}$-linear abelian monoidal category.

Applying Theorem A.3.8, $(\mathcal{C}_\hbar, \omega_\hbar)$ gives us a (topological) $\mathbb{C}[\![\hbar]\!]$-Hopf algebra $H_\hbar$. Since $\mathcal{C}_\hbar$ and $\mathcal{C}$ shares the same set of generators, and the construction of those Hopf algebras as $\mathbb{C}[\![\hbar]\!]$-modules only involves generators of corresponding categories, so $H_\hbar$ is isomorphic to the completion of $H \otimes \mathbb{C}[\![\hbar]\!]$ in the $\hbar$-adic topology:

$$\begin{aligned} H_\hbar := \varprojlim_{i\in I} H_{X_i} \otimes \mathbb{C}[\![\hbar]\!] &\cong \varprojlim_{i\in I} \varprojlim_{n} H_{X_i} \otimes \mathbb{C}[\hbar]/(\hbar^n) \\ &\cong \varprojlim_{n} \varprojlim_{i\in I} H_{X_i} \otimes \mathbb{C}[\hbar]/(\hbar^n) \\ &\cong \varprojlim_{n} H \otimes \mathbb{C}[\hbar]/(\hbar^n) \,. \end{aligned}$$

For the same reason, tensor product of two copies of $H_\hbar$ and completed in the inverse limit topology is isomorphic to the completion of $H_\hbar \otimes_{\mathbb{C}[\![\hbar]\!]} H_\hbar$ in the $\hbar$-adic topology:

$$H_\hbar \widehat{\otimes} H_\hbar \cong \varprojlim_{n} H_\hbar \otimes_{\mathbb{C}[\![\hbar]\!]} H_\hbar/(\hbar^n)$$

From the construction of those Hopf algebras and the condition that a morphism in $\mathcal{C}_\hbar$ modulo $\hbar$ is a morphism in $\mathrm{Rep}_{\mathbb{C}}(H)$, it is easy to see that modulo $\hbar$ respects all structure homomorphisms, thus $H_\hbar$ modulo $\hbar$ and H are isomorphic as Hopf algebras. Finally, structure homomorphisms of $H_\hbar$ are continuous in the $\hbar$-adic topology because they are $\hbar$-linear. Thus we conlude that:

Theorem A.3.9. *$H_\hbar$ is a quantization of H in the sense of Definition 6.1.1 of [24], i.e. it is a topological Hopf algebra over $\mathbb{C}[\![\hbar]\!]$ with $\hbar$-adic topology, such that*

(i) $H_\hbar$ is isomorphic to $H[\![\hbar]\!]$ as a $\mathbb{C}[\![\hbar]\!]$-module;

(ii) $H_\hbar$ modulo $\hbar$ is isomorphic to H as Hopf algebras.

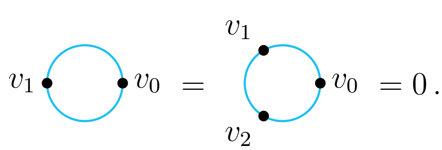
In our case, $H = U(g[z])$ for $g = \mathfrak{gl}_K[z]$, so $H_\hbar$ is a quantization of $U(\mathfrak{gl}_K[z])$, and according to Theorem 12.1.1 of [24], this is unique up to isomorphisms. This proves Proposition (2.4.1).

A.4 Technicalities of Witten Diagrams

A.4.1 Vanishing lemmas

We introduce some lemmas to allow us to readily declare several Witten diagrams in the 4D Chern-Simons theory to be zero.

Lemma A.4.2. *The product of two or three bulk-to-bulk propagators vanish when attached cyclically, diagrammatically this means:*



(A.14)

Proof. Two propagators: We can choose one of the two bulk points, say v_0, to be at the origin and denote v_1 simply as v. This amounts to taking the projection (2.108), namely: $\mathbb{R}^4_{v_0} \times \mathbb{R}^4_{v_1} \ni (v_0, v_1) \mapsto v_1 - v_0 =: v \in \mathbb{R}^4$. Then the product of the two propagators become:

$$P(v_0, v_1) \wedge P(v_1, v_0) \mapsto \overline{P}(v) \wedge \overline{P}(-v) = -\overline{P}(v) \wedge \overline{P}(v)\,. \tag{A.15}$$

This is a four form at v, however, P does not have any $\mathrm{d}z$ component, therefore the four form $P(v) \wedge P(v)$ necessarily contains repetition of a one form and thus vanishes.

Three propagators: By choosing v_0 to be the origin of our coordinate system we can turn the product to the following:

$$\overline{P}(v_1) \wedge \overline{P}(v_2) \wedge P(v_1, v_2)\,. \tag{A.16}$$

We now need to look closely at the propagators (see (2.108) and (2.111)):

$$\overline{P}(v_i) = \frac{\hbar}{2\pi} \frac{x_i\, \mathrm{d}y_i \wedge \mathrm{d}\overline{z}_i + y_i\, \mathrm{d}\overline{z}_i \wedge \mathrm{d}x_i + 2\overline{z}_i\, \mathrm{d}x_i \wedge \mathrm{d}y_i}{d(v_i, 0)^4}\,, \tag{A.17a}$$

$$P(v_1, v_2) = \frac{\hbar}{2\pi} \frac{x_{12}\, \mathrm{d}y_{12} \wedge \mathrm{d}\overline{z}_{12} + y_{12}\, \mathrm{d}\overline{z}_{12} \wedge \mathrm{d}x_{12} + 2\overline{z}_{12}\, \mathrm{d}x_{12} \wedge \mathrm{d}y_{12}}{d(v_1, v_2)^4}\,, \tag{A.17b}$$

where $v_i := (x_i, y_i, z_i, \overline{z}_i)$, $x_{ij} := x_i - x_j$, $y_{ij} := y_i - y_j, \cdots$, and $d(v_i, v_j)^2 := (x_{ij}^2 + y_{ij}^2 + z_{ij}\overline{z}_{ij})$. Since the propagators don't have any $\mathrm{d}z$ component the product (A.16) must be proportional to $\omega := \bigwedge_{i\in\{1,2\}} \mathrm{d}x_i \wedge \mathrm{d}y_i \wedge \mathrm{d}\overline{z}_i$. In the product there are six terms that are proportional to ω. For example, we can pick $\mathrm{d}x_1 \wedge \mathrm{d}y_1$ from $\overline{P}(v_1)$, $\mathrm{d}\overline{z}_2 \wedge \mathrm{d}x_2$ from $\overline{P}(v_2)$ and $\mathrm{d}y_{12} \wedge \mathrm{d}\overline{z}_{12}$ from $P(v_1, v_2)$, this term is proportional to:

$$\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \mathrm{d}\overline{z}_2 \wedge \mathrm{d}x_2 \wedge \mathrm{d}y_{12} \wedge \mathrm{d}\overline{z}_{12} = -\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \mathrm{d}\overline{z}_2 \wedge \mathrm{d}x_2 \wedge \mathrm{d}y_2 \wedge \mathrm{d}\overline{z}_1 = +\omega\,. \tag{A.18}$$

The other five such terms are:

$$\begin{aligned}
\mathrm{d}y_1 \wedge \mathrm{d}\overline{z}_1 \wedge \mathrm{d}\overline{z}_2 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_{12} \wedge \mathrm{d}y_{12} &= -\,\omega\,,\\
\mathrm{d}y_1 \wedge \mathrm{d}\overline{z}_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}y_2 \wedge \mathrm{d}\overline{z}_{12} \wedge \mathrm{d}x_{12} &= +\,\omega\,,\\
\mathrm{d}\overline{z}_1 \wedge \mathrm{d}x_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}\overline{z}_2 \wedge \mathrm{d}x_{12} \wedge \mathrm{d}y_{12} &= +\,\omega\,,\\
\mathrm{d}\overline{z}_1 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}y_2 \wedge \mathrm{d}y_{12} \wedge \mathrm{d}\overline{z}_{12} &= -\,\omega\,,\\
\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}\overline{z}_2 \wedge \mathrm{d}\overline{z}_{12} \wedge \mathrm{d}x_{12} &= -\,\omega\,.
\end{aligned} \tag{A.19}$$

These signs can be determined from a determinant, stated differently, we have the following equation:

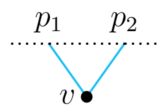
$$\det \begin{pmatrix} \mathrm{d}y_1 \wedge \mathrm{d}\overline{z}_1 & \mathrm{d}\overline{z}_1 \wedge \mathrm{d}x_1 & \mathrm{d}x_1 \wedge \mathrm{d}y_1 \\ \mathrm{d}y_2 \wedge \mathrm{d}\overline{z}_2 & \mathrm{d}\overline{z}_2 \wedge \mathrm{d}x_2 & \mathrm{d}x_2 \wedge \mathrm{d}y_2 \\ \mathrm{d}y_{12} \wedge \mathrm{d}\overline{z}_{12} & \mathrm{d}\overline{z}_{12} \wedge \mathrm{d}x_{12} & \mathrm{d}x_{12} \wedge \mathrm{d}y_{12} \end{pmatrix} = -6\omega\,, \tag{A.20}$$

where the product used in taking determinant is the wedge product. The above equation implies that in the product (A.16) the coefficient of $-\omega$ is given by the same determinant if we replace the two forms with their respective coefficients as they appear in (A.17). Therefore, the coefficient is:

$$\frac{1}{8\pi^3 d(v_1, 0)^4 d(v_2, 0)^4 d(v_1, v_2)^4} \det \begin{pmatrix} x_1 & y_1 & \overline{z}_1 \\ x_2 & y_2 & \overline{z}_2 \\ x_{12} & y_{12} & \overline{z}_{12} \end{pmatrix} = 0\,. \tag{A.21}$$

The determinant vanishes because the three rows of the matrix are linearly dependent. Thus we conclude that the product (A.16) vanishes. ☐

Lemma A.4.3. *The product of two bulk-to-bulk propagators joined at a bulk vertex where the other two endpoints are restricted to the Wilson line, vanishes, i.e., in any Witten diagram:*



$$= 0\,. \tag{A.22}$$

Proof. This simply follows from the explicit form of the bulk-to-bulk propagator. Computation verifies that:

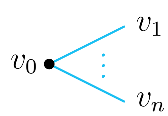
$$\iota_{\partial_{x_1}\wedge\partial_{x_2}}\left(P(v,p_1)\wedge P(v,p_2)\right) = 0\,, \tag{A.23}$$

where x_1 and x_2 are the x-coordinates of the points p_1 and p_2 respectively. ☐

The world-volume on which the CS theory is defined is $\mathbb{R}^2_{x,y}\times\mathbb{C}_z$, which in the presence of the Wilson line at $y = z = 0$ we view as $\mathbb{R}_x\times\mathbb{R}_+\times S^2$. When performing integration over this space we approximate the non-compact direction by a finite interval and then taking the length of the interval to infinity. In doing so we introduce boundaries of the world-volume, namely the two components $B_{\pm D} := \{\pm D\}\times\mathbb{R}_+\times S^2$ at the two ends of the interval $[-D, D]$. Our next lemma concerns some integrals over these boundaries.

Lemma A.4.4. *The integral over a bulk point vanishes when restricted to the spheres at infinity, in diagram:*

$$\lim_{D\to\infty}\int_{v_0\in B_{\pm D}}$$



$$= 0\,. \tag{A.24}$$

Proof. Symbolically, the integration can be written as:

$$\lim_{D\to\infty}\int_{B_{\pm D}} \mathrm{dvol}_{B_{\pm D}}\,\iota_{\partial_y\wedge\partial_{\overline{z}}}\left(P(v_0,v_1)\wedge\cdots\wedge P(v_0,v_n)\right)\,, \tag{A.25}$$

where y and $\overline{z}$ are coordinates of v_0. Note that the dz required for the volume form on $B_{\pm D}$ comes from the structure constant at the interaction vertex, not from the propagators. In the above integration the x-component of v_0 is fixed at $\pm D$, which introduces D dependence in the integrand. The bulk-to-bulk propagator has the following asymptotic scaling behavior:[10]

$$P((D,y,z,\overline{z}),v_j) \overset{D\to\infty}{\sim} D^{-2} + \mathcal{O}(D^{-3})\,. \tag{A.26}$$

[10]Keep in mind that $\hbar$ has a (length) scaling dimension 1.

