Comparing Intersection Cut Closures using Simple Families of Lattice-Free Convex Sets

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

The results in this thesis are based on joint work with my supervisors B. Guenin and L. Tunçel. The results in Chapter 5 were advanced in discussion with G. Cornuéjols. I was responsible for drafting the results, and received help and feedback for multiple revisions.

Abstract

Mixed integer programs are a powerful mathematical tool, providing a general model for expressing both theoretically difficult and practically useful problems. One important subroutine of algorithms solving mixed integer programs is a cut generation procedure. The job of a cut generation procedure is to produce a linear inequality that separates a given infeasible point x^* (usually a basic feasible solution of the linear programming relaxation) from the set of feasible solutions for the problem at hand. Early and well-known cut generation procedures rely on analyzing a single row of the simplex tableau for x^*. The authors of [4] renewed interest in d-row cuts (i.e. cuts derived from d rows of the simplex tableau) by showing that these cuts afford some theoretical benefit.

One lens from which to study d-row cuts is in the context of the intersection cuts of Balas [12] and, in particular, intersection cuts obtained from lattice-free convex sets. The strongest d-row intersection cuts are obtained from maximal lattice-free convex sets in $\mathbb{R}^d$ - all of which are polyhedra with at most 2^d facets. This thesis is concerned with theoretical comparison of the d-row cuts generated by different families of maximal lattice-free convex sets. We use the gauge measure (following [47]) to appraise the quality of the approximation. The main area of focus is 2-row cuts.

The problem of generating 2-row cuts can be re-posed as generating valid inequalities for a mixed integer linear set $\mathcal{F}$ with two free integer variables and any number of non-negative continuous variables, where there are two defining equations. Every minimal valid inequality for the convex hull of $\mathcal{F}$ is an intersection cut generated by a maximal lattice-free split, triangle or quadrilateral. The family of maximal lattice-free triangles can be subdivided into the families of type 1, type 2, and type 3 triangles.

Previous results [15, 11] compare how well the inequalities from one of these families approximates the convex hull of $\mathcal{F}$ (a.k.a. the *corner polyhedron*). In particular, the closure of all type 2 triangle inequalities is shown to be within a factor of $\frac{3}{2}$ of the corner polyhedron. The authors also provide an instance where all type 2 triangles inequalities cannot approximate the corner polyhedron better than a factor of $\frac{9}{8}$. The same bounds are shown for type 3 triangles and quadrilaterals. These results are obtained not by directly comparing the given closures to the convex hull of $\mathcal{F}$, but rather to each other.

In this thesis, we tighten one of the underlying bounds, showing that the closure of all type 2 triangle inequalities are within a factor of $\frac{5}{4}$ of the closure of all quadrilateral inequalities. We also consider the sub-family of quadrilaterals where opposite edges have equal slope. We show that these parallelogram cuts can be approximated by all type 2 triangle inequalities within a factor of $\frac{9}{8}$, and there exist instances where no better approximation is possible.

In proving both these bounds, we use a subset of the family of type 2 triangles; we call the members of this sub-family *ray-sliding triangles*.

A secondary area of focus in this thesis is d-row cuts for $d \geq 3$. For d-row cuts in general, the underlying maximal lattice-free convex sets in $\mathbb{R}^d$ are not easily classified. Absent a classification, the authors of [8] show that all inequalities generated by lattice-free convex sets with at most i facets approximate the corner polyhedron within a finite factor only when $i > 2^{d-1}$. Here we take a different tact and try to prove analogues of 2-row cut results. We extend the proof techniques to obtain a constant factor approximation between two structured families of maximal lattice-free convex sets in $\mathbb{R}^d$ for $d \geq 3$.

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$\mathbb{1}$	A vector of ones of appropriate dimension.
$\mathbb{0}$	A vector of zeros of appropriate dimension.
$\mathbb{R}$	The set of Real numbers.
$\mathbb{R}_+$	The set of non-negative Real numbers.
$\mathbb{R}^n$	The space of n-dimensional vectors with Real coordinates.
$\|\|x\|\|_2$	The standard Euclidean norm of vector $x \in \mathbb{R}^n$ defined by $\|\|x\|\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$.
$\operatorname{conv} S$	The convex hull of set $S \subseteq \mathbb{R}^d$ - the smallest convex set $C \subseteq \mathbb{R}^d$ such that $S \subseteq C$.
$\mathbb{Q}^n$	The space of n-dimensional vectors with rational coordinates.
$\mathbb{Z}$	The set of integers.
$\mathbb{Z}_+$	The set of non-negative integers.
$\mathbb{Z}_{++}$	The set of positive integers.
$\mathbb{Z}^n$	The space of n-dimensional vectors with integer coordinates.
$\mathcal{S}$	The family of maximal lattice-free splits in $\mathbb{R}^2$.
$\mathcal{T}$	The family of maximal lattice-free triangles in $\mathbb{R}^2$.
$\mathcal{T}_1$	The family of type 1 triangles in $\mathbb{R}^2$.
$\mathcal{T}_2$	The family of type 2 triangles in $\mathbb{R}^2$.
$\mathcal{T}_3$	The family of type 3 triangles in $\mathbb{R}^2$.
$\mathcal{Q}$	The family of maximal lattice-free quadrilaterals in $\mathbb{R}^2$.
$\mathcal{Q}_2$	The family of maximal lattice-free parallelograms in $\mathbb{R}^2$.
$\mathcal{C}$	The family of all maximal lattice-free convex sets in $\mathbb{R}^2$.

$\hat{\mathcal{L}}$	A set of representatives for the family $\mathcal{L}$ under the equivalence relation "obtainable via unimodular transformation".
$\mathcal{F}_d$	The family of facet-separable unit-core octahedra in $\mathbb{R}^d$.
$\mathcal{P}_d$	The family of type 2 pyramids in $\mathbb{R}^d$.

Chapter 1

Introduction

A *mixed integer linear program* (MIP henceforth) is the problem of optimizing a linear objective function of the variables subject to linear constraints on the variables and the restriction on some variables to take integer values. MIPs have broad modelling power, ranging from classical NP-hard problems such as the Boolean satisfiability problem to industrial applications modelling physical constraints. For example, MIPs can model the scheduling of nurses to shifts, the operations of shipping ports, or the production plan for a factory.

Given their wide applicability, algorithms for MIPs have received considerable research. An algorithm solving a MIP is a series of steps that is given the MIP as input and outputs the assignment of values to variables optimizing the objective function. A classical approach to efficient algorithm design requires a provably fast algorithm producing the best assignment of values to the variables of the MIP. The speed of the algorithm is measured by the number of *steps* - often basic arithmetic operations - it performs. Certain classes of MIPs such as network flows admit efficient algorithms. However, general MIPs can model provably difficult problems in complexity theory and are thought not to admit efficient algorithms.

In "real world" applications, however, the speed at which a MIP needs to be solved may be more related to clocktime. The hours, minutes, or seconds a decision maker has until the solution is no longer useful is more relevant than the number of steps the algorithm performed to find the solution. In fact, often times a close to optimal solution may suffice. From this perspective an algorithm may be practical if it can produce a "good enough solution fast enough". The codes solving MIPs (a.k.a. *solvers*) marry the theoretical and practical aspects of MIPs; the underlying algorithmic theory guides practical algorithms.

A *linear program* is the problem of optimizing a linear objective function of the variables subject to linear constraints on the variables - unlike a MIP, no variables can be restricted to take integer values. For a given mixed integer program, its *linear programming (LP) relaxation* is obtained by removing the restriction that certain variables take integer values. Unlike MIPs, linear programs can be solved efficiently in theory and quickly in practice. Many techniques for solving MIPs leverage this fact.

One such technique is cut generation based on the LP relaxation. The idea is as follows. First, solve the LP relaxation of the MIP to obtain a solution x^*. If x^* also satisfies the integrality restrictions of the MIP, then it is in fact optimal for the MIP. Otherwise, find a linear inequality that is violated by x^* but holds for all points satisfying the linear constraints and integrality restrictions of the MIP (i.e. all *feasible solutions*). Add this inequality to the MIP constraints. The idea is illustrated in Figure 1.1 for the mixed integer program $(Ex1)$ in variables x_1 and x_2 given by

$$\min \quad x_1 + x_2 \tag{1.1}$$

$$\text{subject to} \quad 2x_1 + x_2 \geq 3 \tag{1.2}$$

$$-20x_1 + 25x_2 \geq -23 \tag{1.3}$$

$$-35x_1 + 15x_2 \geq -69 \tag{1.4}$$

$$x_1, x_2 \geq 0 \quad , \quad x_1 \in \mathbb{Z}. \tag{1.5}$$

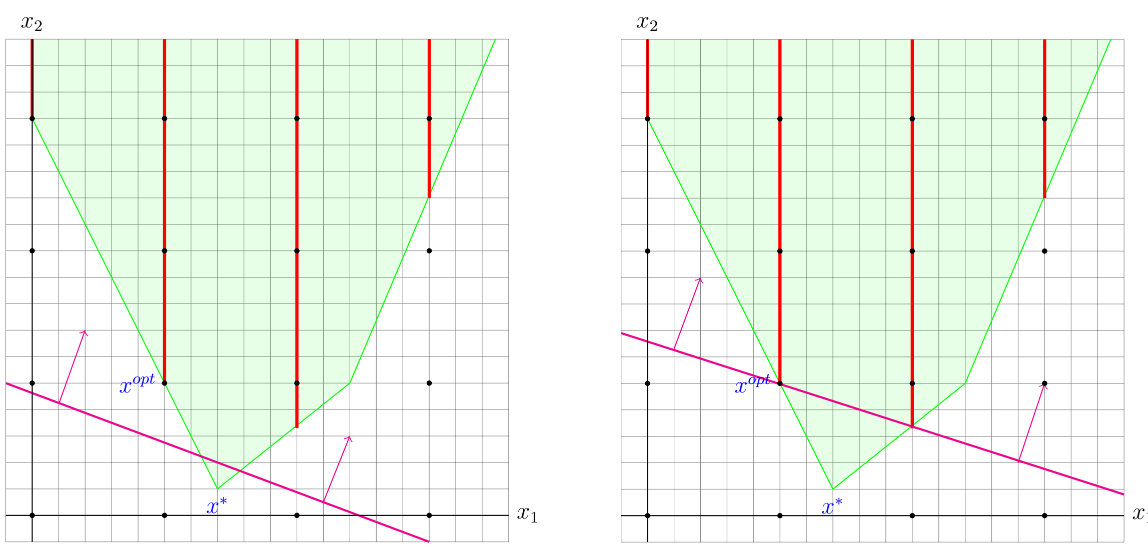


Figure 1.1: Illustration of cut generation based on LP relaxation for mixed integer program $(Ex1)$. All images drawn on grid paper within this thesis were produced using TikZ [69].

The points satisfying the linear constraints of $(Ex1)$ are shaded in green. The points also satisfying $x_1 \in \mathbb{Z}$ are marked thick red. The optimal solution of $(Ex1)$ is x^{opt} and the optimal solution of its LP relaxation is x^*. The magenta inequalities in the left and right images are both examples of linear inequalities that are violated by x^* but satisfied by all feasible solutions of the MIP. Now, intuitively, the inequality in the right image is better than the inequality in the left image - it excludes a larger portion of the shaded region. In fact, adding the right inequality to $(Ex1)$, and solving the LP relaxation of the new problem yields the optimal solution of $(Ex1)$. Of course, we're ahead of ourself in discussing which inequality is better (a.k.a. *stronger*) before discussing how to find such inequalities, or even if they always exist.

Consider an arbitrary mixed integer program and point x^* violating its constraints. A linear inequality that is violated by x^* but satisfied by all feasible solutions of the MIP is called a *cutting plane*. A *cut generation procedure* is a sequence of steps that produces a cutting plane. Often such a procedure requires x^* to be an optimal *basic* solution to the LP relaxation. Figure 1.2 illustrates a "geometric cutting plane procedure" to obtain the right inequality in Figure 1.1. Section 1.1.1 provides one specific sequence of steps and affiliated formulas that yields the inequality via an algebraic argument.

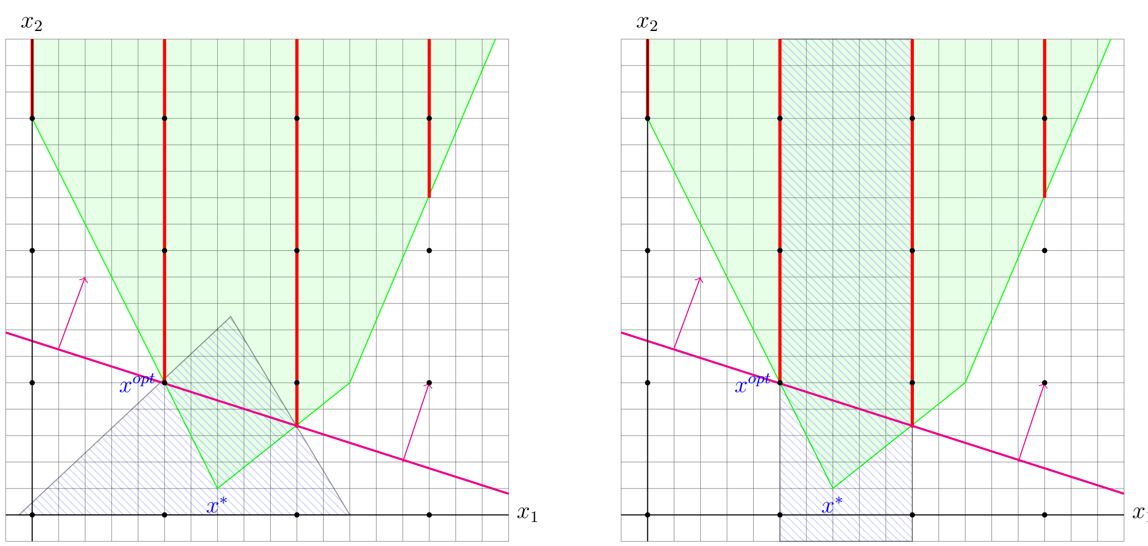


Figure 1.2: The patched blue sets identify a convex set B that does not contain any feasible points of $(Ex1)$ in its interior. Let F be the shaded green set indicating the points satisfying the linear constraints of $(Ex1)$. The magenta inequality is the unique linear inequality needed to reconvexify F after removing the interior of B.

This thesis aims to compare the strength of cutting planes that can be produced by a procedure akin to the geometric one illustrated above. The property differentiating the cutting planes we compare will be the types of convex sets used to produce them. Section 1.1, Section 1.2, and Section 1.3 provide the necessary technical preliminaries to pose these questions rigorously, placing the questions we address in a broader context. Section 1.3.2 gives an overview of past results directly related to the problem at hand while Section 1.3.3 provides the new results. Section 1.4 orients the reader to the remainder of the thesis.

Before moving on, we note the importance of cut generation procedures in practice. MIP solvers are complicated codes and part of their success comes from using many diverse techniques in concert. However, cut generation is an important subroutine. Cut generation procedures played a key role in speeding up commercially available solvers in the early 2000s; "turning off" the cutting planes feature in CPLEX lead to the worst performance hit, compared to turning off other features [22].

Throughout, we assume familiarity with the fundamentals of mixed integer linear programming, and mathematical optimization more generally. For a recent treatment of mixed integer linear programming theory, the reader is referred to [27]. Although we endeavour to define terminology as it is used, the reader is referred to [66] for standard definitions pertaining to convex sets and convex analysis.

1.1 Cut Generation Procedures

A set $P \subseteq \mathbb{R}^k$ is a *polyhedron* if it is the intersection of a finite number of halfspaces, each of the form $\{x \in \mathbb{R}^k : \alpha^T x \leq \beta\}$ for some $\alpha \in \mathbb{R}^k$ and $\beta \in \mathbb{R}$. A polyhedron is *rational* if each of the halfspaces is of the form $\{x \in \mathbb{R}^k : \alpha^T x \leq \beta\}$ where $\alpha \in \mathbb{Q}^k$ and $\beta \in \mathbb{Q}$. Polyhedron P is a *polytope* if it is bounded - i.e. if there exists $L \in \mathbb{R}_+$ such that $||x||_2 \leq L$ for all $x \in P$.

A *mixed integer linear program* (MIP) is the problem of optimizing a linear function of vector variable $x \in \mathbb{R}^k$ subject to x being a member of some polyhedron P, and some entries of x being restricted to take integer values. Throughout, we will consider MIPs with rational constraints given in a specific way.

Let $m \in \mathbb{Z}_{++}$ and $n \in \mathbb{Z}_+$ be integers. Let Q be the rational polyhedron given by

$$Q := \{x \in \mathbb{R}^{m+n} : Ax = b, x \geq \mathbb{0}\} \tag{1.6}$$

for $A \in \mathbb{Q}^{n\times(m+n)}$ a full row rank matrix and $b \in \mathbb{Q}^n$ a vector. Let $c \in \mathbb{Q}^{m+n}$ be a vector and $p \in \{1,\ldots,m+n\}$. We will consider the MIP in variables $x_1,\ldots,x_{m+n}$ described by

$$\max\{c^T x : x \in Q \cap (\mathbb{Z}^p \times \mathbb{R}^{m+n-p})\}. \tag{1.7}$$

An *optimal solution* to this optimization problem is an assignment of values to variables (i.e. $x_i := \hat{x}_i$ for $i \in \{1,\ldots,m+n\}$) that maximizes $c^T x$ over all feasible x (i.e. all $x \in \mathbb{Z}^p \times \mathbb{R}^{m+n-p}$ such that $Ax = b$ and $x \geq \mathbb{0}$). An algorithm *solving* this optimization problem takes data c, A, and b as input and either: produces an optimal solution, identifies there are no feasible solutions, or determines the problem is unbounded.

1.1.1 Cuts Through the Lens of Pure Cutting Plane Algorithms

A pure cutting plane strategy is one approach to designing an algorithm that solves any MIP given as above. First, assume that the MIP has a feasible solution and is not unbounded. Under this assumption, the strategy adheres to the following framework.

1. Solve the LP relaxation $\max\{c^T x : x \in Q\}$. Since (by assumption) the MIP has an optimal solution, so also does its LP relaxation.
2. Find an optimal *basic* solution x^* to the LP relaxation.
3. If $x^* \in \mathbb{Z}^p \times \mathbb{R}^{m+n-p}$, then return x^* as an optimal solution for the original problem.
4. Otherwise, find a linear inequality $\alpha^T z \leq \beta$ for $\alpha \in \mathbb{Q}^{m+n}, \beta \in \mathbb{Q}$ such that $\alpha^T x^* < \beta$ but $\alpha^T x \geq \beta$ for all $x \in Q \cap (\mathbb{Z}^p \times \mathbb{R}^{m+n-p})$.
5. Update $Q := Q \cap \{x : \alpha^T x \geq \beta\}$. To keep the problem in the same form add a non-negative continuous slack variable $s \geq 0$ and add the constraint $\alpha x - s = \beta$. Return to the first step.

The inequality generated in the fourth step is called a *cutting plane* or *cut*. Methods for calculating cuts are called *cut generation procedures*.

Many cut generation procedures are designed to cut off an optimal basic solution of the LP relaxation using information from the affiliated simplex tableau. A *basis* of the linear system $Ax = b$ is a set $B \subseteq \{1,\ldots,m+n\}$ with $|B| = n$ such that column sub-matrix of A restricted to the columns in B (denoted A_B) is invertible. A solution of $Ax = b$ is *basic* if it is the unique solution to $A_B x_B = b, x_N = 0$ for some basis B where $N := \{1,\ldots,m+n\}\setminus B$. If the linear program $\max\{c^T x : x \in Q\}$ has an optimal solution, then it has an optimal solution x^* that is a basic solution of $Ax = b$. The linear system $Ax = b$ can be rewritten so the basic variables x_B are expressed with respect to the non-basic variables x_N as follows:

$$x_i + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{b}_i \quad \text{for all } i \in B \tag{1.8}$$

where $\bar{b}_i \in \mathbb{Q}$ for all $i \in B$ and $\bar{a}_{ij} \in \mathbb{Q}$ for all $i \in B, j \in N$. System (1.8) is the *simplex tableau* for basis B. If x^* is an optimal basic solution corresponding to basis B then $\bar{b}_i \geq 0$ for all $i \in B$ since x^* is feasible. For example, we can rewrite optimization problem $(Ex1)$ as problem $(Ex2)$ given by

$$\begin{aligned}
\max \quad & -x_1 - x_2 \\
\text{subject to} \quad & \begin{bmatrix} -2 & -1 & 1 & 0 & 0 \\ 20 & -25 & 0 & 1 & 0 \\ 35 & -15 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3 \\ 23 \\ 69 \end{bmatrix} \\
& x_1, x_2, x_3, x_4, x_5 \geq 0 \\
& x_1 \in \mathbb{Z}.
\end{aligned}$$

The unique optimal solution to its LP relaxation is $x^* = \left(\frac{7}{5}, \frac{1}{5}, 0, 0, 23\right)$, which is the basic solution for basis $B = \{1, 2, 5\}$. The corresponding simplex tableau is

$$\begin{bmatrix} 1 & 0 & -\frac{5}{14} & \frac{1}{70} & 0 \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{35} & 0 \\ 0 & 0 & \frac{115}{14} & -\frac{13}{14} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{7}{5} \\ \frac{1}{5} \\ 23 \end{bmatrix}.$$

One of the earliest cut generation procedures was proposed by Gomory [48] and works for any pure integer program ($p = m+n$). We illustrate the procedure on problem $(Ex2)$ with the additional restrictions $x_2, x_3, x_4, x_5 \in \mathbb{Z}$. Since all variables are non-negative, we can round down the coefficients of the first row of the simplex tableau to obtain the inequality $x_1 - x_3 \leq \frac{7}{5}$. Then, since all variables take integer values, we can round down the right hand side to obtain the inequality $x_1 - x_3 \leq 1$. Substituting $x_3 = -3 + 2x_1 + x_2$ and rearranging we obtain the inequality $x_2 + x_2 \geq 2$. One can verify in Figure 1.1 that this inequality holds for all solutions to (P) with the additional restriction $x_2 \in \mathbb{Z}$. In general, the procedure selects a row $\hat{\imath} \in B$ of the simplex tableau (1.8) and outputs the cutting plane

$$x_{\hat{\imath}} + \sum_{j \in N} \lfloor \bar{a}_{\hat{\imath}j} \rfloor x_j \leq \lfloor \bar{b}_{\hat{\imath}} \rfloor. \tag{1.9}$$

This inequality is called a *Chvátal-Gomory (CG) cut*. Note that this inequality is always violated by the basic solution for basis B. If the optimal basic solution and row of the simplex tableau used to derive a cut are chosen carefully in each iteration, then the pure cutting plane strategy using CG cuts produces an optimal solution for any pure integer program in a finite number of steps [48].

For mixed integer programs ($p < m + n$), there also exist cut generation procedures that use a single row of the simplex tableau to derive a cut. For example, the *Gomory mixed integer (GMI)* cut [48] for row $\hat{\imath} \in B \cap \{1, \ldots, p\}$ with $b_{\hat{\imath}}$ fractional is

$$\sum_{j \in N \cap \{1,\ldots,p\}: f_j \leq f_0} \left(\frac{f_j}{f_0}\right) x_j + \sum_{j \in N \cap \{1,\ldots,p\}: f_j > f_0} \left(\frac{1 - f_j}{1 - f_0}\right) x_j + \sum_{j \in N \cap \{p+1,\ldots,m+n\}: \bar{a}_{\hat{\imath}j} \geq 0} \left(\frac{\bar{a}_{\hat{\imath}j}}{f_0}\right) x_j$$
$$- \sum_{j \in N \cap \{p+1,\ldots,m+n\}: \bar{a}_{\hat{\imath}j} < 0} \left(\frac{\bar{a}_{\hat{\imath}j}}{1 - f_0}\right) x_j \geq 1 \tag{1.10}$$

where $f_j := \bar{a}_{\hat{\imath}j} - \lfloor \bar{a}_{\hat{\imath}j} \rfloor$ for $j \in N$ is the fractional part of the coefficient of x_j and $f_0 := b_{\hat{\imath}} - \lfloor b_{\hat{\imath}} \rfloor$ is the fractional part of the right hand side. Note that this inequality is always violated by the basic solution for basis B. A bit of algebra is required to show that the inequality is satisfied by all feasible solutions of the underlying MIP; we point to [27, Chapter 5.3] for one derivation. Returning to $(Ex2)$ and applying this formula to the first row of the simplex tableau yields the inequality

$$\left(\frac{1}{70}\right)\left(\frac{1}{\frac{2}{5}}\right) x_4 + \left(\frac{5}{14}\right)\left(\frac{1}{1 - \frac{2}{5}}\right) x_3 \geq 1,$$

or, equivalently $50x_3 + 3x_4 \geq 84$. Substituting $x_3 = -3 + 2x_1 + x_2$ and $x_4 = 23 - 20x_1 + 25x_2$, yields the inequality $8x_1 + 25x_2 \geq 33$, which is exactly the magenta inequality in the right image of Figure 1.1.

Unlike in the pure integer case, there is no way to choose the optimal basic solution and simplex tableau row so that a pure cutting plane strategy based on GMI cuts solves every mixed integer program in a finite number of steps [29]. Obtaining a general pure cutting plane algorithm for MIPs requires more complicated cuts; see [37], [54], and [33]. The problem is more straightforward under various assumptions of boundedness: see [45], [24], [63], [64], and [13].

Note that the cutting plane strategy can be revised to handle instances where the underlying MIP is unbounded or infeasible. If the LP relaxation is infeasible in step 1, the MIP is also infeasible. If the LP relaxation is unbounded in step 1, first note the unbounded direction r and then run the procedure replacing Q with $Q \cap \{x \in \mathbb{R}^{m+n} : -L\mathbb{1} \leq x \leq L\mathbb{1}\}$ for large enough L depending on the entries of A and b (see [68, Corollary 17.1b], for example, for bounds). This new problem is necessarily bounded. If it has a feasible solution

x then the original MIP is also unbounded (using direction r) and if it is infeasible, then the original MIP is infeasible. For a discussion of the practicality of pure cutting plane algorithms and underlying numerical issues see, for example, [75].

1.1.2 The Corner Polyhedron and k-Row Cuts

A cut generation procedure produces a cutting plane separating feasible region F from some point $x^* \notin F$. Formally, it solves the following *separation problem*: given a mixed integer linear set $F := \{x \in \mathbb{Z}^p \times \mathbb{R}^{m+n-p} : Ax = b, x \geq \mathbb{0}\}$ and a point x^* that is not a member of F, find $(\alpha, \beta) \in \mathbb{Q}^{m+n} \times \mathbb{Q}$ such that $\alpha^T x^* < \beta$ and $\alpha^T x \geq \beta$ for all $x \in F$. The inequality $\alpha^T x \geq \beta$ is said to *separate* x^* from F.

Such a procedure may use any characteristic of the structure of F to derive a cut. Most common approaches, however, tend to generate cuts for a relaxation R of F. Set $R \subseteq \mathbb{R}^{m+n}$ is a *relaxation* of F if $F \subseteq R$. Suppose $x^* \notin R$ and let $(\alpha, \beta) \in \mathbb{Q}^{m+n} \times \mathbb{Q}$ be such that $\alpha^T x^* < \beta$ and $\alpha^T r \geq \beta$ for all $r \in R$. As $F \subseteq R$, it follows that $\alpha x \geq \beta$ separates F from x^*.

A common choice for relaxation R is a corner polyhedron. Let B be a basis for the system $Ax = b$ for which the corresponding solution is non-negative. The corresponding simplex tableau is of the form $x_B + \bar{A}x_N = \bar{b}$, where $\bar{A} \in \mathbb{Q}^{n \times m}$, $\bar{b} \in \mathbb{Q}^n_+$, and x_B and x_N index the basic and non-basic variables, respectively. The affiliated *corner polyhedron* is obtained by dropping the non-negativity constraints on the basic variables and then taking the *convex hull*. A set $C \subseteq \mathbb{R}^d$ is *convex* if $\lambda x + (1 - \lambda)y \in C$ whenever $x, y \in C$ and $\lambda \in [0, 1]$. The *convex hull* of a set $S \subseteq \mathbb{R}^d$, denoted $\operatorname{conv} S$, is the smallest convex set $C \subseteq \mathbb{R}^d$ such that $S \subseteq C$. See [66, Section 2] for standard facts regarding convex sets and convex hulls. So the corner polyhedron for basis B is given by

$$\operatorname{corner}(B) := \operatorname{conv}\{x \in (\mathbb{Z}^p \times \mathbb{R}^{m+n-p}) : x_B + \bar{A}x_N = \bar{b}, x_N \geq \mathbb{0}\}. \tag{1.11}$$

By construction $F \subseteq \operatorname{corner}(B)$ and therefore $\operatorname{corner}(B)$ is a relaxation of the feasible region F. As the name suggests, the corner polyhedron is indeed a polyhedron (the intersection of finitely many halfspaces). In fact, it follows from Meyer's Theorem [62, Theorem 3.9] that $\operatorname{corner}(B)$ is a rational polyhedron. Note that any continuous basic variable appears in exactly one equality constraint and therefore can be removed without loss of generality. Thus, we may assume $B \subseteq \{1, \ldots, p\}$ in $\operatorname{corner}(B)$ and the further relaxations that follow.

Another common choice for relaxation R is a k-row relaxation of the corner polyhedron. Let $\mathcal{I} \subseteq B$. An $|\mathcal{I}|$*-row relaxation* of the corner polyhedron is obtained by dropping all equality constraints except those including basic variable x_i for all $i \in \mathcal{I}$. That is,

$$\text{corner}(B, \mathcal{I}) := \text{conv}\{x \in (\mathbb{Z}^p \times \mathbb{R}^{m+n-p}) : x_i + \bar{a}_i^T x_N = \bar{b}_i \ \forall i \in \mathcal{I},\ x_N \geq \mathbb{0}\} \tag{1.12}$$

where $\bar{a}_1^T, \ldots, \bar{a}_n^T$ are the rows of $\bar{A}$. Note that $\text{corner}(B) \subseteq \text{corner}(B, \mathcal{I})$. The set $\text{corner}(B, \mathcal{I})$ can be further relaxed by removing the integrality restrictions on the non-basic variables. If so, we will denote the corresponding relaxation by $\text{corner}_R(B, \mathcal{I})$. It is given by

$$\text{corner}_R(B, \mathcal{I}) := \text{conv}\{x \in (\mathbb{Z}^B \times \mathbb{R}^N) : x_i + \bar{a}_i^T x_N = \bar{b}_i \ \forall i \in \mathcal{I},\ x_N \geq \mathbb{0}\}. \tag{1.13}$$

We refer to such a relaxation as a *continuous k-row relaxation*. Note that $\text{corner}(B, \mathcal{I}) \subseteq \text{corner}_R(B, \mathcal{I})$ and in general the inclusion is strict. In this thesis, we are mostly concerned with continuous relaxations.[1]

We refer to cuts that can be derived from a k-row relaxation of the corner polyhedron as *k-row cuts*. Any cut that can be derived from a continuous k-row relaxation is (trivially) a k-row cut; the converse does not hold. However, one common approach to obtaining k-row cuts is to modify the cuts derived from the continuous k-row relaxation by *lifting* the coefficients of the integer non-basic variables (i.e. to somehow incorporate the integrality information ex-post). See [27, Chapter 6.3.4] and the references therein for formal definitions; we don't require them herein.

The CG cuts and GMI cuts introduced in Section 1.1.1 are both examples of 1-row cuts; split cuts [29] are another classical example of a 1-row cut. Although 1-row cuts are most commonly used in practice, the example in [29] is of a MIP instance where a pure cutting plane strategy based only on split cuts (or GMI cuts since all GMI cuts are split cuts [63]) cannot terminate in a finite numbers of steps. See [56] for a generalization of this example. In [4], Andersen et al. show that a 2-row cut generated from the first simplex tableau for this instance leads to immediate termination of a pure cutting plane strategy. Note that in this example all the non-basic variables of the first simplex tableau are continuous and thus the 2-row relaxation and continuous 2-row relaxation coincide. The 2-row cut derived for this example is an *intersection cut*; this class of cuts will be introduced in detail in the next section.

[1] In other expositions, a k-row relaxation may also be obtained by first aggregating rows of the simplex tableau into k equations and treating the aggregation of the original variables as a new variable.

1.2 Comparing Families of Intersection Cuts

Recall that general MIPs can model provably difficult problems in complexity theory and are thought not to admit *efficient* algorithms. An algorithm solving a MIP of the form (1.7) is *efficient* if it runs in time polynomial in the size of inputs c, A, and b. For a discussion of algorithm runtime in the context of MIPs see [50] or [27, Chapter 1.3]. Pure cutting plane strategies are not proven to be theoretically efficient; they also tend to be less successful in practice than the state of the art. The code solving MIPs in practice (a.k.a. *solvers*) provide practical algorithms, making use of cut generation procedures, but not at the exclusion of other techniques.

One can think[2] of a solver as starting with the given instance and iteratively using information at hand to choose a strategy/subroutine to run. The goal is to identify a better solution, prove the incumbent one is good enough, or learn more information about the problem structure. Under this lens, the running time of the solver depends on how well it chooses its next strategy and how long the underlying subroutine takes. Two choices a solver makes when it comes to running a cut generation procedure are:

1. when it should run the procedure (as opposed to apply a different technique), and
2. which procedure to use (i.e. which cut - or often cuts - to generate).

Here we are mostly interested in the latter choice. We study different families of intersection cuts that can be derived from continuous k-row relaxations of the corner polyhedron and compare them using the gauge (or "blow up") measure.

1.2.1 Intersection Cuts

One way to generate cuts for continuous k-row relaxations of the corner polyhedron is by calculating the *intersection cuts* for lattice-free convex sets. A systematic framework for studying corner polyhedra (absent the MIP context) is provided in [49] and [30], and used extensively in related work. We will adopt this framework (and affiliated notation and conventions) to define intersection cuts. We consider the mixed integer linear set

$$\mathcal{F} := \left\{ (x, s) \in \mathbb{Z}^d \times \mathbb{R}^k_+ : x = f + \sum_{j=1}^{k} r^j s_j \right\} \tag{1.14}$$

where $d, k \in \mathbb{Z}_{++}$, $f \in \mathbb{Q}^d \setminus \mathbb{Z}^d$ and $r^j \in \mathbb{Q}^d \setminus \{\mathbb{0}\}$ for all $j \in \{1, \ldots, k\}$. Note that $corner_R(B, \mathcal{I})$ defined in (1.13) is a mixed linear set of the form $\operatorname{conv} \mathcal{F}$; take $r^j := (-\bar{a}_{ij})_{i \in \mathcal{I}}$

[2] For exposition purposes, but consult [57] or [1] for a more nuanced discussion.

for $j \in N$ and $f = (\bar{b}_i)_{i \in \mathcal{I}}$. Let Γ denote the ordered set $(r^1, \dots, r^k)$. Define $R(f;\Gamma) \subseteq \mathbb{R}^k_+$ by

$$R(f;\Gamma) := \operatorname{conv}\left\{ s \in \mathbb{R}^k_+ \ : f + \sum_{j=1}^{k} r^j s_j \in \mathbb{Z}^d \right\}. \tag{1.15}$$

Remark 1.2.1. *If there exists $s \in \mathbb{R}^k$ such that $f + \sum_{j=1}^k r^j s_j \in \mathbb{Z}^d$, then $R(f;\Gamma)$ is a full-dimensional[3] rational polyhedron. Otherwise $R(f;\Gamma) = \emptyset$.*

Proof. It is straightforward to show that

$$R(f;\Gamma) = \operatorname{proj}_s(\operatorname{conv} \mathcal{F}) := \{s \in \mathbb{R}^k_+ : \exists x \in \mathbb{R}^d \text{ s.t. } (x,s) \in \operatorname{conv} \mathcal{F}\}.$$

As $f, r^1, \dots, r^k$ have rational data, it follows from Meyer's Theorem [62, Theorem 3.9] that $\operatorname{conv} \mathcal{F}$ is a rational polyhedron. The projection of a rational polyhedron is again a rational polyhedron and therefore $R(f;\Gamma)$ is a rational polyhedron. Note that the rationality assumption here is required; if $r^1 := (r^1_x, r^1_y)$ is such that r^1_x / r^1_y is irrational, $r^1 \notin \{\sum_{j=2}^k \lambda_j r^j : \lambda_j \geq 0 \ \forall j\}$, and the ray $\{f + \lambda r^1 : \lambda \geq 0\}$ contains no integral point, then $R(f;\Gamma)$ is not polyhedral.

If there does not exist $s \in \mathbb{R}^k$ such that $f + \sum_{j=1}^k r^j s_j \in \mathbb{Z}^d$ then clearly $R(f;\Gamma) = \emptyset$.

Suppose $\hat{s} \in \mathbb{R}^k$ is such that $f + \sum_{j=1}^k r^j \hat{s}_j \in \mathbb{Z}^d$. Since $r^1, \dots, r^k \in \mathbb{Q}^d$ there exists $D \in \mathbb{Z}_+$ such that $Dr^1, \dots, Dr^k \in \mathbb{Z}^d$. Let $K^- = \{j \in \{1, \dots, k\} \text{ s.t. } \hat{s}_j < 0\}$ and let $\hat{s}_{min} = |\min_{j \in K^-} \hat{s}_j|$. If e^i denotes the i-th standard basis vector then $s' := \hat{s} + D \sum_{j \in K^-} \lceil \frac{\hat{s}_{min}}{D} \rceil e^j \geq \mathbb{O}$ and $f + \sum_{j=1}^k r^j s'_j \in \mathbb{Z}^d$. Therefore $R(f;\Gamma) \neq \emptyset$. Moreover $\{s', s' + De^1, \dots, s' + De^k\}$ are a set of affinely independent points in $R(f;\Gamma)$ and therefore $R(f;\Gamma)$ is full-dimensional. $\square$

Throughout we will assume that $R(f;\Gamma) \neq \emptyset$. An inequality $\sum_{j=1}^k \alpha_j s_j \geq \beta$ is *valid* for $R(f;\Gamma)$ if it is satisfied by every $\bar{s} \in R(f;\Gamma)$. A valid inequality for $R(f;\Gamma)$ is *trivial* if it is of the form $s_j \geq 0$ for all $j \in \{1, \dots, k\}$, or implied by such inequalities.

Remark 1.2.2. *Every non-trivial valid inequality for $R(f;\Gamma)$ is of the form $\sum_{j=1}^k \gamma_j s_j \geq 1$ where $\gamma \geq \mathbb{O}$.*

[3] The *dimension* of a set $S \subseteq \mathbb{R}^d$ is the dimension of its *affine hull*, the smallest affine set containing S. See [66, Section 1] for additional information regarding dimension, affine spaces, and affine independence.

Proof. Let $\sum_{j=1}^{k} \alpha_j s_j \geq \beta$ be a valid inequality for $R(f;\Gamma)$. By way of contradiction suppose that $\alpha_{\hat{\jmath}} < 0$ for $\hat{\jmath} \in \{1,\ldots,k\}$. Let $\hat{s} \in R(f;\Gamma)$ and let $D \in \mathbb{Z}_+$ be such that $Dr^{\hat{\jmath}} \in \mathbb{Z}^d$. Then $s'(\ell) := \hat{s} + \ell D e^{\hat{\jmath}} \in R(f;\Gamma)$ for all $\ell \in \mathbb{Z}_+$ and so for ℓ large enough $\sum_{j=1}^{k} \alpha_j [s'(\ell)]_j < \beta$. Thus $\alpha_j \geq 0$ for all $j \in \{1,\ldots,k\}$. Since the inequality is non-trivial, $\beta > 0$. Multiplying by $\frac{1}{\beta}$ we conclude the inequality can be written in the form $\sum_{j=1}^{k} \gamma_j s_j \geq 1$ for $\gamma \geq \mathbb{0}$. $\square$

Now, every linear inequality separating $\operatorname{conv}(\mathcal{F})$ from $(f,\mathbb{0})$ can be expressed with respect to the variables s_j for $j \in \{1,\ldots,k\}$ by substituting the equality constraints. So, to generate cuts separating $(f,\mathbb{0})$ from $\operatorname{conv}(\mathcal{F})$ we can generate cuts separating $\mathbb{0}$ from $R(f;\Gamma)$ - that is to say, we can find non-trivial valid inequalities for $R(f;\Gamma)$.

One way to obtain such valid inequalities is using lattice-free convex sets. Set $B \subseteq \mathbb{R}^d$ is *lattice-free* if it contains no integral points in its interior - that is, if $(\operatorname{int} B) \cap \mathbb{Z}^d = \emptyset$ where $\operatorname{int} B$ denotes the interior of B. Let $B \subseteq \mathbb{R}^d$ be a closed lattice-free convex set with f in its interior. Let $\psi_{f;B} : \mathbb{R}^d \to \mathbb{R}$ be defined by

$$\psi_{f;B}(r) := \left\{ \begin{array}{ll} \frac{1}{\lambda} & \text{if } \lambda \in \mathbb{R}_+ \text{ is such that } f + \lambda r \text{ on the boundary of } B \\ 0 & \text{if no such } \lambda \text{ exists} \end{array} \right\}.$$

This function is well defined because B is closed and convex. It is the *gauge* function [52, Definition 1.2.4] of $B - f$ and measures the reciprocal of the distance from f to the boundary of B along r. A function $h : \mathbb{R}^d \to \mathbb{R}$ is:

- *non-negative* if $h(r) \geq 0$ for all $r \in \mathbb{R}^d$,
- *positively homogeneous* if $h(\lambda r) = \lambda h(r)$ for all $\lambda \geq 0$ and $r \in \mathbb{R}^d$,
- *subadditive* if $h(r^1) + h(r^2) \geq h(r^1 + r^2)$ for all $r^1, r^2 \in \mathbb{R}^d$, and
- *convex* if $\lambda h(r^1) + (1-\lambda)h(r^2) \geq h(\lambda r^1 + (1-\lambda)r^2)$ for all $\lambda \in [0,1]$ and $r^1, r^2 \in \mathbb{R}^d$.

Remark 1.2.3. *[52, Theorem 1.2.5] Let $B \subseteq \mathbb{R}^d$ be a closed lattice-free convex set with f in its interior. Then $\psi_{f;B}$ is non-negative, positively homogeneous, subadditive, and (therefore) convex.*

The *intersection cut* generated by B for given $(f;\Gamma)$ is

$$\sum_{j=1}^{k} \psi_{f;B}(r^j) s_j \geq 1. \tag{1.16}$$

This inequality is valid for $R(f;\Gamma)$.

Theorem 1.2.4 ([12])**.** *Let $f \in \mathbb{Q}^d \setminus \mathbb{Z}^d$ and let Γ be the ordered set $(r^1, \ldots, r^k)$ with $r^j \in \mathbb{Q}^d \setminus \{\mathbb{0}\}$ for all $j \in \{1, \ldots, k\}$. Let B be a lattice-free convex set with f in its interior. The intersection cut generated by B for given $(f; \Gamma)$ is valid for $R(f; \Gamma)$.*

Proof. For the first proof see [12]. This original proof also handles intersection cuts generated by sets that are not lattice-free. The proof here is akin to the one in [27].

Let $\sum_{j=1}^k \psi_j s_j \geq 1$ be the intersection cut generated by B. Let $\bar{s} \in \mathbb{R}^k_+$ be such that $f + \sum_{j=1}^k r^j \bar{s}_j = \bar{x} \in \mathbb{Z}^d$. It suffices to show $\bar{s}$ satisfies $\sum_{j=1}^k \psi_j \bar{s}_j \geq 1$ since $R(f; \Gamma)$ is the convex hull of all such $\bar{s}$. By way of contradiction, suppose $\sum_{j=1}^k \psi_j \bar{s}_j < 1$. Let $J_{\neq 0} = \{j \in \{1, \ldots, k\} : \psi_j \neq 0\}$ and $J_{=0} = \{j \in \{1, \ldots, k\} : \psi_j = 0\}$. We can express $\bar{x}$ as

$$
\begin{aligned}
\bar{x} &= f + \sum_{j \in J_{\neq 0}} \bar{s}_j r^j + \sum_{j \in J_{=0}} \bar{s}_j r^j \\
&= f + \sum_{j \in J_{\neq 0}} \psi_j \bar{s}_j \left(\frac{r^j}{\psi_j} \right) + \sum_{j \in J_{=0}} \bar{s}_j r^j \\
&= \left(1 - \sum_{j=1}^k \psi_j \bar{s}_j \right) f + \sum_{j \in J_{\neq 0}} \psi_j \bar{s}_j \left(f + \frac{r^j}{\psi_j} \right) + \sum_{j \in J_{=0}} \bar{s}_j r^j
\end{aligned}
$$

Since $\sum_{j=1}^k \psi_j \bar{s}_j < 1$ and $\psi_j \bar{s}_j \geq 0$ for all $j \in \{1, \ldots, k\}$, it follows that $\bar{x} = p + q$ where p is in the interior of the convex hull of f and $\{f + \frac{r^j}{\psi_j} : j \in J_{\neq 0}\}$ and q is in the cone generated by $\{r^j : j \in J_{=0}\}$. By the definition of ψ, each of $f + \frac{r^j}{\psi_j}$ for $j \in J_{\neq 0}$ is on the boundary of B and each of r^j for $j \in J_{=0}$ is in the recession cone of B. It follows that $\bar{x}$ is in the interior of B. This is a contradiction because the interior of B contains no integral points. It follows that $\bar{s}$ satisfies $\sum_{j=1}^k \psi_j \bar{s}_j \geq 1$, completing the proof. □

A valid inequality $\sum_{j=1}^k \gamma_j s_j \geq 1$ for $\gamma \geq \mathbb{0}$ is a *minimal constraint* of $R(f; \Gamma)$ if for every $\gamma' \leq \gamma$ that is distinct from γ there exists $\bar{s} \in R(f; \Gamma)$ such that $\sum_{j=1}^k \gamma'_j \bar{s}_j < 1$. From a cut generation point of view, minimal constraints are appealing because they guarantee there is no constraint achieving the same trade-offs; to obtain a smaller or *stronger* coefficient for one variable, the coefficient for another must be increased.[4]

Lattice-free convex $B \subseteq \mathbb{R}^d$ is a *maximal lattice-free convex set* if there does not exist lattice-free convex B' distinct from B such that $B \subseteq B'$. Every lattice-free convex set is

[4]Being strong from this point of view and being useful in practice may not always agree. See, for example, [28], [2] and [39].

contained in a maximal lattice-free convex set; for a proof (not necessarily the first) see [17]. Not only are the intersection cuts generated by maximal lattice-free convex sets valid, but all minimal constraints of $R(f;\Gamma)$ can be obtained in this way.

Theorem 1.2.5 ([25, Theorem 1])**.** *Let $f \in \mathbb{Q}^d \setminus \mathbb{Z}^d$ and let Γ be the ordered set $(r^1, \ldots, r^k)$ with $r^j \in \mathbb{Q}^d \setminus \{\mathbb{O}\}$ for all $j \in \{1, \ldots, k\}$. If $R(f;\Gamma) \neq \emptyset$, then every minimal constraint of $R(f;\Gamma)$ is the intersection cut of $(f;\Gamma)$ for some maximal lattice-free convex set B with f in its interior.*

Connections to the Semi-Infinite Relaxation

Although the semi-infinite relaxation is not required directly in this thesis, we discuss it briefly for completeness. The *continuous semi-infinite relaxation* R_f [49] [26] is given by the set of vectors $(x, s) \in \mathbb{R}^d \times \mathbb{R}^{\mathbb{Q}^d}$ satisfying

$$x = f + \sum_{r \in \mathbb{Q}^d : s_r \neq 0} r s_r \tag{1.17}$$

$$x \in \mathbb{Z}^d \tag{1.18}$$

$$s \geq \mathbb{O} \text{ and has a finite support.} \tag{1.19}$$

An inequality $\sum_{r \in \mathbb{Q}^d : s_r \neq 0} \gamma_r s_r \geq \beta$ is *valid* for R_f if it satisfied by every vector in R_f. All valid inequalities that separate $(f, \mathbb{O})$ from R_f are of the form $\sum_{r \in \mathbb{Q}^d : s_r \neq 0} \psi(r) s_r \geq 1$ where $\psi : \mathbb{Q}^d \to \mathbb{R} \cup \{+\infty\}$. A valid inequality of this form is *minimal* if there is no valid inequality $\sum_{r \in \mathbb{Q}^d : s_r \neq 0} \psi'(r) s_r \geq 1$ such that $\psi' \leq \psi$ and $\psi'(r) \neq \psi(r)$ for some $r \in \mathbb{Q}^d$.

Function $\psi : \mathbb{Q}^d \to \mathbb{R} \cup \{+\infty\}$ is a *valid* or *minimal* function, respectively, if the corresponding inequality is valid or minimal, respectively. Every minimal valid function is non-negative, piece-wise linear, positively homogeneous, subadditive, and convex [23]. A full description of $R(f;\Gamma)$ can be obtained from the valid inequalities for R_f by restriction; if $\sum_{r \in \mathbb{Q}^d : s_r \neq 0} \psi(r) s_r \geq 1$ is valid for R_f then $\sum_{j=1}^k \psi(r^j) s_{r^j} \geq 1$ is valid for $R(f;\Gamma)$. A sort of converse of this statement is also true; see [17]. For more details on these important connections see [26].

1.2.2 Maximal Lattice-Free Convex Sets

Since intersection cuts for maximal lattice-free convex sets provide the minimal inequalities for $R(f;\Gamma)$, the structural properties of such sets is a natural topic of interest for a couple reasons. Firstly, one must identify a maximal lattice-free convex set B to generate such an intersection cut. Secondly, to calculate the intersection cut $\sum_{j=1}^{k} \psi_{f;B}(r^j)s_j \geq 1$ generated by B, one must evaluate the gauge $\psi_{f;B} : \mathbb{R}^d \to \mathbb{R}$.

A theorem of Lovász is the typical launching point into the structure of maximal lattice-free convex sets. Before stating it, we require some definitions. Let $P \subseteq \mathbb{R}^d$ be a polyhedron. The set $F \subseteq P$ is a *face* of P if $F = \{x \in P : c^T x = k\}$ for some $c \in \mathbb{R}^d$ and $k \in \mathbb{R}$ such that $P \subseteq \{x \in \mathbb{R}^d : c^T x \leq k\}$. Note that $\emptyset$ and P itself are both faces of P. A face of dimension 0 is called a *vertex* of P, while a face of dimension $\dim(P) - 1$ is called a *facet* of P. The *recession cone* of P is given by $\operatorname{rec}(P) := \{r \in \mathbb{R}^d : x + \lambda r \in P \ \forall x \in P,\ \forall \lambda \geq 0\}$ and the *lineality space* of P is given by $\operatorname{lin}(P) := \{r \in \mathbb{R}^d : x + \lambda r \in P \ \forall x \in P,\ \forall \lambda \in \mathbb{R}\}$.

Theorem 1.2.6 ([60]; see [17] or [6] for a proof)**.** *Full dimensional $C \subseteq \mathbb{R}^d$ is a maximal lattice-free convex set if and only if C is a polyhedron with an integral point in the relative interior of each facet and $\operatorname{int} C \cap \mathbb{Z}^d = \emptyset$. Furthermore, if full-dimensional $C \subseteq \mathbb{R}^d$ is a maximal lattice-free convex set, then its recession cone and lineality space coincide.*

An immediate consequence of this theorem is that $\psi_{f;B}$ is a piecewise-linear function whose number of pieces is equal to the number of facets, say t, of B. This is easy to see. Since $f \in \operatorname{int} B$, we can express B as $B = \{x \in \mathbb{R}^d : d_i x \leq 1 + d_i f \ \forall i \in \{1, \ldots, t\}\}$ where d_i is a row vector in $\mathbb{R}^d$ for $i \in \{1, \ldots, t\}$. Then

$$\psi_{f;B}(r) = \max_{i \in \{1,\ldots,t\}} d_i r.$$

This expression for the gauge of a full-dimensional polyhedron is well-known; we mention [25, Corollary 2] where it is used to show every minimal valid function for the continuous semi-infinite relaxation R_f is continuous and piecewise-linear.

Pursuant to the above discussion, one natural way to distinguish families of maximal lattice-free convex sets is by number of facets. Now, if $x^1, x^2 \in \mathbb{Z}^d$ are such that x_i^1 and x_i^2 are congruent modulo two for all $i \in \{1, \ldots, d\}$, then $\frac{1}{2}(x^1 + x^2) \in \mathbb{Z}^d$. Hence two distinct facets F_1 and F_2 of a lattice-free convex set cannot contain points $x_1 \in \operatorname{relint}(F_1)$ and $x_2 \in \operatorname{relint}(F_2)$ such that x^1 and x^2 are congruent modulo two. Applying the Pigeonhole Principle, it follows that any maximal lattice-free polyhedron has at most 2^d facets.

Remark 1.2.7 ([67] [21] [42])**.** *If full dimensional $C \subseteq \mathbb{R}^d$ is a maximal lattice-free convex set then C has at most 2^d facets.*

In two dimensions, all maximal lattice-free convex sets have two, three or four facets. The following theorem of Lovász summarizes the classification nicely.

Theorem 1.2.8 ([60]). *In the plane, a full dimensional maximal lattice-free convex set is one of the following.*

1. *A* split $\{x \in \mathbb{R}^2 : c \leq ax_1 + bx_2 \leq c+1\}$ *for* $a, b, c \in \mathbb{Z}$*, and* a, b *coprime.*
2. *A triangle with at least one integral point in the relative interior of each of its edges.*
3. *A quadrilateral containing exactly four integral points with exactly one of them in the relative interior of each of its edges. The four integral points are vertices of a parallelogram of area one.*

In three dimensions, all maximal lattice-free convex sets have at least two and at most eight facets. However, no analogue of Theorem 1.2.8 is known; see [17] for some partial results on this topic. The work in [71], [10], [9], and [7] is also relevant; however, the authors focus on the closely related problem of characterizing maximal lattice-free *lattice polytopes* - those polytopes having integral vertices. In d dimensions for $d \geq 3$, it seems to be hard to give a nice classification of families of maximal lattice-free convex sets according to number of facets and the position of the integral points in those facets.

Note that when we say $Q \subseteq \mathbb{R}^2$ is a *maximal lattice-free quadrilateral* we mean that Q is a polygon with four facets (i.e. is a quadrilateral) and Q is a maximal lattice-free convex set. It is not always true that a lattice-free polyhedron C_1 with k facets is contained in a maximal lattice-free polyhedron C_2 again with k facets. For example $Q := \operatorname{conv}\{(0,0)^T, (0,2)^T, (1,0)^T, (1,2)^T\}$ is a lattice-free quadrilateral strictly contained in the split $\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1\}$. So Q is not a maximal lattice-free quadrilateral. Any set containing Q must contain at least six integral points but, per Theorem 1.2.8, all maximal lattice-free quadrilaterals contain exactly four integral points.

Note that there exist lattice-free quadrilaterals that are not contained in any maximal lattice-free quadrilateral (or as an element of a limiting family thereof such as a split). However, every lattice-free triangle is contained in a maximal lattice-free triangle or an element of a limiting family thereof. Later we see that splits appear as the limiting set of an infinite family of lattice-free triangles.

1.2.3 The Gauge Measure

Let $\mathcal{L}$ be a family of lattice-free convex sets in $\mathbb{R}^d$. One might think of $\mathcal{L}$ as the family of all maximal lattice-free convex sets in $\mathbb{R}^d$ with exactly t facets for some $t \in \{1, \ldots, 2^d\}$, but the following definitions apply for any well-defined family. The intersection cut generated by $L \in \mathcal{L}$ with $f \in \operatorname{int} L$ is a valid inequality for $R(f;\Gamma)$. The $\mathcal{L}$-*closure* for $(f;\Gamma)$, denoted

$\mathcal{L}(f;\Gamma)$, is the set of $s \in \mathbb{R}^k_+$ satisfying all such intersection cuts. That is, for $\bar{s} \in \mathbb{R}^k_+$, we have $\bar{s} \in \mathcal{L}(f;\Gamma)$ if and only if $\sum_{j=1}^k \psi_{f;B}(r^j)\bar{s}_j \geq 1$ for all $B \in \mathcal{L}$ with $f \in \operatorname{int} B$.

Note that the $\mathcal{L}$-closure is a closed convex set but it is not necessarily a polyhedron. The most well-known example for which the $\mathcal{L}$-closure is a polyhedron is when $\mathcal{L}$ is the family of maximal lattice-free convex sets in $\mathbb{R}^d$ with exactly two facets (i.e. $\mathcal{L}$ is the split closure). There exist other families for which the $\mathcal{L}$-closure is known to be a polyhedron; we'll encounter some later. We point to [3] and [5] for some sufficient conditions on the family $\mathcal{L}$ to conclude the $\mathcal{L}$-closure is a polyhedron.

Given $C \subseteq \mathbb{R}^k_+$, we call C *upper comprehensive* if for all $x \in C$, $x' \geq x$ implies $x' \in C$.

Remark 1.2.9. *If $\mathcal{L}$ is a family of lattice-free convex sets, then $\mathcal{L}(f;\Gamma)$ is upper comprehensive.*

Proof. Let $s \in \mathbb{R}^k_+$ be such that $s \in \mathcal{L}(f;\Gamma)$. Let $s' \in \mathbb{R}^k_+$ be such that $s' \geq s$. Since $\psi_{f;B}(r^j) \geq 0$ for all $j \in \{1, \ldots, k\}$ and $s' \geq s \geq \mathbb{0}$ we have

$$\sum_{j=1}^k \psi_{f;B}(r^j)s'_j \geq \sum_{j=1}^k \psi_{f;B}(r^j)s_j \geq 1$$

for all $B \in \mathcal{L}$ with $f \in \operatorname{int} B$. Thus $s' \in \mathcal{L}(f;\Gamma)$. $\square$

Theorem 1.2.5 says that if $\mathcal{C}$ is the set of all maximal lattice-free convex sets in $\mathbb{R}^d$, then $\mathcal{C}(f;\Gamma) = R(f;\Gamma)$. It is clear that $R(f;\Gamma) \subseteq \mathcal{L}(f;\Gamma)$ whenever $\mathcal{L}$ is a family of lattice-free convex sets; that is, $\mathcal{L}(f;\Gamma)$ is a relaxation of $R(f;\Gamma)$. To compare the intersection cuts generated by two families of lattice-free convex sets $\mathcal{L}_1$ and $\mathcal{L}_2$, we will compare the sets $\mathcal{L}_1(f;\Gamma)$ and $\mathcal{L}_2(f;\Gamma)$. Inherently, this comparison considers *all cuts* from one family against *all cuts* from another.

We compare arbitrary relaxations of $R(f;\Gamma)$ - either to each other or to $R(f;\Gamma)$ itself - using the "blow-up" measure for upper comprehensive sets. This measure was first used to compare cutting plane strength in the integer programming context in [47]. Suppose C_1 and C_2 are convex upper comprehensive sets. Define $\alpha C_2 := \{\alpha c_2 : c_2 \in C_2\}$. Since C_2 is upper comprehensive, $C_2 \subseteq \alpha C_2$ for all $\alpha \leq 1$ and a small α corresponds to inflating C_2 towards the origin. Define

$$\theta[C_1, C_2] := \inf_{\alpha \leq 1} \left\{ \frac{1}{\alpha} : C_1 \subseteq \alpha C_2 \right\}; \tag{1.20}$$

if no such α exists, define $\theta[C_1, C_2] := +\infty$.[5] We think of $\theta[C_1, C_2]$ as indicating how much C_2 must be inflated to contain C_1. If $C_1 \subseteq C_2$, then $\theta[C_1, C_2] = 1$ (i.e. no inflation is required).

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be families of lattice-free convex sets in $\mathbb{R}^d$. We compare the intersection cuts generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ on a "worst case" basis. We compare $\mathcal{L}_1$ to $\mathcal{L}_2$ by looking at the largest amount (over all $(f; \Gamma)$) by which $\mathcal{L}_2(f; \Gamma)$ must be inflated to contain $\mathcal{L}_1(f; \Gamma)$. Accordingly, we define

$$\rho[\mathcal{L}_1, \mathcal{L}_2] := \sup \left\{ \theta[\mathcal{L}_1(f; \Gamma), \mathcal{L}_2(f; \Gamma)] : \begin{array}{l} k \in \mathbb{Z}_{++}, \\ f \in \mathbb{Q}^d \setminus \mathbb{Z}^d, \\ \Gamma = (r^1, \ldots r^k), \text{ and} \\ r^j \in \mathbb{Q}^d \setminus \{\mathbb{O}\} \ \forall j \in \{1, \ldots, k\} \end{array} \right\}. \tag{1.21}$$

Note that $\rho[\mathcal{L}_1, \mathcal{L}_2] \neq \rho[\mathcal{L}_2, \mathcal{L}_1]$. To keep this straight, it is a useful shortcut to think that $\rho[\#_1, \#_2]$ indicates how well family $\#_1$ approximates family $\#_2$; a smaller number indicates a better approximation. By definition $\theta[\mathcal{L}_1(f; \Gamma), \mathcal{L}_2(f; \Gamma)] \geq 1$ and therefore $\rho[\#_1, \#_2] \geq 1$. This means that the measure does not capture whether family $\#_1$ is better than $\#_2$, only whether it is at least as good.

On Which Cut Generation Procedure to Use

As mentioned at the start of Section 1.2, one reason to study intersection cut closures is to provide theoretical insight into which cuts might be useful to generate in practice. For a discussion of the various theoretical frameworks under which cut quality is evaluated, as well as some of the challenges in using theoretically good cuts in practice, see [39]. Some of the gaps between theoretical measures and practical advice are particularly relevant to the setup used in this thesis. The choice to compare all cuts from one family to all cuts from another has some drawbacks. The choice of a worst case measure also has limits; probabilistic analysis may be a fitting theoretical alternative, see [19], [36], and [51]. The choice to consider a continuous relaxation (i.e. drop the integrality information for the non-basic variables) also has limits; in practice, coefficients for integer variables can be strengthened via lifting.

[5] Note that we require $\alpha \leq 1$, which may differ from definitions used elsewhere. We make this choice because we only want to understand the extent to which C_1 is weaker than C_2. In past work this distinction was often irrelevant because a relaxation C_1 of C_2 was compared to C_2 itself (i.e. $C_2 \subseteq C_1$ held trivially). Here we compare different relaxations of $R(f; \Gamma)$ to each other (as opposed to directly to $R(f; \Gamma)$ itself) so the two sets involved may be incomparable with respect to set inclusion.

Computational experiments are also a key piece in comparing the usefulness of cut generation procedures in practice; see [43], [38], [58], [59], and [14]. The results in this thesis provide some insight on d-row cuts that might have good trade-off (in terms of complexity to generate vs. strength of cut) based on the relative strength of the underlying closures. Although we compare different classes of d-row cuts to each other for fixed d, such results can be used to identify candidate strong families of d-row cuts to compare to ℓ-row cuts for $d \neq \ell$, as in [14] and [38]. In Section 6.6, we return to this topic and discuss the theoretical bounds proved in this thesis in the context of past computational work.

1.3 Comparing Families of 2-Row Intersection Cuts

We now turn our attention to the $d = 2$ case. That is, we are interested in cuts separating $\mathbb{O}$ from $R(f;\Gamma)$ where

$$R(f;\Gamma) := \operatorname{conv}\left\{s \in \mathbb{R}^k_+ : f + \sum_{j=1}^{k} r^j s_j \in \mathbb{Z}^2\right\} \tag{1.22}$$

for *fractional point* $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$, *ray directions* $r^1, \ldots, r^k \in \mathbb{Q}^2 \setminus \{\mathbb{O}\}$ and Γ as the ordered set $(r^1, \ldots, r^k)$. The notation $f, r^1, \ldots, r^k$, and Γ will be defined as such in the remainder of this section.

Per Theorem 1.2.5 and Theorem 1.2.8, every minimal constraint of $R(f;\Gamma)$ is the intersection cut of $(f;\Gamma)$ for some maximal lattice-free split, triangle or quadrilateral with f in its interior. A valid inequality $\sum_{j=1}^k \gamma_j s_j \geq 1$ for $\gamma \geq \mathbb{O}$ is a *facet-defining inequality* of $R(f;\Gamma)$ if the dimension of $R(f;\Gamma) \cap \{s \in \mathbb{R}^k : \sum_{j=1}^k \gamma_j s_j = 1\}$ is $k-1$. A valid inequality $\sum_{j=1}^k \gamma_j s_j \geq 1$ for $\gamma \geq \mathbb{O}$ is *rational* if $\gamma_j \in \mathbb{Q}$ for all $j \in \{1, \ldots, k\}$. Since f and $\{r^j : j \in \{1, \ldots, k\}\}$ are rational, every non-trivial facet-defining inequality for $R(f;\Gamma)$ is rational and minimal and therefore is the intersection cut for a maximal lattice-free convex set.

Given Theorem 1.2.8, it follows that every facet-defining inequality of $R(f;\Gamma)$ in $\mathbb{R}^2$ is an intersection cut generated by a maximal lattice-free split, triangle, or quadrilateral. This characterization was first proved in [4]. Although we present it above as a consequence of later results, the first proof came by studying the structure of the set

$$B_\psi := \{x \in \mathbb{R}^2 : x = f + \sum_{j=1}^{k} r^j s_j \text{ for } s \in \mathbb{R}^k_+ \text{ such that } \sum_{j=1}^{k} \psi_j s_j \leq 1\}. \tag{1.23}$$

where $\sum_{j=1}^{k} \psi_j s_j \geq 1$ is some facet defining inequality. The interior of B_ψ gives a two-dimensional representation of the points affected by adding the inequality $\sum_{j=1}^{k} \psi_j s_j \geq 1$ to the linear relaxation of $R(f;\Gamma)$. Exactly which splits, triangles, and quadrilaterals actually define facets of $R(f;\Gamma)$ is provided by [30].

Theorem 1.3.1. *Let $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ and let Γ be the ordered set $(r^1, \ldots, r^k)$ with $r^j \in \mathbb{Q}^2 \setminus \{\mathbb{O}\}$ for all $j \in \{1, \ldots, k\}$. The facets of $R(f;\Gamma)$ are*

1. *all intersection cuts generated by a split S where the unbounded direction of S is r^j for some $j \in \{1, \ldots, k\}$, or where S satisfies the "ray condition";*
2. *all intersection cuts generated by a maximal lattice-free triangle T where each vertex of T is on one of the rays $\{f + \lambda r^j : \lambda \geq 0\}$ for some $j \in \{1, \ldots, k\}$, or where T satisfies the "ray condition"; and*
3. *all intersection cuts generated by a maximal lattice-free quadrilateral Q where each vertex of Q is on one of the rays $\{f + \lambda r^j : \lambda \geq 0\}$ for some $j \in \{1, \ldots, k\}$, or where Q satisfies the "ratio condition".*

Part of this characterization was needed in the first paper comparing intersection cuts for different families of lattice-free convex sets in two dimensions [15]. Since we will not require this characterization in our techniques, we decline to define the ray and ratio conditions.

1.3.1 Families of Maximal Lattice-Free Convex Sets in $\mathbb{R}^2$

Let $\mathcal{C}$ denote the family of all maximal lattice-free convex sets in $\mathbb{R}^2$. Theorem 1.2.8 partitions $\mathcal{C}$ into three subfamilies: splits, maximal lattice-free triangles, and maximal lattice-free quadrilaterals. In this section, we define further subfamilies, identify "representatives" for each family, and also provide the affiliated notation.

A function $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is an *affine unimodular transformation* if there exist $c \in \mathbb{Z}^d$ and *unimodular* $M \in \mathbb{Z}^{d \times d}$ (i.e. $\det M = \pm 1$) such that ϕ is given by $\phi(x) = Mx + c$. We say $\mathcal{L}$ is *closed under unimodular transformation* if whenever $B \in \mathcal{L}$ so are all B' obtained from B by an affine unimodular transformation. Each family of maximal lattice-free convex sets $\mathcal{L}$ defined here is closed under unimodular transformation. Since "obtainable via affine unimodular transformation" is an equivalence relation, $\mathcal{L}$ can be partitioned into equivalence classes. A *set of representatives* for $\mathcal{L}$ is a subset of $\mathcal{L}$ containing at least one lattice-free convex set for each equivalence class. We will define for each family $\mathcal{L}$ a *normalized* member of the family such that the set of all normalized members, denoted $\hat{\mathcal{L}}$, is a set of representatives for $\mathcal{L}$.

For the most part, the families and normalized versions thereof introduced here appear in [15] and [11], as well as [41] and [32] where the terminology "standard form" is used in the place of "normalized". The List of Symbols (page xvi) provides a handy reference for the notation introduced herein.

Splits

Recall that $S \subseteq \mathbb{R}^2$ is a split if there exist $a, b, c \in \mathbb{Z}$ with a, b coprime such that $S = \{x \in \mathbb{R}^2 : c \leq ax_1 + bx_2 \leq c + 1\}$. Let $\mathcal{S}$ denote the family of splits in $\mathbb{R}^2$. By our choice of definition, all splits are maximal lattice-free convex sets. The split $VS := \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1\}$ will be referred to as the *vertical split* and the split $HS := \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_2 \leq 1\}$ will be referred to as the *horizontal split*. We take VS to be the (unique) *normalized* split.

