ALGEBRAIC DEGREE IN SPATIAL MATRICIAL NUMERICAL RANGES OF LINEAR OPERATORS

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Abstract. We study the maximal algebraic degree of principal ortho-compressions of linear operators that constitute spatial matricial numerical ranges of higher order. We demonstrate (amongst other things) that for a (possibly unbounded) operator L on a Hilbert space, every principal m-dimensional ortho-compression of L has algebraic degree less than m if and only if $rank(L - \lambda I) \leq m - 2$ for some $\lambda \in \mathbb{C}$.

1. Introduction

In this paper we study the algebraic degrees of the elements of the spatial matricial numerical ranges of higher order. The origin of the study was a conjecture of the result appearing in Corollary 17. It asserts that a matrix has no cyclic m-dimensional ortho-compressions ($m \geq 2$) exactly when it has rank at most $m - 2$ up to a translation by a scalar multiple of the identity.

This inquiry is part of a general program of studying how various properties of ortho-compressions of an operator can be translated into the (global) properties of the operator as a whole. Similar questions are asked of semigroups and algebras of operators. For example, in [9], having fixed a natural number n, the set $CN \subseteq \mathbb{M}_n(\mathbb{C})$ of operators T for which $P^\perp TP$ and $PTP^\perp$ share a common (operator) norm for all orthogonal projections P was characterised. In particular, this set coincides with the set $CR \subseteq \mathbb{M}_n(\mathbb{C})$ of operators T for which $\operatorname{rank} P^\perp TP = \operatorname{rank} PTP^\perp$ for all such projections P and, when $n \geq 4$, is the set of matrices which can be expressed as a linear combination of the identity matrix and a matrix L which is either self-adjoint or unitary. A complete characterization of the set CN was also obtained in the infinite-dimensional setting. In [10], those integers j and k for which there exist normal matrices $D \in \mathbb{M}_n(\mathbb{C})$ such that $\operatorname{rank} P^\perp DP = j$ while $\operatorname{rank} PDP^\perp = k$ are characterized. In [3] and [2], unital algebras $\mathcal{A} \subseteq \mathbb{M}_n(\mathbb{C})$ for which $P\mathcal{A}P|_{\operatorname{ran} P}$ is an algebra for all orthogonal projections P were described, while in [11], dimension estimates and structure theorems for an algebra $\mathcal{A} \subseteq \mathbb{M}_n(\mathbb{C})$ admitting an off-diagonal corner $\mathcal{L} \stackrel{\text{def}}{=} P^\perp \mathcal{A}|_{\operatorname{ran} P}$ with trivial common kernel and co-kernel were obtained, based purely on the dimension of $\mathcal{L}$.

In such studies the underlying field can make a difference. For example, over $\mathbb{C}$, a matrix all of whose principal ortho-compressions to 1-dimensional subspaces

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are zero – that is, whose numerical range is the singleton $\{0\}$ – is the zero matrix, while over $\mathbb{R}$, the matrix $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ enjoys this property.

In this paper we focus on (linear) operators on finite-dimensional inner product spaces $\mathcal{V}$ over $\mathbb{C}$. For an operator T on such a $\mathcal{V}$, the notation $[\![T]\!]_{\mathfrak{G}}$ refers to the matrix representation of T with respect to a given (ordered) basis $\mathfrak{G}$ of $\mathcal{V}$.

The algebraic degree of a non-zero operator L is the degree of its (monic) minimal polynomial. This degree equals the sum of the sizes of the largest Jordan blocks corresponding to each of the distinct eigenvalues of L.

Ortho-projections are the self-adjoint idempotents, and *scalar operators* are those that are scalar multiples of the identity.

We use the symbol $\oplus$ to indicate direct sums for both spaces and operators. We emphasize that the direct sum of spaces need not be orthogonal.

Unless specified otherwise, the norm $\| \ \|$ on $\mathcal{V}$ and the resulting metric are presumed to be those generated by the inner product. The associated operator norm $\| \ \|_{\mathrm{op}}$ for the operators is given by the Lipschitz constant.

All norms on a finite-dimensional linear space are equivalent, and for this reason, norm-convergence and entry-wise convergence are equivalent for sequences of matrices.

An operator L on an n-dimensional vector space $\mathcal{V}$ is said to be *cyclic* when it has a cyclic vector; i.e. a vector v such that $\operatorname{span}\Big(v, L(v), L^2(v), \ldots, L^{n-1}(v)\Big) = \mathcal{V}$. The property of cyclicity is clearly invariant under translations by scalar operators.

L is cyclic exactly when its minimal polynomial has degree equal to $\dim \mathcal{V}$; i.e. when the minimal polynomial of L coincides with its characteristic polynomial (see, e.g., [6], Section 7.1). This happens exactly when a Jordan form of A has a single Jordan block for each of its eigenvalues, or equivalently, if and only if all eigenvalues of L have geometric multiplicity 1. Such operators are often referred to in the literature as *non-derogatory.*

The proof of the following result may be found on p. 499 of [5]. Clearly a corresponding result holds for the set of cyclic operators on $\mathcal{V}$.

1. **Proposition.** *The set of $n \times n$ cyclic matrices is norm-open in $\mathbb{M}_n(\mathbb{C})$.*

Given an operator L on $\mathcal{V}$, and $1 \le m \le \dim \mathcal{V}$, the *$m$-th spatial matricial numerical range $\mathcal{W}_m(L)$ of L*, defined in [1] where it is attributed to an unpublished work of S. K. Parrott, is the set of $m \times m$ matrices of the form V^*LV, where $V : \mathbb{C}^m \longrightarrow \mathcal{V}$ is a linear isometry.[1] In particular, unitarily equivalent operators have identical spatial matricial numerical ranges. Since we will only consider spatial matricial numerical ranges in this article, we will usually drop off "spatial" from our terminology henceforth.

$\mathcal{W}_m(L)$ coincides with the set of the $m \times m$ matrices $[\![PL_{|_{\mathrm{ran}\,P}}]\!]_{\mathfrak{V}}$, where P is an ortho-projection of rank m on $\mathcal{V}$, and $\mathfrak{V}$ is an orthonormal basis of the range of P. The linear operator $PL_{|_{\mathrm{ran}\,P}}$ is *the ortho-compression* of L to the range of P.

Of course $\mathcal{W}_1(L)$ is nothing more than the numerical range of L. We do not define $\mathcal{W}_m(L)$ when m exceeds the dimension of the underlying space.

A systematic study of the spatial matricial numerical ranges of operators on finite-dimen-sional inner product spaces was presented in [8], and [4] gives a nice survey of this and other generalizations of the classical numerical range.

[1] Since we only deal with linear isometries in this paper, we shall omit the reference to their linearity henceforth.

It is not difficult to check that for any $M \in \mathcal{W}_m(L)$:

$$\mathcal{W}_k(M) \subseteq \mathcal{W}_k(L) \ , \quad \text{for all} \quad 1 \leq k \leq m \ . \tag{1}$$

It is also obvious that the rank of an element of a matricial numerical range of L does not exceed the rank of L.

By the Spectral Theorem, $\mathcal{W}_r(A)$ contains an invertible element whenever A is a non-zero positive semi-definite matrix of rank r. In fact, one of the $r \times r$ principal submatrices of such an A is invertible ([7]). It is shown in [7] that every non-zero matrix A of rank r is unitarily equivalent to a matrix that has an invertible $r \times r$ principal submatrix.

