Clique minors in dense matroids

by

Fernanda Rivera Omana

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2022

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The objective of this thesis is to bound the number of points a $U_{2,\ell+2}$- and $M(K_{k+1})$-minor-free matroid has. We first prove that a sufficiently large matroid will contain a structure called a tower. We then use towers to find a complete minor in a matroid with no $U_{2,\ell+2}$-minor.

Acknowledgements

I would like to thank my supervisor Peter Nelson for all his guidance and support. I'm extremely grateful to him for all his availability and his patience.

I would like to thank Jim Geelen and Bruce Richter for taking the time to read this thesis and for all their suggestions.

I am also grateful to my family and friends for all their moral support during this process. Specially when it involved sweets and tea.

Table of Contents

List of Figures

Chapter 1

Preliminaries

This thesis studies the maximal density of matroids that do not contain the complete graphic matroid $M(K_{k+1})$ and the rank-2 uniform matroid on $\ell + 2$ elements $U_{2,\ell+2}$ as a minor. The results in chapters 2 and 3 use some of the ideas from a proof for the case of binary matroids given in personal correspondence between Sergey Norin and Peter Nelson. We will start with the basic definitions.

The following definitions are standard and can be found in [12] by Oxley. A *matroid* M is a pair (E, r) where E is a finite *ground* set and $r : 2^E \to \mathbb{Z}$ is a *rank function* that satisfies the following axioms:

(R1) If $X \subseteq E$, then $0 \leqslant r(X) \leqslant |X|$.

(R2) If $X \subseteq Y \subseteq E$, then $r(X) \leqslant r(Y)$.

(R3) $r(X \cup Y) + r(X \cap Y) \leqslant r(X) + r(Y)$ for all $X, Y \subseteq E$.

We use r_M to refer to the rank function of M if needed.

The *rank of the matroid* M is defined as $r(E)$. A set $X \subseteq E$ is *independent* if $r(X) = |X|$, otherwise we say a set is *dependent*. A maximal independent set of M is called a *basis* and a minimal dependent set is called a *circuit*. A circuit of size one is called a loop and if two elements form a circuit of M, they are called a *parallel pair*. A circuit of size three is called a *triangle*. Given a matroid M, the *dual* of M is the matroid M^* with rank function $r_{M^*}(A) := |A| - r(M) + r_M(E \setminus A)$. The circuits of the dual are called cocircuits of M.

The *closure* of $X \subseteq E$ is the set $\mathrm{cl}_M(X) = \{x \in E : r(X \cup x) = r(X)\}$. A set $F \subseteq E$ is called a *flat* of M if $\mathrm{cl}_M(F) = F$. A *hyperplane* is a flat of rank $r(M) - 1$, a *line* of M is a flat of rank 2 and a flat of rank 1 is called a *point*; we will use $\varepsilon(M)$ to denote the number of points of M. We say $X \subseteq E(M)$ *spans* $Y \subseteq E(M)$ if $Y \subseteq \mathrm{cl}_M(X)$. We will denote by $\epsilon(M)$ the number of points of a matroid M.

Given $X \subseteq E$, the *restriction* of M to X is the pair (X, r_X) where r_X is the restriction of the function r to the power set of X. The *deletion* of X in M is the restriction of M to $E \setminus X$. The *contraction* of X in M is the matroid $M/X = (E \setminus X, r_{M/X})$ where $r_{M/X}(A) = r_M(A \cup X) - r_M(X)$ for all $A \subseteq E \setminus X$. We say N is a *minor* of M if $N = M/C \backslash D$ for some $C, D \subseteq E$. The *simplification* of M, denoted by $\mathrm{si}(M)$, is the restriction of M to its set of points.

Let E be the set of edges of a graph G. The matroid $M(G)$ is the matroid with ground set E and which circuits consist of all the the cycles of G. In particular, we will be interested in the matroids $M(K_{k+1})$, where K_{k+1} is a complete graph on $k+1$ vertices. We define the uniform matroid $U_{m,n}$ to be the matroid on n elements where a set A is independent if and only if $|A| \leqslant m$. Note that when $m = 2$, the ground set of $U_{2,n}$ is a line with n elements.

A natural question that arises is the following; What is the maximum number of elements a simple matroid can have without containing $M(K_{k+1})$ and $U_{2,\ell+2}$ as a minor? In the case that a rank-n matroid does not contain $M(K_3)$ as a minor, the matroid cannot contain any circuit and hence is isomorphic to $U_{n,n}$. Kung in [8] gave a singly exponential bound for the case when excluding $M(K_4)$ and $U_{2,\ell+2}$ as minor. Latter, Geelen and Whittle in [6] prove the existence of a linear function in the rank, λ, dependent on k and ℓ such that every simple matroid M without an $U_{2,\ell+2}$- or $M(K_{k+1})$-minor contains at most $\lambda r(M)$ elements.

Theorem 1.1 (Geelen, Whittle)**.** *For any positive integers k and ℓ, there exists an integer $\lambda = \lambda(k, \ell)$ such that every simple matroid M with no $U_{2,\ell+2}$- or $M(K_{k+1})$-minor satisfies $|E(M)| \leqslant \lambda r(M)$.*

Geelen proved in [3] that $\lambda(k, \ell) \leqslant \ell^{\ell^{3k}}$, a doubly exponential bound. In the next section, for each prime power q we will show via a simple construction an example of a $U_{2,q+2}$-minor-free and $M(K_{k+1})$-minor-free matroid M that satisfies $|E(M)| = \frac{q^{k-2}-1}{q-1} + (n-k+2)q^{k-2}$. In particular this implies that any upper bound on λ is at least singly exponential. The main result of this thesis gives a singly exponential upper bound;

Theorem 1.2. *If M is a $U_{2,\ell+2}$- and $M(K_{k+1})$-minor-free simple matroid, then*

$$|E(M)| \leqslant 2(720)^2 k^4 (\ell + 1)^2 \log(k) r(M).$$

Where the logarithm here and all the other logarithms are natural. One can rewrite Theorem 1.1 as follows;

Theorem 1.1 (Geelen, Whittle [6]). *Given a minor-closed class of matroids $\mathcal{M}$, either*

1. *there exists $\lambda \in \mathbb{R}$ such that $\varepsilon \leqslant \lambda r(M)$ for all simple matroids $M \in \mathcal{M}$,*
2. *$\mathcal{M}$ contains all graphic matroids, or*
3. *$\mathcal{M}$ contains all simple rank-2 matroids.*

This theorem is generalized to the Growth rate theorem in [4, 5] by Geelen, Kabell, Kung and Whittle.

Growth rate theorem 1.3 (Geelen, Kabell, Kung and Whittle). *If $\mathcal{M}$ is a minor-closed class of matroids, then either*

1. *there exists $c \in \mathbb{R}$ such that $|E(M)| \leqslant cr(M)$ for all simple matroids $M \in \mathcal{M}$,*
2. *$\mathcal{M}$ contains all graphic matroids and there exists $c \in \mathbb{R}$ such that $|E(M)| \leqslant c(r(M))^2$ for all simple matroids $M \in \mathcal{M}$,*
3. *there is a prime-power q and $c \in \mathbb{R}$ such that $\mathcal{M}$ contains all $GF(q)$-representable matroids and $|E(M)| \leqslant cq^{r(M)}$ for all simple matroids $M \in \mathcal{M}$, or*
4. *$\mathcal{M}$ contains all simple rank-2 matroids.*

Note that if a matroid M is $M(H)$-minor-free for a simple graph H, then M is $M(K_{|V(H)|})$-minor-free. Our main result implies the following corollaries which improve the bound on c in Theorem 1.3.

Corollary 1.3.1. *Let $\ell \geqslant 2$ and $k \geqslant 4$, if H is a simple graph on k vertices and $\mathcal{M}$ is the class of matroids with no $U_{2,\ell+2}$-minor or $M(H)$-minor, then*

$$\epsilon(M) \leqslant \ell^{4(10)^6(\ell+1)^2k^4\log(k)}r(M)$$

for all simple matroids $M \in \mathcal{M}$.

Corollary 1.3.2. *Let $\ell \geqslant 2$ and $k \geqslant 3$, if H is a simple graph on k vertices and $\mathcal{M}$ is the class of matroids with no $U_{2,\ell+2}$-minor or $M(H)$-minor, then*

$$\epsilon(M) \leqslant \ell^{(4\alpha+o_k(1))(\ell+1)^2k^4\log(k)}r(M)$$

for all simple matroids $M \in \mathcal{M}$ and where $\alpha = 0.319\ldots$ is an explicit constant.

When our main theorem is specialized to the case of graphic matroids we obtain the following;

Corollary 1.3.3. *Let $t \geqslant 4$, if G is a simple graph on n vertices with no $M(K_t)$-minor, then $|E(G)| \leqslant \ell^{c(\ell+1)^2 t^4 \log(t)}(n-1)$ for some constant c.*

The bound for the graphic case has been studied before by bounding the density needed for a graph to contain K_t as a minor, where the *density* of a graph G is defined as $\frac{|E(G)|}{|V(G)|}$. Mader[10] proved that all graphs with density at least 2^{t-3} contain a complete graph on t vertices as a minor. The bound given in Corollary 1.3.3, although it has the best possible order for matroids, is still much weaker than the best known bounds for graphic matroids. It has been proven in [7] that the density needed for the graphic case is of the order of $t\sqrt{\log t}$, see Proposition 1.11. In fact, we use the stronger bound in the proof of Corollary 1.3.3.

With this in mind, for any graph H, we define the function $d(H)$ to be the infimum number d of the set of positive real numbers such that if G is a graph with $\frac{|E(G)|}{|V(G)|} \geqslant d$, then G contains H as a minor. In the case of the complete graph, we will simply write $d(t)$ instead of $d(K_t)$. Thomason [14] prove that $d(t) = (\alpha + o(1))t\sqrt{\log t}$, where $\alpha = 0.319\ldots$ is explicitly given. Using random graphs, Bollobás, Catlin and Erdös in [1] proved that this bound is optimal for K_t-minor-free graphs up to a constant factor for t large enough. For the more general setting of excluding any graph rather than a complete graph, Reed and Wood [13] gave an upper bound for sufficiently large densities. Latter, Norin, Reed, Thomason and Wood [11] proved this is the best bound possible up to a constant factor.

1.1 Round matroids and Crowns

A matroid M *splits* if $E(M) = F_1 \cup F_2$ where F_1 and F_2 are proper flats of M. We will say M is round if it does not split. Note that if M splits, we may assume without loss of generality that F_1 and F_2 are hyperplanes. The following two theorems regarding split matroids are well known.

Proposition 1.4. *If $M = M(G)$ is a simple graphic matroid and G does not contain any isolated vertices, then M is round if and only if G is a complete graph.*

Proof. Let G be the graph associated to M. Suppose that there exist $v, u \in V(G)$ such that $uv \notin E(G)$. Let $F_1 = E(G[V(G) \setminus \{u\}])$ and $F_2 = E(G[V(G) \setminus \{v\}])$. Note that if

$e \in E(G)$, then $e \in F_1$ or $e \in F_2$. Furthermore, by construction F_1, F_2 are hyperplanes of $M(G)$. Therefore, the matroid M splits.

