Convex Optimization, Stochastic Approximation, and Optimal Contract Management in Real-time Bidding

by

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Author's Declaration

Abstract

This thesis studies problems at the intersection of monotone and convex optimization, auction theory, and electronic commerce. Convex optimization and the theory of stochastic approximation serve as the basic practical and theoretical tools we have drawn upon. We solve important problems facing Demand Side Platforms (DSPs) and other demand aggregators (to be defined in the main body) in the e-commerce space, particularly in the field of real-time bidding (RTB). RTB is a real-time auction market, the primary application of which is the selling advertising space. Our main contribution to this field, at its most basic, is to recognize that certain optimal bidding problems can be re-cast as convex optimization problems. Particular focus will be placed upon the second price auction mechanism due to the strikingly simple structural results that hold in this case; but many results generalize to the first price auction mechanism under additional assumptions. We will also touch upon formal connections between these auction problems and two important problems in finance, namely the dark pool problem, and optimal portfolio construction.

Acknowledgements

First of all, I must thank my advisor Ravi Mazumdar for inspiring me to pursue this path, and for helping me through my many years at the University of Waterloo[1]. The input and discussion with our co-author, and my more or less informal co-advisor Peter Marbach has also been invaluable, thank you for asking so many insightful questions. Many thanks to my committee members Patrick Mitran, Andrew Heunis, and Henry Wolkowicz not only for serving on my committee, but for playing an important role in my education. Discussions and suggestions from fellow students Syamantak Datta Gupta, Arpan Mukhopadhyay, Thirupathaiah Vasantam, and Samira Ghanbarian have played an important part in my intellectual development. I must also acknowledge an old high school teacher, Mr. Lloyd, whose physics class had a major impact on my interest in the sciences, thank you for challenging us.

One of the additional example applications presented in this thesis, namely, the Dark Pool Problem, arose from connections I recognized while working as an intern in Morgan Stanley's machine learning research group. I would like to thank Yuriy Nevmyvaka for the opportunity to contribute to this problem.

My loving parents have been instrumental in enabling me to pursue a PhD. They have my unending love and appreciation.

Finally, I want to acknowledge many fellow friends from *UWat*, you have made my years in Waterloo a joy. Adi, for your positivity; Ali and Anil, for getting us outside; Amit, Brad, and Nathan, for being model researchers; Becca, Dani, Julia, and Jeremy, for the music; Camilla, for your inspiring talents and hobbies; Deep, Egon, and Omar, for your (online) company over the Covid-19 lockdowns; Erik and Sajin, for always laughing at my lousy jokes; Gillian, for your humour; Indi and Sophia, for all the lunch dates; Mukhtar, for the rhymes; Nate, Nupur, and Sabria, for helping to bring us together; Raisa, for your kindness, generosity, and talents as a DJ; Savio, for being such a great host; Tom and Steph, for all the adventures in Montreal; and to everyone else I have met along the way, thank you all.

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Dedication

This thesis is dedicated to my friends and family.

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Some Humour and Other Things

"The other, was a scheme for entirely abolishing all words whatsoever: and this was urged as a great advantage in point of health as well as brevity [...] An expedient was therefore offered, that since words are only names for things, it would be more convenient for all men to carry about them, such things as were necessary to express the particular business they are to discourse on [...] Many of the most learned and wise adhere to the new scheme of expressing themselves by things; which hath only this inconvenience attending it; that if a man's business be very great, and of various kinds, he must be obliged in proportion to carry a greater bundle of things upon his back, unless he can afford one or two strong servants to attend him. I have often beheld two of those sages almost sinking under the weight of their packs, like pedlars among us; who, when they met in the streets, would lay down their loads, open their sacks, and hold conversation for an hour together; then put up their implements, help each other to resume their burdens, and take their leave."
Jonathan Swift, Gulliver's Travels, 1726

"No man is an island, entire of itself; every man is a piece of the continent, a part of the main. If a clod be washed away by the sea, Europe is the less, as well as if a promontory were, as well as if a manor of thy friend's or of thine own were: any man's death diminishes me, because I am involved in mankind, and therefore never send to know for whom the bells tolls; it tolls for thee."
John Donne, Devotions Upon Emergent Occasions, 1624

"This is the task of a liberal education: to give a sense of the value of things other than domination, to help to create wise citizens of a free community, and through the combination of citizenship with liberty in individual creativeness to enable men to give to human life that splendour which some few have shown that it can achieve."
Bertrand Russell, Power: A New Social Analysis, 1938

Chapter 1

Introduction

1.1 Auctions

In 2020, Paul Milgrom and Robert Wilson won the *Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel* "for improvements to auction theory and inventions of new auction formats." [199, 123] (for a substantive review of the contributions of these researchers see [169]). According to Milgrom [122], one of the earliest examples of an auction is a bridal auction described by Herodotus (and formally analyzed by [16]) around 500 B.C. Since this time, various auction formats have been applied to allocate an array of different types of scarce resources: mining rights, wireless spectrum, art, electricity, *etc.* and these auctions are often carefully designed to elicit certain behaviour from participants. Indeed, Wilson emphasized in his lecture the importance of what he referred to as *economic engineering*, *i.e.,* the design of auctions, markets, and trading rules that lead to the outcomes the economic engineer intends.

One of the most famous examples of successful auction market design is in spectrum auctions [120] where governments need to decide how to allocate scarce wireless spectrum licenses to firms for, *e.g.,* TV, cell phones, radio, *etc.* This allocation problem is challenging both economically and computationally [111]. From an economic point of view, the government needs to design an auction mechanism which seeks to maximize revenue while still respecting established property rights law, promoting competition, preventing collusion, *etc.* Computationally, determining if a collection of licenses satisfies interference rules is a large and complicated combinatorial problem and optimization over the set of feasible licensing arrangements is computationally infeasible. These challenges necessitated the invention of efficient algorithms and the design of completely new auction mechanisms.

Another area where auctions have played an important role is in electricity markets [198, 190]. In the primary market for electricity (there is also a derivatives market for futures) governments solicit bids from energy suppliers to commit to supplying a certain amount of power at a certain time in the future (*e.g.,* day ahead, month ahead, year ahead, *etc.*). Ontario, for example, has recently updated their *demand response auction* into what they claim is a broader and more competitive *capacity auction* soliciting commitments for one year in advance from a variety of power producers and consumers [1]. The moment-to-moment *spot market* for electricity operates as a *double auction* (at least in Ontario), where producers and consumers of electricity place a bid to buy or sell electricity and a centralized operator (in Ontario, the *Independent Electricity System Operator* or *IESO*) calculates a price that clears the market [1].

Similarly to the spot market for electricity, securities markets (include electricity futures markets) operate in real time as *continuous* double auctions [163, 27] where bids (firm commitments to buy a certain quantity at a certain price) and offers (commitments to sell a quantity at a price) are continually updated by competing participants with exchange occurring only when two orders *cross*, *i.e.,* a bid to buy comes in higher than an offer to sell.

One key aspect illustrated by these examples is that what constitutes an appropriate market or auction mechanism is dependent upon the characteristics of the resources being sold or traded. A continuous double auction is not appropriate for the provisioning of electricity since the stability of the grid depends on the exact balance between supply and demand, necessitating a central operator for fast time scale production decisions. Likewise, an auction for stocks or electricity futures taking place once every year is unacceptable as it would undermine the risk management and price discovery purposes of the market, not to mention that there is not just a single seller of such items, as in the case of spectrum auctions.

1.1.1 Auction Formats

The most famous auction format, and historically most popular, is the *English auction* [119]. In an English auction, there is a single item up for sale and the auction proceeds continuously through *open-outcry* and *ascending* bidding. The seller of the item sets some initial price, called the *reserve price*, at which the bidding process starts, and then the participants continually state their willingness to pay a greater and greater price for the item, until finally no one is willing to go any higher and the item is awarded to the bidder who placed the last (and necessarily highest) bid. The winning bidder pays the winning bid to the seller and is awarded the item.

A much simpler auction format is the *Vickrey auction* [184], which is a *sealed-bid* (as opposed to open-outcry) auction that occurs in just one shot, *i.e.,* the bidders don't get the opportunity to place a higher bid if they find out someone outbid them. The defining characteristic of the Vickrey auction is that, while the item is still awarded to the participant that submit the highest bid, the winner pays the *second highest* bid to the seller of the item. For this reason, the auction is also referred to as a *sealed-bid second-price auction*.

In a Vickrey auction, it is a (weakly) *dominant strategy*[1] to *bid your valuation*. That is, if the item up for sale is worth v to you, then it is, regardless of the actions taken by any competitors, an optimal course of action to place the bid v. To see why, we can recognize that the optimal bid $b^\star$ necessarily satisfies $b^\star \leq v$, since if you bid $b > v$ there would be a chance of paying more for the item than it is worth to you (this occurs if a competing bidder placed a bid c such that $v < c < b$); as well, the optimal bid necessarily satisfies $b^\star \geq v$ since you would rather pay $v - \epsilon$ (for any $\epsilon > 0$) than not win the item (*i.e.,* if you bid $v - \epsilon$ and a competitor placed a bid c satisfying $v - \epsilon < c < v$, then you would regret not placing a bid $c < b \leq v$). Thus, ($v \leq b^\star \leq v \implies b^\star = v$) it is optimal to bid your valuation. This means that the second price auction is a *truth revealing* mechanism, *i.e.,* agents have the incentive to reveal their true private value to the auctioneer.

Remarkably, if bidders all have private and independent valuations for the item being sold (*e.g.,* if the item is a work of art and each bidder has independent tastes) this auction format is theoretically equivalent to the English auction [184]. The reason for this equivalence is that in the English auction, if again your valuation of the item is v, you will be willing to bid $c+\epsilon$, for some $\epsilon > 0$ against any competing bid $c < v$. Thus, if your valuation is the highest amongst all competitors, and p is the highest any competitor is willing to bid (which will be called the price), then the auction will stop when this competitor announces the bid p and you announce some bid $p + \epsilon$.

Another, natural pair of auction mechanisms is the open-outcry descending *Dutch auction* and the sealed-bid *first-price auction*. In the former, the auctioneer begins with some initial very high price for the item, and descends downwards offering lower and lower prices until a participant finally speaks up and claims the item at the last stated price. In the latter, agents simply submit a secret bid, and the highest bidder wins and pays the seller whatever it is that they bid. Again, these two auctions are strategically similar [103].

Some work has also been done to understand how these auction mechanisms can be blended together as a "soft floor" [209]. In such an auction if the highest bid b is greater

[1] In Game Theory, a dominant strategy (which need not exist) is a course of action which is always optimal, regardless of the actions or behaviours of any other participant [66].

than the soft floor threshold s the winning bidder pays the maximum between s and the second highest bid (as in the second price auction); if the highest bid lies below s but above a hard floor threshold $h < s$ the winner pays b itself (as in the first price auction), and if $b < h$, the seller simply keeps the item (*i.e.,* h is the reserve price). There are of course a wide variety of alternative auction mechanisms arising in various contexts or simply as theoretical curiosities, for example, *third price auctions* where the winner pays the third highest bid; *all-pay auctions* where every participant has to pay their bid, whether or not they win; the *Vickrey-Clarke-Groves auction*, which is a generalization of the Vickrey auction [182]; and *French auctions* used for pricing initial public offerings where some of the highest bids are actually excluded from the auction [53, 117].

Remark 1.1.1 (Optimal Auction Design)**.** It is a natural question to ask: "what is the *optimal* auction format?" [129]. The notion of *optimal* of course depends upon who is asking the question, but the seller of the item, for example, may take *optimal* to mean *revenue maximizing*. It was shown by [122] that, of the four auction mechanisms discussed here (English, Dutch, Vickrey, and sealed-bid first-price) it is the ascending English auction that, under realistic assumptions, generates the greatest revenue for the seller. The innovation of this result is in how the prevailing assumptions in auction theory were challenged; indeed, under some rather strong assumptions (primarily: private and independent values), all of these auctions result in the *same revenue*: the revenue equivalence principle [103, Ch. 3]. When these assumptions are relaxed, revenue equivalence no longer holds. For general markets, determining what the revenue maximizing auction actually is can be very difficult [57]. Moreover, the *Wilson doctrine* [197] [160, Ch. 11] argues that mechanisms should make minimal assumptions about the common knowledge of participants (in terms of functional forms, probability distributions, *etc.*) which revenue optimality may hinge upon. Thus, optimal auctions rarely appear in practice, and the four above described auction formats cover a very wide range of applications.

Auctions for Multiple Items When there are multiple items that need to be allocated it is natural to either allocate them all at once as in a *combinatorial auction* [150, 48] or *simultaneous auction* [121, 47] or to allocate them in a sequence of individual auctions. The original application of the combinatorial auction was for allocating airport takeoff and landing slots [150] where bidders can place bids on *combinations* of items where they are only interested in winning if they can obtain the entire package, since a time slot for take off has limited value without a corresponding time slot to land. For an example of a sequential mechanism, a simultaneous ascending auction where bidders can submit bids on any number of items in a series of bidding rounds was the method famously applied to spectrum auctions [121]. Additionally, the idea of allocating items in a sequence of single-item auctions is natural in markets where items are produced sequentially and perish rapidly (*e.g.,* electricity or advertising markets) and therefore where the items need to be allocated immediately. This is the primary setting of this thesis and constitutes *real-time bidding* (RTB).

This thesis, focusing on RTB, exclusively studies auctions wherein multiple items (in fact, a continuous and un-ending stream of items) are sold sequentially in sealed-bid auctions using either the first or second price mechanism.

1.1.2 Real-time Bidding Markets

Online advertising is another enormous industry that now leverages auctions in a fundamental way, and is the main motivating example behind the work carried out in this thesis. Online advertising constitutes a significant part of today's advertising landscape: the total amount of money spent directly on internet advertising (the largest advertising segment, far surpassing competitors like TV and print) in 2020 in the US, according to the Interactive Advertising Bureau [2] exceeded $130b. Moreover, year on year growth rates remain high: revenue in 2020 grew by 12.2% over 2019, despite the Coronavirus disease (COVID-19) pandemic. As well, the proportion of *programmatic* placements (involving advertising agreements reached by algorithms) is the vast majority of this total, over 90%, much of which is allocated by auction mechanisms including sponsored search and *real-time bidding*.

In computational advertising [205] *real-time bidding* (RTB) refers to a specific type of advertising auction where advertisers (or their representatives) bid for the right to display their content in some available ad slot to a particular user [189, 207, 44]. In more detail, a sale in RTB consists of a group of advertisers (*advertisers, bidders, agents* and *participants* will all be synonyms) who are looking to purchase ad space, and at least three additional parties: the *publisher*, who is the party that owns the advertising slot, *e.g.,* the operator of a website, mobile app, *etc.*; the *user*, who is the person visiting the publisher's website (say) and will view the ad placed into the slot; and the *auctioneer*, who coordinates the bidding process and selects the winning bidder. When an ad is served to a user this is referred to as an ad *impression* and therefore we may also say that advertisers *purchase impressions*, or that RTB is a *market for impressions*. More generically, we will also use the term *item*, for whatever is being sold in RTB.

Of course, the RTB ecosystem can be quite a bit more complicated than this simple description suggests. A slightly more detailed depiction of RTB is provided in Figure 1.1.

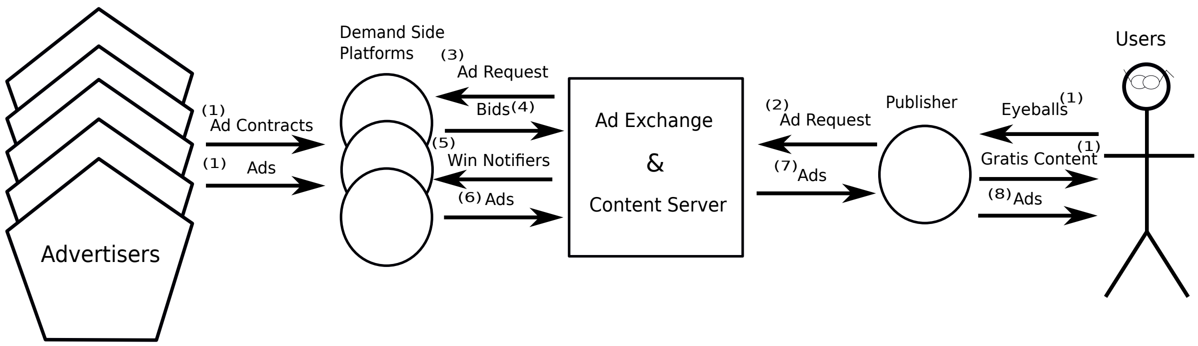


Figure 1.1: Real-time Bidding Process
The basic aspects of the RTB process. Numbering indicates an ordering of events. Advertisers contract with DSPs and provide to them the ads they want served as part of a contractual deal. As users visit publisher pages, being compensated for their attention with the content provided gratis by the publisher, publishers (if choosing to send the request to RTB, as opposed to an alternative advertising channel, which we do not discuss in detail) generate ad requests which are sent (generally through a network of SSPs) to ad auctions. The request, along with accompanying data, is forwarded to DSPs. These DSPs analyze the bid request and then send a bid to the ad auction. The winning DSP chooses an ad (from among their contracts) to forward back to the publisher to be displayed to the user.

Indeed, there can be a multitude of ad-exchanges, as well as various intermediaries sitting between the user and the advertiser including supply side platforms (SSPs) responsible for aggregating available publisher inventories, demand side platforms (DSPs) responsible for aggregating the demand of advertisers, agencies that coordinate advertising efforts on behalf of brands and businesses, as well as various servers, IT, and data infrastructure providers that facilitate the process.

Remark 1.1.2 (Terminology)**.** The terminology "real-time bidding" may be confusing since there are numerous markets where bidding occurs in real time. However, RTB refers to the specific form of online advertising described in this section. If we have need to refer to the more generic notion of "bidding in real time", we will write that.

Remark 1.1.3 (Sponsored Search)**.** Sponsored search [59] is another type of online advertising auction. In some ways it was a precursor to RTB. In sponsored search, originally employed at least by Yahoo! and Google, advertisers pay to have links to their content displayed to users searching for relevant terms. For example, a user searching for the very obvious key words "used car" is likely to attract the interest of used car dealerships, who may be willing to pay to have a link to their website displayed to the user. The auction format employed in sponsored search, a generalized second price (GSP) auction, was itself innovative and addressed the unique needs of the market. In a GSP auction, there are multiple winners: the agent with the highest bid is allocated the most prominent advertising slot (usually the first search result) with a few runner-up bidders allocated to progressively less valuable slots [180, 181, 145].

Similarly to the examples given earlier, there are important structural aspects that lead to the RTB market being structured as an auction in real time. Firstly, the items being sold (ad impressions) are perishable and cannot be re-sold. This feature is shared with electricity for immediate delivery, but is in contrast to electricity derivatives markets where the re-sale of futures is a key part of the market's purpose. Secondly, the items up for sale in RTB may often be close substitutes of one and other (although this depends on the targeting granularity, see Remark 1.1.4). This is in contrast to spectrum auctions (or, *e.g.,* auctions for mineral rights) but is a feature shared with securities markets where a share of a given company's stock is the same as any other share. A third key point is that the auctions are, naturally, taking place in *real time*, and in fact very rapidly (the round trip time from a user visiting a publisher page, the impression auction being carried out, and the winner's ad being shown to the user, is generally no more than 100ms). This, along with the very large number of participants, precludes the possibility of auctioning large lots of items simultaneously (but see Remark 1.3.1)

Remark 1.1.4 (Privacy)**.** Real-time bidding, by its nature, raises serious privacy concerns [137] [195]. Indeed, much of what makes the RTB technology valuable to advertisers results from what may be considered intrusive monitoring of user behaviour. A great deal user data, including browser histories, also easily leaks throughout the system to unintended parties [15]. Europe's General Data Protection Regulation (GDPR) [77, 186] is designed to help protect user privacy and give users more control over their data and how it is used. This regulation presents significant problems for publisher's current models and for RTB [183, 93]. The problems formulated in this thesis (see Chapter 2 for formal descriptions) do not rely on invasive tracking technology and are formulated entirely in terms of generic user *types*. It is technically possible for these types to encode arbitrarily detailed user information; however, they can of course also represent much coarser information. Indeed, the abstract formulation in types provides a plausible mechanism for privacy protection wherein only a certain agreed upon collection of types is even *allowed* to be reported to advertisers.

1.2 Aims, Scope, and Perspective: Contract Management Problems

In the real-time bidding literature (see Section 1.3) we can recognize a categorization based on two broad perspectives that approach the field in different (though often complementary) ways: *descriptive* and *normative*. The descriptive approach seeks to understand *how* the various participants (publishers, users, advertisers, auction platforms *etc.*) behave and why, while the normative approach analyzes how these participants *should* behave. The descriptive approach includes empirical analysis of actual market data, as well as the game theoretic analysis of different market rules under various stylized assumptions. This game theoretic line of inquiry may also often be meta-analytic and include a normative component that seeks to design the rules of the market themselves (*i.e.,* to perform economic engineering), based on how the various agents are expected to behave given a specified set of rules. On the other hand, the purely normative approach takes the rules of the market for granted and attempts to determine the optimal behaviour of a particular agent whose perspective they take. This approach typically makes fewer stylized assumptions and attempts to leverage insights from normative analysis to design practical algorithms, either as a proof-of-concept, or which have been tested in live markets.

The approach taken in this thesis is primarily *normative*. Specifically, we will take the perspective of an intermediating firm that aggregates the demand of a multitude of marketing agencies and participates in the RTB markets on their behalf. Intermediaries of this type are referred to as *Demand Side Platforms* (DSPs), though this term can also be used quite broadly to refer to any intermediating firm that works with marketing agencies to facilitate their acquisition of ad space, which may be through RTB or alternative channels. We treat this normative demand aggregation problem as a *contract management* problem where the intermediary has entered into a set of contracts which obligate them to acquire some volume of items, via the RTB market, on behalf of the contract counter party. The DSP will seek to obtain these items at minimum cost, and are not budget constrained (since the contract is an obligation). The most elementary formulation of this problem can be written as

$$
\begin{aligned}
\underset{x,\gamma}{\text{minimize}} \quad & \sum_{j=1}^{M} f_j(x_j) \\
\text{subject to} \quad & \sum_{j=1}^{M} \gamma_{ij} W_j(x_j) \geq C_i \\
& \sum_{i=1}^{N} \gamma_{ij} \leq 1, \gamma_{ij} \geq 0,
\end{aligned} \tag{P^m}
$$

where $W_j(x)$ is a function determining the rate at which items of "type" j are obtained by placing the bid x, $f_j(x)$ is a function determining the expected cost of placing this bid (which depends upon the rules of the auction), γ_{ij} is an "allocation array" indicating the rate at which items of type j should be allocated towards the fulfillment of contract i, and $C_i > 0$ are the target requirements of that contract. A much more detailed explanation, and more general problem formulation, will appear in Chapter 2.

There is a natural economic niche occupied by these DSPs, analyzed formally by [11], see also the discussion of [213]. First of all, individual advertisers range all the way from small firms and individuals to the world's largest multinational institutions. DSPs offer, first of all, the technical infrastructure and talent necessary to effectively participate in the online advertising space; small firms are simply unable afford this expense, and most large businesses can be expected to benefit from outsourcing this capacity due to comparative advantages. Secondly, the intermediary may serve a risk hedging function. Indeed, the contract, once accepted, *must* be fulfilled, even if it is no longer profitable for the DSP to do so, *i.e.,* the DSP bears the risk of adverse changes in the market. Finally, as is explained by [11], since the intermediary is not budget constrained, they have at their disposal a wider array of bidding strategies than does a budget-constrained individual advertiser. This leads to cost reductions and profitability even if DSPs are not technologically more sophisticated.

The study of Problem (P^m) in the context of RTB arose with [170] and was formulated based on a problem faced by a Canadian DSP. Thus, our problem is of practical industrial importance. However, our approach is not purely practical; indeed, the empirical analysis of algorithm performance, either with real data or in live markets, does not feature prominently in this work (though it is not absent either, see Chapter 5). Ultimately, we seek to develop a thorough understanding of the elegant structure arising in Problem (P^m) and to show how this structure can be leveraged to develop practical bidding algorithms.

In addition, we recognize that the insight into Problem (P^m), both in terms of analytic techniques, and through an understanding of the functions W, f arising therein, can be further leveraged to tackle seemingly disparate problems in other fields (see Section 2.5 and 3.4). Finally, while the description of Problem (P^m) is given through the language of RTB, similar problem formulations may be applicable to alternative auction based markets, *e.g.,* electricity markets, or processor and cloud scheduling [196, 151, 211, 217, 208]; the key aspect is that the nature of the "items" be such that they are generated (and necessarily sold) rapidly in sequence, and that the items be rapidly perishable (thus precluding storage or secondary markets; but not necessarily derivatives markets).

1.2.1 Computational Methods

The main focus of this this thesis is not intended to be upon numerical computations, but throughout the thesis we will touch upon some of the practical computational methods drawn upon for calculations and simulations. All of our code is implemented in Python [179], primarily using `numpy` and `scipy` [185] for numerically calculations, and `matplotlib` for our figures [89]. Optimization problems are solved through some combination of custom algorithms, the convex modelling software `cvxpy` [54], and the interior point solver `CVXOPT` [178]. Some important and illustrative code listings are given in Appendix C. We remark further upon computation and algorithms in Sections 2.4 and 3.2.1.

1.3 Literature Review

The literature associated with auction theory, which we have already touched upon, and with computational advertising, broadly speaking, is vast [88]. We cannot provide a comprehensive review of this broad field, but instead primarily focus here upon the specific sub field of real-time bidding. The closely related, yet distinct, field of sponsored search (see Remark 1.1.3) will also receive little attention in this review, but see, *e.g.,* [60], for early work along similar lines as the bid optimization literature in RTB.

We categorize our overview roughly into three categories: game theoretic analysis, estimation and machine learning problems, and optimal bidding. Papers which focus on game theoretic analysis provide insightful and foundational theoretical understanding for how the market operates (or how it should be structured). Practical problems in estimation are important for applications since algorithms designed for practical problems generally assume some knowledge about prevailing market statistics (*e.g.,* prices, volume, item characteristics, *etc.*) which must be estimated. Finally, problems of optimal bidding answer normative questions about how agents in RTB should bid in order to, *e.g.,* maximize the value of items obtained, respect budget constraints, *etc.* This thesis is primarily concerned with problems of optimal bidding.

Game Theoretic and Mean Field Analysis There is a rich literature focusing on the application of game theoretic tools to analyze certain types of equilibria arising in RTB and sponsored search auctions. The papers of [157] and [35] apply a "classical" game theoretic analysis to the problem. However, since RTB auctions consist of a large number of players (say N), an exact game theoretic analysis of N player RTB games is generally both intractable and implausible as a model of reality since the information processing required for a player to best respond to their $N-1$ competitors is immense. Therefore, fluid limits [51, 156] (where the "importance" of each auction is scaled towards zero simultaneously as the number of auctions occurring goes to infinity), mean field limits (where $N \to \infty$) [107, 71, 34], or some combination thereof, are used to approximate the behaviour of agents in large repeated auctions.

In particular, [84] analyzes budget-constrained sequential Vickrey auctions as a Markov Decision Process (MDP) with known market statistics, using a fluid limit to obtain an approximately optimum bid-your-*shaded*-value policy. This policy is structurally similar to many results in optimum bidding for second price auctions where if the agent's (random) budget is B and (random) item valuation is v, the optimum bid is given by $\frac{v}{1+s(B)}$ for some *shading factor* $s(B)$. Often, this shading factor is derived directly from Lagrange multiplier associated to the budget constraint. Since it is optimal to bid your valuation in a one-shot second price auction, one may interpret the bid shading that occurs in budget constrained auctions as encoding the value of saving some budget in the hope of coming across a great opportunity in the future.

A mean-field equilibrium can be obtained from the optimum strategy of [84] by considering a mapping $\Gamma : \mathcal{P}(U) \to \mathcal{P}(U)$, where U is the action space (a finite set of possible bids) and $\mathcal{P}(U)$ is the space of probability measures on this set, such that Γ maps from some supposed distribution of the maximum competing bid, call it μ, into the distribution over the maximum bid induced by the optimum bidding strategy, namely $\Gamma(\mu)$. An equilibrium in this case, as described by [84], is a fixed point of this mapping, *i.e.,* a price distribution μ such that $\Gamma(\mu) = \mu$. This constitutes a simplified equilibrium concept since agents are best responding only to the distribution of the maximum bid, and not to the entire history of the game and every action taken by every opposing player as in, *e.g.,* perfect Bayesian equilibrium [67].

Along similar lines, [91] develops a mean-field model that includes agent *learning.* That is, in the case of [84], agents are assumed to know their valuation (but not that of other agents). This is not always realistic since the valuation in the case of RTB may represent, *e.g.,* the probability of a purchasing decision, and the agent needs to estimate this probability by participating in the market. The result of [91] is structurally similar to that of [84] where the agents bid their valuation (there is no shading as [91] does not model budgets), but in this case the valuation of an item includes an additional factor that accounts for the information value of winning an item (and hence learning more about the agent's true value). This situation again leads, under appropriate assumptions, to a mean field equilibrium where agents best responding to the stationary distribution of prices reproduce that same stationary distribution.

The work of [10] combines the concepts of mean field limit (large number of agents) and the fluid limit (large number of auctions) into a fluid mean-field equilibrium solution concept for repeated auctions. As well, in contrast to [91], [10] includes a budget constraint, but does not model the effects of learning. An optimal bidding problem that conditions on prevailing market statistics, closely related to [84], is solved to again obtain a bid-your-shaded-value policy, and the loop is closed into a strategic equilibrium by showing the existence of a fixed point of the mapping from market statistics, through optimal bids, and back to induced market statistics. These ideas are further developed by [13] through the application of stochastic approximation in a setting that includes both learning and budgets.

The existence of stationary mean field equilibria motivate and justify many assumptions relating to stationarity and independence of competing bids in RTB. Indeed, the market model outlined in Definition 2.2.1 is essentially an interpretation of the equilibrium state of an auction market. However, real market data is known to not be entirely stationary [206, 215, 113], a point we address in Chapter 4 and 5

Estimation and Machine Learning Problems This thesis works primarily with very simple estimation methods (see Section 2.4.2), or algorithms which combine optimization and estimation simultaneously (see Chapter 4). However, estimation problems are still of interest and we occasionally point out places where superior estimation algorithms may be combined with our developments in practice; the astute reader is likely to recognize numerous others.

One of the earliest papers (published in 1956) applying statistical learning methods to problems of optimal bidding is [65] where they suggest using historical data to estimate the bids that will be placed by competitors, and then to optimize one's bid based both on this data as well as upon one's estimate of the value of the item. This is an elementary application of statistical learning to the problem of optimum bidding, and since this time, learning algorithms have benefited from enormous research efforts and enjoyed profound success on many important problems; we draw in part from [128, 80] for general reference, and hold up AlphaGo [161] as one of the field's most remarkable successes.

Of course, these advancements have been applied to RTB. Of particular interest, and similarly to [65], is the problem of estimating the "bid landscape" [49], *i.e.,* the prevailing statistics of competing bids. This is in many cases successfully carried out by fairly standard generalized linear models [219] or mixtures thereof [74]. Deep learning algorithms have also been applied to this problem [204].

An important aspect of RTB auction data is that you are often only informed of the price at which an item sold for if *you yourself are the winner*. That is, RTB data is often *censored*. This issue is often tackled by some modification of the Kaplan-Meier estimator [94], *e.g.,* [192, 216], but this may not be ideal for RTB data since the distribution of prices may be statistically dependent upon whether or not you are the winner of the auction (similarly to the idea of the *winner's curse* [122]) and some papers apply alternatives to the Kaplan-Meier estimate [203, 204].

Aside from estimating the statistics of competing prices, learning models are also applied to the problem of estimating the value of items. It is sometimes taken for granted that agents in auction markets know the value of the object they are bidding for, but this is not always the case. In RTB, the value of an impression (*i.e.,* "item") is often synonymous with the user's *click through* or *conversion* rate, *i.e.,* the probability that the user will actually click the ad, or make a purchase. This estimation is an important part of item valuation in RTB [110, 5, 36, 36], and has also been combined with simultaneous bid optimization [152].

Optimal Bidding The learn-then-bid approach advocated by [65] is criticized by [119] for assuming that valuations are private and independent, which is a dubious assumption in many auction markets. Nevertheless, this is also the approach taken by many optimum bidding papers in RTB markets, early examples being given by [76, 106]. This approach is often justified by appealing to the aforementioned equilibrium analysis and the mean-field nature of the market. Indeed, in the finance literature, purely *i.i.d.* statistical "zero-intelligence" models [68] and modest "ϵ-intelligence" [172, 171, 79] modifications, often via Markov models or differential equations, (*c.f.,* "population games" [156]), are able to capture salient aspects of market data. Optimal trading and portfolio construction algorithms based on these models are successfully employed in practice [70, 3], despite the fact that they treat what is ultimately a strategic game as a statistical entity.

For model-free approaches to optimal bidding, recent papers [32, 202] apply the reinforcement learning framework where optimal bids are learned directly as a result of participating in the market (or simulating participation on historical data); a related model-free approach is that of [55] which develops a budget constrained multi-armed bandit algorithm, though not specifically for application in RTB.

A number of papers which apply classical feedback control tools have also arisen. The first paper that we are familiar with[2] applying this technique for online advertising is [42]. The work of [109] applies similar ideas to try to maintain a consistent budget depletion rate. Further papers directly applying feedback control are given by [212, 96, 97]. We view these methods as being complementary to the analysis we carry out in Chapter 5 where we calculate a forward looking *plan* for item win rates designed to account for the forecastable periodic fluctuations in market statistics; feedback control algorithms may apply to adapt to higher frequency changes (*c.f.,* Chapter 4). Stochastic optimal control algorithms have also been developed [62, 83] for similar problems (see Section 5.2 for further discussion of these papers).

Further papers along these lines are [214] which calculates optimal budget constrained bidding functions (mapping from estimates of value into a bid) based upon estimates of the competing bid landscape (this paper is discussed further in Example 2.1.6). Randomized bidding is advocated by [95], which was also employed in our earlier work [101] to provide a smoothness guarantees for a win probability function. As well, [213] studies an interesting arbitrage problem and shares some features with Problem (P^m). Another paper sharing some features with the Problems studied in this thesis is [40] since it involves a transportation network (*c.f.,* Section 3.2.3).

Remark 1.3.1 (Publisher Decisions)**.** There is yet another angle from which RTB can be studied: the decisions made by *publishers*, *i.e.,* the sell-side. In particular, publishers have some control over the reserve prices set for advertising space (see Section 6.1.3 for a discussion of accounting for this competitive effect), and these decisions can have a significant impact on revenue [10, 126]. As well, publishers face a decision between whether they want to sell their ad inventory openly through RTB, or instead allocate it towards traditional guaranteed contracts (where advertisers buy a guaranteed number of impressions from the publisher) [148, 112, 39, 12]; they also face the problem of how to price those contracts [75, 24]. These problems are important in the broader context of RTB, but are not treated in this thesis.

[2]There may be much earlier papers applying control theory in this domain, but our literature review has focused on real-time bidding, which did not arise until around *c.* 2009

Remark 1.3.2 (Derivatives Markets)**.** As discussed in Section 1.1, many electricity markets employ auction mechanisms to organize the purchasing and provisioning of electricity. In addition to this primary market, there is a derivatives market dealing in futures contracts. This market provides a mechanism for participants in the primary market to hedge risk, and speculators are compensated for bearing this risk. While there is not, to our knowledge, a developed derivatives market associated with RTB, a number of papers have studied advertising options contracts [127, 38, 37]. The intermediary contract problem we study provides, in part, a similar risk bearing purpose.

1.4 Outline and Contributions

The remainder of this thesis proceeds as follows. In Chapter 2 we lay the formal foundations for the remainder of the thesis beginning with the definition of a *supply curve* (essentially nothing but a cumulative distribution function modelling the probability of winning an item given some bid) and an analysis of the expected costs of placing that bid in both first-price and second-price (sealed-bid) auctions. Following this basic introduction, we define the *acquisition cost* function, which encodes a one-to-one transformation of variables from the bid $x \in \mathbb{R}$ to the probability of winning $q \in [0, 1]$. We will show that this function is convex for second-price auctions, and convex under weak assumptions (a generalized type of concavity of the supply curve) for first-price auctions. The remainder of the thesis is, at the most abstract level, about exploiting the properties of this transformation of variables for the solution of practical optimal contract management problems.

The contract management problem is introduced in Section 2.3 (in particular, Section 2.3.3) where we begin with a number of simplified examples that illustrate key structural themes seen throughout the thesis. The most important result of this section is Proposition 2.3.1, which establishes that the contract management problem we consider is in fact a convex optimization problem. This transformation into a convex program results from straightforward application of the properties of the acquisition cost function, but we believe that the utility of this technique is not fully appreciated in the literature, see Example 2.1.6. To further illustrate this method, we cover some additional example problems from the field of finance in Section 2.5. These problems are tangential, but serve to further illustrate some of the key features encountered in our main application. We review some well known practical computational methods that we have used in our experimentation in Section 2.4.

There are a couple of salient features in our model which contrast with problems considered elsewhere in the literature. Firstly, the agent we study is not budget constrained.

While many papers study the problem of determining how to acquire items of maximum value subject to a budget constraint, our problem is in some sense dual to this: we attempt to acquire a specified quantity of items at a minimum cost. Secondly, the problem we formulate does not explicitly model the *value* of an item, at least not in the same way as does the existing literature. Indeed, our model will place a value $v_{ij} \in \mathbb{R}_+$ on items, but this is not used in the same way as is the typical value to acquiring an item (*i.e.,* it does not appear in the objective function). Instead, actual item valuations can only be *derived* as a result of the interaction between available supply in the market and the quantity demanded by the contractual obligations. This is discussed further in Chapter 3.

Since the main problem studied in this thesis is equivalent to a convex problem, it admits a rich duality theory. Chapter 3 is dedicated to an analysis of the implications of this duality. We analyze the theoretical structural implications as well as intuitive interpretations (and formal regularity results) in Section 3.1.2. As well, these results are seen (Section 3.2) to have important computational implications, and enable the derivation of efficient algorithms. These results are of particular practical importance since, except for special cases, the problem formulated in Chapter 2, while convex, does not fall into any "famous" class of convex programs. That is, it is not a linear program (unless we consider polyhedral approximations: Section 2.4.3), it is not a quadratic program (unless optimum bids are known: Section 3.2.3), it is not a semidefinite program, *etc.* Indeed, the cost functions we face are more or less arbitrary monotone convex functions. Section 3.3 illustrates the structural facts uncovered through duality with computational examples and also demonstrates the scalability of algorithms to large problem instances. In this chapter we also return to the additional problems from the end of Chapter 2 in order to, again, further illustrate the common threads of our analysis.

Chapter 4 deals with stochastic approximation algorithms. Here, we make only the weakest possible assumptions about the knowledge the bidding agent has at their disposal and show how the agent can simultaneously calculate an optimum bidding policy *and* learn everything needed about the prevailing market statistics. These algorithms can be *adaptive* in the sense that the market need not be completely stationary. As a result of the duality analysis of Chapter 3 we also uncover further duality relationships between stochastic approximation algorithms operating over different sets of variables. Algorithms are illustrated on simulated market data.

The final main chapter, Chapter 5, expands the definition of the main contract management problem to incorporate time dynamics and deadlines for contract fulfillment. In terms of the history of the development of this thesis, our final chapter was in fact the first formulation of the problem we began to study (see [99]). Much of the structure in the case with time deadlines follows similarly as do the duality results of Chapter 3. However, there is an additional consequence in this case: a duality analysis enables the reduction of the *infinite dimensional* primal optimization problem into a *finite dimensional* dual problem. In section 5.2 we consider two further approaches to adaptive bidding (complementary to Chapter 4), and study the practical performance of these methods using real market data in Section 5.4.

Chapter 6 includes our final remarks, as well as extended discussions of further research directions.

The Appendices contain some additional materials, as well as brief statements or reviews of known facts (Appendix A). Some proofs have been deferred to Appendix B, which also contains (organized by chapter) some additional discussion and results that may have been only briefly touched on in the main body. Finally, since we have often emphasized the importance of practical computational methods and representations, some actual program code listings are given in Appendix C, along with brief commentary on optimization technology.

The thesis is based primarily upon the papers [101, 102]. Another paper presenting nascent ideas developed further by the aforementioned will remain available only as a preprint [99], as the results therein have been essentially superseded. The germ of the ideas developed in this thesis began with the work of [170]. Additional publications based on the work presented here are under development. A further publication in a distinct field [100] has also arisen from work carried out during the author's PhD, but will not be touched upon in this thesis.

Chapter 2

Supply Curves, Real-time Bidding, and Contract Management Problems

The purpose of this chapter is to provide a technical introduction to the problems studied in this thesis. We begin in Section 2.1 with the basic building blocks for a single auction, *supply curves* and *cost functions*, which model, respectively, the probability of winning an item in an auction, and the expected cost of doing so. Additionally, we introduce the *acquisition cost function*, which models the expected cost of winning an item with some specified probability. This function will be the key ingredient for the entire thesis: every subsequent chapter can, at the highest level, be seen as understanding and exploiting the properties of this function for the solution of practical problems. Elementary examples are used to illustrate the properties of these functions.

In Section 2.3 we introduce a basic *market model*, which will play an important role in Chapter 4. There is a rich literature focusing on the application of game theoretic tools to analyze certain types of strategic equilibria arising in RTB auctions, see *e.g.,* [84, 91, 10, 13]. For the market to be *in equilibrium* means that no agent has the incentive to modify their bidding strategy, given the current strategies of their competitors. The existence of such equilibria motivate and justify many assumptions relating to the stationarity and independence of competing bids in RTB. The statistics of prices are stationary since agents do not have the incentive to modify their bidding strategy, and prices are statistically independent as a result of mean-field interactions: agents best respond to the *distribution* of competing bids. Indeed, the market model we outline is essentially an interpretation of the equilibrium state of an auction market. While our market model does not employ any game theoretic concepts directly, we point out that the finance literature is replete with purely stochastic or so called ϵ-*intelligence* models [163, 172, 171, 79] which successfully model real world market data.

The main problem considered in this thesis is introduced in Section 2.3 where we combine the previously discussed concepts into a concrete contract management problem. The problem can be formally modelled in two ways: the most natural is as a *monotone optimization problem* [177], which is an optimization problem wherein the objective and the constraints have certain monotonicity properties, to be specified more clearly in the sequel. The second formulation, which can be obtained under certain restrictions, is as a *convex* optimization problem [29, 21, 19, 86, 30]. The study of this convex program and its various extensions will constitute the majority of the thesis, and we again illustrate basic issues with elementary examples.

In Section 2.4 we review some well known computational methods, namely the method of bisection and kernel density estimation, which are of fundamental importance for practical implementations.

We conclude this chapter in Section 2.5 with some additional examples from the field of finance to illustrate the ubiquity of some of the functions encountered in this thesis. These problems will appear again in Chapter 3, but these sections are not essential to the thesis as they primarily serve as additional examples and to hopefully inspire further research directions. Concluding remarks are given in Section 2.6.

2.1 Supply Curves and Cost Functions

In Section 2.1.1 and Definition 2.1.1 we introduce the fundamental notion of a Supply curve. Section 2.1.2 and Equations (2.1), (2.2) introduces two pertinent cost functions which are associated to supply curves under different auction rules (first and second price auctions) and Equation 2.5 defines the acquisition cost curve, which is a particular composition of these functions.

2.1.1 Supply Curves

Consider an auction market with indistinguishable items. Throughout, the discussion will generally be from the perspective of a particular DSP who places a bid (or bids) x for an item that becomes available in the auction. The *supply curve*, which is ultimately nothing but a cumulative distribution function (*c.d.f.*), characterizes the DSP's competition by quantifying the probability of winning an item given that the DSP places the bid x [65]. This function will be denoted by $W : \mathbb{R} \to [0, 1]$ and is such that the probability of winning an item with the bid $x \in \mathbb{R}$ is $W(x)$. We formally allow for the possibility that $x < 0$, but such a bid will always have zero probability of winning the item, and in practice, such a bid would either be rejected, or treated as being equal to zero by the rules of the market.

The bidder may or may not have quantitative knowledge about W. In Chapter 3 and 5 we will assume the function is either fully known, or at least estimated. On the other hand, in Chapter 4, the agent will have no quantitative knowledge of W and must learn to bid simultaneously while estimating W.

We assume that the *price* p_n, of the n^{th} item arriving at the auction is distributed according to a *c.d.f.* W, and that prices are always non-negative: $p_n \geq 0$ almost surely. The DSP wins the item if the bid, x, that they place satisfies $x \geq p$ as in $W(x) = \mathbb{P}\{p_n \leq x\}$ (further discussion will appear in Section 2.2). Additionally, we will assume that prices are sequentially independent so that if items arrive at a rate $\lambda > 0$, the average rate that items will be won is $\lambda W(x)$. If the inter-arrival times are exponential, then both the arrival times and the win rate (for a constant bid placed on each arriving item) are Poisson processes by the Poisson thinning property.

Remark 2.1.1 (Supply Curves from Market Models)**.** The assumption of *i.i.d.* prices may appear strong, but can in fact arise as a consequence of competitive equilibrium among agents. This setting is typically analyzed through the methods of mean field games [34] or population games [156, 141], which use certain limiting arguments for large populations of agents. An analysis of this form is carried out by [91, 10, 13] where, in stylized bidding markets, there exist competitive equilibria where prices become *i.i.d.*. Indeed, taking the *i.i.d.* prices assumption as a starting point is typical in many previous papers [42, 84, 214], and can serve as adequate models for deriving bidding algorithms.

Remark 2.1.2 (Comparisons in Finance)**.** By way of comparison, in financial markets, it has been observed that simple *zero-intelligence* (*i.e.,* purely statistical) models [163] are capable of explaining some important aspects of price dynamics, and only marginally more complicated ϵ-intelligence models [172, 79] (which are statistical and dynamic, but still not strategic) are capable of capturing many of the most salient aspects of market dynamics. These models are used to derive, among other things, transaction cost models (*i.e.,* models of the excess expense, over and above the current market price, of trading a large batch of securities) that form key components in modern portfolio management [8, 70]. We view the statistical modelling approach adopted in this thesis as being analogous to this approach to finance.

Formally, a *supply curve*, *i.e.,* the function $W : \mathbb{R} \to [0, 1]$, will be a cumulative distribution function, but the amount of smoothness that is to be assumed of the supply curve is a delicate consideration. It is absolutely essential for our results that supply curves be continuous (though see [170], which studied a similar problem without assuming continuity). However, Example 2.1.1 and Section 2.4 shows that even the assumption of completely smooth supply curves would not be unreasonable. Ultimately, the reader may think of W as being as differentiable as necessary (usually only at most a single derivative is used) but generally, no differentiability is required when dealing with second price auctions.

Definition 2.1.1 (Supply Curve and Differentiable Supply Curve)**.** A *supply curve* $W(x)$ is a continuous cumulative distribution function on $\mathbb{R}_\infty \triangleq \mathbb{R} \cup \{\infty\}$ such that $\forall x \leq 0 : W(x) = 0$, that W is strictly monotone up to some maximum bid $\bar{x} \leq \infty$, and the first two moments $\bar{p} \triangleq \int_0^\infty u \mathsf{d}W(u) < \infty$ and $\int_0^\infty u^2 \mathsf{d}W(u) < \infty$ are finite. In the context of first price auctions, the additional assumption of differentiability will be added, wherein W will be differentiable on the interval $(0, \bar{x})$ and will be referred to as a *differentiable supply curve*. The prices (*i.e.,* the highest competing bids) $p_1, p_2, \ldots$ of items (of the same type) are distributed *i.i.d.* according to W such that $\mathbb{P}\{p_n \leq x\} = W(x)$ models the probability of winning an item given a bid of x. The inverse of W is denoted $W^{-1}(q)$ and is extended such that $W^{-1}(q) = 0$ if $q \leq 0$ and $W^{-1}(q) = \infty$ if $q > 1$.

The integrals in Definition 2.1.1 are Lebesgue-Stieltjes integrals (see, *e.g.,* [164]), satisfying, *e.g.,* $\int_0^\infty u \mathsf{d}W(u) = \int_0^\infty u W'(u) \mathsf{d}u$ if W is differentiable.

Examples of supply curves estimated (by the methods of Section 2.4) from market data are given in Figure 2.1. These curves allow a maximum bid of $\bar{x} = 300$, and are valid for different times of day. We deal with time constraints and time varying supply curves in Chapter 5.

Example 2.1.1 (Randomized Bidding)**.** Let $p \sim \widetilde{W}$ be a price distributed according to an *arbitrary* cumulative distribution function $\widetilde{W}$, let $Z \sim \mathcal{N}(0, 1)$ be a Gaussian random variable independent of p, and let $\sigma > 0$ be a parameter. Then, the function $W_\sigma(x) = \mathbb{E}\widetilde{W}(x + \sigma Z)$ can be shown to be a smooth cumulative distribution function on $\mathbb{R}$ and models the probability of winning the item when the *nominal* bid is x, but the *actual* bid is randomized $\mathcal{N}(x, \sigma^2)$. That is, by injecting an arbitrarily small amount of noise into the bidding process, the *effective* supply curve becomes a smooth supply curve. See [95, 96] for further discussion of randomized bidding.

Example 2.1.2 (Analytic Example)**.** One of the simplest reasonable analytic examples of a supply curve is furnished by the exponential distribution. That is, by assuming that

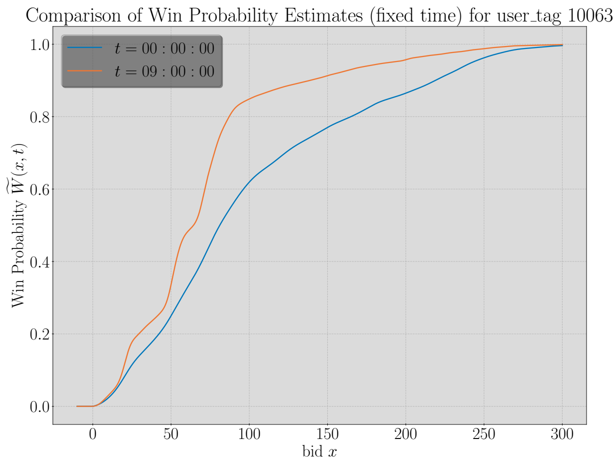


Figure 2.1: Win Probability Estimates
Price distribution estimate based on Gaussian KDE of a particular item type at two different times of day.

$p_n \sim \exp(\gamma)$ and $W(x) = (1 - e^{-\gamma x})\mathbf{1}_{\mathbb{R}_+}(x)$, which has $\bar{x} = \infty$. The utility of this example is that the inverse function $W^{-1}(q) = \frac{1}{\gamma}\ln(1-q)$ is easily available for $q \in [0, 1)$. This function models the bid x that must be placed in order to win the item with probability q, and has an unbounded maximum bid $\bar{x} = \infty$. Another simple example, with $\bar{x} < \infty$ is given by $W(x) = x^2\mathbf{1}_{[0,1]}(x)$. Finally, we mention the Gamma distribution (see also [219]) as being convenient for simulation when more control over the variance of prices is desired.

Example 2.1.3 (Extreme Value Distributions)**.** The price of an item arises as the maximum bid amongst competitors, *e.g.*, $p = \max_{k\in[K]} b_k$. This suggest that the supply curve W may be sensibly modelled by an extreme value distribution according to the Fisher-Tippet-Gnedenko Theorem [52, Thm 1.1.3], *i.e.*, that $W(x) = \exp\big(-(1+\gamma x)^{-1/\gamma}\big)$ for some parameter $\gamma > 0$.

Remark 2.1.3 (Extensions)**.** There are two extensions to this model that will be encountered in the thesis. The first is supply curves that depend on *types* $j \in [M] = \{1, 2, \ldots, M\}$ (for a finite number of types) or $\phi \in \Phi$ (when the type is random or if the type space is uncountable) as in $W_j(x)$ or $W_\phi(x)$ (see Section 2.3). The second is when supply curves are time dependent as in $W(x,t)$ (see Chapter 5).

2.1.2 Cost Functions

There are two types of cost functions that will be encountered in this thesis — the first is the expected cost of bidding x on an item, which will be denoted by $f(x)$, and the second is the *acquisition cost*, denoted by $\Lambda(q)$, that models the expected cost of bidding such that you win an item with probability q. We consider these in turn.

Bidding Costs The bidding cost f is the expected cost of bidding x on an item. This function depends upon the particular auction mechanism that is employed by the platform. The most common mechanisms are *first price* and *second price* auctions [184, 103, 118]. In the former case, the winner of the auction pays their bid, and in the latter, the winner pays the *price*, *i.e.,* the highest competing bid (which is necessarily less than or equal to the winning bid). Additionally, the seller of the item may incorporate a *reserve price* for the auction which is such that any bid below the reserve price is immediately rejected. Reserve prices can arise if the seller has a personal valuation for the item (*e.g.,* a website may associate some value with advertising their own premium service to their own users) or the seller may have alternative platforms where they are guaranteed to be able to sell their items above some specified price. Since there are a large number of sellers, and the sellers may have additional private information about the items that affects their personal value of the items, we simply treat the reserve price as though it arises from another competing bidder.

Due to these auction rules, the cost functions associated to the first and second price auctions are given by

$$f^{1st}(x) \triangleq \mathbb{E}[x\mathbf{1}(p \leq x)] = xW(x), \tag{2.1}$$

$$f^{2nd}(x) \triangleq \mathbb{E}[p\mathbf{1}(p \leq x)] = \int_0^x u\mathsf{d}W(u), \tag{2.2}$$

respectively.

The interpretation of the function $f^{1st}(x) = xW(x)$ is quite simple: when you bid x you win the item with probability $W(x)$, and the amount that you pay if you win is exactly your bid x. The function $f^{2nd}(x) = \int_0^x u\mathsf{d}W(u)\mathsf{d}u$ is less intuitive, but the integrand $u\mathsf{d}W(u)$ can be interpreted as the marginal cost of *convincing* all the competing bidders with a valuation of u to part with the item.

By default, we focus upon the second price auction, and f (without a subscript or superscript specifying the auction type) should be understood to be referring to f^{2nd}. This case is in many ways simpler, and a critical property of second price auctions, not available in the first price case, enables the application of stochastic approximation to certain optimal bidding problems (see Chapter 4).

Both of the above functions are strictly monotone increasing. As well, via integration by parts, it can be recognized that $f^{2nd}(x) = xW(x) - \int_0^x W(u)\mathsf{d}u = f^{1st}(x) - \int_0^x W(u)\mathsf{d}u$ and therefore $f^{1st} \geq f^{2nd}$, as well

$$f^{1st} - f^{2nd} = \int_0^x W(u)\mathsf{d}u. \tag{2.3}$$

The Equation (2.3) represents the additional costs paid in a first price auction, over and above what would be paid in a second price auction *all else being equal*. Moreover, we can also write

$$\begin{aligned} f^{1st}(x) - f^{2nd}(x) &= xW(x) - \int_0^x u\mathsf{d}W(u) \\ &= \int_0^x (x-u)\mathsf{d}W(u) \\ &= \mathbb{E}(x-p)_+, \end{aligned} \tag{2.4}$$

where $p \sim W$. This again represents the amount by which the bidder overpays for the item in a first price auction. This function also arises in a problem in finance called the dark pool problem [69], and is briefly touched on in Section 2.5.2.

Acquisition Costs Given some $q = W(x)$, which models the probability of winning the item given the bid x, we also have the expected cost $f(x)$ of bidding x. We can also, since W is strictly monotone over an interval $[0, \bar{x}] \subseteq [0, \infty]$, calculate what bid is needed to acquire the item with probability $q \in [0, 1]$, that is, $x = W^{-1}(q)$. Formally, $W^{-1}(q) \triangleq \infty$ if $\bar{x} = \infty$ and $q = 1$. It is then natural to ask a related question: "what is the expected cost to win the item with probability q?" The answer to this question is the *acquisition cost function*

$$\Lambda(q) = f \circ W^{-1}(q). \tag{2.5}$$

This function naturally depends upon the type of auction, but by direct substitution of $W^{-1}(q)$ into f_{1st} or f_{2nd} we can see that:

$$\Lambda^{1st}(q) = qW^{-1}(q)$$
$$\Lambda^{2nd}(q) = \int_0^{W^{-1}(q)} u\mathsf{d}W(u). \tag{2.6}$$

In order to extend the definition of Λ to all of $\mathbb{R}$, we let $\Lambda(q) = \infty$ on $q > 1$ and $\Lambda(q) = 0$ for $q < 0$.

We turn now to an analysis of these acquisition cost functions. The remarkable of $\Lambda(q)$ is that it tends to be *convex*. Indeed, for second price auctions it is always convex (under Definition 2.1.1), and remains convex for the first price auction under some additional (fairly weak) assumptions on W. A sufficient condition in the latter case is that W be log-concave, *i.e.*, $x \mapsto \ln \circ W(x)$ is a concave function. However, we will see that a slightly weaker condition of *2-concavity* is sufficient. We first state the second price case in Proposition 2.1.1 and then the first price case in Proposition 2.1.2.

Proposition 2.1.1 (Convex Acquisition Costs — Second Price Case [101])**.** *Let $W(x)$ be a supply curve. Then, in a second price auction, the acquisition cost function $\Lambda^{2nd}(q) = f^{2nd} \circ W^{-1}(q)$ is given by $\int_0^q W^{-1}(u)\mathsf{d}u$ on $q \in [0,1]$. If this is extended to:*

$$\Lambda^{2nd}(q) \triangleq \begin{cases} \infty; & q > 1 \\ 0; & q \le 0 \\ \int_0^q W^{-1}(u)\mathsf{d}u; & \text{otherwise} \end{cases}, \tag{2.7}$$

then Λ^{2nd} is a proper[1]*, lower semi-continuous, non-decreasing, and convex function on $\mathbb{R}$. Moreover, Λ^{2nd} is strictly convex over $[0,1]$, differentiable on $(0,1)$, and the derivative can be extended continuously to $[0,1]$ if $\bar{x} < \infty$.*

Proof. We first calculate $\Lambda = f^{2nd} \circ W^{-1}$ for $q \in [0,1]$ by

$$\Lambda(q) \overset{(a)}{=} \int_0^{W^{-1}(q)} x\mathsf{d}W(x)$$
$$\overset{(b)}{=} \int_0^q W^{-1}(u)\mathsf{d}u,$$

where (a) follows by definition of f^{2nd} (Equation (2.2)) and (b) is by the substitution of variables $v = W(x)$ which results in $x = W^{-1}(v)$, $\mathsf{d}W(x) = \mathsf{d}v$ and transforms the bounds

[1]Recall that a function $f : \mathbb{R} \to (-\infty, \infty]$ is *proper* if it is not everywhere equal to $+\infty$.

of integration from $[0, W^{-1}(q)]$ into $[0, q]$ since $W(0) = 0$ and $W \circ W^{-1}(q) = q$. We see from this latter formula that Λ^{2nd} is differentiable on $(0,1)$ with derivative $W^{-1}(q)$, which is continuous over $[0,1]$ when $\bar{x} < \infty$. Since W is strictly monotone increasing, W^{-1} is also strictly monotone increasing, and functions with strictly monotone increasing derivatives are strictly convex, it follows that Λ^{2nd} is strictly convex over $(0,1)$. The extension as given maintains convexity, and ensures that Λ^{2nd} is lower semicontinuous. It is proper since, *e.g.*, $\Lambda(1) = \int_0^\infty u \mathsf{d}W(u) < \infty$. □

Some additional assumptions are needed to establish convexity in the first price case.

Definition 2.1.2 (α-concave)**.** Define, for $\alpha \geq 0$, $x > 0$ the function

$$\ell_\alpha(x) \triangleq \int_1^x \frac{1}{t^\alpha} \mathsf{d}t = \begin{cases} \ln x & \alpha = 1 \\ \frac{x^{1-\alpha}-1}{1-\alpha} & \text{otherwise} \end{cases},$$

where in particular $\ell_2(x) = 1 - 1/x$. We will say that a positive function $W : \mathbb{R} \to (0, \infty)$ is (strictly) α-concave if $\ell_\alpha \circ W$ is (strictly) concave. In particular, W is log-concave if $\alpha = 1$ and concave if $\alpha = 0$.

It is shown in Proposition B.2.1 that 2-concavity is a *weaker* condition than is log-concavity. It is the former notion that is needed to establish convexity of Λ^{1st}.

Proposition 2.1.2 (Convex Acquisition Costs — First Price Case [101])**.** *Suppose that the supply curve $W(x)$ is strictly 2-concave (* c.f., *Definition 2.1.2),* i.e., *$\ell_2 \circ W$ is strictly concave on $(0, \infty)$ where $\ell_2(x) = 1 - 1/x$. Then in a first price auction, the extended acquisition cost function*

$$\Lambda^{1st}(q) \triangleq \begin{cases} \infty; & q > 1 \\ 0; & q \leq 0 \\ qW^{-1}(q); & \text{otherwise} \end{cases}, \tag{2.8}$$

is a proper, lower semi-continuous, non-decreasing, and convex function on $\mathbb{R}$. Moreover, Λ^{1st} is strictly convex over $[0,1]$.

