ON SPECHT'S THEOREM IN UHF C^*-ALGEBRAS

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ABSTRACT. Specht's Theorem states that two matrices A and B in $\mathbb{M}_n(\mathbb{C})$ are unitarily equivalent if and only if $\operatorname{tr}(w(A, A^*)) = \operatorname{tr}(w(B, B^*))$ for all words $w(x, y)$ in two non-commuting variables x and y. In this article we examine to what extent this trace condition characterises approximate unitary equivalence in uniformly hyperfinite (UHF) C^*-algebras. In particular, we show that given two elements a, b of the universal UHF-algebra $\mathcal{Q}$ which generate C^*-algebras satisfying the UCT, they are approximately unitarily equivalent if and only if $\tau(w(a, a^*)) = \tau(w(b, b^*))$ for all words $w(x, y)$ in two non-commuting variables (where τ denotes the unique tracial state on $\mathcal{Q}$), while there exist two elements a, b in the UHF-algebra $\mathbb{M}_{2^\infty}$ which fail to be approximately unitarily equivalent despite the fact that they satisfy the trace condition. We also examine a consequence of these results for ampliations of matrices.

1. INTRODUCTION

A standard and important strategy in mathematics is to classify the elements of a given set up to some form of equivalence. For elements A and B of the algebra $\mathbb{M}_n(\mathbb{C})$ of $n \times n$ complex matrices, the two most important forms of equivalence are similarity: $B = S^{-1}AS$ for some invertible matrix $S \in \mathbb{M}_n(\mathbb{C})$, and unitary equivalence: $B = U^*AU$ where $U \in \mathbb{M}_n(\mathbb{C})$ is unitary. As is well-known, the Jordan form provides a complete invariant for the similarity orbit $\mathcal{S}(A) := \{S^{-1}AS : S \text{ invertible in } \mathbb{M}_n(\mathbb{C})\}$ of $A \in \mathbb{M}_n(\mathbb{C})$. For unitary equivalence, the invariant is somewhat more delicate, and is due to Specht [30]. In particular, he shows that two matrices A and B in $\mathbb{M}_n(\mathbb{C})$ are unitarily equivalent if and only if $\operatorname{tr}(w(A, A^*)) = \operatorname{tr}(w(B, B^*))$ for all words $w(x, y)$ in two non-commuting variables x and y. A result of Pearcy [20] shows that in fact one need only consider words of length at most $2n^2$. A nice and complete (up to 1990) survey of Specht-Pearcy trace invariants can be found in [27].

Our goal in the present article is to examine to what extent Specht's trace condition above characterises approximate unitary equivalence in C^*-algebras. Because Specht's trace condition requires a family of traces that separates projections, we shall focus our attention on uniformly hyperfinite C^*-algebras (UHF-algebras), or very closely related approximately finite C^*-algebras (AF-algebras). Recall that the **universal**

UHF-algebra $\mathcal{Q}$ is the UHF-algebra whose supernatural number is divisible by any positive integer – or equivalently, it is the UHF-algebra whose K_0-group is divisible. As we shall see below (Theorem 3.2), in the case of the universal UHF-algebra $\mathcal{Q}$, given two elements $a, b \in \mathcal{Q}$, if a and b satisfy Specht's trace condition and $C^*(a)$ satisfies the UCT, then a is approximately unitarily equivalent to b. We shall also see that this fails for more general UHF-algebras (even under the assumption that *both* $C^*(a)$ and $C^*(b)$ satisfy the UCT), and in particular for the CAR algebra $\mathbb{M}_{2^\infty}$ (Theorem 3.7).

The idea of characterising approximate unitary equivalence of single elements in various C^*-algebras is not new. Of course, in $\mathcal{B}(\mathcal{H})$, this problem has a long history, of which we cite only [13, 34], and characterising the (necessarily closed) unitary orbits of normal elements of the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the content of the celebrated Brown-Douglas-Fillmore Theorem [6]. Passing to approximate unitary equivalence in subalgebras of $\mathcal{B}(\mathcal{H})$, Sherman [28] obtained a description of the norm- and strong*-closures of a normal operator in a von Neumann algebra, and Skoufranis [29] has characterised approximate unitary equivalence classes of normal elements of unital, simple, purely infinite C^*-algebras whose K_1-groups are trivial. We emphasise that while many of these results focus on approximate unitary equivalence of normal elements of the algebra under consideration, our results will apply to general elements of $\mathcal{Q}$. Having said this, it is worth noting that there is a vast and relevant literature dealing with approximate unitary equivalence of *-homomorphisms between C^*-algebras, led in large part by H. Lin (see [17] and its references).

In Section 4 we establish a relation, which we refer to as the *approximate absolutely value condition* (AAVC), which coincides with Specht's trace condition for UHF-algebras, but which may be formulated in any C^*-algebra, regardless of the presence or absence of a trace. Theorem 4.7 below shows that two bounded linear operators A and B acting on a complex, separable Hilbert space are approximately unitarily equivalent if and only if they satisfy the AAVC.

In Section 5, we produce an interesting consequence of our work on UHF-algebras by demonstrating the existence of positive integers n and k, and two $n \times n$ complex matrices A and B such that the distance between the unitary orbits of $A^{(k)} := I_k \otimes A$ (where $I_k \in \mathbb{M}_k(\mathbb{C})$ is the identity matrix) and $B^{(k)}$ in $\mathbb{M}_{nk}(\mathbb{C})$ is strictly smaller than the distance between the unitary orbits of A and B in $\mathbb{M}_n(\mathbb{C})$.

Let us now establish some definitions and notations which will be used throughout the remainder of the paper.

Given a unital C^*-algebra $\mathcal{A}$, we shall denote by $\mathcal{U}(\mathcal{A})$ the unitary group of $\mathcal{A}$; that is, $\mathcal{U}(\mathcal{A}) = \{u \in \mathcal{A} : u^*u = 1 = uu^*\}$. If $a \in \mathcal{A}$, the **unitary orbit** of a is the set $\mathcal{U}(a) := \{u^*au : u \in \mathcal{U}(\mathcal{A})\}$. When $\mathcal{A}$ is finite-dimensional, $\mathcal{U}(\mathcal{A})$ is compact and $\mathcal{U}(a)$ is necessarily closed. In general, however, $\mathcal{U}(a)$ is not closed. By $\mathcal{A}_{\mathrm{sa}}$ we denote the set of all hermitian elements of $\mathcal{A}$.

We denote by $\mathcal{W}_2$ the set of all words in two non-commuting variables x and y. There is a natural action of $\mathcal{W}_2$ on $\mathcal{A}$ given by $[w(x, y)] \bullet a := w(a, a^*)$ for all $w \in \mathcal{W}_2$, $a \in \mathcal{A}$. This action naturally extends to an action of the complex algebra $\mathcal{P}_2$ of all polynomials in two non-commuting variables spanned by $\mathcal{W}_2$ on $\mathcal{A}$.

We shall assume that the reader is familiar with the notion of a UHF-algebra $\mathcal{A}$ as an inductive limit of full matrix algebras $\mathbb{M}_{n_k}(\mathbb{C})$, $k \geq 1$ and that UHF-algebras are classified up to isometric $*$-isomorphism by their *supernatural numbers* $s(\mathcal{A})$; equivalently by their ordered K_0-group $K_0(\mathcal{A})$. We also assume that the reader is familiar with approximately finite, or AF-algebras as inductive limits of finite-dimensional C^*-algebras and their classification [11] in terms of K-theory. The reader may refer to [10] for more details if required. We shall also assume that the reader is familiar with KK- and KL-theory as found in [17, 21].

Specht's Theorem mentioned above says that given two matrices A and B in $\mathbb{M}_n(\mathbb{C})$, A is unitarily equivalent to B if and only if

$$\operatorname{tr}(w(A, A^*)) = \operatorname{tr}(w(B, B^*)) \quad \text{for all } w \in \mathcal{W}_2. \tag{$*$}$$

We shall refer to equation $(*)$ and its analogue for a pair of elements a, b in a general C^*-algebra $\mathcal{A}$ possessing a unique tracial state τ as **Specht's trace condition**.

Let us also remind the reader that if $\mathcal{A}$ is a C^*-algebra and τ is a tracial state on $\mathcal{A}$, then one can always extend τ to a tracial state τ_n on $\mathbb{M}_n(\mathcal{A})$ by setting $\tau_n := \tau \otimes \operatorname{tr}$, i.e., $\tau_n([a_{i,j}]) = \frac{1}{n} \sum_{i=1}^{n} \tau(a_{i,i})$.

We require the following definition.

Definition 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras and $\varphi : \mathcal{A} \to \mathcal{B}$ be a linear map. The n**-fold ampliation**

$$\varphi^{(n)} := \operatorname{id}_{\mathbb{M}_n} \otimes \varphi : \begin{array}{ccc} \mathbb{M}_n(\mathcal{A}) & \to & \mathbb{M}_n(\mathcal{B}) \\ [a_{i,j}] & \mapsto & [\varphi(a_{i,j})] \end{array}$$

is also linear. It is well-known and easy to verify that if φ is a $*$-homomorphism, then $\varphi^{(n)}$ is also a $*$-homomorphism.

2. Preliminary results

The principal result of this section is Theorem 2.5, which will be the key to proving our main theorems in Section 3. Recall that given an algebra $\mathcal{A}$ and an invertible element $s \in \mathcal{A}$, we denote by $\operatorname{ad}_s : \mathcal{A} \to \mathcal{A}$ the continuous linear map $\operatorname{ad}_s(a) = s^{-1}as$.

Definition 2.1. Let $\mathcal{A}$ be a separable, unital C^*-algebra and $a, b \in \mathcal{A}$. We say that a and b are

(a) **unitarily equivalent**, written $a \simeq b$, if $\operatorname{ad}_u(a) = b$ for some unitary $u \in \mathcal{A}$, and that they are

(b) **approximately unitarily equivalent**, in symbols $a \simeq_a b$, if there is a sequence $(u_n)_{n=1}^{\infty}$ of unitaries in $\mathcal{A}$ such that $\lim_{n \to \infty} \operatorname{ad}_{u_n}(a) = b$.

(c) If $\mathcal{B}$ is another unital C^*-algebra and φ and ψ are two unital $*$-homorphisms from $\mathcal{B}$ to $\mathcal{A}$, we say that φ and ψ are **approximately unitarily equivalent** if there exists a sequence $(u_n)_{n=1}^{\infty}$ of unitaries in $\mathcal{A}$ such that $\psi(b) = \lim_{n \to \infty} \operatorname{ad}_{u_n}(\varphi(b))$ for all $b \in \mathcal{B}$.