Now suppose that there exist two proper flats F_1, F_2 of M such that $E(M) = F_1 \cup F_2$. As F_i is a flat of a graphic matroid M for each $i = 1, 2$, there exists $S_i \subseteq V(G)$ such that $F_i = E(G[S_i])$. Furthermore, there exists an edge $e_1 = u_1v_1$ such that $e_1 \in F_1 \setminus F_2$. Since $e_1 \notin E(G[S_2])$, either $u_1 \notin S_2$ or $v_1 \notin S_2$. Without loss of generality suppose that $v_1 \notin S_2$. Similarly, there exists a vertex $v_2 \in S_2 \setminus S_1$. Finally note that the edge $v_1v_2 \notin E(G[S_1]), E(G[S_2])$. Therefore, we have $v_1v_2 \notin E(G)$. We conclude G is not a complete graph. □

Lemma 1.5. *If M is a round matroid, then so is M/e.*

Proof. First note that as the complement of a hyperplane is a cocircuit M is the union of two different hyperplanes if and only if M has two disjoint cocircuit. That is, the matroid M is round if and only if M^* has no disjoint circuits.

Suppose that M is a round matroid, then M^* has no disjoint circuits. Since, this property is preserved by deleting elements, then $M^* \setminus e$ has no disjoint circuits. Thus, we get that $(M^* \setminus e)^* = M/e$ has no disjoint cocircuits. It follows that M/e is round. □

We will now construct an example of a $U_{2,\ell+2}$-minor-free and $M(K_{k+1})$-minor-free matroid with singly exponential number of elements, as mentioned earlier. Given a basis B of the projective geometry $G = PG(n-1, q)$, an $(n, k, q)-$*crown* is the matroid M obtained from a set $K \subseteq B$ of size k by considering the restriction of G to the union of the closures of $K \cup e$ for all $e \in B \setminus K$. That is,

$$M = G| \left(\bigcup_{e \in B \setminus K} \mathrm{cl}_G(K \cup e) \right).$$

An example of a $(8, 6, q)$-crown can be seen in Figure 1.1.

Proposition 1.6. *If M is an (n, k, q)-crown and $e \in E(M)$, then $si(M/e)$ is an $(n-1, k', q)-$crown, where $k' \in \{k-1, k\}$.*

Proof. Let $e \in M$, then G/e which is isomorphic to $PG(n-2, q)$. If $e \notin \mathrm{cl}_G(K)$, then there exists $b \in B \setminus K$ such that $e \in \mathrm{cl}_G(K \cup b)$. In this case, we have that $\mathrm{cl}_G(K \cup b)$ is projected into B. Thus, the set $B' := B - b$ is a basis of G/e and $M/e = \bigcup_{x \in B' \setminus K} \mathrm{cl}_{G/e}(K \cup x)$, hence M/e is an $(n-1, k, q)-$crown. Now, if $e \in \mathrm{cl}_G(K)$, then $B' = B - e$ is a basis of G/e and $M/e = \bigcup_{x \in B' \setminus (K-e)} \mathrm{cl}_{G/e}((K - e) \cup x)$ where $|K - e| = k - 1$. Therefore, the matroid M/e is an $(n-1, k-1, q)-$crown. □

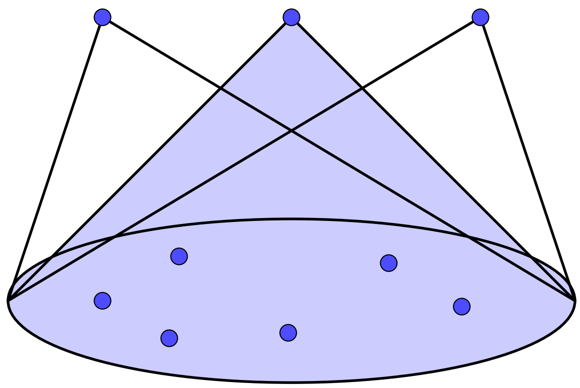

Figure 1.1: $(8, 6, q)$-Crown.

Proposition 1.7. *An (n, k, q)-crown is not round for $k < n-1$.*

Proof. Let $e, f \in B \setminus K$ and define $F_1 = \bigcup_{x \in B \setminus (K \cup e)}$ and $F_2 = \bigcup_{x \in B \setminus (K \cup f)}$. Note that $r(F_1), r(F_2) \leqslant n-1$ and $E(M) = F_1 \cup F_2$. Thus, we get that M splits and hence is not round. □

Crowns will give us a lower bound to the number of elements of a $U_{2,q+2}$ and $M(K_{k+1})$-minor-free matroid. To do so, we will need to prove the following lemma.

Lemma 1.8. *An $(n, k, q)-$crown has no round minor of rank at least $k+2$.*

Proof. Let M be an (n, k, q)-crown and suppose by way of contradiction there exists a round minor $M/C \backslash D$ of M with $r(M/C \backslash D) > k+1$. We may assume that C is independent and D is coindependent. This way, we have that $r(M/C \setminus D) = r(M/C)$. In particular, any hyperplane of M/C is a hyperplane of $M/C \setminus D$. Thus, M/C is round with rank greater than $k+1$. Finally note that by Proposition 1.6 M/C is an $(n', k', q)-$crown for some $k' \leqslant (k+1) - 1 < r(M/C) - 1$, a contradiction as crowns are not round by Proposition 1.7. □

By Lemma 1.4, we know that $M(K_{k+1})$ is a round matroid. It follows from Lemma 1.8 that an $(n, k-2, q)$-crown M is $M(K_{k+1})$-minor-free. Furthermore, we know that

$$\varepsilon(M) = \frac{q^{k-2} - 1}{q - 1} + (n - k + 2)q^{k-2}.$$

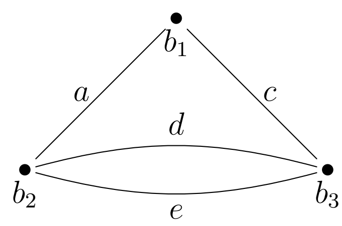

Figure 1.2: The graph G associated to the framed matroid represented by M_1 with basis $B = \{b_1, b_2, b_3\}$.

As $(n, k-2, q)$-crowns are representable, they are $U_{2,q+2}$-minor-free. Thus, the best we can hope for Theorem 1.1 is a singly exponential bound. It is worth noticing that contrary to the graphic case where the best lower bound is given by a random graph; in the case of matroids, the best known lower bounds are archived by $(n, k-2, q)$-crown which are are concrete structures that are very symmetrical.

1.2 Framed matroids

Framed matroids are a class of matroids that closely resemble graphic matroids but are not generally graphic. Nonetheless, in the second chapter our argument will show that a frame matroid will contain a complete graph as a minor. A matroid M is *framed* by a basis B if for each element $e \in E(M)$ there is a set of at most two elements of B that spans e. If M is framed by B, there is a naturally associated graph G, which vertices are the elements of B and where the set of edges are the elements of $E(M) \setminus B$ such that each edge b, b' corresponds to an element of $E(M) \setminus B$ that is spanned by the set $\{b, b'\}$. Note that the graph does not need to be simple. An example of a framed matroid is the $GF(3)$-representable matroid given in Definition 1.1. M_1 is framed by $B = \{b_1, b_2, b_3\}$; its associated graph is shown in figure 1.2.

$$\begin{array}{cc} & \begin{array}{ccccccc} b_1 & b_2 & b_3 & a & c & d & e \end{array} \\ M_1 = & \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 2 \end{pmatrix} \end{array} \tag{1.1}$$

A Θ-*subgraph* of a graph G consists of the union of three edge-disjoint xy-paths for distinct vertices x, y of G. Note that a Θ-subgraph contains exactly three cycles. See

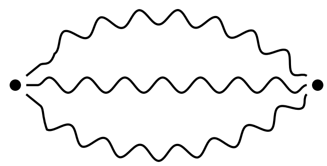

Figure 1.3: Θ-subgraph.

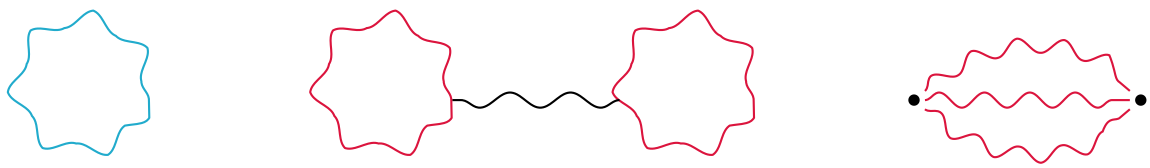

Figure 1.4: From left to right: balanced cycle, unbalanced handcuffs and unbalanced Θ-graph.

Figure 1.3 for an example of a Θ-subgraph. A *bias* of a graph G as defined in [15], is a collection $\mathcal{B}$ of *balanced* cycles of G satisfying the Θ*-property*: In any Θ-subgraph H, if two cycles of H are balanced, then the third one is balanced. Given a biased graph $(G, \mathcal{B})$, the matroid $FM(G, \mathcal{B})$ is the matroid on $E(G)$ whose circuits are all balanced cycles, or pairs of disjoint unbalanced cycles joined by a path (*'handcuffs'*) and any Θ-subgraph with all cycles unbalanced. Figure 1.4 shows the three types of circuits, where balanced cycles are blue and unbalanced cycles are red. Zaslavsky in [16] proved the following theorem:

Theorem 1.9 (Zaslavsky)**.** *If M is a framed matroid with underlying graph G, then there exists a collection $\mathcal{B}$ of cycles of G such that $M = FM(G, \mathcal{B})$.*

An important subclass of framed matroids is the class of B-cliques. A B*-clique* is a framed matroid M with a basis B such that for all distinct elements $b, b' \in B$, there is a triangle containing b and b'. Note that as all pairs of basis elements span at least one element, then the graph associated to a B-clique is a complete graph with possibly multiple edges and loops. Furthermore, by Theorem 1.9, if $C \subseteq E(G)$ is a cycle, then $\{e \in E(M) : e \in E(C)\}$ is either an independent set or a circuit.

1.3 Finding complete minors in dense subgraphs

As previously defined for an integer $t \geqslant 3$, the function $d(t)$ is the infimum d of all positive real numbers such that all graphs with density strictly higher than d contain K_t as a minor. In [7], Kostochka gives the following theorem that can be used to lower bound $d(t)$;

Theorem 1.10 (Kostochka)**.** *For any $c \geqslant 2$, if G is a simple graph with $|E(G)| \geqslant c|V(G)|$, then G has a K_t-minor where $t = \lceil \frac{c}{270\sqrt{\log c}} \rceil$.*

By setting $c = 500\sqrt{2}t\sqrt{\log t}$, we can use Kostochka's result to bound $d(t)$ for all $t \geqslant 4$.

Proposition 1.11. *For all integers $t \geqslant 4$, $d(t) \leqslant (500\sqrt{2})t\sqrt{\log t}$.*

Proof. Let $h(c) = \frac{c}{270\sqrt{\log c}}$, which we will evaluate in $c = 500\sqrt{2}t\sqrt{\log t}$. The real-valued function

$$f(t) = \frac{50\sqrt{2}\sqrt{\log t}}{27\sqrt{\log(500\sqrt{2}t\sqrt{\log t})}}$$

is such that $f(4) \geqslant 1$ and

$$f'(t) = \frac{50(-2\log(t) + 2\log(t\sqrt{\log(t)}) - 1 + \log(5 \cdot 10^5))}{27t\sqrt{\log t}(2\log(t\sqrt{\log t}) + \log(5 \cdot 10^5))^{\frac{3}{2}}} \geqslant 0.$$

This way, we get that $f(t) \geqslant 1$ for all $t \geqslant 4$ and hence

$$h(500\sqrt{2}t\sqrt{\log t}) = \frac{50\sqrt{2}t\sqrt{\log t}}{27\sqrt{\log(500\sqrt{2}t\sqrt{\log t})}} \geqslant t,$$

as desired. Finally, given that $\eta(c) \geqslant h(c)$, it follows that for all $t \geqslant 4$ if G is a graph with $\frac{|E(G)|}{|V(G)|} \geqslant 500\sqrt{2}t\sqrt{\log t}$, then G contains K_t as a minor. $\square$

Chapter 2

Building Towers

2.1 A function

In this section we will define a structure in a matroid called a tower. The main proof is an adaptation of the proof given in [3]. Geelen uses the huge density of a matroid to find a stratified round minor of large size, from which a clique is obtained. On both steps all but a logarithmic number of elements are lost. This gives rise to a doubly exponential bound. In our approach, we start with a huge density matroid from which we obtain a tower of large size by losing all but a logarithmic number of elements. Later, by exploiting the concrete structure of towers, we obtain a clique in exchange of only losing all but a polynomial number of elements. This will give rise to a singly exponential bound.