Proof. Since $f(x) = xW(x)$ we have, for $q \in (0,1)$, $\Lambda(q) = qW^{-1}(q)$. On $q \leq 0$ we define $\Lambda(q) = 0$, on $q > 1$ we define $\Lambda(q) = \infty$, and finally $\Lambda(1) = \bar{x} \leq \infty$.

Convexity will therefore follow if Λ is convex on $(0,1)$. To this end, we use the 2-concavity of W to see that $1 - 1/W(x)$ is concave on its domain and therefore that the inverse, $W^{-1}\big(1/(1-x)\big)$ is convex on $0 < \frac{1}{1-x} < 1$. It is well known that for a convex function g, the function

$$(cx+d)g\big(\frac{ax+b}{cx+d}\big)$$

is convex on $cx+d>0$ (see e.g. [29, Ex. 3.20]). Therefore, by setting $a=c=1$, $b=-1$ and $d=0$ we obtain convexity of

$$\begin{aligned}(cq+d)W^{-1}\Big(\frac{1}{1-\frac{aq+b}{cq+d}}\Big) &= qW^{-1}\big(\frac{1}{1-\frac{q-1}{q}}\big)\\ &= qW^{-1}\big(q\big)\end{aligned}$$

which is the function $\Lambda(s)$. Since $qW^{-1}(q)$ is strictly monotone increasing on $(0,1)$, Λ^{1st} is strictly convex on this interval, and evidently non-decreasing on all of $\mathbb{R}$. □

Some examples of acquisition costs, and methods to guarantee (for computational purposes) the convexity of Λ_{1st} when using estimated curves from real data, are provided in Section 2.4.

For any function, there is associated a dual function: the Fenchel conjugate [45, Sec 4.2]. When the original function is convex, this is derived from the hyperplanes supporting that function's epigraph. There is an appealing duality relationship between Λ_{2nd} and the conjugate function

$$\Lambda^{\star}_{2nd}(\mu) \triangleq \sup_q\ [\mu q - \Lambda_{2nd}(q)],$$

namely that Λ_{2nd} is the integral of W^{-1} and that $\Lambda^{\star}_{2nd}$ is the integral of W, see [85] for further analysis of integrated quantile functions. This relationship is used for the derivation of dual stochastic approximations in Chapter 4, and appears in the duality analysis of Chapter 3.

Proposition 2.1.3 (Fenchel Conjugate — Second Price Case[102, 85])**.** *The Fenchel duality relationship between Λ_{2nd} and $\Lambda^{\star}_{2nd}$ is between the integrated c.d.f. W and the integrated quantile function W^{-1}:*

$$\Lambda^{2nd}(q) \triangleq \begin{cases} \infty; & q>1 \\ 0; & q \le 0 \\ \int_0^q W^{-1}(u)\mathsf{d}u; & \text{otherwise} \end{cases}, \tag{2.9}$$

$$\Lambda^{\star}_{2nd}(\mu) = \begin{cases} \infty, & \textit{if } \mu < 0 \\ \int_0^{\mu} W(u)\mathsf{d}u, & \textit{if } \mu \in [0, \bar{x}] \\ \mu - \bar{x}, & \textit{if } \mu > \bar{x}. \end{cases} \tag{2.10}$$

The function $\Lambda^{\star}_{2nd}$ is a proper, convex, and lower-semicontinuous function, which is strictly convex and strictly monotone increasing $\mathbb{R}_+$.

Proof. By definition,

$$\Lambda^{\star}(\mu) = \sup_{q \in (-\infty, 1]} \big[\mu q - \Lambda(q)\big],$$

where the domain is restricted to $(-\infty, 1]$ since $\Lambda(q) = \infty$ for $q > 1$. If $\mu < 0$ then $\Lambda^{\star}(\mu) = \infty$ since $\Lambda(q) = 0$ for $q \leq 0$. If[2] $\mu > \bar{x}$ then $\Lambda^{\star}(\mu) = \mu - \bar{x}$ since μ is greater than the maximal slope of Λ. Finally, if $\mu \in [0, \bar{x}]$ we can differentiate $q \mapsto \mu q - \Lambda(q)$ to find that, at optimality, $q = W(\mu)$ and hence

$$\begin{aligned} \Lambda^{\star}(\mu) &= \mu W(\mu) - \Lambda \circ W(\mu) \\ &= f^{1st}(\mu) - f^{2nd}(\mu) \\ &= \int_0^{\mu} W(u)\mathsf{d}u, \end{aligned}$$

where the final equality was pointed out in Equation (2.3) and follows through integration by parts. It is proper, convex, and lower-semicontinuous since it is a conjugate function, and strictly convex on $\mathbb{R}_+$ since W is strictly monotone on $[0, \bar{x}]$ and the function is linear thereafter. It remains strictly convex on the rest of $\mathbb{R}$ since it is affine for $\mu > \bar{x}$. It is strictly monotone on $\mathbb{R}_+$ by inspection. □

We postpone the derivation of the Fenchel conjugate in the first price case until Section 2.1.3. For later use, we explicitly point out the form of the derivatives of Λ.

Lemma 2.1.1 (Derivatives)**.** *In a second price auction $\Lambda'_{2nd}(q) = W^{-1}(q)$ for $q \in (0, 1)$ and in a first price auction, $\Lambda'_{1st}(q) = W^{-1}(q) + \frac{q}{W' \circ W^{-1}(q)}$.*

[2] This case is excluded if $\bar{x} = \infty$ since then $\mu > \bar{x}$ is impossible.

2.1.3 Bid Mapping Function

The last important function that arises in this work is what we refer to as the *bid-mapping function*. This is a function g, which depends on the auction type, and is *defined* such that $\Lambda'(q) = g \circ W^{-1}(q)$. That is, g is such that the inverse of the derivative of the acquisition cost curve is given through the composition of the supply curve with the function g^{-1}. In a second price auction, since $\Lambda'(q) = W^{-1}(q)$ (*c.f.,* Equation2.7), and therefore $\Lambda'^{-1}(x) = W(x)$, the function $g^{2nd}(x) = x$. Since in the second price case g is the identity, it will only be relevant when discussing first price auctions and we will write g without specifying g^{1st} or g^{2nd}.

Proposition 2.1.4 (Bid Mapping Function (1st Price Case))**.** *Let W be a differentiable supply curve. The bid mapping function*

$$g(x) = \begin{cases} x + \frac{W(x)}{W'(x)}, & \text{if } x \in (0, \bar{x}) \\ \bar{x} + \left(\lim\limits_{x \to \bar{x}} W'(x)\right)^{-1}, & \text{if } x = \bar{x} \\ 0, & \text{if } x \leq 0 \\ \infty, & \text{if } x > \bar{x}. \end{cases}$$

is a continuous function defined such that $\Lambda'_{1st}(q) = g \circ W^{-1}(q)$ whenever the derivative exists, and takes the value of ∞ otherwise. Moreover, if W is strictly 2-concave (See Proposition 2.1.2) then g is a strictly monotone increasing function on $[0, \bar{x}]$ with range $[0, g(\bar{x})]$, where $g(\bar{x}) \triangleq \infty$ if $\bar{x} = \infty$. In second price auctions, $g(x) = x$.

Proof. For $q \in (0, 1)$ we have the calculations (*c.f.,* Equation 2.8)

$$\begin{aligned} \frac{\mathsf{d}}{\mathsf{d}q}\Lambda(q) &= \frac{\mathsf{d}}{\mathsf{d}q} q W^{-1}(q) \\ &= W^{-1}(q) + \frac{q}{W' \circ W^{-1}(q)}. \end{aligned}$$

Therefore, when $x \in (0, \bar{x})$ we have $\Lambda'_{1st}(q) = g_{1st} \circ W^{-1}(q)$ where $g_{1st}(x) = x + \frac{W(x)}{W'(x)}$. On $q \leq 0$ the derivative Λ'_{1st} is zero (by definition of Λ_{1st}, we extend the value of $g(x)$ on $x > \bar{x}$ to take the value ∞, and fill in the value of g at $\bar{x}$ by continuity. Since $\frac{d}{\mathsf{d}x} \ln W(x) = W'(x)/W(x)$ and $\ln W(x) \to -\infty$ as $x \to 0$, it must be that the derivative of $\ln W(x)$ is converging to ∞ and therefore $W(x)/W'(x) \to 0$ as $x \to 0$, so g is continuous at 0.

Moreover, when $\Lambda_{1st}(q)$ is convex (see Proposition 2.1.2) the derivative $\Lambda_{1st}(q)$ is monotone increasing and therefore, since $g_{1st} = \Lambda'_{1st} \circ W$, it is also monotone increasing, since the composition of monotone increasing functions is monotone increasing.

The statement for second price auctions was pointed out in the paragraph prior to the statement of the proposition. □

Part of the purpose of defining g is to facilitate a more aesthetic expression for $\Lambda^\star_{1st}$, as follows.

Proposition 2.1.5 (Fenchel Conjugate — First Price Case[102, 85])**.** *Let $\Lambda(q) = f_{1st} \circ W^{-1}(q)$ where W is a strictly 2-concave differentiable supply curve. The Fenchel conjugate $\Lambda^\star(\mu)$ is given by*

$$\Lambda^\star(\mu) = \begin{cases} \infty, & \text{if } \mu < 0 \\ \big(\mu - g^{-1}(\mu)\big) W \circ g^{-1}(\mu), & \text{if } \mu \in [0, g(\bar{x})] \\ \mu - \bar{x}, & \text{if } \mu > g(\bar{x}). \end{cases} \tag{2.11}$$

The function $\Lambda^\star$ is a proper, convex, and lower-semicontinuous function, which is strictly convex and strictly monotone increasing on $\mathbb{R}_+$.

Proof. By definition, we need to calculate

$$\Lambda^\star(\mu) = \sup_{q \in (-\infty, 1]} [\mu q - \Lambda(q)].$$

For $\mu \in [0, g(\bar{x})]$ we apply Fermat's rule (that maximizers of concave functions occur at points where the derivative is zero) to see that we need to solve $\mu = \Lambda'(q)$ for q and whence we obtain $q = W \circ g^{-1}(\mu)$ since $\Lambda'(q) = g \circ W^{-1}(q)$ and the inverse g^{-1} exists by Proposition 2.1.4. Substituting this into the definition we have

$$\begin{aligned} \mu q - \Lambda(q) &= \mu W \circ g^{-1}(\mu) - f_{1st} \circ W^{-1} \circ W \circ g^{-1}(\mu) \\ &= \mu W \circ g^{-1}(\mu) - g^{-1}(\mu) W \circ g^{-1}(\mu) \\ &= \big(\mu - g^{-1}(\mu)\big) W \circ g^{-1}(\mu). \end{aligned}$$

Finally, if $\mu > g(\bar{x})$ there is no solution to this system and we have $q = 1$ by monotonicity.

That $\Lambda^\star$ is strictly convex on $\mathbb{R}_+$ follows since Λ^{1st} is differentiable on $(0, 1)$ and Theorem [154, p. 11.13] establishing a relationship between differentiability and strict convexity of convex functions and their conjugates.

The strict monotonicity of $\Lambda^\star$ on $\mu \geq \bar{x}$ is clear, so it remains to show that it is strictly monotone on $[0, \bar{x})$. To this end, recognize that $\Lambda^\star \circ g(x) = \big(g(x) - x\big)W(x)$ and this is equal to $W(x)^2/W'(x)$ on the interval $[0, \bar{x})$ by the definition of $g(x)$. Then, since W is strictly $2-$*concave* it must be that the derivative of $1 - 1/W(x)$ is strictly decreasing, and thus $W'(x)/W(x)^2$ is a decreasing function. Therefore, $\Lambda^\star \circ g(x)$ is strictly increasing $[0, \bar{x})$. Since g is strictly increasing, and the composition $\Lambda^\star \circ g$ is strictly increasing, it follows that $\Lambda^\star$ is as well. □

2.1.4 Analytic Examples

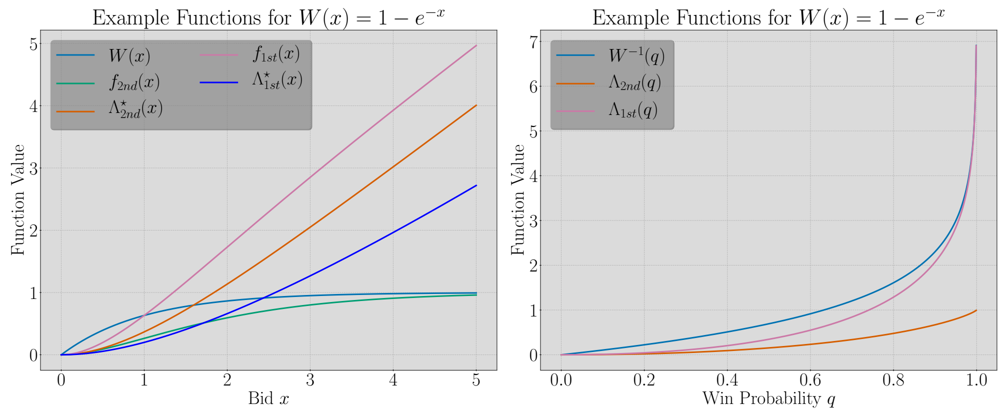
Here we briefly provide some examples of supply functions which are analytically tractable, which were first mentioned in Example 2.1.2.

Example 2.1.4 (Exponential)**.** Consider the *c.d.f.* of the exponential distribution $W(x) = 1 - e^{-\gamma x}$, plots of the relevant functions are provided in Figure 2.2. In this case, we find that the bid required to win with probability q is the function $W^{-1}(q) = -\frac{1}{\gamma}\ln(1-q)$, with domain $[0, 1)$ and the second price acquisition cost is $\Lambda_{2nd}(q) = \frac{1}{\gamma}[q + (1-1)\ln(1-q)]$ with conjugate $\Lambda^\star_{2nd}(\mu) = \mu + \frac{1}{\gamma}[e^{-\gamma\mu} - 1]$. The quantities relating to the first price case are also tractable. Indeed, we have $g(x) = x + W(x)/W'(x)$ for $W'(x) = \gamma e^{-\gamma x}$ so that $g(x) = x + \frac{1}{\gamma}[e^{\gamma x} - 1]$ and then $\Lambda_{1st}(q) = -\frac{q}{\gamma}\ln(1-q)$. We can also calculate $\Lambda^\star_{1st}(\mu)$ as follows. In order to maximize $\mu q - \Lambda_{1st}(q)$ we have to solve the equation:

$$
\begin{aligned}
\Lambda'_{1st}(q) &= \mu \\
\iff g \circ W^{-1}(q) &= \mu \\
\iff \frac{1}{\gamma}\big[\ln\frac{1}{1-q} + \frac{1}{1-q} - \big] &= \mu \\
\iff \frac{1}{1-q}\exp\big(\frac{1}{1-q}\big) &= e^{1+\gamma\mu} \\
\iff q = 1 - 1/\mathcal{W}_0\big(e^{1+\gamma\mu}\big),
\end{aligned}
$$

where $\mathcal{W}_0$ is the principal branch of the Lambert-W function [46]. If we call this function $q_\gamma(\mu)$ then we have $\Lambda^\star_{1st}(\mu) = q_\gamma(\mu)\big(\mu - W^{-1} \circ q_\gamma(\mu)\big)$.

Subfigure 2.2a plots functions of the bid x. In this subfigure it should be recognized that the second price cost f_{2nd} is bounded, but the first price cost is not.



(a) Functions of the Bid x (b) Functions of Win Probability q

Figure 2.2: Illustrative Example Functions
Important example functions for the simple case that $W(x) = 1 - e^{-x}$, *i.e.*, where the price is $\exp(1)$ distributed.

Subfigure (b) plots functions of the win probability q. The function $W^{-1}(q)$ converges to ∞ as $q \to 1$ since $\exp(1)$ random variables have unbounded support. Yet, $\Lambda_{2nd}(q)$ remains bounded, since the maximum expected cost is the expected value of the price. However, $\Lambda_{1st}(q)$ *does* converge to ∞ since you need to place very high bids in order to win with a high probability, and in a first price auction, you pay your bid.

Example 2.1.5 (Quadratic)**.** Consider now $W(x) = x^2 \mathbf{1}_{[0,1]}(x)$. The calculations involved are elementary, and we are loose in precisely specifying the domain. We have $f^{2nd}(x) = \frac{2}{3}x^3 \mathbf{1}_{[0,1]}(x)$ and $\Lambda_{2nd}(q) = \frac{2}{3}q^{3/2}$, then by integrating W (*c.f.*, Proposition 2.1.3) we obtain $\Lambda^{\star}_{2nd}(\mu) = \frac{1}{3}\mu^3$.

In the first price case, we have $f^{1st}(x) = x^3 \mathbf{1}_{[0,1]}(x)$ and $\Lambda^{1st}(q) = q^{3/2}$. As well, the bid mapping function is $g(x) = x + \frac{1}{2}x^2$ and then $\Lambda^{\star}_{1st}(\mu) = \frac{1}{9}\mu^3$. In contrast to the case with $\bar{x} = \infty$ where some functions converge to ∞ (particularly W^{-1}), each of these functions are quite well behaved.

Example 2.1.6 (Budget Constrained Optimum Bidding)**.** The work [214] is an important paper in RTB. The authors of this work formulated a budget constrained optimal bidding problem where the agent has a goal of finding an optimal bidding function $b : \Phi \to \mathbb{R}$ to maximize the total expected value of items won $\mathbb{E}[v(\phi)W(b(\phi))]$ (where $v(\phi)$ is the value of an item of type $\phi \in \Phi$) subject to an expected budget constraint in a first price auction $\mathbb{E}[b(\phi)W(b(\phi))] \leq B$ (the expectation is taken over a distribution on Φ). In [214], the convexifying transformation $q = W^{-1}(b)$ is not used, and the general solution for

b is described implicitly through $\mu W(b) = (v(\phi) - \mu b)W'(b)$, where μ is the optimum Lagrange multiplier. However, applying the convex transformation here results in the problem to optimize $\mathbb{E}[v(\phi)q(\phi)]$ over the win probability function $q : \Phi \to [0, 1]$ subject to the constraint $\mathbb{E}[\Lambda_{1st}(q(\phi))] \leq B$. Using the results of this chapter, we can specify the optimal bidding function as $b(\phi) = g^{-1}(v(\phi)/\mu)$, and reproduce the results of [214] given particular examples of the W function. The optimum multiplier μ is obtained, as usual, through the monotone root-finding problem for the function $\mu \mapsto \mathbb{E}[\Lambda_{1st} \circ q_\mu(\phi)] - B$. In the second price case, we have $b(\phi) = v(\phi)/\mu$. This is exactly an instance of a bid-your-*shaded*-valuation policy, a common structural result in the case of budget constrained optimum bidding [92, 84, 10]. Finally, a similar problem arises as Problem $(1a), (1b)$ in [10]; the convex transformation discussed here seems to explain why [10] is able to show strong duality for their seemingly non-convex problem.

2.2 The Market Model

2.2.1 Heterogeneous Item Types

The supply curves specified in Section 2.1 were developed entirely for a single heterogeneous collection of items. However, in actuality, items in RTB can be highly distinguished, and indeed, no two people are the same. Yet, items may still exhibit some substantial similarities, particularly with respect to the characteristics that bidders are interested in. As well, publishers may simply not be able to keep track of everything that makes one user distinct from another; indeed, there are relevant legal restrictions (*e.g.,* GDPR[3] [186]) and privacy concerns [137, 15, 90, 176, 175].

To model differing item types, we start by considering some *type space* Φ, where Φ is generic notation for a space of possible item types. This set may be finite (the typical case considered in this thesis) in which case we have $\Phi = [M] \triangleq \{1, 2, \ldots, M\}$, a subset of euclidean space $\Phi \subseteq \mathbb{R}^d$ (which may be a space into which item characteristics are embedded by machine learning models [128]), or just an arbitrary set. Then, each $\phi \in \Phi$ (or $j \in [M]$, for the finite case) has it's own associated supply curve, $W(x; \phi)$, or $W_j(x)$.

We focus almost exclusively on the case of a finite type space $\Phi = [M]$, and provide rigorous definitions for this case only. The only other places where a continuum of item

[3]GDPR is the European Union's *General Data Protection Regulation* and relates to privacy law and data protection.

types is encountered is in Section 2.3.5, where we provide some justification for the assumption that Φ is finite, and in Section 6.1 where we discuss extensions and future work. The calculation of appropriate segments of Φ into a finite type space is studied by [147], see also [207] for a discussion on the effects of segmentation granularity.

The following market model should be thought of roughly as an interpretation of a market in equilibrium, in the sense of the mean field models of [84, 91, 10, 13].

Definition 2.2.1 (Market Model)**.** There is an auction market with item types $[M] = \{1, \dots, M\}$, where items arrive according to a marked Poisson processes with rate $\lambda > 0$ and (exponential) inter-arrival times $\tau_1, \tau_2, \dots$ Each arrival process $j \in [M]$ is associated with a supply curve W_j (*c.f.,* Definition 2.1.1). The mark of arrival number $n \in \mathbb{N}$ is given by a (*type, price*) pair, (ϕ_n, p_n), where $\phi_n \sim \text{Cat}(\eta)$ describes the probability distribution of types and $p_n|(\phi_n = j) \sim W_j$ describes the probability distribution of prices. The vector $\eta = (\eta_1, \eta_2, \dots, \eta_M) \in (0,1)^M$ are probabilities of the categorical distribution such that $\mathbb{P}\{\phi_n = j\} = \eta_j$. The arrival rates of each item type are therefore given by $\lambda_j \overset{\Delta}{=} \lambda\eta_j$, and are also Poisson processes by the thinning property. The inter-arrival times of these types will sometimes also be written $\tau_j(n)$, and a counter $\nu_j(n) = \sum_{u=1}^{n} \mathbf{1}_j(\phi_n)$ will sometimes be used to count the number of arrivals of a particular type.

Remark 2.2.1. Most of the details of Definition 2.2.1 are only used explicitly in Chapter 4 where the distributional assumptions on item arrival rates are used to facilitate the proof of convergence results for stochastic approximations. For most of the problems considered in the thesis, all that is needed is a supply curve $W_j(x)$ and an arrival rate $\lambda_j > 0$ which characterize, as $\lambda_j W_j(x)$, the average number of items of type j won by an exogenous bidder bidding x on each arrival of type j. This is essentially a *fluid model* for a stochastic optimization problem [51, 71], a topic upon which we will remark further in Section 6.1.

2.3 Contract Management

The main problem studied in this thesis is one of *contract management*, and variations thereon. This is a problem faced by intermediaries in RTB markets who form agreements to participate in RTB on behalf of some counter parties; in our context, these counter parties are advertisers.

The contracts we will consider come in some slightly different formulations, but essentially stipulate that the intermediary acquire either a certain fixed number of items (possibly with a time deadline, see Chapter 5), or acquire items at some average rate, and that these items either simply belong to a particular subset of available types of items, or more generally, that the *value* to the counter party of items acquired adds up to some desired amount.

There are a myriad economic reasons for such intermediaries to exist, and a formal analysis of some of the economic benefits of such intermediation is given by [11]. Essentially, the profit of the intermediary can be expected to arise from aggregating the risk of adverse price movements from the multitude of counter parties, as well as from maintaining the technical infrastructure needed for bidding efficiently in RTB.

We introduce these problems through a series of examples. We begin with a problem involving only a single item type in Section 2.3.1, and extend this to multiple item types in Section 2.3.2. We then state the general formulation with multiple contracts and multiple item types, and demonstrate the reformulation of these problems into convex programs in Section 2.3.3. We comment on the case of continuum item types in Section 2.3.5. For most of this thesis we focus on problems without time deadlines or time dynamics – our analysis is adapted to this case in Chapter 5.

Remark 2.3.1 (Notation)**.** Throughout the thesis we will encounter numerous optimization problems. The decision variables of these problems will be indicated underneath the word *minimize*, and the space in which these variables live should be clear from the context. The following will be encountered frequently: $x \in \mathbb{R}^M, s \in \mathbb{R}^M, \mu \in \mathbb{R}^M, \rho \in \mathbb{R}^N$ as well as matrices $R \in \mathbb{R}^{N\times M}$ and $\gamma \in \mathbb{R}^{N\times M}$. Alternatively, R, γ may be *ragged arrays* where $R_{ij} \in \mathbb{R}$ for $i \in [N], j \in \mathcal{A}_i$ for some $\mathcal{A}_i \subseteq [M]$. Any unqualified indices in the constraints of these optimization problems are to be understood as ranging over all their possible values, *i.e.,* rather than writing $R_{ij} \geq 0\ \forall i \in [N], j \in [M]$ we will simply write $R_{ij} \geq 0$.

2.3.1 Example: Single Item Type

Suppose that it is required to win items of a single type (with supply curve W) at the average rate $C > 0$. We are faced with the following optimization problem in $x \in \mathbb{R}$:

$$\begin{aligned} \underset{x\in\mathbb{R}}{\text{minimize}} \quad & \lambda f(x) \\ \text{subject to} \quad & \lambda W(x) \geq C. \end{aligned} \tag{2.12}$$

Since the cost function f is monotone increasing, the solution to this problem is given simply by the minimum bid that attains items at the required rate C. That is, $x = W^{-1}(C/\lambda)$.

Recall from Definition 2.1.1 that $W^{-1}(q)$ is defined to take the value $\bar{x}$ whenever $q > 1$, which may be $\bar{x} = \infty$ if the support of W is all of $[0, \infty)$. Placing the bid $\bar{x}$ may be interpreted as a *best effort* attempt to to solve Problem (2.12) when $C > \lambda$. If $C < \lambda$ (*i.e.,* there is enough supply available to satisfy the constraint) then the optimum bid x is guaranteed to be finite (an important generalization of this requirement will appear as Assumption 3.0.1). The acquisition cost function arises as the total average cost $\lambda f \circ W^{-1}(C/\lambda) = \lambda\Lambda(C/\lambda)$ of fulfilling these rate requirements.

2.3.2 Example: Multiple Item Types

Suppose now that there is a finite number M of types $j \in [M] = \{1, 2, \ldots, M\}$ with supply curves W_j and arrival rates $\lambda_j > 0$. As well, suppose that the contract has differing valuations $v_j \geq 0$ for each of these items, and some target *conversion rate* $C > 0$ (see Remark 2.3.2 for the notion of conversion rate). We then formulate the problem over the M variables $x_1, \ldots, x_M$ indicating the bids to be placed on items of the various different types:

$$\begin{aligned} \underset{x \in \mathbb{R}^M}{\text{minimize}} \quad & \sum_{j=1}^{M} \lambda_j f_j(x_j) \\ \text{subject to} \quad & \sum_{j=1}^{M} \lambda_j v_j W_j(x_j) \geq C. \end{aligned}$$

For the sake of this example, let us further suppose that each W_j and f_j are continuously differentiable on all of $\mathbb{R}$ and that there exists some bid $\tilde{x} < \bar{x}$ such that $\sum_{j=1}^{M} \lambda_j v_j W(\tilde{x}) > C$ (*c.f.,* Assumption 3.0.1). Then, since the objective function is monotone increasing in each x_j, any solution must lie within the set

$$\widetilde{X} = \big\{x \in \mathbb{R}^M \mid \sum_{j=1}^{M} \lambda_j v_j W_j(x_j) \geq C\big\} \cap [0, \tilde{x}]^M,$$

which is compact by the continuity of each W_j, and non-empty by the assumption. Thus, a solution exists by Weirstrass' theorem. As well, since $C > 0$ and $W_j(x) = 0$ for $x \leq 0$, there must be at least one $x_j > 0$, which is enough to establish the existence of a Lagrange multiplier associated to any solution[4] (see [45, Thm. 9.1] or [18], which relies on our present additional assumption that W is differentiable on all of $\mathbb{R}$), that is, any optimal solution must satisfy

$$\frac{\partial}{\partial x_j}\Big[\sum_{k=1}^{M}\big(\lambda_k f_k(x_k) - \rho\lambda_k v_k W_k(x_k)\big) - \rho C\Big] = 0,$$

for each j and for some multiplier ρ.

In the second price case $f_j(x) = \int_0^x uW_j'(u)\mathsf{d}u$ and, the above derivative is given by $\lambda_j(x_j - \rho v_j)W_j'(x_j)$. Hence, we must have an optimal bid of either $x_j = v_j\rho$, or some $x_j \le 0$ which is such that $W_j'(x_j) = 0$. If $x_j = v_j\rho$, then the j^{th} component of the summation takes the value propotionate to $f_j^{2nd}(\rho v_j) - \rho v_j W_j(\rho v_j) = -\mathbb{E}(\rho v_j - p)_+ \le 0$ (see Equation 2.4), which can be no worse than the value of 0 obtained by $x_j \le 0$. Hence, $x_j = \rho v_j \ge 0$.

In the first price case, differentiation results in the requirement

$$W_j(x_j) + (x_j - v_j\rho)W_j'(x_j) = 0$$

and therefore, we can either select some $x_j < 0$ or, after rearranging, solve $x_j + \frac{W_j(x_j)}{W_j'(x_j)} = v_j\rho$. This latter expression is the bid mapping function (Section 2.1.3) g_j, *i.e.*, $g_j(x_j) = v_j\rho$. Under the assumptions of Proposition 2.1.2, this function is strictly monotone and therefore we obtain the candidate solution $x_j = g_j^{-1}(v_j\rho)$. Similarly as before, it cannot be that $x_j < 0$ is optimum since the value of the Lagrangian corresponding to such a bid is $-\rho C$ and it can be shown, since $g^{-1}(x) \le x$, that the value obtained by $g_j^{-1}(v_j\rho)$ can only be less than that.

Thus, in both the first and second price auction, the optimal bid x_j can be written in terms of the single scalar variable ρ as in $x_j(\rho)$, where $x_j(\rho)$ is a monotone increasing function of ρ. This results in the univariate optimization problem

[4] Theorem [45, Thm. 9.1] states the Fritz-John optimality conditions and establishes the existience of $(\eta, \rho) \neq 0$ such that $\eta \in \{0, 1\}$ and $\frac{\partial}{\partial x_j}\Big[\sum_{j=1}^{M}\big(\eta\lambda_j f_j(x_j) - \rho\lambda_j v_j W_j(x_j)\big) - \rho C\Big] = 0$ for each j. If $\eta = 0$ then necessarily $\rho = 0$ since there must be some $x_j > 0$, a contradiction, and thus $\eta = 1$.

$$\begin{aligned} \underset{\rho \in \mathbb{R}}{\text{minimize}} \quad & \sum_{j=1}^{M} \lambda_j f_j(x_j(\rho)) \\ \text{subject to} \quad & \sum_{j=1}^{M} \lambda_j v_j W_j(x_j(\rho)) \geq C. \end{aligned}$$

Following similar reasoning as in the previous section, the optimal ρ is the smallest such that the constraints are attained. That is,

$$\rho^\star = \boldsymbol{W}^{-1}(C),$$

where $\boldsymbol{W}(\rho) = \sum_{j=1}^{M} \lambda_j v_j W_j \circ g_j^{-1}(v_j \rho)$. The optimal bids are then given by $x_j(\rho^\star)$, and thus the problem is completely reduced simply to the the task of computing a root of a monotone function. Roots of monotone functions are a common theme throughout and we discuss this computationally in Section 2.4.

Remark 2.3.2 (Conversions)**.** The valuations v_j, if they satisfy $v_j \in [0, 1]$, may be interpreted as *conversion probabilities* [110, 5], that is, the probability that a user of type j will go on to make a purchase (or take an action desirable to the advertiser). In this case, the contract stipulates that the DSP must acquire *conversions* at the rate $C_i > 0$ at minimum cost.

2.3.3 The Main Problem: Multiple Item Types and Multiple Contracts

Suppose now that there is both a collection $j \in [M]$ as well as a collection of contracts $i \in [N]$. As in Section 2.3.2, these item types have arrival rates λ_j and supply curves W_j. As well, each contract may have a different valuation of different item types, i.e., $v_{ij} \geq 0$ is the valuation of type j by contract i.

As opposed to the problems of Section 2.3.1 and Section 2.3.2, we now need to do more than calculate bids which will win a specified number of items. Indeed, we must *also* determine towards which contract items which are won should be allocated. In order to tackle this problem, we introduce an additional set of variables $\gamma_{ij} \in [0, 1]$, which are to be interpreted as the proportion of items of type j that, when won, will be allocated towards contract i. In practice, this can be implemented by allocating an item of type j, if won, towards contract i with probability γ_{ij}. The following optimization problem models this situation.

$$\begin{aligned}
\underset{x,\gamma}{\text{minimize}} \quad & \sum_{j=1}^{M}\sum_{i=1}^{N}\gamma_{ij} f_j(x_{ij}) \\
\text{subject to} \quad & \sum_{j=1}^{M}\gamma_{ij}\lambda_j v_{ij} W_j(x_{ij}) \geq C_i \\
& \sum_{i=1}^{N}\gamma_{ij} \leq 1, \gamma_{ij} \geq 0,
\end{aligned} \tag{P^m}$$

where x_{ij} is the bid to be placed on items of type j when they will be allocated towards contract i. A version of this problem was first studied, without the assumption of continuity of supply curves or differing valuations v_{ij}, by [170].

As in Sections 2.3.1 and 2.3.2, Problem (P^m) has a monotone objective function and constraints which involve monotone functions (hence the superscript m), where in this case they are monotone in the sense that a function $h : \mathbb{R}^d \to \mathbb{R}$ is monotone if $x \leq y \implies h(x) \leq h(y)$, and where $x \leq y$ means that $\forall i \in [d] : \; x_i \leq y_i$. Therefore, Problem ($P^m$) is an instance of a monotonic optimization problem [177], and there are convergent algorithms (particularly the *Polyblock algorithm*) which can find solutions. However, these algorithms do not have convergence *rate* guarantees (indeed, general monotone optimization is computationally hard). Fortunately, the cost f is not just an arbitrary monotone function as there is a close relationship between f and W, which provides useful structure.

The Convex Transformation

We will see that Problem (P^m) can actually be reformulated as a *convex* optimization problem. The key to this transformation is a change of variables $q_j = W_j(x_j)$, which is the probability of winning an item of type j. A key lemma for establishing this result is that for any solution x_{ij} of Problem (P^m), there exists (if each Λ_j is convex) another solution (abusing notation) x_j such that $x_{ij} = x_j$ for each i, *i.e.,* the value of x_{ij} does not depend upon i. This was first recognized by [170] and in Lemma B.2.1 we derive this fact as a consequence of the convexity of Λ. Moreover, we show in Lemma B.2.2 that any solution to Problem (P^m) is equivalent to a solution wherein $x_{ij} \geq 0$ and that for any solution the conversion constraints will be binding: $\sum_{j=1}^{M}\gamma_{ij}\lambda_j v_{ij} W_j(x_{ij}) = C_i$. This is a consequence of the monotonicity of the objective and constraint functions, as well as that $W_j(x) = f_j(x) = 0$ for any $x \leq 0$.

In the following, Problem (P) has a similar interpretation as does Problem (P^m), except now the decision variables s_j, R_{ij} indicate *rates* (rather than bids or probabilities). That is, R_{ij} takes the place of γ_{ij} and is such that, while γ_{ij} is the proportion of items of type j allocated towards contract i, R_{ij} is now the unnormalized rate at which items of type j are allocated towards contract i. Similarly, s_j is the absolute rate at which items of type j are to be acquired, and $q_j = s_j/\lambda_j$ is the target probability of winning items of type j. The optimal bid is thus easily obtained as $x_j = W_j^{-1}(q_j)$.

First, some notation is in order.

Definition 2.3.1 (Notation)**.** Given valuations v_{ij} of contract $i \in [N]$ for items of type $j \in [M]$, we denote $\mathcal{A}_i = \{j \in [M] \mid v_{ij} > 0\}$ the set of *useful types* for contract i. Dually, we denote $\mathcal{B}_j = \{i \in [N] \mid v_{ij} > 0\}$ the set of *fulfillable contracts* for item type j. These sets have the property that $j \in \mathcal{A}_i \iff i \in \mathcal{B}_j$. The cardinality of the sets $\mathcal{A}_i$ are relevant for the number of variables we need to optimize over and we denote $d_i = |\mathcal{A}_i|$ and $d = \sum_{i=1}^N d_i$. We assume throughout that $d_i \geq 1$.

Theorem 2.3.1 ([101])**.** *In a first or second price auction, suppose that for each $j \in [M]$ the acquisition cost curve $\Lambda_j(q)$ is convex. Then, Problem (P^m) can be reformulated as*

$$\begin{aligned} \underset{s,R}{\text{minimize}} \quad & \sum_{j=1}^{M} \lambda_j \Lambda_j(s_j/\lambda_j) \\ \text{subject to} \quad & \sum_{j \in \mathcal{A}_i} v_{ij} R_{ij} = C_i \\ & \sum_{i \in \mathcal{B}_j} R_{ij} = s_j, R_{ij} \geq 0. \end{aligned} \tag{P}$$

If a solution exists, then a solution to the original problem (P^m) is obtained via $x_{ij} = W_j^{-1}(s_j/\lambda_j) < \infty$ for each $i \in [N]$ and $\gamma_{ij} = R_{ij}/s_j$ (with $0/0 \triangleq 0$). As well, given a solution (x, γ) to Problem (P^m), a solution (s, R) to Problem (P) is obtained by $s_j = \lambda_j W_j(x_j)$ and $R_{ij} = \gamma_{ij} s_j$. Moreover, Problem (P) is a convex optimization problem (in the sense of [29]).

Proof. We apply Lemma B.2.1 to first eliminate the dependence of the bid on i, since if a solution exists it can be assumed to have the property $x_{ij} = x_j$. As well, Lemma B.2.2

tells us that the conversion constraint inequality will always be binding and that we can choose $x \geq 0$. We therefore have the equivalent problem:

$$\begin{aligned}
\underset{x\geq 0,\gamma}{\text{minimize}} \quad & \sum_{i=1}^{N}\sum_{j=1}^{M}\gamma_{ij}\lambda_j f_j(x_j) \\
\text{subject to} \quad & \sum_{j=1}^{M}\gamma_{ij}\lambda_j v_{ij} W_j(x_j) = C_i \\
& \sum_{i=1}^{N}\gamma_{ij} \leq 1, \gamma_{ij} \geq 0,
\end{aligned}$$

Due to the bids' independence of i, we can rearrange the objective by swapping the order of summation:

$$\sum_{i=1}^{N}\sum_{j=1}^{M}\gamma_{ij}\lambda_j f_j(x_j) = \sum_{j=1}^{M}\lambda_j f_j(x_j)\sum_{i=1}^{N}\gamma_{ij} \overset{(a)}{=} \sum_{j=1}^{M}\lambda_j f_j(x_j), \tag{2.13}$$

where (a) follows since $\sum_{i=1}^{N}\gamma_{ij} \in \{0,1\}$ (see Lemma B.2.1), and if $\sum_{i=1}^{N}\gamma_{ij} = 0$ we can take $x_j = 0$ since $f_j(0) = 0$. This results in the equivalent problem

$$\begin{aligned}
\underset{x\geq 0,\gamma}{\text{minimize}} \quad & \sum_{j=1}^{M}\lambda_j f_j(x_j) \\
\text{subject to} \quad & \sum_{j=1}^{M}\gamma_{ij}\lambda_j v_{ij} W_j(x_j) = C_i \\
& \sum_{i=1}^{N}\gamma_{ij} = 1, \gamma_{ij} \geq 0,
\end{aligned}$$

We consider the change of variables $s_j = \lambda_j W_j(x_j)$ and $R_{ij} = \gamma_{ij} s_j$, each of which is invertible by the relation $x_j = W_j^{-1}(s_j/\lambda_j)$ (the inverse exists since W_j is strictly monotone), and it is shown that x_j obtained from this inverse satisfies $x_j < \infty$ (*i.e.*, $s_j < \lambda_j$ if $\bar{x}_j = \infty$) in Proposition 3.1.2. This results in the problem

$$
\begin{aligned}
\underset{s,\gamma,R}{\text{minimize}} \quad & \sum_{j=1}^{M} \lambda_j \Lambda_j(s_j/\lambda_j) \\
\text{subject to} \quad & \sum_{j=1}^{M} R_{ij} v_{ij} = C_i \\
& \sum_{i=1}^{N} R_{ij} = s_j \\
& R_{ij} \geq 0 \\
& R_{ij} = \gamma_{ij} s_j .
\end{aligned} \tag{2.14}
$$

Since the objective function is independent of γ, for any value of s, R, a feasible γ can be constructed which satisfies the constraints and $R_{ij} = \gamma_{ij} s_j$ without affecting the objective function value. Therefore, we can drop γ from this problem without affecting the solutions in s, R to obtain Problem (P). Since this transformation of variables is invertible, the optimal objective values of Problem (P^m) and Problem (P) are equal, and a solution to one can be converted into a solution to the other as described.

That Problem (P) is a convex optimization problem is because the objective function is convex, the inequality constraints are specified by a finite number of convex (in fact, linear) functions, and the equality constraints are linear. □

Problem (P) is the basic version of the problem studied in the remainder of this thesis. The key to the reformulation of the intractable Problem (P^m) into the convex Problem (P^m) is the simple substitution of variables $s_j = \lambda_j W_j(x_j)$, optimizing the probability of winning items of type j rather than optimizing the bid directly. While a similar transformation appears in a proof of [62], and the perspective of working with (win probability, expected payment) pairs is well known in auction theory [122], the convexity properties of this transformation may not be widely understood. Indeed, the problems studied by [214, 213], for example, are convex under this same transformation (see Example 2.1.6). As well, this important property does not appear to be explicitly mentioned by textbooks on auction theory [118, 103], or by [122].

A detailed analysis of this basic problem is carried out in Chapter 3, and extensions to cases with certain types of dynamics are given in Chapter 4 and Chapter 5.

Remark 2.3.3 (Transportation Production Problems)**.** Formally, Problem (P) is an instance of a *transportation-production* problem [108, 159]. These problems are analogous to transportation problems [187, 63] (through the calculation of allocation rates R) except that, in addition, one needs to choose the production levels at each of the production nodes (in our case, these are $j \in [M]$ and s_j). Indeed, there can be seen similarities between the decomposition method of [170], for the monotone programming formulation, and the work of [159] on the production-transportation problem.

Remark 2.3.4 (Blended Auction Types and Alternative Channels)**.** Throughout the thesis, we generally think of problems as being situated in *either* a first price auction, *or* a second price auction. However, it is possible in practice to formulate contract management problems where some item types are associated with second price cost functions and some item types are associated with first price cost functions. Such problems may arise if items are acquired across multiple auction exchanges. Indeed, additional advertising channels can be included through separate item types and further cost function specifications, even if they are not based on auction mechanisms.

2.3.4 Regularization and Constraint Elimination

Another form of Problem (P) will be useful in later developments, particularly in Chapter 4. This modified problem is obtained simply by eliminating the equality constraint $\sum_{i\in\mathcal{B}_j} R_{ij} = s_j$ to obtain the problem

$$\begin{aligned} \underset{R}{\text{minimize}} \quad & \sum_{j=1}^{M} \lambda_j \Lambda_j \Big(\frac{1}{\lambda_j} \sum_{i\in\mathcal{B}_j} R_{ij}\Big) \\ \text{subject to} \quad & \sum_{j\in\mathcal{A}_i} v_{ij} R_{ij} = C_i \\ & R_{ij} \geq 0. \end{aligned} \tag{P_R}$$

which involves only the variables R. The Problem (P_R) can be further modified by eliminating the remaining equality constraint $\sum_{j\in\mathcal{A}_i} v_{ij} R_{ij} = C_i$. This equality constraint forces R to be confined to a subspace of $d - N$ dimensions and therefore we should be able to search for the solution directly over some vector $u \in \mathbb{R}^{d-N}$ rather than over $\mathbb{R}^d$.

This reduction step will find use in Chapter 4 where it plays a role in constructing stochastic approximation algorithms. Essentially, the methods of that chapter are most natural when the only constraints are linear *inequality* constraints, without any implied equalities.

Proposition 2.3.1 (Reduced Problem)**.** *Let $\bar{R}$ be a particular solution to the linear system $\sum_{j\in\mathcal{A}_i} v_{ij}R_{ij} = C_i$,* e.g.,

$$\bar{R}_{ij} = \frac{C_i}{\sum_{\ell\in\mathcal{A}_i} v_{i\ell}}.$$

Additionally, let $\bar{\boldsymbol{s}} \in \mathbb{R}^M$ be given by $\bar{\boldsymbol{s}}_j = \sum_{i\in\mathcal{B}_j} \bar{R}_{ij}$. Then, there exists a full-rank matrix $P \in \mathbb{R}^{d\times(d-N)}$, a permutation matrix $T \in \mathbb{R}^{d\times d}$, and a matrix $H \in \mathbb{R}^{M\times d}$ with rows $H_j = \big(0, 0, \ldots, \boldsymbol{e}^{\mathsf{T}}_{|\mathcal{B}_j|}, 0, \ldots, 0\big)$ such that Problem (P_R) can be equivalently written as

$$\begin{aligned} \underset{u\in\mathbb{R}^{d-N}}{\text{minimize}} \quad & \boldsymbol{\Lambda}\big(\bar{\boldsymbol{s}} + HTPu\big) \\ \text{subject to} \quad & Pu \geq -\bar{\boldsymbol{r}} \\ & HTPu \leq \boldsymbol{\lambda} - \bar{\boldsymbol{s}}, \end{aligned} \tag{2.15}$$

where $\boldsymbol{\Lambda}(s) = \sum_{j=1}^{M} \lambda_j \Lambda_j(s_j/\lambda_j)$, $\bar{\boldsymbol{r}} \in \mathbb{R}^d$ contains the entries of $\bar{R}$ arranged in appropriate order, and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M)$. As well, there exists a full rank matrix $G \in \mathbb{R}^{(d-N+M)\times(d-N)}$, a matrix $B \in \mathbb{R}^{M\times(d-N)}$, and a vector $h \in \mathbb{R}^{d+M}$ such that this problem is equivalent to the reduced problem

$$\begin{aligned} \underset{u\in\mathbb{R}^{d-N}}{\text{minimize}} \quad & \boldsymbol{\Lambda}(\bar{s} + Bu) \\ \text{subject to} \quad & Gu \leq h. \end{aligned} \tag{P^u}$$

The polytope $P_{G,h} = \{u \in \mathbb{R}^{d-N} \mid Gu \leq h\}$ is compact. Solutions to Problem (P_R) are reconstructed from u by calculating $\boldsymbol{r} = \bar{\boldsymbol{r}} + Pu$ and then rearranging the entries of the vector $\boldsymbol{r} \in \mathbb{R}^d$ into a the sparse matrix $R \in \mathbb{R}^{N\times M}$. Finally, neither Problem (P^u) or (2.15) have any implicit constraints in relative interior of $P_{G,h}$ i.e., *for any u such that $Gu < h$ we have $\boldsymbol{\Lambda}(\bar{s} + Bu) < \infty$.*

Proof. Consider the particular solution $\bar{R}$ to the system $\sum_{j\in\mathcal{A}_i} v_{ij}R_{ij} = C_i, i \in [N]$. The subspace of solutions to this system can be parameterized in terms of $\bar{R}$ and an additional vector $u \in \mathbb{R}^{d-N}$. To do so, consider a particular equation for some fixed i and let $u \in \mathbb{R}^{d_i-1}$. Pick some arbitrary $j^\star \in \mathcal{A}_i$ and let $R_{ij} = \bar{R}_{ij} + u_{ij}$ for each $j \in \mathcal{A}_i$ except

$$R_{ij^\star} = \bar{R}_{ij^\star} - \frac{1}{v_{ij^\star}} \sum_{j\in\mathcal{A}_i\setminus\{j^\star\}} v_{ij}u_{ij},$$

so that

$$\sum_{j\in\mathcal{A}_i} v_{ij}R_{ij} = \sum_{j\in\mathcal{A}_i} v_{ij}\bar{R}_{ij} + \sum_{j\in\mathcal{A}_i\setminus\{j^\star\}} v_{ij}u_{ij} - \frac{v_{ij^\star}}{v_{ij^\star}} \sum_{j\in\mathcal{A}_i\setminus\{j^\star\}} v_{ij}u_{ij}$$
$$= C_i.$$

Therefore, the array R_{ij}, constructed in terms of $u_{ij} \in \mathbb{R}$, is guaranteed to exactly satisfy the contract equality constraint. Notice that if $\mathcal{A}_i = \{j\}$ then C_i/v_{ij} is the only viable solution. These linear operations can be described by a matrix $P \in \mathbb{R}^{d\times(d-N)}$ as in $\boldsymbol{r} = \bar{\boldsymbol{r}} + Pu$ where $\boldsymbol{r} \in \mathbb{R}^d$ contains the entries of R_{ij} and $\bar{\boldsymbol{r}}$ contains the entries of the particular solution $\bar{R}$. The entries of matrix P are either 1, 0, or $-\frac{v_{ij}}{v_{ij^\star}} < 0$, and since $v_{ij} > 0$ for $j \in \mathcal{A}_i$, the sparsity pattern of P is such that, by inspection, P has full rank.

The inequality constraint $R_{ij} \geq 0$, which is equivalent to $\boldsymbol{r} \geq 0$, can be written $Pu \geq -\bar{\boldsymbol{r}}$.

Then, since $\Lambda_j(s) = \infty$ for any $s > 1$, there is an implicit constraint $\sum_{i\in\mathcal{B}_j} R_{ij} \leq \lambda_j$. By appropriately permuting the entries of $\boldsymbol{r}$ with a matrix $T \in \mathbb{R}^{d\times d}$ and then multiplying by a "summation matrix" H having rows $(0, \dots, \boldsymbol{e}_{|\mathcal{B}_j|}, \dots, 0)$ *etc.*this constraint can be written $HTPu \leq \boldsymbol{e} - \bar{s}$ where $\bar{s}_j = \sum_{i\in\mathcal{B}_j} \bar{R}_{ij}$. We then let $B = HTP \in \mathbb{R}^{M\times(d-N)}$ and the objective can be written $\boldsymbol{\Lambda}(\bar{s}+Bu)$. This establishes the equivalence between Problem (P_R) and Problem (2.15).

Define $G \in \mathbb{R}^{(d+M)\times(d-N)}$ and $h \in \mathbb{R}^{d+M}$ by

$$G = \begin{bmatrix} -P \\ B \end{bmatrix}, h = \begin{bmatrix} \bar{\boldsymbol{r}} \\ \boldsymbol{\lambda} - \bar{s} \end{bmatrix}.$$

This then gives us the equivalence with Problem (P^u).

The matrix G has full rank since P has full rank (*i.e.*, $\mathcal{N}(G) = \{0\}$), and the polytope $P_{G,h}$ is compact since it encodes the constraints $R_{ij} \geq 0$ and $\sum_{i\in\mathcal{B}_j} R_{ij} \leq \lambda_j < \infty$. □

Regularization

In the general case, it can be expected that $M < d - N$ and hence the matrix B has a non-trivial nullspace. This implies that solutions to Problem (P^u) need not be unique (even if $\boldsymbol{\Lambda}$ is strictly or strongly convex). The interpretation of this fact is that there may be multiple allocation matrices R which satisfy the contracts and which also need to acquire items at the exact same rate. In order to select a particular solution, it is reasonable to choose the *least norm* allocation matrix, but unfortunately, because of the constraint $R \geq 0$, there is no analytic formula for this matrix. Instead, it can be estimated by solving problems of the form

$$\begin{aligned} \underset{u \in \mathbb{R}^d}{\text{minimize}} \quad & \mathbf{\Lambda}(\bar{s} + Bu) + \frac{1}{2\beta}||u||_2^2 \\ \text{subject to} \quad & Gu \leq h, \end{aligned} \tag{P^u_β}$$

where $\beta > 0$ serves as a regularization parameter. The term $\frac{1}{2\beta}||u||_2^2$ encourages u to have a small norm (and ensures that (P^u_β) has a unique solution).

There are conditions [64] under which Problem (P^u_β) is *exact* in the sense that there exists some finite $\bar{\beta}$ such that for any $\beta > \bar{\beta}$, the unique minimizer of (P^u_β) is the the least norm solution of (P^u). Further discussion of this point is delegated to Section 3.2.3.

2.3.5 A Continuum of Types

Finally, we consider the case where there is a continuum of item types available. We generically denote these types by $\phi \in \Phi$, where Φ is some abstract space of possible item types. These types may be derived, for example, through an embedding of detailed item characteristics into euclidean space by a machine learning algorithm. Throughout the thesis, we do *not* place a significant emphasis on this case, but provide additional discussion in Section 6.1.

One of the reasons that we restrict our attention to finite type spaces $[M]$ is that, in second price auctions, a target supply rate cannot be obtained at lower cost by placing a type-dependent bid.

Proposition 2.3.2 (Sufficiency of Finite Type Spaces in Second Price Auctions)**.** *Consider an arbitrary distribution P over item characteristics Φ. Suppose that supply curves depend on ϕ as in $W(x;\phi)$ being the probability of winning an item with characteristics ϕ given a bid of x. Similarly, let $\Lambda(q;\phi)$ be the acquisition cost function $\Lambda(q;\phi) = f^{2nd} \circ W^{-1}(q;\phi)$ associated to ϕ, where the arrival rate of items is $\lambda = 1$ (without loss) and function inverse is to be understood as applying to the first argument. Then, the optimal bid function $x : \Phi \to \mathbb{R}$ for the problem*

$$\begin{aligned} \underset{x:\Phi \to \mathbb{R}}{\textit{minimize}} \quad & \mathbb{E} f(x(\phi);\phi) \\ \textit{subject to} \quad & \mathbb{E} W(x(\phi);\phi) = \bar{s}, \end{aligned}$$

is a constant $x(\phi) = \bar{x} \in \mathbb{R}$. *The expectation is taken over the distribution of* ϕ *as in* $\mathbb{E}z(\phi) = \int_\Phi z(\phi)\mathsf{d}P(\phi)$. *Moreover, the cost is given by* $\overline{\Lambda}(\bar{s}) = \mathbb{E}\Lambda(\bar{s};\phi)$.

Proof. First, apply the transformation of variables $s(\phi) = W(x(\phi);\phi)$ to reduce the problem into the convex problem

$$\begin{aligned} \underset{s:\Phi\to\mathbb{R}}{\text{minimize}} \quad & \mathbb{E}\Lambda(s(\phi);\phi) \\ \text{subject to} \quad & \mathbb{E}s(\phi) = \bar{s}. \end{aligned}$$

This problem can be solved by optimizing $s(\phi)$ pointwise. To this end, consider the Lagrangian:

$$\mathcal{L}(s(\phi),\mu) = \mathbb{E}\Lambda(s(\phi);\phi) + (\mathbb{E}s(\phi) - \bar{s})\mu.$$

Recalling that $\Lambda'_{2nd}(\cdot;\phi) = W^{-1}(\cdot;\phi)$ (see Lemma 2.1.1) the Lagrangian is minimized (since it is convex) pointwise over ϕ by the choice $s^\star(\phi) = W(\mu;\phi)$, which is the optimal rate of obtaining items with characteristic ϕ. Therefore, inverting the transformation of variables we determine the optimal bid is $x^\star(\phi) = W^{-1}(W(\mu;\phi);\phi) = \mu$, a constant.

To calculate the optimal cost, we recognize that meeting the constraint requires that $\mathbb{E}s(\phi) = \bar{s}$, which therefore requires $\mathbb{E}W(\mu;\phi) = \bar{s}$ or $\mu = \overline{W}^{-1}(\bar{s})$, where $\overline{W}(\cdot) = \mathbb{E}W(\cdot;\phi)$. Substituting this into the objective function we obtain

$$\begin{aligned} \mathbb{E}\Lambda(s(\phi);\phi) &= \mathbb{E}\Lambda(W(\mu;\phi);\phi) \\ &= \mathbb{E}f(\mu;\phi) \\ &= \mathbb{E}f(\overline{W}^{-1}(\bar{s}),\phi) \\ &= \overline{\Lambda}(\bar{s}). \end{aligned}$$

□

Another way in which a finite collection of item types may arise is if contracts are specified more coarsely by simply stipulating that any items of types $S_i \subseteq \Phi$ can be used towards fulfilling the contract, and that there are no distinguishing values[5](this is the case

[5]some distinction in valuations can still be modelled by formulating multiple contracts

studied in [170, 101, 102]). Or, these sets may be optimized while estimating a piecewise model of the form $W(x;\phi) = \sum_{j=1}^{M} W_j(x)\mathbf{1}_{S_j}(\phi)$. We denote the collection of all targeting sets by $\mathcal{S} = \{S_1, S_2, \ldots, S_N\}$. Since the targeting specifications S_i for each contract may be overlapping, it is necessary to find the coarsest partition $\mathcal{R}$ of the set $\bigcup \mathcal{S}$. This partitioning step is illustrated in Figure 2.3

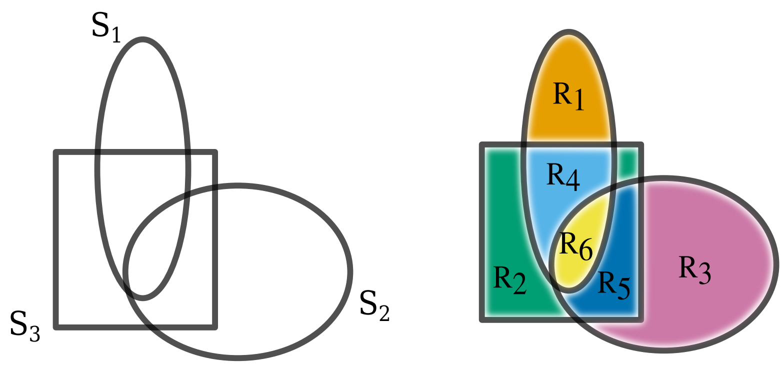


Figure 2.3: Set Partitioning Example
An example of set partitioning, best viewed in colour. In this case, $\mathcal{S} = \{S_1, S_2, S_3\}$, $M = 6$ and $\mathcal{R} = \{R_m\}_{m=1}^{M}$ contains subsets such that $\mathcal{R}$ is a partition of $\bigcup \mathcal{S}$. Moreover, for any $S_i \in \mathcal{S}$ we have some $\mathcal{A}_i \subseteq [M]$ such that $\bigcup_{j \in \mathcal{A}_i} R_j = S_i$. For example, $S_2 = R_3 \cup R_5 \cup R_6$. That is, $\mathcal{A}_2 = \{3, 5, 6\}$. Likewise, we have sets $\mathcal{B}_j$ such that $j \in \mathcal{A}_i \iff i \in \mathcal{B}_j$. For example, $\mathcal{B}_1 = \{1\}$ and $\mathcal{B}_6 = \{1, 2, 3\}$.

2.4 Computational Methods

In this section, we review a basic yet fundamental algorithm for monotone functions in Section 2.4.1, discuss the practical methods we have used to work with and represent supply curves in Section 2.4.2, and describe a primal algorithms for solving Problem (P) in Section 2.4.3. A primal algorithm will be further discussed in Chapter 3 and Chapter 5.

2.4.1 The Method of Bisection

The method of bisection is a simple algorithm for finding zeros of continuous monotone functions. That is, if $h : \mathbb{R} \to \mathbb{R}$ is strictly monotone and continuous, the method of bisection will find the point $x \in \mathbb{R}$ such that $h(x) = 0$, if it exists. This algorithm plays a central role in many of the examples and applications in this thesis, as many problems are either completely reduced to finding a zero of a monotone function, or require this as a subroutine. We describe the algorithm in Algorithm 1, and provide a concrete code example in Appendix C.1.

Algorithm 1: Bisection

input : A strictly monotone increasing function $h : \mathbb{R} \to \mathbb{R}$, bounds $x_l < x_r$ such that $h^{-1}(0) \in [x_l, x_r]$, and a stopping criteria $\epsilon > 0$.
output : `Bisection`(h, ϵ) returns a point $x \in [x_l, x_r]$ such that $|x - h^{-1}(0)| < \epsilon$.