It is routine to verify that both unitary equivalence and approximate unitary equivalance are indeed equivalence relations on $\mathcal{A}$, and that two elements $a, b \in \mathcal{A}$ satisfy $a \simeq_a b$ if and only if $\overline{\mathcal{U}(a)} = \overline{\mathcal{U}(b)}$. When $\mathcal{A}$ is finite-dimensional, the unitary group

of $\mathcal{A}$ is compact, in which case unitary equivalence and approximate unitary equivalence coincide. That these concepts are in general different is demonstrated by the following example, the existence of which is surely known to the experts in the field. Since it is difficult to trace the origin or a reference for this example, we provide the construction for the benefit of the reader.

Example 2.2. We shall produce two positive elements a and b of a UHF C^*-algebra which are approximately unitarily equivalent but not unitarily equivalent.

Let $(m_i)_{i \geq 1}$ be a sequence of integers each greater than or equal to three, and define $k_n := \prod_{i=1}^{n} m_i$, $n \geq 1$. Consider the UHF C^*-algebra

$$\mathcal{A} = \overline{\underset{n \geq 1}{\cup} \mathbb{M}_{k_n}(\mathbb{C})},$$

where $\mathbb{M}_{k_n}(\mathbb{C})$ is identified with the subalgebra $\mathbb{M}_{k_n}(\mathbb{C}) \otimes I_{m_{n+1}}$ of $\mathbb{M}_{k_{n+1}}(\mathbb{C})$ for each $n \geq 1$. Let $\{e_{i,j}^{(n)} : 1 \leq i, j \leq k_n\}$ denote the standard matrix units of $\mathbb{M}_{k_n}(\mathbb{C})$, and for $n \geq 2$, set

$$q_1^{(n)} = e_{k_n-1,k_n-1}^{(n)}, \qquad q_2^{(n)} = e_{k_n-2,k_n-2}^{(n)}.$$

Then $q_1^{(n)}, q_2^{(n)}$ are projections in $\mathcal{A}$ – unitarily equivalent to each other via a unitary in $\mathbb{M}_{k_n}(\mathbb{C})$ – and

$$\tau(q_i^{(n)}) = \frac{1}{k_n}, \ i = 1, 2.$$

Next, define

$$a = \sum_{n \geq 2} \frac{1}{2^n} q_1^{(n)}, \ b = \sum_{n \geq 2} \frac{1}{2^n} q_2^{(n)}.$$

Then a, b are positive elements in $\mathcal{A}$ with $\sigma(a) = \sigma(b) = \{\frac{1}{2^n} : 2 \leq n\} \cup \{0\}$. Moreover, $q_1^{(n)}$ (resp. $q_2^{(n)}$) is the spectral projection for a (resp. for b) corresponding to the measurable set $E_n := \{\frac{1}{2^n}\}$.

To see that a and b are approximately unitarily equivalent, we set, for each $N \geq 2$, $a_N = \sum_{n=2}^{N} \frac{1}{2^n} q_1^{(n)}$ and $b_N = \sum_{n=2}^{N} \frac{1}{2^n} q_2^{(n)}$. Since $\operatorname{tr}(q_1^{(n)}) = \operatorname{tr}(q_2^{(n)})$ for all $n \geq 2$, we see that – viewing a_N and b_N as elements of $\mathbb{M}_{k_N}(\mathbb{C})$ – they are selfadjoint, have the same spectrum, and each of the eigenvalues is repeated with the same multiplicity. As such, they are unitarily equivalent in $\mathbb{M}_{k_N}(\mathbb{C})$ via a unitary element, say $u_N^* a_N u_N = b_N$. It is routine to verify that $a = \lim_{N \to \infty} a_N$ and similarly that $b = \lim_{N \to \infty} b_N$. A standard application of the triangle inequality

$$\|b - u_N^* a u_N\| \leq \|b - b_N\| + \|b_N - u_N^* a_N u_N\| + \|u_N^*(a_N - a)u_N\|$$

then shows that $a \simeq_a b$ in $\mathcal{A}$.

We claim that a and b are not unitarily equivalent in $\mathcal{A}$. Otherwise, there exists a unitary $u \in \mathcal{U}(\mathcal{A})$, such that $u^* a u = b$. By the continuous functional calculus for normal elements in $\mathcal{A}$, it is then easy to show that

$$u^* q_1^{(n)} u = q_2^{(n)}, \ \forall n \in \mathbb{N}.$$

Since $u \in \mathcal{A}$, there exists $N \in \mathbb{N}$, such that

$$\operatorname{dist}(u, \mathbb{M}_{k_N}(\mathbb{C})) < \frac{1}{2}.$$

Choose such a $t = [t_{i,j}] \in \mathbb{M}_{k_N}(\mathbb{C}) \subseteq \mathcal{A}$ for which $\|u - t\| < \frac{1}{2}$. As noted above, in $\mathbb{M}_{k_{N+1}}(\mathbb{C}) \subseteq \mathcal{A}$, we identify t with $[t_{i,j} \otimes 1_{m_{N+1}}]$.

Now $q_1^{(N+1)}u = uq_2^{(N+1)}$, and so

$$1 = \|uq_2^{(N+1)}\| = \|(uq_2^{(N+1)})q_2^{(N+1)}\| = \|(q_1^{(N+1)}u)q_2^{(N+1)}\|.$$

On the other hand, a simple computation reveals that

$$q_1^{(N+1)}[t_{i,j} \otimes I_{m_{N+1}}]q_2^{(N+1)} = 0.$$

But then

$$\frac{1}{2} > \|u - t\| \geq \|q_1^{(N+1)}(u - [t_{i,j} \otimes I_{m_{N+1}}])q_2^{(N+1)}\| \geq 1 - 0 = 1,$$

an obvious contradiction. Thus a and b are not unitarily equivalent.

Given a tracial state τ on a C^*-algebra $\mathcal{A}$, $a \in \mathcal{A}$ and $u \in \mathcal{U}(\mathcal{A})$, we see that $\tau(u^*au) = \tau((au)u^*) = \tau(a)$, so that τ is constant on unitary orbits. In fact, by continuity, τ is constant on closures of unitary orbits. From this observation, we obtain the following Proposition.

Proposition 2.3. *Let $\mathcal{A}$ be a unital C^*-algebra, $a, b \in \mathcal{A}$, and τ be a tracial state on $\mathcal{A}$. If $a \simeq_a b$, then for each word $w \in \mathcal{W}_2$, we have that $\tau(w(a, a^*)) = \tau(w(b, b^*))$.*

Proof. Bearing in mind the statement preceding this Proposition, we see that it suffices to prove that if $a \simeq_a b$, and if $w \in \mathcal{W}_2$, then $w(a, a^*) \simeq_a w(b, b^*)$. This is a relatively simple consequence of the continuity of words as functions on $\mathcal{A} \times \mathcal{A}$.

Indeed, fix $w \in \mathcal{W}_2$, and choose a sequence $(u_n)_{n=1}^\infty \in \mathcal{U}(\mathcal{A})$ such that $b = \lim_{n \to \infty} \mathrm{ad}_{u_n}(a)$. As w defines a continuous function on $\mathcal{A} \times \mathcal{A}$, and as $b^* = \lim_{n \to \infty} \mathrm{ad}_{u_n}(a^*)$, we see that

$$w(b, b^*) = \lim_{n \to \infty} w(u_n^*au_n, u_n^*a^*u_n) = \lim_{n \to \infty} u_n^*w(a, a^*)u_n.$$

In other words, $w(a, a^*) \simeq_a w(b, b^*)$, completing the proof. $\blacksquare$

In light of the above Proposition, the best one might hope for in trying to generalise Specht's Theorem to the C^*-algebra setting would be to replace "unitary equivalence" by "approximate unitary equivalence". That is, the most straightforward conjecture might be that the converse of Proposition 2.3 holds. As the next example shows, however, even when the ambient C^*-algebra is simple, unital and admits a unique, faithful tracial state τ, this generalisation of Specht's Theorem may fail.

Example 2.4. As shown in [21, p24], there exists a simple, unital AF algebra $\mathcal{A}$ with unique faithful tracial state τ, and a pair of projections p, q in $\mathcal{A}$, such that $\tau(p) = \tau(q)$, and yet p and q are not unitarily equivalent. Observe that for each word w,

$$\tau(w(p, p^*)) = \tau(p) = \tau(q) = \tau(w(q, q^*)).$$

On the other hand, it is well-known that two projections in a C^*-algebra are approximately unitarily equivalent if and only if they are unitarily equivalent. Thus p and q are not approximately unitarily equivalent either.

The problem highlighted by the above example is that in many (simple, unital) C^*-algebras, the tracial states (even if there happens to be only one and it is faithful) need not distinguish inequivalent projections. Without this, a direct generalisation of Specht's Theorem is impossible. As we shall now see, however, Specht's trace condition on two elements a and b of a C^*-algebra $\mathcal{A}$ with a faithful tracial state is sufficient to show that a and b are algebraically equivalent. (Recall that two elements a and b of a C^*-algebra $\mathcal{A}$ are said to be **algebraically equivalent** if there exists a *-isomorphism $\Phi : C^*(a) \to C^*(b)$ satisfying $\Phi(a) = b$.)

Theorem 2.5. *Let $\mathcal{A}$ be a unital C^*-algebra with a faithful tracial state τ, and $a, b \in \mathcal{A}$. Assume that for each two-variable word $w \in \mathcal{W}_2$, $\tau(w(a, a^*)) = \tau(w(b, b^*))$. For each polynomial $p \in \mathcal{P}_2$ in two non-commuting variables, define $\Phi(p(a, a^*)) = p(b, b^*)$. The following conclusions hold.*

(a) *$\|p(a, a^*)\| = \|p(b, b^*)\|$ for all polynomials $p \in \mathcal{P}_2$.*
(b) *Φ is well-defined and extends in a unique way to an isomorphism from $C^*(a)$ onto $C^*(b)$ which implements the algebraic equivalence of a and b.*
(c) *$\sigma(a) = \sigma(b)$.*
(d) *If $1 \le k \in \mathbb{N}$ and $[a_{i,j}] \in \mathbb{M}_k(C^*(a))$, then*

$$\tau_k([a_{i,j}]) = \tau_k(\Phi^{(k)}([a_{i,j}])).$$

(e) *Suppose furthermore that a and b are normal and denote $X := \sigma(a) = \sigma(b)$. If $F \in \mathbb{M}_k(\mathcal{C}(X))$ then*

$$\tau_k(\varphi^{(k)}(F)) = \tau_k(\psi^{(k)}(F)),$$

where $\varphi, \psi : \mathcal{C}(X) \to \mathcal{A}$ are defined via $\varphi(f) := f(a)$ and $\psi(f) = f(b)$ for all $f \in \mathcal{C}(X)$.