In our Theorem 16 we present a direct analogue of this statement for rank modulo the scalars, but at this point we would like to observe that the quoted result of [7] can be strengthened (Proposition 2) with an elementary proof, which is relegated to the last section of the paper. (To enable a better flow of our presentation, we have relegated most of the proofs to the end.)

2. **Proposition.** *For every $A \in \mathbb{M}_n(\mathbb{C})$ of positive rank r there exists a matrix $W \in \mathbb{M}_n(\mathbb{Z})$ with non-zero orthogonal columns, such that $W^T A W$ has an invertible principal $r \times r$ submatrix.*

We remark that in the above result, W can be replaced with a unitary $U \in \mathbb{M}_n(\mathbb{F})$, where $\mathbb{F}$ is the smallest subfield of $\mathbb{R}$ extending $\mathbb{Q}$ that contains a square root of each of its elements.

2. Preliminary Results

3. **Lemma.** *Suppose that L is an operator on $\mathcal{V}$, and a non-trivial proper subspace $\mathcal{Z}$ of $\mathcal{V}$ is invariant under L. If $\mathcal{X}$ and $\mathcal{Y}$ are two subspaces of $\mathcal{V}$ complementary to $\mathcal{Z}$, and L is expressed as*

$$\begin{bmatrix} A_o & B \\ O & A_1 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} A_o & C \\ O & A_2 \end{bmatrix}$$

with respect to the decompositions $\mathcal{V} = \mathcal{Z} \oplus \mathcal{X}$ and $\mathcal{V} = \mathcal{Z} \oplus \mathcal{Y}$ respectively, the operators A_1 and A_2 are similar.

4. **Proposition.** *If L is a cyclic operator on a finite-dimensional inner product space, then every matricial numerical range of L contains a cyclic matrix.*

5. **Corollary.** *Suppose that L is an operator on $\mathcal{V}$ such that, for some $d < m$, all matrices in $\mathcal{W}_m(L)$ are algebraic of degree at most d. Then L is algebraic of degree at most d.*

We shall complement this result in Corollary 17, which states that for $d = m-1$ the hypothesis of Corollary 5 is equivalent to the claim that L is a scalar perturbation of an operator of rank at most $m-2$.

6. **Observation.** The converse of Corollary 5 fails dramatically: for any $n \times n$ matrix A, the block-matrix $\left[\begin{smallmatrix} A & A \\ -A & -A \end{smallmatrix}\right]$ is nilpotent of degree 2.

3. Main Results

Our study of the algebraic degrees of the elements of matricial numerical ranges requires a strengthening of Proposition 4 in the form of Theorem 12. The proof of this theorem relies on several approximation results, which precede it.

7. **Proposition.** *For any matrix $M \in \mathbb{M}_{n\times k}(\mathbb{C})$ without zero rows and $k \geq 2$, there exist positive scalars $\gamma_2, \ldots, \gamma_k$ such that for all large enough natural numbers m, $M \begin{pmatrix} m \\ \gamma_2 \\ \vdots \\ \gamma_k \end{pmatrix}$ has no zero entries.*

8. **Corollary.** *For any matrix $M \in \mathbb{M}_{n\times k}(\mathbb{C})$ without zero rows there is a sequence $[u_i]_{i\in\mathbb{N}}$ of entry-wise positive unit vectors in $\mathbb{C}^k$ convergent to $(1, 0, \ldots, 0)$, such that each $M(u_i)$ has no zero entries.*

Proof of Corollary 8. By Proposition 7 there exist positive scalars $\gamma_2, \ldots, \gamma_k$ such that $M \begin{pmatrix} m \\ \gamma_2 \\ \vdots \\ \gamma_k \end{pmatrix}$ has no zero entries for all large enough natural numbers m. Let g_m be the normalization of $(m, \gamma_2, \ldots, \gamma_k)$:

$$g_m = \frac{(m, \gamma_2, \ldots, \gamma_k)}{\sqrt{m^2 + \gamma_2^2 + \cdots + \gamma_k^2}} \ .$$

Then an appropriate tail of the sequence $[g_m]_{m\in\mathbb{N}}$ will serve as the desired $[u_i]_{i\in\mathbb{N}}$. $\square$

9. **Corollary.** *Let M be the $n\times n$ Gramian matrix for non-zero vectors $f_1, f_2, \ldots, f_n$ $(n \geq 2)$ in $\mathcal{V}$; (i.e. let $M_{ij} = \langle f_j, f_i \rangle$).*

Then there exists a sequence $[u_k]_{k\in\mathbb{N}}$ of entry-wise positive unit n-tuples convergent to $(1, 0, \ldots, 0)$, such that each $M(u_k)$ has no zero entries, or equivalently

$$\Big\langle u_k(1)f_1 + \cdots + u_k(n)f_n \ , \ f_i \Big\rangle \neq 0, \quad \textit{for all} \quad k, i.$$

Proof of Corollary 9. The positive definiteness of the inner product implies that all diagonal entries of M are positive, and so the desired conclusion follows from Proposition 7. $\square$

10. **Lemma.** *Let $V : \mathbb{C}^m \longrightarrow \mathcal{V}$ be an isometry and $u \in \mathcal{V}$ be a unit vector. Then there exists an isometry $Z : \mathbb{C}^m \longrightarrow \mathcal{V}$ such that*

$$u = Z(e_1) \qquad \textit{and} \qquad \|V - Z\|_{op} = \|V(e_1) - Z(e_1)\| \ .$$

11. **Proposition.** *Suppose that $[h_k]_{k\in\mathbb{N}}$ is a sequence of non-zero vectors in an n-dimensional $\mathcal{V}$, and $[h_k]_{k\in\mathbb{N}}$ converges to a non-zero vector h.*

If $V : \mathbb{C}^{n-1} \longrightarrow \mathcal{V}$ is an isometry with range $h^\perp$, then there exist isometries $V_k : \mathbb{C}^{n-1} \longrightarrow \mathcal{V}$ with range$(V_k) = h_k^\perp$, such that $[V_k]_{k\in\mathbb{N}}$ converges to V.

Consequently, if $[L_k]_{k\in\mathbb{N}}$ is a sequence of operators on $\mathcal{V}$ which converges to an operator L, then

$$V_k^* L_k V_k \rightarrow V^* L V \ .$$

At this point we can strengthen Proposition 4.

12. **Theorem.** *Suppose that L is a cyclic operator on $\mathcal{V}$, where $\dim\mathcal{V} = n \geq 2$. Then $W_{n-1}(L)$ contains a cyclic matrix whose spectrum is disjoint from the spectrum of L.*

Theorem 12 leads to a complete characterization of cyclicity in matricial numerical ranges, which we present in Theorem 16. Our method relies upon showing that for a non-cyclic non-scalar operator L on $\mathcal{V}$, proper matricial numerical ranges of

L contain operators that are closer to being cyclic than L (Theorem 15). To this end we need to introduce some terminology.

Given an operator L on $\mathcal{V}$, let $\operatorname{rank}_{\mathbb{C}I}(L)$ denote *the rank of* L *modulo the scalars*, defined by

$$\operatorname{rank}_{\mathbb{C}I}(L) \stackrel{\text{def}}{=} \min_{\gamma\in\mathbb{C}} \Big(\operatorname{rank}(L-\gamma I)\Big) . \tag{2}$$

Since $\mathbb{C}$ is algebraically closed, $0 \leq \operatorname{rank}_{\mathbb{C}I}(L) \leq \dim\mathcal{V} - 1$.