The integration measure on $B_{\pm D}$ is independent of D, therefore the integral behaves as D^{-2n} for large D, and consequently vanishes in the limit $D \to \infty$. □

A.4.5 Comments on integration by parts

Finally, let us make a few general remarks about the integrals involved in computing Witten diagrams. Since the boundary-to-bulk propagators are exact and the bulk-to-bulk propagators behave nicely when acted upon by differential (see (2.109)), we want to use Stoke's theorem to simplify any given Witten diagram. Suppose we have a Witten diagram with m propagators connected to the boundary, n propagators connected to the Wilson line, and l bulk points. Let us denote the bulk points by v_i for $i = 1, \cdots, l$, the points on the Wilson line by p_j for $j = 1, \cdots, n$, and the points on the boundary as x_k for $k = 1, \cdots, m$. The domain of integration for the diagram is then $M^l \times \Delta_n$, where $M = \mathbb{R} \times \mathbb{R}_+ \times S^2$ and Δ_n is an n-simplex defined as:

$$\Delta_n := \{(p_1, \cdots, p_n) \in \mathbb{R}^n \,|\, p_1 \leq p_2 \leq \cdots \leq p_n\}\,. \tag{A.27}$$

This domain may need to be modified in some Witten diagrams due to the integral over this domain having UV divergences. UV divergences can occur when some points along the Wilson line collide with each other. To avoid such divergences we shall use a *point splitting* regulator, i.e., we shall cut some corners from the simplex Δ_n. Let us denote the regularized simplex as $\widetilde{\Delta}_n$. The exact description of $\widetilde{\Delta}_n$ will vary from diagram to diagram, and we shall describe them as we encounter them.

When we do integration by parts with respect to the differential in a boundary-to-bulk propagator, we get the following three types of terms:

1. A boundary term. Boundaries of our integration domain comes from boundaries of M and $\widetilde{\Delta}_n$. For M we get:
$$\partial M = B_{+\infty} \sqcup B_{-\infty}\,. \tag{A.28}$$
Due to Lemma A.4.4, integrations over ∂M will vanish. Therefore, nonzero contribution to the boundary integration, when we do integration by parts, will only come from the boundary of the regularized simplex, namely $\partial\widetilde{\Delta}_n$. Schematically, the appearance of such a boundary integral will look like:
$$\int_{M^l \times \widetilde{\Delta}_n} \mathrm{d}\theta \wedge (\cdots) = \int_{M^l \times \partial\widetilde{\Delta}_n} \theta \wedge (\cdots) + \cdots\,. \tag{A.29}$$

2. The differential acts on a bulk-to-bulk propagator. Due to (2.109), this identifies the two end points of the propagator, schematically:

$$\mathsf{b} \in \{0,1\}\,, \qquad \int_{M^l \times \partial^{\mathsf{b}} \widetilde{\Delta}_n} \mathrm{d}\theta \wedge P \wedge (\cdots) = \int_{M^{l-1} \times \partial^{\mathsf{b}} \widetilde{\Delta}_n} \theta \wedge (\cdots) + \cdots\,. \tag{A.30}$$

3. The differential acts on a step function left by a previous integration by parts. This does not change the domain of integration.

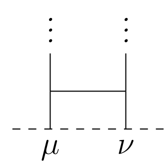
The third option does not to lead a simplification of the domain of integration. Therefore, at the present abstract level, our strategy to simplify an integration is: first go to the boundary of the simplex, and then keep collapsing bulk-to-bulk propagators until we have no more differential left or when no more bulk-to-bulk propagator can be collapsed without the diagram vanishing due the vanishing lemmas from §A.4.1.

A.5 Proof of Lemma 2.4.8

All the diagrams that we draw in this section only exist to represent color factors, their numerical values are irrelevant. Which is why we also ignore the color coding we used in the diagrams in the chapter 2.

We start with yet another lemma:

Lemma A.5.1. *The color factor of any Witten diagram with two boundary-to-bulk propagators connected by a single bulk-to-bulk propagator, that is any Witten diagrams with the following configuration:*



(A.31)

upon anti-symmetrizing the color labels of the boundary-to-bulk propagators, involves the following factor:

$$f_{\mu\nu}{}^{\xi} X_{\xi}\,, \tag{A.32}$$

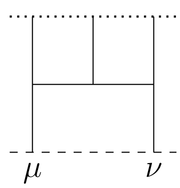
for some matrix X_ξ that transforms under the adjoint representation of $\mathfrak{gl}_K$. In particular, this color factor is the image in $\mathrm{End}(V)$ of some element of $\mathfrak{gl}_K$ where V is the representation of some distant Wilson line.

Proof. The two bulk vertices in the diagram results in the following product of structure constants: $f_{\mu o}{}^{\pi} f_{\nu\rho}{}^{o}$ where the indices π and ρ are contracted with the rest of the diagram. Anti-symmetrizing the indices μ and ν we get $f_{\mu o}{}^{\pi} f_{\nu\rho}{}^{o} - f_{\nu o}{}^{\pi} f_{\mu\rho}{}^{o}$, which using the Jacobi identity becomes $-f_{\mu\nu}{}^{o} f_{\rho o}{}^{\pi}$. Once π and ρ are contracted with the rest of the diagram we get an expression of the general form (A.32). Furthermore, any expression of the form (A.32) is an image in $\mathrm{End}(V)$ of some element in $\mathfrak{gl}_K$, since the structure constant $f_{\mu\nu}{}^{\xi}$ can be viewed as a map:

$$f : \wedge^2 \mathfrak{gl}_K \to \mathfrak{gl}_K \,, \qquad f : t_\mu \wedge t_\nu \mapsto f_{\mu\nu}{}^{\xi} t_\xi \,. \tag{A.33}$$

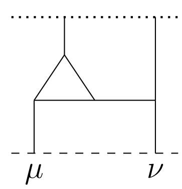
Now composing the above map with a representation of $\mathfrak{gl}_K$ on V gives the aforementioned image. □

Let us now look at the color factor (2.158) of the diagram (2.157), both of which we repeat here:



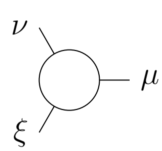
$$, \qquad f_{\mu}{}^{\xi o} f_{\xi}{}^{\pi\rho} f_{\nu\pi}{}^{\sigma} \varrho(t_o)\varrho(t_\rho)\varrho(t_\sigma) \,. \tag{A.34}$$

By commuting $\varrho(t_o)$ and $\varrho(t_\rho)$ in the color factor we create a difference which is the color factor of the following diagram:



$$. \tag{A.35}$$

The key feature of the above diagram is the loop with three propagators attached to it. Such a loop produces a color factor which is a $\mathfrak{gl}_K$-invariant inside $(\mathfrak{gl}_K)^{\otimes 3}$, explicitly we can write a loop and its associated color factor respectively as:



$$\text{and} \quad f_{\mu o}{}^{\pi} f_{\nu\rho}{}^{o} f_{\xi\pi}{}^{\rho} \,. \tag{A.36}$$

The color factor is $\mathfrak{gl}_K$-invariant since the structure constant itself is such an invariant. To find the invariants in $(\mathfrak{gl}_K)^{\otimes 3}$ we start by writing $\mathfrak{gl}_K$ as:

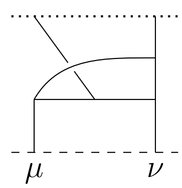
$$\mathfrak{gl}_K = \mathfrak{sl}_K \oplus \mathbb{C} \,, \tag{A.37}$$

where by $\mathfrak{sl}_K$ we mean the complexified algebra $\mathfrak{sl}(K, \mathbb{C})$. This gives us the decomposition

$$(\mathfrak{gl}_K)^{\otimes 3} = (\mathfrak{sl}_K)^{\otimes 3} \oplus \cdots \,, \tag{A.38}$$

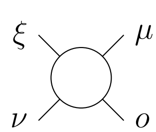
where the "$\cdots$" contains summands that necessarily include at leas one factor of the center $\mathbb{C}$. However, none of the three indices that appear in the diagram in (A.36) can correspond to the center, because each of these indices belong to an instance of the structure constant, which vanishes whenever one of its indices correspond to the center.[11] This means that the $\mathfrak{gl}_K$ invariant we are looking for must lie in $(\mathfrak{sl}_K)^{\otimes 3}$. For $K > 2$, there are exactly two such invariants [121], one of them is the structure constant itself, which is totally anti-symmetric. The other invariant is totally symmetric. However the structure constant is even (invariant) under the $\mathbb{Z}_2$ outer automorphism of $\mathfrak{sl}_K$ whereas the symmetric invariant is odd. Since our theory has this $\mathbb{Z}_2$ as a symmetry, only the structure constant can appear as the invariant in a diagram.[12] This means, as far as the color factor is concerned, we can collapse a loop such as the one in (A.36) to an interaction vertex. As soon as we do this operation to the diagram (A.35), Lemma A.5.1 tells us that the color factor of the diagram is an image in $\mathrm{End}(V)$ of an element in $\mathfrak{gl}_K$. This shows that we can swap the positions of any of the two pairs of the adjacent matrices in the color factor in (A.34) and the difference we shall create is an image of a map $\mathfrak{gl}_K \to \mathrm{End}(V)$. To achieve all permutations of the three matrices wee need to be able to keep swaping positions, let us therefore keep looking forward.

Suppose we commute $\varrho(t_o)$ and $\varrho(t_\rho)$ in (A.34), then we end up with the color factor of the diagram (2.156). Now if we commute $\varrho(t_o)$ and $\varrho(t_\sigma)$, we create a difference that corresponds the color factor of the following diagram:



. (A.39)

The key feature of this diagram is a loop with four propagators attached to it. The loop and its associated color factor can be written as:



$$, \qquad f_{\mu\pi}{}^{\tau} f_{o\tau}{}^{\sigma} f_{\nu\sigma}{}^{\rho} f_{\xi\rho}{}^{\pi} \,. \tag{A.40}$$

[11]In other words, the central abelian photon in $\mathfrak{gl}_K$ interacts with neither itself nor the non-abelian gluons and therefore can not contribute to the diagrams we are considering.

[12]This is also apparent from the way this invariant is written in (A.36), since the structure constant is invariant under this $\mathbb{Z}_2$, certainly a product of them is invariant as well.

As before, the color factor is a $\mathfrak{gl}_K$-invariant in $(\mathfrak{gl}_K)^{\otimes 4}$. This time, it will be more convenient to write the color factor as a trace. Noting that the structure constants are the adjoint representations of the generators of the algebra we can write the above color factor as:

$$\mathrm{tr}_{\mathrm{ad}}(t_\mu t_o t_\nu t_\xi)\,. \tag{A.41}$$

The adjoint representation of $\mathfrak{gl}_K$ factors through $\mathfrak{sl}_K$, and the adjoint representation of $\mathfrak{sl}_K$ has a non-degenerate metric with which we can raise and lower adjoint indices. Suitably changing positions of some of the indices in the color factor we can conclude:

$$\mathrm{tr}_{\mathrm{ad}}(t_\mu t_o t_\nu t_\xi) = \mathrm{tr}_{\mathrm{ad}}(t_\mu t_\xi t_\nu t_o)\,. \tag{A.42}$$

Using the cyclic symmetry of the trace we then find that the color factor is symmetric under the exchange of μ and ν, therefore when we anti-symmetrize the diagram with respect to μ and ν it vanishes.

In summary, starting from the color factor in (A.34), we can keep swapping any two adjacent matrices and the difference can always be written as an image of some map $\mathfrak{gl}_K \to \mathrm{End}(V)$. The same argument applies to the color factors of all the diagrams in (2.156). This proves the lemma.

Appendix B

Techniques in the Computation of Hilbert Series of $\mathbb{C}[\mathcal{M}(N,K)]$

B.1 Hall-Littlewood Polynomials

In this appendix we review some background on symmetric functions, following section 3 of [91].