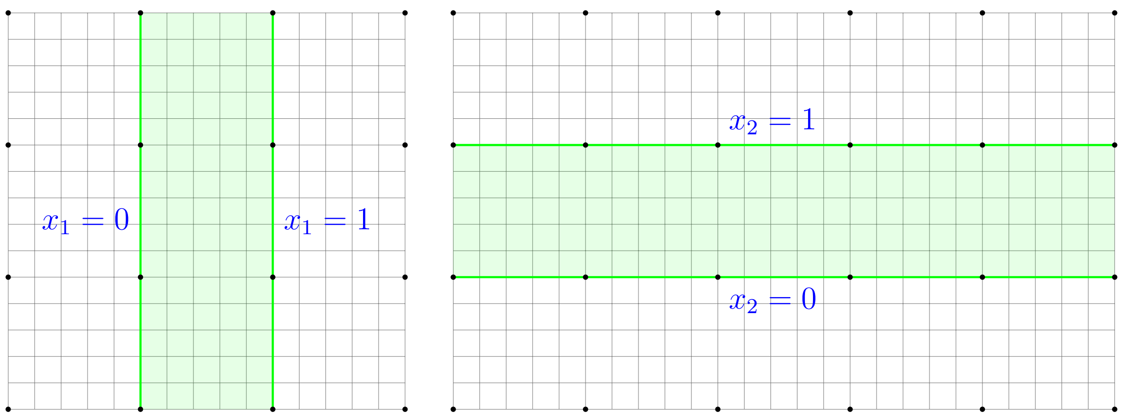


Figure 1.3: Vertical split (left) and horizontal split (right).

Remark 1.3.2. *Let $S \in \mathcal{S}$ be a split. Then there exists an affine unimodular transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(S) = VS$ where $VS = \{(x_1, x_2)^T \in \mathbb{R}^2 : 0 \leq x_1 \leq 1\}$.*

Given $(f; \Gamma)$, the *split closure* is

$$\mathcal{S}(f;\Gamma) := \left\{ s \in \mathbb{R}^k_+ : \sum_{j=1}^{k} \psi_{f;S}(r^j) s_j \geq 1 \text{ for all } S \in \mathcal{S} \text{ with } f \in \operatorname{int} S \right\}. \tag{1.24}$$

The split closure is a polyhedron [29].

Triangles

Let $\mathcal{T}$ denote the family of maximal lattice-free triangles in $\mathbb{R}^2$. This family can be partitioned into three subfamilies [40, Proposition 1]. We say that triangle $T \in \mathcal{T}$ is either a

1. *type 1 triangle* if T has integral vertices and exactly one integral point in the relative interior of each edge; or
2. *type 2 triangle* if T has at least one fractional vertex v, exactly one integral point in the relative interior of the edges incident to v and at least two integral points on the edge opposite v; or
3. *type 3 triangle* if T has exactly three integral points on its boundary - one in the relative interior of each edge.

Let $\mathcal{T}_1$ denote the family of type 1 triangles in $\mathbb{R}^2$. A type 1 triangle $T \in \mathcal{T}_1$ is *normalized* if its vertices are $(0,0)^T$, $(2,0)^T$ and $(0,2)^T$. Let $\mathcal{T}_2$ denote the family of type 2 triangles in $\mathbb{R}^2$. A type 2 triangle $T \in \mathcal{T}_2$ is *normalized* if one of its edges contains $(0,0)^T$ and $(0,1)^T$ and the other two edges contain in their relative interior $(1,1)^T$ and $(1,0)^T$ respectively. Let $\mathcal{T}_3$ denote the family of type 3 triangles in $\mathbb{R}^2$. A type 3 triangle $T \in \mathcal{T}_3$ is *normalized* if each of $(0,0)^T$, $(1,0)^T$ and $(0,1)^T$ is in the relative interior of a different edge of the triangle and if as well the line defining the edge of T containing $(1,0)^T$ separates $(0,0)^T$ and $(1,1)^T$. A triangle $T \in \mathcal{T}$ is *normalized* if it is a normalized type 1, type 2, or type 3 triangle.

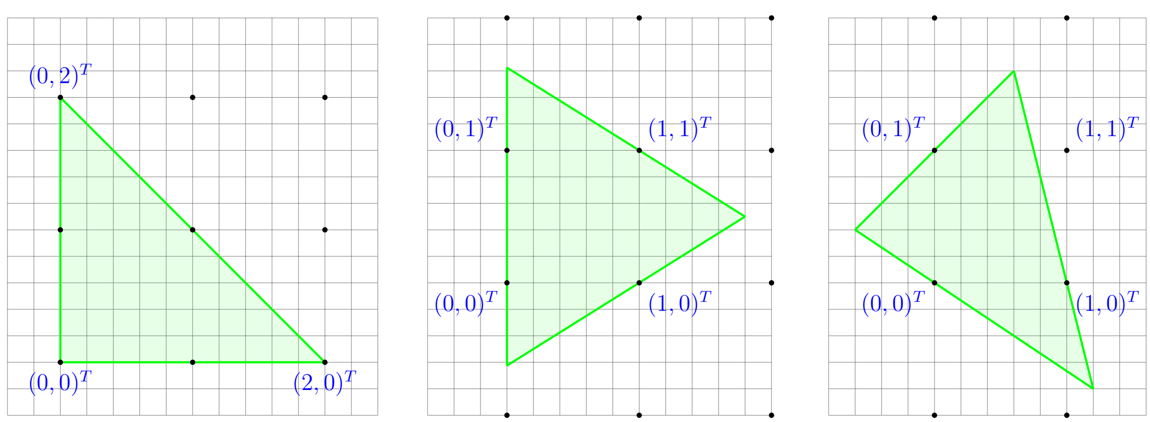


Figure 1.4: The unique normalized type 1 triangle (left), and examples of normalized type 2 (center) and type 3 (right) triangles.

Remark 1.3.3. *Let triangle $T \in \mathcal{T}$ be a type i triangle for $i \in \{1,2,3\}$. Then, there exists an affine unimodular transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(T)$ is a normalized type i triangle.*

Given $(f;\Gamma)$, the *type i triangle closure* for $i \in \{1,2,3\}$ is

$$\mathcal{T}_i(f;\Gamma) := \left\{ s \in \mathbb{R}^k_+ : \sum_{j=1}^{k} \psi_{f;T}(r^j) s_j \geq 1 \text{ for all } T \in \mathcal{T}_i \text{ with } f \in \operatorname{int} T \right\}. \tag{1.25}$$

Given $(f;\Gamma)$, the *triangle closure* is

$$\mathcal{T}(f;\Gamma) := \left\{ s \in \mathbb{R}^k_+ : \sum_{j=1}^{k} \psi_{f;T}(r^j) s_j \geq 1 \text{ for all } T \in \mathcal{T} \text{ with } f \in \operatorname{int} T \right\}. \tag{1.26}$$

Note that $\mathcal{T}(f;\Gamma) = \mathcal{T}_1(f;\Gamma) \cap \mathcal{T}_2(f;\Gamma) \cap \mathcal{T}_3(f;\Gamma)$. The triangle closure is a polyhedron [20]. The type 1 triangle closure is a polyhedron; this follows directly from Remark 1.3.3 and [5, Corollary 1.4]. It is open as to whether the type 2 and type 3 triangle closures are polyhedra.

Quadrilaterals

Let $\mathcal{Q}$ denote the family of maximal lattice-free quadrilaterals in $\mathbb{R}^2$. A quadrilateral $Q \in \mathcal{Q}$ is *normalized* if each of the points $(0,0)^T$, $(0,1)^T$, $(1,0)^T$, and $(1,1)^T$ is in the relative interior of a different edge of the quadrilateral.

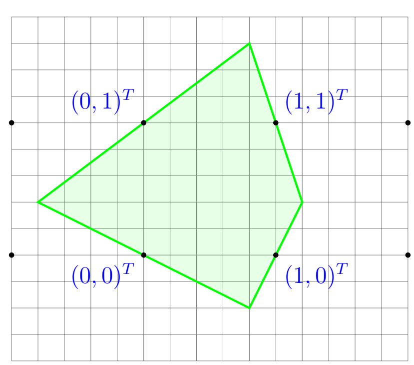


Figure 1.5: Example of a normalized quadrilateral.

Remark 1.3.4. *For every maximal lattice-free quadrilateral $Q \in \mathcal{Q}$, there exists an affine unimodular transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(Q)$ is a normalized quadrilateral.*

Proof. We prove this remark since we require it within this thesis. The proofs for Remark 1.3.2 and Remark 1.3.3 are similar.

Let Q be a maximal lattice-free quadrilateral. By Theorem 1.2.8, there exist $\ell_1, \ell_2, \ell_3, \ell_4 \in \mathbb{Z}^2$ each contained in a unique edge of Q such that $\operatorname{conv}\{\ell_1, \ell_2, \ell_3, \ell_4\}$ is a parallelogram P of area one. We may assume $\ell_1, \ell_2, \ell_3, \ell_4$ are labelled so that they occur on the boundary of P in that order. Let $U_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the unimodular transformation given by $U_1(x) = Ix - \ell_1$. Let $U_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the unimodular transformation given by $U_2(x) = M^{-1}x$ where M is the matrix with columns $\ell_2 - \ell_1$ and $\ell_4 - \ell_1$. The entries of M are integer and the absolute value of the determinant of M is the area of P, which is one. Hence M^{-1} has determinant ± 1 and integer entries and therefore U_2 is unimodular. Then $U : \mathbb{R}^2 \to \mathbb{R}^2$ given by $U(x) = U_2(U_1(x)) = M^{-1}(x - \ell_1)$ is an affine unimodular transformation. We will show the image of Q under U is a normalized quadrilateral.

By construction, U maps $\operatorname{conv}\{\ell_1, \ell_2, \ell_3, \ell_4\}$ to $[0,1]^2$. To verify $U(Q)$ is a quadrilateral: note that U maps each of the vertices of Q to distinct points, U maps edges of Q to edges of $U(Q)$ and the interior of Q to the interior of $U(Q)$. To verify $U(Q)$ is normalized, the last thing to check is that $U(Q)$ is lattice-free. Suppose to the contrary that $v \in \mathbb{Z}^2$ is an interior point of $U(Q)$. Since U is unimodular, U^{-1} is unimodular and therefore it preserves the integer lattice. So $U^{-1}v$ would be an integer point in the interior of Q, which would contradict the fact Q is lattice-free. Hence $U(Q)$ is lattice-free, completing the proof. $\square$

Given $(f;\Gamma)$, the *quadrilateral closure* is

$$\mathcal{Q}(f;\Gamma) := \left\{ s \in \mathbb{R}^k_+ : \sum_{j=1}^{k} \psi_{f;Q}(r^j) s_j \geq 1 \text{ for all } Q \in \mathcal{Q} \text{ with } f \in \operatorname{int} Q \right\}. \tag{1.27}$$

The quadrilateral closure is a polyhedron [35].

Throughout, we will refer to an intersection cut generated by a split, maximal lattice-free triangle, type i triangle, or maximal lattice-free quadrilateral, respectively, as a *split cut*, *triangle cut*, *type i triangle cut*, or *quadrilateral cut*, respectively.

1.3.2 Previously Established Bounds and Techniques

We would like to know $\rho[\mathcal{L}_1, \mathcal{L}_2]$ for all pairs of families $(\mathcal{L}_1, \mathcal{L}_2)$ where each family is one of $\{\mathcal{S}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}, \mathcal{Q}\}$. Here we review the best bounds on such $\rho[\mathcal{L}_1, \mathcal{L}_2]$ as calculated first in [15] and improved subsequently in [11]. These bounds will be referred to as "previously established bounds" to distinguish them from any bounds appearing in this thesis.

Each of the following subsections outlines a technique to calculate exactly or bound $\rho[\mathcal{L}_1, \mathcal{L}_2]$ for some pairs of families $(\mathcal{L}_1, \mathcal{L}_2)$. Along the way we keep a "running table" of bounds. At the end of a subsection we display the just discussed bounds in regular typeface in the table, and the already introduced values in faded typeface. The final subsection summarizes the discussed bounds and implications thereof, and uses them to bound $\rho[\mathcal{L}_1, \mathcal{C}]$ for $\mathcal{L}_1 \in \{\mathcal{S}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}, \mathcal{Q}\}$.

Using Set Inclusion

When one family is a subfamily of the other, a ρ value of one follows from the definitions. We provide a proof as a warm-up to using the definitions.

Remark 1.3.5. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are families of lattice-free convex sets in $\mathbb{R}^d$ and $\mathcal{L}_1 \subseteq \mathcal{L}_2$ then $\rho[\mathcal{L}_2, \mathcal{L}_1] = 1$.*

Proof. Consider arbitrary $k \in \mathbb{Z}_{++}$, $f \in \mathbb{Q}^d \setminus \mathbb{Z}^d$, and $\Gamma = (r^1, \dots, r^k)$ where $r^j \in \mathbb{Q}^d \setminus \{\mathbb{O}\}$ for all $j \in \{1, \dots, k\}$. Let $s \in \mathcal{L}_2(f;\Gamma)$; by definition, this means $\sum_{j=1}^k \psi_{f;B}(r^j)s_j \geq 1$ for every $B \in \mathcal{L}_2$ with $f \in \operatorname{int} B$. In particular, $\sum_{j=1}^k \psi_{f;B}(r^j)s_j \geq 1$ for every $B \in \mathcal{L}_1$ with $f \in \operatorname{int} B$. So, $s \in \mathcal{L}_1(f;\Gamma)$ and thus $\mathcal{L}_2(f;\Gamma) \subseteq \mathcal{L}_1(f;\Gamma)$. By the definition of θ, this gives $\theta[\mathcal{L}_2(f;\Gamma), \mathcal{L}_1(f;\Gamma)] = 1$. The result follows from the definition of ρ. $\square$

$\rho[\mathcal{L}_1, \mathcal{L}_2]$	$\mathcal{S}$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}$	$\mathcal{Q}$
$\mathcal{S}$	-					
$\mathcal{T}_1$		-				
$\mathcal{T}_2$			-			
$\mathcal{T}_3$				-		
$\mathcal{T}$		1	1	1	-	
$\mathcal{Q}$						-

Table 1.1: Values of $\rho[\mathcal{L}_1, \mathcal{L}_2] = \rho[\text{row set}, \text{column set}]$ determined using set inclusion.

Using ϵ-Relaxations

In fact, exact set containment is not required in Remark 1.3.5. A ρ value of one also follows when members of one family can be approximated "arbitrarily well" by members of the other. The presentation here unifies the analysis in [15] and [11] and requires a strengthened version of [11, Proposition 2.1]. We first state a proposition required to prove the strengthened result.

Proposition 1.3.6. *Let $k \in \mathbb{Z}_{++}$, $f \in \mathbb{Q}^d \backslash \mathbb{Z}^d$, and $\Gamma = (r^1, \ldots, r^k)$ where $r^j \in \mathbb{Q}^d \backslash \{\mathbb{O}\}$ for all $j \in \{1, \ldots, k\}$. Let B be a lattice-free convex set with f in its interior. Suppose $\bar{s} \in \mathbb{R}^k_+$ violates the intersection cut generated by B. Then there exists $M = M(\bar{s}, r^1, \ldots, r^k)$ such that $\bar{s}$ violates the intersection cut generated by $\bar{B} := B \cap \{x \in \mathbb{R}^d : ||x - f||_2 \leq M\}$.*

Proof. Let $\delta := 1 - \sum_{i=1}^k \psi_{f;B}(r^i)\bar{s}_i > 0$ be the amount by which $\bar{s}$ violates the constraint generated by B. We choose

$$M := \frac{2k \left(\max_{j \in \{1,\ldots,k\}} ||r^j||_2\right) \left(\max_{j \in \{1,\ldots,k\}} \bar{s}_j\right)}{\delta}$$

and $\bar{B} := B \cap \{x \in \mathbb{R}^d : ||x - f||_2 \leq M\}$. Let $S := \{i \in \{1, \ldots, k\} : \psi_{f;B}(r^i) = \psi_{f;\bar{B}}(r^i)\}$ and $D := \{1, \ldots, k\} \setminus S$. Now for all $i \in D$ we have $\psi_{f;\bar{B}}(r^i) = \frac{||r^i||_2}{M} \geq \psi_{f;B}(r^i) \geq 0$. Thus

$$\begin{aligned}
\sum_{i=1}^k \psi_{f;\bar{B}}(r^i)\bar{s}_i &= \sum_{i=1}^k \psi_{f;B}(r^i)\bar{s}_i + \sum_{i \in D} [\psi_{f;\bar{B}}(r^i) - \psi_{f;B}(r^i)]\bar{s}_i \\
&\leq \sum_{i=1}^k \psi_{f;B}(r^i)\bar{s}_i + \sum_{i \in D} \frac{||r^i||_2}{M}\bar{s}_i \\
&\leq \sum_{i=1}^k \psi_{f;B}(r^i)\bar{s}_i + \frac{\delta}{2k}\left(\sum_{i \in D} \frac{||r^i||_2 \bar{s}_i}{\left(\max_{j \in \{1,\ldots,k\}} ||r^j||_2\right)\left(\max_{j \in \{1,\ldots,k\}} \bar{s}_j\right)}\right) \\
&\leq \sum_{i=1}^k \psi_{f;B}(r^i)\bar{s}_i + \frac{\delta}{2} \\
&= 1 - \delta + \frac{\delta}{2} \\
&< 1.
\end{aligned}$$

and so $\bar{s}$ violates the constraint generated by $\bar{B}$. The result follows. □

Now we return to specifying what it means for members of one family to approximate members of another "arbitrary well". For $B \subseteq \mathbb{R}^d$ and $\epsilon > 0$, the *ϵ-relaxation* of B is defined by

$$\text{relax}(B; \epsilon) := \{x \in \mathbb{R}^d : ||x - \bar{x}||_2 \leq \epsilon \text{ for some } \bar{x} \in B\}. \tag{1.28}$$

Proposition 1.3.7. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be families of lattice-free convex sets in $\mathbb{R}^d$. Let $k \in \mathbb{Z}_{++}$, $f \in \mathbb{Q}^d \setminus \mathbb{Z}^d$, and $\Gamma = (r^1, \ldots, r^k)$ where $r^j \in \mathbb{Q}^d \setminus \{\mathbb{0}\}$ for all $j \in \{1, \ldots, k\}$. Suppose that for every $M, \epsilon > 0$ and every $B_1 \in \mathcal{L}_1$ with $f \in \operatorname{int} B_1$ there exists $B_2 \in \mathcal{L}_2$ such that $B_1 \cap \{x \in \mathbb{R}^d : ||x - f||_2 \leq M\} \subseteq relax(B_2; \epsilon)$. Then the inclusion $\mathcal{L}_2(f; \Gamma) \subseteq \mathcal{L}_1(f; \Gamma)$ holds, and (therefore) $\rho[\mathcal{L}_2, \mathcal{L}_1] = 1$.*

Proof. The result follows from applying Proposition 1.3.6 at the beginning of the proof of [11, Proposition 2.1]. □

It suffices to verify Proposition 1.3.7 for an arbitrary normalized member of family $\mathcal{L}_1$ because the families of sets considered are closed under unimodular transformation. Figure 1.7 shows the proposition holds when $\mathcal{L}_1 = \mathcal{S}$ and $\mathcal{L}_2 \in \{\mathcal{T}_2, \mathcal{T}_3, \mathcal{Q}\}$. The left image illustrates that a vertical split can be approximated by a normalized type 2 triangle with vertex v as close to $(1, \frac{1}{2})^T$ as is necessary and opposite edge $\{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 = 0\}$. The right image illustrates that a vertical split can be approximated by a normalized quadrilateral with two opposite vertices as close, and equally close, to $\left(0, \frac{1}{2}\right)^T$ and $\left(1, \frac{1}{2}\right)^T$ as necessary. Figure 1.6 gives an analogous "proof by picture" when $\mathcal{L}_1 = \mathcal{T}_1$ and $\mathcal{L}_2 \in \{\mathcal{T}_2, \mathcal{T}_3, \mathcal{Q}\}$. Figure 1.8 gives an analogous "proof by picture when $\mathcal{L}_1 = \mathcal{T}_2$, and $\mathcal{L}_2 \in \{\mathcal{T}_3, \mathcal{Q}\}$. The shaded set represents a normalized member of $\mathcal{L}_1$ and the dashed polygons provide the subfamily of $\mathcal{L}_2$ from which to choose B_2 for given ϵ and M.

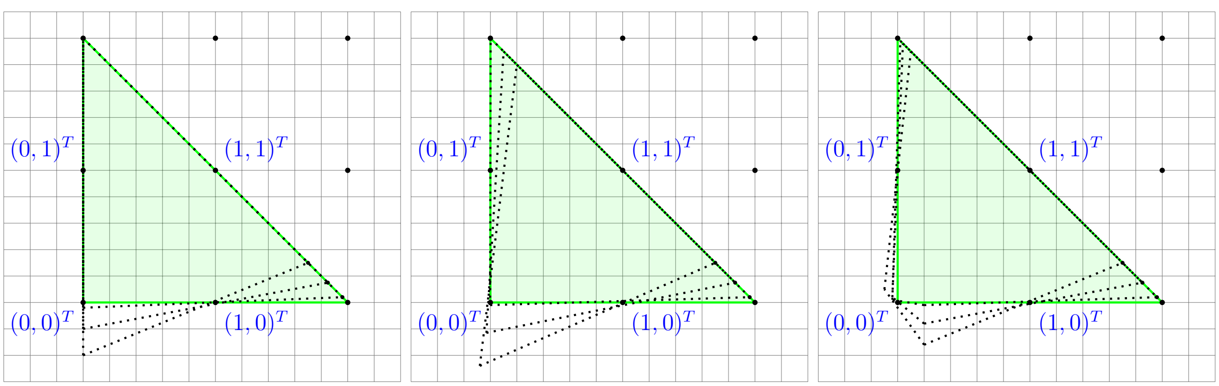


Figure 1.6: Approximate type 1 triangle by type 2 triangle (left), type 3 triangle (center), and quadrilateral (right). These constructions appear in [11, Theorem 1.3].

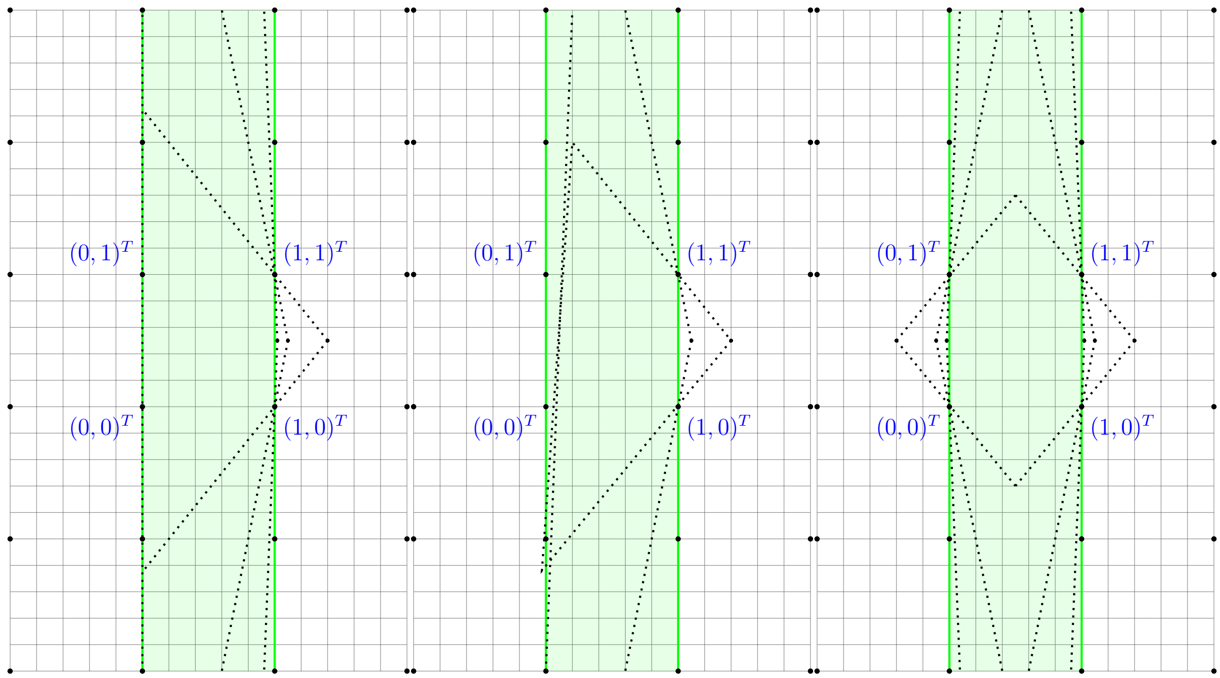


Figure 1.7: Approximate split by type 2 triangle (left), type 3 triangle (center), and quadrilateral (right). These constructions appear in [15, Theorem 1.4].

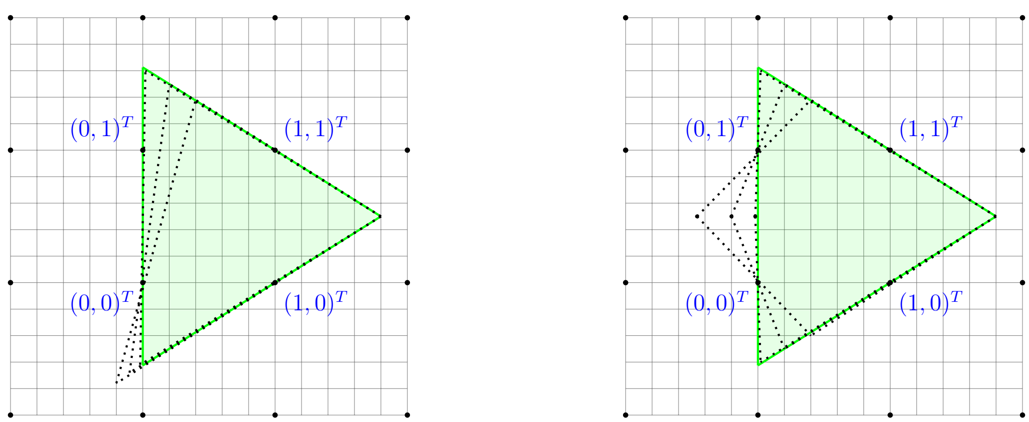


Figure 1.8: Approximate type 2 triangle by type 3 triangle (left) and quadrilateral (right). These constructions appear in [11, Theorem 1.4].

$\rho[\mathcal{L}_1, \mathcal{L}_2]$	$\mathcal{S}$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}$	$\mathcal{Q}$
$\mathcal{S}$	-					
$\mathcal{T}_1$		-				
$\mathcal{T}_2$	1	1	-			
$\mathcal{T}_3$	1	1	1	-		
$\mathcal{T}$	1	1	1	1		
$\mathcal{Q}$	1	1	1			-

Table 1.2: Values of $\rho[\mathcal{L}_1, \mathcal{L}_2] = \rho[\text{row set}, \text{column set}]$ determined using ϵ-relaxations.

Upper Bounds Using a Representative Cut

Bounds on $\rho[\mathcal{L}_1, \mathcal{L}_2]$ can be calculated by optimizing the inequality generated by an arbitrary member of $\mathcal{L}_2$ subject to all inequalities generated by members of $\mathcal{L}_1$. The following three theorems provide lower bounds on the optimal value of the relevant optimization problems. We explain the resulting upper bounds on $\rho[\mathcal{L}_1, \mathcal{L}_2]$ afterwards.

Theorem 1.3.8 (Follows from proof of [15, Theorem 1.6])**.** *Let T be a normalized type 1 triangle. Let $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ be in the interior of T. Define ray directions $r^1, r^2, r^3 \in \mathbb{Q}^2 \setminus \{\mathbb{O}\}$ so that the vertices of T are $\{f + r^i : i \in \{1, 2, 3\}\}$. Hence, for $\Gamma := (r^1, r^2, r^3)$ the intersection cut generated by T for $(f; \Gamma)$ is $\sum_{j=1}^{3} \psi_{f;T}(r^j)s_j = s_1 + s_2 + s_3 \geq 1$. Then*

$$\inf\{s_1 + s_2 + s_3 : s \in \mathcal{S}(f; \Gamma)\} \geq \frac{1}{2}.$$

Proof Sketch. Consider the relaxation (P') of the given optimization problem obtained by replacing the constraint $s \in \mathcal{S}(f; \Gamma)$ with the constraint $s \geq \mathbb{O}$ and two or three intersection cuts generated by splits. The chosen splits are illustrated in Figure 1.9; use splits *VS*, *HS*, and S_3 for which f is in the interior of the split. Then (P') is a linear program in non-negative variables s_1, s_2, s_3 with two or three constraints of the form $\psi^T s \geq 0$ for $\psi \geq 0$. The coefficients for each constraint can be calculated explicitly by computing the corresponding gauge functions. Note that these coefficients are a function of f. Lower-bounding the optimal value of the objective function as function of f and then minimizing the lower bound over all f yields the bound of $\frac{1}{2}$. □

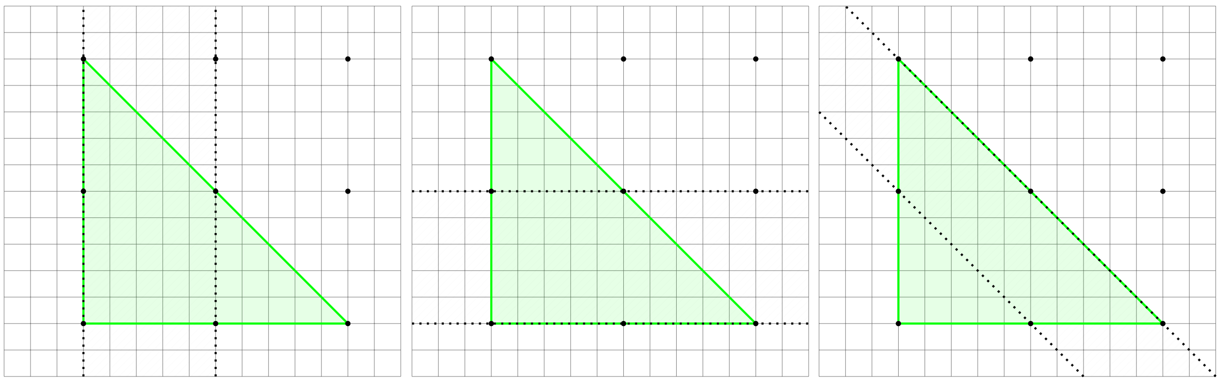


Figure 1.9: The splits VS (left), HS (center), and S_3 (right) with dotted facets, imposed on the shaded normalized type 1 triangle.

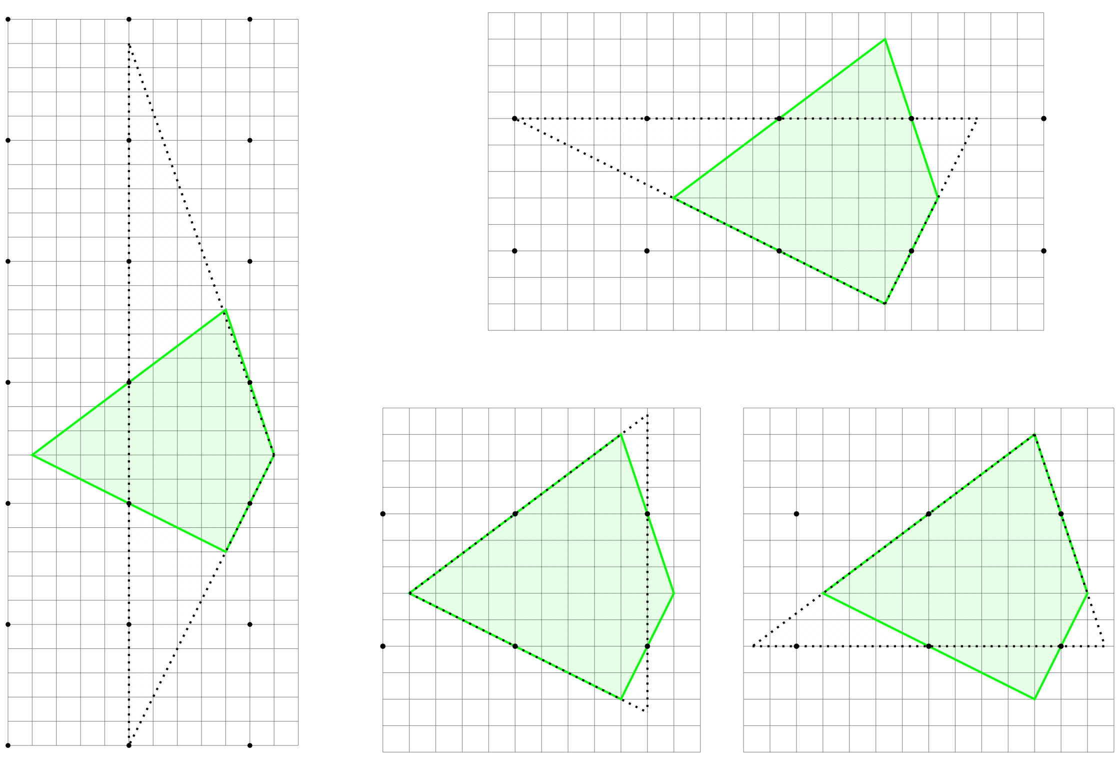


Figure 1.10: The fixed triangles F_1 (left), F_2 (top right), F_3 (bottom middle), F_4 (bottom right) with dotted facets, imposed on the shaded normalized quadrilateral.

Theorem 1.3.9 (Follows from proof of [11, Theorem 1.7], Weaker Bound of $\frac{1}{2}$ given by [15, Theorem 1.7])**.** *Let Q be a normalized quadrilateral with rational vertices. Let $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ be in the interior of Q. Define ray directions $r^1, r^2, r^3, r^4 \in \mathbb{Q}^2 \setminus \{\mathbb{0}\}$ so that the vertices of Q are $\{f + r^i : i \in \{1, 2, 3, 4\}\}$. Hence, for $\Gamma := (r^1, r^2, r^3, r^4)$ the intersection cut generated by Q for $(f; \Gamma)$ is $s_1 + s_2 + s_3 + s_4 \geq 1$. Then*

$$\inf\{s_1 + s_2 + s_3 + s_4 : s \in \mathcal{T}_2(f; \Gamma)\} \geq \frac{2}{3}.$$

Proof Sketch. The proof is analogous to the proof of Theorem 1.3.8, except three or four intersection cuts generated by type 2 triangles are used. The chosen type 2 triangles are illustrated in Figure 1.10; use *fixed triangles* F_1, F_2, F_3, and F_4 for which f is in the interior of the triangle. To get the weaker bound of $\frac{1}{2}$ choose two of F_1, F_2, F_3, and F_4, depending on position of f inside Q. □

Theorem 1.3.10 (Follows from proof of [11, Theorem 1.8], Weaker Bound of $\frac{1}{2}$ given by [15, Theorem 1.7])**.** *Let T be a type 3 triangle with rational vertices. Let $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ be in the interior of T. Define ray directions $r^1, r^2, r^3 \in \mathbb{Q}^2 \setminus \{\mathbb{0}\}$ so that the vertices of T are $\{f + r^i : i \in \{1, 2, 3\}\}$. Hence, for $\Gamma := (r^1, r^2, r^3)$ the intersection cut generated by T for $(f; \Gamma)$ is $s_1 + s_2 + s_3 \geq 1$. Then*

$$\inf\{s_1 + s_2 + s_3 : s \in \mathcal{T}_2(f; \Gamma)\} \geq \frac{2}{3}.$$

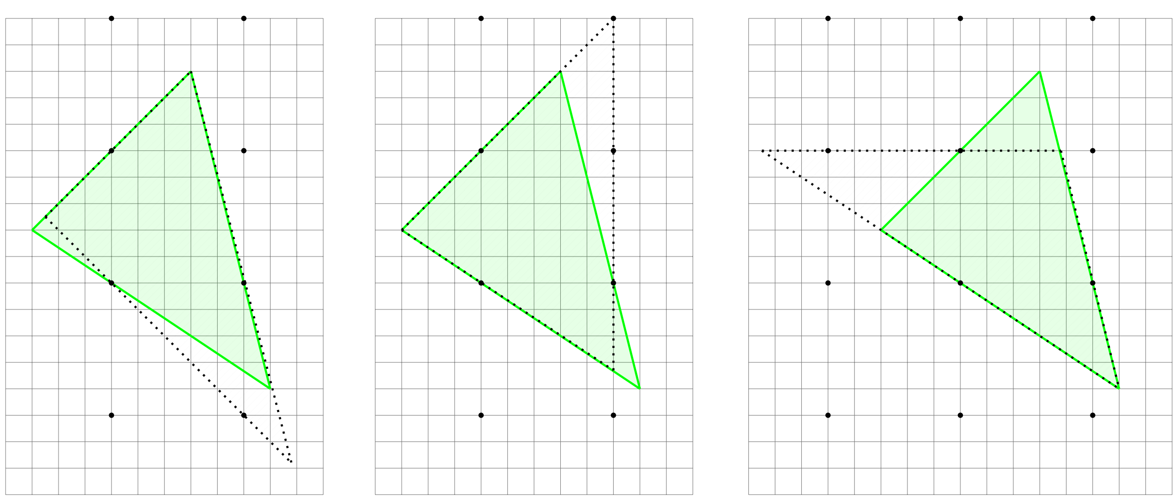


Figure 1.11: The type 2 triangles F_1 (left), F_2 (center), and F_3 (right) with dotted facets, imposed on the shaded normalized type 3 triangle.

Proof Sketch. The proof is analogous to the proof of Theorem 1.3.8, except two or three intersection cuts generated by type 2 triangles are used. The chosen type 2 triangles are illustrated in Figure 1.11; use triangles F_1, F_2, and F_3 for which f is in the interior of the triangle. □

Although the conditions in the preceding theorems may seem a bit contrived, a theorem of the above persuasion can be revised to upper bound the ρ value for the two families involved. If the bound in the theorem is LB, then an upper bound of $\frac{1}{LB}$ on ρ holds. In particular, Theorem 1.3.8 gives $\rho[\mathcal{S}, \mathcal{T}_1] \leq 2$, Theorem 1.3.9 gives $\rho[\mathcal{T}_2, \mathcal{Q}] \leq \frac{3}{2}$, and Theorem 1.3.10 gives $\rho[\mathcal{T}_2, \mathcal{T}_3] \leq \frac{3}{2}$. A bit of technical machinery is required; see the details in Section 2.1.1.

$\rho[\mathcal{L}_1, \mathcal{L}_2]$	$\mathcal{S}$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}$	$\mathcal{Q}$
$\mathcal{S}$	-	[1,**2**]				
$\mathcal{T}_1$		-				
$\mathcal{T}_2$	1	1	-	[1,**1.5**]		[1,**1.5**]
$\mathcal{T}_3$	1	1	1	-		
$\mathcal{T}$	1	1	1	1	-	
$\mathcal{Q}$	1	1	1			-

Table 1.3: **Upper bounds** on $\rho[\mathcal{L}_1, \mathcal{L}_2] = \rho[\text{row set}, \text{column set}]$ determined using a representative cut. Entries of the form $[LB, UB]$ provide a lower bound and upper bound on the value. Recall that a lower bound of 1 follows from the definition.

Inapproximability by the Split Closure and Type 1 Triangle Closure

A similar approach to the above strategy of using a representative cut can show $\rho[\mathcal{L}_1, \mathcal{L}_2]$ is unbounded. It suffices to construct for every $\beta > 1$ a member of $L(\beta) \in \mathcal{L}_2$ and fractional point $f(\beta)$ such that optimizing the inequality generated by $L(\beta)$ over the $\mathcal{L}_1$-closure gives a value at most $\frac{1}{\beta}$. The following theorem and below examples can be obtained by a straightforward modification of the proof of [15, Theorem 8.6].

Theorem 1.3.11. *For any $\beta > 1$, let $T(\beta)$ be the normalized Type 2 triangle with vertices*

$$v^1 = \begin{pmatrix} 0 \\ 2\beta + 1 \end{pmatrix}, \quad v^2 = \begin{pmatrix} 1 + \frac{1}{4\beta} \\ \frac{1}{2} \end{pmatrix}, \text{ and } v^3 = \begin{pmatrix} 0 \\ -2\beta \end{pmatrix}.$$

Let $f = (1, \frac{1}{2})^T$. Define ray directions $r^1, r^2, r^3 \in \mathbb{Q}^2 \setminus \{\mathbb{O}\}$ so that the vertices of $T(\beta)$ are $\{f + r^i : i \in \{1,2,3\}\}$. Hence, for $\Gamma := (r^1, r^2, r^3)$ the intersection cut generated by $T(\beta)$ for $(f;\Gamma)$ is $s_1 + s_2 + s_3 \geq 1$. Then

$$\inf\{s_1 + s_2 + s_3 : s \in \mathcal{S}(f;\Gamma)\} \leq \frac{1}{\beta}.$$

Families of type 3 triangles and quadrilaterals can be constructed based on the type 2 triangle $T(\beta)$ by applying the construction in Figure 1.8. To construct the type 3 triangle for $\beta > 1$, perturb $T(\beta)$ by tilting the edge of $T(\beta)$ defined by $\{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 = 0\}$ clockwise using the integral point in the interior of the edge closest to v^3 as the pivot. To construct the quadrilateral for $\beta > 1$, perturb $T(\beta)$ by removing the edge of $T(\beta)$ defined by $\{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 = 0\}$ and replacing it with a vertex $v^4 = (\epsilon, \frac{1}{2})^T$ for small enough ϵ and two edges defined by the lines joining v^4 with $(0,0)^T$ and $(0,1)^T$ respectively. See Figure 1.12. More families of examples are provided in [15, Section 8.4.2]; for instance, there exist examples for every $f = (1, f_2)^T$ with $0 < f_2 < 1$.

Another way to show $\rho[\mathcal{L}_1, \mathcal{L}_2]$ is unbounded is directly through the definition; i.e. give $(f;\Gamma)$ for which there does not exist $\alpha > 0$ satisfying $\mathcal{L}_1(f;\Gamma) \subseteq \alpha\mathcal{L}_2(f;\Gamma)$. As one might expect, these two approaches are equivalent - see the discussion in Section 2.1.1. We use this approach for $\mathcal{L}_1 = \mathcal{T}_1$ and $\mathcal{L}_2 = \mathcal{S}$ and then use the discussion following Proposition 1.3.7 to obtain results for $\mathcal{L}_2 \in \{\mathcal{T}_2, \mathcal{T}_3, \mathcal{T}, \mathcal{Q}\}$.

Proposition 1.3.12 ([11, Theorem 1.5(1)]). *Let*

$$f = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \ , \ r^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \ r^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and $\Gamma = (r^1, r^2)$. Then, there does not exist $\alpha > 0$ such that $\mathcal{T}_1(f;\Gamma) \subseteq \alpha\mathcal{S}(f;\Gamma)$.

Proof Sketch. Show that $\mathcal{S}(f, \Gamma) = \{s \in \mathbb{R}^2_+ : s_1 \geq \frac{1}{2}\}$ but that $(0, \frac{3}{2})^T \in \mathcal{T}_1(f;\Gamma)$. □

Corollary 1.3.13. *For all families $\mathcal{L} \in \{\mathcal{S}, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}, \mathcal{Q}\}$, we have $\rho[\mathcal{T}_1, \mathcal{L}] = +\infty$.*

Proof. Let f, r^1, r^2, and Γ be as in Proposition 1.3.12. Per the discussion in the ϵ-Relaxations section and Proposition 1.3.7, we have $\mathcal{L}(f;\Gamma) \subseteq \mathcal{S}(f;\Gamma)$ for all $\mathcal{L} \in \{\mathcal{T}_2, \mathcal{T}_3, \mathcal{T}, \mathcal{Q}\}$. If there exists $\alpha > 0$ such that $\mathcal{T}_1(f;\Gamma) \subseteq \alpha\mathcal{L}(f;\Gamma)$, then also $\mathcal{T}_1(f;\Gamma) \subseteq \alpha\mathcal{S}(f;\Gamma)$. This contradicts Proposition 1.3.12 and ergo no such α exists. The result follows by the definition of ρ. □

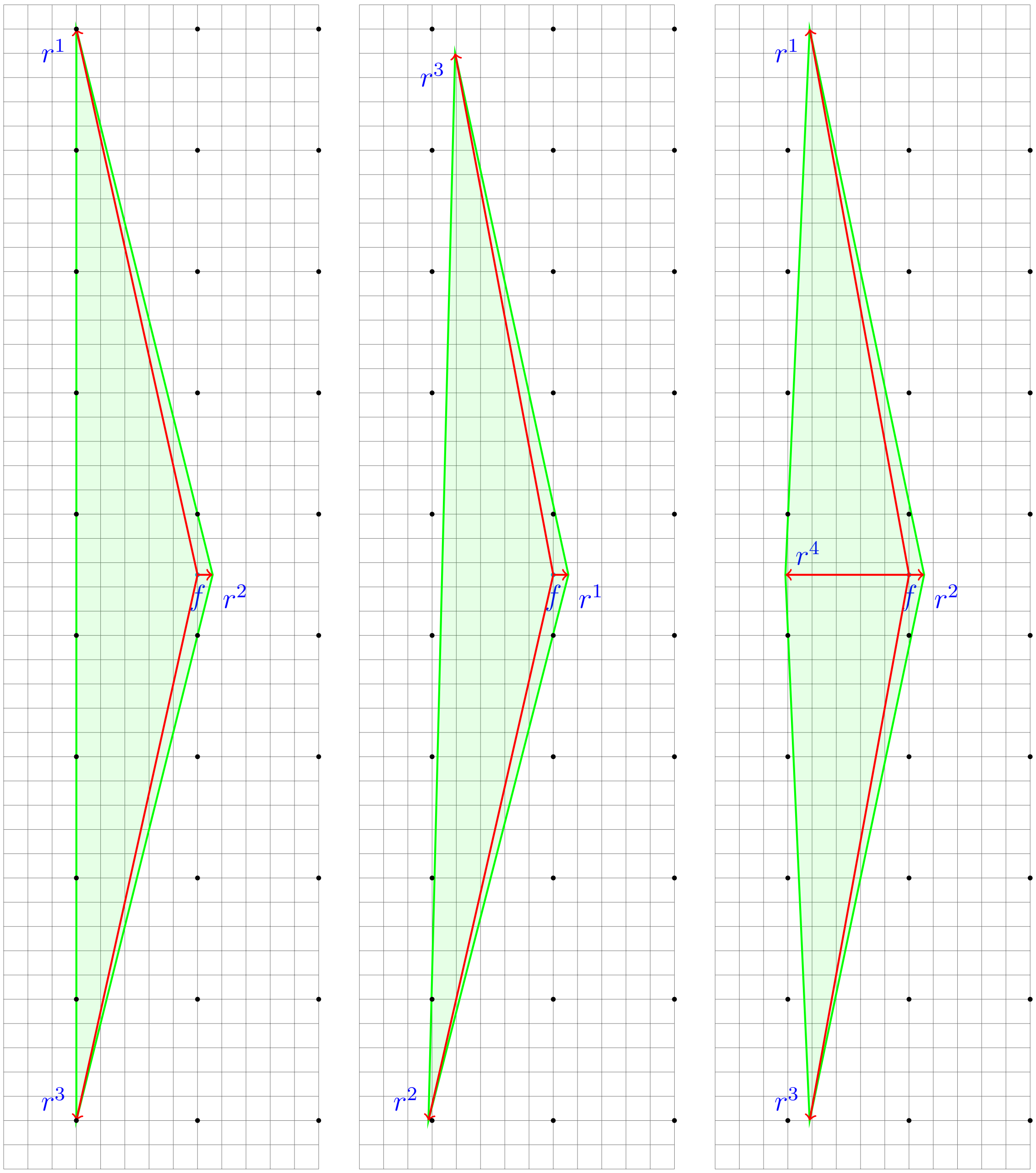


Figure 1.12: The type 2 triangle appearing in Theorem 1.3.11 for $\beta = 2$ (left), perturbing that triangle to obtain a type 3 triangle (center), and the quadrilateral (right).

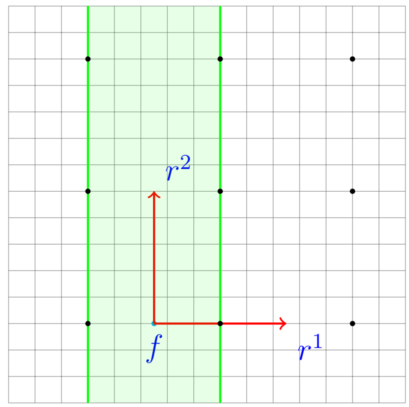


Figure 1.13: The dominating split and f, r^1, and r^2 from Proposition 1.3.12.

$\rho[\mathcal{L}_1, \mathcal{L}_2]$	$\mathcal{S}$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}$	$\mathcal{Q}$
$\mathcal{S}$	-	[1,2]	$+\infty$	$+\infty$	$+\infty$	$+\infty$
$\mathcal{T}_1$	$+\infty$	-	$+\infty$	$+\infty$	$+\infty$	$+\infty$
$\mathcal{T}_2$	1	1	-	[1,1.5]		[1,1.5]
$\mathcal{T}_3$	1	1	1	-		
$\mathcal{T}$	1	1	1	1	-	
$\mathcal{Q}$	1	1	1			-

Table 1.4: Lower Bounds on $\rho[\mathcal{L}_1, \mathcal{L}_2] = \rho[\text{row set}, \text{column set}]$ determined using inapproximability by splits and type 1 triangles.

Lower Bounds using a Specific Cut

As in the preceding two sections, we bound $\rho[\mathcal{L}_1, \mathcal{L}_2]$ by optimizing an inequality generated by a member of $\mathcal{L}_2$ subject to all inequalities generated by members of $\mathcal{L}_1$. The following three theorems provide finite upper bounds on the optimal value of the relevant optimization problems. These problems consider an inequality generated by a specific "hard to approximate" member of $\mathcal{L}_2$. We explain the resulting lower bounds on $\rho[\mathcal{L}_1, \mathcal{L}_2]$ afterwards.

Theorem 1.3.14 (Follows from proof of [15, Theorem 1.6])**.** *Let T be a normalized type 1 triangle. Let $f = \left(\frac{2}{3}, \frac{2}{3}\right)^T$. Define ray directions $r^1, r^2, r^3 \in \mathbb{Q}^2 \setminus \{\mathbb{0}\}$ so that the vertices* *of T are $\{f + r^i : i \in \{1,2,3\}\}$. Hence, for $\Gamma := (r^1, r^2, r^3)$ the intersection cut generated by T for $(f;\Gamma)$ is $s_1 + s_2 + s_3 \geq 1$. Then*

$$\inf\{s_1 + s_2 + s_3 : s \in \mathcal{S}(f;\Gamma)\} \leq \frac{1}{2}.$$

Proof Sketch. First, show that $\hat{s} = \frac{1}{6}\mathbb{1}$ satisfies the intersection cuts generated by *VS*, *HS* and S_3 as given in Theorem 1.3.8. Second, show if $s \in \mathbb{R}^3_+$ satisfies the cuts generated by *VS*, *HS* and S_3, then $s \in \mathcal{S}(f;\Gamma)$. Then the result follows because $\hat{s}$ is a feasible solution to the optimization problem with objective function value $\frac{1}{2}$. $\square$

Theorem 1.3.15 ([11, Theorem 1.6(1)])**.** *Let Q be the normalized quadrilateral with vertices*

$$v^1 = \frac{1}{5}\begin{pmatrix}7\\4\end{pmatrix}\ ,\ v^2 = \frac{1}{5}\begin{pmatrix}4\\-2\end{pmatrix}\ ,\ v^3 = \frac{1}{5}\begin{pmatrix}-2\\1\end{pmatrix}\ , \text{and } v^4 = \frac{1}{5}\begin{pmatrix}1\\7\end{pmatrix}.$$

Let $f = \left(\frac{1}{2}, \frac{1}{2}\right)^T$. Define ray directions $r^1, r^2, r^3, r^4 \in \mathbb{Q}^2 \setminus \{\mathbb{0}\}$ so that the vertices of Q are $\{f + r^i : i \in \{1,2,3,4\}\}$. Hence, for $\Gamma := (r^1, r^2, r^3, r^4)$ the intersection cut generated by Q for $(f;\Gamma)$ is $s_1 + s_2 + s_3 + s_4 \geq 1$. Then

$$\inf\{s_1 + s_2 + s_3 + s_4 : s \in \mathcal{T}(f;\Gamma)\} \leq \frac{8}{9}.$$

Proof Sketch. Show that $\hat{s} = \frac{2}{9}\mathbb{1}$ satisfies every cut generated by a maximal lattice-free triangle. For proof of a more general version of this theorem see Section 2.3. $\square$

Theorem 1.3.16 ([11, Theorem 1.6(2)])**.** *Let T be the normalized type 3 triangle with vertices*

$$v^1 = \frac{1}{3}\begin{pmatrix}4\\-2\end{pmatrix}\ ,\ v^2 = \frac{1}{3}\begin{pmatrix}1\\4\end{pmatrix}\ , \text{and } v^3 = \frac{1}{3}\begin{pmatrix}-2\\1\end{pmatrix}.$$

Let $f = \left(\frac{1}{3}, \frac{1}{3}\right)^T$. Define ray directions $r^1, r^2, r^3 \in \mathbb{Q}^2 \setminus \{\mathbb{0}\}$ so that the vertices of T are $\{f + r^i : i \in \{1,2,3\}\}$. Hence, for $\Gamma := (r^1, r^2, r^3)$ the intersection cut generated by T for $(f;\Gamma)$ is $s_1 + s_2 + s_3 \geq 1$. Then

$$\inf\{s_1 + s_2 + s_3 : s \in \mathcal{Q}(f;\Gamma)\} \leq \frac{8}{9}.$$

Proof Sketch. Show that $\hat{s} = \frac{8}{27}\mathbb{1}$ satisfies every cut generated by a maximal lattice-free quadrilateral. $\square$

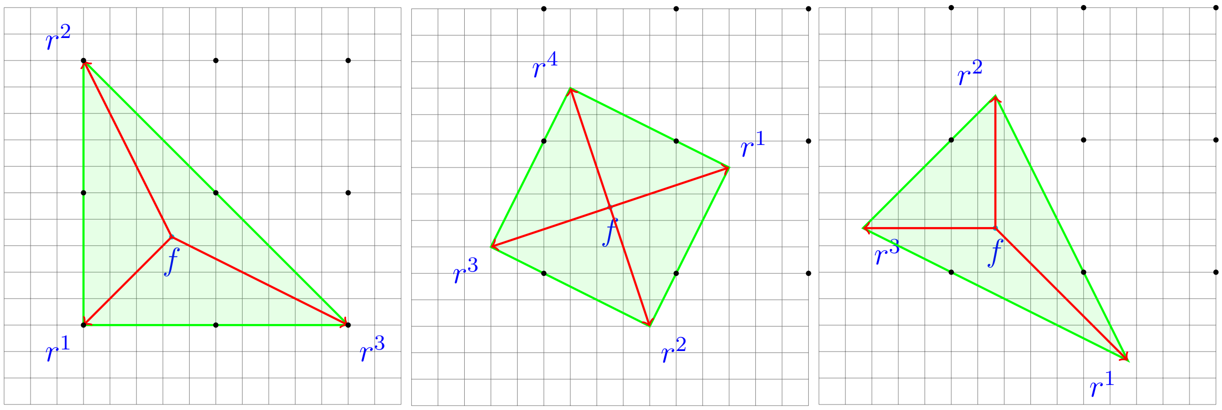


Figure 1.14: The instances for Theorem 1.3.14 (left), Theorem 1.3.15 (center), and Theorem 1.3.16 (right).

A theorem of the above persuasion implies a lower bound on the ρ value for the two families involved. The only extra step required is to argue the ρ value can be calculated by solving the given optimization problem; this is in contrast to using a representative cut to upper bound where one must also argue it is enough to consider $(f;\Gamma)$ with certain properties. If the bound in the theorem is UB then a lower bound of $\frac{1}{UB}$ on ρ holds. In particular, Theorem 1.3.14 gives $\rho[\mathcal{S},\mathcal{T}_1] \geq 2$, Theorem 1.3.15 gives $\rho[\mathcal{T},\mathcal{Q}] \geq \frac{9}{8}$, and Theorem 1.3.16 gives $\rho[\mathcal{Q},\mathcal{T}_3] \geq \frac{9}{8}$.

$\rho[\mathcal{L}_1,\mathcal{L}_2]$	$\mathcal{S}$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}$	$\mathcal{Q}$
$\mathcal{S}$	-	[**2**,2]	$+\infty$	$+\infty$	$+\infty$	$+\infty$
$\mathcal{T}_1$	$+\infty$	-	$+\infty$	$+\infty$	$+\infty$	$+\infty$
$\mathcal{T}_2$	1	1	-	[1,1.5]		[1,1.5]
$\mathcal{T}_3$	1	1	1	-		
$\mathcal{T}$	1	1	1	1	-	[**1.125**,$+\infty$]
$\mathcal{Q}$	1	1	1	[**1.125**,$+\infty$]		-

Table 1.5: Finite **lower bounds** on $\rho[\mathcal{L}_1,\mathcal{L}_2] = \rho[\text{row set}, \text{column set}]$ determined using a specific cut.

Previously Established Bounds for All Pairs of Families

All previously established bounds are summarized in Table 1.6; this table can be found in [11, Table 2]. Some of these bounds were accounted for in the preceding subsections; the balance can be derived from the definition of ρ. The most pertinent example of such a derived bound is the bound on $\rho[\mathcal{T}_2, \mathcal{C}]$.

Proposition 1.3.17. $\rho[\mathcal{T}_2, \mathcal{C}] \leq \frac{3}{2}$.

Proof. Fix $(f;\Gamma)$ such that $f \in \mathbb{Q}^2 \backslash \mathbb{Z}^2$ and $\Gamma = (r^1, \ldots, r^k)$ for $k \in \mathbb{Z}_{++}$ where $r^j \in \mathbb{Q}^2 \backslash \{\mathbb{O}\}$ for all $j \in \{1, \ldots, k\}$.

1. Since $\rho[\mathcal{T}_2, \mathcal{Q}] \leq \frac{3}{2}$, it follows that $\mathcal{T}_2(f;\Gamma) \subseteq \frac{2}{3}\mathcal{Q}(f;\Gamma)$.
2. Since $\rho[\mathcal{T}_2, \mathcal{T}_3] \leq \frac{3}{2}$, it follows that $\mathcal{T}_2(f;\Gamma) \subseteq \frac{2}{3}\mathcal{T}_3(f;\Gamma)$.
3. Since $\mathcal{T}_2(f;\Gamma)$ is upper comprehensive and $\frac{2}{3} \leq 1$, it follows that $\mathcal{T}_2(f;\Gamma) \subseteq \frac{2}{3}\mathcal{T}_2(f;\Gamma)$.
4. Since $\rho[\mathcal{T}_2, \mathcal{T}_1] = 1$, it follows that $\mathcal{T}_2(f;\Gamma) \subseteq \mathcal{T}_1(f;\Gamma)$ and therefore $\mathcal{T}_2(f;\Gamma) \subseteq \frac{2}{3}\mathcal{T}_1(f;\Gamma)$ because $\mathcal{T}_1(f;\Gamma)$ is upper comprehensive and $\frac{2}{3} \leq 1$.
5. Since $\rho[\mathcal{T}_2, \mathcal{S}] = 1$, it follows that $\mathcal{T}_2(f;\Gamma) \subseteq \mathcal{S}(f;\Gamma)$ and therefore $\mathcal{T}_2(f;\Gamma) \subseteq \frac{2}{3}\mathcal{S}(f;\Gamma)$ because $\mathcal{S}(f;\Gamma)$ is upper comprehensive and $\frac{2}{3} \leq 1$.

Therefore $\mathcal{T}_2(f;\Gamma) \subseteq \frac{2}{3}\left[\mathcal{S}(f;\Gamma) \cap \mathcal{T}_1(f;\Gamma) \cap \mathcal{T}_2(f;\Gamma) \cap \mathcal{T}_3(f;\Gamma) \cap \mathcal{Q}(f;\Gamma)\right] = \frac{2}{3}\mathcal{C}(f;\Gamma)$ where the second equality holds because $\mathcal{C} = \mathcal{S} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{Q}$. The bound on $\rho[\mathcal{T}_2, \mathcal{C}]$ follows. $\square$

The remainder of the bounds can be derived using similar arguments. For example, one can use the fact upper bounds on ρ are transitive; if $\rho[\mathcal{L}_1, \mathcal{L}_2] \leq k$ and $\rho[\mathcal{L}_2, \mathcal{L}_3] \leq \ell$ then $\rho[\mathcal{L}_1, \mathcal{L}_3] \leq k\ell$.

$\rho[\mathcal{L}_1, \mathcal{L}_2]$	$\mathcal{S}$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}$	$\mathcal{Q}$	$\mathcal{C}$
$\mathcal{S}$	-	2	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
$\mathcal{T}_1$	$+\infty$	-	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
$\mathcal{T}_2$	1	1	-	[**1.125**,1.5]	**[1.125,1.5]**	[**1.125**,1.5]	**[1.125,1.5]**
$\mathcal{T}_3$	1	1	1	-	**1**	**[1.125,1.5]**	**[1.125,1.5]**
$\mathcal{T}$	1	1	1	1	-	**[1.125,1.5]**	**[1.125,1.5]**
$\mathcal{Q}$	1	1	1	[1.125,1.5]	**[1.125,1.5]**	-	**[1.125,1.5]**

Table 1.6: Previously established best bounds for $\rho[\mathcal{L}_1, \mathcal{L}_2] = \rho[\text{row set}, \text{column set}]$ for dimension two. Derived bounds are indicated in **bold**.

1.3.3 New Bounds

Let $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ and $\Gamma = (r^1, \ldots, r^k)$ for $k \in \mathbb{Z}_{++}$ where $r^j \in \mathbb{Q}^2 \setminus \{\mathbb{O}\}$ for all $j \in \{1, \ldots, k\}$. Since $R(f;\Gamma) = \mathcal{C}(f;\Gamma)$, the bounds in the previous section provide insight into how well intersection cuts generated by a single family can approximate $R(f;\Gamma)$.

1. If all intersection cuts from any one of $\mathcal{T}_2$,$\mathcal{T}_3$ or $\mathcal{Q}$ are used, then the corresponding relaxation $\mathcal{T}_2(f;\Gamma), \mathcal{T}_3(f;\Gamma)$ or $\mathcal{Q}(f;\Gamma)$ is within a factor of $\frac{3}{2}$ of $R(f;\Gamma)$.
2. To the contrary, if all intersection cuts from any one of $\mathcal{T}_2$, $\mathcal{T}_3$ or $\mathcal{Q}$ are used, then there is a gap of $\frac{1}{8}$ between the corresponding relaxation $\mathcal{T}_2(f;\Gamma), \mathcal{T}_3(f;\Gamma)$ or $\mathcal{Q}(f;\Gamma)$ and $R(f;\Gamma)$.

These bounds point to type 2 triangles as a natural compromise: the family is simple and isn't provably better or worse at approximating $R(f;\Gamma)$ based on established bounds. Tighter bounds on $\rho[\mathcal{T}_2, \mathcal{L}]$ for $\mathcal{L} \in \{\mathcal{T}_3, \mathcal{Q}\}$ would provide more insight into the strength of cuts generated by type 2 triangles.

We prove a tighter bound on $\rho[\mathcal{T}_2, \mathcal{Q}]$ in this thesis by improving the bound in Theorem 1.3.9. To do so, we choose a different relaxation of $\mathcal{T}_2(f;\Gamma)$. Rather than using the intersection cuts generated by fixed triangles F_1, F_2, F_3, and F_4, we select triangles from a family of type 2 triangles we call *ray-sliding triangles*. This family is illustrated in Figure 1.15; the main idea is to choose a parameterized family that captures the trade-off in cut coefficient on three rays. Unlike fixed triangles, the family of ray-sliding triangles depends on the fractional point f in the interior of the quadrilateral. Following the proof sketch of Theorem 1.3.9 above, but relaxing the constraint $s \in \mathcal{T}_2(f;\Gamma)$ to require $s \geq \mathbb{0}$ and s satisfy the intersection cuts generated by certain ray-sliding triangles, we can obtain the following result.

Theorem 1.3.18. *Let Q be a normalized quadrilateral with rational vertices. Let $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ be in the interior of Q. Define ray directions $r^1, r^2, r^3, r^4 \in \mathbb{Q}^2 \setminus \{\mathbb{0}\}$ so that the vertices of Q are $\{f + r^i : i \in \{1,2,3,4\}\}$. Hence, for $\Gamma := (r^1, r^2, r^3, r^4)$ the intersection cut generated by Q for $(f;\Gamma)$ is $s_1 + s_2 + s_3 + s_4 \geq 1$. Then*

$$\inf\{s_1 + s_2 + s_3 + s_4 : s \in \mathcal{T}_2(f;\Gamma)\} \geq \frac{4}{5}.$$

A tighter upper bound on $\rho[\mathcal{T}_2, \mathcal{Q}]$ follows.

Theorem 1.3.19. $\rho[\mathcal{T}_2, \mathcal{Q}] \leq \frac{5}{4}$.

An upper bound on $\rho[\mathcal{T}_2, \mathcal{Q}]$ is of particular interest because quadrilaterals have one more facet than type 2 triangles and, moreover, the family of type 2 triangles are a strict subset of the family of maximal lattice-free triangles. We would like to understand this bound better to extend such a result in $\mathbb{R}^2$ to $\mathbb{R}^d$ - comparing intersection cut closures of simple families with fewer facets to closures of complicated families with more facets. To obtain

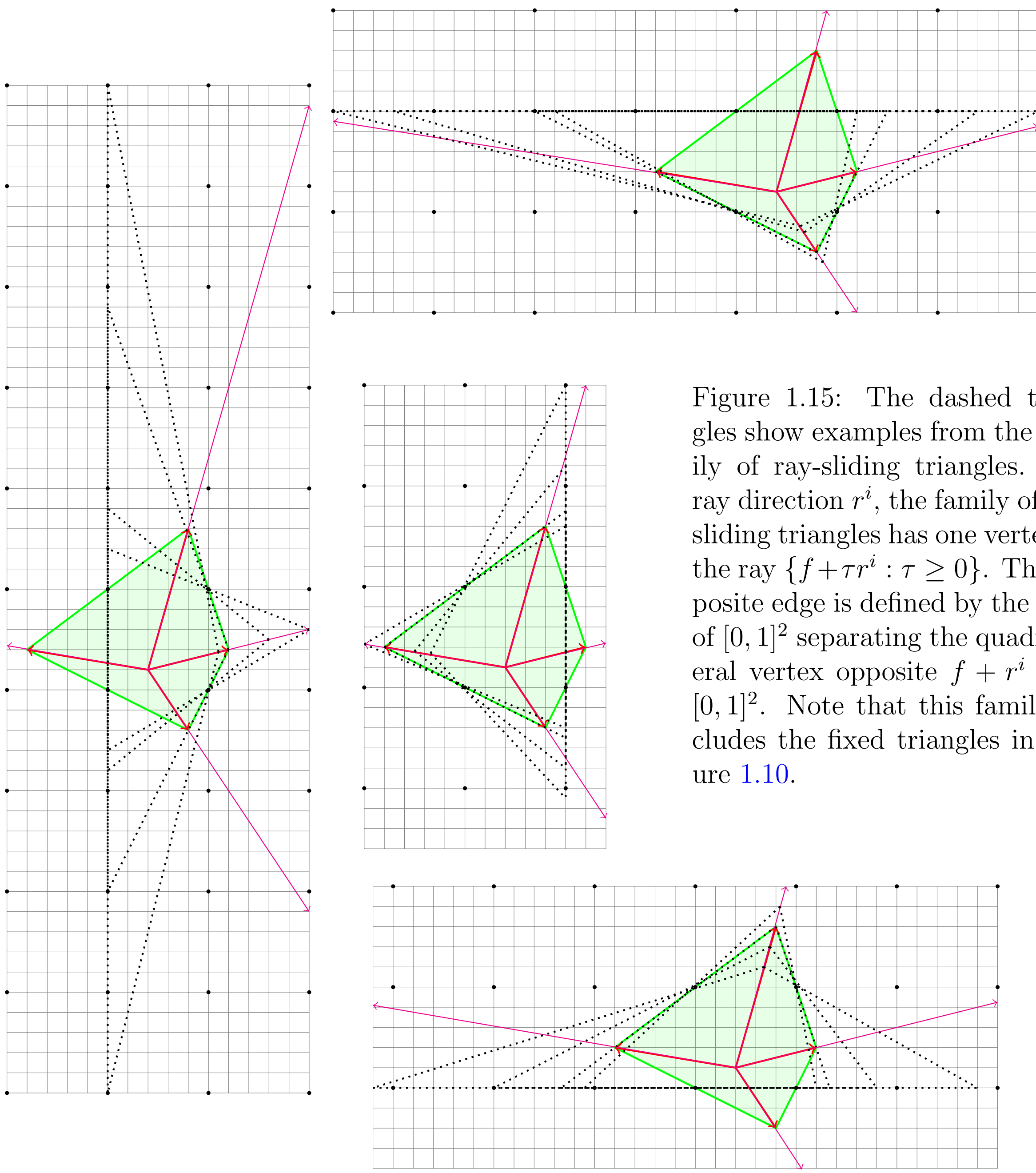


Figure 1.15: The dashed triangles show examples from the family of ray-sliding triangles. For ray direction r^i, the family of ray-sliding triangles has one vertex on the ray $\{f + \tau r^i : \tau \geq 0\}$. The opposite edge is defined by the facet of $[0, 1]^2$ separating the quadrilateral vertex opposite $f + r^i$ from $[0, 1]^2$. Note that this family includes the fixed triangles in Figure 1.10.

a tighter bound on $\rho[\mathcal{T}_2, \mathcal{C}]$, however, a tighter bound on $\rho[\mathcal{T}_2, \mathcal{T}_3]$ would also be required. With respect to a type 3 triangle, we can similarly define a family of ray-sliding triangles capturing the trade-off in cut coefficient on two rays. We expect that using this family and similar techniques to the proof of Theorem 1.3.19, the upper bound on $\rho[\mathcal{T}_2, \mathcal{T}_3]$ can be improved. See the discussion in Section 6.2.

On our way to proving Theorem 1.3.19, we consider the specialization to the family of maximal lattice-free parallelograms. We've already encountered a member of this family once in Theorem 1.3.15. A quadrilateral is a *parallelogram* if each pair of opposite edges has the same slope. Let $\mathcal{Q}_2 \subseteq \mathcal{Q}$ denote the family of maximal lattice-free parallelograms in $\mathbb{R}^2$. A parallelogram is *normalized* if it is a normalized quadrilateral.