Proof. (a)   Firstly, consider the case where a, b are normal elements of $\mathcal{A}$. Suppose that there exists $\lambda \in \sigma(b) \setminus \sigma(a)$.

Let $Y = \sigma(a) \cup \sigma(b)$, and note that Y is a compact set of $\mathbb{C}$. As $\lambda \notin \sigma(a)$, by Urysohn's Lemma, there exists $f \in \mathcal{C}(Y)$, such that

$$f(\mu) = 0, \ \forall \mu \in \sigma(a)$$

and

$$f(\lambda) = 1, \ f(Y) \subset [0, 1].$$

By the continuous functional calculus, $f(a) = 0$, while $f(b)$ is normal with

$$1 \in \sigma(f(b)) = f(\sigma(b)) \subset [0, 1].$$

In particular, $f(b)$ is nonzero positive element. Since τ is linear, the hypothesis that $\tau(w(a, a^*)) = \tau(w(b, b^*))$ for all $w \in \mathcal{W}_2$ implies that $\tau(p(a, a^*)) = \tau(p(b, b^*))$ for all $p \in \mathcal{P}_2$. Since τ is continuous on $\mathcal{A}$, it follows that

$$\tau(f(a)) = \tau(f(b)).$$

On the other hand, clearly $\tau(f(a)) = \tau(0) = 0$, while the fact that τ is faithful and b is a non-zero positive element implies that $\tau(f(b)) > 0$, a contradiction. It follows that $\sigma(b) \subseteq \sigma(a)$. By symmetry,

$$\sigma(a) = \sigma(b).$$

Next, given a general pair $a, b \in \mathcal{A}$ with $\tau(w(a, a^*)) = \tau(w(b, b^*))$ for each word $w \in \mathcal{W}_2$, fix an element $p \in \mathcal{P}_2$ and define

$$x = p(a, a^*)^* p(a, a^*) \qquad y = p(b, b^*)^* p(b, b^*).$$

It follows that x and y are positive elements of $\mathcal{A}$, and for each word $w \in \mathcal{W}_2$,

$$\tau(w(x, x^*)) = \tau(w(y, y^*)).$$

By the above argument, $\sigma(x) = \sigma(y)$, from which it follows that $\|x\| = \|y\|$. From the C^*-equation we deduce that

$$\|p(a, a^*)\| = \|p(b, b^*)\|,$$

as required.

(b) Suppose that $p(a, a^*) = q(a, a^*)$. Then

$$\Phi(p(a, a^*)) - \Phi(q(a, a^*)) = p(b, b^*) - q(b, b^*) = (p - q)(b, b^*).$$

Observe that

$$(p - q)(a, a^*) = 0,$$

and

$$\|(p - q)(a, a^*)\| = \|(p - q)(b, b^*)\|,$$

whence

$$(p - q)(b, b^*) = 0.$$

Hence

$$\Phi(p(a, a^*)) = \Phi(q(a, a^*)).$$

Thus, Φ is well-defined and isometric, and as such, it extends in a unique way to an isomorphism from $C^*(a)$ onto $C^*(b)$. That $\Phi(a) = b$ is clear.

(c) Note that Φ is a unital isomorphism, and unital isomorphisms always preserve spectrum.

(d) Given $[a_{i,j}]$ in $\mathbb{M}_k(C^*(a))$,

$$\tau_k([a_{i,j}]) = \sum_{i=1}^{k} \tau(a_{i,i}) = \sum_{i=1}^{k} \tau(\Phi(a_{i,i})) = \tau_k(\Phi^{(k)}([a_{i,j}])).$$

(e) Let $F = [F_{i,j}] \in \mathbb{M}_k(\mathcal{C}(X))$. Then

$$\tau_k(\varphi^{(k)}(F)) = \sum_{i=1}^{k} \tau(\varphi(F_{i,i})) = \sum_{i=1}^{k} \tau(\psi(F_{i,i})) = \tau_k(\psi^{(k)}(F)).$$

∎

The above Theorem yields the following mild improvement of a result of Schafhauser's [25, Corollary 6.6]. In his case, he required $\sigma(a) = \sigma(b)$ and $\tau(f(a)) = \tau(f(b))$ for all $f \in \mathcal{C}(\sigma(a))$. These conditions follow automatically from our weaker assumption. The hypothesis that τ is faithful is also automatic.

We recall that an abelian group $(G, +)$ is said to be **divisible** if for each $n \in \mathbb{N}$, $G = nG$; i.e. given $g \in G$, there exists $h \in G$ such that $g = h + h + \cdots + h$ (n times). In the case of UHF-algebras $\mathcal{A}$, $K_0(\mathcal{A})$ is divisible if and only if $\mathcal{A} = \mathcal{Q}$, the universal UHF-algebra.

Corollary 2.6. *Suppose that $\mathcal{B}$ is a simple, unital AF-algebra with a unique trace τ and divisible K_0-group. Two normal operators a and b in $\mathcal{B}$ are approximately unitarily equivalent if and only if $\tau(w(a, a^*)) = \tau(w(b, b^*))$ for all $w \in \mathcal{W}_2$, and for every compact, open set $U \subset \sigma(a)$, the spectral projections $\chi_U(a)$ and $\chi_U(b)$ are unitarily equivalent.*

Proof. Since $\mathcal{B}$ is a simple unital AF-algebra with a unique trace τ, $\mathcal{B}$ is both nuclear and has stable rank one. Thus τ is faithful by [19, Theorem 5].

By Theorem 2.5, $\sigma(a) = \sigma(b)$ and $\tau(f(a)) = \tau(f(b))$ for all $f \in \mathcal{C}(\sigma(a))$. The remainder of the proof therefore reduces to that of Schafhauser's. ∎

Example 2.4 shows that even in a relatively nice C^*-algebra (i.e. a simple, unital AF-algebra with a unique trace), we cannot expect Specht's trace condition to characterise approximate unitary equivalence, even for normal elements. If we turn our attention to UHF C^*-algebras, however, there is still some hope. In particular, if $\mathcal{A}$ is a UHF C^*-algebra with unique tracial state τ, then two projections p and q in $\mathcal{A}$ are known to be unitarily equivalent in $\mathcal{A}$ if and only if $\tau(p) = \tau(q)$. Thus, in Corollary 2.6 above, Specht's trace condition implies that $\tau(\chi_U(a)) = \tau(\chi_U(b))$, whence $\chi_U(a) \simeq \chi_U(b)$ for all compact, open subsets $U \subseteq \sigma(a)$.

In the UHF-algebra setting, when dealing with normal elements, it turns out that the divisibility of the K_0-group is not essential, as we shall now see.

Proposition 2.7. *Let $\mathcal{A}$ be a UHF-algebra. Suppose that $m, n, \in \mathcal{A}$, and that m is normal. Then $m \simeq_a n$ if and only if $\tau(w(m, m^*)) = \tau(w(n, n^*))$ for all $w \in \mathcal{W}_2$.*

Proof. That Specht's trace condition is necessary follows from Proposition 2.3.

As for its sufficiency, by Theorem 2.5 (b), n is normal and $\sigma(m) = \sigma(n)$. Let $X := \sigma(m)$. Observe that by Corollary 7.5.4 of [15], $K_0(\mathcal{C}(X))$ and $K_1(\mathcal{C}(X))$ are free abelian groups. We may therefore apply Theorem 23.1.1 of [3] (the Universal Coefficient Theorem) to conclude that (up to isomorphism),

$$KK^*(\mathcal{C}(X), \mathcal{A}) = \mathrm{Hom}(K_*(\mathcal{C}(X)), K_*(\mathcal{A})).$$

Note, however that since $\mathcal{A}$ is a UHF-algebra, $K_1(\mathcal{A}) = 0$ [23], and thus (up to isomorphism)

$$KK(\mathcal{C}(X), \mathcal{A}) = \mathrm{Hom}(K_0(\mathcal{C}(X)), K_0(\mathcal{A})).$$

Define two maps φ and ψ from $\mathcal{C}(X)$ to $\mathcal{A}$ via

$$\varphi(f) = f(m), \qquad \psi(f) = f(n), \qquad f \in \mathcal{C}(X).$$

Then φ and ψ are monomorphisms and by Theorem 2.5(e),

$$K_0(\varphi) = K_0(\psi) \text{ in } \mathrm{Hom}(K_0(\mathcal{C}(X), K_0(\mathcal{A}))).$$

We conclude that $KK(\varphi) = KK(\psi)$. As $KL(\mathcal{C}(X), \mathcal{A})$ is a quotient of $KK(\mathcal{C}(X), \mathcal{A})$, *a fortiori*, $KL(\varphi) = KL(\psi)$. Also note that

$$\tau(f(m)) = \tau(f(n)) \text{ for all } f \in \mathcal{C}(X).$$

By [12, Theorem 2.15], $\varphi \simeq_a \psi$, and in particular,

$$m \simeq_a n.$$

∎

One of the keys to the proof of Proposition 2.7 is that projections in UHF-algebras are determined (up to unitary equivalence) by their trace. If $\mathcal{A}$ is a simple, unital AF-algebra with a unique tracial state τ, and if $\mathcal{A}$ satisfies Blackadar's FCQ1 [4] (that is, given two projections p and q in $\mathcal{A}$, $\tau(p) \leq \tau(q)$ if and only if $p \preceq q$), then this is also the case.

As such, Proposition 2.7 extends *mutatis mutandis* to this setting.

3. The good, and the ugly

In trying to extend Proposition 2.7 to more general elements of a UHF-algebra, we shall appeal to an interesting recent result of Schafhauser:

Theorem 3.1. [25, Theorem D] *If $\mathcal{A}$ is a separable, unital, exact C^*-algebra satisfying the UCT and having a faithful, amenable trace, and $\mathcal{B}$ is a simple, unital AF-algebra with a unique trace and divisible K_0-group, then the unital, trace-preserving *-homomorphisms $\mathcal{A} \to \mathcal{B}$ are classified up to approximate unitary equivalence by their behaviour on the K_0-group.*

As previously mentioned, for UHF-algebras, divisibility of the K_0-group implies that we are dealing with the universal UHF-algebra $\mathcal{Q}$. The following is our main result.