Definition B.1.1. For a partition $\underline{\lambda} = (1^{\alpha_1}, 2^{\alpha_2}, \cdots)$, the Hall-Littlewood polynomial $P_{\underline{\lambda}}(x;q)$ is defined in $n \geq l(\underline{\lambda}) = \sum_{i\geq 1} \alpha_i$ variables $x_1, \cdots, x_n$ by the formula

$$P_{\underline{\lambda}}(x;q) = \frac{1}{\prod_{i\geq 0}[\alpha_i]_q!} \sum_{w\in S_n} w\left(x^{\underline{\lambda}} \prod_{i<j} \frac{1 - qx_j/x_i}{1 - x_j/x_i}\right). \tag{B.1}$$

Here α_0 is defined so that $\sum_{i\geq 0} \alpha_i = \alpha_0 + l(\underline{\lambda}) = n$, and $x^{\underline{\lambda}} = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$, and we use the standard q-number notation

$$[n]_q = \frac{1-q^n}{1-q},\ [n]_q! = [n]_q[n-1]_q \cdots [1]_q,\ \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

The Hall-Littlewood polynomial $P_{\underline{\lambda}}(x;q)$ is an interpolation between Schur symmetric functions $s_{\underline{\lambda}}(x)$ and monomial symmetric functions $m_{\underline{\lambda}}(x)$, in fact we have

$$P_{\underline{\lambda}}(x;0) = s_{\underline{\lambda}}(x),\ P_{\underline{\lambda}}(x;1) = m_{\underline{\lambda}}(x). \tag{B.2}$$

Definition B.1.2. The Kostka-Foulkes functions are coefficients of the expansion

$$s_{\underline{\lambda}}(x) = \sum_{\underline{\lambda},\underline{\mu}} K_{\underline{\lambda}\underline{\mu}}(q) P_{\underline{\mu}}(x;q). \tag{B.3}$$

In particular, by (B.2) we have

$$K_{\underline{\lambda}\underline{\mu}}(0) = \delta_{\underline{\lambda}\underline{\mu}}.$$

B.1.3 Jing operators and transformed Hall-Littlewood polynomials

Naihuan Jing found a definition of Hall-Littlewood polynomials using vertex algebra [101]. Before giving his definition, we recall some plethystic notations.

The ring of symmetric functions over a base field $\mathbb{F}$ (assuming characteristic zero) is freely generated by power sum functions p_k, that is

$$\Lambda_{\mathbb{F}} = \mathbb{F}[p_1, p_2, \cdots].$$

Let R be a ring containing $\mathbb{F}$, A be a formal Laurent series with R coefficients in indeterminates $a_1, a_2, \cdots$, we define $p_k[A]$ to be the result of replacing each indeterminate a_i in A by a_i^k. Then for any $f \in \Lambda_{\mathbb{F}}$, the plethystic substitution of A into f, denoted $f[A]$, is the image of f under the homomorphism sending p_k to $p_k[A]$.

Example B.1.4. We list some special cases here.

- Let $A = a_1 + \cdots + a_n$, then $p_k[A] = a_1^k + \cdots + a_n^k = p_k(a_1, \cdots, a_n)$, and thus for any $f \in \Lambda_{\mathbb{F}}$, we have $f[A] = f(a_1, \cdots, a_n)$.
- Let A, B be formal Laurent series with R coefficients, then $p_k[A \pm B] = p_k[A] \pm p_k[B]$.
- Let $\mathrm{PE} = \exp\left(\sum_{k=1}^{\infty} p_k/k\right)$, then we have

$$\mathrm{PE}[A + B] = \mathrm{PE}[A]\mathrm{PE}[B],\ \mathrm{PE}[A - B] = \mathrm{PE}[A]/\mathrm{PE}[B].$$

For a single variable x, we have $\mathrm{PE}(x) = \frac{1}{1-x}$, thus for a summation $X = x_1 + x_2 + \cdots$,

$$\mathrm{PE}(X) = \prod_{i\geq 1} \frac{1}{1 - x_i},\ \mathrm{PE}(-X) = \prod_{i \geq 1}(1 - x_i).$$

For the rest of this section, we fix the notation $X = x_1 + x_2 + \cdots$.

Definition B.1.5. The Jing operators are the coefficients $S^q_m = [u^m]S^q(u)$ of the operator generating function $S^q(u)$ defined by

$$S^q(u)f = f[X + (q-1)u^{-1}]\mathrm{PE}[uX]. \tag{B.4}$$

Proposition B.1.6. *Jing operators S^q_m satisfy relations:*

$$S^q_n S^q_{m+1} - qS^q_{m+1}S^q_n = qS^q_{n+1}S^q_m - S^q_m S^q_{n+1}. \tag{B.5}$$

For a proof, see [101, Proposition 2.12], mind that our q is denoted by t there and our S^q_m is denoted by H_{-m} there.

Definition B.1.7. Let $\underline{\mu} = (\mu_1 \geq \cdots \geq \mu_l)$ be a Young tableaux (partition), define the transformed Hall-Littlewood polynomial by

$$H_{\underline{\mu}}(x;q) = S^q_{\mu_1}S^q_{\mu_2}\cdots S^q_{\mu_l}(1). \tag{B.6}$$

For a general array $\underline{\mu} = (\mu_1, \cdots, \mu_l) \in \mathbb{Z}^l_{\geq 0}$, we define the generalized transformed Hall-Littlewood polynomial by the same formula above.

Using relations (B.5) recursively, we can bring a product of operators $S^q_{\mu_1}\cdots S^q_{\mu_l}$ for an array $\underline{\mu} = (\mu_1, \cdots, \mu_l) \in \mathbb{Z}^l_{\geq 0}$ into a linear combination of operators $S^q_{\mu'_1}\cdots S^q_{\mu'_l}$ such that $\mu'_1 \geq \cdots \geq \mu'_l$, in other words, a generalized transformed Hall-Littlewood polynomial can be written as linear combination of usual transformed Hall-Littlewood polynomials.

The following proposition summarizes the fundamental properties of transformed Hall-Littlewood polynomials, for a proof, see [91, 3.4.3].

Proposition B.1.8. *The transformed Hall-Littlewood polynomials $H_{\underline{\mu}}$ are related to the classical Hall-Littlewood polynomials $P_{\underline{\mu}}$ by*

$$H_{\underline{\mu}}[(1-q)X;q] = (1-q)^{l(\underline{\mu})}\prod_{i=1}^{\mu_1}[\alpha_i(\underline{\mu})]_q! P_{\underline{\mu}}(x;q). \tag{B.7}$$

They are uniquely characterized by the following properties.

(i) $H_{\underline{\mu}}(x;q) \in s_{\underline{\mu}}(x) + \mathbb{Z}[q]\cdot\{s_{\underline{\lambda}}(x) : \underline{\lambda} > \underline{\mu}\}$,

(ii) $H_{\underline{\mu}}[(1-q)x;q] \in \mathbb{Z}[q]\cdot\{s_{\underline{\lambda}}(x) : \underline{\lambda} \leq \underline{\mu}\}$.

And $H_{\underline{\mu}}$ *is related to Schur functions by*

$$H_{\underline{\mu}}(x;q) = \sum_{\underline{\lambda\mu}} K_{\underline{\lambda\mu}}(q) s_{\underline{\lambda}}(x). \tag{B.8}$$

It turns out that we can rewrite the definition of Jing operators without referring to the generating function $S^q(u)$.

Lemma B.1.9. *For an* n*-variable function* $f \in \mathbb{F}[p_1, \cdots, p_n](q)$*, where* $p_k(x) = x_1^k + \cdots + x_n^k$*, Jing operator* S^q_m *acts on it as*

$$(S^q_m f)(x;q) = \sum_{i=1}^{n} f(x_1, \cdots, qx_i, \cdots, x_n; q) \frac{x_i^m}{\prod_{j\neq i}(1 - x_j/x_i)}. \tag{B.9}$$

Proof. Notice that

$$\mathrm{PE}(uX) = \prod_{i=1}^{n} \frac{1}{1-ux_i} = \sum_{i=1}^{n} \frac{1}{1-ux_i} \prod_{j\neq i} \frac{1}{1-x_j/x_i}. \tag{B.10}$$

Without loss of generality, we assume that $f = p_{k_1} \cdots p_{k_s}$, then by definition, S^q_m is the coefficient of u^m in the series expansion

$$(p_{k_1} + (q^{k_1} - 1)u^{-k_1}) \cdots (p_{k_s} + (q^{k_s} - 1)u^{-k_s}) \sum_{i=1}^{n} \frac{1}{1-ux_i} \prod_{j\neq i} \frac{1}{1-x_j/x_i}.$$

Let us fix an index i in the summation, then for this summand, its $[u^m]$ coefficient is

$$\begin{aligned}
&(p_{k_1} + (q^{k_1} - 1)x_i^{k_1}) \cdots (p_{k_s} + (q^{k_s} - 1)x_i^{k_s}) \frac{x_i^m}{\prod_{j\neq i}(1 - x_j/x_i)} \\
&= (x_1^{k_1} + \cdots + q^{k_1}x_i^{k_1} + \cdots + x_n^{k_1}) \cdots (x_s^{k_s} + \cdots + q^{k_s}x_i^{k_s} + \cdots + x_n^{k_s}) \frac{x_i^m}{\prod_{j\neq i}(1 - x_j/x_i)} \\
&= f(x_1, \cdots, qx_i, \cdots, x_n; q) \frac{x_i^m}{\prod_{j\neq i}(1 - x_j/x_i)}.
\end{aligned}$$

Summing over i gives the desired formula (B.9). □

B.2 Affine Grassmannians and Geometrization of Jing Operators

In this section we give a geometric definition of Jing operators S^q_m. Recall that

$$K_{\mathrm{GL}_n\times\mathbb{C}^\times}(\mathrm{pt})=\mathbb{Q}[x_1^\pm,\cdots,x_n^\pm,q^\pm]^{S_n}. \tag{B.11}$$

Here we take rationalized coefficients in the K-theory. Notice that $\mathbb{Q}[p_1,\cdots,p_n,q^\pm]\subset K_{\mathrm{GL}_n\times\mathbb{C}^\times}(\mathrm{pt})$ is a subalgebra.

Consider the affine Grassmannian $\mathrm{Gr}_{\mathrm{GL}_n}=\mathrm{GL}_n(\mathcal{K})/\mathrm{GL}_n(\mathcal{O})$, and let $\omega_1=(1,0,\cdots,0)$ be the first fundamental coweight of GL_n, then the $\mathrm{GL}_n(\mathcal{O})$-orbit Gr^{ω_1} is isomorphic to $\mathbb{P}^{n-1}$ and it is fixed by the $\mathbb{C}^\times$-rotation.

The category that we are interested in is $D^b_{\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{\mathrm{GL}_n})$, the $\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{C}^\times$-equivariant bounded derive category of coherent sheaves on $\mathrm{Gr}_{\mathrm{GL}_n}$. Here coherent sheaves on ind-scheme like $\mathrm{Gr}_{\mathrm{GL}_n}$ are defined to have finite type support, so for any $\mathcal{F}\in D^b_{\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{\mathrm{GL}_n})$, we have $\chi(\mathcal{F})\in K_{\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{pt})=K_{\mathrm{GL}_n\times\mathbb{C}^\times}(\mathrm{pt})$.

There is a convolution product on affine Grassmannian, defined as:

$$m:\mathrm{Gr}_{\mathrm{GL}_n}\widetilde{\times}\mathrm{Gr}_{\mathrm{GL}_n}=\mathrm{GL}_n(\mathcal{K})\overset{\mathrm{GL}_n(\mathcal{O})}{\times}\mathrm{GL}_n(\mathcal{K})/\mathrm{GL}_n(\mathcal{O})\to\mathrm{GL}_n(\mathcal{K})/\mathrm{GL}_n(\mathcal{O}). \tag{B.12}$$

Here the map sends (g_1,g_2) to g_1g_2. The convolution map of $\mathrm{Gr}_{\mathrm{GL}_n}$ induces a functor $\star:D^b_{\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{\mathrm{GL}_n})\times D^b_{\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{\mathrm{GL}_n})\to D^b_{\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{\mathrm{GL}_n})$ defined as

$$\mathcal{F}\star\mathcal{G}=Rm_*(\mathcal{F}\widetilde{\boxtimes}\mathcal{G}).$$

Passing to the K-theory, we obtain an map

$$\star:K_{\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{\mathrm{GL}_n})\otimes K_{\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{\mathrm{GL}_n})\longrightarrow K_{\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{\mathrm{GL}_n}). \tag{B.13}$$

In fact, the $\star$-product on $K_{\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{\mathrm{GL}_n})$ is associative, and moreover we have the following.

Theorem B.2.1. *The algebra $K_{\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{\mathrm{GL}_n})$ endowed with $\star$-product is isomorphic to $\mathbb{SH}_n$, the spherical part of double affine Hecke algebra of GL_n.*

The part of story which is relevant to us is that the convolution between $\mathrm{Gr}_{\mathrm{GL}_n}$ and the identity point makes $K_{\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{pt})=\mathbb{Q}[x_1^\pm,\cdots,x_n^\pm,q^\pm]^{S_n}$ into a module of

$K_{\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{\mathrm{GL}_n})$, and we can realize Jing operators S^q_m geometrically from $K_{\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{\mathrm{GL}_n})$ as follows.