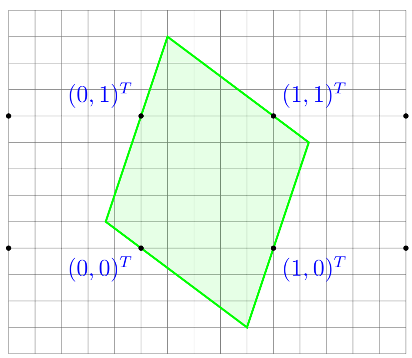


Figure 1.16: Normalized parallelogram.

Remark 1.3.20. *For every parallelogram $P \in \mathcal{Q}_2$, there exists an affine unimodular transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(P)$ is a normalized parallelogram.*

Proof. The remark follows from Remark 1.3.4 and the fact affine unimodular transformations preserve parallel lines. $\square$

Given $(f; \Gamma)$, the *parallelogram closure* is

$$\mathcal{Q}_2(f; \Gamma) := \left\{ s \in \mathbb{R}^k_+ : \sum_{j=1}^{k} \psi_{f;Q}(r^j) s_j \geq 1 \text{ for all } Q \in \mathcal{Q}_2 \text{ with } f \in \operatorname{int} Q \right\}. \tag{1.29}$$

Since the quadrilateral in Theorem 1.3.15 is a parallelogram (a square, in fact), we already know $\rho[\mathcal{T}_2, \mathcal{Q}_2] \geq \frac{9}{8}$. We prove this bound is in fact tight.

Theorem 1.3.21. $\rho[\mathcal{T}_2, \mathcal{Q}_2] \leq \frac{9}{8}$.

Our study of the cuts generated by maximal lattice-free parallelograms was motivated by treating such cuts as a restriction of those generated by maximal lattice-free quadrilaterals. However, the family of maximal lattice-free parallelograms may be interesting in its own right. Parallelograms are in some sense a "simpler set" - described by only two slopes. We can construct parallelograms as the intersection of (not necessarily lattice-free) splits. The family generalizes naturally to higher dimensions as the family of centrally symmetric octahedra. We will return to parallelograms in Section 6.1 and, in particular, discuss comparison of the $\mathcal{Q}_2$-closure to others.

1.4 Overview of Remainder of the Thesis

The next three chapters are dedicated to proving Theorem 1.3.21 and Theorem 1.3.19. Chapter 2 reviews the existing techniques and introduces new concepts required to prove Theorem 1.3.21 in Chapter 3 and Theorem 1.3.19 in Chapter 4. Chapter 5 investigates extending these results to higher dimensions, providing an analogue of Theorem 1.3.9 for d-row intersection cuts for $d \geq 3$. Although an analogue can be obtained, the families of intersection cuts being compared are less natural, echoing the difficulty in characterizing maximal lattice-free convex sets in dimension $d \geq 3$. Chapter 6 summarizes the results and highlights some open problems.

Chapter 2

Existing Techniques and New Families of Triangles

Before bounding $\rho[\mathcal{T}_2, \mathcal{Q}_2]$ in Chapter 3 and $\rho[\mathcal{T}_2, \mathcal{Q}]$ in Chapter 4, we present the tools and techniques required. As always, let $\mathcal{T}_2$ denote the family of type 2 triangles in $\mathbb{R}^2$, let $\mathcal{T}$ denote the family of maximal lattice-free triangles in $\mathbb{R}^2$, let $\mathcal{Q}$ denote the family of maximal lattice-free quadrilaterals in $\mathbb{R}^2$, and let $\mathcal{Q}_2$ denote the family of maximal lattice-free parallelograms in $\mathbb{R}^2$. Our strategy to calculate a target approximation measure - in this section we choose target $\rho[\mathcal{T}_2, \mathcal{Q}]$ - closely follows [11] and [15]. Roughly, the strategy is as follows.

1. Reduce calculating $\rho[\mathcal{T}_2, \mathcal{Q}]$ to solving the optimization problem $(S(\mathcal{T}_2, \mathcal{Q}))$ in variables $s_1, s_2, s_3, s_4 \in \mathbb{R}$ and $r^1, r^2, r^3, r^4, f \in \mathbb{R}^2$ given by

$$\begin{aligned} \inf \quad & s_1 + s_2 + s_3 + s_4 && (2.1)\\ \text{subject to} \quad & s \in \mathcal{T}_2(f; r^1, r^2, r^3, r^4) && (2.2)\\ & f + \operatorname{conv}\{r^1, r^2, r^3, r^4\} \in \hat{\mathcal{Q}} && (2.3)\\ & f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2 && (2.4)\\ & r^1, r^2, r^3, r^4 \in \mathbb{Q}^2 \setminus \{\mathbb{O}\} && (2.5)\\ & \operatorname{cone}\{r^1, r^2, r^3, r^4\} = \mathbb{R}^2. && (2.6) \end{aligned}$$

2. To any feasible solution of $(S(\mathcal{T}_2, \mathcal{Q}))$, there is an affiliated normalized quadrilateral $Q \in \hat{\mathcal{Q}}$ and fractional point f in the interior of Q. Normalized quadrilateral Q can be described by the slopes of its edges $-a, b, c, -d$ for $a, b, c, d > 0$. Fractional point f can be described by its coordinates $(g, h)^T$. Reparameterize $(S(\mathcal{T}_2, \mathcal{Q}))$ in terms of $a, b, c, d, g,$ and h.

3. Find a lower bound for the optimal value of $(S(\mathcal{T}_2, \mathcal{Q}))$ as a function of parameters $a, b, c, d, g,$ and h. To find the lower bound, consider a relaxation obtained by replacing constraint (2.2) with the constraint $s \geq \mathbb{0}$ and the intersection cuts for a small number of type 2 triangles. The resulting relaxation is a parameterized linear program. Find a feasible solution to its dual to obtain a lower bound for the optimal value of $(S(\mathcal{T}_2, \mathcal{Q}))$ as a function of $a, b, c, d, g,$ and h.
4. Vary $a, b, c, d, g,$ and h to determine for which choices of these parameters the lower bound is the weakest. Denote this value by $\hat{LB}$.
5. Find a feasible solution to $(S(\mathcal{T}_2, \mathcal{Q}))$ and denote its objective function value by $\hat{UB}$.
6. If $\hat{LB} = \hat{UB}$ then $(S(\mathcal{T}_2, \mathcal{Q}))$ is solved and $\rho[\mathcal{T}_2, \mathcal{Q}] = \frac{1}{\hat{LB}}$. If $\hat{LB} < \hat{UB}$, even though $(S(\mathcal{T}_2, \mathcal{Q}))$ is not solved exactly, the bounds $\frac{1}{\hat{UB}} \leq \rho[\mathcal{T}_2, \mathcal{Q}] \leq \frac{1}{\hat{LB}}$ follow.

To calculate the lower bound in step 3 as in the proof of Theorem 1.3.9 ([15] and [11]), constraint (2.2) was replaced with the intersection cuts for two, three, or four fixed triangles. The resulting parameterized linear program was solved exactly in both analyses, even though a dual feasible solution suffices for the purposes of lower-bounding. Hence the analyses are in some sense "tight" - no better bound can be calculated by replacing the constraint (2.2) with the intersection cuts for the given triangles. In Chapter 3 and Chapter 4, we will improve the lower bounds calculated in step 3 by choosing different intersection cuts generated by ray-sliding triangles to replace constraint (2.2). In our analysis, we also solve the resulting parametric linear programs exactly. This guarantees the analysis is again "tight" and demonstrates that certain simple families of triangle cuts are limited in how well they approximate the quadrilateral closure.

To calculate the upper bound in step 5 as in Theorem 1.3.15, specific values for a, b, c, d, g, h, s_1, s_2, s_3, and s_4 that provide a feasible solution to $(S(\mathcal{T}_2, \mathcal{Q}))$ were given. This work can be extended directly to provide a family of feasible solutions to $(S(\mathcal{T}_2, \mathcal{Q}))$ where the corresponding normalized quadrilateral can be fixed to any parallelogram. In light of the parameterized lower bound in step 3, a parameterized family of feasible solutions in step 5 can provide a "tightness" guarantee as well. If the corresponding objective function value matches the lower bound for the given parameters, then the intersection cuts chosen to replace the triangle closure constraint are best possible.

The preceding observations will be detailed in the remainder of this chapter. First we refine the technical details of the proof strategy above and prove two simple lemmas which will be applied often to solve the optimization problems at hand. We use the proof strategy to rederive the previously established upper bound on $\rho[\mathcal{T}_2, \mathcal{Q}]$ and calculate a first upper bound on $\rho[\mathcal{T}_2, \mathcal{Q}_2]$. Then, we provide a parameterized family of feasible solutions to $(S(\mathcal{T}, \mathcal{Q}))$

and use them to re-derive the previously established lower bounds on $\rho[\mathcal{T}, \mathcal{Q}]$ and $\rho[\mathcal{T}, \mathcal{Q}_2]$[1]. Lastly, we define the family of ray-sliding triangles comprehensively.

2.1 Proof Strategy Detail

2.1.1 Step 1: Identify Optimization Problem ($S(\mathcal{T}_2, \mathcal{Q})$)

Let $\mathcal{L}_1$ denote a family of lattice-free convex sets in $\mathbb{R}^d$. Let $\mathcal{L}_2$ denote a family of lattice-free polytopes in $\mathbb{R}^d$ such that each member has exactly $\ell \in \mathbb{Z}_{++}$ vertices and $\mathcal{L}_2$ is closed under unimodular transformation. For such families, we can reduce the problem of calculating $\rho[\mathcal{L}_1, \mathcal{L}_2]$ to solving a related semi-infinite optimization problem. The optimization problem captures the fact $\theta[C_1, C_2]$ can be calculated by minimizing the defining inequalities of C_2 over C_1 . It also incorporates some simplifying assumptions about the fractional points and ray directions we must consider. The optimization problem is provided by the below theorem.

Theorem 2.1.1 ([11, Theorem 3.1])**.** *Let $\mathcal{L}_1, \mathcal{L}_2$ denote families of lattice-free convex sets in $\mathbb{R}^d$. Suppose that all sets in $\mathcal{L}_2$ are polytopes with exactly ℓ vertices and that $\mathcal{L}_2$ is closed under unimodular transformation. Let $\hat{\mathcal{L}}_2$ denote a set of representatives for $\mathcal{L}_2$. Then $\frac{1}{\rho[\mathcal{L}_1, \mathcal{L}_2]}$ is equal to the maximum of one and the optimal value of the optimization problem ($S(\mathcal{L}_1, \mathcal{L}_2)$) given by*

$$\begin{aligned}
\inf \quad & \sum_{i=1}^{\ell} s_i && (2.7)\\
\text{subject to} \quad & s \in \mathcal{L}_1(f; r^1, \ldots, r^\ell) && (2.8)\\
& f + \operatorname{conv}\{r^1, \ldots, r^\ell\} \in \hat{\mathcal{L}}_2 && (2.9)\\
& f \in \mathbb{Q}^d \setminus \mathbb{Z}^d && (2.10)\\
& r^1, \ldots, r^\ell \in \mathbb{Q}^d \setminus \{\mathbb{O}\} && (2.11)\\
& \operatorname{cone}\{r^1, \ldots, r^\ell\} = \mathbb{R}^d. && (2.12)
\end{aligned}$$

Proof Sketch. We provide the main observations required (loosely and without proof). To compute $\frac{1}{\theta[\mathcal{L}_1(f;\Gamma), \mathcal{L}_2(f;\Gamma)]}$ for any $(f; \Gamma)$ where $\Gamma = (r^1, \ldots, r^k)$ for some $k \in \mathbb{Z}_{++}$, it suffices to take the minimizer of $\psi_{f;B}^T s$ over all $B \in \mathcal{L}_2$ and $s \in \mathcal{L}_1(f; \Gamma)$. In other words, there exists

[1]Observe here we are solving ($S(\mathcal{T}, \mathcal{Q})$) to obtain a stronger result.

some $B \in \mathcal{L}_2$ such that the minimizer of $\psi^T_{f;B}s$ over $s \in \mathcal{L}_1(f;\Gamma)$ gives $\frac{1}{\rho[\mathcal{L}_1(f;\Gamma),\mathcal{L}_2(f;\Gamma)]}$. As $\mathcal{L}_2$ is closed under affine unimodular transformation and the definition of ρ takes a supremum over all $(f;\Gamma)$ we may assume $B \in \hat{\mathcal{L}}_2$. As the definition of ρ takes a supremum over all $(f;\Gamma)$, we may rescale the rays r^j in Γ so that $\psi_{f;B}(r^j) = 1$ for all $j \in \{1,\ldots,k\}$. Since $\psi_{f;B}$ is convex, if $\kappa = \frac{1}{\theta[\mathcal{L}_1(f;\Gamma),\mathcal{L}_2(f;\Gamma)]}$ is determined by the constraint for B, then for given fractional point f, the value of κ is smallest (and therefore ρ is largest) for $\Gamma = (r^1, r^2, \ldots, r^\ell)$ such that $f + \operatorname{conv}\{r^1,\ldots,r^\ell\} = B$. □

This theorem shows that how well the $\mathcal{L}_1$-closure approximates the $\mathcal{L}_2$-closure is captured completely by how well the $\mathcal{L}_1$-closure approximates the "hardest to approximate" intersection cut generated by a member of $\mathcal{L}_2$. So, it provides the machinery required to obtain bounds from results such as Theorem 1.3.9 and Theorem 1.3.10. Problem $(S(\mathcal{L}_1,\mathcal{L}_2))$ is easiest to parse as a two-stage optimization problem: think first of choosing a member B of $\hat{\mathcal{L}}_2$ and fractional point f in its interior. These choices impute a choice for ℓ rays $\Gamma = (r^1,\ldots,r^\ell)$. Then, minimize $\mathbb{1}^T s$ over the $\mathcal{L}_1(f;\Gamma)$ to calculate a bound. The value for $\frac{1}{\rho[\mathcal{L}_1,\mathcal{L}_2]}$ is obtained from the choices for B and f for which this bound is weakest.

Now, in general, the restriction $s \in \mathcal{L}_1(f;r^1,\ldots,r^\ell)$ may require infinitely many linear constraints and so we refer to this problem as a semi-infinite optimization problem. When the $\mathcal{L}_1$-closure is a polyhedron - for example, when $\mathcal{L}_1 \in \{\mathcal{S},\mathcal{T},\mathcal{Q}\}$ - only finitely many constraints are required. Note that the bounds we prove here do not rely on the underlying closure being a polyhedron.

Applying Theorem 2.1.1 for $\mathcal{L}_1 = \mathcal{T}_2$ and $\mathcal{L}_2 = \mathcal{Q}$ we see that to compute $\rho[\mathcal{T}_2,\mathcal{Q}]$, we can solve the semi-infinite optimization problem $(S(\mathcal{T}_2,\mathcal{Q}))$ given above by the infimum of (2.1) subject to (2.2) - (2.6).

2.1.2 Step 2: Parameterize Normalized Quadrilateral and Fractional Point

To solve $(S(\mathcal{T}_2,\mathcal{Q}))$, we first describe $\hat{\mathcal{Q}}$ so that each $Q \in \hat{\mathcal{Q}}$ is determined by a small number of parameters. We choose to parameterize each $Q \in \hat{\mathcal{Q}}$ by the slopes of its edges. Given a normalized quadrilateral Q, there is a natural bijection between the edges of Q and the vertices of $[0,1]^2$ given by containment.

1. Exactly one edge of Q contains $(0,0)^T$; this edge must have negative slope; let its slope be $-a$ for $a > 0$.
2. Exactly one edge of Q contains $(1,0)^T$; this edge must have positive slope; let its slope be b for $b > 0$.

3. Exactly one edge of Q contains $(0,1)^T$; this edge must have positive slope; let its slope be c for $c > 0$.
4. Exactly one edge of Q contains $(1,1)^T$; this edge must have negative slope; let its slope be $-d$ for $d > 0$.

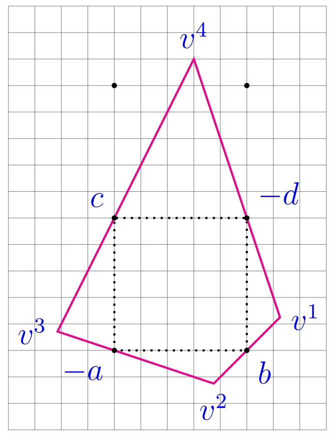


Figure 2.1: Normalized quadrilateral $Q(a,b,c,d)$ with edge slopes $-a, b, c,$ and $-d$ for $a,b,c,d > 0$ and vertices $v^1, v^2, v^3,$ and v^4.

In fact, any normalized quadrilateral is completely described by values for $a, b, c,$ and d. We denote by $Q(a,b,c,d)$ the normalized quadrilateral with edge slopes $-a, b, c, -d$ as given above. For notational convenience, we also fix a labelling of the vertices of normalized quadrilateral $Q = Q(a,b,c,d)$ so that

1. v^1 is at the intersection of the edges of Q containing $(1,1)^T$ and $(1,0)^T$,
2. v^2 is at the intersection of the edges of Q containing $(0,0)^T$ and $(1,0)^T$,
3. v^3 is at the intersection of the edges of Q containing $(0,0)^T$ and $(0,1)^T$, and
4. v^4 is at the intersection of the edges of Q containing $(0,1)^T$ and $(1,1)^T$.

The coordinates of $v^1, v^2, v^3,$ and v^4 are (necessarily) determined completely by the values of $a, b, c,$ and d. In particular

$$v^1 := (v^1_x, v^1_y)^T = \left(1 + \frac{1}{b+d}\,,\ \frac{b}{b+d}\right)^T \quad , \quad v^2 := (v^2_x, v^2_x)^T = \left(\frac{b}{a+b}\,,\ \frac{-ab}{a+b}\right)^T \quad ,$$
$$v^3 := (v^3_x, v^3_y)^T = \left(\frac{-1}{a+c}\,,\ \frac{a}{a+c}\right)^T \quad , \quad \text{and} \quad v^4 := (v^4_x, v^4_y)^T = \left(\frac{d}{c+d}\,,\ 1 + \frac{cd}{c+d}\right)^T .$$

We can restrict $\hat{\mathcal{Q}}$ to $\{Q(a,b,c,d) : a,b,c,d > 0, ad \leq bc\}$ and maintain a set of representatives (with respect to unimodular transformation) for the family $\mathcal{Q}$ of maximal lattice-free quadrilaterals.

Remark 2.1.2. *Let $Q_1 := Q(a,b,c,d)$ for $a,b,c,d > 0$ with $ad > bc$ be a normalized quadrilateral. Then there exist $\hat{a},\hat{b},\hat{c},\hat{d} > 0$ with $\hat{a}\hat{d} \leq \hat{b}\hat{c}$ and affine unimodular transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(Q_1) = Q(\hat{a},\hat{b},\hat{c},\hat{d})$.*

Proof. Choose $(\hat{a},\hat{b},\hat{c},\hat{d}) = (b,a,d,c)$ and ϕ as the map $\phi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ corresponding to reflection in the line $\{(x,y)^T \in \mathbb{R}^2 : x = \frac{1}{2}\}$. □

Therefore, we can use the above parameterization to rewrite $(S(\mathcal{T}_2, \mathcal{Q}))$ as the following semi-infinite program in variables a,b,c,d,g,h,s_1,s_2,s_3, and s_4:

$$\inf \quad s_1 + s_2 + s_3 + s_4 \tag{2.13}$$

$$\text{s.t.} \quad s \in \mathcal{T}_2\left(\begin{pmatrix} g \\ h \end{pmatrix}; \begin{pmatrix} 1 + \frac{1}{b+d} - g \\ \frac{b}{b+d} - h \end{pmatrix}, \begin{pmatrix} \frac{b}{a+b} - g \\ \frac{-ab}{a+b} - h \end{pmatrix}, \begin{pmatrix} \frac{-1}{a+c} - g \\ \frac{a}{a+c} - h \end{pmatrix}, \begin{pmatrix} \frac{d}{c+d} - g \\ 1 + \frac{cd}{c+d} - h \end{pmatrix}\right) \tag{2.14}$$

$$\begin{pmatrix} g \\ h \end{pmatrix} \in \operatorname{int}\left(\operatorname{conv}\left\{\begin{pmatrix} 1 + \frac{1}{b+d} \\ \frac{b}{b+d} \end{pmatrix}, \begin{pmatrix} \frac{b}{a+b} \\ \frac{-ab}{a+b} \end{pmatrix}, \begin{pmatrix} \frac{-1}{a+c} \\ \frac{a}{a+c} \end{pmatrix}, \begin{pmatrix} \frac{d}{c+d} \\ 1 + \frac{cd}{c+d} \end{pmatrix}\right\}\right) \tag{2.15}$$

$$a,b,c,d > 0 \quad ; \quad a,b,c,d \in \mathbb{Q} \quad ; \quad ad \leq bc \tag{2.16}$$

$$\begin{pmatrix} g \\ h \end{pmatrix} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2. \tag{2.17}$$

Treating $a,b,c,d > 0$ with $ad \leq bc$ and $(g,h)^T \in \operatorname{int} Q(a,b,c,d)$ as data in the optimization problem above, we obtain a semi-infinite linear program in variables s_1, s_2, s_3, and s_4. We denote this program by $(P(a,b,c,d,g,h))$. It is given by

$$\inf \quad s_1 + s_2 + s_3 + s_4 \tag{2.18}$$

$$\text{s.t.} \quad s \in \mathcal{T}_2\left(\begin{pmatrix} g \\ h \end{pmatrix}; \begin{pmatrix} 1 + \frac{1}{b+d} - g \\ \frac{b}{b+d} - h \end{pmatrix}, \begin{pmatrix} \frac{b}{a+b} - g \\ \frac{-ab}{a+b} - h \end{pmatrix}, \begin{pmatrix} \frac{-1}{a+c} - g \\ \frac{a}{a+c} - h \end{pmatrix}, \begin{pmatrix} \frac{d}{c+d} - g \\ 1 + \frac{cd}{c+d} - h \end{pmatrix}\right). \tag{2.19}$$

For $i \in \{1,2,3,4\}$ we will denote $v^i - f$ as appearing in (2.19) by r^i.

2.1.3 Step 3: Replace the Type 2 Triangle Closure Constraint with a Small Number of Intersection Cuts

For fixed $a,b,c,d > 0$ with $ad \leq bc$ and $(g,h)^T \in \operatorname{int} Q(a,b,c,d)$, we would like to lower bound the optimal value of $(P(a,b,c,d,g,h))$. To do so, we replace constraint (2.19) with the intersection cuts for at least two and at most four type 2 triangles. We can lower bound the optimal value of the resulting parameterized linear program by finding a dual feasible solution to obtain a lower bound $LB : \mathbb{R}^6 \to \mathbb{R}$. In practice, the parameterized linear program we must solve is often in the form provided by the following lemma.

Lemma 2.1.3. *Let $n \in \mathbb{Z}_+$. Consider the linear program $(LP1)$ given by*

$$\begin{aligned} \min \quad & \mathbb{1}^T s && (2.20) \\ \text{subject to} \quad & \left[\mathbb{1}\mathbb{1}^T + A\right] s \geq \mathbb{1} && (2.21) \\ & s \geq \mathbb{0} && (2.22) \end{aligned}$$

where $A \in \mathbb{R}^{n \times n}$. If A is invertible and the entries of A^{-1} are non-negative, then

$$\hat{s} := \left(\frac{1}{1 + \mathbb{1}^T A^{-1} \mathbb{1}}\right) A^{-1} \mathbb{1}$$

is an optimal solution of $(LP1)$ and its optimal value is

$$\frac{\mathbb{1}^T A^{-1} \mathbb{1}}{1 + \mathbb{1}^T A^{-1} \mathbb{1}} = 1 - \frac{1}{1 + \mathbb{1}^T A^{-1} \mathbb{1}}.$$

Proof. The dual of of $(LP1)$ is $(DP1)$ given by

$$\begin{aligned} \max \quad & \mathbb{1}^T v && (2.23) \\ \text{subject to} \quad & \left[\mathbb{1}\mathbb{1}^T + A^T\right] v \leq \mathbb{1} && (2.24) \\ & v \geq \mathbb{0} && (2.25) \end{aligned}$$

Note that $\hat{s} \geq \mathbb{0}$ because the entries of A^{-1} are non-negative. Computing

$$\begin{aligned} (\mathbb{1}\mathbb{1}^T + A)\hat{s} &= \left(\frac{1}{1 + \mathbb{1}^T A^{-1} \mathbb{1}}\right) \left(\mathbb{1}\mathbb{1}^T A^{-1} \mathbb{1} + A A^{-1} \mathbb{1}\right) \\ &= \left(\frac{1}{1 + \mathbb{1}^T A^{-1} \mathbb{1}}\right) \left(\mathbb{1}^T A^{-1} \mathbb{1} + 1\right) \mathbb{1} \\ &= \mathbb{1} \end{aligned}$$

we see that $\hat{s}$ is feasible for $(LP1)$ and satisfies the n constraints (2.21) with equality. Let

$$\hat{v} := \left(\frac{1}{1 + \mathbb{1}^T A^{-1} \mathbb{1}}\right) A^{-T} \mathbb{1}$$

Note that $\hat{v} \geq \mathbb{0}$ because the entries of A^{-1} and therefore also A^{-T} are non-negative. Computing

$$\begin{aligned}
(\mathbb{1}\mathbb{1}^T + A^T)\hat{v} &= \left(\frac{1}{1 + \mathbb{1}^T A^{-1}\mathbb{1}}\right)\left(\mathbb{1}\mathbb{1}^T A^{-T}\mathbb{1} + A^T A^{-T}\mathbb{1}\right) \\
&= \left(\frac{1}{1 + \mathbb{1}^T A^{-1}\mathbb{1}}\right)\left(\mathbb{1}^T A^{-T}\mathbb{1} + 1\right)\mathbb{1} \\
&= \left(\frac{1}{1 + \mathbb{1}^T A^{-1}\mathbb{1}}\right)\left(\mathbb{1}^T A^{-1}\mathbb{1} + 1\right)\mathbb{1} \\
&= \mathbb{1}
\end{aligned}$$

we see that $\hat{v}$ is feasible for $(DP1)$ and satisfies the n constraints (2.24) with equality. Therefore $(\hat{s}, \hat{v})$ are a pair of primal and dual feasible solutions satisfying complementary slackness and $\hat{s}$ is an optimal solution for $(LP1)$. Calculating $\mathbb{1}^T\hat{s}$ we see the optimal value is as claimed. □

2.1.4 Step 4: Vary Parameters to Find Weakest Bound

In step 4, we vary LB over all $a, b, c, d > 0$ such that $ad \leq bc$ and $(g, h)^T \in \operatorname{int} Q(a, b, c, d)$ to find the weakest lower bound $\hat{LB}$ on the optimal value of $(S(\mathcal{T}_2, \mathcal{Q}))$. When Lemma 2.1.3 is used to find the optimal solution, the corresponding expression is of the form $1 - \frac{1}{1+\sum_{i=1}^m a_i}$ where a_i for $i \in \{1, \ldots, m\}$ are the entries of A^{-1}. In these instances, the following lemma can be used in finding the weakest lower bound.

Lemma 2.1.4. *Let $m \in \mathbb{Z}_+$. Let $a := (a_1, a_2, \ldots, a_m)$. Then $a := \hat{a}$ is an assignment of real values to non-negative variables a_i for $i \in \{1, \ldots, m\}$ minimizing the expression $1 - \frac{1}{1+\sum_{i=1}^m a_i}$ if and only if $a := \hat{a}$ is an assignment of real values to non-negative variables a_i for $i \in \{1, \ldots, m\}$ minimizing the expression $\sum_{i=1}^m a_i$.*

Proof. Let $x \geq 0$. Minimizing $1 - \frac{1}{x}$ is equivalent to maximizing $\frac{1}{x}$. Maximizing $\frac{1}{x}$ is equivalent to minimizing x. Setting $x = 1 + \sum_{i=1}^m a_i$, the result follows. □

After calculating the weakest lower bound $\hat{LB}$ we conclude $\rho[\mathcal{T}_2, \mathcal{Q}] \leq \frac{1}{\hat{LB}}$.

Step 3 and step 4 are best illustrated by way of example, which is provided in the next section.

2.2 Previously Established Upper Bound on $\rho[\mathcal{T}_2, \mathcal{Q}]$

We present a proof that $\rho[\mathcal{T}_2, \mathcal{Q}] \leq \frac{3}{2}$ following [11] and the proof strategy outlined above. We lower bound the optimal value of $(P(a,b,c,d,g,h))$ for all $a,b,c,d > 0$ with $ad \leq bc$ and $(g,h)^T \in \operatorname{int} Q(a,b,c,d)$. Since we vary a, b, c, and d restricted only by $a,b,c,d > 0$ and $ad \leq bc$, the below remark shows that we can consider only the parameterizations where $(g,h)^T \in (0,1)^2$ or $h \leq 0$.

Remark 2.2.1. *Let $Q_1 := Q(a,b,c,d)$ for $a,b,c,d > 0$ with $ad \leq bc$ be a normalized quadrilateral. Let $f = (g,h)^T \in \operatorname{int} Q_1 \setminus (0,1)^2$. Then there exist $\hat{a}, \hat{b}, \hat{c}$, and $\hat{d}$ with $\hat{a}\hat{d} \leq \hat{b}\hat{c}$ and affine unimodular transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that:*

1. *$\phi(Q_1) = Q(\hat{a}, \hat{b}, \hat{c}, \hat{d})$, and*
2. *$\phi(f) = (\hat{g}, \hat{h})^T$ where $\hat{h} \leq 0$.*

Proof. If $h \leq 0$, then no transformation is required. If $h \geq 0$, then either (1) $g \geq 1$, (2) $h \geq 1$, or (3) $g \leq 0$ because $(g,h)^T \notin (0,1)^2$. In all three cases, the affine unimodular transformation required is a clockwise rotation (by $\frac{\pi}{2}$, π or $\frac{3\pi}{2}$, respectively).

Case 1: $g \leq 1$: Choose $(\hat{a}, \hat{b}, \hat{c}, \hat{d}) := \left(\frac{1}{b}, \frac{1}{d}, \frac{1}{a}, \frac{1}{c}\right)$. Note that $\hat{a}\hat{d} = \frac{1}{bc} \leq \frac{1}{da} = \hat{b}\hat{c}$ where the inequality holds because $ad \leq bc$. Consider the affine unimodular transformation $\phi_{\frac{\pi}{2}}$ given by

$$\phi_{\frac{\pi}{2}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\right) + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Then $\phi_{\frac{\pi}{2}}(Q(a,b,c,d)) = Q(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ and $\phi_{\frac{\pi}{2}}\left(\begin{bmatrix} g \\ h \end{bmatrix}\right) = \begin{bmatrix} h \\ 1-g \end{bmatrix}$ and so $\hat{h} = 1 - g \leq 0$.

Case 2: $h \leq 1$: Choose $(\hat{a}, \hat{b}, \hat{c}, \hat{d}) := (d, c, b, a)$. Note that $\hat{a}\hat{d} = da \leq cb = \hat{b}\hat{c}$ where the inequality holds because $ad \leq bc$. Consider the affine unimodular transformation ϕ_π given by

$$\phi_{\pi}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\right) + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Then $\phi_\pi(Q(a,b,c,d)) = Q(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ and $\phi_\pi\left(\begin{bmatrix} g \\ h \end{bmatrix}\right) = \begin{bmatrix} 1-g \\ 1-h \end{bmatrix}$ and so $\hat{h} = 1 - h \leq 0$.

Case 3: $g \leq 0$: Choose $(\hat{a}, \hat{b}, \hat{c}, \hat{d}) := (\frac{1}{c}, \frac{1}{a}, \frac{1}{d}, \frac{1}{b})$. Note that $\hat{a}\hat{d} = \frac{1}{cb} \leq \frac{1}{ad} = \hat{b}\hat{c}$ where the inequality holds because $ad \leq bc$. Consider the affine unimodular transformation $\phi_{\frac{3\pi}{2}}$ given by

$$\phi_{\frac{3\pi}{2}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\right) + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Then $\phi_{\frac{3\pi}{2}}(Q(a,b,c,d)) = Q(\hat{a},\hat{b},\hat{c},\hat{d})$ and $\phi_{\frac{3\pi}{2}}\left(\begin{bmatrix} g \\ h \end{bmatrix}\right) = \begin{bmatrix} 1-h \\ g \end{bmatrix}$ and so $\hat{h} = g \leq 0$. $\square$

First we consider the case where $(g,h)^T \in (0,1)^2$ and find a parameterized lower bound when using the intersection cuts generated by the four fixed triangles F_1, F_2, F_3, and F_4. For $i \in \{1,2,3,4\}$, fixed triangle F_i is the type 2 triangle containing $[0,1]^2$ with vertex v^i whose opposite edge contains at least two integer points and is defined by the facet of $[0,1]^2$ separating $v^{i+2 \mod 4}$ from $[0,1]^2$. See Figure 2.2.

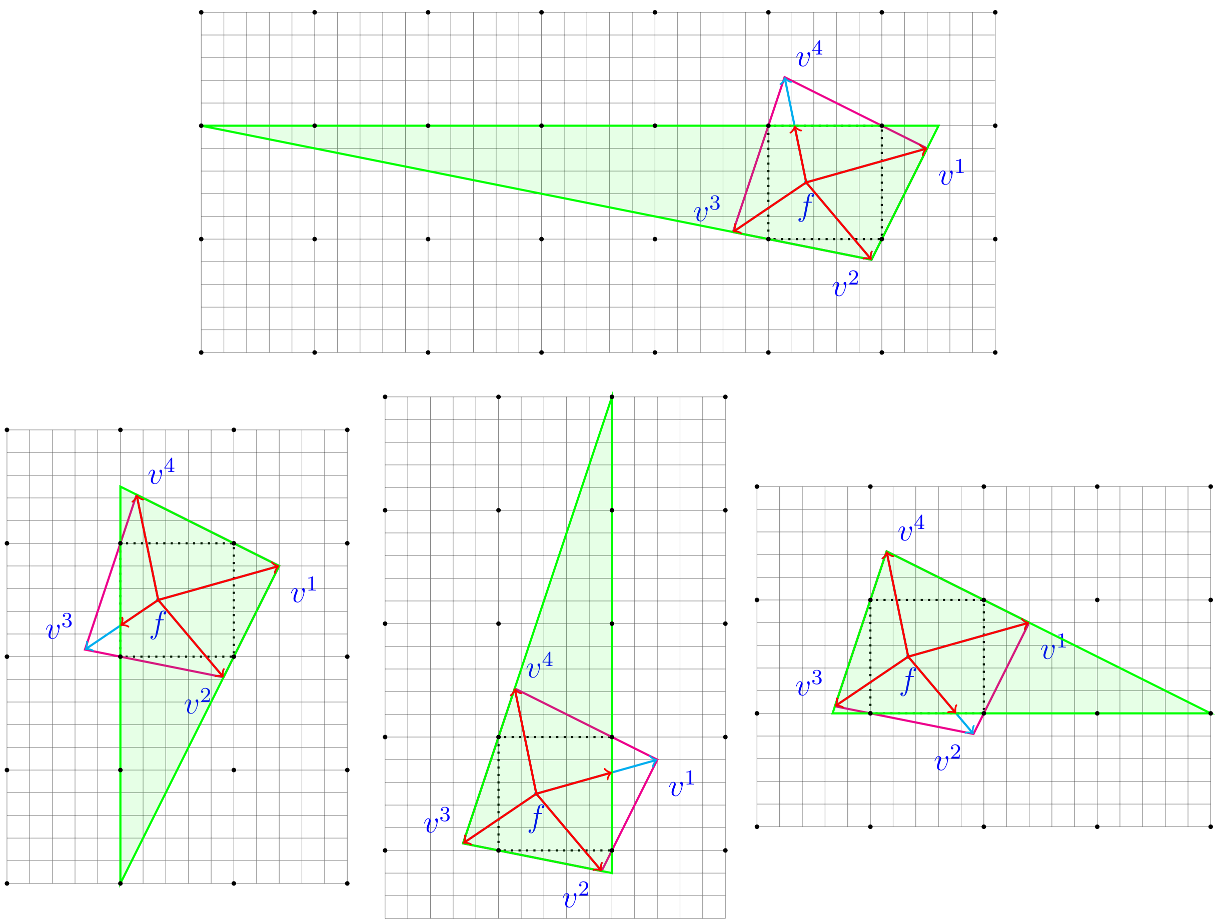


Figure 2.2: Fixed triangles F_1 (bottom left), F_2 (top), F_3 (bottom center), and F_4 (bottom right) for $a = \frac{1}{5}$, $b = 2$, $c = 3$, $d = \frac{1}{2}$, $g = \frac{1}{3}$, and $h = \frac{1}{2}$.

Lemma 2.2.2. *For fixed $a, b, c, d > 0$ such that $ad \leq bc$ and $(g, h)^T \in (0,1)^2$, let (P') be the parameterized linear program obtained from (P(a, b, c, d, g, h)) by replacing constraint (2.19) with the constraint $s \geq \mathbb{0}$ and the intersection cuts for fixed triangles F_1, F_2, F_3, and F_4. Then, the optimal value of (P') is equal to*

$$LB_0^Q(a,b,c,d,g,h) := 1 - \frac{1}{1 + t_1 + t_2 + t_3 + t_4}$$

$$\text{where} \qquad t_1 = (b+d)(1-g) \quad , \quad t_2 = \frac{(a+b)h}{ab}$$
$$t_3 = (a+c)g \quad , \quad \text{and} \quad t_4 = \frac{(c+d)(1-h)}{cd}.$$

Proof. To calculate the constraints of optimization problem (P'), we must calculate the intersection cuts generated by F_1, F_2, F_3, and F_4. First, we calculate the cut coefficients $\psi_{f;F_1}(r^j)$ for $j \in \{1,2,3,4\}$. By construction, v^1, v^2, and v^4 are on the boundary of F_1 and therefore $\psi_{f;F_1}(r^1) = \psi_{f;F_1}(r^2) = \psi_{f;F_1}(r^4) = 1$. To calculate $\psi_{f;F_1}(r^3)$ we need to find $\lambda \geq 0$ such that $f + \lambda(v^3 - f)$ is on the line $x = 0$. So, we need to solve $g + \lambda(\frac{-1}{a+c} - g) = 0$ for λ. The solution is $\lambda = \frac{(a+c)g}{1+(a+c)g}$ and therefore $\psi_{f;F_1}(r^3) = 1 + \frac{1}{(a+c)g}$. The computations for F_2, F_3, and F_4 are analogous.

So optimization problem (P') is given by

$$\begin{aligned} \min \quad & \mathbb{1}^T s \\ \text{subject to} \quad & \left[\mathbb{1}\mathbb{1}^T + A\right] s \geq \mathbb{1} \\ & s \geq \mathbb{0} \end{aligned}$$

where

$$A := \begin{bmatrix} \left(\frac{1}{(b+d)(1-g)}\right) & 0 & 0 & 0 \\ 0 & \left(\frac{ab}{(a+b)h}\right) & 0 & 0 \\ 0 & 0 & \left(\frac{1}{(a+c)g}\right) & 0 \\ 0 & 0 & 0 & \left(\frac{cd}{(c+d)(1-h)}\right) \end{bmatrix}.$$

We compute

$$A^{-1} = \begin{bmatrix} (b+d)(1-g) & 0 & 0 & 0 \\ 0 & \frac{(a+b)h}{ab} & 0 & 0 \\ 0 & 0 & (a+c)g & 0 \\ 0 & 0 & 0 & \frac{(c+d)(1-h)}{cd} \end{bmatrix}.$$

Note that the entries of A^{-1} are non-negative because $a, b, c, d > 0$ and $(g, h)^T \in (0, 1)^2$. Apply Lemma 2.1.3 to conclude the optimal value of (P') is

$$1 - \frac{1}{1 + \mathbb{1}^T A^{-1} \mathbb{1}} = 1 - \frac{1}{1 + t_2 + t_2 + t_3 + t_4}$$

as required. $\square$

The analysis is similar for the case where $h \leq 0$. However, the intersection cut for fixed triangle F_4 cannot be used because $(g, h)^T \notin \operatorname{int} F_4$. We instead find the parameterized lower bound when using the intersection cuts for F_1, F_2, and F_3.

Lemma 2.2.3. *For fixed $a, b, c, d > 0$ such that $ad \leq bc$, and $(g, h)^T \in \operatorname{int} Q(a, b, c, d)$ such that $h \leq 0$, let (P') be the parameterized linear program obtained from $(P(a, b, c, d, g, h))$ by replacing constraint (2.19) with the constraint $s \geq \mathbb{0}$ and the intersection cuts for fixed triangles F_1, F_2, and F_3. Then, the optimal value of (P') is equal to*

$$LB_1^Q(a, b, c, d, g, h) := 1 - \frac{1}{1 + t_1 + t_3 + t_4}$$

$$\textit{where} \qquad t_1 = (b + d)(1 - g) \quad , \quad t_3 = (a + c)g \quad , \quad \textit{and} \quad t_4 = \frac{(c + d)(1 - h)}{cd}.$$

Proof. The optimization problem (P') is given by

$$\begin{aligned} \min \quad & \mathbb{1}^T s \\ \text{subject to} \quad & \left[\mathbb{1}\mathbb{1}^T + A\right] s \geq \mathbb{1} \\ & s \geq \mathbb{0} \end{aligned}$$

where

$$A := \begin{bmatrix} \left(\frac{1}{(b+d)(1-g)}\right) & 0 & 0 & 0 \\ 0 & 0 & \left(\frac{1}{(a+c)g}\right) & 0 \\ 0 & 0 & 0 & \left(\frac{cd}{(c+d)(1-h)}\right) \end{bmatrix}.$$

Note that A is a row sub-matrix of the constraint matrix for Lemma 2.2.2. We denote the dual of this linear program by (D'). It is given by

$$\begin{aligned} \max \quad & \mathbb{1}^T v \\ \text{subject to} \quad & \left[\mathbb{1}\mathbb{1}^T + A^T\right] v \leq \mathbb{1} \\ & v \geq \mathbb{0}. \end{aligned}$$

Consider the (possibly infeasible) solution $\hat{s}$ obtained by setting the intersection cut constraints tight and setting $s_2 = 0$. We can compute

$$\begin{bmatrix} \hat{s}_1 \\ \hat{s}_3 \\ \hat{s}_4 \end{bmatrix} = \left(\mathbb{1}\mathbb{1}^T + \begin{bmatrix} \left(\frac{1}{(b+d)(1-g)}\right) & 0 & 0 \\ 0 & \left(\frac{1}{(a+c)g}\right) & 0 \\ 0 & 0 & \left(\frac{cd}{(c+d)(1-h)}\right) \end{bmatrix} \right)^{-1} \mathbb{1}.$$

That is,

$$\begin{bmatrix} \hat{s}_1 \\ \hat{s}_3 \\ \hat{s}_4 \end{bmatrix} = \left(\frac{1}{1 + \mathbb{1}^T B^{-1} \mathbb{1}} \right) B^{-1} \mathbb{1}$$

where

$$B = \begin{bmatrix} \left(\frac{1}{(b+d)(1-g)}\right) & 0 & 0 \\ 0 & \left(\frac{1}{(a+c)g}\right) & 0 \\ 0 & 0 & \left(\frac{cd}{(c+d)(1-h)}\right) \end{bmatrix}.$$

and thus

$$B^{-1}\mathbb{1} = \begin{bmatrix} (b+d)(1-g) \\ (a+c)g \\ \frac{(c+d)(1-h)}{cd} \end{bmatrix} = \begin{bmatrix} t_1 \\ t_3 \\ t_4 \end{bmatrix}.$$

Note that $\hat{s} \geq \mathbb{0}$ because $g \in [0,1]$, $h \leq 0$ and $a, b, c, d > 0$; therefore it is feasible for (P'). It is straightforward to check that $\hat{v} := \begin{bmatrix} \hat{s}_1 & \hat{s}_3 & \hat{s}_4 \end{bmatrix}^T$ is feasible for (D'). The first, third and fourth constraints hold at equality by construction. The second constraint requires $\hat{v}_1 + \hat{v}_2 + \hat{v}_3 \leq 1$; since $\hat{v}_1 + \hat{v}_2 + \hat{v}_3 = \frac{\mathbb{1}^T B^{-1} \mathbb{1}}{1 + \mathbb{1}^T B^{-1} \mathbb{1}}$ where $\mathbb{1}^T B^{-1} \mathbb{1} \geq 0$, this constraint holds. So $(\hat{s}, \hat{v})$ are a pair of primal-dual feasible solutions satisfying complementary slackness and therefore $\hat{s}$ is optimal for (P') and $\hat{v}$ is optimal for (D'). Therefore the optimal value of (P') is

$$\mathbb{1}^T \hat{s} = 1 - \frac{1}{1 + t_1 + t_3 + t_4}$$

as required. $\square$

Now, we can find the worst case lower bound on the optimal value of $(S(\mathcal{T}_2, \mathcal{Q}))$ following [11] closely.

Theorem 2.2.4 (Theorem 1.7 in [11])**.** *The optimal value of* $(P(a,b,c,d,g,h))$ *is at least* $\frac{2}{3}$ *for all* $a,b,c,d > 0$ *such that* $ad \le bc$ *and* $(g,h)^T \in \operatorname{int} Q(a,b,c,d)$*. Therefore* $\rho[\mathcal{T}_2, \mathcal{Q}] \le \frac{3}{2}$*.*

Proof. By Remark 2.2.1, it suffices to show that the optimal value of $(P(a,b,c,d,g,h))$ is at least $\frac{2}{3}$ for all $a,b,c,d > 0$ such that $ad \le bc$ and $(g,h)^T \in \operatorname{int} Q(a,b,c,d)$ where either: (1) $(g,h)^T \in (0,1)^2$, or (2) $h \le 0$.

<u>Case 1: $(g,h)^T \in (0,1)^2$</u>

The optimal value of $(P(a,b,c,d,g,h))$ is lower-bounded by $LB_0^Q := LB_0^Q(a,b,c,d,g,h)$ as provided by Lemma 2.2.2. Now $LB_0^Q = 1 - \frac{1}{1+T(a,b,c,d,g,h)}$ where

$$\begin{aligned} T(a,b,c,d,g,h) &:= (b+d)(1-g) + \frac{(a+b)h}{ab} + (a+c)g + \frac{(c+d)(1-h)}{cd} \\ &= b + \frac{1}{c} + d + \frac{1}{d} + (a-b+c-d)\,g + \left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c} - \frac{1}{d}\right) h. \end{aligned}$$

Since $T(a,b,c,d,g,h) \ge 0$, LB_0^Q is smallest whenever T is smallest by Lemma 2.1.4. Define $T' : \mathbb{R}_{++} \to \mathbb{R}$ by

$$T'(\epsilon) := \inf \left\{ T(a,b,c,d,g,h) : (g,h)^T \in [0,1]^2, a,b,c,d \in \left[\epsilon, \frac{1}{\epsilon}\right] \right\}.$$

For fixed $\epsilon > 0$, $T'(\epsilon)$ is the infimum of a continuous function over a compact set. It therefore attains its minimum at some fixed values for $a,b,c,d,g,$ and h. For fixed $a,b,c,$ and d, $T(a,b,c,d,g,h)$ is affine in g and h and therefore attains its minimum for $g = 0$ or $g = 1$ and also $h = 0$ or $h = 1$. Now

1. $T(a,b,c,d,0,0) = b + \frac{1}{c} + d + \frac{1}{d} \ge 2$ because $x + \frac{1}{x} \ge 2$ for all $x > 0$,
2. $T(a,b,c,d,1,0) = a + c + \frac{1}{c} + \frac{1}{d} \ge 2$,
3. $T(a,b,c,d,0,1) = \frac{1}{a} + b + \frac{1}{b} + d \ge 2$, and
4. $T(a,b,c,d,1,1) = a + \frac{1}{a} + \frac{1}{b} + c \ge 2$.

Therefore $T'(\epsilon) \ge 2$ for all $\epsilon > 0$. It follows that $T(a,b,c,d,g,h) \ge 2$ for all $a,b,c,d > 0$ such that $ad \le bc$ and $(g,h)^T \in (0,1)^2$. Therefore $LB_0^Q \ge 1 - \frac{1}{1+2} = \frac{2}{3}$.

<u>Case 2: $h \le 0$</u>

The optimal value of $(P(a,b,c,d,g,h))$ is lower-bounded by $LB_1^Q := LB_1^Q(a,b,c,d,g,h)$ as provided by Lemma 2.2.3. Now $LB_1^Q = 1 - \frac{1}{1+R(a,b,c,d,g,h)}$ where

$$\begin{aligned} R(a,b,c,d,g,h) &= (b+d)(1-g) + (a+c)g + \frac{(c+d)(1-h)}{cd} \\ &= b + \frac{1}{c} + d + \frac{1}{d} + (a-b+c-d)g - \left(\frac{1}{c} + \frac{1}{d}\right) h \end{aligned}$$

Since $R(a,b,c,d,g,h) \geq 0$, LB_1^Q is smallest whenever R is smallest by Lemma 2.1.4. Note that

$$R(a,b,c,d,g,h) \geq R(a,b,c,d,g,0) = b + \frac{1}{c} + d + \frac{1}{d} + (a-b+c-d)g$$

since $h \leq 0$ and the coefficient of h is non-positive. Define $R' : \mathbb{R}_{++} \to \mathbb{R}$ by

$$R'(\epsilon) := \inf\left\{ R(a,b,c,d,g,0) : g \in [0,1], a,b,c,d \in \left[\epsilon, \frac{1}{\epsilon}\right]\right\}.$$

For fixed $\epsilon > 0$, $R'(\epsilon)$ is the infimum of a continuous function over a compact set. It therefore attains its minimum at some fixed values for a,b,c,d and g. For fixed a,b,c, and d, $R(a,b,c,d,g,h)$ is affine in g. Now

1. $R(a,b,c,d,0,0) = b + \frac{1}{c} + d + \frac{1}{d} \geq 2$ because $x + \frac{1}{x} \geq 2$ for all $x > 0$, and
2. $R(a,b,c,d,1,0) = a + c + \frac{1}{c} + \frac{1}{d} \geq 2$.

Therefore $R'(\epsilon) \geq 2$ for all $\epsilon > 0$. It follows that $R(a,b,c,d,g,h) \geq 2$ for all $a,b,c,d > 0$ such that $ad \leq bc$ and $(g,h)^T \in \operatorname{int} Q(a,b,c,d) \cap \{(x,y)^T \in \mathbb{R}^2 : y \leq 0\}$. Therefore $LB_1^Q \geq 1 - \frac{1}{1+2} = \frac{2}{3}$.

Therefore the optimal value of $(S(\mathcal{T}_2, \mathcal{Q}))$ is at least $\frac{2}{3}$ and the bound on $\rho[\mathcal{T}_2, \mathcal{Q}]$ follows immediately from Theorem 2.1.1. $\square$

The preceding bound is tight insofar as the linear program (P') was solved to optimality in Lemma 2.2.2 and Lemma 2.2.3. It is also tight insofar as it gives a family of quadrilateral and fractional point pairs whose corresponding lower bound in Lemma 2.2.2 tends to $\frac{2}{3}$. This family of quadrilateral and fractional point pairs is given by

$$\left\{ \left(Q\left(1, \sqrt{\frac{\epsilon}{1-\epsilon}}, \sqrt{\frac{1-\epsilon}{\epsilon}}, 1\right), (\epsilon,\epsilon)^T \right) : \epsilon \in (0,1) \right\}.$$

Evaluating T as defined in the previous proof, we have

$$T\left(1, \sqrt{\frac{\epsilon}{1-\epsilon}}, \sqrt{\frac{1-\epsilon}{\epsilon}}, 1, \epsilon, \epsilon\right) = 2 + 4\sqrt{\epsilon(1-\epsilon)}$$

which tends to 2 for $\epsilon \to 0$.

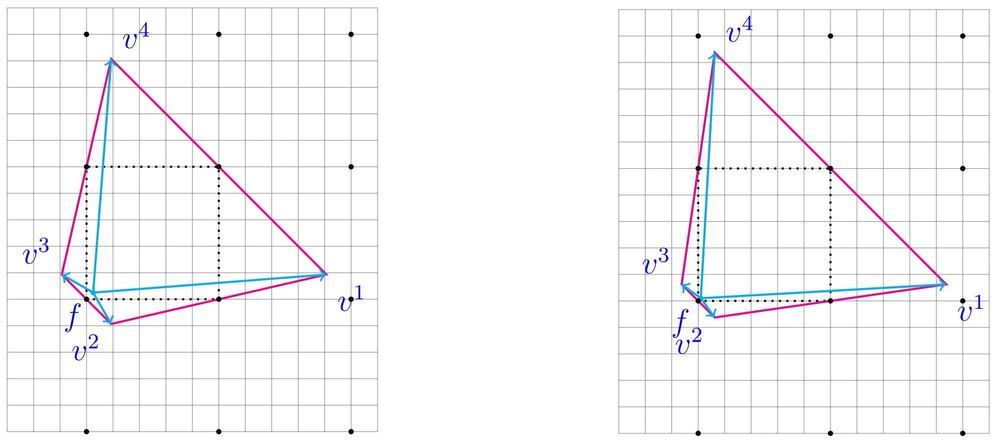


Figure 2.3: Examples of quadrilateral and fixed point pairs from the "weakest lower bound" family for $\epsilon = \frac{1}{20}$ (left) and $\epsilon = \frac{1}{50}$ (right).

Although the previous proof used affineness in g and h to analyze $T(a, b, c, d, g, h)$ for fixed a, b, c, and d, note that $T(a, b, c, d, g, h)$ is also a separable function of a, b, c, and d for fixed g and h. It can be expressed as

$$T(a, b, c, d, g, h) = \left(ga + \frac{h}{a}\right) + \left((1-g)b + \frac{h}{b}\right) + \left(gc + \frac{1-h}{c}\right) + \left((1-g)d + \frac{1-h}{d}\right).$$

Remark 2.2.5 (Remark 4.4 in [11]). *For every pair of constants $c_1, c_2 > 0$, the function $x \to c_1 x + c_2 \frac{1}{x}$ attains it minimum value $2\sqrt{c_1 c_2}$ uniquely at $x = \sqrt{\frac{c_2}{c_1}}$.*

So given fixed $(g, h)^T \in (0, 1)^2$, the quadrilateral with the corresponding weakest lower bound is $Q(a, b, c, d)$ for $a = \sqrt{\frac{h}{g}}$, $b = \sqrt{\frac{h}{1-g}}$, $c = \sqrt{\frac{1-h}{g}}$, and $d = \sqrt{\frac{1-h}{1-g}}$. The corresponding lower bound is $1 - \frac{1}{1+V}$ where

$$V = 2\left(\sqrt{gh} + \sqrt{(1-g)h} + \sqrt{g(1-h)} + \sqrt{(1-g)(1-h)}\right) = 2\left(\sqrt{g} + \sqrt{1-g}\right)\left(\sqrt{h} + \sqrt{1-h}\right).$$

This expression is largest for $(g, h)^T = \left(\frac{1}{2}, \frac{1}{2}\right)$ where the bound is equal to $1 - \frac{1}{1+4} = \frac{4}{5}$ and corresponds to the square $a = b = c = d = 1$. It is smallest for $(g, h)^T$ tending towards one of the vertices $[0, 1]^2$. The family of "weakest lower bound" quadrilateral and fractional point pairs was constructed by taking $(g,h)^T \to (\epsilon,\epsilon)^T$ and $\epsilon \to 0$. Tending the values of g and h to 0 at different rates can give rise to different, though qualitatively similar, families of quadrilaterals with weak lower bounds. In these families, the value of b will tend to 0, the value of c will grow without bound, and the value of d will be around 1. In the limit, these families of quadrilaterals all tend to a triangle with vertex $(0,0)^T$ where the fractional point is equal to (or as close as possible to) the vertex.

2.2.1 Specialization for Parallelograms

The families of quadrilateral and fractional point pairs with "weakest lower bound" constructed in the previous section do not include any parallelograms. In fact, specializing Lemma 2.2.2 to parallelograms, the resulting optimal value of (P') does not depend on g and h and can be lower-bounded more tightly. Before stating the specialized results, we note that $\{Q(\alpha,\beta,\beta,\alpha) : \beta \geq \alpha > 0\}$ is a set of representatives (with respect to unimodular transformation) for the family of normalized parallelograms.

Lemma 2.2.6. *For fixed $\alpha, \beta > 0$ such that $\alpha \leq \beta$ and $(g,h)^T \in (0,1)^2$, let (P') be the parameterized linear program obtained from $(P(\alpha,\beta,\beta,\alpha,g,h))$ by replacing constraint (2.19) with the constraint $s \geq \mathbb{0}$ and the intersection cuts for fixed triangles F_1, F_2, F_3, and F_4. Then, the optimal value of (P') is equal to*

$$LB_0^P(\alpha,\beta) := 1 - \frac{1}{1+\alpha+\beta+\frac{1}{\alpha}+\frac{1}{\beta}}.$$

Proof. Substituting into the bound from Lemma 2.2.2 we get

$$\begin{aligned} LB_0^Q(\alpha,\beta,\beta,\alpha,g,h) &= 1 - \frac{1}{1+(\alpha+\beta)(1-g)+\frac{(\alpha+\beta)h}{\alpha\beta}+(\alpha+\beta)g+\frac{(\alpha+\beta)(1-h)}{\alpha\beta}} \\ &= 1 - \frac{1}{1+(\alpha+\beta)+\frac{(\alpha+\beta)}{\alpha\beta}} \\ &= 1 - \frac{1}{1+\alpha+\beta+\frac{1}{\alpha}+\frac{1}{\beta}} \end{aligned}$$

as required. □

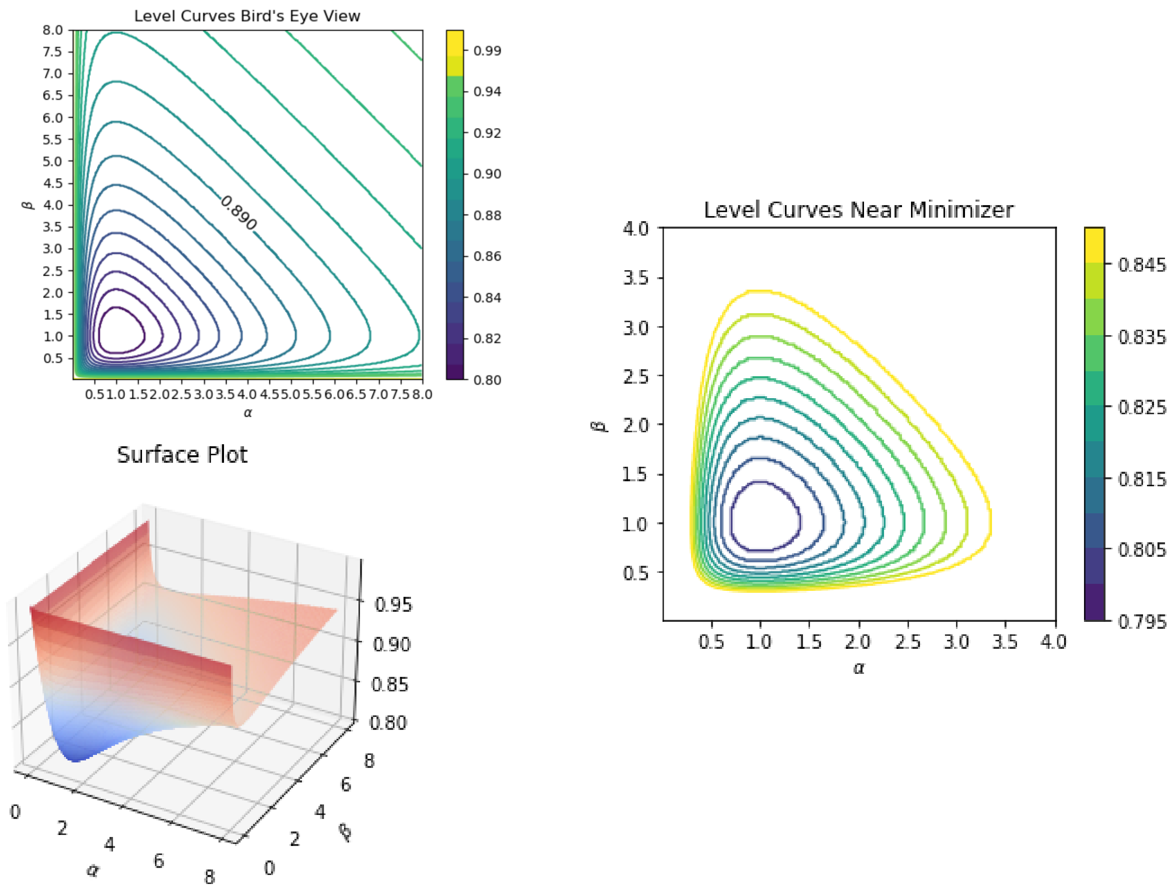


Figure 2.4: Contour and surface plots of $(\alpha, \beta) \to 1 - \frac{1}{1+\alpha+\beta+\frac{1}{\alpha}+\frac{1}{\beta}}$ for $\alpha, \beta \in [0, 8]$. All surface plots, contour plots, 2D plots, and images of polyhedra in $\mathbb{R}^3$ in this thesis were produced using Matplotlib [53].

Lemma 2.2.7. *For fixed $\alpha, \beta > 0$ such that $\alpha \leq \beta$, and $(g, h)^T \in \operatorname{int} Q(\alpha, \beta, \beta, \alpha)$ such that $h \leq 0$, let (P') be the parameterized linear program obtained from $(P(\alpha, \beta, \beta, \alpha, g, h))$ by replacing constraint (2.19) with the constraint $s \geq \mathbb{0}$ and the intersection cuts for fixed triangles F_1, F_2, and F_3. Then, the optimal value of (P') is equal to*

$$LB_1^P(\alpha, \beta, h) := 1 - \frac{1}{1 + \alpha + \beta + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)(1 - h)}.$$

Proof. Substituting into the bound from Lemma 2.2.3 we get

$$\begin{aligned}
LB_1^Q(\alpha,\beta,\beta,\alpha,g,h) &= 1 - \frac{1}{1+(\alpha+\beta)(1-g)+(\alpha+\beta)g+\frac{(\alpha+\beta)(1-h)}{\alpha\beta}} \\
&= 1 - \frac{1}{1+(\alpha+\beta)+\frac{(\alpha+\beta)(1-h)}{\alpha\beta}} \\
&= 1 - \frac{1}{1+\alpha+\beta+\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)(1-h)}
\end{aligned}$$

as required. $\square$

Theorem 2.2.8 (Specialization of Theorem 2.2.4.)**.** *The optimal value of* $(P(\alpha,\beta,\beta,\alpha,g,h))$ *is at least* $\frac{4}{5}$ *for all* $\alpha,\beta>0$ *such that* $\alpha\le\beta$ *and* $(g,h)^T\in\operatorname{int}Q(\alpha,\beta,\beta,\alpha)$*. Therefore* $\rho[\mathcal{T}_2,\mathcal{Q}_2]\le\frac{5}{4}$*.*

Proof. By Remark 2.2.1, it suffices to show that the optimal value of $(P(\alpha,\beta,\beta,\alpha,g,h))$ is at least $\frac{4}{5}$ for all $\alpha,\beta>0$ such that $\alpha\le\beta$ and $(g,h)^T\in\operatorname{int}Q(\alpha,\beta,\beta,\alpha)$ where either (1) $(g,h)^T\in(0,1)^2$ or (2) $h\le 0$.

<u>Case 1: $(g,h)^T\in(0,1)^2$</u>: The optimal value of $(P(\alpha,\beta,\beta,\alpha,g,h))$ is lower-bounded by $LB_0^P(\alpha,\beta)$ as given in Lemma 2.2.6. By Lemma 2.1.4, $LB_0(\alpha,\beta)$ is smallest whenever $\alpha+\beta+\frac{1}{\alpha}+\frac{1}{\beta}$ is smallest. Since this is a separable expression in α and β and $x+\frac{1}{x}$ is smallest for $x>0$ when $x=1$ the weakest lower bound is given by $LB_0^P(1,1)=1-\frac{1}{5}=\frac{4}{5}$.

<u>Case 2: $h\le 0$</u>: The optimal value of $(P(\alpha,\beta,\beta,\alpha,g,h))$ is lower-bounded by $LB_1^P(\alpha,\beta,h)$ as given in Lemma 2.2.7. This expression is smallest for $h\le 0$ for $h=0$ where it equals $LB_0^P(\alpha,\beta)$. The bound follows as in Case 1.

The bound on $\rho[\mathcal{T}_2,\mathcal{Q}_2]$ follows immediately from Theorem 2.1.1. $\square$

Corollary 2.2.9. *The optimal value of* $(P(\alpha,\beta,\beta,\alpha,g,h))$ *is at least* $\frac{8}{9}$ *for all* $\alpha,\beta>0$ *with* $\alpha\le\beta$ *such that* $\alpha+\beta+\frac{1}{\alpha}+\frac{1}{\beta}\ge 8$ *and* $(g,h)^T\in\operatorname{int}Q(\alpha,\beta,\beta,\alpha)$*.*

Proof. Following the proof of Theorem 2.2.8 we can check when $LB_0(\alpha,\beta)\ge\frac{8}{9}$. This holds whenever $\alpha+\beta+\frac{1}{\alpha}+\frac{1}{\beta}\ge 8$. $\square$

So, if we use fixed triangles to approximate parallelograms, then the "weakest lower bound" quadrilateral and fractional point pairs are the quadrilateral $Q(1,1,1,1)$ and any fractional point. In particular, the fractional points $\left(\frac{1}{2},\frac{1}{2}\right)^T$ and $(\epsilon,\epsilon)^T$ for $\epsilon\to 0$ provide equal bounds of $\frac{4}{5}$. This is in contrast to the general quadrilateral case where if we fix $(g,h)^T$ and vary a, b, c, and d to find the weakest bound, choosing $(g,h)^T = \left(\frac{1}{2}, \frac{1}{2}\right)^T$ gives the largest bound of $\frac{4}{5}$ and choosing $(g,h)^T \to (\epsilon, \epsilon)^T$ for $\epsilon \to 0$ gives the smallest bound of $\frac{2}{3}$.

Keep in mind that these families of "weak lower bound" examples give instances where replacing the type 2 triangle closure constraint with the intersection cuts for fixed triangles is worst case. The next section addresses how to show a family of instances is provably "hard to approximate" within a given factor for any choices of triangle cuts.

2.3 Previously Established Lower Bound on $\rho[\mathcal{T}, \mathcal{Q}]$

We would like to find a feasible solution to $(S(\mathcal{T}, \mathcal{Q}))$ of objective function value $\frac{8}{9}$. Like $(S(\mathcal{T}_2, \mathcal{Q}))$, this optimization problem can again be parameterized by a, b, c, d, g, h - the triangle closure replaces the type 2 triangle closure in constraint (2.14). Let $(P^{\mathcal{T}}(a, b, c, d, g, h))$ be the optimization problem

$$\begin{aligned} \inf \quad & s_1 + s_2 + s_3 + s_4 && (2.26) \\ \text{s.t.} \quad & s \in \mathcal{T}\left(\begin{pmatrix} g \\ h \end{pmatrix}; \begin{pmatrix} 1 + \frac{1}{b+d} - g \\ \frac{b}{b+d} - h \end{pmatrix}, \begin{pmatrix} \frac{b}{a+b} - g \\ \frac{-ab}{a+b} - h \end{pmatrix}, \begin{pmatrix} \frac{-1}{a+c} - g \\ \frac{a}{a+c} - h \end{pmatrix}, \begin{pmatrix} \frac{d}{c+d} - g \\ 1 + \frac{cd}{c+d} - h \end{pmatrix}\right). && (2.27) \end{aligned}$$

We look for a feasible solution $\hat{s}$ to $(P^{\mathcal{T}}(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{g}, \overline{h}))$ for some choice of $\overline{a}, \overline{b}, \overline{c}, \overline{d} > 0$ with $\overline{a}\overline{d} \leq \overline{b}\overline{c}$ and $(\overline{g}, \overline{h})^T \in \operatorname{int} Q(\overline{a}, \overline{b}, \overline{c}, \overline{d})$. The corresponding objective function value $\mathbb{1}^T\hat{s}$ gives an upper bound on the optimal value of $(S(\mathcal{T}, \mathcal{Q}))$. The specific feasible solution described in Theorem 1.3.15 comes from the family of feasible solutions in the following lemma.

Lemma 2.3.1. *For every $\beta > \alpha > 0$ and $(g,h)^T \in \left(\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta}\right)^2$ the point*

$$\hat{s}(\alpha, \beta, g, h) := \left(\frac{\alpha\beta}{\alpha(\beta - \alpha) + \alpha\beta(1+\beta) + \beta(1+\alpha)}\right) \begin{bmatrix} \beta(1-g) + h \\ \frac{h}{\alpha} + g \\ \beta g + (1-h) \\ \frac{(1-h)}{\alpha} + (1-g) \end{bmatrix}$$

is feasible for $(P^{\mathcal{T}}(\alpha, \beta, \beta, \alpha, g, h))$. Thus

$$1 + \frac{\alpha(\beta - \alpha)}{\alpha\beta(1+\beta) + \beta(1+\alpha)}$$

is a lower bound on $\rho[\mathcal{T}, \mathcal{Q}]$ and $\rho[\mathcal{T}, \mathcal{Q}_2]$ for all $\beta > \alpha > 0$.