Definition 2.1. *An n-tower in a matroid M is a function $\varphi : 2^{[n]} \setminus \{\varnothing\} \to E(M) \setminus \mathrm{cl}_M(\varnothing)$ such that:*

(i) The set $V_0 = \{\varphi(\{i\}) : i \in [n]\}$ is an n-element independent set in M,

(ii) For all $i, j \in [n]$ with $i < j$, $\varphi(\{i, j\})$ is not parallel to $\varphi(\{j\})$,

(iii) For all $S \subseteq 2^{[n]} \setminus \{\varnothing\}$ and $k > \max(S)$,

$$\varphi(S \cup \{k\}) \in \mathrm{cl}_M(\{\varphi(S), \varphi(\{k\})\}) \cap \mathrm{cl}_M(\{\varphi(\{s, k\}) : s \in S\}).$$

(iv) For all $j > 1$, there exists some $i < j$ such that $\varphi(\{i, j\})$ is not parallel to $\varphi(\{i\})$.

Note that we do not require φ to be injective. Deviating from Oxley, we consider equal elements to be parallel. Figure 2.1 shows an example of condition (iii) for $S = \{s, s', s''\}$ and $k > \max(S)$. One can see $\varphi(S \cup k)$ is spanned by the set $\{\varphi(k), \varphi(S)\}$ and by the set

$$\{\varphi(\{s,k\}), \varphi(\{s',k\}), \varphi(\{s'',k\})\}.$$

In this example, the element $\varphi(\{s',k\})$ is parallel to $\varphi(s')$. If $1 \leqslant i < j \leqslant n$ are such that $\varphi(i)$ is not parallel to $\varphi(\{i,j\})$ we will say j *lifts* i.

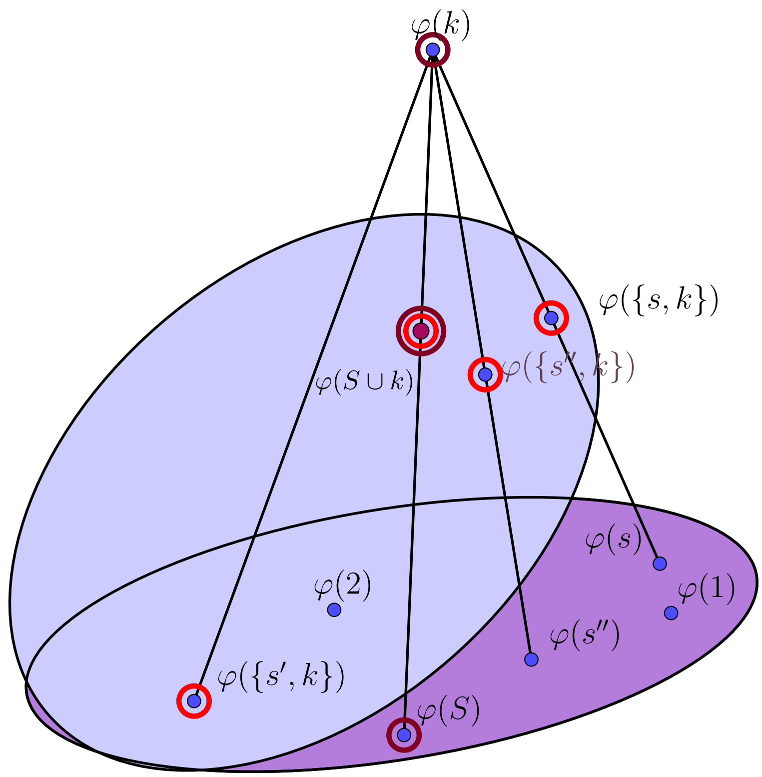

Figure 2.1: $\varphi(S \cup k) \in \mathrm{cl}_M(\{\varphi(k), \varphi(S)\}) \cap \mathrm{cl}_M(\{\varphi(\{s,k\}), \varphi(\{s',k\}), \varphi(\{s'',k\})\})$

For every $i \in [n]$, we will write $v_i := \varphi(\{i\}) = \varphi(i)$ and we will refer to the elements $\varphi(i)$ as *joints*. Additionally, for an n-tower φ and $\varnothing \neq X \subseteq [n]$, we define the set

$$J_X^{\varphi} := \{\varphi(x) : x \in X\}$$

and for every n-tower φ we define the set

$$E(\varphi) := \{\varphi(S) : \varnothing \neq S \subseteq [n]\}.$$

We will say a matroid M *contains* an n-tower, if there exists an n-tower $\varphi : 2^{[n]} \setminus \{\varnothing\} \to E(M) \setminus \mathrm{cl}_M(\varnothing)$.

We will now prove some properties of towers.

Lemma 2.2. *For all $\varnothing \neq X \subseteq [n]$, if φ is an n-tower on M, then $\varphi(X) \in \mathrm{cl}_M(J_X)$.*

Proof. For the case that $X = \{i\}$, we have $\varphi(i) = v_i \in \mathrm{cl}_M(\{v_i\})$. Suppose $X = \{x, y\}$ for some $x, y \in [n]$ with $x < y$ then, by (iii), it follows that $\varphi(X) = \varphi(\{x\} \cup \{y\}) \in \mathrm{cl}_M(\{v_x, v_y\})$.

Let $\varnothing \neq X \subseteq [n]$ and let $k = \max(X)$. By (iii) and the previous case,

$$\begin{aligned}\varphi((X-k) \cup \{k\}) &\in \mathrm{cl}_M(\{\varphi(\{x,k\}) : x \in X - k\}) \\ &\subseteq \mathrm{cl}_M(\{\mathrm{cl}_M(v_x, v_k) : x \in X - k\}) \\ &\subseteq \mathrm{cl}_M(J_X).\end{aligned}$$

□

Lemma 2.3. *Let H_0, H_1 are hyperplanes of a simple matroid M. If $e \notin H_0 \cup H_1$ is such that for every $x \in H_0$ there exists $y \in H_1$ with $r_M(\{x, y, e\}) = 2$ and for every $y \in H_1$ there exists $x \in H_0$ with $r_M(\{x, y, e\}) = 2$, then $M \mid H_0$ is isomorphic to $M \mid H_1$.*

Proof. Let $f : H_0 \to H_1$ be a function that assigns every $x \in H_0$ and element $y = f(x)$ such that $r_M(\{x, f(x), e\}) = 2$. Similarly, let $g : H_1 \to H_0$ be a function such that $r_M(\{g(y), y, e\}) = 2$. We claim g is the inverse function of f. Let $x \in H_0$ and note that by definition $g(f(x)) \in H_0$. Furthermore, $x \in \mathrm{cl}_M(\{f(x), e\})$ and $g(f(x)) \in \mathrm{cl}_M(\{f(x), e\})$, implying that $g(f(x)) \in H_0 \cap \mathrm{cl}_M(\{x, e\})$. As $e \notin H_0$, then $r_M(H_0 \cap \mathrm{cl}_M(\{x, e\})) \leqslant 1$. This way, since $x, g(f(x)) \in H_0 \cap \mathrm{cl}_M(\{x, e\})$ and M is simple, then $x = g(f(x))$. Similarly we obtain that for every $y \in H_1$, we have that $f(g(y)) = y$.

Finally note that for any $X \subseteq H_0$ we have

$$r_{M/e}(X) = r_M(f(X)).$$

Furthermore, since $e \notin H_0 \cup H_1$, it follows that $r_M(X) = r_{M/e}(X)$ and $r_M(f(X)) = r_{M/e}(f(X))$. Therefore, $r_M(X) = r_M(f(X))$ and $M|H_0$ is isomorphic to $M|H_1$. □

Proposition 2.4. *If φ is an n-tower on a simple matroid M, then the sets*

$$H_0 = \{\varphi(S) : \varnothing \neq S \subseteq [n-1]\} \cap E(\varphi)$$

and

$$H_1 = \{\varphi(S \cup n) : \varnothing \neq S \subseteq [n-1]\} \cap E(\varphi)$$

are hyperplanes of $M|E(\varphi)$, and $M|H_0$ is isomorphic to $M|H_1$.

Proof. We will first prove that H_0 is a hyperplane; to do so it is enough to prove that

$$H_0 = \mathrm{cl}_M(J^{\varphi}_{[n-1]}) \cap E(\varphi).$$

By Lemma 2.2, we get $H_0 \subseteq \mathrm{cl}_M(J^{\varphi}_{[n-1]})$. Suppose that $H_0 \subsetneq \mathrm{cl}_M(J^{\varphi}_{[n-1]}) \cap E(\varphi)$; then there exists $\varnothing \neq T \subseteq [n]$ such that $\varphi(T) \in \mathrm{cl}_M(J^{\varphi}_{[n-1]}) \setminus H_0$. By the definition of H_0, we get $n \in T$. Since $\varphi(T \setminus \{n\}) \in J^{\varphi}_{[n-1]}$ and $\varphi(n) \in \mathrm{cl}_M(\{\varphi(T \setminus \{n\}), \varphi(T)\})$, it follows that $\varphi(n) \in \mathrm{cl}_M(J^{\varphi}_{[n-1]})$, which contradicts that $J^{\varphi}_{[n]}$ is independent. Therefore, H_0 is a flat of rank $n-1$ in $M|E(\varphi)$, that is, a hyperplane of $M|E(\varphi)$.

For every $i \in [n-1]$ let $f_i = \varphi(\{i, n\})$. We claim that

$$H_1 = \mathrm{cl}_M(\{f_1, \ldots, f_{n-1})\}) \cap E(\varphi).$$

By Definition 2.1(iii), for all $\varnothing \neq S \subseteq [n-1]$, we have $\varphi(S \cup n) \in \mathrm{cl}(\{f_s : s \in S\})$. Thus, it is enough to prove that if $\varphi(T) \in \mathrm{cl}_M(\{f_1, \ldots, f_{n-1})\})$, then $n \in T$ and $T \neq \{n\}$ or $\varphi(T \cup n) \in \mathrm{cl}_M(\varphi(T))$. First suppose that $\varphi(n) \in \mathrm{cl}_M(\{f_1, \ldots, f_{n-1})\})$. Given that $f_i \in \mathrm{cl}_M(\{\varphi(i), \varphi(n)\})$, then $J^{\varphi}_{[n]} \subseteq \mathrm{cl}_M(\{f_1, \ldots, f_{n-1})\})$. This is a contradiction as

$$r_M(J^{\varphi}_{[n]}) = n > r_M(\mathrm{cl}_M(\{f_1, \ldots, f_{n-1})\})).$$

Thus, we may assume that $\varphi(n) \notin \mathrm{cl}_M(\{f_1, \ldots, f_{n-1})\})$. Now suppose that $n \notin T$ and $\varphi(T \cup n) \notin \mathrm{cl}_M(\varphi(T))$. Similarly to before, by Definition 2.1(iii) we have that

$$\varphi(T \cup n) \in \mathrm{cl}_M(\{f_t : t \in T\}).$$

Therefore, $\varphi(T), \varphi(T \cup n) \in \mathrm{cl}_M(\{f_1, \ldots, f_{n-1})\})$. Finally note that $\varphi(n) \in \mathrm{cl}_M(\varphi(T), \varphi(T \cup n))$, which contradicts $\varphi(n) \notin \mathrm{cl}_M(\{f_1, \ldots, f_{n-1})\})$. Furthermore, we know that the set $\{f_1, \ldots, f_{n-1})\}$ is projected into $J^{\varphi}_{[n-1]}$ in $M/\varphi(n)$. Thus, $\{f_1, \ldots, f_{n-1})\}$ is an independent set of M and H_1 is a hyperplane of M.