There is a distinguished line bundle $\mathcal{O}(1)$ (determinant line bundle) on $\mathrm{Gr}_{\mathrm{GL}_n}$ [164, 1.5], and from the construction of $\mathcal{O}(1)$ we know that it is $\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{C}^\times$-equivariant. Let us use $\mathcal{O}(m)|_{\mathrm{Gr}^{\omega_1}}$ to denote $i_*i^*\mathcal{O}(1)^{\otimes m}$ where $i:\mathrm{Gr}^{\omega_1}\hookrightarrow\mathrm{Gr}_{\mathrm{GL}_n}$ is the natural embedding. Since i is $\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{C}^\times$-equivariant, $\mathcal{O}(m)|_{\mathrm{Gr}^{\omega_1}}$ is also $\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{C}^\times$-equivariant.

Proposition B.2.2. *For $\mathcal{F}\in D^b_{\mathrm{GL}_n(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{\mathrm{GL}_n})$, let $\chi=\chi(\mathcal{F})\in\mathbb{Q}[x_1^\pm,\cdots,x_n^\pm,q^\pm]^{S_n}$ be the equivariant Euler characteristic of $\mathcal{F}$, similarly let $\widetilde{\chi}=\chi(\mathcal{O}(m)|_{\mathrm{Gr}^{\omega_1}}\star\mathcal{F})$. Then*

$$\widetilde{\chi}(x;q)=\sum_{i=1}^n\chi(x_1,\cdots,qx_i,\cdots,x_n;q)\frac{x_i^m}{\prod_{j\neq i}(1-x_j/x_i)}. \tag{B.14}$$

Proof. Let $p:\mathrm{Gr}_{\mathrm{GL}_n}\widetilde{\times}\mathrm{Gr}_{\mathrm{GL}_n}\to\mathrm{Gr}_{\mathrm{GL}_n}$ be the projection to the first component map, i.e. $p(g_1,g_2)=g_1$, this is a fibration with fibers isomorphic to $\mathrm{Gr}_{\mathrm{GL}_n}$. Then by the projection formula we have

$$\chi(\mathcal{O}(m)|_{\mathrm{Gr}^{\omega_1}}\star\mathcal{F})=\chi(\mathbb{P}^{n-1},\mathcal{O}(m)\otimes Li^*Rp_*\widetilde{\mathcal{F}}). \tag{B.15}$$

Here $\widetilde{\mathcal{F}}=\mathcal{O}\widetilde{\boxtimes}\mathcal{F}$ is the twist of $\mathcal{F}$ on $\mathrm{Gr}_{\mathrm{GL}_n}\widetilde{\times}\mathrm{Gr}_{\mathrm{GL}_n}$. We use the localization on $\mathbb{P}^{n-1}$ to compute the right hand side of (B.15) as following. Let the maximal torus of GL_n be T, then T-fixed points of $\mathbb{P}^{n-1}$ are $[1,0,\cdots,0],\cdots,[0,\cdots,1,\cdots,0],\cdots,[0,\cdots,1]$ (in homogeneous coordinates of $\mathbb{P}^{n-1}$), label these points by $e_1,\cdots,e_n$. Observe that

(1) The fiber of determinant line bundle $\mathcal{O}(1)$ at e_i has T-weight x_i,

(2) The tangent space at e_i has T-weights $x_i/x_j, j\in\{1,\cdots,n\}\backslash\{i\}$,

(3) The fiber of $Li^*Rp_*\widetilde{\mathcal{F}}$ at e_i has the same T-weights as $\chi(\mathcal{F})$, but the $\mathbb{C}^\times$-action is different, because the fiber $p^{-1}(e_i)$ is is identified with $\mathrm{Gr}_{\mathrm{GL}_n}$ via a translation $g\mapsto z^{\omega_i-\omega_{i-1}}g$ and the new $\mathbb{C}^\times$ acts through the diagonal of $\mathbb{C}^\times_{\mathrm{rotation}}\times T_i$, where T_i is the i'th $\mathbb{C}^\times$-component of T. In other word, the fiber of $Li^*Rp_*\widetilde{\mathcal{F}}$ at e_i has the $T\times\mathbb{C}^\times$-weights
$$\chi(\mathcal{F})(x_1,\cdots,qx_i,\cdots,x_n;q).$$

Then (B.14) follows from applying localization to $\mathcal{O}(m)\otimes Li^*Rp_*\widetilde{\mathcal{F}}$ using three observations made above. □

Comparing (B.14) and (B.9), we have the following

Corollary B.2.3. *If* $\chi(\mathcal{F}) \in \mathbb{Q}[p_1, \cdots, p_n, q^{\pm}] \subset K_{\mathrm{GL}_n \times \mathbb{C}^\times}(\mathrm{pt})$*, then*

$$\chi(\mathcal{O}(m)|_{\mathrm{Gr}^{\omega_1}} \star \mathcal{F}) = S^q_m \chi(\mathcal{F}). \tag{B.16}$$

From this corollary we see that the operator $\mathcal{O}(m)|_{\mathrm{Gr}^{\omega_1}} \star (-)$ is a geometrization of the Jing operator S^q_m. In fact, it extends the domain of S^q_m to $K_{\mathrm{GL}_n \times \mathbb{C}^\times}(\mathrm{pt}) = \mathbb{Q}[p_1, \cdots, p_n, h_n^{-1}, q^{\pm}]$, and negative m is also allowed.

Corollary B.2.4. *Let* $\underline{\mu} = (\mu_1, \cdots, \mu_l)$ *be an array of nonnegative integers, then*

$$H_{\underline{\mu}}(x; q) = \chi(\mathrm{Gr}_{\mathrm{GL}_n}, \mathcal{O}(\mu_1)|_{\mathrm{Gr}^{\omega_1}} \star \cdots \star \mathcal{O}(\mu_l)|_{\mathrm{Gr}^{\omega_1}}). \tag{B.17}$$

Proof. Combine (B.16) with the definition of $H_{\underline{\mu}}$ in terms of iterative action of $S^q_{\mu_i}$ (B.6). □

Corollary B.2.5. *Let* $\overline{\mathrm{Gr}}^{N\omega_1}$ *be the closure of the* $\mathrm{GL}_n(\mathcal{O})$*-orbit through* $z^{N\omega_1}$*, then*

$$\chi(\overline{\mathrm{Gr}}^{N\omega_1}, \mathcal{O}(k)) = H_{(k^N)}(x; q). \tag{B.18}$$

Here (k^N) *is the partition consisting of* N *copies of* k*, i.e.* $(k, k, \cdots, k)$*.*

Proof. Let $m : \mathrm{Gr}_{\mathrm{GL}_n} \widetilde{\times} \mathrm{Gr}_{\mathrm{GL}_n} \cdots \widetilde{\times} \mathrm{Gr}_{\mathrm{GL}_n} \to \mathrm{Gr}_{\mathrm{GL}_n}$ be the convolution map of N-copies of $\mathrm{Gr}_{\mathrm{GL}_n}$, it is easy to see from the definition of determinate line bundle that there is a $\mathrm{GL}_n(\mathcal{O}) \rtimes \mathbb{C}^\times$-equivariant isomorphism

$$m^* \mathcal{O}(1) \cong \mathcal{O}(1) \widetilde{\boxtimes} \cdots \widetilde{\boxtimes} \mathcal{O}(1).$$

It is known that $m(\mathrm{Gr}^{\omega_1} \widetilde{\times} \cdots \widetilde{\times} \mathrm{Gr}^{\omega_1}) = \overline{\mathrm{Gr}}^{N\omega_1}$, and it is birational, thus m is a resolution of singularities. It is also known that $\overline{\mathrm{Gr}}^{N\omega_1}$ has rational singularities (this is true for all $G(\mathcal{O})$-orbit closure on affine Grassmannian of any reductive group G, see [105, Theorem 2.7]), therefore $Rm_* \mathcal{O} \cong \mathcal{O}$ and $Rm_* m^* \mathcal{O}(k) \cong \mathcal{O}(k)$, thus

$$\begin{aligned}
\chi(\overline{\mathrm{Gr}}^{N\omega_1}, \mathcal{O}(k)) &= \chi(\mathrm{Gr}^{\omega_1} \widetilde{\times} \cdots \widetilde{\times} \mathrm{Gr}^{\omega_1}, m^* \mathcal{O}(k)) \\
&= \chi(\mathrm{Gr}_{\mathrm{GL}_n}, \mathcal{O}(k)|_{\mathrm{Gr}^{\omega_1}} \star \cdots \star \mathcal{O}(k)|_{\mathrm{Gr}^{\omega_1}}) \\
&= H_{(k^N)}(x; q).
\end{aligned}$$

□

Appendix C

Quantization of Quiver Varieties

C.1 Quantum Moment Map and Quantum Hamiltonian Reduction

Fix the base field to be $\mathbb{C}$. Let $\mathfrak{g}$ be a Lie algebra with an action on an associative algebra A, i.e. a Lie homomorphism $\phi : \mathfrak{g} \to \mathrm{Der}(A)$.

Definition C.1.1. A Lie homomorphism $\mu : \mathfrak{g} \to A$ is called a *quantum moment map* if $\forall a \in \mathfrak{g}, b \in A$,

$$[\mu(a), b] = \phi(a) \cdot b. \tag{C.1}$$

Lemma C.1.2. *Let J be the left ideal $A \cdot \mu(\mathfrak{g})$, then $J^{\mathfrak{g}}$ is a two-sided ideal of $A^{\mathfrak{g}}$.*

Proof. Let $x = \sum b_i \mu(a_i) \in J^{\mathfrak{g}}$ and $y \in A^{\mathfrak{g}}$, then

$$\begin{aligned} xy = \sum b_i \mu(a_i) y = \sum b_i y \mu(a_i) + \sum b_i [\mu(a_i), y] &= \sum b_i y \mu(a_i) + \sum b_i \phi(a_i) \cdot y \\ &= \sum b_i y \mu(a_i) \in J^{\mathfrak{g}}. \end{aligned}$$

□

Definition C.1.3. Define the quantum Hamiltonian reduction $A /\!\!/ \mathfrak{g}$ to be $A^{\mathfrak{g}}/J^{\mathfrak{g}}$.

Assumption C.1.4. Assume that A is filtered such that

(1) $[F_k A, F_l A] \subset F_{k+l-d} A$ for a fixed positive integer d,

(2) $\mathrm{gr}A$ is commutative,

(3) $\mu(\mathfrak{g}) \subset F_d A$.

Under the above assumptions $\mathrm{gr}A$ obtains a Poisson structure of degree $-d$ defined as

$$\{a, b\} = [\tilde{a}, \tilde{b}],$$

where $\tilde{a}$ is a lift of a, $\tilde{b}$ is a lift of b. It follows that the action of $\mathfrak{g}$ on A preserves the filtration, therefore the Poisson algebra $\mathrm{gr}A$ inherits a $\mathfrak{g}$-action (denoted by $\overline{\phi}$) which preserves the Poisson structure. Note that the image of $\overline{\mu}$ in $\mathrm{gr}_d A$ (denoted by $\overline{\mu}$) is a classical moment map, i.e. for $a \in \mathfrak{g}, b \in \mathrm{gr}A$,

$$\{\overline{\mu}(a), b\} = \overline{\phi}(a) \cdot b. \tag{C.2}$$

We define the ideal $I = \mathrm{gr}A \cdot \overline{\mu}(\mathfrak{g})$, it is easy to see that $I^{\mathfrak{g}}$ is a Poisson ideal of $(\mathrm{gr}A)^{\mathfrak{g}}$, i.e. $\{I^{\mathfrak{g}}, (\mathrm{gr}A)^{\mathfrak{g}}\} \subset I^{\mathfrak{g}}$.

Definition C.1.5. Define the classical Hamiltonian reduction $\mathrm{gr}A /\!\!/ \mathfrak{g}$ to be $(\mathrm{gr}A)^{\mathfrak{g}}/I^{\mathfrak{g}}$.