13. **Observation.** The algebraic degree of an operator L of rank r is at most $r+1$, and the equality holds if and only if $L = C \oplus O$ where C is cyclic and not invertible; (the zero direct summand may be absent).

Since perturbing L by a scalar operator has no effect on the algebraic degree, it follows that the algebraic degree of L does not exceed $\operatorname{rank}_{\mathbb{C}I}(L) + 1$, and the equality holds if and only if $L = C \oplus \lambda I$, where C is cyclic, λ is an eigenvalue of C, and the summand λI may be absent.

The ranks of matrices in a matricial numerical range of L cannot exceed the rank of L, and consequently the same holds true for rank modulo the scalars. In particular, the algebraic degrees of matrices A in a matricial numerical range of L cannot exceed $\operatorname{rank}_{\mathbb{C}I}(L) + 1$, with the equality holding if and only if A has the same rank modulo the scalars as L, and $A = C \oplus \lambda I$, where C is cyclic, λ is an eigenvalue of C, and the summand λI may be absent.

Consequently $\mathcal{W}_m(L)$ contains no cyclic matrices when $m > \operatorname{rank}_{\mathbb{C}I}(L) + 1$. In Theorem 16 we will show that the converse implication holds true as well.

If we define $\operatorname{co-rank}_{\mathbb{C}I}(L)$ by

$$\operatorname{co-rank}_{\mathbb{C}I}(L) \stackrel{\text{def}}{=} \dim\mathcal{V} - \operatorname{rank}_{\mathbb{C}I}(L) \ ,$$

then $\operatorname{co-rank}_{\mathbb{C}I}(L)$ is the largest of the dimensions of the eigenspaces of L, i.e. the largest among the geometric multiplicities of the eigenvalues of L. In other words, $\operatorname{co-rank}_{\mathbb{C}I}(L)$ is the largest number of the Jordan blocks corresponding to a single eigenvalue of L in its Jordan form. Obviously $1 \leq \operatorname{co-rank}_{\mathbb{C}I}(L) \leq \dim\mathcal{V}$.

The number $\operatorname{co-rank}_{\mathbb{C}I}(L)$ can be thought of as a *cyclicity defect* of L; that is to say, an index of non-cyclicity of L. The cyclic operators L are exactly those with $\operatorname{co-rank}_{\mathbb{C}I}(L) = 1$, while the equation $\operatorname{co-rank}_{\mathbb{C}I}(L) = \dim\mathcal{V}$ identifies the scalar operators.

14. **Definition.** Let $A \in \mathbb{M}_n(\mathbb{C})$ be a matrix with m distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ listed in a *decreasing order of geometric multiplicity*. We associate to A a *Jordan profile matrix* $\mathrm{JP}(A)$ [2] having non-negative integer entries representing the sizes of the Jordan blocks in a Jordan Canonical Form of A as follows:

(1) $\mathrm{JP}(A)$ has m rows and $\operatorname{co-rank}_{\mathbb{C}I}(A)$ columns;
(2) the i-th row of $\mathrm{JP}(A)$ is an *increasing* list of the sizes of the Jordan blocks of A corresponding to the eigenvalue λ_i, if necessary padded with zeros *on the left* to bring it to the proper length.

We can associate a matrix $\mathrm{JP}(L)$ to an operator L on $\mathcal{V}$ in an obvious fashion.
The following theorem shows that matricial numerical ranges of a non-cyclic non-scalar operator L contain matrices that have a smaller cyclicity defect than L.

[2] $\mathrm{JP}(A)$ is sensitive to the ordering of the eigenvalues, when several have the same geometric multiplicity.

15. **Theorem.** *If $n \geq 3$, and L is a non-scalar non-cyclic operator on an n-dimensional complex inner product space, then $W_{n-1}(L)$ contains a matrix K satisfying the following conditions:*

(1) $\operatorname{rank}_{\mathbb{C}I}(K) = \operatorname{rank}_{\mathbb{C}I}(L)$; $\Big($*i.e.* $\text{co-rank}_{\mathbb{C}I}(K) = \text{co-rank}_{\mathbb{C}I}(L) - 1\Big)$;

(2) *a Jordan profile matrix* $\mathrm{JP}(K)$ *can be constructed from a given* $\mathrm{JP}(L)$ *by first deleting the last column of that* $\mathrm{JP}(L)$*, then deleting any resulting zero rows, and finally appropriately appending (at the bottom) some number of rows that have a single positive entry in the last column;*

It follows that

(a) *K is not a scalar matrix;*

(b) *the sum of the appended positive entries is one less than the algebraic degree of L; and*

(c) *the algebraic degree of K, which is the last column sum of* $\mathrm{JP}(K)$*, is one less than the sum of the last <u>two</u> column sums of* $\mathrm{JP}(L)$.

Our next theorem gives a complete characterization of the existence of cyclic elements in matricial numerical ranges of a non-zero operator L on $\mathcal{V}$, as well as of the existence of elements having the same rank modulo the scalars as L.

For convenience, we shall write $\operatorname{maxdeg}_m(L)$ for the maximum of the algebraic degrees of the elements of $\mathcal{W}_m(L)$. Clearly $\operatorname{maxdeg}_m(L) \leq \min\left(m,\ \operatorname{rank}_{\mathbb{C}I}(L) + 1\right)$.

16. **Theorem.** *If L is a non-zero operator on $\mathcal{V}$ and* $\mathrm{JP}(L)$ *is a Jordan profile matrix for L, then*

(1) *$\mathcal{W}_m(L)$ contains a cyclic matrix if and only if* $1 \leq m \leq \operatorname{rank}_{\mathbb{C}I}(L) + 1$*; in other words:* $\operatorname{maxdeg}_m(L) = m \iff 1 \leq m \leq \operatorname{rank}_{\mathbb{C}I}(L) + 1$.

(2) *$\mathcal{W}_m(L)$ contains a matrix whose rank modulo the scalars equals* $\operatorname{rank}_{\mathbb{C}I}(L)$ *if and only if* $\operatorname{rank}_{\mathbb{C}I}(L) + 1 \leq m \leq \dim\mathcal{V}$.

In fact, $\mathcal{W}_m(L)$ contains such a matrix whose algebraic degree is p less than the sum of the last $p+1$ column sums of $\mathrm{JP}(L)$*, where $p = \dim\mathcal{V} - m$.*

The following result serves as a complement to Corollary 5.

17. **Corollary.** *For an operator L on $\mathcal{V}$ with $2 \leq m \leq \dim\mathcal{V}$, the following claims are equivalent:*

(1) $\operatorname{maxdeg}_m(L) < m$;

(2) *$L = \gamma I + F$, for some scalar γ and some operator F of rank at most $m-2$.*

Next we explore the conditions under which we have the equality in the inequality $\operatorname{maxdeg}_m(L) \leq \operatorname{rank}_{\mathbb{C}I}(L)+1$. Let us note that a matrix of the form $(J_o \oplus O)+\lambda I_n$, where J_o is a singular Jordan block, demonstrates that the equality $\operatorname{maxdeg}_m(L) = \operatorname{rank}_{\mathbb{C}I}(L) + 1$ may hold for all m such that $\operatorname{rank}_{\mathbb{C}I}(L) + 1 \leq m \leq n$.

In Theorem 19 we will show that when L has an eigenspace that is large enough in proportion to $\mathcal{V}$, the equality $\operatorname{maxdeg}_m(L) = \operatorname{rank}_{\mathbb{C}I}(L) + 1$ holds for a range of m exceeding $\operatorname{rank}_{\mathbb{C}I}(L) + 1$.