Theorem 3.2. *Let $a, b \in \mathcal{Q}$ and suppose that $C^*(a)$ satisfies the UCT. Then $a \simeq_a b$ if and only if $\tau(w(a, a^*)) = \tau(w(b, b^*))$ for all $w \in \mathcal{W}_2$.*

Proof. The necessity of Specht's trace condition follows from Proposition 2.3.

Consider now the sufficiency of this condition. Since $\mathcal{Q}$ is a nuclear C^*-algebra, the unique (faithful) tracial state τ on $\mathcal{Q}$ is amenable [7, Prop. 6.3.4]. Next, $C^*(a)$ is exact, being a C^*-subalgebra of a nuclear C^*-algebra, and by [7, Prop. 6.3.5 (a)], $\tau|_{C^*(a)}$ is amenable (and clearly faithful) as well. For each polynomial $p \in \mathcal{P}_2$, define $\Phi(p(a, a^*)) = p(b, b^*)$. By Theorem 2.5, Φ is well-defined and extends uniquely to an isomorphism from $C^*(a)$ onto $C^*(b)$. Let $\iota : C^*(a) \to \mathcal{Q}$ denote the inclusion map. Notice that Φ and ι are trace-preserving and by Theorem 2.5(d), $\Phi_* = \iota_*$ as homomorphisms from $K_0(C^*(a))$ to $K_0(\mathcal{Q})$. By Theorem 3.1, $\iota \simeq_a \Phi$. In particular,

$$a = \iota(a) \simeq_a \Phi(a) = b.$$

∎

As was the case with Proposition 2.7, the above result also extends to all simple, unital AF-algebras satisfying Blackadar's FCQ1, possessing a unique tracial state, and admitting a divisible K_0-group. But a result of Blackadar's [2, Thm 3.9] shows that a simple, unital AF-algebra $\mathcal{A}$ with a unique tracial state τ and a divisible K_0 group satisfies FCQ1 if and only if there exists a countable (possibly finite) set of positive real numbers $\{r_n\}_n$, linearly independent over $\mathbb{Q}$, such that $K_0(\mathcal{A}) = < \cup_n r_n \mathbb{Q} >$ (the smallest additive subgroup of $\mathbb{R}$ containing $r_n \mathbb{Q}$ for all n), equipped with the natural order it inherits as a subset of $\mathbb{R}$.

Unlike the case for normal elements of a UHF-algebra $\mathcal{A}$, when dealing with a pair a, b of not-necessarily normal elements of $\mathcal{A}$, the divisibility of $\mathcal{A}$'s K_0-group now presents a *bona fide* obstruction to extending Specht's Theorem in the manner of

Theorem 3.2. Our present goal is to exhibit, in the CAR algebra $\mathbb{M}_{2^\infty}$, two elements a and b which satisfy Specht's trace condition but which fail to be approximately unitarily equivalent.

Notation 3.3. We define $\mathbb{I}_3 := \{f \in \mathbb{M}_3(\mathcal{C}[0,1], \mathbb{C}) : f(0), f(1) \in \mathbb{C}I_3\}$. Given $f \in \mathbb{I}_3$, there exist $\lambda, \mu \in \mathbb{C}$ such that $f(0) = \lambda I_3$ and $f(1) = \mu I_3$. We also define $\underline{f}(0) = \lambda$, and $\underline{f}(1) = \mu$.

As always, we denote the identity matrix in $\mathbb{M}_n(\mathbb{C})$ by I_n, $n \geq 1$.

Theorem 3.4. *There exist two injective homomorphisms* $\Phi, \Psi : \mathbb{I}_3 \to \mathbb{M}_{2^\infty}$ *such that*

(a) *for each* $f \in \mathbb{I}_3$, $\tau(\Phi(f)) = \tau(\Psi(f))$, *although*
(b) Φ, Ψ *are not approximately unitarily equivalent.*

Proof.

STEP 1.

Let $\{f_1, f_2, \cdots\}$ be a countable dense set of the unit ball of $\mathbb{I}_3$. For $n \in \mathbb{N}$, set $\varepsilon_n := \frac{1}{2^n}$. Since each f_i is uniformly continuous, we may choose $2 \leq d_n \in \mathbb{N}$ such that for $1 \leq i \leq n$,

$$\|f_i(t) - f_i(s)\| < \varepsilon_n$$

whenever $s, t \in [0,1]$ and $|s - t| \leq \frac{1}{d_n}$. Without loss of generality, we may assume that the sequence $(d_n)_{n=1}^\infty$ is strictly increasing.

STEP 2.

Set, for each $n \in \mathbb{N}$,

$$l_n = \frac{4^{d_n} - 1}{3} \qquad\qquad r_n = \frac{1}{l_n + 1} = \frac{3}{4^{d_n} + 2}$$
$$t_j^{(n)} = j \cdot r_n \text{ for } j = 0, 1, 2, \cdots, l_n \qquad\qquad m_n = 4^{d_{n+1} - d_n},$$

and observe that $l_n \in \mathbb{N}$ for all n, and $0 \leq t_j^{(n)} \leq 1$ for all $n \geq 1$, $0 \leq j \leq l_n$.

Define $\varphi_n : \mathbb{I}_3 \to \mathbb{M}_{4^{d_n}}(\mathbb{C})$ by

$$\varphi_n(f) = \mathrm{diag}(\underline{f}(0); f(t_1^{(n)}), \cdots, f(t_{l_n}^{(n)})),$$

and $\alpha_n : \mathbb{M}_{4^{d_n}}(\mathbb{C}) \to \mathbb{M}_{4^{d_{n+1}}}(\mathbb{C})$ by

$$\alpha_n(a) = u_{n+1}^*(I_{m_n} \otimes a)u_{n+1},$$

where u_{n+1} is a unitary in $\mathbb{M}_{4^{d_{n+1}}}(\mathbb{C})$ such for each $a \in \mathbb{M}_{4^n}(\mathbb{C})$ of the form

$a = \mathrm{diag}(\lambda; a_1, a_2, \cdots, a_{l_n})$, where $\lambda \in \mathbb{C}$, $a_j \in \mathbb{M}_3(\mathbb{C})$, $j = 1, 2, \cdots, l_n$,

we have

$$u_{n+1}^*(I_{m_n} \otimes a)u_{n+1} = \mathrm{diag}(I_{m_n} \otimes \lambda; I_{m_n} \otimes a_1, I_{m_n} \otimes a_2, \ldots, I_{m_n} \otimes a_{l_n}).$$

It is easy to check that $\mathcal{A} := \lim_{n \to \infty} (\mathbb{M}_{4^{d_n}}(\mathbb{C}), \alpha_n) \simeq^* \mathbb{M}_{2^\infty}$

STEP 3.

Define $s_n := \frac{m_n - 1}{3}$. Observe that for $f \in \{f_1, f_2, \cdots, f_n\}$,

$$\|\alpha_n \circ \varphi_n(f) - \varphi_{n+1}(f)\| =$$
$$\| \operatorname{diag}(\underline{f}(0); f(0) \otimes I_{s_n}, f(t_1^{(n)}) \otimes I_{m_n}, \ldots, f(t_{l_n}^{(n)}) \otimes I_{m_n})$$
$$- \operatorname{diag}(\underline{f}(0); f(t_1^{(n+1)}), \cdots, f(t_{l_{n+1}}^{(n+1)}))\|.$$

Our immediate goal is to show that this quantity is at most ε_n.

The following three calculations, while tedious, are routine, and are left to the reader.

(i) For $1 \leq j \leq s_n$, $|t_j^{(n+1)} - 0| \leq t_{s_n}^{(n+1)} < r_n < \frac{1}{d_n}$.

(ii) For $0 \leq j \leq l_n$, $t_j^{(n)} \leq t_{s_n + jm_n}^{(n+1)}$.

(iii) For $0 \leq j \leq l_n$, $t_{s_n + jm_n}^{(n+1)} \leq t_{j+1}^{(n)}$.

It follows from item (i) that for $1 \leq j \leq s_n$,

$$\| \operatorname{diag}(\underline{f}(0); f(0) \otimes I_{s_n}) - \operatorname{diag}(\underline{f}(0); f(t_1^{(n+1)}), f(t_2^{(n+1)}), \ldots, f(t_{s_n}^{(n+1)}))\| \leq \varepsilon_n.$$

Meanwhile, from (ii) and (iii) we find that for $1 \leq j \leq l_n$,

$$t_{j-1}^{(n)} \leq t_{s_n + (j-1)m_n}^{(n+1)} \leq t_{s_n + (j-1)m_n + 1}^{(n+1)} \leq \cdots \leq t_{s_n + jm_n}^{(n+1)} \leq t_{j+1}^{(n)}.$$

Since $t_{j-1}^{(n)} \leq t_j^{(n)} \leq t_{j+1}^{(n)}$ and $t_{j+1}^{(n)} - t_{j-1}^{(n)} = 2r_n < \frac{1}{d_n}$, this allows us to infer that

$$|t_{s_n + (j-1)m_n + i}^{(n+1)} - t_j^{(n)}| < \frac{1}{d_n} \text{ for all } 1 \leq i \leq m_n,$$

whence

$$\| \operatorname{diag}(f(t_{s_n + (j-1)m_n + 1}^{(n+1)}), \cdots, f(t_{s_n + jm_n}^{(n+1)})) - (f(t_j^{(n)}) \otimes I_{m_n})\| < \varepsilon_n.$$

Thus

$$\|\alpha_n \circ \varphi_n(f) - \varphi_{n+1}(f)\| < \varepsilon_n.$$

From this, one easily deduces that the diagram

$$
\begin{array}{ccccccccc}
\mathbb{I}_3 & \xrightarrow{\text{id}} & \mathbb{I}_3 & \xrightarrow{\text{id}} & \mathbb{I}_3 & \xrightarrow{\text{id}} & \cdots & \longrightarrow & \mathbb{I}_3 \\
\downarrow{\varphi_1} & & \downarrow{\varphi_2} & & \downarrow{\varphi_3} & & & & \\
\mathbb{M}_{4^{d_1}}(\mathbb{C}) & \xrightarrow{\alpha_1} & \mathbb{M}_{4^{d_2}}(\mathbb{C}) & \xrightarrow{\alpha_2} & \mathbb{M}_{4^{d_3}}(\mathbb{C}) & \xrightarrow{\alpha_3} & \cdots & \longrightarrow & \mathbb{M}_{2^\infty}
\end{array}
$$

is approximately commutative.