Proof. The proof follows [11], where the authors analyze the $(\alpha, \beta, g, h) := (\frac{1}{2}, 2, \frac{1}{2}, \frac{1}{2})$ case. Fix $\beta > \alpha > 0$. Let $f = (g, h)^T$ and let $Q := Q(\alpha, \beta, \beta, \alpha)$ so that

$$r^1 = \left(1 + \frac{1}{\alpha+\beta} - g \ , \ \frac{\beta}{\alpha+\beta} - h\right)^T \quad , \quad r^2 = \left(\frac{\beta}{\alpha+\beta} - g \ , \ \frac{-\alpha\beta}{\alpha+\beta} - h\right)^T$$

$$r^3 = \left(\frac{-1}{\alpha+\beta} - g \ , \ \frac{\alpha}{\alpha+\beta} - h\right)^T \quad , \quad \text{and} \quad r^4 = \left(\frac{\alpha}{\alpha+\beta} - g \ , \ 1 + \frac{\alpha\beta}{\alpha+\beta} - h\right)^T .$$

Given the bounds on α, β, g, and h, clearly $\hat{s}(\alpha, \beta, g, h) \geq \mathbb{0}$. Now, to show that $\hat{s}(\alpha, \beta, g, h)$ is feasible for $(P(\alpha, \beta, \beta, \alpha, g, h))$, we need to show that $\hat{s}(\alpha, \beta, g, h)$ satisfies every intersection cut generated by a maximal lattice-free triangle. Let T be an arbitrary maximal lattice-free triangle containing f in its interior. Let $\psi_i \geq 0$ denote the coefficient of s_i in the intersection cut generated by T. To show $\hat{s}(\alpha, \beta, g, h) := (\hat{s}_1, \hat{s}_2, \hat{s}_3, \hat{s}_4)^T$ satisfies this intersection cut, we will show that $\psi_1\hat{s}_1 + \psi_2\hat{s}_2 + \psi_3\hat{s}_3 + \psi_4\hat{s}_4 \geq 1$.

Now, because T is lattice-free and has three edges, one of the lines defining the edges of T must separate f from two of: $(0,0)^T$, $(0,1)^T$, $(1,0)^T$, and $(1,1)^T$. By rotational symmetry (analogous to Remark 2.2.1) we may assume this line, which we will denote L, separates $(1,0)^T$ and $(1,1)^T$ from f. Then, ψ_1 is given by $\frac{1}{\lambda_1}$ where

$$g + \lambda_1\left(1 + \frac{1}{\alpha+\beta} - g\right) \leq 1 \quad \Longrightarrow \quad \lambda_1 \leq \frac{(\alpha+\beta)(1-g)}{(\alpha+\beta)(1-g)+1}$$

Thus, $\psi_1 \geq 1 + \frac{1}{(\alpha+\beta)(1-g)}$.

Let $r := \left(\frac{\alpha}{1+\alpha}\right) r^1 + \left(\frac{1}{1+\alpha}\right) r^2$. Then

$$\left(\frac{\alpha}{1+\alpha}\right)\psi_1 + \left(\frac{1}{1+\alpha}\right)\psi_2 \geq \psi_{f;T}(r) \geq 1 \tag{2.28}$$

where the first inequality holds because $\psi_{f;T}$ is convex (Remark 1.2.3) and the second inequality holds because $f + r = (1,0)^T$ and there are no integral points in the interior of T.

Let $r' := \left(\frac{\alpha}{1+\alpha}\right) r^3 + \left(\frac{1}{1+\alpha}\right) r^4$. Then

$$\left(\frac{\alpha}{1+\alpha}\right)\psi_3 + \left(\frac{1}{1+\alpha}\right)\psi_4 \geq \psi_{f;T}(r') \geq 1 \tag{2.29}$$

where the first inequality holds because $\psi_{f;T}$ is convex and the second inequality holds because $f + r' = (0,1)^T$ and there are no integral points in the interior of T.

The rest of the proof is divided into two cases, depending on the coefficient ψ_3.

Case 1: $\psi_3 \geq 1 + \frac{\alpha}{\beta(\alpha+\beta)(1-g)}$

We check

$$
\begin{aligned}
&\left(\frac{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+\beta(1+\alpha)}{\alpha\beta}\right)\psi^T\hat{s} \\
&= \left[\beta(1-g)+h\right]\psi_1 + \left[\frac{h}{\alpha}+g\right]\psi_2 + \left[\beta g+(1-h)\right]\psi_3 + \left[\frac{(1-h)}{\alpha}+(1-g)\right]\psi_4 \\
&\geq \left[\beta(1-g)+h\right]\psi_1 + \left(\frac{1+\alpha}{\alpha}\right)h - h\psi_1 + (1+\alpha)g - \alpha g\psi_1 + \ldots \\
&\qquad \ldots + \left[\beta g+(1-h)\right]\psi_3 + \left[\frac{(1-h)}{\alpha}+(1-g)\right]\psi_4 && \text{by (2.28)} \\
&= \left[\beta-(\alpha+\beta)g\right]\psi_1 + \left(\frac{1+\alpha}{\alpha}\right)h + (1+\alpha)g + \ldots \\
&\qquad \ldots + \left[\beta g+(1-h)\right]\psi_3 + \left[\frac{(1-h)}{\alpha}+(1-g)\right]\psi_4 \\
&\geq \left[\beta-(\alpha+\beta)g\right]\psi_1 + \left(\frac{1+\alpha}{\alpha}\right)h + (1+\alpha)g + \left[\beta g+(1-h)\right]\psi_3 \ldots \\
&\qquad \ldots + \left(\frac{1+\alpha}{\alpha}\right)(1-h) - (1-h)\psi_3 + (1+\alpha)(1-g) - \alpha(1-g)\psi_3 && \text{by (2.29)} \\
&= \left[\beta-(\alpha+\beta)g\right]\psi_1 + \left(\frac{1+\alpha}{\alpha}\right)h + (1+\alpha)g + \ldots \\
&\qquad \ldots + \left[(\alpha+\beta)g-\alpha\right]\psi_3 + \left(\frac{1+\alpha}{\alpha}\right)(1-h) + (1+\alpha)(1-g) \\
&= \left[\beta-(\alpha+\beta)g\right]\psi_1 + \left[(\alpha+\beta)g-\alpha\right]\psi_3 + \left(\frac{1+\alpha}{\alpha}\right) + (1+\alpha) \\
&= \left[(\alpha+\beta)(1-g)-\alpha\right]\psi_1 + \left[\beta-(\alpha+\beta)(1-g)\right]\psi_3 + \left(\frac{1+\alpha}{\alpha}\right) + (1+\alpha) \\
&\geq (\alpha+\beta)(1-g) - \alpha + 1 - \frac{\alpha}{(1-g)(\alpha+\beta)} + \ldots \\
&\qquad \ldots + \beta - (\alpha+\beta)(1-g) + \frac{\alpha}{(1-g)(\alpha+\beta)} - \frac{\alpha}{\beta} + \left(\frac{1+\alpha}{\alpha}\right) + (1+\alpha) && \text{by } \psi_1 \text{ \& } \psi_3 \text{ bnds} \\
&= 1 + \beta - \frac{\alpha}{\beta} + \left(\frac{1+\alpha}{\alpha}\right) + 1 \\
&= \left(\frac{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+\beta(1+\alpha)}{\alpha\beta}\right)
\end{aligned}
$$

and therefore $\psi_1\hat{s}_1 + \psi_2\hat{s}_2 + \psi_3\hat{s}_3 + \psi_4\hat{s}_4 \geq 1$ as required.

Case 2: $\psi_3 \leq 1 + \frac{\alpha}{\beta(\alpha+\beta)(1-g)}$

It is straightforward to check that the x-coordinate of $f + \kappa_3 r^3$ is exactly zero for $\kappa_3 = \frac{(\alpha+\beta)g}{(\alpha+\beta)g+1}$. Since $\psi_3 \leq \frac{1}{\kappa_3}$, it follows that the intersection $w = (w_1, w_2)^T$ of T with $\{f + \lambda r^3 : \lambda \geq 0\}$ is such that $w_1 < 0$. Now, since T is convex, two distinct sides of T (not equal to L) separate the segment wf from the points $(0,0)^T$ and $(0,1)^T$. Let line L_1 define the side separating wf from $(0,0)^T$ and line L_2 define the side separating wf from $(0,1)^T$. Because T is maximally lattice-free, L_1 must contain $(0,0)^T$ and L_2 must contain $(0,1)^T$.

We may assume w is the vertex of T at the intersection of L_1 and L_2. Otherwise, whichever of L_1 or L_2 intersects the ray $\{f + \lambda r^3 : \lambda \geq 0\}$ with a smaller x-coordinate can be tilted on its integral point towards the other to obtain an inequality at least as strong. So, $w = f + \tau_3 r^3$ for some $\tau_3 \geq 1 - \frac{\alpha}{\beta(\alpha+\beta)(1-g)+\alpha}$ gives the vertex of T at the intersection of L_1 and L_2. In particular, L_1 is the line through w and $(0,0)^T$ and L_2 is the line through w and $(0,1)^T$.

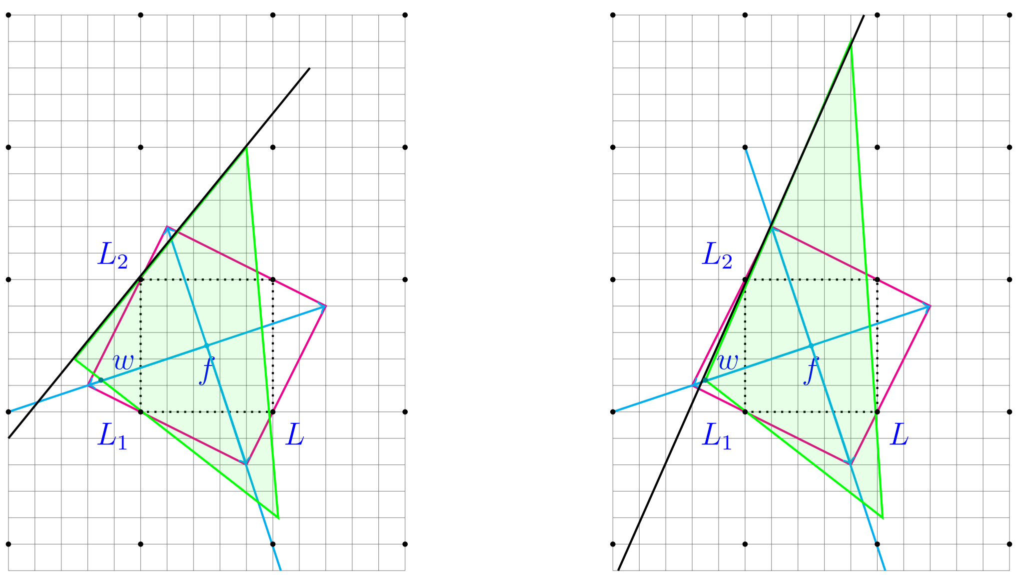


Figure 2.5: In the left image, w is not at the intersection of L_1 and L_2. Line L_2 intersects the ray from f in direction r^3 at a lesser x-coordinate. Tilting L_2 on $(0,1)^T$ gives a new triangle whose intersection cut has the same coefficient on every ray except r^4 where the coefficient is smaller.

Note that the cut coefficient ψ_4 is always determined by the intersection of $\{f + \lambda r^4 : \lambda \geq 0\}$ with L_2 because $g > \frac{\alpha}{\alpha+\beta} = v_x^4$. This intersection has x-coordinate at most one. Checking which facet of T determines the cut coefficient ψ_2 is a bit more work. The intersection point of $\{f + \lambda r^2 : \lambda \geq 0\}$ with $\{(x, y)^T \in \mathbb{R}^2 : x = 1\}$ is $\left(1 \ , \ \frac{-\alpha(h+\beta(1-g))}{\beta-(\alpha+\beta)g}\right)^T$. For $\tau_3 = 1 - \frac{\alpha}{\beta(\alpha+\beta)(1-g)+\alpha}$, the intersection point of line L_1 with $\{(x, y)^T \in \mathbb{R}^2 : x = 1\}$ is also $\left(1 \ , \ \frac{-\alpha(h+\beta(1-g))}{\beta-(\alpha+\beta)g}\right)^T$. Since increasing τ_3 will increase the y-coordinate of the intersection point of L_1 with $\{(x, y) \in \mathbb{R}^2 : x = 1\}$, the cut coefficient ψ_2 is always determined by the intersection of $\{f + \lambda r^2 : \lambda \geq 0\}$ with L_1. This intersection has x-coordinate at most one.

The above two observations show we may assume line L is given by $\{(x, y)^T \in \mathbb{R}^2 : x = 1\}$ because L only determines the cut coefficient ψ_1, and such choice will give the strongest coefficient. So, $\psi_1 = 1 + \frac{1}{(1-g)(\alpha+\beta)}$ exactly.

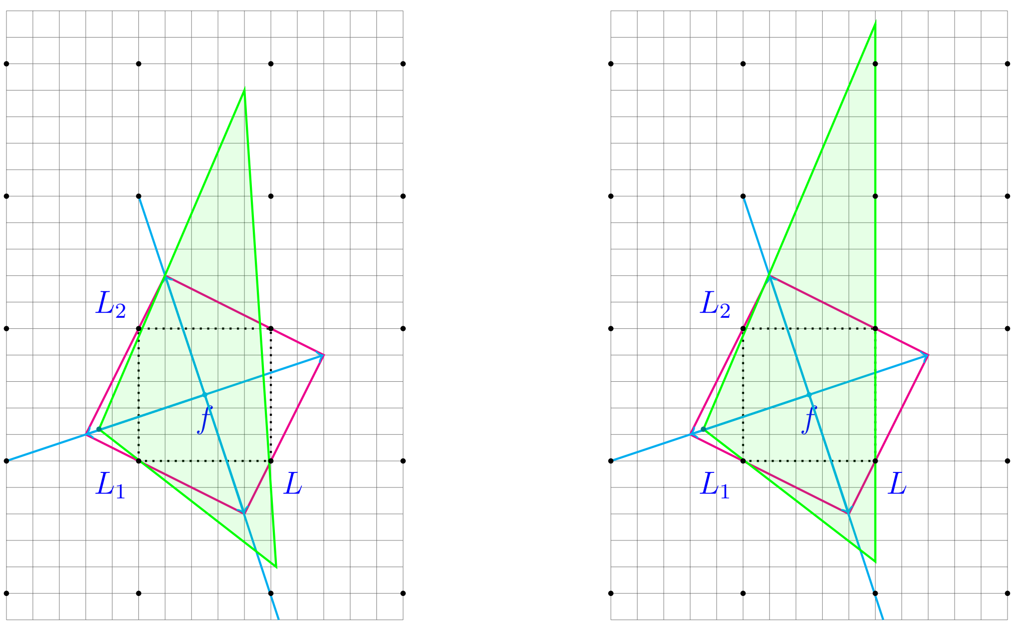


Figure 2.6: In the left image, the facet separating $(1, 0)^T$ and $(1, 1)^T$ from f does not determine the intersection cut coefficient for r^2. Replacing this facet with $\{(x, y)^T : x = 1\}$ gives a new triangle whose intersection cut has the same coefficient on every ray except r^1 where the coefficient is smaller.

Let $r'' := \left(\frac{1}{1+\beta}\right) r^2 + \left(\frac{\beta}{1+\beta}\right) r^3$. Then

$$\left(\frac{1}{1+\beta}\right)\psi_2 + \left(\frac{\beta}{1+\beta}\right)\psi_3 \geq \psi_{f;T}(r'') = 1 \tag{2.30}$$

where the inequality holds because $\psi_{f;T}$ is convex and the equality holds because $f + r'' = (0,0)^T$ and $(0,0)^T$ is on the boundary of T.

Then we check

$$\begin{aligned}
&\left(\frac{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+\beta(1+\alpha)}{\alpha\beta}\right)\psi^T\hat{s} \\
&= \left[\beta(1-g)+h\right]\psi_1 + \left[\frac{h}{\alpha}+g\right]\psi_2 + \left[\beta g+(1-h)\right]\psi_3 + \left[\frac{(1-h)}{\alpha}+(1-g)\right]\psi_4 \\
&\geq \left[\beta(1-g)+h\right]\psi_1 + \left[\frac{h}{\alpha}+g\right](1+\beta-\beta\psi_3) + \ldots \\
&\qquad \ldots + \left[\beta g+(1-h)\right]\psi_3 + \left[\frac{(1-h)}{\alpha}+(1-g)\right](1+\alpha-\alpha\psi_3) && \text{by (2.29) and (2.30)} \\
&= \left[\beta(1-g)+h\right]\psi_1 + \left[\frac{h}{\alpha}+g\right](1+\beta) - \frac{\beta h}{\alpha}\psi_3 + \ldots \\
&\qquad \ldots + \left[\frac{(1-h)}{\alpha}+(1-g)\right](1+\alpha) - (1-g)\alpha\psi_3 \\
&= \left[\beta(1-g)+h\right]\psi_1 + \left[\frac{\beta h}{\alpha}+(1-g)\alpha\right](1-\psi_3) + \left(\frac{1+\alpha}{\alpha}\right) + g\beta + (1-h) \\
&= \left[\beta(1-g)+h\right]\left(1+\frac{1}{(\alpha+\beta)(1-g)}\right) + \left[\frac{\beta h}{\alpha}+(1-g)\alpha\right](1-\psi_3) + \ldots \\
&\qquad \ldots + \left(\frac{1+\alpha}{\alpha}\right) + g\beta + (1-h) && \text{by value of } \psi_1 \\
&= \left(\frac{\beta(1-g)+h}{(\alpha+\beta)(1-g)}\right) + \left[\frac{\beta h}{\alpha}+(1-g)\alpha\right](1-\psi_3) + \left(\frac{1+\alpha}{\alpha}\right) + 1 + \beta \\
&\geq \left(\frac{\beta(1-g)+h}{(\alpha+\beta)(1-g)}\right) - \left[\frac{\beta h}{\alpha}+(1-g)\alpha\right]\left(\frac{\alpha}{\beta(\alpha+\beta)(1-g)}\right) + \left(\frac{1+\alpha}{\alpha}\right) + 1 + \beta && \text{by } \psi_3 \text{ bnd} \\
&= \left(\frac{(\beta^2-\alpha^2)(1-g)}{\beta(\alpha+\beta)(1-g)}\right) + \left(\frac{1+\alpha}{\alpha}\right) + 1 + \beta \\
&= 1 - \frac{\alpha}{\beta} + \left(\frac{1+\alpha}{\alpha}\right) + 1 + \beta \\
&= \left(\frac{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+\beta(1+\alpha)}{\alpha\beta}\right)
\end{aligned}$$

and therefore $\psi_1\hat{s}_1 + \psi_2\hat{s}_2 + \psi_3\hat{s}_3 + \psi_4\hat{s}_4 \geq 1$ as required. □

The lower bound on $\rho[\mathcal{T}, \mathcal{Q}]$ is strongest for $(\alpha, \beta, g, h) = (\frac{1}{2}, 2, \frac{1}{2}, \frac{1}{2})$ - the instance in Theorem 1.3.15. The behaviour of the bound as a function of values α and β will be investigated more closely in Chapter 3.

2.4 Ray-Sliding Triangles

To improve the bounds calculated in step 3 of the proof strategy, our plan is to replace constraint (2.19) with the intersection cuts for different type 2 triangles. We will select these triangles from the family of *ray-sliding triangles*. Here we define the family of ray-sliding triangles for each ray $i \in \{1, 2, 3, 4\}$. For each ray i, this provides an infinite family of type 2 triangles. For each ray i, we also identify a finite subfamily where each member has a "geometric" interpretation.

A ray-sliding triangle is defined with respect to a given normalized quadrilateral $Q(a, b, c, d)$ and a point $(g, h)^T$ in the interior of $Q(a, b, c, d)$. For such fixed quadrilateral and interior point, define the following *ray directions* and corresponding *rays*.

1. Ray direction 1 is given by $r^1 := \begin{pmatrix} 1 + \frac{1}{b+d} - g \\ \frac{b}{b+d} - h \end{pmatrix}$. Ray 1 is $\{(g, h)^T + \lambda r^1 : \lambda \geq 0\}$.
2. Ray direction 2 is given by $r^2 := \begin{pmatrix} \frac{b}{a+b} - g \\ \frac{-ab}{a+b} - h \end{pmatrix}$. Ray 2 is $\{(g, h)^T + \lambda r^2 : \lambda \geq 0\}$.
3. Ray direction 3 is given by $r^3 := \begin{pmatrix} \frac{-1}{a+c} - g \\ \frac{a}{a+c} - h \end{pmatrix}$. Ray 3 is $\{(g, h)^T + \lambda r^3 : \lambda \geq 0\}$.
4. Ray direction 4 is given by $r^4 := \begin{pmatrix} \frac{d}{c+d} - g \\ 1 + \frac{cd}{c+d} - h \end{pmatrix}$. Ray 4 is $\{(g, h)^T + \lambda r^4 : \lambda \geq 0\}$.

The following definitions are made in the context of fixed a, b, c, d, g, and h and therefore also fixed r^1, r^2, r^3, and r^4. To avoid cumbersome notation, we allow these parameters to be implicit from context.

2.4.1 Motivation

Recall that the *fixed triangle for ray* 1 F_1 is constructed so that it generates an intersection cut with the same cut coefficients for r^1, r^2, and r^4 as the intersection cut generated by $Q(a, b, c, d)$. The edges of fixed triangle F_1 are defined by:

1. $\{(x, y)^T \in \mathbb{R}^2 : x = 0\}$;
2. the line joining $(g, h)^T + r^1$ and $(1, 1)^T$; and

3. the line joining $(g,h)^T + r^1$ and $(1,0)^T$.

Hence one vertex of F_1 is $v^1 = (g,h)^T + r^1$ and the opposite facet is defined by $\{(x,y)^T \in \mathbb{R}^2 : x \geq 0\}$.

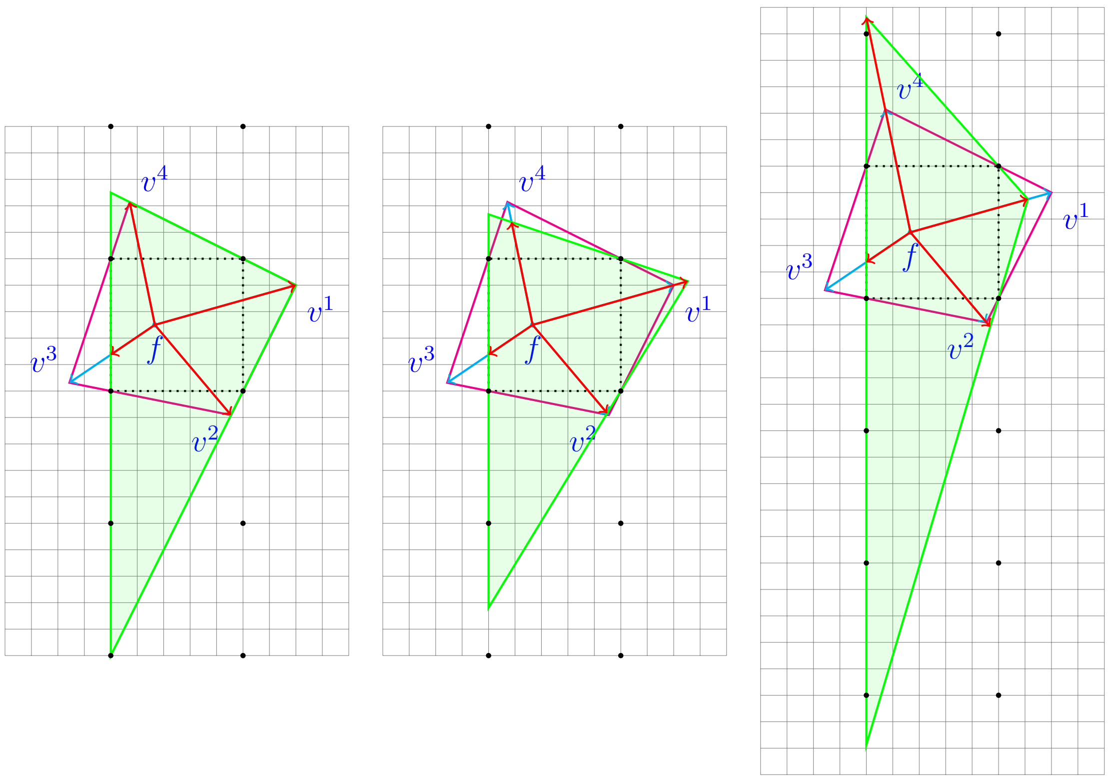


Figure 2.7: (left) Fixed Triangle F_1 and τ-ray-sliding triangle for Ray 1 $RS_1(\tau)$ for $\tau > 1$ (center) and $\tau < 1$ (right) for parameters $a = \frac{1}{5}$, $b = 2$, $c = 3$, $d = \frac{1}{2}$, $g = \frac{1}{3}$, and h $= \frac{1}{2}$.

Suppose we are willing to accept the intersection cut coefficient for r^3 as determined by the edge $\{(x,y)^T \in \mathbb{R}^2 : x = 0\}$, but would like to obtain stronger (i.e. smaller) coefficients for some of r^1, r^2 and r^4. Thinking specifically of improving the coefficient for r^1, we could replace the vertex v^1 with the vertex $(g,h)^T + \tau r^1$ for some $\tau > 1$, but still enforce the opposite facet be defined by $\{x,y)^T \in \mathbb{R}^2 : x \geq 0\}$. One triangle meeting this requirement is the type 2 triangle $RS_1(\tau)$ with edges defined by:

1. $\{(x,y)^T \in \mathbb{R}^2 : x = 0\}$;
2. the line joining $(g,h)^T + \tau r^1$ and $(1,1)^T$; and

3. the line joining $(g,h)^T + \tau r^1$ and $(1,0)^T$.

Choosing $\tau > 1$ will decrease the cut coefficient for r^1, but necessarily increase the coefficients for r^2 and r^4. Conversely, choosing $\tau < 1$ will increase the cut coefficient for r^1 and decrease the coefficient of r^2 and r^4. For some breakpoint $\tau^{bp} < 1$ (depending on the direction of r^2 or r^4), the corresponding coefficient will no longer decrease since it will be determined by the edge $\{(x,y)^T \in \mathbb{R}^2 : x = 0\}$ for all $\tau \leq \tau^{bp}$. One such breakpoint is illustrated in the right image in Figure 2.7. We call $RS_1(\tau)$ a "ray-sliding triangle" because we think of sliding the vertex of our triangle along ray 1. Hence we call ray 1 the *sliding-ray*, ray 3 the *base-ray*, and rays 2 and 4 the *off-rays*. We define the family of ray-sliding triangles for ray i for all $i \in \{1,2,3,4\}$ formally in the next section.

Although this construction may seem a bit unnatural, we've actually seen this family of triangles before in Case 2 of the proof of Lemma 2.3.1. For the given quadrilateral and fractional point, the proof shows that the strongest triangle cuts are generated by triangles of the above construction.

2.4.2 Formal Definition

For $\tau \geq 0$ and $i \in \{1,2,3,4\}$ the *τ-ray-sliding triangle for ray i*, denoted $RS_i(\tau)$, is the maximal lattice-free triangle such that

1. $[0,1]^2 \subseteq RS_i(\tau)$ and
2. $(g,h)^T + \tau r^i$ is a vertex of the triangle,

provided such a triangle exists.

Remark 2.4.1. *$RS_1(\tau)$ is well-defined for appropriate bounds on τ.*

Proof. We denote $(g,h)^T$ by f.

There exists some $\tau_1^{min} \geq 0$ such that the x-coordinate of $f + \tau_1^{min} r^1$ is 1. For all $\tau \leq \tau_1^{min}$, no triangle that contains $[0,1]^2$ and has vertex $f + \tau r^1$ exists because $f + \tau r^1$ is inside $[0,1]^2$ but not a vertex thereof.

There exists some $\tau_1^{max} > 1$ such that the y-coordinate of $f + \tau_1^{max} r^1$ is either 0 or 1. We consider the case where τ_1^{max} is such that the y-coordinate of τ_1^{max} is 1; the other case is symmetric. For all $\tau > \tau_1^{max}$, no triangle that contains $[0,1]^2$ and has vertex $f + \tau r^1$ exists because the convex hull of $\{f + \tau r^1, (1,0)^T, (0,1)^T\}$ contains $(1,1)^T$ in its interior.

Now consider $\tau \in (\tau_1^{min}, \tau_1^{max}]$. Let $L_{(1,0)}(\tau)$ be the line through $f + \tau r^1$ and $(1,0)^T$. Let $L_{(1,1)}(\tau)$ be the line through $f + \tau r^1$ and $(1,1)^T$. Let T be the triangle with edges defined by $L_{(1,0)}(\tau)$, $L_{(1,1)}(\tau)$ and $\{(x,y) \in \mathbb{R}^2 : x = 0\}$. Then T is a maximal lattice-free triangle with vertex $f + \tau r^1$ such that $[0,1]^2 \subseteq T$.

In fact, T is the only such maximal lattice-free triangle. Consider any other maximal lattice-free triangle T' with vertex $f + \tau r^1$. Note that the y-coordinate of $f + \tau r^1$ is between zero and one. Hence the edges of T' incident to $f + \tau r^1$ cannot contain $(0,0)^T$ or $(0,1)^T$ (since either such edge would then separate $(1,0)^T$ from $(1,1)^T$). So $(0,0)^T$ and $(0,1)^T$ must be on the edge of T' opposite $f + \tau r^1$ and this edge must be defined by $\{(x,y)^T \in \mathbb{R}^2 : x = 0\}$. Then, the two edges of T' incident with $f + \tau r^1$ must contain $(1,0)^T$ and $(1,1)^T$ respectively. It follows that $T' = T$. Note that that triangle T matches $RS_1(\tau)$ as described in Section 2.4.1. $\square$

It follows by symmetry that for all $i \in \{1,2,3,4\}$ triangle $RS_i(\tau)$ is well-defined for appropriate bounds on τ. Moreover, $RS_i(\tau)$ is a type 2 triangle.

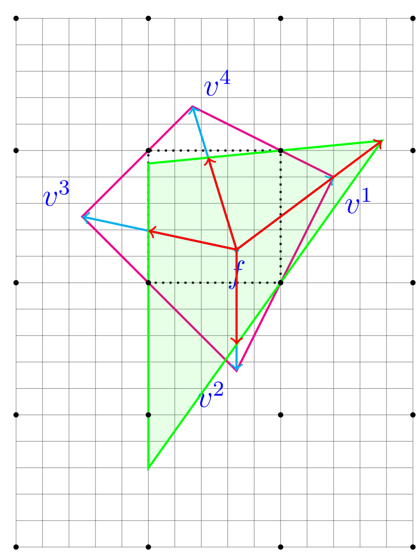


Figure 2.8: An alternative definition would include this τ-Ray-Sliding triangle for Ray 1 $RS_1(\tau)$ for $\tau = \frac{3}{2}$ and parameters $a = 1$, $b = 2$, $c = 1$, $d = \frac{1}{2}$, $g = \frac{2}{3}$, and $h = \frac{1}{4}$.

Note that the requirement $[0,1]^2 \subseteq RS_1(\tau)$ could be considered unnecessarily restrictive. An alternative definition might require: $f + \tau r^1$ be a vertex of $RS_1(\tau)$, $\{(x,y)^T \in \mathbb{R}^2 : x = 0\}$ define the opposite facet of $RS_1(\tau)$, and $RS_1(\tau)$ be maximally lattice-free. Although enlarging the family in this way may provide additional candidate cuts, we have chosen to require $[0,1]^2 \subseteq RS_1(\tau)$ for a few reasons. Firstly, it ensures τ_1^{max} is well-defined and it is easy to verify $RS_1(\tau)$ is lattice-free within this bound. Secondly, it ensures the coefficient of r^3 is always determined by its intersection with $\{(x,y)^T \in \mathbb{R}^2 : x = 0\}$. Lastly, it ensures the coefficients of r^2 and r^4 are no weaker than those determined by $\{(x,y)^T \in \mathbb{R}^2 : y = 0\}$ and $\{(x,y)^T \in \mathbb{R}^2 : y = 1\}$, respectively.

2.4.3 Standard Breakpoints

For ray-sliding triangle $RS_1(\tau)$ we can identify values for τ that correspond to natural geometric breakpoints. As in the proof of Remark 2.4.1, denote by $\tau_1^{min} < 1$ the smallest allowable choice for τ and denote by $\tau_1^{max} > 1$ the largest allowable choice for τ.

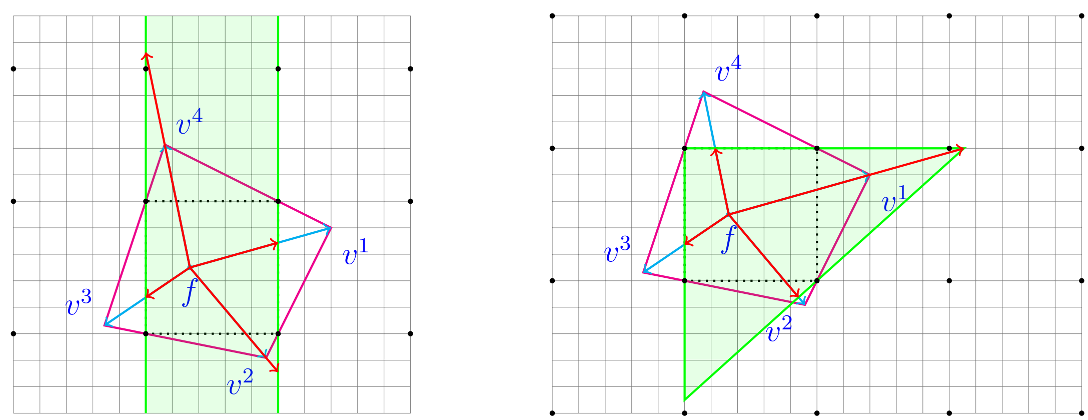


Figure 2.9: τ-ray-sliding triangle for Ray 1 for $\tau = \tau_1^{min}$ (left) and $\tau = \tau_1^{max}$ (right) for parameters $a = \frac{1}{5}$, $b = 2$, $c = 3$, $d = \frac{1}{2}$, $g = \frac{1}{3}$, and h $= \frac{1}{2}$.

The cut coefficient improvement realized on r^2 and r^4 for $\tau < 1$ is limited by the underlying geometry. First consider r^2.

1. If the x-coordinate of r^2 is non-negative, then for all $\tau \in [\tau^{min}, 1]$ the intersection cut coefficient for r^2 is determined by the intersection of ray 2 with the line joining $f + \tau r^1$ and $(1,0)^T$. Note that if the x-coordinate of r^2 is 0 and $\tau = \tau^{min}$, then ray 2 is in the recession cone of the corresponding split and the intersection cut coefficient for r^2 is 0.
2. If the x-coordinate of r^2 is negative, then there exists $\tau^{bp} \in [\tau^{min}, 1]$ such that

- for $\tau \in [\tau^{bp}, 1]$, the intersection cut coefficient for r^2 is determined by the intersection of ray 2 with the line joining $f + \tau r^1$ and $(1, 0)^T$. Decreasing τ in this range will improve the coefficient for r^2; and
- for $\tau \in [\tau^{min}, \tau^{bp}]$, the intersection cut coefficient for r^2 is determined by the intersection of ray 2 with the line $\{(x, y)^T \in \mathbb{R}^2 : x = 0\}$. This coefficient does not depend on τ and so decreasing τ in this range does not improve the coefficient for r^2.

The analysis for r^4 is analogous.

For example, in Figure 2.7, the x-coordinate of r^2 is positive and so the intersection cut coefficient is always determined by the intersection of ray 2 with the line joining $f + \tau r^1$ and $(1, 0)^T$. Thus for $\tau < 1$, the intersection cut coefficient for r^2 improves. However, the x-coordinate of r^4 is negative and so the intersection cut coefficient can be determined by the intersection of ray 4 with either: the line joining $f + \tau r^1$ and $(1, 1)^T$, or, the facet $\{(x, y)^T \in \mathbb{R}^2 : x = 0\}$. The image on the right illustrates the τ^{bp} for which the facet of $RS_1(\tau)$ determining the coefficient of r^4 changes. For any $\tau < \tau^{bp}$ the intersection cut coefficient for r^4 does not improve. We consider τ^{bp} to define a natural geometric breakpoint for τ since it indicates a change in the cut coefficient formula for an off-ray.

2.4.4 Summary of Families of Ray-Sliding Triangles

There are four families of ray-sliding triangles: $\{RS_i(\tau) : \tau \in [\tau_i^{min}, \tau_i^{max}]\}$ for $i \in \{1, 2, 3, 4\}$. For each family $RS_i(\tau)$, there are always three breakpoints; they are

1. τ_i^{min}: the smallest allowable choice for τ (defining a split cut in the limit);
2. $\tau_i^{fixed} = 1$: corresponding to the fixed triangle for ray i; and
3. τ_i^{max}: the largest allowable choice for τ (corresponding to where $[0, 1]^2$ would no longer be contained in the triangle).

There may be two additional *ray breakpoints*: $\tau_i^{bp2}, \tau_i^{bp1} \in [\tau_i^{min}, 1]$. These breakpoints correspond to changes in the cut coefficient formula for off-rays where the facet of the triangle determining the coefficient changes. These (at least three and at most five) breakpoints will be called the *standard breakpoints* for a family of τ-ray-sliding triangles. Note that if both off-rays have a corresponding breakpoint, the underlying geometry determines which breakpoint is smaller. We adopt the convention that if only one additional breakpoint applies then it is denoted τ_i^{bp1} and that if two additional breakpoints apply, they are denoted $\tau_i^{bp1} \geq \tau_i^{bp2}$. The subfamily $\{RS_i(\tau) : \tau \in \{\tau_i^{min}, \tau_i^{bp2}, \tau_i^{bp1}, \tau_i^{fixed}, \tau_i^{max}\}\}$ comprised of the ray-sliding triangles at standard breakpoints is a natural finite subfamily of $\{RS_i(\tau) : \tau \in [\tau_1^{min}, \tau_1^{max}]\}$.

Appendix A provides the exact values of the standard breakpoints as well as the intersection cut coefficients for $RS_i(\tau)$ as a function of a, b, c, d, g, h, and τ for all $i \in \{1,2,3,4\}$. Since the coefficients can be calculated in a straightforward way by intersecting $\{f + \lambda r^i : \lambda \geq 0\}$ with the appropriate facet of the underlying triangle, we aggregate the results of these computations in Appendix A for easy reference, and treat the appendix as a catalogue of intersection cut coefficients. These computations are tedious to do by hand (and it's easy to make an error), so we also checked the computations using Sage[2]. For pedagogical purposes, we provide one example of the necessary computations here.

We will compute the intersection cuts generated by the family $\{RS_1(\tau) : \tau \in [\tau_1^{min}, \tau_1^{max}]\}$ for instances where $v_x^4 \leq g \leq v_x^2$ (for example, as in Figure 2.7). Under this assumption, there is one ray breakpoint, determined by the intersection of the ray $\{f + \lambda r^4 : \lambda \geq 0\}$ with $\{(x,y) \in \mathbb{R}^2 : x = 0\}$. For $i \in \{1,2,3,4\}$, let ψ_i denote $\psi_{f;RS_1(\tau)}(r^i)$. By the construction of $RS_1(\tau)$ we know that $\psi_1 = \frac{1}{\tau}$ and $\psi_3 = 1 + \frac{1}{(a+c)g}$ as computed for fixed triangle F_1 in the proof of Lemma 2.2.2. To calculate ψ_2, we calculate the value of $\lambda_2 \geq 0$ such that the point $f + \lambda_2 r^2$ is on the line joining $f + \tau r^1$ and $(1,0)^T$. The equation of the line $L_2(\tau)$ joining $f + \tau r^1$ and $(1,0)^T$ is given by

$$y = \left[\frac{(b+d)h(1-\tau) + b\tau}{(b+d)(g-1)(1-\tau) + \tau}\right](x-1)$$

Then for $\lambda_2 := \frac{(a+b)\tau}{(a+b)\tau - a(b+d)(1-\tau)}$, we calculate

$$p^2 := f + \lambda_2 r^2 = \left(g + \frac{\tau[b-(a+b)g]}{(a+b)\tau - a(b+d)(1-\tau)}\ ,\ h - \frac{\tau[ab+(a+b)h]}{(a+b)\tau - a(b+d)(1-\tau)}\right)^T.$$

Then we can check p^2 is on line $L_2(\tau)$ by calculating

$$\begin{aligned}
&\left[\frac{(b+d)h(1-\tau) + b\tau}{(b+d)(g-1)(1-\tau) + \tau}\right](p_x^2 - 1) \\
&\quad = \left[\frac{(b+d)h(1-\tau) + b\tau}{(b+d)(g-1)(1-\tau) + \tau}\right]\left(g + \frac{\tau[b-(a+b)g]}{(a+b)\tau - a(b+d)(1-\tau)} - 1\right) \\
&\quad = \left[\frac{(b+d)h(1-\tau) + b\tau}{(b+d)(g-1)(1-\tau) + \tau}\right]\left(\frac{a(b+d)(1-g)(1-\tau) - a\tau)}{(a+b)\tau - a(b+d)(1-\tau)}\right) \\
&\quad = \frac{-a(b+d)h(1-\tau) - ab\tau}{(a+b)\tau - a(b+d)(1-\tau)} \\
&\quad = p_y^2
\end{aligned}$$

[2]SageMath [70] was used extensively to perform and verify many of the symbolic computations within this thesis.

as required. Thus $\psi_2 = \frac{1}{\lambda_2} = 1 - \frac{a(b+d)(1-\tau)}{(a+b)\tau}$.

Next we calculate the value of τ_1^{bp1}. The first step is to calculate the value of $\lambda_4 \geq 0$ such that $f + \lambda_4 r^4$ is on the line $x = 0$. For $\lambda_4 := \frac{(c+d)g}{(c+d)g-d}$, we calculate the x-coordinate of $f + \lambda_4 r^4$ as

$$g + \left[\frac{(c+d)g}{(c+d)g-d}\right]\left(\frac{d}{c+d} - g\right) = g + \left[\frac{(c+d)g}{(c+d)g-d}\right]\left(\frac{d-g(c+d)}{c+d}\right) = g - g$$

and verify that it is indeed 0 as required. We'll need the y-coordinate of $f + \lambda_4 r^4$ later and calculate it now as

$$\begin{aligned} h + \left[\frac{(c+d)g}{(c+d)g-d}\right]\left(1 + \frac{cd}{c+d} - h\right) &= h + \left[\frac{(c+d)g}{(c+d)g-d}\right]\left(\frac{(c+d)(1-h)+cd}{c+d}\right) \\ &= \frac{(c+d)gh - dh + (c+d)g(1-h) + cdg}{(c+d)g-d} \\ &= \frac{-dh + (c+d)g - cdg}{(c+d)g-d} \\ &= 1 + \frac{d(1-h)+cdg}{(c+d)g-d}. \end{aligned}$$

Then, we calculate the value of τ_1 for which the line joining $f + \tau_1 r^1$ and $(1,1)^T$ contains $f + \lambda_4 r^4$. We obtain the value $\tau_1^{bp1} = 1 - \frac{d}{c(b+d)g+d}$. To check this is correct, first we compute the equation of the line $L_4(\tau)$ joining $f + \tau r^1$ and $(1,1)^T$. It is given by

$$y = \left[\frac{(b+d)(h-1)(1-\tau) - d\tau}{(b+d)(g-1)(1-\tau)+\tau}\right](x-1) + 1.$$

We check $f + \lambda_4 r^4$ is on $L_4(\tau_1^{bp1})$. Recalling the x-coordinate of $f + \lambda_4 r^4$ is 0, we check

$$\begin{aligned} \left[\frac{(b+d)(h-1)(\frac{d}{c(b+d)g+d}) - d\frac{c(b+d)g}{c(b+d)g+d}}{(b+d)(g-1)(\frac{d}{c(b+d)g+d}) + \frac{c(b+d)g}{c(b+d)g+d}}\right](-1) + 1 &= 1 - \left[\frac{d(b+d)(h-1) - dc(b+d)g}{d(b+d)(g-1) + c(b+d)g}\right] \\ &= 1 + \frac{d(1-h) + gcd}{(c+d)g-d}. \end{aligned}$$

which is the y-coordinate of $f + \lambda_4 r^4$ as required. Now, for $\tau \in [\tau_1^{min}, \tau_1^{bp1}]$, we have $\psi_4 = \frac{1}{\lambda_4} = 1 - \frac{d}{(c+d)g}$. For $\tau \in [\tau_1^{bp1}, \tau_1^{max}]$, we calculate the value of $\hat{\lambda}_4 \geq 0$ such that the point $f + \hat{\lambda}_4 r^4$ is on the line joining $f + \tau r^1$ and $(1,1)^T$. The computations are similar to calculating λ_2 as above, so we will omit them. We get $\hat{\lambda}_4 = \frac{(c+d)\tau}{(c+d)\tau - c(b+d)(1-\tau)}$. Thus for $\tau \in [\tau_1^{bp1}, \tau_1^{max}]$ we have $\psi_4 = \frac{1}{\hat{\lambda}_4} = 1 - \frac{c(b+d)(1-\tau)}{(c+d)\tau}$. The results of these computations can be found in Appendix A.3 Case 2.

2.5 Observations and Summary

In this chapter we introduced the techniques we require to prove bounds on $\rho[\mathcal{T}_2, \mathcal{Q}_2]$ and $\rho[\mathcal{T}_2, \mathcal{Q}]$. Before moving on to this task we make a couple observations.

Firstly, note that there is a disconnect between the family of instances that

1. give weak bounds when replacing constraint (2.19) with fixed triangle cuts (Figure 2.3 and surrounding discussion), and
2. were proven to be hard to approximate better than a given factor with any triangle cuts (Lemma 2.3.1).

In fact, the latter family is comprised of parallelograms and therefore Theorem 2.2.8 gives stronger bounds when using fixed triangles. In Chapter 3 we will see that the bound in Lemma 2.3.1 is the optimal value of $(P(\alpha, \beta, \beta, \alpha, g, h))$ for all $0 < \alpha < \beta$ and $(g, h)^T \in (\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta})^2$.

Secondly, observe that the bounds in Lemma 2.2.2 and Lemma 2.2.3 are continuous functions of a, b, c, d, g, and h. Moreover, the bounds agree for $(g, h)^T$ on the boundary of the two regions: i.e) for $(g, h)^T \in \{(x, y) \in \mathbb{R}^2 : y = 0\}$. These properties arise because the bounds come from dual solutions of a parameterized linear program for the same optimal basis. It would be nice to improve these bounds and maintain the same smoothness properties.

Lastly, we mention one technique to prove a target bound:

1. sequentially identify regions of the parameter set where the target bound holds using a given set of type 2 triangle cuts; and
2. then focus on choosing different sets of type 2 triangle cuts for the other regions.

For example, Corollary 2.2.9 indicates target bound $\frac{8}{9}$ holds for $\{(a, b, c, d, g, h) : a = d, b = c, a + b + \frac{1}{a} + \frac{1}{b} \geq 8\}$ using fixed triangle cuts. One might hesitate to employ this technique iteratively: it could lead to a disjointed analysis where different triangle cuts provide bounds in different regions, with no consistency on the boundaries or intersections. The practical obstacle we ran into was different: given a target lower bound $\hat{LB}$ and a parameter set, there may be many choices of triangle cuts giving the target bound. So after accounting for one set of parameters, the triangles we chose to analyze the next set of parameters often worked for the first as well. Eventually, this technique lead to finding parameter sets for which it is difficult to prove the target bound. Once triangle cuts that work for these "hard to bound" parameter sets were identified, they could be used everywhere. We'll see these choices of triangles in the next two chapters.

Chapter 3

Approximating Parallelogram Cuts with Type 2 Triangle Cuts

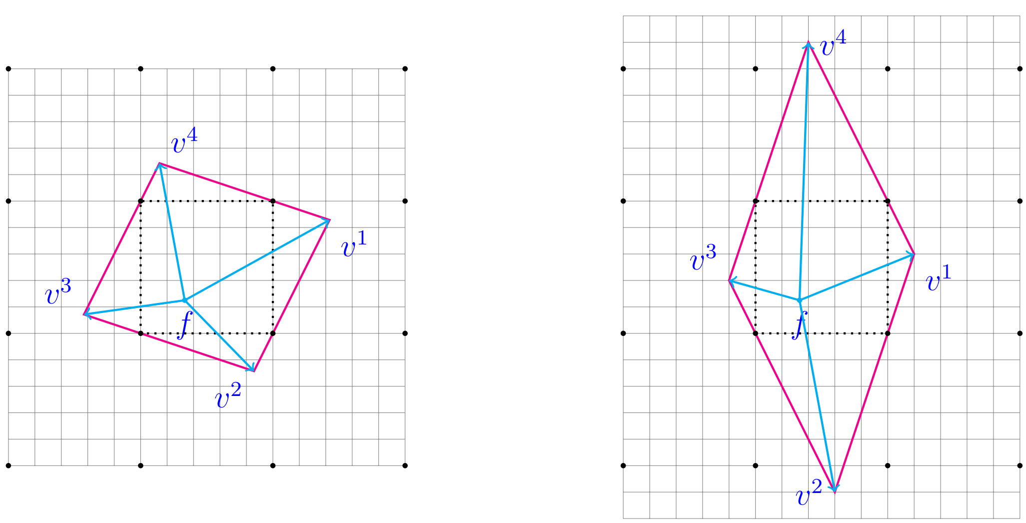


Figure 3.1: Parallelograms for parameters $\alpha = \frac{1}{3}$, $\beta = 2$, $g = \frac{1}{3}$, and $h = \frac{1}{4}$ (left) and $\alpha = 2$, $\beta = 3$, $g = \frac{1}{3}$, and $h = \frac{1}{4}$ (right).

The main purpose of this chapter is to prove Theorem 1.3.21 showing that $\rho[\mathcal{T}_2, \mathcal{Q}_2] = \frac{9}{8}$. We follow the proof strategy in Section 2.1. Let $Q_2(\alpha, \beta) := Q(\alpha, \beta, \beta, \alpha)$. Let $\hat{\mathcal{Q}}_2 := \{Q_2(\alpha, \beta) : \alpha > 0, \beta > 0\}$ denote the family of normalized parallelograms. The value of $\frac{1}{\rho[\mathcal{T}_2, \mathcal{Q}_2]}$ is equal to the optimal value of semi-infinite program $(S(\mathcal{T}_2, \mathcal{Q}_2))$ in variables $\alpha, \beta, g, h, s_1, s_2, s_3$, and s_4 given by

$$\inf \quad s_1 + s_2 + s_3 + s_4 \tag{3.1}$$

$$\text{s.t.} \quad s \in \mathcal{T}_2\left(\begin{pmatrix} g \\ h \end{pmatrix}; \begin{pmatrix} 1 + \frac{1}{\alpha+\beta} - g \\ \frac{\beta}{\alpha+\beta} - h \end{pmatrix}, \begin{pmatrix} \frac{\beta}{\alpha+\beta} - g \\ \frac{-\alpha\beta}{\alpha+\beta} - h \end{pmatrix}, \begin{pmatrix} \frac{-1}{\alpha+\beta} - g \\ \frac{\alpha}{\alpha+\beta} - h \end{pmatrix}, \begin{pmatrix} \frac{\alpha}{\alpha+\beta} - g \\ 1 + \frac{\alpha\beta}{\alpha+\beta} - h \end{pmatrix}\right) \tag{3.2}$$

$$\begin{pmatrix} g \\ h \end{pmatrix} \in \operatorname{int}\left(\operatorname{conv}\left\{\begin{pmatrix} 1 + \frac{1}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} \end{pmatrix}, \begin{pmatrix} \frac{\beta}{\alpha+\beta} \\ \frac{-\alpha\beta}{\alpha+\beta} \end{pmatrix}, \begin{pmatrix} \frac{-1}{\alpha+\beta} \\ \frac{\alpha}{\alpha+\beta} \end{pmatrix}, \begin{pmatrix} \frac{\alpha}{\alpha+\beta} \\ 1 + \frac{\alpha\beta}{\alpha+\beta} \end{pmatrix}\right\}\right) \tag{3.3}$$

$$\alpha, \beta > 0 \quad ; \quad \alpha, \beta \in \mathbb{Q} \quad ; \quad \alpha \leq \beta \tag{3.4}$$

$$\begin{pmatrix} g \\ h \end{pmatrix} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2. \tag{3.5}$$

Note that the assumption $\alpha \leq \beta$ follows from Remark 2.1.2. Treating $\alpha, \beta > 0$ with $\alpha \leq \beta$ and $(g, h)^T \in \operatorname{int} Q_2(\alpha, \beta)$ as data in the optimization problem above, we obtain a semi-infinite linear program in variables s_1, s_2, s_3, and s_4 given by

$$\inf \quad s_1 + s_2 + s_3 + s_4 \tag{3.6}$$

$$\text{s.t.} \quad s \in \mathcal{T}_2\left(\begin{pmatrix} g \\ h \end{pmatrix}; \begin{pmatrix} 1 + \frac{1}{\alpha+\beta} - g \\ \frac{\beta}{\alpha+\beta} - h \end{pmatrix}, \begin{pmatrix} \frac{\beta}{\alpha+\beta} - g \\ \frac{-\alpha\beta}{\alpha+\beta} - h \end{pmatrix}, \begin{pmatrix} \frac{-1}{\alpha+\beta} - g \\ \frac{\alpha}{\alpha+\beta} - h \end{pmatrix}, \begin{pmatrix} \frac{\alpha}{\alpha+\beta} - g \\ 1 + \frac{\alpha\beta}{\alpha+\beta} - h \end{pmatrix}\right). \tag{3.7}$$

We will denote this program by $(P(\alpha, \beta, g, h))$.

Section 2.2.1 showed the optimal value of $(P(\alpha, \beta, g, h))$ is at least $\frac{4}{5}$ for all applicable α, β, g, and h by:

1. fixing α and β, sub-dividing the interior of $Q_2(\alpha, \beta)$, and identifying representative regions $R_1 := (0, 1)^2$ and $R_2 := \operatorname{int} Q_2(\alpha, \beta) \cap \{(x, y)^T \in \mathbb{R}^2 : x \leq 0\}$;
2. for $(g, h)^T \in R_1$, replacing constraint (3.7) with the intersection cuts for four fixed triangles and solving the resulting parametric linear program to obtain a bound;
3. for $(g, h)^T \in R_2$, replacing constraint (3.7) with the intersection cuts for three fixed triangles and solving the resulting parametric linear program to obtain a bound; and then
4. varying the parameters to find the weakest lower bound.

Here we subdivide the interior of $Q_2(\alpha, \beta)$ differently to identify three representative regions: the Central Region, the South-West Region, and the South Region. This subdivision is illustrated in Figure 3.2. One reason this subdivision is natural is that the set of standard breakpoints is the same for all $(g, h)^T$ in a region. For example, each family of ray-sliding triangles has exactly one ray breakpoint in the Central Region.

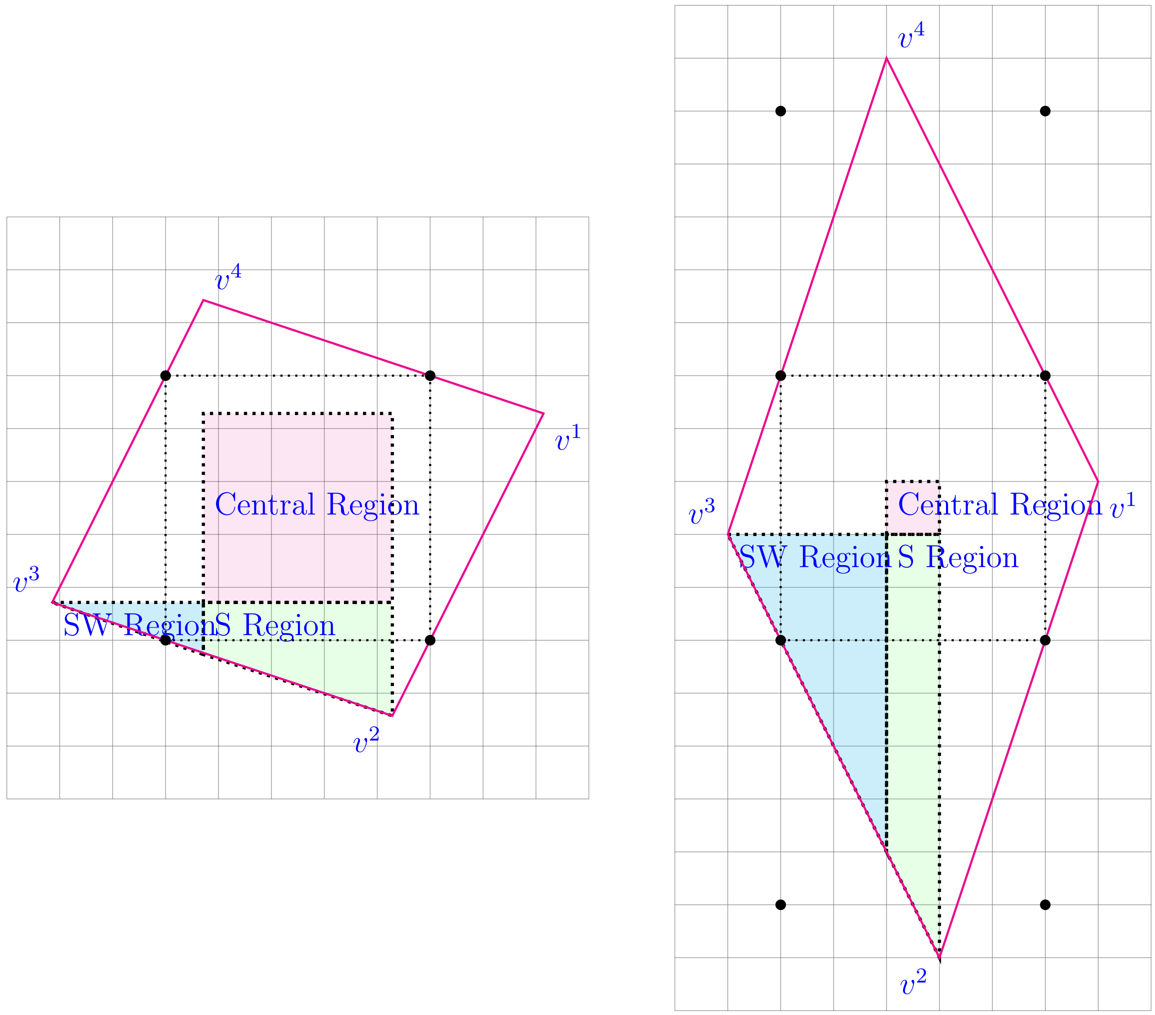


Figure 3.2: Central, South-West, and South Regions of base parallelograms for parameters $\alpha = \frac{1}{3}$, $\beta = 2$ (left) and $\alpha = 2$, $\beta = 3$ (right).

In Section 3.2 we analyze the Central Region. We replace constraint (3.7) with the intersection cuts for four ray-sliding triangles at their unique ray breakpoint and solve the resulting parametric linear program. The resulting weakest lower bound is $\frac{8}{9}$.

Section 3.3 is investigative in nature; it probes the ray-sliding triangles to use to replace constraint (3.7) for the South-West and South Regions. We learn that some natural choices of ray-sliding triangles at standard breakpoints can improve on fixed triangles, but won't lead to a bound of $\frac{8}{9}$. The investigation points us to the triangles we eventually choose for the South-West and South Regions. A reader on the shortest path (within this thesis) to proving $\rho[\mathcal{T}_2, \mathcal{Q}_2] = \frac{9}{8}$ may comfortably skip this section.

In Section 3.4 we analyze the South-West Region. We replace constraint (3.7) with the intersection cuts for $RS_2(\hat{\tau}_2)$ and $RS_3(\hat{\tau}_3)$ where $\hat{\tau}_2$ and $\hat{\tau}_3$ fall between the two ray breakpoints, serving as a sort of geometric average. In Section 3.5 we analyze the South Region, using $RS_2(\hat{\tau}_2)$, as well as the ray-sliding triangles for ray 1 and ray 3 at their unique ray breakpoint. We amalgamate the results for the Central, South-West, and South Regions to conclude $\rho[\mathcal{T}_2, \mathcal{Q}_2] \leq \frac{9}{8}$ in Section 3.6.

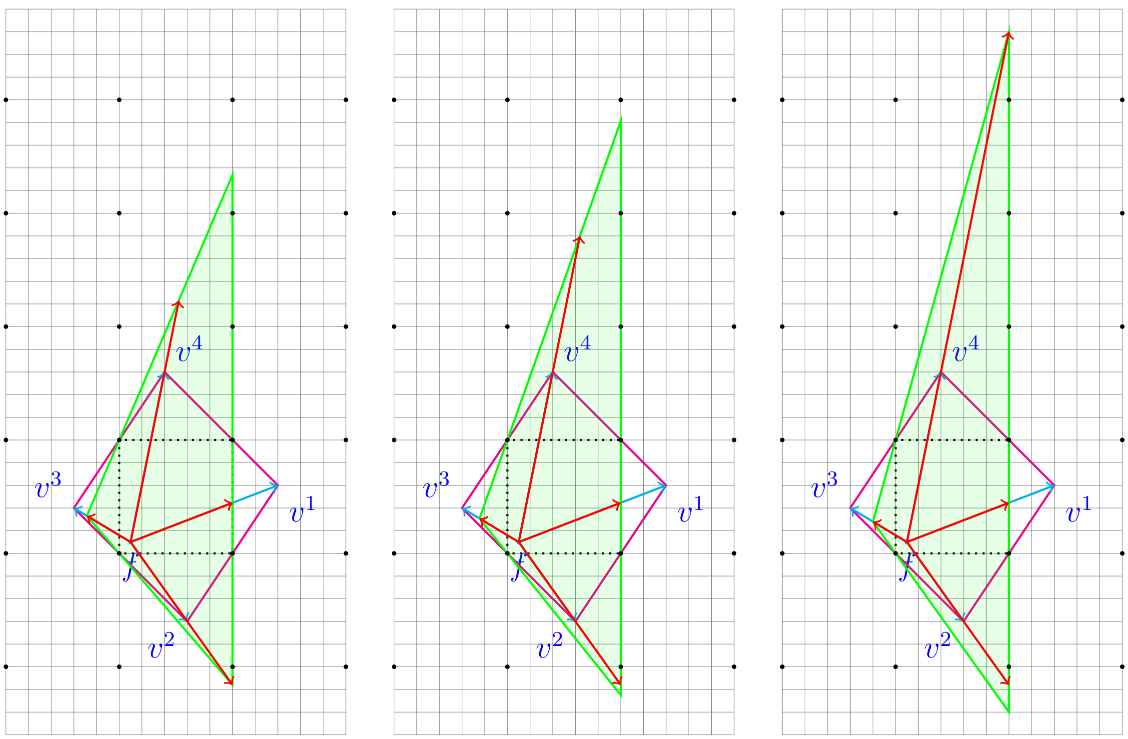


Figure 3.3: τ-ray-sliding triangle for Ray 3 for $\alpha = 1$, $\beta = \frac{3}{2}$, $g = \frac{1}{10}$, and $h = \frac{1}{10}$ and (left) $\tau = \tau_3^{bp1}$, (center) $\tau = \hat{\tau}_3$, and (right) $\tau = \tau_3^{bp2}$.

3.1 Identify Regions of $\operatorname{int} Q_2(\alpha, \beta)$ to Analyze

As in Section 2.2, we identify sub-regions of $\operatorname{int} Q_2(\alpha, \beta)$ and use a symmetry argument to show it suffices to analyze only these regions. The same regions and symmetry argument are used in Chapter 4 and thus we present them for an arbitrary quadrilateral. For fixed $a, b, c, d > 0$ with $ad \le bc$ consider the following regions of $\operatorname{int} Q(a, b, c, d)$:

1. *South-West Region*: $R_{SW}(a, b, c, d) := \left\{(g, h)^T \in \operatorname{int} Q(a, b, c, d) : g \le \frac{d}{c+d}\ ,\ h \le \frac{a}{a+c}\right\}$,
2. *South Region*: $R_S(a, b, c, d) := \left\{(g, h)^T \in \operatorname{int} Q(a, b, c, d) : g \in \left[\frac{d}{c+d}, \frac{b}{a+b}\right]\ ,\ h \le \frac{a}{a+c}\right\}$,
3. and *Central Region*:
   $R_{Central}(a, b, c, d) := \left\{(g, h)^T \in \operatorname{int} Q(a, b, c, d) : g \in \left[\frac{d}{c+d}, \frac{b}{a+b}\right]\ ,\ h \in \left[\frac{a}{a+c}, \frac{b}{b+d}\right]\right\}$.

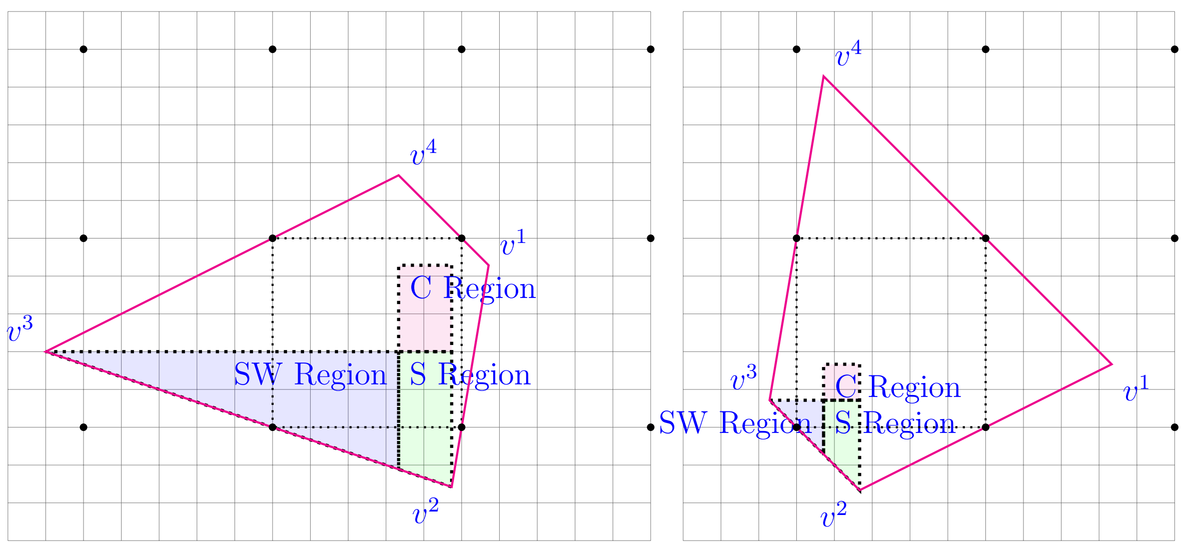


Figure 3.4: South-West, South, and Central Regions of base quadrilaterals for parameters $a = \frac{1}{3}$, $b = 6$, $c = \frac{1}{2}$, and $d = 1$ (left) and $a = \frac{1}{3}$, $b = \frac{1}{2}$, $c = 6$, and $d = 1$ (right).

Since we vary a, b, c, d restricted only by $a, b, c, d > 0$ and $ad \le bc$, the below remark shows that we can consider only the parameterizations where: $(g, h)^T \in R_{Central}(a, b, c, d)$, $(g, h)^T \in R_{SW}(a, b, c, d)$, or $(g, h)^T \in R_S(a, b, c, d)$.

Remark 3.1.1. *Let $Q_1 := Q(a, b, c, d)$ for $a, b, c, d > 0$ with $ad \le bc$ be a normalized quadrilateral. Let $f = (g, h)^T \in \operatorname{int} Q_1$. Then there exist $\hat{a}, \hat{b}, \hat{c}$, and $\hat{d}$ with $\hat{a}\hat{d} \le \hat{b}\hat{c}$ and affine unimodular transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that:*

1. $\phi(Q_1) = Q(\hat{a}, \hat{b}, \hat{c}, \hat{d})$, *and*

2. $\phi(f) \in R_{SW}(\hat{a}, \hat{b}, \hat{c}, \hat{d}) \cup R_S(\hat{a}, \hat{b}, \hat{c}, \hat{d}) \cup R_{Central}(\hat{a}, \hat{b}, \hat{c}, \hat{d})$.

Proof. If $f \in R_{SW}(a,b,c,d) \cup R_S(a,b,c,d) \cup R_{Central}(a,b,c,d)$, then no transformation is required. If $f \notin R_{SW}(a,b,c,d) \cup R_S(a,b,c,d) \cup R_{Central}(a,b,c,d)$, the result follows by rotating the quadrilateral clockwise $\frac{\pi}{2}$, π or $\frac{3\pi}{2}$ as described below.