18. **Lemma.** *If L is an operator on $\mathcal{V}$ such that*

$$(\dim\mathcal{V} + 2) - m \leq \operatorname{rank}_{\mathbb{C}I}(L) + 1 \leq m \quad (\leq \dim\mathcal{V}),$$

then $\operatorname{maxdeg}_m(L) \geq (\dim\mathcal{V} + 2) - m$.

The following theorem complements Theorem 16.

19. **Theorem.** *If L is an operator on $\mathcal{V}$ and* $\operatorname{maxdeg}_m(L) < m$, [3]
then $\operatorname{maxdeg}_m(L) \geq \min\left(\operatorname{rank}_{\mathbb{C}I}(L) + 1, \quad (\dim\mathcal{V} + 2) - m\right)$.

This inequality can be interpreted as follows.

If $\operatorname{rank}_{\mathbb{C}I}(L) < \frac{\dim\mathcal{V}}{2}$, *then*

(1) $\operatorname{maxdeg}_m(L) = \operatorname{rank}_{\mathbb{C}I}(L) + 1$,
when $\operatorname{rank}_{\mathbb{C}I}(L) + 1 < m \leq \dim\mathcal{V} + 1 - \operatorname{rank}_{\mathbb{C}I}(L)$;[4]

(2) $\operatorname{maxdeg}_m(L) \geq (\dim\mathcal{V} + 2) - m$,
when $\dim\mathcal{V} + 1 - \operatorname{rank}_{\mathbb{C}I}(L) \leq m \leq \dim\mathcal{V}$.

If $\operatorname{rank}_{\mathbb{C}I}(L) \geq \frac{\dim\mathcal{V}}{2}$, *then*

(3) $\operatorname{maxdeg}_m(L) \geq (\dim\mathcal{V} + 2) - m$,
when $\operatorname{rank}_{\mathbb{C}I}(L) + 1 < m \leq \dim\mathcal{V}$.

The following result is a companion to Corollary 17.

20. **Corollary.** *If L is an operator on $\mathcal{V}$, and*

$$\operatorname{maxdeg}_m(L) < \min\left(m, \ (\dim\mathcal{V} + 2) - m\right),$$

then $L = \gamma I + F$, for some scalar γ and some operator F of rank at most $\operatorname{maxdeg}_m(L) - 1$.

We conclude by presenting a version of Theorems 16 and 19 for (possibly unbounded) operators L on infinite-dimensional Hilbert spaces. The notions of $\mathcal{W}_m(L)$ and $\operatorname{maxdeg}_m(L)$ $(m \in \mathbb{N})$ extend naturally to this setting. L is said to have a *finite scalar rank* if $rank(L - \lambda I) < \infty$ for some $\lambda \in \mathbb{C}$, and in such a case $\operatorname{rank}_{\mathbb{C}I}(L)$ is defined according to the formula (2). Setting $\operatorname{rank}_{\mathbb{C}I}(L) = \infty$ otherwise, we observe that the inequality $\operatorname{maxdeg}_m(L) \leq \min\left(m, \ \operatorname{rank}_{\mathbb{C}I}(L) + 1\right)$ holds in all cases.

21. **Theorem.** *Suppose that L is a (not possibly unbounded) operator on an infinite-dimensional Hilbert space.*

(1) *If L has finite scalar rank then*

$$\operatorname{maxdeg}_m(L) = \begin{cases} m, & \text{if } 1 \leq m \leq \operatorname{rank}_{\mathbb{C}I}(L) + 1 \ ; \\ \operatorname{rank}_{\mathbb{C}I}(L) + 1, & \text{if } \operatorname{rank}_{\mathbb{C}I}(L) + 1 < m \ . \end{cases}$$

(2) *If L does not have finite scalar rank, then* $\operatorname{maxdeg}_m(L) = m$, *for all* $m \in \mathbb{N}$.

Proof of Theorem 21. If L has finite scalar rank, let $k = \operatorname{rank}_{\mathbb{C}I}(L)$, and for a given m let $n = m + 2k$. There is an element T of $\mathcal{W}_n(L)$ such that $\operatorname{rank}_{\mathbb{C}I}(T) = \operatorname{rank}_{\mathbb{C}I}(L)$. Then $\mathcal{W}_m(T) \subseteq \mathcal{W}_m(L)$ (see formula (1)), and so $\operatorname{maxdeg}_m(T) \leq \operatorname{maxdeg}_m(L) \leq \min\left(m, \ k+1\right)$. By Theorem 16, $m = \operatorname{maxdeg}_m(T)$, when $m \leq k+1$, and by Theorem 19, $k+1 = \min\left(k+1, (n+2) - m\right) \leq \operatorname{maxdeg}_m(T)$, when $k+1 < m$. This settles part (1). If L does not have finite scalar rank, then for any given m, there is an $n > m$ such that $\mathcal{W}_n(L)$ has an element T satisfying $\operatorname{rank}_{\mathbb{C}I}(T) \geq m$. By Theorem 19, $m = \operatorname{maxdeg}_m(T) \leq \operatorname{maxdeg}_m(L) \leq m$, and the proof is complete. $\square$

[3] i.e. $\operatorname{rank}_{\mathbb{C}I}(L) + 1 < m$.

[4] In this case $\mathcal{W}_m(L)$ contains a matrix A with algebraic degree $\operatorname{rank}_{\mathbb{C}I}(L) + 1$, such that with respect to an appropriate basis, A can be expressed as $C \oplus \lambda I$, where $C \in \mathcal{W}_{(\operatorname{rank}_{\mathbb{C}I}(L)+1)}(L)$ is cyclic, and λ is an eigenvalue of C.

22. **Corollary.** *If L is a (possibly unbounded) operator on an infinite-dimensional Hilbert space and* $\operatorname{maxdeg}_m(L) < m$ *for some* $m \in \mathbb{N}$*, then*

$$\operatorname{rank}_{\mathbb{C}I}(L) \leq \operatorname{maxdeg}_m(L) - 1.$$

4. Proofs

Proof of Proposition 2. Since the claim is trivially true when A is invertible, we shall assume that $0 < r < n$.

When $\mathfrak{B}$ is a basis of a subspace $\mathcal{Z}$ of $\mathbb{C}^n$, and $\mathfrak{B}$ is comprised of elements of $\mathbb{Z}^n$, the Gram-Schmidt process (and intermittent scaling) can be used to convert $\mathfrak{B}$ to an orthogonal basis of $\mathcal{Z}$ comprised of elements of $\mathbb{Z}^n$. Since an orthogonal list in $\mathbb{Z}^n$ can be enlarged to a basis of $\mathbb{C}^n$ through an addition of some number of the standard basis n-tuples e_i, every such orthogonal list can be enlarged to an orthogonal basis of $\mathbb{C}^n$ comprised of elements of $\mathbb{Z}^n$.

To establish the result in the proposition, it is sufficient to demonstrate the existence of a matrix $W_o \in \mathbb{M}_{n\times r}(\mathbb{Z})$ with orthogonal columns, such that $W_o^* A\, W_o$ is invertible, or equivalently, injective. The latter condition is equivalent to the conjunction

$$kernel(A) \cap range(W_o) = \{O\} \quad \text{and} \quad range(A) \cap \Big(range(W_o)\Big)^{\perp} = \{O\}\ .$$

Thus, all that one needs to demonstrate is the existence of an r-dimensional subspace $\mathcal{Z}$ of $\mathbb{C}^n$, with a basis comprised of elements of $\mathbb{Z}^n$, such that

$$kernel(A) \cap \mathcal{Z} = \{O\}, \qquad range(A) \cap \mathcal{Z}^{\perp} = \{O\}\ . \tag{3}$$

Note that by the Rank-Nullity Theorem, conditions (3) imply that

$$\dim \mathcal{Z} \leq r \quad \text{and} \quad n - \dim \mathcal{Z} \leq n - r\ ,$$

so that $\dim \mathcal{Z} = r$, and therefore this condition does not need to be included as a requirement beforehand.