We will now use Lemma 2.3 to prove that $M|H_0$ is isomorphic to $M|H_1$. Note that for every $\varnothing \neq T \subseteq [n-1]$, we have that $r_M(\{\varphi(T), \varphi(T \cup n), \varphi(n)\}) = 2$. Thus, for every $x = \varphi(S) \in H_0$ there exists $y = \varphi(S \cup n) \in H_1$ with $r_M(\{x, y, \varphi(n)\}) = 2$, where $\varnothing \neq S \subseteq [n-1]$. Additionally, if $\varnothing \neq S \subseteq [n-1]$ we have that for every $y = \varphi(S \cup n) \in H_1$ there exists $x = \varphi(S) \in H_0$ with $r_M(\{x, y, \varphi(n)\}) = 2$. Thus, the matroid $M|H_0$ is isomorphic to $M|H_1$. □

Proposition 2.5. *Let φ be an n-tower on a matroid M and for each $\varnothing \neq S \subseteq [n-1]$ define φ_1 by*

$$\varphi_1(S) := \varphi(S \cup n),$$

then φ_1 is an $(n-1)$-tower.

Proof. We may assume that M is simple. Note that the set $J^{\varphi_1}_{[n-1]} = \{\varphi(\{i,n\}) : i \in [n-1]\}$ and for any $\varnothing \neq S \subseteq [n-1]$, we have $\varphi(S) \in \mathrm{cl}_M(J^{\varphi_1}_{[n-1]})$. Thus, we get

$$M|(J^{\varphi_1}_{[n-1]}) = M|(\mathrm{cl}_M(\{\varphi(\{i,n\}) : i \in [n-1]\}) \cap \{\varphi(S) : S \subseteq [n], S \neq \varnothing\}).$$

Properties (i)-(iv) from Definition 2.1 now follow from Proposition 2.4 as φ restricted to $\varphi : 2^{[n-1]} \setminus \{\varnothing\}$ is an $(n-1)$-tower. $\square$

We say a set $S \subseteq [n]$ is a *tree* of an n-tower φ if for all $j \in S$ with $j > \min(S)$ there exists a unique $i \in S$ with $i < j$ such that $\varphi(\{i,j\})$ is not parallel to $\varphi(i)$. We will now need the following lemma about circuits.

Lemma 2.6. *If C and C' are circuits of a matroid M such that $C \cap C' = \{e\}$ and $r_M(C \cup C') = r_M(C) + r_M(C') - 1$, then $(C \cup C') \setminus \{e\}$ is a circuit.*

Proof. It is enough to consider $M' = M|C \cup C'$, note that $|E(M')| = |C| + |C'| - 1$ and $r(M') = r_M(C) + r_M(C') - 1 = |C| + |C'| - 3$. Thus,

$$\begin{aligned} r((M')^*) &= |E(M')| - r(M') \\ &= (|C| + |C'| - 1) - (|C| + |C'| - 3) = 2. \end{aligned}$$

Now, note that for all $x \in (C \cup C') \setminus \{e\}$, the set $\{e, x\}$ is independent in $(M')^*$. As $r((M')^*) = 2$ and e is not a loop, the set $\{e\}$ is a hyperplane of $(M')^*$ and hence $(C \cup C') \setminus \{e\}$ is a cocircuit of $(M')^*$. It follows that $(C \cup C') \setminus \{e\}$ is a circuit of M' as desired. $\square$

We can now prove the following lemma about trees in towers.

Proposition 2.7. *For an n-tower φ on M and $S \subseteq [n]$ with $|S| \geqslant 2$, if S is a tree of φ, then the fundamental circuit of $\varphi(S)$ with respect to V_0 is $J^{\varphi}_S \cup \varphi(S)$.*

Proof. The proof is by induction on $|S|$. If $S = \{i,j\}$, then $\varphi(\{i,j\})$ is spanned by $\{\varphi(i), \varphi(j)\}$. As S is a tree of φ, then $\varphi(\{i,j\})$ is not parallel to either $\varphi(i)$ or $\varphi(j)$. Therefore, the set $\{\varphi(\{i,j\}), \varphi(i), \varphi(j)\}$ is a triangle.

Let $k = \max(S)$ and let $S_0 = S - k$. First note that for all $j < k$, we get $\varphi(\{j,k\}) \notin \mathrm{cl}_M(\varphi(k))$ and thus S_0 is also a tree of φ. Hence, by induction $C = J^{\varphi}_{S_0} \cup \varphi(S_0)$ is a circuit.

Claim 2.8. $C' := \{\varphi(S), \varphi(S_0), \varphi(k)\}$ *is a triangle.*

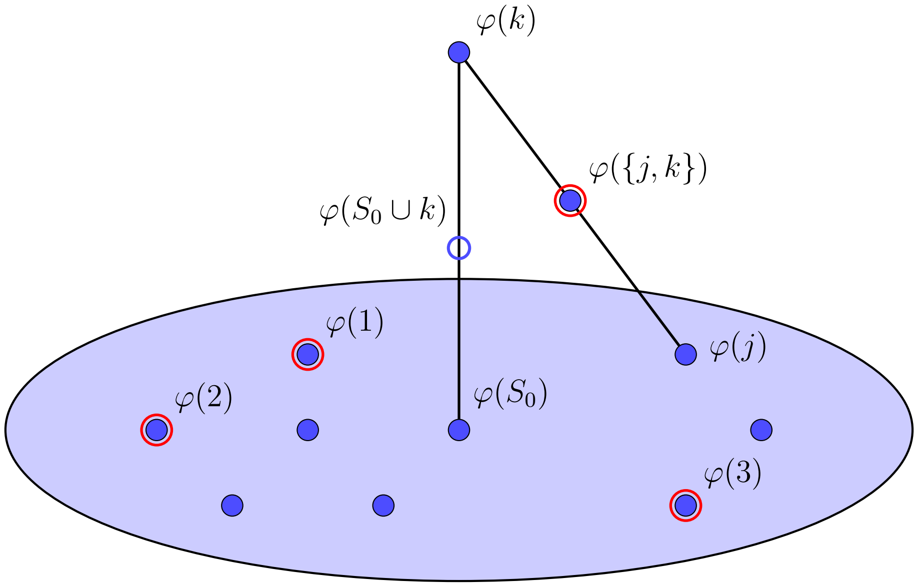

Figure 2.2: Triangle $\{\varphi(k), \varphi(S_0 \cup k), \varphi(S_0)\}$.

Proof. By (iii) $\varphi(S) \in \mathrm{cl}_M(\{\varphi(\{s,k\}) : s \in S_0\}) \cap \mathrm{cl}_M(\{\varphi(k), \varphi(S_0)\})$. Since S is a tree of φ, there exists a unique $j \in S$ such that $\varphi(\{j,k\})$ is not parallel to $\varphi(j)$. Hence, we get

$$\varphi(S) \in \mathrm{cl}_M(J^{\varphi}_{S_0-j} \cup \varphi(\{j,k\})) \cap \mathrm{cl}_M(\{\varphi(k), \varphi(S_0)\}),$$

as shown in Figure 2.2. Let F_0 be the flat spanned by $J^{\varphi}_{S_0}$ and consider $M' = M/J^{\varphi}_{S_0-j}$. Then

$$r_{M'}(F_0) = r_M(F_0 \cup J^{\varphi}_{S_0-j}) - r_M(J^{\varphi}_{S_0-j}) = 1.$$

As $\varphi(S_0), \varphi(j) \in F_0$, it follows that $\varphi(S_0)$ is parallel to $\varphi(j)$ in M'. Furthermore,

$$r_{M'}(\mathrm{cl}_M(J^{\varphi}_{S_0-j} \cup \varphi(\{j,k\}))) = |S_0| - (|S_0| - 1) = 1.$$

Therefore, the element $\varphi(S_0 \cup k)$ is parallel to $\varphi(\{j,k\})$ in M'. Given that $\varphi(\{j,k\})$ is neither parallel to $\varphi(k)$ or $\varphi(j)$ and $\varphi(S_0)$ is parallel to $\varphi(j)$, then

$$\varphi(S_0 \cup k) \notin \mathrm{cl}_M(\varphi(S_0)), \mathrm{cl}_M(\varphi(k)).$$

It follows that $\{\varphi(S), \varphi(S_0), \varphi(k)\}$ is a triangle. ∎

Therefore, the sets C and C' are two circuits such that $C \cap C' = \varphi(S_0)$ and

$$r_M(C \cup C') = |S_0| + 1 = |S_0| + 2 - 1 = r_M(C) + r_M(C') - 1.$$

By Proposition 2.6 the set $J^{\varphi}_{S_0} \cup \{\varphi(k), \varphi(S)\} = J^{\varphi}_{S} \cup \varphi(S)$ is a circuit as desired. $\square$

We say two n-towers φ_1 and φ_2 are *equivalent*, if there exists a permutation σ of $[n]$ such that $\varphi_1(S) = \varphi_2(\sigma(S))$ for all $\varnothing \neq S \subseteq [n]$.

Given $\varphi_1, \varphi_2 : 2^{[n]} \setminus \{\varnothing\} \to E(M)$ and $e \in E(M)$, we define the function $\varphi_1 \oplus_e \varphi_2 : 2^{[n+1]} \setminus \{\varnothing\} \to E(M)$ by

$$\varphi_1 \oplus_e \varphi_2(S) = \begin{cases} \varphi_1(S) & \text{if } n+1 \notin S \\ \varphi_2(S \setminus \{n+1\}) & \text{if } n+1 \in S, S \neq \{n+1\} \\ e & \text{if } S = \{n+1\} \end{cases}$$

Lemma 2.9. *If $e \in E(M)$, and the functions φ_1, φ_2 are inequivalent n-towers in M and equivalent in M/e, then $\varphi_1 \oplus_e \varphi_2$ is an $(n+1)$-tower.*

Proof. Let $\varphi := \varphi_1 \oplus_e \varphi_2$. We first need to check the set $J^{\varphi}_{[n+1]} = J^{\varphi_1}_{[n]} \cup e$ is independent. Since φ_1 is a tower, the set $J^{\varphi}_{[n+1]}$ is independent. If $e \in \mathrm{cl}_M(J^{\varphi}_{[n+1]})$, then $r_{M/e}(J^{\varphi}_{[n+1]}) = n - 1$, a contradiction. Therefore, we get that $J^{\varphi}_{[n+1]}$ is independent.

Let $i, j \in [n+1]$ with $i < j$. We may assume $j = n+1$, otherwise

$$\varphi(\{i,j\}) = \varphi_1(\{i,j\}) \notin \mathrm{cl}_M(\varphi_1(j)) = \mathrm{cl}_M(\varphi(j)).$$

Now, if $j = n+1$,

$$\varphi(\{i, n+1\}) = \varphi_2(i) \notin \mathrm{cl}_M(\varphi(n+1)) = \mathrm{cl}_M(e).$$

Thus, the function φ satisfies (ii).