Proposition C.1.6 (Classical Limit Commutes with Hamiltonian Reduction). *Suppose that $\mathfrak{g}$ is reductive, and $\{e_1, \cdots, e_n\}$ is a basis of $\mathfrak{g}$ such that $\{\overline{\mu}(e_i)\}_{i=1}^n$ is a regular sequence in* $\mathrm{gr}A$*, then*

$$\mathrm{gr}A /\!\!/ \mathfrak{g} \cong \mathrm{gr}\left(A /\!\!/ \mathfrak{g}\right). \tag{C.3}$$

Proof. By definition, $\mathrm{gr}\,(A /\!\!/ \mathfrak{g}) = \mathrm{gr}(A^{\mathfrak{g}})/\mathrm{gr}(J^{\mathfrak{g}})$, and since $\mathfrak{g}$ is reductive, we have

$$\mathrm{gr}\,(A /\!\!/ \mathfrak{g}) = \mathrm{gr}(A)^{\mathfrak{g}}/\mathrm{gr}(J)^{\mathfrak{g}}.$$

Therefore it suffices to show that $I = \mathrm{gr}J$, in other words, if $\sum f_i \mu(e_i) \in F_m A$ then $\exists g_i \in F_{m-d}A$ such that $\sum f_i \mu(e_i) = \sum g_i \mu(e_i)$.

Suppose that $\sum f_i \mu(e_i) \in F_m A$ and $f_i \in F_k A$ such that $k > m - d$, we claim that $\exists f_i' \in F_{k-1}A$ such that $\sum f_i \mu(e_i) = \sum f_i' \mu(e_i)$. To prove this claim, we can assume that $\forall i, f_i \notin F_{k-1}A$, and denote their image in $\mathrm{gr}_k A$ by $\overline{f}_i$. Then we have

$$\sum \overline{f}_i \overline{\mu}(e_i) = 0 \text{ in } \mathrm{gr}_{k+d}A.$$

By the assumption that $\{\overline{\mu}(e_i)\}_{i=1}^n$ is a regular sequence in $\mathrm{gr}A$, we see that $\exists \overline{h}_{ij} \in \mathrm{gr}_{k-d}A$ such that

$$\overline{f}_i = \sum_j \overline{h}_{ij} \overline{\mu}(e_j),\ \overline{h}_{ij} + \overline{h}_{ji} = 0,$$

because the Koszul complex associated to $\{\overline{\mu}(e_i)\}_{i=1}^n$ is exact. Let us fix a lift $h_{ij} \in F_{k-d}A$ such that $h_{ij} + h_{ji} = 0$, and take $h_i = f_i - \sum_j h_{ij}\mu(e_j)$, then

- $h_i \in F_{k-1}A$, since its image $\overline{h}_i$ in $\mathrm{gr}_k A$ is zero;
- $\sum_i (f_i - h_i)\mu(e_i) = \frac{1}{2}\sum_{ij} h_{ij}\mu([e_i, e_j]) = \frac{1}{2}\sum_{ijk} h_{ij}C_{ijk}\mu(e_k)$, where C_{ijk} is the structure constant of $\mathfrak{g}$.

Hence we see that $f_i' = h_i + \frac{1}{2}\sum_{jk} h_{jk}C_{jki}$ is in $F_{k-1}A$ and $\sum f_i\mu(e_i) = \sum f_i'\mu(e_i)$, which proves the claim. □

C.1.7 Shift of quantum moment map

Let $\chi : \mathfrak{g} \to \mathbb{C}$ be a character, then

$$\mu_\chi : \mathfrak{g} \to A,\ a \mapsto \mu(a) - \chi(a) \cdot 1$$

is a new quantum moment map. More generally, we can put the family of characters $\mathbb{C} \cdot \chi$ together as following:

$$\mu_{t\chi} : \mathfrak{g} \to A[t],\ a \mapsto \mu(a) - \chi(a) \cdot t.$$

Definition C.1.8. Denote by J_χ the left ideal of A generated by $\mu_\chi(\mathfrak{g})$, and denote by $J_{t\chi}$ the left ideal of $A[t]$ generated by $\mu_{t\chi}(\mathfrak{g})$. Define the shifted quantum Hamiltonian reduction $A /\!\!/_\chi\, \mathfrak{g}$ to be $A^{\mathfrak{g}}/J_\chi^{\mathfrak{g}}$, and also define the 1-parameter family $A /\!\!/_{t\chi}\, \mathfrak{g}$ to be $A^{\mathfrak{g}}[t]/J_{t\chi}^{\mathfrak{g}}$.

It is elementary to see that

$$A /\!\!/_\chi\, \mathfrak{g} \cong \left(A /\!\!/_{t\chi}\, \mathfrak{g}\right)/(t-1). \tag{C.4}$$

Proposition C.1.9. *Under the Assumption C.1.4, and additionally assume that $\mathfrak{g}$ is reductive, and $\{e_1, \cdots, e_n\}$ is a basis of $\mathfrak{g}$ such that $\{\overline{\mu}(e_i)\}_{i=1}^n$ is a regular sequence in $\mathrm{gr}A$, then*

$$\left(\mathrm{gr}A /\!\!/ \mathfrak{g}\right)[t] \cong \mathrm{gr}\left(A /\!\!/_{t\chi}\, \mathfrak{g}\right). \tag{C.5}$$

In particular, $A /\!\!/_{t\chi}\, \mathfrak{g}$ is a free $\mathbb{C}[t]$-module.

Proof. Apply Proposition C.1.6 to $A[t]$, where we take the filtration degree of t to be zero. □

C.1.10 Quantum quiver variety

Let $(Q, \mathbf{v})$ be a quiver, where Q is a finite directed graph with vertex set Q_0 and edge set Q_1, and $\mathbf{v} \in \mathbb{N}^{Q_0}$ is called the dimension vector. Define the representation space of $(Q, \mathbf{v})$ as

$$R(Q, \mathbf{v}) = \bigoplus_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\mathbf{v}_{t(a)}}, \mathbb{C}^{\mathbf{v}_{h(a)}}), \tag{C.6}$$

where $h(a)$ and $t(a)$ are head and tail of an arrow $a \in Q_1$. The reductive group $G(\mathbf{v}) = \prod_{i \in Q_0} \mathrm{GL}(\mathbf{v}_i)/\mathbb{C}^\times$ naturally acts on $R(Q, \mathbf{v})$, where $\mathbb{C}^\times$ is the diagonal embedding of central tori, which acts trivially on $R(Q, \mathbf{v})$. Define $D(R(Q, \mathbf{v}))$ be the ring of differential operators on $R(Q, \mathbf{v})$. The action of $G(\mathbf{v})$ on $R(Q, \mathbf{v})$ naturally extends to $D(R(Q, \mathbf{v}))$.

Let us introduce a quantum moment map $\mu : \mathfrak{g}(\mathbf{v}) \to D(R(Q, \mathbf{v}))$ as following. Fix a coordinate $X_{a\alpha}^{\beta}$ of $R(Q, \mathbf{v})$, where $a \in Q_1$, $\alpha = 1, \cdots, \mathbf{v}_{h(a)}$, and $\beta = 1, \cdots, \mathbf{v}_{t(a)}$, then for a basis element e_ρ^ν of $\mathfrak{gl}(\mathbf{v}_i)$, define

$$\mu(e_\rho^\nu) = \sum_{\substack{a \in Q_1, h(a)=i \\ 1 \le \gamma \le \mathbf{v}_{t(a)}}} X_{a\rho}^{\gamma} \frac{\partial}{\partial X_{a\nu}^{\gamma}} - \sum_{\substack{a \in Q_1, t(a)=i \\ 1 \le \gamma \le \mathbf{v}_{t(a)}}} X_{a\rho}^{\gamma} \frac{\partial}{\partial X_{a\nu}^{\gamma}}. \tag{C.7}$$

It is elementary to check that μ is indeed a quantum moment map. Note that in the classical limit by taking associated graded of $D(R(Q, \mathbf{v}))$ with respect to the filtration degree $\deg X_{a\alpha}^{\beta} = 0, \deg \partial_{X_{a\alpha}^{\beta}} = 1$, $\mathrm{gr} D(R(Q, \mathbf{v}))$ is the ring of polynomials on $T^* R(Q, \mathbf{v})$, in particular $D(R(Q, \mathbf{v}))$ satisfies the Assumption C.1.4 with $d = 1$. Moreover $\overline{\mu}$ is the classical moment map in the context of quiver varieties, and in the classical setting, it is conventional to write the Hamiltonian reduction as

$$\mathcal{M}(Q, \mathbf{v}) = \mathrm{Spec}\left(\mathbb{C}[T^* R(Q, \mathbf{v})] /\!\!/ \mathfrak{g}(\mathbf{v})\right). \tag{C.8}$$

Definition C.1.11. Define the quantum quiver variety as $\mathbb{C}[\hbar]$-algebra

$$\mathbb{C}^\hbar[\mathcal{M}(Q, \mathbf{v})] = \mathrm{Rees}(D(R(Q, \mathbf{v})) /\!\!/ \mathfrak{g}(\mathbf{v})), \tag{C.9}$$

the RHS is the Rees algebra of $D(R(Q, \mathbf{v})) /\!\!/ \mathfrak{g}(\mathbf{v})$ with respect to the filtration degree $\deg X_{a\alpha}^{\beta} = 0, \deg \partial_{X_{a\alpha}^{\beta}} = 1$.

We can also define another algebra

$$\mathbb{C}_\hbar[\mathcal{M}(Q, \mathbf{v})] = D_\hbar(R(Q, \mathbf{v}))^{\mathfrak{g}(\mathbf{v})} / \left(D_\hbar(R(Q, \mathbf{v})) \cdot \mu(\mathfrak{g}(\mathbf{v}))\right)^{\mathfrak{g}(\mathbf{v})}, \tag{C.10}$$

where $D_\hbar(R(Q,\mathbf{v}))$ is the ring of $\hbar$-differential operators on $R(Q,\mathbf{v})$, i.e. the Rees algebra of $D(R(Q,\mathbf{v}))$ with respect to the filtration degree $\deg X^\beta_{a\alpha}=0, \deg \partial_{X^\beta_{a\alpha}}=1$. We observe that there is an isomorphism

$$\mathbb{C}^\hbar[\mathcal{M}(Q,\mathbf{v})]/(\hbar-1)\cong D(R(Q,\mathbf{v}))/\!\!/\,\mathfrak{g}(\mathbf{v})\cong \mathbb{C}_\hbar[\mathcal{M}(Q,\mathbf{v})]/(\hbar-1). \tag{C.11}$$

Then it follows from Rees construction that there is a graded $\mathbb{C}[\hbar]$-algebra homomorphism

$$\Phi:\mathbb{C}_\hbar[\mathcal{M}(Q,\mathbf{v})]\longrightarrow \mathbb{C}^\hbar[\mathcal{M}(Q,\mathbf{v})],$$

such that $\Phi[\hbar^{-1}]$ is isomorphism. We claim that $\Phi:\mathbb{C}_\hbar[\mathcal{M}(Q,\mathbf{v})]\to\mathbb{C}^\hbar[\mathcal{M}(Q,\mathbf{v})]$ is surjective. In fact,

$$\mathbb{C}_\hbar[\mathcal{M}(Q,\mathbf{v})]/(\hbar)=\mathrm{gr}\left(D(R(Q,\mathbf{v}))^{\mathfrak{g}(\mathbf{v})}\right)/\left(\mathrm{gr}D(R(Q,\mathbf{v}))\cdot\mu(\mathfrak{g}(\mathbf{v}))\right)^{\mathfrak{g}(\mathbf{v})},$$
$$\mathbb{C}^\hbar[\mathcal{M}(Q,\mathbf{v})]/(\hbar)=\mathrm{gr}\left(D(R(Q,\mathbf{v}))^{\mathfrak{g}(\mathbf{v})}\right)/\mathrm{gr}\left(\left(D(R(Q,\mathbf{v}))\cdot\mu(\mathfrak{g}(\mathbf{v}))\right)^{\mathfrak{g}(\mathbf{v})}\right),$$

and it is easy to see that $\left(\mathrm{gr}D(R(Q,\mathbf{v}))\cdot\mu(\mathfrak{g}(\mathbf{v}))\right)^{\mathfrak{g}(\mathbf{v})}\subset\mathrm{gr}\left(\left(D(R(Q,\mathbf{v}))\cdot\mu(\mathfrak{g}(\mathbf{v}))\right)^{\mathfrak{g}(\mathbf{v})}\right)$.