<u>Case 1: $g > \frac{b}{a+b}$ and $h \le \frac{b}{b+d}$:</u> Choose $(\hat{a}, \hat{b}, \hat{c}, \hat{d}) := \left(\frac{1}{b}, \frac{1}{d}, \frac{1}{a}, \frac{1}{c}\right)$. Note that $\hat{a}\hat{d} = \frac{1}{bc} \le \frac{1}{da} = \hat{b}\hat{c}$ where the inequality holds because $ad \le bc$. Consider the affine unimodular transformation $\phi_{\frac{\pi}{2}}$ given by

$$\phi_{\frac{\pi}{2}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\right) + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

It is straightforward to verify that $\phi_{\frac{\pi}{2}}(Q(a,b,c,d)) = Q(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ and

$$\phi_{\frac{\pi}{2}}\left(\begin{bmatrix} g \\ h \end{bmatrix}\right) = \begin{bmatrix} h \\ 1-g \end{bmatrix} \in R_{SW}(\hat{a}, \hat{b}, \hat{c}, \hat{d}) \cup R_S(\hat{a}, \hat{b}, \hat{c}, \hat{d}).$$

<u>Case 2: $g \ge \frac{d}{c+d}$ and $h > \frac{b}{b+d}$:</u> Choose $(\hat{a}, \hat{b}, \hat{c}, \hat{d}) := (d,c,b,a)$. Note that $\hat{a}\hat{d} = da \le cb = \hat{b}\hat{c}$ where the inequality holds because $ad \le bc$. Consider the affine unimodular transformation ϕ_π given by

$$\phi_{\pi}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\right) + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

It is straightforward to verify that $\phi_\pi(Q(a,b,c,d)) = Q(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ and

$$\phi_{\pi}\left(\begin{bmatrix} g \\ h \end{bmatrix}\right) = \begin{bmatrix} 1-g \\ 1-h \end{bmatrix} \in R_{SW}(\hat{a}, \hat{b}, \hat{c}, \hat{d}) \cup R_S(\hat{a}, \hat{b}, \hat{c}, \hat{d}).$$

<u>Case 3: $g < \frac{d}{c+d}$ and $h > \frac{a}{a+c}$:</u> Choose $(\hat{a}, \hat{b}, \hat{c}, \hat{d}) := (\frac{1}{c}, \frac{1}{a}, \frac{1}{d}, \frac{1}{b})$. Note that $\hat{a}\hat{d} = \frac{1}{cb} \le \frac{1}{ad} = \hat{b}\hat{c}$ where the inequality holds because $ad \le bc$. Consider the affine unimodular transformation $\phi_{\frac{3\pi}{2}}$ given by

$$\phi_{\frac{3\pi}{2}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\right) + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

It is straightforward to verify that $\phi_{\frac{3\pi}{2}}(Q(a,b,c,d)) = Q(\hat{a},\hat{b},\hat{c},\hat{d})$ and

$$\phi_{\frac{3\pi}{2}}\left(\begin{bmatrix} g \\ h \end{bmatrix}\right) = \begin{bmatrix} 1-h \\ g \end{bmatrix} \in R_{SW}(\hat{a},\hat{b},\hat{c},\hat{d}) \cup R_S(\hat{a},\hat{b},\hat{c},\hat{d}).$$

The affine unimodular transformation described has the required property in all three cases and therefore the result holds. $\square$

For the parallelogram case ($a = d = \alpha$ and $b = c = \beta$) the three representative regions are:

1. *South-West Region*: $R_{SW}(\alpha,\beta) := \left\{(g,h)^T \in \operatorname{int} Q_2(\alpha,\beta) : g \le \frac{\alpha}{\alpha+\beta}\ ,\ h \le \frac{\alpha}{\alpha+\beta}\right\}$
2. *South Region*: $R_S(\alpha,\beta) := \left\{(g,h)^T \in \operatorname{int} Q_2(\alpha,\beta) : g \in \left[\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta}\right]\ ,\ h \le \frac{\alpha}{\alpha+\beta}\right\}$,
3. and *Central Region*:
   $R_{Central}(\alpha,\beta) := \left\{(g,h)^T \in \operatorname{int} Q_2(\alpha,\beta) : g \in \left[\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta}\right]\ ,\ h \in \left[\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta}\right]\right\}$.

See Figure 3.2. Note that the Central Region always contains the point $\left(\frac{1}{2},\frac{1}{2}\right)^T$ in the parallelogram case because $\alpha \le \beta$ implies $\frac{\alpha}{\alpha+\beta} \le \frac{1}{2}$ and $\frac{\beta}{\alpha+\beta} \ge \frac{1}{2}$.

3.2 Central Region

Focusing first on the Central Region, we use the intersection cuts for four ray-sliding triangles at the first (unique) ray breakpoint, as illustrated in Figure 3.5

Lemma 3.2.1. *For fixed $\alpha,\beta > 0$ with $\alpha \le \beta$ and g,h with $(g,h)^T \in R_{Central}(\alpha,\beta)$, let (P') be the parameterized linear program obtained from $(P(\alpha,\beta,g,h))$ by replacing constraint (3.7) with the constraint $s \ge \mathbb{0}$ and the intersection cuts for ray-sliding triangles $RS_1(\tau_1^{bp1})$, $RS_2(\tau_2^{bp1})$, $RS_3(\tau_3^{bp1})$, and $RS_4(\tau_4^{bp1})$. Then, the optimal value of (P') is equal to*

$$LB_{Central}(\alpha,\beta) := 1 - \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+\beta(1+\alpha)}.$$

Proof. We order the cuts $RS_3(\tau_3^{bp1})$, $RS_4(\tau_4^{bp1})$, $RS_1(\tau_1^{bp1})$, $RS_2(\tau_2^{bp1})$ in the constraint matrix. Looking up the cuts in Appendix A, the constraint matrix for a general quadrilateral is

$$\begin{bmatrix} \left(1+\frac{1}{(b+d)(1-g)}\right) & \left(1-\frac{a}{(a+b)(1-g)}\right) & \left(1+\frac{a}{b(a+c)(1-g)}\right) & \left(1-\frac{ad}{b(c+d)(1-g)}\right) \\ \left(1-\frac{ad}{c(b+d)h}\right) & \left(1+\frac{ab}{(a+b)h}\right) & \left(1-\frac{a}{(a+c)h}\right) & \left(1+\frac{ad}{(c+d)h}\right) \\ \left(1+\frac{d}{c(b+d)g}\right) & \left(1-\frac{ad}{c(a+b)g}\right) & \left(1+\frac{1}{(a+c)g}\right) & \left(1-\frac{d}{(c+d)g}\right) \\ \left(1-\frac{d}{(b+d)(1-h)}\right) & \left(1+\frac{ad}{(a+b)(1-h)}\right) & \left(1-\frac{ad}{b(a+c)(1-h)}\right) & \left(1+\frac{cd}{(c+d)(1-h)}\right) \end{bmatrix}.$$

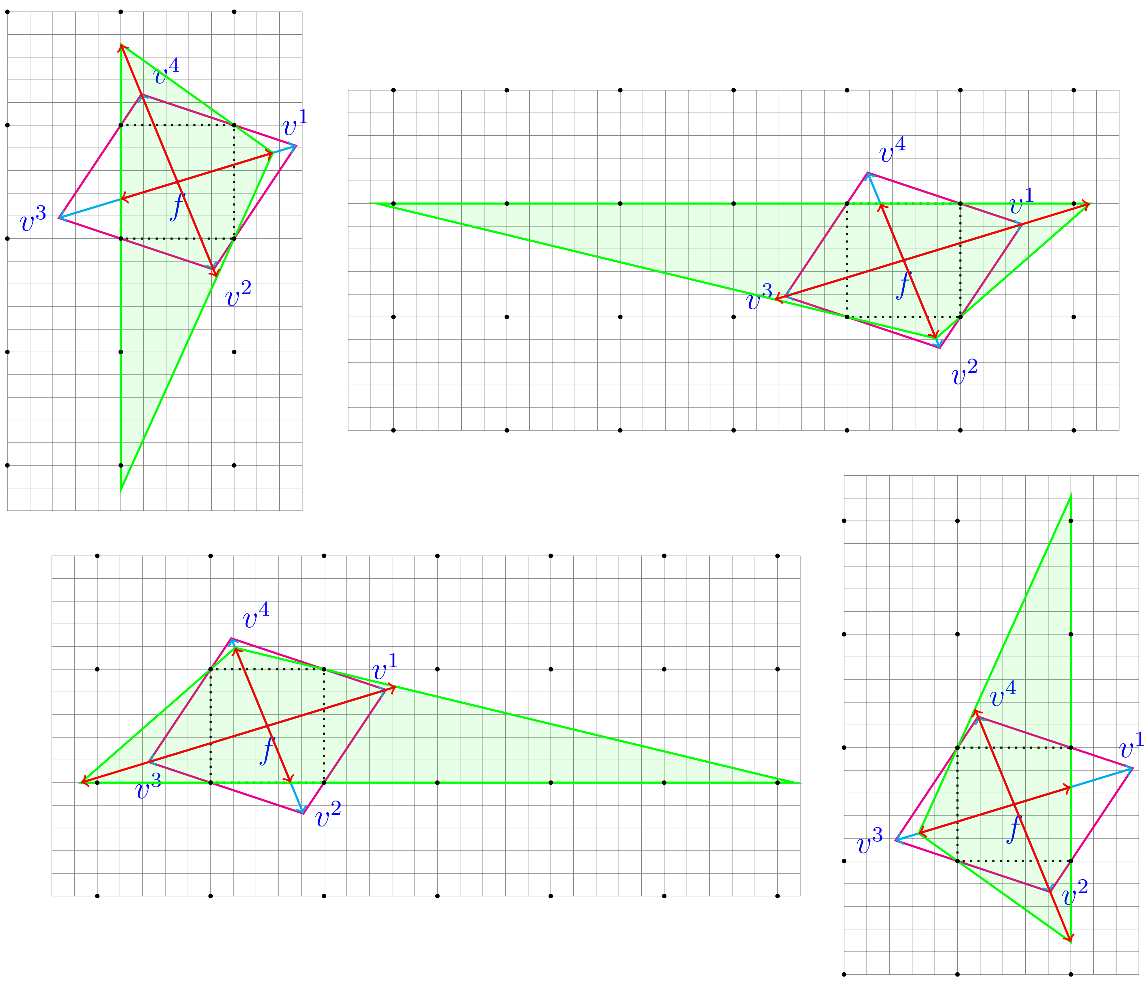


Figure 3.5: Clockwise from top left : $RS_1(\tau_1^{bp1}), RS_2(\tau_2^{bp1}), RS_3(\tau_3^{bp1})$, and $RS_4(\tau_4^{bp1})$ for $\alpha = \frac{1}{3}$, $\beta = \frac{3}{2}$, $g = \frac{1}{2}$ and $h = \frac{1}{2}$.

Setting $b = c = \beta$ and $a = d = \alpha$ the constraint matrix is

$$\begin{bmatrix} \left(1+\frac{1}{(\alpha+\beta)(1-g)}\right) & \left(1-\frac{\alpha}{(\alpha+\beta)(1-g)}\right) & \left(1+\frac{\alpha}{\beta(\alpha+\beta)(1-g)}\right) & \left(1-\frac{\alpha^2}{\beta(\alpha+\beta)(1-g)}\right) \\ \left(1-\frac{\alpha^2}{\beta(\alpha+\beta)h}\right) & \left(1+\frac{\alpha\beta}{(\alpha+\beta)h}\right) & \left(1-\frac{\alpha}{(\alpha+\beta)h}\right) & \left(1+\frac{\alpha^2}{(\alpha+\beta)h}\right) \\ \left(1+\frac{\alpha}{\beta(\alpha+\beta)g}\right) & \left(1-\frac{\alpha^2}{\beta(\alpha+\beta)g}\right) & \left(1+\frac{1}{(\alpha+\beta)g}\right) & \left(1-\frac{\alpha}{(\alpha+\beta)g}\right) \\ \left(1-\frac{\alpha}{(\alpha+\beta)(1-h)}\right) & \left(1+\frac{\alpha^2}{(\alpha+\beta)(1-h)}\right) & \left(1-\frac{\alpha^2}{\beta(\alpha+\beta)(1-h)}\right) & \left(1+\frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right) \end{bmatrix}.$$

So, optimization problem (P') is given by

$$\begin{aligned} \min \quad & \mathbb{1}^T s \\ \text{subject to} \quad & \left[\mathbb{1}\mathbb{1}^T + A\right] s \geq \mathbb{1} \\ & s \geq \mathbb{0} \end{aligned}$$

where

$$A := \left(\frac{1}{\alpha+\beta}\right) \begin{bmatrix} \frac{1}{1-g} & 0 & 0 & 0 \\ 0 & \frac{1}{h} & 0 & 0 \\ 0 & 0 & \frac{1}{g} & 0 \\ 0 & 0 & 0 & \frac{1}{1-h} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix} \begin{bmatrix} 1 & -1 & \frac{\alpha}{\beta} & -\frac{\alpha}{\beta} \\ -\frac{\alpha}{\beta} & \frac{\beta}{\alpha} & -1 & 1 \\ \frac{\alpha}{\beta} & -\frac{\alpha}{\beta} & 1 & -1 \\ -1 & 1 & -\frac{\alpha}{\beta} & \frac{\beta}{\alpha} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix}.$$

First we assume $\alpha \neq \beta$. We compute

$$A^{-1} = \left(\frac{\beta}{\beta-\alpha}\right) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha} \end{bmatrix} \begin{bmatrix} \beta & \alpha & 0 & 0 \\ 0 & \alpha & \alpha & 0 \\ 0 & 0 & \beta & \alpha \\ \alpha & 0 & 0 & \alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha} \end{bmatrix} \begin{bmatrix} 1-g & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1-h \end{bmatrix},$$

$$A^{-1}\mathbb{1} = \left(\frac{\beta}{\beta-\alpha}\right) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha} \end{bmatrix} \begin{bmatrix} \beta & \alpha & 0 & 0 \\ 0 & \alpha & \alpha & 0 \\ 0 & 0 & \beta & \alpha \\ \alpha & 0 & 0 & \alpha \end{bmatrix} \begin{bmatrix} (1-g) \\ \frac{h}{\alpha} \\ g \\ \frac{1-h}{\alpha} \end{bmatrix} = \left(\frac{\beta}{\beta-\alpha}\right) \begin{bmatrix} \beta(1-g)+h \\ \frac{h}{\alpha}+g \\ \beta g + (1-h) \\ (1-g) + \frac{1-h}{\alpha} \end{bmatrix},$$

and

$$\mathbb{1}^T A^{-1}\mathbb{1} = \left(\frac{\beta}{\beta-\alpha}\right)\left(1+\beta+1+\frac{1}{\alpha}\right) = \left(\frac{1}{\alpha(\beta-\alpha)}\right)\left(\alpha\beta(1+\beta)+\beta(1+\alpha)\right).$$

Note that the entries of A^{-1} are non-negative because $\alpha, \beta > 0$, $\alpha < \beta$ and $(g, h)^T \in [0, 1]^2$. We apply Lemma 2.1.3 to conclude the optimal value of (P') is

$$1 - \frac{1}{1 + \mathbb{1}^T A^{-1} \mathbb{1}} = 1 - \frac{\alpha(\beta - \alpha)}{\alpha(\beta - \alpha) + \alpha\beta(1 + \beta) + \beta(1 + \alpha)}$$

whenever $\alpha \neq \beta$.

If $\alpha = \beta$, then the optimal primal and dual solutions can still be constructed in the same way and the objective function value is 1. We handle this case in detail in the general quadrilateral analysis; see Lemma 4.1.1. □

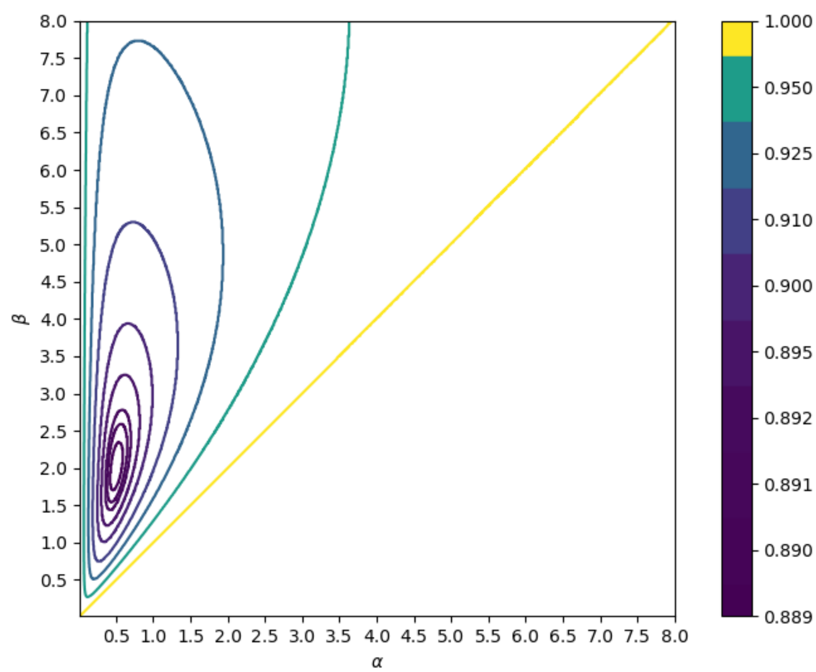


Figure 3.6: Contour plot of $LB_{Central}(\alpha, \beta)$.

We vary α and β to find the weakest bound.

Lemma 3.2.2. *For all $\alpha, \beta > 0$ with $\alpha \leq \beta$, $LB_{Central}(\alpha, \beta) \geq \frac{8}{9}$.*

Proof. For $\alpha = \beta$ we have $LB_{Central}(\alpha, \beta) = 1$ so we may assume $\alpha < \beta$. Now

$$LB_{Central}(\alpha, \beta) = 1 - \frac{1}{1 + g(\alpha, \beta)} \quad \text{where} \quad g(\alpha, \beta) := \frac{\alpha\beta(1+\beta) + \beta(1+\alpha)}{\alpha(\beta - \alpha)}.$$

So, for $\alpha, \beta > 0$, $LB_{Central}(\alpha, \beta) \geq \frac{8}{9}$ if and only if $g(\alpha, \beta) \geq 8$ and

$$\begin{aligned} g(\alpha, \beta) \geq 8 &\iff \alpha\beta + \alpha\beta^2 + \beta + \alpha\beta \geq 8\alpha\beta - 8\alpha^2 \\ &\iff \alpha\beta^2 + (1 - 6\alpha)\beta + 8\alpha^2 \geq 0. \end{aligned}$$

This inequality holds trivially for $\alpha \leq \frac{1}{6}$ since all the terms are non-negative. Fixing $\alpha \geq \frac{1}{6}$, we can view $\alpha\beta^2 + (1 - 6\alpha)\beta + 8\alpha^2$ as a degree two polynomial in β. Its discriminant is $(1 - 6\alpha)^2 - 4\alpha(8\alpha^2) = (-1 + 2\alpha)^2(1 - 8\alpha)$ which is negative because $\alpha \geq \frac{1}{6}$. Since $\alpha\beta^2 + (1 - 6\alpha)\beta + 8\alpha^2$ is positive for $\beta = 0$, it follows that the polynomial is positive everywhere. Therefore $g(\alpha, \beta) \geq 8$ for all $\alpha, \beta > 0$. □

Note that $LB_{Central}(\alpha, \beta)$ exactly matches the objective function value of the feasible solution to $(P(\alpha, \beta, g, h))$ given in Lemma 2.3.1. In fact, if we were to calculate the optimal solution to $(P(\alpha, \beta, g, h))$ in Lemma 3.2.1 using Lemma 2.1.3, it is exactly equal to $\hat{s}(\alpha, \beta, g, h)$ in Lemma 2.3.1. Therefore, we have solved $(P(\alpha, \beta, g, h))$ for all $\beta > \alpha > 0$ and $(g, h)^T \in \operatorname{int} R_{Central}(\alpha, \beta)$.

3.3 Investigate Triangle Choices for the South-West and South Regions

In the South-West Region, the ray-sliding triangles on rays 1 and 4 have no ray breakpoints while the ray-sliding triangles on rays 2 and 3 have two ray breakpoints. In the South Region, the ray-sliding triangle on ray 2 has two ray breakpoints while the ray-sliding triangle on ray 4 has no ray breakpoints. Since each ray-sliding triangle has exactly one ray breakpoint in the Central Region, it is not immediately obvious how to choose ray-sliding triangles consistently with the Central Region analysis.

3.3.1 First Breakpoints

As a first attempt, we choose the ray-sliding triangle for ray i at the first breakpoint if it is defined and use a fixed triangle otherwise. We restrict our analysis to the family of parallelograms where $\alpha = \frac{1}{\beta}$ and fractional points $(g, h)^T \in [0, 1]^2$. This restricted case suffices to show both how the bound can be improved and limits on the gains. This restriction isn't too unnatural - it stems from the example showing $\rho[\mathcal{T}, \mathcal{Q}] \geq \frac{9}{8}$ in Section 2.3.

South-West Region

First we consider the South-West Region. The candidate choices of ray-sliding triangles are illustrated in Figure 3.7.

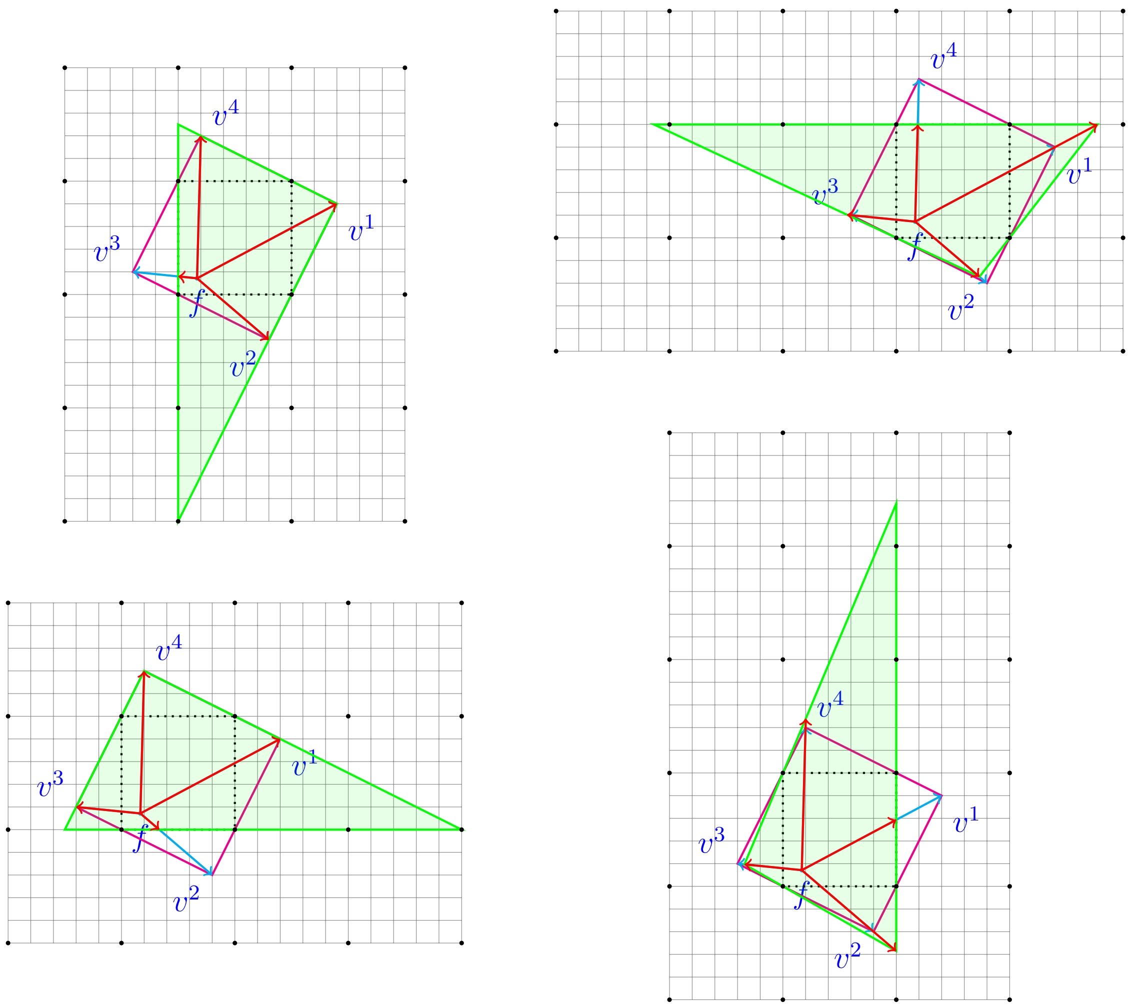


Figure 3.7: Clockwise from top left: $F_1, RS_2(\tau_2^{bp1}), RS_3(\tau_3^{bp1})$, and F_4 for $\alpha = \frac{1}{2}$, $\beta = 2$, $g = \frac{1}{6}$ and $h = \frac{1}{7}$.

Lemma 3.3.1. *For fixed $\beta \geq 1$ and g, h such that $(g,h)^T \in R_{SW}\left(\frac{1}{\beta}, \beta\right) \cap [0,1]^2$, let (P') be the parameterized linear program obtained from $(P(\frac{1}{\beta}, \beta, g, h))$ by replacing constraint (3.7) with the constraint $s \geq \mathbb{0}$ and the intersection cuts for ray-sliding triangles $F_1, RS_2(\tau_2^{bp1}), RS_3(\tau_3^{bp1})$, and F_4. Then, the optimal value of (P') is equal to*

$$LB_{SW1}(\beta, g, h) := 1 - \frac{\beta^3(-1+\beta)}{2\beta^5 + (1-\beta-\beta^4)g - \beta(1+\beta^2-\beta^3)h}.$$

Proof. We order the cuts $RS_3(\tau_3^{bp1}), F_4, F_1, RS_2(\tau_2^{bp1})$. Looking up the cuts in Appendix A, the constraint matrix for a general quadrilateral is

$$\begin{bmatrix} \left(1+\frac{1}{(b+d)(1-g)}\right) & \left(1-\frac{a}{(a+b)(1-g)}\right) & \left(1+\frac{a}{b(a+c)(1-g)}\right) & \left(1-\frac{ad}{b(c+d)(1-g)}\right) \\ 1 & \left(1+\frac{ab}{(a+b)h}\right) & 1 & 1 \\ 1 & 1 & \left(1+\frac{1}{(a+c)g}\right) & 1 \\ \left(1-\frac{d}{(b+d)(1-h)}\right) & \left(1+\frac{ad}{(a+b)(1-h)}\right) & \left(1-\frac{ad}{b(a+c)(1-h)}\right) & \left(1+\frac{cd}{(c+d)(1-h)}\right) \end{bmatrix}.$$

Setting $a = d = \frac{1}{\beta}$ and $b = c = \beta$, optimization problem (P') is given by

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 + s_4 \\ \text{s. t.} \quad & \begin{bmatrix} \left(1+\frac{\beta}{(1+\beta^2)(1-g)}\right) & \left(1-\frac{1}{(1+\beta^2)(1-g)}\right) & \left(1+\frac{1}{\beta(1+\beta^2)(1-g)}\right) & \left(1-\frac{1}{\beta^2(1+\beta^2)(1-g)}\right) \\ 1 & \left(1+\frac{\beta}{(1+\beta^2)h}\right) & 1 & 1 \\ 1 & 1 & \left(1+\frac{\beta}{(1+\beta^2)g}\right) & 1 \\ \left(1-\frac{1}{(1+\beta^2)(1-h)}\right) & \left(1+\frac{1}{\beta(1+\beta^2)(1-h)}\right) & \left(1-\frac{1}{\beta^2(1+\beta^2)(1-h)}\right) & \left(1+\frac{\beta}{(1+\beta^2)(1-h)}\right) \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \geq \mathbb{1} \\ & s_1, s_2, s_3, s_4 \geq 0. \end{aligned}$$

The dual (D') of (P') is given by

$$\begin{aligned} \max \quad & v_1 + v_2 + v_3 + v_4 \\ \text{s. t.} \quad & \begin{bmatrix} \left(1+\frac{\beta}{(1+\beta^2)(1-g)}\right) & 1 & 1 & \left(1-\frac{1}{(1+\beta^2)(1-h)}\right) \\ \left(1-\frac{1}{(1+\beta^2)(1-g)}\right) & \left(1+\frac{\beta}{(1+\beta^2)h}\right) & 1 & \left(1+\frac{1}{\beta(1+\beta^2)(1-h)}\right) \\ \left(1+\frac{1}{\beta(1+\beta^2)(1-g)}\right) & 1 & \left(1+\frac{\beta}{(1+\beta^2)g}\right) & \left(1-\frac{1}{\beta^2(1+\beta^2)(1-h)}\right) \\ \left(1-\frac{1}{\beta^2(1+\beta^2)(1-g)}\right) & 1 & 1 & \left(1+\frac{\beta}{(1+\beta^2)(1-h)}\right) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \leq \mathbb{1} \\ & v_1, v_2, v_3, v_4 \geq 0. \end{aligned}$$

It is straightforward (enough) algebra to show that

$$\hat{s} := \begin{bmatrix} \frac{\beta^3+\beta^6+(1-\beta^4-\beta^6)g-\beta(1+\beta^2-\beta^4)h}{(1+\beta)[2\beta^5+(1-\beta-\beta^4)g-\beta(1+\beta^2-\beta^3)h]} \\ \frac{\beta^2(-1+\beta)(1+\beta^2)h}{2\beta^5+(1-\beta-\beta^4)g-\beta(1+\beta^2-\beta^3)h} \\ \frac{\beta^2(-1+\beta)(1+\beta^2)g}{2\beta^5+(1-\beta-\beta^4)g-\beta(1+\beta^2-\beta^3)h} \\ \frac{\beta^6(1-h)+\beta^5(1-g)}{(1+\beta)[2\beta^5+(1-\beta-\beta^4)g-\beta(1+\beta^2-\beta^3)h]} \end{bmatrix}$$

and

$$\hat{v} := \begin{bmatrix} \frac{\beta^5(1-g)}{2\beta^5+(1-\beta-\beta^4)g-\beta(1+\beta^2-\beta^3)h} \\ \frac{\beta(-1+\beta^4)h}{2\beta^5+(1-\beta-\beta^4)g-\beta(1+\beta^2-\beta^3)h} \\ \frac{(1-\beta)(1-\beta^4)g}{2\beta^5+(1-\beta-\beta^4)g-\beta(1+\beta^2-\beta^3)h} \\ \frac{\beta^3(1-\beta+\beta^2)(1-h)}{2\beta^5+(1-\beta-\beta^4)g-\beta(1+\beta^2-\beta^3)h} \end{bmatrix}$$

are a pair of primal and dual optimal solutions. Therefore the optimal value of (P') is

$$\mathbb{1}^T\hat{s} = 1 - \frac{\beta^3(-1+\beta)}{2\beta^5+(1-\beta-\beta^4)g-\beta(1+\beta^2-\beta^3)h} = LB_{SW1}(\beta, g, h)$$

as required. □

Corollary 3.3.2. *The optimal value of* $(P(\frac{1}{\beta}, \beta, g, h))$ *is at least* 0.867 *for all* $\beta \geq 1$ *and* g, h *such that* $(g,h)^T \in R_{SW}(\frac{1}{\beta}, \beta) \cap [0,1]^2$.

Proof. First note that $R_{SW}(\frac{1}{\beta}, \beta) \cap [0,1]^2 \subseteq \left[0, \frac{1}{1+\beta^2}\right]^2$. Now, by Lemma 3.3.1, the optimal value of $(P(\frac{1}{\beta}, \beta, g, h))$ is greater than or equal to $LB_{SW1}(\beta, g, h) = 1 - \frac{\beta^3(-1+\beta)}{D(\beta,g,h)}$ where $D(\beta, g, h) = 2\beta^5 + (1-\beta-\beta^4)g - \beta(1+\beta^2-\beta^3)h$. Note that $\beta^3(-1+\beta) \geq 0$ and

$$\begin{aligned} D(\beta, g, h) &\geq 2\beta^5 + (1-\beta-\beta^4)g - \beta(1+\beta^2)h \\ &\geq \frac{2\beta^5(1+\beta^2) + (1-\beta-\beta^4) - \beta(1+\beta^2)}{1+\beta^2} \\ &= \frac{2\beta^7 + 2\beta^5 + 1 - 2\beta - \beta^3 - \beta^4}{1+\beta^2} \\ &> 0 \end{aligned}$$

for all $\beta \geq 1$ and $(g,h)^T \in \left[0, \frac{1}{1+\beta^2}\right]^2$. Hence for fixed $\beta \geq 1$ the lower bound $LB(\beta, g, h)$ is smallest whenever $D(\beta, g, h)$ is smallest.

For fixed $\beta \geq 1$ the denominator $D(\beta, g, h)$ is affine in g and h. The coefficient of g in $D(\beta, g, h)$ is $(1 - \beta - \beta^4)$, which is negative for all $\beta \geq 1$. The coefficient of h in $D(\beta, g, h)$ is $-\beta(1 + \beta^2 - \beta^3)$, which is non-positive for all $\beta \leq \frac{3}{2}$ and non-negative for all $\beta \geq \frac{3}{2}$.

For fixed $\beta \leq \frac{3}{2}$, it follows that the smallest bound is given by

$$
\begin{aligned}
LB_{SW1}\left(\beta\ ,\ \frac{1}{1+\beta^2}\ ,\ \frac{1}{1+\beta^2}\right) &= 1 - \frac{\beta^3(-1+\beta)(1+\beta^2)}{2\beta^5(1+\beta^2) + (1-\beta-\beta^4) - \beta(1+\beta^2-\beta^3)} \\
&= 1 - \frac{-\beta^3 + \beta^4 - \beta^5 + \beta^6}{1 - 2\beta - \beta^3 + 2\beta^5 + 2\beta^7}.
\end{aligned}
$$

For fixed $\beta \geq \frac{3}{2}$, it follows that the smallest bound is given by

$$
\begin{aligned}
LB_{SW1}\left(\beta\ ,\ \frac{1}{1+\beta^2}\ ,\ 0\right) &= 1 - \frac{\beta^3(-1+\beta)(1+\beta^2)}{2\beta^5(1+\beta^2) + (1-\beta-\beta^4)} \\
&= 1 - \frac{-\beta^3 + \beta^4 - \beta^5 + \beta^6}{1 - \beta - \beta^4 + 2\beta^5 + 2\beta^7}.
\end{aligned}
$$

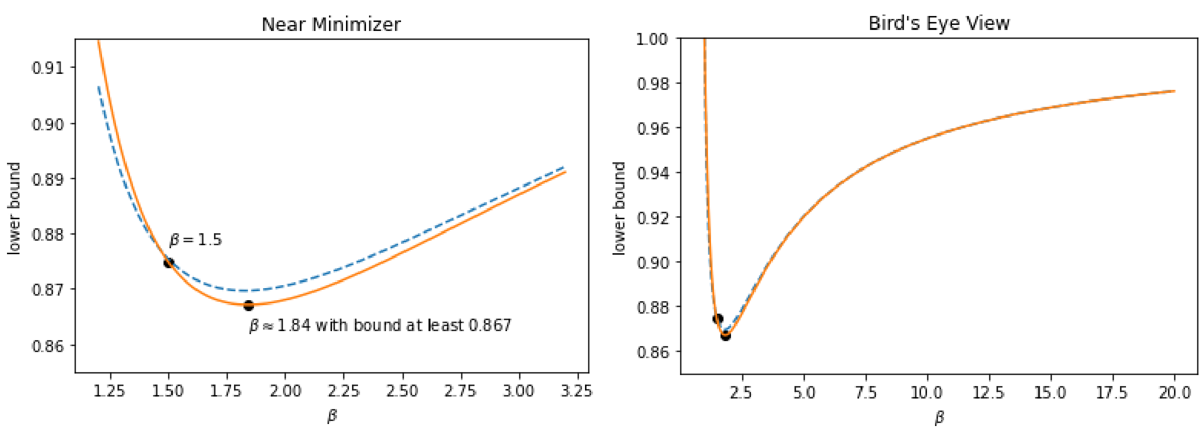


Figure 3.8: Plot of $LB_{SW1}\left(\beta, \frac{1}{1+\beta^2}, 0\right)$ in solid orange and $LB_{SW1}\left(\beta, \frac{1}{1+\beta^2}, \frac{1}{1+\beta^2}\right)$ in dashed blue.

By inspection, the smallest bound is at least 0.867 and it occurs for β around 1.84, $g = \frac{1}{1+\beta^2}$, and $h = 0$. $\square$

South Region

Now we consider the South Region. The candidate choices of ray-sliding triangles are illustrated in Figure 3.9.

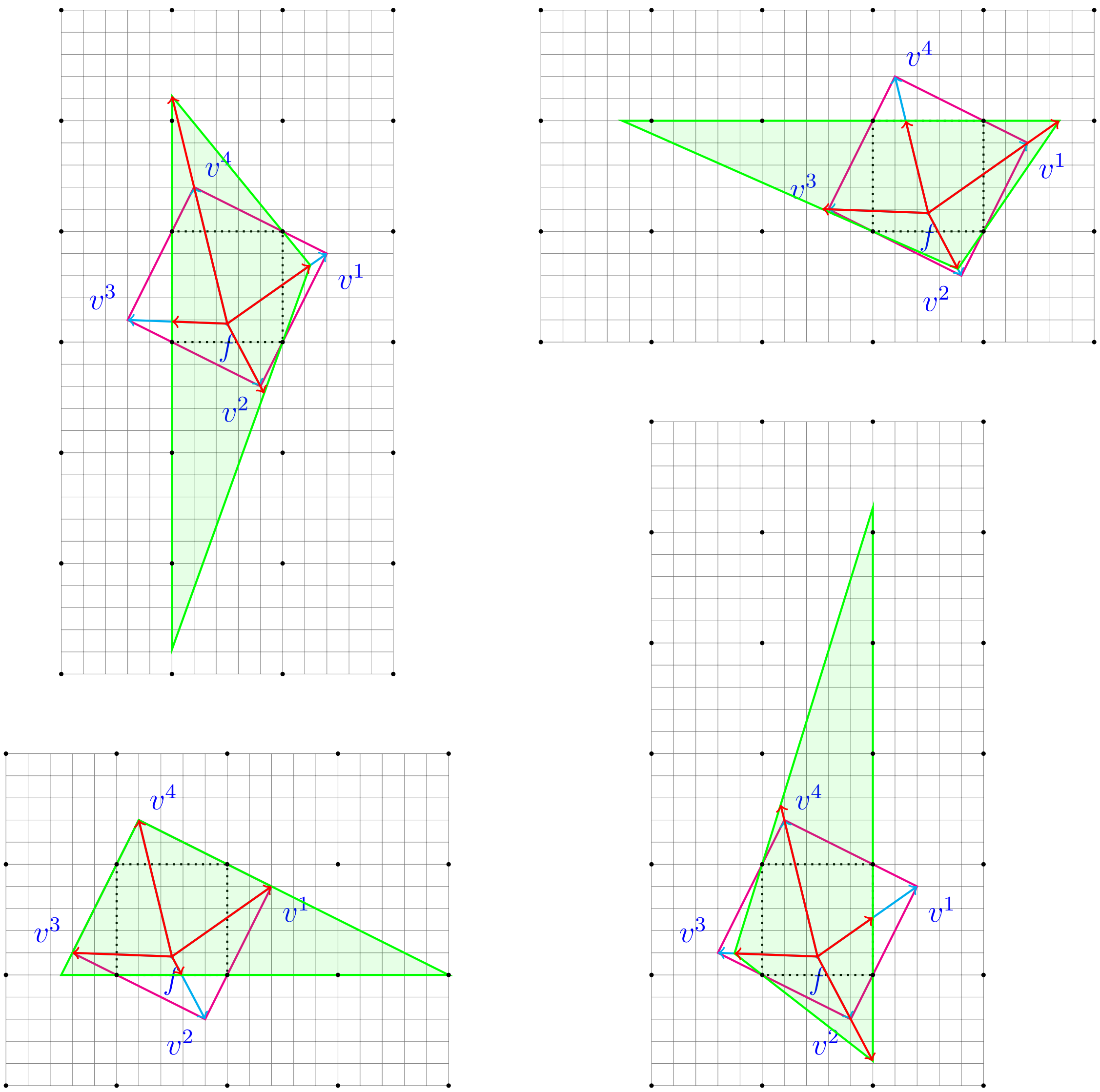


Figure 3.9: Clockwise from top left: $RS_1(\tau_1^{bp1})$, $RS_2(\tau_2^{bp1})$, $RS_3(\tau_3^{bp1})$, and F_4 for $\alpha = \frac{1}{2}$, $\beta = 2$, $g = \frac{1}{2}$ and $h = \frac{1}{6}$.

Lemma 3.3.3. *For fixed $\beta \geq 1$ and g, h such that $(g,h)^T \in R_S\left(\frac{1}{\beta}, \beta\right) \cap [0,1]^2$, let (P') be the parameterized linear program obtained from $(P(\frac{1}{\beta}, \beta, g, h))$ by replacing constraint (3.7) with the constraint $s \geq \mathbb{0}$ and the intersection cuts for ray-sliding triangles $RS_1(\tau_1^{bp1})$, $RS_2(\tau_2^{bp1})$, $RS_3(\tau_3^{bp1})$, and F_4. Then, the optimal value of (P') is equal to*

$$LB_{S1}(\beta, g, h) := 1 - \frac{\beta^2(-1+\beta)}{-\beta^2 + \beta^3 + 2\beta^4 - \beta^3 g - h}.$$

Proof. We order the cuts $RS_3(\tau_3^{bp1})$, F_4, $RS_1(\tau_1^{bp1})$, $RS_2(\tau_2^{bp1})$. Looking up the cuts in Appendix A, the constraint matrix for a general quadrilateral is

$$\begin{bmatrix} \left(1+\frac{1}{(b+d)(1-g)}\right) & \left(1-\frac{a}{(a+b)(1-g)}\right) & \left(1+\frac{a}{b(a+c)(1-g)}\right) & \left(1-\frac{ad}{b(c+d)(1-g)}\right) \\ 1 & \left(1+\frac{ab}{(a+b)h}\right) & 1 & 1 \\ \left(1+\frac{d}{c(b+d)g}\right) & \left(1-\frac{ad}{c(a+b)g}\right) & \left(1+\frac{1}{(a+c)g}\right) & \left(1-\frac{d}{(c+d)g}\right) \\ \left(1-\frac{d}{(b+d)(1-h)}\right) & \left(1+\frac{ad}{(a+b)(1-h)}\right) & \left(1-\frac{ad}{b(a+c)(1-h)}\right) & \left(1+\frac{cd}{(c+d)(1-h)}\right) \end{bmatrix}.$$

Setting $a = d = \frac{1}{\beta}$ and $b = c = \beta$, optimization problem (P') is given by

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 + s_4 \\ \text{s.t.} \quad & \begin{bmatrix} \left(1+\frac{\beta}{(1+\beta^2)(1-g)}\right) & \left(1-\frac{1}{(1+\beta^2)(1-g)}\right) & \left(1+\frac{1}{\beta(1+\beta^2)(1-g)}\right) & \left(1-\frac{1}{\beta^2(1+\beta^2)(1-g)}\right) \\ 1 & \left(1+\frac{\beta}{(1+\beta^2)h}\right) & 1 & 1 \\ \left(1+\frac{1}{\beta(1+\beta^2)g}\right) & \left(1-\frac{1}{\beta^2(1+\beta^2)g}\right) & \left(1+\frac{\beta}{(1+\beta^2)g}\right) & \left(1-\frac{1}{(1+\beta^2)g}\right) \\ \left(1-\frac{1}{(1+\beta^2)(1-h)}\right) & \left(1+\frac{1}{\beta(1+\beta^2)(1-h)}\right) & \left(1-\frac{1}{\beta^2(1+\beta^2)(1-h)}\right) & \left(1+\frac{\beta}{(1+\beta^2)(1-h)}\right) \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \geq \mathbb{1} \\ & s_1, s_2, s_3, s_4 \geq 0. \end{aligned}$$

The dual (D') of (P') is given by

$$\begin{aligned} \max \quad & v_1 + v_2 + v_3 + v_4 \\ \text{s.t.} \quad & \begin{bmatrix} \left(1+\frac{\beta}{(1+\beta^2)(1-g)}\right) & 1 & \left(1+\frac{1}{\beta(1+\beta^2)g}\right) & \left(1-\frac{1}{(1+\beta^2)(1-h)}\right) \\ \left(1-\frac{1}{(1+\beta^2)(1-g)}\right) & \left(1+\frac{\beta}{(1+\beta^2)h}\right) & \left(1-\frac{1}{\beta^2(1+\beta^2)g}\right) & \left(1+\frac{1}{\beta(1+\beta^2)(1-h)}\right) \\ \left(1+\frac{1}{\beta(1+\beta^2)(1-g)}\right) & 1 & \left(1+\frac{\beta}{(1+\beta^2)g}\right) & \left(1-\frac{1}{\beta^2(1+\beta^2)(1-h)}\right) \\ \left(1-\frac{1}{\beta^2(1+\beta^2)(1-g)}\right) & 1 & \left(1-\frac{1}{(1+\beta^2)g}\right) & \left(1+\frac{\beta}{(1+\beta^2)(1-h)}\right) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \leq \mathbb{1} \\ & v_1, v_2, v_3, v_4 \geq 0. \end{aligned}$$

It is straightforward (enough) algebra to show that

$$\hat{s} := \begin{bmatrix} \frac{\beta^5-\beta^3(1+\beta^2)g-(1-\beta^4)h}{(1+\beta)(-\beta^2+\beta^3+2\beta^4-\beta^3 g-h)} \\ \frac{-\beta(1-\beta)(1+\beta^2)h}{(-\beta^2+\beta^3+2\beta^4-\beta^3 g-h)} \\ \frac{\beta^4(1+\beta g-h)}{(1+\beta)(-\beta^2+\beta^3+2\beta^4-\beta^3 g-h)} \\ \frac{\beta^4(1-g)+\beta^5(1-h)}{(1+\beta)(-\beta^2+\beta^3+2\beta^4-\beta^3 g-h)} \end{bmatrix}$$

and

$$\hat{v} := \begin{bmatrix} \frac{\beta^4(1-g)}{-\beta^2+\beta^3+2\beta^4-\beta^3 g-h} \\ \frac{(-1+\beta^4)h}{-\beta^2+\beta^3+2\beta^4-\beta^3 g-h} \\ \frac{\beta^3(-1+\beta)g}{-\beta^2+\beta^3+2\beta^4-\beta^3 g-h} \\ \frac{\beta^4(1-h)}{-\beta^2+\beta^3+2\beta^4-\beta^3 g-h} \end{bmatrix}$$

are a pair of primal and dual optimal solutions. Therefore the optimal value of (P') is

$$\mathbb{1}^T \hat{s} = 1 - \left(\frac{\beta^2(-1+\beta)}{-\beta^2+\beta^3+2\beta^4-\beta^3 g-h} \right) = LB_{S1}(\beta, g, h)$$

as required. $\square$

Corollary 3.3.4. *The optimal value of* $(P(\frac{1}{\beta}, \beta, g, h))$ *is at least* 0.862 *for all* $\beta \geq 1$ *and* g, h *such that* $(g,h)^T \in R_S(\frac{1}{\beta}, \beta) \cap [0,1]^2$.

Proof. First note that $R_S(\frac{1}{\beta}, \beta) \cap [0,1]^2 \subseteq \left\{ (x,y)^T \in \mathbb{R}^2 : x \in \left[\frac{1}{1+\beta^2}, \frac{\beta^2}{1+\beta^2} \right], y \in \left[0, \frac{1}{1+\beta^2} \right] \right\}$. Now, by Lemma 3.3.3, the optimal value of $(P(\frac{1}{\beta}, \beta, g, h))$ is greater than or equal to $LB_{S1}(\beta, g, h) = 1 - \frac{\beta^2(-1+\beta)}{D(\beta,g,h)}$ where $D(\beta, g, h) = -\beta^2 + \beta^3 + 2\beta^4 - \beta^3 g - h$. Note that $\beta^2(-1+\beta) \geq 0$ and

$$\begin{aligned} D(\beta, g, h) &\geq \frac{(-\beta^2+\beta^3+2\beta^4)(1+\beta^2) - \beta^5 - 1}{1+\beta^2} \\ &= \frac{-1-\beta^2+\beta^3+\beta^4+2\beta^6}{1+\beta^2} \\ &> 0 \end{aligned}$$

for all $\beta \geq 1$, $g \in \left[\frac{1}{1+\beta^2}, \frac{\beta^2}{1+\beta^2} \right]$ and $h \in \left[0, \frac{1}{1+\beta^2} \right]$. Hence for fixed $\beta \geq 1$ the lower bound $LB(\beta, g, h)$ is smallest whenever $D(\beta, g, h)$ is smallest.

For fixed $\beta \geq 1$ the denominator $D(\beta, g, h)$ is affine in g and h. The coefficient of g in $D(\beta, g, h)$ is $-\beta^3$, which is negative for all $\beta \geq 1$. The coefficient of h in $D(\beta, g, h)$ is -1, which is negative for all $\beta \geq 1$. For fixed $\beta \geq 1$, it follows that the smallest bound is given by

$$\begin{aligned} LB_{S1}\left(\beta\ ,\ \frac{\beta^2}{1+\beta^2}\ ,\ \frac{1}{1+\beta^2}\right) &= 1 - \frac{\beta^2(-1+\beta)(1+\beta^2)}{(-\beta^2+\beta^3+2\beta^4)(1+\beta^2)-\beta^5-1} \\ &= 1 - \frac{-\beta^2+\beta^3-\beta^4+\beta^5}{-1-\beta^2+\beta^3+\beta^4+2\beta^6}. \end{aligned}$$

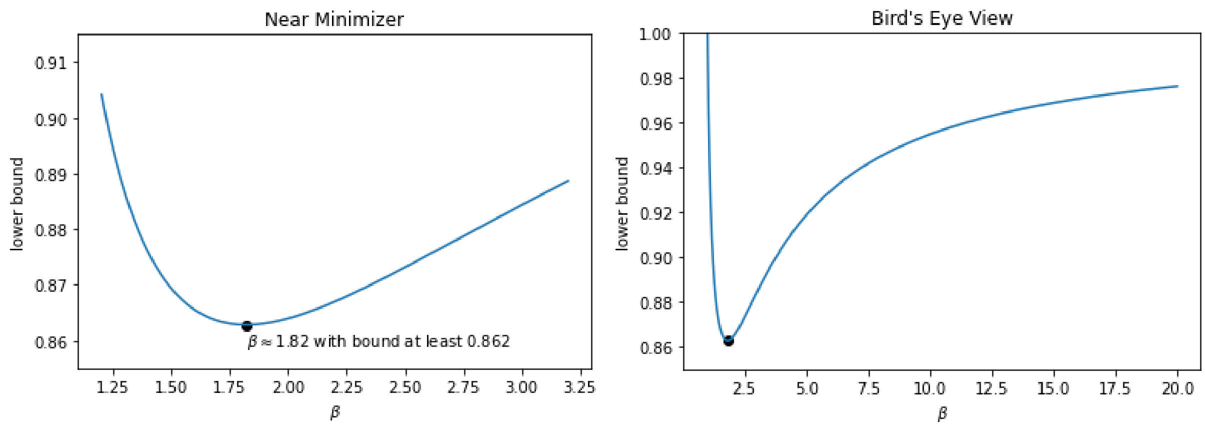


Figure 3.10: Plot of $LB_{S1}(\beta, \frac{\beta^2}{1+\beta^2}, \frac{1}{1+\beta^2})$.

By inspection, the smallest bound is at least 0.862 and it occurs for β around 1.82, $g = \frac{\beta^2}{1+\beta^2}$ and $h = \frac{1}{1+\beta^2}$. $\square$

Observations

The above choices improve on the bound provided in Lemma 2.2.6. For example, the optimal value of $(P(1, 1, g, h))$ for any $(g, h)^T \in [0, 1]^2$ can only be lower-bounded by $\frac{4}{5}$ using fixed triangles but it can be lower-bounded by 1 using the triangles described above. However, the analysis also shows that the selected triangles are insufficient to lower bound the optimal value of $(P(\alpha, \beta, g, h))$ by $\frac{8}{9}$ for all applicable choices of α, β, g, and h. Moreover, the bound calculated is not consistent for $(g, h)^T$ at the meeting of the Central, South-West, and South Regions, nor on the boundary of the South and South-West Regions. Together, these two facts suggest a better analysis may be possible.

3.3.2 Testing Other Choices of Standard Breakpoints

Recall that in the South-West Region there are:

1. three standard breakpoints for the ray-sliding triangles for each of rays 1 and 4; and
2. five standard breakpoints for the ray-sliding triangles for each of rays 2 and 3.

If we aim to use one ray-sliding triangle at a standard breakpoint per ray to prove the bound of $\frac{8}{9}$, there are $3 \cdot 5 \cdot 5 \cdot 3 = 225$ possible combinations. One might ask:

1. The first combination we guessed didn't prove the $\frac{8}{9}$ bound, but can another combination work?; and
2. Even if there isn't one "global best choice", is it possible that for every applicable α, β, g, and h, there exists a combination that works?

Empirical evidence says that the answer to both of these questions is "no". Although other choices may improve on the bounds in Section 3.3.1, a bound of $\frac{8}{9}$ cannot be obtained. To show this, we fix the parameters α, β, g, and h to specific numerical values, solve 225 linear programs, and observe that each has optimal value strictly less than $\frac{8}{9}$. We will solve these linear programs using a numerical solver and, accordingly, standard caveats about numerical round-off, and the distance from the true optimal value apply (hence this evidence is said to be "empirical" rather than "mathematically rigorous").

We choose the parameters $(\alpha, \beta, g, h) = \left(\frac{1}{2}, 2, \frac{1}{10}, \frac{1}{10}\right)$ for our experiments. Note that $\left(\frac{1}{10}, \frac{1}{10}\right)^T \in R_{SW}\left(\frac{1}{2}, 2\right)$. Let A_i for $i \in \{1,2,3,4\}$ denote the set of possible intersection cut coefficients obtained from a ray-sliding triangle for a standard breakpoint for ray i. For example $|A_1| = 3$ and its elements are $(\psi_{f;T}(r^1), \psi_{f;T}(r^2), \psi_{f;T}(r^3), \psi_{f;T}(r^4))^T$ for $T \in \{RS_1(\tau_1^{min}), F_1, RS_1(\tau_1^{max})\}$. Looking up the cuts in Appendix A and substituting our choice of parameters $a = d = \frac{1}{2}$, $b = c = 2$, $g = h = \frac{1}{10}$ we calculate

$$A_1 := \left\{ \begin{bmatrix} \frac{13}{9} \\ \frac{7}{9} \\ 5 \\ \frac{1}{9} \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{7}{9} \\ \frac{10}{9} \\ 5 \\ \frac{13}{9} \end{bmatrix} \right\}, \; A_2 := \left\{ \begin{bmatrix} \frac{7}{9} \\ 5 \\ \frac{1}{9} \\ \frac{13}{9} \end{bmatrix}, \begin{bmatrix} \frac{7}{9} \\ \frac{25}{9} \\ \frac{1}{9} \\ \frac{13}{9} \end{bmatrix}, \begin{bmatrix} \frac{7}{9} \\ \frac{10}{9} \\ \frac{17}{18} \\ \frac{13}{9} \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ \frac{13}{9} \end{bmatrix}, \begin{bmatrix} \frac{13}{9} \\ \frac{7}{9} \\ \frac{10}{9} \\ \frac{13}{9} \end{bmatrix} \right\},$$

$$A_3 := \left\{ \begin{bmatrix} \frac{13}{9} \\ \frac{7}{9} \\ 5 \\ \frac{1}{9} \end{bmatrix}, \begin{bmatrix} \frac{13}{9} \\ \frac{7}{9} \\ \frac{25}{9} \\ \frac{1}{9} \end{bmatrix}, \begin{bmatrix} \frac{13}{9} \\ \frac{7}{9} \\ \frac{10}{9} \\ \frac{17}{18} \end{bmatrix}, \begin{bmatrix} \frac{13}{9} \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{13}{9} \\ \frac{25}{9} \\ \frac{1}{9} \\ \frac{13}{9} \end{bmatrix} \right\}, \text{and } A_4 := \left\{ \begin{bmatrix} \frac{7}{9} \\ 5 \\ \frac{1}{9} \\ \frac{13}{9} \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{13}{9} \\ 5 \\ \frac{25}{9} \\ \frac{1}{9} \end{bmatrix} \right\}.$$

Then for all $a_1 \in A_1$, $a_2 \in A_2$, $a_3 \in A_3$, and $a_4 \in A_4$ we need to solve

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 + s_4 \\ \text{subject to} \quad & \begin{bmatrix} (a_1)^T \\ (a_2)^T \\ (a_3)^T \\ (a_4)^T \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \geq \mathbb{1} \\ & s_1, s_2, s_3, s_4 \geq 0. \end{aligned}$$

The problem data is reasonable and we use Python and scipy.optimize to do the heavy lifting of solving these problems.[1] The linear program with largest objective function value is given by the assignment

$$a_1 := \begin{bmatrix} \frac{13}{9} \\ \frac{7}{9} \\ 5 \\ \frac{1}{9} \end{bmatrix}, \; a_2 := \begin{bmatrix} \frac{7}{9} \\ \frac{10}{9} \\ \frac{17}{18} \\ \frac{13}{9} \end{bmatrix}, \; a_3 := \begin{bmatrix} \frac{13}{9} \\ \frac{7}{9} \\ \frac{10}{9} \\ \frac{17}{18} \end{bmatrix}, \text{and } a_4 := \begin{bmatrix} \frac{13}{9} \\ 5 \\ \frac{25}{9} \\ \frac{1}{9} \end{bmatrix},$$

corresponding to using the cuts for $RS_1(\tau_1^{min})$, $RS_2(\tau_2^{bp1})$, $RS_3(\tau_3^{bp1})$ and $RS_4(\tau_4^{max})$. The objective function value calculated is a little below 0.883.

Although these experiments suggest that using one ray-sliding triangle per ray won't lead to a bound of $\frac{8}{9}$, they can be modified to give insight into alternative candidate ray-sliding triangle sets. Firstly, we solve the linear program

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 + s_4 \\ \text{subject to} \quad & (a_1)^T s \geq 1 \quad \forall a_1 \in A_1 \\ & (a_2)^T s \geq 1 \quad \forall a_2 \in A_2 \\ & (a_3)^T s \geq 1 \quad \forall a_3 \in A_3 \\ & (a_4)^T s \geq 1 \quad \forall a_4 \in A_4 \\ & s_1, s_2, s_3, s_4 \geq 0. \end{aligned}$$

It's optimal value is around 0.90. For the optimal solution found, the tight primal constraints are those generated by the intersection cuts for $RS_2(\tau_2^{bp2})$, $RS_2(\tau_2^{bp1})$, $RS_3(\tau_3^{bp2})$, and $RS_3(\tau_3^{bp1})$. So, if we are willing to use more than one ray-sliding triangle per ray, these

[1]Scipy [72] was used to perform numerical experiments that informed the choice of intersection cuts made throughout this thesis.

four triangles would be a reasonable candidate set to try. Alternatively, if we are willing to consider choices for τ other than standard breakpoints, we might want to choose $RS_2(\tau_2)$ for $\tau_2 \in (\tau_2^{bp2}, \tau_2^{bp1})$ and $RS_3(\tau_3)$ for $\tau_3 \in (\tau_3^{bp2}, \tau_3^{bp1})$. To further investigate this option, we choose to fix F_1 and F_4, vary τ_2 between τ_2^{bp2} and τ_2^{bp1}, and vary τ_3 between τ_3^{bp2} and τ_3^{bp1}. The results are illustrated in Figure 3.11. The optimal value of the corresponding linear program is largest for some unique choice of τ_2 and τ_3, indicated in the figure by the black dot. Denoting these values as $\hat{\tau}_2$ and $\hat{\tau}_3$, then these two triangles (or these two triangles plus F_1 and F_4) are a reasonable candidate set to try.

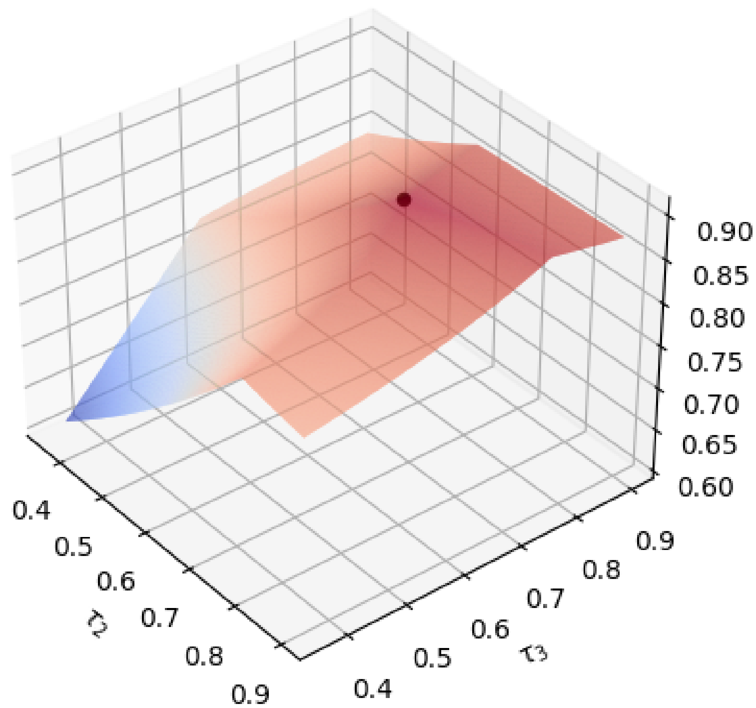


Figure 3.11: Optimal value of linear program $\min s_1 + s_2 + s_3 + s_4$ subject to $s \geq \mathbb{0}$ and s satisfying the intersection cuts generated by F_1, $RS_2(\tau_2)$, $RS_3(\tau_3)$, and F_4, as a function of $\tau_2 \in [\tau_2^{bp2}, \tau_2^{bp1}]$ and $\tau_3 \in [\tau_3^{bp2}, \tau_3^{bp1}]$.

This discussion has led us to the choices for ray-sliding triangles we use for the South and South-West Regions in Chapter 3 and Chapter 4. When these triangles appear later, we define and analyze them analytically, and do not depend on the discussion here.

3.4 South-West Region

We return to bounding the optimal value of $(P(\alpha,\beta,g,h))$ analytically, focusing next on the South-West Region. We use the intersection cuts for the two triangles $RS_2(\hat{\tau}_2)$ and $RS_3(\hat{\tau}_3)$ where

$$\hat{\tau}_2 := \frac{(\alpha+\beta)(1-h)}{\alpha\beta+(\alpha+\beta)(1-h)} \quad \text{and} \quad \hat{\tau}_3 := \frac{(\alpha+\beta)(1-g)}{1+(\alpha+\beta)(1-g)}.$$

These triangles are illustrated in Figure 3.12.

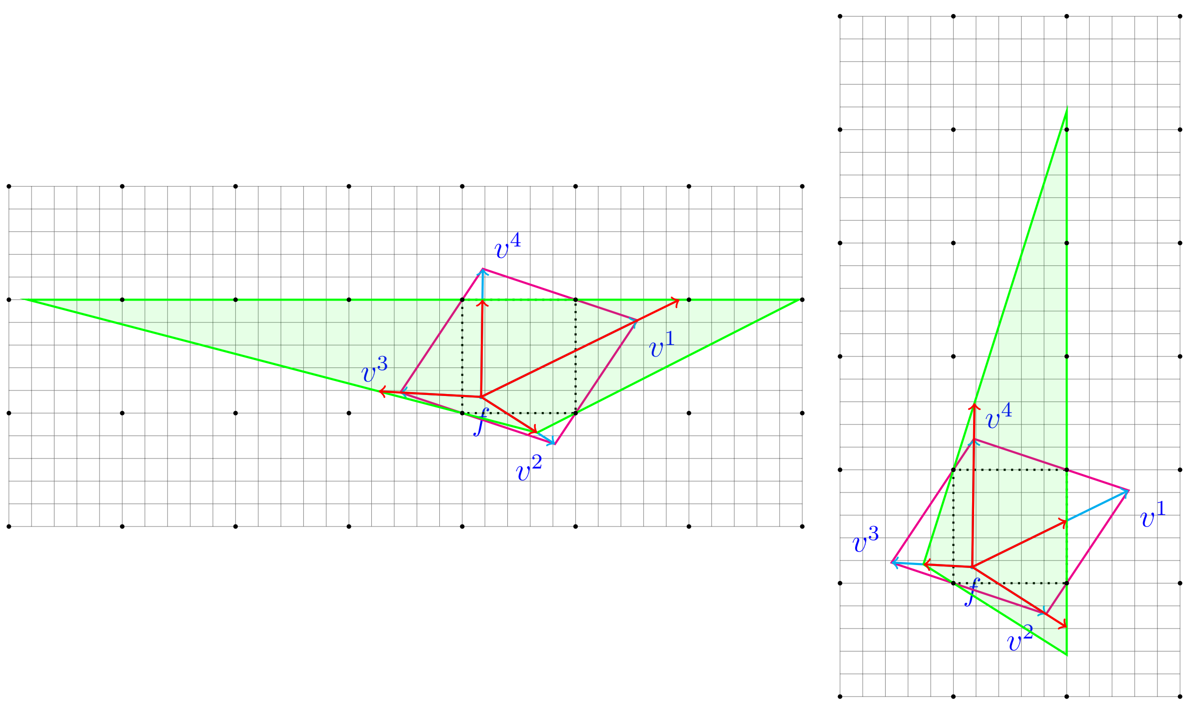


Figure 3.12: $RS_2(\hat{\tau}_2)$ (left) and $RS_3(\hat{\tau}_3)$ (right) for $\alpha=\frac{1}{3}$, $\beta=\frac{3}{2}$, $g=\frac{1}{6}$ and $h=\frac{1}{7}$.

Lemma 3.4.1. *For fixed $\alpha, \beta > 0$ such that $\alpha \le \beta$ and g, h such that $(g,h)^T \in R_{SW}(\alpha,\beta)$, let (P') be the parameterized linear program obtained from $(P(\alpha,\beta,g,h))$ by replacing constraint (3.7) with the constraint $s \ge \mathbb{0}$ and the intersection cuts for ray-sliding triangles $RS_2(\hat{\tau}_2)$ and $RS_3(\hat{\tau}_3)$. Then, the optimal value of (P') is greater than or equal to*

$$LB_{SW}(\alpha,\beta,g,h) := 1 - \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}.$$

Proof. Looking up the cuts in Appendix A and specializing for $a = d = \alpha$ and $b = c = \beta$, the constraint matrix is

$$M(\alpha,\beta,g,h) := \begin{bmatrix} \begin{pmatrix} 1 - \frac{\alpha}{(\alpha+\beta)(1-h)} \\ 1 + \frac{1}{(\alpha+\beta)(1-g)} \end{pmatrix} & \begin{pmatrix} 1 + \frac{\alpha\beta}{(\alpha+\beta)(1-h)} \\ 1 - \frac{\alpha}{(\alpha+\beta)(1-g)} \end{pmatrix} & \begin{pmatrix} 1 - \frac{\alpha}{(\alpha+\beta)(1-h)} \\ 1 + \frac{1}{(\alpha+\beta)(1-g)} \end{pmatrix} & \begin{pmatrix} 1 + \frac{\alpha\beta}{(\alpha+\beta)(1-h)} \\ 1 - \frac{\alpha}{(\alpha+\beta)(1-g)} \end{pmatrix} \end{bmatrix}.$$

So, optimization problem (P') is given by

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 + s_4 \\ \text{s.t.} \quad & \begin{bmatrix} \begin{pmatrix} 1 - \frac{\alpha}{(\alpha+\beta)(1-h)} \\ 1 + \frac{1}{(\alpha+\beta)(1-g)} \end{pmatrix} & \begin{pmatrix} 1 + \frac{\alpha\beta}{(\alpha+\beta)(1-h)} \\ 1 - \frac{\alpha}{(\alpha+\beta)(1-g)} \end{pmatrix} & \begin{pmatrix} 1 - \frac{\alpha}{(\alpha+\beta)(1-h)} \\ 1 + \frac{1}{(\alpha+\beta)(1-g)} \end{pmatrix} & \begin{pmatrix} 1 + \frac{\alpha\beta}{(\alpha+\beta)(1-h)} \\ 1 - \frac{\alpha}{(\alpha+\beta)(1-g)} \end{pmatrix} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \ge \mathbb{1} \\ & s_1, s_2, s_3, s_4 \ge 0. \end{aligned}$$

The dual (D') of (P') is given by

$$\begin{aligned} \max \quad & v_1 + v_2 \\ \text{s.t.} \quad & \begin{bmatrix} \begin{pmatrix} 1 - \frac{\alpha}{(\alpha+\beta)(1-h)} \end{pmatrix} & \begin{pmatrix} 1 + \frac{1}{(\alpha+\beta)(1-g)} \end{pmatrix} \\ \begin{pmatrix} 1 + \frac{\alpha\beta}{(\alpha+\beta)(1-h)} \end{pmatrix} & \begin{pmatrix} 1 - \frac{\alpha}{(\alpha+\beta)(1-g)} \end{pmatrix} \\ \begin{pmatrix} 1 - \frac{\alpha}{(\alpha+\beta)(1-h)} \end{pmatrix} & \begin{pmatrix} 1 + \frac{1}{(\alpha+\beta)(1-g)} \end{pmatrix} \\ \begin{pmatrix} 1 + \frac{\alpha\beta}{(\alpha+\beta)(1-h)} \end{pmatrix} & \begin{pmatrix} 1 - \frac{\alpha}{(\alpha+\beta)(1-g)} \end{pmatrix} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \le \mathbb{1} \\ & v_1, v_2 \ge 0. \end{aligned}$$

Note that the first and third dual constraints coincide, as do the second and fourth dual constraints.

We will show that

$$\hat{v} := \begin{bmatrix} \frac{(1+\alpha)(\alpha+\beta)(1-h)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)} \\ \frac{\alpha(1+\beta)(\alpha+\beta)(1-g)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)} \end{bmatrix}$$

is feasible for (D'). Thus, the optimal value of (P') is bounded below by $LB_{SW}(\alpha, \beta, g, h)$ because

$$\begin{aligned}\mathbb{1}^T \hat{v} &= \frac{(1+\alpha)(\alpha+\beta)(1-h) + \alpha(1+\beta)(\alpha+\beta)(1-g)}{\alpha(\beta-\alpha) + \alpha(1+\beta)(\alpha+\beta)(1-g) + (1+\alpha)(\alpha+\beta)(1-h)} \\ &= 1 - \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha) + \alpha(1+\beta)(\alpha+\beta)(1-g) + (1+\alpha)(\alpha+\beta)(1-h)}.\end{aligned}$$

Now we show that $\hat{v}$ is feasible for (D'). Each entry of $\hat{v}$ is of the form $C \cdot \frac{1}{K}$ for some constant C where

$$K := \alpha(\beta-\alpha) + \alpha(1+\beta)(\alpha+\beta)(1-g) + (1+\alpha)(\alpha+\beta)(1-h) > 0$$

for any fixed $\alpha, \beta > 0$ with $\alpha \leq \beta$, $g \leq \frac{\alpha}{\alpha+\beta}$, and $h \leq \frac{\alpha}{\alpha+\beta}$. So, to verify $\hat{v} \geq \mathbb{0}$, we check that the corresponding constant C is non-negative for each entry. To verify $\hat{v}_1 \geq 0$, note that $(1+\alpha)(\alpha+\beta)(1-h) \geq 0$ for any fixed $\alpha, \beta > 0$, and $h \leq \frac{\alpha}{\alpha+\beta}$. To verify $\hat{v}_2 \geq 0$, note that $\alpha(1+\beta)(\alpha+\beta)(1-g) \geq 0$ for any fixed $\alpha, \beta > 0$, and $g \leq \frac{\alpha}{\alpha+\beta}$.