If $\mathcal{X}$ and $\mathcal{Y}$ are proper subspaces of $\mathbb{C}^n$ so that $\mathcal{X} \cup \mathcal{Y} \subsetneq \mathbb{C}^n$, then $\mathcal{X} \cup \mathcal{Y}$ lacks one of the sums $e_i + e_j$ of the standard basis n-tuples. This is so because neither $\mathcal{X}$ nor $\mathcal{Y}$ contains all of the e_k's, and if $\mathcal{X} \cup \mathcal{Y}$ contains all of them, then it does not contain $e_i + e_j$, where $e_i \in \mathcal{X}\backslash\mathcal{Y}$ and $e_j \in \mathcal{Y}\backslash\mathcal{X}$.

Claim: If $\mathcal{X}$ and $\mathcal{Y}$ are subspaces of $\mathbb{C}^n$ and $\dim(\mathcal{X}) \leq \dim(\mathcal{Y})$, then there exists a subspace $\mathcal{Z}$ of $\mathbb{C}^n$ such that

$$\mathcal{X} \cap \mathcal{Z} = \{O\}, \qquad \mathcal{Y} + \mathcal{Z} = \mathbb{C}^n\ ,$$

and $\mathcal{Z}$ is either trivial or has a basis comprised of elements of $\mathbb{Z}^n$.

To verify this Claim, we induct on the co-dimension of $\mathcal{Y}$. If $\mathcal{Y} = \mathbb{C}^n$, $\mathcal{Z} = \{O\}$ will do. If the result is true whenever the co-dimension of $\mathcal{Y}$ is at most k ($< n$), and we consider $\mathcal{Y}$ of co-dimension $k+1$, then

$$\mathcal{X} \cup \mathcal{Y} \neq \mathbb{C}^n\ ,$$

since a finite union of subspaces of $\mathbb{C}^n$ is a subspace only when one of them contains all of the others. As noted above, $\mathcal{X} \cup \mathcal{Y}$ lacks some $e_i + e_j$ ($\stackrel{\text{def}}{=} h$). Apply the inductive hypothesis to

$$\mathcal{X}_o \stackrel{\text{def}}{=} \mathcal{X} + \operatorname{span}(h) \quad \text{and} \quad \mathcal{Y}_o \stackrel{\text{def}}{=} \mathcal{Y} + \operatorname{span}(h)\ ,$$

to assert the existence of a subspace $\mathcal{Z}_o$ such that

$$\Big(\mathcal{X} + \operatorname{span}(h)\Big) \cap \mathcal{Z}_o = \{O\}, \qquad \Big(\mathcal{Y} + \operatorname{span}(h)\Big) + \mathcal{Z}_o = \mathbb{C}^n \ ,$$

and $\mathcal{Z}_o$ is either trivial or has a basis comprised of elements of $\mathbb{Z}^n$. In this case $\mathcal{Z} \stackrel{\text{def}}{=} \mathcal{Z}_o + \operatorname{span}(h)$ is the subspace we seek, and the proof of the claim is complete.

From our Claim we can conclude that for any subspaces $\mathcal{X}$ and $\mathcal{Y}$ of $\mathbb{C}^n$ satisfying

$$\dim(\mathcal{X}) \leq \dim\left(\mathcal{Y}^{\perp}\right) = n - \dim(\mathcal{Y}) \ ,$$

there exists a subspace $\mathcal{Z}$ of $\mathbb{C}^n$ such that

$$\mathcal{X} \cap \mathcal{Z} = \{O\}, \qquad \mathcal{Y} \cap \mathcal{Z}^{\perp} = \{O\} \ ,$$

and $\mathcal{Z}$ is either trivial or has a basis comprised of elements of $\mathbb{Z}^n$.

Since $\dim(kernel(A)) + \dim(range(A)) = n$, and neither dimension is zero by the assumptions, taking $\mathcal{X} = kernel(A)$ and $\mathcal{Y} = range(A)$ establishes the claim. □

Proof of Lemma 3. If we start with a basis $\mathfrak{B}$ of $\mathcal{Z}$ and concatenate it with a basis of $\mathcal{X}$ to produce a basis of $\mathcal{V}$, and then similarly concatenate $\mathfrak{B}$ with a basis of $\mathcal{Y}$, the corresponding change of basis matrix for the two bases of $\mathcal{V}$ thus created has a partitioned form $\left[\begin{smallmatrix} I & M \\ O & S \end{smallmatrix}\right]$. By the change of basis formula we have:

$$\begin{bmatrix} [\![A_0]\!] & [\![B]\!] \\ O & [\![A_1]\!] \end{bmatrix} = \begin{bmatrix} I & M \\ O & S \end{bmatrix}^{-1} \begin{bmatrix} [\![A_0]\!] & [\![C]\!] \\ O & [\![A_1]\!] \end{bmatrix} \begin{bmatrix} I & M \\ O & S \end{bmatrix} = \begin{bmatrix} I & -MS^{-1} \\ O & S^{-1} \end{bmatrix} \begin{bmatrix} [\![A_0]\!] & [\![C]\!] \\ O & [\![A_1]\!] \end{bmatrix} \begin{bmatrix} I & M \\ O & S \end{bmatrix} ,$$

where the brackets $[\![\ ]\!]$ indicate matrix representations of the linear maps with respect to the appropriate bases. Therefore

$$[\![A_1]\!] = S^{-1} [\![A_2]\!] S,$$

which is sufficient to establish the required result. □

Proof of Proposition 4. The case $n = 1$ is trivial. Let us treat the case $n = 2$. If L is cyclic, it is not a scalar operator, and its numerical range is not a singleton. Being convex, the numerical range of L is an infinite set, and therefore it contains a non-zero number α. Then $[\alpha]_{1\times 1}$ is the required element of the 1-st matricial numerical range of L.

For the rest of the proof we assume that $n \geq 3$.

Given a cyclic L, let $\tilde{\mathfrak{F}} \stackrel{\text{def}}{=} (f_1, f_2, \ldots, f_n)$ be a basis of the underlying space $\mathcal{V}$ with respect to which L is represented by a direct sum of Jordan blocks, one for each distinct eigenvalue of L.

It is easy to see that with respect to the decomposition $\mathcal{V} = \operatorname{span}(f_1) \oplus \operatorname{span}(f_2, \ldots, f_n)$, L has the form $\left[\begin{smallmatrix} \lambda & B \\ O & D \end{smallmatrix}\right]$, where D is cyclic.

By Lemma 3, L has the form $\left[\begin{smallmatrix} \lambda & S \\ O & T \end{smallmatrix}\right]$ with respect to the decomposition $\mathcal{V} = \operatorname{span}(f_1) \oplus (f_1)^{\perp}$, where T is cyclic. Since the ortho-compression T of L to $(f_1)^{\perp}$ is cyclic, $\mathcal{W}_{n-1}(L)$ contains a cyclic matrix.