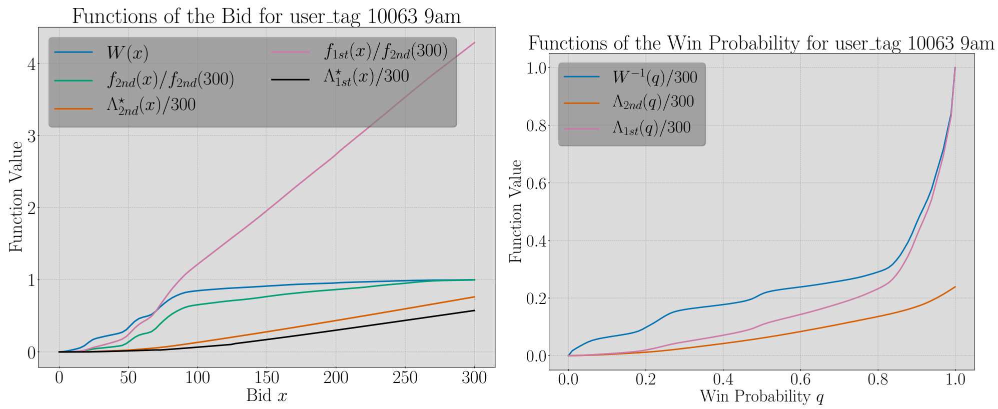
Function `bisection`(h, ϵ)
  repeat
    $x \leftarrow x_l + (x_r - x_l)/2$
    if $h(x) \geq 0$ **then**
      $x_l \leftarrow x$
    else
      $x_r \leftarrow x$
  until $|x_r - x_l| < \epsilon$
  return x

Given bounds $x_l < x_r$ such that the solution is known to lie in $[x_l, x_r]$, Algorithm (1) returns a point x such that $h(x) \approx 0$ in the sense that $|x - h^{-1}(0)| < \epsilon$. The bounds x_l, x_r can be found, for example, by searching for n such that $h(2^n) \geq 0$ and $h(-2^n) \leq 0$.

Since each iteration of the algorithm divides the interval in half, the length of the interval $[x_l, x_r]$ after n iterations, call it ℓ_n, will be given by $\ell_n = 2^{-n}\ell_0$. Since the algorithm terminates when $\ell_n < \epsilon$, it will terminate after $\mathcal{O}(\log_2 \ell_0/\epsilon)$ iterations. The number of iterations needed if the stopping criteria is instead based on $|h(x)| < \epsilon$ depends on the smoothness h. Indeed, if h is L-Lipschitz, then the bound for this alternative stopping criteria is $\mathcal{O}(\log_2 L\ell_0/\epsilon)$.

2.4.2 Working with Supply Curves

All of the mathematical functions relevant to the contract management problem (*e.g.,* $f, \Lambda, \Lambda^\star$ etc.) are ultimately derived from the supply curve W. This curve may be available in closed form (*e.g.,* see Example 2.1.2) if it is estimated, for example, by maximum likelihood of some known parameterized distribution. However, the natural simple distributions for this



(a) Functions of the Bid x (b) Functions of Win Probability q

Figure 2.4: Representative Functions from Market Data
Relevant functions constructed from real market data. Values are normalized for scale and to facilitate comparison to Figure 2.2.

type of pricing data (*e.g.,* Exponential, Gaussian, Gamma) do not fit well to market data, and we typically use these distributions only for illustration, or in some simulations.

Instead of fitting a named distribution, kernel density estimation (KDE) [162] [194, Sec 21.3] can be used to generically fit cumulative distribution functions by smoothing the data histogram. That is, if we have a dataset of L prices $p_1, p_2, \ldots, p_L$ of a single item type, we can estimate the supply curve via

$$\widehat{W}(x) = \frac{1}{L} \sum_{\ell=1}^{L} K\Big(\frac{x - p_\ell}{\sigma}\Big),$$

where $\sigma > 0$ is some bandwidth parameter (chosen *e.g.,* by cross-validation or Silverman's criteria [162]), and $K : \mathbb{R} \to \mathbb{R}$ is the kernel function. Usually, K is either a Gaussian or a Gamma *c.d.f.* [41]. This estimation method ensures the strict monotonicity and smoothness of $\widehat{W}$, and hence the assumptions made of W in Definition 2.1.1 could easily be strengthened substantially if the need were to arise.

The downside of using kernel density estimates is that simple evaluation of W takes time proportional to the number L of price samples. For performance reasons, we have often started with a KDE step, and followed this by another step where the function $\widehat{W}$ is given a coarser representation as a piecewise linear function. This step poses no problems for algorithms that do not require evaluation of W'. If evaluation of W' is necessary, the

KDE estimate of $\widehat{W}$ can be differentiated analytically, and the resulting derivative function in turn represented with a piecewise linear approximation.

Using the KDE method, we plot the same functions as in the analytic case of Figure 2.2 in Figure 2.4, except that in the latter case everything has been computed for the orange supply curve of Figure 2.1, which was obtained by KDE estimation with real data. These curves are obtained from real data, yet, they are qualitatively similar to those derived for the simple example $W(x) = 1 - e^{-x}$.

Close inspection of Figure 2.4 shows that Λ_{1st} is *not* convex. This poses an issue in practice, and the caveat on the convexity condition (Proposition 2.1.2) for the first price case is part of what makes the first price auction considerably more difficult than the second price case. This convexity issue is addressed further in Appendix B.2.1.

Remark 2.4.1 (Censored Data)**.** In real-time bidding, price data tends to be *censored* [216, 9, 203], that is, you only observe the price when you are in fact the winner of the item. The dataset we are working with was constructed so as to avoid the problem of censoring, and this is not an issue we focus upon in this thesis. However, the stochastic approximation algorithms of Chapter 4 naturally operate only on censored data.

2.4.3 Polyhedral Approximation Methods

Problem (P) is also an instance of a *monotropic program* [20, 158] and hence polyhedral approximation algorithms are natural methods [21, Ch. 4] [22]. A straightforward approximation method is simply to break the compact interval $[0, 1]$ into K segments with points $q_1, \ldots, q_K$ and approximate Λ as a max-affine function, *i.e.,*

$$\widetilde{\Lambda}(q) = \max_k \, [\Lambda(q_k) + \Lambda'(q_k)(q - q_k)], \tag{2.16}$$

which is well known to approximate $\Lambda(q)$ since the epigraph of $\Lambda(q)$ can be written as the intersection of half spaces defined through supporting hyperplanes. The derivative Λ' is available through Lemma 2.1.1. Supposing that a max-affine approximation of each acquisition cost function is in hand, Problem (P) can be approximated by a linear program [29, Sec. 4.3.1] by converting the approximations into their epigraph forms:

$$
\begin{aligned}
\underset{t,s,R}{\text{minimize}} \quad & \sum_{j=1}^{M} \lambda_j t_j \\
\text{subject to} \quad & \Lambda_j(q_k) + \Lambda_j'(q_k)(s_j/\lambda_j - q_k) \le t_j \\
& \sum_{j \in \mathcal{A}_i} v_{ij} R_{ij} = C_i, R_{ij} \ge 0 \\
& \sum_{i \in B_j} R_{ij} = s_j, s_j \le \lambda_j.
\end{aligned}
\tag{2.17}
$$

We have often made use of this approximation method (see also Chapter 5) due to its simplicity: it requires no substantial numerical or implementation expertise as it can be encoded directly into `cvxpy`.

Remark 2.4.2 (Simplicial Decomposition)**.** An alternative computational method to employ for this problem is a *simplicial decomposition* where we approximate solutions with convex combinations of extreme points of the constraint region. Since we have not experimented with this algorithm, we do not report the details. We mention simply that it involves two sub-problems: firstly, the minimization of $\sum_{j=1}^{M} \lambda_j \Lambda_j \big(\frac{1}{\lambda_j} \sum_{k=1}^{K} \alpha_k \tilde{s}_j(k) \big)$ over α constrained to a K-simplex and where $\tilde{s}_j(k)$ are fixed constants (this can be carried out, *e.g.,* by exponentiated gradient descent); and secondly, a linear program over the constraint region of Problem (P) with an objective derived again from Λ_j' similarly to Problem (2.17). This algorithm has stronger convergence guarantees for our problem, namely, that it converges in a finite number of iterations [21, Prop 4.2.1].

2.5 Additional Examples

This section gives a brief overview of some applications outside of real-time bidding where the functions in this chapter make natural appearances, and where the methods studied in this thesis may also be useful. The purpose of the additional examples is simply to illustrate the ubiquity of the important functions studied in this thesis, to establish some formal connections with different problems and thus provide further intuitive understanding, and to potentially inspire future applications. These examples will be briefly touched on in later chapters, but such sections are not essential to the thesis and can be skipped without serious loss.

2.5.1 Volume Costs in Limit Order Books

Modern financial markets are organized around a *limit order book* (LOB) that keeps track of the willingness of market participants to buy or sell securities at a certain price, and up to a certain volume [27]. Because there is no single seller, the "price" of a security is ambiguous, but generally the quoted price is given by the *mid price*, call it m. That is, the average between the seller with the lowest willingness to sell (called the best ask), and the buyer with the greatest willingness to buy (called the best bid). However, it is not in general possible to actually transact at this quoted mid price, since there is no willing counter-party (if there were, then it wouldn't be the midprice) and the amount actually paid to acquire (or the amount obtained by selling) v shares must be greater than the nominal value vm. This difference is the *transaction cost*, and for concreteness, we will focus on the case of *buying* — the case of selling being exactly symmetric.

There are numerous sources of transaction costs — the first is the spread (simply the difference between the midprice and the best bid), but there is also a "volume cost" that arises when there are fewer than v shares available in the LOB at the best bid. To establish an analogy with the functions introduced in Section 2.1.1, suppose that we approximate the availability of shares at specified prices in the LOB by a density function $w(p)$ such that there is $\mathsf{d}v = w(p)\mathsf{d}p$ volume available at the price $m + p$, *i.e.,* $p \geq 0$ is an offset from the midprice. Then, the *total* volume available at or below price p is given by $W(p) = \int_0^p w(x)\mathsf{d}x$. This is an unnormalized supply curve. If we decide to buy all the available shares up to and including price $m + p$, then we will need to pay a total of $f(p) = \int_0^p xw(x)\mathsf{d}x$, which is exactly the expected cost incurred by bidding p in a second price auction. As well, the transaction costs associated with purchasing a volume v of shares is given by $\Lambda(v) = f \circ W^{-1}(v)$, which is exactly the second price acquisition cost function. As a corollary to Proposition 2.1.1, these transaction costs are convex functions of volume and hence can be incorporated into tractable portfolio construction problems (see Section 3.4).

2.5.2 The Dark Pool Problem

As a second example in finance, the *dark pool problem* [69], is the problem of how to allocate shares across multiple "dark pool" (DP) exchanges in order to maximize the number of shares that are sold. A DP is a type of financial exchange providing an alternative to the limit order book. In a DP, prices and volumes are not quoted, rather, one simply announces their willingness to transact a certain volume of securities and agree to a transaction, if there is an available counter party, at the mid price quoted on some exogenous LOB exchange. The purpose of these markets is to enable institutions to liquidate large blocks of shares without revealing this large order to the rest of the market, and hence adversely impacting the price.

The dark pool problem is, formally, a stochastic optimization problem

$$\begin{aligned}
\underset{a_1,a_2,\ldots,a_K}{\text{minimize}} \quad & s - \sum_{k=1}^{K} \left[a_k - \mathbb{E}(a_k - \xi_k)_+\right] \\
\text{subject to} \quad & \sum_{k=1}^{K} a_k \leq s \\
& a_k \geq 0,
\end{aligned}$$

where $\xi_k \sim V_k$ is the (unknown) volume available to transact at DP $k \in [K]$ and $s > 0$ is the quantity of shares that the agent is seeking to sell. The cost incurred by the agent is the total number of shares left unsold, which would be s if no action were taken. Then, because the agent sends a_k shares to DP k, they hope to be left over with only $s - \sum_{k=1}^{K} a_k$ shares. However, since there is only ξ_k volume available at this DP, $\mathbb{E}(a_k - \xi_k)_+$ shares will be left un-traded and sent back to the agent, leaving them with $\sum_{k=1}^{K} \mathbb{E}(a_k - \xi_k)_+$ extra shares than they had hoped.

Since the objective function is monotone in a, it is optimal for the constraint $\sum_{k=1}^{K} a_k \leq s$ to be binding, and it is clearly never optimal to have any $a_k < 0$. Hence, the DP problem is equivalent to

$$\begin{aligned}
\underset{a_1,a_2,\ldots,a_K}{\text{minimize}} \quad & \sum_{k=1}^{K} \mathbb{E}(a_k - \xi_k)_+ \\
\text{subject to} \quad & \sum_{k=1}^{K} a_k = s,
\end{aligned}$$

where the objective function $\mathbb{E}(a_k - \xi_k)_+$ has been seen in Equation (2.4) to be exactly equal to the difference in costs $f_{1st}(x) - f_{2nd}(x)$ between a first and a second price auction. That is, the number of shares left un-executed is formally comparable to the amount by which one overpays in a first price auction.

Further connections between this problem and the auction problems studied in this chapter will be elucidated in Section 3.4.

2.6 Conclusion

This chapter has introduced the main building blocks, namely the functions $W, f, \Lambda, \Lambda^\star$, which play prominent roles throughout the thesis. The function W_j, called the *supply curve* for items of type j models the probability of a bidder winning an item of that type with some fixed bid. In the market model, Definition 2.2.1, these functions serve simply as cumulative distribution functions from which prices p_n are drawn *i.i.d.* and where these prices model the aggregate behaviour of competing bidders in equilibrium and the reserve prices set by sellers.

The strong assumptions in this model are not essential to the formulation of the main contract management problem, Problem (P^m). Rather, this problem can be viewed as a fluid approximation of a stochastic optimization problem where the bidding agent seeks to acquire items with adequate total valuations $\sum_{i \in B_j} v_{ij} R_{ij} = C_i$ in order to fulfill the requirements, $C_i > 0$, of contract i. This is the basic problem studied in the remainder of this work.

The key insight of this Chapter is that the acquisition cost function $\Lambda = f \circ W^{-1}$ is *always* a convex function for second price auctions (Proposition 2.1.1) and is convex under weak assumptions in the first price case as well (Proposition 2.1.2, where W must satisfy a requirement slightly weaker than log-concavity). The interpretation is that, rather than optimizing bids, one should optimize their expected probability of winning an item (and then later transform that probability back into a bid). This recognition enables the transformation of Problem (P^m) (which is nominally merely a monotone optimization problem) into a convex program, Problem (P), which opens up a wide range of possibilities that we explore in the sequel. As well, while this transformation is elementary, its use may not be widely appreciated: we have pointed to examples in the literature which are convex under the same transformation, but which were not analyzed as such.

In section 2.4 we have discussed practical computational methods, namely, the method of bisection, estimation and interpolation methods for supply curves, and polyhedral approximation algorithms for Problem (P). These methods have been used extensively in our own computational experiments. Since the functions W are arbitrary continuous distribution functions, the question of how to practically compute with and represent them (as well as derivative functions f, Λ *etc.*) can have subtle difficulties. Thus, care has been taken to discuss computational algorithms which take these difficulties into account.

Finally, two additional problems in finance, the calculation of volume costs in limit order books, and the dark pool problem, are discussed in Section 2.5. These problems are touched upon since they are similarly formulated using some of the basic functions in auction theory, and we thus recognize that some of the insights of this thesis can also be applied to these problems and more broadly.

Chapter 3

Duality and Its Consequences

The consequences of duality for the contract management problem are deep and have far reaching computational implications.

Recall from Proposition 2.3.1 that the optimal contract management problem can (assuming the supply curve is 2-concave and differentiable in the first price case) be cast as a convex program:

$$\begin{aligned} \underset{s,R}{\text{minimize}} \quad & \sum_{j=1}^{M} \lambda_j \Lambda_j(s_j/\lambda_j) \\ \text{subject to} \quad & \sum_{j\in\mathcal{A}_i} v_{ij} R_{ij} = C_i \\ & \sum_{i\in\mathcal{B}_j} R_{ij} = s_j, R_{ij} \geq 0. \end{aligned} \tag{P}$$

Recall that $\mathcal{A}_i = \{j \in [M] \mid v_{ij} > 0\}$ is the set of items that are useful for contract i, and that the cardinality of this set is $d_i \triangleq |\mathcal{A}_i|$. The total number of variables in Problem (P) is exactly $d \triangleq \sum_{i=1}^{N} d_i$, along with $N + M$ equality constraints and d non-negativity constraints. In general, we have $d_i \leq M$ and hence $d \leq MN$, so that the worst case number of variables scales quadratically. However, in practice, it can be expected that the number of items M is large, and that most contracts can only be usefully fulfilled by a small subset of all available items. That is, the valuation matrix $v \in \mathbb{R}^{N\times M}$ should be *sparse* and we can expect $d_i \ll M$ and $d \ll MN$.

Before proceeding, we need to make a basic assumption that will guarantee existence of solutions to Problem (P):

Assumption 3.0.1 (Adequate Supply)**.** There exists an array R_{ij} of allocations such that

$$\forall i \in [N], j \in \mathcal{A}_i : \; R_{ij} \geq 0,$$

$$\forall j \in [M] : \; \sum_{i \in \mathcal{B}_j} R_{ij} < \lambda_j,$$

and

$$\forall i \in [N] : \; \sum_{j \in \mathcal{A}_i} v_{ij} R_{ij} = C_i.$$

This assumption is natural in the sense that if it were not satisfied, the contracts would not be fulfillable, and they should not have been accepted in the first place. Essentially, if Assumption 3.0.1 holds, then Problem (P) admits a *Slater Point* (see *e.g.,* [29, Sec. 5.2.3] [19, Prop 5.3.1]). It should be noted that this R can also be chosen to satisfy $R_{ij} > 0$ since $C_i > 0$. An assumption similar in spirit, but much stronger, appeared in [108]:

Assumption 3.0.2 (Abundant Supply [108])**.** For each item type $j \in [M]$ we have

$$\lambda_j \min_{i \in \mathcal{B}_j} v_{ij} > \sum_{i \in \mathcal{B}_j} C_i.$$

Assumption 3.0.2 clearly implies Assumption 3.0.1 but may be much easier to verify. As well, in some applications (*e.g.,* in advertising where a single intermediary may account for only a small proportion of all purchases) Assumption 3.0.1 may not be unreasonable. We also remark that existence and duality theorems can be obtained in some cases where the strict inequality in Assumption (3.0.1) is actually binding (*i.e.,* the strict inequality can be replaced by $\leq$). However, the assumption is used for purposes beyond strong duality in Chapter 4, and keeping precise account of when this inequality can or cannot be binding is of only marginal interest.

Chapter Outline The purpose of this chapter is firstly to derive a convex dual (D) program to Problem (P). This is carried out in Section 3.1. In Section 3.1.1 we show that Assumption 3.0.1 implies the existence of both primal and dual solutions, as well as some regularity conditions (*i.e.,* to eliminate the possibility that $s_j = \lambda_j$ when $\bar{x}_j = \infty$, which would necessitate an unbounded bid). In Sections 3.1.2 and 3.1.3 we derive some important consequences of duality (which hearkens back to [170]) and then derive and explain important interpretations of these consequences. Section 3.2 focuses on computational methods and examples. We derive an Alternating Direction Method of Multipliers (ADMM) algorithm with an elegant parallelizable structure that computes a primal-dual optimal pair for Problem (P_R) in both the first and second price case.

3.1 Duality Analysis

In this section, we carry out the exercise of deriving a dual of Problem (P) and then proceed to an analysis and interpretation of the dual.

We begin with the Lagrangian function associated to Problem (P)

$$\begin{aligned}\mathcal{L}(s, R, \mu, \rho, \theta) &= \sum_{j=1}^{M} \lambda_j \Lambda_j\big(s_j/\lambda_j\big) + \sum_{i=1}^{N} \rho_i\big(C_i - \sum_{j\in\mathcal{A}_i} R_{ij}v_{ij}\big) + \sum_{j=1}^{M} \mu_j\big(\sum_{i\in\mathcal{B}_j} R_{ij} - s_j\big) - \sum_{j=1}^{M}\sum_{i\in\mathcal{B}_j} \theta_{ij}R_{ij} \\ &= \sum_{i=1}^{N} \rho_i C_i + \sum_{j=1}^{M}\Big[\lambda_j\Lambda_j\big(s_j/\lambda_j\big) - \mu_j s_j + \sum_{i\in\mathcal{B}_j} R_{ij}\big(\mu_j - \theta_{ij} - v_{ij}\rho_i\big)\Big],\end{aligned}$$

where $\mu \in \mathbb{R}^M$ is associated with the equality constraints $\sum_{i\in\mathcal{B}_j} R_{ij} = s_j$, $\rho \in \mathbb{R}^N$ is associated with the contract fulfillment constraints, and θ to the non-negativity constraints. The dual constraints are $\theta \geq 0$ and the dual problem is derived through determining the form of

$$\underset{\rho,\mu,\theta\geq 0}{\text{maximize}} \inf_{s,R} \mathcal{L}(s, R, \mu, \rho, \theta).$$

To this end, we minimize $\mathcal{L}$ pointwise over s, which results in the appearance of the Fenchel conjugate (see Proposition 2.1.1):

$$\mathcal{L}(s^\star, R, \mu, \rho, \theta) = \sum_{i=1}^{N} \rho_i C_i - \sum_{j=1}^{M} \lambda_j \Lambda_j^\star(\mu_j) + \sum_{j=1}^{M}\sum_{i\in\mathcal{B}_j} R_{ij}\big(\mu_j - \theta_{ij} - v_{ij}\rho_i\big). \tag{3.1}$$

Then, minimizing over R induces the dual equality constraint $\mu_j = \theta_{ij} + v_{ij}\rho_i$ (since otherwise the infimum would be $-\infty$), which must hold over all (i, j) such that $i \in [N], j \in \mathcal{A}_i$, or equivalently, $j \in [M], i \in \mathcal{B}_j$. Moreover, since we must have the dual cone constraint $\theta_{ij} \geq 0$ (*i.e.*, θ_{ij} is simply a slack variable) this is equivalent to the constraint $\forall i \in [N], j \in \mathcal{A}_i :\ \mu_j \geq \rho_i v_{ij}$. Thus, we have obtained:

Proposition 3.1.1 (Duality)**.** *A dual of Problem (P) can be formulated as*

$$\begin{aligned} \underset{\rho,\mu}{\text{maximize}} \quad & \sum_{i=1}^{N} \rho_i C_i - \sum_{j=1}^{M} \lambda_j \Lambda_j^\star(\mu_j) \\ \text{subject to} \quad & v_{ij}\rho_i \le \mu_j. \end{aligned} \tag{D}$$

Problem (D) is dual to Problem (P) in the sense that if $D^\star$ and $P^\star$ are their respective values (possibly ∞ or $-\infty$), then $D^\star \le P^\star$.

Problem (D) has $N+M$ variables, and $d = \sum_{i=1}^{N} |\mathcal{A}_j|$ linear inequality constraints. This is in contrast to the primal (P) which has d variables, $N+M$ linear equality constraints and d non-negativity constraints. Two further forms of this dual involving only ρ or only μ (but not both), will be seen in Chapter 4, where we will analyze the relevance to stochastic approximation.

Remark 3.1.1 (Implicit Constraints and Non-negativity)**.** In the primal, Problem (P), there is an implicit constraint $s_j \le \lambda_j$ due to the domain of Λ_j. Similarly, in the dual (D) there is an implicit constraint that $\mu_j \ge 0$ due to the domain of $\Lambda_j^\star$. Additionally, as a consequence of the monotonicity of the objective (*i.e.,* $\rho_i C_i$ is strictly monotone increasing since $C_i > 0$), it is easy to observe that at the optimal solution the dual variables ρ_i will also be non-negative: $\rho \ge 0$. It is not formally necessary to include these as constraints in (D), but we have observed improved numerical optimization performance (with `cvxpy`) when they are made explicit, particularly for large problem instances (1000s of contracts and/or item types).

3.1.1 Existence and Regularity

In this section we rigorously establish the existence of primal and dual optimal solutions. The main technical concern is that, since we allow for the distribution of prices to have a non-compact support, it could be the case that, at optimality $s_j = \lambda_j$, and then the bid required to win items of type j with probability 1 would formally be $x_j = \infty$. Under Assumption 3.0.1, this cannot happen, as the following proposition shows. It is in the sense that the corresponding optimum bids are finite that the solution is *regular*.

Proposition 3.1.2 (Existence and Regularity)**.** *Suppose Assumption 3.0.1 holds. Then, there exists an optimal allocation array $R^\star$ for Problem (P), the win rates $s_j^\star = \sum_{i \in \mathcal{B}_j} R_{ij}^\star$ are unique and there exist unique finite bids $x_j = W_j^{-1}(s_j/\lambda_j)$ which win items at the* *optimal rate. Moreover, there exists a unique solution $\rho^\star, \mu^\star$ to the dual, Problem (D), and there is zero duality gap.*

Proof. By Assumption 3.0.1, the value of Problem (P) is finite since there exists a feasible s, R. First, Λ_j is lower semicontinuous (see Proposition 2.1.1 and 2.1.2). As well, the feasible region, namely $R_{ij} \geq 0$ and $\sum_{i\in\mathcal{B}_j} R_{ij} = s_j$, and $0 \leq s_j \leq \lambda_j$ (which is implicit in the domain of Λ_j), is compact (it is a closed and bounded polytope). Therefore, there exists a solution $s^\star, R^\star$ to Problem (P) by Weierstrass' theorem. The acquisition rates $s_j^\star = \sum_{i\in\mathcal{B}_j} R_{ij}^\star$ are unique since Λ_j is strictly convex.

Now, by the strong duality theorem (*e.g.,* [19, Prop 5.3.1]) and the Slater point of Assumption 3.0.1, there exists a solution $\mu^\star, \rho^\star$ (along with the slack variables $\theta_{ij}^\star = \mu_j^\star - v_{ij}\rho_i^\star$) to the dual, Problem (D), and the value of the dual, which is finite, is equal to the value of the primal (*i.e.,* there is zero duality gap).

Finally, by strong duality and the existence of solutions, the Lagrangian optimality conditions (*e.g.,* [19, Prop 5.3.2]) establish that $s^\star \in \operatorname*{argmin}_{s\in\mathbb{R}^M} \mathcal{L}(s, R^\star, \mu^\star, \rho^\star, \theta^\star)$ and

$$\mathcal{L}(s, R^\star, \mu^\star, \rho^\star, \theta^\star) = \sum_{i=1}^{N} \rho_i^\star C_i + \sum_{j=1}^{M} [\lambda_j \Lambda_j(s_j/\lambda_j) - \mu_j^\star s_j],$$

where we have used complementary slackness [19, prop 5.3.2] to eliminate terms involving $\theta^\star, R^\star$. Then, by the subgradient optimality conditions [19, Prop. 5.4.3, Prop 5.4.4], it must be that $\forall j \in [M]: \ 0 \in \partial_{s_j}\mathcal{L}(s^\star, R^\star, \mu^\star, \rho^\star, \theta^\star) = \lambda_j \partial\Lambda_j(s_j^\star/\lambda_j) - \mu_j^\star$. We treat two cases. First, if the support of the distribution of the prices of type j is compact (*i.e.,* $\bar{x}_j < \infty$), then since $s_j \leq \lambda_j$ we have a finite optimal bid $x_j = W_j^{-1}(s_j/\lambda_j)$. If $\bar{x}_j = \infty$ then $\partial\Lambda_j(q)$ is non-empty for $q \in [0, 1)$ and thus, since we know this subgradient is indeed non-empty (solutions exist), we must have $s_j < \lambda_j$ and there is a unique optimal bid $x_j = W_j^{-1}(s_j/\lambda) < \infty$. □

3.1.2 Consequences of Duality

There are two main consequences of duality. Firstly, the dual variables induce subsets $\mathcal{A}_i^\star \subseteq \mathcal{A}_i$ and $\mathcal{B}_j^\star \subseteq \mathcal{B}_j$ which have the effect of further reducing the set of useful and usable items, in the sense that for an optimal allocation array R, it holds that $j \notin \mathcal{A}_i^\star \implies R_{ij} = 0$. These sets can again considerably reduce the dimensionality of the allocation array R that needs to be calculated, if dual solutions are known (see Section 3.2.3). Secondly, the optimal bids to be placed (but not the optimal allocation array R) are obtained by mapping the dual solution through the known bid-mapping function g_j (Proposition 2.1.4) as in $x_j = g_j^{-1}(\mu_j)$. These optimum bids can be characterized either directly by variables μ_j associated to each item type, or indirectly through the vector of N *pseudo-bids* [101] $\rho \in \mathbb{R}^N$, with each ρ_i being associated with a contract, rather than an item type. Moreover, the optimal bid to be placed across all items belonging to the set $\mathcal{A}_i^\star$ are equal, and can be obtained through the relation $j \in \mathcal{A}_i^\star \implies \mu_j = v_{ij}\rho_i$. We will shortly summarize these facts in Proposition 3.1.3. Similar results are reported by [170], where they are derived directly, rather than as consequences of duality. We summarize these results in the following proposition.

Proposition 3.1.3 (Consequences of Duality)**.** *Suppose Assumption 3.0.1 holds and let each W_j be a supply curve or a differentiable 2-concave supply curve in the first price case. Let $R \in \mathbb{R}^{N\times M}$ and $s \in \mathbb{R}^M$ be optimal primal solutions of Problem (P) such that $\sum_{i\in\mathcal{B}_j} R_{ij} = s_j$ and $\sum_{j\in\mathcal{A}_i}^M R_{ij}v_{ij} = C_i$. As well, Let $\rho \in \mathbb{R}^N$ and $\mu \in \mathbb{R}^M$ constitute optimal dual solutions to Problem (D) and let $\theta_{ij} = \mu_j - v_{ij}\rho_i$ be the slack for $i \in [N], j \in \mathcal{A}_i$. Finally, let $x_j = W_j^{-1}(s_j/\lambda_j)$ be the optimal bids for acquiring items of type j. Then,*

1. *(Optimal Bids for $j \in [M]$): The dual variables μ_j are exactly equal to the optimal bids $x_j = \mu_j$ for second price auctions, and are obtained through inverting the bid mapping function $x_j = g_j^{-1}(\mu_j)$ in first price auctions.*
2. *(Pseudo Bids for $i \in [N]$): The dual variables ρ_i, μ_j satisfy the equalities $\mu_j = \max_{i\in\mathcal{B}_j} v_{ij}\rho_i$, and $\rho_i = \min_{j\in\mathcal{A}_i} \left(\frac{\mu_j}{v_{ij}}\right)$.*
3. *(Useful Items): The dual slack variables induce a collection $\mathcal{A}_i^\star = \{j \in \mathcal{A}_i \mid \theta_{ij} = 0\}$ which is such that $j \notin \mathcal{A}_i^\star \implies R_{ij} = 0$. As well, the contrapositive: $R_{ij} > 0 \implies j \in \mathcal{A}_i^\star$*
4. *(Fulfilled Contracts): The dual slack variables induce a collection $\mathcal{B}_j^\star = \{i \in \mathcal{B}_j \mid \theta_{ij} = 0\}$ which is such that $i \notin \mathcal{B}_j^\star \implies R_{ij} = 0$. As well, the contrapositive: $R_{ij} > 0 \implies i \in \mathcal{B}_j^\star$.*

Proof. Item 1 follows by inspecting the derivation of $\mathcal{L}_s$ in Equation (3.1), the Lagrangian after minimizing over s. The optimum value of s_j is obtained by minimizing $s \mapsto \lambda_j\Lambda_j(s/\lambda_j) - \mu_j s$, which occurs at $\Lambda_j'(s/\lambda_j) = \mu_j$ (see Lemma 2.1.1 for differentiability). Making the substitution $s = \lambda_j W_j(x)$ for the bid required to obtain supply s, we obtain the optimum bid through $\Lambda_j' \circ W_j(x) = \mu_j$ which results in $x_j = g_j^{-1}(\mu_j)$ (see Lemma 2.1.4), and this is nothing but $x_j = \mu_j$ in the second price case. If $\mu_j = 0$ these calculations remain consistent as $s \mapsto \lambda_j \Lambda_j(s/\lambda_j)$ is minimized at $s = 0$ (by monotonicity, see Chapter 2). Similarly if $\bar{x} < \infty$ and $s = \bar{x}$.

Item 2 follows by the monotonicity in the objective of Problem (D). Indeed, since $-\Lambda_j^\star(\mu_j)$ is monotone decreasing in μ_j, and the goal of the program is to *maximize* the objective, is is optimal that μ_j be as small as possible while still satisfying the constraints $\forall j \in [M], i \in \mathcal{B}_j :\ v_{ij}\rho_i \leq \mu_j$, which is exactly $\mu_j = \max\limits_{i \in \mathcal{B}_j} v_{ij}\rho_i$. Similar reasoning applies for ρ_i since $C_i > 0$, resulting in the requirement that ρ_i be as large as possible while still satisfying the constraints $\forall i \in [N], j \in \mathcal{A}_i :\ \rho_i \leq \frac{\mu_j}{v_{ij}}$ which is exactly $\rho_i = \min\limits_{j \in \mathcal{A}_i} \frac{\mu_j}{v_{ij}}$.

The conclusions of Items 3 and 4 follow from complementary slackness. Recall the dual variable $\theta_{ij} = \mu_j - v_{ij}\rho_i$ from Equation 3.1, which was eliminated before the statement of Problem (D). We have $\theta_{ij} > 0 \iff \mu_j > v_{ij}\rho_i \iff \rho_i < \frac{\mu_j}{v_{ij}}$ (*i.e.*, if i or j do not attain the respective maxima or minima from Item 2). The variable θ_{ij} is associated with the constraint $R_{ij} \geq 0$ and hence by complementary slackness $\theta_{ij}R_{ij} = 0$ so that $\theta_{ij} > 0 \implies R_{ij} = 0$. □

Let us consider a number of corollaries to this result. We will focus upon the second price auction where statements are slightly simpler; the case of first price auctions are similar but require mapping μ_j through g_j^{-1} to obtain optimal bids. We will discuss interpretations of these corollaries of Proposition 3.1.3 further in Section 3.1.3.

Corollary 3.1.1 (Induced Valuations)**.** *For each fixed $i \in [N]$, for any item type $j \in \mathcal{A}_i^\star$ the optimal bid is a positive multiple of the pseudo-bid ρ_i: $\mu_j = v_{ij}\rho_i$.*

If item valuations are simpler, then there is just a single optimum bid across each of the sets $\mathcal{A}_i^\star$.

Corollary 3.1.2 (Uniform Bids)**.** *Suppose that $v_{ij} \in \{0, 1\}$. Then, the optimal bids placed in a second price auction across all items of types $j \in \mathcal{A}_i^\star$ are equal and given by $x_j = \rho_i$.*

Proof. The dual variables μ_j are equal to the optimal bids for items of type j: $x_j = \mu_j$. Then, by the definition of $\mathcal{A}_i^\star$ we must have $\theta_{ij} = \mu_j - \rho_i = 0$. □

In the particularly simple case where items do not have differing valuations, and any item can be used to fulfill any contract, then we can recognize that there is just a single optimal pseudo bid characterizing all of the bids, as well as that, for second price auctions, there is just one single optimal bid.

Corollary 3.1.3 (Uniform Bid)**.** *If $v_{ij} = 1$ for each i, j, then there exists a single pseudo bid $\rho^\star$ such that the optimal bids satisfy $x_j = g_j^{-1}(\rho^\star)$ across all $j \in [M]$.*

Proof. We must have $\rho_i = \min_{j\in[M]}\mu_j$, and since the r.h.s. does not depend on i, there must be some $\rho^\star$ such that $\rho_i = \rho^\star$ for each i. □

Remark 3.1.2. In the special case where Corollary (3.1.3) holds, the problem again reduces into a monotone function root-finding problem for $\rho \mapsto \sum_{j=1}^{M} \lambda_j W_j \circ g_j^{-1}(\rho) - \sum_{i=1}^{N} C_i$. This is analogous to the example problem with a single contract in Section 2.3.2.

3.1.3 Interpretations

Induced Valuations and Supply and Demand In many game theoretic analyses of auction markets, a *valuation* over items is assumed to exist for market participants, and this valuation is essential to the analysis. While our problem formulation contains item valuations v_{ij}, these are not valuations in the same game theoretic sense. Indeed, the intermediary does not care at all about the items themselves, only about fulfilling their contractual obligations. However, in second price auctions with independent valuations (*i.e.,* each agent has their own valuation which is independent of all other agents' valuations) it is a dominant strategy simply to *bid your valuation* [103]. Thus, Corollary 3.1.1 can be understood as a way in which item valuations are *induced* by the contractual requirements. For example, if we have $v_{ij} \in [0, 1]$ and we interpret these quantities as conversion probabilities (*e.g.,* sales *etc.*) then the equation $\mu_j = v_{ij}\rho_i$, with μ_j being the optimal bid for items of type j, indicates that ρ_i is the value of a single conversion of type j for contract i.

Moreover, if we recall from the Lagrangian that the dual variables ρ_i are associated to the constraint $\sum_{j=1}^{M} \lambda_j R_{ij} v_{ij} = C_i$, the *shadow price interpretation* tells us that ρ_i is equal to the marginal cost to obtaining additional item values for contract i.

Similarly, the conclusion 1 drawn from Proposition 3.1.3, that the dual multiplier μ_j associated with the constraint $\sum_{i=1}^{N} R_{ij} = s_j$ is exactly the optimal bid $\mu_j = x_j$ can be understood, as in the proof, by examining the analysis of the Lagrangian function. In particular, there arises the minimization over s_j of the function $s_j \mapsto \lambda_j \Lambda_j(s_j/\lambda_j) - \mu_j s_j$, the solution of which occurs when $\Lambda_j'(s_j/\lambda_j) = \mu_j$ and since $\Lambda_j' = W_j^{-1}$ we must have $s_j = \lambda_j W_j(\mu_j)$. Indeed, these relations indicate that μ_j is exactly equal to the marginal cost of obtaining items of type j at the acquisition rate s_j. And, in the case of a second price auction, this is what one must bid in order to win the item against the marginal competing bidder.

To summarize some of this discussion, one may keep in mind that, for second price auctions and with $v_{ij} \in [0, 1]$ being conversion probabilities, we can interpret μ_j as being the value to the intermediary of obtaining an item of type j, and ρ_i as the value to contract i of obtaining a conversion. These valuations are *induced* through an optimal balance between the supply available (as characterized by λ_j, W_j) and the supply demanded (as characterized through v_{ij}, C_i).

Graph Partitioning Continuing the discussion to look at conclusion 3 and 4, these establish existence of subsets $\mathcal{A}_i^\star \subseteq \mathcal{A}_i$ and $\mathcal{B}_j^\star \subseteq \mathcal{B}_j$ constituting, respectively, the items which are *actually used* for the fulfillment of contract i, and the contracts towards which items of type j are *actually sent*. These sets are a result of complementary slackness applied to the quantity $\theta_{ij} = \mu_j - v_{ij}\rho_i$. If $\theta_{ij} > 0$ it indicates that the cost for contract i to bid on items of type j is greater (by the amount θ_{ij}) than the minimum cost at which value can be acquired for that contract, and hence type j should not be used to fulfill contract i: $R_{ij} = 0$.

These sets $\mathcal{A}_i^\star, \mathcal{B}_j^\star$ (more directly, the slack variables θ_{ij}) induce partitions of the graph $\mathcal{G} = \{(i, j) \mid i \in [N], j \in \mathcal{A}_j\}$ as in $\mathcal{G}_i^\star \triangleq \{(i, j) \mid j \in \mathcal{A}_i^\star\}$ being the subgraph of $\mathcal{G}$ to which contract i belongs. Within this subgraph, bids are determined entirely by the scalar ρ_i and the values $\big(v_{ij}\big)_{j \in \mathcal{A}_i^\star}$. In fact, due to Corollary 3.1.2 if $v_{ij} \in \{0, 1\}$ the bids placed on items in this subgraph are all equal to ρ_i. This graph partitioning was also recognized by [170], but not as a consequence of convex duality.

Finally, in Corollary 3.1.3, we recognize that if there is no differentiation between item valuations, and any item can be used to fulfill any contract, *i.e.,* $v_{ij} = 1$, then there is no reason to place different bids for different items. The existence of such a $\rho^\star$ holds in general only in this very special case. Indeed, if $v_{ij} \in \{0, 1\}$, the availability to contract i of item types with lower costs can induce differing pseudo bids across contracts. However, we have observed in some computational examples that, particularly when there is little margin in Assumption 3.0.1 (*i.e.,* there is just barely enough supply available to fulfill the contracts), the optimal solution may still collapse to the case $\forall i \in [N]\ \rho_i = \rho^\star$ (see Section 3.3.1).

Remark 3.1.3 (Sets $\mathcal{A}_i^\star, \mathcal{B}_j^\star$ in Practice)**.** When obtaining solutions to Problem (D) numerically, it may be the case only that $\theta_{ij} \approx 0$, up to the tolerance parameters set for the algorithm. In this case, and considering the interpretation provided in the previous paragraph, it may be reasonable to fix some parameter $\epsilon > 0$ and treat the slack variables θ_{ij} as if they are equal to zero whenever they satisfy $\theta_{ij} \leq \epsilon$. Then, the intermediary would be overpaying by at most ϵ.

The Objective Function Considering the objective function $\sum_{i=1}^{N} \rho_i C_i - \sum_{j=1}^{M} \lambda_j \Lambda_j^\star(\mu_j)$ and suppose we are bidding in a second price auction. With the interpretations of ρ_i, μ_j in mind, we may think of the terms $\rho_i C_i$ as measuring what we would need to pay to fulfill the contracts *had the auction actually been a first price auction* (all else being equal). As well, it was observed in Chapter 2 (through integration by parts) that $\lambda_j \Lambda_j^\star(\mu_j) = \lambda_j \int_0^{\mu_j} W_j(x) \mathsf{d}x$ is exactly equal to

$$\sum_{j=1}^{M} \lambda_j \big(\mu_j W_j(\mu_j) - \int_0^{\mu_j} x \mathsf{d}W_j(x)\big) = \sum_{j=1}^{M} \lambda_j \big(f_j^{1st}(\mu_j) - f_j^{2nd}(\mu_j)\big),$$

where $f_j^{1st}(x) = xW_j(x)$ is the cost function associated with a first price auction and $f_j^{2nd}(x) = \int_0^x u \mathsf{d}W(u)$ is the cost function associated to the second price auction. Thus, given the interpretation of $\rho_i C_i$ (and this can be shown through elementary calculation), the cost is equal to $\sum_{j=1}^{M} \lambda_j \int_0^{\mu_j} x \mathsf{d}W_j(x)x$, which is exactly the cost of bidding μ_j on items of type j in a second price auction. Given that strong duality holds, this is the result we should expect. Similarly, we see that $\Lambda_j^\star(x) = xW_j(x) - \int_0^{\mu} u \mathsf{d}W(u)$ measures the expected margin by which one wins items of type j, indeed, $\Lambda_j^\star(x) = \int_0^x (x-u)W'(u)\mathsf{d}u = \mathbb{E}(x-p)_+$ where $p \sim W_j$. So the objective function in the dual (D) consists of the contract fulfillment costs in a first price auction less the savings obtained in the second price auction.

Remark 3.1.4 (Revenue Equivalence)**.** One of the cornerstones of auction theory is *revenue equivalence* (*e.g.,* [103, Ch. 3]). Roughly, the revenue of a seller is equal regardless of whether the auction mechanism is first price or second price, and this is a result of how agents are expected to modify their behaviour based on the auction mechanism. Hence, the comparisons made in this section cannot be interpreted to actually quantify the differences in payments that would be made were the auction mechanism to change, but rather are intended only to provide a more thorough intuitive understanding of the quantities involved.

3.2 Computational Methods

In this section we derive algorithms applicable to the solution of the main problem, Problem (P), through the solution of the dual, Problem (D). We begin with the dual problem itself in Section 3.2.2, and then methods by which we can calculate an allocation array R_{ij} from specified acquisition rates $s \in \mathbb{R}^M$ in Section 3.2.3. However, we begin this section with some historical remarks, and remarks upon our motivation and the value of these contributions.

3.2.1 Some Remarks on History and Algorithms

Algorithms for convex optimization have revolutionized engineering applications. A reasonable starting point in a brief account of the history would be the discovery of the simplex method for linear programming by George Dantzig in 1947 [50]. These methods can take one quite far, since many general convex programs can be approximated by linear programming through polyhedral approximation (*c.f.,* Section 2.4.3). Surprisingly, it was not until the interior point revolution of the 1980s [201] that linear and (convex) nonlinear programming came to be seen as part of the same subject: convex optimization. In the decades since this time, enormous advances have been made to the point that solving convex optimization problems in practice is often thought of as being "easy".

Indeed, convex modelling software like `cvxpy` [54] commodifies many aspects of solving convex optimization problems. We have used this tool to solve various example problems when supply curves are known in closed form (this is one reason we are so fond of the example $W_j(x) = 1 - e^{-\gamma_j x}$), and for solving the linear programming approximation of Section 2.4.3. However, these tools are not silver bullets. `cvxpy` for instance, is, in its current form, largely limited to solving problems which fall into some predefined classes (*i.e.,* certain types of cone programs [135], and various subclasses including semidefinite programs, quadratic programs, linear programs, *etc.* [4]).

An alternative tool, `cvxopt` [178], allows for the specification of problems simply in terms of first and second order derivative information. We have also used this tool for the practical solution of Problem (P) with infinitely differentiable curves estimated from kernel smoothing (Section 2.4.2). But, `cvxopt` is thus limited by the need for differentiability, which, since it is not necessary for the theoretical analysis of Problem (P) (except for first derivatives in the case of first price auctions), is preferable to avoid, even if for nothing more than aesthetics.

Moreover, we have come to discover that there is often a remarkable amount of structure inherent to particular optimization problems which is brought to light through the application of various different generic algorithms to optimization problems, as well as to different faces of the same problems (*e.g.,* the primal and the dual, whether certain equalities are eliminated, representation of constraints explicitly or through the domain of the objective, *etc.*).

Particularly for the problem studied in this thesis, where the functions W_j are allowed to be almost arbitrary continuous distribution functions (*c.f.,* Definition 2.1.1) we have come to recognize that the derivation of specialized algorithms is essential and forms an important part of this work's contributions. We turn towards the derivation of such an algorithm in Section 3.2.2.

3.2.2 Primal Dual Solutions

There are many possible algorithms that can be applied to the solution of Problem (D). In this section, we apply the Alternating Direction Method of Multipliers (ADMM) [21, Sec. 5.4] [28] which will be seen to lead to elegant subproblems which make essential use of the structure of the problem (see also Appendix B.3 for an overview of ADMM). In particular, it will be seen that we can split the problem into a sequences of simple and trivially parallel iterations. Moreover, the algorithm will only require access to the supply curve W, (and the bid mapping function g in the first price case), without any of its derivatives, and without the need to evaluate the cost functions $f, \Lambda, \Lambda^\star$ *etc.*This is a great advantage when W does not have an analytic representation, and hence the calculation of these functions in and of themselves may be non-trivial.

In order to derive an ADMM algorithm for this problem, we reintroduce the additional slack variables θ_{ij} for $i \in [N], j \in \mathcal{A}_i$:

$$\begin{aligned} \underset{\rho,\mu,\theta}{\text{minimize}} \quad & \sum_{j=1}^{M} \lambda_j \Lambda_j^\star(\mu_j) - \sum_{i=1}^{N} \rho_i C_i \\ \text{subject to} \quad & \mu_j - v_{ij}\rho_i = \theta_{ij} \; \forall i \in [N] \; j \in \mathcal{A}_i \\ & \theta_{ij} \geq 0, \end{aligned}$$

which implements the inequality constraint $v_{ij}\rho_i \leq \mu_j$ as an equality. Then, application of ADMM to this problem proceeds through the following iterations (these iterations are trivial to derive from standard statements of the algorithm, see Appendix B.3), for some parameter $\nu > 0$:

$$\begin{aligned} \rho(t+1) &= \text{argmin}_\rho \; \sum_{i=1}^{N} \Big[-C_i\rho_i + \frac{1}{2\nu} \sum_{j \in \mathcal{A}_i} \big(\mu_j(t) - v_{ij}\rho_i - \theta_{ij}(t) + \Gamma_{ij}(t)\big)^2 \Big] \\ \mu(t+1) &= \text{argmin}_\mu \; \sum_{j=1}^{M} \Big[\lambda_j \Lambda_j^\star(\mu_j) + \frac{1}{2\nu} \sum_{i \in \mathcal{B}_j} \big(\mu_j - v_{ij}\rho_i(t+1) - \theta_{ij}(t) + \Gamma_{ij}(t)\big)^2 \Big] \\ \theta(t+1) &= \text{argmin}_\theta \; \sum_{i=1}^{N} \sum_{j \in \mathcal{A}_i} \Big[\mathbf{1}_{\mathbb{R}_+}(\theta_{ij}) + \frac{1}{2\nu} \big(\mu_j(t+1) - v_{ij}\rho_i(t+1) - \theta_{ij} + \Gamma_{ij}(t)\big)^2 \Big] \\ \Gamma(t+1) &= \Gamma(t) + \mu_j(t+1) - v_{ij}\rho_i(t+1) - \theta_{ij}(t+1), \end{aligned}$$

where the quadratic terms serve as penalties on the linear equality constraint $\mu_j - v_{ij}\rho_i = \theta_{ij}$. We have written the summations in such a way as to make clear that each of these subproblems are *separable* – *i.e.,* the problems can be split into parallel optimization problems involving only a single variable at a time. We solve these in turn.

The ρ Update

Consider the quadratic form $-C_i\rho_i + \frac{1}{2\nu}\sum_{j\in\mathcal{A}_i}\big(\mu_j - v_{ij}\rho_i - \theta_{ij} + \Gamma_{ij}\big)^2$. As long as $v_{ij} > 0$ for at least one j (which is the case but for pathological problems, excluded by Assumption 3.0.1, where contract i cannot be satisfied by any item j) then this function has a unique minimizer in ρ_i. It can be found by simple differentiation:

$$\frac{\partial}{\partial\rho_i}\Big[-C_i\rho_i + \frac{1}{2\nu}\sum_{j\in\mathcal{A}_i}\big(\mu_j - v_{ij}\rho_i - w_{ij}\big)^2\Big] = -C_i - \frac{1}{\nu}\sum_{j\in\mathcal{A}_i} v_{ij}(\mu_j - w_{ij}) + \frac{1}{\nu}\rho_i\sum_{j\in\mathcal{A}_i} v_{ij}^2,$$

where $w_{ij} = \theta_{ij} - \Gamma_{ij}$. The zero of this derivative is obtained as

$$\rho_i = \frac{\nu C_i + \sum_{j\in\mathcal{A}_i} v_{ij}(\mu_j - \theta_{ij} + \Gamma_{ij})}{\sum_{j\in\mathcal{A}_i} v_{ij}^2},$$

which is a closed form expression for the minimizer.

The μ Update

Similarly, we can characterize the unique minimizer of the function

$$\mu_j \mapsto \lambda_j\Lambda_j^\star(\mu_j) + \frac{1}{2\nu}\sum_{i\in\mathcal{B}_j}\big(\mu_j - v_{ij}\rho_i - \theta_{ij} + \Gamma_{ij}\big)^2$$

by the roots of the derivative. Using g_j to denote the bid mapping function (Section 2.1.3) which is simply $g_j(x) = x$ in the second price case, we can write this root finding problem as (see Lemma 2.1.1)

$$\begin{aligned}
&\lambda_j W_j \circ g_j^{-1}(\mu_j) + \frac{1}{\nu}\sum_{i\in\mathcal{B}_j}\big(\mu_j - v_{ij}\rho_i - \theta_{ij} + \Gamma_{ij}\big) = 0 \\
&\qquad\qquad\iff \lambda_j W_j \circ g_j^{-1}(\mu_j) + \frac{1}{\nu}|\mathcal{B}_j|\mu_j = \frac{1}{\nu}\sum_{i\in\mathcal{B}_j}(v_{ij}\rho_i + \theta_{ij} - \Gamma_{ij}).
\end{aligned}$$

In the second price case, we simply have the function $\mu \mapsto \lambda_j W_j(\mu) + \frac{1}{\nu}|\mathcal{B}_j|\mu$ on the left hand side, which is monotone increasing and only requires the evaluation of the supply curve W_j (rather than, *e.g.,* $\Lambda_j, \Lambda_j^\star$, *etc.*), so this root finding problem can be solved via the method of bisection, see Section 2.4.1.

In the first price case, where g_j is not simply the identity function, this equation is monotone under the 2-concavity of W (Proposition 2.1.4). However, it's solution now requires *nested* bisection, since g_j^{-1} is itself be merely *evaluated* via bisection (unless some pre-processing step is taken to exactly represent g_j^{-1} itself). This makes the computational complexity of obtaining an ϵ accurate solution for μ_j $\mathcal{O}\big((\ln 1/\epsilon)^2\big)$ in the first price case, but only $\mathcal{O}\big(\ln 1/\epsilon\big)$ in the second price case.

The θ Update

The update to θ is again completely separable – we need to solve

$$\min_{\theta_{ij}} \left[\mathbf{1}_{\mathbb{R}_+}(\theta_{ij}) + \frac{1}{2\nu}\big(\mu_j - v_{ij}\rho_i - \theta_{ij} + \Gamma_{ij}\big)^2\right],$$

but this is nothing but a projection onto $\mathbb{R}_+$. That is, the solution to this problem is given simply by the positive part:

$$\theta_{ij} = \big(\mu_j - v_{ij}\rho_i + \Gamma_{ij}\big)_+.$$

The Final Algorithm

Putting these steps together into the ADMM iterations, we have the result in Algorithm 2:

Given arbitrary initialization, and under the adequate supply assumption 3.0.1, these iterations converge to an optimal dual solution [21, Prop 5.4.1] [28, Sec 3.2.1]. Implementation details and convergence criteria are also obtained in these references. See also [133].

Proposition 3.2.1 (Convergence)**.** *Let W be a supply curve (or a differentiable, strictly 2-concave supply curve in the first price case) and suppose that Assumption 3.0.1 holds. Then, the iterates $\mu_j(t), \rho_i(t)$ of Algorithm 2 converge to an optimal solution of Problem* (D) *as $t \to \infty$. Moreover, the scaled iterates $\nu\Gamma_{ij}(t)$ converge to an optimal solution R_{ij} of Problem* (P_R) *as $t \to \infty$,* i.e., *an optimal allocation array.*

Algorithm 2: ADMM Dual Algorithm

input : Sets $\mathcal{A}_i, \mathcal{B}_j$, supply curves W_j, targets C_i, rates λ_j, values v, and parameter ν.

output : A solution μ, ρ of Problem (D).

repeat

  for $i \in [N]$ **do**

    $\rho_i(t+1) = \frac{\nu C_i + \sum_{j \in \mathcal{A}_i} v_{ij}(\mu_j(t) - \theta_{ij}(t) + \Gamma_{ij}(t))}{\sum_{j \in \mathcal{A}_i} v_{ij}^2}$

  for $j \in [M]$ **do**

    $\mu_j(t+1) = \underset{\mu}{\texttt{bisection}}\Big\{ W_j \circ g_j^{-1}(\mu) + \frac{1}{\nu\lambda_j}|\mathcal{B}_j|\mu = \frac{1}{\nu\lambda_j}\sum_{i \in \mathcal{B}_j}(v_{ij}\rho_i + \theta_{ij} - \Gamma_{ij}) \Big\}$

  for $(i,j) : i \in [N], j \in \mathcal{A}_i$ **do**

    $\theta_{ij}(t+1) = \big(\mu_j(t+1) - v_{ij}\rho_i(t+1) + \Gamma_{ij}(t)\big)_+$

  for $(i,j) : i \in [N], j \in \mathcal{A}_i$ **do**

    $\Gamma_{ij}(t+1) = \Gamma_{ij}(t) + \mu_j(t+1) - v_{ij}\rho_i(t+1) - \theta_{ij}(t+1)$

until *convergence*

return (μ, ρ)

Proof. First, we show that Problem (P_R) is a dual of Problem (D) and relate the dual variables of Problem (D) back to optimal allocation arrays (*i.e.*, simply check that the double dual recovers the primal). We treat the problem as a minimization problem by negating the objective and we have the Lagrangian with dual variables Γ and dual constraint $\Gamma \geq 0$:

$$\begin{aligned} \mathcal{L}(\rho, \mu, \Gamma) &= -\sum_{i=1}^{N} \rho_i C_i + \sum_{j=1}^{M} \lambda_j \Lambda_j^\star(\mu_j) + \sum_{j=1}^{M} \sum_{i \in \mathcal{B}_j} \Gamma_{ij}(\rho_i v_{ij} - \mu_j) \\ &= \sum_{j=1}^{M} \lambda_j \big[\Lambda_j^\star(\mu_j) - \frac{\mu_j}{\lambda_j} \sum_{i \in \mathcal{B}_j} \Gamma_{ij}\big] + \sum_{i=1}^{N} \rho_i \big(\sum_{j \in \mathcal{A}_i} \Gamma_{ij} v_{ij} - C_i\big). \end{aligned}$$

Minimizing this over μ_j results in the function

$$\mathcal{L}(\rho, \mu^\star, R) = -\sum_{j=1}^{M} \lambda_j \Lambda_j \big(\frac{1}{\lambda_j} \sum_{i \in \mathcal{B}_j} \Gamma_{ij}\big) + \sum_{i=1}^{N} \rho_i \big(\sum_{j \in \mathcal{A}_i} \Gamma_{ij} v_{ij} - C_i\big),$$

since $\left(\Lambda_j^\star\right)^\star = \Lambda_j$ (Fenchel-Moreau Theorem). Minimizing this function over ρ induces the constraint $\sum_{j\in\mathcal{A}_i} \Gamma_{ij} v_{ij} = C_i$ and thus we have recovered Problem (P_R) with the variables $\Gamma = R$.

We apply the convergence theorem of [28, Sec. 3], which tells us that, under strong duality and the existence of primal and dual solutions: (1) the iterates are asymptotically feasible, (2) that the objective function value converges to the optimal value and (3) that $\nu\Gamma(t)$ converges to an optimal dual solution.

The objective functions $\rho \mapsto C^\mathsf{T}\rho$ and $\mu \mapsto \sum_{j=1}^M \lambda_j \Lambda_j^\star(\mu_j)$ are both proper, convex, and lower-semicontinuous, and by Assumption 3.0.1 combined with Proposition 3.1.2 there exist primal and dual optimal solutions. Therefore, the iterates $\mu_j(t), \rho_i(t), \theta_{ij}(t)$ are asymptotically feasible (*i.e.,* satisfy $\mu_j(t) - v_{ij}\rho_i(t) = \theta_{ij}(t)$) and thus, asymptotically, $v_{ij}\rho_i \le \mu_j(t)$. As well, the objective value $\sum_{j=1}^M \lambda_j \Lambda_j^\star(\mu_j(t)) - C^\mathsf{T}\rho(t)$ converges to the optimal value of the problem, and, since this function is strictly convex, (Proposition 2.1.5) the iterates themselves converge to optimal solutions. Finally, the scaled iterates $\nu\Gamma_{ij}(t)$ converge to an optimal dual solution of (D), which we have established is exactly Problem (P). □

Remark 3.2.1 (Non-Convex Problems)**.** The ADMM algorithm has convergence guarantees for convex programs, and since conjugate functions are always convex, the ADMM algorithm is generally convergent for Problem (D), regardless of the convexity of Λ. However, in the first price case, the functions g_j may not be strictly monotone, unless W is strictly 2-concave (which is the convexity condition for Λ^{1st}, Proposition 2.1.4), and hence the calculation of the inverse g_j^{-1} in the μ step would not be well defined. As a *heuristic*, one could run this same algorithm and simply pick some arbitrary element in the set of inverses (See [28, Sec. 9] and [125] for examples of ADMM applied to non-convex problems). However, the optimal inverse of g to pick is through the values which attain the maximizer in the definition of $\Lambda^\star$. Since our functions are only in $\mathbb{R}$, it may be possible in practice to find these minimizers, *e.g.,* by discretizing an interval and applying a golden-section search within each sub-interval, particularly if some smoothness (*i.e.,* Lipschitz continuity) assumptions can be made about W^{-1}. Hence, even non-convex versions of Problem (P) may be tractable through a modification of Algorithm (2).

Remark 3.2.2 (Primal Algorithms)**.** Since the μ iterations of ADMM involve root finding problems involving W, one may suspect that some primal algorithm, *e.g.,* applying the Augmented Lagrangian algorithm (ALM) directly to the primal (P), would result in root-finding problems involving W^{-1}. This intuition is correct. As well, it is suggestive of the possibility of skipping the bisection iterations to find roots of functions like $\mu \mapsto W(\mu) + a\mu - b$, since the inverse of W^{-1} is simply W, which can easily be evaluated. Unfortunately, the iterations of the ALM requires finding roots of $q \mapsto W^{-1}(q) + aq - b$, rather than of $W^{-1} - b$. Thus, while a nested bisection step can be avoided by recognizing that $W^{-1}(q) + aq - b \leq 0 \iff x \leq W(ax - b)$, bisection iterations are not avoided all together.