Having shown $\hat{v}$ is non-negative, we next verify that $\hat{v}$ satisfies the constraints of (D'). As each constraint is of the form $(\mathbb{1}^T + [k_1, k_2])v \leq 1$, it suffices to check

$$k_1\hat{v}_1 + k_2\hat{v}_2 \leq \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha) + \alpha(1+\beta)(\alpha+\beta)(1-g) + (1+\alpha)(\alpha+\beta)(1-h)}.$$

To verify $\hat{v}$ satisfies the first constraint, we calculate

$$\begin{aligned}LHS &= \left(\frac{1}{\alpha(\beta-\alpha) + \alpha(1+\beta)(\alpha+\beta)(1-g) + (1+\alpha)(\alpha+\beta)(1-h)}\right) \cdot \\ &\quad \left[-\left(\frac{\alpha}{(\alpha+\beta)(1-h)}\right)((1+\alpha)(\alpha+\beta)(1-h)) + \left(\frac{1}{(\alpha+\beta)(1-g)}\right)(\alpha(1+\beta)(\alpha+\beta)(1-g))\right] \\ &= \frac{-\alpha(1+\alpha) + \alpha(1+\beta)}{\alpha(\beta-\alpha) + \alpha(1+\beta)(\alpha+\beta)(1-g) + (1+\alpha)(\alpha+\beta)(1-h)} \\ &= \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha) + \alpha(1+\beta)(\alpha+\beta)(1-g) + (1+\alpha)(\alpha+\beta)(1-h)}\end{aligned}$$

and conclude that $\hat{v}$ satisfies the first constraint; moreover, it satisfies this constraint with equality. To verify $\hat{v}$ satisfies the second constraint, we calculate

$$\begin{aligned}
LHS &= \left(\frac{1}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}\right)\cdot \\
&\quad \left[\left(\frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right)((1+\alpha)(\alpha+\beta)(1-h)) - \left(\frac{\alpha}{(\alpha+\beta)(1-g)}\right)(\alpha(1+\beta)(\alpha+\beta)(1-g))\right] \\
&= \frac{\alpha\beta(1+\alpha)-\alpha^2(1+\beta)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)} \\
&= \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}
\end{aligned}$$

and conclude that $\hat{v}$ satisfies the second constraint; moreover, it satisfies this constraint with equality. As the first and third dual constraints coincide, $\hat{v}$ satisfies the third constraint with equality. As the second and fourth dual constraints coincide $\hat{v}$ satisfies the fourth constraint with equality. Therefore, $\hat{v}$ is feasible for (D') and thus the optimal value of (P') is at least $LB_{SW}(\alpha, \beta, g, h)$. □

Proposition 3.4.2. *The optimal value of (P') in Lemma 3.4.1 is exactly $LB_{SW}(\alpha, \beta, g, h)$.*

Proof. See Appendix B.1 where a primal feasible solution of objective function value $LB_{SW}(\alpha, \beta, g, h)$ is provided. □

3.5 South Region

In the South Region, we use the intersection cuts for the three triangles $RS_1(\tau_1^{bp1})$, $RS_2(\hat{\tau}_2)$, and $RS_3(\tau_3^{bp1})$ where

$$\hat{\tau}_2 := \frac{(\alpha+\beta)(1-h)}{\alpha\beta+(\alpha+\beta)(1-h)},$$

consistently with our choice of $\hat{\tau}_2$ in the South-West Region. These triangles are illustrated in Figure 3.13.

Lemma 3.5.1. *For fixed $\alpha, \beta > 0$ such that $\alpha \le \beta$ and g, h such that $(g,h)^T \in R_S(\alpha, \beta)$, let (P') be the parameterized linear program obtained from $(P(\alpha, \beta, g, h))$ by replacing constraint (3.7) with the constraint $s \ge \mathbb{0}$ and the intersection cuts for ray-sliding triangles $RS_1(\tau_1^{bp1}), RS_2(\hat{\tau}_2)$, and $RS_3(\tau_3^{bp1})$. Then, the optimal value of (P') is greater than or equal to*

$$LB_S(\alpha, \beta, h) := 1 - \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}.$$

Proof. Looking up the cuts in Appendix A and specializing for $a = d = \alpha$ and $b = c = \beta$, the constraint matrix is

$$M(\alpha,\beta,g,h) := \begin{bmatrix} \left(1+\frac{\alpha}{\beta(\alpha+\beta)g}\right) & \left(1-\frac{\alpha^2}{\beta(\alpha+\beta)g}\right) & \left(1+\frac{1}{(\alpha+\beta)g}\right) & \left(1-\frac{\alpha}{(\alpha+\beta)g}\right) \\ \left(1-\frac{\alpha}{(\alpha+\beta)(1-h)}\right) & \left(1+\frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right) & \left(1-\frac{\alpha}{(\alpha+\beta)(1-h)}\right) & \left(1+\frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right) \\ \left(1+\frac{1}{(\alpha+\beta)(1-g)}\right) & \left(1-\frac{\alpha}{(\alpha+\beta)(1-g)}\right) & \left(1+\frac{\alpha}{\beta(\alpha+\beta)(1-g)}\right) & \left(1-\frac{\alpha^2}{\beta(\alpha+\beta)(1-g)}\right) \end{bmatrix}.$$

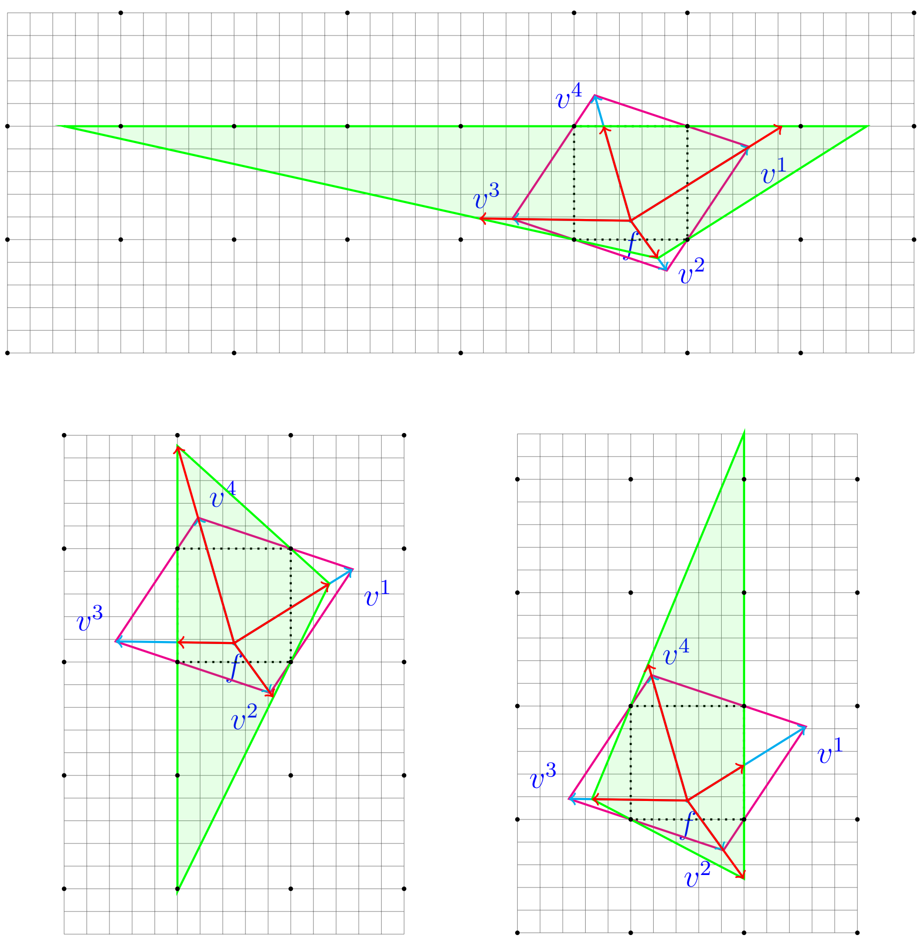


Figure 3.13: Clockwise from top: $RS_2(\hat{\tau}_2), RS_3(\tau_3^{bp1})$, and $RS_1(\tau_1^{bp1})$ for $\alpha = \frac{1}{3}$, $\beta = \frac{3}{2}$, $g = \frac{1}{2}$ and $h = \frac{1}{6}$.

So, optimization problem (P') is given by

$$\begin{array}{ll} \min & s_1+s_2+s_3+s_4 \\ \text{s.t.} & \begin{bmatrix} \left(1+\frac{\alpha}{\beta(\alpha+\beta)g}\right) & \left(1-\frac{\alpha^2}{\beta(\alpha+\beta)g}\right) & \left(1+\frac{1}{(\alpha+\beta)g}\right) & \left(1-\frac{\alpha}{(\alpha+\beta)g}\right) \\ \left(1-\frac{\alpha}{(\alpha+\beta)(1-h)}\right) & \left(1+\frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right) & \left(1-\frac{\alpha}{(\alpha+\beta)(1-h)}\right) & \left(1+\frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right) \\ \left(1+\frac{1}{(\alpha+\beta)(1-g)}\right) & \left(1-\frac{\alpha}{(\alpha+\beta)(1-g)}\right) & \left(1+\frac{\alpha}{\beta(\alpha+\beta)(1-g)}\right) & \left(1-\frac{\alpha^2}{\beta(\alpha+\beta)(1-g)}\right) \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \geq \mathbb{1} \\ & s_1,s_2,s_3,s_4 \geq 0. \end{array}$$

The dual (D') of (P') is given by

$$\begin{array}{ll} \max & v_1+v_2+v_3 \\ \text{subject to} & \begin{bmatrix} \left(1+\frac{\alpha}{\beta(\alpha+\beta)g}\right) & \left(1-\frac{\alpha}{(\alpha+\beta)(1-h)}\right) & \left(1+\frac{1}{(\alpha+\beta)(1-g)}\right) \\ \left(1-\frac{\alpha^2}{\beta(\alpha+\beta)g}\right) & \left(1+\frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right) & \left(1-\frac{\alpha}{(\alpha+\beta)(1-g)}\right) \\ \left(1+\frac{1}{(\alpha+\beta)g}\right) & \left(1-\frac{\alpha}{(\alpha+\beta)(1-h)}\right) & \left(1+\frac{\alpha}{\beta(\alpha+\beta)(1-g)}\right) \\ \left(1-\frac{\alpha}{(\alpha+\beta)g}\right) & \left(1+\frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right) & \left(1-\frac{\alpha^2}{\beta(\alpha+\beta)(1-g)}\right) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \leq \mathbb{1} \\ & v_1,v_2,v_3 \geq 0. \end{array}$$

We will show that

$$\hat{v} := \begin{bmatrix} \frac{\alpha\beta(1+\beta)g}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)} \\ \frac{(1+\alpha)(\alpha+\beta)(1-h)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)} \\ \frac{\alpha\beta(1+\beta)(1-g)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)} \end{bmatrix}$$

is feasible for (D'). Thus, the optimal value of (P') is bounded below by $LB_S(\alpha,\beta,g,h)$ because

$$\begin{aligned} \mathbb{1}^T\hat{v} &= \frac{\alpha\beta(1+\beta)g+(1+\alpha)(\alpha+\beta)(1-h)+\alpha\beta(1+\beta)(1-g)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)} \\ &= \frac{\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)} \\ &= 1-\frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}. \end{aligned}$$

Now we show that $\hat{v}$ is feasible for (D'). Each entry of $\hat{v}$ is of the form $C\cdot\frac{1}{K}$ for some constant C where

$$K := \alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h) > 0$$

for any fixed $\alpha, \beta > 0$ with $\alpha \le \beta$, and $h \le \frac{\alpha}{\alpha+\beta}$. So, to verify $\hat{v} \ge \mathbb{0}$, we check that the corresponding constant C is non-negative for each entry. To verify $\hat{v}_1 \ge 0$, note that $\alpha\beta(1+\beta)g \ge 0$ for any fixed $\alpha, \beta > 0$, and $g \in \left[\frac{\alpha}{\alpha+\beta}\ ,\ \frac{\beta}{\alpha+\beta}\right]$. To verify $\hat{v}_2 \ge 0$, note that $(1+\alpha)(\alpha+\beta)(1-h) \ge 0$ for any fixed $\alpha, \beta > 0$, and $h \le \frac{\alpha}{\alpha+\beta}$. To verify $\hat{v}_3 \ge 0$, note that $\alpha\beta(1+\beta)(1-g) \ge 0$ for any fixed $\alpha, \beta > 0$, and $g \in \left[\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta}\right]$.

Having shown $\hat{v}$ is non-negative, we next verify that $\hat{v}$ satisfies the constraints of (D'). As each constraint is of the form $(\mathbb{1}^T + [k_1, k_2, k_3])v \le 1$, it suffices to check

$$k_1\hat{v}_1 + k_2\hat{v}_2 + k_3\hat{v}_3 \le \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha) + \alpha\beta(1+\beta) + (1+\alpha)(\alpha+\beta)(1-h)}.$$

To verify $\hat{v}$ satisfies the first constraint, we calculate

$$\begin{aligned}
LHS &= \left(\frac{1}{\alpha(\beta-\alpha) + \alpha\beta(1+\beta) + (1+\alpha)(\alpha+\beta)(1-h)}\right) \cdot \Bigg[\left(\frac{\alpha}{\beta(\alpha+\beta)g}\right)(\alpha\beta(1+\beta)g) + \dots \\
&\qquad - \left(\frac{\alpha}{(\alpha+\beta)(1-h)}\right)((1+\alpha)(\alpha+\beta)(1-h)) + \left(\frac{1}{(\alpha+\beta)(1-g)}\right)(\alpha\beta(1+\beta)(1-g))\Bigg] \\
&= \left(\frac{1}{\alpha(\beta-\alpha) + \alpha\beta(1+\beta) + (1+\alpha)(\alpha+\beta)(1-h)}\right) \cdot \left[\left(\frac{\alpha^2(1+\beta)}{(\alpha+\beta)}\right) - \alpha(1+\alpha) + \left(\frac{\alpha\beta(1+\beta)}{(\alpha+\beta)}\right)\right] \\
&= \left(\frac{\alpha(1+\beta) - \alpha(1+\alpha)}{\alpha(\beta-\alpha) + \alpha\beta(1+\beta) + (1+\alpha)(\alpha+\beta)(1-h)}\right) \\
&= \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha) + \alpha\beta(1+\beta) + (1+\alpha)(\alpha+\beta)(1-h)}
\end{aligned}$$

and conclude that $\hat{v}$ satisfies the first constraint; moreover, it satisfies this constraint with equality. To verify $\hat{v}$ satisfies the second constraint, we calculate

$$\begin{aligned}
LHS &= \left(\frac{1}{\alpha(\beta-\alpha) + \alpha\beta(1+\beta) + (1+\alpha)(\alpha+\beta)(1-h)}\right) \cdot \Bigg[-\left(\frac{\alpha^2}{\beta(\alpha+\beta)g}\right)(\alpha\beta(1+\beta)g) + \dots \\
&\qquad + \left(\frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right)((1+\alpha)(\alpha+\beta)(1-h)) - \left(\frac{\alpha}{(\alpha+\beta)(1-g)}\right)(\alpha\beta(1+\beta)(1-g))\Bigg] \\
&= \left(\frac{1}{\alpha(\beta-\alpha) + \alpha\beta(1+\beta) + (1+\alpha)(\alpha+\beta)(1-h)}\right) \cdot \left[-\left(\frac{\alpha^3(1+\beta)}{(\alpha+\beta)}\right) + \alpha\beta(1+\alpha) - \left(\frac{\alpha^2\beta(1+\beta)}{(\alpha+\beta)}\right)\right] \\
&= \frac{-\alpha^2(1+\beta) + \alpha\beta(1+\alpha)}{\alpha(\beta-\alpha) + \alpha\beta(1+\beta) + (1+\alpha)(\alpha+\beta)(1-h)} \\
&= \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha) + \alpha\beta(1+\beta) + (1+\alpha)(\alpha+\beta)(1-h)}
\end{aligned}$$

and conclude that $\hat{v}$ satisfies the second constraint; moreover, it satisfies this constraint with equality. To verify $\hat{v}$ satisfies the third constraint, we calculate

$$\begin{aligned}
LHS &= \left(\frac{1}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}\right)\cdot\left[\left(\frac{1}{(\alpha+\beta)g}\right)(\alpha\beta(1+\beta)g)+\ldots\right.\\
&\qquad\left.-\left(\frac{\alpha}{(\alpha+\beta)(1-h)}\right)((1+\alpha)(\alpha+\beta)(1-h))+\left(\frac{\alpha}{\beta(\alpha+\beta)(1-g)}\right)(\alpha\beta(1+\beta)(1-g))\right]\\
&= \left(\frac{1}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}\right)\cdot\left[\left(\frac{\alpha\beta(1+\beta)}{(\alpha+\beta)}\right)-\alpha(1+\alpha)+\left(\frac{\alpha^2(1+\beta)}{(\alpha+\beta)}\right)\right]\\
&= \frac{\alpha(1+\beta)-\alpha(1+\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}\\
&= \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}
\end{aligned}$$

and conclude that $\hat{v}$ satisfies the third constraint; moreover, it satisfies this constraint with equality. To verify $\hat{v}$ satisfies the fourth constraint, we calculate

$$\begin{aligned}
LHS &= \left(\frac{1}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}\right)\cdot\left[-\left(\frac{\alpha}{(\alpha+\beta)g}\right)(\alpha\beta(1+\beta)g)+\ldots\right.\\
&\qquad\left.+\left(\frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right)((1+\alpha)(\alpha+\beta)(1-h))-\left(\frac{\alpha^2}{\beta(\alpha+\beta)(1-g)}\right)(\alpha\beta(1+\beta)(1-g))\right]\\
&= \left(\frac{1}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}\right)\cdot\left[-\left(\frac{\alpha^2\beta(1+\beta)}{(\alpha+\beta)}\right)+\alpha\beta(1+\alpha)-\left(\frac{\alpha^3(1+\beta)}{(\alpha+\beta)}\right)\right]\\
&= \frac{-\alpha^2(1+\beta)+\alpha\beta(1+\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}\\
&= \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}
\end{aligned}$$

and conclude that $\hat{v}$ satisfies the fourth constraint; moreover, it satisfies this constraint with equality. Therefore, $\hat{v}$ is feasible for (D') and thus the optimal value of (P') is at least $LB_S(\alpha,\beta,g,h)$. $\square$

Proposition 3.5.2. *The optimal value of* (P') *in Lemma 3.5.1 is exactly* $LB_S(\alpha,\beta,g,h)$*.*

Proof. See Appendix B.2 where a primal feasible solution of objective function value $LB_S(\alpha,\beta,g,h)$ is provided. $\square$

3.6 Proof of Theorem 1.3.21

We find the worst case bound on the optimal value of $(P(\alpha, \beta, g, h))$ over all $0 < \alpha \leq \beta$ and $(g, h)^T \in \operatorname{int} Q_2(\alpha, \beta)$ to bound $\rho[\mathcal{T}_2, \mathcal{Q}_2]$.

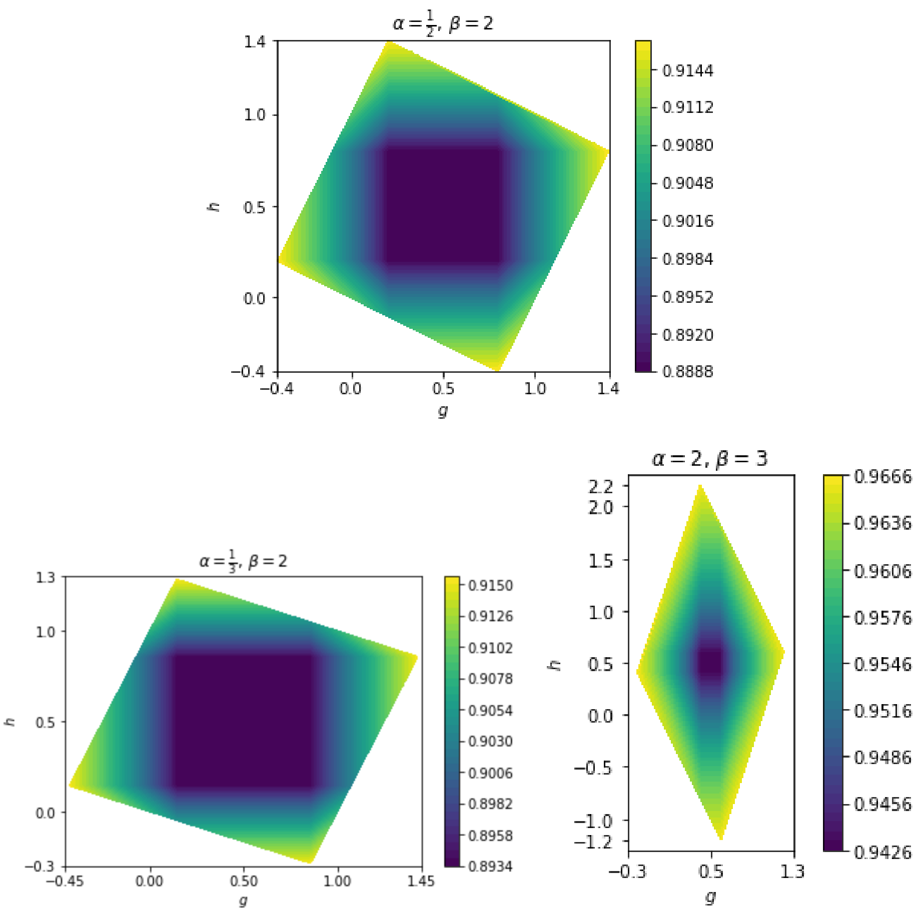


Figure 3.14: Contour plots of the lower bound on the optimal value of $(P(\frac{1}{2}, 2, g, h))$ (top), $(P(\frac{1}{3}, 2, g, h))$ (left) and $(P(2, 3, g, h))$ (right) as a function of g and h. To obtain the bounds outside of the South-West, South and Central Regions, the mapping in Remark 3.1.1 is used.

Theorem 3.6.1 (Theorem 1.3.21)**.** *The optimal value of* $(P(\alpha, \beta, g, h))$ *is at least* $\frac{8}{9}$ *for all* $\alpha, \beta > 0$ *with* $\alpha \leq \beta$ *and* $(g, h)^T \in \operatorname{int} Q_2(\alpha, \beta)$*. Therefore* $\rho[\mathcal{T}_2, \mathcal{Q}_2] \leq \frac{9}{8}$*.*

Proof. By Remark 3.1.1, it suffices to show that the optimal value of $(P(\alpha,\beta,g,h))$ is at least $\frac{8}{9}$ for all $\alpha,\beta > 0$ with $\alpha \leq \beta$ and $(g,h)^T \in \operatorname{int} Q_2(\alpha,\beta)$ where either: (1) $(g,h)^T \in R_{Central}(\alpha,\beta)$, (2) $(g,h)^T \in R_{SW}(\alpha,\beta)$, or (3) $(g,h)^T \in R_S(\alpha,\beta)$. If $\alpha = \beta$, then the bounds provided by Lemma 3.2.1, Lemma 3.4.1, and Lemma 3.5.1 are equal to $1 \geq \frac{8}{9}$. Thus we may assume that $\alpha < \beta$. We show that in all three cases the optimal value of $(P(\alpha,\beta,g,h))$ is lower-bounded by $LB_{Central}(\alpha,\beta)$.

Case 1: $(g,h)^T \in R_{Central}(\alpha,\beta)$

The optimal value of $(P(\alpha,\beta,g,h))$ is lower-bounded by $LB_{Central}(\alpha,\beta)$ as provided by Lemma 3.2.1.

Case 2: $(g,h)^T \in R_{SW}(\alpha,\beta)$

By Lemma 3.4.1, the optimal value of $(P(\alpha,\beta,g,h))$ is lower-bounded by

$$LB_{SW}(\alpha,\beta,g,h) = 1 - \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}.$$

Now $LB_{SW}(\alpha,\beta,g,h) = 1 - \frac{1}{1+T(\alpha,\beta,g,h)}$ where

$$T(\alpha,\beta,g,h) := \left(\frac{\alpha(1+\beta)(\alpha+\beta)}{\alpha(\beta-\alpha)}\right)(1-g) + \left(\frac{(1+\alpha)(\alpha+\beta)}{\alpha(\beta-\alpha)}\right)(1-h)$$

Since $T(\alpha,\beta,g,h) \geq 0$, LB_{SW} is smallest whenever T is smallest by Lemma 2.1.4. Note that the coefficients of g and h in $T(\alpha,\beta,g,h)$ are non-positive. Therefore

$$T(\alpha,\beta,g,h) \geq T\left(\alpha,\beta,\frac{\alpha}{\alpha+\beta},\frac{\alpha}{\alpha+\beta}\right) = \left(\frac{\alpha\beta(1+\beta)}{\alpha(\beta-\alpha)}\right) + \left(\frac{(1+\alpha)\beta}{\alpha(\beta-\alpha)}\right)$$

and

$$LB_{SW}(\alpha,\beta,g,h) \geq 1 - \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+\beta(1+\alpha)} = LB_{Central}(\alpha,\beta).$$

Therefore the optimal value of $(P(\alpha,\beta,g,h))$ is lower-bounded by $LB_{Central}(\alpha,\beta)$

Case 3: $(g,h)^T \in R_S(\alpha,\beta)$

By Lemma 3.5.1, the optimal value of $(P(\alpha,\beta,g,h))$ is lower-bounded by

$$LB_S(\alpha,\beta,h) := 1 - \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}.$$

Now $LB_S(\alpha,\beta,h) = 1 - \frac{1}{1+T(\alpha,\beta,h)}$ where

$$T(\alpha,\beta,h) := \left(\frac{\alpha\beta(1+\beta)}{\alpha(\beta-\alpha)}\right) + \left(\frac{(1+\alpha)(\alpha+\beta)}{\alpha(\beta-\alpha)}\right)(1-h)$$

Since $T(\alpha, \beta, h) \geq 0$, LB_{SW} is smallest whenever T is smallest by Lemma 2.1.4. Note that the coefficient of h in $T(\alpha, \beta, h)$ is non-positive. Therefore

$$T(\alpha, \beta, h) \geq T\left(\alpha, \beta, \frac{\alpha}{\alpha+\beta}\right) = \left(\frac{\alpha\beta(1+\beta)}{\alpha(\beta-\alpha)}\right) + \left(\frac{(1+\alpha)\beta}{\alpha(\beta-\alpha)}\right)$$

and

$$LB_S(\alpha, \beta, g, h) \geq 1 - \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha) + \alpha\beta(1+\beta) + \beta(1+\alpha)} = LB_{Central}(\alpha, \beta).$$

Therefore the optimal value of $(P(\alpha, \beta, g, h))$ is lower-bounded by $LB_{Central}(\alpha, \beta)$ for all $\alpha, \beta > 0$ with $\alpha \leq \beta$ and $(g, h)^T \in \operatorname{int} Q_2(\alpha, \beta)$. Since $LB_{Central}(\alpha, \beta) \geq \frac{8}{9}$ for all $\alpha, \beta > 0$ with $\alpha \leq \beta$ by Lemma 3.2.2, it follows that the optimal value of $(S(\mathcal{T}_2, \mathcal{Q}_2))$ is at least $\frac{8}{9}$. The bound on $\rho[\mathcal{T}_2, \mathcal{Q}_2]$ follows immediately from Theorem 2.1.1. $\square$

3.7 Observations and Summary

Together Theorem 3.6.1 and Lemma 2.3.1 imply that $\rho[\mathcal{T}_2, \mathcal{Q}_2] = \frac{9}{8}$. Hence we have proved the main result in this chapter. Before moving on to general quadrilaterals, we note a couple similarities in the proof that $\rho[\mathcal{T}_2, \mathcal{Q}_2] \leq \frac{5}{4}$ using fixed triangles (Theorem 2.2.8) and the stronger bound proved here.

In both the analyses, finding the weakest lower bound relied on the lower bound being of the form $1 + \frac{1}{1+T(\alpha,\beta,g,h)}$ where T is an affine function of g and h. This is not an accident. We choose τ values so that for $T = RS_i(\tau)$ there is some $A(g,h) \in \{g, 1-g, h, 1-h\}$ and constants κ_j such that $\psi_{f;T}(r^j) = 1 \pm \frac{1}{\kappa_j A(g,h)}$ for all $j \in \{1, 2, 3, 4\}$. When the constraint matrix is of the form in Lemma 2.1.3, matrix A can therefore be expressed as DA' where D is a diagonal matrix with each diagonal entry in $\{\frac{1}{g}, \frac{1}{1-g}, \frac{1}{h}, \frac{1}{1-h}\}$. The corresponding objective function provided by Lemma 2.1.3 is $1 + \frac{1}{1+\mathbb{1}^T A^{-1}\mathbb{1}}$. Now A^{-1} is $A'^{-1}D^{-1}$ and D^{-1} is a diagonal matrix with each diagonal entry in $\{g, 1-g, h, 1-h\}$. From here it is easy to see that the bound will be of the form $1 + \frac{1}{1+T(\alpha,\beta,g,h)}$ where T is affine in g and h.

In both the analyses, if we fix α and β, the weakest lower bound occurs for f on the boundary lines of the partition of $\operatorname{int} Q_2(\alpha, \beta)$. Using fixed triangles, this gave the weakest bound occurring for f tending to the corners of $[0, 1]^2$. The bound is constant across $[0, 1]^2$. Using ray-sliding triangles as described in this chapter, this gave the weakest bound occurring for f tending towards the corners of $R_{Central}(\alpha, \beta)$. Interestingly, the bound is constant across $R_{Central}(\alpha, \beta)$.

Chapter 4

Approximating General Quadrilateral Cuts with Type 2 Triangle Cuts

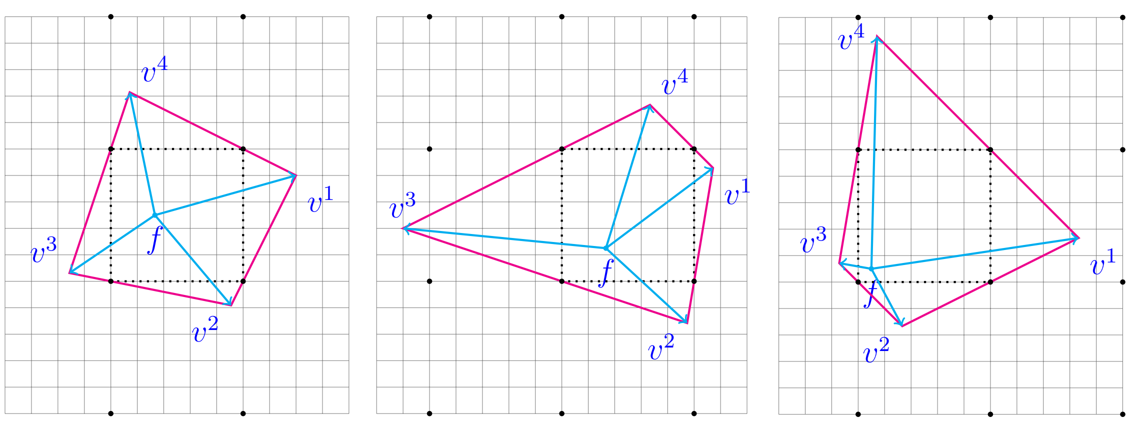


Figure 4.1: Quadrilaterals for parameters $a = \frac{1}{5}$, $b = 2$, $c = 3$, $d = \frac{1}{2}$, $g = \frac{1}{3}$, and $h = \frac{1}{2}$ (left) ; $a = \frac{1}{3}$, $b = 6$, $c = \frac{1}{2}$, $d = 1$, $g = \frac{1}{3}$, and $h = \frac{1}{4}$ (center) ; and $a = 1$, $b = \frac{1}{2}$, $c = 6$, $d = 1$, $g = \frac{1}{10}$, and $h = \frac{1}{10}$ (right).

In this chapter we prove Theorem 1.3.19 showing that $\rho[\mathcal{T}_2, \mathcal{Q}] \leq \frac{5}{4}$. We follow the proof strategy in Section 2.1. Per Section 2.1.1 the value of $\frac{1}{\rho[\mathcal{T}_2,\mathcal{Q}]}$ is equal to the optimal value of semi-infinite program $(S(\mathcal{T}_2, \mathcal{Q}))$ in variables $a, b, c, d, g, h, s_1, s_2, s_3$, and s_4 given by

$$\begin{aligned}
\inf \quad & s_1 + s_2 + s_3 + s_4 && (4.1)\\
\text{s. t.} \quad & s \in \mathcal{T}_2\left(\begin{pmatrix} g \\ h \end{pmatrix}; \begin{pmatrix} 1 + \frac{1}{b+d} - g \\ \frac{b}{b+d} - h \end{pmatrix}, \begin{pmatrix} \frac{b}{a+b} - g \\ \frac{-ab}{a+b} - h \end{pmatrix}, \begin{pmatrix} \frac{-1}{a+c} - g \\ \frac{a}{a+c} - h \end{pmatrix}, \begin{pmatrix} \frac{d}{c+d} - g \\ 1 + \frac{cd}{c+d} - h \end{pmatrix}\right) && (4.2)\\
& \begin{pmatrix} g \\ h \end{pmatrix} \in \operatorname{int}\left(\operatorname{conv}\left\{ \begin{pmatrix} 1 + \frac{1}{b+d} \\ \frac{b}{b+d} \end{pmatrix}, \begin{pmatrix} \frac{b}{a+b} \\ \frac{-ab}{a+b} \end{pmatrix}, \begin{pmatrix} \frac{-1}{a+c} \\ \frac{a}{a+c} \end{pmatrix}, \begin{pmatrix} \frac{d}{c+d} \\ 1 + \frac{cd}{c+d} \end{pmatrix} \right\}\right) && (4.3)\\
& a, b, c, d > 0 \quad ; \quad a, b, c, d \in \mathbb{Q} \quad ; \quad ad \le bc && (4.4)\\
& \begin{pmatrix} g \\ h \end{pmatrix} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2. && (4.5)
\end{aligned}$$

Treating $a, b, c, d > 0$ with $ad \le bc$ and $(g, h)^T \in \operatorname{int} Q(a, b, c, d)$ as data in the optimization problem above, we obtain a semi-infinite linear program in variables s_1, s_2, s_3, and s_4 given by

$$\begin{aligned}
\inf \quad & s_1 + s_2 + s_3 + s_4 && (4.6)\\
\text{s. t.} \quad & s \in \mathcal{T}_2\left(\begin{pmatrix} g \\ h \end{pmatrix}; \begin{pmatrix} 1 + \frac{1}{b+d} - g \\ \frac{b}{b+d} - h \end{pmatrix}, \begin{pmatrix} \frac{b}{a+b} - g \\ \frac{-ab}{a+b} - h \end{pmatrix}, \begin{pmatrix} \frac{-1}{a+c} - g \\ \frac{a}{a+c} - h \end{pmatrix}, \begin{pmatrix} \frac{d}{c+d} - g \\ 1 + \frac{cd}{c+d} - h \end{pmatrix}\right). && (4.7)
\end{aligned}$$

We will denote this program by $(P(a, b, c, d, g, h))$.

We will lower bound the optimal value of $(P(a, b, c, d, g, h))$ for all $a, b, c, d > 0$ with $ad \le bc$ and $(g, h)^T \in \operatorname{int} Q(a, b, c, d)$. Following Section 3.1, we identify the following three subregions of $\operatorname{int} Q(a, b, c, d)$:

1. *South-West Region*: $R_{SW}(a, b, c, d) := \left\{(g, h)^T \in \operatorname{int} Q(a, b, c, d) : g \le \frac{d}{c+d} \ , \ h \le \frac{a}{a+c}\right\}$,
2. *South Region*: $R_S(a, b, c, d) := \left\{(g, h)^T \in \operatorname{int} Q(a, b, c, d) : g \in \left[\frac{d}{c+d}, \frac{b}{a+b}\right] \ , \ h \le \frac{a}{a+c}\right\}$,
3. and *Central Region*:
   $R_{Central}(a, b, c, d) := \left\{(g, h)^T \in \operatorname{int} Q(a, b, c, d) : g \in \left[\frac{d}{c+d}, \frac{b}{a+b}\right] \ , \ h \in \left[\frac{a}{a+c}, \frac{b}{b+d}\right]\right\}$.

The symmetry argument in Remark 3.1.1 shows it suffices to analyze $(P(a, b, c, d, g, h))$ for given $a, b, c, d > 0$ with $ad \le bc$ only for $(g, h)^T$ in one of these three regions.

In Section 4.1 we analyze the Central Region. We replace constraint (4.7) with the intersection cuts for four ray-sliding triangles at their unique ray breakpoint. Solving the resulting parametric linear program exactly we obtain a lower bound $LB_{Central}(a, b, c, d, g, h)$. Lower bound $LB_{Central}(a, b, c, d, g, h)$ is at least $\frac{4}{5}$ for all $a, b, c, d > 0$ with $ad \le bc$ and g, h such that $(g, h)^T \in [0, 1]^2$.

In Section 4.2 we analyze the South-West Region. We replace constraint (4.7) with the intersection cuts for four ray-sliding triangles: two ray-sliding triangles for each of rays

2 and 3. For each ray $i \in \{2,3\}$ we choose the ray-sliding triangles for both ray breakpoints. Solving the resulting parametric linear program exactly we obtain a lower bound $LB_{SW}(a,b,c,d,g,h)$. For fixed a,b,c,d, the lower bound is affine in g and h and weakest for $(g,h)^T = \left(\frac{d}{c+d}, \frac{a}{a+c}\right)^T = R_{Central}(a,b,c,d) \cap R_{SW}(a,b,c,d)$. Therefore a lower bound of $\frac{4}{5}$ follows from the bound on $LB_{Central}(a,b,c,d,g,h)$.

In Section 4.3 we analyze the South Region. We replace constraint (4.7) with the intersection cuts for four ray-sliding triangles: one ray-sliding triangle for each of rays 1 and 3 and two ray-sliding triangles for ray 2. For each ray $i \in \{1,3\}$ we choose the ray-sliding triangle at its unique ray breakpoint. For ray 2, we choose the ray-sliding triangles for both ray breakpoints. Solving the resulting parametric linear program exactly we obtain a lower bound $LB_S(a,b,c,d,g,h)$. For fixed a,b,c,d, the lower bound is affine in g and h and weakest for $(g,h)^T \in \{(x,y)^T \in R_S(a,b,c,d) : y = \frac{a}{a+c}\} = R_{Central}(a,b,c,d) \cap R_S(a,b,c,d)$. Therefore a lower bound of $\frac{4}{5}$ follows from the bound on $LB_{Central}(a,b,c,d,g,h)$.

We amalgamate the results for the Central, South-West and South Regions to conclude $\rho[\mathcal{T}_2, \mathcal{Q}] \leq \frac{5}{4}$ in Section 4.4.

4.1 Central Region

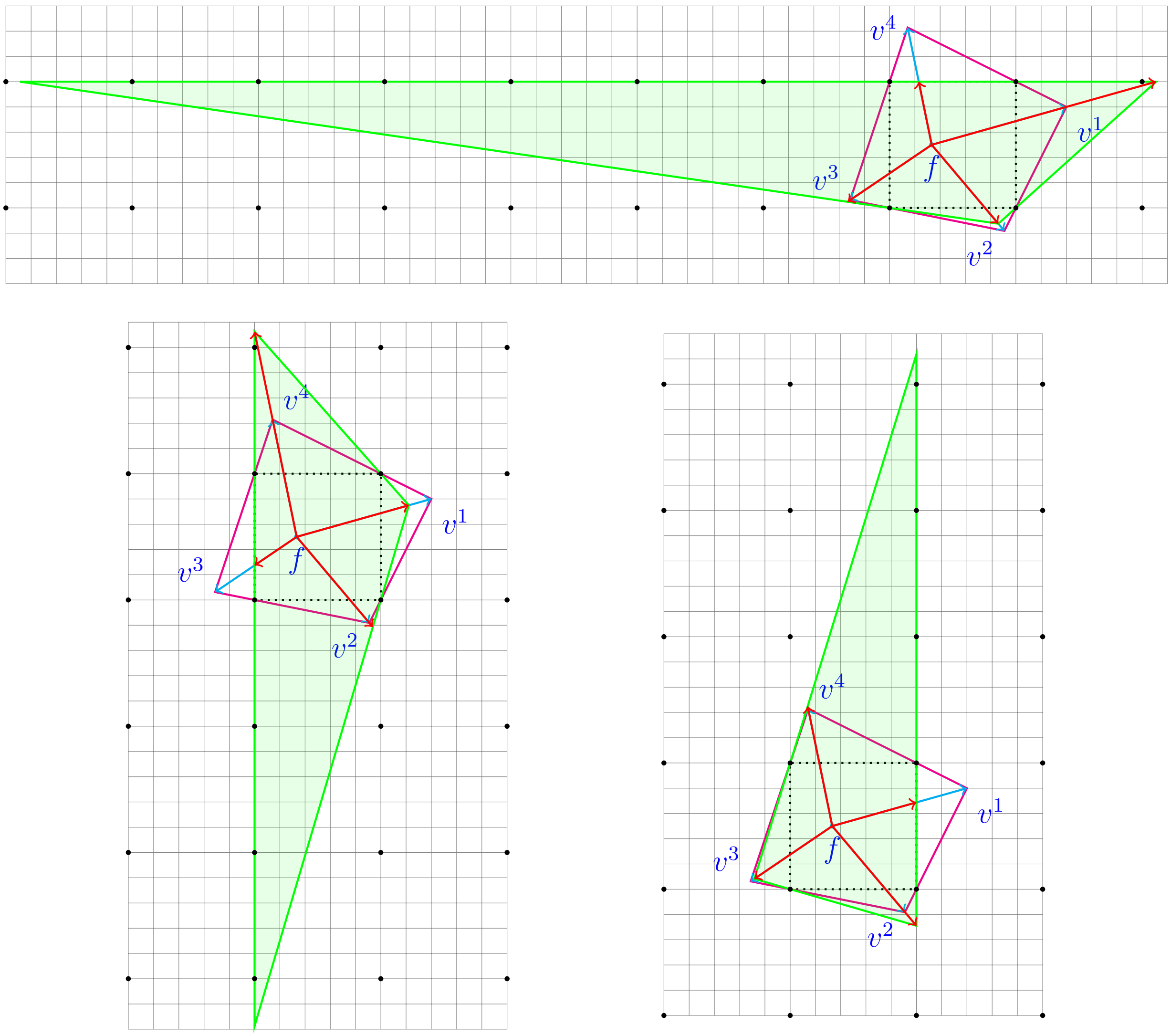
Focusing first on the Central Region, we use the intersection cuts for four ray-sliding triangles at the first (unique) ray breakpoint, as illustrated in Figure 4.2.

Lemma 4.1.1. *For fixed $a,b,c,d > 0$ with $ad \leq bc$ and g,h with $(g,h)^T \in R_{Central}(a,b,c,d)$, let (P') be the parameterized linear program obtained from $(P(a,b,c,d,g,h))$ by replacing constraint (4.7) with the constraint $s \geq \mathbb{0}$ and the intersection cuts for ray-sliding triangles $RS_1(\tau_1^{bp1})$, $RS_2(\tau_2^{bp1})$, $RS_3(\tau_3^{bp1})$, and $RS_4(\tau_4^{bp1})$. Then, the optimal value of (P') is equal to*

$$LB_{Central}(a,b,c,d,g,h) := 1 - \frac{bc-ad}{bc-ad+t_1^C+t_2^C+t_3^C+t_4^C}$$

where

$$t_1^C = (b+d)[bc(1-g)+ch] \quad , \quad t_2^C = \left(\frac{a+b}{a}\right)[ch+acg],$$
$$t_3^C = (a+c)[bcg+b(1-h)] \quad , \quad \text{and} \quad t_4^C = \left(\frac{c+d}{d}\right)[b(1-h)+bd(1-g)].$$



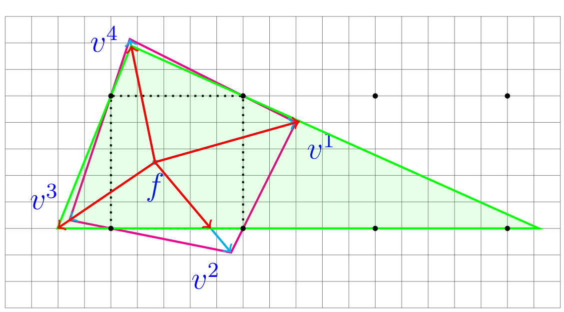


Figure 4.2: Triangles $RS_1(\tau_1^{bp1})$ (middle left), $RS_2(\tau_2^{bp1})$ (top), $RS_3(\tau_3^{bp1})$ (middle right), and $RS_4(\tau_4^{bp1})$ (bottom) for $a = \frac{1}{5}$, $b = 2$, $c = 3$, $d = \frac{1}{2}$, $g = \frac{1}{3}$, and $h = \frac{1}{2}$.

Proof. We order the cuts $RS_3(\tau_3^{bp1})$, $RS_4(\tau_4^{bp1})$, $RS_1(\tau_1^{bp1})$, $RS_2(\tau_2^{bp1})$ so the constraint matrix is nicer to work with. Looking up the cuts in Appendix A, the constraint matrix is

$$\begin{bmatrix} \left(1+\frac{1}{(b+d)(1-g)}\right) & \left(1-\frac{a}{(a+b)(1-g)}\right) & \left(1+\frac{a}{b(a+c)(1-g)}\right) & \left(1-\frac{ad}{b(c+d)(1-g)}\right) \\ \left(1-\frac{ad}{c(b+d)h}\right) & \left(1+\frac{ab}{(a+b)h}\right) & \left(1-\frac{a}{(a+c)h}\right) & \left(1+\frac{ad}{(c+d)h}\right) \\ \left(1+\frac{d}{c(b+d)g}\right) & \left(1-\frac{ad}{c(a+b)g}\right) & \left(1+\frac{1}{(a+c)g}\right) & \left(1-\frac{d}{(c+d)g}\right) \\ \left(1-\frac{d}{(b+d)(1-h)}\right) & \left(1+\frac{ad}{(a+b)(1-h)}\right) & \left(1-\frac{ad}{b(a+c)(1-h)}\right) & \left(1+\frac{cd}{(c+d)(1-h)}\right) \end{bmatrix}.$$

So, optimization problem (P') is given by

$$\begin{aligned} \min \quad & \mathbb{1}^T s \\ \text{subject to} \quad & \left[\mathbb{1}\mathbb{1}^T + A\right] s \geq \mathbb{1} \\ & s \geq \mathbb{0} \end{aligned}$$

where

$$A := \begin{bmatrix} \frac{1}{1-g} & 0 & 0 & 0 \\ 0 & \frac{a}{h} & 0 & 0 \\ 0 & 0 & \frac{1}{g} & 0 \\ 0 & 0 & 0 & \frac{d}{1-h} \end{bmatrix} \begin{bmatrix} 1 & -1 & \frac{a}{b} & -\frac{a}{b} \\ -\frac{d}{c} & \frac{b}{a} & -1 & 1 \\ \frac{d}{c} & -\frac{d}{c} & 1 & -1 \\ -1 & 1 & -\frac{a}{b} & \frac{c}{d} \end{bmatrix} \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{a}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{d}{c+d} \end{bmatrix}.$$

First we assume $ad \neq bc$. We compute

$$A^{-1} = \left(\frac{1}{bc-ad}\right) \begin{bmatrix} b+d & 0 & 0 & 0 \\ 0 & \frac{a+b}{a} & 0 & 0 \\ 0 & 0 & a+c & 0 \\ 0 & 0 & 0 & \frac{c+d}{d} \end{bmatrix} \begin{bmatrix} bc & ac & 0 & 0 \\ 0 & ac & ac & 0 \\ 0 & 0 & bc & bd \\ bd & 0 & 0 & bd \end{bmatrix} \begin{bmatrix} 1-g & 0 & 0 & 0 \\ 0 & \frac{h}{a} & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & \frac{1-h}{d} \end{bmatrix},$$

$$A^{-1}\mathbb{1} = \left(\frac{1}{bc-ad}\right) \begin{bmatrix} (b+d)[bc(1-g)+ch] \\ \left(\frac{a+b}{a}\right)(ch+acg) \\ (a+c)[bcg+b(1-h)] \\ \left(\frac{c+d}{d}\right)[b(1-h)+bd(1-g)] \end{bmatrix} = \left(\frac{1}{bc-ad}\right) \begin{bmatrix} t_1^C \\ t_2^C \\ t_3^C \\ t_4^C \end{bmatrix}, \text{ and}$$

$$\mathbb{1}^T A^{-1}\mathbb{1} = \left(\frac{1}{bc-ad}\right)\left(t_1^C + t_2^C + t_3^C + t_4^C\right).$$

Note that the entries of A^{-1} are non-negative because $a, b, c, d > 0$, $ad < bc$, and $(g, h)^T \in R_{Central}(a, b, c, d) \subseteq (0, 1)^2$. Apply Lemma 2.1.3 to conclude the optimal value of (P') is

$$1 - \frac{1}{1 + \mathbb{1}^T A^{-1} \mathbb{1}} = 1 - \frac{bc - ad}{bc - ad + t_1^C + t_2^C + t_3^C + t_4^C} = LB_{Central}(a, b, c, d, g, h)$$

whenever $ad < bc$.

If $ad = bc$, then A isn't invertible. However, we can still construct primal and dual feasible solutions with equal objective function value of 1 as required. We refer the reader to Appendix C where we reprove this lemma by explicitly constructing the solutions and checking primal feasibility, dual feasibility and optimality, rather than relying on Lemma 2.1.3. $\square$

Lemma 4.1.2. *For all $a, b, c, d > 0$ with $ad \le bc$ and g, h with $(g, h)^T \in [0, 1]^2$,*

$$LB_{Central}(a, b, c, d, g, h) \ge \frac{4}{5}.$$

Proof. If $ad = bc$, then $LB_{Central}(a, b, c, d, g, h) = 1 \ge \frac{4}{5}$. So, we may assume $ad < bc$. Now $LB_{Central}(a, b, c, d, g, h) = 1 - \frac{1}{1+T(a,b,c,d,g,h)}$ where

$$T(a, b, c, d, g, h) := \frac{t_1^C + t_2^C + t_3^C + t_4^C}{bc - ad}.$$

Now $t_i^C \ge 0$ for all $i \in \{1, 2, 3, 4\}$ and $bc - ad > 0$. Therefore $T(a, b, c, d, g, h) \ge 0$ and $LB_{Central}(a, b, c, d, g, h)$ is smallest whenever $T(a, b, c, d, g, h)$ is smallest by Lemma 2.1.4. Define $T' : \mathbb{R}_{++} \to \mathbb{R}$ by

$$T'(\epsilon) = \inf \left\{ T(a, b, c, d, g, h) : (g, h)^T \in [0, 1]^2, a, b, c, d \in \left[\epsilon, \frac{1}{\epsilon}\right] \right\}.$$

For fixed $\epsilon > 0$, $T'(\epsilon)$ is the infimum of a continuous function over a compact set. It therefore attains its minimum at some fixed values for a, b, c, d, g, and h. For fixed a, b, c, and d, $T(a, b, c, d, g, h)$ is affine in g and h and therefore attains its minimum for $g = 0$ or $g = 1$ and also $h = 0$ or $h = 1$.

Case 1 : $(g, h)^T = (0, 0)^T$

$$\begin{aligned} T(a, b, c, d, 0, 0) &= \frac{(b + d)[bc] + (a + c)[b] + \left(\frac{c+d}{d}\right)[b + bd]}{bc - ad} \\ &= \frac{bc\left(b + d + 2 + \frac{1}{d}\right) + ab + b + bd}{bc - ad} \\ &\ge \frac{bc(b + d + 2 + \frac{1}{d})}{bc} \\ &\ge 4 \end{aligned}$$

where the last inequality holds because $d + \frac{1}{d} \geq 2$ for $d > 0$.

Case 2 : $(g, h)^T = (0, 1)^T$

$$\begin{aligned}
T(a, b, c, d, 0, 1) &= \frac{(b+d)[bc+c] + \left(\frac{a+b}{a}\right)[c] + \left(\frac{c+d}{d}\right)[bd]}{bc - ad} \\
&= \frac{bc(b + d + 2 + \frac{1}{b} + \frac{1}{a}) + cd + bd}{bc - ad} \\
&\geq \frac{bc(b + d + 2 + \frac{1}{b} + \frac{1}{a})}{bc} \\
&\geq 4
\end{aligned}$$

where the last inequality holds because $b + \frac{1}{b} \geq 2$ for $b > 0$.

Case 3 : $(g, h)^T = (1, 0)^T$

$$\begin{aligned}
T(a, b, c, d, 1, 0) &= \frac{\left(\frac{a+b}{a}\right)[ac] + (a+c)[bc+b] + \left(\frac{c+d}{d}\right)[b]}{bc - ad} \\
&= \frac{bc\left(1 + a + c + 1 + \frac{1}{d} + \frac{1}{c}\right) + ac + ab}{bc - ad} \\
&\geq \frac{bc\left(1 + a + c + 1 + \frac{1}{d} + \frac{1}{c}\right)}{bc} \\
&\geq 4
\end{aligned}$$

where the last inequality holds because $c + \frac{1}{c} \geq 2$ for $c > 0$.

Case 4 : $(g, h)^T = (1, 1)^T$

$$\begin{aligned}
T(a, b, c, d, 1, 1) &= \frac{(b+d)[c] + \left(\frac{a+b}{a}\right)[c + ac] + (a+c)[bc]}{bc - ad} \\
&= \frac{bc(2 + \frac{1}{a} + 1 + a + c) + dc + d + ac}{bc - ad} \\
&\geq \frac{bc(2 + \frac{1}{a} + 1 + a + c)}{bc} \\
&\geq 4
\end{aligned}$$

where the last inequality holds because $a + \frac{1}{a} \geq 2$ for $a > 0$.

Therefore $T'(\epsilon) \geq 4$ for all $\epsilon > 0$. It follows that $T(a, b, c, d, g, h) \geq 4$ for all $a, b, c, d > 0$ such that $ad \leq bc$ and $(g, h)^T \in [0, 1]^2$. Therefore $LB_{Central}(a, b, c, d, g, h) \geq 1 - \frac{1}{1+4} = \frac{4}{5}$. $\square$

4.2 South-West Region

In the South-West Region, we use the intersection cuts for ray-sliding triangles for rays 2 and 3 at both ray breakpoints, as illustrated in Figure 4.3.

Lemma 4.2.1. *For fixed $a, b, c, d > 0$ with $ad \leq bc$ and g, h with $(g,h)^T \in R_{SW}(a,b,c,d)$, let (P') be the parameterized linear program obtained from $(P(a,b,c,d,g,h))$ by replacing constraint (4.7) with the constraint $s \geq \mathbb{0}$ and the intersection cuts for ray-sliding triangles $RS_2(\tau_2^{bp2})$, $RS_2(\tau_2^{bp1})$, $RS_3(\tau_3^{bp2})$, and $RS_3(\tau_3^{bp1})$. Then, the optimal value of (P') is equal to*

$$LB_{SW}(a,b,c,d,g,h) := 1 - \frac{bc - ad}{bc - ad + t_1^{SW} + t_2^{SW} + t_3^{SW} + t_4^{SW}}$$

where

$$t_1^{SW} = (b+d)[a(1-h) + bc(1-g)] \quad , \quad t_2^{SW} = (a+b)[(1-h) + d(1-g)],$$
$$t_3^{SW} = (a+c)[b(1-h) + bd(1-g)] \quad , \quad and \;\; t_4^{SW} = (c+d)\left[\frac{b(1-h)}{d} + b(1-g)\right].$$

Proof. Looking up the cuts in Appendix A the constraint matrix is

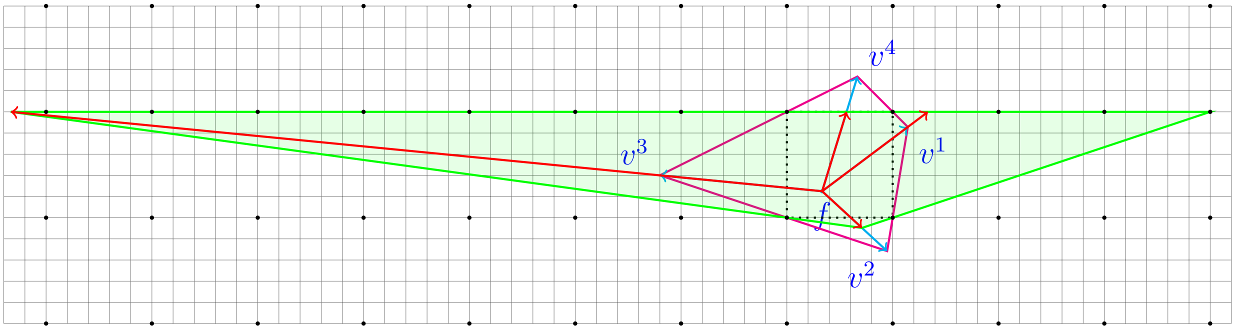
$$\begin{bmatrix} \left(1 - \frac{d}{(b+d)(1-h)}\right) & \left(1 + \frac{bc}{(a+b)(1-h)}\right) & \left(1 - \frac{c}{(a+c)(1-h)}\right) & \left(1 + \frac{cd}{(c+d)(1-h)}\right) \\ \left(1 - \frac{d}{(b+d)(1-h)}\right) & \left(1 + \frac{ad}{(a+b)(1-h)}\right) & \left(1 - \frac{ad}{b(a+c)(1-h)}\right) & \left(1 + \frac{cd}{(c+d)(1-h)}\right) \\ \left(1 + \frac{1}{(b+d)(1-g)}\right) & \left(1 - \frac{a}{(a+b)(1-g)}\right) & \left(1 + \frac{c}{(a+c)d(1-g)}\right) & \left(1 - \frac{c}{(c+d)(1-g)}\right) \\ \left(1 + \frac{1}{(b+d)(1-g)}\right) & \left(1 - \frac{a}{(a+b)(1-g)}\right) & \left(1 + \frac{a}{b(a+c)(1-g)}\right) & \left(1 - \frac{ad}{b(c+d)(1-g)}\right) \end{bmatrix}$$

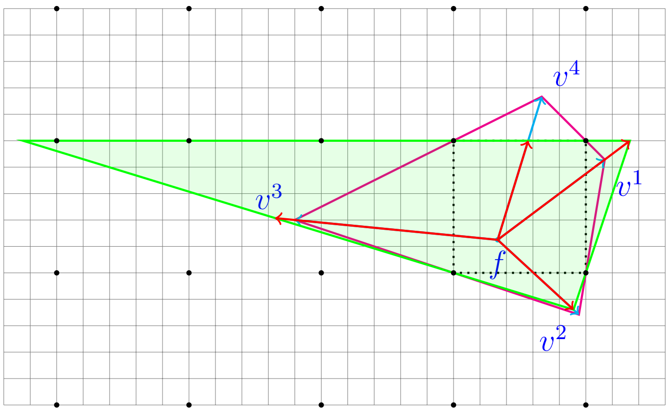
So, optimization problem (P') is given by

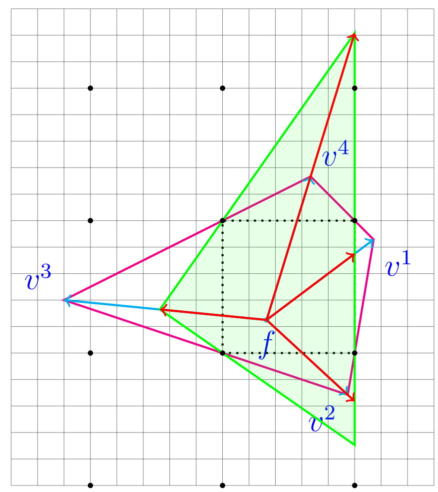
$$\begin{aligned} \min \quad & \mathbb{1}^T s \\ \text{subject to} \quad & \left[\mathbb{1}\mathbb{1}^T + A\right] s \geq \mathbb{1} \\ & s \geq \mathbb{0} \end{aligned}$$

where

$$A := \begin{bmatrix} \frac{1}{1-h} & 0 & 0 & 0 \\ 0 & \frac{1}{1-h} & 0 & 0 \\ 0 & 0 & \frac{1}{1-g} & 0 \\ 0 & 0 & 0 & \frac{1}{1-g} \end{bmatrix} \begin{bmatrix} -d & bc & -c & cd \\ -d & ad & -\frac{ad}{b} & cd \\ 1 & -a & \frac{c}{d} & -c \\ 1 & -a & \frac{a}{b} & -\frac{ad}{b} \end{bmatrix} \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{1}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{1}{c+d} \end{bmatrix}.$$







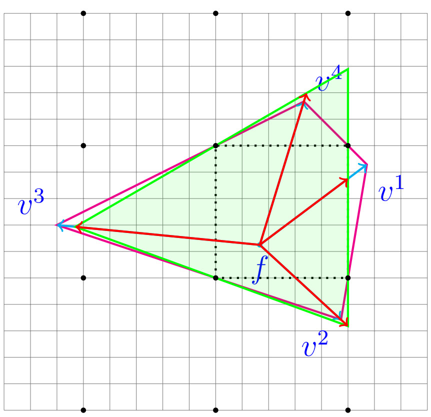


Figure 4.3: Triangles $RS_2(\tau_2^{bp2})$ (top left), $RS_2(\tau_2^{bp1})$ (middle left), $RS_3(\tau_3^{bp2})$ (middle right), and $RS_3(\tau_3^{bp1})$ (bottom) for $a = \frac{1}{3}$, $b = 6$, $c = \frac{1}{2}$, $d = 1$, $g = \frac{1}{3}$, and $h = \frac{1}{4}$. To provide examples where the corresponding triangles fit on the page at a reasonable scale, we use different parameters a, b, c, d, g and h for some examples. Ray-sliding triangles tend to a split as τ approaches τ^{min}, so its easy to construct instances where some vertex of the triangle has a very large coordinate.

First we assume $ad \neq bc$. We compute

$$A^{-1} = \left(\frac{1}{bc-ad}\right) \begin{bmatrix} b+d & 0 & 0 & 0 \\ 0 & a+b & 0 & 0 \\ 0 & 0 & a+c & 0 \\ 0 & 0 & 0 & c+d \end{bmatrix} \begin{bmatrix} a & 0 & 0 & bc \\ 1 & 0 & d & 0 \\ 0 & b & bd & 0 \\ 0 & \frac{b}{d} & 0 & b \end{bmatrix} \begin{bmatrix} 1-h & 0 & 0 & 0 \\ 0 & 1-h & 0 & 0 \\ 0 & 0 & 1-g & 0 \\ 0 & 0 & 0 & 1-g \end{bmatrix},$$

$$A^{-1}\mathbb{1} = \left(\frac{1}{bc-ad}\right) \begin{bmatrix} (b+d)[a(1-h)+bc(1-g)] \\ (a+b)[(1-h)+d(1-g)] \\ (a+c)[b(1-h)+bd(1-g)] \\ (c+d)\left[\frac{b(1-h)}{d}+b(1-g)\right] \end{bmatrix} = \left(\frac{1}{bc-ad}\right) \begin{bmatrix} t_1^{SW} \\ t_2^{SW} \\ t_3^{SW} \\ t_4^{SW} \end{bmatrix}, \text{ and}$$

$$\mathbb{1}^T A^{-1}\mathbb{1} = \left(\frac{1}{bc-ad}\right)\left(t_1^{SW}+t_2^{SW}+t_3^{SW}+t_4^{SW}\right).$$

Note that the entries of A^{-1} are non-negative because $a, b, c, d > 0$, $ad < bc$ and $(g, h)^T \in R_{SW}(a, b, c, d) \subseteq \{(x, y)^T \in \mathbb{R}^2 : x < 1, y < 1\}$. Apply Lemma 2.1.3 to conclude the optimal value of (P') is

$$1 - \frac{1}{1+\mathbb{1}^T A^{-1}\mathbb{1}} = 1 - \frac{bc-ad}{bc-ad+t_1^{SW}+t_2^{SW}+t_3^{SW}+t_4^{SW}} = LB_{SW}(a, b, c, d, g, h)$$

whenever $ad < bc$.

If $ad = bc$, then A isn't invertible. However, we can still construct primal and dual feasible solutions with equal objective function value of 1 as required. We refer the reader to Appendix C where we reprove this lemma by explicitly constructing the solutions and checking primal feasibility, dual feasibility and optimality, rather than relying on Lemma 2.1.3. $\square$

Lemma 4.2.2. *For all $a, b, c, d > 0$ with $ad \leq bc$ and g, h with $(g, h)^T \in R_{SW}(a, b, c, d)$,*

$$LB_{SW}(a, b, c, d, g, h) \geq \frac{4}{5}.$$

Proof. Note that $t_i^{SW} \geq 0$ for all $i \in \{1, 2, 3, 4\}$ but the coefficients of g and h in t_i^{SW} are negative for all $i \in \{1, 2, 3, 4\}$. Since $bc \geq ad$ we have

$$LB_{SW}(a, b, c, d, g, h) \geq LB_{SW}\left(a, b, c, d, \frac{d}{c+d}, \frac{a}{a+c}\right)$$

because $g \leq \frac{d}{c+d}$ and $h \leq \frac{a}{a+c}$ whenever $(g,h)^T \in R_{SW}(a,b,c,d)$.

Note that $LB_{Central}\left(a,b,c,d,\frac{d}{c+d},\frac{a}{a+c}\right) = 1 - \frac{bc-ad}{bc-ad+T}$ where

$$\begin{aligned} T = (b+d)\left[\left(\frac{bc^2}{c+d}\right) + \left(\frac{ac}{a+c}\right)\right] + \left(\frac{a+b}{a}\right)\left[\left(\frac{ac}{a+c}\right) + \left(\frac{acd}{c+d}\right)\right] + \dots \\ \dots + (a+c)\left[\left(\frac{bcd}{c+d}\right) + \left(\frac{bc}{a+c}\right)\right] + \left(\frac{c+d}{d}\right)\left[\left(\frac{bc}{a+c}\right) + \left(\frac{bcd}{c+d}\right)\right]. \end{aligned}$$

Note that $LB_{SW}(a,b,c,d,\frac{d}{c+d},\frac{a}{a+c}) = 1 - \frac{bc-ad}{bc-ad+T}$ where

$$\begin{aligned} T = (b+d)\left[\left(\frac{ac}{a+c}\right) + \left(\frac{bc^2}{c+d}\right)\right] + (a+b)\left[\left(\frac{c}{a+c}\right) + \left(\frac{cd}{c+d}\right)\right] + \dots \\ \dots + (a+c)\left[\left(\frac{bc}{a+c}\right) + \left(\frac{bcd}{c+d}\right)\right] + (c+d)\left[\left(\frac{bc}{d(a+c)}\right) + \left(\frac{bc}{c+d}\right)\right]. \end{aligned}$$

So $LB_{SW}\left(a,b,c,d,\frac{d}{c+d},\frac{a}{a+c}\right) = LB_{Central}\left(a,b,c,d,\frac{d}{c+d},\frac{a}{a+c}\right)$. As $\frac{d}{c+d} \in [0,1]$ and $\frac{a}{a+c} \in [0,1]$, it follows from Lemma 4.1.2 that $LB_{SW}\left(a,b,c,d,\frac{d}{c+d},\frac{a}{a+c}\right) \geq \frac{4}{5}$ and therefore $LB_{SW}(a,b,c,d,g,h) \geq \frac{4}{5}$. □

4.3 South Region

In the South Region, we use the intersection cuts for ray-sliding triangles for rays 1 and 3 at their unique ray breakpoint and ray 2 at both ray breakpoints, as illustrated in Figure 4.4.