Lemma C.1.12. *The following statements are equivalent:*

(1) $\mathbb{C}_\hbar[\mathcal{M}(Q,\mathbf{v})]$ *is flat over* $\mathbb{C}[\hbar]$.

(2) $\Phi:\mathbb{C}_\hbar[\mathcal{M}(Q,\mathbf{v})]\to\mathbb{C}^\hbar[\mathcal{M}(Q,\mathbf{v})]$ *is isomorphism.*

(3) The natural map $\mathbb{C}[T^*R(Q,\mathbf{v})]/\!\!/\,\mathfrak{g}(\mathbf{v})\to\mathrm{gr}(D(R(Q,\mathbf{v}))/\!\!/\,\mathfrak{g}(\mathbf{v}))$ *is isomorphism.*

Proof. As we have explained, $\Phi[\hbar^{-1}]$ is isomorphism, so the kernel of Φ is $\hbar$-torsion. Since $\mathbb{C}^\hbar[\mathcal{M}(Q,\mathbf{v})]$ is $\mathbb{C}[\hbar]$-free by the Rees construction, the equivalence between (1) and (2) is obvious. The statement (3) is equivalent to $\Phi/(\hbar)$ being an isomorphism, thus (2) implies (3). Suppose that (3) holds, then the long exact sequence associated to $-\otimes^L_{\mathbb{C}[\hbar]}\mathbb{C}$ implies that $\ker(\Phi)=\hbar\cdot\ker(\Phi)$. However $D_\hbar(R(Q,\mathbf{v}))$ is graded with $\deg\hbar=1$ and the grading is bounded below, thus $\ker(\Phi)\supsetneq\hbar\cdot\ker(\Phi)$ unless $\ker(\Phi)=0$. This shows that (3) implies (2). ☐

Definition C.1.13. We say that **quantization commutes with reduction** if one of the equivalent statements in Lemma C.1.12 holds.

To state the next proposition, we need the following convention: let $p(\mathbf{v})$ be the function

$$p(\mathbf{v})=1+\sum_{a\in Q_1}\mathbf{v}_{h(a)}\mathbf{v}_{t(a)}-\sum_{i\in Q_0}\mathbf{v}_i\mathbf{v}_i. \tag{C.12}$$

Definition C.1.14. We say that $(Q, \mathbf{v})$ is *good* if

- $p(\mathbf{v}) \geq \sum_{t=1}^{r} p(\mathbf{v}^{(t)})$ for any decomposition $\mathbf{v} = \mathbf{v}^{(1)} + \cdots + \mathbf{v}^{(r)}$ into nonzero $\mathbf{v}^{(t)} \in \mathbb{N}^{Q_0}$.

Proposition C.1.15. *If $(Q, \mathbf{v})$ is good, then quantization of $\mathcal{M}(Q, \mathbf{v})$ commutes with reduction. In particular*

$$\mathbb{C}^{\hbar}[\mathcal{M}(Q, \mathbf{v})]/(\hbar) \cong \mathbb{C}[\mathcal{M}(Q, \mathbf{v})], \tag{C.13}$$

i.e. $\mathbb{C}^{\hbar}[\mathcal{M}(Q, \mathbf{v})]$ is a flat deformation of $\mathbb{C}[\mathcal{M}(Q, \mathbf{v})]$.

Proof. Since $(Q, \mathbf{v})$ is good, and Crawley-Boevey [46, Theorem 1.1] shows that in this case the classical moment map $\overline{\mu} : S^*(\mathfrak{g}(\mathbf{v})) \to \mathbb{C}[T^*R(Q, \mathbf{v})]$ ($\overline{\mu}$ is extended to the symmetric algebra of $\mathfrak{g}(\mathbf{v})$) is flat, in particular any sequence of basis of $\mathfrak{g}(\mathbf{v})$ is mapped to a regular sequence in $\mathbb{C}[T^*R(Q, \mathbf{v})]$, which is exactly the condition in Proposition C.1.6, so we have

$$\mathbb{C}^{\hbar}[\mathcal{M}(Q, \mathbf{v})]/(\hbar) = \mathrm{gr}\,(D(R(Q, \mathbf{v})) /\!\!/ \mathfrak{g}(\mathbf{v})) \cong \mathbb{C}[T^*R(Q, \mathbf{v})] /\!\!/ \mathfrak{g}(\mathbf{v}) = \mathbb{C}[\mathcal{M}(Q, \mathbf{v})].$$

□

C.1.16 Quantum Nakajima quiver variety

Let $(Q, \mathbf{v}, \mathbf{w})$ be a framed quiver with framing vector $\mathbf{w}$ (assume $\mathbf{w} \neq 0$). Following Crawley-Boevey, we define the associated unframed quiver $(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}})$ as $Q_0^{\mathbf{w}} = Q_0 \coprod \{\infty\}$ (union of vertices of Q with an extra vertex denoted by ∞), and arrows in $Q^{\mathbf{w}}$ are those from Q and for each vertex $i \in Q_0$ attach $\mathbf{w}_i$-copies of arrows from ∞ to i, and set $\mathbf{v}_i^{\mathbf{w}} = \mathbf{v}_i$ if $i \in Q_0$ and $\mathbf{v}_{\infty}^{\mathbf{w}} = 1$. From the construction we see that the group

$$G(\mathbf{v}^{\mathbf{w}}) = \prod_{i \in Q_0^{\mathbf{w}}} \mathrm{GL}(\mathbf{v}_i^{\mathbf{w}})/\mathbb{C}^{\times} \cong \prod_{i \in Q_0} \mathrm{GL}(\mathbf{v}_i) =: \mathrm{GL}(\mathbf{v}) \tag{C.14}$$

acts on $R(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}})$. Then we have the quantum moment map $\mu : \mathfrak{gl}(\mathbf{v}) \to D(R(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}}))$ defined by the equation (C.7).

We can consider the shift of μ. Namely there is a character $\chi : \mathrm{GL}(\mathbf{v}) \to \mathbb{C}^{\times}$ sending $(g_i \mid g_i \in \mathrm{GL}(\mathbf{v}_i))$ to $\prod_i \det(g_i)$. It gives rise to a $|Q_0|$-dimensional family of quantum moment maps $\mu_{\mathbf{t}\chi} : \mathfrak{gl}(\mathbf{v}) \to D(R(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}}))[\mathbf{t}]$, here $\mathbb{C}[\mathbf{t}] = \mathbb{C}[t_i \mid i \in Q_0]$, and $\mu_{\mathbf{t}\chi}$ acts on basis element e_{ρ}^{ν} of $\mathfrak{gl}(\mathbf{v}_i)$ as

$$\mu_{\mathbf{t}\chi}(e_{\rho}^{\nu}) = \mu(e_{\rho}^{\nu}) - \delta_{\rho}^{\nu} t_i.$$

Definition C.1.17. Define the deformed quantum Nakajima quiver variety as $\mathbb{C}[\hbar, \mathbf{t}]$-algebra

$$\mathbb{C}^{\hbar}[\mathcal{M}_{\mathbf{t}}(Q, \mathbf{v}, \mathbf{w})] = \mathrm{Rees}(D(R(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}})) /\!\!/_{\mathbf{t}\chi}\, \mathfrak{gl}(\mathbf{v})), \tag{C.15}$$

the RHS is the Rees algebra of $D(R(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}})) /\!\!/_{\mathbf{t}\chi}\, \mathfrak{gl}(\mathbf{v})$ with respect to the filtration degree $\deg X^{\beta}_{a\alpha} = 0, \deg \partial_{X^{\beta}_{a\alpha}} = \deg t_i = 1$.

Similarly, given a vector $\lambda \in \mathbb{C}^{Q_0}$, Define the λ-specialized quantum Nakajima quiver variety as $\mathbb{C}[\hbar]$-algebra

$$\mathbb{C}^{\hbar}[\mathcal{M}_{\lambda}(Q, \mathbf{v}, \mathbf{w})] = \mathrm{Rees}(D(R(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}})) /\!\!/_{\lambda\chi}\, \mathfrak{gl}(\mathbf{v})), \tag{C.16}$$

the RHS is the Rees algebra of $D(R(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}})) /\!\!/_{\lambda\chi}\, \mathfrak{gl}(\mathbf{v})$ with respect to the filtration degree $\deg X^{\beta}_{a\alpha} = 0, \deg \partial_{X^{\beta}_{a\alpha}} = 1$. It is elementray to see that

$$\mathbb{C}^{\hbar}[\mathcal{M}_{\mathbf{t}}(Q, \mathbf{v}, \mathbf{w})]/(t_i - \lambda_i \mid i \in Q_0) \cong \mathbb{C}^{\hbar}[\mathcal{M}_{\lambda}(Q, \mathbf{v}, \mathbf{w})]. \tag{C.17}$$

Proposition C.1.18. *If $(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}})$ is good, then $\mathbb{C}^{\hbar}[\mathcal{M}_{\mathbf{t}}(Q, \mathbf{v}, \mathbf{w})]$ is a free $\mathbb{C}[\hbar, \mathbf{t}]$-module, and*

$$\begin{aligned} \mathbb{C}^{\hbar}[\mathcal{M}_{\mathbf{t}}(Q, \mathbf{v}, \mathbf{w})]/(\hbar) &\cong \mathbb{C}[\mathcal{M}_{\mathbf{t}}(Q, \mathbf{v}, \mathbf{w})], \\ \mathbb{C}^{\hbar}[\mathcal{M}_{\lambda}(Q, \mathbf{v}, \mathbf{w})]/(\hbar) &\cong \mathbb{C}[\mathcal{M}_{\lambda}(Q, \mathbf{v}, \mathbf{w})]. \end{aligned} \tag{C.18}$$

In other words, $\mathbb{C}^{\hbar}[\mathcal{M}_{\mathbf{t}}(Q, \mathbf{v}, \mathbf{w})]$ and $\mathbb{C}^{\hbar}[\mathcal{M}_{\lambda}(Q, \mathbf{v}, \mathbf{w})]$ are quantizations of $\mathbb{C}[\mathcal{M}_{\mathbf{t}}(Q, \mathbf{v}, \mathbf{w})]$ and $\mathbb{C}[\mathcal{M}_{\lambda}(Q, \mathbf{v}, \mathbf{w})]$.

Proof. Isomorphisms (C.18) follow from Proposition C.1.15. In order to show the freeness of $\mathbb{C}^{\hbar}[\mathcal{M}_{\mathbf{t}}(Q, \mathbf{v}, \mathbf{w})]$ (equivalently $\mathbb{C}_{\hbar}[\mathcal{M}_{\mathbf{t}}(Q, \mathbf{v}, \mathbf{w})]$), as a $\mathbb{C}[\hbar, \mathbf{t}]$-module, we modify the Proposition C.1.6 by changing the definition of quantum moment map to its Rees version, i.e. $\forall a \in \mathfrak{g}, b \in A$,

$$[\mu(a), b] = \hbar\phi(a) \cdot b,$$

here $A = D_{\hbar}(R(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}}))[\mathbf{t}]$ and $\mathfrak{g} = \mathfrak{gl}(\mathbf{v})$. Introduce a filtration $F_{\bullet}D_{\hbar}(R(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}}))[\mathbf{t}]$ by setting $\deg X^{\beta}_{a\alpha} = \deg \hbar = \deg t_i = 0, \deg \partial_{X^{\beta}_{a\alpha}} = 1$, then

$$\mathrm{gr}_F D_{\hbar}(R(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}}))[\mathbf{t}] = \mathbb{C}[T^*R(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}})][\hbar, \mathbf{t}],$$

and $\overline{\mu}$ agrees with the classical moment map for $T^*R(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}})$, which maps a basis of $\mathfrak{gl}(\mathbf{v})$ to a regular sequence because we assume that $(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}})$ is good, therefore the same argument of Proposition C.1.6 shows that

$$\mathbb{C}[T^*R(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}})][\hbar, \mathbf{t}] /\!\!/ \mathfrak{gl}(\mathbf{v}) \cong \mathrm{gr}_F\left(\mathbb{C}_{\hbar}[\mathcal{M}_{\mathbf{t}}(Q, \mathbf{v}, \mathbf{w})]\right).$$

The LHS is a undeformed Hamiltonian reduction, since the deformation parameter t_i is in zeroth filtration part and is modulo out in the associated graded $\overline{\mu}$, therefore

$$\mathbb{C}[T^*R(Q^{\mathbf{w}},\mathbf{v}^{\mathbf{w}})][\hbar,\mathbf{t}] /\!\!/ \mathfrak{gl}(\mathbf{v}) = \left(\mathbb{C}[T^*R(Q^{\mathbf{w}},\mathbf{v}^{\mathbf{w}})] /\!\!/ \mathfrak{gl}(\mathbf{v})\right)[\hbar,\mathbf{t}],$$

thus $\mathrm{gr}_F\left(\mathbb{C}_\hbar[\mathcal{M}_{\mathbf{t}}(Q,\mathbf{v},\mathbf{w})]\right)$ is a free $\mathbb{C}[\hbar,\mathbf{t}]$-module. □

The following theorem is a special case of the main result of [163].