3.2.3 Calculating Optimal Allocations

Sometimes (see Chapter 4) we may have access to an optimal dual solution ρ, μ of Problem (D), without also having been able to calculate an allocation matrix R. The optimal bids x_j derived from this dual solution are enough to calculate $s_j = \lambda_j W_j(x_j)$, the rate at which items of type j need to be obtained, but not to directly calculate R_{ij}, the rate at which they are to be allocated from type j towards contract i. That is, we still need to find an adequate *allocation* from items to contracts. This essentially requires finding an array R satisfying the summation constraints $\sum_{i \in \mathcal{B}_j} R_{ij} = s_j$, $\sum_{j \in \mathcal{A}_i} R_{ij} v_{ij} = C_i$, as well as the sparsity constraints induced by the sets $\mathcal{A}_i^\star, \mathcal{B}_j^\star$. That is, $(j \notin \mathcal{A}_i^\star \iff i \notin \mathcal{B}_j^\star) \implies R_{ij} = 0$.

Remark 3.2.3 (Sparsity from s)**.** Suppose that a purported optimal allocation vector $s \in \mathbb{R}^M$ is given. We can calculate the sparse sets $\mathcal{A}_i^\star, \mathcal{B}_j^\star$ as follows. First, calculate the bids $\mu_j = W_j^{-1}(s_j)$ (Item 1 of Proposition 3.1.3) and then pseudo-bids $\rho_i = \min_{j \in \mathcal{A}_i} \left(\frac{\mu_j}{v_{ij}}\right)$ (by Item 2). Then, the slack variables $\theta_{ij} = \mu_j - v_{ij}\rho_i$ are available and induce the sparse sets according to Item's 3 and 4. These sets should be calculated up to some tolerance $\epsilon > 0$ *c.f.,* Remark 3.1.3.

These summations are closely analogous to the column and row sums of a matrix $R \in \mathbb{R}^{N \times M}$. We first consider a simple example where $v_{ij} = 1$ for every i, j, so that the uniform bid principle Corollary 3.1.3 holds. In this case, we need to simply find a fully dense and non-negative R matrix[1].

Example 3.2.1. Suppose that $v_{ij} = 1$ for each i, j. Then, by Corollary 3.1.3, there is a single optimal bid $x^\star$ applied to each contract. Hence, the optimal acquisition rates are given by $s_j = \lambda_j W_j(x^\star)$ and it is necessary to find a non-negative matrix $R \in \mathbb{R}_+^{N \times M}$ such that the column sums satisfy $\sum_{i=1}^N R_{ij} = s_j$ and the row sums satisfy $\sum_{j=1}^M R_{ij} = C_i$. This is a very simple transportation problem, and a natural solution (although there are many others) is simply given by

$$R_{ij} = \frac{C_i s_j}{T},$$

[1]It is also possible to solve the integer-constrained version of this problem: https://leetcode.com/problems/find-valid-matrix-given-row-and-column-sums/

where $T = \sum_{i=1}^{N} C_i = \sum_{j=1}^{M} s_j$ is the total number of items needed. This matrix satisfies $R_{ij} \geq 0$ and has the needed row and column sums. It is therefore an appropriate allocation rate for Problem (P).

In the case of general valuations $v_{ij} \geq 0$, the sparsity constraints induced by the dual solution must be respected, and there is no longer a closed form solution to the problem. Moreover, solutions may not be unique; or, if acquisition rates s are not properly specified, there may not exist a solution at all. In order to deal with these possibilities, we consider the regularized problem

$$\begin{aligned} \underset{R}{\text{minimize}} \quad & \sum_{i=1}^{N} \big(C_i - \sum_{j \in \mathcal{A}_i^\star} R_{ij} v_{ij}\big)_+ + \frac{1}{2} t ||R||_F^2 \\ \text{subject to} \quad & \sum_{i \in \mathcal{B}_j^\star} R_{ij} = s_j \\ & R_{ij} \geq 0, \end{aligned} \tag{T_t}$$

where $||R||_F^2 = \sum_{i=1}^{N} \sum_{j \in \mathcal{A}_i} R_{ij}^2$ is the Frobenius norm of the matrix.

We require that constraint $\sum_{i \in \mathcal{B}_j^\star} R_{ij} = s_j$ be satisfied exactly since s_j is the rate of type j supply that is acquired, and exactly that much should be allocated. The objective term $\sum_{i=1}^{N} \big(C_i - \sum_{j \in \mathcal{A}_i^\star} R_{ij} v_{ij}\big)_+$ penalizes allocations which fail to satisfy contracts. Penalizing the $\sum_{j \in \mathcal{A}_i^\star} R_{ij} v_{ij} \geq C_i$ in the objective, rather than as a linear constraint, reflects the fact that if s_j is not properly or accurately specified then there need not exist a matrix satisfying this constraint.

When solutions *do* exist (e.g., if $s_j = \lambda_j$, c.f. Assumption 3.0.1), there may be *many* solutions. To this end, we will seek the *least norm* solution (again, having a direct analogy with least norm solutions of under-determined linear systems). Intuitively, this will tend to produce solutions which are uniform, *i.e.,* $(1/2, 1/2)$ has a lower euclidean norm than does $(1, 0)$, despite having the same sum. Precisely, we want to pick $R_{ln} \in \mathcal{R}_{OT} = \{R \mid R \text{ solves } (T_0)\}$ such that $\forall R \in \mathcal{R}_{OT}$ $||R_{ln}||_F \leq ||R||_F$. In fact, under certain regularity conditions (See [64]) there exists some $t > 0$ such that the solution of Problem (T_t) is the least-norm solution of the same problem with $t = 0$, *i.e.,* the regularization is *exact*. Indeed, since all of the constraints of Problem (T_t) are linear and the objective can be re-written with linear constraints in epigraph form, the constraint regularity conditions of [64] are immediately satisfied. As a remark, it should be noted that this will *not* work if the objective penalizes the squared terms $(C_i - \sum_j v_{ij} R_{ij})_+^2$ *etc.*instead. In fact, we can determine a sufficient condition on t, establishing that contracts will never be over-provisioned if it is possible to increase the fulfillment of another contract, and this value of t ensures the regularization is exact.

Proposition 3.2.2 (Exact Least Norm Regularization)**.** *Let* $\lambda_{max} = \max_{j \in [M]} \lambda_j$*. If the value of* $t \geq 0$ *is such that*

$$t < \frac{1}{\lambda_{max}} \min_{i \in [N]} \min_{j \in \mathcal{A}_i} v_{ij},$$

then regularization of Problem (T_t) *is exact and the solution* $R^\star$ *of this problem is the least norm solution of Problem* (T_0)*.*

Proof. Let $t > 0$ and let R be the unique solution to Problem (T_t). We consider two cases. First, suppose that there does *not* exist a pair of contracts i, u such that $\sum_{j \in \mathcal{A}_i^\star} v_{ij} R_{ij} > C_i$ and $\sum_{w \in \mathcal{A}_u^\star} v_{uw} R_{uw} < C_u$ and $R_{ik} > 0$ for $k \in \mathcal{A}_i^\star \cap \mathcal{A}_u^\star$. Then, R must also be a solution to (T_0) since $\sum_{i=1}^N \big(C_i - \sum_{j \in \mathcal{A}_i^\star} v_{ij} R_{ij}\big)_+$ cannot be reduced. Denote the value of this sum by $p^\star$. Then, R is also feasible for

$$\begin{aligned} \underset{R}{\text{minimize}} \quad & \frac{1}{2} t ||R||_F^2 \\ \text{subject to} \quad & \sum_{i \in \mathcal{B}_j^\star} R_{ij} = s_j, R_{ij} \geq 0 \\ & \sum_{i=1}^N \big(C_i - \sum_{j \in \mathcal{A}_i^\star} R_{ij} v_{ij}\big)_+ \leq p^\star, \end{aligned} \tag{3.2}$$

and, in fact, it must necessarily be the *solution* to this program as well. If this were not the case, the cost for Problem (T_t) could have been reduced, hence R is the least norm solution of (T_0).

Now, suppose that there *does* exist a pair of contracts i, u such that $\sum_{j \in \mathcal{A}_i^\star} v_{ij} R_{ij} > C_i$ and $\sum_{w \in \mathcal{A}_u^\star} v_{uw} R_{uw} < C_u$ and $R_{ik} > 0$ for $k \in \mathcal{A}_i^\star \cap \mathcal{A}_u^\star$. Then, consider some parameter ζ, and let $\widetilde{R}(\zeta)$ be a new array equal to R except that $\widetilde{R}(\zeta)_{ik} = R_{ik} - \zeta$ and $\widetilde{R}(\zeta)_{uk} = R_{uk} + \zeta$. Since $R_{ik} > 0$ and $\sum_{j \in \mathcal{A}_i^\star} v_{ij} R_{ij} > C_i$ there exists some ζ small enough, and an open neighbourhood $\mathcal{N}_\zeta$ thereof such that $\widetilde{R}(\phi)_{ik} \geq 0$, $\sum_{j \in \mathcal{A}_i^\star} v_{ij} \widetilde{R}_{ij}(\phi) > C_i$, and (by construction) $\sum_{i \in \mathcal{B}_j^\star} \widetilde{R}_{ij}(\phi) = s_j$, for any $\phi \in \mathcal{N}_\zeta$. Thus, $\widetilde{R}(\phi)$ is feasible for Problem (T_t) over the neighbourhood $\mathcal{N}_\zeta$. Finally, consider the derivative of the objective function at ζ:

$$\frac{\mathsf{d}}{\mathsf{d}\zeta}\Big[\sum_{i=1}^{N}\big(C_i - \sum_{j\in\mathcal{A}_i^\star} v_{ij}\widetilde{R}_{ij}(\zeta)\big)_+ + \frac{1}{2}t||\widetilde{R}(\zeta)||_F^2\Big] = -v_{uk} - t(R_{ik}-\zeta) + t(R_{uk}+\zeta)$$

$$= -v_{uk} + t[R_{uk} - R_{ik}] + 2t\zeta$$

$$\leq -\min_{i\in[N]}\min_{j\in\mathcal{A}_i} v_{ij} + t\lambda_{\max} + 2t\zeta,$$

where the last inequality follows since $R_{uk} - R_{ik} \leq s_k$ and $s_k \leq \lambda_k \leq \lambda_{\max}$.

This derivative is therefore negative for small ζ if $t < \frac{1}{\lambda_{\max}} \min_{i\in[N]}\min_{j\in\mathcal{A}_i} v_{ij}$. Therefore, there exists a feasible descent direction for Problem (T_t) and the supposed R was not optimal. Therefore, such a pair of contracts i, u does not exist. □

An exact regularization parameter which applies uniformly over any s can be easily computed according to Proposition 3.2.2 (as well as updated in practice if λ_j or v_{ij} change) since the parameter there specified does not depend on s.

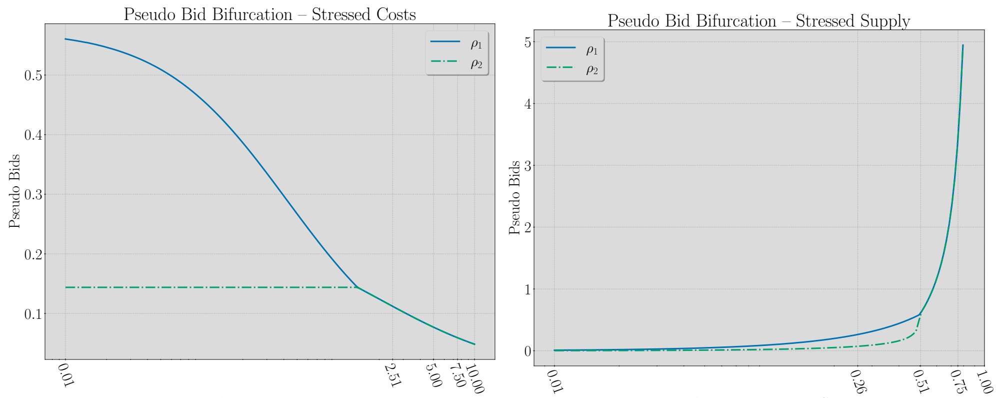
3.3 Examples

In this section we examine some computational simulation examples. The primary purpose of this section is to further develop an intuitive and qualitative understand of the contract management problem. Secondarily, the example of Section 3.3.2 demonstrates the scalability of algorithms for the dual problem[2], as well as how solutions of the dual induces sparsity in the transportation problem for the calculation of R, which further enables computational scalability.

3.3.1 Bidding Bifurcations

Even though Corollary 3.1.3 holds only in very special circumstances, it turns out that there is still just a single optimal bid applying across all item types in some practical scenarios as well. Typically this arises when the available rate of items is only just adequate the fulfill contracts (*i.e.,* there is little margin available in Assumption 3.0.1), but we also observe that it can occur when contracts have shared access to a cheap source of items. These cases are exemplified in Figure 3.1.

[2]An example with over 1000 item types is comfortably solved in under a minute on a personal laptop.



(a) Stressing Cost Differences (b) Stressing Available Supply

Figure 3.1: Bifurcation Examples of Pseudo Bids

Examples of how optimal bids can naturally differ between contracts and items, as well as examples of the cases where the Uniform Bid Principle (Corollary 3.1.3) tends to hold.

To construct Figure 3.1 we have considered a simple case when $M = 3$ item types and $N = 2$ contracts. In both subfigures we have $v_{ij} \in \{0, 1\}$ with $\mathcal{A}_1 = \{1, 2\}$, and $\mathcal{A}_2 = \{2, 3\}$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$. As well, the supply curves are given by simple exponentials $W_j(x) = 1 - e^{-\gamma_j x}$ (see Example 2.1.2 and Figure 2.2). The average cost of these items is given by $1/\gamma_j$, and hence become cheaper as γ becomes larger.

In Figure 3.1a we have $\gamma_1 = 1/2, \gamma_3 = 2$ but $\gamma_2 = \gamma$ is a parameter. We see that, since the average price of an item is $1/\gamma$, as item type 2 gets cheaper, it is eventually the case that both contracts draw the majority of their items from type 2. Prior to this, the majority of items are drawn from the cheaper types 1 for contract 1 or 3 for contract 2. It should be pointed out that if $\gamma_1 = \gamma_3$, the contracts would still draw the majority of their items from types 1 and 3, respectively, for small values of γ_2, but their bids would still be identical as a coincidence of having access to different items that happen to have the same costs. *i.e.*, the reason that bids coincide in that case would be a simple pathology of particular numerical values, not a consistent structural feature.

In Figure 3.1b we have $\gamma_1 = 1/10$ (expensive), $\gamma_2 = 1$, and $\gamma_3 = 10$ (cheap). As well, we modify the contract item requirements $C_1 = C$ and $C_2 = 2C$. We see that, as C approaches 1, and the available supply is stressed, the bids are eventually equal (and large), even though there are large differences between the costs of items available to the contracts.

For an example on a larger scale, consider a case where $M = 30$, $N = 20$ and we

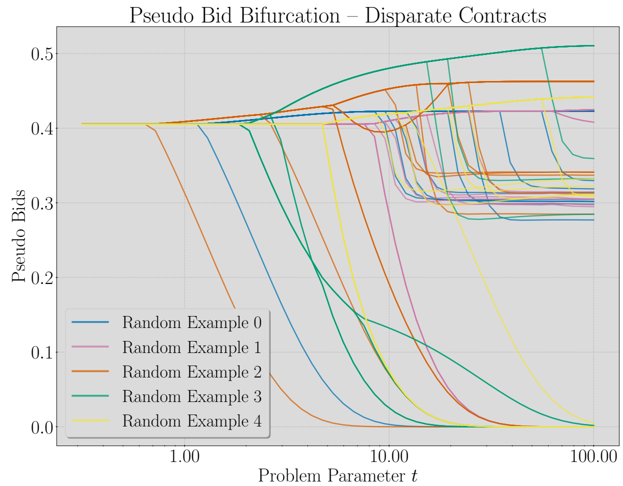


Figure 3.2: Randomized Medium Scale Bid Bifurcation Example
A larger scale example of bid bifurcations where contracts become more distinct as $t \to \infty$ but are all identical when $t = 0$. All item types are identical and variability in values $v_{ij} \sim \text{Ber}(1/4)$ is depicted through differing colours. Each $\rho_i(t)$ is given the same colour across randomized simulation runs.

have $\gamma_j = \lambda_j = 1$ for every j (*i.e.*, each of the item types are identical), and the values are random $v_{ij} \sim \text{Ber}(1/4)$. We again sweep through a parameter t which is such that contracts C_i will have differing requirements. Precisely, $C_i(t) = (1 + e^{-\ell_i t})^{-1}$ where $\ell_i \in [-1, 1]$ takes values in a uniform grid. In this case, $C_i(t) \to \{0, 1\}$ as $t \to \infty$ depending on whether $\ell_i > 0$ or $\ell_i < 0$ and $C_i(0) = 1/2$. Hence, for $t = 0$ the contracts are all identical, but distinguished into two distinct groups for large t, with some relatively smooth distinction when t is "modest". Figure 3.2 provides 5 separate examples of this scenario (in order to show the variability with respect to the randomized v). We can observe that the UBP holds when contracts are all similar (*i.e.*, $t \approx 0$), and bids become highly distinct as $t \to \infty$. The particular values of v have a significant quantitative impact on the results (determining when and where bifurcations occur)), but behaviour is qualitatively similar across v.

3.3.2 Large Scale Example and Dual Induced Sparsity

In this section we provide a large scale randomized example for the purposes of qualitatively comparing item valuations v_{ij} and the slack variable θ_{ij}, as well as comparing sets $\mathcal{A}_i$ and $\mathcal{A}_i^\star$.

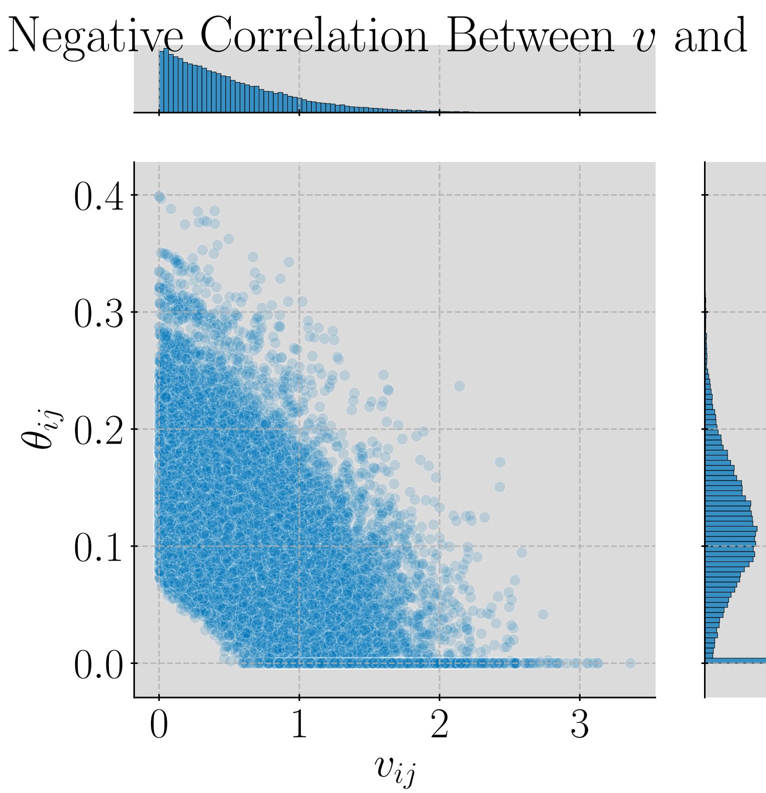
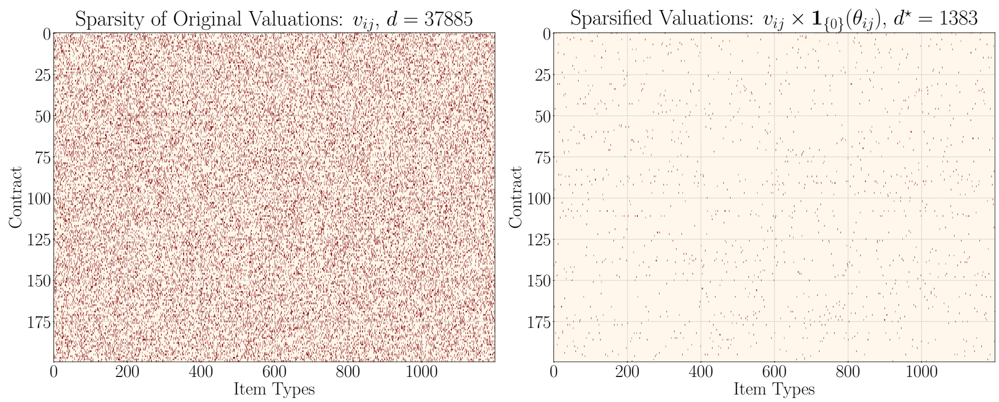


Figure 3.3: Scatterplot of (v_{ij}, θ_{ij}) for $v_{ij} \neq 0$
When v_{ij} is small, it is expected that there should be a wider margin between the value of items of type j, namely μ_j, and $v_{ij}\rho_i$. This difference is $\theta_{ij} = \mu_j - v_{ij}\rho_i$. Hence, v and θ should be negatively correlated.

Due to the definition $\theta_{ij} = \mu_j - v_{ij}\rho_i$, we should expect that when v_{ij} is small, θ_{ij} should be large. This effect is depicted in Figure 3.3

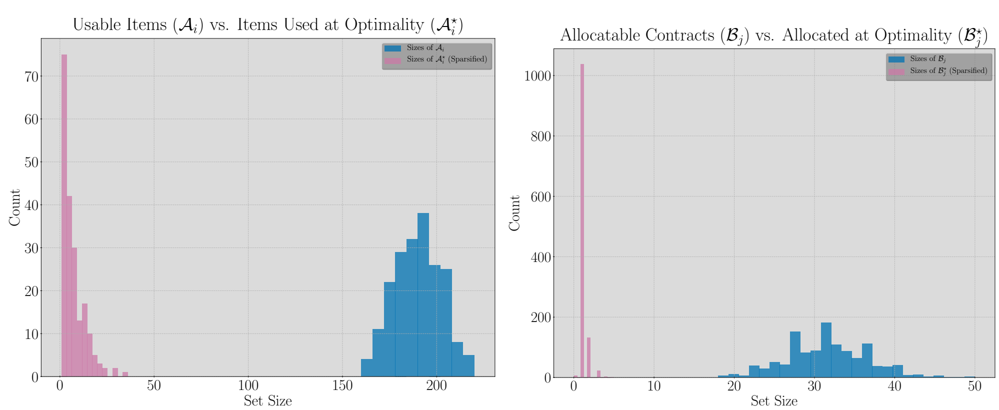
Moreover, when $\theta_{ij} > 0$, the item j is not used for contract i, which induces the subsets $\mathcal{A}_i^\star \subseteq \mathcal{A}_i$. To get a sense of this effect, consider Figure 3.4 where we display the sparsity pattern of v_{ij} as well as $v_{ij} \times \mathbf{1}_{\{0\}}(\theta_{ij})$. Additionally, the figure points out the difference between $d = \sum_{i=1}^N |\mathcal{A}_i|$ and $d^\star = \sum_{i=1}^N |\mathcal{A}_i^\star|$. The former is the *basic* sparsity of the problem (which is already $d \ll MN$, in this case $MN = 240,000$) and $d^\star$ is the *induced* sparsity of the problem, and is the dimensionality of the quadratic program that would need to solve to calculate the allocation matrix R (see Section 3.2.3). While it naturally depends on the particulars of v, λ, W etc., it can be reasonably expected again that $d^\star \ll d$, and hence the dual program can induce substantial additional sparsity into the matrix R.



(a) Original Valuations (b) Sparsified Valuations

Figure 3.4: Dual Induced Sparsification

A large scale example with $M = 1200$, $N = 200$. The parameters λ_j, γ_j (for $W_j(x) = 1 - e^{-\gamma_j x}$) and C_i are all $\exp(1)$ distributed. As well, we construct valuations via $v_{ij} \sim \mathcal{N}(0,1) - 1$ and then clipping values between $[0, \infty)$.



(a) Sparsification of $\mathcal{A}_i$ (b) Sparsification of $\mathcal{B}_j$

Figure 3.5: Sparsification of sets $\mathcal{A}_i$ and $\mathcal{B}_j$

Under the same simulation settings as Figure 3.4 we plot the sizes of the sets $\mathcal{A}_i, \mathcal{A}_i^{\star}, \mathcal{B}_j, \mathcal{B}_j^{\star}$.

Similarly, Figure 3.5 displays the sizes of the sets $\mathcal{A}_i, \mathcal{A}_i^\star$ (Figure 3.5a) and $\mathcal{B}_j, \mathcal{B}_j^\star$ (Figure 3.5b). We observe that every contract i is supplied by at least one item (*i.e.,* $|\mathcal{A}_i^\star| \geq 1$, necessary for the contract to be fulfilled). On the other hand, there are some sets $\mathcal{B}_j^\star$ which are empty, *i.e.,* it is inefficient to use item type j at all. Of course, these specific results are highly dependent upon the particulars of the simulation setup, but the examples serve to illustrate the predictions made by Proposition 3.1.3.

3.4 Additional Examples

We return here to the two additional examples from Section 2.5 and examine how the same techniques applied for the optimal contract management problem can also be applied there. As with Section 2.5, this section is not essential to understanding the overall contributions of the thesis.

3.4.1 Volume Costs in Limit Order Book: Dual Algorithms and Portfolio Construction

Recall the model of volume costs in a limit order book from Section 2.5.1. An important problem in finance is to construct, based on an estimate of risk and future returns, a portfolio that optimally trades off between these two aspects. One of the first formulations of this problem was famously carried out via mean-variance optimization (quadratic programming) by Markowitz [116]. The book [82] is a textbook introduction, but this remains an active area of research, *e.g.,* [174, 115, 3], and many others.

We can formulate a simple instance of this problem that takes into account order book volume costs as

$$\underset{x \in \mathbb{R}^N}{\text{minimize}} \quad \frac{1}{2}\lambda x^\mathsf{T} \Sigma x + \sum_{i=1}^{N} \big[\Lambda_i(x_i) - \alpha_i x_i\big], \tag{3.3}$$

where $x \in \mathbb{R}^N$ is the portfolio allocation across N risky assets[3], α_i is the forecasted future (proportionate) returns of asset i, and $\Sigma \succ 0$ is the covariance matrix of returns. This covariance matrix arises from the calculation $\operatorname{Var} \alpha^\mathsf{T} x$, which is often used as a measurement of the "risk" of portfolio x, and is scaled by the risk-aversion parameter $\lambda > 0$. In order to analyze this problem, we introduce a new variable and a trivial constraint

[3] In this simple example, there is no costs associated with leverage or with borrowing shares to short.

$$\begin{aligned} \underset{x\in\mathbb{R}^N, u\in\mathbb{R}^N}{\text{minimize}} \quad & \frac{1}{2}\lambda u^\mathsf{T}\Sigma u + \sum_{i=1}^{N}\big[\Lambda_i(x_i) - \alpha_i x_i\big] \\ \text{subject to} \quad & x = u, \end{aligned}$$

solely for the purpose of deriving a particular dual. Carrying out the usual calculations for the Lagrangian function, we obtain

$$\mathcal{L}(x, u, \phi) = \frac{1}{2}\lambda u^\mathsf{T}\Sigma u + \sum_{i=1}^{N}\big[\Lambda_i(x_i) - (\alpha_i - \phi_i)x_i\big] - \sum_{i=1}^{N}\phi_i u_i,$$

and minimizing with respect to x, we obtain $\mathcal{L}(x^\star, u, \phi) = \frac{1}{2}\lambda u^\mathsf{T}\Sigma u - \sum_{i=1}^{N}\phi_i u_i - \sum_{i=1}^{N}\Lambda_i^\star(\alpha_i - \phi_i)$, which, when minimizing again over u results in $u^\star = -\frac{1}{\lambda}\Sigma^{-1}\phi$, and the dual function $g(\phi) = -\frac{1}{2\lambda}\phi^\mathsf{T}\Sigma^{-1}\phi - \sum_{i=1}^{N}\Lambda_i^\star(\alpha_i - \phi_i)$. Incidentally, the quantity $x = -\frac{1}{\lambda}\Sigma^{-1}\alpha$ is exactly the optimal portfolio when $\Lambda_i \equiv 0$ (*c.f.,* $u^\star$), so that the dual vector ϕ can be understood as a sort of *cost-adjusted returns* vector.

Thus, through solving the dual, the optimal portfolio can be obtained by solving

$$\underset{\phi}{\text{minimize}} \quad \frac{1}{2\lambda}\phi^\mathsf{T}\Sigma^{-1}\phi + \sum_{i=1}^{N}\Lambda_i^\star(\alpha_i - \phi_i).$$

This can be further simplified by using the Cholesky decomposition $\Sigma = LL^\mathsf{T}$ (and hence $\Sigma^{-1} = L^{-\mathsf{T}}L^{-1}$) and the change of variables $\zeta = L^{-1}\phi$ to obtain

$$\underset{\zeta}{\text{minimize}} \quad \boldsymbol{\Lambda}^\star(\alpha - L\zeta) + \frac{1}{2\lambda}||\zeta||_2^2,$$

where $\boldsymbol{\Lambda}^\star(z) = \sum_{j=1}^{M}\Lambda_j^\star(z_j)$. This is now nothing but a regularized minimization problem of the Fenchel conjugate. The advantage of this formulation is that derivatives of $\Lambda^\star$ (see Section 2.1.2) are in terms of the volume available in the order book (*i.e.,* W and w – *c.f.,* Section 2.5.1) without involving the inverse of W, which occurs in derivatives of the primal objective.

3.4.2 Dark Pool Liquidation Problem: Dual Algorithms and Interpretation

Recall the dark pool liquidation problem from Section 2.5. We will again analyze this problem by means of Lagrangian duality. We have:

$$
\begin{aligned}
\mathcal{L}(u,\lambda,\mu) &= \sum_{k=1}^{K} \mathbb{E}(u_k - \xi_k)_+ + \lambda(\sum_{k=1}^{K} u_k - s) - \sum_{k=1}^{K} \mu_k u_k \\
&= \sum_{k=1}^{K} \big[u_k F_k(u_k) - \int_0^{u_k} x f_k(x) \mathsf{d}x + (\lambda - \mu_k) u_k \big] - \lambda s,
\end{aligned}
$$

where λ is the multiplier associated with the *send all shares* constraint $\sum_{k=1}^{K} u_k = s$ and μ_k are associated with the non-negativity constraints $u_k \geq 0$.

Since this function is convex, minimizers of $\mathcal{L}$ with respect to u can be characterized by Fermat's rule:

$$
\begin{aligned}
\frac{\partial}{\partial u_k} \mathcal{L}(u,\lambda,\mu) &= F_k(u_k) + u_k f_k(u_k) - u_k f_k(u_k) + (\lambda - \mu_k) \\
&= F_k(u_k) + (\lambda - \mu_k).
\end{aligned}
$$

So that $\mathsf{D}_u \mathcal{L}(u,\lambda,\mu) = 0 \iff \forall k \in [K]: \; u_k = F_k^{-1}(\lambda - \mu_k)$ and where $\mu_k \geq 0$ is the dual constraint. Substituting this optimal allocation back into $\mathcal{L}$ we find the dual function

$$
\begin{aligned}
g(\lambda,\mu) &= -\sum_{k=1}^{K} \int_0^{\lambda - \mu_k} F_k^{-1}(x) \mathsf{d}x - \lambda s \\
&= -\sum_{k=1}^{K} \Lambda_k(\lambda - \mu_k) - \lambda s,
\end{aligned}
$$

which involves a function of exactly the same form as the acquisition cost function for second price auctions (*c.f.,* Section 2.1.1). Since we know that Λ_k is a monotone increasing function, it must be that $\mu_k = 0$, and the problem can be solved by minimizing the monotone function $\lambda \mapsto \sum_{k=1}^K \Lambda_k(\lambda) - \lambda s$, or, after differentiating (due to the convexity of Λ_k), by solving the equation $\sum_{k=1}^K F_k^{-1}(\lambda) = s$ for λ. After solving this equation (by monotonicity, this can be carried out via bisection), we obtain the optimal allocation by $u_k = F_k^{-1}(\lambda)$. This simple solution is comparable to the contract management problem with multiple item types and a single contract, *c.f.,* Section 2.3.2.

Interpretation The dual variable λ, using the shadow price interpretation, measures the marginal of the number of shares that we will fail to liquidate, when we increase s and attempt to liquidate more shares. As well, it allows us to calculate the probability that we successfully liquidate all s shares. Indeed:

$$\begin{aligned}\mathbb{P}\Big\{\bigcap_{k=1}^K (\xi_k > F_k^{-1}(\lambda))\Big\} &= \prod_{k=1}^K (1 - F_k(F_k^{-1}(\lambda))) \\ &= (1-\lambda)^K.\end{aligned}$$

Thus, if $\lambda \approx 0$ we are highly likely to liquidate all of the shares (the marginal cost of having to liquidate more shares is low), and if $\lambda \approx 1$ we are highly unlikely to do so (and the marginal cost of having to liquidate more shares is high). This provides a measure of how difficult the problem instance is, since we need λ to satisfy $\sum_{k=1}^K F_k^{-1}(\lambda) = s$ it means that if λ needs to be large for this to occur, then u_k must be so large that $\mathbb{P}(\xi_k \leq u_k) = \lambda \approx 1$. i.e., it is highly unlikely for there to be enough supply available.

As well, even if $\lambda \approx 0$, $(1-\lambda)^K \to 0$ as $K \to \infty$. Thus, even if there is a large amount of aggregate supply available, if it is dispersed across a large number of DPs, then, since it is not easily accessible, we are still unlikely to liquidate all s shares. This can be interpreted as an effect of opportunity costs: when sending shares to DP 1, we lose the opportunity to have sent those shares to DP 2, even if the latter happened to have more supply available.

3.5 Conclusion

This chapter has carried out a duality analysis of the contract management problem, Problem (P). We find that strong duality holds under the natural condition, Assumption 3.0.1, which simply asks that there be enough supply available in the market to fulfill the constraints. This assumptions is weaker than the coarse sufficient condition Assumption 3.0.2 appearing earlier in [108].

The consequences of duality, summarized in Proposition 3.1.3, are far reaching and enable the characterization of optimal bids in terms of dual variables μ_j associated to each item type $j \in [M]$, but as well as through the additional set of variables ρ_i associated to each contract $i \in [N]$. These dual variables induce, as a consequence of the slack in the dual inequality $v_{ij}\rho_i \leq \mu_j$, additional subsets $\mathcal{A}_i^\star \subseteq \mathcal{A}_i$ and $\mathcal{B}_j^\star \subseteq \mathcal{B}_j$ that further restrict the items that are to be allocated towards contracts, *i.e.*, $j \notin \mathcal{A}_i^\star \implies R_{ij} = 0$. In addition, Corollaries 3.1.2 and 3.1.3 establish the further implication that, in second price auctions with $v_{ij} \in \{0, 1\}$ (a common special case), the bid placed for any item which can be usefully allocated towards contract i must be equal. When v_{ij} is general, it is still the case that the bid placed for an item j which can be usefully allocated towards contract i is nothing but a constant multiple v_{ij} of a common *pseudo-bid* ρ_i. Some of these consequences were established in [170] by alternative methods, but are here recognized as being naturally derived from duality.

In Section 3.2 we have derived a specialized ADMM algorithm for the solution of Problem (P) through the dual (D). This algorithm separates into elegant univariate and trivially parallel subproblems that rely only upon array arithmetic and the bisection algorithm (1). This algorithm is likely more efficient than a comparable augmented Lagrangian algorithm applied directly to the primal problem, as it avoids a nested bisection step (see Remark 3.2.2). In addition, Section 3.2.3 constructs a quadratic program appropriate for calculating the allocation matrix R, if dual variables μ, ρ or optimal acquisition rates s are known. This program always admits a solution, even if the allocation rates s are misspecified. Moreover, this formulation can benefit from the additional sparsity associated with the sets $\mathcal{A}_i^\star, \mathcal{B}_j^\star$ derived from solving the dual.

Section 3.3 discusses a number of computational examples, and empirically explores some of the properties of optimal solutions with small and carefully crafted examples, as well as large scale randomized problems.

Finally, Section 3.4 revisits the additional examples encountered first in Chapter 2 and further explores how the special properties of the function Λ, and its dual $\Lambda^\star$, play a role in other important application areas.

Chapter 4

Adaptive Bidding and Stochastic Approximation

An important aspect of real-time bidding markets (and markets in general) is that relevant statistics (*i.e.,* the supply curves and arrival rates) change over time. Thus, the static problem formulations developed in previous chapters are not yet completely adequate for applications as it is important for the DSP to be able to adapt their bids to these changing statistics.

Part of the time-varying nature of market prices are a result of natural and predictable daily and weekly fluctuations in human activity. These fluctuations can be accounted for by forecasting, which we consider in Chapter 5. However, some fluctuations are completely unpredictable, and we need algorithms that can adapt to change.

Throughout this chapter, we focus exclusively on second price auctions, and any appearance of the function Λ is to be understood as referring to Λ_{2nd}. The reason for this is that the simple relationship between Λ_{2nd} and W^{-1}, and between $\Lambda^{\star}_{2nd}$ and W seen in Proposition 2.1.3 does not hold in the first price case, and it is this duality that makes the simple stochastic approximation methods developed in this chapter possible. It is a topic for future consideration whether and to what extent these methods can be naturally generalized.

Outline We begin this chapter in Section 4.1 with a brief overview of stochastic approximation theory in order to establish the basic ideas behind the tools we will use to adapt to changing market statistics. In this section we also introduce elementary convergence theorems and definitions. In section 4.2 we will study a more abstract convex program, and the results for this case will be specialized to our application. We study stochastic approximation algorithms for finding solutions of this abstract problem when parts of the objective function are not known, and we will state convergence theorems general enough to apply to the primal and dual contract management problems: Problem (P) and Problem (D). The reader who is only interested in the results pertaining to the RTB application can skip to Section 4.3. In Section 4.4 we apply stochastic approximation directly to the primal problem (P), and, dually, Section 4.5 deals with stochastic approximation algorithms applied to the dual problem (D).

What makes these algorithms interesting is the peculiar way in which derivatives of the objective functions arise for second price auctions, as well as a further duality appearing in the solution of Problem (D) by stochastic approximation. Precisely, since the derivative of Λ is, roughly speaking, W^{-1}, and that of $\Lambda^\star$ is W, the derivatives of the objective functions are obtained by solving the equation $s = \lambda W(x)$ either for x, in the primal case, or for s, in the dual case. Therefore, the derivative of $\lambda\Lambda(s/\lambda)$ (the term arising in the primal objective) is exactly equal to the bid which is required to obtain items at the rate s. Dually, the derivative of $\lambda\Lambda^\star(\mu)$ is exactly equal to the amount of supply that is obtained through the bid μ. For this reason, Section 4.3 is dedicated to the solution of the equation $s = \lambda W(x)$ by the methods of stochastic approximation.

4.1 Introduction

Stochastic approximation (SA) began with the work of [153] which derived an algorithm designed to find a zero of a monotone function, when only noisy evaluations of the function are available. Since this groundbreaking work, stochastic approximation has developed into a profoundly influential field which ultimately forms the theoretical basis for innumerable practical algorithms including stochastic gradient descent [155] (the workhorse method for deep learning [80] and scalable machine learning) and Q-learning [173, 166], an important algorithm in the theory of Markov decision processes.

Abstractly, stochastic approximation studies algorithms of the form

$$x_{n+1} = x_n + a_n\big[h(x_n) + M_{n+1}\big], \tag{SA}$$

where $x_n \in \mathbb{R}^d$ is a sequence of iterates (with $x_0 \in \mathbb{R}^d$ some fixed initial point), $a_n > 0$ are step-sizes, $h : \mathbb{R}^d \to \mathbb{R}^d$ is a Lipschitz mapping, and $M_{n+1} \in \mathbb{R}^d$ is a noise sequence. The noise M_n is generally assumed to be a Martingale difference sequence[1] with respect to it's own filtration (possibly enlarged with the σ-algebra generated by x_0, a_n if any of these are random)[2].

Equation (SA) is analogous to a noisy discretization of the ordinary differential equation (ODE) $\dot{x} = h(x)$ [43, 98]. Indeed, the approach of [25] is to analyze the properties of the iteration (SA) in terms of the asymptotic properties of this ODE. Thus, the approach taken in the present chapter is to construct SAs where the associated ODE has a unique and globally asymptotically stable (GAS) equilibrium[3] that has desirable properties. The iterations of (SA), as long as they are stable, will converge towards these equilibria.

While more general step-size sequences are possible, we will assume that a_n satisfies the *Robbins-Monro conditions*

$$\sum_{n=1}^{\infty} a_n = \infty, \sum_{n=1}^{\infty} a_n^2 < \infty. \tag{4.1}$$

Remark 4.1.1 (Robbins-Monro Step-size Sequence)**.** The intuition behind Equation (4.1) is that $\tau_a(n) = \sum_{k=0}^{n-1} a_k$ is the simulation time of the ODE $\dot{x}(t) = h(x(t))$ and $\tau_n \to \infty$ is necessary to ensure that the iterates of Equation (SA) converge to the same point as the ODE solution converges to as $t \to \infty$. The condition $\sum_{n=1}^{\infty} a_n^2 < \infty$ is used to ensure that the noise M_n is adequately averaged away, in the sense that random variables of the form $\zeta_m = \sum_{n=0}^{m} a_n M_{n+1}$ must converge. The square summability of a_n is combined with a square-integrability assumption on M_n in order to apply a Martingale convergence theorem. Weaker summability assumptions on a_n can be traded for stronger integrability assumptions on M_n, but this is of only tangential interest for our purposes.

Our primary reference for basic results in SA is [25], see also [104, 17]. The results in these references are enough to construct proofs of a primal stochastic approximation algorithm (Section 4.4). For the dual algorithms studied in Section 4.5 we refer to [144] for results on asynchronous[4] stochastic recursive inclusions on multiple timescales, the general

[1] A Martingale difference sequence is a stochastic process M_n such that $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = 0$ where $\mathcal{F}_n$ is the filtration of M_n, and serves as a general noise model. For example, an *i.i.d.*sequence is a Martingale difference sequence.

[2] The iteration index n in Equation (SA) is such that variables with index n are "known" at step n, and variables with index $n+1$ are "unknown".

[3] Recall that a GAS equilibrium is a point $x_e \in \mathbb{R}^d$ such that from any initial point x_0 solutions $x(t)$ of the ODE converge to x_e as in $x(t) \to x_e$ as $t \to \infty$.

[4] An *asynchronous* stochastic approximation is one wherein only a random subset of the components of the vector x_n are updated on each iteration. In our application, this will correspond to the arriving item types.

results of which require only minor modification.

The main convergence theorem is the following:

Theorem 4.1.1 ([25]). *Consider the stochastic approximation (SA) beginning with an arbitrary $x_0 \in \mathbb{R}^N$. Let $\mathcal{F}_n = \sigma(M_1, M_2, \ldots, M_n)$ be the σ-algebra generated by the noise sequence M_n. Assume the following conditions*

1. *(ODE Solutions) $h : \mathbb{R}^N \to \mathbb{R}^N$ is Lipschitz continuous.*
2. *(Robbins-Monro[5]) $\sum_{n=0}^{\infty} a_n = \infty, \sum_{n=0}^{\infty} a_n^2 < \infty$ and $\exists N \in \mathbb{N}$ such that $\forall n \geq N$: $a_{n+1} \leq a_n$.*
3. *(Variance Bounds) M_n is a square integrable martingale difference sequence,* i.e., $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = 0$, *and* $\mathbb{E}[||M_{n+1}||^2 \mid \mathcal{F}_n] \leq K(1 + ||x_n||^2)$ *a.s. for some* $K \geq 0$
4. *(ODE Convergence) The ODE $\dot{x} = h(x)$ has a unique globally asymptotically stable equilibrium $x^\star$.*
5. *(Stability) $\sup_n ||x_n|| < \infty$ a.s.*

Then, $x_n \stackrel{a.s.}{\to} x^\star$ as $n \to \infty$.

An important generalization of the stochastic approximation (SA) is the following stochastic *recursive inclusion* ([25, Ch.5]):

$$x_{n+1} \in x_n + a_n\big[H(x_n) + M_{n+1}\big], \tag{SRI}$$

where $H : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is a set valued mapping with non-empty values. Notationally, what Equation SRI means is that

$$x_{n+1} = x_n + a_n\big[\tilde{H}_n + M_{n+1}\big]$$

for some random sequence $\tilde{H}_n$ satisfying $\tilde{H}_n \in H(x_n)$ for every n. In this case, it is important that the σ-algebra to which M_n is adapted now includes the σ-algebra generated by $\tilde{H}_n$, which measures the choice of element in $H(x_n)$ used in the iterations.

Similarly to how (SA) asymptotically approximates the differential equation $\dot{x} = H(x)$, Equation (SRI) can be expected to asymptotically approximate the differential inclusion $\dot{x} \in h(x)$. Indeed, this will be established after the following definitions.

[5] The monotonicity condition is not part of the usual Robbins-Monro conditions, and is not necessary for most proofs. However, subtle use of this monotonicity is used by [25, Ch. 6] and [144] in establishing a weak convergence theorem and which we rely upon for asynchronous updates.

Definition 4.1.1 (Marchaud Map)**.** A set valued mapping $H : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is a *Marchaud map* if (1) for each $x \in \mathbb{R}^d$, $H(x) \subset \mathbb{R}^d$ is convex and compact, (2) there exists some $K \geq 0$ such that for each $x \in \mathbb{R}^d$ we have $\sup_{z \in H(x)} ||z||_2 \leq K(1 + ||z||)$ and (3) the graph $\mathcal{G}_H \triangleq \{(x, y) \mid y \in H(x)\}$ is closed.

The structure of the application problems we study motivates the following restriction.

Definition 4.1.2 (Bounded Marchaud Map)**.** A Marchaud map $H : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ will be called a *bounded* Marchaud Map if there exists a bounded set $B \subset \mathbb{R}^d$ such that $\forall x \in \mathbb{R}^d :$ $H(x) \subseteq B$.

A basic convergence theorem for stochastic recursive inclusions, which generalizes Theorem 4.1.1 is the following:

Theorem 4.1.2 (Cor 6.4 [25])**.** *Let $H : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be a Marchaud map. Suppose that the DI $\dot{x} \in H(x)$ has a unique GAS equilibrium $x^\star$, and Items 2, 3 and 5 from Theorem 4.1.1 hold. Then, if x_n is a sequence satisfying* (SRI) *we have $x_n \to x^\star$* a.s.

Remark 4.1.2 (Constant Step Size Algorithms)**.** The Robbins-Monro step-size rule is designed so as to result in stochastic approximations which converge almost surely to some fixed value. However, we are often interested in algorithms which attempt to approximately *track* a time varying solution. In such cases, it is common in practice to instead apply a small *constant* step size $a \in (0, 1)$, rather than a decreasing step size $a_n \to 0$ (see [25, Ch. 9]). However, formally developing models for time varying environments becomes extremely cumbersome (but see [101]) and distracts from our primary purpose.

4.2 Stochastic Approximation for Linearly Constrained Convex Programs

We review some important stability theorems in Section 4.2.1 and then establish general results for stochastic approximation algorithm applied to a linearly constrained convex program in Section 4.2.2.

4.2.1 Stability Theorems

The primary difficulty in applying the general convergence results of [25] is that of *stability* (*i.e.,* Item 5 of Theorem 4.1.1). That is, one must verify *a-priori* that $\sup_n ||x_n|| \overset{a.s.}{<} \infty$.

The most easily applicable general theorem for establishing this stability criterion is the *Borkar-Meyn* theorem [25, Theorem 3.7]:

Theorem 4.2.1 (Borkar-Meyn [26]). *Consider a stochastic approximation (Equation (SA)) with limiting ODE $\dot{x} = h(x)$ that satisfies all the assumptions of Theorem 4.1.1, except for stability. Let $h_c(x) \triangleq \frac{1}{c}h(cx)$. If there exists a Lipschitz continuous function $h_\infty : \mathbb{R}^d \to \mathbb{R}^d$ such that $h_c \to h_\infty$ as $c \to \infty$, uniformly on compacts, and the ODE $\dot{x} = h_\infty(x)$ has the origin as its unique globally asymptotically stable equilibrium, then $sup_n \; ||x_n|| \overset{a.s.}{<} \infty$.*

The work of [149] generalizes Theorem 4.2.1 to the case of stochastic recursive inclusions. In the following, we combine a special case of [149] with a simple limiting argument.

Proposition 4.2.1. *Consider a stochastic recursive inclusion (Equation (SRI))*

$$x_{n+1} \in x_n + a_n\big[H(x_n) + h(x_n) + \epsilon_n + M_{n+1}\big], \tag{4.2}$$

where $H : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is a bounded Marchaud map, h satisfies the conditions of Theorem 4.2.1, and $\epsilon_n \in \mathbb{R}^d$ is a stable random sequence, i.e., *$sup_n \; ||\epsilon||_2 < \infty$* a.s. *and where M_n is a Martingale difference sequence with respect to the σ-algebra enlarged with the filtration of ϵ_n. Then, almost any sequence of iterates satisfying (4.2) are stable,* i.e., *$sup_n \; ||x_n|| < \infty$ a.s.*

Proof. We work on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define the random variable $Z = \sup_n ||\epsilon_n||$. First, suppose there is some $B \in \mathbb{R}_+$ such that $Z \leq B$ almost surely. We apply the main theorem of [149], which is analogous to the Borkar-Meyn stability theorem 4.2.1. The recursive inclusion (4.2) asymptotically approximates (in the sense of Theorem 4.1.2) the differential inclusion (DI) $\dot{x} \in H(x) + h + \mathcal{B}_B$ where $\mathcal{B}_B = \{x \in \mathbb{R}^d \mid ||x|| \leq B\}$. This set-valued mapping is still a Marchaud map. Under our assumptions, the conditions required by [149] are immediate and apply to the limiting differential inclusion (which is in fact a bona-fide differential equation)

$$\begin{aligned} \dot{x} &\in \lim_{c\to\infty} \frac{1}{c}\big[H(cx) + h(cx) + \mathcal{B}_B\big] \\ &= \lim_{c\to\infty} \frac{1}{c}h(cx) \\ &\triangleq h_\infty(x), \end{aligned}$$

since $H + \mathcal{B}$ are bounded. This establishes the stability of x_n.

Now, if Z is not uniformly bounded, let B_N be a sequence such that $B_N \to \infty$ and let $S_N = \{\omega \in \Omega \mid Z(\omega) \leq B_N\}$. Then, as above, $\mathbb{P}\{\sup_n \ \|x_n\| < \infty\} \geq \mathbb{P}\{S_N\}$, since the even that x_n is stable is a subset of S_N, by the earlier part of this proof. Moreover, since $\sup_n \ ||\epsilon_n|| < \infty$ *a.s.* we have

$$\begin{aligned}1 &= \mathbb{P}\{Z < \infty\} \\ &= \mathbb{P}\{\bigcup_{N=1}^{\infty} S_N\} \\ &= \lim_{N\to\infty} \mathbb{P}\{S_N\},\end{aligned}$$

since $S_N \subseteq S_{N+1}$. Therefore, taking a limit over N of $\mathbb{P}\{\sup_n \ \|x_n\| < \infty\} \geq \mathbb{P}\{S_N\}$ we obtain $\mathbb{P}\{\sup_n \ \|x_n\| < \infty\} = 1$. □

Theorem 4.2.1 can often fail. In particular, we will encounter a problem wherein $\frac{1}{c}h(cx) \to 0$ as $c \to \infty$, so that $h_\infty \equiv 0$ certainly does not have 0 as a GAS equilibrium. Examples of this behaviour come from functions which are not coercive, *e.g.*, $h(x) = -\mathsf{tanh}(x)$, or indeed, functions involving supply curves $W(x)$. In these cases, we will supply ad-hoc stability theorems.

4.2.2 Linearly Constrained Convex Program

Before moving on to our application, we study a more general stochastic approximation algorithm that solves the following linearly constrained convex optimization problem

$$\begin{aligned}\underset{x\in\mathbb{R}^d}{\text{minimize}} \quad & f(x) + \theta(x) \\ \text{subject to} \quad & Gx \leq h.\end{aligned} \tag{4.3}$$

Recall that for a convex function $\theta : \mathbb{R}^n \to \mathbb{R}$ (say), the *subdifferential* $\partial\theta(x)$ generalizes the notion of a derivative to a set-valued mapping for points which are not classically differentiable:

$$\partial\theta(x) = \{\phi \in \mathbb{R}^n \mid \forall y \in \mathbb{R}^n : \ \theta(y) \geq \theta(x) + \langle \phi, y - x\rangle\}.$$

We adopt the following assumptions throughout this section.

Assumption 4.2.1. In Problem (4.3), the function $f : \mathbb{R}^d \to \mathbb{R}$ is a proper, strictly convex function, bounded below by $-B_f$, and is continuously differentiable with a uniformly bounded derivative, *i.e.,* there is some $B \in \mathbb{R}$ such that $\forall x \in \mathbb{R}^d : f'(x) \le B < \infty$. The function $\theta \ge 0$ is real-valued, convex, and $\partial\theta(x)$ is uniformly bounded over x, *i.e.,* there is some bounded set $B \subset \mathbb{R}^d$ such that $\forall x \in \mathbb{R}^d : \ \partial\theta(x) \subseteq B$. The matrix $G \in \mathbb{R}^{m\times d}$, where $m \le d$, is full-rank (*i.e.,* $\mathcal{N}(G) = \{0\}$) and the polytope $P_{G,h} \triangleq \{x \in \mathbb{R}^d \mid Gx \le h\}$ is nonempty and compact.

Lemma 4.2.1. *Let θ satisfy the assumptions of 4.2.1. Then, $\partial\theta(x)$ is a bounded Marchaud map.*

Proof. Since θ is convex and everywhere real-valued, the subdifferential $\partial\theta(x)$ is a convex and compact set ([19, Prop. 5.4.1.]). Since $\partial\theta(x)$ is uniformly bounded, it satisfies the growth condition, *i.e.,* for some $K \ge 0$ we have, for any $x \in \mathbb{R}^d$, that $||z||_2 \le K(1+||x||_2)$. Finally, let $(z_n, \phi_n) \in \mathcal{G}_\theta$ be a convergent sequence. Using the definition of the subgradient, we must have, for any n, $\forall y \in \mathbb{R}^d : \ \theta(y) \ge \theta(z_n) + \langle \phi_n, y - z_n\rangle$, which, after taking the limit on the right (using joint continuity of the inner product and the fact that convex functions are continuous, in fact, locally Lipschitz [45, Thm. 2.34]) results in $\forall y \in \mathbb{R}^d : \ \theta(y) \ge \theta(z) + \langle \phi, y - z\rangle$ and hence $(z, \phi) \in \mathcal{G}$. That $\partial\theta$ is bounded is part of Assumption 4.2.1. □

Equality constraints, *i.e.,* $Ax = b$, are not explicitly included in Problem (4.3). While the inequalities $Gx \le h$ are technically general enough to handle such constraints, it is preferable to treat them separately. Since Proposition 2.3.1 provides a means of eliminating equality constraints from our main application, we defer an abstract discussion of equality constraints, along with a connection between *projected* gradient descent and the method used in Proposition 2.3.1, to Appendix B.4.1.

To deal with the inequality constraints, we apply a quadratic penalty function and derive an algorithm along similar lines as [167, 188]. Consider the function

$$\mathcal{L}_\mu(x) = f(x) + \theta(x) + \frac{1}{2}\mu||(Gx - h)_+||_2^2, \tag{4.4}$$

where $(x)_+ = \max(0, x)$ is the non-negative part of x. This penalty function is similar to the standard augmented Lagrangian method [21, Ch. 5], but does not include any dual variables. We exclude the use of dual multiplier updates since, in the context of our application, this would lead to a stochastic approximation on three time-scales; this will become clear throughout this chapter. Even though the extension of the results here to a bona-fide augmented Lagrangian algorithm is straightforward, [188] reports similar performance between stochastic augmented Lagrangian methods and simple quadratic penalties.

Since the function f is strictly convex, $\mathcal{L}_\mu$ admits a unique minimizer over $\mathbb{R}^d$ which we will denote $x_\mu \triangleq \operatorname*{argmin}_{x\in\mathbb{R}^d} \mathcal{L}_\mu(x)$. We proceed to an analysis of the approximation properties of x_μ to the Problem (4.3).

Proposition 4.2.2 (Asymptotic Feasibility)**.** *Let $\bar{\mu} > 0$ and $\mu \geq \bar{\mu}$. Under Assumption 4.2.1, minimizers x_μ of $\mathcal{L}_\mu$ are asymptotically feasible as $\mu \to \infty$. Precisely, $\exists M_{\bar{\mu}} < \infty$ such that*

$$||(Gx_\mu - h)_+||_2 \leq \sqrt{\frac{2LM_{\bar{\mu}}}{\mu}}, \tag{4.5}$$

where L is the Lipschitz constant of $f + \theta$.

Proof. From the definitions, we have for any $x \in P_{G,h}$ that $\mathcal{L}_\mu(x_\mu) \leq \mathcal{L}_\mu(x)$ and hence

$$\begin{aligned} f(x_\mu) - f(x) + \frac{1}{2}\mu\Big[||(Gx_\mu - h)_+||_2^2 - ||(Gx - h)_+||_2^2\Big] &\leq 0 \\ \overset{(a)}{\Longrightarrow} -L||x_\mu - x|| + \frac{1}{2}\mu||(Gx_\mu - h)_+||_2^2 &\leq 0 \\ \overset{(b)}{\Longrightarrow} ||(Gx_\mu - h)_+||_2 &\leq \sqrt{\frac{2LM_{\bar{\mu}}}{\mu}}, \end{aligned}$$

where in (a) we use the Lipschitz constant of $f + \theta$ and the fact that x is feasible, in (b) the constant $M_{\bar{\mu}} < \infty$ is a bound on $||x_\mu - x||$ which exists by the compactness of $P_{G,h}$ and the boundedness of $||x_\mu||$ by Lemma B.4.1. □

Proposition 4.2.3 (Asymptotic Optimality)**.** *The sequence x_μ of minimizers of $\mathcal{L}_\mu$ converges to the minimizer $x^\star$ of Problem (4.3) as $\mu \to \infty$.*

Proof. We follow the method of [18] and begin with the inequality $\mathcal{L}_\mu(x_\mu) \leq \mathcal{L}_\mu(x^\star)$. Then, calculate

$$\begin{aligned} &\mathcal{L}_\mu(x_\mu) \leq \mathcal{L}_\mu(x^\star) \\ \Longrightarrow\ & f(x_\mu) + \theta(x_\mu) + \frac{1}{2}\mu||(Gx_\mu - h)_+||_2^2 \leq f(x^\star) + \theta(x^\star) + \frac{1}{2}\mu||(Gx^\star - h)_+||_2^2 \\ \overset{(a)}{\Longrightarrow}\ & f(\bar{x}) + \theta(\bar{x}) + \frac{1}{2}\limsup_{\mu\to\infty} \mu||(Gx_\mu - h)_+||_2^2 \leq f(x^\star) + \theta(x^\star), \end{aligned}$$

where in (a) $\bar{x}$ is a cluster point of x_μ (using the continuity of $f + \theta$ and boundedness of x_μ) and $||(Gx^\star - h)_+||_2 = 0$ since $x^\star \in P_{G,h}$. Since $P_{G,h}$ is closed and x_μ is asymptotically feasible, we know that $\bar{x} \in P_{G,h}$ and hence $f(\bar{x}) + \theta(\bar{x}) \le f(x^\star) + \theta(x^\star)$. Therefore,

$$\limsup_{\mu \to \infty} \mu ||(Gx_\mu - h)_+||_2^2 = 0,$$

and $\bar{x}$ must be a minimizer for $f + \theta$ over $P_{G,h}$. Since $x^\star$ is a unique minimizer, it must be that $\bar{x} = x^\star$. We conclude that $x_\mu \to x^\star$. □

4.2.3 Two Timescale Stochastic Approximation

The subgradient of the function $\mathcal{L}_\mu(x)$ is given by

$$\partial \mathcal{L}_\mu(x) = \nabla f(x) + \partial \theta(x) + \mu G^\mathsf{T}(Gx - h)_+. \tag{4.6}$$

Therefore, if $\nabla f(x) + \partial\theta(x)$ is available to us, the function $\mathcal{L}_\mu(x)$ is easy to minimize through the method of subgradient descent: $x_{n+1} \in x_n - a_n \partial \mathcal{L}_\mu(x_n)$. However, in the applications of interest in this Chapter, $\nabla f(x)$ is not available (as we do not assume knowledge of f). A common scenario when f is unknown is when instead it is available through *i.i.d.* samples, *i.e.,* a sequence g_n such that $\mathbb{E} g_n = \nabla f(x_n)$. Plugging this in as an approximation leads, if $\theta \equiv 0$, to stochastic gradient descent: $x_{n+1} = x_n - a_n g_n$ [155].