Lemma 4.3.1. *For fixed $a,b,c,d > 0$ with $ad \leq bc$ and g,h with $(g,h)^T \in R_S(a,b,c,d)$, let (P') be the parameterized linear program obtained from $(P(a,b,c,d,g,h))$ by replacing constraint (4.7) with the constraint $s \geq \mathbb{0}$ and the intersection cuts for ray-sliding triangles $RS_1(\tau_1^{bp1})$, $RS_2(\tau_2^{bp2})$, $RS_2(\tau_2^{bp1})$, and $RS_3(\tau_3^{bp1})$. Then, the optimal value of (P') is equal to*

$$LB_S(a,b,c,d,g,h) := 1 - \frac{bc-ad}{bc-ad+t_1^S+t_2^S+t_3^S+t_4^S}$$

where

$$\begin{aligned} t_1^S &= (b+d)[a(1-h)+bc(1-g)] \quad , \quad t_2^S = (a+b)[cg+(1-h)], \\ t_3^S &= (a+c)[bcg+b(1-h)] \quad , \quad \textit{and} \quad t_4^S = (c+d)\left[\frac{b}{d}(1-h)+b(1-g)\right]. \end{aligned}$$

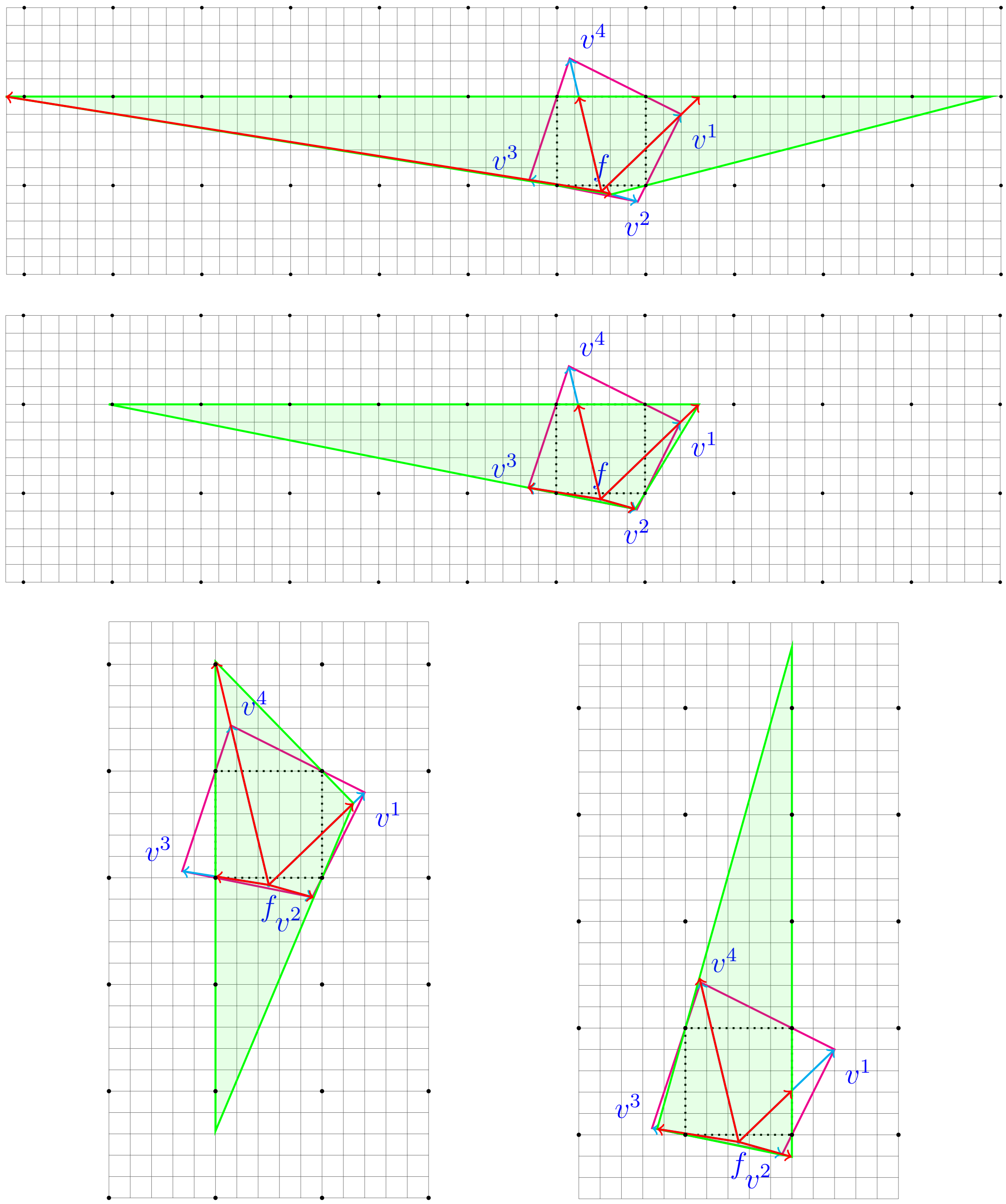


Figure 4.4: Triangles $RS_1(\tau_1^{bp1})$ (bottom left), $RS_2(\tau_2^{bp2})$ (top), $RS_2(\tau_2^{bp1})$ (middle), and $RS_3(\tau_3^{bp1})$ (bottom right) for $a = \frac{1}{5}$, $b = 2$, $c = 3$, $d = \frac{1}{2}$, $g = \frac{1}{2}$, and $h = -\frac{1}{15}$.

Proof. Looking up the cuts in Appendix A the constraint matrix is

$$\begin{bmatrix} \left(1+\frac{d}{c(b+d)g}\right) & \left(1-\frac{ad}{c(a+b)g}\right) & \left(1+\frac{1}{(a+c)g}\right) & \left(1-\frac{d}{(c+d)g}\right) \\ \left(1-\frac{d}{(b+d)(1-h)}\right) & \left(1+\frac{bc}{(a+b)(1-h)}\right) & \left(1-\frac{c}{(a+c)(1-h)}\right) & \left(1+\frac{cd}{(c+d)(1-h)}\right) \\ \left(1-\frac{d}{(b+d)(1-h)}\right) & \left(1+\frac{ad}{(a+b)(1-h)}\right) & \left(1-\frac{ad}{b(a+c)(1-h)}\right) & \left(1+\frac{cd}{(c+d)(1-h)}\right) \\ \left(1+\frac{1}{(b+d)(1-g)}\right) & \left(1-\frac{a}{(a+b)(1-g)}\right) & \left(1+\frac{a}{b(a+c)(1-g)}\right) & \left(1-\frac{ad}{b(c+d)(1-g)}\right) \end{bmatrix}$$

So, optimization problem (P') is given by

$$\begin{aligned} \min \quad & \mathbb{1}^T s \\ \text{subject to} \quad & \left[\mathbb{1}\mathbb{1}^T + A\right] s \geq \mathbb{1} \\ & s \geq \mathbb{0} \end{aligned}$$

where

$$A := \begin{bmatrix} \frac{1}{g} & 0 & 0 & 0 \\ 0 & \frac{1}{1-h} & 0 & 0 \\ 0 & 0 & \frac{1}{1-h} & 0 \\ 0 & 0 & 0 & \frac{1}{1-g} \end{bmatrix} \begin{bmatrix} \frac{d}{c} & -\frac{ad}{c} & 1 & -d \\ -d & bc & -c & cd \\ -d & ad & -\frac{ad}{b} & cd \\ 1 & -a & \frac{a}{b} & -\frac{ad}{b} \end{bmatrix} \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{1}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{1}{c+d} \end{bmatrix}.$$

First we assume $ad \neq bc$. We compute

$$A^{-1} = \left(\frac{1}{bc-ad}\right) \begin{bmatrix} b+d & 0 & 0 & 0 \\ 0 & a+b & 0 & 0 \\ 0 & 0 & a+c & 0 \\ 0 & 0 & 0 & c+d \end{bmatrix} \begin{bmatrix} 0 & a & 0 & bc \\ c & 1 & 0 & 0 \\ bc & 0 & b & 0 \\ 0 & 0 & \frac{b}{d} & b \end{bmatrix} \begin{bmatrix} g & 0 & 0 & 0 \\ 0 & 1-h & 0 & 0 \\ 0 & 0 & 1-h & 0 \\ 0 & 0 & 0 & 1-g \end{bmatrix},$$

$$A^{-1}\mathbb{1} = \left(\frac{1}{bc-ad}\right) \begin{bmatrix} (b+d)[a(1-h)+bc(1-g)] \\ (a+b)[cg+(1-h)] \\ (a+c)[bcg+b(1-h)] \\ (c+d)[\frac{b}{d}(1-h)+b(1-g)] \end{bmatrix} = \left(\frac{1}{bc-ad}\right) \begin{bmatrix} t_1^S \\ t_2^S \\ t_3^S \\ t_4^S \end{bmatrix}, \text{ and}$$

$$\mathbb{1}^T A^{-1} \mathbb{1} = \left(\frac{1}{bc-ad}\right) \left(t_1^S + t_2^S + t_3^S + t_4^S\right).$$

Note that the entries of A^{-1} are non-negative because $a, b, c, d > 0$, $ad < bc$, and $(g, h)^T \in R_S(a, b, c, d) \subseteq \{(x, y)^T \in \mathbb{R}^2 : 0 < x < 1, y < 1\}$. Apply Lemma 2.1.3 to conclude the optimal value of (P') is

$$1 - \frac{1}{1 + \mathbb{1}^T A^{-1} \mathbb{1}} = 1 - \frac{bc - ad}{bc - ad + t_1^S + t_2^S + t_3^S + t_4^S} = LB_S(a, b, c, d, g, h)$$

whenever $ad < bc$.

If $ad = bc$, then A isn't invertible. However, we can still construct primal and dual feasible solutions with equal objective function value of 1 as required. We refer the reader to Appendix C where we reprove this lemma by explicitly constructing the solutions and checking primal feasibility, dual feasibility and optimality, rather than relying on Lemma 2.1.3. $\square$

Lemma 4.3.2. *For all $a, b, c, d > 0$ with $ad \leq bc$ and g, h with $(g, h)^T \in R_S(a, b, c, d)$,*

$$LB_S(a, b, c, d, g, h) \geq \frac{4}{5}.$$

Proof. Note that $t_i^S \geq 0$ for all $i \in \{1, 2, 3, 4\}$ but the coefficient of h is negative for all $i \in \{1, 2, 3, 4\}$. Since $bc \geq ad$ we have

$$LB_S\left(a, b, c, d, g, h\right) \geq LB_S\left(a, b, c, d, g, \frac{a}{a + c}\right)$$

because $h \leq \frac{a}{a+c}$ whenever $(g, h)^T \in R_S(a, b, c, d)$.

Note that $LB_{Central}\left(a, b, c, d, g, \frac{a}{a+c}\right) = 1 - \frac{bc-ad}{bc-ad+T}$ where

$$\begin{aligned}
T = (b + d)\left[bc(1 - g) + \left(\frac{ac}{a + c}\right)\right] + \left(\frac{a + b}{a}\right)\left[\left(\frac{ac}{a + c}\right) + acg\right] + \dots \\
\dots + (a + c)\left[bcg + \left(\frac{bc}{a + c}\right)\right] + \left(\frac{c + d}{d}\right)\left[\left(\frac{bc}{a + c}\right) + bd(1 - g)\right].
\end{aligned}$$

Note that $LB_S\left(a, b, c, d, g, \frac{a}{a+c}\right) = 1 - \frac{bc-ad}{bc-ad+T}$ where

$$\begin{aligned}
T = (b + d)\left[\left(\frac{ac}{a + c}\right) + bc(1 - g)\right] + (a + b)\left[cg + \left(\frac{c}{a + c}\right)\right] + \dots \\
\dots + (a + c)\left[bcg + \left(\frac{bc}{a + c}\right)\right] + (c + d)\left[\left(\frac{bc}{d(a + c)}\right) + b(1 - g)\right].
\end{aligned}$$

So $LB_S\left(a, b, c, d, g, \frac{a}{a+c}\right) = LB_{Central}\left(a, b, c, d, g, \frac{a}{a+c}\right)$. As $\left[\frac{d}{c+d}, \frac{b}{a+b}\right] \subset [0, 1]$ and $\frac{a}{a+c} \in [0, 1]$, it follows from Lemma 4.1.2 that $LB_S\left(a, b, c, d, g, \frac{a}{a+c}\right) \geq \frac{4}{5}$ and therefore $LB_S\left(a, b, c, d, g, h\right) \geq \frac{4}{5}$. $\square$

4.4 Proof of Theorem 1.3.19

We combine the lower bounds obtained in the preceding sections to quickly arrive at Theorem 1.3.19.

Theorem 4.4.1 (Theorem 1.3.19)**.** *The optimal value of* $(P(a,b,c,d,g,h))$ *if at least* $\frac{4}{5}$ *for all* $a,b,c,d>0$ *with* $ad \leq bc$ *such that* $(g,h)^T \in \operatorname{int} Q(a,b,c,d)$*. Therefore* $\rho[\mathcal{T}_2, \mathcal{Q}] \leq \frac{5}{4}$*.*

Proof. By Remark 3.1.1, it suffices to show that the optimal value of $(P(a,b,c,d,g,h))$ is at least $\frac{4}{5}$ for all $a,b,c,d>0$ with $ad \leq bc$ and $(g,h)^T \in \operatorname{int} Q(a,b,c,d)$ where either: (1) $(g,h)^T \in R_{Central}(a,b,c,d)$, (2) $(g,h)^T \in R_{SW}(a,b,c,d)$, or (3) $(g,h)^T \in R_S(a,b,c,d)$. The bound holds in case (1) by Lemma 4.1.2, in case (2) by Lemma 4.2.2, and in case (3) by Lemma 4.3.2. It follows that the optimal value of $(S(\mathcal{T}_2,\mathcal{Q}))$ is at least $\frac{4}{5}$. The bound on $\rho[\mathcal{T}_2,\mathcal{Q}]$ follows immediately from Theorem 2.1.1. $\square$

4.5 Observations and Summary

We have proved the main result in this chapter, showing that $\rho[\mathcal{T}_2,\mathcal{Q}] \leq \frac{5}{4}$. See Section 6.3 for a broader discussion of these results and related open problems for 2-row intersection cuts. The following observations, however, are best understood alongside the technicalities in this chapter.

Unlike in the proofs of Theorem 2.2.4 and Theorem 1.3.21, the analysis we provide here is not tight. The optimization problems in Lemma 4.1.1, Lemma 4.2.1 and Lemma 4.3.1 are solved to optimality. The bound provided in Lemma 4.1.2 is tight as stated: take $(a,b,c,d,g,h) := (\epsilon^2, 1, 1, \frac{1}{\epsilon}, 1, 0)$ for $\epsilon \to 0$. However, we did not construct a family of quadrilateral and fractional points pairs which demonstrate that the ray-sliding triangles selected in this chapter do not achieve a better approximation than $\frac{4}{5}$. The gap here is that creating an example with $(g,h)^T \in R_{Central}(a,b,c,d)$ would push the fractional point away from $\{0,1\}^2$ where the bound in Lemma 4.1.2 is tight. Running some numerical experiments, it is easy to construct instances where the bound in Lemma 4.1.1 is less than $\frac{8}{9}$. The weakest bound the experiment found was around 0.8284, but this provides little analytical insight into a tight bound on $LB_{Central}(a,b,c,d,g,h)$ over $R_{Central}(a,b,c,d)$.

The optimal values calculated here can be specialized for the parallelogram case (i.e. $a=d=\alpha$ and $b=c=\beta$) to recover those calculated in Chapter 3. Since the same intersection cuts were used in the Central Region in both analyses (i.e. in Lemma 3.2.1 and Lemma 4.1.1), the optimal values obviously coincide. However, the optimal values in the South-West Region (i.e. Lemma 3.4.1 and Lemma 4.2.1) agree even though different sets of ray-sliding triangles were used. In the parallelogram analysis, we used a total of two ray-sliding triangles, each at a geometric average of their respective ray breakpoints. In the general quadrilateral analysis, we used a total of four ray-sliding triangles: for the same rays, we used the ray-sliding triangles for both ray breakpoints. Similarly, the optimal values in the South Region (i.e. Lemma 3.5.1 and Lemma 4.3.1) also agree.

Finding the weakest lower bound in this chapter again relied on the lower bound being of the form $1 + \frac{1}{1+T(a,b,c,d,g,h)}$ where T is an affine function of g and h. This phenomenon is explained in Section 3.7. However, in contrast to the parallelogram case, the bound is not constant across $R_{Central}(a, b, c, d, g, h)$.

Chapter 5

Extensions to d-Row Intersection Cuts

We are interested in extending results to the mixed integer linear set

$$R(f;\Gamma) := \operatorname{conv}\left\{ s \in \mathbb{R}^k_+ \ : f + \sum_{j=1}^{k} r^j s_j \in \mathbb{Z}^d \right\} \tag{5.1}$$

for $d \geq 3$. As for $d = 2$, all minimal valid inequalities are intersection cuts generated by maximal lattice-free convex sets. However, the structure of maximal lattice-free convex sets in $\mathbb{R}^d$ is not well understood as it is in $\mathbb{R}^2$. Giving a classification of families of maximal lattice-free convex sets in $\mathbb{R}^d$ has proven difficult. Even for $d = 3$, the classification is complicated; see the discussion in Section 1.2.2.

Here we take a different tact that does not require a classification result; we try to prove an analogue of Theorem 2.2.4 that holds for all dimensions $d \geq 2$. First, we provide a complete statement and proof of the proposed analogue in Section 5.1. We define two families of maximal lattice-free convex sets in $\mathbb{R}^d$, denoted $\mathcal{F}_d$ and $\mathcal{P}_d$, and prove a theorem upper-bounding $\rho[\mathcal{P}_d, \mathcal{F}_d]$. For $d = 2$, the theorem implies Theorem 2.2.4 from [11].

The families we choose are not arbitrary. $\mathcal{F}_d$ is a family of octahedra and its members have the largest possible number of facets for a maximal lattice-free convex set in $\mathbb{R}^d$. In fact, this family has been studied before in computational experiments for $d \in \{2, \ldots, 10\}$. The experiments showed that adding an inequality from a specific canonical member of $\mathcal{F}_d$ within a solver can have a positive impact on the branch and cut performance and the closed gap at the root node. See [43] for more information on these experiments and [57] for more information on measures of solver progress.

The families $\mathcal{F}_d$ and $\mathcal{P}_d$ are chosen not just by facet structure but also so that certain properties of maximal lattice-free quadrilaterals and type 2 triangles in $\mathbb{R}^2$ extend directly to $\mathbb{R}^d$. In Section 5.2, we discuss these properties in greater detail. In Section 5.3, we discuss the limitations and strengths of the bound proved and provide a contrast to other results known to hold for arbitrary dimension d.

5.1 Approximating Octahedra Cuts with Pyramid Cuts: A Generalization of Theorem 2.2.4

Throughout we assume $d \in \mathbb{Z}_+$ is at least 2.

5.1.1 Families of Octahedra and Pyramids in $\mathbb{R}^d$

The *canonical octahedron* $\hat{O}_d$ in $\mathbb{R}^d$ is the polyhedron defined by

$$\hat{O}_d := \{x \in \mathbb{R}^d : \sigma^T x \leq 1 \; \forall \sigma \in \{-1, +1\}^d\}.$$

For $i \in \{1, \ldots, d\}$, let e^i denote the i-th standard basis vector; i.e. the i-th entry of e^i is 1 and the other entries are 0. It is straightforward to check that $\hat{O}_d$ is the convex hull of $\{\pm e^1, \pm e^2, \ldots, \pm e^d\}$ and therefore that $\hat{O}_d$ is a polytope (i.e. a bounded polyhedron). The canonical octahedron $\hat{O}_d$ is the polar of the d-dimensional hypercube with vertices $\{+1, -1\}^d$, where the *polar* of set $A \subseteq \mathbb{R}^d$ is defined by $A^0 := \{y \in \mathbb{R}^d : y^T x \leq 1 \; \forall \; x \in A\}$.

Let $P \subseteq \mathbb{R}^d$ be a polytope. A linear inequality $c^T x \leq k$ is *valid* for P if it is satisfied by every $x \in P$. Recall that the set $F \subseteq P$ is a *face* of P if $F = \{x \in P : c^T x = k\}$ for some $c \in \mathbb{R}^d$ and $k \in \mathbb{R}$ such that $c^T x \leq k$ is a valid inequality for P. Note that $\emptyset$ and P are both faces of P. A face of dimension 0 is called a *vertex* of P, while a face of dimension $\dim(P) - 1$ is called a *facet* of P. For example, the vertices of $\hat{O}_d$ are $\{\pm e^1, \pm e^2, \ldots, \pm e^d\}$ and the facets of $\hat{O}_d$ are $\{F_\sigma : \sigma \in \{-1, +1\}^d\}$ where $F_\sigma := \{x \in \hat{O}_d : \sigma^T x = 1\}$. Now, ordering the faces of arbitrary polytope P by inclusion, we obtain a lattice[1], which we call the *face lattice* of P. Two lattices are the same if there is an order preserving bijection between them.

A polytope $O \subseteq \mathbb{R}^d$ is an *octahedron* if O (is full-dimensional and) has the same face lattice as the canonical octahedron $\hat{O}_d$. Per our usual convention, the family of maximal lattice

[1]A bounded partially ordered set where each pair of elements has a unique minimal upper bound and a unique maximal lower bound. We refer the reader to [76, Chapter 2.2] for details.

free octahedra in $\mathbb{R}^d$ is the set of all polytopes $P \subseteq \mathbb{R}^d$ such that P is an octahedron and P is a maximal lattice-free convex set. We will denote this family by $\mathcal{O}_d$. The canonical octahedron is not lattice-free, but scaling and translating we can obtain the maximal lattice-free octahedron $\tilde{O}_d = \frac{d}{2}\hat{O}_d + \frac{1}{2}\mathbb{1}$.

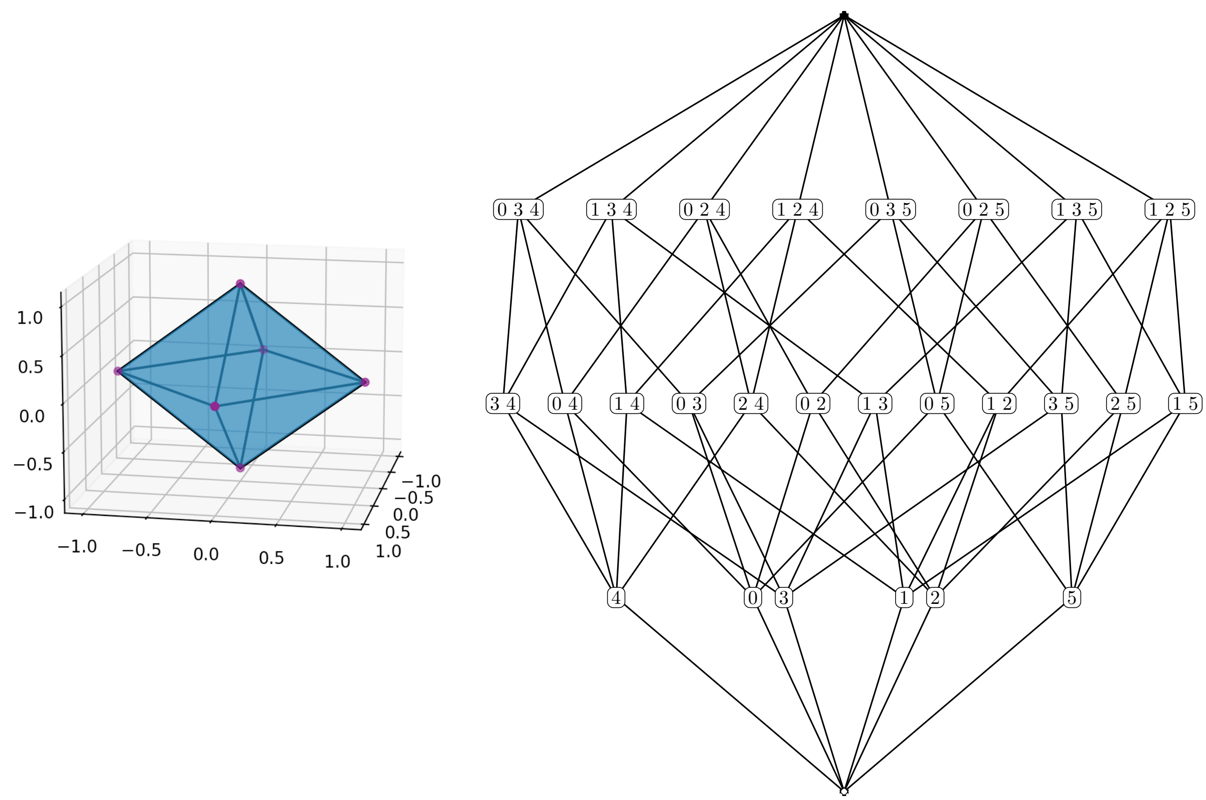


Figure 5.1: Canonical octahedron $\hat{O}_3$ (left) and its face lattice (right). Face lattice images in this thesis were generated using polymake [46].

A polytope $P \subseteq \mathbb{R}^d$ is a *parallelepiped* if there exist $c \in \mathbb{R}^d$ and linearly independent $z^1, \ldots, z^d \in \mathbb{R}^d$ such that $P = \{x \in \mathbb{R}^d : \exists \lambda \in [0,1]^d \text{ for which } x = c + \sum_{i=1}^d \lambda_i z^i\}$. We call such $z^1, \ldots, z^d$ the *edges* of P. The *volume* of P is given by $\prod_{i=1}^d ||z^i||_2$. Note that a parallelepiped in $\mathbb{R}^d$ has 2^d vertices and $2d$ facets. Its face lattice is the same as that of the unit hypercube $[0,1]^d$.

A straightforward extension of Theorem 1.2.6 shows that any maximal lattice-free polyhedron in $\mathbb{R}^d$ with 2^d facets has exactly one integral point in the relative interior of each facet.

We defer the proof to Section 5.2.2. Therefore $|O \cap \mathbb{Z}^d| = 2^d$ for any maximal lattice-free octahedron $O \in \mathcal{O}_d$. A maximal lattice-free octahedron $O \in \mathcal{O}_d$ is a *unit-core octahedron* if $\text{conv}(O \cap \mathbb{Z}^d)$ is a parallelepiped of volume one. Let $\mathcal{U}_d$ denote the set of all unit-core octahedra in $\mathbb{R}^d$. A unit-core octahedron $O \in \mathcal{U}_d$ is a *normalized octahedron* if $\text{conv}(O \cap \mathbb{Z}^d)$ is the unit hypercube $[0,1]^d$.

Remark 5.1.1. *Let $U \in \mathcal{U}_d$ be a unit-core octahedron. Then there exists an affine unimodular transformation $\phi : \mathbb{R}^d \to \mathbb{R}^d$ such that $\phi(U) = N$ where N is a normalized octahedron.*

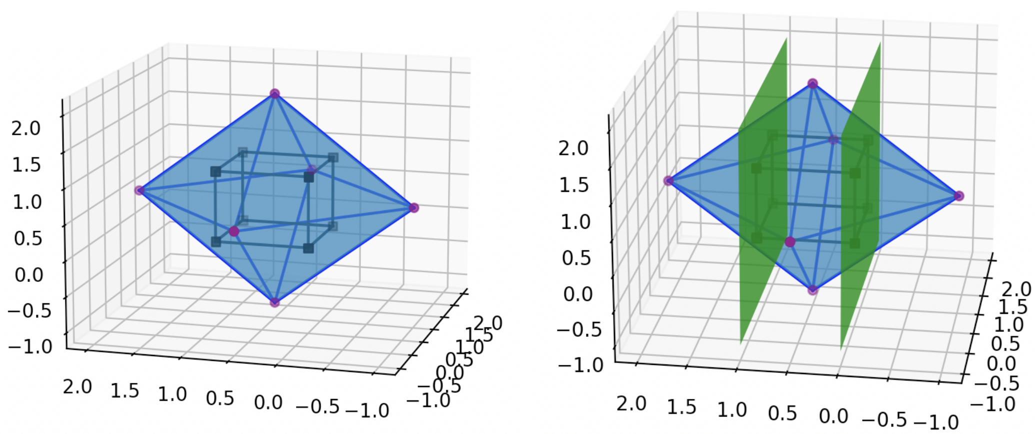


Figure 5.2: Normalized octahedron $\tilde{O}_3$ with embedded hypercube $[0,1]^3$ illustrated on the left and facet separation property illustrated on the right.

Let $\hat{\mathcal{U}}_d$ denote the set of all normalized octahedra in $\mathbb{R}^d$. Let $O \in \hat{\mathcal{U}}_d$ be a normalized octahedron with vertices $\{v^1, \ldots, v^{2d}\}$. We say O has the *facet separation property* (or is *facet-separable*) if there is a bijection $b : \{v^1, \ldots, v^{2d}\} \to \{1, \ldots, d\} \times \{0, 1\}$ such that if $b(v^i) = (j, \triangle)$ then

1. $0 < v^i_k < 1$ for all $k \in \{1, \ldots, d\} \setminus \{j\}$, and
2. if $\triangle = 0$ then $v^i_j < 0$ and if $\triangle = 1$ then $v^i_j > 1$.

That is, $O \in \hat{\mathcal{U}}_d$ has the facet separation property if and only if there is a unique facet defining inequality of $[0,1]^d$ separating each vertex of O from $[0,1]^d$. If O has this property then we can label the vertices of O as $v^1, \ldots, v^{2d}$ so that for all $i \in \{1, \ldots, d\}$:

1. $v^i_i < 0$,
2. $v^{d+i}_i > 1$, and

3. $0 < v_j^i, v_j^{d+i} < 1$ for $j \in \{1, \ldots, d\} \setminus \{i\}$.

Under such a labeling, we refer to v^i and v^{d+i} as an *antipodal pair* of vertices. Necessarily, v^i is contained in the 2^{d-1} facets of O that contain $\{x \in \{0,1\}^d : x_i = 0\}$ and v^{d+i} is contained in the 2^{d-1} facets of O that contain $\{x \in \{0,1\}^d : x_i = 1\}$. In particular, there is no facet of O that contains v^i and v^{d+i}.

We call an octahedron $O \in \mathcal{O}_d$ a *facet-separable unit-core octahedron* if there exists an affine unimodular transformation $\phi : \mathbb{R}^d \to \mathbb{R}^d$ such that $\phi(O)$ is a normalized octahedron with the facet separation property. Let $\mathcal{F}_d$ denote the set of all facet-separable unit-core octahedra in $\mathbb{R}^d$. By Remark 1.3.4, $\mathcal{F}_2$ is the family of all maximal lattice-free quadrilaterals in $\mathbb{R}^2$. In dimension $d \geq 3$, $\mathcal{F}_d$ is non-empty (it contains $\tilde{O}_d$), but it may not contain all maximal lattice-free octahedra. See Section 5.2 for further discussion of this class of octahedra.

Let $V := \{v_1, \ldots, v_k\} \subseteq \mathbb{R}^d$ be a set of $k \geq d$ points such that $\dim(V) = d-1$ and $\operatorname{conv} V$ has k vertices. Let hyperplane H be the affine hull of V (i.e. the unique hyperplane containing every element of V). Let $x \in \mathbb{R}^d$ be some point not on H. Any polytope P of the form $\operatorname{conv}\{v_1, \ldots, v_k, x\}$ for some $v_1, \ldots, v_k$ and x as described is called a *pyramid*. Pyramid P is of dimension d and has $k+1$ vertices and $k+1$ facets - one facet is defined by H, and the remainder contain x. We call pyramid P a *type 2 pyramid* if

(i) P is lattice-free,
(ii) P has $2^{d-1}+1$ facets,
(iii) P contains one integer point in the relative interior of each facet that contains x, and
(iv) P contains at least 2^{d-1} integer points in the relative interior of the facet defined by H.

It follows from Theorem 1.2.6 that any type 2 pyramid is a maximal lattice-free convex set. Let $\mathcal{P}_d$ denote the family of type 2 pyramids in $\mathbb{R}^d$. Observe that the family $\mathcal{P}_2$ is the family of all type 2 triangles in $\mathbb{R}^2$.

We will prove the following theorem, bounding $\rho[\mathcal{P}_d, \mathcal{F}_d]$.

Theorem 5.1.2. *Let $d \in \mathbb{Z}_+$ be at least 2. Let $\mathcal{F}_d$ denote the family of facet-separable unit-core octahedra in $\mathbb{R}^d$. Let $\mathcal{P}_d$ denote the family of type 2 pyramids in $\mathbb{R}^d$. Then $\rho[\mathcal{P}_d, \mathcal{F}_d] \leq 2 - \frac{1}{d}$.*

5.1.2 Proof of Theorem 5.1.2

Let $\hat{\mathcal{F}}_d$ denote the set of all facet-separable normalized octahedra in $\mathbb{R}^d$. It follows from Theorem 2.1.1 that $\frac{1}{\rho[\mathcal{P}_d, \mathcal{F}_d]}$ is equal to the maximum of one and the optimal value of the optimization problem $(S(\mathcal{P}_d, \mathcal{F}_d))$ given by

$$\begin{aligned}
\inf \quad & \sum_{i=1}^{2d} s_i && (5.2)\\
\text{subject to} \quad & s \in \mathcal{P}_d(f; r^1, \ldots, r^{2d}) && (5.3)\\
& f + \operatorname{conv}\{r^1, \ldots, r^{2d}\} \in \hat{\mathcal{F}}_d && (5.4)\\
& f \in \mathbb{Q}^d \setminus \mathbb{Z}^d && (5.5)\\
& r^1, \ldots r^{2d} \in \mathbb{Q}^d \setminus \{\mathbb{0}\} && (5.6)\\
& \operatorname{cone}(\{r^1, \ldots, r^{2d}\}) = \mathbb{R}^d. && (5.7)
\end{aligned}$$

Now, we reparameterize this problem with respect to data $v^1, \ldots, v^{2d} \in \mathbb{Q}^d$ and $f \in \mathbb{Q}^d \setminus \mathbb{Z}^d$ for which $O := \operatorname{conv}(\{v^1, \ldots, v^{2d}\}) \in \hat{\mathcal{F}}_d$ and $f \in \operatorname{int} O$. As discussed in Section 5.1.1, since O is facet-separable, we may label vertices $v^1, \ldots, v^{2d}$ so that for all $i \in \{1, \ldots, d\}$: $v^i_i < 0$, $v^{d+i}_i > 1$, and $0 < v^i_j, v^{d+i}_j < 1$ for $j \in \{1, \ldots, d\} \setminus \{i\}$. Under this labeling, (v^i, v^{d+i}) are an antipodal pair. For all $i \in \{1, \ldots, 2d\}$, let $r^i = v^i - f$. For given data $v^1, \ldots, v^{2d}$ and f, consider the semi-infinite program

$$\begin{aligned}
\inf \quad & \sum_{i=1}^{2d} s_i && (5.8)\\
\text{subject to} \quad & s \in \mathcal{P}_d(f; r^1, \ldots, r^{2d}). && (5.9)
\end{aligned}$$

Denoting this problem by $(P(f, v^1, \ldots, v^{2d}))$, our goal is to lower bound its optimal value. To do so, we will lower bound the optimal value of the parameterized linear program obtained by replacing constraint (5.9) with the constraint $s \geq \mathbb{0}$ and the intersection cuts for a finite number of type 2 pyramids.

We will choose these type 2 pyramids to be *fixed pyramids*, which are defined with respect to a given octahedron $O = \operatorname{conv}(\{v^1, \ldots, v^{2d}\})$ as above. For $i \in \{1, \ldots, 2d\}$, let C_i be the cone with apex v^i that is defined by the 2^{d-1} facets of O containing v^i. For $i \in \{1, \ldots, d\}$, fixed pyramid T_i is equal to $C_i \cap \{x \in \mathbb{R}^d : x_i \leq 1\}$. For $i \in \{d+1, \ldots, 2d\}$, fixed pyramid T_i is equal to $C_i \cap \{x \in \mathbb{R}^d : x_i \geq 0\}$. Note that T_i is a type 2 pyramid by construction.

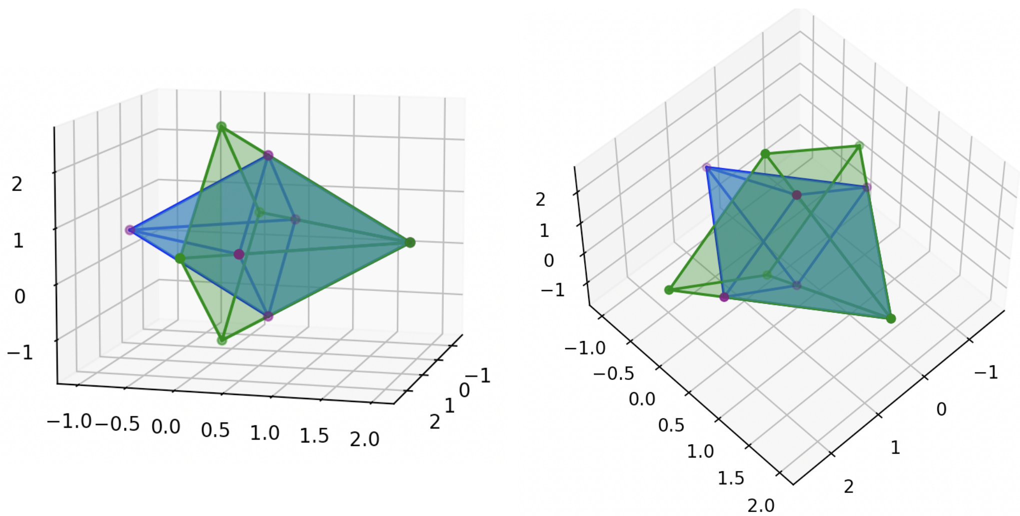


Figure 5.3: Fixed pyramid of normalized octahedron $\tilde{O}_3$, from two points of view.

Proposition 5.1.3. *Let $i \in \{1, \dots, 2d\}$. Suppose $f \in \operatorname{int}(T_i \cap O)$.*

1. *If $i \in \{1, \dots, d\}$, then*
   (a) *$\psi_{f;T_i}(r^j) = 1$ for all $j \in \{1, \dots, 2d\} \setminus \{i + d\}$, and*
   (b) *$\psi_{f;T_i}(r^{i+d}) = 1 + \frac{1}{a_{i+d}}$ where $a_{i+d} := a_{i+d}(f, v^{i+d}) = \frac{1-f_i}{v_i^{i+d}-1}$.*
2. *If $i \in \{d + 1, \dots, 2d\}$, then*
   (a) *$\psi_{f;T_i}(r^j) = 1$ for all $j \in \{1, \dots, 2d\} \setminus \{i - d\}$, and*
   (b) *$\psi_{f;T_i}(r^{i-d}) = 1 + \frac{1}{a_{i-d}}$ where $a_{i-d} := a_{i-d}(f, v^{i-d}) = \frac{-f_{i-d}}{v_{i-d}^{i-d}}$.*

Proof. We will prove 1. where $i \in \{1, \dots, d\}$. The proof of 2. is analogous.

Since O has the facet separation property, all the vertices of O except for v^{i+d} are on the boundary of T_i. As $r^j = v^j - f$, 1(a) follows.

As $v^{i+d} > 1$ and by our choice of T_i, we know $\psi_{f;T_i}(r^{i+d})$ is determined by the intersection of $\{f + \lambda r^{i+d} : \lambda \geq 0\}$ with $\{x \in \mathbb{R}^d : x_i = 1\}$. Hence $\psi_{f;T_i}(r^{i+d}) > 1$ and therefore it can be expressed as $1 + \frac{1}{a_{i+d}}$ for $a_{i+d} > 0$. So, by the definition of the gauge function, to find a_{i+d} so that $\psi_{f;T_i}(r^{i+d}) = 1 + \frac{1}{a_{i+d}}$ we can find a_{i+d} such that the x_i-coordinate of

$f + \left(\frac{a_{i+d}}{1+a_{i+d}}\right) r^{i+d}$ is equal to 1. Now

$$\begin{aligned}\left[f + \left(\frac{a_{i+d}}{1+a_{i+d}}\right) r^{i+d}\right]_i = 1 &\Leftrightarrow f_i + \left(\frac{a_{i+d}}{1+a_{i+d}}\right)(v_i^{i+d} - f_i) = 1\\ &\Leftrightarrow (1+a_{i+d})f_i + a_{i+d}(v_i^{i+d} - f_i) = 1 + a_{i+d}\\ &\Leftrightarrow a_{i+d}(v_i^{i+d} - 1) = 1 - f_i\\ &\Leftrightarrow a_{i+d} = \frac{1-f_i}{v_i^{i+d} - 1}.\end{aligned}$$

By our assumption on the labelling of $v^1, \ldots, v^{2d}$, since $i \in \{1, \ldots, d\}$ we have $v_i^{i+d} - 1 > 0$. As $f \in \operatorname{int}(T_i \cap O)$ and $i \in \{1, \ldots, d\}$ we have $f_i < 1$. So we can see that $a_{i+d} > 0$. Therefore 1(b) follows.

Note that in 2. where $i \in \{d+1, \ldots, 2d\}$ we have $v_{i-d}^{i-d} < 0$ and $f_{i-d} > 0$ so it is clear that $a_{i-d} > 0$ in this case as well. □

Observe that for all $i \in \{1, \ldots, 2d\}$, the function $a_i(f, v^i)$ is affine in f. For $f \in (0,1)^d$, we calculate the optimal value of the optimization problem obtained from $(P(f, v^1, \ldots, v^{2d}))$ using the intersection cuts for fixed pyramids $T_1, \ldots, T_{2d}$.

Lemma 5.1.4. *Let $v^1, \ldots, v^{2d} \in \mathbb{Q}^d$ and $f \in \mathbb{Q}^d \setminus \mathbb{Z}^d$ be such that $\operatorname{conv}(\{v^1, \ldots, v^{2d}\}) \in \hat{\mathcal{F}}_d$ and $f \in (0,1)^d$. Let (P') be the linear program obtained from $(P(f, v^1, \ldots, v^{2d}))$ by replacing constraint (5.9) with the constraint $s \geq \mathbb{0}$ and the intersection cuts for $T_1, \ldots, T_{2d}$. Then, the optimal value of (P') is equal to*

$$LB_0^d(f, v^1, \ldots, v^{2d}) := 1 - \frac{1}{1 + \sum_{i=1}^{2d} a_i(f, v^i)}$$

for a_i as given in Proposition 5.1.3.

Proof. Using Proposition 5.1.3, the optimization problem (P') is given by

$$\begin{array}{ll}\min & s_1 + s_2 + \ldots + s_{2d}\\ \text{subject to} & \begin{pmatrix} 1+\frac{1}{a_1} & 1 & 1 & \ldots & 1\\ 1 & 1+\frac{1}{a_2} & 1 & \ldots & 1\\ 1 & 1 & 1+\frac{1}{a_3} & \ldots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & 1 & \ldots & 1+\frac{1}{a_{2d}} \end{pmatrix} s \geq \mathbb{1}\\ & s \geq \mathbb{0}.\end{array}$$

This optimization problem is of the form in Lemma 2.1.3 where A is the diagonal matrix with diagonal entries $\frac{1}{a_i}$ for all $i \in \{1, \dots, 2d\}$. Note that A^{-1} is the diagonal matrix with entries a_i, and therefore A^{-1} is non-negative because $a_i > 0$ for all $i \in \{1, \dots, 2d\}$. We can therefore apply Lemma 2.1.3 to conclude that the optimal value of (P') is equal to

$$1 - \frac{1}{1 + \mathbb{1}^T A^{-1} \mathbb{1}} = 1 - \frac{1}{1 + \sum_{i=1}^{2d} a_i}$$

as required. □

We vary $f, v^1, \dots, v^{2d}$ to find the weakest bound.

Lemma 5.1.5. *Let $v^1, \dots, v^{2d} \in \mathbb{Q}^d$ and $f \in \mathbb{Q}^d \setminus \mathbb{Z}^d$ be such that* $\operatorname{conv}(\{v^1, \dots, v^{2d}\}) \in \hat{\mathcal{F}}_d$ *and $f \in [0,1]^d$. Then $LB_0^d(f, v^1, \dots, v^{2d}) \geq \frac{d}{2d-1}$.*

Proof. Let $T(f, v^1, \dots, v^{2d}) := \sum_{i=1}^{2d} a_i(f, v^i)$. By Lemma 2.1.4, we can lower bound $LB_0^d(f, v^1, \dots, v^{2d})$ over all $f \in [0,1]^d$ and $v^1, \dots, v^{2d}$ such that $\operatorname{conv}\{v^1, \dots, v^{2d}\} \in \hat{\mathcal{F}}_d$ by lower-bounding $T(f, v^1, \dots, v^{2d})$ over the same set. So, we want to lower bound the optimal value of the optimization problem $(P1)$ in variables $f, v^1, \dots, v^{2d}$ given by

$$\begin{aligned} \inf \quad & T(f, v^1, \dots, v^{2d}) && (5.10) \\ \text{subject to} \quad & \operatorname{conv}\{v^1, \dots, v^{2d}\} \in \hat{\mathcal{F}}_d && (5.11) \\ & f \in [0,1]^d. && (5.12) \end{aligned}$$

Since $\hat{\mathcal{F}}_d \neq \emptyset$, $(P1)$ is feasible. Note that $a_i(f, v^i) \geq 0$ for all $i \in \{1, \dots, 2d\}$ and therefore the optimal value of $(P1)$ is bounded below by 0. Suppose the optimal value is $K \geq 0$ and consider a sequence

$$\{S^{(\ell)}\}_{\ell \geq 1} = \{[\hat{f}^{(\ell)}, (\hat{v}^1, \dots, \hat{v}^{2d})^{(\ell)}]\}_{\ell \geq 1}$$

of feasible solutions with objective function value tending to K. Then for $\ell \geq 1$, we may fix the assignment $(v^1, \dots, v^{2d}) := (\hat{v}^1, \dots, \hat{v}^{2d})^{(\ell)}$ in $(P1)$ to obtain a problem whose objective function is affine in the variable f. Therefore there exists some $\tilde{f}^{(\ell)} \in \{0,1\}^d$ such that the objective function value of $(\tilde{f}^{(\ell)}, (\hat{v}^1, \dots, \hat{v}^{2d})^{(\ell)})$ is less than or equal to the objective function value of $(\hat{f}^{(\ell)}, (\hat{v}^1, \dots, \hat{v}^{2d})^{(\ell)})$. So we may construct a sequence

$$\{S_2^{(\ell)}\}_{\ell \geq 1} = \{[\tilde{f}^{(\ell)}, (\hat{v}^1, \dots, \hat{v}^{2d})^{(\ell)}]\}_{\ell \geq 1}$$

of feasible solutions whose objective function value tends to K such that $\tilde{f}^{(\ell)} \in \{0,1\}^d$ for all $\ell \geq 1$. In the sequence $\{\tilde{f}^{(\ell)}\}_{\ell \geq 1}$ some $f^* \in \{0,1\}^d$ occurs infinitely often. So, we can construct a subsequence of feasible solutions with objective function value tending to K where the assignment of values to variable f is some fixed element of $\{0,1\}^d$. By symmetry, we may assume that $f = \mathbb{0}$.

Substituting $f := \mathbb{0}$, we see that we can lower bound the optimal value of $(P1)$ by calculating a lower bound on the optimal value of $(P2)$ given by

$$\inf \quad \sum_{i=1}^{d} \frac{1}{v_i^{i+d} - 1} \tag{5.13}$$

$$\text{subject to} \quad \operatorname{conv}\{v^1, \ldots, v^{2d}\} \in \hat{\mathcal{F}}_d. \tag{5.14}$$

Relabeling the variables $x^i := v^{i+d}$ for all $i \in \{1, \ldots, d\}$ and introducing additional variable $\lambda \in \mathbb{R}^d$, we consider the relaxation $(P3)$ of $(P2)$ given by

$$\begin{aligned}
\inf \quad & \sum_{i=1}^{d} \frac{1}{x_i^i - 1} && && (5.15)\\
\text{subject to} \quad & \sum_{i=1}^{d} \lambda_i x^i = \mathbb{1} && && (5.16)\\
& \mathbb{1}^T \lambda = 1 && && (5.17)\\
& \lambda > \mathbb{0} && && (5.18)\\
& x_i^i > 1 && \forall i \in \{1, \ldots, d\} && (5.19)\\
& 0 \leq x_j^i < 1 && \forall i \in \{1, \ldots, d\}, j \in \{1, \ldots, d\} \setminus \{i\}. && (5.20)
\end{aligned}$$

Note that $(P3)$ is indeed a relaxation of $(P2)$ because it enforces three necessary conditions for $\operatorname{conv}\{v^1, \ldots, v^{2d}\} \in \hat{\mathcal{F}}_d$: $v_i^{i+d} > 1$ for all $i \in \{1, \ldots, d\}$, $0 \leq v_j^{i+d} < 1$ for all $i \in \{1, \ldots, d\}$, $j \in \{1, \ldots, d\} \setminus \{i\}$ and $\mathbb{1} \in \operatorname{conv}\{v^{d+1}, \ldots, v^{2d}\}$. We may restrict $\lambda_i > 0$ for all $i \in \{1, \ldots, d\}$ because only x^i has i-th coordinate greater than or equal to one, and therefore x^i must have non-zero coefficient in any convex combination of $\{x^1, \ldots, x^d\}$ summing to $\mathbb{1}$.

Next, we show that restricting x^i to be of the form $x^i = \mu_i e^i$ for $\mu_i > 0$ for all $i \in \{1, \ldots, d\}$ does not change the optimal value of $(P3)$. Let $(\hat{\lambda}, \hat{x}^1, \ldots, \hat{x}^d)$ be a feasible solution to $(P3)$ such that $t > 0$ entries $\hat{x}_j^i$ for $j \neq i$ are non-zero. We construct a solution $(\tilde{\lambda}, \tilde{x}^1, \ldots, \tilde{x}^d)$ such that $t-1$ such entries are non-zero and the objective function value is strictly smaller. Given $(\hat{\lambda}, \hat{x}^1, \ldots, \hat{x}^d)$, let $\hat{\imath} \neq \hat{\jmath} \in \{1, \ldots, d\}$ be such that $\hat{x}_{\hat{\jmath}}^{\hat{\imath}} = \kappa > 0$. Define $(\tilde{\lambda}, \tilde{x}^1, \ldots, \tilde{x}^d)$

as follows:

$$\begin{aligned}
\tilde{\lambda} &:= \hat{\lambda}, \\
\tilde{x}^i &:= \hat{x}^i \qquad \forall i \in \{1, \ldots, d\} \setminus \{\hat{\imath}, \hat{\jmath}\}, \\
\tilde{x}^{\hat{\imath}} &:= \hat{x}^{\hat{\imath}} - \kappa e^{\hat{\jmath}}, \quad \text{and} \\
\tilde{x}^{\hat{\jmath}} &:= \hat{x}^{\hat{\jmath}} + \kappa \left(\frac{\hat{\lambda}_{\hat{\imath}}}{\hat{\lambda}_{\hat{\jmath}}} \right) e^{\hat{\jmath}}.
\end{aligned}$$

First we verify $(\tilde{\lambda}, \tilde{x}^1, \ldots, \tilde{x}^d)$ satisfies the constraints of $(P3)$. Since $\tilde{\lambda} = \hat{\lambda}$, and $\hat{\lambda}$ is feasible for $(P3)$, clearly $\mathbb{1}^T \tilde{\lambda} = 1$ and $\tilde{\lambda} > \mathbb{0}$ and so $\tilde{\lambda}$ satisfies (5.17) and (5.18). For all $i \in \{1, \ldots, d\}$, $\tilde{x}^i_i \geq \hat{x}^i_i > 1$ where the first inequality holds by construction and the second inequality holds because $(\hat{x}^1, \ldots, \hat{x}^d)$ satisfies (5.19). Therefore $(\tilde{x}^1, \ldots, \tilde{x}^d)$ satisfies (5.19). For all $i \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, d\} \setminus \{i\}$, $0 \leq \tilde{x}^i_j \leq \hat{x}^i_j < 1$ where the first two inequalities hold by construction and the second inequality holds because $(\hat{x}^1, \ldots, \hat{x}^d)$ satisfies (5.20). Therefore $(\tilde{x}^1, \ldots, \tilde{x}^d)$ satisfies (5.20). Lastly, we calculate

$$\sum_{i=1}^d \tilde{\lambda}_i \tilde{x}^i = \sum_{i=1}^d \hat{\lambda}_i \hat{x}^i - \hat{\lambda}_{\hat{\imath}} \kappa e^{\hat{\jmath}} + \hat{\lambda}_{\hat{\jmath}} \kappa \left(\frac{\hat{\lambda}_{\hat{\imath}}}{\hat{\lambda}_{\hat{\jmath}}} \right) e^{\hat{\jmath}} = \sum_{i=1}^d \hat{\lambda}_i \hat{x}^i = 1$$

and therefore $(\tilde{\lambda}, \tilde{x^1}, \ldots, \tilde{x^d})$ satisfies (5.16). The objective function value changed by $\frac{1}{\hat{x}^{\hat{\jmath}}_{\hat{\jmath}} + \kappa \frac{\hat{\lambda}_{\hat{\imath}}}{\hat{\lambda}_{\hat{\jmath}}} - 1} - \frac{1}{\hat{x}^{\hat{\jmath}}_{\hat{\jmath}} - 1}$, a strict decrease. So we may restrict $x^i = \mu_i e^i$ for $i \in \{1, \ldots, d\}$ and not change the optimal value of $(P3)$.

Setting $x_i = \mu_i e^i$, and solving constraint (5.16) gives $\mu_i = \frac{1}{\lambda_i}$. We can substitute this identity into the objective function and eliminate constraints (5.16), (5.19), and (5.20) to obtain a new optimization problem. Thus we can lower bound the optimal value of $(P1)$ by lower-bounding the optimal value of $(P4)$ with variable $\lambda \in \mathbb{R}^d$ given by

$$\begin{aligned}
\inf \quad & \sum_{i=1}^d \frac{\lambda_i}{1 - \lambda_i} && (5.21) \\
\text{subject to} \quad & \mathbb{1}^T \lambda = 1 && (5.22) \\
& \lambda > \mathbb{0}. && (5.23)
\end{aligned}$$

Note that the constraints imply $\lambda \in (0,1)^d$ for all feasible solutions of $(P4)$. Let $h(\lambda) = \sum_{i=1}^d \frac{\lambda_i}{1-\lambda_i}$. For $i \in \{1, \ldots, d\}$ we calculate $\frac{\partial h}{\partial \lambda_i} = \frac{1}{(1-\lambda_i)^2}$ and $\frac{\partial^2 h}{\partial x_i^2} = \frac{2}{(1-\lambda_i)^3}$. Since h is a separable function of the λ_i, its Hessian is the diagonal matrix with entries $\frac{2}{(1-\lambda_i)^3}$

for $i \in \{1, \ldots, d\}$. Since $\frac{2}{(1-\lambda_i)^3} > 0$ for $\lambda_i \in (0,1)$ it follows that the Hessian of h is positive definite on $(0,1)^d$ and therefore h is strictly convex over $(0,1)^d$. Noting that the constraints of $(P4)$ are linear, we see that $(P4)$ is a convex optimization problem whose objective function is strictly convex on the feasible region.

We can solve $(P4)$ exactly using standard tools from convex optimization. Noting that h is a C^1 function on $(0,1)^d$, we have $\bar{\lambda}$ is a minimizer of $h(\lambda)$ over $\{\lambda \in (0,1)^d : \mathbb{1}^T\lambda = 1\}$ if and only if $\nabla h(\bar{\lambda})^T(\lambda - \bar{\lambda}) \geq 0$ for all $\lambda > 0$ such that $\mathbb{1}^T\lambda = 1$. We claim $\lambda^* = \frac{1}{d}\mathbb{1}$ satisfies this condition. Note that $\nabla h(\lambda^*) = \frac{d^2}{(d-1)^2}\mathbb{1}$. Then for any $\lambda > \mathbb{0}$ such that $\mathbb{1}^T\lambda = 1$

$$\begin{aligned}
\nabla(h(\lambda^*))^T(\lambda - \lambda^*) &= \left(\frac{d^2}{(d-1)^2}\right)\sum_{i=1}^{d}\left(\lambda - \frac{1}{d}\right) \\
&= \left(\frac{d^2}{(d-1)^2}\right)\left(\sum_{i=1}^{d}\lambda - \sum_{i=1}^{d}\frac{1}{d}\right) \\
&= \left(\frac{d^2}{(d-1)^2}\right)(1-1) \\
&= 0.
\end{aligned}$$

So λ^* is an optimal solution to of $(P5)$; its objective function value is $\frac{d}{d-1}$. We conclude that $\frac{d}{d-1}$ is a lower-bound on the optimal value of $(P1)$. Thus

$$LB_0^d(f, v^1, \ldots, v^{2d}) \geq 1 - \frac{1}{1 + \frac{d}{d-1}} = 1 - \frac{d-1}{2d-1} = \frac{d}{2d-1}$$

as claimed. $\square$

For $f \notin (0,1)^d$ with $f_1 < 0$, we calculate the optimal value of the optimization problem obtained from $(P(f, v^1, \ldots, v^{2d}))$ using the intersection cuts for fixed pyramids $T_1, \ldots, T_d$, and $T_{d+2}, \ldots, T_{2d}$.

Lemma 5.1.6. *Let $v^1, \ldots, v^{2d} \in \mathbb{Q}^d$ and $f \in \mathbb{Q}^d \setminus \mathbb{Z}^d$ be such that $\operatorname{conv}(\{v^1, \ldots, v^{2d}\}) \in \hat{\mathcal{F}}_d$ and $f \in \operatorname{conv}(\{v^1, \ldots, v^{2d}\}) \cap \{x \in \mathbb{R}^d : x_1 \leq 0\}$. Let (P') be the linear program obtained from $(P(f, v^1, \ldots, v^{2d}))$ by replacing constraint (5.9) with the constraint $s \geq \mathbb{0}$ and the intersection cuts for $T_1, \ldots, T_d$ and $T_{d+2}, \ldots, T_{2d}$. Then, the optimal value of (P') is equal to*

$$LB_1^d(f, v^1, \ldots, v^{2d}) := 1 - \frac{1}{1 + \sum_{i=2}^{2d} a_i(f, v^i)}$$

for a_i as given in Proposition 5.1.3.

Proof. The optimization problem (P') is given by

$$\begin{aligned}
\min \quad & s_1 + s_2 + \ldots + s_{2d} \\
\text{subject to} \quad & \left[\mathbb{1}\mathbb{1}^T + A\right] s \geq \mathbb{1} \\
& s \geq \mathbb{0}
\end{aligned}$$

where

$$A := \begin{pmatrix} 1 & 1 + \frac{1}{a_2} & 1 & \ldots & 1 \\ 1 & 1 & 1 + \frac{1}{a_3} & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 1 + \frac{1}{a_{2d}} \end{pmatrix}.$$

Note that A is a row sub-matrix of the constraint matrix for Lemma 5.1.4. We denote the dual of this linear program by (D'). It is given by

$$\begin{aligned}
\max \quad & s_1 + s_2 + \ldots + s_{2d} \\
\text{subject to} \quad & \left[\mathbb{1}\mathbb{1}^T + A^T\right] v \leq \mathbb{1} \\
& v \geq \mathbb{0}.
\end{aligned}$$

Consider the (possibly infeasible) solution $\hat{s}$ obtained by setting the intersection cut constraints tight and setting $s_1 = 0$. We can compute

$$\begin{bmatrix} \hat{s}_2 \\ \hat{s}_3 \\ \ddots \\ \hat{s}_{2d} \end{bmatrix} = \left(\frac{1}{1 + \sum_{i=2}^{2d} a_i} \right) \begin{bmatrix} a_2 \\ a_3 \\ \ddots \\ a_{2d} \end{bmatrix}.$$

Since $f_i \in (0, 1)$ for all $i \in \{2, \ldots, d\}$ we have $a_2, \ldots, a_d, a_{d+2}, \ldots, a_{2d} \geq 0$. Since $f_1 < 0$ we have $a_{d+1} \geq 0$. Therefore $\hat{s} \geq \mathbb{0}$.

It is straightforward to check that $\hat{v} := \begin{bmatrix} \hat{s}_2 \\ \hat{s}_3 \\ \ddots \\ \hat{s}_{2d} \end{bmatrix}$ is feasible for (D'). The second through last constraints hold at equality by construction. The first constraint holds because $\frac{\sum_{i=2}^{2d} a_i}{1+\sum_{i=2}^{2d} a_i} <$

1. So $(\hat{s}, \hat{v})$ are a pair of primal-dual feasible solutions satisfying complementary slackness and therefore $\hat{s}$ is optimal for (P') and $\hat{v}$ is optimal for (D'). The optimal value of (P') is

$$\mathbb{1}^T \hat{s} = \frac{\sum_{i=2}^{2d} a_i}{1 + \sum_{i=2}^{2d} a_i} = 1 - \frac{1}{1 + \sum_{i=2}^{2d} a_i}$$

as required. □

Theorem 5.1.2 follows quickly from the above lemmas.

Theorem 5.1.7 (Theorem 5.1.2)**.** *The optimal value of* $(P(f, v^1, \ldots, v^{2d}))$ *is at least* $\frac{d}{2d-1}$ *for all* $v^1, \ldots, v^{2d} \in \mathbb{Q}^d$ *and* $f \in \mathbb{Q}^d \setminus \mathbb{Z}^d$ *such that* $O := \mathrm{conv}(\{v^1, \ldots, v^{2d}\}) \in \hat{\mathcal{F}}_d$ *and* $f \in \mathrm{int}\, O$*. Therefore* $\rho[\mathcal{P}_d, \mathcal{F}_d] \leq 2 - \frac{1}{d}$*.*

Proof. Since we vary over all possible $v^1, \ldots, v^{2d}$ we may use rotational symmetry to consider two cases: (1) $f \in (0,1)^d$ and (2) $f \in \mathrm{conv}(\{v^1, \ldots, v^{2d}\}) \cap \{x \in \mathbb{R}^d : x_1 \leq 0\}$. The bound holds in case (1) by Lemma 5.1.5. For (2), consider the bound $LB_1^d(f, v^1, \ldots, v^{2d})$ provided by Lemma 5.1.6. Note that f_1 only appears in $\sum_{i=2}^{2d} a_i$ in the term a_{d+1} where it has negative coefficient $\frac{-1}{v_1^{d+1}-1}$. Given f, let f' be the vector obtained from f by setting the first coordinate to 0. Then

$$LB_1^d(f, v^1, \ldots, v^{2d}) \geq LB_1^d(f', v^1, \ldots, v^{2d}) \geq \frac{d}{2d-1}$$

where the second inequality holds by Lemma 5.1.5 since $f' \in [0,1]^d$.

It follows that the optimal value of $(S(\mathcal{P}_d, \mathcal{F}_d))$ is at least $\frac{d}{2d-1}$. The bound on $\rho[\mathcal{P}_d, \mathcal{F}_d]$ follows immediately from Theorem 2.1.1. □

5.2 Octahedra as a Generalization of Quadrilaterals

The family of all maximal lattice-free octahedra in $\mathbb{R}^2$ is exactly the family of all maximal lattice-free quadrilaterals. Every maximal lattice-free octahedron in $\mathbb{R}^2$ is a unit-core octahedron. Every normalized octahedron in $\mathbb{R}^2$ has the facet separation property. Here we investigate if these characterizations extend to $\mathbb{R}^d$.

In Section 5.2.1, we provide additional definitions. Notably, we define the family of symmetric octahedra in $\mathbb{R}^d$, which generalizes the family of parallelograms in $\mathbb{R}^2$. In Section 5.2.2

we investigate the question: are all maximal lattice-free octahedra in $\mathbb{R}^d$ unit-core octahedra? In Section 5.2.3 we investigate the question: do all normalized octahedra in $\mathbb{R}^d$ have the facet separation property? We investigate these questions for general octahedra in $\mathbb{R}^d$, as well as the restrictions to symmetric octahedra and to octahedra in $\mathbb{R}^3$; not to bury the lede, the answers to these questions remain open.

5.2.1 Antipodal Facets and Vertices, Symmetric Octahedra

As in Section 5.1.1, let $\hat{O}_d$ denote the canonical octahedron in $\mathbb{R}^d$.

The vertices of $\hat{O}_d$ are $\{\pm e^1, \pm e^2, \ldots, \pm e^d\}$. For $i \in \{1, \ldots, d\}$, let F_i and G_i, respectively, denote the facets of $\hat{O}_d$ containing e^i and $-e^i$, respectively. It is easy to verify that (F_i, G_i) is a partition of the facets of $\hat{O}_d$. We call e^i and $-e^i$ *antipodal vertices* of $\hat{O}_d$. Since any octahedron O has the same face lattice as $\hat{O}_d$, for each vertex v of O there exists a unique vertex $\hat{v}$ of O such that the facets containing v and $\hat{v}$ are a partition of the facets of O. In this case we call v and $\hat{v}$ *antipodal vertices* of O. For example, in Figure 5.1, the vertices labeled 4 and 5 in the face lattice image are antipodal vertices.

The facets of $\hat{O}_d$ are $\{F_\sigma : \sigma \in \{-1, +1\}^d\}$ where $F_\sigma := \{x \in \hat{O}_d : \sigma^T x = 1\}$. For $\sigma \in \{-1, +1\}^d$, let V_σ and $V_{-\sigma}$, respectively, denote the vertices of $\hat{O}_d$ contained in F_σ and $F_{-\sigma}$. It is easy to verify that $(V_\sigma, V_{-\sigma})$ is a partition of the vertices of $\hat{O}_d$. We call F_σ and $F_{-\sigma}$ *antipodal facets* of $\hat{O}_d$. Since any octahedron O has the same face lattice as $\hat{O}_d$, for each facet F of O there exists a unique facet $\hat{F}$ of O such that the vertices in F and the vertices in $\hat{F}$ are a partition of the vertices of O. In this case we call F and $\hat{F}$ *antipodal facets* of O. For example, in Figure 5.1, the facets labeled 034 and 125 in the face lattice image are antipodal facets.

Polytope $O \subseteq \mathbb{R}^d$ is a *symmetric octahedron* if there exists an affine invertible map L such that $O = L(\hat{O}_d)$. Note that symmetric octahedra are indeed octahedra because a polytope and its image under an affine invertible map have the same face lattice. For $i \in \{1, \ldots, d\}$, $L(e^i)$ and $L(-e^i)$ are antipodal vertices of O. For $\sigma \in \{-1, +1\}^d$, $L(F_\sigma)$ and $L(F_{-\sigma})$ are antipodal facets of O. The family of all (maximal lattice-free) symmetric octahedra in $\mathbb{R}^2$ is exactly the family of all (maximal lattice-free) parallelograms.

5.2.2 Unit-Core Octahedra in $\mathbb{R}^d$

Let B be a lattice-free convex set. The *lattice core* of B is the convex hull of the integral points in B; i.e. $\mathrm{conv}(B \cap \mathbb{Z}^d)$. This is typically called the *integer hull* of B. We use different terminology here to highlight one distinction - the integer hull of an arbitrary convex set may contain integral points in its interior, but the lattice core of a lattice-free convex set will never contain integral points in its interior. Using this terminology, an octahedron in $\mathbb{R}^d$ is a *unit-core octahedron* if its lattice core is a parallelepiped of volume one.

By Theorem 1.2.8 the lattice core of a maximal lattice-free octahedron in $\mathbb{R}^2$ is a parallelogram of area (i.e. volume) one. One way to prove this characterization holds in $\mathbb{R}^2$ is using *parity bases* as an intermediate object. The *parity function* parity : $\mathbb{Z}^d \to \{-,+\}^d$ is given by

$$[\text{parity}(w)]_j = \begin{cases} +, & \text{if } w_j \text{ is even} \\ -, & \text{if } w_j \text{ is odd.} \end{cases}$$

A finite set $W \subseteq \mathbb{Z}^d$ is a *parity basis* if

(i) $|W| = 2^d$,
(ii) $\{\text{parity}(w) : w \in W\} = \{-,+\}^d$, and
(iii) $\text{conv}(W) \cap \mathbb{Z}^d = W$.

It is straightforward to show that parity bases are closed under affine unimodular transformation. The set of integral points in a maximal lattice-free octahedron form a parity basis.

Proposition 5.2.1. *Let $P \subseteq \mathbb{R}^d$ be a maximal lattice-free polyhedron with 2^d facets. Then $W := P \cap \mathbb{Z}^d$ is a parity basis.*

Proof. Let F_1 and F_2 be any two distinct facets of P. By Theorem 1.2.6, there exist integral points $w_1 \in \text{relint}\, F_1$ and $w_2 \in \text{relint}\, F_2$. If $\text{parity}(w^1) = \text{parity}(w^2)$ then $\frac{1}{2}w^1 + \frac{1}{2}w^2$ is an integral point in the interior of P. This contradicts the fact P is lattice-free. Since P has 2^d facets it follows by the Pigeonhole Principle that all the integral points in a given facet of P have the same parity.

Let F be a facet of P. All the elements of $F \cap \mathbb{Z}^d$ have the same parity. Next we show that $|F \cap \mathbb{Z}^d| = 1$. For a contradiction, suppose $|F \cap \mathbb{Z}^d| \geq 2$. Pick $z^1 \in F \cap \mathbb{Z}^d$ arbitrarily. Then we can choose $z^2 \neq z^1 \in F \cap \mathbb{Z}^d$ such that $||z^1 - z^2||$ is smallest (breaking a tie arbitrarily). Since every element of $F \cap \mathbb{Z}^d$ has the same parity, $\text{parity}(z^1) = \text{parity}(z^2)$. Let $\overline{z} := \frac{1}{2}z^1 + \frac{1}{2}z^2$. Now $\overline{z} \in F \cap \mathbb{Z}^d$ and $||z^1 - \overline{z}|| < ||z^1 - z^2||$. This contradicts our choice of z^2. We conclude $|F \cap \mathbb{Z}^d| = 1$.

Now P has 2^d facets and each facet of P contains exactly one integral point; thus $|W| = 2^d$. Moreover, since the integral point in each facet has a different parity, $\{\text{parity}(w) : w \in$

$W\} = \{-,+\}^d$. Lastly we note that since P is lattice-free, there are no integral points in the interior of $\operatorname{conv}(W)$. Thus, $\operatorname{conv}(W) \cap \mathbb{Z}^d = W$. We conclude that W is indeed a parity basis. $\square$

Corollary 5.2.2. *Let $O \in \mathcal{O}_d$ be a maximal lattice-free octahedron. Then $W := O \cap \mathbb{Z}^d$ is a parity basis and therefore $|W| = 2^d$.*

The following argument shows that if $O \subseteq \mathbb{R}^2$ is a maximal lattice-free quadrilateral, then its lattice core is a parallelogram of area one.

1. By Corollary 5.2.2, O contain four integral points $\{v^1, v^2, v^3, v^4\}$ - one in the relative interior of each of its facets. Moreover, $\{v^1, v^2, v^3, v^4\}$ form a parity basis.
2. For any parity basis $\{w^1, w^2, w^3, w^4\}$ in $\mathbb{R}^2$, the set $\operatorname{conv}(\{w^1, w^2, w^3, w^4\})$ is a parallelogram of area one (follows from Pick's Theorem, for example).
3. Therefore, the lattice core of O, which is $\operatorname{conv}(\{v^1, v^2, v^3, v^4\})$, is a parallelogram of area one.