By iterating the argument via equality (1), we arrive at the desired result for all $1 \leq m \leq n-1$. □

Proof of Corollary 5. Suppose, for the sake of contradiction, that L is algebraic of degree k, where $k > d$. There is a vector $v \in \mathcal{V}$, such that the (local) minimal polynomial of L at v equals the (global) minimal polynomial of L. Consider the k-dimensional cyclic invariant subspace generated by v:

$$\mathcal{Z} \stackrel{\text{def}}{=} \operatorname{span}\Big(v, L(v), L^2(v), \ldots, L^{k-1}(v)\Big) \ .$$

$L_{|\mathcal{Z}}$ is algebraic of degree k, and therefore is a cyclic operator on $\mathcal{Z}$.

Since $\mathcal{Z}$ is invariant under L, it is invariant under an ortho-compression of L to any subspace that contains $\mathcal{Z}$. In particular the degree of the minimal polynomial of any such ortho-compression is at least k, and hence exceeds d. Thus, m must be less than the dimension of $\mathcal{Z}$; i.e. less than k.

By Proposition 4, $\mathcal{W}_m\big(L_{|\mathcal{Z}}\big)$ contains a cyclic matrix, i.e. a matrix with the algebraic degree m. Yet $\mathcal{W}_m\big(L_{|\mathcal{Z}}\big) \subset \mathcal{W}_m(L)$, which leads to a contradiction. $\square$

Proof of Proposition 7. Our proof is by induction on k. Let us consider the base case $k=2$. It is clear that for any matrix $M \in \mathbb{M}_{n\times 2}(\mathbb{C})$ without zero rows, the first entry of $M\binom{m}{1}$ is not zero for all large enough natural numbers m. The same claim can made about the second entry; etc. So, $M\binom{m}{1}$ has no zero entries for all large enough natural numbers m.

Suppose that the claim holds for some natural $k_o \geq 2$ and all $n \in \mathbb{N}$. Consider a matrix $M \in \mathbb{M}_{n\times(k_o+1)}(\mathbb{C})$ without zero rows, with the columns $c_1, c_2, \ldots, c_{k_o+1}$. Let $T \in \mathbb{M}_{n\times k_o}(\mathbb{C})$ be the matrix with columns $c_2, \ldots, c_{k_o+1}$. By the inductive hypothesis, there exist positive scalars $m_2, \gamma_3, \ldots, \gamma_{k_o+1}$ such that the zero entries of $m_2c_2 + \gamma_3c_3 + \cdots + \gamma_{k_o+1}c_{k_o+1}$ correspond to the zero rows of T. In particular, the $n \times 2$ matrix with the columns c_1 and $m_2c_2 + \gamma_3c_3 + \cdots + \gamma_{k_o+1}c_{k_o+1}$ has no zero rows, and hence by the base case, the linear combination

$$mc_1 + m_2c_2 + \gamma_3c_3 + \cdots + \gamma_{k_o+1}c_{k_o+1}$$

has no zero entries for all large enough natural numbers m. $\square$

Proof of Lemma 10. First we shall demonstrate that given unit vectors v and u in a complex inner-product space $\mathcal{V}$, there exists a unitary operator Y with $Yv = u$ and $\|U - I\| = \|v - u\|$.

If $u = \alpha v$ for some $\alpha \in \mathbb{C}$, then it suffices to let $Yv = u$, $Yx = x$ for all vectors x perpendicular to v, and to extend Y by linearity to all of $\mathcal{V}$.

If $\{v, u\}$ is linearly independent, then using the Gram-Schmidt process, we can write

$$u = \alpha v + \beta y$$

where $\|y\| = 1$, $\langle v, y\rangle = 0$, and $\beta \geq 0$. Consider the operator Y_o defined on $\mathcal{V}_o := \operatorname{span}\{v, u\} = \operatorname{span}\{v, y\}$ whose matrix is

$$\begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix}$$

relative to the orthonormal basis $\{v, y\}$. Clearly $Y_ov = u$, and a routine calculation shows that if I_2 denotes the identity operator acting on $\mathcal{V}_o$, then

$$\|Y_0 - I_2\|_{\text{op}}^2 = |\alpha - 1|^2 + \beta^2 = \|(Y_0 - I_2)v\|^2 .$$

We then define a linear map Y by setting $Y|_{\mathcal{V}_o} = Y_o$, $Yz = z$ for all $z \in \mathcal{V}_o^{\perp}$, and extending Y to all of $\mathcal{V}$ by linearity. A second routine calculation shows that

$$\|Y - I\|_{\text{op}} = \|Y_o - I_2\|_{\text{op}} = \|Y_ov - v\| = \|u - v\| .$$

Next, if $V : \mathbb{C}^m \to \mathcal{V}$ is an isometry and $u \in \mathcal{V}$ is a unit vector, let $v = Ve_1$ and choose a unitary operator Y acting on $\mathcal{V}$ such that $Yv = u$ and $\|Y - I\|_{\text{op}} = \|v - u\|$. Then $Z \stackrel{\text{def}}{=} YV$ is an isometry, $Ze_1 = YVe_1 = Yv = u$, and

$$\|Z - V\|_{\text{op}} \leq \|Y - I\|_{\text{op}} \|V\|_{\text{op}} = \|Y - I\|_{\text{op}} = \|v - u\| ,$$

as required. $\square$

Proof of Proposition 11. Given an integer $r \geq 1$ and list $g_1, g_2, \ldots, g_r$ in $\mathcal{V}$, let us write

$$[g_1\ g_2 \cdots g_r]$$

for the linear function $L : \mathbb{C}^r \longrightarrow \mathcal{V}$ such that $L(e_i) = g_i$.

Express V as $[f_1\ f_2 \cdots f_{n-1}]$, and let

$$Z = \left[\frac{h}{\|h\|}\ f_1\ f_2 \cdots f_{n-1}\right] : \mathbb{C}^n \longrightarrow \mathcal{V}\ .$$

Then Z is a surjective isometry, and

$$\frac{h_k}{\|h_k\|} \to \frac{h}{\|h\|} = Z(e_1)\ .$$

By Lemma 10, there exists a sequence $[Z_j]$ of surjective isometries $Z_j : \mathbb{C}^n \longrightarrow \mathcal{V}$ such that

$$Z_j(e_1) = \frac{h_j}{\|h_j\|} \quad \text{and} \quad \|Z_j - Z\|_{\text{op}} = \left\|\frac{h_k}{\|h_k\|} - Z(e_1)\right\|.$$

Obviously $[Z_j]_j \to Z$, and so $[Z_j(e_i)]_j \to Z(e_i)$ for every i. In particular,

$$V_j \stackrel{\text{def}}{=} [Z_j(e_2)\ \ldots\ Z_j(e_n)] \to [Z(e_2)\ \ldots\ Z(e_n)] = [f_1\ f_2 \cdots f_{n-1}] = V$$

Since $V_i : \mathbb{C}^{n-1} \longrightarrow \mathcal{V}$ is an isometry with $range(V_i) = h_i^{\perp}$, the proof is complete. $\square$

Proof of Theorem 12. The beginning of the proof follows the same path as the proof of Proposition 4. Let us treat the case $n = 2$ first. If L is cyclic, it is not a scalar multiple of the identity, and its numerical range is not a singleton. Being convex, the numerical range of L is an infinite set, and therefore it contains a non-zero number α distinct from the eigenvalues of L. Then $[\alpha]_{1\times 1}$ is the required element of the 1-st matricial numerical range of L.

For the rest of the proof we shall assume that $n \geq 3$.