However, in our application, even assuming the availability of an unbiased gradient estimate is too strong. Instead, we suppose only that, given some fixed $x \in \mathbb{R}^d$, we have in hand a standard stochastic approximation with function ψ and noise M_n

$$g_{n+1} = g_n + a_n[\psi(g_n, x) + M_{n+1}];\ g_0 \in \mathbb{R}^d,$$

such that g_n asymptotically approximates the gradient: $g_n \overset{a.s.}{\to} \nabla f(x)$. It will be seen in the sequel how this stochastic approximation is constructed, we merely assume for now that it is available. We consider the stochastic recursive inclusion on two timescales:

$$\begin{aligned} g_{n+1} &= g_n + a_n[\psi(g_n, x_n) + M_{n+1}];\ g_0 \in \mathbb{R}^d \\ x_{n+1} &\in x_n - b_n[\partial\theta(x_n) + g_n + \mu G^\mathsf{T}(Gx_n - h)_+];\ x_0 \in \mathbb{R}^d. \end{aligned} \tag{4.7}$$

The idea that these iterations are on *two timescales* is that the step-size sequences a_n, b_n are designed so as to ensure faster convergence of g_n than of x_n. Specifically, we require that $b_n / a_n \to 0$ as $n \to \infty$. The reason that this is to be understood as operating on two timescales follows from Remark 4.1.1: the simulation time of the ODE $\dot{g} = \psi(g)$ approximated by g_n is given by $\tau_a(n) \triangleq \sum_{k=0}^{n-1} a_n$ and the simulation time of x_n is given by $\tau_b(n) \triangleq \sum_{k=0}^{n-1} b_n$. If $b_n/a_n \to 0$, then $\tau_a(n)/\tau_b(n) \to \infty$ so that the recursion in x_n effectively *sees* only the equilibrium values of g_n. That is, larger step-sizes (in this case, a_n) simulate the corresponding ODE *faster* than do the small step-sizes b_n.

We will also assume a somewhat stronger stability condition on g_n, namely, that it is almost surely bounded for any possible x_n sequence. This assumption may appear extremely strong, but it will turn out to be satisfied by the algorithm used in our application (see Section 4.3) We establish the stability of Algorithm (4.7) as follows.

Proposition 4.2.4 (Stability)**.** *For the stochastic recursive inclusion* (4.7)*, suppose that* $\sup_n \|g_n\| < \infty$ *a.s. uniformly over any sequence* x_n *and that* a_n, b_n *satisfy the Robbins-Monro conditions (Item 2 of Theorem 4.1.1). Then, under Assumption 4.2.1* $\sup_n ||x_n|| < \infty$ *a.s..*

Proof. We will first verify the conditions of Theorem 4.2.1 for the differential equation $h(x) = -\mu G^\mathsf{T}(Gx_n - h)_+$. We have

$$\begin{aligned} h_c(x) &\triangleq \frac{1}{c} h(cx) \\ &= -\frac{\mu}{c} G^\mathsf{T}(cGx - h)_+ \\ &= -\mu G^\mathsf{T}(Gx - h/c)_+ \\ &\xrightarrow{c\to\infty} -\mu G^\mathsf{T}(Gx)_+ \triangleq h_\infty(x), \end{aligned}$$

where the convergence is uniform.

Consider the Lyapunov function candidate $V(x) = ||(Gx)_+||_2^2$, which naturally satisfies $V \geq 0$. Since $P_{G,h}$ is compact and G is full-rank (by Assumption 4.2.1), the cone $C_G = \{x \in \mathbb{R}^d \mid Gx \leq 0\}$ is necessarily the origin alone $C_G = \{0\}$. Therefore, $V(x)$ is coercive, *i.e.*, $V(x) \to \infty$ as $||x|| \to \infty$. Now, if $x(t)$ is a solution path for $\dot{x} = h_\infty(x)$ (such paths are well defined since h_∞ is Lipschitz and grows linearly) then the time derivative of $V(x(t))$ satisfies

$$\begin{aligned} \dot{V}(x) &\triangleq \langle \nabla V(x), h_\infty(x) \rangle \\ &= \langle G^\mathsf{T}(Gx)_+, -G^\mathsf{T}(Gx)_+ \rangle \\ &= -||(Gx)_+||_2^2, \end{aligned}$$

that is, $\dot{V} \leq 0$. Since G is full rank it follows that $\dot{V}(x) < 0$ and $V(x) > 0$ except at $x = 0$. Thus, by Lyapunov's stability theorem (Theorem A.0.1 $x = 0$ is a globally asymptotically stable equilibrium for $\dot{x} = h_\infty(x)$.

Now, we can apply Theorem 4.2.1. Precisely, we us the uniform stability of g_n and uniform boundedness of $\partial\theta$ (which is included in Assumption 4.2.1) so that

$$\lim_{c\to\infty} \frac{1}{c}\Big[\partial\theta(x) + g_n + \mu G^\mathsf{T}(Gx - h)\Big] = h_\infty(x)$$

which we have shown has 0 as its unique GAS equilibrium. Therefore, x_n is stable, *i.e.,* $\sup_n ||x_n|| < \infty$ *a.s.*. □

Remark 4.2.1 (Regularization)**.** The use of a regularization term $\frac{1}{2\mu}||x||_2^2$ in the objective function $\mathcal{L}_\mu$ would be enough on its own to prove the stability of the stochastic approximation, and does not require compactness etc. of $P_{G,h}$. We prove Proposition 4.2.4 using only this compactness firstly to emphasize that stability does not depend upon the use of regularization, and secondly because the stability proposition is insightful on it's own.

This will be enough to establish the convergence to the minimizer of $\mathcal{L}_\mu$ over $\mathbb{R}^d$ of a stochastic recursive inclusion on two time scales. We start with the case of an ordinary stochastic approximation, *i.e.,* where $\theta \equiv 0$.

Proposition 4.2.5 (Convergence I)**.** *Consider the stochastic approximation* (4.7) *where* $\theta \equiv 0$ *and with* a_n, b_n *satisfying the Robbins-Monro conditions and are such that* $b_n/a_n \to 0$ *as* $n \to \infty$*. Suppose that all of the conditions of Proposition 4.2.4 are satisfied, as well as all of those of Theorem 4.1.1 for the recursion in* g_n*. Finally, suppose that for every* $x \in \mathbb{R}^d$*,* $\nabla f(x)$ *is the GAS equilibrium for the ordinary differential equation* $\dot{g}(t) = \psi(g(t), x)$*. Then, for* $x_\mu \triangleq \operatorname*{argmin}_{x\in\mathbb{R}^d} \mathcal{L}_\mu(x)$*, we have* $g_n, x_n \to (\nabla f(x_\mu), x_\mu)$ *almost surely*

Proof. The iterates of x_n approximate the ODE $\dot{x} = -\nabla\mathcal{L}_\mu(x)$. Using the Lyapunov function $V(x) = \mathcal{L}_\mu(x)$ we have $\dot{V}(x) = -||\nabla\mathcal{L}_\mu(x)||_2^2$ which satisfies $V \geq 0$ and $\dot{V} \leq 0$ with strict equality at every point other than x_μ, by the strict convexity of $\mathcal{L}_\mu$. By reasoning similarly as in Proposition 4.2.4, using the compactness of $P_{G,h}$ we find that V is also coercive. By the Lyapunov stability theorem A.0.1 it follows that x_μ is a GAS equilibrium.

To show that x_n converges to this equilibrium, use Proposition 4.2.4 to obtain the stability of the iterates $\sup_n ||x_n|| < \infty$ almost surely. Then, following [25], we can write the SA (4.7) as

$$g_{n+1} = g_n + a_n[\psi(g_n, x_n) + M_{n+1}];\;\; g_0 \in \mathbb{R}^d$$
$$x_{n+1} \in x_n - a_n \frac{b_n}{a_n}[g_n + \mu G^\mathsf{T}(Gx_n - h)_+];\;\; x_0 \in \mathbb{R}^d,$$

which is an ordinary SA which approximates, by Theorem 4.1.1 and that $b_n/a_n \to 0$, the pair of ODEs $\dot{x} = 0$ and $\dot{g} = \psi(g, x)$. Since $\nabla f(x)$ is a GAS of the latter, we must have $g_n \to \{\nabla f(x) \mid x \in \mathbb{R}^d\}$ (this follows by a slightly more general theorem than Theorem 4.1.1, that x_n, g_n will converge to invariant sets of the associated ODE, see [25, Theorem 2.2]). This is the conclusion of Lemma 6.1 of [25]. This is the key lemma for Theorem [25, Theorem 6.2], which concludes that that $x_n \to x_\mu$ and $g_n \to \nabla f(x_\mu)$ almost surely as $n \to \infty$. $\square$

Asynchronous Updates

For the application described in the remainder of this chapter, the gradient estimates g_n will be updated *asynchronously*. Precisely, the j component of the gradient will be updated upon the arrival of an item of type j, see Definition 2.2.1. This stochastic approximation will be separable across item types, and a generalization of Proposition 4.2.5 will continue to hold in this case, including with a non-zero θ function. In order to formally model this situation, we follow [25, Ch. 7].

Let $\phi_n \in [M]$ be distributed *i.i.d.* across the set $[M]$ according to a categorical distribution, *i.e.*, $\mathbb{P}\{\phi_n = j\} \triangleq \eta_j$ for some $\eta \in \mathsf{int}\ \mathbb{P}^M$, where $\mathbb{P}^M$ is the probability simplex. Let $\nu_j(n) \triangleq \sum_{k=1}^n \mathbf{1}_j(\phi_n)$ count the number of occurrences of j up to occurrence n. We henceforth suppose that the function $\psi(g, x)$ is separable across components of g, that is, the approximation of Equation (4.7) takes the form:

$$\begin{aligned} g^j_{n+1} &= g^j_n + a_{\nu_j(n)}\mathbf{1}_j(\phi_n)[\psi_j(g^j_n, x_n) + M_{n+1}] \;\forall\; j \in [M], \\ x_{n+1} &\in x_n - b_n[\partial\theta(x_n) + g_n + \mu G^\mathsf{T}(Gx_n - h)_+]. \end{aligned} \tag{4.8}$$

We have $\sum_{n=1}^\infty a_{\nu_j(n)}\mathbf{1}_j(\phi_n) \overset{\text{a.s.}}{=} \sum_{n=1}^\infty a_n = \infty$ since $\mathbb{P}\{\phi_n = j \text{ i.o.}\} = 1$ (since ϕ_n is an *i.i.d.* sequence with $\mathbb{P}\{\phi_n = j\} \triangleq \eta_j > 0$). Likewise,

$$\sum_{n=1}^\infty a^2_{\nu_j(n)}\mathbf{1}_j(\theta_n) < \infty \text{ a.s.}$$

so the modified step-size sequences $a_{\nu_j(n)}\mathbf{1}_j(\theta_n)$ *etc.* satisfy the Robbins-Monro conditions almost surely. Moreover, $\mathbf{1}_j(\phi_n)M_{n+1}$ remains a Martingale difference sequence (w.r.t. the enlarged σ-algebra measuring ϕ_n), since ϕ_n is independent of M_{n+1}.

In order to establish the convergence of (4.8) we begin by applying the general results of [144] to establish the convergence of g_n.

Proposition 4.2.6 (Convergence II)**.** *Consider the stochastic recursive inclusion* (4.8)*. Suppose that all of the conditions of Proposition 4.2.4 are satisfied (other than $\theta \equiv 0$), as well as all of those of Theorem 4.1.1 for each $j \in [M]$ of the recursion g_n^j. That is, that $\psi_j(g^j, x)$ has $\partial f(x)/\partial x_j$ as the unique GAS equilibrium of $\dot{g} = \psi_j(g, x)$. Finally, suppose that for every $x \in \mathbb{R}^d$, $\nabla f(x)$ is the GAS equilibrium for the ordinary differential equation $\dot{g}(t) = \psi(g(t), x)$. Then, $(g_n, x_n) \to \{(\nabla f(x), x) \mid x \in \mathbb{R}^d\}$ almost surely.*

Proof. Firstly, the results of [144] allow for the asynchronous update schedule to be driven by a Markov chain, *i.e.,* at each iteration there is some random set of coefficients $J_n \subseteq [M]$ which are updated in Equation (4.7) and J_n is a Markov Chain over the power set $2^{[M]}$, and the transition kernel additionally depends on the iterates x_n. In both [144] and [25, Ch. 6] it is necessary that there is some $\nu_0 > 0$ such that $\liminf_{n\to\infty} \frac{\nu_j(n)}{n} \geq \nu_0$ almost surely[6]. In this context, an assumption stronger than boundedness is used by [144], namely that there is some compact set $C \subset \mathbb{R}^d$ such that $\forall n: \ x_n \in C$ almost surely. This assumption is used in [144, Lemma A.1] to select some $\nu_0 > 0$ through an application of Weierstrass' theorem over the compact C to ensure that there is some uniform lower bound on the stationary probabilities of the Markov Chain J_n. In Equation (4.7) the update index does not depend upon x and by the law of large numbers we have $\nu_j(n)/n \to \eta_j \geq \min_v \eta_v > 0$. Thus, we can replace Assumption (B1a) of [144] (that $x_n \in C$ for compact C) with the weaker assumption that the iterates are merely stable, Proposition 4.2.4. Moreover, the same reasoning allows us to drop the assumption made by [144] that $\sup_n \frac{a_{\lfloor vn \rfloor}}{a_n} < \infty$ for $v \in (0, 1)$. The remaining assumptions (B1), (B2), (B3), (B4), (B5) of [144] are then immediate from the assumptions we have made, with $q = 2$ in (B5) corresponding to the usual Robbin's Monro conditions.

Now, since each component j of the iterates (4.7) are updated separately and have the assumed GAS equilibria, the convergence towards the set $\{(\nabla f(x), x) \mid x \in \mathbb{R}^d\}$ follows through [144, Corollary 4.4]. □

[6] This uniform bound guarantees that each component is updated "comparably often".

Proposition 4.2.7 (Convergence III). *Consider the stochastic recursive inclusion* (4.7). *Suppose that all of the conditions of Proposition 4.2.4 are satisfied (other than $\theta \equiv 0$), as well as all of those of Proposition 4.2.6. Then, where $x_\mu \triangleq \underset{x\in\mathbb{R}^d}{\text{argmin}}\, \mathcal{L}_\mu(x)$, we have $x_n \to x_\mu$ almost surely.*

Proof. We follow the methods of [25, Ch. 5]. Since we have $x_{n+1} \in x_n - b_n[\partial\theta(x_n) + g_n + \mu G^\mathsf{T}(Gx_n - h)_+]$ we can instead write this as $x_{n+1} = x_n + b_n[z_n + \epsilon_n]$ where $z_n \in -\partial\mathcal{L}_\mu(x_n)$ and $\epsilon_n = g_n - \nabla f(x_n)$. From Proposition 4.2.6 we have $\epsilon_n \to 0$ almost surely.

Now, following [25], let $\tau_n = \sum_{k=0}^{n-1} b_k$ and $\bar{z}(t) = z_n$ for $t \in [\tau_n, \tau_{n+1})$ be a piecewise constant interpolation of the sequence z_n. As well, let $\bar{x}(t)$ be a linear interpolation of the x_n sequence, *i.e.*,

$$\bar{x}(t) = x_n + (t - \tau_n)[z_n + \epsilon_n],\ t \in [\tau_n, \tau_{n+1}),$$

and x^s be defined through integration of $\bar{z}(t)$ as in $x^s(t) = \bar{x}(s) + \int_s^t \bar{z}(u)\mathsf{d}u$. Then, $x^{\tau_n}(\tau_{n+m} + t) = \bar{x}(\tau_n) + \sum_{k=0}^{m-1} b_{n+k} z_{n+k} + (t - \tau_{n+m})z_{n+m}$ for $t \in [\tau_{n+m}, \tau_{n+m+1})$ and similarly $\bar{x}(\tau_{n+m} + t) = \bar{x}(\tau_n) + \sum_{k=0}^{m-1} b_{n+k}z_{n+k} + (t - \tau_{n+m})z_{n+m} + \sum_{k=0}^{m-1} b_{n+k}\epsilon_{n+k}$. Since $b_n \to 0$ and $\epsilon_n \to 0$ we have $\sum_{k=0}^{m-1} b_{n+k}\epsilon_{n+k} \to 0$ as $n \to \infty$ for any finite m. Thus, for any finite T, since $\tau_n \to \infty$ as $n \to \infty$ we have

$$\lim_{t\to\infty} \sup_{s\in[0,T]} ||\bar{x}(t + s) - x^t(t + s)|| \to 0.$$

This is the appropriate analogy to [25, Lemma 5.1]. It then follows from [25, Theorem 5.2] and [25, Corollary 5.4] that if the DI $\dot{x} \in -\partial\mathcal{L}_\mu(x)$ has a GAS equilibrium, then x_n converges to it almost surely.

We turn now to analyze the attractors of this DI. To do so, consider the coercive Lyapunov function $V(x) = \frac{1}{2}||x - x_\mu||_2^2$. The time derivatives satisfy the DI $\dot{V}(x) \in -(x - x_\mu)^\mathsf{T}\partial\mathcal{L}_\mu(x)$. Then, since for any $\phi \in \partial\mathcal{L}_\mu(x)$ we have $\forall x \in \mathbb{R}^d :\ \mathcal{L}_\mu(x_\mu) \geq \mathcal{L}_\mu(x) + \langle\phi, x_\mu - x\rangle$ and thus for any $x \neq x_\mu$ we have $\langle\phi, x_\mu - x\rangle \leq \mathcal{L}_\mu(x_\mu) - \mathcal{L}_\mu(x) < 0$. Therefore, since $\dot{V}(x) = (x_\mu - x)^\mathsf{T}\phi$ for some $\phi \in \partial\mathcal{L}_\mu(x)$, we have $\dot{V}(x) < 0$ for every $x \neq x_\mu$. It follows by Theorem A.0.2 that x_μ is a GAS equilibrium. □

4.3 Stochastic Approximation of Supply Curves

This Section reviews stochastic approximation methods for the equations $s = \lambda W(x)$ and $W^{-1}(s/\lambda) = x$, given observations of the indicator functions $\mathbf{1}(p_n \leq x)$ for prices p_n. Since

W are monotone functions, this situation is closely related to the classical problem of Robbins and Monro [153]. The results of this section will correspond a *fast timescale* in a combined SA which solves Problem (P).

We first focus on a single item type, and therefore we specialize Definition 2.2.1 for this section to treat just a single stream of items arriving according to a Poisson process of rate $\lambda > 0$, having inter-arrival times $\tau_n \sim \exp(\lambda)$ and with prices drawn from a single supply curve $p_n \sim W$.

Additionally, this chapter makes use of some additional regularity assumptions for $W(x)$.

Assumption 4.3.1 (Additional Regularity)**.** The supply curves $W_j(x)$ (*c.f.,* Definition 2.1.1) are Lipschitz and continuously differentiable.

4.3.1 Estimating $\lambda W(x)$

Let us suppose that we have some fixed bid x. If W is unknown, how do we estimate $s = \lambda W(x)$, the rate at which items are obtained? This is naturally an important quantity, and will turn out to be equal to an important derivative used in Section 4.5. Consider the stochastic approximation with $x \in \mathbb{R}$ held fixed

$$s_{n+1} = s_n + a_n[\mathbf{1}(p_{n+1} \leq x) - \tau_{n+1} s_n], \tag{4.9}$$

We can rewrite this equation as

$$s_{n+1} = s_n + a_n\big[W(x) - \bar{\tau} s_n + \big(\bar{\tau} s_n - \tau_{n+1} s_n + \mathbf{1}(p_{n+1} \leq x) - W(x)\big)\big], \tag{4.10}$$

and, recognizing that the term in parentheses is constructed to have mean 0, we recognize the ODE

$$\dot{s} = W(x) - \bar{\tau} s.$$

The analysis of this ODE and algorithm (4.9) is fairly straightforward: we verify the assumptions of Theorem 4.1.1 using the Borkar-Meyn theorem 4.2.1 to establish stability. This is the simplest analysis we can hope for.

Remark 4.3.1 (Censored Data)**.** Note that the price of an item does not need to be known, only whether or not the bidder won the auction, *i.e.,* $\mathbf{1}(p_{n+1} \leq x_n)$. This an even weaker requirement than the situation faced in reality wherein only the winning bidder learns the price of the item, *c.f.,* Remark 2.4.1.

Remark 4.3.2 (Averaging)**.** The rate $s = \lambda W(x)$ at which items are obtained can be calculated through simple averaging. That is, calculate the empirical probability of winning $W(x) \approx \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{1}(p_{n+1} \leq x)$ and the average inter-arrival time $1/\lambda \approx \frac{1}{N}\sum_{n=0}^{N-1} \tau_{n+1}$. These empirical averages converge by the law of large numbers. However, we make use of the formalism of stochastic approximation in order to later combine the results, using the analysis of Section 4.2, with gradient-based optimization algorithms on slower time scales.

We now establish the almost sure convergence of this algorithm.

Proposition 4.3.1 (Convergence)**.** *Let $x \in \mathbb{R}$ be some fixed bid. Suppose prices $p_n \overset{\text{i.i.d.}}{\sim} W$, inter-arrivals τ_n, and the supply curve W are generated according to the market model of Definition 2.2.1. Then, if a_n satisfies the Robbins-Monro conditions, the iterates s_n of Equation (4.9) converge almost surely $s_n \overset{\text{a.s.}}{\to} \lambda W(x)$.*

Proof. The convergence will follow by Theorem 4.1.1. Clearly, Item 2 is satisfied by assumption.

Consider now Item 3. Denote by $\mathcal{F}_n$ the σ-algebra generated by $\mathbf{1}(p_{n+1} \leq x)$ and τ_n then $\mathbb{E}[\bar{\tau}s_n - \tau_{n+1}s_n \mid \mathcal{F}_n] = 0$ and $\mathbb{E}[\mathbf{1}(p_{n+1} \leq x) - W(x) \mid \mathcal{F}_n] = 0$ so that the noise term of Equation (4.10) is a Martingale difference sequence. Additionally, we have

$$\begin{aligned}\mathbb{E}\Big[\big(\bar{\tau}s_n - \tau_{n+1}s_n + \mathbf{1}(p_{n+1} \leq x) - W(x)\big)^2 \mid \mathcal{F}_n\Big] &= s_n^2 \operatorname{Var}\tau_{n+1} + \operatorname{Var}\mathbf{1}(p_{n+1} \leq x) \\ &\leq K(1 + s_n^2),\end{aligned}$$

where there exists some appropriate constant K since τ_n has finite variance.

The remainder of the proof is concerned with the ODE $\dot{s} = W(x) - \bar{\tau}s$. This function is Lipschitz continuous by Assumption 4.3.1 (establishing item 1). To recognize that $\lambda W(x)$ is a globally asymptotically stable equilibrium, consider the Lyapunov function $V(s) = \frac{1}{2}\big(W(x) - \bar{\tau}s\big)^2$ which is coercive, satisfies $V \geq 0$, and $\dot{V}(s) = -\big(W(x) - \bar{\tau}s\big)^2 < 0$ except at $s = \lambda W(x)$ where $\dot{V}(s) = 0$. Thus, Lyapunov's stability theorem (Theorem A.0.1) establishes that $\lambda W(x)$ is a GAS equilibrium for the ODE, and we have established item 4.

Finally, let $h_c(s) = \frac{1}{c}W(x) - \frac{1}{c}\bar{\tau}(cs)$ so that $h_\infty \triangleq \lim_{c\to\infty} h_c(s) = -\bar{\tau}s$, where the convergence is uniform. The function $h_\infty(s)$ evidently has 0 as its unique GAS equilibrium, so the algorithm is stable (Item 5) by Theorem 4.2.1. □

It will also be important for later that the algorithm is stable over any sequence of inputs.

Lemma 4.3.1 (Uniform Stability)**.** *Let a_n be a step-size sequence satisfying the Robbins-Monro conditions and suppose that $\mu(n) \in \mathbb{R}$ is an arbitrary sequence and $p(n), \tau(n)$ are i.i.d. random variables in $\mathbb{R}_+$ with finite variance and mean $\mathbb{E}\tau(n) = \bar{\tau}$. Then, the iterates $w(n+1) = w(n) + a_n[\mathbf{1}(p(n+1) \leq \mu(n)) - \tau(n+1)w(n)]$, for arbitrary $w(0) \in \mathbb{R}$ are stable,* i.e., $sup_n |w_j(n)| < \infty$ a.s.

Proof. Consider the stochastic inclusion $w(n+1) \in w(n) + a_n[Z - \tau(n+1)w(n)]$ where $Z = [0, 1]$ is the unit interval. These iterations approximate the differential inclusion $\dot{w} \in h(w)$ where $h(w) = Z - \bar{\tau}w$ since we can equivalently write $w(n+1) \in w(n) + a_n[Z - \bar{\tau}w(n) + (\bar{\tau} - \tau(n+1))]$ with noise term $\bar{\tau} - \tau(n+1)$. The set-valued map h has compact and convex values, a closed graph, and satisfies $|z| \leq 1 + \bar{\tau}|w|$ for any $z \in h(w)$. It is thus a Marchaud map.

Let $h_c(w) = \frac{1}{c}h(cw)$, which, since h is bounded, converges uniformly $h_c \to h_\infty$ for $h_\infty(w) = -\bar{\tau}w$. This latter function has 0 as its unique globally asymptotically stable equilibria. Therefore, the stochastic iterates $w(n)$ are bounded almost surely by Proposition 4.2.1. □

4.3.2 Estimating $W^{-1}(s/\lambda)$

The dual problem to the stochastic approximation of $\lambda W(x)$ is that of $W^{-1}(s/\lambda)$, *i.e.,* solving $s = \lambda W(x)$ for s or for x, respectively. The former is a question of simply estimating the rate at which items are acquired given a bid of x (carried out in Section 4.3.1), and the latter is a question of finding the bid needed to obtain items at the rate s.

We here suppose that there is some fixed target rate $s > 0$ (not to be confused with the rates R_{ij} with which we allocate items) at which we want to obtain items. That is, we want to find an appropriate x such that $\lambda W(x) = s$; equivalently $x = W^{-1}(s/\lambda)$. This is a problem of *learning to bid* and falls into a similar problem domain as that studied by [212, 96].

To this end, and similarly to Section 4.3.1, we may consider the stochastic approximation

$$x_{n+1} = x_n + a_n[s\tau_{n+1} - \mathbf{1}(p_{n+1} \leq x_n)], \tag{4.11}$$

starting at some arbitrary initial point $x_0 \in \mathbb{R}$, and with step sizes a_n, as usual, satisfying the Robbins-Monro conditions. By re-writing this recursion as

$$x_{n+1} = x_n + a_n\big[s\bar{\tau} - W(x_n) + \big(s\tau_{n+1} - s\bar{\tau} + W(x_n) - \mathbf{1}(p_{n+1} \le x_n)\big)\big], \tag{4.12}$$

it can be recognized that this approximates the ordinary differential equation

$$\dot{x} = s\bar{\tau} - W(x).$$

This algorithm is a natural dual to that of Equation (4.9) and again, convergence towards $W^{-1}(s/\lambda)$ can be established. In fact, the convergence of a slightly more sophisticated algorithm will be established, namely:

$$\begin{aligned} \bar{\tau}_{n+1} &= \bar{\tau}_n + a_n\big[\tau_{n+1} - \bar{\tau}_n\big], \\ x_{n+1} &= x_n + a_n\big[s\bar{\tau}_n - \mathbf{1}(p_{n+1} \le x_n)\big], \\ L_{n+1} &= L_n + a_n\big[p_{n+1}\mathbf{1}(p_{n+1} \le x_n) - \bar{\tau}_n L_n\big], \end{aligned} \tag{BA}$$

where we simultaneously estimate the inter-arrival time $\bar{\tau}_n$ and the average cost of biding L_n as well. The label (BA) stands for *bid adaptation*. In our simulations, we have observed that this approximation converges faster than the univariate algorithm in x_n alone (likely as a result of maintaining an estimate of $\bar{\tau}$, as in Polyak averaging [146]), and the bid cost estimate is an important practical statistic. Thus, while the simpler iterations of (4.12) are theoretically adequate, the iterations of (BA) are more practically relevant and our proofs of this section will cover this case.

The convergence of this algorithm depends upon an analysis of the system

$$\begin{aligned} \dot{\tau}(t) &= 1/\lambda - \tau(t) \\ \dot{x}(t) &= s\tau(t) - W\big(x(t)\big) \\ \dot{L}(t) &= \bar{\tau} f\big(x(t)\big) - \tau(t)L(t), \end{aligned}$$

which, by inspection, can be seen to have equilibria $\tau(\infty) = 1/\lambda$, $x(\infty) = W^{-1}(s/\lambda)$ and $L(\infty) = \lambda\Lambda(s/\lambda)$. The formal arguments used to establish that this equilibrium is globally asymptotically stable are relegated to the appendix. As well, the stability of the iterations (BA) cannot be established through simple application of Theorem 4.2.1, and instead relies upon an ad-hoc argument that, as $x_n \to \infty$, it eventually behaves as a random walk with negative drift, and must therefore be bounded.

Proposition 4.3.2 (Bid Adaptation)**.** *Suppose that τ_n, p_n are drawn according to the market model Definition* (2.2.1) *and for a fixed type $j \in [M]$ (omitted from the notation) with* *supply curve W that also satisfies Assumption 4.3.1. Let $f(x) = \int_0^x uW'(u)\mathsf{d}u$ be the corresponding cost curve, and $\Lambda(q) = f \circ W^{-1}(q)$ the acquisition cost. If a_n satisfies the Robbins-Monro conditions and $s \in (0, \lambda)$ then the stochastic approximation described in Equation (BA), converges almost surely:*

$$(\bar{\tau}_n, x_n, L_n) \stackrel{a.s.}{\to} (\bar{\tau}, W^{-1}(s/\lambda), \lambda\Lambda(s/\lambda)) \text{ as } n \to \infty, \tag{4.13}$$

where $\bar{\tau} = 1/\lambda$.

Proof. Consider the stochastic approximation (BA). If we again denote $\bar{\tau} = 1/\lambda$, it can be rewritten in the canonical form

$$\begin{aligned}
\bar{\tau}_{n+1} &= \bar{\tau}_n + a_n\big[\bar{\tau} - \bar{\tau}_n + \big(\tau_{n+1} - \bar{\tau}\big)\big],\\
x_{n+1} &= x_n + a_n\big[s\bar{\tau}_n - W\big(x_n\big) + \big(W\big(x_n\big) - \mathbf{1}(p_{n+1} \leq x_n)\big)\big],\\
L_{n+1} &= L_n + a_n\big[f\big(x_n\big) - \bar{\tau}_n L_n + \big(p_{n+1}\mathbf{1}(p_{n+1} \leq x_n) - f\big(x_n\big)\big)\big].
\end{aligned}$$

In more abstract terms, the recursions can be written simply as

$$z_{n+1} = z_n + a_n[h(z_n) + M_{n+1}],$$

where $z_n = (\tau_n, x_n, L_n)$, h is the function

$$h(\tau, x, L) = (\bar{\tau} - \tau, s\tau - W\big(x\big), f\big(x\big) - \tau L),$$

which summarizes the dynamics, and

$$M_{n+1} = \big(\tau_{n+1} - \bar{\tau}, W\big(x_n\big) - \mathbf{1}(p_{n+1} \leq x_n), p_{n+1}\mathbf{1}(p_{n+1} \leq x_n) - f\big(x_n\big)\big)$$

is the remainder term. In order to establish convergence, we need only verify the assumptions required by Theorem 4.1.1. In particular, we must show that h is Lipschitz, that M_n is a uniformly square integrable Martingale difference sequence, and *stability*: $\sup_n ||z_n||_2 < \infty$ a.s. The stability condition is established in Corollary B.4.1. We verify the remaining conditions in turn. The function h is Lipschitz by Assumption 4.3.1.

It is also clear that $\sup_n \mathbb{E}||M_n||_2^2$ is bounded by the assumption that p_n has finite variance. To see that M_{n+1} is a Martingale difference sequence we must show that $\mathbb{E}[M_{n+1} \,|\mathcal{F}_n] = 0$ where $\mathcal{F}_n$ is the $\sigma-$algebra generated by M_n. Now, it in fact becomes clear that $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = 0$ by construction since x_n is $\mathcal{F}_n-$measurable and thus $\mathbb{E}[\mathbf{1}(p_{n+1} \leq x_n) \mid \mathcal{F}_n] = W\big(x_n\big)$ and $\mathbb{E}[p_{n+1}\mathbf{1}(p_{n+1} \leq x_n) \mid \mathcal{F}_n] = f\big(x_n\big)$.

Finally, since the ODE $\dot{z} = h(z)$ has a unique globally asymptotically stable equilibrium (Lemma B.4.5) consisting of $\bar{\tau}(t) \to \bar{\tau}$, $x(t) \to W^{-1}(s/\lambda)$, $L(t) \to \Lambda(r)$ as $t \to \infty$, the convergence announced in Equation (4.13) follows by Theorem 4.1.1. □

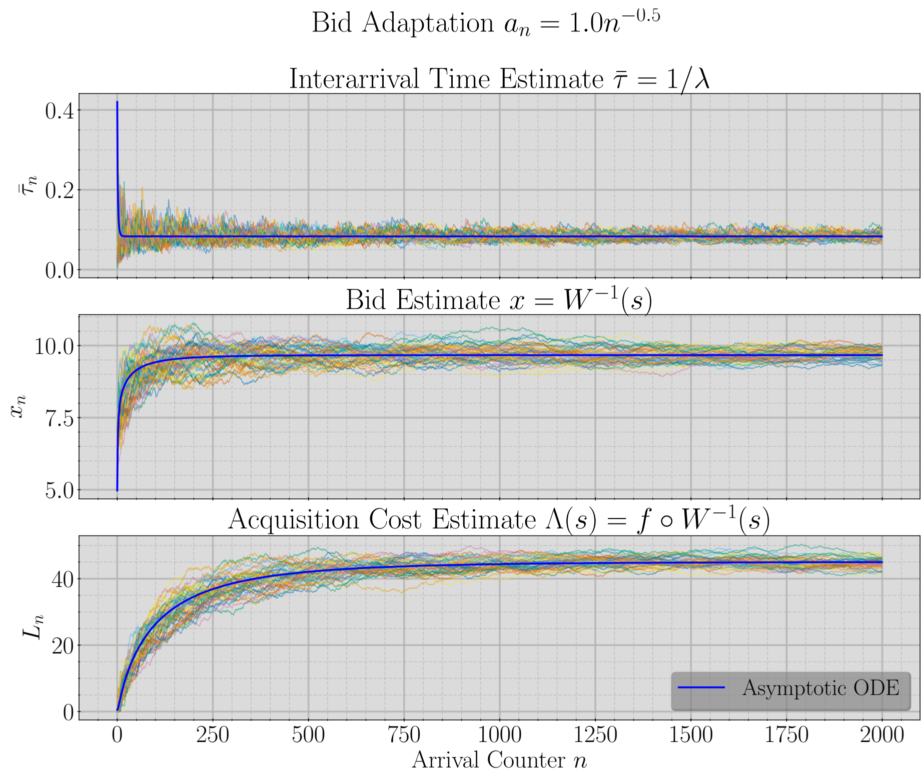


Figure 4.1: Bid Adaptation

Numerical simulation of the process of finding the bid x corresponding to a target supply s such that $\lambda W(x) = s$ (middle subfigure). A total of 50 simulations are layered in the figure to show the variation.

Computational Example We have carried out an illustrative Monte Carlo simulation of the bid adaptation algorithm (BA) for a single supply curve and fixed supply target $s = \lambda/2$. Prices are drawn *i.i.d.* from a Gamma distribution with mean 10 and variance 10. The results of this simulation are depicted in Figure 4.1, where separate colours indicate separate simulation runs. In each case, the simulation is initialized at $\bar{\tau}_0, x_0, L_0 = 1, 1, 1$. We can observe that the estimation of $\bar{\tau} = 1/\lambda$ converges extremely rapidly, as this estimator is essentially nothing more than a type of running average. The bid x_n, which approximates the inverse $x_n \approx W^{-1}(s/\lambda)$, and the cost $L_n \approx \lambda f \circ W^{-1}(s/\lambda)$, converge to within a modest factor of the asymptotic values within a few hundred to a thousand iterations.

4.4 Primal Algorithms

Section 4.3.2 describes a method which, given a desired target supply rate s_j for items of type j, estimates the cost $\lambda_j \Lambda_j(s_j/\lambda_j)$ of obtaining this rate, as well as the bid x_j which attains it, *i.e.,* a bid such that $\lambda_j W_j(x_j) = s_j$. Using this information, and a key fact about derivatives of Λ in second price auctions (Lemma 4.4.1), we will estimate the solution $R \in \mathbb{R}^{N\times M}$ of the main Problem (P), and hence derive the supply rate requirement $s_j = \sum_{i\in\mathcal{B}_j} \lambda_j R_{ij}$ which feeds into the stochastic approximation of Section 4.3.2. This primal stochastic approximation is analyzed in Section 4.4.1.

4.4.1 Primal Stochastic Approximation

The key Lemma that enables the solution of the primal problem by stochastic approximation is the key fact that derivatives of Λ are obtained automatically from the process of calculating bids.

Lemma 4.4.1 (Cost Derivative)**.** *Given a target supply rate $s \in (0, \lambda_j)$, the bid x having the property that $\lambda_j W_j(x) = s$ additionally satisfies the derivative property $\frac{\mathsf{d}}{\mathsf{d}s}\lambda\Lambda_j(s/\lambda) = x_j$.*

Proof. We know from Proposition 2.1.1 that $\lambda\Lambda'(s/\lambda) = \lambda W^{-1}(s/\lambda)$ on $s \in (0, 1)$. Thus, we see that since $\lambda W(x) = s$ we have in fact $x = W^{-1}(s/\lambda) = \lambda\Lambda'(s/\lambda)$. $\square$

Remark 4.4.1 (Alternative Auction Mechanisms)**.** Lemma 4.4.1 is particular to the second price auction mechanism. While most of the methods of the previous Sections can easily be adapted to a first price auction, this does not appear to be the case for the methods of the present section, since the derivative of Λ, in general, depends on the derivative W' of the supply curve itself.

Since this derivative becomes available to us automatically through the bidding process, we are motivated to study a first order (*i.e.,* using only first derivatives) method for the solution of Problem (P_R) which can be applied via a second stochastic approximation on a slower time scale.

For the purposes of applying stochastic approximation for the calculation of R, it is desirable to allow iterations of R_{ij} to take values in all of $\mathbb{R}$, rather than just inside the interior of the constraints of Problem (P_R). However, it is also important, for the stability of the iterations of Equation (BA) (c.f., Proposition 4.3.2) and the existence of the derivative Λ' (c.f., Lemma 4.4.1) that s_j be contained within the open interval $(0, \lambda_j)$; indeed, if the support of a supply curve W is all of $\mathbb{R}_+$ then it requires a bid $x \to \infty$ to obtain *all* of the available supply. With this in mind, we fix some small boundary approximation parameter $\epsilon \in (0, 1)$ and consider the function $Z_\epsilon : \mathbb{R}^M \to \mathbb{R}^M$ which truncates components of s:

$$Z_\epsilon(s)_j = \max\Big(\min\big(s_j, \lambda_j - \epsilon\big), \epsilon\Big). \tag{4.14}$$

In general, ϵ could be chosen to depend upon j, but this distinction is merely a change in notation. As well, it is to be understood that the function Z_ϵ need not use the exact arrival rate λ_j, but needs to only have access to a lower bound. This is not a strong assumption as the average arrival rate λ_j is the easiest parameter to estimate, *c.f.,* Figure 4.1, and is essentially independently estimated from all other parameters.

We then specify a function $\mathcal{L}^\epsilon_\beta$ by writing down the gradient we want it to have, namely:

$$\nabla \mathcal{L}^\epsilon_\beta(u) = B^\mathsf{T} \nabla \mathbf{\Lambda}(Z_\epsilon(\bar{s} + Bu)) + \frac{1}{\beta} u + \beta G^\mathsf{T} \big(Gu - h - \epsilon\big)_+, \tag{4.15}$$

where $\mathbf{\Lambda}, B, G, h$ were defined in Chapter 2 Problem (2.15) and essentially serve to write Problem (P_R) in more compact notation and without equality constraints. This gradient corresponds to that of the function

$$\mathcal{L}^\epsilon_\beta(u) = \widetilde{\mathbf{\Lambda}}^\epsilon(Z_\epsilon(\bar{s} + Bu)) + \frac{1}{2\beta}||u||_2^2 + \frac{1}{2}\beta||\big(Gu - h - \epsilon\big)_+||_2^2, \tag{4.16}$$

where $\widetilde{\mathbf{\Lambda}}^\epsilon$ is equal to $\mathbf{\Lambda}$ within the ϵ-tightened bounds of s, and is linearly extended (*i.e.,* by matching derivatives) beyond that point. This ensures that the derivative

$$\widetilde{\Lambda}_j'(\frac{1}{\lambda_j} Z_\epsilon(s)_j) = W_j^{-1}(\frac{\lambda_j - \epsilon}{\lambda_j})$$

at $s > \lambda_j - \epsilon$ corresponds to the asymptotic value of x_n in the stochastic approximation (4.13) with target supply $s = \lambda_j - \epsilon$ and similarly for $s < \epsilon$. Aside from the ϵ-tightening and the linear extension of $\mathbf{\Lambda}$, the function $\mathcal{L}^\epsilon_\beta$ corresponds exactly to the objective of the regularized and penalized Problem

$$\underset{u}{\text{minimize}}\ \mathbf{\Lambda}(\bar{\boldsymbol{s}} + Bu) + \frac{1}{2\beta}||u||_2^2 + \frac{1}{2}\beta||\big(Gu - h - \epsilon\big)_+||_2^2. \tag{4.17}$$

The minimizers of (4.17) approximate the least norm solution of Problem (P^u), *c.f.,* Problem (P^u_β) in Chapter 2.

4.4.2 Primal Convergence

The parameter ϵ has a desirable *regularizing effect* on solutions: each contract must use at least some items of every type available to it; and it will never seek to obtain all the supply available for any particular item – thus ensuring stability of the bidding stochastic approximation *c.f.,* Proposition 4.3.2. Moreover, contracts must be slightly over-provisioned (by the amount ϵ), which can be tuned to reduce the probability of failing to meet contracts in the short run (this is explored further in Chapter 5). Moreover, the ϵ parameter can be used to ensure that minimizers of $\mathcal{L}^{\epsilon}_{\beta}$ are *strictly* feasible for the original problem, as is seen in the following proposition.

Proposition 4.4.1 (Feasibility)**.** *Suppose that there is adequate supply (c.f. Assumption 3.0.1) for Problem (P). Denote by u^{ϵ}_{β} the minimizer of $\mathcal{L}^{\epsilon}_{\beta}$. Then, there exists $\epsilon > 0$ and a corresponding $\bar{\beta}_{\epsilon} > 0$ such that $\forall \beta \geq \bar{\beta}_{\epsilon}$ the minimizers u^{ϵ}_{β} exist and are strictly feasible for Problem (2.15).*

Proof. Since the constraints have non-empty interior (by the adequate supply assumption), there exists an ϵ such that the tightened constraints still admit at least one feasible point. Now, we know from Proposition 4.2.2 that u^{ϵ}_{β} is asymptotically feasible for the ϵ-tightened problem and hence there must be a point $\bar{\beta}_{\epsilon}$ such that u^{ϵ}_{β} is strictly feasible for the original un-tightened constraints for every $\beta \geq \bar{\beta}_{\epsilon}$. □

Finally, we combine these results with those of Section 4.3.2 to derive a stochastic approximation on two time scales which is capable of dynamically managing a collection of impression contracts without requiring accurate knowledge of the supply curves W_j.

Recall from Definition 2.2.1 that we have an *i.i.d.* sequence (ϕ_n, p_n) of (type, price) pairs modelling the items arriving at auction. Specifically, we have $\phi_n \overset{\text{i.i.d.}}{\sim} \text{Cat}(\boldsymbol{\eta})$ and $p_n \mid \phi_n \sim W_{\phi_n}$, for $\boldsymbol{\eta} \in \mathbb{R}^M_{++}$. The stochastic approximation estimating the bid x^j_n for items of type j will be updated upon each arrival of an item of type j. This is modelled as in Section 4.2.3: let $\nu_j(n) = \sum_{k=1}^{n} \mathbf{1}_j(\phi_n)$ count the number of arrivals of type j up to (and including) time n.

Then, to keep notation compact, we let $\psi(\bar{\tau}, x, s)$ be the mapping from $\mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^M$ into $\mathbb{R}^M \times \mathbb{R}^M$ which implements the bid-adaptation stochastic approximation of Equation (BA). That is, such that, using M^j_n to denote the j^{th} component of the corresponding noise, we have

$$(\bar{\tau}^j_{n+1}, x^j_{n+1}) = (\bar{\tau}^j_n, x^j_n) + a_{\nu_j(n)} \mathbf{1}_j(\phi_n)[\psi_j(\bar{\tau}^j_n, x^j_n, s_j) + M^j_{n+1}],$$

where we are careful to note that each component of this algorithm operates independently of the others. Thus, for any s having $0 < s_j < \lambda_j$, the convergence $\bar{\tau}_n^j \to 1/\lambda_j$ and $x_n^j \to W^{-1}(s_j/\lambda_j)$ almost surely by Proposition 4.3.2.

We then combine this recursion with stochastic gradient descent applied to $\mathcal{L}_\beta^\epsilon$ and consider the algorithm

$$\begin{aligned}(\bar{\tau}_{n+1}^j, x_{n+1}^j) &= (\bar{\tau}_n^j, x_n^j) + a_{\nu_j(n)} \mathbf{1}_j(\phi_n)[\psi_j(\bar{\tau}_n^j, x_n^j, Z_\epsilon(\bar{s} + Bu_n)) + M_{n+1}^j], j \in [M] \\ u_{n+1} &= u_n - b_n\big(B^\mathsf{T} x_n + \frac{1}{\beta} u + \beta G^\mathsf{T}(Gu_n - b - \epsilon)_+\big),\end{aligned} \tag{P-SA}$$

where b_n is another Robbins-Monro step size sequence which additionally satisfies $b_n/a_n \to 0$, *c.f.,* Section 4.2.3. The first recursion is used to provide adaptive estimates of the gradient of $\mathbf{\Lambda}$ (via Lemma 4.4.1 and Equation (4.15)).

As well, recall from Proposition 2.3.1 that the sequence of iterates u_n are confined to the $(d - N)$-dimensional subspace induced by the $\sum_{j\in\mathcal{A}_i} v_{ij} R_{ij} = C_i$ constraint. The iterates u_n asymptotically minimize $\mathcal{L}_\beta^\epsilon$. Approximate solutions, R_n, of the ϵ-tightened version of the original problem (P) are reconstructed from u_n according to the description of Proposition 2.3.1. The minimizers of $\mathcal{L}_\beta^\epsilon$ converge to the least-norm ϵ-tightened solution as $\beta \to \infty$.

Theorem 4.4.1. *Suppose there is adequate supply (Assumption 3.0.1) for Problem (P) and that $\epsilon > 0$ is such that the ϵ-tightened constraints remain non-empty. As well, suppose that both a_n, b_n satisfy the Robbins-Monro conditions and $b_n/a_n \to 0$ as $n \to \infty$. Then, the iterates $u_n \overset{\text{a.s.}}{\to} u_\beta^\epsilon = \operatorname{argmin} \mathcal{L}_\beta^\epsilon$ converge almost surely as $n \to \infty$.*

Proof. The projection Z_ϵ ensures that $Z_\epsilon(s_n)$ is always strictly feasible for Problem (P) and therefore satisfies the convergence requirements of Proposition 4.3.2. Moreover, this projection, combined with the linear extension $\widetilde{\mathbf{\Lambda}}$, *c.f.,* Equation (4.15), and Lemma 4.4.1 ensures that $B^\mathsf{T} x_n + \frac{1}{\beta} u + \beta G^\mathsf{T}(Gu - h - \epsilon)_+$ is an appropriate (for Proposition 4.2.6) approximation of $\nabla \mathcal{L}_\beta^\epsilon$ – see Equation (4.16).

We proceed to verify the requirements of Proposition 4.2.6. Firstly, the iterates x_n are bounded by Lemma B.4.1. Lemma B.4.5 established the global asymptotic equilibria of the first approximation. As well, the matrix G is full rank and the polytope $P_{G,h}$ is compact by Proposition 2.3.1. This establishes the announced convergence by Proposition 4.2.6. ☐

4.4.3 Primal Computational Examples

In order to illustrate the performance of our methods, we have carried out numerical simulations for an example contract management problem with $M = 5$ distinct item types, $N = 6$ contracts, and $v_{ij} \in \{0, 1\}$. The prices (denominated in arbitrary monetary units) p_n for each type are drawn *i.i.d.* from Gamma distributions and arrive according to Poisson processes of rates (having units of Hz) $\boldsymbol{\lambda} = \big(\lambda_j, j \in [M]\big)$. The specific parameters are given by, where $\boldsymbol{p}(n) = \big(p_n \mid \phi_n = 1, \ldots, p_n \mid \phi_n = M\big)$

$$
\begin{aligned}
\boldsymbol{\lambda} &= (3.0, 16.0, 18.0, 20.0, 22.0), \\
\mathbb{E}[\boldsymbol{p}(n)] &= (20.0, 23.0, 26.0, 29.0, 32.0), \\
\mathrm{Var}[\boldsymbol{p}(n)] &= (20.0, 18.0, 16.0, 14.0, 12.0),
\end{aligned}
\tag{4.18}
$$

which is enough to fully specify the market model (see Definition 2.2.1). The item type $j = 1$ is the cheapest, yet has by far the lowest supply and has a high variance, these parameters stress the stochastic approximation since the optimal solution is likely to require nearly all of the supply of type $j = 1$ that is available. As well, the parameters are chosen such that there is not a single optimal bid $\rho^\star$ as in Corollary 3.1.3, as can often happen when $v_{ij} \in \{0, 1\}$. The $N = 6$ contracts are specified through the sets $\mathcal{A} = \big(\mathcal{A}_i, i \in [N]\big)$ and $\boldsymbol{C} = \big(C_i, i \in [N]\big)$ with $v_{ij} \in \{0, 1\}$ as

$$
\begin{aligned}
\mathcal{A} &= \big(\{1, 5\}, \{2, 4\}, \{3\}, \{1, 2, 3\}, \{3, 4, 5\}, \{2, 3, 4\}\big), \\
\boldsymbol{C} &= \big(4.0, 3.4, 1.8, 6.6, 4.4, 5.2\big).
\end{aligned}
\tag{4.19}
$$

This indicates, for example, that contract 1 can be fulfilled by obtaining, on average, 4 items per second of types 1 or 5 (which arrive at rates $\lambda_1 = 3.0$ and $\lambda_5 = 22.0$, respectively). These items have an average price of 20.0 and 32.0, as well as a variance in prices of 20.0 and 12.0, respectively. As well, the dimensionality of R is $d = 14$ and $u \in \mathbb{R}^9$.

Initialization, Parameter Selection, and Other Considerations

In order to initialize Algorithm (P-SA), a vector u_0 needs to be specified. In principle, since the Algorithm is convergent for any starting point, the algorithm can be initialized at random or arbitrarily (e.g., at $u_0 = 0$). However, superior initialization methods are available.

Firstly, if prior information is available, then an ϵ-tightened formulation of Problem (P) can be solved, using an appropriate model of $\boldsymbol{\Lambda}$ which captures this prior knowledge, and then the solution of this program can be used as the initialization u_0.

Absent any prior knowledge, we propose to use the Chebyshev center of the constraint polytope, defined as a center of a ball having maximum radius that is contained in the constraint polytope $P_{G,h}$ (see [29, Sec. 8.5]). Since $P_{G,h}$ is a polytope, the Chebyshev center can be obtained via the solution of a linear program. Moreover, the *value* of this program, called the *depth* δ of the polytope, can be used to choose the constraint tightening parameter, e.g., via $\epsilon = \delta/100$ (the choice made in our simulations). It is therefore guaranteed that this ϵ will still result in a feasible optimization problem after the constraints are tightened.

In the absence of prior information, there is no means for initializing the bid estimate x_0. For this reason, we do not perform any updates of the matrix R_n until 250 iterations of the bid adaptation procedure have occurred. This reduces any large initial fluctuations in the algorithm caused by extremely inaccurate estimates of the gradient of $\mathbf{\Lambda}$. A great deal of extreme fluctuations are also avoided by eliminating the equality constraints $\sum_{j\in\mathcal{A}_i} R_{ij} = C_i$ (see Proposition 2.3.1); if these constraints are instead implemented with a penalty, the algorithm will tend to hug this face of the polytope and make slow progress – constantly oscillating between feasible and infeasible points with corresponding large fluctuations in the gradient estimates (see Appendix B.4.1 for further discussion).

The step size schedule a_n, b_n can have a significant impact on the convergence rate of the algorithm, but tuning these sequences is largely a matter of empirical experimentation (though see [78] and the commentary of [25, Ch. 2]). The only strict requirement placed on these sequences is that $b_n/a_n \to 0$, in order to ensure adequate separation of the two time scales. For the purposes of our simulation we have used $a_n = 2n^{-0.55}$ and $b_n = n^{-0.95}$. In practice, these are often parameterized by $a_n = a_0 n^{-\phi_a}$, and time-scale separation simply requires that $\phi_a < \phi_b$.

Finally, the parameter μ needs to be large (*c.f.,* Proposition 4.4.1), but using too large of a value of μ can easily cause numerical overflow in early iterations. One method to deal with this is to project iterates of the approximation back onto either the feasible region, or a ball of arbitrarily chosen (but relatively small) radius whenever they drift too far from any reasonable values. This procedure allows for the step sizes a_n, b_n to decay sufficiently to prevent unreasonable excursions. As long as these projections are only carried out a finite number of times, the convergence analysis is not impacted. Alternatively, μ can be increased (e.g., via $\mu \leftarrow (1+\kappa)\mu, \kappa > 0$) throughout the simulation (perhaps up to some large maximum value) whenever the iterates are detected to be infeasible. This latter method has been used in our simulations in order to avoid any unusual discontinuous jumps in the algorithm, with the value $\kappa = 0.01$, up to a maximum of $\mu \leq 10^4$. This maximum value seems to be reasonable based on simulation evidence of [64].

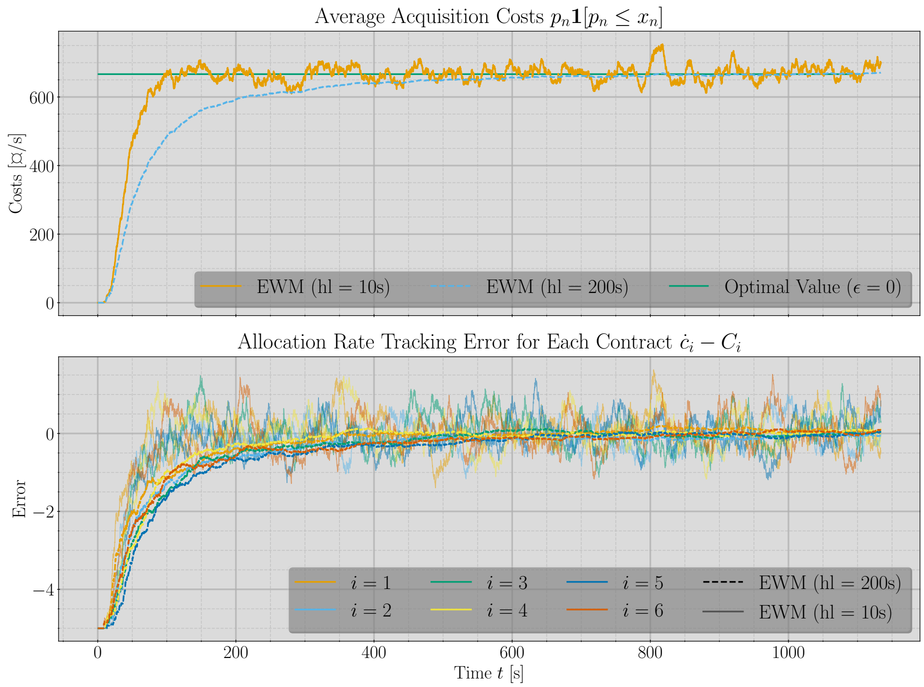


Figure 4.2: Costs and Contract Fulfillment

Convergence of the *empirical* (i.e., external to the algorithm) costs and item acquisition rates. The iterates are extremely noisy, but convergence is clear when averaged over a long enough time period.

Results Numerical results of the stochastic approximation (P-SA) are given in Figure 4.2 and 4.3. With reference to Figure 4.2, we have calculated exponential moving averages (with half lives of both $10s$ and $200s$ of simulation time) of the sequence $p_n\mathbf{1}[p_n \leq x_n]$ which provides an approximation to the long-term average cost, as well as averages of the number of items assigned to each contract, providing an empirical estimate of $\dot{c}_i \approx C_i$. The results indicate that there is, in the short run, a substantial degree of variation in these empirical averages, but which is averaged out over longer time horizons. This variation is a result of the high degree of price variation we have included in our simulation. But, this accords accurately with realistic data, where price variation can be quite substantial [215, 113]. Some of this variation can be damped by methods of feedback control [212, 96], at the expense of higher long term average costs.

Inspecting the bottom subfigure of Figure 4.3, it should be clear that the wide variations in acquisition rates are *not* an inherent aspect of the stochastic approximation algorithm, since the target acquisition rates s_j remain stable after a brief initial learning period.

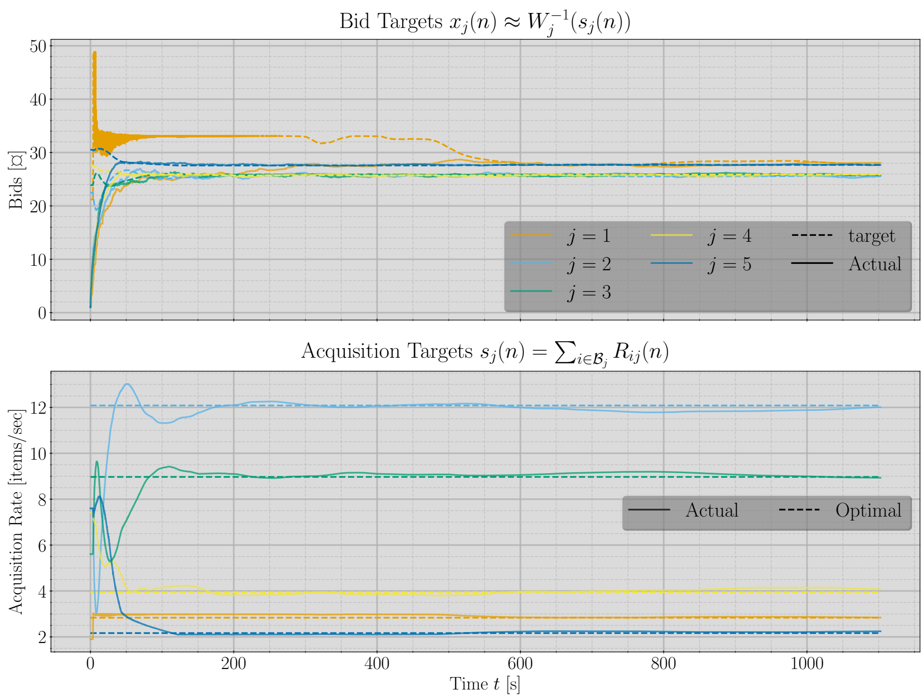


Figure 4.3: Supply Rates and Bids

Convergence of $s_j = \sum_{i \in \mathcal{B}_j} R_{ij}$ and of the bid $x_j = W_j^{-1}(s_j)$, these quantities are internal to the algorithm. The optimal bid and item acquisition rates are calculated by solving Problem (P) with exactly known supply curves. The poor behaviour of the bid estimate associated to type $j = 1$ is explained further in the main text.

However, the actual bid $x_n^j \approx W_j^{-1}(s_n^j)$ can exhibit greater instability in regions where W_j has flat curvature, since large differences in bids can result only in small differences in empirical supply rates. The item type $j = 1$ in our simulation is deliberately constructed to illustrate this challenge: early in the simulation x_n^1 exhibits wild oscillations since the constraint $s_1 \leq \lambda_1$ is nearly binding and is hence subject to the quadratic penalty as well as being in a region of low curvature. As well, the arrival rate λ_1 is slow, hence impeding rapid convergence.

We finally see from both of these figures that the algorithms have converged after about 300 seconds of simulation time, which corresponds to approximately $24,000$ item arrivals. According to statistics reported by [215], these arrival rates should be accurate within reasonable orders of magnitude (particularly considering the growth of the industry since the appearance of this paper). Still, the actual arrival rates will be highly dependent upon what how the type of an item is characterized.

4.5 Dual Algorithms

Dually to Section 4.3.2, Section 4.3.1 describes how to estimate the rate at which items are obtained $s_j = \lambda_j W_j(x)$ given some fixed bid $x \in \mathbb{R}$. This will be similarly combined with properties of the derivative of $\Lambda^\star$ to derive dual stochastic approximations for the solution of (D). In fact, it will be seen that there is a further duality induced by the properties that these dual variables must necessarily satisfy, and induces two equivalent unconstrained problems (D_μ), whose sole variables are μ, and (D_ρ), whose sole variables are ρ. We derive similar stochastic approximation algorithms for both of these problems, confining our theoretical analysis to Problem (D_μ).