To prove (iii), first note that if $n+1 \in S$, then there does not exists $k > \max(S)$. Thus, suppose $n+1 \notin S$ and let $k > \max(S)$. As $n+1 \notin S$ and φ_1 is an n-tower, if $k \neq n+1$ then the result follows. Assume $k = n+1$, this way, we get that $\varphi(S \cup \{n+1\}) = \varphi_2(S)$ and $\mathrm{cl}_M(\{\varphi(S), \varphi(n+1)\}) = \mathrm{cl}_M(\{\varphi_1(S) \cup e\})$. Given that φ_1, φ_2 are equivalent in M/e, it follows that

$$\varphi_2(S) \in \mathrm{cl}_M(\{\varphi_1(S) \cup e\}).$$

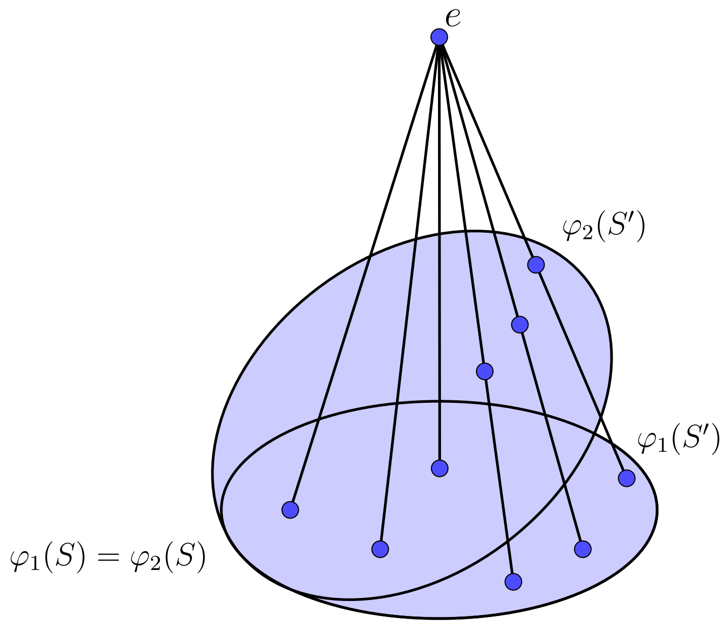

Figure 2.3: Towers φ_1 and φ_2 are equivalent in M/e.

Similarly, we have

$$\mathrm{cl}_M(\{\varphi(\{s, n+1\}) : s \in S\}) = \mathrm{cl}_M(J_S^{\varphi_2}).$$

By (iii) for φ_2, we obtain $\varphi_2(S) \in \mathrm{cl}_M(J_S^{\varphi_2})$.

Note that as φ_1 satisfies Definition 2.1(iv), it is sufficient to prove this condition for $\varphi(n+1)$. Since φ_1 is not equivalent to φ_2 in M, then there exists $i \in [n]$ such that $\varphi_1(i)$ is not parallel to $\varphi_2(i)$. This way, the element $\varphi(\{n+1, i\}) = \varphi_2(i)$ is not parallel to $\varphi_1(i) = \varphi(i)$. Therefore, the function φ is an $(n+1)$-tower. $\square$

It is important to note that the construction defined in the previous lemma is asymmetric, as $\varphi_1 \oplus_e \varphi_2 \neq \varphi_2 \oplus_e \varphi_1$, where

$$\varphi_2 \oplus_e \varphi_1(S) = \begin{cases} \varphi_2(S) & \text{if } n+1 \notin S \\ \varphi_1(S \setminus \{n+1\}) & \text{if } n+1 \in S, S \neq \{n+1\} \\ e & \text{if } S = \{n+1\}. \end{cases}$$

We will use this property in chapter 3. If φ_1 and φ_2 are two different towers of M that are equivalent in M/e, we will say φ_1 *collapses into* φ_2 with respect to e. For every $e \in E(M)$, collapsing induces an equivalence relationship. With this in mind, we will denote by $\mathcal{T}^n_{M/e}$ the equivalence class of an n-tower φ in M with respect to the collapsing relationship when contracting e.

Proposition 2.10. *Let φ, φ' be n-towers in a simple matroid M for which $\varphi(i) = \varphi'(i)$ for all $i \in [n]$. If $x \in E(M)$ and φ and φ' are equivalent n-towers in M/x, then they are equivalent in M.*

Proof. Note that as $r_{M/x}(J^{\varphi}_{[n]}) = r_{M/x}(J^{\varphi'}_{[n]}) = n$, it follows that $x \notin \mathrm{cl}_M(J^{\varphi}_{[n]}) \cup \mathrm{cl}_M(J^{\varphi'}_{[n]})$. Given that φ and φ' are equivalent in M/x, then for every $S \subseteq [n]$, we have $\varphi'(S) \in \mathrm{cl}_M(\{\varphi(S), x\}) \setminus \{x\}$. By the definition of n-tower, for every $\varnothing \neq S \subseteq [n]$ we have $\varphi'(S) \in \mathrm{cl}_M(J^{\varphi'}_{[n]}) = \mathrm{cl}_M(J^{\varphi}_{[n]})$. Similarly, by definition, we get $\varphi(S) \in \mathrm{cl}_M(J^{\varphi}_{[n]}) \cap \mathrm{cl}_M(\{\varphi(S), x\})$. Since $x \notin \mathrm{cl}_M(J^{\varphi}_{[n]})$ it follows that $r_M(\mathrm{cl}_M(J^{\varphi}_{[n]}) \cap \mathrm{cl}_M(\{\varphi(S), x\})) \leqslant 1$. Given that M is simple, it follows that

$$|E(M|(\mathrm{cl}_M(J^{\varphi}_{[n]}) \cap \mathrm{cl}_M(\{\varphi(S), x\})))| \leqslant 1.$$

Therefore, $\varphi(S) = \varphi'(S)$. □

Proposition 2.11. *Let ℓ be a positive integer and let M be a simple matroid with no $U_{2,\ell+2}$-restriction. If $x \in E(M)$ and φ is an n-tower in both M and M/x, then there are at most ℓ^n n-towers of M that are n-towers of M/x and equivalent to φ.*

Proof. For every $i \in [n]$, let $L_i = \mathrm{cl}_M(\{x, \varphi(i)\}) \setminus \{x\}$. Since M is $U_{2,\ell+2}$-restriction-free, we have that $|L_i| \leqslant \ell$. By Proposition 2.10, if φ' and φ'' are n-towers with joint set $J^{\varphi'}_{[n]}$ and such that φ' and φ'' are equivalent n-towers to φ in M/x, then φ'' is isomorphic to φ'. As there are at most $\prod_{i\in[n]} |L_i| \leqslant \ell^n$ possible choices of joint set, there are at most ℓ^n n-towers that are equivalent to φ in M/x. □

2.2 The two towers

In this section we will consider two important structures contained in n-towers. We say a set $S \subseteq [n]$ is a *clique* or *complete* of φ if for all $i, j \in S$ with $i < j$, the element $\varphi(\{i, j\})$ is not parallel to $\varphi(i)$. The second structure occurs when j only lifts i if $j = i + 1$ for all $i, j \in [n]$.

Lemma 2.12. *If S is a clique of an n-tower φ in M, then the matroid M restricted to $J^{\varphi}_S \cup \{\varphi(\{i, j\}) : i, j \in S\} \subseteq E(M)$ is a J^{φ}_S-clique.*

Proof. Take any $\varphi(i), \varphi(j) \in J_S^\varphi$ with $i, j \in S$ and $i < j$. Note that $\varphi(\{i,j\}) \notin \mathrm{cl}_M(\varphi(i))$ and by Definition 2.1(ii), we get $\varphi(\{i,j\}) \notin \mathrm{cl}_M(\varphi(j))$. Thus, as $j > \max\{i\}$, by Definition 2.1(iii), the set $\{\varphi(\{i,j\}), \varphi(i), \varphi(j)\}$ is a triangle.

Let $E_0 := \{\varphi(\{i,j\}) : i, j \in S\}$; we need to prove that any $e \in E_0$ is spanned by two elements of J_S^φ. Note that by Definition 2.1(iii), we have $\varphi(\{i,j\}) \in \mathrm{cl}_M(\varphi(i), \varphi(j))$ for any $i < j$, with $i, j \in S$. Thus, the matroid $M|(J_S^\varphi \cup E_0)$ is a J_S^φ-clique. □

Given a B-clique M, by Theorem 1.9, there exists a complete biased graph $(G, \mathcal{B})$ such that M is the framed matroid obtained from $(G, \mathcal{B})$. It is now convenient to talk about $(G, \mathcal{B})$ instead of a clique S on an n-tower. Given a clique S in an n-tower φ, let G be the graph associated to the B-clique $M|(J_S^\varphi \cup E_0)$, where the vertices are the elements of J_S^φ and the edges are E_0.

Lemma 2.13. *Let $(G, \mathcal{B})$ be a biased graph from which G is a complete graph and let M be its associated framed matroid. If $X \subseteq E(G)$ contains a unique cycle, then X is independent or contains a unique circuit in M.*

Proof. Follows from M being a framed matroid with underlying graph $(G, \mathcal{B})$. □

Lemma 2.14. *For any integer $\ell \geqslant 2$, if M is a B-clique of size $2n$ with no $U_{2,\ell+2}$-minor, then the associated biased graph G contains a set $X \subseteq E(G)$ and independent sets $J_1, J_2 \subseteq E(G)$ such that:*

1. *$r(J_i) = n - 1$ for $i = 1, 2$,*
2. *$|X| \geqslant \frac{n^2}{l+1}$, and*
3. *$M|(X \cup (J_1 \cup J_2))) = M(G|(X \cup (J_1 \cup J_2)))$.*

Proof. Let (V_1, V_2) be a partition of the vertices of G into equal size sets such that $G[V_1]$ and $G[V_2]$ contain (respectively) a spanning subgraph G_i, isomorphic to $K_{1,n-1}$ which center is v_i. Let J_i be the $n-1$ edges of G_i for $i = 1, 2$ and let $F = E[V_1, V_2]$. This way, we have $|F| = n^2$. Furthermore,

$$\begin{aligned} r_{M/(J_1 \cup J_2)}(F) &\leqslant r(M/J_1 \cup J_2) \\ &= r(M) - r(J_1 \cup J_2) \\ &= 2n - 2(n-2) = 2. \end{aligned}$$

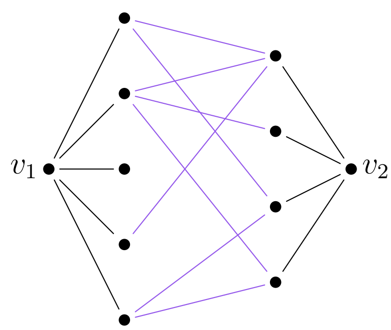

Figure 2.4: Stars G_1 and G_2 with an equivalence class of parallel edges in $M/(J_1 \cup J_2)$.