Theorem C.1.19. *If $(Q^{\mathbf{w}},\mathbf{v}^{\mathbf{w}})$ is good, then $\mathcal{M}_\lambda(Q,\mathbf{v},\mathbf{w})$ is reduced for all $\lambda\in\mathbb{C}^{Q_0}$.*

Remark C.1.20. It can be shown that if $\mathbb{C}[\mathcal{M}_{\mathbf{t}}(Q,\mathbf{v},\mathbf{w})]$ is a flat $\mathbb{C}[\mathbf{t}]$-module, then $(Q^{\mathbf{w}},\mathbf{v}^{\mathbf{w}})$ is good. This is a corollary of Crawley-Boevey's criterion on the flatness of classical moment map for quivers [46, Theorem 1.1]. We give a sketch of proof to this corollary. Let $\mu_{\mathrm{cl}}:T^*R(Q^{\mathbf{w}},\mathbf{v}^{\mathbf{w}})\to\mathfrak{gl}(\mathbf{v})^*$ be the geometric version of classical moment map. The affine space $Z:=\mathrm{Spec}\mathbb{C}[\mathbf{t}]$ embeds into $\mathfrak{gl}(\mathbf{v})^*$ as the dual of $\mathfrak{gl}(\mathbf{v})/[\mathfrak{gl}(\mathbf{v}),\mathfrak{gl}(\mathbf{v})]$. $\mathbb{C}[\mathcal{M}_{\mathbf{t}}(Q,\mathbf{v},\mathbf{w})]$ being a flat $\mathbb{C}[\mathbf{t}]$-module implies that $\mu_{\mathrm{cl}}^{-1}(Z)\to Z$ is dominant. It is easy to see that the set of $\lambda\in Z$ such that $\mathbf{v}^{\mathbf{w}}\in\Sigma_\lambda$ (see comments after [46, Theorem 1.2] for notation) contains the complement of union of finitely many hyperplanes in Z, therefore $\exists\lambda\in Z$ such that $\mathbf{v}^{\mathbf{w}}\in\Sigma_\lambda$ and $\mu_{\mathrm{cl}}^{-1}(\lambda)$ is nonempty. According to [46, Corollary 1.4], this implies that

$$\dim\mu_{\mathrm{cl}}^{-1}(\lambda)/\mathrm{GL}(\mathbf{v})=2p(\mathbf{v}^{\mathbf{w}}),$$

and by the flatness of $\mathcal{M}_{\mathbf{t}}(Q,\mathbf{v},\mathbf{w})$, we see that $\dim\mu_{\mathrm{cl}}^{-1}(0)/\mathrm{GL}(\mathbf{v})=2p(\mathbf{v}^{\mathbf{w}})$, and according to [48, Theorem 1.1], this in turn implies that $(Q^{\mathbf{w}},\mathbf{v}^{\mathbf{w}})$ is good.

Remark C.1.21. We can write the condition for $(Q^{\mathbf{w}},\mathbf{v}^{\mathbf{w}})$ being good in terms of framed quiver $(Q,\mathbf{v},\mathbf{w})$. Note that $p(\mathbf{v}^{\mathbf{w}})=p(\mathbf{v})+\mathbf{w}\cdot\mathbf{v}-1$, here $\mathbf{w}\cdot\mathbf{v}=\sum_{i\in Q_0}\mathbf{w}_i\mathbf{v}_i$, and $p(\mathbf{v})=1-\frac{1}{2}\mathbf{v}\cdot C_Q\mathbf{v}$, here C_Q is the Cartan matrix of Q. Any decomposition of $\mathbf{v}^{\mathbf{w}}$ into nonzero element in $\mathbb{N}^{Q_0^{\mathbf{w}}}$ is of the form $\mathbf{v}^{(0)\mathbf{w}}+\mathbf{v}^{(1)}+\cdots+\mathbf{v}^{(r)}$, where $\mathbf{v}=\mathbf{v}^{(0)}+\mathbf{v}^{(1)}+\cdots+\mathbf{v}^{(r)}$ is a decomposition of $\mathbf{v}$ into elements in $\mathbb{N}^{Q_0}$ such that $\mathbf{v}^{(t)}\neq0$ for $t>0$. Thus $(Q^{\mathbf{w}},\mathbf{v}^{\mathbf{w}})$ is good iff

$$\sum_{t=1}^{r}\mathbf{w}\cdot\mathbf{v}^{(t)}\geq r+\sum_{0\leq t<u\leq r}\mathbf{v}^{(t)}\cdot C_Q\mathbf{v}^{(u)},\tag{C.19}$$

for all decomposition $\mathbf{v}=\mathbf{v}^{(0)}+\mathbf{v}^{(1)}+\cdots+\mathbf{v}^{(r)}$ into elements in $\mathbb{N}^{Q_0}$ such that $\mathbf{v}^{(t)}\neq0$ for $t>0$.

Remark C.1.22. Nakajima gave a sufficient condition for $(Q^{\mathbf{w}},\mathbf{v}^{\mathbf{w}})$ being good in some cases [127, Proposition 10.5]. Namely, assume that Q has no edge loop, and

(1) $\mathbf{w}\cdot\delta \geq 2$ for all imaginary root δ of Q,

(2) $\mathbf{w} - C_Q\mathbf{v} \in \mathbb{Z}_{\geq 0}^{Q_0}$,

then $(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}})$ is good. For example, if Q is of finite type (i.e. ADE quivers), then $(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}})$ is good if $\mathbf{w} - C_Q\mathbf{v} \in \mathbb{Z}_{\geq 0}^{Q_0}$.

C.1.23 Sheaf version of quantization

Let $(Q, \mathbf{v}, \mathbf{w})$ be a framed quiver such that $(Q^{\mathbf{v}}, \mathbf{v}^{\mathbf{w}})$ is good, choose a generic stability $\theta \in \mathbb{Q}^{Q_0}$, then the stable moduli space $\mathcal{M}^\theta_{\mathbf{t}}(Q, \mathbf{v}, \mathbf{w})$ is smooth over the base $\mathrm{Spec}\mathbb{C}[\mathbf{t}]$. Moreover the natural projection $p : \mathcal{M}^\theta_{\mathbf{t}}(Q, \mathbf{v}, \mathbf{w}) \to \mathcal{M}_{\mathbf{t}}(Q, \mathbf{v}, \mathbf{w})$ is projective, and for all $\lambda \in \mathbb{C}^{Q_0}$, $p_\lambda : \mathcal{M}^\theta_\lambda(Q, \mathbf{v}, \mathbf{w}) \to \mathcal{M}_\lambda(Q, \mathbf{v}, \mathbf{w})$ is a symplectic resolution. By the construction in [115], there is a sheaf of flat $\mathbb{C}[\![\hbar]\!]$-algebras on $\mathcal{M}^\theta_{\mathbf{t}}(Q, \mathbf{v}, \mathbf{w})$, denote by $\widetilde{\mathcal{O}}_{\mathcal{M}^\theta_{\mathbf{t}}}$, such that $\widetilde{\mathcal{O}}_{\mathcal{M}^\theta_{\mathbf{t}}}/(\hbar)$ is the structure sheaf $\mathcal{O}_{\mathcal{M}^\theta_{\mathbf{t}}}$. By [115] there is a natural $\mathbb{C}[\hbar, \mathbf{t}]$-algebra homomorphism

$$\mathbb{C}_\hbar[\mathcal{M}^\theta_{\mathbf{t}}(Q, \mathbf{v}, \mathbf{w})] \to \Gamma\left(\mathcal{M}^\theta_{\mathbf{t}}(Q, \mathbf{v}, \mathbf{w}), \widetilde{\mathcal{O}}_{\mathcal{M}^\theta_{\mathbf{t}}}\right),$$

and moreover we have

Proposition C.1.24. *Under the above homomorphism $\mathbb{C}_\hbar[\mathcal{M}^\theta_{\mathbf{t}}(Q, \mathbf{v}, \mathbf{w})]$ is identified with $\mathbb{C}^\times$-finite elements in $\Gamma\left(\mathcal{M}^\theta_{\mathbf{t}}(Q, \mathbf{v}, \mathbf{w}), \widetilde{\mathcal{O}}_{\mathcal{M}^\theta_{\mathbf{t}}}\right)$, where $\mathbb{C}^\times$ acts on quiver path generators with weight one and on $\hbar, \mathbf{t}$ with weight two.*

C.1.25 Calogero representation

Let $(Q, \mathbf{v}, \mathbf{w})$ be a framed quiver such that $(Q^{\mathbf{v}}, \mathbf{v}^{\mathbf{w}})$ is good, and moreover assume that the θ-stable locus of $R(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}})$, denote by $R^\theta(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}})$, is nonempty, then $T^*R^\theta(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}})$ is contained in the θ-stable locus of $T^*R(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}})$, and it is $\mathrm{GL}(\mathbf{v})$-stable, therefore there is an open embedding

$$T^*R^\theta(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}}) /\!\!/_{\mathbf{t}\chi} \mathrm{GL}(\mathbf{v}) \hookrightarrow \mathcal{M}^\theta_{\mathbf{t}}(Q, \mathbf{v}, \mathbf{w}).$$

Note that $T^*R^\theta(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}}) /\!\!/_{\mathbf{t}\chi} \mathrm{GL}(\mathbf{v})$ is the $\mathbf{t}$-twisted cotangent bundle $T^*_{\mathbf{t}}\mathcal{N}^\theta(Q, \mathbf{v}, \mathbf{w})$ of the affine quotient

$$\mathcal{N}^\theta(Q, \mathbf{v}, \mathbf{w}) = R^\theta(Q^{\mathbf{w}}, \mathbf{v}^{\mathbf{w}})/\mathrm{GL}(\mathbf{v}).$$

Here $\mathbf{t}$-twisted cotangent bundle is defined as the affine bundle over $\mathcal{N}^\theta(Q,\mathbf{v},\mathbf{w})\times\mathrm{Spec}\mathbb{C}[\mathbf{t}]$ modelled on $T^*\mathcal{N}^\theta(Q,\mathbf{v},\mathbf{w})\times\mathrm{Spec}\mathbb{C}[\mathbf{t}]$ and determined by the class $\sum_{i\in Q_0} c_1(\mathcal{L}_i)\otimes t_i \in \mathrm{H}^1(\Omega^1)$, where $\mathcal{L}_i$ is the tautological line bundle associated to the i'th node.

After passing to quantization, $\widetilde{\mathcal{O}}_{\mathcal{M}^\theta_{\mathbf{t}}}|_{T^*_{\mathbf{t}}\mathcal{N}^\theta(Q,\mathbf{v},\mathbf{w})}$ is naturally identified with $\hbar$-adic completion of the sheaf of $\mathbf{t}$-twisted $\hbar$-differential operators on $\mathcal{N}^\theta(Q,\mathbf{v},\mathbf{w})$. Since $T^*_{\mathbf{t}}\mathcal{N}^\theta(Q,\mathbf{v},\mathbf{w})$ is open and dense in $\mathcal{M}^\theta_{\mathbf{t}}(Q,\mathbf{v},\mathbf{w})$, composing the embedding in Proposition C.1.24 with the restriction map $\Gamma\left(\mathcal{M}^\theta_{\mathbf{t}}(Q,\mathbf{v},\mathbf{w}),\widetilde{\mathcal{O}}_{\mathcal{M}^\theta_{\mathbf{t}}}\right)\hookrightarrow\Gamma\left(T^*_{\mathbf{t}}\mathcal{N}^\theta(Q,\mathbf{v},\mathbf{w}),\widetilde{\mathcal{O}}_{\mathcal{M}^\theta_{\mathbf{t}}}\right)$, we obtain an embedding of $\mathbb{C}[\hbar,\mathbf{t}]$-algebras

$$\mathbb{C}_\hbar[\mathcal{M}^\theta_{\mathbf{t}}(Q,\mathbf{v},\mathbf{w})]\hookrightarrow D^{\mathbf{t}}_\hbar(\mathcal{N}^\theta(Q,\mathbf{v},\mathbf{w})), \tag{C.20}$$

where the right-hand-side is the ring of $\mathbf{t}$-twisted $\hbar$-differential operators on $\mathcal{N}^\theta(Q,\mathbf{v},\mathbf{w})$. We call such embedding a Calogero representation of $\mathbb{C}_\hbar[\mathcal{M}^\theta_{\mathbf{t}}(Q,\mathbf{v},\mathbf{w})]$.