Now $\sum_{n=1}^{\infty} \varepsilon_n = 1 < \infty$. By Theorem 1.10.14 of [17], this diagram induces a *-homomorphism $\Phi : \mathbb{I}_3 \to \mathbb{M}_{2^\infty}$. Suppose that $f \in \mathbb{I}_3$ satisfies $\Phi(f) = 0$. Let $\varepsilon > 0$. Since f is uniformly continuous on $[0,1]$, as before, we may find $\delta > 0$ such that $s, t \in [0,1]$ and $|s - t| < \delta$ implies that $\|f(s) - f(t)\| < \frac{\varepsilon}{2}$. Choose $n^* \in \mathbb{N}$ such that

(iv) $\|\varphi_{n^*}(f)\| < \frac{\varepsilon}{2}$, and

(v) $[0,1] \subseteq \cup_{j=0}^{l_n}(t_j^{(n^*)} - \delta, t_j^{(n^*)} + \delta)$.

If $t \in [0,1]$, say
$$t_{j_0}^{(n^*)} - \delta < t < t_{j_0}^{(n^*)} + \delta,$$
then
$$\|f(t) - f(t_{j_0}^{(n^*)})\| < \frac{\varepsilon}{2},$$
and thus
$$\|f(t)\| \leq \|f(t_{j_0}^{(n^*)})\| + \frac{\varepsilon}{2} < \|\varphi_{n^*}(f)\| + \frac{\varepsilon}{2} = \varepsilon.$$

Since $t \in [0,1]$ is arbitrary,
$$\|f\| < \varepsilon.$$

But $\varepsilon > 0$ was also arbitrary, and so $f = 0$. In other words, Φ is injective.

STEP 4.

Similarly, define $\psi_n : \mathbb{I}_3 \to \mathbb{M}_{4^{d_n}}(\mathbb{C})$ by
$$\psi_n(f) = \operatorname{diag}(f(t_1^{(n)}), \cdots, f(t_{l_n}^{(n)}); \underline{f}(1)).$$

Define $\beta_n : \mathbb{M}_{4^{d_n}}(\mathbb{C}) \to \mathbb{M}_{4^{d_{n+1}}}(\mathbb{C})$ by
$$\beta_n(a) = v_{n+1}^* \operatorname{diag}(I_{m_n} \otimes a) v_{n+1},$$

where v_{n+1} is a unitary in $\mathbb{M}_{4^{n+1}}(\mathbb{C})$ such for each $a \in \mathbb{M}_{4^n}(\mathbb{C})$ of the form

$a = \operatorname{diag}\{a_1, a_2, \cdots, a_{l_n}; \lambda\}$, where $\lambda \in \mathbb{C}$, $a_j \in \mathbb{M}_3(\mathbb{C})$, $j = 1, 2, \cdots, l_n$,

we have
$$v_{n+1}^* \operatorname{diag}(I_{m_n} \otimes a) v_{n+1} = \operatorname{diag}(I_{m_n} \otimes a_1, I_{m_n} \otimes a_2, \ldots, I_{m_n} \otimes a_{l_n}; I_{m_n} \otimes \lambda)$$

Once again, it is routine to check that $\lim_{n \to \infty}(\mathbb{M}_{4^{d_n}}(\mathbb{C}), \beta_n) \simeq^* \mathbb{M}_{2^\infty}$

Using an analogous argument to that used in STEP 3 above, one constructs the following approximately commutative diagram:

$$
\begin{array}{ccccccccc}
\mathbb{I}_3 & \xrightarrow{\text{id}} & \mathbb{I}_3 & \xrightarrow{\text{id}} & \mathbb{I}_3 & \xrightarrow{\text{id}} & \cdots & \longrightarrow & \mathbb{I}_3 \\
\downarrow{\psi_1} & & \downarrow{\psi_2} & & \downarrow{\psi_3} & & & & \\
\mathbb{M}_{4^{d_1}}(\mathbb{C}) & \xrightarrow{\beta_1} & \mathbb{M}_{4^{d_2}}(\mathbb{C}) & \xrightarrow{\beta_2} & \mathbb{M}_{4^{d_3}}(\mathbb{C}) & \xrightarrow{\beta_3} & \cdots & \longrightarrow & \mathbb{M}_{2^\infty}
\end{array}
$$

As before, since $\sum_{n=1}^\infty \varepsilon_n = 1 < \infty$, [17, Theorem 1.10.14] shows that this diagram induces an injective *-homomorphism $\Psi : \mathbb{I}_3 \to \mathbb{M}_{2^\infty}$.

STEP 5.

Note that for any unit vector $f \in \mathbb{I}_3$ (using the unique normalised trace τ on $\mathbb{M}_{2^\infty}$ restricted to $\mathbb{M}_{4^{d_n}}(\mathbb{C}) \subseteq \mathbb{M}_{2^\infty}$) and any $n \geq 1$,
$$|\tau(\varphi_n(f)) - \tau(\psi_n(f))| \leq \frac{|\underline{f}(0) - \underline{f}(1)|}{4^{d_n}} \leq \frac{2}{4^{d_n}}.$$

Therefore
$$\tau(\Phi(f)) = \tau(\Psi(f)) \quad \text{for all } f \in \mathbb{I}_3.$$

STEP 6.

Define $\delta_j : \mathbb{I}_3 \to \mathbb{C}$, $\delta_j(f) = f(j)$. By [8, Page 605], $KK(\delta_0) \neq KK(\delta_1)$ in $KK(\mathbb{I}_3, \mathbb{C})$. Define $\theta_n : \mathbb{I}_3 \to M_{4^{d_n}-1}(\mathbb{C})$ by

$$\theta_n(f) = \mathrm{diag}(f(t_1^{(n)}), \cdots, f(t_{l_n}^{(n)})),$$

and $\tilde{\psi}_n : \mathbb{I}_3 \to M_{4^{d_n}}(\mathbb{C})$ by

$$\tilde{\psi}_n(f) = \delta_1(f) \oplus \theta_n(f).$$

Since the unitary group of $\mathbb{M}_{k_n}(\mathbb{C})$ is path connected, it follows that $\tilde{\psi}_n$ is homotopic to ψ_n, and thus $KK(\tilde{\psi}_n) = KK(\psi_n)$ in $KK(\mathbb{I}_3, \mathbb{M}_{4^{d_n}}(\mathbb{C})) = KK(\mathbb{I}_3, \mathbb{C})$.

Note that

$$\varphi_n = \delta_0 \oplus \theta_n, \tilde{\psi}_n = \delta_1 \oplus \theta_n.$$

Hence $KK(\varphi_n) \neq KK(\tilde{\psi}_n) = KK(\psi_n)$ in $KK(\mathbb{I}_3, M_{4^{d_n}}(\mathbb{C})) = KK(\mathbb{I}_3, \mathbb{C})$.

Suppose that $KK(\Phi) = KK(\Psi)$ in $KK(\mathbb{I}_3, \mathbb{M}_{2^\infty})$. Noting that $KK(\alpha_n) = KK(\beta_n)$ in $KK(\mathbb{M}_{4^{d_n}}(\mathbb{C}), \mathbb{M}_{4^{d_{n+1}}}(\mathbb{C}))$ for all $n \in \mathbb{N}$, we may apply [8, Proposition 2.5] to obtain the existence of $n^* \in \mathbb{N}$ such that $KK(\varphi_{n^*}) = KK(\psi_{n^*})$ in $KK(\mathbb{I}_3, \mathbb{C})$, a contradiction. We conclude that $KK(\Phi) \neq KK(\Psi)$ in $KK(\mathbb{I}_3, \mathbb{M}_{2^\infty})$.

As $K_0(\mathbb{I}_3) = 0$, $K_1(\mathbb{I}_3) = \mathbb{Z}/3$ are both finitely generated, by [9, Proposition 2.4], $KK(\mathbb{I}_3, \mathbb{M}_{2^\infty}) = KL(\mathbb{I}_3, \mathbb{M}_{2^\infty})$. This implies that $KL(\Phi) \neq KL(\Psi)$ in $KL(\mathbb{I}_3, \mathbb{C})$. By [22, Proposition 5.4], Φ, Ψ are not approximately unitarily equivalent.

∎

We wish to translate the result above to a result about singly-generated subalgebras of $\mathbb{M}_{2^\infty}$. To that end, we consider the following definition.

Definition 3.5. Let $\mathcal{A}$ be a C^*-algebra. We denote by $\mathrm{gen}(\mathcal{A})$ the minimal number of *self-adjoint* generators of $\mathcal{A}$; i.e., $\mathrm{gen}(\mathcal{A})$ the smallest number $n \in \{1, 2, \cdots, \infty\}$ such that $\mathcal{A}$ contains a generating subset $\mathcal{S} \subset \mathcal{A}_{\mathrm{sa}}$ of cardinality n.

Two self-adjoint elements a, b generate the same C^*-algebra as the single (non-self-adjoint) element $a + \mathrm{i}b$. Therefore, a C^*-algebra $\mathcal{A}$ is said to be **singly generated** if $\mathrm{gen}(\mathcal{A}) \leq 2$.

Proposition 3.6. *For $n \geq 1$, $\mathbb{M}_{2^n}(\mathbb{I}_3)$ is singly generated.*

Proof. Note that by [32, Remark 2.3], $\mathrm{gen}(\mathcal{C}_0(0,1), \mathbb{C}) = 2$, and by [32, §2.1(4)],

$$\mathrm{gen}(\mathbb{M}_3(\mathcal{C}_0(0,1), \mathbb{C})) = 1.$$

Consider $h, g \in \mathbb{M}_3(\mathcal{C}[0,1], \mathbb{C})$ defined by $h(t) = (1-t)I_3$, $g(t) = tI_3$. Then, for all $f \in \mathbb{I}_3$, we have

$$f \in \mathrm{span}\{g, h, k : k \in \mathbb{M}_3(\mathcal{C}_0(0,1), \mathbb{C})\}.$$

Hence, $\mathrm{gen}(\mathbb{I}_3) \leq 3$. Again, by [32, §2.1(4)], for each $n \geq 1$, $\mathbb{M}_{2^n}(\mathbb{I}_3)$ is singly generated.

∎

We are now in a position to show that the generalisation of Specht's Theorem to the universal UHF-algebra $\mathcal{Q}$ which we presented in Theorem 3.2 does not extend to all UHF-algebras. We note that $\mathbb{M}_2(\mathbb{I}_3)$ satisfies the UCT, by virtue of the fact that it is a type I C^*-algebra [26].