As M is $U_{2,\ell+2}$-minor-free, the matroid $(M/(J_1 \cup J_2))|F$ contains at most $\ell + 1$ points. Note that the elements of $(M/(J_1 \cup J_2))|F$ are not loops as the set $J_1 \cup J_2 \cup f$ does not contain a circuit for any $f \in F$. Thus, there exists a parallel class X of $M/(J_1 \cup J_2)$ with at least $\frac{n^2}{\ell+1}$ points of M. We claim $M|(X \cup J_1 \cup J_2)) = M(G|(X \cup J_1 \cup J_2))$. To see this, it is enough to see all the cycles of $M|(X \cup J_1 \cup J_2))$ are circuits of M, that is, all cycles are balanced. Suppose there exists an M-independent cycle $C \subseteq X \cup J_1 \cup J_2$. By extending C to a spanning pseudo-tree, we obtain a spanning set without balanced cycles. That is an independent set of $M|E(G)$ with size $2n$. This is a contradiction as

$$\begin{aligned} r_M(X \cup (J_1 \cup J_2)) &= r_{M/J_1 \cup J_2}(X) + |J_1 \cup J_2| \\ &= 1 + 2(n-1) = 2n - 1 < 2n. \end{aligned}$$

Therefore, we have $M|(X \cup (J_1 \cup J_2))) = M(G|X \cup (J_1 \cup J_2))$. $\square$

As previously defined, for each $t > 1$ we will use $d(t)$ to denote the infimum of all all positive real numbers d such that if G is a graph with $\frac{|E(G)|}{|V(G)|} \geqslant d$, then G contains K_t as a minor. Since there are $2n$ vertices in $G \setminus \{v_1, v_2\}$ and at least $\frac{n^2}{\ell+1}$ edges in $G|(X \cup (J_1 \cup J_2))$, the average degree of $G|X \cup (J_1 \cup J_2)$ is at least $\frac{n^2}{(\ell+1)(2n)} = \frac{n}{2(\ell+1)}$. As a result we obtain the following theorem.

Theorem 2.15. *For any integer $\ell \geqslant 2$, if M is a B-clique and the rank of M is at least $4(\ell+1)\lceil d(k) \rceil$, then M contains $M(K_k)$ as a minor.*

Given an n-tower φ, we say the r-tuple $S = (s_1, \ldots, s_r)$ where $1 \leqslant s_1 < \ldots < s_r \leqslant n$ is a *path* if for all $1 < i \leqslant r$, the element $\varphi(\{s_i, s_{i-1}\})$ is not parallel to $\varphi(s_i)$ but for all

$j < i-1 \leqslant k$, we have that $\varphi(\{s_i, s_j\})$ is parallel to $\varphi(s_j)$. If $S = (1, \ldots, n)$, we will say the path has *length* n. For simplicity we may refer to S as a set. An example of a 6-path is shown in Figure 2.5. Similarly to paths in graphs, if $S = (s_1, \ldots, s_r)$ is a path on φ then for any $1 \leqslant i < j \leqslant r$, the tuple $(s_i, \ldots, s_j)$ is also a path on φ.

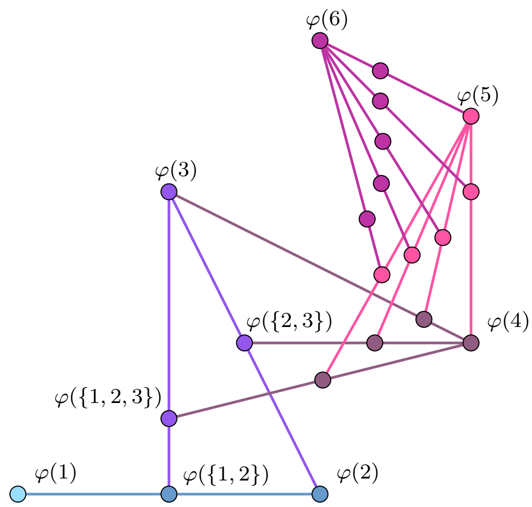

Figure 2.5: 6-path

We will use the following proposition to find $M(K_n)$ as a minor in a matroid that contains an n-path.

Proposition 2.16. *Let M be a matroid, H_0, H_1 be hyperplanes of M and $e \notin H_0 \cup H_1$ such that $M|H_0$ and $M|H_1$ are isomorphic to $M(K_{n-1})$, the matroid $F = M|(H_0 \cap H_1)$ is isomorphic to $M(K_{n-2})$ and $E(M) = H_0 \cup H_1 \cup e$. If for each $x \in H_0 - (H_0 \cap H_1)$ there is some $y \in H_1 - (H_0 \cap H_1)$ such that $\{e, x, y\}$ is a triangle, then $M|(H_0 \cup H_1 \cup e)$ is isomorphic to $M(K_n)$.*

To prove this, we will need the following lemma found in [3].

Lemma 2.17. *Let M be a matroid with ground set $B \cup H$ where $B = b_1, \ldots, b_n$ is a basis of M, $H = \{e_{i,j} : 1 \leqslant i < j \leqslant n\}$ is a hyperplane of M disjoint from B, and $\{b_i, e_{i,j}, b_j\}$ is a triangle of M for each $i < j$. Then M is isomorphic to $M(K_{n+1})$.*

Proof of Proposition 2.16. We first claim that if M is isomorphic to $M(K_{n-1})$ and K is a cocircuit of M of size $n-2$, then K is a star of the clique. As K is a cocircuit of a graphic matroid, there exists $X \subseteq V(K_{n-1})$ such that $K = \delta(X)$. Given that $|K| = n-2$, then X or $V \setminus X$ is a vertex. Therefore, we have that K is a star of the clique.

Since H_0 is a hyperplane of M, to use Lemma 2.17, it is enough to prove that the cocircuit $B = (H_1 \setminus H_0) \cup e$ is a basis of M, where every element of H_0 is spanned by two elements of B. Take an element $x \in E(M) \setminus B$. If $x \in H_0 \setminus H_1$, by hypothesis there exists $y \in H_1 \setminus H_0$ such that $x \in \mathrm{cl}_M(\{e, y\})$, where $e, y \in B$. Now suppose that $x \in H_0 \cap H_1$, given that $H_0 \cap H_1$ is isomorphic to $M(K_{n-2})$ and H_1 is isomorphic to $M(K_{n-1})$, then $H_0 \cap H_1$ is a hyperplane of H_1. Thus, by the previous claim, the set $K = H_1 \setminus H_0$ is a star of H_1. Hence, there exists $b, b' \in K$ such that $x \in \mathrm{cl}_{H_1}(\{b, b'\})$. Therefore, we have that B is a basis of M disjoint from H_0 and such that $E(M) = H_0 \cup B$. By Lemma 2.17, the matroid M is isomorphic to $M(K_n)$. $\square$

We now have all the lemmas we need to prove that if X is a path in an n-tower of a matroid M, then M contains a clique as a restriction.

Lemma 2.18. *If $X \subseteq [n]$ is a path in an n-tower φ on M, then*

$$M|(\mathrm{cl}_M(J_X^{\varphi}) \cap E(\varphi))$$

is isomorphic to $M(K_{|X|+1})$.

Proof. We may assume that M is simple. Note that when $X = (x_1, x_2)$, as X is a path on φ, the set $\{\varphi(x_1), \varphi(x_2), \varphi(\{x_1, x_2\})\}$ is a triangle and hence isomorphic to K_3. Suppose inductively the statement is true for paths with length less than r. Let $X = (x_1, \dots, x_r)$ be a path on φ.

By induction, the sets $X_0 = (x_1, \dots, x_{r-1})$ and $F = (x_1, \dots, x_{r-2})$ are both paths of M and thus,

$$M|(\mathrm{cl}_M(J_{X_0}^{\varphi}) \cap E(\varphi))$$

and

$$M|(\mathrm{cl}_M(J_F^{\varphi}) \cap E(\varphi))$$

are isomorphic to $M(K_r)$ and $M(K_{r-1})$ respectively. Furthermore, by Property 2.4

$$M|(\mathrm{cl}_M(\{\varphi(x_i, x_r) : i \in [r-1]\}) \cap E(\varphi))$$

is isomorphic to $M|(\mathrm{cl}(J_{X_0}^{\varphi}) \cap E(\varphi)))$ and hence to $M(K_r)$.

Since X is a path in φ, we have that $\varphi(\{x_i, x_r\})$ is parallel to $\varphi(\{x_i\})$ for any $i < r-1$ and

$$\mathrm{cl}_M(\{\varphi(x_i, x_r) : i \in [r-2]\}) = \mathrm{cl}_M(J_{[r-2]}^{\varphi} \cup \{\varphi(x_{r-1}, x_r)\}).$$

Therefore,

$$(\mathrm{cl}_M(J^{\varphi}_{X_0}) \cap \mathrm{cl}_M(\{\varphi(x_i, x_r) : i \in [r-1]\})) \cap E(\varphi)$$

is equal to

$$\mathrm{cl}_M(J^{\varphi}_F) \cap E(\varphi).$$

Note that if $S \subseteq F$ or $S \subseteq X$ with $x_{r-1} \notin S$, then

$$\varphi(S) \notin (\mathrm{cl}_M(\{\varphi(x_i, x_r) : i \in [r-1]\} \setminus \mathrm{cl}_M(J^{\varphi}_{X_0})) \cap E(\varphi)$$

Thus, for any

$$\varphi(S) \in (\mathrm{cl}_M(\{\varphi(x_i, x_r) : i \in [r-1]\} \setminus \mathrm{cl}_M(J^{\varphi}_{X_0})) \cap E(\varphi),$$

we get that $x_{r-1}, x_r \in S$. In particular $\{\varphi(S), \varphi(S - x_r), \varphi(x_r)\}$ is a triangle of M, where $\varphi(S - x_r) \in \mathrm{cl}_M(J^{\varphi}_{X_0}) \cap E(\varphi)$. Hence, by Proposition 2.16, the matroid

$$M|(\mathrm{cl}_M(J^{\varphi}_X) \cap E(\varphi))$$

is isomorphic to $M(K_{|X|+1})$. □

2.3 Finding a complete graph

We will now use both the previous structures to prove the following result.

Lemma 2.19. *For any integer $\ell \geqslant 2$, if M is a $U_{2,\ell+2}$-minor-free matroid that contains a $4(\ell+1)(k-1)\lceil d(k+1)\rceil$-tower, then M contains an $M(K_{k+1})$-minor.*

Proof. Let $t = 4(\ell+1)(k-1)\lceil d(k+1)\rceil$ and φ be a t-tower in M. As φ satisfies Definition 2.1(iv), for every $2 \leqslant j \leqslant n$, the set $\{i \in [j-1] : \varphi(\{i,j\}) \notin \mathrm{cl}_M(\varphi(i))\}$ is non-empty. We define $\alpha(1) = 1$ and for every $2 \leqslant j \leqslant n$ we define $\alpha(j)$ as follows:

$$\alpha(j) := \min(\{i \in [j-1] : \varphi(\{i,j\}) \notin \mathrm{cl}_M(\varphi(i))\}).$$

We call an r-tuple $V = (v_1, \ldots, v_r)$ an *α-path* with respect to φ if the sequence $1 \leqslant v_1 < v_2 < \ldots < v_r \leqslant t$ is such that $\alpha(v_j) = v_{j-1}$ for all $j \in \{2, \ldots, r\}$. Note that an α-path is a path in M with respect to φ. Indeed, by the minimality of $\alpha(s)$, for any $1 \leqslant k < s-1 \leqslant r$, we get $\alpha(s) \neq \varphi(v_k)$ and thus $\varphi(v_k)$ is not parallel to $\varphi(\{v_k, v_s\})$. Therefore, by Lemma 2.18, if there exists an α-path of length at least k in M, then there is a path of length k in the tower φ and hence M contains $M(K_{k+1})$. Suppose therefore that there is no α-path of length k.