4.5.1 Dual Stochastic Approximations

In this section, we will consider stochastic approximation algorithms for the solution of the dual problem (D). We will see that this approach is, naturally, dual to that of Section 4.4, but in addition that there is a further symmetry between the dual variables μ and ρ. Indeed, by Proposition 3.1.3 it is possible to write one set of variables in terms of the other, and this will result in two unconstrained optimization problems (D_μ) and (D_ρ). Stochastic approximation algorithms can be derived for either of these unconstrained problems and thus there is a choice between approximating μ, the bids associated to item types, or ρ, the pseudo-bids associated to contracts.

Remark 4.5.1 (Stochastic Subgradient Ascent)**.** In an abstract setting where we seek to solve the unconstrained convex maximization problem $\max_x J(x)$, if the function J is differentiable then an appropriate method is to iterate $x_{n+1} = x_n + a_n \nabla J(x_n)$. That is, gradient ascent. If J is not differentiable, since it is convex it still has a subdifferential $\partial J(x) = \{\phi \in \mathbb{R}^d \mid J(z) \geq J(z) + \langle \phi, z - x\rangle\}$, which can be used in a subgradient ascent algorithm (see, *e.g.,* [30, Ch. 3]). This algorithm is described by a recursive inclusion $x_{n+1} \in x_n + a_n \partial J(x_n)$. What this notation means is that $x_{n+1} = x_n + a_n g_n$ for some $g_n \in \partial J(x_n)$. This is the algorithm that is implemented as a stochastic approximation in this chapter.

Combining the result of item 2 from Proposition 3.1.3 to write μ in terms of ρ and vice-versa, we can obtain two further *unconstrained* forms of the dual (D). Specifically, we have $\mu_j = \max\limits_{i \in \mathcal{B}_j} v_{ij}\rho_i$ and $\rho_i = \min\limits_{j \in \mathcal{A}_i} \left(\frac{\mu_j}{v_{ij}}\right)$, which results in:

$$\underset{\mu}{\text{maximize}} \quad \sum_{i=1}^{N} C_i \min_{j \in \mathcal{A}_i} (\mu_j / v_{ij}) - \sum_{j=1}^{M} \lambda_j \Lambda_j^\star(\mu_j), \tag{D_μ}$$

and

$$\underset{\rho}{\text{maximize}} \quad \sum_{i=1}^{N} \rho_i C_i - \sum_{j=1}^{M} \lambda_j \Lambda_j^\star(\max_{i \in \mathcal{B}_j} \rho_i v_{ij}). \tag{D_ρ}$$

Although it should already be clear from the fact that these are convex duals, we can separately recognize that these are convex programs as follows: Problem (D_μ) is convex since the pointwise minimum over affine functions is concave, and Problem (D_ρ) is convex since the pointwise maximum of convex functions is convex and thus, since $\Lambda_j^\star$ is convex and monotone increasing, the composition of these functions is convex in ρ.

In terms of the generic convex program studied in Section 4.2, and for the Problem (D_μ) the function "f" is given by $-\sum_{j=1}^M \lambda_j \Lambda_j^\star(\mu_j)$ and the function "θ" is given by $\sum_{i=1}^N C_i \min\limits_{j \in \mathcal{A}_i} (\mu_j / v_{ij})$. A stochastic subgradient ascent algorithm will be applied to this unconstrained problem, similarly as the stochastic gradient descent method applied in Section 4.4.

The most important observation here is that the dual program can be solved by optimizing the bids $\mu \in \mathbb{R}^M$ *or* the pseudo-bids $\rho \in \mathbb{R}^N$, they are mathematically equivalent.

Let us consider the subdifferentials of the objectives in Problem (D_μ) and (D_ρ) separately. We have (see, *e.g.,* [19, Ex. 5.4.5] for the subdifferential of the maximum[7]):

$$\partial_{\mu_\ell}\Big[\sum_{i=1}^{N} C_i \min_{j\in\mathcal{A}_i}(\mu_j/v_{ij}) - \sum_{j=1}^{M}\lambda_j\Lambda_j^\star(\mu_j)\Big] = \sum_{i\in\mathcal{B}_\ell^\star(\mu)} C_i\,\mathsf{conv}\bigcup_{j\in\mathcal{A}_i^\star(\mu)}\{\frac{1}{v_{ij}}\} - \lambda_\ell W_\ell(\mu_\ell), \quad (4.20)$$

and

$$\partial_{\rho_\ell}\Big[\sum_{i=1}^{N} C_i\rho_i - \sum_{j=1}^{M}\lambda_j\Lambda_j^\star(\max_{i\in\mathcal{B}_j}\rho_i v_{ij})\Big] = C_\ell - \sum_{j\in\mathcal{A}_\ell^\star(\rho)}\lambda_j W_j(\max_{i\in\mathcal{B}_j}\rho_i v_{ij})\,\mathsf{conv}\bigcup_{i\in\mathcal{B}_j^\star(\rho)}\{v_{ij}\}, \quad (4.21)$$

where $\mathsf{conv}\,S$ indicates the convex hull of the set S. Abusing notation, we use $\mathcal{A}_\ell^\star(\rho)$ and $\mathcal{A}_\ell^\star(\mu)$ (similarly for $\mathcal{B}$) to denote the same set, namely $\{j \in \mathcal{A}_i \mid \theta_{ij} = 0\}$, where we must recall from Proposition 3.1.3 that the slack variables $\theta_{ij} = \mu_j - v_{ij}\rho_i$ can be derived from either ρ or μ alone; these sets are simply the sets of indices where the corresponding maxima or minima is attained. We use this notation to emphasize that Equation (4.20) depends only upon μ and (4.21) depends only upon ρ.

Both of these differentials require accurate estimates of the functions W_j, which may not realistically be available. In the case of the primal approximation, the function needed for the derivative estimate is $W_j^{-1}(q)$, which is exactly the bid needed to win with probability q. Dually, the derivatives in the present case require the function $\lambda W_j(\mu)$, which is simply the rate at which items of type j are obtained with the bid μ. A stochastic approximation algorithm for this quantity was analyzed in Section 4.3.1 and is displayed formally as Algorithm 3.

This derivative estimator can be combined with subderivatives in order to construct two time scale stochastic subgradient ascent algorithms for the solution of the dual Problem (D). To this end, consider Algorithm 4, which is used for the solution of (D_μ) and Algorithm 5, which is constructed to solve (D_ρ). We use the notation $g_i^\mu(n) \in \mathsf{conv}\bigcup_{j\in\mathcal{A}_i^\star(n)}\{\frac{1}{v_{ij}}\}$ and $g_j^\rho(n) \in \mathsf{conv}\bigcup_{i\in\mathcal{B}_j^\star(n)}\{v_{ij}\}$ for subgradients that are maintained throughout execution, and $\mathcal{A}_i^\star(n), \mathcal{B}_j^\star(n)$ as the sets of extremizing indices which are also maintained throughout execution. Algorithms 4 and 5, respectively, are carrying out stochastic approximations of the following subgradient ascent algorithms

[7]Recall that for $f(x) = \max\big(f_1(x), \ldots, f_M(x)\big)$ we have $\partial f(x) = \mathsf{conv}\cup_{j\in B(x)}\partial f_j(x)$ where $B(x) = \{j \in [M] \mid f(x) = f_j(x)\}$ are the set of binding indices.

Algorithm 3: Derivative Estimate

input : A bidding opportunity of type j, with inter-arrival time τ^j_{n+1}, gradient estimates w^j_n, and arrival count $\nu_j(n)$.

output : Updated derivative estimates

Function `Derivative-Update(`x`,`τ`):`

  $\nu_j(n+1) = \nu_j(n) + 1$ `# Update arrival count`

  Place bid x and observe $\mathbf{1}\big(p^j_{n+1} \le \mu_j\big)$

  $w^j_{n+1} = w^j_n + a_{\nu_j(n)}\big[\mathbf{1}\big(p^j_{n+1} \le \mu^j_n\big) - \tau^j_{n+1} w^j_n\big]$

  return w^j_{n+1}

$$\mu^\ell_{n+1} \in \mu^j_n + b_n \partial_{\mu_\ell}\big[\sum_{i=1}^N C_i \min_{j\in\mathcal{A}_i} (\mu_j/v_{ij}) - \sum_{j=1}^M \lambda_j \Lambda^\star_j(\mu_j)\big]$$
$$\rho^\ell_{n+1} \in \rho^\ell_n + b_n \partial_{\rho_\ell}\big[\sum_{i=1}^N C_i\rho_i - \sum_{j=1}^M \lambda_j \Lambda^\star_j(\max_{i\in\mathcal{B}_j} \rho_i v_{ij})\big],$$

but with derivatives of each Λ_j replaced by estimtes from Algorithm 3, and an additional loop to maintain g^μ_i, g^ρ_j for computational efficiency.

The subgradients g^μ_i and g^ρ_j need only be arbitrary elements of the subgradients given in Equations (4.20) and (4.21), respectively. Since these subderivative sets depend upon the extremizing index sets $\mathcal{B}^\star_\ell(\mu)$ and $\mathcal{A}^\star_i(\mu)$ (similarly for ρ) we maintain these sets throughout the execution of the algorithm (this corresponds to the inner loops of each algorithm). The reason for this is computational efficiency: constructing the set $\mathcal{B}^\star_j(\mu) \triangleq \{i \in \mathcal{B}_j \mid \mu_j/v_{ij} = \min_{\ell\in\mathcal{A}_i} \mu_\ell/v_{i\ell}\}$ takes time $\sum_{i\in\mathcal{B}_j} |\mathcal{A}_i|$ which is $\mathcal{O}(MN)$. Instead, the set can be updated in time $\mathcal{O}(N)$ when one of the entries of μ is updated. This is done by scanning through each $i \in \mathcal{B}_j$ and checking to see if it still obtains the minimum, where the minimum is similarly stored and maintained between iterations.

The additional term arising in the Algorithm 4 and Algorithm 5

$$\mu \mapsto t\big((\mu - \bar{\mu})_+ - (-\mu)_+\big), \tag{4.22}$$

corresponds to the derivative of a quadratic penalty term $P_t(\mu) = \frac{1}{2}t\left\|\begin{bmatrix}\mu-\bar{\mu}\\ -\mu\end{bmatrix}\right\|_2^2$. This term is used to ensure the stability of the algorithm by penalizing iterates outside of the interval $[0, \bar{\mu}]$. As long as the true solution lies in this interval, this penalty term does not impact the point to which the algorithm converges. This is introduced ad-hoc to take advantage of the stability theorem, Proposition 4.2.1.

Algorithm 4 has M variables and a per-iteration complexity of $\mathcal{O}\big(\max_j |\mathcal{B}_j|\big)$. In general, we have the simple bound $|\mathcal{B}_j| \leq N$. However, in typical cases, it should be expected that $|\mathcal{B}_j| \ll N$.

Algorithm 4: Stochastic Gradient Ascent Step for μ

---

input : A bidding opportunity of type j, with inter-arrival time τ^j_{n+1}, the dual variables ρ_n, μ_n, gradient estimates $w_j(n)$, arrival count $\nu_j(n)$, allocation R_{ij}.

output : An updated set of dual variables ρ_{n+1}, μ_{n+1}

$w^j_{n+1} =$ `Derivative-Update`(μ^j_n, τ^j_{n+1}) `# Update` w_j

Allocate item, if won, towards contract sampled from $\mathrm{Cat}(q_1, \ldots, q_N)$ for $q_i \propto R_{ij}$

$\mu_{n+1} \leftarrow \mu_n$ `# Update counters`

$\rho_{n+1} \leftarrow \rho_n$

$g^\mu(n+1) \leftarrow g^\mu(n)$

`# gradient step:`

$\mu^j_{n+1} = \mu^j_n + b_{\nu_j(n)} \left[\sum_{i \in \mathcal{B}^\star_j(n)} C_i g^\mu_i(n) - w_j(n+1) - t\big((\mu_j(n) - \bar{\mu})_+ - (-\mu_j(n))_+\big)\right]$

$\mathcal{B}^\star_j(n+1) = \emptyset$ `# Initialize new set`

for $i \in \mathcal{B}_j$ **do**

  `# Maintain data structures`

  if $\mu^j_{n+1}/v_{ij} \leq \rho^i_n$ **then**

    $\rho^i_{n+1} = \mu^j_{n+1}/v_{ij}$

    $\mathcal{B}^\star_j(n+1) \leftarrow \{i\} \cup \mathcal{B}^\star_j(n+1)$

    $g^\mu_i(n+1) = 1/v_{ij}$

Algorithm 5 is comparable to Algorithm 4, except that it takes gradient steps along the pseudo-bids ρ_i rather than the bids μ_j. Therefore, it has N variables, and the per-iteration complexity is given by $\mathcal{O}\big(\max_i |\mathcal{A}_i|\big)$. This is bounded by the number M of item types, but again, in practice, it is expected that $|\mathcal{A}_i| \ll M$. While this algorithm keeps track of both sets of dual variables μ, ρ, it only applies a gradient update to μ, the corresponding ρ is derived from μ.

In Algorithm 5, we sample i uniformly from $\mathcal{B}_j$ and update ρ_i. An alternative update rule may be to update the ρ_i corresponding to the contract that the item was allocated

Algorithm 5: Stochastic Gradient Step for ρ

input : A bidding opportunity of type j, with inter-arrival time $\tau_j(n+1)$, the dual variables ρ_n, μ_n, gradient estimates $w_j(n)$, arrival count $\nu_j(n)$, allocation R_{ij}.

output : An updated set of dual variables ρ_{n+1}, μ_{n+1}

$w_j(n+1) =$ `Derivative-Update`(μ_n^j, τ_{n+1}^j) `# Update` w_j

Allocate item, if won, towards contract sampled from $\mathrm{Cat}(q_1, \ldots, q_N)$ for $q_i \propto R_{ij}$

Sample $i \sim \mathcal{U}(\mathcal{B}_j)$ `# Randomly choose which` ρ_i `to update`

$\mu_{n+1} = \mu_n$ `# Update counters`

$\rho_{n+1} = \rho_n$

$g^\rho(n+1) = g^\rho(n+1)$

`# Gradient Step`

$\rho_{n+1}^i = \rho_n^i + b_{\nu_j(n)}\big[C_i - \sum_{\ell \in \mathcal{A}_i^\star(n)} w_\ell(n+1) g_\ell^\rho(n) - t\big((\rho_n^i - \bar{\rho})_+ - (-\rho_n^i)_+\big)\big]$

$\mathcal{A}_i^\star(n+1) = \emptyset$ `# Initialize new set`

for $\ell \in \mathcal{A}_i$ **do**

  if $v_{i\ell}\rho_{n+1}^i \geq \mu_\ell(n)$ **then**

    $\mu_{n+1}^\ell = v_{i\ell}\rho_{n+1}^i$

    $\mathcal{A}_i^\star(n+1) = \{\ell\} \cup \mathcal{A}_i^\star(n+1)$

    $g_\ell^\rho(n+1) = v_{i\ell}$

to. However, this would introduce a dependence between the current allocation array R_{ij}, and the ascent rule, which we wish to avoid.

In contrast to the primal stochastic approximation algorithms (Section 4.4), the algorithms in the present section result only in the optimal dual variables ρ_i or μ_j. These dual variables can be converted into optimum bids according to Proposition 3.1.3, and the rate s_j at which supply is attained by these bids is estimated according to Algorithm 3. These supply rate estimates then need to be fed into Problem (T_t) (Chapter 3) to calculate an appropriate allocation matrix R. Since s_j is calculated through a stochastic algorithm, it has absolutely no guarantee of being feasible for Problem (P). This makes it clear why it is so important that Problem (T_t) implements the contract fulfillment via a penalty, rather than a strict constraint. A new R matrix can be calculated periodically and independently from the stochastic approximations.

4.5.2 Dual Algorithm Convergence

We will analyze the convergence of Algorithm (4), the algorithm that purports to solve (D_μ), as Algorithm (5) for Problem (D_ρ) is analogous.

The key to establishing the convergence of these algorithms to optimal dual multipliers is the a-priori stability of the iterates. That is, it should be shown that $\sup_n ||\mu_j(n)|| < \infty$ *a.s.* and $\sup_n ||\rho_i(n)|| < \infty$ *a.s.*. The way we establish this is through the penalty term $P_t(\mu)$ (see Equation 4.22) which penalizes iterates outside of the interval $[0, \bar{\mu}]$. Since these constraints encode a compact polytope, the result [102, Prop C.3] can be applied to obtain stability. The intuition is essentially to add a term to the stochastic iterates which facilitates the application of the Borkar-Meyn stability theorem [26] [25, Thm 3.7], *i.e.,* the iterates are attracted back to some compact set when they drift too far away. Indeed, the following is a corollary of Proposition 4.2.4.

Proposition 4.5.1 (Stability). *Let $t > 0, \bar{\mu} > 0$ and a_n, b_n satisfy the Robbins-Monro conditions. The iterates μ_n of Algorithm (4) are stable,* i.e., *sup_n* $||\mu(n)||_2 < \infty$ a.s.

It is also important to verify that Algorithm 4 correctly calculates subgradients in Equation (4.20).

Lemma 4.5.1 (Correctness). *With reference to Algorithm 4, if ρ_n, μ_n satisfies $\rho_n^i = \min_{j \in \mathcal{A}_i} (\mu_n^j / v_{ij})$ then the same relationship holds for ρ_{n+1} and μ_{n+1}. Moreover,*

$$\mathcal{B}_j^\star(n+1) = \{i \in \mathcal{B}_j \mid \rho_{n+1}^i = \mu_{n+1}^j / v_{ij}\}$$

and $\sum_{i \in \mathcal{B}_j^\star(n+1)} C_i g_i^\mu(n+1)$ *is a subgradient for the function* $\mu \mapsto \sum_{i=1}^N C_i \min_{\ell \in \mathcal{A}_i} (\mu_\ell / v_{i\ell})$ *at* μ_{n+1}^j.

Proof. Suppose that $\forall i \in [N] : \rho_n^i = \min_{j \in \mathcal{A}_i} (\mu_n^j / v_{ij})$ and pick some arbitrary $j \in [M]$, $i \in \mathcal{B}_j$ and consider the updated μ_{n+1}. We have $\mu_{n+1}^\ell = \mu_n^\ell$ for each $\ell \neq j$. If $\mu_{n+1}^j / v_{ij} \leq \rho_n^i$ then $\rho_{n+1}^i = \min_{\ell \in \mathcal{A}_i} (\mu_\ell(n+1)/v_{i\ell}) = \mu_{n+1}^j / v_{ij}$ and hence ρ_{n+1} satisfies the stated loop invariant, since j, i were arbitrary.

By definition, the set $\mathcal{B}_j^\star(\mu)$ is exactly

$$\mathcal{B}_j^\star(\mu) = \{i \in \mathcal{B}_j \mid \rho_i(\mu) = \mu_j / v_{ij}\},$$

where $\rho_i(\mu) = \min_{\ell \in \mathcal{A}_i} (\mu_\ell / v_{i\ell})$, which is the set of contracts where the minimization in the definition of ρ_i is attained by type j. Since the algorithm checks each possible $i \in \mathcal{B}_j$ for whether or not it is included in this set, we have $\mathcal{B}_j^\star(n+1) = \mathcal{B}_j^\star(\mu_{n+1})$.

That $\sum_{i \in \mathcal{B}_j^\star(n+1)} C_i g_i^\mu(n+1)$ is a subgradient now follows from Equation (4.20) since we have correctly constructed the set $\mathcal{B}_j^\star(\mu_{n+1})$ and $j \in \mathcal{A}_i^\star(\mu_{n+1})$ since

$$\mathcal{A}_i^\star(\mu) = \{\ell \in \mathcal{A}_i \mid \rho_i(\mu) = \mu_\ell / v_{i\ell}\},$$

and by the above described loop invariant. □

Thus, we have the convergence theorem, a corollary of Proposition 4.2.7.

Theorem 4.5.1 (Convergence)**.** *Let* $t > 0, \bar{\mu} > 0$ *and* a_n, b_n *satisfy the Robbins-Monro conditions with* $b_n / a_n \to 0$, *and let* p_n, τ_n, ϕ_n *be generated according to the market model Definition 2.2.1. Let* μ_n *be a sequence generated by iteration of Algorithm 4 where* μ_n^j *is updated whenever* $\phi_n = j$. *Suppose that* $\mu^\star$ *is optimal for Problem* (D_μ) *and* $\bar{\mu}$ *is chosen such that* $0 < \mu^\star < \bar{\mu}$. *Then, the iterations converge almost surely* $\mu_n \to \mu^\star$ a.s. *as* $n \to \infty$.

4.6 Conclusion

In this chapter we have seen how the particular properties of second price auctions lead naturally to the construction of stochastic approximation algorithms for learning solutions to Problem (P) and (D). The key enabling properties are the integral representations of Λ_{2nd} and $\Lambda_{2nd}^\star$ (Proposition 2.1.1, and 2.1.3). These properties imply that the derivatives of the cost functions are obtained by $\Lambda'_j(q) = W_j^{-1}(q)$, which is nothing but the bid needed to win items of type j with probability q, and $\Lambda_j^{\star\prime}(\mu) = W_j(\mu)$, which is the rate at which items are won with a bid of μ. In addition to the convex duality discussed in Chapter 3, there is evidently a further duality between $W(q)$ and its inverse $W^{-1}(\mu)$. Moreover, these quantities would need to be calculated whether or not this derivative relationship were satisfied, and thus the derivatives come for free. This key property holds only for the second price auction.

While the primal algorithm (Section 4.4) calculates the full solution (s, R) (*i.e.,* both the rates of supply s and the allocation array R) the dual approach (Section 4.5) obtains only the bids μ_j or pseudo bids ρ_i. However, given these quantities, it is possible to calculate an appropriate array R according to the methods of Section 3.2.3.

As a consequence of the relationship between μ, ρ (Proposition 3.1.3) there is a further duality between forms of the dual, given precisely by Problems (D_μ) and (D_ρ). This results in two separate forms of a dual stochastic approximation algorithm (Algorithms 4 and 5). Algorithm (4) has M variables and a per-iteration time complexity of $\mathcal{O}(N)$ whereas Algorithm (5) has N variables and a per-iteration time complexity of $\mathcal{O}(M)$.

Since the primal algorithm has a large number, $d - N \leq (M-1)N$ variables, it is likely to be the slowest to converge in practice and has a high per-iteration complexity; its only advantage being that it provides a completely solution to the problem simultaneously. Since in practice it is expected that $N \ll M$, Algorithm (5) which applies gradient steps to ρ has the fewest variables and is likely to converge the most rapidly. However, Algorithm (4) for solving Problem (D_μ), should have the lowest per-iteration complexity and may be applicable for situations with a very fast arrival rate.

Future Work and Convergence Rates Following the discussion of the previous paragraph, there remains a need to study the convergence rate of the three algorithms developed in this chapter. Based on general stochastic approximation theory (*e.g.,* [25, Ch. 8]), we conjecture that the algorithms in this chapter can be expected to converge, roughly speaking, at the rate $\mathcal{O}(\sqrt{a_n})$. Gradient based convex optimization algorithms can obtain linear convergence rates (*i.e.,* $\mathcal{O}(e^{-n})$) with adequate regularity assumptions [30], and these results can be carried over to stochastic gradient descent with unbiased gradients as well [131]. However, it does not seem likely that the two time-scale algorithms studied here (or any other method) can attain rates faster than $\mathcal{O}(1/\sqrt{n})$ as the gradient estimates are biased and cannot themselves be expected to converge faster than $\mathcal{O}(1/\sqrt{n})$. Still, a more thorough analytic and computational comparison between the three algorithms studied in this chapter is a topic of future and present ongoing work.

Chapter 5

Time Constraints and Bidding with Forecasted Supply Curves

In this chapter, we extend the problems previously considered throughout the thesis to a more general case where contracts have time deadlines, and supply curves include time-dependent forecasts. This work is based primarily upon [101] as well as in part upon the early unpublished work [99].

Overall, the problem formulations found in this chapter are the most practically realistic. However, a great deal of the analysis for static problems (Chapters 2 and 3) generalizes in natural ways to the time-dependent case. Moreover, stochastic approximation (Chapter 4) methods can potentially be applied to adapt to specific market conditions while still staying close to a planned bidding path.

The outline is as follows. First, in Section 5.1.1 we modify the definitions of supply curves (and the associated cost functions) to account for time-dependency, as in $W(x,t)$. Using this time-dependent function, we define an infinite dimensional optimization problem (Section 5.1.3) which extends the contract management problem to account for time deadlines (Section 5.1.2). In Section 5.2 we discuss two modifications to the basic problem (namely, receding horizon control and over-provisioning) which are constructed to adapt to stochastic changes in the environment and to increase the probability of completely fulfilling contracts. A number of computational examples are given in this same section. Finally, we discuss computational methods for working with time-dependent forecasts and carry out simulations with real market data in Section 5.3. We conclude the chapter in Section 5.4.

Market Data The IPinYou dataset [113, 215] is a publicly available RTB dataset consisting of price and bidding information from a single DSP. The dataset consists of timestamped bidding logs consisting of the bid that was placed, the price that was paid (if the item was won), and a set of *tags* indicating part of the DSP's segmentation of user types, among other data. We have made some use of this data throughout the thesis, and it plays a more significant role in the simulations of this chapter. We have used the tag as the item type j, and the timestamps of bidding opportunities to measure arrival rates of items. We make use of data available for a 168 hour period (*i.e.,* one week). Some additional details of how we have used this data is available in the appendix.

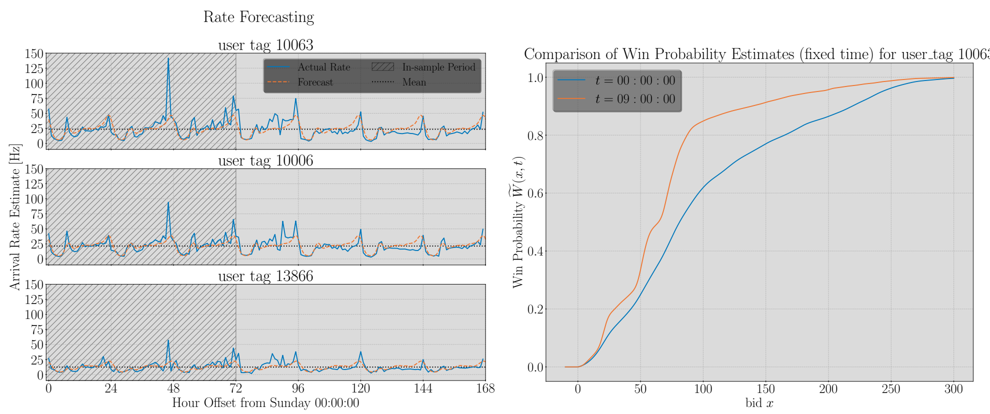
5.1 Continuous Time Contract Management

In the following subsections, we extend the notion of a supply curve to include time $t \in \mathbb{R}_+$ (Section 5.1.1) and then adapt the definition of the contract management problem to include a time deadline $T_i \in \mathbb{R}_+$ (Section 5.1.2) and then analyze this problem in Section 5.1.3. Much of the analysis mirrors the work of Chapter 3, however, as an additional consequence of duality in this section, we are able to reduce the infinite dimensional planning problem into a simple finite dimensional optimization problem (albeit involving integration): Propositions 5.1.3 and 5.1.4.

5.1.1 Time Varying Supply Curves

Recall from Section 2.1 that a supply curve simply models the probability of winning an item of a specified type with a specified and fixed bid. However, this definition is not completely adequate since the statistics of market prices are not stationary. Firstly, they exhibit predictable daily and weekly seasonality (as well as predictable holiday effects, *etc.*) and unpredictable stochastic variations. This is illustrated clearly in Figure 5.1a and Figure 5.1b (the latter of which is reproduced from Chapter 2).

In Figure 5.1a we have, for three separate tags in the IPinYou dataset, calculated hourly average item arrival rates (measured in Hertz) by taking simple windowed averages and then interpolating the results into smooth curves. The curves are constructed to be 24-hour periodic functions $\lambda_j(t)$ using averages calculated for only the first 72 hours of available data (the in-sample period). The results of this procedure are depicted qualitatively in the figure, and are able, broadly speaking, to capture the daily seasonality of item arrival rates. These simple forecasts are adequate for our purposes, but it is to be understood that,



(a) Rate Forecasting (b) Win Probability Estimates

Figure 5.1: Illustration of $W_j(x,t)$ Estimation Methods

given larger datasets or additional factors (*e.g.,* weather, holiday schedules, measurements of media "hype", *etc.*) superior predictive forecasts can be constructed. As long as these forecasts result in a function $\lambda_j(t)$, providing an expectation of the item arrival rates at time t, the methods of this section are applicable.

Similarly, Figure 5.1a demonstrates the variability in the *prices* of items in the bidding market. These estimates are obtained by KDE smoothing (see Section 2.4) data contained within a small window surrounding the specified time. Motivated by these figures, we will now consider item arrival rates $\lambda_j(t) > 0$ which is a positive and continuous function indicating the rate, instantaneously at time t, that items of type j arrive to auction. Similarly, $W_j(x,t)$ is a function such that $x \mapsto W_j(x,t)$ satisfies Definition 2.1.1 for every t, and such that $t \mapsto W_j(x,t)$ is measurable for every x.

Every other function (*c.f.,* Section 2.1) has similar t-pointwise analogs, *i.e.,* $f_j(x,t)$, $\Lambda_j(x,t)$, $\Lambda_j^\star(x,t)$, $W_j'(x,t)$ (if the derivative exists) *etc.* where the modifiers * and $'$ (indicating convex conjugation and differentiation, respectively) are to be understood to apply to x, pointwise for each t.

5.1.2 Contracts with Deadlines

In Chapter 2, we defined a contract to be a tuple consisting of a target item valuation $C_i > 0$ as well as a collection of non-negative item valuations $\left(v_{ij}\right)_{j=1}^{M}$. We now introduce a *time deadline* $T_i > 0$ into the contract, indicating that items summing to C_i value need to be obtained by the deadline. This is a clearly distinct, but closely related goal as has been considered thus far. The technical problem formulation, analogous to Problem (P^m), is given in the next section.

As a potential extension, we may also specify a *start* time of the contract S_i (say). This extension can be modelled within our framework since for each item type $j \in \mathcal{A}_i$ we can introduce another pseudo-type j' with the same supply curve for $t \geq S_i$ but with $W_{j'}(x,t) = 0$ for $t < S_i$ and then replace each $j \in \mathcal{A}_i$ with j' instead. We then treat the contract as if $S_i = 0$. Even though the contract is technically part of the problem at time 0, there are no types which can fulfill it until time S_i. Still, even tackling this extension directly is itself trivial, and would essentially come down to replacing integrals over $[0, T_i]$ with integrals over $[S_i, T_i]$.

5.1.3 Analysis of the Continuous Time Problem

The basic set up of our problem in this chapter is exactly analogous to Problem (P^m), except that now we are seeking to calculate a *bid path* $x_{ij}(t)$ and an *allocation path* $\gamma_{ij}(t)$. These quantities have similar interpretations as do x_{ij}, γ_{ij} in Section 2.3, except that the particular time instant of the arriving item must be taken into account. To be specific, suppose an item of type j arrives at time t: a contract $i \in [N]$ such that $t < T_i$ would be chosen according to the probability $\gamma_{ij}(t)$, and then the bid $x_{ij}(t)$ should be placed for the item; if it is won, it should be allocated towards contract i.

The following problem provides the basic formulation for the time constrained instance:

$$\begin{aligned} \underset{x,\gamma}{\text{minimize}} \quad & \sum_{i=1}^{N} \int_0^{T_i} \Big[\sum_{j \in \mathcal{A}_i} \gamma_{ij}(t) \lambda_j(t) f_j(x_{ij}(t), t) \Big] \mathsf{d}t \\ \text{subject to} \quad & \sum_{j \in \mathcal{A}_i} \int_0^{T_i} \gamma_{ij}(t) v_{ij} \lambda_j(t) W_j(x_{ij}(t), t) \mathsf{d}t \geq C_i \\ & \sum_{i \in \mathcal{B}_j} \gamma_{ij}(t) \leq 1, \gamma_{ij}(t) \geq 0. \end{aligned} \tag{P^m_T}$$

As opposed to Problem (P^m), Problem (P^m_T) is an *infinite dimensional* optimization problem. However, we are careful to note that it is *not* an optimal control problem (and thus does not warrant the use of either dynamic programming or the maximum principle [99]) since the dynamics do not need to be described by a differential equation, and it is also *not* a problem in the calculus of variations, since the objective does not depend on derivatives of x or γ (and thus does not warrant the application of the Euler-Lagrange equations).

Definition 5.1.1 (Additional Notation and Conventions)**.** To avoid any notational confusion, we will assume, without loss of generality, that contracts are sorted in order of their time deadlines as in $0 = T_0 \leq T_1 \leq \cdots \leq T_N \triangleq T$ and where we have introduced the extra notation T_0, T for convenience in some expressions. The time interval $[0, T)$ will sometimes need to be broken into K segments denoted $[0, \widetilde{T}_1)$, $[\widetilde{T}_1, \widetilde{T}_2)$, ..., $[\widetilde{T}_{K-1}, \widetilde{T}_K)$, and indexed by k. The time T_i can be viewed as the time that contract i "exits the problem". Similarly, we denote $T^j \triangleq \max_{i \in \mathcal{B}_j} T_i$ to be the time that item type j exits (*i.e.,* is no longer useful). And $\mathcal{T}_t \triangleq \{i \in [N] \mid t < T_i\}$ to be the set of contracts that remain active at time t.

Remark 5.1.1 (Speed Scaling and Processor Scheduling)**.** It has been pointed out by anonymous reviewers that Problem (P^m_T) has a tenuous connection with *speed scaling* [73, 14]. This is a problem of dynamically tuning the clock speed of a processor in order to minimize power consumption and respect temperature bounds, while still trying to complete incoming jobs by their specified deadlines. Problem (P^m_T) is clearly much more general than vanilla speed scaling problems since there are multiple objectives and a sum of multiple terms in the cost functions. However, there may be reasonable analogies between Problem (P^m_T) and multiprocessor speed scaling, or distributed scheduling [6, 7].

Similarly to Problem (P^m), it is possible to reformulate Problem (P^m_T) as an infinite dimensional *convex* optimization problem, by following essentially the same steps as were carried out in Chapter 2. Indeed, all of the results with respect to the convexity of $\Lambda_j(q, t) \triangleq f \circ W_j^{-1}(x, t)$ continue to hold t-pointwise, where function compositions, inversion, differentiation, *etc.* are understood to be with respect to the first argument.

Proposition 5.1.1 (Convex Reformulation [101])**.** *In a first or second price auction, suppose that for each $j \in [M]$, the acquisition cost curve $\Lambda_j(q, t)$ is convex. Then, Problem (P^m_T) can be reformulated as*

$$\begin{aligned}
\underset{s,R}{\textit{minimize}} \quad & \sum_{j=1}^{M} \int_0^{T^j} \lambda_j(t) \Lambda_j(s_j(t)/\lambda_j(t), t) \mathsf{d}t \\
\textit{subject to} \quad & \sum_{j \in \mathcal{A}_i} v_{ij} \int_0^{T_i} R_{ij}(t) \mathsf{d}t \geq C_i \\
& \sum_{i \in \mathcal{B}_j \cap \mathcal{T}_t} R_{ij}(t) = s_j(t), R_{ij}(t) \geq 0.
\end{aligned} \tag{P_T}$$

If a solution exists, then a solution to the original problem (P_T^m) *is obtained via* $x_{ij}(t) = W_j^{-1}(s_j(t)/\lambda_j(t), t)$ *for each* $i \in [N]$ *and* $\gamma_{ij}(t) = R_{ij}(t)/s_j(t)$ *(with* $0/0 \triangleq 0$*). Moreover, Problem* (P_T) *is a convex optimization problem.*

Proof. The proof is mutatis-mutandis of the proof of Proposition 2.3.1. □

Similarly to Assumption 3.0.1, we need to assume that there exists enough supply in the market to fulfill all of the contracts, this enables a duality analysis.

Assumption 5.1.1 (Adequate Supply)**.** There exists an array $\gamma_{ijk} \geq 0$ of allocation probabilities such that $\forall j \in [M], k \in [K] : \sum_{i \in \mathcal{B}_j} \gamma_{ijk} < 1$ and

$$\sum_{k:T_k \leq T_i} \sum_{j \in \mathcal{A}_i} v_{ij} \gamma_{ijk} \int_{T_{k-1}}^{T_k} \lambda_j(t) \mathsf{d}t \geq C_i,$$

for each contract $i \in [N]$.

The most important conclusion to be drawn for Problem (P_T) is that, even though it is an infinite dimensional problem, exact solutions can be represented with only a *finite* number of parameters, which can be calculated through the dual, which is also a finite convex optimization problem.

Proposition 5.1.2 (Duality)**.** *A dual of Problem of* (P_T) *can be formulated as*

$$\begin{aligned} \underset{\rho,\mu}{\text{maximize}} \quad & \sum_{i=1}^{N} \rho_i C_i - \sum_{j=1}^{M} \sum_{k:T_k \leq T^j} \int_{T_{k-1}}^{T_k} \lambda_j(t) \Lambda_j^\star(\mu_{jk}, t) \mathsf{d}t \\ \text{subject to} \quad & v_{ij}\rho_i \leq \mu_{jk} \; \forall i \in \mathcal{B}_j \cap \mathcal{T}_{T_k} \end{aligned} \tag{D_T}$$

which is a finite convex program. Problem (D_T) *is dual to Problem* (P_T) *in the sense that if* $D_T^\star$ *and* $P_T^\star$ *are their respective values (possibly* ∞ *or* $-\infty$*), then* $D_T^\star \leq P_T^\star$*. Moreover, under Assumption 5.1.1 there exists a solution* $(s, R) \in L_2([0,T])^M \times L_2([0,T])^d$ *to Problem* (P_T) *and a solution* $(\rho, \mu) \in \mathbb{R}^N \times \mathbb{R}^M$ *to Problem* (D_T) *and* $-\infty < D_T^\star = P_T^\star < \infty$*.*

See proof on page 195.

Similarly to the results of Chapter 3 (Proposition 3.1.3), optimal bids are obtained as $x_j(t) = g_j^{-1}(\mu_{jk}, t)$ for $t \in [T_{k-1}, T_k)$. That is, optimal bid paths, which are in general infinite dimensional objects, can be fully specified by the finite collection of dual variables

μ_{jk} in the second price case, and by these variables along with the function g_j^{-1} in the first price case.

Now, let us define $\lambda_{jk}\overline{W}_{jk}(x) \triangleq \int_{T_{k-1}}^{T_k} \lambda_j(t)W_j(x,t)\mathsf{d}t$ where λ_{jk} is selected so as to normalize $\overline{W}_{jk}$ into a cumulative distribution function. We have

$$\begin{aligned}\int_{T_{k-1}}^{T_k} \lambda_j(t) f_j^{2nd}(x,t)\mathsf{d}t &= \int_{T_{k-1}}^{T_k} \lambda_j(t) \int_0^x u\mathsf{d}W_j(u,t)\\ &= \lambda_{jk}\int_0^x u\mathsf{d}\overline{W}_{jk}(u)\\ &\triangleq \lambda_{jk}\overline{f}_{jk}^{2nd}(x).\end{aligned}$$

That is, the time integral of the second price cost function is equal to the function obtained by combining the formula for expected cost in second price auctions (Equation 2.2) with the averaged supply curve. A similar property is obtained for $\overline{\Lambda}_{jk}^{2nd}(q) = \int_{T_{k-1}}^{T_k} \Lambda_j^{2nd}(q,t)\mathsf{d}t$ by swapping the order of integration in Equation 2.6

Moreover, since valuations v_{ij} are time independent, this implies that we can further restrict the search for an optimal allocation path $\gamma_{ij}(t)$ to another finite set of variables γ_{ijk}. Thus, we have established the following propositions.

Proposition 5.1.3 (Finite Reduction – Second Price Case)**.** *In the second price case, the infinite dimensional convex optimization problem* (P_T) *can be reduced into an instance of the finite dimensional convex optimization problem* (P) *with the averaged supply curves* $\overline{W}_{jk}$ *and rates* λ_{jk}*, along with the expanded valuations*

$$v_{ijk} \triangleq \begin{cases} v_{ij} & \textit{if } T_k \leq T_i \\ 0 & \textit{otherwise.}\end{cases} \tag{5.1}$$

The convex acquisition cost functions are obtained, as usual, by $\overline{\Lambda}_{jk} \triangleq f_{jk}^{2nd} \circ \overline{W}_{jk}$.

A finite dimensional problem can be constructed for the first price case as well, however, it does not appear to be possible to obtain a convex program in the same way. Precisely, we may define new supply curves $\lambda_{jk}\widetilde{W}_{jk}(x) = \int_{T_{k-1}}^{T_k} \lambda_j(t)W_j(g_j^{-1}(x,t),t)\mathsf{d}t$, but the cost function *can not* simply be constructed as $\widetilde{f}_{jk}(x) \stackrel{?}{=} x\widetilde{W}_{jk}(x)$, since the bid is $g_j^{-1}(x,t)$, rather than simply x. Indeed, we need to consider the cost function

$\widetilde{f}_{jk}(x) \triangleq \lambda_{jk} \int_{T_{k-1}}^{T_k} g_j^{-1}(x,t) W_j(g_j^{-1}(x,t),t)\mathsf{d}t$. This results in the following reduction, stated formally.

Proposition 5.1.4 (Finite Reduction – First Price Case)**.** *In the first price case, the infinite dimensional optimization problem* (P_T^m) *can be reduced into an instance of the finite dimensional optimization problem* (P^m) *with the supply curves* $\widetilde{W}_{jk}(x)$ *and rates* λ_{jk}*, along with the expanded valuations*

$$v_{ijk} \triangleq \begin{cases} v_{ij} & \textit{if } T_k \leq T_i \\ 0 & \textit{otherwise,} \end{cases} \tag{5.2}$$

and the cost functions $\widetilde{f}_{jk}(x) \triangleq \lambda_{jk} \int_{T_{k-1}}^{T_k} g_j^{-1}(x,t) W_j(g_j^{-1}(x,t),t)\mathsf{d}t$.

Remark 5.1.2 (Convexity – First Price Case)**.** The difficulty in obtaining a finite primal convex program in the first price, in the same simple way as in the second price case, is with establishing and verifying conditions under which the function $\widetilde{\Lambda}_{jk} = \widetilde{f}_{jk} \circ \widetilde{W}_{jk}^{-1}$ is convex. To see the difficulty, it must be recognized that the definition of $\widetilde{W}_{jk}^{-1}$ involves an integral over time, and so too does $\widetilde{f}_{jk}$; these functions simply do not combine in the same simple way as in the second price case. Still, convexity conditions for this function can be obtained by analyzing functions of the form $\widetilde{\Lambda} = \widetilde{f} \circ f \circ \Lambda \circ W \circ \widetilde{W}^{-1}$. This is a perturbation of a convex function by functions which "should" be "approximately the identity". Unfortunately, the conditions resulting from this analysis are quite complicated and unlikely to be of practical utility. An alternative approach is to recover the primal problem through the dual of the dual, however, this requires the calculation of the convex conjugate of functions of the form $\mu \mapsto \int_0^T \lambda(t)\Lambda_j^\star(\mu,t)\mathsf{d}t$.

Despite the difficulty raised in Remark 5.1.2, the first price case can still be fully solved through the application of finite convex optimization. Given a solution μ, ρ to the dual problem, we can calculate the total supply acquired $s_{jk}^\star \triangleq \int_{T_{k-1}}^{T_k} W_j(g_j^{-1}(\mu_{jk},t),t)\mathsf{d}t$ by an optimal bid path. With this optimal supply in hand, an exactly analogous instance of Problem (T_t) (Section 3.2.3) can be used to calculate the rates R_{ijk} at which to allocate supply towards contracts.

Thus, we have formally reduced the solution of the infinite dimensional monotone optimization problem (P_T^m) into a form that can be fully solved by the methods of Chapter 3.

5.2 Dynamic Algorithms

The bid paths $x_j(t)$ and allocations $\gamma_{ij}(t)$ computed according to the methods of Section 5.1.3 (see also 5.3 for more on computational methods) result in completely *open loop* bidding algorithms. That is, at time t, the bid $x_j(t)$ is optimal only so long as the total allocation towards contract i is actually equal to $c_i(t) = \sum_{j \in \mathcal{A}_i} v_{ij} \int_0^t \gamma_{ij}(u)\lambda_j(u)W_j(x_j(u), u)\mathsf{d}u$, and that this model of acquired items remains accurate until the time deadline T.

Of course, since the environment is stochastic, the calculated bids and allocations cannot be fully optimal going forward. Indeed, $x_j(t), \gamma_{ij}(t)$ constitutes an *open loop* control for the problem. Put another way, $x_j(t), \gamma_{ij}(t)$ is an optimal solution to a *fluid limit* of some appropriate Markov Decision Process [51, 71, 166, 23]. We discuss this further in Section 6.1. In the following sections, we consider two heuristic methods for adapting to stochastic changes to the environment: Section 5.2.1 on receding horizon control (also referred to as model predictive control or the re-solving heuristic [191]) [105, 33], and Section 5.2.2 on probabilistic constraints. We illustrate these methods with simple examples.

The methods considered in this section are distinct from, but complementary to, those of Chapter 4. As well, the discussion of this section is relatively *informal*, as a complete, detailed, and rigorous discussion of this content would bring us far from the main point of this thesis.

5.2.1 Model Predictive Control

Recall the simple single item type Example (2.3.1) from Chapter 2 where we need to minimize the cost $\lambda f(x)$ of bidding subject to the constraint that $\lambda W(x) \geq C$. Let us augment this slightly with a time deadline T and consider the problem of minimizing $T\lambda f(x)$ subject to the constraint that $T\lambda W(x) \geq C$. This problem has two parameters, namely, $C > 0$ and $T > 0$. Since the objective is monotone, the solution mapping for this problem is $x^\star(C, T) = W^{-1}\left(\frac{1}{\lambda}\frac{C}{T}\right)$ where, for notational simplicity, define $W^{-1}(q) = \bar{x}$ (the maximum bid) if $q \geq 1$ (that is, we bid based on our *best effort* if the problem is not feasible).

If over the time period $[0, T]$ there are stochastic fluctuations in the actual number of item arrivals, or in the prices of those items, or if the supply curve is not perfectly known, then it is not certain that the constant bid $x^\star(C, T)$ will actually fulfill the contract by time T, and if it does, it may not have done so at the lowest possible cost.

Therefore, it is natural to convert our solution into a receding horizon (RH) [33] algorithm in order to adapt the bid as time progresses. To do so, suppose that after t time has elapsed, we have accumulated $c(t)$ supply. In this situation, we are faced with a problem identical to the original one at time $t = 0$ except that we now have a $T - t$ time available to acquire $C - c(t)$ items. The mapping $x^\star$ for these parameters is defined as the receding horizon control for this problem:

$$x^\star(C - c(t), T - t) = W^{-1}\big(\frac{1}{\lambda}\frac{C - c(t)}{T - t}\big). \tag{5.3}$$

The RH framework accounts for unexpected supply shortages or surpluses and also enables us to naturally incorporate a case wherein new contracts arrive before the set of current contracts have been fulfilled.

Example 5.2.1. We consider an illustrative example where the DSP forecasts supply with $W(x) = 1 - e^{-\gamma x}$ (*c.f.,* Example 2.1.2), and the constant supply rate λ_0. We know that $W^{-1}(q) = -\frac{1}{\gamma}\ln(1 - q)$ for $q \in [0, 1)$, and thus, the RH bidding function is easily available in closed form as

$$x_{\mathrm{rh}}(c(t), t) = \begin{cases} -\frac{1}{\gamma}\ln\big[1 - \frac{1}{\lambda_0}\frac{C-c(t)}{T-t}\big] & \frac{1}{\lambda_0}\frac{C-c(t)}{T-t} < 1 \\ \infty & \text{otherwise} \end{cases}. \tag{5.4}$$

Suppose now that the realized supply over the period $[0, T]$ obeyed the law $\lambda(t)W(x)$, i.e., the DSP's estimate of the arrival rate is in error by $\lambda(t) - \lambda_0$. For the receding horizon case, the supply *actually* attained can be described by the differential equation

$$\dot{c}_{\mathrm{rh}}(t) = \lambda(t)W(x^\star(C - c_{\mathrm{rh}}(t), T - t));\ c_{\mathrm{rh}}(0) = 0, \tag{5.5}$$

and the analogous equations for the static case $c(t)$.

Since the optimal bid $x^\star$ involves the inverse of the win probability W^{-1}, substituting it into Equation (5.5) results in a separable ordinary differential equation

$$\begin{aligned} \dot{c}_{\mathrm{rh}}(t) &= \frac{\lambda(t)}{\lambda_0}\frac{C - c_{\mathrm{rh}}(t)}{T - t};\ c_{\mathrm{rh}}(0) = 0 \\ \implies c_{\mathrm{rh}}(t) &= C\Big[1 - \exp\big(-\frac{1}{\lambda_0}\int_0^t \frac{\lambda(s)\mathsf{d}s}{T - s}\big)\Big], \end{aligned}$$

which reduces to the straight line $c_{\mathrm{rh}}(t) = \frac{Ct}{T}$ if the estimate is accurate, *i.e.,* $\lambda(t) = \lambda_0$. An illustrative simulation example is seen in Figure 5.2.

5.2.2 Probabilistic Constraints and Overprovisioning

The receding horizon method described in Section 5.2.1 is only one of many possible methods for adapting to a stochastic environment. Another method is *over provisioning*, where we *aim* to obtain more than C supply, in order to avoid the risk of available supply being lower than was expected.

To understand over provisioning, suppose that we want to calculate a bid that will allow us to fulfill a contract with high probability. Probabilistic constraints (also referred to as *chance constrained programming*) is often intractable (but see [124, 139, 132, 143]). However, in the present simple single-item single-contract case we are able to leverage the monotonicity of f, W to obtain a solution. This method of risk management differs significantly from, *e.g.,* [83], where the authors consider a stochastic control problem. However, the end result is qualitatively similar: the bid is inflated, "front-loading" the acquisition of items.

For the time being, suppose that supply curves are time-independent and consider the following problem with constant bid $x \in \mathbb{R}_+$

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \lambda T f(x) \\ \text{subject to} \quad & \mathbb{P}\Big\{ \int_0^T \mathbf{1}[p(s) \leq x] \mathsf{d}A(s) < C \Big\} \leq \delta, \end{aligned} \tag{5.6}$$

and where $A(t)$ is a point process describing the arrivals of items, and $p(s)$ are their prices. Ultimately, the quantity

$$c(T) = \int_0^T \mathbf{1}[p(s) \leq x] \mathsf{d}A(s), \tag{5.7}$$

is nothing but a random variable parameterized by the bid x. The expectation of this quantity is the supply curve $\mathbb{E}c(T) = T\lambda W(x)$.

Assuming that the DSP has some model[1], $F_{c(T)}(\cdot; x)$ for the number of items that will be won given a bid of x, in the form of a cumulative distribution function for the distribution of this random variable. Then, the constraint above is equivalent to (assuming F is continuous) the inequality $F_{c(T)}(C; x) \leq \delta$. It is clear from inspection that $x \mapsto$

[1]This model is distinct from the supply curve as ordinarily discussed in this work as the distribution function F is now used to model the distribution of items *actually won* given a bid of x. On the other hand, the supply curve W measures the expected number of items won with that bid, it just so happens to also be a distribution function since $W(x) = \mathbb{E}\mathbf{1}[x \leq p]$ for a price p.

$F_{c(T)}(C;x)$ is monotone decreasing in x, and therefore we conclude that the optimal bid is obtained by finding the smallest bid x which verifies the inequality $1 - F_{c(T)}(C;x) \geq 1-\delta$, which can be obtained (if $1-\delta$ is in the range) by finding an inverse the function $x \mapsto 1 - F_{c(T)}(C;x)$.

An alternative perspective can be obtained by instead parameterizing $F_{c(T)}$ through Equation (5.3), the solution $x^\star(C,T)$ to the mean constrained problem. Since this map is itself monotone in C, finding a minimal x to verify $F_{c(T)}(C;x) \leq \delta$ is equivalent to finding a minimal *supply inflation* $C_{\text{infl}} \geq C$ which verifies $F_{c(T)}(C;x^\star(C_{\text{infl}},T)) \leq \delta$. We emphasize that, owing to the monotonicity of the functions involved, this procedure is a simple computation and the supply inflation is unique (if the problem is feasible), regardless of the complexity of the model $F_{c(T)}$. The model F, for some fixed "risk level" δ therefore serves a purpose equivalent to that of a supply curve $W(x)$.

It follows that the DSP is free to construct a highly sophisticated model for $c(T)$ and then substitute the supply inflation C_{infl} implied by this model and their risk constraint δ into the simple form of Problem 2.12 involving only the *average* supply curve W. Similarly to [83], this algorithm will front-load the purchase of items earlier in the period, and is therefore more robust to supply shocks. We have informally established the following:

Proposition 5.2.1 (Risk Management). *Given a model $F_{c(T)}$ in the form of a continuous cumulative distribution function for the supply $c(T)$ (Equation 5.7), the risk constrained problem 5.6 is equivalent to the average-case problem with the inflated supply*

$$C_{\text{infl}} = \min\{z \mid F_{c(T)}(C;x^\star(z,T)) \leq \delta, C \leq z\}.$$

Example 5.2.2. Let us take the simple Poisson model $c(T) \sim \text{Po}(\lambda T W(x))$ for the supply attained over the period $[0,T]$. Though the previous discussion makes clear that it is not necessary, we will approximate the Poisson c.d.f. via a Chernoff bound [2]:

$$F_{\text{Po}(\lambda T W(x))}(C) \leq \big(\frac{e\lambda T W(x)}{C}\big)^C e^{-\lambda T W(x)} \text{ if } C < \lambda T W(x),$$

which will allow us to obtain an illustrative closed form expression that can be used as C_{infl}. Using the Chernoff bound, solutions of the problem

[2] Taking the Chernoff bound $\mathbb{P}\{\mathcal{X} \leq x\} \leq e^{tx}\mathbb{E}[e^{-t\mathcal{X}}]$ for any $t > 0$, and using the Poisson moment generating function $\mathbb{P}\{\mathcal{X} \leq x\} \leq \exp(tx + \ell(e^{-t}-1))$ and taking $t = -\ln(x/\ell) > 0$ for $x < \ell$, we obtain the bound.

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & \lambda T f(x) \\
\text{subject to} \quad & \big(\frac{e\lambda TW(x)}{C}\big)^C \exp(-\lambda TW(x)) \leq \delta \\
& \lambda TW(x) \geq C,
\end{aligned} \tag{5.8}
$$

are feasible for Problem (5.6) under the model $c(T) \sim \text{Po}(\lambda TW(x))$.

Making the temporary substitutions $w \triangleq \lambda TW(x)$ and $u \triangleq -w/C$ we solve the inequality

$$
\begin{aligned}
& (\frac{e}{C}w)^C e^{-w} \leq \delta \\
& \iff ue^u \geq -\frac{1}{e}\delta^{1/C} \\
& \overset{(a)}{\iff} u \leq \mathcal{W}_0\big(-\frac{1}{e}\delta^{1/C}\big) \\
& \iff \lambda TW(x) \geq -C\mathcal{W}_0\big(-\frac{1}{e}\delta^{1/C}\big)
\end{aligned}
$$

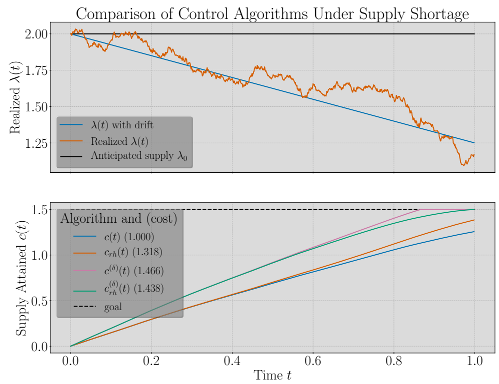
where in implication (a) $\mathcal{W}_0$ is the lower branch of the Lambert-W function, which is monotone decreasing[3] [46], and therefore the mean constraint (necessary for the Chernoff bound to apply) $\lambda TW(x) \geq C$ must still be satisfied.

It is the case that $-\mathcal{W}_0(-x/e) \geq 0$ for $x \in [0,1]$ (with $-\mathcal{W}_0(0) \triangleq \infty$) and therefore $-\mathcal{W}_0\big(-\frac{1}{e}\delta^{1/C}\big)$ serves as a "risk-inflation factor" from which we obtain an inflated supply target C_{infl} which when substituted into the constraints of Problem 2.12 will result in a feasible approximate solution of Problem 5.6

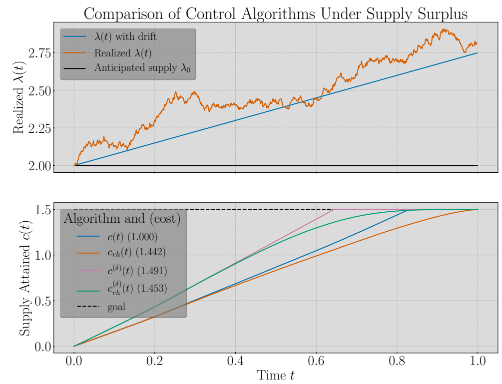
$$
C_{\text{infl}} = -C\mathcal{W}_0\big(-\frac{1}{e}\delta^{1/C}\big). \tag{5.9}
$$

Example 5.2.3. This example combines both the receding horizon example 5.2.1 and the Poisson distribution-based risk-adjustment procedure of Example 5.2.2. The DSP has the supply curve estimate $W(x) = 1 - e^{-\gamma x}$ and rate estimate λ_0. Combining the risk adjustment of Equation 5.9 with the receding horizon policy (5.4) we obtain

[3]The Lambert-W function is the inverse of $x \mapsto xe^x$. $\mathcal{W}_0(ue^u) = u$ if $u \leq -1$, i.e., if $\lambda TW(x) \geq C$



(a) Shortage



(b) Surplus

Figure 5.2: Receding Horizon Acquisition Paths

Simulated acquisition paths $c(t)$ for the case $M = N = 1$ comparing the behaviour of different algorithms in the presence of supply shortages or surpluses in comparison to expectation λ_0. Best viewed in colour. Qualitatively, when there is oversupply, the receding horizon smooths the acquisition rate to reduce costs, and when there is undersupply, it increases the bid in reaction to the shortage. The risk parameter δ is given by $1/2$ and serves to front-load the item acquisition rate.

$$x_{\text{rh}}^{(\delta)}(c(t), t) = -\frac{1}{\gamma} \ln\big[1 - \frac{1}{\lambda_0} \frac{C - c(t)}{T - t} \mathcal{W}_0\big(-\frac{1}{e} \exp(\frac{\ln \delta}{C - c(t)})\big)\big]. \tag{5.10}$$

A simple computational illustration is provided in Section 5.2.3.

5.2.3 Contract Fulfillment Example

Combining the results of the examples given in Section 5.2.1 and Section 5.2.2 results in Figure 5.2. To generate these figures, we bid according to the supply curve $W(x) = 1 - e^{-x}$ (which is known precisely) but encounter the actual supply $\lambda(t)$, which we treat as a stochastic process.

We consider two scenarios: a supply shortage (Figure 5.2a) and a supply surplus (Figure 5.2b). The expectation of $\lambda(t)$ in the shortage case is $\lambda(t) = \lambda_0 - 3t/4$ and the expectation in the surplus case is given by $\lambda(t) = \lambda_0 + 3t/4$. In addition to these drifts, we add some stochastic fluctuations (by simulating a random walk). The upper subfigures depict the supply that was actually available to the DSP over the period $[0, 1]$, over which the agent seeks to obtain $3/2$ items. The bottom set of subfigures displays the number of items that have actually been obtained by time t by various different algorithms. The costs are all relative to the cost of the basic constant bid $c(t)$.