Given a cyclic L, let $\tilde{\mathfrak{F}} \stackrel{\text{def}}{=} (f_1, f_2, \ldots, f_n)$ be a basis of the underlying space $\mathcal{V}$ with respect to which L is represented by a direct sum of Jordan blocks, one for each distinct eigenvalue of L. In particular, each eigenvector of L is a scalar multiple of one of the f_i's.

With respect to the decomposition $\mathcal{V} = \text{span}(f_1) \oplus \text{span}(f_2, \ldots, f_n)$, L has the form $\left[\begin{smallmatrix} \beta & B \\ O & D \end{smallmatrix}\right]$, where D is cyclic and β is the eigenvalue corresponding to the eigenvector f_1. Hence, by Lemma 3, with respect to the decomposition $\mathcal{V} = \text{span}(f_1) \oplus (f_1)^{\perp}$, L has the form $\left[\begin{smallmatrix} \beta & S \\ O & T \end{smallmatrix}\right]$, where T is cyclic.

By Corollary 9 there exists a sequence $[u_k]$ of entry-wise positive unit n-tuples convergent to $(1, 0, \ldots, 0)$, such that

$$\Big\langle u_k(1)f_1 + \cdots + u_k(n)f_n\ ,\ f_i\Big\rangle \neq 0\ , \quad \text{for all} \quad k, i. \tag{4}$$

Let $h_k \stackrel{\text{def}}{=} u_k(1)f_1 + \cdots + u_k(n)f_n$. In view of (4), $[h_k]_k$ is a sequence of non-zero vectors convergent to f_1. By Proposition 11, for any isometry $V : \mathbb{C}^{n-1} \longrightarrow \mathcal{V}$ with the range $(f_1)^{\perp}$, there exist isometries $V_k : \mathbb{C}^{n-1} \longrightarrow \mathcal{V}$, with respective ranges $(h_k)^{\perp}$, such that

$$V_k^* L V_k \to V^* L V\ .$$

Since V^*LV is an $(n-1)\times(n-1)$ matrix which represents T with respect to an appropriate orthonormal basis of $(f_1)^\perp$, V^*LV is cyclic.

By Proposition 1, $V_k^*LV_k$ is a cyclic element of $\mathcal{W}_{n-1}(L)$ for all large k.

To complete the proof, let us show that $V_k^*LV_k$ shares no eigenvalues with L.

To this end, temporarily fix k, and suppose that λ is an eigenvalue of $V_k^*LV_k$ with a corresponding eigenvector w. So,

$$\mathbf{0} = \left(V_k^*LV_k - \lambda I_{n-1}\right)(w) = \Big(V_k^*\left(L-\lambda I\right)V_k\Big)(w)\ .$$

Set $z \overset{\text{def}}{=} V_k(w)$. Then

$$\left(L-\lambda I\right)(z) \in kernel(V_k^*) = \operatorname{span}(h_k)\ .$$

Let us argue that $\left(L-\lambda I\right)(z) \neq \mathbf{0}$. If z were an eigenvector of L, it would be a non-zero scalar multiple of one of the f_i's (as noted above). Yet $z = V_k(w) \in (h_k)^\perp$, and no h_k is orthogonal to an f_i by (4). It follows that $\left(L-\lambda I\right)(z) = \rho h_k$, for some non-zero ρ.

If λ were an eigenvalue of L, its generalized eigenspace would be the span of some consecutive elements $f_{i_1},\ldots,f_{i_m}$ of $\mathfrak{F}$. In particular the representation of any element of the range of $L-\lambda I$ as a linear combination of the basis vectors f_i will have a zero coefficient attached to the vector f_{i_m}. This would hold true for $\left(L-\lambda I\right)(z)$, and hence for ρh_k. Yet $\rho h_k = \rho u_k(1)f_1 + \cdots + \rho u_k(n)f_n$, where every $u_k(i)$ is positive, and ρ is not zero. Therefore λ is not an eigenvalue of L, and so $V_k^*LV_k$ shares no eigenvalues with L. □

Proof of Theorem 15. For each eigenvalue α of L, choose a Jordan block J_α of maximal length corresponding to that eigenvalue in the Jordan form of L, and decompose the underlying inner product space $\mathcal{V}$ non-trivially as $\mathcal{Z}\oplus\mathcal{X}$ in such a way that L is expressed as $A_0\oplus A_1$ with respect to this decomposition, where A_1 is cyclic and its Jordan form is the direct sum of the J_α's chosen above. In particular the spectrum of A_1 coincides with that of L, and the dimension of $\mathcal{X}$ equals the algebraic degree of L.

Since L is not a scalar operator, the dimension of $\mathcal{X}$ is at least 2, and

$$\text{co-rank}_{\mathbb{C}I}(A_0) = \text{co-rank}_{\mathbb{C}I}(L) - 1\ .$$

By Lemma 3, L has the form $\left[\begin{smallmatrix} R & S\\ O & T\end{smallmatrix}\right]$ with respect to the decomposition $\mathcal{V} = \mathcal{Z}\oplus\mathcal{Z}^\perp$, where T is cyclic and has the same spectrum as L. Let us say that the dimension of $\mathcal{Z}$ is m, and let us fix a surjective linear isometry $U:\mathbb{C}^m \longrightarrow \mathcal{Z}$. As we have already noted, $n-m = \dim\mathcal{X} \geq 2$.

By Theorem 12, there is a linear isometry $V:\mathbb{C}^{n-m-1}\longrightarrow\mathcal{Z}^\perp$ such that V^*TV is cyclic and has the spectrum disjoint from that of T, i.e. from that of L.

Note that $U\oplus V:\mathbb{C}^{n-1}\longrightarrow\mathcal{V}$ is a linear isometry, and let

$$K \overset{\text{def}}{=} (U\oplus V)^*\,L\,(U\oplus V) = (U^*\oplus V^*)\,L\,(U\oplus V) = \begin{bmatrix} U^*RU & U^*SV\\ O & V^*TV\end{bmatrix}_{(n-1)\times(n-1)},$$

so that $K\in\mathcal{W}_{n-1}(L)$.

The spectrum of U^*RU equals that of R, which is a subset of the spectrum of L, and so is disjoint from the spectrum of V^*TV. It follows that K is similar to

$$\begin{bmatrix} U^*RU & O\\ O & V^*TV\end{bmatrix}.$$

In particular, a direct sum of Jordan forms of U^*RU (i.e. of R) and of V^*TV gives a Jordan form of K. The validity of the claims of the theorem is an immediate consequence of these facts and of the definition of $\mathcal{Z}$. We offer brief comments for each individual claim.

(1) Since V^*TV and T are cyclic,
$$\text{co-rank}_{\mathbb{C}I}(K) = \text{co-rank}_{\mathbb{C}I}(U^*RU) = \text{co-rank}_{\mathbb{C}I}(R) = \text{co-rank}_{\mathbb{C}I}(L) - 1\ ,$$
which, in view of the dimensions, means that $\text{rank}_{\mathbb{C}I}(K) = \text{rank}_{\mathbb{C}I}(L)$.

(2) A Jordan form of R is obtained by removing the largest Jordan block of L for each eigenvalue of L. So, a Jordan profile of R can be obtained by removing the last column of the $\text{JP}(L)$, dropping off any resulting zero rows. Since V^*TV is cyclic, its Jordan profile is a one-column matrix with positive integer entries.