Appendix D

Integrals in the 5d CS Perturbative Calculations

Lemma 1.
We will compute the following integral.

$$\epsilon_1\epsilon_2^2\int_{v_1} dw_1 \wedge dz_1 \wedge \partial_{z_1}P_1(v_0,v_1) \wedge \partial_{z_2}\partial_{w_1}P_2(v_1,v_2)(z_1^2 w_1 \partial_{z_1}^2\partial_{w_1}A) \tag{D.1}$$

Computing the partial derivatives, we can re-write it as

$$\epsilon_1\epsilon_2^2\left(\frac{\bar{z}_1}{d_{01}^2}\frac{\bar{w}_1}{d_{12}^4}(w_1 z_1 \bar{z}_2)\right)[P(v_0,v_1)\wedge dw_1 \wedge z_1 dz_1 \wedge P(v_1,v_2)]$$

Note that we ignore all constant factors here. We see that

$$\begin{aligned}P(v_0,v_1)\wedge P(v_1,v_2) = \frac{d\bar{z}_1 d\bar{w}_1 dt_1}{d_{01}^5 d_{12}^5}(&\bar{z}_{01}\bar{w}_{12}dt_2 - \bar{z}_{01}t_{12}d\bar{w}_2 + \bar{w}_{01}t_{12}d\bar{z}_2 \\ &- \bar{w}_{01}\bar{z}_{12}dt_2 + t_{01}\bar{z}_{12}d\bar{w}_2 - t_{01}\bar{w}_{12}d\bar{z}_2)\end{aligned}$$

Including $\wedge dw_1 \wedge (z_1 dz_1)\wedge$, we can simplify it:

$$\begin{aligned}&P(v_0,v_1)\wedge P(v_1,v_2)\wedge(w_1 dw_1)\wedge(z_1 dz_1) = d\bar{z}_1 dz_1 dw_1 d\bar{w}_1 dt_1\left(|z_1|^2|w_1|^2\bar{z}_2\right)\times \\ &\left[\partial_{\bar{z}_0}\left(\frac{\bar{z}_{01}\bar{w}_{12}dt_2 - \bar{z}_{01}t_{12}d\bar{w}_2 + \bar{w}_{01}t_{12}d\bar{z}_2 - \bar{w}_{01}\bar{z}_{12}dt_2 + t_{01}\bar{z}_{12}d\bar{w}_2 - t_{01}\bar{w}_{12}d\bar{z}_{12}}{d_{01}^5 d_{12}^9}\right)\right. \\ &\left. - \frac{\partial_{\bar{z}_0}(\bar{z}_{01}\bar{w}_{12}dt_2 - \bar{z}_{01}t_{12}d\bar{w}_2 + \bar{w}_{01}t_{12}d\bar{z}_2 - \bar{w}_{01}\bar{z}_{12}dt_2 + t_{01}\bar{z}_{12}d\bar{w}_2 - t_{01}\bar{w}_{12}d\bar{z}_{12})}{d_{01}^5 d_{12}^9}\right]\end{aligned}$$

By integration by parts, the the integral over t_1, z_1, $\bar{z}_1$, w_1, $\bar{w}_1$ of all terms in the first two lines vanishes.

Therefore, we are left with

$$-\int_{v_1} dt_1 dz_1 d\bar{z}_1 dw_1 d\bar{w}_1 \frac{|z_1|^2|w_1|^2\bar{z}_2(\bar{w}_{12}dt_2 - t_{12}d\bar{w}_2)}{d_{01}^5 d_{12}^9} \tag{D.2}$$

Lemma 2.

We can use Feynman integral technique to convert (D.2) to the following:

$$\begin{aligned}
&\int_{v_1}\int_0^1 dx \frac{\Gamma(7)}{\Gamma(5/2)\Gamma(9/2)} \frac{\sqrt{x^3(1-x)^7}|z_1|^2|w_1|^2\bar{z}_2(\bar{w}_{12}dt_2 - t_{12}d\bar{w}_2)}{((1-x)(|z_1|^2+|w_1|^2+t_1^2)+x(|z_{12}|^2+|w_{12}|^2+t_{12}^2))^7}\\
&= \int_{v_1}\int_0^1 dx \frac{(\Gamma \text{ factors})\sqrt{x^3(1-x)^7}|z_1|^2|w_1|^2\bar{z}_2(\bar{w}_{12}dt_2 - t_{12}d\bar{w}_2)}{(|z_1 - xz_2|^2+|w_1 - xw_2|^2+(t_1 - xt_2)^2 + x(1-x)(|z_2|^2+|w_2|^2+t_2^2))^7}
\end{aligned}$$

Shift the integral variables as

$$z_1 \to z_1 + xz_2, \quad w_1 \to w_1 + xw_2, \quad t_1 \to t_1 + xt_2$$

Then the above becomes

$$\begin{aligned}
\int_{v_1}\int_0^1 dx \frac{\Gamma(7)}{\Gamma(5/2)\Gamma(9/2)} &\frac{\sqrt{x^3(1-x)^7}|z_1 + xz_2|^2|w_1 + xw_2|^2\bar{z}_2}{(|z_1|^2+|w_1|^2+t_1^2+x(1-x)(|z_2|^2+|w_2|^2+t_2^2))^7}\\
&\times ((\bar{w}_1 + (x-1)\bar{w}_2)dt_2 - (t_1 + (x-1)t_2)d\bar{w}_2)
\end{aligned}$$

Drop terms with odd number of t_1 and terms that has holomorphic or anti-holomorphic dependence on z_1 or w_1:

$$\int_{v_1}\int_0^1 dx \frac{\Gamma(7)}{\Gamma(5/2)\Gamma(9/2)} \frac{\sqrt{x^3(1-x)^9}(|z_1|^2 + x^2|z_2|^2)(|w_1|^2 + x^2|w_2|^2)\bar{z}_2(\bar{w}_2 dt_2 - t_2 d\bar{w}_2)}{(|z_1|^2+|w_1|^2+t_1^2+x(1-x)(|z_2|^2+|w_2|^2+t_2^2))^7}$$

After doing the v_1 integral using Mathematica with the integral measure $dt_1 dz_1 d\bar{z}_1 dz_2 d\bar{z}_2$, we get

$$\bar{z}_2(\bar{w}_2 dt_2 - t_2 d\bar{w}_2)\left(\frac{c_1}{d_{02}^5} + \frac{c_2 w_2^2}{d_{02}^7} + \frac{c_3 z_2^2}{d_{02}^7} + \frac{c_4 z_2^2 w_2^2}{d_{02}^9}\right) \tag{D.3}$$

Lemma 3.

We will compute the integral over the second vertex.

$$\begin{aligned}&\int_{v_2} \mathcal{P} \wedge \partial_{w_2} P_3(v_2, v_3) \wedge dz_2 \wedge dw_2 (z_2 w_2^2 \partial_{z_2} \partial_{w_2}^2 A) \\ =&\int_{v_2} \mathcal{P} \wedge \frac{\bar{w}_2(\bar{z}_{23} d\bar{w}_2 dt_2 - \bar{w}_{23} d\bar{z}_2 dt_2 + t_{23} d\bar{z}_2 d\bar{w}_2)}{d_{23}^7} \wedge dw_2 \wedge dz_2\end{aligned} \tag{D.4}$$

Now, compute the integrand:

$$\begin{aligned}&\frac{\bar{z}_2(\bar{w}_2 dt_2 - t_2 d\bar{w}_2)\bar{w}_2(\bar{z}_{23} d\bar{w}_2 dt_2 - \bar{w}_{23} d\bar{z}_2 dt_2 + t_{23} d\bar{z}_2 d\bar{w}_2)}{d_{02}^5 d_{23}^7} \wedge dw_2 \wedge dz_2 \\ =&\frac{|z_2|^2 |w_2|^4 (t_2 - t_3 - t_2)}{d_{02}^5 d_{23}^7} dt_2 d\bar{z}_2 d\bar{w}_2 dw_2 dz_2 \\ =&-\frac{|z_2|^2 |w_2|^4 t_3}{d_{02}^5 d_{23}^7} dt_2 d\bar{z}_2 d\bar{w}_2 dw_2 dz_2 \quad \text{substitute } t_3 = \epsilon, \text{ then,} \\ =&-\frac{|z_2|^2 |w_2|^4 \epsilon}{d_{02}^5 d_{23}^7} dt_2 d\bar{z}_2 d\bar{w}_2 dw_2 dz_2\end{aligned} \tag{D.5}$$

We can rescale $\epsilon \to 1$, without loss of generality, then it becomes

$$-\frac{|z_2|^2 |w_2|^4}{d_{02}^5 d_{23}^7} dt_2 d\bar{z}_2 d\bar{w}_2 dw_2 dz_2 \tag{D.6}$$

Lemma 4.
Now, it remains to evaluate the delta function at the third vertex. In other words, substitute

$$w_3 \to 0, \quad z_3 \to 0, \quad t_3 \to \epsilon = 1 \tag{D.7}$$

and use Feynman technique to convert the above integral into

$$\begin{aligned}&-\frac{\Gamma(6)}{\Gamma(5/2)\Gamma(7/2)} \int_0^1 dx \int_{v_2} \frac{\sqrt{x^3(1-x)^5} |z_2|^2 |w_2|^4}{(x(z_2^2 + w_2^2 + (t_2 - 1)^2) + (1-x)(z_2^2 + w_2^2 + t_2^2))^6} \\ =&-\frac{\Gamma(6)}{\Gamma(5/2)\Gamma(7/2)} \int_0^1 dx \int_{v_2} \frac{\sqrt{x^3(1-x)^5} |z_2|^2 |w_2|^4}{(z_2^2 + w_2^2 + (t_2 - x)^2 + x(1-x))^6} \\ =&-\frac{\Gamma(6)}{\Gamma(5/2)\Gamma(7/2)} \int_0^1 dx \int_{v_2} \frac{\sqrt{x^3(1-x)^5} |z_2|^2 |w_2|^4}{(z_2^2 + w_2^2 + t_2^2 + x(1-x))^6}\end{aligned}$$

In the second equality, we shift t_2 to $t_2 + x$. After doing v_2 integral, it reduces to

$$\frac{\Gamma(6)}{\Gamma(5/2)\Gamma(7/2)} \frac{\pi}{2880} \int_0^1 dx x(1-x)^2 = \frac{\Gamma(6)}{\Gamma(5/2)\Gamma(7/2)} \frac{\pi}{2880}$$

Finally, re-introduce all omitted constants:

$$(\text{First Term}) = \frac{\Gamma(6)}{\Gamma(5/2)\Gamma(7/2)} \frac{\Gamma(7)}{\Gamma(5/2)\Gamma(9/2)} (2\pi)^2 (2\pi)^2 \frac{\pi}{2880} \tag{D.8}$$

Similarly, we can compute all the others without any divergence.

$$\begin{aligned}
(\text{Second Term}) &= \frac{\Gamma(6)}{\Gamma(5/2)\Gamma(7/2)} \frac{\Gamma(7)}{\Gamma(5/2)\Gamma(9/2)} (2\pi)^2 (2\pi)^2 \frac{\pi}{5760} \\
(\text{Third Term}) &= \frac{\Gamma(6)}{\Gamma(5/2)\Gamma(7/2)} \frac{\Gamma(7)}{\Gamma(5/2)\Gamma(9/2)} (2\pi)^2 (2\pi)^2 \frac{\pi}{8640} \\
(\text{Fourth Term}) &= \frac{\Gamma(6)}{\Gamma(5/2)\Gamma(7/2)} \frac{\Gamma(7)}{\Gamma(5/2)\Gamma(9/2)} (2\pi)^2 (2\pi)^2 \frac{\pi}{20160}
\end{aligned} \tag{D.9}$$

Hence, every terms in (D.3) are integrated into finite terms.