We illustrate four different algorithms. First, $c(t)$ is obtained through a constant bid *c.f.,* Section 2.3.1. When there is a supply *shortage*, $c(t)$ has the lowest cost, however, it fails to meet the supply targets. When there is a supply *surplus* $c(t)$ comfortably fulfills the contract (reaching fulfillment well before the time deadline), but has a higher cost than did $c_{rh}(t)$, the basic receding horizon bid policy (Equation 5.3), which also fulfills the contract. However, under supply shortages, the receding horizon bid policy still fails to fulfill the contract – indeed, regardless of how high the receding horizon policy bids near the end of the period, there may simply not be enough supply available. In contrast to $c_{rh}(t)$, the bidding policies of $c^{(\delta)}(t)$ and $c_{rh}^{(\delta)}(t)$ are the risk-adjusted constant bids (a constant bid targeting the inflated supply of Equation 5.9) and the receding horizon bids with risk adjustment (Equation (5.10)), respectively. The bids resulting in acquisitions $c^{(\delta)}(t)$ successfully fulfill the contracts in both the surplus and shortage cases, since the over-provisioning target is large enough. Indeed, similarly to [83], the risk-adjusted acquisition paths *front load* the acquisition of items. The downside of the risk-adjustment is that costs are much higher, even in the case of supply surpluses, when the simple receding horizon method fulfilled the contracts at a low cost. The combined method $c_{rh}^{(\delta)}$ blends the best aspects of both: when supply is short, it front loads the acquisition of items and ensures that the contract is fulfilled; when there is a surplus of supply, $c_{rh}^{(\delta)}$ *smooths* the item acquisition rate near the end of the period and fulfills the contract at lower cost than does $c^{(\delta)}(t)$.

These basic observations manifest again in the real-data simulations carried out in Section 5.3.

5.3 Computational Methods and Examples

In this section we further describe some computational methods for the solution of the continuous contract management problem, Problem (P_T). We then apply this algorithm to real bidding data and compare the results with the basic simulations of Section 5.2.3.

5.3.1 Computation

The results of Section 5.1.3 establish that the continuous contract management problem can ultimately be solved by the same methods as were developed in Chapter 3: by the use of Proposition 5.1.3 to completely reduce the problem into the finite case (for second price auctions), or by first solving the dual, Problem (D_T), and then computing an appropriate allocation array via Problem (3.2.3).

However, both of these approaches require computing and representing averaged functions of the form, *e.g.,* $\overline{\Lambda}(x) = \int_0^T \Lambda(x,t)\mathsf{d}t$. In practice, we achieve this by discretizing $[0,1]$ into finite intervals $[0,q_1),[q_1,q_2),\ldots$, calculating each $w_k \triangleq \overline{\Lambda}(q_k)$, and then reconstructing an estimate of the function $\overline{\Lambda}(x)$ by interpolation. This process is extremely time consuming, and does not apply easily to the first price auction. Instead, we develop here an alternative method based on a trapezoidal approximation of the integral. This method is applicable to both the first and second price auctions, and is essentially a polyhedral approximation algorithm (see [21, Ch. 4] and Section 2.4).

Additionally, we will see that the linearity of the constraints allows us to construct *feasible* approximate solutions for (P_T), and that we can also estimate the sub-optimality of the approximation.

To this end, choose a sequence of $K \geq N+1$ points $0 \triangleq \widetilde{T}_0 < \widetilde{T}_1 < \cdots < \widetilde{T}_K \triangleq T$ such that $\{T_i\}_{i=1}^N \subset \{\widetilde{T}_k\}_{k=1}^K$. We use the notation $\Delta_k \triangleq \widetilde{T}_k - \widetilde{T}_{k-1}$ and $\bar{\lambda}_{jk} = \int_{\widetilde{T}_{k-1}}^{\widetilde{T}_k} \lambda_j(t)\mathsf{d}t$. Then, define

$$\overline{\Lambda}_{jk}(q) \triangleq \frac{1}{2}\Delta_k\big[\lambda_j(\widetilde{T}_k)\Lambda_j(q,\widetilde{T}_k) + \lambda_j(\widetilde{T}_{k-1})\Lambda_j(q,\widetilde{T}_{k-1})\big], \tag{5.11}$$

i.e., a trapezoidal approximation[4] of the integral $\int_{\widetilde{T}_{k-1}}^{\widetilde{T}_k} \lambda_j(t)\Lambda_j(q,t)\mathsf{d}t$. The following finite approximation of Problem P_T is natural:

Proposition 5.3.1 (Finite Primal Problem)**.** *The finite optimization problem over the variables $s_{ij}[k], R_{ij}[k]$ defined by*

$$\begin{aligned} \underset{s,R}{\text{minimize}} \quad & \sum_{j=1}^{M} \sum_{k:\widetilde{T}_k \leq T^j} \overline{\Lambda}_{jk}(s_j[k]/\lambda_{jk}) \\ \text{subject to} \quad & \sum_{j\in\mathcal{A}_i} \sum_{k:\widetilde{T}_k\leq T_i} \Delta_k v_{ij} R_{ij}[k] = C_i \\ & \sum_{i\in\mathcal{B}_j\cap\mathcal{T}_{T_k}} R_{ij}[k] = s_j[k] \\ & R_{ij}[k] \geq 0, \end{aligned} \tag{P_K}$$

[4]The midpoint rule provides a constant factor improvement in the error bound over the trapezoidal rule. However, since using linear interpolation to approximate $\Lambda(x,t)$ along t is a natural method, we specify the trapezoidal rule, since it is exact for such a function.

is a finite approximation of (P_T) *in the following sense.*

For any solution $(s_j[k], R_{ij}[k])$ *of* (P_K)*, the functions* $R_{ij}(t) = q_{jk}\gamma_{ijk}\lambda_j(t)$*, where* $q_{jk} = s_j[k]/\bar{\lambda}_{jk}$ *and* $\gamma_{ijk} = R_{ij}[k]/s_j[k]$*, and* $s_j(t) = \sum_{i\in\mathcal{B}_j} R_{ij}(t)$ *are feasible for Problem* (P_T)*.*

If, in addition, Δ_k *is* $\mathcal{O}(1/K)$ *and each cost function* $\Lambda_j(x,t)$ *is Lipschitz in* x *(uniformly in* t *a.e.) and twice continuously differentiable almost everywhere in* t *(uniformly in* x *a.e.); each* $W_j(x,t)$ *and* $g_j^{-1}(x,t)$ *are Lipschitz in* t *(uniformly in* x *a.e.); then the cost difference between the optimal solution to* (P_T) *and the feasible piecewise constant approximation obtained from Problem* (P_K) *is* $\mathcal{O}(1/K)$*. A more precise bound is provided in the proof.*

See proof on page 197.

The conclusion of our developments is provided in Algorithm 6. The `error` condition in Algorithm 6 encountered when supply is inadequate can be handled, for example, by modifying the objective of Problem (P_K) to instead penalize supply shortfalls, rather than attempting to enforce them as constraints, resulting in a "best effort" solution, which is the approach taken in our simulations. A summary of our proposed bidding methods are provided in Algorithm 6. The cost functions must correspond to either a second or a first price auction, and the supply curves must be strictly 2−concave in the latter case, see Proposition 2.1.2.

Algorithm 6: Computing Optimal Bids

input : Contracts $\{(\{v_{ij}\}_{j=1}^M, C_i, T_i)\}_{i=1}^N$, supply curves $\{W_j(x,t)\}_{j=1}^M$, cost functions $f_j(x,t)$, parameter $K \geq N+1$
output : Bid path $x(t)$ and allocation path $\gamma(t)$

$(\widetilde{T}_k)_{k=0}^{K-N} \leftarrow$ `segment`$([0,T], K-N)$ `# cut` $[0,T]$ `into` $K-N$ `equal segments`
$(\widetilde{T}_k)_{k=0}^{K} \leftarrow$ `sort`$(\{\widetilde{T}_k\}_{k=0}^{K-N} \cup \{T_i\}_{i=1}^N)$ `# incorporate contract deadlines`
`Let` $\Lambda_j(s,t) = f_j \circ W_j^{-1}(s,t)$ `# acquisition function`
if *Assumption 5.1.1 holds (adequate supply is available)* **then**
  $s_j[k], R_{ij}[k] \leftarrow$ `solve`(P_K) `# solve discretized problem`
else
  return `error`
$x_j(t) \leftarrow W_j^{-1}(s_j[k], t)\ \forall t \in [\widetilde{T}_{k-1}, \widetilde{T}_k), k : \widetilde{T}_k \leq T^j, j \in [M]$ `# construct bid path`
$\gamma_{ij}(t) \leftarrow R_{ij}[k]/s_j[k]\ \forall t \in [\widetilde{T}_{k-1}, \widetilde{T}_k), k : \widetilde{T}_k \leq T^j, i \in \mathcal{B}_j, j \in [M]$
return $(x(t), \gamma(t))$

5.3.2 Monte Carlo Simulations

We have run experiments to empirically evaluate the performance of our methods on real data from the IPinYou dataset [113, 215]. Further details on the specific simulation methodology are provided in Appendix Section B.5.

All our computations have been carried out with Python's scientific computing stack [185] and `cvxpy` [54] [4]. To summarize our setup (see also Section 2.4), we have estimated supply rate curves $W_j(x, t)$ via Gaussian kernel density estimation using the one week of available data stratified into twenty-four one hour intervals. The bandwidth is chosen via Silverman's rule (see e.g., [193, ch. 6]), which results in smooth estimates, and is not likely to be overfit. The hourly stratification results in a supply rate curve estimate which accounts for daily periodic trends, but not weekly ones.

The problems we simulate in this section are small: containing $N = 3$ contracts and $M = 3$ item types. Each contract deadline T_i in the collection is randomly sampled uniformly at random within prescribed bounds. In particular, the starting "real" time point (i.e., time $T_0 = 0$) is sampled from anywhere between the bounds of available data, the length of contracts is uniformly random between 0 and 70 hours, and the number of required items are sampled uniformly between a small (easily fulfilled) lower bound, and a large (near the maximum available supply) upper bound – random contracts which are not feasible are re-sampled. A total of 500 Monte Carlo iterations are carried out. We use the same estimated supply rate curves in each simulation[5], but the auction prices and item arrival times are sampled directly from real data. We constraint the valuations as $v_{ij} \in \{0, 1\}$.

Whenever an adequate number of items to fulfill a contract are acquired, that contract is removed from simulation and a completely new set of bids are calculated given the new (reduced) set of contracts. Not doing so would result in *overfilling* some contracts, which would never be done in practice. Thus, since there are a total of 6 contracts, we solve at least 6 instances of Problem (P_K) over the course of each simulation run, as well as many more when applying a receding horizon.

The main results of our simulations are provided in Figure 5.3 (which quantifies the levels of fulfillment) and Figure 5.4 (which quantifies the costs). The adjustment parameter $\delta \geq 0$ (constituting the abscissa for the figures) is used for over-provisioning (*c.f.,* Section 5.2.2) where if the contract requires C_i supply to be fulfilled, we supply a *target* of

[5] This allows some data leakage. However, the estimation methods are crude, not likely to overfit, and an abundance of data would ordinarily be available in practice, so we do not consider this to be a relevant concern.

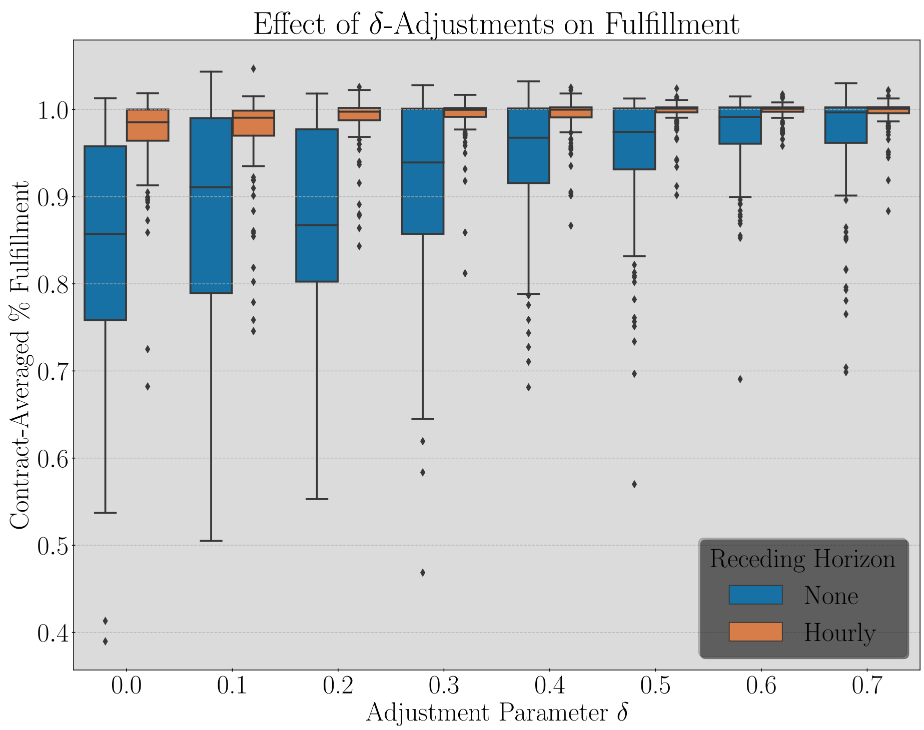


Figure 5.3: Risk Adjustment Simulations (Fulfillment)

As the over-provisioning parameter δ increases, the probability of completely fulfilling a contract naturally increases concomitantly. Hourly (in simulation time) receding horizon re-calculations also serve to dramatically increase fulfillment.

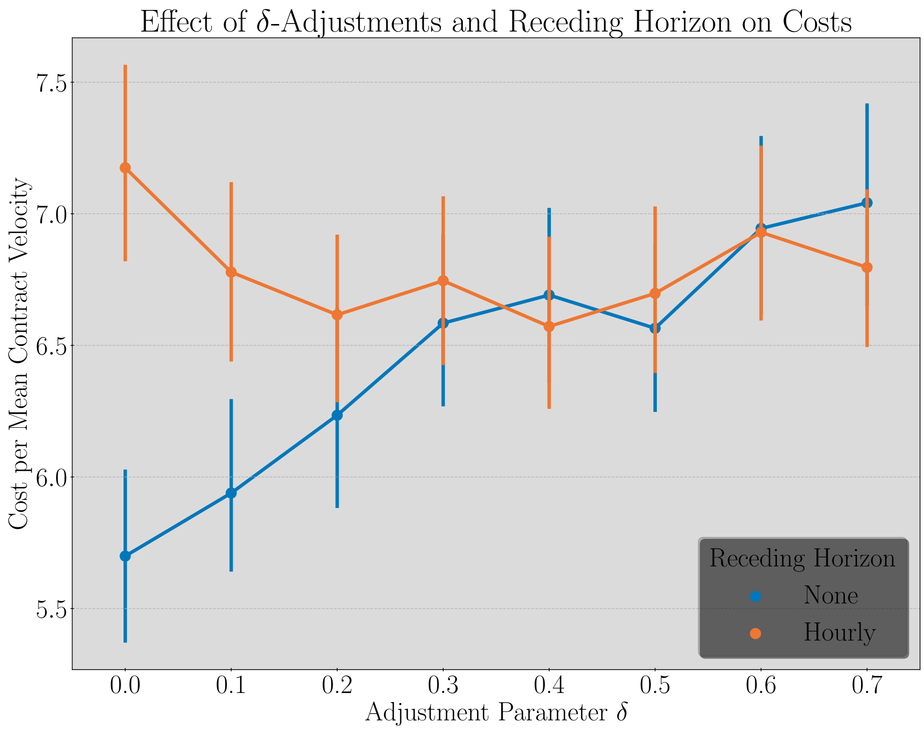


Figure 5.4: Risk Adjustment Simulations (Costs)

Increases in the adjustment parameter δ result in higher costs since we necessarily place larger bids. Receding horizon re-calculations serve to keep these costs under-control for large values of δ. The cost increase is not monotone in the presence of a receding horizon since the front-loading of supply acquisition eliminates the need to place very large bids near the contract deadlines.

$(1+\delta)C_i$ supply. This results in front-loading the acquisition of supply and increases the probability that a contract will be fulfilled completely.

Figure 5.3 quantifies the average contract fulfillment of simulation ℓ by the quantity

$$C_{\text{avg}}^{(\ell)} \triangleq \frac{1}{N}\sum_{i=1}^{N}\frac{c_i^{(k)}(T_i)}{C_i}, \tag{5.12}$$

and Figure 5.3 plots each of these quantities over $\ell \in [L]$, for $L = 500$ simulations. As is expected, the proportion of contracts which are completely fulfilled increases with δ. In addition, the plot provides pairs of boxes comparing the results when the bids are updated only after a contract is fulfilled (this is the minimal update schedule that doesn't overfill contracts), and when the bids are also updated every hour of simulation time (typically, about 20 updates are calculated over the simulation). An hourly receding horizon has a dramatic affect on the fulfillment – even without a δ adjustment, more than half of the contracts are fulfilled to at least 98%, and with a risk adjustment, almost all contracts are completely fulfilled. The occasional over-fulfilled contract arises as a result of some loose numerical tolerance parameters in the simulation, so set to reduce the run time.

Figure 5.4 provides a line plot similar to Figure 5.3 except the ordinate now quantifies the cost paid for items. Points are mean values across the $L = 500$ simulations, and error bars are 98% confidence intervals. In order to create a reasonable cost metric which is comparable across different contracts, we have normalized the total cost of attempting to fulfill a contract by that contract's "mean velocity" requirement. To understand what we refer to as the velocity requirement, suppose that we are obligated to obtain C items in T time. Then, we define the velocity of this requirement by $v = C/T$. We take the mean velocity of the whole contract to be the average of the velocities of each individual requirement in the contract (in this case, over the $N = 3$ requirements). To merely normalize by, e.g., the total number of items, would not result in a fair comparison since one contract may require obtaining the same number of items in less time (since they are generated randomly), which necessitates paying higher prices.

The main conclusions to be drawn from 5.4 are as follows. Firstly, without a receding horizon, there is naturally a tendency for costs to increase, since the supply acquisition is front-loaded. Secondly, as δ increases, the same upward pressure on costs is applicable as in the case without any receding horizon, but at the same time, for contracts which are difficult to fulfill, the receding horizon will drastically increase bids, which also increases the cost. However, for contracts which are easy to fulfill, or which get close to fulfillment early in the bidding period, the receding horizon smooths out the acquisition rate by reducing the bid, and therefore reducing costs (*c.f.,* Section 5.2.1). These are competing effects, but the use of the receding horizon is still able to reduce costs when δ is large; an ultimately small cost for achieving near complete fulfillment.

To summarize this discussion, the receding horizon is able to achieve a much higher fulfillment proportion by increasing the bids (and incurring higher costs) for contracts which are difficult to fulfill (e.g., when supply rates are overestimated), while at the same time reducing the costs of fulfilling the remaining contracts by smoothing out bids that are too large.

5.4 Conclusion

This chapter has extended the basic contract management problem introduced in Chapter 2 to a continuous time (but still deterministic) optimization problem involving supply rate forecasts and contract time deadlines. Elementary observations from real market data indicate that this formulation is more realistic, and enables bidding algorithms to take advantage of the approximate cyclo-stationarity of arrival rates and supply curves.

We have shown that even though the main problem, Problem (P_T) is infinite dimensional, it is solvable through the *finite* dual (Problem (D_T)) and a finite transportation problem (Section (3.2.3)), as long as time integrals can be calculated. Moreover, in the second price case, the problem can be completely reduced to an instance of the static optimal contract management problem studied in the rest of this thesis. However, while the former finite solution method still applies to first price auctions, this latter result *does not*. Still, a discretization of the time-integrals can be applied to construct finite approximations of Problem (P_T) for both auction mechanisms, see Section 5.3.1.

The consideration of time deadlines also necessitates the consideration of more stochastic effects. That is, while the optimal bids computed through the solution of Problem (P_T) are optimal in an *average* sense, they still represent simplistic open loop control policies. Practical methods for dealing with stochastic fluctuations (distinct from those considered in Chapter 4), namely over-provisioning and receding horizon control, are developed through stylized examples in Section 5.2.

A Monte Carlo simulation, using realistic market data from the IPinYou dataset, is carried out in Section 5.3. The results of this simulation demonstrate the effectiveness of the aforementioned adaptive methods for increasing the probability of contract fulfillment, and for controlling costs. Further commentary on stochastic effects is given in Section 6.1.

Chapter 6

Conclusion

In this concluding chapter we touch upon extensions to the models considered in this thesis in Section 6.1 and provide a summary with concluding remarks and summary of our contributions in Section 6.2.

6.1 Extensions and Future Work

6.1.1 Continuum of Item Types

The possibility of modelling a continuum of item types $\phi \in \Phi$ was briefly discussed in Section 2.2.1, where we immediately argued, in Proposition 2.3.2, that it is not essential to do so. However, this proposition may not give a completely satisfying conclusion since it does not apply to first price auctions, and only tells us that we cannot obtain a specified average supply rate at a lower cost by considering more detailed item types, which does not fully address the real contract management problem with type dependent valuations. The appropriate optimization problem in this case is

$$\begin{aligned} \underset{R}{\text{minimize}} \quad & \mathbb{E}\lambda(\phi)\Lambda_\phi\big(\frac{1}{\lambda(\phi)}\sum_{i=1}^{N} R_i(\phi)\big) \\ \text{subject to} \quad & \mathbb{E}R_i(\phi)v_i(\phi) = C_i \\ & R_i \geq 0 \end{aligned}$$

where the expectation is taken over a distribution on item types Φ. A duality analysis (*c.f.,* Chatper 3) produces another *finite* problem for the pseudo-bids ρ_i, similarly to the problem (D_ρ):

$$\underset{\rho}{\text{maximize}} \quad \sum_{i=1}^{N} \rho_i C_i - \mathbb{E}\lambda(\phi)\Lambda_\phi^\star\big(\max_i \ \rho_i v_i(\phi)\big). \tag{6.1}$$

If this problem can be solved, the optimal bid for an item of type ϕ is given by, in a second price auction, $x(\phi) = \max_i \ \rho_i v_i(\phi)$. Thus, many of the key structural results derived in Chapter 3 continue to hold in this case, and, similarly to the continuous time problem in Chapter 5, we can fully represent the optimal bids with a finite vector $\rho \in \mathbb{R}^N$.

The main challenges beyond the finite case are computational. In particular, how should $v_i(\phi)$ be represented? How should one calculate $\Lambda_\phi^\star$ (and can a version of ADMM, as in Algorithm 2, be applied to circumvent this calculation)? Can the associated infinite dimensional transportation problem (*c.f.,* Section 3.2.3) be solved? Many of the algorithms for estimating bid landscapes [49, 74, 192] and click through rates [218, 81, 140] are generic machine learning algorithms where a parameterized function is estimated from data. In our context therefore, it may be a reasonable model to treat W_ϕ as arising from some parameterized function (*e.g.,* a neural network) $W_\phi(q) = F(q, \phi; \theta)$ with parameter vector θ, to be estimated from data. Studying computational methods for Problem (6.1) under this model is therefore of practical interest. This approach would be comparable to [61, 57, 130].

6.1.2 Stochastic Optimal Control

The contract management problems studied in this thesis are, from one perspective, natural optimal bidding problems that make minimal assumptions about market dynamics (*i.e.,* we work almost exclusively with first order statistics). Alternatively, our models may be viewed as fluid approximations to stochastic optimal control problems, which may also be worth studying directly (*c.f.,* [84, 62]). A stochastic optimal control problem would necessitate the calculation of an optimum bidding *policy* $x(c, t)$ (mapping from the current number of items obtained and the current time), similarly to the receding horizon policies of Section 5.2.1, but applied directly to the discrete market model Definition 2.2.1 (see also Chapter 4).

A simple example with a single contract and single item type $N = 1, M = 1$ can be formulated, see also [84, 62] for similar budget constrained problems. Drawing experience from the convexity of Λ (*c.f.,* Chapter 2), we formulate the problem in terms of calculating an optimal probability of winning $q \in [0,1]$ (*i.e.,* a policy $q(c,t)$). As has been argued throughout, we believe this formulation to be much more insightful than a formulation directly in terms of the bids. The dynamics for the the number of items obtained can then be modelled through the recursion $c(t+\tau_n) = c(t) + \mathbf{1}_{\mathbb{R}_+}(q - U_k)$ where $U_k \sim \mathcal{U}[0,1]$ are *i.i.d.* uniform and τ_n are the inter-arrival times. If the cost-to-go function for having c items in inventory at time t is denoted by $J(c,t)$, then we can inuit the appropriate Dynamic Programming recursions

$$J(c,t) = \min_{q\in[0,1]} \big[\Lambda(q) + \mathbb{E}J(c + \mathbf{1}_{\mathbb{R}_+}(q-U), t+\tau)\big], \tag{DP}$$

with terminal conditions $J(c,T) = L(C-c)$ for some loss function L associated with attaining fewer than C items. This recursive equation can be analytically simplified into a problem involving the expectation $\mathcal{J}(c,t) = \mathbb{E}J(c, t+\tau_k)$ and the conjugate $\Lambda^\star$. Precisely,

$$J(c,t) = \mathcal{J}(c,t) - \Lambda^\star\Big(\mathcal{J}(c,t) - \mathcal{J}(c-1,t)\Big).$$

A further analysis reveals that, in a second price auction, it is optimal to bid $\mathcal{J}(c,t) - \mathcal{J}(c-1,t)$, which, unsurprisingly, is the value to obtaining one additional item. We have observed in computational examples[1] that $J(c,t)$ is a convex function of c, but a proof is not obvious since $-\Lambda^\star$ is a concave function.

The model can be extended to a multitude of item types by including a type indicator j in the cost function $J(c,t,j)$ indicating what item type is under consideration, and extended to multiple contracts by treating c as a vector in $\mathbb{R}^N$. A similar set of dynamic programming recursions, which now also include optimizing over $i \in [N]$, are given by

$$J(c,t,j) = \min_{s\in[0,1]} \Big[\Lambda_j(s) + \mathbb{E}J^+\big(c, t+\tau, j\mathbf{1}_{\mathbb{R}_+}(s-U)\big)\Big],$$
$$J^+(c,t,j) = \begin{cases} \mathbb{E}J(c,t,\phi); & j = 0 \\ \min\limits_{i\in\mathcal{A}_j} \mathbb{E}J(c+\boldsymbol{e}_i, t, \phi); & j \in [M] \end{cases},$$

where $\boldsymbol{e}_i$ is a canonical basis vector and ϕ is a random variable distributed over $[M]$.

[1]Computationally, if τ_k are exponentially distributed, Equation (DP) can be discretized by approximating these interarrival times with scaled geometric random variables: $\tau \approx X_\eta/\eta$ for $X_\eta \sim \mathrm{Ge}(\lambda/\eta)$, and an efficient dynamic programming algorithm can be derived for this discretized problem.

This problem does not appear to be analytically tractable, but it is clear that it still admits some interesting structure. In particular, we conjecture that the function J is again convex in c, as a result of the convexity of Λ.

6.1.3 Price Impact

Throughout this thesis, we have treated the competitive auction market simply as a statistical, rather than strategic, entity. The optimization problems studied here do not directly take account of how competing bidders (or platforms, or sellers) may respond to our own actions (by placing higher bids themselves, or raising the reserve price), an assumption motivated by the mean-field nature of the markets. However, there is a wealth of research in finance which studies *market impact* [56, 72], *i.e.,* how one's own activity affects prices, and how to account for this impact [70, 3]. Even simple linear models of price impact have been effective for this purpose [168]. Since RTB is a competitive market, a similar approach may be worthwhile in this case as well (see, *e.g.,* [130]).

Given the benefits that convexity of the optimization problem confers, it is desirable that modifications to Problem (P) designed to account for market impact preserve this convexity, even if approximations need to be made. For instance, we would conjecture that augmenting the objective of Problem (P) with a simple L_2 regularization term proportional to $||s||_2^2$ would confer a benefit to the DSP (as long as the constant of proportionality were tuned appropriately). The reason is that this would, all else being equal, tend to *diversify* the acquisition of items across types and spread any impact out further across the market. Unless price impact is completely linear, which is implausible, this may serve to reduce total costs. Moreover, it can serve to relax the assumptions on W in the first price case, since, roughly speaking, $\Lambda_{1st}(q)$ is expected to be "almost convex" (see Proposition 2.1.2), the sum $\lambda\Lambda_{1st}(s/\lambda) + \frac{1}{2}s^2$ can be expected to be "more convex".

As in the finance literature, the appropriate form of any regularization functions or price impact models need to be derived through a blend of theoretical motivation and empirical analysis. As was emphasized in Chapter 1, the structure of RTB markets is much different than that of ordinary financial markets, and therefore this may yet be a worthwhile avenue for further study.

6.2 Conclusion and Final Remarks

In this thesis, we have focused on problems facing intermediaries in real-time bidding markets. Specifically, Problem (P^m), reproduced here:

$$\begin{aligned} \underset{x,\gamma}{\text{minimize}} \quad & \sum_{j=1}^{M}\sum_{i=1}^{N} \gamma_{ij} f_j(x_{ij}) \\ \text{subject to} \quad & \sum_{j=1}^{M} \gamma_{ij}\lambda_j v_{ij} W_j(x_{ij}) = C_i \\ & \sum_{i=1}^{N} \gamma_{ij} \leq 1, \gamma_{ij} \geq 0. \end{aligned} \tag{P^m}$$

This is a computationally intractable monotone programming problem encoding the goals of a DSP to fulfill N contracts $\left((v_{ij})_{j=1}^{M}, C_i\right)_{i=1}^{N}$, given access to M distinct item types. These contracts specify a value target $C_i > 0$ and the values v_{ij} of items of type j to contract i. The goal of the DSP is to calculate a bid vector $x \in \mathbb{R}^M$ and an allocation vector $\gamma \in \mathbb{R}^{N\times M}$ such that bidding x_j on all items of type j, and allocating γ_{ij} proportion of such items towards contract i results in the acquisition of items having total value C_i. The characteristics of item type j, say, is described by the supply curve $W_j(x)$, indicating the probability of winning the item with a bid of x, the rate $\lambda_j > 0$ at which these items become available, and finally the cost function f_j which is derived from the type of auction (either first or second price).

Broadly speaking, a variety of related problems have received substantial academic and industrial attention over the past decade. The specific Problem (P^m) (with $v_{ij} \in \{0, 1\}$) first arose as a novel formulation in [170] and was deeply motivated by a real industrial problem. We have broadened the applicability of this formulation through the well-motivated introduction of type and contract dependent valuations v_{ij}, as well as in the discussion of an infinite number of item types (*c.f.,* Section 6.1.1), the consideration of both second price *and* first price auctions, and the analysis of a continuous time formulation of the problem.

Aside from this additional modelling, the principal contribution we have made towards the solution of Problem (P^m) is the recognition of its equivalence to a convex optimization problem: Problem (P) (see Chapter 2). This convexity is obtained through the transformation of variables

$$s_j = \lambda_j W_j(x_j)$$

and leads unconditionally to convexity in the second price auction (Proposition 2.1.1) and under mild additional assumptions for first price auctions (Proposition 2.1.2).

This convex transformation is a powerful tool. Using this technique, we have recognized formal connections to important problems in finance and have provided brief analysis of these additional example problems. Precisely, the Dark Pool Problem, and a transaction cost aware portfolio construction problem (see Section 2.5 and Section 3.4) are both recognized as being convex programs involving many of the same functions as in Problem (P). Moreover, some existing problems in the literature on real-time bidding can be transformed into convex programs using this method (see Example 2.1.6). These examples suggest that this transformation may be applicable to a variety of important problems in diverse fields.

Chapter 3 was dedicated to a duality analysis of the convex program (P). This analysis provides deep insights into the structure of the problem's solutions (see Proposition 3.1.3). In particular, in both first and second price auctions, the optimal bids $x \in \mathbb{R}_+^M$ can be recovered directly from the dual solution $\mu \in \mathbb{R}_+^M, \rho \in \mathbb{R}_+^N$, and furthermore, can be represented solely in terms of $\mu \in \mathbb{R}_+^M$ (the dual variables associated with the M item types) *or* $\rho \in \mathbb{R}_+^N$ (the dual variables associated with the N contracts). These consequences recover many of the results of [170] (originally derived ad-hoc from first principles), in particular, Corollary 3.1.1, which shows that, for the $v_{ij} \in \{0, 1\}$ case, all the bids placed across items used to fulfill a given contract must be equal at an optimal solution. These consequences are illustrated in Section 3.3 by numerical examples.

In Section 3.2 (as well as Section 2.4) we examine practical computational algorithms for the solution of Problem (P) and its dual, Problem (D). One of the main computational difficulties in solving Problem (2.4) is that the supply curves W do not necessarily belong to any simple class of parameterized distributions, and are often to be represented by, *e.g.,* piecewise affine functions or as weighted sums of kernel functions. As well, the acquisition cost function $\Lambda(q) = \int_0^q W^{-1}(u)\mathsf{d}u$ is itself a complicated and expensive to evaluate function of W. Thus, Problem (P) is not an instance of any particular class of convex program (linear, quadratic, semidefinite *etc.*), and the specialized algorithm we provide are essential for realistic applications.

The derivations of Section 3.2 shows that an application of ADMM to Problem (D) results in profound simplifications. Firstly, it is necessary, due to the relationship between Λ and its conjugate $\Lambda^\star$ *c.f.,* Proposition 2.1.3, only to evaluate W itself, and not the more complicated functions Λ or $\Lambda^\star$. And secondly, each of the steps in the ADMM algorithm are completely separable, *i.e.,* can be trivially parallelized across multiple processors. These properties are not, to our knowledge, enjoyed by any other algorithm applicable to Problem (P). These results have important consequences for practical problems, particularly when there is need to scale to large problem instances.

In Chapter 4 we have applied the method of stochastic approximation to learn optimal solutions to Problem (P) without any prior knowledge about the supply curves W. While stochastic approximation is a well known methodology, and the theoretical results of this chapter are largely derived from more general existing results, the application of these methods to Problem (P) is non-trivial, and the feasibility of doing so relies on a very special property of the second price auction: that the derivative of Λ_{2nd} is W^{-1} (and dually, that the derivative of $\Lambda_{2nd}^{\star}$ is W). This implies that, given an acquisition rate matrix R (the primal variables for Problem (P)), the bids x needed to acquire items at the specified rates are exactly the derivatives of the objective function. Thus, given an approximately optimal R matrix, the DSP can refine this matrix towards a better solution naturally through the process of bidding. This is an important result as the supply curves are not completely stationary over time, and due to the censored nature of prices in RTB, high quality and recent data may not be easily available.

Chapter 5 expands the definition of the DSP's contract management problem to include time deadlines and time-dependent supply curves. This turns the contract management problem into an optimal control problem (even simpler, an infinite dimensional optimization problem with trivial dynamics). Most results from Chapter 2 and Chapter 3 continue to hold with appropriate modification in this case. Additionally, further structure results from the dual problem in this case: the dual, Problem (D_T), is *finite*. Indeed, in both the first and second price cases, the optimal bids $x_j(t)$ to be placed at time t on items of type j can be parameterized via the vector $\rho \in \mathbb{R}_+^N$ of dual variables, and for second price auctions $x_j(t)$ is *piecewise constant*. In this chapter we have also discussed practical methods, separate from those of Chapter 4, for adapting the solution $x_j(t)$ to changes in market statistics. This analysis is tied together with a simulation example using real market data.

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APPENDICES

Appendix A

Useful Results

We here provide statements of a select few important theorems used in our analysis and not given in the main text.

A.0.1 Lyapunov Stability

The following stability theorems are used frequently in the convergence analysis of stochastic approximation algorithms in Chapter 4.

Theorem A.0.1 (Lyapunov Stability [98, 43])**.** *Let $\dot{x} = h(x)$ be an ODE where $h : D \to \mathbb{R}^d$ is locally Lipschitz on an open domain $D \subseteq \mathbb{R}^d$. Let x_e be an equilibrium point: $h(x_e) = 0$. If $V : \mathbb{R}^d \to \mathbb{R}$ is a continuously differentiable function satisfying*

$$\begin{aligned}
&V(x_e) = 0, \forall x \neq x_e : \; V(x) > 0 \\
&V(x) \to \infty \text{ as } ||x|| \to \infty \\
&\forall x \neq x_e : \; \dot{V}(x) \triangleq \langle \mathsf{D}V(x), h(x) \rangle < 0,
\end{aligned}$$

then x_e is globally asymptotically stable for the ODE. That is, if $x_{x_0}(t)$ is the solution of $\dot{x} = h(x)$ for some initial condition $x(0) = x_0 \in \mathbb{R}^d$ then $x_{x_0}(t) \to x_e$ as $t \to \infty$.

Theorem A.0.2 (Lyapunov Stability for DIs)**.** *Let $\dot{x} \in h(x)$ be a DI where h is a Marchaud map (Definition 4.1.1) on $\mathbb{R}^d$. Let x_e be an equilibrium point: $0 \in h(x_e)$. If $V : \mathbb{R}^d \to \mathbb{R}$ is a continuously differentiable mapping which satisfies*

$$V(x_e) = 0, \forall x \neq x_e : \ V(x) > 0$$
$$V(x) \to \infty \ \textit{as} \ ||x|| \to \infty$$
$$\forall x \neq x_e : \ \dot{V}(x) \triangleq \max_{\phi \in h(x)} \langle \mathsf{D}V(x), \phi \rangle < 0,$$

then x_e is globally asymptotically stable for the DI. That is, if $x_{x_0}(t)$ is a solution of $\dot{x} \in h(x)$ for some initial condition $x(0) = x_0 \in \mathbb{R}^d$ then $x_{x_0}(t) \to x_e$ as $t \to \infty$.

Proof. We follow a similar method as [98, Theorem 4.1]. Let $x_{x_0}(t)$ be a solution of the DI with initial condition $x(0) = x_0$. For any $r > 0$, since V is continuous and coercive, $U_r = \{x \in \mathbb{R}^d \mid V(x) \leq r\}$ is a compact set and $\{U_r\}_{r>0}$ constitutes a decreasing family of sets such that $\bigcap_{r>0} U_r = \{x_e\}$. Moreover, since $\dot{V}(x) < 0$ for $x \neq x_e$ each set U_r is an invariant set for the DI. Let $t_1, t_2, \ldots$ be a sequence of times such that $t_n \to \infty$ and consider the sequence $x_1 = x_{x_0}(t_1), x_2 = x_{x_0}(t_2), \ldots$. By compactness, this sequence admits a convergent subsequence $\tilde{x}_k \to \tilde{x}_\infty$ and $V(x_\infty) \geq 0$. If $V(x_\infty) = 0$ then necessarily $x_\infty = x_e$ and by invariance $x(t) \to x_e$. Suppose by way of contradiction that $V(x_\infty) = \alpha > 0$. Then, let

$$\sup_{x \in \mathsf{bd}\ U_\alpha} \max_{\phi \in h(x)} \langle \mathsf{D}V(x), \phi \rangle = -\beta \leq 0.$$

Since $\dot{V}(x) < 0$ for every $x \neq x_e$ it must be that $-\beta < 0$ since U_α is compact, the graph of h is closed, and $x_e \neq\in U_\alpha$. But then $x(t)$ must escape $\mathsf{bd}\ U_\alpha$ and $V(x_\infty) < \alpha$. □

A.0.2 A Brief Introduction to ADMM

It is seen through the derivations in Section 3.2.2 that ADMM serves as a natural algorithm to apply to the dual problem (D), as it leads to a complete separation of the variables in the problem into a collection of trivial univariate problems.

The Alternating Direction Method of Multipliers (ADMM) [21, Sec. 5.4] [28] is an algorithm for separable, linearly constrained convex optimization. It's applicability is incredibly general as it serves essentially as the default algorithm for the convex modelling software `cvxpy` [135, 54]. ADMM is generally described as an algorithm for the following convex program with separable objective[1] [28]:

[1]The functions f, g can be very general, and the problem formulation easily encodes general convex programs $\inf_C f(x)$ via indicator functions $g(z) = \chi_C(z)$ and the equality constraint $x = z$ (corresponding to $c = 0$, $A = I$, $B = -I$).

$$\begin{aligned} \underset{x,z}{\text{minimize}} \quad & f(x) + g(z) \\ \text{subject to} \quad & Ax + Bz = c, \end{aligned}$$

for $A \in \mathbb{R}^{p \times n}$, $B \in \mathbb{R}^{p \times m}$. The algorithm proceeds by solving two separated minimization problems and then accumulating the error in the equality constraints:

$$\begin{aligned} x_{t+1} &\leftarrow \underset{x}{\text{argmin}} \left(f(x) + \frac{1}{2\nu} \|Ax + Bz_t - c + u_t\|_2^2 \right) \\ z_{t+1} &\leftarrow \underset{z}{\text{argmin}} \left(g(z) + \frac{1}{2\nu} \|Ax_{t+1} + Bz - c + u_t\|_2^2 \right) \\ u_{t+1} &\leftarrow u_t + Ax_{t+1} + Bz_{t+1} - c. \end{aligned}$$

Convergence If the functions f, g are proper, convex, lower semicontinuous, and strong duality holds, then the objective value converges to the optimal value $f(x_t) + g(z_t) \to p^\star$, the iterates are asymptotically feasible $Ax_t + Bz_t \to c$, and νu_t converges to an optimal dual solution. Naturally, if f, g are strictly convex, then the primal iterates x_t, z_t must also converge to primal solutions. These results can be found in [28, 21, 58].

Appendix B

Proofs and Auxiliary Material

B.1 Chapter 1

This section intentionally left blank.

B.2 Chapter 2

B.2.1 Convex Envelopes

In practice, the functions $W(x)$ will be estimated from available historical data. Therefore, depending on the method used to carry out this estimation, the resulting $\Lambda(x)$ may not necessarily be guaranteed to be convex in the first price case (recall Proposition 2.1.2 requires 2-concavity of W). If the estimate of $W(x)$ is simply carried out by fitting a particular parameterized distribution (e.g., a normal approximation) to a dataset, then there is unlikely to be any issue since most distributions commonly employed for this purpose do in fact have log-concave cumulative distribution functions.

However, our empirical data (see also [113, 215]) suggests that such simple models are *not* necessarily good estimates for supply rate curves, and that the curves have a tendency towards some multi-modality. For this reason, as described in Section 2.4.2, we generally use KDE estimation for W.

Unfortunately, KDE estimates of $W(x)$ from data need not be (and often aren't) log-concave or α-concave. In order to deal with this problem, we consider calculating a convex and piecewise affine majorant of the function $\Lambda_{1st}(s)$. A similar minorizing envelope can also be calculated through the methods of [136], and a piecewise minorant thereof computed by outer linear approximations. There is no obvious reason to prefer one approximation over the other and our experience has not demonstrated a clear benefit either way.

Alternatively, taking the log-concave envelope of $W(x)$ through similar methods is guaranteed to result in convex acquisition functions. The considerations of the previous paragraph suggest that we should at least expect the KDE estimates of W to be "almost" log-concave, and indeed, this is what we have observed in our own experiments; the convex envelopes are only slight perturbations of the original supply rate curve estimate.

Piecewise Affine Approximation

Let us denote by $\widetilde{\Lambda}(x)$ the acquisition cost function attained from an estimated supply rate curve and by $\Lambda^{\mathsf{U}}(x)$ the minimal convex majorant $\widetilde{\Lambda}(x)$, i.e.,

$$\Lambda^{\mathsf{U}}(x) = \inf\{\lambda(x) \mid \lambda(x) \geq \widetilde{\Lambda}(x), \lambda \text{ is convex and monotone increasing}\}. \tag{B.1}$$

We emphasize that λ must be monotone increasing, but this would also follow as a consequence of the monotonicity of $\widetilde{\Lambda}$. The maximal minorant can be defined similarly. Moreover, α-concave envelopes can be calculated by requiring that $\ell_\alpha \circ \lambda$ is convex.

A piecewise affine approximation of Λ^{U} can be found by discretizing a compact interval $[a, b] \subset \mathbb{R}$ into $n+1$ points $x_0, x_1, \ldots, x_n$ and solving the following convex quadratic program where convexity and monotonicity are enforced via finite differences

$$\begin{aligned} \underset{\lambda}{\text{minimize}} \quad & \frac{1}{n+1} \sum_{i=0}^{n} \big(\lambda_i - \widetilde{\Lambda}(x_i)\big)^2 \\ \text{subject to} \quad & \lambda_i \geq \widetilde{\Lambda}(x_i) \\ & \lambda_i - \lambda_{i-1} \geq 0 \ \forall i \in [n] \\ & \lambda_{i+1} - 2\lambda_i + \lambda_{i-1} \geq 0 \ \forall i \in [n-1]. \end{aligned} \tag{E^{U}}$$

An accurate approximation of the convex majorant is recovered via linearly interpolating λ_i. In fact, λ_i will result in a *strictly* monotone function (and therefore a continuous inverse) whenever $\widetilde{\Lambda}$ is strictly monotone.

Remark B.2.1 (Sparse Approximations)**.** It is desirable to use a fine discretization in Problem (E^{U}), otherwise the resulting function may fail to majorize $\widetilde{\Lambda}$ in regions of high curvature. However, each piecewise adds complexity to the representation of Λ^{U}. Therefore, it may be desirable to use a coarser approximation obtained by linearly interpolating samples of $\Lambda^{\mathsf{U}}(s)$. Since $\Lambda^{\mathsf{U}}(s)$ is convex, this process is guaranteed to produce another piecewise affine convex function which further majorizes $\Lambda^{\mathsf{U}}(s)$.

B.2.2 An Example

We consider an illustrative example of calculating log-concave envelopes of the supply rate curve in a simple market model. We suppose that each participant is characterized by a bid and rate pair (b_i, r_i) indicating that they will bid b_i with probability r_i on any arriving item. We sample $\{(b_i, r_i)\}_{i=1}^{30}$ randomly as $b \sim \exp(0.5)$ and $r_i \sim \beta(11.1, 10)$ which represents a market with 30 participants whose average bid is 0.5 and have an average probability of $(11.1 - 10)/11.1 \approx 0.1$ of bidding. The *bid landscape* in this situation is given by

$$W(x) = \prod_{i:b_i > x} (1 - r_i),$$

indicating the probability of winning an item if the bid x is placed.

We let $\widetilde{W}(x)$ be a KDE smoothed (with $\sigma^2 = 1/4$) version of W which corresponds either to the true supply rate curve under randomized bidding, or a reasonable estimate (from historical data) thereof. We denote supply rate curve estimates $\widetilde{W}^{\mathsf{L}}(x)$ and $\widetilde{W}_{\mathcal{N}}$ which are derived from $W(x)$ by solving Problem (B.1) for the function $-\log \circ W$ and moment matching a Gaussian c.d.f., respectively. Note that this procedure produces *minorants* of the supply rate curve (and therefore majorants of the acquisition cost curve), since using a *convex* majorant procedure results in minorants of *concave* functions. Figure B.1 plots examples of these functions and their associated cost and acquisition counterparts.

The examples of Figure B.1 are chosen to deliberately exaggerate the differences between the supply curve estimates and the envelopes. For the simulation examples of Section 5.3.2 the two curves are often indistinguishable.

B.2.3 Proofs

Definition B.2.1 (α-concavity)**.** Define, for $\alpha \geq 0$, $x > 0$ the function

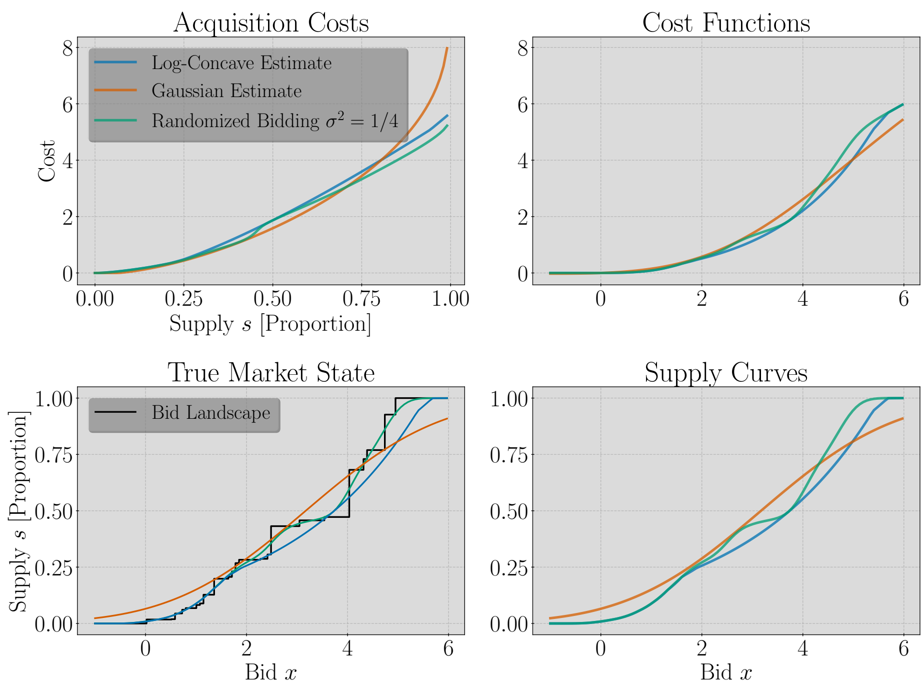

Figure B.1: Example Supply Rate Functions
Comparison of different methods of estimating supply rate curves. Lower left: Comparison of KDE smoothing, the maximal log-concave minorant thereof, and a Gaussian *c.d.f.*(fit by moment matching) overlaid upon a true market state. Lower right: The three corresponding supply rate curves. Upper right: Corresponding cost curves. Upper left: Corresponding acquisition cost functions where we see that KDE smoothing does not lead to convexity, and that a Gaussian estimate is not a consistent minorant or majorant.

$$\ell_\alpha(x) \triangleq \int_1^x \frac{1}{t^\alpha}\mathsf{d}t = \begin{cases} \ln x & \alpha = 1 \\ \frac{x^{1-\alpha}-1}{1-\alpha} & \text{otherwise} \end{cases},$$

where in particular $\ell_2(x) = 1 - 1/x$. We will say that a positive function $W : \mathbb{R} \to (0, \infty)$ is (strictly) α-concave if $\ell_\alpha \circ W$ is (strictly) concave. In particular, W is log-concave if $\alpha = 1$ and concave if $\alpha = 0$.

Proposition B.2.1 (Hierarchy of α-concavity)**.** *For $0 \leq \alpha < \beta$, if W is α-concave, then it is also β-concave.*

Proof of Proposition B.2.1. First we check that $\ell_\beta \circ \ell_\alpha^{-1}(x)$ is both monotone increasing and concave. This follows if the first derivative

$$\frac{\mathsf{d}}{\mathsf{d}x}\ell_\beta \circ \ell_\alpha^{-1}(x) = \frac{\ell'_\beta \circ \ell_\alpha^{-1}(x)}{\ell'_\beta \circ \ell_\alpha^{-1}(x)},$$

is positive and monotone non-increasing; which, since ℓ_α^{-1} is itself monotone non-decreasing, follows if

$$\frac{\ell'_\beta(x)}{\ell'_\alpha(x)}$$

is both positive and monotone decreasing. Since $\ell'_\alpha(x) = x^{-\alpha}$ this function is $x^{\alpha-\beta}$, which is positive on the domain $x > 0$ and decreasing if $\alpha < \beta$.

Now, we check concavity of $\ell_\beta \circ W$ directly from the definition, for $t \in (0, 1)$:

$$\begin{aligned} \ell_\beta \circ W(tx + (1-t)y) &= \ell_\beta \circ \ell_\alpha^{-1} \circ \ell_\alpha \circ W(tx + (1-t)y) \\ &\overset{(a)}{\geq} \ell_\beta \circ \ell_\alpha^{-1}\big(t\ell_\alpha \circ W(x) + (1-t)\ell_\alpha \circ W(y)\big) \\ &\overset{(b)}{\geq} t\ell_\beta \circ \ell_\alpha \circ \ell_\alpha^{-1} \circ W(x) + (1-t)\ell_\beta \circ \ell_\alpha^{-1} \circ \ell_\alpha \circ W(y) \\ &= t\ell_\beta \circ W(x) + (1-t)\ell_\beta \circ W(y), \end{aligned}$$

where (a) follows by the assumed concavity of $\ell_\alpha \circ W$ and the monotonicity of $\ell_\beta \circ \ell_\alpha^{-1}$ while (b) from the concavity of $\ell_\beta \circ \ell_\alpha^{-1}$. □

Lemma B.2.1 ([101])**.** *In a first or second price auction, suppose that for each $j \in [M]$, the acquisition cost curve $\Lambda_j(q)$ is convex. Then, if a solution x_{ij}, γ_{ij} to Problem (P^m) exists, there is also a solution with the property that $\forall i \in [N]: \ x_{ij} = x_j$ and such that $\sum_{i \in \mathcal{B}_j} \gamma_{ij}(t) \in \{0, 1\}$.*

Proof of Lemma B.2.1. Suppose (x, γ) is a solution of Problem (P^m), and with total cost J. Let $(\tilde{x}, \tilde{\gamma})$ be another candidate solution with total cost $\tilde{J}$ defined by

$$
\begin{aligned}
\tilde{x}_j &\triangleq W_j^{-1}\Big(\sum_{i=1}^{N} \gamma_{ij} W_j(x_{ij})\Big), \\
\tilde{\gamma}_{ij} &\triangleq \frac{\gamma_{ij} W_j(x_{ij})}{\sum_{u=1}^{N} \gamma_{uj} W_j(x_{uj})},
\end{aligned}
$$

where $0/0 \triangleq 0$ in the definition of $\tilde{\gamma}$. We proceed to show that $(\tilde{x}, \tilde{\gamma})$ is also a solution and we note that the definition of $\tilde{\gamma}$ satisfies $\sum_{i=1}^{N} \tilde{\gamma}_{ij} \in \{0, 1\}$.

The pair $\tilde{x}, \tilde{\gamma}$ is feasible by construction. Indeed, $\tilde{\gamma}_{ij} \geq 0$ and $\sum_{i=1}^{N} \tilde{\gamma}_{ij} \leq 1$ by definition. Moreover, we have

$$
\begin{aligned}
\sum_{j=1}^{M} \tilde{\gamma}_{ij} \lambda_j v_{ij} W_j(\tilde{x}_j) &= \sum_{j=1}^{M} \Big[\frac{\gamma_{ij} \lambda_j v_{ij} W_j(x_{ij}) \sum_{v=1}^{N} \gamma_{vj} W_j(x_{vj})}{\sum_{u=1}^{N} \gamma_{uj} W_j(x_{uj})} \Big] \\
&= \sum_{j=1}^{M} \gamma_{ij} \lambda_j v_{ij} W_j(x_{ij}) \\
&= C_i,
\end{aligned}
$$

where the last equality follows since x, γ is assumed to be a solution.

Now, we see that the cost of $(\tilde{x}, \tilde{\gamma})$ satisfies $\tilde{J} = J$ since J is the minimal cost and

$$\begin{aligned}\tilde{J} &\triangleq \sum_{i=1}^{N} \sum_{j=1}^{M} \tilde{\gamma}_{ij} f_j(\tilde{x}_j) \\ &\stackrel{(a)}{=} \sum_{i=1}^{N} \sum_{j=1}^{M} \tilde{\gamma}_{ij} \Lambda_j \Big(\sum_{u \in \mathcal{B}_j} \gamma_{uj} W_j(x_{uj}), t \Big) \\ &\stackrel{(b)}{\leq} \sum_{i=1}^{N} \sum_{j=1}^{M} \tilde{\gamma}_{ij} \sum_{u=1}^{N} \gamma_{uj} \Lambda_j(W_j(x_{uj})) \\ &\stackrel{(c)}{=} \sum_{j=1}^{M} \sum_{u=1}^{N} \gamma_{uj} f_j(x_{uj}) \sum_{i=1}^{N} \tilde{\gamma}_{ij} = J\end{aligned}$$

where (a) is just the definition of $\Lambda_j = f_j \circ W_j^{-1}$ (see Section 2.1.2), (b) follows by the convexity of Λ_j and that $\Lambda_j(0) = 0$ (since γ_{ij} need not necessarily sum to 1), and (c) follows again by $\Lambda_j = f_j \circ W_j^{-1}$ and then by swapping the order of summation.

Since $\tilde{x}, \tilde{\gamma}$ is feasible and has optimal cost, it is a solution. □

Lemma B.2.2. *In a first or second price auction, suppose that (x, γ) is a solution for Problem (P^m). If for any i, j we have that $x_{ij} < 0$ then $(\tilde{x}, \gamma)$ is also a solution where $\tilde{x}_{ij} = 0$ if $x_{ij} < 0$ and $\tilde{x}_{ij} = x_{ij}$ otherwise. Moreover, the constraint inequalities are binding: $\sum_{j=1}^{M} \gamma_{ij} \lambda_j v_{ij} W_j(x_{ij}) = C_i$.*

Proof of Lemma B.2.2. Let (x, γ) be a solution to Problem (P^m). If $x_{ij} < 0$ then $w_j(x_{ij}) = f_j(x_{ij}) = 0$ and since $W_j(0) = f_j(0) = 0$, the point $(\tilde{x}, \gamma)$ is also a solution as it remains feasible and has the same objective value.

Let us now suppose, by way of contradiction, that for some $i^\star \in [N]$ the constraint is not binding: $\sum_{j=1}^{M} \gamma_{i^\star j} \lambda_j v_{i^\star j} W_j(x_{i^\star j}) > C_{i^\star}$. Since the supply curves W_j are continuous and strictly monotone increasing with $W(0) = 0$, there exists (by the intermediate value theorem) some bid array $\tilde{x}$ such that $\tilde{x} \leq x$ and with $\tilde{x}_{i^\star j} < x_{i^\star j}$ and $\tilde{x}_{ij} = x_{ij}$ otherwise, such that $\sum_{j=1}^{M} \gamma_{i^\star j} \lambda_j v_{i^\star j} W_j(\tilde{x}_{i^\star j}) = C_{i^\star}$. Since the cost functions f_j are monotone increasing as well, $\tilde{x}$ must attain lower cost while still satisfying the constraints. Thus, (x, γ) was not a solution and the constraint so the constraint must have been binding. □

B.3 Chapter 3

This section intentionally left blank.

B.4 Chapter 4

B.4.1 Eliminating Equality Constraints

We describe here in an abstract setting the idea used in Proposition 2.3.1 to eliminate equality constraints of the form $Ax = b$ (where $A \in \mathbb{R}^{k \times d}$ with $k \leq d$, and A being full rank, *i.e.,*, $\mathsf{rk}\, A = k$) from an optimization problem. Consider the abstract linearly constrained convex program (*c.f.,*(4.3) (where we maintain Assumption 4.2.1)

$$\begin{aligned} \underset{x \in \mathbb{R}^d}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & Ax = b \\ & Gx \leq h. \end{aligned} \tag{B.2}$$

The constraints $Gx \leq h$ are general enough that they can technically absorb the equality constraints $Ax = b$ by including both inequalities $Ax \leq b$ and $Ax \geq b$. However, encoding equality constraints in this way is problematic for our stochastic approximation scheme since the random iterates will never exactly fulfill the equalities. This results in ungraceful performance of the stochastic approximation as the iterates oscillate around the subspace to which they should be confined. An example of this behaviour is provided in Figure B.2, and is carried out for a problem similar to that of Section 4.4.3.

The most straightforward alternative means of dealing with equality constraints is by simply eliminating them: let $Q \in \mathbb{R}^{N \times (N-k)}$ be an orthogonal basis for $\mathcal{N}(A)$. This basis can be constructed by, *e.g.,* a full QR decomposition. As well, let $\bar{x} = A^{\dagger} h$ be the least-norm solution to the equality system, with $A^{\dagger}$ denoting the Moore-Penrose pseudo-inverse of A.

Using the particular solution $\bar{x}$ and the basis Q we can parameterize the subspace of solutions as $V = \{\bar{x} + Qu \mid u \in \mathbb{R}^{N-k}\}$. Hence, Problem (B.2) can be equivalently written as the reduced problem

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & f(\bar{x} + Qu) \\ \text{subject to} \quad & GQu \leq h - G\bar{x}, \end{aligned} \tag{$\mathcal{P}_r'$}$$

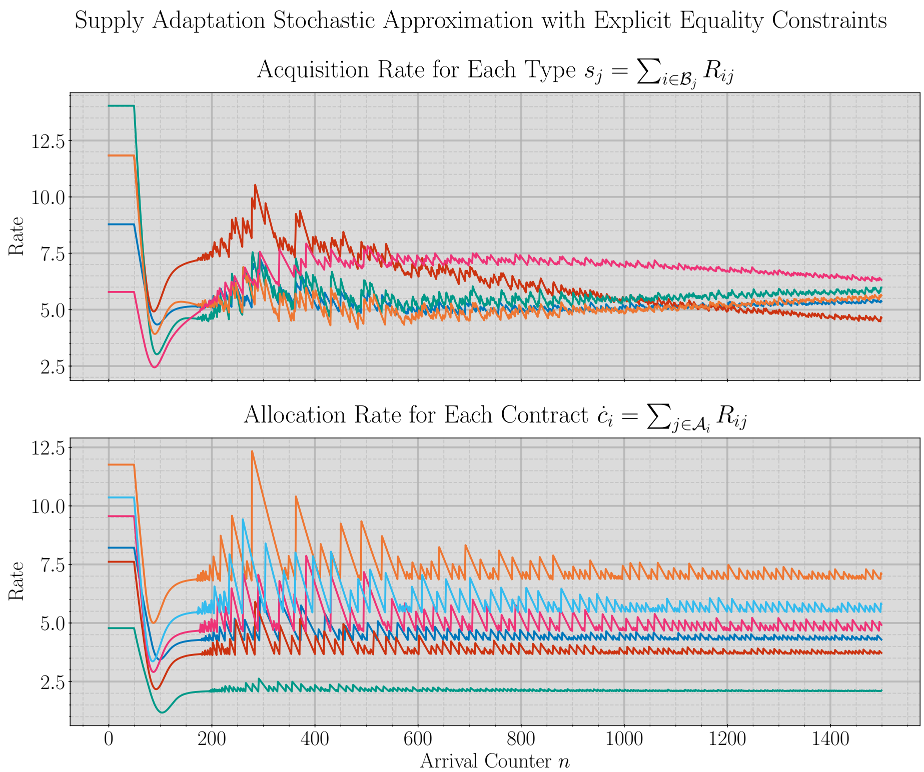

Figure B.2: Oscillatory Iterations

Numerical example of carrying out the quadratic penalty stochastic approximation algorithm without eliminating any explicit equalities $Ax = b$. The iterates oscillate across the subspace V of solutions. Compare to the smooth curves of Figures 4.3.

which an instance of Problem (B.2), involving only inequality constraints. As well, assuming that a Slater point for Problem (B.2) exists, and since the polytope $P_{G,h}$ is compact, then necessarily the intersection of this polytope with the subspace V must also be compact and non-empty and Problem ($\mathcal{P}'_r$) still admits a Slater point.