Claim 2.20. *There exists a set $L \subseteq [t]$, with $|L| \geqslant 4(\ell+1)\lceil d(k+1)\rceil$ such that for all $v \in L$ and $j \in [t] \setminus \{v\}$ we have $v \neq \alpha(j)$.*

Proof. Consider the collection of all maximal α-paths. Note that by maximality, if $(v_1, \ldots, v_r)$ is a maximal α-path then each $j \in [t] \setminus \{v_r\}$ satisfies $v_r \neq \alpha(j)$. Furthermore, every element of $E(M)$ is in a maximal α-path and any α-path has length at most $k-1$. Thus, there are at least $\frac{t}{k-1} = 4(\ell+1)\lceil d(k+1)\rceil$ maximal α-paths. Let L be the set consisting of the highest element of a maximal path. ■

Claim 2.21. *If $C = V_0 \setminus J_L^{\varphi}$, then the matroid M/C contains a J_L^{φ}-clique as a restriction.*

Proof. Fix $i < j$ with $i, j \in L$. We need to prove there is a triangle of M/C containing $\varphi(i)$ and $\varphi(j)$. Let P_i and P_j be the α-paths in M that start in i and j respectively and end in 1. This way, there exists $v_i \in P_i$ and $v_j \in P_j$ such that $\varphi(\{v_i, v_j\}) \notin \{\mathrm{cl}_M(\varphi(v_i)), \mathrm{cl}_M(\varphi(v_j))\}$ but for all $v \in P_i$ with $v > v_i$, and $w \in P_j$ with $w > v_j$, we have $\varphi(\{v, w\}) \in \{\mathrm{cl}_M(\varphi(v)), \mathrm{cl}_M(\varphi(w))\}$.

Let $P_i' = \{v \in P_i : i > v > v_i\}$ and $P_j' = \{v \in P_j : j > v > v_j\}$. We will prove there exists an element $e_{i,j} \notin V_0$ and a circuit $C_{i,j}$ such that

$$\{e_{i,j}, \varphi(i), \varphi(j)\} \subseteq C_{i,j} \subseteq (V_0 \setminus \{\varphi(k) : k \in L \setminus \{i,j\}\}) \cup e_{i,j}.$$

Note that if such circuit exists, then in M/C the set $\{e_{i,j}, \varphi(i), \varphi(j)\}$ is a triangle. Let $e_{i,j} = \varphi(\{v_i, \ldots, i, v_j \ldots, j\})$. By our choice of v_i and v_j, the set $P_i' \cup P_j'$ is a path of φ. Hence, by Proposition 2.7,

$$C_{i,j} = \{\varphi(k) : k \in P_i' \cup P_j'\} \cup e_{i,j}$$

is a circuit. Therefore, the matroid M/C contains as a restriction a J_L^{φ}-clique. ■

By Theorem 2.15, given that $|L| \geqslant 4(\ell+1)(k-1)\lceil d(k+1)\rceil$, then M contains $M(K_{k+1})$ as a minor. □

Chapter 3

Finding Towers

In this chapter, by taking advantage of the density of a matroid, we will inductively construct a t-tower for some large t. Using the fact that $\varepsilon(M)$ is the number of 1-towers and $\varepsilon(M)$ is huge compared to the rank of M, we will find a minor of M with a large number of 2-towers. Recursively, for each $1 \leqslant i \leqslant t$ we will exploit the huge number of i-towers to find a minor with a large number of $(i+1)$-towers. Our final objective is to show the number of t-towers is positive for a minor of M. We will lastly combine this result with the results of chapter 2 to find a K_{k+1}-minor.

We will first use the following theorem by Kung [9] to bound the number of points in a $U_{2,\ell+2}$-free matroid. Note that as a consequence of this theorem, a t-tower spans ℓ^t points.

Theorem 3.1 (Kung[9]). *For any integer $\ell \geqslant 2$, If M is a simple $U_{2,\ell+2}$-minor-free matroid, then*

$$|M| \leqslant 1 + \ell + \ell^2 + \ldots + \ell^{r(M)-1} = \frac{\ell^{r(M)} - 1}{\ell - 1}.$$

For $t \geqslant 1$ denote by $N_t(M)$ the number of pairwise inequivalent t-towers in a matroid M and let $N_0(M) = r(M)$. The following theorem will prove the existence of a t-tower for any big enough matroid.

Lemma 3.2. *For all integers $\ell \geqslant 2$, $a \geqslant 1$, $t \geqslant 0$, if M is a matroid with no $U_{2,\ell+2}$-minor and $N_t(M) > \ell^t(a+\ell^{t-1})N_{t-1}(M)$, then M has a minor M_0 such that $N_{t+1}(M_0) > aN_t(M_0)$.*

Proof. Fix $\ell \geqslant 2$, $a \geqslant 1$ and $t \geqslant 0$ and let M_0 be a minor-minimal minor of M such that $N_t(M_0) > \ell^t(a + \ell^{t-1})N_{t-1}(M_0)$. Note that by minimality, the matroid M_0 is simple. We may assume by contradiction that $N_{t+1}(M_0) \leqslant aN_t(M_0)$. Furthermore, by the minimality of M_0, for every $x \in E(M_0)$, we have $N_t(M_0/x) \leqslant \ell^t(a+\ell^{t-1})N_{t-1}(M_0/x)$. Thus,

$$\sum_{x\in E(M_0)} N_t(M_0/x) \leqslant \sum_{x\in E(M_0)} \ell^t(a+\ell^{t-1})N_{t-1}(M_0/x).$$

Moreover, we get

$$\sum_{x\in E(M_0)} (N_t(M_0) - N_t(M_0/x)) > \ell^t(a+\ell^{t-1}) \sum_{x\in E(M_0)} (N_{t-1}(M_0) - N_{t-1}(M_0/x)).$$

We define the set $T^t_{M_0}$ to be the set of t-towers of M_0 and for every $x \in E(M_0)$ we define $T^t_{M_0/x}$ to be the set of t-towers in M_0/x. Let $\mathcal{T}^t_{M_0/x}$ denote the set of equivalent classes of $T^t_{M_0/x}$ with respect to the equivalence in M_0/x. Additionally, for every natural number i and for every $x \in E(M_0)$ let $A_i(x)$ be the set of pairs (x,φ) where φ is a i-tower of M_0 but not of M_0/x. Note that if $\varphi \in A_t(x)$ then $x \in \mathrm{cl}_M(J^\varphi_{[t]})$. Given that $M_0|\,\mathrm{cl}_M(J^\varphi_{[t]})$ does not contain $U_{2,\ell+2}$ as a minor, by Theorem 3.1 for each tower φ there are at most ℓ^t such x. Hence $\sum_{x\in E(M_0)} |A_t(x)| \leqslant \ell^t N_t(M_0)$. Let $B_t(x)$ denote the set of triplets (x,φ_1,φ_2) where φ_1 and φ_2 are distinct t-towers of M_0 that are equivalent t-towers in M_0/x, that is, if φ_1 collapses into φ_2. Now, if $\varphi \in T^t_{M_0}$ and $x \in E(M_0)$, either φ spans x and is not a t-tower in M_0/x or φ belongs to an equivalence class of M_0/x. Thus, for each $x \in E(M)$

$$N_t(M_0) = |A_t(x)| + \sum_{c\in\mathcal{T}^t_{M_0/x}} |c|.$$

If φ is a t-tower and $x \notin \mathrm{cl}_M(E(\varphi))$ then φ is a t-tower of M_0/x. Hence, we obtain $N_t(M_0/x) \geqslant |\mathcal{T}^t_{M_0/x}|$. Therefore, for every $x \in E(M_0)$

$$\begin{aligned}
N_t(M_0) - N_t(M_0/x) &\leqslant \left(A_t(x) + \sum_{c\in\mathcal{T}^t_{M_0/x}} |c| \right) - |\mathcal{T}^t_{M_0/x}| \\
&= A_t(x) + \sum_{c\in\mathcal{T}^t_{M_0/x}} (|c|-1) \\
&\leqslant |A_t(x)| + \sum_{c\in\mathcal{T}^t_{M_0/x}} \binom{|c|}{2} \\
&\leqslant |A_t(x)| + \frac{1}{2}|\{(\varphi_1,\varphi_2) : \varphi_1 \text{ collapses into } \varphi_2 \text{ in } M_0/x\}|
\end{aligned}$$

where we are using the fact that $(|c|-1) \leqslant \binom{|c|}{2}$ and the last inequality holds because for any two t-towers φ, φ' in an equivalence class with respect to M_0/x, the pairs (φ, φ') and (φ', φ) are in the set $\{(\varphi_1, \varphi_2) : \varphi_1 \text{ collapses into } \varphi_2\}$. Furthermore, by Lemma 2.9, given a pair (φ, φ') of t-towers such that φ collapses into φ' in M_0/x, then there exists two different $(t+1)$-towers $\varphi \oplus_x \varphi'$ and $\varphi' \oplus_x \varphi$ containing φ and φ'. Therefore

$$\begin{aligned}
&\sum_{x \in E(M_0)} (N_t(M_0) - N_t(M_0/x)) \\
&\leqslant \sum_{x \in E(M_0)} (|A_t(x)| + \frac{1}{2}|\{(\varphi_1, \varphi_2) : \varphi_1 \text{ collapses into } \varphi_2 \text{ in } M_0/x\}|) \\
&\leqslant \ell^t N_t(M_0) + \sum_{x \in E(M_0)} \left(\frac{1}{2}|\{(\varphi_1, \varphi_2) : \varphi_1 \text{ collapses into } \varphi_2 \text{ in } M_0/x\}|\right) \\
&\leqslant \ell^t N_t(M_0) + N_{t+1}(M_0) \\
&\leqslant (\ell^t + a) N_t(M_0).
\end{aligned}$$

Now consider the equivalence class of φ, a $(t-1)$-tower with $t > 1$ and with respect to M_0/x. By Lemma 2.9, for any two towers φ_1, φ_2 in the same equivalence class as φ, the functions $\varphi_1 \oplus_x \varphi_2$ and $\varphi_2 \oplus_x \varphi_1$ are two t-towers in M_0. For any $x \in E(M_0)$ define the function $f_x : B_{t-1}(x) \to T^t_{M_0}$ as

$$f_x((x, \varphi_1, \varphi_2)) = \varphi_1 \oplus_x \varphi_2.$$

Note that f_x is injective. Furthermore, given any distinct $x, y \in E(M_0)$, and triples $(x, \varphi_1^x, \varphi_2^x)$, $(y, \varphi_1^y, \varphi_2^y)$, we have

$$\begin{aligned}
f_x((x, \varphi_1^x, \varphi_2^x))(n) = x \\
f_y((y, \varphi_1^y, \varphi_2^y))(n) = y
\end{aligned}$$

Thus, the towers given by $f_x((x, \varphi_1^x, \varphi_2^x))$ and $f_y((y, \varphi_1^y, \varphi_2^y))$ are distinct because there are no equivalent towers in M. Therefore, the function $f : \bigcup_{x \in E(M_0)} B_{t-1}(x) \to T^t_{M_0}$ defined as follows; For $x \in E(M_0)$ and $(x, \varphi_1, \varphi_2) \in B_{t-1}(x)$;

$$f((x, \varphi_1, \varphi_2)) := f_x((x, \varphi_1, \varphi_2))),$$

is injective. Hence, we get $|\bigcup_{x \in E(M_0)} B_{t-1}(x)| \leqslant N_t(M_0)$.