Theorem 3.7. *Let τ denote the unique, faithful tracial state on $\mathbb{M}_{2^\infty}$. There exist $a, b \in \mathbb{M}_{2^\infty}$ such that $C^*(a)$ and $C^*(b)$ both satisfy the UCT, $\tau(p(a, a^*)) = \tau(p(b, b^*))$ for each polynomial $p \in \mathcal{P}_2$, and yet a and b are not approximately unitarily equivalent in $\mathbb{M}_{2^\infty}$.*

Proof. By Proposition 3.6, we can choose $f_0 \in \mathbb{M}_2(\mathbb{I}_3)$ to be a generator and define $a = (\mathrm{id}_{\mathbb{M}_2} \otimes \Phi)(f_0), b = (\mathrm{id}_{\mathbb{M}_2} \otimes \Psi)(f_0) \in \mathbb{M}_2(\mathbb{M}_{2^\infty})$, where Φ and Ψ are the injective, unital *-homomorphisms from Proposition 3.4. By that Proposition, we have the following (recall that $\tau_2 = \mathrm{tr} \otimes \tau$):

(I) for each two variable polynomial p, $\tau_2(p(a, a^*)) = \tau_2(p(b, b^*))$;

(II) the injectivity of Φ and Ψ (and hence of $\mathrm{id}_{\mathbb{M}_2} \otimes \Phi$ and of $\mathrm{id}_{\mathbb{M}_2} \otimes \Psi$) implies that $C^*(a) \cong^* \mathbb{M}_2(\mathbb{I}_3) \cong^* C^*(b)$. As such, $C^*(a)$ and $C^*(b)$ satisfy the UCT;

(III) a and b are not approximately unitarily equivalent in $\mathbb{M}_2(\mathbb{M}_{2^\infty})$.

To see this, we argue by contradiction. Suppose to the contrary that there exists a sequence $u_n = \begin{bmatrix} u_{n,1} & u_{n,2} \\ u_{n,3} & u_{n,4} \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_{2^\infty})$ such that $b = \lim_{n \to \infty} u_n^* a u_n$.

Since f_0 is a generator of $\mathbb{M}_2(\mathbb{I}_3)$, we see that for all $g_1, g_2, g_3, g_4 \in \mathbb{I}_3$,

$$\begin{bmatrix} \Psi(g_1) & \Psi(g_2) \\ \Psi(g_3) & \Psi(g_4) \end{bmatrix} = \lim_{n \to \infty} \begin{bmatrix} u_{n,1}^* & u_{n,3}^* \\ u_{n,2}^* & u_{n,4}^* \end{bmatrix} \begin{bmatrix} \Phi(g_1) & \Phi(g_2) \\ \Phi(g_3) & \Phi(g_4) \end{bmatrix} \begin{bmatrix} u_{n,1} & u_{n,2} \\ u_{n,3} & u_{n,4} \end{bmatrix}.$$

In particular, taking $g_1 = 1$, $g_2 = g_3 = g_4 = 0$, and recalling that both Φ and Ψ are unital, we find that

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \lim_{n \to \infty} \begin{bmatrix} u_{n,1}^* & u_{n,3}^* \\ u_{n,2}^* & u_{n,4}^* \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_{n,1} & u_{n,2} \\ u_{n,3} & u_{n,4} \end{bmatrix},$$

and

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \lim_{n \to \infty} \begin{bmatrix} u_{n,1}^* & u_{n,3}^* \\ u_{n,2}^* & u_{n,4}^* \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{n,1} & u_{n,2} \\ u_{n,3} & u_{n,4} \end{bmatrix}.$$

A routine calculation shows that this implies that $\lim_{n \to \infty} u_{n,2} = 0 = \lim_{n \to \infty} u_{n,3}$, and thus there exist unitary elements $v_{n,1}$ and $v_{n,4}$ of $\mathbb{M}_{2^\infty}$ such that $\lim_{n \to \infty} u_{n,1} - v_{n,1} = 0 = \lim_{n \to \infty} u_{n,4} - v_{n,4}$.

From this we deduce that for all $g_1 \in \mathbb{I}_3$, $\Psi(g_1) = \lim_{n \to \infty} v_{n,1}^* \Phi(g_1) v_{n,1}$, implying that Φ and Ψ are approximately unitarily equivalent, a contradiction of Theorem 3.4.

Of course, since $\mathbb{M}_2(\mathbb{M}_{2^\infty}) \simeq^* \mathbb{M}_{2^\infty}$, we may view a and b as elements of $\mathbb{M}_{2^\infty}$ under this identification.

$\blacksquare$

4. A non-tracial formulation of Specht's condition

Obviously, to state Specht's trace condition for a pair a, b of elements in a C^*-algebra $\mathcal{A}$ requires that the C^*-algebra admit a tracial state. Many interesting C^*-algebras, including the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on a complex, separable Hilbert space $\mathcal{H}$, do not. We now consider the generalisation of a relation first studied in [18] which we shall demonstrate to coincide with Specht's trace condition for UHF-algebras, but which can be formulated in an arbitrary C^*-algebra.

Definition 4.1. Let $\mathcal{A}$ be a C^* algebra and $a, b \in \mathcal{A}$. We shall say a, b satisfy the **approximate absolute value condition (AAVC)** if for any polynomial $p \in \mathcal{P}_2$, $|p(a, a^*)|$ is approximately unitarily equivalent to $|p(b, b^*)|$ in $\mathcal{A}$.

We emphasise the fact that in the definition of the AAVC, the sequence $(u_n)_n$ of unitaries implementing the approximate unitary equivalence of a given pair $|p(a, a^*)|$ and $|p(b, b^*)|$ depends upon the polynomial p.

It is routine to verify that if $a, b \in \mathcal{A}$ are approximately unitarily equivalent in $\mathcal{A}$, then a and b satisfy the AAVC.

If $\mathcal{A}$ is finite-dimensional, then this relation agrees with the "absolute value condition" (AVC) studied in [18], which replaces approximate unitary equivalence by unitary equivalence in the above definition. It was shown there [18, Prop. 4.2 and Thm 4.6] that two matrices $A, B \in \mathbb{M}_n(\mathbb{C})$ satisfy the AVC if and only if they are unitarily equivalent. *A fortiori*, two matrices $A, B \in \mathbb{M}_n(\mathbb{C})$ satisfy the AAVC if and only if they are unitarily equivalent.

We recall that two elements a and b of a C^*-algebra $\mathcal{A}$ are said to be *algebraically equivalent* if there exists a *-isomorphism $\varphi : C^*(a) \to C^*(b)$ satisfying $\varphi(a) = b$.

Proposition 4.2. *Let $\mathcal{A}$ be a C^* algebra, and suppose that $a, b \in \mathcal{A}$ satisfy the AAVC. Then the well-defined map $\Phi : p(a, a^*) \mapsto p(b, b^*)$ extends in a unique way to a unital isomorphism from $C^*(a)$ onto $C^*(b)$ which send a to b. In other words, a and b are algebraically equivalent.*

Proof. Let $p \in \mathcal{P}_2$ be a fixed polynomial. Since $|p(a, a^*)^* p(a, a^*)|$ and $|p(b, b^*)^* p(b, b^*)|$ are approximately unitarily equivalent by hypothesis, it follows that

$$\|p(a, a^*)\|^2 = \||p(a, a^*)^* p(a, a^*)|\| = \||p(b, b^*)^* p(b, b^*)|\| = \|p(b, b^*)\|^2.$$

To see that Φ is well-defined, note that for general $p, q \in \mathcal{P}_2$, if $p(a, a^*) = q(a, a^*)$, we then have

$$0 = \|p(a, a^*) - q(a, a^*)\| = \|(p - q)(a, a^*)\| = \|(p - q)(b, b^*)\| = \|p(b, b^*) - q(b, b^*)\|.$$

Hence $p(b, b^*) = q(b, b^*)$, as required.

Clearly Φ is linear, isometric and has dense range in $C^*(b)$, and thus Φ extends in a unique way to a unital isomorphism from $C^*(a)$ onto $C^*(b)$ which send a to b. ∎

Proposition 4.3. *Let $\mathcal{A}$ be a C^*-algebra with a tracial state τ, and suppose that $a, b \in \mathcal{A}$ satisfy the AAVC. Then a, b satisfy Specht's trace condition, i.e.,*

$$\tau(p(a, a^*)) = \tau(p(b, b^*)) \qquad \text{for all } p \in \mathcal{P}_2.$$

Proof. By Proposition 4.2 above, we know that the map $\Phi : p(a, a^*) \mapsto p(b, b^*)$ extends in a unique way to a unital isomorphism from $C^*(a)$ onto $C^*(b)$ which sends a to b.

Let $0 \neq x = x^* \in C^*(a)$. There exists a sequence $(p_n)_n$ of polynomials in $\mathcal{P}_2$ such that

$$x = \lim_{n \to \infty} p_n(a, a^*).$$

Without loss of generality, we may assume that $p_n(a, a^*)$ is self-adjoint, and

$$\|p_n(a, a^*)\| < 2\|x\|.$$

(Otherwise, we replace $p_n(a, a^*)$ by $\operatorname{Re} p_n(a, a^*)$ for all $n \geq 1$ and scale as required.) It follows that $(p_n(a, a^*) + 2\|x\|)_n$ is a sequence of positive elements in $C^*(a)$. Meanwhile,

$$\Phi(x) + 2\|x\| = \Phi(x) + 2\|\Phi(x)\| = \Phi(\lim_{n \to \infty} p_n(a, a^*)) + 2\|\Phi(x)\| = \lim_{n \to \infty} p_n(b, b^*) + 2\|\Phi(x)\|.$$

Since a, b satisfy the AAVC,

$$p_n(b, b^*) + 2\|x\| = |p_n(b, b^*) + 2\|x\||$$

and

$$p_n(a, a^*) + 2\|x\| = |p_n(a, a^*) + 2\|x\||$$

are approximately unitarily equivalent, and so $x + 2\|x\|$ and $\Phi(x + 2\|x\|)$ are also approximately unitarily equivalent. As τ is a tracial state,

$$\tau(x + 2\|x\|) = \tau(\Phi(x) + 2\|x\|),$$

that is,

$$\tau(x) = \tau(\Phi(x)).$$

From this we immediately see that for arbitrary $y \in C^*(a)$,

$$\tau(y) = \tau(\Phi(y)).$$

In particular,

$$\tau(p(a, a^*)) = \tau(p(b, b^*)) \qquad \text{for all } p \in \mathcal{P}_2.$$

∎

Corollary 4.4. *Let $\mathcal{A}$ be a UHF-algebra, $a, b \in A$, and τ denote the unique trace of $\mathcal{A}$. The following are equivalent:*

  (a) *a, b satisfy the AAVC; and*
  (b) *a and b satisfy Specht's trace condition: i.e. $\tau(w(a, a^*)) = \tau(w(b, b^*))$ for all $w \in \mathcal{W}_2$.*

Proof. That (a) implies (b) is the content of Proposition 4.3.