Connections to Projected Gradient Descent Let $\mathcal{L}_\mu(x) = f(x) + \frac{1}{2}\mu||(Gx-h)_+||_2^2$ and consider gradient descent applied to $\mathcal{L}_\mu(\bar{x}+Qz)$ as in $z_{n+1} = z_n - \alpha Q^\mathsf{T}\nabla\mathcal{L}_\mu(\bar{x}+Qz_n)$. Consider as well $x_n = \bar{x}+Qz_n$ and substitute this into the iterations to obtain $x_{n+1} = x_n - \alpha QQ^\mathsf{T}\nabla\mathcal{L}_\mu(x_n)$. Since Q is orthogonal, the matrix QQ^T is a projection onto $\mathcal{R}(Q) = \mathcal{N}(G)$, where this equality follows since Q is a basis for $\mathcal{N}(G)$. Hence, we can write $x_{n+1} = x_n - \alpha\Pi_{\mathcal{N}(G)}\big(\nabla\mathcal{L}_\mu(x_n)\big)$. This can be further re-written as $x_{n+1} = \bar{x} + \Pi_{\mathcal{N}(G)}\big(x_n - \alpha\nabla\mathcal{L}_\mu(x_n)\big)$ since $x_n = \bar{x}+z$ for some $z\in\mathcal{N}(G)$ and $\bar{x}\in\mathcal{N}(G)^\perp$. Finally, this is exactly $x_{n+1} = \Pi_V\big(x_n - \alpha\nabla\mathcal{L}_\mu(x_n)\big)$, i.e., gradient descent with projection onto the subspace V.

Remark B.4.1 (Accumulation of Numerical Errors)**.** While indeed, the algorithms $x_{n+1} = \Pi_V\big(x_n - \alpha\nabla\mathcal{L}_\mu(x_n)\big)$ and $\tilde{x}_{n+1} = \tilde{x}_n - \alpha\Pi_{\mathcal{N}(A)}\big(\nabla\mathcal{L}_\mu(\tilde{x}_n)\big)$ are mathematically equivalent, the former is preferable for numerical reasons. Suppose that the projections are only carried out up to some numerical tolerance ϵ. In this case, the iterates x_n are guaranteed to be within that tolerance for each step. That is, $|x_n - \Pi_V(x_n)| \le \epsilon$, where $\Pi_V(x_n)$ here represents an exact projection. However, these errors will *accumulate* for $\tilde{x}_n$ such that we can only guarantee $|\tilde{x}_n - \Pi_V(\tilde{x}_n)| \le n\epsilon$.

B.4.2 Proofs

Lemma B.4.1 (boundedness)**.** *Fix some $\bar{\mu} > 0$ and let $x_\mu \triangleq \underset{x\in\mathbb{R}^d}{\text{argmin}}\ \mathcal{L}_\mu(x)$ for $\mu \ge \bar{\mu}$. Then, the set $\{x_\mu \mid \mu \ge \bar{\mu}\}$ is bounded, that is, $\exists B_{\bar{\mu}}\ \forall\mu\ge\bar{\mu}:\ ||x_\mu|| < B_{\bar{\mu}} < \infty$.*

Proof of Lemma B.4.1. Let $x\in\mathbb{R}^d$ and consider the inequality

$$\mathcal{L}_\mu(x) \ge -B_f + \mu||(Gx-h)_+||_2^2,$$

which uses the lower bound on f and that $\theta \ge 0$. Then, since $P_{G,h}$ is compact, we must have $\mathcal{L}_\mu(x)\to\infty$ as $||x||\to\infty$. But, there exists a feasible point, for instance, the Slater point x_s, such that $\mathcal{L}_\mu(x_s) < \infty$ for any $\mu\ge\bar{\mu}$. Hence, x_μ must remain bounded, for otherwise it would not minimize $\mathcal{L}_\mu$. □

Lemma B.4.2 (Existence and Equilibria)**.** *Consider the system of ordinary differential equations*

$$\begin{aligned}\dot{\tau}(t) &= \bar{\tau} - \tau(t)\\ \dot{x}(t) &= s\tau(t) - W\big(x(t)\big)\\ \dot{L}(t) &= f\big(x(t)\big) - \tau(t)L(t),\end{aligned} \tag{B.3}$$

where $\bar{\tau} \triangleq 1/\lambda$*, and* $s \in (0, \lambda)$ *is fixed.*

Suppose that W *is a supply curve (Definition 2.1.1) which also satisfies the regularity assumption 4.3.1. Then, this system admits a unique solution for any given set of initial conditions, as well as a unique asymptotically stable equilibrium* $(\bar{\tau}, W^{-1}(s/\lambda), \lambda\Lambda(s/\lambda))$*.*

Proof of Lemma B.4.2. We can summarize the ODE (B.3) with the function

$$h(\tau, x, L) \triangleq (\bar{\tau} - \tau, s\tau - W(x), f(x) - \tau L)$$

as in

$$(\dot{\tau}, \dot{x}, \dot{L}) = h(\tau, x, L).$$

The function h (and hence the right hand side of the ODE) by our assumptions on W, which in turn imply the same for f. It follows by the Picard-Lindelöf Theorem that the ODE (B.3) admits a unique solution, and moreover, since W and f are bounded (W is a c.d.f., and $f(x) \leq E[p_n] < \infty$) h grows at most linearly, so the solution is uniquely well defined over all of $\mathbb{R}^3$.

We now proceed to establish that the equilibrium $(\bar{\tau}, W^{-1}(s/\lambda), \Lambda(s))$ is unique. This follows since $\bar{\tau}$ is the only solution of $\tau - \bar{\tau} = 0$, $W^{-1}(s/\lambda)$ is the unique solution of $\bar{\tau}s - W(x) = 0$ since W is strictly monotone (and $\bar{\tau} = 1/\lambda$), and that $\lambda\Lambda(s/\lambda)$ is the only zero of $\Lambda(s) - \bar{\tau}L = f \circ W^{-1}(r/\lambda) - \bar{\tau}L = 0$, again since f is strictly monotone. The assumption $s \in (0, \lambda)$ ensures that s/λ is in the domain of W^{-1}.

Then, asymptotic stability follows since the Jacobian J of the system (for $x > 0$) is given by

$$J = \begin{bmatrix} -1 & 0 & 0 \\ s & -W'(x) & 0 \\ -L & f'(x) & -\tau \end{bmatrix}, \tag{B.4}$$

and therefore has eigenvalues $(-1, -W'(W^{-1}(r/\lambda)), -\bar{\tau})$ at the equilibrium, where W is differentiable at $W^{-1}(r/\lambda)$ since $r \in (0, \lambda)$. Then, since $W' > 0$ (strict monotonicity), and $\bar{\tau} > 0$, this is a Hyperbolic equilibrium and its asymptotic stability follows by the Hartman-Grobman theorem. □

Lemma B.4.3 (Lyapunov Functions)**.** *Recall the ODE system* (B.3) *from Lemma B.4.2, and that $s \in (0, \lambda)$. Then, fix any $x_0 > 0$ and $x_\infty < \bar{x}$ such that $W^{-1}(s/\lambda) \in (x_0, x_\infty)$ and let $U = (-\infty, \infty) \times (x_0, x_\infty) \times (-\infty, \infty)$. There exists a strict Lyapunov function $V : U \to \mathbb{R}$ for the system* (B.3)*. That is, there exists a $C^1(U)$ function such that $V \geq 0$ and $\dot{V} \leq 0$ on all of U and where strict inequality holds everywhere on U except at the equilibrium point.*

Proof of Lemma B.4.3. Consider the family of Lyapunov function candidates parameterized by α, γ:

$$V_{\alpha,\gamma}(\tau, x, L) = \frac{1}{2}\alpha(\tau - \bar{\tau})^2 + \frac{1}{2}(s\tau - W(x))^2 + \frac{1}{2}\gamma(f(x) - \tau L)^2.$$

We will calculate $\dot{V}_{\alpha,\gamma} \triangleq \langle \nabla V_{\alpha,\gamma}, h \rangle$, which is differentiable on U by the differentiability assumptions on W

$$\nabla V_{\alpha,\gamma}(\tau, x, L) = \begin{bmatrix} \alpha(\tau - \bar{\tau}) + s(s\tau - W(x)) - \gamma L(f(x) - \tau L) \\ -W'(x)(s\tau - W(x)) + \gamma f'(x)(f(x) - \tau L) \\ -\gamma\tau(f(x) - \tau L), \end{bmatrix}$$

so that the inner product $\dot{V}_{\alpha,\gamma}$

$$\langle \nabla V_{\alpha,\gamma}, h \rangle = \nabla V_{\alpha,\gamma}^{\mathsf{T}} \begin{bmatrix} \bar{\tau} - \tau \\ s\tau - W(x) \\ f(x) - \tau L \end{bmatrix}$$

can be written as a quadratic form

$$\dot{V}_{\alpha,\gamma} = -\begin{bmatrix} \tau - \bar{\tau} & s\tau - W(x) & f(x) - \tau L \end{bmatrix}^{\mathsf{T}} P_{\alpha,\gamma} \begin{bmatrix} \tau - \bar{\tau} \\ s\tau - W(x) \\ f(x) - \tau L \end{bmatrix}, \tag{B.5}$$

where

$$P_{\alpha,\gamma} = \begin{bmatrix} \alpha & -s/2 & \frac{1}{2}\gamma L \\ -s/2 & W'(x) & -\frac{1}{2}\gamma f'(x) \\ \frac{1}{2}\gamma L & -\frac{1}{2}\gamma f'(x) & \gamma\tau \end{bmatrix}. \tag{B.6}$$

Applying the Schur complement condition, we have that $P_{\alpha,\gamma} \succ 0$ (and hence $\dot{V}_{\alpha,\gamma} < 0$) if and only if $A_\alpha \succ 0$ and

$$\gamma\tau - \frac{1}{4}\begin{bmatrix} \gamma L & -\gamma f'(x) \end{bmatrix}^\mathsf{T} A_\alpha^{-1} \begin{bmatrix} \gamma L \\ -\gamma f'(x) \end{bmatrix} > 0, \tag{B.7}$$

where

$$A_\alpha = \begin{bmatrix} \alpha & -s/2 \\ -s/2 & W'(x) \end{bmatrix}.$$

Considering first $A_\alpha \succ 0$, we apply the Schur complement condition again, to conclude that $A_\alpha \succ 0$ if and only if $\alpha > 0$ and $\alpha - \frac{1}{4\alpha}s^2 W'(x) > 0$. Since $W'(x) < \infty$ uniformly, there exists an $\alpha > 0$ such that this inequality holds over U and therefore that $A_\alpha \succ 0$ eventually as $\alpha \to \infty$.

To verify Inequality (B.7), we directly calculate

$$A_\alpha^{-1} = \frac{1}{\alpha W'(x) - s^2/4}\begin{bmatrix} W'(x) & s/2 \\ s/2 & \alpha \end{bmatrix},$$

and hence we need to verify

$$\gamma\tau - \frac{\gamma^2}{4\alpha W'(x) - s^2}\big(L^2 W'(x) + \alpha f'(x)^2 - sLf'(x)\big) > 0.$$

Taking the limit $\alpha \to \infty$ (which incidentally guarantees $A_\alpha \succ 0$ by our earlier calculations) we find that this inequality reduces to

$$\tau - \frac{\gamma}{4}\frac{f'(x)^2}{W'(x)} > 0,$$

which holds eventually as $\gamma \to 0$, whenever $\tau > 0$, uniformly over U since $W'(x) > 0$.

Therefore, $V_{\alpha,\gamma}$ is eventually a strict Lyapunov function over U as $\alpha \to \infty$ and $\gamma \to 0$. □

Lemma B.4.4 (Invariant Sets)**.** *Recall the setting of Lemma B.4.2. Fix some $\epsilon > 0$ small enough such that $s \in [\epsilon, \lambda - \epsilon]$ and $\epsilon < 2s$. Then, the closed set $S_\epsilon = S_\epsilon^\tau \times S_\epsilon^x \times S_\epsilon^L$ where*

$$S_\epsilon^\tau = \big[\bar{\tau}(1 - \epsilon/(2s)), \bar{\tau}(1 + \epsilon/(2s))\big],$$
$$S_\epsilon^x = \Big[W^{-1}\big(\frac{s - \epsilon}{\lambda}\big), W^{-1}\big(\frac{s + \epsilon}{\lambda}\big)\Big],$$

and

$$S_\epsilon^L = \Big[\frac{\lambda\Lambda(\frac{s-\epsilon}{\lambda}) - \lambda}{1 + \epsilon/(2s)}, \frac{\lambda\Lambda(\frac{s+\epsilon}{\lambda}) + \lambda}{1 - \epsilon/(2s)}\Big],$$

is invariant for the ODE (B.3), and contains the equilibrium $(\bar{\tau}, W^{-1}(s/\lambda), \lambda\Lambda(s/\lambda))$. Moreover, if $\tau \in S_\epsilon^\tau$ then S_ϵ^x is attractive for x and $|\dot{x}| \geq min\big(\epsilon/(2\lambda), \frac{s-\epsilon/2}{\lambda}\big)$. As well, if $\tau \in S_\epsilon^\tau$ and $x \in S_\epsilon^x$, then S_ϵ^L is attractive for L and $|\dot{L}| \geq 1$.

Proof of Lemma B.4.4. That S_ϵ contains the equilibrium is evident by its construction. As well, that S_ϵ^τ is invariant for τ is also clear since $\dot{\tau} = \bar{\tau} - \tau$, any interval containing $\bar{\tau}$ is invariant for τ.

Suppose now that $\tau \in S_\epsilon^\tau$. Then, if $0 \leq x \leq W^{-1}\big(\frac{s-\epsilon}{\lambda}\big)$ we have

$$\begin{aligned}\dot{x} &= r\tau - W(x)\\ &\overset{(a)}{\geq} s\tau - \frac{s-\epsilon}{\lambda}\\ &\overset{(b)}{\geq} s\frac{(1-\epsilon/(2s))}{\lambda} - \frac{s-\epsilon}{\lambda}\\ &= \frac{\epsilon}{2\lambda},\end{aligned}$$

where in (a) we apply the monotonicity of W and that $x \leq W^{-1}\big(\frac{s-\epsilon}{\lambda}\big)$, and in (b) we use the fact that $\tau \in S_\epsilon^\tau$. If $x < 0$ then $\dot{x} = s\tau \geq s\bar{\tau}(1 - \epsilon/(2s)) = (s - \epsilon/2)/\lambda$. The inequality corresponding to $x \geq W^{-1}\big(\frac{s+\epsilon}{\lambda}\big)$ is exactly analogous.

Suppose now that $\tau \in S_\epsilon^\tau$ and $x \in S_\epsilon^x$, but that $L \leq S_\epsilon^L$. Then,

$$
\begin{aligned}
\dot{L} &= f(x) - \tau L \\
&\overset{(a)}{\geq} f \circ W^{-1}\big(\frac{s-\epsilon}{\lambda}\big) - \tau L \\
&\overset{(b)}{\geq} \Lambda(\frac{s-\epsilon}{\lambda}) - \bar{\tau}(1+\epsilon/(2s))L \\
&\overset{(c)}{\geq} \Lambda(\frac{s-\epsilon}{\lambda}) - \big(\Lambda(\frac{s-\epsilon}{\lambda}) - 1\big) \\
&= 1,
\end{aligned}
$$

where in (a) we use the monotonicity of f and W and the fact that $x \in S_\epsilon^x$, in (b) we use the fact that $\tau \in S_\epsilon^\tau$ as well as the definition of $\Lambda(q) = f \circ W^{-1}(q)$, and in (c) we use the fact that $L \leq S_\epsilon^L$. The direction $L \geq S_\epsilon^L$ is exactly analogous. □

Lemma B.4.5 (Global Asymptotic Stability)**.** *Recall the setup of Lemma B.4.2, and that the target rate r satisfies $r \in (0, \lambda)$. Then, the equilibrium $(\bar{\tau}, W^{-1}(r/\lambda), \Lambda(r))$ announced in Lemma B.4.2 is globally asymptotically stable.*

Proof of Lemma B.4.5. Fix some $\epsilon > 0$ such that $r \in [\epsilon, \lambda - \epsilon]$ and $\epsilon < 2r$. Then, the equilibrium is contained in the invariant set S_ϵ announced in Lemma B.4.4. Now, using Lemma B.4.3, we can select an open set U such that $S_\epsilon \subseteq U$ and $\dot{V}$ is non-negative over all of S_ϵ, and strictly so except at the equilibrium point. It follows from the LaSalle invariance principle (see, *e.g.*, [98, Thm. 4.4]) that from any initial point $(\tau_0, x_0, L_0) \in S_\epsilon$, the ODE converges to the equilibrium $(\bar{\tau}, W^{-1}(r/\lambda), \Lambda(r))$.

Then, by Lemma B.4.4, we know that if $\tau \in S_\epsilon^\tau$ then S_ϵ^x is attractive for x and $|\dot{x}| > 0$ uniformly over $x \notin S_\epsilon^x$, so $x \to S_\epsilon^x$ in finite time. Finally, if $(\tau, x) \in S_\epsilon^\tau \times S_\epsilon^x$ then $|\dot{L}| \geq 1$ for $L \notin S_\epsilon^L$ so $L \to S_\epsilon^L$ in finite time. The Equilibrium announced in Lemma (B.4.2) is therefore globally asymptotically stable. □

Lemma B.4.6 (Uniform Stability)**.** *Generalize the iterations of Equation (BA) to allow for r to depend arbitrarily upon the iteration number, as in r_n and fix some $\epsilon > 0$. Then, the iterates $z_n = (\bar{\tau}_n, x_n)$ of the stochastic approximation*

$$
\begin{aligned}
\bar{\tau}_{n+1} &= \bar{\tau}_n + a_n\big[\tau_{n+1} - \bar{\tau}_n\big] \\
x_{n+1} &= x_n + a_n\big[r_n\bar{\tau}_n - \mathbf{1}\big(p_{n+1} \leq x_n\big)\big]
\end{aligned}
$$

are uniformly stable over all sequences r_n such that $r_n \in [0, \lambda - \epsilon]$. That is,

$$\underset{r:r_n\in[0,\lambda-\epsilon]}{sup} \quad sup_n ||z_n|| < \infty \ a.s.$$

Proof of Lemma B.4.6. Since the sequence $\bar{\tau}_n$ does not depend upon x_n or L_n it can be analyzed separately. To this end, recall that the limiting ODE for the iterates $\bar{\tau}_n$ is simply given by $\dot{\tau}(t) = \bar{\tau} - \tau(t)$. In order to apply Theorem 4.2.1 we consider $h_c(\tau) = \frac{1}{c}(\bar{\tau} - c\tau)$ and observe that $h_c \to h_\infty$ where $h_\infty(\tau) = -\tau$. Clearly, the system $\dot{\tau} = h_\infty(\tau)$ has the origin as a globally asymptotically stable equilibrium, and hence $\bar{\tau}_n$ is stable, and by Theorem 4.1.1 $\bar{\tau}_n \to \bar{\tau}$ a.s.

We proceed to establish the boundedness of x_n. Since $p_n \geq 0$, it is evident that $\inf_n x_n = -\infty$ is impossible, so we focus only on the possibility $\sup_n x_n = \infty$. We will show that if x_n were to diverge, it must eventually behave as a random walk with negative drift, contradicting its divergence. To this end, using the convergence of $\bar{\tau}_n \to \bar{\tau}$ and the fact that $r_n \leq \lambda - \epsilon$, there exists some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ $r_n \bar{\tau}_n < 1$ a.s., so (except possibly on sets of measure 0) there exists $\delta > 0$ (depending upon ϵ) and another n_0 (possibly depending on δ) such that for any $n > n_0$ we have $r_n \bar{\tau}_n < 1 - \delta$ and hence

$$a_n[r_n \bar{\tau} - \mathbf{1}_{(p_{n+1} \leq x_n)}] \leq a_n[(1 - \delta) - \mathbf{1}_{(p_{n+1} \leq x_n)}].$$

Moreover, $\exists x_\delta$ large enough such that $\mathbb{P}\{p_{n+1} \leq x_\delta\} \geq 1 - \delta/2$. We then let

$$X_n = a_n[(1 - \delta) - \mathbf{1}_{(p_{n+1} \leq x_\delta)}],$$

which is a sequence of independent random variables with $\mathbb{E}[X_n] = -\delta a_n/2$ and $X_n \in [-a_n\delta, a_n(1 - \delta)]$. If $n_1 > n_0$ and $n > n_1$ then $x_n < x_{n_1} + \sum_{k=n_1}^{n} X_k$, so it is sufficient to show that $S_N = \sum_{n=1}^{N} X_n$ is bounded above. The sequence S_N is a random walk with negative drift, and hence cannot diverge to ∞. Precisely, we can apply Hoeffding's inequality to establish

$$\mathbb{P}\Big\{S_N + \sum_{n=1}^{N} \frac{\delta a_n}{2} > t + \frac{\delta}{2}\sum_{n=1}^{N} a_n\Big\} \leq \exp\Big(-\frac{2}{a^2}(t + \frac{\delta}{2}\sum_{n=1}^{N} a_n)\Big), \tag{B.8}$$

where $a^2 = \sum_{n=1}^{\infty} a_n^2 < \infty$. The infinite sum of terms

$$\sum_{N=1}^{\infty} \exp\Big(-\frac{2}{a^2}(t + \frac{\delta}{2}\sum_{n=1}^{N} a_n)\Big) < \infty \tag{B.9}$$

converges by the ratio test and hence by the Borel-Cantelli lemma the probability of an infinite number of the events $E_N = \{\omega \mid S_N > 0\}$ is 0. We conclude that $\sup_N S_N < \infty$ a.s. and hence $\sup_n x_n < \infty$ a.s. The result now follows since $s_n \in [0, \lambda - \epsilon]$ was arbitrary. □

Corollary B.4.1 (Stability)**.** *Suppose that the target supply r is fixed and that $r \in [0, \lambda)$, then the iterates $z_n = (\tau_n, x_n, L_n)$ of Equation (BA) are stable. Conversely, if $r > \lambda$ then $x_n \to \infty$ a.s.*

Proof of Corollary B.4.1. From Lemma B.4.6 we know that (τ_n, x_n) are stable. Since $\bar{\tau}_n$ and x_n do not depend upon L_n, Proposition 4.3.2 applies to the stochastic approximation involving only $(\bar{\tau}_n, x_n)$ and hence $x_n \to W^{-1}(r/\lambda)$ a.s. Therefore, $\exists \epsilon \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that $x_n \leq (1 + \epsilon)W^{-1}(r/\lambda)$ and $\bar{\tau}_n \geq (1 - \epsilon)\bar{\tau}$ for all $n > n_0$. Thus, for any such n we have

$$\begin{aligned} p_{n+1}\mathbf{1}(p_{n+1} \leq x_n) - \bar{\tau}_n L_n &\leq p_{n+1}\mathbf{1}(p_{n+1} \leq (1 + \epsilon)W^{-1}(r/\lambda)) - (1 - \epsilon)\bar{\tau} L_n \\ &\leq (1 + \epsilon)W^{-1}(r/\lambda) - (1 - \epsilon)\bar{\tau} L_n, \end{aligned}$$

and hence for $L_n \geq \frac{1+\epsilon}{1-\epsilon}\frac{W^{-1}(r/\lambda)}{\bar{\tau}}$ the drift in L_n is negative almost surely, thus by similar reasoning as for x_n we have $\sup_n L_n < \infty$. Similarly, if $L_n < 0$ and $n > n_0$ the drift in L_n is necessarily positive and hence $\inf_n L_n > -\infty$.

In the case that $r > \lambda$, for n large enough, the iterates satisfy $x_{n+1} = x_n + a_n[r\bar{\tau}_n - \mathbf{1}(p_{n+1} \leq x_n)] \geq x_n + a_n[(1 + \epsilon) - 1]$ and hence $x_n \to \infty$. □

B.5 Chapter 5

B.5.1 Linear Approximations of Primal Problem

From Section 5.3.1 we have the time discretized primal problem:

$$\begin{aligned}
\underset{s,r}{\text{minimize}} \quad & \sum_{j=1}^{M} \sum_{k:\widetilde{T}_k < T^j} \overline{\Lambda}_{jk}(s_j[k]) \\
\text{subject to} \quad & \sum_{j \in \mathcal{A}_i} \sum_{k:\widetilde{T}_k < T^j} \Delta_k r_{ij}[k] \geq C_i \\
& \sum_{i \in \mathcal{B}_j \cap \mathcal{T}_{T_k}} r_{ij}[k] = s_j[k] \\
& r_{ij}[k] \geq 0.
\end{aligned} \tag{P_K}$$

Suppose that the functions $\Lambda_j(s, \widetilde{T}_k)$ have piecewise affine approximations $\widehat{\Lambda}_k(s) = \max_{h \in H_{jk}} \big(m_{hj}[k]s + b_{hj}[k]\big)$. Then, by introducing additional variables α_{jk} we can reformulate P_K in epigraph form as a linear program

$$\begin{aligned}
\underset{s,r}{\text{minimize}} \quad & \frac{1}{2} \sum_{j=1}^{M} \sum_{k:\widetilde{T}_k \leq T^j} \Delta_k(\alpha_{jk} + \alpha_{j,k-1}) \\
\text{subject to} \quad & \sum_{j \in \mathcal{A}_i} \sum_{k:\widetilde{T}_k \leq T^j} \Delta_k r_{ij}[k] \geq C_i \\
& m_{hj}[k] s_j[k] + b_{hj}[k] \leq \alpha_{jk} \\
& \sum_{i \in \mathcal{B}_j \cap \mathcal{T}_{T_k}} r_{ij}[k] = s_j[k] \\
& r_{ij}[k] \geq 0,
\end{aligned} \tag{B.10}$$

which is the formulation we have employed for our simulations in Section 5.3.2.

B.5.2 Simulating the Bidding Process

The simulations of Section 5.3.2 are obtained by storing the hour-by-hour inter-arrival and price data for each item type $j \in [M]$ and sampling uniformly from these datasets. At simulation time $t \in \mathbb{R}_+$ we sample an inter-arrival time Δt and price P from the data for hour $\lfloor t \rfloor + 1$ with probability $t - \lfloor t \rfloor$ and otherwise from the data for hour $\lfloor t \rfloor$. A bid is solicited from a bidder (an implementation of (P_K)) and if the bid exceeds P the bidder allocates that item to the fulfillment of a contract. The simulation time is them updated to $t + \Delta t$ and the process continues.

Algorithm 7: Bidding Simulation

input : A Bidder derived from Section 5.3.1.

output : Recording of Bidder's item allocations to process into normalized acquisition curves.

```
// Initialize:
Q ← Priority-Queue([ ])  // Sort by time
t ← 0  // The ``current'' time
for j ∈ [M] do
    // Sample an inter-arrival time and a price
    (Δt, P) ← Sample-Dataset(t,j)
    Q.push((t + Δt, P, j))

// Simulate bidding process:
while t < T_end do
    t, P, j ← Q.pop()
    b ← Bidder.solicit_bid(t,j)  // Ask for a bid on type j at time t
    if b ≥ P then
        Bidder.award_item(t,j)  // Allocate items for winning bids
    (Δt, P) ← Sample-Dataset(t,j)  // Append next (t, P) pair for j to Q
    Q.push((t + Δt, P, j))

Function Sample-Dataset(t,j):
    p ← t − ⌊t⌋
    U ∼ 𝒰(0, 1)  // Interpolate between hours
    if p ≤ U then
        h ← ⌊t⌋
    else
        h ← ⌊t⌋ + 1
    Δt ← Sample-Inter-arrivals(hour=h,type=j)
    P ← Sample-Prices(hour=h,type=j)
    return (Δt, P)
```

B.5.3 Proofs

Proposition 5.1.2 (Duality)**.** *A dual of Problem of* (P_T) *can be formulated as*

$$\begin{aligned} \underset{\rho,\mu}{\textit{maximize}} \quad & \sum_{i=1}^{N} \rho_i C_i - \sum_{j=1}^{M} \sum_{k:T_k \le T^j} \int_{T_{k-1}}^{T_k} \lambda_j(t) \Lambda_j^\star(\mu_{jk}, t) \mathsf{d}t \\ \textit{subject to} \quad & v_{ij}\rho_i \le \mu_{jk} \ \forall i \in \mathcal{B}_j \cap \mathcal{T}_{T_k} \end{aligned} \tag{D_T}$$

which is a finite convex program. Problem (D_T) *is dual to Problem* (P_T) *in the sense that if $D_T^\star$ and $P_T^\star$ are their respective values (possibly ∞ or $-\infty$), then $D_T^\star \le P_T^\star$. Moreover, under Assumption 5.1.1 there exists a solution $(s, R) \in L_2([0,T])^M \times L_2([0,T])^d$ to Problem* (P_T) *and a solution $(\rho, \mu) \in \mathbb{R}^N \times \mathbb{R}^M$ to Problem* (D_T) *and $-\infty < D_T^\star = P_T^\star < \infty$.*

Proof of Proposition 5.1.2. The proof is similar to the analysis of Section 3.1.2, but requires additional care due to the infinite dimensional nature of the problem. We begin with the Lagrangian, whose domain we will restrict to $R_{ij}(t) \ge 0$ (rather than introducing slack variables $\theta_{ij}(t)$):

$$\begin{aligned} \mathcal{L}(s, R, \mu, \rho) &= \sum_{j=1}^{M} \int_0^{T^j} \lambda_j(t) \Lambda_j(s_j(t)/\lambda_j(t), t) \mathsf{d}t + \sum_{i=1}^{N} \rho_i \big(C_i - \sum_{j \in \mathcal{A}_i} v_{ij} \int_0^{T_i} R_{ij}(t) \mathsf{d}t\big) \\ &\quad + \sum_{j=1}^{M} \int_0^{T^j} \mu_j(t) \big(\sum_{i \in \mathcal{B}_j \cap \mathcal{T}_t} R_{ij}(t) - s_j(t)\big) \mathsf{d}t \\ &\overset{(a)}{=} \sum_{i=1}^{N} \rho_i C_i + \sum_{j=1}^{M} \int_0^{T^j} \Big[\lambda_j(t) \Lambda_j(s_j(t)/\lambda_j(t), t) - \mu_j(t) s_j(t)\Big] \mathsf{d}t \\ &\quad + \sum_{j=1}^{M} \int_0^{T^j} \sum_{i \in \mathcal{B}_j \cap \mathcal{T}_t} R_{ij}(t) \big(\mu_j(t) - v_{ij}\rho_i\big) \mathsf{d}t, \end{aligned}$$

where in (a) we have combined the restriction that $i \in \mathcal{T}_t \triangleq \{i \in [N] \mid t < T_i\}$ in the summation to expand the bounds of integration from $[0, T_i]$ to $[0, T^j]$.

To obtain the dual, we take the pointwise infimum of this function. This infimum over $R_{ij}(t) \ge 0$ induces the pointwise inequality $\mu_j(t) \ge v_{ij}\rho_i$ since otherwise the value would be $-\infty$, and taking a similar infimum over $s_j(t)$ results in the convex conjugate $\Lambda_j^\star(\mu_j(t), t)$ (by definition) and hence $\text{maximize}_{\rho,\mu} \inf_{s,R} \mathcal{L}(s, R, \mu, \rho)$ can be written

$$\begin{aligned} \underset{\rho,\mu}{\text{maximize}} \quad & \sum_{i=1}^{N} \rho_i C_i - \sum_{j=1}^{M} \int_0^{T^j} \lambda_j(t) \Lambda_j^\star(\mu_j(t), t) \mathsf{d}t \\ \text{subject to} \quad & v_{ij}\rho_i \leq \mu_j(t) \ \forall i \in \mathcal{B}_j \cap \mathcal{T}_t. \end{aligned}$$

Since $\Lambda_j^\star$ is monotone increasing in its first argument, we obtain that $\mu_j(t) = \max_{i \in \mathcal{B}_j \cap \mathcal{T}_t} v_{ij}\rho_i$. Since the right hand side of this equation is piecewise constant (changing only at contract deadlines T_i) it must be that there exists a finite collection μ_{jk} such that $\mu_j(t) = \mu_{jk}$ for $t \in [T_{k-1}, T_k)$ and therefore we can equivalently formulate the dual as

$$\begin{aligned} \underset{\rho,\mu}{\text{maximize}} \quad & \sum_{i=1}^{N} \rho_i C_i - \sum_{j=1}^{M} \sum_{k: T_k \leq T^j} \int_{T_{k-1}}^{T_k} \lambda_j(t) \Lambda_j^\star(\mu_{jk}, t) \mathsf{d}t \\ \text{subject to} \quad & v_{ij}\rho_i \leq \mu_{jk} \ \forall i \in \mathcal{B}_j \cap \mathsf{T}_{T_k}. \end{aligned}$$

The weak duality statement $D_T^\star \leq P_T^\star$ is well known and follows from an elementary analysis of $\mathcal{L}$.

We must establish the existence of solutions to these problems. Firstly, since $\Lambda_j(q, t) = \infty$ for $q > 1$, Problem (P_T) includes the implicit constraint $s_j(t) \leq \lambda_j(t) \leq B$ where $B = \max_j \sup_{t \in [0,T]} \lambda_j(t) < \infty$ which is bounded since $\lambda_j(t)$ is assumed continuous and $[0, T]$ is compact. It follows that over $L_2([0, T])$ the feasible region of the primal problem is a convex and norm closed and bounded set. Hence, since closed convex sets are also weakly closed ([45, Thm. 3.6]) the constraints of Problem (P_T) describes a *weakly closed* and bounded set. Since the objective is convex (as a functional of $s_j(t), R_{ij}(t)$) and lower semicontinuous (hence weakly lower semicontinuous), these properties combine with the *direct method*[1] [45, Thm. 5.51] to establish the existence of a solution to (P_T).

[1] The *direct method* is a well known program for establishing existence of solutions. We construct a minimizing sequence and use weak compactness (weak closedness combined with weak boundedness) to obtain a weakly convergent subsequence. The point of convergence belongs to the constraint set of the problem since it is weakly closed. The lower semicontinuity of the objective is then used to show that this point is in fact a minimizer. Weak closedness and weak lower semicontinuity is obtained by combining convexity with norm closedness and norm lower semicontinuity. That is, the convexity of the problem now plays a key role even just in the *existence* of solutions.

Now, combining Assumption 5.1.1 (a Slater point) with [45, Thm. 9.8] (existence of a *normal* dual multiplier) and [45, Thm. 9.13] (strong duality) we conclude that there must also exist a dual solution and $D_T^\star = P_T^\star$. □

Proposition 5.3.1 (Finite Primal Problem)**.** *The finite optimization problem over the variables $s_{ij}[k], R_{ij}[k]$ defined by*

$$\begin{aligned} \underset{s,R}{\text{minimize}} \quad & \sum_{j=1}^{M} \sum_{k:\widetilde{T}_k \le T^j} \overline{\Lambda}_{jk}(s_j[k]/\lambda_{jk}) \\ \text{subject to} \quad & \sum_{j \in \mathcal{A}_i} \sum_{k:\widetilde{T}_k \le T_i} \Delta_k v_{ij} R_{ij}[k] = C_i \\ & \sum_{i \in \mathcal{B}_j \cap \mathcal{T}_{T_k}} R_{ij}[k] = s_j[k] \\ & R_{ij}[k] \ge 0, \end{aligned} \tag{P_K}$$

is a finite approximation of (P_T) in the following sense.

For any solution $(s_j[k], R_{ij}[k])$ of (P_K), the functions $R_{ij}(t) = q_{jk}\gamma_{ijk}\lambda_j(t)$, where $q_{jk} = s_j[k]/\bar{\lambda}_{jk}$ and $\gamma_{ijk} = R_{ij}[k]/s_j[k]$, and $s_j(t) = \sum_{i \in \mathcal{B}_j} R_{ij}(t)$ are feasible for Problem (P_T).

If, in addition, Δ_k is $\mathcal{O}(1/K)$ and each cost function $\Lambda_j(x,t)$ is Lipschitz in x (uniformly in t a.e.) and twice continuously differentiable almost everywhere in t (uniformly in x a.e.); each $W_j(x,t)$ and $g_j^{-1}(x,t)$ are Lipschitz in t (uniformly in x a.e.); then the cost difference between the optimal solution to (P_T) and the feasible piecewise constant approximation obtained from Problem (P_K) is $\mathcal{O}(1/K)$. A more precise bound is provided in the proof.

Proof of Proposition 5.3.1. Let $(s^\star[k], R^\star[k])$ be an optimal solution to Problem (P_K) and $(s^\star(t), R^\star(t))$ an optimal solution to (P_T) with corresponding (piecewise constant) dual multipliers $\mu_j^\star(t)$. To see that $R_{ij}(t) = q_{jk}\gamma_{ijk}\lambda_j(t)$ is feasible for Problem (P_T) we first recognize that $R_{ij}(t) \ge 0$ and $\sum_{i \in \mathcal{B}_j \cap \mathcal{T}_{T_k}} R_{ij}(t) = q_{jk}\lambda_j(t) \le \lambda_j(t)$. Then, we have

$$\sum_{j\in\mathcal{A}_i} v_{ij}\int_0^{T_i} R_{ij}(t)\mathsf{d}t = \sum_{j\in\mathcal{A}_i} v_{ij}\sum_{k:\widetilde{T}_k\le T_i}\int_{\widetilde{T}_{k-1}}^{\widetilde{T}_k} q_{jk}\gamma_{ijk}\lambda_j(t)\mathsf{d}t$$
$$= \sum_{j\in\mathcal{A}_i} v_{ij}\sum_{k:\widetilde{T}_k\le T_i}\frac{s_j[k]}{\bar{\lambda}_{jk}}\frac{R_{ij}[k]}{s_j[k]}\Delta_k\bar{\lambda}_{jk}$$
$$= \sum_{j\in\mathcal{A}_i}\sum_{k:\widetilde{T}_k\le T_i}\Delta_k v_{ij}R_{ij}[k]$$
$$= C_i.$$

Bounding the integral approximation error is a simple application of the well known error bound for the trapezoidal rule

We now place a bound on the integral approximation error. This follows from application of the well known error bound for the trapezoidal rule (see e.g. [138, ch. 7]), i.e., for any k, q there exists some $\hat{t}\in[\widetilde{T}_{k-1},\widetilde{T}_k]$ such that

$$\left|\overline{\Lambda}_{jk}(q) - \int_{\widetilde{T}_{k-1}}^{\widetilde{T}_k}\lambda_j(t)\Lambda_j(q,t)\mathsf{d}t\right| \le \frac{\Delta_k^3}{12}\left|\frac{\partial^2\Lambda_j(q,\hat{t})}{\partial t^2}\right|.$$

Thus, since Δ_k is $\mathcal{O}(1/K)$ we have the bounds

$$\left|\sum_{j=1}^{M}\sum_{k:\widetilde{T}_k\le T^j}\int_{\widetilde{T}_{k-1}}^{\widetilde{T}_k}\Lambda_j(q_k,t)\mathsf{d}t - \sum_{j=1}^{M}\sum_{k:\widetilde{T}_k\le T^j}\overline{\Lambda}_{jk}(q_k)\right| \le B\sum_{k=1}^{K}\Delta_k^3$$
$$= \mathcal{O}(1/K^2).$$

Using this bound, and the notation $q_j^\star(t) = s_j^\star(t)/\lambda_j(t)$, $q_{jk}(t) = \frac{s_j[k]}{\bar{\lambda}_{jk}}\lambda_j(t)$ for $t\in[\widetilde{T}_{k-1},\widetilde{T}_k)$ we have the bounds

$$\sum_{j\in\mathcal{A}_i}\int_0^{T^j}\lambda_j(t)\Lambda_j(q_j^\star(t),t)\mathsf{d}t \overset{(a)}{\leq} \sum_{j\in\mathcal{A}_i}\sum_{k:\widetilde{T}_k\leq T^j}\int_{\widetilde{T}_{k-1}}^{\widetilde{T}_k}\lambda_j(t)\Lambda_j(q_{jk},t)\mathsf{d}t$$
$$\overset{(b)}{=} \sum_{j\in\mathcal{A}_i}\sum_{k:\widetilde{T}_k\leq T^j}\overline{\Lambda}_j(q_{jk}) + \mathcal{O}(1/K^2)$$
$$\overset{(c)}{\leq} \sum_{j\in\mathcal{A}_i}\sum_{k:\widetilde{T}_k\leq T^j}\overline{\Lambda}_j(q_j^\star(\tau_k)) + \mathcal{O}(1/K^2)$$
$$\overset{(d)}{=} \sum_{j\in\mathcal{A}_i}\sum_{k:\widetilde{T}_k\leq T^j}\int_{\widetilde{T}_{k-1}}^{\widetilde{T}_k}\lambda_j(t)\Lambda_j(q_j^\star(\tau_k),t) + \mathcal{O}(1/K^2),$$

where (a) follows since s_{jk} (hence q_{jk}) is feasible for (P_T) and $q_j^\star(t)$ is optimal, (b) follows by applying the bound for trapezoid integration error, in (c) we choose a sequence of points $\tau_k \in [\widetilde{T}_{k-1}, \widetilde{T}_k)$ such that $q_j^\star(\tau_k)$ is feasible for Problem (P_K) (such a sequence exists, *e.g.*, by maximizing $q_j^\star(t)$ in each interval), and (d) again applies the integral approximation bound, absorbing a factor of 2 into $\mathcal{O}$.

We will now bound the distance between the above two integrals, using L_Λ as the Lipschitz constant of Λ_j in its first argument, and Γ_W, Γ_g are those of W, g in t:

$$\left|\int_{\widetilde{T}_{k-1}}^{\widetilde{T}_k}\lambda_j(t)\Lambda_j(q_j^\star(t),t)\mathsf{d}t - \int_{\widetilde{T}_{k-1}}^{\widetilde{T}_k}\lambda_j(t)\Lambda_j(q_j^\star(\tau_k),t)\mathsf{d}t\right|$$
$$\overset{(a)}{\leq} \hat{\lambda}_k\int_{\widetilde{T}_{k-1}}^{\widetilde{T}_k}\left|\Lambda_j(q_j^\star(t),t) - \Lambda_j(q_j^\star(\tau_k),t)\right|\mathsf{d}t$$
$$\overset{(b)}{=} \hat{\lambda}_k L_\Lambda\int_{\widetilde{T}_{k-1}}^{\widetilde{T}_k}\left|W_j\circ g_j^{-1}(\mu_j^\star(t),t) - W_j\circ g_j^{-1}(\mu_j^\star(\tau_k),\tau_k)\right|\mathsf{d}t$$
$$\overset{(c)}{=} \hat{\lambda}_k L_\Lambda\int_{\widetilde{T}_{k-1}}^{\widetilde{T}_k}\left|W_j\circ g_j^{-1}(\mu_j^\star(t),t) - W_j\circ g_j^{-1}(\mu_j^\star(t),\tau_k)\right|\mathsf{d}t$$
$$\overset{(d)}{=} \hat{\lambda}_k L_\Lambda\Gamma_W\Gamma_g\int_{\widetilde{T}_{k-1}}^{\widetilde{T}_k}|t-\tau_k|\mathsf{d}t$$
$$\overset{(d)}{=} \frac{1}{4}\hat{\lambda}_k L_\Lambda\Gamma_W\Gamma_g\Delta_k^2$$

where in (a) $\hat{\lambda}_k = \max_{t \in [\widetilde{T}_{k-1}, \widetilde{T}_k]} \lambda_j(t)$, in (b) we use the Lipschitz constant of Λ_j in its first argument, in (c) we have $\mu_j^\star(t) = \mu_j^\star(\tau_k)$ since this quantity is piecewise constant on each $[T_{i-1}, T_i]$ (which are contained in $\{\widetilde{T}_k\}_{k=1}^K$) and in (d) we apply the Lipschitz constants (w.r.t. t) of W, g^{-1}.

If we now sum this bound across k and j, using $\Delta_k = \mathcal{O}(1/K)$ we obtain the stated bound. If this error is ϵ_K, a more precise bound is given by

$$\epsilon_K \leq \frac{M}{4} L_\Lambda \Gamma_W \Gamma_g \sum_{k=1}^{K} \hat{\lambda}_k \Delta_k^2 + \mathcal{O}(1/K^2).$$

□

Appendix C

Code Listings

This section provides concise listings of select Python programs used in core parts of the thesis. Various non-essential pieces have been omitted for brevity.

C.1 The Method of Bisection

First, we have a simple implementation of the fundamental bisection algorithm, see Section 2.4.1. This serves as a key subroutine for nearly any specialized algorithm for solving RTB contract management problems.

```
def binary_search_f(f, y, xl, xr, max_iter=100, eps=1e-6):
    """
    Given a function f which is monotone /increasing/, find an x
    in the interval [xl, xr] such that f(x) = y.
    """
    for _ in range(max_iter):
        x = xl + (xr - xl) / 2.0
        if f(x) >= y:
            xr = x
        else:
            xl = x

        err = xr - xl
```

```
        if err < eps:
            break
    return xl + (xr - xl) / 2.0
```

And a convenient method for constructing inverses

```
def make_inverse(f, xl, xr, eps=1e-6, max_iter=100):
    def inverse_f(y):
        return binary_search_f(f, y, xl, xr, max_iter=max_iter, eps=eps)
    return inverse_f
```

C.2 Primal Solver in cvxpy

Recall that the sets $\mathcal{A}_i, \mathcal{B}_j$ are derived simply from the sparsity pattern of the valuations v_{ij}.

```
def AB_from_v(v):
    N, M = v.shape
    A = {i: {j for j in range(M) if v[i, j] > 0} for i in range(N)}
    B = {j: {i for i in range(N) if v[i, j] > 0} for j in range(M)}
    return A, B
```

The simple example problem with $W_j(x) = 1 - e^{-\gamma_j x}$ serves as the canonical example throughout the thesis. Although much of the interesting computational results arise as part of a duality analysis, the following listing provides a `cvxpy` program to solve the convex primal Problem (P_R) given these supply curves.

```
"""Solves the particular primal problem where W(x) = 1 - exp(-gamma * x)"""

import numpy as np
import cvxpy as cvx

def acq(q, gamma):
    return (1.0 / gamma) * (q - cvx.entr(1 - q))
```

```
def primal_objective(R, B, lmbda, gamma):
    ret = 0.0
    for j in B:
        s_j = cvx.sum([R[i][j] for i in B[j]])
        ret += lmbda[j] * acq(s_j / lmbda[j], gamma[j])
    return ret

def primal_constraints(R, A, v, C):
    N, M = v.shape
    constraints = []
    for i in A:
        constraints.append(sum([R[i][j] for j in A[i]]) == C[i])

    for i, A_j in A.items():
        for j in A[i]:
            constraints.append(R[i][j] >= 0)
    return constraints

def construct_primal_program(v, C, lmbda, gamma):
    A, B = AB_from_v(v)
    N, M = v.shape
    R = {i: {j: cvx.Variable(name=f"R[{i}][{j}]")
        for j in A[i]} for i in range(N)}
    obj = cvx.Minimize(primal_objective(R, B, lmbda, gamma))
    constraints = primal_constraints(R, A, v, C)
    problem = cvx.Problem(objective=obj, constraints=constraints)
    return problem, R

def solve_primal(v, C, lmbda, gamma, eps=1e-3, verbose=True):
    problem, R = construct_primal_program(v, C, lmbda, gamma)
    problem.solve(solver="SCS", verbose=verbose, eps=eps)
    R = {i: {j: float(R[i][j].value) for j in R[i]} for i in R}
    return (problem.value, R)
```

The ADMM based *Splitting Conic Solver* `SCS` [134, 135] performs well on this problem.

C.3 Dual Solver in cvxpy

The simple example problem for $W_j(x) = 1 - e^{-\gamma_j x}$ is also one of the few cases where closed form for the conjugate acquisition cost function $\Lambda_j^\star$ can be derived in terms of elementary functions. We are thus able to use `cvxpy` as a solver for the dual problem (D).

```python
"""Solve the particular dual problem for W(x) = 1 - exp(-gamma * x)."""

import numpy as np
import cvxpy as cvx

def conjugate_acq(mu, gamma):
    return mu + (1 / gamma) * (cvx.exp(-gamma * mu) - 1)

def mu_to_s(mu, gamma):
    return 1 - np.exp(-gamma * mu)

def dual_objective(mu, rho, C, lmbda, gamma):
    M, N = len(gamma), len(C)
    return cvx.sum(cvx.multiply(C, rho)) - cvx.sum(
        [lmbda[j] * conjugate_acq(mu[j], gamma[j]) for j in range(M)]
    )

def dual_constraints(mu, rho, v):
    N, M = v.shape
    constraints = []
    for i in range(N):
        for j in (j for j in range(M) if v[i, j] > 0):
            constr = v[i, j] * rho[i] <= mu[j]
            constraints.append(constr)
    return constraints

def construct_dual_program(M, N, v, C, lmbda, gamma):
    rho = cvx.Variable(shape=(N,), name="rho")
    mu = cvx.Variable(shape=(M,), name="mu")

    obj = cvx.Maximize(dual_objective(mu, rho, C, lmbda, gamma))
```

```
    constraints = dual_constraints(mu, rho, v)
    constraints = constraints + [mu >= 0, rho >= 0]

    problem = cvx.Problem(objective=obj, constraints=constraints)
    return problem, rho, mu

def solve_dual(M, N, v, C, lmbda, gamma, eps=1e-3, verbose=True):
    problem, rho, mu = construct_dual_program(M, N, v, C, lmbda, gamma)
    problem.solve(solver="SCS", verbose=verbose, eps=eps)
    return rho.value, mu.value
```

Again, the SCS solver performs well on this problem. As we have remarked in Chapter 3, the constraints $\mu_j \geq 0, \rho_i \geq 0$ are not formally necessary due to the monotonicity properties of $\Lambda_j^\star$. However, explicitly including them in the optimization model drastically improves numerical performance on large problem instances. In some cases, the solver fails completely if these constraints are not included.

C.4 Transportation Solver in cvxpy

The following program will solve Problem (T_t) from Section 3.2.3 for obtaining an optimal R_{ij} array when an optimal acquisition rate s_j is known. This is in general a simple convex quadratic program (as it does not involve supply curves) and can leverage existing high quality solvers. In particular, we have found the ADMM based *Operator Splitting QP Solver* OSQP [165] to perform well, and is generally more reliable than SCS for this problem.

```
"""Find an allocation array R given fixed supply rates s."""

import numpy as np
import cvxpy as cvx

class RaggedArray:
    def __init__(self, R_dict, M, N):
        """Make a nested dictionary behave like an array."""
        self.R = R_dict
        self.M = M
        self.N = N
```

```
        return

    def __getitem__(self, ix):
        return self.R[ix[0]][ix[1]]

    @property
    def value(self):
        R = np.zeros((self.N, self.M))
        for i in range(self.N):
            for j in self.R[i]:
                R[i, j] = self[i, j].value
        return R

def construct_objective(R, C, v, lmbda, s, A, t=1e-6):
    N = len(C)
    obj = 0.0
    for i in range(N):
        sval = sum(lmbda[j] * v[i, j] * R[i, j] for j in A[i])
        obj += cvx.pos(C[i] - sval)
    obj += t * sum(sum(R[i, j] ** 2 for j in A[i]) for i in range(N))
    return obj

def construct_constraints(R, B, s):
    constraints = []
    M = len(B)
    for j in range(M):
        for i in B[j]:
            constraints.append(R[i, j] >= 0)
        constraints.append(sum(R[i, j] for i in B[j]) == s[j])
    return constraints

def construct_transportation_program(M, N, A, B, v, C, s, lmbda):
    R = RaggedArray(
        {i: {j: cvx.Variable(name=f"R[{i}, {j}]")
            for j in A[i]} for i in range(N)},
        M, N)
    s = cvx.Parameter(shape=M, name="s")
```

```python
    t = t_exact(lmbda, v, A)
    obj = construct_objective(R, C, v, lmbda, s, A, t=t)
    constraints = construct_constraints(R, B, s)
    problem = cvx.Problem(objective=cvx.Minimize(obj), constraints=constraints)
    return problem, R, s

def t_exact(lmbda, v, A):
    N = v.shape[0]
    lmbda_max = max(lmbda)
    return 0.99 * min((max(lmbda[j] * v[i, j] for j in A[i]) for i in range(N)))

def solve_transportation_problem(M, N, A, B, v, C, s, lmbda, eps=1e-4):
    problem, R, s_param = construct_transportation_program(
        M, N, A, B, v, C, s, lmbda)
    s_param.value = s
    problem.solve(solver="OSQP", eps=eps)
    return R.value
```

Generic Mathematical Notation

$\mathbb{R}$ The set of real numbers.

$\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ The extended real numbers.

$\triangleq$ Equal by definition.

$\mathbb{R}_+ = [0, \infty)$ The set of non-negative real numbers.

$\mathbb{N} = \{1, 2, \ldots\}$ The Natural numbers.

$[N] = \{1, 2, \ldots, N\}$ Set of N objects.

$f \equiv a$ The function f always evaluates to the constant value a.

$\mathsf{int}\, S$ Interior of the set S.

$\mathsf{cl}\, S$ Closure of the set S.

$\mathsf{conv}\, S$ Convex hull of the set S.

$f \circ g(x) = f(g(x))$ Function composition. For functions of two variables, we occasionally write $f \circ g(x, y) = f(g(x, y), y)$, *i.e.*, composition with respect to the first argument.

∇f The gradient (column) vector of the function f.

∂f The subgradient set valued map for the convex function f.

$\mathsf{D} f$ The derivative (row) vector of the function f.

f' Derivative of a scalar function $f : \mathbb{R} \to \mathbb{R}$.

$\mathsf{dg}(A)$ The vector formed from the diagonal of the square matrix A.

$\mathsf{Dg}(a)$ A square diagonal matrix consisting of the entries of the vector a along the main diagonal and 0 elsewhere.

$\boldsymbol{e}$ A vector containing all 1.

$\mathbf{vec}\, A$ Stacks the columns of A into a vector.

a^{T} Transpose of the vector a.

$\mathcal{N}(A)$ Nullspace of the matrix A.

$\langle x, y \rangle = x^{\mathsf{T}} y$ Inner product between vectors x, y.

$||x||$ A norm of x. $||x||_2$ is the Euclidean norm.

List of Symbols

$\mathcal{A}_i^\star \subseteq \mathcal{A}_i$ Item types actually allocated towards contract i at an optimal solution.

$\mathcal{B}_j^\star \subseteq \mathcal{B}_j$ Contracts towards which items of type j are actually allocated at an optimal solution.

N Number of contracts to be fulfilled.

$i \in [N]$ Denotes the i^{th} contract.

M Number of available discrete item types.

$j \in [M]$ Used to denote the j^{th} item type.

$\eta_j > 0$ Probability that an arriving item is of type j.

$\lambda > 0$ Arrival rate of items. As well, $\lambda_j = \eta_j \lambda$.

Φ Occasionally used when referring to a continuum of item types.

$\phi \in \Phi$ Generic notation for item type. Occasionally used for elements in a dual vector space $V^\star$, such as in the subgradient $\phi \in \partial f(x)$.

x Usually used to denote a bid. x_j is the bid placed on items of type j.

$s \geq 0$ The rate s of acquiring items. s_j is the acquisition rate for items of type j.

q Used to denote a probability of winning an item. q_j is the probability to win an item of type j.

$\mu_j \geq 0$ Dual variable associated to the constraint $s_j \leq \lambda_j$. May appear in similar places as x, due to the relationship between this dual variable and a bid.

$\rho_i \geq 0$ Dual variable associated to contract fulfillment constraints, the "pseudo-bids". May appear in similar places as x or μ, due to the relationship between this dual variable and a bid.

$\theta_{ij} \geq 0$ Dual variables associated to the constraint $R_{ij} \geq 0$, $\theta_{ij} = \mu_j - v_{ij}\rho_i$. Also used generically to denote a convex (but not differentiable) function in Section 4.2.2.

$W(x)$ A supply curve *c.f.*,Definition 2.1.1.

$f(x)$ A cost function derived from a supply curve. f_{2nd} specifies the second price auction (the default) and f_{1st} specifies the first price auction. See Section 2.1.2.

$\Lambda(q)$ Acquisition cost function – see Section 2.1.2.

$\Lambda^\star(\mu)$ Conjugate acquisition cost function – see Section 2.1.2.

$C_i > 0$ Amount of supply that needs to be allocated towards contract $i \in [N]$.

$v_{ij} \geq 0$ The value of items of type j towards the fulfillment of contract i.

$\mathcal{A}_i \subseteq [M]$ Item types that can fulfill contract i: $j \in \mathcal{A}_i \iff v_{ij} > 0$.

$\mathcal{B}_j \subseteq [N]$ Contracts towards which items of type j can be usefully allocated: $i \in \mathcal{B}_j \iff v_{ij} > 0$.

T_i Time deadline for contract $i \in [N]$.

T Final time deadline $T = T_N = \max_{i \in [N]} T_i$.

$\mathcal{T}_t$ Set of contracts active at time t: $\mathcal{T}_t \triangleq \{i \in [N] \mid t < T_i\}$.

$\widetilde{T}_k$ Times used to segment the interval $[0, T)$.

List of Optimization Problems

(P^m) The main problem in the original monotone programming formulation involving W, f.

(P) The main problem reformulated as a convex optimization problem.

(P_R) The main problem in convex form with the variable s eliminated.

(P^u) The main problem in compact form with $u \in \mathbb{R}^{d-M}$ and constraints $Gu \leq h$.

(P^u_β) Problem (P^u) with Tikhonov regularization $\frac{1}{2\beta}||u||_2^2$.

(D) The dual of Problem (P).

(T_t) The pure transportation problem with s fixed and penalty $\frac{1}{2}t||R||_2^2$.

(SA) The generic stochastic approximation algorithm.

(SRI) The generic stochastic recursive inclusion algorithm.

(D_μ) Dual problem with variables ρ eliminated.

(D_ρ) Dual problem with variables μ eliminated.

(P^m_T) The main problem for contracts with time deadlines in monotone programming form.

(P_T) The main problem for contracts with time deadlines in convex form.

(D_T) The dual problem for contracts with time deadlines.

Definitions of Select Terms

Auction A mechanism for selling items. The item is awarded to the agent offering to pay the most, but their payment depends on the type of the auction.

Bidder Agent bidding for (*i.e.,* attempting to buy) items in RT.

Censored An observation is censored if values outside of a certain raange cannot be distinguished. For example, $Z = \min(X, 10)$ is a censored observation of the value X.

Contract An agreement to acquire a certain total value of items in the RTB market. Formally, the i^{th} contract is the triple $\big((v_{ij})_{j\in[M]}, C_i\big)$.

First Price Auction An auction where the winner pays what they bid.

Impression When an ad is served to a user, it is considered an "impression".

Intermediary An agent that aggregates the demand of multiple bidders and participates in RTB on their behalf.

Item A generic term for whatever is being sold in RTB. Usually, an impression.

Item type The *type* of an item specifies certain identifiable properties. Generically, items belong either to a finite collection of M types, or more generally belong to an abstract (possibly uncountable) space Φ. All items of the same type are completely fungible.

Publisher Operator of a website or app, usually themselves a seller.

Real-time Bidding An auction mechanism for sequentially selling multiple items. Items arrive and are immediately sold through a sealed bid auction using either a second price or first price auction..

Reserve (price) In an auction, the reserve price is the minimum allowed bid.

Sealed Bid An auction is sealed bid if agents do not observe the bids placed by other bidders. All auction mechanisms employed in RTB are sealed bid.

Second Price Auction An auction where the winner pays the second highest bid.

Seller Agent selling items in RTB.

User A visitor to a webpage, app, etc.

Vickrey Auction A sealed bid second price auction.