Since $\text{rank}_{\mathbb{C}I}(K) = \text{rank}_{\mathbb{C}I}(L) \neq 0$, by claim (1), K is not a scalar operator. The sum of the appended positive entries is the algebraic degree of the cyclic matrix V^*TV, and so is $n - m - 1$, which is one less than the algebraic degree of L. □

Proof of Theorem 16. Since the claims are clearly true when L is a scalar multiple of the identity or is cyclic (see Proposition 4), we shall restrict our attention to a non-scalar non-cyclic L. In other words, we shall focus on the case $2 \leq \text{co-rank}_{\mathbb{C}I}(L) \leq n - 1$, and $n \geq 3$.

Let us begin by verifying that $m \geq \text{rank}_{\mathbb{C}I}(L) + 2$ implies that $\mathcal{W}_m(L)$ contains no cyclic matrices.

The ranks of the elements of a matricial numerical range of an operator cannot exceed its rank, and consequently the same is true for rank modulo the scalars. If $\text{rank}_{\mathbb{C}I}(L) \leq m - 2$, then $\text{rank}_{\mathbb{C}I}(M) \leq m - 2$ for any $M \in \mathcal{W}_m(L)$. So, for any $M \in \mathcal{W}_m(L)$, there is a scalar γ such that $\text{rank}(M + \gamma I) \leq m - 2$, and therefore $M + \gamma I$ is annihilated by a polynomial of degree at most $m - 1$. This shows that no $M \in \mathcal{W}_m(L)$ is cyclic.

Now let us consider the remaining claims. Let k stand for the $\text{co-rank}_{\mathbb{C}I}(L)$ to unburden the formulas. By applying Theorem 15 recursively, starting with $K_0 \stackrel{\text{def}}{=} L$, we can generate matrices K_i from K_{i-1}, $1 \leq i \leq k - 1$ with the properties described in that Theorem. In particular, each K_i is a non-scalar matrix in $\mathcal{W}_{n-i}(L)$ (which we have shown not to contain any cyclic matrices when $1 \leq i \leq k - 2$). Moreover, $\text{rank}_{\mathbb{C}I}(K_i) = \text{rank}_{\mathbb{C}I}(K_0) = \text{rank}_{\mathbb{C}I}(L)$ for all i. Select a Jordan profile matrix $\text{JP}(L)$ for L, and from this construct $\text{JP}(K_i)$ from $\text{JP}(K_{i-1})$, $1 \leq i \leq k - 1$ via the procedure described in Theorem 15.

Since $\text{rank}_{\mathbb{C}I}(K_i) = \text{rank}_{\mathbb{C}I}(L) = n - k$, we have the equality
$$\text{co-rank}_{\mathbb{C}I}(K_i) = k - i.$$
In particular K_{k-1} is a cyclic matrix in $\mathcal{W}_{n-k+1}(L)$. Since $n-k+1 = \text{rank}_{\mathbb{C}I}(L)+1$, we have established the validity of claim 1 of the theorem via Proposition 4.

Next we turn to the second claim in the statement of the theorem. Since the rank modulo the scalars of any operator is always strictly less than the dimension of the underlying space, for $m \leq \text{rank}_{\mathbb{C}I}(L)$, no matrix in $\mathcal{W}_m(L)$ has the same rank modulo the scalars as L.

The matrices $K_1, K_2, \ldots, K_{k-2}$ demonstrate the validity of the other implication in the claim.

To establish the remaining portion of that claim, note that removing the last column of $\mathrm{JP}(K_i)$ and removing the resulting zero rows produces the same matrix as removing the last two columns of $\mathrm{JP}(K_{i-1})$, and removing the resulting zero rows.

The algebraic degree of K_i is the sum of the last column of $\mathrm{JP}(K_i)$, and is 1 less than the sum of the last two column sums of $\mathrm{JP}(K_{i-1})$. □

Proof of Lemma 18. Our proof will rely on the following observation.

Suppose that $m_1, m_2, \ldots, m_k$ are positive integers such that

$$m_1 \leq m_2 \leq \cdots \leq m_k \ .$$

If the sum of the largest $p+1$ of these integers is at most $2p+1$, then

$$m_1 + m_2 + \ldots + m_k \leq k + p \ .$$

Indeed, let $n_i := m_i - 1$, $1 \leq i \leq k$. Then

$$0 \leq n_1 \leq n_2 \leq \cdots \leq n_k \ ,$$

and the sum of the $p+1$ largest n_i's is at most p, which indicates that the smallest among these is zero. Hence all other (smaller) n_i's are also zero, and so

$$n_1 + n_2 + \cdots + n_k \leq p \ .$$

Thus

$$m_1 + m_2 + \ldots + m_k \leq k + p \ .$$

Now let us prove Lemma 18.

Note that the hypothesis of the lemma implies that $m > \text{co-rank}_{\mathbb{C}I}(L)$. Let us say that $\dim\mathcal{V} = n$. By letting $p = n - m$, we can restate the claim as follows:

if

$$0 \leq p \leq \text{co-rank}_{\mathbb{C}I}(L) - 1 \leq m - 2 \ ,$$

then $\underset{m}{\mathrm{maxdeg}}(L) \geq p + 2$.

Suppose, for the sake of contradiction, that $0 \leq p \leq \text{co-rank}_{\mathbb{C}I}(L) - 1 \leq m - 2$, but $\underset{m}{\mathrm{maxdeg}}(L) \leq p + 1$.

Let $\mathrm{JP}(L)$ be a Jordan profile matrix for L. By Theorem 16, p less than the sum of the last $p+1$ column sums of $\mathrm{JP}(L)$ is at most $p+1$. In other words, the sum of the last p column sums of $\mathrm{JP}(L)$ is at most $2p+1$.

The column sums of $\mathrm{JP}(L)$ form an increasing list of $\text{co-rank}_{\mathbb{C}I}(L)$ positive integers that add up to n. So, by our initial observation, we can conclude that

$$n \leq \text{co-rank}_{\mathbb{C}I}(L) + p \ ,$$

which leads to $m \leq \text{co-rank}_{\mathbb{C}I}(L)$, in contradiction to our hypothesis. □

Proof of Theorem 19. We consider two cases, and write n for $\dim\mathcal{V}$.

Case 1: $m \geq n + 2 - \Big(\text{rank}_{\mathbb{C}I}(L) + 1\Big)$.

In this case

$$2 \leq n + 2 - m \leq \text{rank}_{\mathbb{C}I}(L) + 1 \leq m = (n+2) - (n+2-m) \ ,$$

and by Lemma 18, $\mathcal{W}_m(L)$ contains a matrix whose algebraic degree is at least $n - m + 2$.

Case 2: $m < n + 2 - \Big(\text{rank}_{\mathbb{C}I}(L) + 1\Big)$.

In this case we define $n_o \stackrel{\text{def}}{=} m + \text{rank}_{\mathbb{C}I}(L) - 1$, so that

$$m = n_o + 2 - \Big(\text{rank}_{\mathbb{C}I}(L) + 1\Big) .$$

Note that $m \le n_o < n$. By Theorem 16, $\mathcal{W}_{n_o}(L)$ contains a matrix L_o such that

$$\text{rank}_{\mathbb{C}I}(L_o) + 1 = \text{rank}_{\mathbb{C}I}(L) + 1 .$$

It is easy to see that L_o satisfies the hypotheses of the present theorem and of Case 1 of this proof. Hence $\mathcal{W}_m(L_o)$ contains a matrix whose algebraic degree is at least $n_o - m + 2$, and the latter number is $\text{rank}_{\mathbb{C}I}(L) + 1$, by the definition of n_o.

Since $\mathcal{W}_m(L_o) \subseteq \mathcal{W}_m(L)$, the proof is complete. □

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