To see that (b) implies (a), let $p \in \mathcal{P}_2$, and define

$$\tilde{a} := |p(a^*, a)|^2, \quad \text{and} \quad \tilde{b} := |p(b^*, b)|^2.$$

Note that for any word $w \in \mathcal{W}_2$,

$$\tau(w(a, a^*)) = \tau(w(b, b^*)),$$

and hence it is routine to check that

$$\tau(v(\tilde{a}, \tilde{a}^*)) = \tau(v(\tilde{b}, \tilde{b}^*)),$$

for any word $v \in \mathcal{W}_2$. By Proposition 2.7, $\tilde{a} \simeq_a \tilde{b}$, whence

$$|p(a^*, a)| \simeq_a |p(b^*, b)|.$$

∎

Corollary 4.5. *There exist a pair $a, b \in \mathbb{M}_{2^\infty}$ which satisfy the AAVC but which are not approximately unitarily equivalent in $\mathbb{M}_{2^\infty}$. Furthermore, $C^*(a)$ and $C^*(b)$ satisfy the UCT.*

Proof. This follows immediately from Corollary 4.4 and Theorem 3.7. ∎

As mentioned above, the AAVC makes sense in all C^*-algebras, including $\mathcal{B}(\mathcal{H})$, when $\mathcal{H}$ is a complex, separable Hilbert space. To examine it there, it will be useful to first recall the following result of Hadwin based on Voiculescu's non-commutative Weyl-von Neumann Theorem [34].

Proposition 4.6. [13, Corollary 3.7] *Suppose $A, B \in \mathcal{B}(\mathcal{H})$. Then $A \simeq_a B$ if and only if there is a representation $\pi : C^*(A) \to C^*(B)$ such that $\pi(A) = B$ and $\mathrm{rank}(S) = \mathrm{rank}(\pi(S))$ for every $S \in C^*(A)$.*

The following result may be viewed as a version of Specht's Theorem for $\mathcal{B}(\mathcal{H})$.

Theorem 4.7. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then A and B satisfy the AAVC if and only if A, B are approximately unitarily equivalent.*

Proof. As mentioned following Definition 4.1, if A and B approximately unitarily equivalent, then they satisfy the AAVC.

Conversely, suppose that A and B satisfy the AAVC in $\mathcal{B}(\mathcal{H})$. By Proposition 4.2, A and B are algebraically equivalent via an isomorphism $\Phi : C^*(A) \to C^*(B)$ with $\Phi(A) = B$. There remains only to show the rank condition for elements of $C^*(A)$. To that end, let $S \in C^*(A)$, and suppose that $S = \lim_{n \to \infty} p_n(A, A^*)$. Set $T := \Phi(S) = \lim_{n \to \infty} p_n(B, B^*)$.

It follows that

$$|S| = \lim_{n \to \infty} |p_n(A, A^*)| \quad \text{and} \quad |T| = \lim_{n \to \infty} |p_n(B, B^*)|.$$

For each $n \geq 1$, since $|p_n(A, A^*)| \simeq_a |p_n(B, B^*)|$, we can find a unitary operator $U_n \in \mathcal{B}(\mathcal{H})$ such that

$$\| \, |p_n(B, B^*)| - U_n^* |p_n(A, A^*)| U_n \| < \frac{1}{n}.$$

From this and an easy application of the triangle inequality we see that

$$\lim_{n \to \infty} \| \, |T| - U_n^* |S| U_n \| = 0.$$

Since the rank function is weakly lower semicontinuous (see [14, Appendix]),

$$\mathrm{rank}(|T|) \leq \liminf_{n \to \infty} \mathrm{rank}(U_n |S| U_n^*) = \mathrm{rank}(|S|).$$

Therefore, by symmetry,

$$\mathrm{rank}(|S|) = \mathrm{rank}(|T|),$$

and thus
$$\text{rank}(S) = \text{rank}(T).$$

The theorem now follows from Proposition 4.6. ∎

While Specht's trace condition for two elements a and b in a UHF-algebra $\mathcal{A}$ is not always sufficient to imply approximate unitary equivalence of a and b, it is strong enough to imply approximate unitary equivalence of their images under a unital, faithful *-representation of $\mathcal{A}$.

Corollary 4.8. *Let $\mathcal{A}$ be a UHF-algebra, $a, b \in \mathcal{A}$, and denote by τ the unique trace of $\mathcal{A}$. Suppose that*

$$\tau(w(a, a^*)) = \tau(w(b, b^*)) \qquad \text{for all } w \in \mathcal{W}_2.$$

If $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\rho)$ is a unital, faithful *-representation of $\mathcal{A}$ acting on a separable Hilbert space $\mathcal{H}_\rho$, then

$$\rho(a) \simeq_a \rho(b).$$

Thus, if $\Phi : C^*(a) \to C^*(b)$ implements the algebraic equivalence of a and b as in Theorem 2.5, then $\rho|_{C^*(a)} \simeq_a \rho \circ \Phi|_{C^*(a)}$.

Proof. By Corollary 4.4, a and b satisfy the AAVC in $\mathcal{A}$. It is straightforward to check that this implies that $\rho(a)$ and $\rho(b)$ satisfy the AAVC in $B(\mathcal{H}_\rho)$. The result now follows from Theorem 4.7. ∎

Unfortunately, the situation is not as nice in the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

Example 4.9. Let $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ denote the canonical homomorphism from $\mathcal{B}(\mathcal{H})$ to the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Consider $s := \pi(S)$, and $t := s \oplus s = \pi(S \oplus S)$, where S denotes the unilateral forward shift in $\mathcal{B}(\mathcal{H})$. Then

(a) s and t satisfy the AAVC in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Note that $|p(s, s^*)|$ and $|p(t, t^*)|$ are positive elements in $\mathcal{B}(\mathcal{H})$ and that they have the same spectrum. By the Brown-Douglas-Fillmore Theorem [6], we deduce that $|p(s, s^*)|$ and $|p(t, t^*)|$ are unitarily equivalent in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

(b) s and t are not approximately unitarily equivalent in the Calkin algebra. Indeed, suppose otherwise. Since s and t are normal elements of $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, their unitary orbits are closed (again, by the BDF Theorem) and as such, s and t must be unitarily equivalent. But then they must share the same index function. However,

$$\text{ind}(s - 0) = -1 \neq -2 = \text{ind}(t - 0),$$

a contradiction.

In the Calkin algebra, the obstruction encountered in Example 4.9 is the index, reflected in the fact that $K_1(\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})) = \mathbb{Z}$. Note that for the Cuntz algebra $\mathcal{O}_2$, we have that $K_0(\mathcal{O}_2) = K_1(\mathcal{O}_2) = 0$, which effectively removes any index obstruction. It seems natural to ask whether the AAVC implies approximate unitary equivalence in this setting, and indeed, it does.

Example 4.10. Let $a, b \in \mathcal{O}_2$ be a pair which satisfies the AAVC. By Proposition 4.2, $C^*(a)$ is isomorphic to $C^*(b)$ via a unital *-isomorphism Φ which sends a to b. Since $C^*(a)$ is unital, separable, and exact, by [21, Theorem 6.3.8], ι and Φ are approximately unitarily equivalent, where ι denotes the inclusion map from $C^*(a)$ to $\mathcal{O}_2$. In particular,

$$a = \iota(a) \simeq_a \Phi(a) = b.$$

5. The distance between unitary orbits of ampliations of matrices

One of the more interesting and unexpected (at least to us) consequences of Theorem 3.7 is the following result concerning distances between unitary orbits of ampliations of matrices. Note that for non-empty sets $\mathcal{S}$ and $\mathcal{T}$ of $\mathbb{M}_n(\mathbb{C})$ – or more generally of $\mathcal{B}(\mathcal{H})$ for some complex Hilbert space $\mathcal{H}$ – the distance considered below is the usual metric space distance

$$\mathrm{d}(\mathcal{S}, \mathcal{T}) := \inf\{\|S - T\| : S \in \mathcal{S}, T \in \mathcal{T}\}.$$

When $A, B \in \mathcal{B}(\mathcal{H})$, it is clear that $\mathrm{d}(\mathcal{U}(A), \mathcal{U}(B)) = \mathrm{d}(A, \mathcal{U}(B)) = \mathrm{d}(\mathcal{U}(A), B)$.

Theorem 5.1. *There exist positive integers n and k, and a pair $A, B \in \mathbb{M}_n(\mathbb{C})$ such that*

$$\mathrm{d}(\mathcal{U}(A^{(k)}), \mathcal{U}(B^{(k)})) < \mathrm{d}(\mathcal{U}(A), \mathcal{U}(B)).$$

Proof. The inequality

$$\mathrm{d}(\mathcal{U}(A^{(k)}), \mathcal{U}(B^{(k)})) \leq \mathrm{d}(\mathcal{U}(A), \mathcal{U}(B))$$

clearly holds for all $n, k \geq 1$ and pairs $A, B \in \mathbb{M}_n(\mathbb{C})$. To prove our result, we shall argue by contradiction. Suppose otherwise; that is, suppose that

$$\mathrm{d}(\mathcal{U}(A^{(k)}), \mathcal{U}(B^{(k)})) = \mathrm{d}(\mathcal{U}(A), \mathcal{U}(B))$$

for all integers $n, k \geq 1$ and pairs $A, B \in \mathbb{M}_n(\mathbb{C})$. Let $\mathcal{Q}$ denote the universal UHF-algebra. It is well-known that under the standard trace preserving inclusion map ι from $\mathbb{M}_{2^\infty}$ into $\mathcal{Q}$, $\mathbb{M}_{2^\infty}$ may be viewed as a unital subalgebra of $\mathcal{Q}$. Let $a, b \in \mathbb{M}_{2^\infty} \subseteq \mathcal{Q}$ be the elements in the statement of Theorem 3.7. Without loss of generality, we may assume that $\|a\| = 1 = \|b\|$. Since $C^*(a)$ satisfies the UCT, we know that a and b are approximately unitarily equivalent in $\mathcal{Q}$. Let $\varepsilon > 0$ and choose a unitary element $u \in \mathcal{Q}$ such that $\|b - u^*au\| < \varepsilon$.