Additionally, suppose that φ is a t-tower of M_0 such that $x = \varphi(t)$. Given that φ satisfies Definition 2.1(iv), there exists $\varnothing \neq S \subseteq [t-1]$ such that $\varphi(S \cup t)$ is not parallel to $\varphi(S)$. Then, the $(t-1)$-towers φ_1 and φ_2 are inequivalent in M_0 but collapse into the same tower in M/x. For any $\varnothing \neq S \subseteq [t-1]$ define φ_1 and φ_2 as follows

$$\begin{aligned}\varphi_1(S) &= \varphi(S \cup t)\\ \varphi_2(S) &= \varphi(S).\end{aligned}$$

Note that by Proposition 2.5, the functions φ_1 and φ_2 are $(t-1)$-towers. Therefore, we get $(\varphi(t), \varphi_1, \varphi_2) \in B_{t-1}(\varphi(t))$. We conclude $\sum_{x\in E(M_0)} |B_{t-1}(x)| = N_t(M_0)$, where

$$|B_{t-1}(x)| = \sum_{c\in \mathcal{T}^{t-1}_{M_0/x}} 2\binom{|c|}{2} = \sum_{c\in \mathcal{T}^{t-1}_{M_0/x}} |c|(|c|-1).$$

Note that by Proposition 2.11 we get $|c| \leqslant \ell^{t-1}$. As a consequence;

$$|B_{t-1}(x)| \leqslant \sum_{c\in \mathcal{T}^{t-1}_{M_0/x}} \ell^{t-1}(|c|-1)$$

and thus

$$\frac{|B_{t-1}(x)|}{\ell^{t-1}} \leqslant \sum_{c\in \mathcal{T}^{t-1}_{M_0/x}} (|c|-1).$$

With this in mind we get the following inequalities when $t > 1$;

$$\begin{aligned}\sum_{x\in E(M_0)} N_{t-1}(M_0) - N_{t-1}(M_0/x) &= \sum_{x\in E(M_0)} \left(\left(|A_{t-1}(x)| + \sum_{c\in \mathcal{T}^{t-1}_{M_0/x}} |c|\right) - |\mathcal{T}^{t-1}_{M_0/x}|\right)\\ &= \sum_{x\in E(M_0)} \left(|A_{t-1}(x)| + \sum_{c\in \mathcal{T}^{t-1}_{M_0/x}} (|c|-1)\right)\\ &\geqslant \sum_{x\in E(M_0)} \left(\sum_{c\in \mathcal{T}^{t-1}_{M_0/x}} (|c|-1)\right)\\ &\geqslant \sum_{x\in E(M_0)} \frac{|B_{t-1}(x)|}{\ell^{t-1}}\\ &= \frac{N_t(M_0)}{l^{t-1}}.\end{aligned}$$

Finally note that when $t = 1$ and since there are no loops in $E(M_0)$, the rank of M_0/x is $r(M_0/x) = r(M_0) - 1$ for all $x \in E(M_0)$. Thus,

$$\begin{aligned}
\sum_{x \in E(M_0)} N_{t-1}(M_0) - N_{t-1}(M_0/x) &= \sum_{x \in E(M_0)} r(M_0) - r(M_0/x) \\
&= \sum_{x \in E(M_0) \setminus \mathrm{cl}_{M_0}(\varnothing)} 1 \\
&= |E(M_0) \setminus \mathrm{cl}_{M_0}(\varnothing)| \\
&= N_1(M_0).
\end{aligned}$$

Therefore, we have the following;

$$\begin{aligned}
\sum_{x \in E(M_0)} (N_t(M_0) - N_t(M_0/x)) &\leqslant (\ell^t + a) N_t(M_0), \\
\frac{N_t(M_0)}{l^{t-1}} &\leqslant \sum_{x \in E(M_0)} (N_{t-1}(M_0) - N_{t-1}(M_0/x)), \\
\ell^t(a + \ell^{t-1}) \sum_{x \in E(M_0)} (N_{t-1}(M_0) - N_{t-1}(M_0/x)) &< \sum_{x \in E(M_0)} (N_t(M_0) - N_t(M_0/x))
\end{aligned}$$

which implies

$$\begin{aligned}
\ell(a + \ell^{t-1}) N_t(M_0) = \ell^t(a + \ell^{t-1}) \frac{N_t(M_0)}{\ell^{t-1}} &< (\ell^t + a) N_t(M_0), \\
\ell a N_t(M_0) &< a N_t(M_0),
\end{aligned}$$

a contradiction. Therefore, $N_{t+1}(M_0) \leqslant a N_t(M_0)$. □

Theorem 3.3. *Let M be a $U_{2,\ell+2}$-free matroid. If $\varepsilon(M) > \ell^{t^2 - \binom{t-1}{2}} r(M)$, then there exists a minor M' of M that contains a t-tower.*

Proof. Let $a_{t-1} = 0$ and define the integers $a_{t-2}, \ldots, a_0$ recursively by $a_i = \ell^{i+1}(a_{i+1} + \ell^i)$ for all $0 \leqslant i \leqslant t - 2$.

Claim 3.4. *For all $i \leqslant t - 1$, we have $a_{t-i} \leqslant \ell^{it - \binom{i-1}{2}}$.*

Proof. The proof is by induction on i. The base case $i = 1$ is trivial. Suppose that

$a_{t-(i-1)} \leqslant \ell^{(i-1)t-\binom{i-2}{2}}$. By definition,

$$\begin{aligned} a_{t-i} &= \ell^{t-i+1}(a_{t-i+1} + \ell^{t-i}) \\ &\leqslant \ell^{t-i+1}(\ell^{(i-1)t-\binom{i-2}{2}} + \ell^{t-i}) \end{aligned}$$

The real-valued function $f(x) = (x-1)t - \binom{x-2}{2} - (t-x)$ satisfies $f(2) = 2$ and

$$f'(x) = t - (x - \frac{1}{2}) + 1$$

which is non-negative for $2 \leqslant x \leqslant t$. Hence

$$\ell^{(i-1)t-\binom{i-2}{2}} + \ell^{t-i} \leqslant \ell^{(i-1)t-\binom{i-2}{2}+1}.$$

Therefore,

$$\begin{aligned} a_{t-1} &\leqslant \ell^{t-i+1}(\ell^{(i-1)t-\binom{i-2}{2}} + \ell^{t-i}) \\ &\leqslant \ell^{t-i+1}(\ell^{(i-1)t-\binom{i-2}{2}+1}) \\ &= \ell^{it-((i-2)+\binom{i-2}{2})} \\ &= \ell^{it-\binom{i-1}{2}} \end{aligned}$$

as desired. ■

Let $i \in \{0, \dots, t-1\}$ be maximal such that M has a minor M_0 for which

$$N_{i+1}(M_0) > a_i N_i(M_0).$$

Such an i exists since

$$N_1(M) = \varepsilon(M) > \ell^{t^2-\binom{t-1}{2}} r(M) \geqslant a_0 r(M),$$

where we have $a_0 \leqslant \ell^{t^2-\binom{t-1}{2}}$ by the previous claim.

Suppose that $i = t-1$, then there exists a minor M' of M such that $N_t(M') > a_{t-1}N_{t-1}(M')$. As $a_{t-1} = 0$, then $N_t(M') > 0$ as desired. Suppose otherwise, and let M' minor of M such that $N_{i+1}(M') > a_i N_i(M') = \ell^{i+1}(a_{i+1} + \ell^i)N_i(M')$. By Lemma 3.2, there exists a minor M_0 of M such that $N_{(i+1)+1}(M_0) > a_{i+1}N_{i+1}(M_0)$, contradicting the minimality of i. □

As a direct result we obtain the following theorem, where $d(k+1)$ is the infimum of all positive real numbers d such that all simple graphs G with $\frac{|E(G)|}{|V(G)|} \geqslant d$ contain K_{k+1} as a minor.

Theorem 3.5. *If M is a $U_{2,\ell+2}$-minor-free matroid and $\varepsilon(M) \geqslant \ell^{8((\ell+1)k\lceil d(k+1)\rceil)^2} r(M)$, then M contains $M(K_{k+1})$ as a minor.*

Proof. By Theorem 3.3, since $\frac{1}{2}(t+2)^2 > t^2 - \binom{t-1}{2}$ and $4(\ell+1)k\lceil d(k+1)\rceil) > t+2$ when $t = 4(\ell+1)(k-1)\lceil d(k+1)\rceil$, the matroid M contains a $(4(\ell+1)(k-1)\lceil d(k+1)\rceil)$-tower. Therefore, by Lemma 2.19 the matroid M contain $M(K_{k+1})$ as a minor. □

First recall that Thomason showed in [14] that $d(k) = (\alpha + o(1))k\sqrt{\log(k)}$ where $\alpha = 0.319\ldots$. Combining this result with Theorem 3.5 for large k we obtain the following bound asymptotic in k.

Theorem 3.6. *Let $k \geqslant 3$ and $\ell \geqslant 2$ be integers, if M is a $U_{2,\ell+2}$-minor-free matroid and*

$$\varepsilon(M) \geqslant \ell^{(4\alpha+o_k(1))(\ell+1)^2 k^4 \log(k)} r(M)$$

where $\alpha = 0.319\ldots$, then M contains $M(K_k)$ as a minor.

Additionally, by using the bound for $d(k)$ given by Kostochka and using Proposition 1.11 and Theorem 3.5 we obtain the following bound unconditionally for all k.

Theorem 3.7. *Let $k \geqslant 4$ and $\ell \geqslant 2$ be integers, if M is a $U_{2,\ell+2}$-minor-free matroid and*

$$\varepsilon(M) \geqslant \ell^{4(10)^6(\ell+1)^2 k^4 \log(k)} r(M),$$

then M contains $M(K_k)$ as a minor.

References

[1] B. Bollobás, P. A. Catlin, and P. Erdős. Hadwiger's conjecture is true for almost every graph. *European J. Combin.*, 1(3):195–199, 1980.

[2] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Berlin, fifth edition, 2018. Paperback edition of [MR3644391].

[3] J. Geelen. Small cocircuits in matroids. *European J. Combin.*, 32(6):795–801, 2011.

[4] J. Geelen and K. Kabell. Projective geometries in dense matroids. *J. Combin. Theory Ser. B*, 99(1):1–8, 2009.

[5] J. Geelen, J. P. S. Kung, and G. Whittle. Growth rates of minor-closed classes of matroids. *J. Combin. Theory Ser. B*, 99(2):420–427, 2009.

[6] J. Geelen and G. Whittle. Cliques in dense GF(q)-representable matroids. *J. Combin. Theory Ser. B*, 87(2):264–269, 2003.

[7] A. V. Kostochka. Lower bound of the Hadwiger number of graphs by their average degree. *Combinatorica*, 4(4):307–316, 1984.

[8] J. P. S. Kung. The long-line graph of a combinatorial geometry. I. Excluding $M(K_4)$ and the $(q+2)$-point line as minors. *Quart. J. Math. Oxford Ser. (2)*, 39(154):223–234, 1988.

[9] J. P. S. Kung. Extremal matroid theory. In *Graph structure theory (Seattle, WA, 1991)*, volume 147 of *Contemp. Math.*, pages 21–61. Amer. Math. Soc., Providence, RI, 1993.

[10] W. Mader. Homomorphieeigenschaften und mittlere Kantendichte von Graphen. *Mathematische Annalen*, 174(4):256–268, 1967.

[11] S. Norin, B. Reed, A. Thomason, and D. R. Wood. A lower bound on the average degree forcing a minor. *Electron. J. Combin.*, 27(2):Paper No. 2.4, 9, 2020.

[12] J. Oxley. *Matroid theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011.

[13] B. Reed and D. R. Wood. Forcing a sparse minor. *Combin. Probab. Comput.*, 25(2):300–322, 2016.

[14] A. Thomason. The extremal function for complete minors. *J. Combin. Theory Ser. B*, 81(2):318–338, 2001.

[15] T. Zaslavsky. Biased graphs. I. Bias, balance, and gains. *Journal of Combinatorial Theory, Series B*, 47(1):32–52, 1989.

[16] T. Zaslavsky. Biased graphs. II. The three matroids. *Journal of Combinatorial Theory, Series B*, 51(1):46–72, 1991.