Now $a, b \in \mathbb{M}_{2^\infty}$ implies that we can find an integer $n \geq 1$ and a pair $A_n, B_n \in \mathbb{M}_{2^n}(\mathbb{C}) \subseteq \mathbb{M}_{2^\infty}$ such that $\|A_n - a\| < \varepsilon$ and $\|B_n - b\| < \varepsilon$.

Moreover, there exists a positive integer $k \geq 1$ and a unitary $U_{n,k} \in \mathbb{M}_{2^n \cdot k}(\mathbb{C}) \subseteq \mathcal{Q}$ such that $\|U_{n,k} - u\| < \varepsilon$. Note that (without loss of generality) $\mathbb{M}_{2^n}(\mathbb{C})$ embeds in $\mathbb{M}_{2^n \cdot k}(\mathbb{C})$ via the standard embedding $X \mapsto X^{(k)}$.

A standard (multiple) application of the triangle inequality then shows that

$$\|B_n^{(k)} - U_{n,k}^* A_n^{(k)} U_{n,k}\| < 5\varepsilon.$$

But then from our assumption it follows that

$$\mathrm{d}(\mathcal{U}(A_n), \mathcal{U}(B_n)) = \mathrm{d}(\mathcal{U}(A_n^{(k)}), \mathcal{U}(B_n^{(k)})) < 5\varepsilon.$$

That is, we can now find a unitary $V_n \in \mathbb{M}_{2^n}(\mathbb{C}) \subseteq \mathbb{M}_{2^\infty}$ such that

$$\|B_n - V_n^* A_n V_n\| < 5\varepsilon.$$

Once again, a standard application of the triangle inequality yields

$$\|b - V_n^* a V_n\| < 7\varepsilon.$$

Since ε is arbitrary, this implies that a and b are approximately unitarily equivalent in M_{2^∞}, a contradiction. ∎

Unfortunately, the proof Theorem 5.1 does not give us any idea how to find the integers $n, k \geq 1$ and the pair $A, B \in \mathbb{M}_n(\mathbb{C})$ in the statement above. Indeed, it is an interesting and difficult question to decide for which integers n and k and pairs $A, B \in \mathbb{M}_n(\mathbb{C})$ the distance remains invariant. The following two results were obtained in conjunction with H. Radjavi, and we are grateful to him for allowing us to include their proofs here.

Proposition 5.2. *Let P be a rank one projection in $\mathbb{M}_2(\mathbb{C})$, and let $R \in \mathbb{M}_2(\mathbb{C})$ be a normal matrix with eigenvalues λ, μ. For any $k \in \mathbb{N}$,*

$$\mathrm{d}(R, \mathcal{U}(P)) = \mathrm{d}(R^{(k)}, \mathcal{U}(P^{(k)})).$$

Proof. Since $\mathrm{d}(R, \mathcal{U}(P)) = \mathrm{d}(\mathcal{U}(R), \mathcal{U}(P))$, there is no loss of generality in assuming that $R = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$. If $Q \in \mathbb{M}_{2k}(\mathbb{C})$ is any projection of rank k, then we can find an isometry $W = \begin{bmatrix} X \\ Y \end{bmatrix}$ from $\mathbb{C}^k$ to $\mathbb{C}^{2k}$ such that

$$Q = WW^* = \begin{bmatrix} X \\ Y \end{bmatrix} \begin{bmatrix} X^* & Y^* \end{bmatrix} = \begin{bmatrix} XX^* & XY^* \\ YX^* & YY^* \end{bmatrix}.$$

Of course, $W^*W = X^*X + Y^*Y = I_k$. Decomposing X and Y into their polar decompositions $X = UM$ and $Y = VN$ where $M = |X|$ and $N = |Y|$ and $U, V \in \mathbb{M}_k(\mathbb{C})$ are unitary, we see that $M^2 + N^2 = I_k$ and in particular $0 \leq M$ and $0 \leq N$ commute, since $N = (I_k - M^2)^{1/2}$.

Thus

$$Q = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} M^2 & MN \\ MN & N^2 \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & V^* \end{bmatrix}.$$

Set $Z = \mathrm{diag}(U, V)$. Then $Z \in \mathbb{M}_{2k}(\mathbb{C})$ is unitary and $Z^* R^{(k)} Z = R^{(k)}$. Thus

$$\|Q - R^{(k)}\| = \|Z^* Q Z - R^{(k)}\| = \left\| \begin{bmatrix} M^2 - \lambda I_k & MN \\ MN & N^2 - \mu I_k \end{bmatrix} \right\|.$$

Since M, N are commuting normal matrices, we can simultaneously diagonalise them, say $M = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $N = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_k)$.

Since $M^2 + N^2 = I_k$, each $\alpha_i^2 + \beta_i^2 = 1$, and thus for each $1 \leq i \leq k$, $\begin{bmatrix} \alpha_i^2 & \alpha_i \beta_i \\ \alpha_i \beta_i & \beta_i^2 \end{bmatrix}$ is a rank one projection. Thus

$$\|R^{(k)} - Q\| = \left\| \begin{bmatrix} M^2 - \lambda I_k & MN \\ MN & N^2 - \mu I_k \end{bmatrix} \right\| = \max_{1 \leq i \leq k} \left\| \begin{bmatrix} \alpha_i^2 - \lambda & \alpha_i \beta_i \\ \alpha_i \beta_i & \beta_i^2 - \mu \end{bmatrix} \right\| \geq \mathrm{d}(R, \mathcal{U}(P)).$$

Of course, any element of $\mathcal{U}(P^{(k)})$ is a projection of rank k, and so the above argument shows that

$$\mathrm{d}(R^{(k)}, \mathcal{U}(P^{(k)})) \geq \mathrm{d}(R, \mathcal{U}(P)).$$

As previously mentioned, the reverse inequality is trivial. ∎

The next result shows that the matrices A and B from Theorem 5.1 cannot be normal matrices in $\mathbb{M}_2(\mathbb{C})$.

Theorem 5.3. *Let A and B be normal matrices in $\mathbb{M}_2(\mathbb{C})$. For any $k \in \mathbb{N}$,*

$$\mathrm{d}(A, \mathcal{U}(B)) = \mathrm{d}(A^{(k)}, \mathcal{U}(B^{(k)})).$$

Proof. A moment's thought shows that it suffices to consider the case where $A = \mathrm{diag}\{\alpha, \beta\}$ and $\alpha \neq \beta$. Then $A = \beta I_2 + (\alpha - \beta)P$, where $P = \mathrm{diag}\{1, 0\}$. By the previous proposition,

$$\begin{aligned}
\mathrm{d}(A, \mathcal{U}(B)) &= \mathrm{d}((\alpha - \beta)P, \mathcal{U}(B - \beta I_2)) \\
&= |\alpha - \beta|\, \mathrm{d}(P, \mathcal{U}(\frac{B - \beta I_2}{\alpha - \beta})) \\
&= |\alpha - \beta|\, \mathrm{d}(P^{(k)}, \mathcal{U}((\frac{B - \beta I_2}{\alpha - \beta})^{(k)})) \\
&= \mathrm{d}(((\alpha - \beta)P)^{(k)}, \mathcal{U}((B - \beta I_2)^{(k)})) \\
&= \mathrm{d}(((\alpha - \beta)P + \beta I_2)^{(k)}, \mathcal{U}(B^{(k)})) \\
&= \mathrm{d}(A^{(k)}, \mathcal{U}(B^{(k)})).
\end{aligned}$$

∎

It would be most interesting to know if Theorem 5.3 extends to arbitrary pairs A and B of normal matrices in $\mathbb{M}_n(\mathbb{C})$ for $n \geq 3$. One of the problems with extending such a result lies in the fact that – even in the case of normal operators $A, B \in \mathbb{M}_n(\mathbb{C})$ – when $n \geq 3$, calculating the distance $\mathrm{d}(A, \mathcal{U}(B))$ is a highly non-trivial task. If A and B are normal matrices in $\mathbb{M}_n(\mathbb{C})$ with eigenvalues $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\{\beta_1, \beta_2, \ldots, \beta_n\}$ respectively, we define the **spectral distance** between A and B to be

$$\mathrm{spd}(A, B) := \min_{\varrho \in \mathcal{S}_n} \left(\max_{1 \leq k \leq n} |\alpha_k - \beta_{\varrho(k)}| \right),$$

where $\mathcal{S}_n$ denotes the symmetric group of all permutations of the set $\{1, 2, \ldots, n\}$. An alternative way of viewing $\mathrm{spd}(A, B)$ is to first diagonalise both A and B with respect to a fixed orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ for $\mathbb{C}^n$. The symmetric group $\mathcal{S}_n$ acts on this basis in the obvious way via permutation unitaries: for $\varrho \in \mathcal{S}_n$, $U_\varrho(e_k) = e_{\varrho(k)}$, $1 \leq k \leq n$. Then

$$\mathrm{spd}(A, B) = \min_{\varrho \in \mathcal{S}_n} \|A - U_\varrho^* B U_\varrho\|.$$

While $\mathrm{spd}(A, B) = \mathrm{d}(A, \mathcal{U}(B))$ when $A, B \in \mathbb{M}_2(\mathbb{C})$ are normal, it was shown by Holbrook [16] that there exist normal 3×3 matrices A and B such that

$$\mathrm{d}(A, \mathcal{U}(B)) < \mathrm{spd}(A, B).$$

We remark that Bhatia, Davis, and Koosis [1] have shown that for all integers $n \geq 2$ and all $A, B \in \mathbb{M}_n(\mathbb{C})$ normal matrices,

$$\mathrm{spd}(A, B) \leq 2.91 \ \mathrm{d}(A, \mathcal{U}(B)).$$

We leave the following as an open question for the interested reader.

Question 5.4. Let $n \geq 3$ be an integer and $A, B \in \mathbb{M}_n(\mathbb{C})$ be normal matrices. If $k \geq 2$ is an integer, is

$$\mathrm{d}(A, \mathcal{U}(B)) = \mathrm{d}(A^{(k)}, \mathcal{U}(B^{(k)}))?$$

We remark that if we set $k = \aleph_0$, the question admits a negative answer, as is easily seen by taking $A = \mathrm{diag}(1, 0, 0)$ and $B = \mathrm{diag}(1, 1, 0)$.

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