This argument does not directly generalize to d dimensions. Although 1. holds, 2. does not hold even in $\mathbb{R}^3$.

Howe's Theorem and Parity Bases in Higher Dimensions

Let $\{w^1, w^2, \ldots, w^8\}$ be a parity basis in $\mathbb{R}^3$. Note that $\operatorname{conv}(\{w^1, w^2, \ldots, w^8\})$ is a full-dimensional polytope with integral vertices and no integral points in its interior. The following theorem can be used to construct examples where $\operatorname{conv}(\{w^1, w^2, \ldots, w^8\})$ is not a parallelepiped of volume one.

Theorem 5.2.3 (Howe's Theorem [67])**.** *Let $P \subseteq \mathbb{R}^3$ be a full dimensional polytope that has integral vertices and no integral points in its interior. If P has eight vertices, then there is a unimodular transformation from P to a polytope B with vertices*

$$\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\\beta_1\\\alpha_1\end{pmatrix}, \begin{pmatrix}1\\\beta_2\\\alpha_2\end{pmatrix}, \textit{ and } \begin{pmatrix}1\\\beta_1+\beta_2\\\alpha_1+\alpha_2\end{pmatrix}$$

where $\beta_1, \beta_2, \alpha_1, \alpha_2 \in \mathbb{Z}_+$, $\beta_1(\alpha_1+\alpha_2) - \alpha_1(\beta_1+\beta_2) = 1$, and $\beta_1+\beta_2$ and $\alpha_1+\alpha_2$ are relatively prime.

Let B be the polytope with vertices as given in Howe's Theorem. Note that the vertices of B form a parity basis. Restricting to the vertices on the $x_1 = 0$ hyperplane, we get an

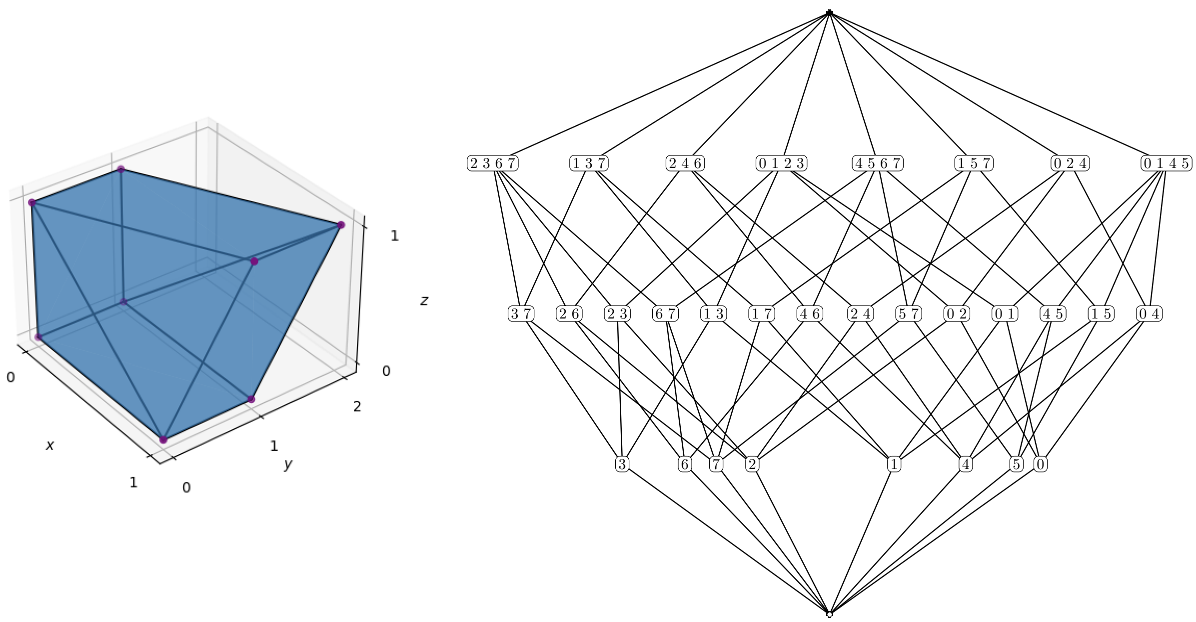


Figure 5.4: Polytope B as in Howe's Theorem for $(\beta_1, \alpha_1, \beta_2, \alpha_2) = (1, 0, 1, 1)$ (left) and its face lattice (right).

embedding of the square $[0, 1]^2$. Restricting to the vertices on the $x_1 = 1$ hyperplane, we get an embedding of a parallelogram with edges $(\beta_1, \alpha_1)^T$ and $(\beta_2, \alpha_2)^T$. Thinking of B as the convex hull of these two-dimensional parallelograms embedded at distance one in three space, we see that polytope B is not a parallelepiped for many choices of $\beta_1, \alpha_1, \beta_2$, and α_2. For example, we can choose $(\beta_1, \alpha_1, \beta_2, \alpha_2) \in \{(1, 0, 1, 1), (1, 1, 5, 3)\}$. See Figure 5.4 and Figure 5.5.

Now, polytope B contains the simplex S with vertices $(1, 0, 0)^T$, $(0, 1, 0)^T$, $(0, 0, 1)^T$ and $(1, \beta_1, \alpha_1)^T$. As noted in [67], S has volume $\frac{1}{6}(\beta_1 + \alpha_1)$. Choosing $(\beta_1, \alpha_1, \beta_2, \alpha_2) = (3, 5, 13, 22)$ satisfying the hypotheses of Howe's Theorem, S has volume $\frac{4}{3} > 1$ and therefore B has volume greater than one.

Now suppose W is a parity basis and $P = \operatorname{conv}(W)$ is a parallelepiped. The image of W under any invertible map is a parallelepiped and therefore the image of W under the unimodular transformation provided by Howe's Theorem is a parallelepiped. Since polytope B in Howe's Theorem is a parallelepiped only when $(\beta_1, \alpha_1, \beta_2, \alpha_2) = (1, 0, 0, 1)$ and unimodular transformations preserve volume, it follows that $\operatorname{conv}(W)$ must have volume one. To summarize: if W is a parity basis in $\mathbb{R}^3$, then $\operatorname{conv}(W)$ needn't be a parallelepiped

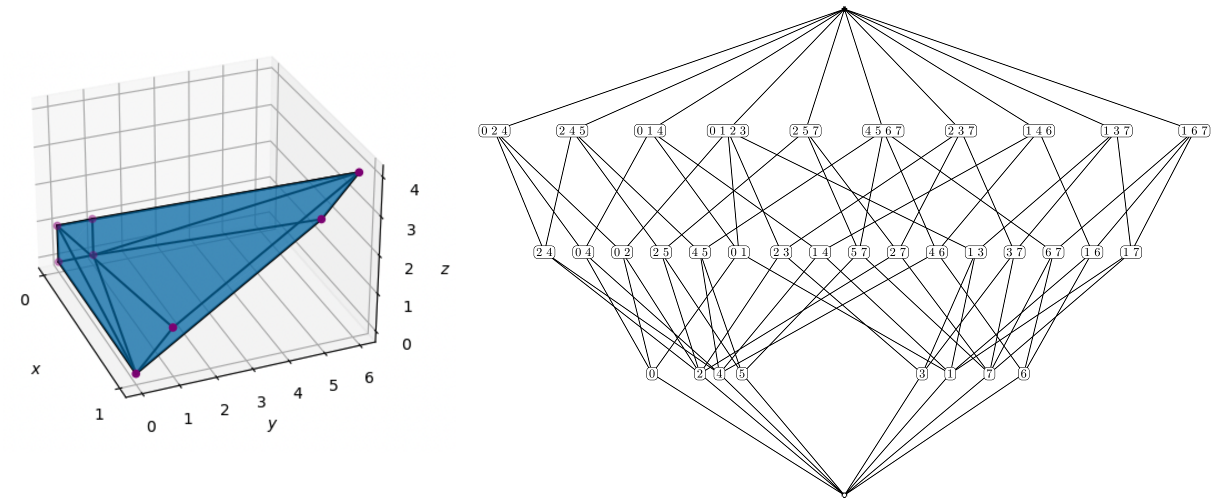


Figure 5.5: Polytope B as in Howe's Theorem for $(\beta_1, \alpha_1, \beta_2, \alpha_2) = (1, 1, 5, 3)$ (left) and its face lattice (right).

and $\operatorname{conv}(W)$ needn't have volume one, but if $\operatorname{conv}(W)$ is a parallelepiped then it must have volume one.

The construction in Howe's Theorem can be extended to $d > 3$. For $d = 4$, take two arbitrary parity bases C_1 and C_2 for $\mathbb{Z}^3$. Let $C := \operatorname{conv}(\{(c_1, 0) : c_1 \in C_1\} \cup \{(c_2, 1) : c_2 \in C_2\})$. Then, it is easy to check C is a parity basis in $\mathbb{Z}^4$ and we can make sure $\operatorname{conv} C$ is not a parallelepiped and $\operatorname{volume}(\operatorname{conv} C) >> 1$. We can recursively construct more complicated parity bases for larger d. However, not all parity bases in $\mathbb{R}^d$ for $d \geq 4$ can be constructed in this way. There exist parity bases in $\mathbb{R}^4$ whose members do not lie on two adjacent lattice planes; for more information see [67].

Are All Maximal Lattice-Free Octahedra Unit-Core Octahedra?

The above discussion shows that we can't generalize the $\mathbb{R}^2$ argument presented after Corollary 5.2.2 to $\mathbb{R}^d$. However, it does not answer the question at hand.

Open Question 5.2.4. *Is every maximal lattice-free octahedron in $\mathbb{R}^d$ a unit-core octahedron? Is every maximal lattice-free octahedron in $\mathbb{R}^3$ a unit-core octahedron? Is every symmetric maximal lattice-free octahedron a unit-core octahedron?*

For symmetric octahedra in $\mathbb{R}^3$, one strategy we used to approach this question was to parameterize the problem and try to determine if the underlying system was consistent.

Parameterize a symmetric octahedron by $(C,b) \in \mathbb{R}^{3\times 3} \times \mathbb{R}^3$ with C non-singular so that

$$O_{(C,b)} := \{x \in \mathbb{R}^3 : (C\sigma)^T x \leq 1 + \sigma^T b \;\; \forall \sigma \in \{-1,+1\}^3\}.$$

Then, Open Question 5.2.4 can be posed as follows: for which $\beta_1, \alpha_1, \beta_2, \alpha_2 \in \mathbb{Z}_{\geq 0}$ with $\beta_1(\alpha_1+\alpha_2) - \alpha_1(\beta_1+\beta_2) = 1$ does there exist $C \in \mathbb{R}^{3\times 3}$ non-singular and $b \in \mathbb{R}^3$ such that $O_{(C,b)}$ is a maximal lattice-free octahedron with

$$O_{(C,b)} \cap \mathbb{Z}^3 = \underbrace{\left\{ \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\\beta_1\\\alpha_1\end{pmatrix}, \begin{pmatrix}1\\\beta_2\\\alpha_2\end{pmatrix}, \begin{pmatrix}1\\\beta_1+\beta_2\\\alpha_1+\alpha_2\end{pmatrix} \right\}}_{Q(\beta_1,\alpha_1,\beta_2,\alpha_2)=:Q}?$$

For $\sigma \in \{-1,+1\}^3$, let $F_\sigma := \{x \in O_{(C,b)} : (C\sigma)^T x = 1 + \sigma^T b\}$; the facets of $O_{(C,b)}$ are exactly $\{F_\sigma : \sigma \in \{-1,+1\}^3\}$. If

$$C := \begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{pmatrix},\; b := \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},\; \text{and } \sigma := \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix},$$

the inequality $(C\sigma)^T x \leq 1 + \sigma^T b$ defining facet F_σ can be expressed as

$$x_1(\sigma_1c_1+\sigma_2c_2+\sigma_3c_3)+x_2(\sigma_1c_4+\sigma_2c_5+\sigma_3c_6)+x_3(\sigma_1c_7+\sigma_2c_8+\sigma_3c_9)-(\sigma_1b_1+\sigma_2b_2+\sigma_3b_3) \leq 1.$$

If $O_{(C,b)} \cap \mathbb{Z}^3 = Q$, then there exists $\phi : Q \to \{-1,+1\}^3$ such that q is on facet $F_{\phi(q)}$ for all $q \in Q$. Suppose without loss of generality that $\phi((0,0,0)^T) = (-1,-1,-1)$. Then

1. $b_1 + b_2 + b_3 = 1$, and
2. $\sigma_1 b_1 + \sigma_2 b_2 + \sigma_3 b_3 < 1$ for all $\sigma \in \{-1,+1\}^3 \setminus \{(-1,-1,-1)\}$.

Here 1. is obtained by substituting $(0,0,0)^T$ into the constraint defining facet $F_{(-1,-1,-1)}$. We obtain 2. since $(0,0,0)^T$ is on none of the facets F_σ for $\sigma \in \{-1,+1\}^3 \setminus \{(-1,-1,-1)\}$ but $(0,0,0)^T \in O_{(C,b)}$. More generally, we can derive such equations and inequalities for all $q \in Q$. Let $q = (q_1,q_2,q_3)^T \in Q$ and suppose $\phi(q) = \sigma$ for $\sigma = (\sigma_1,\sigma_2,\sigma_3)^T \in \{-1,+1\}^3$. Then the equation

$$q_1(\sigma_1c_1+\sigma_2c_2+\sigma_3c_3)+q_2(\sigma_1c_4+\sigma_2c_5+\sigma_3c_6)+q_3(\sigma_1c_7+\sigma_2c_8+\sigma_3c_9)-(\sigma_1b_1+\sigma_2b_2+\sigma_3b_3) = 1$$

must hold. Moreover, for all $r = (r_1,r_2,r_3)^T \in Q \setminus \{q\}$, the strict inequality

$$r_1(\sigma_1c_1+\sigma_2c_2+\sigma_3c_3)+r_2(\sigma_1c_4+\sigma_2c_5+\sigma_3c_6)+r_3(\sigma_1c_7+\sigma_2c_8+\sigma_3c_9) -(\sigma_1b_1+\sigma_2b_2+\sigma_3b_3) < 1$$

must hold. To determine candidate $\beta_1, \alpha_1, \beta_2, \alpha_2 \in \mathbb{Z}_{\geq 0}$, we can first determine candidate bijections $\phi : Q \to \{-1, +1\}^3$ such that the implied equations and strict inequalities could be consistent. When the resulting system must be inconsistent, a version of Farkas' Lemma can be used to provide a certificate of inconsistency. Many of these bijections can be discarded, providing information about the way in which Q must be embedded in $O_{(C,b)}$. We won't be specific; most of the information can be described (and justified) in a geometric way as well. For example, one may limit the pairs of vertices of Q that may be embedded into antipodal facets of O.

Such an approach doesn't generalize directly to $\mathbb{R}^d$ because no simple characterization of parity bases is known. It also doesn't generalize directly to general octahedra in $\mathbb{R}^3$ since the normals of the facets cannot be described with so few parameters. Some candidate bijections may be discarded in this way using the underlying geometric characterization directly. However, restricting possible bijections of elements of Q to facets of O is only the first step in this approach.

Lastly we remark that we require unit core octahedra in Section 5.1 because a set of representatives (with respect to unimodular transformation) for the family of maximal lattice-free octahedra is required. Alternatively, we could construct a different set of representatives. If so, the definition of fixed pyramid would need to be modified; in $\mathbb{R}^3$ this might be possible since Howe's Theorem gives a good description of the embedded lattice polytope. In higher dimensions this seems less tractable.

5.2.3 The Facet Separation Property in $\mathbb{R}^d$

We are interested in the facet separation property in $\mathbb{R}^d$ and specifically the answer to the following question.

Open Question 5.2.5. *Is every unit-core octahedron in $\mathbb{R}^d$ facet-separable? Is every unit-core octahedron in $\mathbb{R}^3$ facet-separable? Is every symmetric unit-core octahedron facet-separable?*

The following counterexample shows that enforcing the local structure of an octahedron at one vertex does not suffice to obtain the facet separation property.

Untruth 5.2.6. *Let $w \in \mathbb{R}^d$. Let $D = \{1\} \times \{0,1\}^{d-1}$. Let $P = \{w\} + C$ be the Minkowski sum of $\{w\}$ and some polyhedral cone C with 2^{d-1} facets. Let the facets of P be denoted $F_1, \dots, F_{2^{d-1}}$. Let $\phi : D \to \{F_i : i \in \{1, \dots, 2^{d-1}\}\}$ be a bijection such that $g \in \operatorname{relint}(\phi(g))$ for all $g \in D$. If $[0,1]^d \subseteq P$, then $w_1 > 1$ and $0 < w_j < 1$ for all $j \in \{2, \dots d\}$.*

Counterexample 5.2.7. *Let*

1. $a_1 = (6, -3, -1)$, $b_1 = 6$,
2. $a_2 = (1, -3, 2)$, $b_2 = 3$,
3. $a_3 = (6, 3, -1)$, $b_3 = 9$, *and*
4. $a_4 = (1, 2, 1)$, $b_4 = 4$.

Let $P = \{x \in \mathbb{R}^3 : a_i x \leq b_i \ \forall i \in \{1, 2, 3, 4\}\}$. *Now* $P = \{w\} + C$ *for* $w = (\frac{3}{2}, \frac{1}{2}, \frac{3}{2})^T$ *and* $C = \{x \in \mathbb{R}^3 : a_i^T x \leq 0 \ \forall i \in \{1, 2, 3, 4\}\}$ *(a polyhedral cone with* $4 = 2^{3-1}$ *facets). Let* $F_1 = (P \cap \{x \in \mathbb{R}^3 : a_1^T x = b_1\})$. *Then* $F_1 \cap \{0, 1\}^3 = \{(1, 0, 0)^T\}$. *Let* $F_2 = (P \cap \{x \in \mathbb{R}^3 : a_2^T x = b_2\})$. *Then* $F_2 \cap \{0, 1\}^3 = \{(1, 0, 1)^T\}$. *Let* $F_3 = (P \cap \{x \in \mathbb{R}^3 : a_3^T x = b_3\})$. *Then* $F_3 \cap \{0, 1\}^3 = \{(1, 1, 0)^T\}$. *Let* $F_4 = (P \cap \{x \in \mathbb{R}^3 : a_4^T x = b_4\})$. *Then* $F_4 \cap \{0, 1\}^3 = \{(1, 1, 1)^T\}$. *The unit cube* $[0, 1]^3 \subseteq P$. *However,* $w_3 > 1$, *showing that this is indeed a counterexample.*

If a normalized octahedron has the facet separation property, then the following (weaker) property holds. Let $O \subseteq \mathbb{R}^d$ be a normalized octahedron with vertices $v^1, \ldots, v^{2d}$. Let $f \in (0, 1)^d$. Let L_i be the line segment joining f to v^i for all $i \in \{1, \ldots, 2d\}$. Let $G_1, \ldots, G_{2d}$ denote the facets of $[0, 1]^d$. We say O has the *ray-piercing property* if there is a bijection $\phi : \{L_1, \ldots, L_{2d}\} \to \{G_1, \ldots, G_{2d}\}$ such that $L_i \cap \phi(L_i) \neq \emptyset$ for all $i \in \{1, \ldots, 2d\}$ and $L_i \cap \phi(L_j) = \emptyset$ for all $i \neq j \in \{1, \ldots, 2d\}$. This property is, in fact, sufficient for the proof in Section 5.1 to work. Either proving the ray-piercing property holds for all unit core octahedra, or showing the ray-piercing property and facet separation property are equivalent would provide more insight into the family $\mathcal{F}_d$.

5.2.4 The Unit-Core and Facet Separation Properties for Other Polyhedra

Both the unit-core and facet separation properties are not unique to octahedra and can be defined more generally.

The unit-core property can be generalized to maximal lattice-free convex sets in $\mathbb{R}^d$ with exactly 2^d facets. Although quadrilaterals are the only maximal lattice-free polyhedra in $\mathbb{R}^2$ with exactly four facets, octahedra are not the only maximal lattice-free polyhedra in $\mathbb{R}^3$ with exactly eight facets. One example is provided in Figure 5.6. For more examples see [18]. As per Proposition 5.2.1, the set of integral points in a maximal lattice-free polyhedron P with 2^d facets is a parity basis. For any such P, we again ask if $\text{conv}(P \cap \mathbb{Z}^d)$ is a parallelepiped of volume one (i.e. if P is *unit-core*).

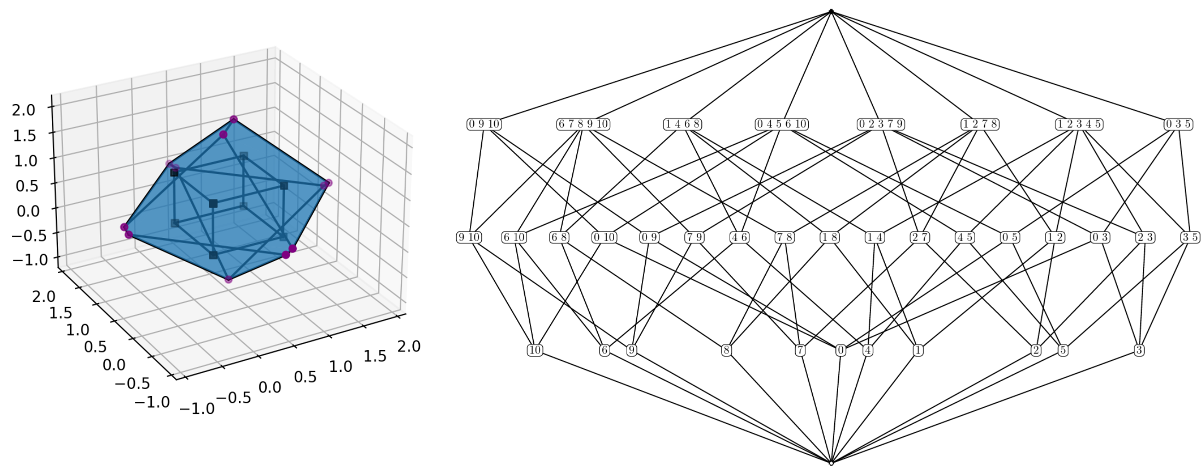


Figure 5.6: The polytope on the left has 8 facets, is maximally lattice-free, and unit core. It is not an octahedron; its face lattice is provided on the right.

Open Question 5.2.8. *Is every maximal lattice-free convex set in $\mathbb{R}^d$ with 2^d facets unit-core? Is every maximal lattice-free convex set in $\mathbb{R}^3$ with 8 facets unit-core? Is every symmetric maximal lattice-free convex set in $\mathbb{R}^d$ with 2^d facets unit-core?*

The facet separation property is not specific to polyhedra which are maximally lattice-free, but rather applies more generally to any polyhedron and its embedded dual. Let $P \subseteq \mathbb{R}^d$ be a polytope with face lattice $\mathcal{F}(P)$. We say polytope $Q \subseteq \mathbb{R}^d$ is a *combinatorial dual* of P if its face lattice $\mathcal{F}(Q)$ is the opposite lattice to $\mathcal{F}(P)$; that is, if $\mathcal{F}(Q)$ is isomorphic to $\mathcal{F}(P)^{\text{op}}$. Suppose the facets of P are $F_1, \dots, F_k$ and the vertices of its combinatorial dual Q are $v_1, \dots, v_k$. Let $\phi : \mathcal{F}(P)^{\text{op}} \to \mathcal{F}(Q)$ be the bijection between $\mathcal{F}(Q)$ and $\mathcal{F}(P)^{\text{op}}$ showing Q is a combinatorial dual of P. Note that ϕ maps facets of P to vertices of Q. By relabelling, we may assume $\phi(F_i) = v_i$ for all $i \in \{1, \dots, k\}$. We say Q is *combinatorially embedded* in P if $v_i \in F_i$ for all $i \in \{1, \dots, k\}$. We say the *facet separation property* holds for (P, Q) if there is a unique facet defining inequality of Q separating each vertex of P from Q.

Open Question 5.2.9. *When does the facet-separation property hold for polytopes and their embedded combinatorial duals?*

5.3 Observations and Summary

Theorem 5.1.2 serves as an analogue to Theorem 2.2.4 in many ways:

1. we consider octahedra, which are maximal lattice-free convex sets in $\mathbb{R}^d$ with the maximum number of facets;
2. to approximate a given octahedron cut, we pick one type 2 pyramid for each vertex of the octahedron; and
3. we obtain an approximation factor which is constant (bounded above by 2 for any d).

One shortcoming, however, is that the expressiveness of the sets involved is not clear. Besides questions regarding the structural requirements on $\mathcal{F}_d$ captured in Section 5.2, it is also unclear how well the $\mathcal{F}_d$-closure approximates the the corner polyhedron.

We mention two existing d-dimensional inapproximability results to put this constant factor approximation in context. The work in [8] compares the i-hedral closures for various $i \in \{1, \dots, 2^d\}$. Here the *i-hedral closure* is obtained from all intersection cuts for lattice-free polyhedra with at most i facets.

Theorem 5.3.1 ([8, Theorem 2]). *For $i \in \{1, \dots, 2^d\}$ let $\mathcal{L}_i^d$ denote the family of lattice-free (not necessarily maximal) polyhedra in $\mathbb{R}^d$ with at most i facets. Let $\mathcal{L}_*^d$ denote the family of all lattice-free (not necessarily maximal) polyhedra in $\mathbb{R}^d$. If $i \leq 2^{d-1}$ then $\rho[\mathcal{L}_i^d, \mathcal{L}_{i+1}^d] = \infty$. If $i \geq 2^{d-1} + 1$ then $\rho[\mathcal{L}_i^d, \mathcal{L}_*^d] \leq 4d^{\frac{5}{2}}$.*[2]

The results in [3] also compare the closures for different families of intersection cuts in $\mathbb{R}^d$ for $d \geq 3$. Colloquially, the main inapproximability result shows that one must use lattice-free convex sets that do not contain a line to obtain a good approximation of the corner polyhedron. Both the families we consider ($\mathcal{P}_d$ and $\mathcal{F}_d$) have at least $2^{d-1} + 1$ facets and are bounded.

[2] Or more accurately $4Flt(d)$ were $Flt(d)$ denotes the flatness constant, the maximum lattice-width of a lattice-free set.

Chapter 6

Conclusions and Future Work

In the first two sections we provide some additional information and open questions about comparing intersection cuts generated by parallelograms and type 3 triangles to other families of cuts. Thereafter, we turn to questions of a broader scope.

6.1 Parallelograms

The results in this thesis show that $\rho[\mathcal{T}_2, \mathcal{Q}_2] = \rho[\mathcal{T}_3, \mathcal{Q}_2] = \rho[\mathcal{T}, \mathcal{Q}_2] = \frac{9}{8}$. Clearly $\rho[\mathcal{Q}, \mathcal{Q}_2] = 1$ by Remark 1.3.5. The values of $\rho[\mathcal{L}, \mathcal{Q}_2]$ for the remainder of the families discussed in Section 1.3.1 follow quickly from existing results.

Remark 6.1.1. *Let $\mathcal{Q}_2$ denote the family of maximal lattice-free parallelograms in $\mathbb{R}^2$. Let $\mathcal{S}$ denote the family of splits in $\mathbb{R}^2$. Let $\mathcal{T}_1$ denote the family of type 1 triangles in $\mathbb{R}^2$. Then: (1) $\rho[\mathcal{Q}_2, \mathcal{S}] = 1$, (2) $\rho[\mathcal{T}_1, \mathcal{Q}_2] = \infty$, and (3) $\rho[\mathcal{S}, \mathcal{Q}_2] = \infty$.*

Proof. The right image of Figure 1.7 illustrates that a vertical split can be approximated by a normalized parallelogram with two opposite vertices as close, and equally close, to $(0, \frac{1}{2})^T$ and $(1, \frac{1}{2})^T$ as necessary. Therefore (1) holds. Then (2) follows by transitivity of bounds on ρ: if $\rho[\mathcal{T}_1, \mathcal{Q}_2] < \infty$ then $\rho[\mathcal{T}_1, \mathcal{S}] < \infty$ but we have $\rho[\mathcal{T}_1, \mathcal{S}] = \infty$ by Proposition 1.3.12. The construction of quadrilaterals which are "hard to approximate by a split" described following Theorem 1.3.11 does not yield a family of parallelograms. However, the techniques in [15] can be extended in a straightforward way to provide such an example, showing (3) holds. $\square$

Other than $\rho[\mathcal{Q}_2, \mathcal{S}]$, however, we cannot immediately obtain $\rho[\mathcal{Q}_2, \mathcal{L}]$ for other families of maximal lattice-free convex sets in $\mathbb{R}^2$ via a straightforward extension of existing techniques. We are particularly interested in these values to understand the strength of the simpler family $\mathcal{Q}_2$ in approximating the corner polyhedron.

Open Question 6.1.2. *Calculate or bound $\rho[\mathcal{Q}_2, \mathcal{L}]$ for $\mathcal{L} \in \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{Q}\}$.*

6.2 Type 3 Triangles

To tighten the upper bound $\rho[\mathcal{T}_2, \mathcal{C}] \leq \frac{3}{2}$, it would suffice to prove $\rho[\mathcal{T}_2, \mathcal{T}_3] \leq K$ for some $K < \frac{3}{2}$. One approach to doing so is to use the proof strategy in Section 2.1 and try to define an analogue of ray-sliding triangles with respect to a given normalized type 3 triangle. Any normalized type 3 triangle can be described by the slopes of its edges as illustrated in Figure 6.1.

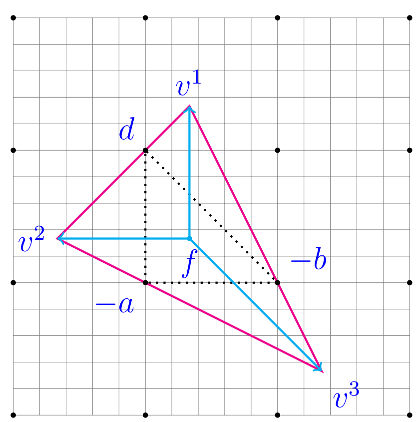


Figure 6.1: A type 3 triangle may be parameterized with respect to the slopes of its edges $-a, -b, d$ for $a, b, d > 0$ with $a < 1$ and $b > 1$ as illustrated here for $a = \frac{1}{2}$, $b = 2$, and $d = 1$. A fractional point in its interior $f = (g, h)^T$ for $g = \frac{1}{3}$ and $h = \frac{1}{3}$ is also shown.

A ray-sliding triangle can be defined with respect to a normalized type 3 triangle in a straightforward way. Only three families of ray-sliding triangles are defined: one for each ray direction $r^i = v^i - f$. Figure 6.2 illustrates two members of the family of ray-sliding triangles for ray 1. One vertex of such a triangle is $f + \tau_1 r^1$ for τ_1 within appropriate bounds and the opposite edge is defined by $\{(x,y)^T \in \mathbb{R}^2 : x+y=0\}$. A ray-sliding triangle for ray 2 has one vertex equal to $f+\tau_2 r^2$ for τ_2 within appropriate bounds and the opposite edge defined by $\{(x,y)^T \in \mathbb{R}^2 : x=1\}$. A ray-sliding triangle for ray 3 has one vertex equal to $f+\tau_3 r^3$ for τ_3 within appropriate bounds and the opposite edge defined by $\{(x,y)^T \in \mathbb{R}^2 : y=1\}$.

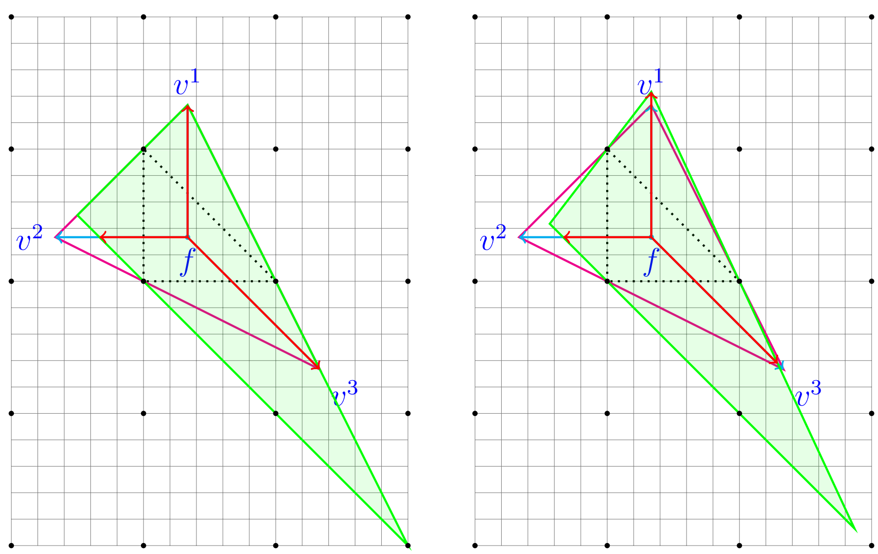


Figure 6.2: Examples of fixed triangle (left) and ray-sliding triangle (right) for ray 1 defined with respect to the base type 3 triangle and fractional point in Figure 6.1.

The next step in the proof strategy is to select the ray-sliding triangles whose intersection cuts will replace the type 2 triangle closure constraint. In the quadrilateral case, the ray-sliding triangles chosen gave a worse intersection cut coefficient on the **one** sliding-ray to obtain better cut coefficients on **two** off-rays (i.e. choosing $\tau < 1$). No analogous choice can be made for type 3 triangles: weakening the coefficient on the **one** sliding-ray strengthens the coefficient on the **one** off-ray. In fact, if we define natural geometric ray breakpoints for these ray-sliding triangles then one such breakpoint corresponds to $\tau > 1$. One ray breakpoint is obtained from decreasing $\tau < 1$ until there is no gain on the coefficient for the unique off-ray. The other ray breakpoint is obtained from increasing $\tau > 1$ until the coefficient for the base-ray is no longer determined by its intersection with the base of the type 2 triangle. See Figure 6.3.

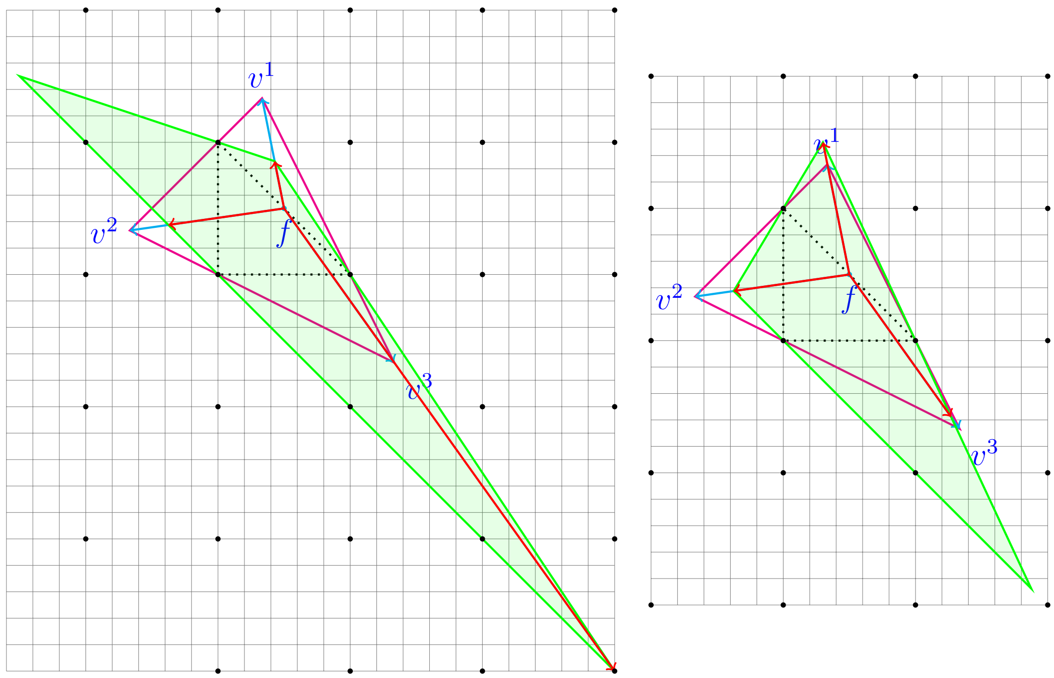


Figure 6.3: Examples of ray-sliding triangles for ray 1 at the breakpoint for $\tau < 1$ (left) and $\tau > 1$ (right). The base type 3 triangle is given in Figure 6.1 and the fractional point is $f = (\frac{1}{2}, \frac{1}{2})^T$.

Computational experiments suggest that choosing the ray-sliding triangles for each ray at the geometric breakpoint for $\tau > 1$ might provide a stronger bound than for $\tau < 1$. However, experience has shown that choosing "close to the right set of type 2 triangles, but not quite the right set" based on computational results is easy to do. When experimenting with quadrilateral cuts, there were frequently many choices of ray-sliding triangle cuts that approximate a given quadrilateral cut within a target bound. We expect the same is true for type 3 triangle cuts as well.

Open Question 6.2.1. *Show* $\rho[\mathcal{T}_2, \mathcal{T}_3] \leq K$ *for some* $K < \frac{3}{2}$.

6.3 2-Row Intersection Cuts

One appeal of type 2 triangle cuts is that the previously established bounds on $\rho[\mathcal{T}_2, \mathcal{C}]$ equalled those for $\rho[\mathcal{T}_3, \mathcal{C}]$ and $\rho[\mathcal{Q}, \mathcal{C}]$. However, the proof $\rho[\mathcal{T}_2, \mathcal{Q}] \leq \frac{5}{4}$ breaks this symmetry. The bound $\rho[\mathcal{T}_3, \mathcal{Q}] \leq \frac{5}{4}$ follows since $\rho[\mathcal{T}_3, \mathcal{T}_2] = 1$, $\rho[\mathcal{T}_2, \mathcal{Q}] \leq \frac{5}{4}$ and the transitivity of the upper bounds. An argument analogous to Proposition 1.3.17 gives that $\rho[\mathcal{T}_3, \mathcal{C}] \leq \frac{5}{4}$. Now, all the established bounds on $\rho[\mathcal{T}_3, \mathcal{Q}]$ and $\rho[\mathcal{Q}, \mathcal{T}_3]$ are calculated using type 2 triangles as an intermediary, but it's not clear if the best possible bound can be calculated in this way. Answering this question would be a good next step in capturing the relative strength of intersection cuts from a single family in $\{\mathcal{T}_2, \mathcal{T}_3, \mathcal{Q}\}$ in approximating $R(f; \Gamma)$.

Open Question 6.3.1. *Does* $\rho[\mathcal{T}_3, \mathcal{Q}] = \rho[\mathcal{T}_2, \mathcal{Q}]$*? Does* $\rho[\mathcal{Q}, \mathcal{T}_3] = \rho[\mathcal{T}_2, \mathcal{T}_3]$*?*

The example in Figure 6.4 gives some insight into approaching this question. It provides a quadrilateral such that for $r^i = v^i - f$, there exists a type 3 triangle cut that is stronger than any one ray-sliding triangle cut. For this example, however, there may be some combination of ray-sliding triangle cuts or other type 2 triangle cuts that do as well as the type 3 triangle cut. Moreover, the values $\rho[\mathcal{T}_2, \mathcal{Q}]$ and $\rho[\mathcal{T}_3, \mathcal{Q}]$ take the worst case over all $(f; \Gamma)$. It may be the case that there exists $(f; \Gamma)$ such that $\theta[\mathcal{T}_2(f; \Gamma), \mathcal{Q}(f; \Gamma)] \neq \theta[\mathcal{T}_3(f; \Gamma), \mathcal{Q}(f; \Gamma)]$ but $\rho[\mathcal{T}_2, \mathcal{Q}]$ and $\rho[\mathcal{T}_3, \mathcal{Q}]$ coincide.

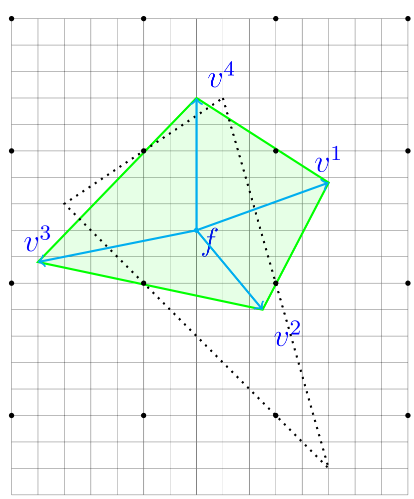


Figure 6.4: Suppose we want to construct a ray-sliding triangle with the same cut coefficient or smaller on each ray as the dotted type 3 triangle. For the base $\{(x, y)^T \in \mathbb{R}^2 : x = 0\}$ the cut coefficient for r^3 is weaker. For the base $\{(x, y)^T \in \mathbb{R}^2 : y = 0\}$ the cut coefficient for r^2 is weaker. For the base $\{(x, y)^T \in \mathbb{R}^2 : y = 1\}$ the cut coefficient for r^4 is weaker. For the base $\{(x, y)^T \in \mathbb{R}^2 : x = 1\}$ the cut coefficient for r^2 is weaker. Since we cannot select a base for the ray-sliding triangle it cannot be constructed.

One nice property of proofs here is that they prescribe a set of at most four ray-sliding triangle cuts to approximate a given quadrilateral cut. So, even though $\rho[\mathcal{T}_2, \mathcal{Q}]$ is a worst case bound, the techniques capture the behaviour of the bound on $\theta[\mathcal{T}_2(f;\Gamma), \mathcal{Q}(f;\Gamma)]$ as we vary parameters. In fact, we calculate the value of $\theta[\mathcal{T}_2(f;\Gamma), \mathcal{Q}_2(f;\Gamma)]$ exactly in the Central Region. For the practical problem of cut generation, exactly providing the triangles might prove useful.

From a mathematical point of view, however, it might be interesting to give a less constructive proof. The proofs used (a great deal of) algebra to derive some global properties of the semi-infinite programs ($\mathcal{S}(\mathcal{T}_2, \mathcal{Q}_2)$) and ($\mathcal{S}(\mathcal{T}_2, \mathcal{Q})$). For example, the bound as a function of parameters a, b, c, and d is affine in f in each region of the partition of $Q(a, b, c, d)$. Moreover, the value of f in the worst case bound tended to the "center" of the underlying quadrilateral - to $(\frac{1}{2}, \frac{1}{2})^T$ in the parallelogram case and to the Central Region in the general quadrilateral case. It would be interesting to try to exploit these properties to provide a global proof that does not rely so heavily on solving the underlying low dimensional optimization problems.

Open Question 6.3.2. *Can we bound $\rho[\mathcal{T}_2, \mathcal{Q}]$ by characterizing directly the instances achieving the supremum?*

We point to Chapter 5 as motivation, where abstracting out the key properties of the bound using fixed triangles led to a generalization to d-row cuts.

Lastly, the values for $\rho[\mathcal{T}_2, \mathcal{Q}]$ and $\rho[\mathcal{T}_2, \mathcal{Q}_2]$ were bounded by considering only the intersection cuts generated by ray-sliding triangles. Since we know $\rho[\mathcal{T}_2, \mathcal{Q}_2]$ exactly, this shows we can achieve the best bound using ray-sliding triangles. In fact, Chapter 3 and Chapter 4 provided two different sets of ray-sliding triangles that work to prove the bound for the South-West and South Regions. However, we do not know if the same is true for bounding $\rho[\mathcal{T}_2, \mathcal{Q}]$.

Open Question 6.3.3. *Can the exact value of $\rho[\mathcal{T}_2, \mathcal{Q}]$ be obtained using intersection cuts from ray-sliding triangles?*

Proving a lower bound of $\rho[\mathcal{T}_2, \mathcal{Q}] \geq \frac{5}{4}$ would suffice to answer "yes"; however, this lower bound seems doubtful given the analysis in Lemma 4.1.2 is not tight. To prove a tighter lower bound, an instance where the corresponding quadrilateral is not a parallelogram is required.

6.4 d-Row Intersection Cuts

Section 5.2.2 and Section 5.2.3 provided some open problems related to the structure of maximal lattice-free octahedra. Section 5.2.4 provided generalizations of these open problems to arbitrary polyhedra. More general questions relating to a classification of maximal lattice-free convex sets in $\mathbb{R}^d$ also remain open.

However, Theorem 5.1.2 provides techniques for proving constant factor approximations in $\mathbb{R}^d$ absent such a classification. Although the result in [8] shows that lattice-free convex sets with at least $2^{d-1}+1$ facets are required to obtain a finite approximation of the corner polyhedron in general, a refined list of the lattice-free convex sets required would be interesting.

Open Question 6.4.1. *Let $\mathcal{F}_{2^d}$ be the family of maximal lattice-free convex sets in $\mathbb{R}^d$ with 2^d facets. Let $\mathcal{O}_d$ be the family of maximal lattice-free octahedra in $\mathbb{R}^d$. Is $\rho[\mathcal{O}_d, \mathcal{F}_{2^d}]$ constant?*

Colloquially, we'd like to understand how rich a family of maximal lattice-free convex sets is required to obtain a finite approximation of the corner polyhedron. For example, a polytope $P \subseteq \mathbb{R}^d$ is *centrally symmetric* if it has a center : a point $x_0 \in \mathbb{R}^d$ such that $x_0 + x \in P$ holds if and only if $x_0 - x \in P$. The family of centrally symmetric octahedra in $\mathbb{R}^2$ is exactly the family $\mathcal{Q}_2$. The problem of bounding $\rho[\mathcal{Q}_2, \mathcal{Q}]$ naturally extends to $\mathbb{R}^d$ via centrally symmetric octahedra.

Open Question 6.4.2. *Let $\mathcal{O}_d$ be the family of maximal lattice-free octahedra in $\mathbb{R}^d$. Let $\mathcal{SO}_d$ be the family of maximal lattice-free centrally symmetric octahedra in $\mathbb{R}^d$. Bound $\rho[\mathcal{SO}_d, \mathcal{O}_d]$.*

These theoretical questions may be amenable to using tools from convex geometry to move past comparisons based only on facet count.

6.5 Other Comparison Measures

To upper-bound $\rho[\mathcal{L}_1, \mathcal{L}_2]$ for various families $\mathcal{L}_1$ and $\mathcal{L}_2$ appearing in this thesis, we often explicitly give a small number of cuts generated by members of $\mathcal{L}_1$ with which to approximate a given cut from $\mathcal{L}_2$. With this in mind, we propose an alternative theoretical measure of the relative strength of the intersection cuts generated by members of $\mathcal{L}_1$ and

$\mathcal{L}_2$. It is designed with the intent of capturing the effect of limiting the number of cuts we may choose from the second family to g. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be families of lattice-free convex sets in $\mathbb{R}^d$. Suppose $\mathcal{L}_2$ is a family of polytopes with exactly ℓ extreme points. Define

$$\frac{1}{\rho_g[\mathcal{L}_1, \mathcal{L}_2]} := \inf_{\substack{f, r^1, \ldots r^\ell \in \mathbb{R}^d: \\ f + \operatorname{conv}(r^1, \ldots, r^\ell) \in \hat{\mathcal{L}}_2, \\ f \in \operatorname{int}(f + \operatorname{conv}(r^1, \ldots, r^\ell))}} \left(\sup_{\substack{\mathcal{F} \subseteq \mathcal{L}_1 : |\mathcal{F}| = g \\ f \in \operatorname{int} B \;\; \forall B \in \mathcal{F}}} \left(\min_{\substack{\sum_{i=1}^{\ell} \psi_{f;B}(r^i) s_i \geq 1 \;\; \forall B \in \mathcal{F} \\ s \geq \mathbb{0}}} \quad \sum_{i=1}^{\ell} s_i \right) \right).$$

The infimum captures choosing a "hard to approximate" member of $\mathcal{L}_2$; the supremum captures choosing the "best" g members of $\mathcal{L}_1$ with which to approximate the given cut; and the inner-most optimization problem computes the corresponding bound. This min-max relationship can be viewed as a "game" of sorts with the following rules:

1. The "$\mathcal{L}_2$ player" chooses $f, r^1, \ldots, r^\ell$ such that $f + \operatorname{conv}\{r^1, \ldots, r^\ell\} \in \hat{\mathcal{L}}_2$.
2. The "$\mathcal{L}_1$ player" chooses $F_1, \ldots, F_g \in \mathcal{L}_1$ attempting to achieve the highest possible value of

$$\min \sum_{i=1}^{\ell} s_i \quad \text{subject to} \quad \sum_{i=1}^{\ell} \psi_{f;F_j}(r^i) s_i \geq 1 \;\; \forall j \in \{1, \ldots, g\}.$$

This measure can be applied to our analysis of approximating parallelogram cuts with type 2 triangle cuts. The proof of Theorem 1.3.21 shows that $\rho_4[\mathcal{T}_2, \mathcal{Q}_2] \leq \frac{9}{8}$. In the South-West Region, we only required two cuts in Lemma 3.4.1. However, we can compute the same bound using four cuts as in Lemma 4.2.1. Hence, for given (f, Γ) (or a, b, c, d, g, h depending on your parameterization), the same bounds can be obtained using different numbers of cuts. The value of $\rho_3[\mathcal{T}_2, \mathcal{Q}_2]$ is not given by the results in this thesis - are exactly four cuts needed to obtain a bound of $\frac{9}{8}$ in the Central Region? In general, it would be interesting to investigate instances where $\rho[\mathcal{L}_1, \mathcal{L}_2]$ is not big, but $\rho_g[\mathcal{L}_1, \mathcal{L}_2]$ is big for small g.

Alternative theoretical constraint measures that move away from the gauge measure have been proposed in other work. These measures include depth of cut, volume cut off, and change in objective function value, among others. Many of the measures of "goodness" are amenable to questions related to limiting the number of cuts used ("if you can only pick at most k cuts, which ones?") as well as limiting the number of rounds of cuts added ("you may only generate one round of cuts at the root node, which ones?", "you may not generate cuts recursively, which ones?"). For more details see, for example, [39] [55] and [65].

6.6 Connections to Computational Results

With the families of intersection cuts of interest firmly in hand, we return to computational work on the strength of multi-row cuts. As far as 2-row cuts are concerned, [38] and [14] are most applicable. Both papers consider the strength of 2-row cuts generated by type 2 triangles relative to 1-row cuts and split cuts. The type 2 triangle cuts are computed using (different) greedy heuristics and also strengthened via lifting where applicable. Both papers report mixed results and highlight the sensitivity of the conclusions to the instances used (ex. 2-row cuts perform better on instances that look like a continuous relaxation, versus arbitrary MIPLIB instances) and the need for careful experimental design. The authors of [59] provide more computational information on the relative strength of the underlying multi-row models - for example, they speak to the impact of dropping variable bounds and integrality restrictions.

The cost of generating the cut and lifting [44] may also degrade the usefulness of multi-row cuts in practice. Accordingly, theoretical bounds on the richness of cuts from simple families of lattice-free convex sets can provide guidance. Good approximations allow that experiments based on "easy to generate" d-row cuts may speak to the strength of all d-row cuts. The discussion in this thesis suggests further experiments based on type 2 triangles and parallelograms, as well as d-row cuts based on type 2 pyramids or symmetric octahedra as in [43].

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APPENDICES

Appendix A

Cut Coefficient Formulae for Approximating Given Quadrilateral Cut

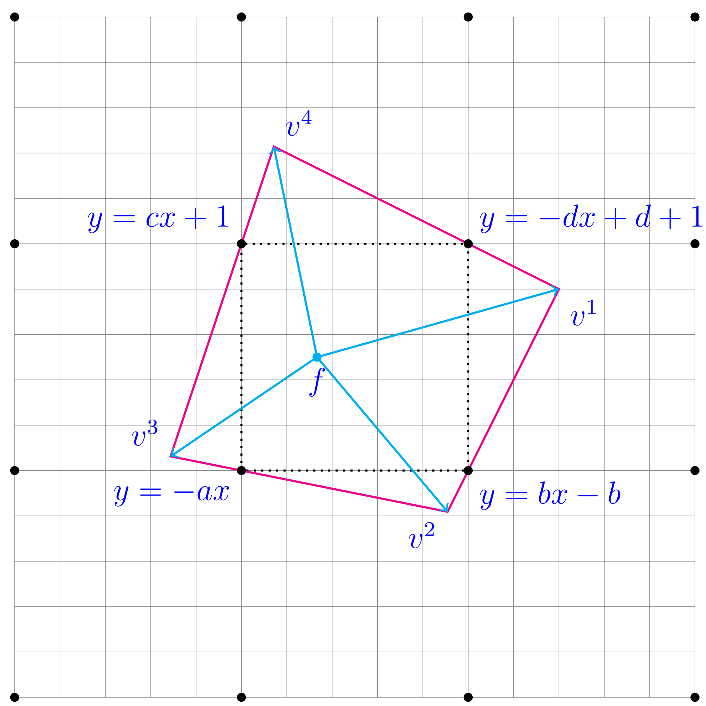

Figure A.1: Base quadrilateral for parameters $a = \frac{1}{5}$, $b = 2$, $c = 3$, $d = \frac{1}{2}$, $g = \frac{1}{3}$, and $h = \frac{1}{2}$.

Consider the normalized quadrilateral $Q(a,b,c,d)$ parameterized by $a,b,c,d>0$ as described in Section 2.1.2. Assume that $ad \le bc$. The vertices of $Q(a,b,c,d)$ are:

$$v^1 := (v_x^1, v_y^1)^T = \left(1+\frac{1}{b+d}\ ,\ \frac{b}{b+d}\right)^T \quad , \quad v^2 := (v_x^2, v_y^2)^T = \left(\frac{b}{a+b}\ ,\ \frac{-ab}{a+b}\right)^T ,$$

$$v^3 := (v_x^3, v_y^3)^T = \left(\frac{-1}{a+c}\ ,\ \frac{a}{a+c}\right)^T , \quad \text{and} \quad v^4 := (v_x^4, v_y^4)^T = \left(\frac{d}{c+d}\ ,\ 1+\frac{cd}{c+d}\right)^T .$$

Note that $v_x^4 \le v_x^2$ and $v_y^3 \le v_y^1$ because $ad \le bc$. Also consider fractional point $f=(g,h)^T$ in the interior of $Q(a,b,c,d)$.

Define ray direction r^i by $r^i = v^i - f$ for $i \in \{1,2,3,4\}$ and $\Gamma := \{r^1, r^2, r^3, r^4\}$. We refer to ray i as the ray $\{f+\lambda r^i : \lambda \ge 0\}$. For B as a vertical split, horizontal split, or τ-ray-sliding triangle, we calculate the intersection cut $\psi_{f,B}(r^1)s_1+\psi_{f,B}(r^2)s_2+\psi_{f,B}(r^3)s_3+\psi_{f,B}(r^4)s_4 \ge 1$ where the coefficients $\psi_{f,B}(r^i)$ are given by functions of a,b,c,d,g,h, and τ.

A.1 Vertical Split

Let VS denote the split $\{(x,y)\in\mathbb{R}^2 : 0 \le x \le 1\}$. Note that VS generates a valid cut for $R(f;\Gamma)$ whenever $f \in \operatorname{int} VS$ - that is, whenever $0<g<1$. The intersection cut coefficient formulae for VS depend on the directions of r^2 and r^4, which in turn depend on the x-coordinate of f, relative to those of v^2 and v^4.

Case 1: $0 < g \le v_x^4$: The intersection cut for VS is

$$\left(1+\frac{1}{(b+d)(1-g)}\right)s_1 + \left(1-\frac{a}{(a+b)(1-g)}\right)s_2 + \left(1+\frac{1}{(a+c)g}\right)s_3 + \left(1-\frac{c}{(c+d)(1-g)}\right)s_4 \ge 1.$$

Case 2: $v_x^4 \le g \le v_x^2$: The intersection cut for VS is

$$\left(1+\frac{1}{(b+d)(1-g)}\right)s_1 + \left(1-\frac{a}{(a+b)(1-g)}\right)s_2 + \left(1+\frac{1}{(a+c)g}\right)s_3 + \left(1-\frac{d}{(c+d)g}\right)s_4 \ge 1.$$

Case 3: $v_x^2 \le g < 1$: The intersection cut for VS is

$$\left(1+\frac{1}{(b+d)(1-g)}\right)s_1 + \left(1-\frac{b}{(a+b)g}\right)s_2 + \left(1+\frac{1}{(a+c)g}\right)s_3 + \left(1-\frac{d}{(c+d)g}\right)s_4 \ge 1.$$

Note that when $g=v_x^4$ or $g=v_x^2$, two of the above cut formulas apply. A straightforward substitution shows that the two formulas agree on these boundary points. When $g=v_x^4$ the coefficient for s_4 is 0 and when $g=v_x^2$ the coefficient for s_2 is 0, as expected.

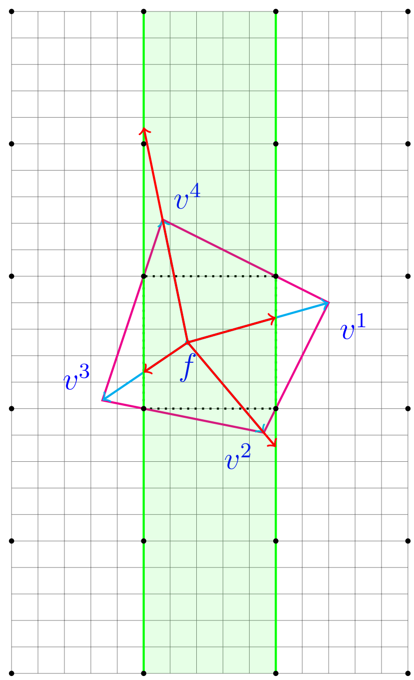

Figure A.2: Vertical split for parameters $a = \frac{1}{5}$, $b = 2$, $c = 3$, $d = \frac{1}{2}$, $g = \frac{1}{3}$, and $h = \frac{1}{2}$.

A.2 Horizontal Split

Let HS denote the split $\{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1\}$. Note that HS generates a valid cut for $R(f; \Gamma)$ whenever $f \in \operatorname{int} HS$ - that is, whenever $0 < h < 1$. The intersection cut coefficient formulae for HS depend on the directions of r^1 and r^3, which in turn depend on the y-coordinate of f, relative to those of v^1 and v^3.

Case 1: $0 < h \leq v^3_y$: The intersection cut for HS is

$$\left(1 - \frac{d}{(b+d)(1-h)}\right) s_1 + \left(1 + \frac{ab}{(a+b)h}\right) s_2 + \left(1 - \frac{c}{(a+c)(1-h)}\right) s_3 + \left(1 + \frac{cd}{(c+d)(1-h)}\right) s_4 \geq 1.$$

Case 2: $v^3_y \leq h \leq v^1_y$: The intersection cut for HS is

$$\left(1 - \frac{d}{(b+d)(1-h)}\right) s_1 + \left(1 + \frac{ab}{(a+b)h}\right) s_2 + \left(1 - \frac{a}{(a+c)h}\right) s_3 + \left(1 + \frac{cd}{(c+d)(1-h)}\right) s_4 \geq 1.$$

Case 3: $v_y^1 \leq h < 1$: The intersection cut for HS is

$$\left(1 - \frac{b}{(b+d)h}\right) s_1 + \left(1 + \frac{ab}{(a+b)h}\right) s_2 + \left(1 - \frac{a}{(a+c)h}\right) s_3 + \left(1 + \frac{cd}{(c+d)(1-h)}\right) s_4 \geq 1.$$

Note that when $h = v_y^3$ or $h = v_y^1$, two of the above cut formulas apply. A straightforward substitution shows that the two formulas agree on these boundary points. When $h = v_y^3$ the coefficient for s_3 is 0 and when $h = v_y^1$ the coefficient for s_1 is 0, as expected.

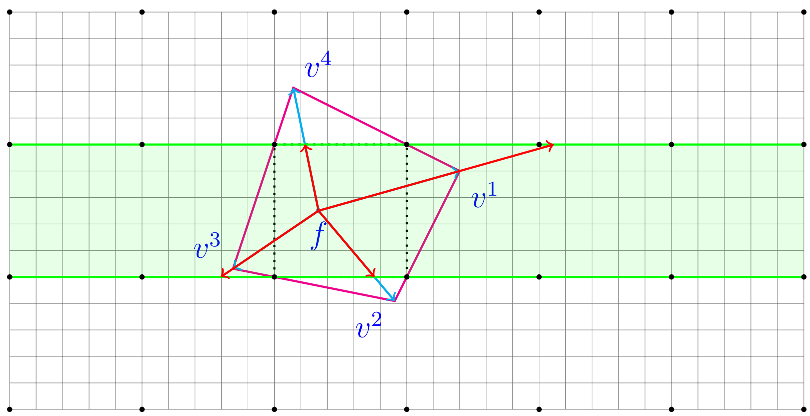

Figure A.3: Horizontal split for parameters $a = \frac{1}{5}$, $b = 2$, $c = 3$, $d = \frac{1}{2}$, $g = \frac{1}{3}$, and $h = \frac{1}{2}$.

A.3 τ-Ray-Sliding Triangle for Ray 1

Let $F = \{(x, y) \in \mathbb{R}^2 : x = 0\}$. Fix $\tau \in \mathbb{R}_+$ such that the x-coordinate of $f + \tau r^1$ is at least 1. Let $L_{(1,0)}(\tau)$ be the line through $f + \tau r^1$ and $(1, 0)^T$. Let $L_{(1,1)}(\tau)$ be the line through $f + \tau r^1$ and $(1, 1)^T$. Suppose $F, L_{(1,0)}(\tau)$, and $L_{(1,1)}(\tau)$ bound a triangle T with points $(0, 0)^T, (0, 1)^T, (1, 0)^T$, and $(1, 1)^T$ on its boundary. Then T is $RS_1(\tau)$ as defined in Section 2.4. There are three, four, or five standard breakpoints; the number of breakpoints depends on the directions of r^2 and r^4, which in turn depend on the x-coordinate of f, relative to those of v^2 and v^4. In all cases, the following three breakpoints apply.

1. If $g < 1$, the smallest τ value for which $RS_1(\tau)$ is defined is $\tau_1^{min} := 1 - \frac{1}{(1-g)(b+d)+1}$. Note that this corresponds to a vertical split. If $g \geq 1$, triangle $RS_1(\tau)$ is defined for all $\tau \in (0, 1]$. We define $\tau_1^{min} = 0$ in this case for convenience, but note that $f \notin \operatorname{int} RS_1(\tau)$ for $\tau = 0$.

2. The value of τ corresponding to not sliding along ray 1 is $\tau_1^{fixed} := 1$. The corresponding intersection cut for fixed triangle F_1 is

$$s_1 + s_2 + \left(1 + \frac{1}{(a+c)g}\right) s_3 + s_4 \geq 1.$$

3. The largest τ value for which $RS_1(\tau)$ is defined is either determined by the intersection of ray 1 with $y = 0$ or the intersection of ray 1 with $y = 1$, depending on the direction r^1. Accordingly we define:

$$\tau_1^{max} := \begin{cases} 1 + \frac{b}{h(b+d)-b} & \text{if } h > v_y^1 \\ 1 + \frac{d}{(1-h)(b+d)-d} & \text{if } h < v_y^1 \\ \infty & \text{otherwise} \end{cases}.$$

Note that $RS_1(\tau)$ generates a valid cut for $R(f;\Gamma)$ whenever $f \in \operatorname{int} RS_1(\tau)$.

1. For $g \in (0, 1]$ and $h \in [0, 1]$, $RS_1(\tau)$ generates a valid cut for all $\tau \in [\tau_1^{min}, \tau_1^{max}]$.
2. For $g > 1$, $RS_1(\tau)$ generates a valid cut for all $\tau \in (0, \tau_1^{max}]$.
3. For $g \leq 0$, $RS_1(\tau)$ never generates a valid cut.
4. For $h > 1$, $RS_1(\tau)$ generates a valid cut whenever $\tau \leq 1$ (and for some $\tau \geq 1$, though these values aren't required for our purposes).
5. For $h < 0$, $RS_1(\tau)$ generates a valid cut whenever $\tau \leq 1$ (and again for some $\tau \geq 1$).

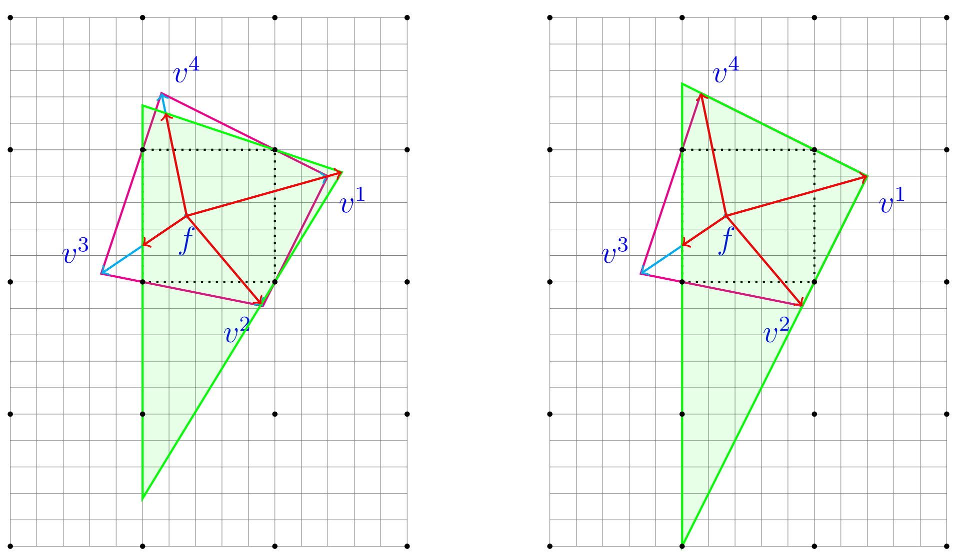

Figure A.4: (left) τ-Ray-Sliding Triangle for Ray 1 $RS_1(\tau)$ for $\tau = \frac{11}{10}$ and (right) Fixed Triangle F_1 for parameters $a = \frac{1}{5}$, $b = 2$, $c = 3$, $d = \frac{1}{2}$, $g = \frac{1}{3}$, and $h = \frac{1}{2}$.

Case 1: $0 < g \le v_x^4$: There are three standard breakpoints: $\tau_1^{min} \le \tau_1^{fixed} \le \tau_1^{max}$. For all $\tau \in [\tau_1^{min}, \tau_1^{max}]$, the intersection cut for ray-sliding triangle $RS_1(\tau)$ is

$$\left(\frac{1}{\tau}\right) s_1 + \left(1 - \frac{a(b+d)(1-\tau)}{(a+b)\tau}\right) s_2 + \left(1 + \frac{1}{(a+c)g}\right) s_3 + \left(1 - \frac{c(b+d)(1-\tau)}{(c+d)\tau}\right) s_4 \ge 1.$$

Case 2: $v_x^4 \le g \le v_x^2$: There are four standard breakpoints: $\tau_1^{min} \le \tau_1^{bp1} \le \tau_1^{fixed} \le \tau_1^{max}$ where

$$\tau_1^{bp1} := 1 - \frac{d}{c(b+d)g + d}.$$

For all $\tau \in [\tau_1^{min}, \tau_1^{bp1}]$, the intersection cut for ray-sliding triangle $RS_1(\tau)$ is

$$\left(\frac{1}{\tau}\right) s_1 + \left(1 - \frac{a(b+d)(1-\tau)}{(a+b)\tau}\right) s_2 + \left(1 + \frac{1}{(a+c)g}\right) s_3 + \left(1 - \frac{d}{(c+d)g}\right) s_4 \ge 1.$$

For all $\tau \in [\tau_1^{bp1}, \tau_1^{max}]$, the intersection cut for ray-sliding triangle $RS_1(\tau)$ is

$$\left(\frac{1}{\tau}\right) s_1 + \left(1 - \frac{a(b+d)(1-\tau)}{(a+b)\tau}\right) s_2 + \left(1 + \frac{1}{(a+c)g}\right) s_3 + \left(1 - \frac{c(b+d)(1-\tau)}{(c+d)\tau}\right) s_4 \ge 1.$$

Note that when $\tau = \tau_1^{bp1}$, two of the above cut formulas apply. A straightforward substitution shows that the two formulas agree on this boundary point and the corresponding intersection cut for ray-sliding triangle $RS_1(\tau_1^{bp1})$ is

$$\left(1 + \frac{d}{c(b+d)g}\right) s_1 + \left(1 - \frac{ad}{c(a+b)g}\right) s_2 + \left(1 + \frac{1}{(a+c)g}\right) s_3 + \left(1 - \frac{d}{(c+d)g}\right) s_4 \ge 1.$$

Case 3: $v_x^2 \le g$: There are five standard breakpoints: $\tau_1^{min} \le \tau_1^{bp2} \le \tau_1^{bp1} \le \tau_1^{fixed} \le \tau_1^{max}$ where

$$\tau_1^{bp2} := 1 - \frac{b}{a(b+d)g + b}, \quad \text{and} \quad \tau_1^{bp1} := 1 - \frac{d}{c(b+d)g + d}.$$

For all $\tau \in [\tau_1^{min}, \tau_1^{bp2}]$, the intersection cut for ray-sliding triangle $RS_1(\tau)$ is

$$\left(\frac{1}{\tau}\right) s_1 + \left(1 - \frac{b}{(a+b)g}\right) s_2 + \left(1 + \frac{1}{(a+c)g}\right) s_3 + \left(1 - \frac{d}{(c+d)g}\right) s_4 \ge 1.$$

For all $\tau \in [\tau_1^{bp2}, \tau_1^{bp1}]$, the intersection cut for ray-sliding triangle $RS_1(\tau)$ is

$$\left(\frac{1}{\tau}\right) s_1 + \left(1 - \frac{a(b+d)(1-\tau)}{(a+b)\tau}\right) s_2 + \left(1 + \frac{1}{(a+c)g}\right) s_3 + \left(1 - \frac{d}{(c+d)g}\right) s_4 \ge 1.$$

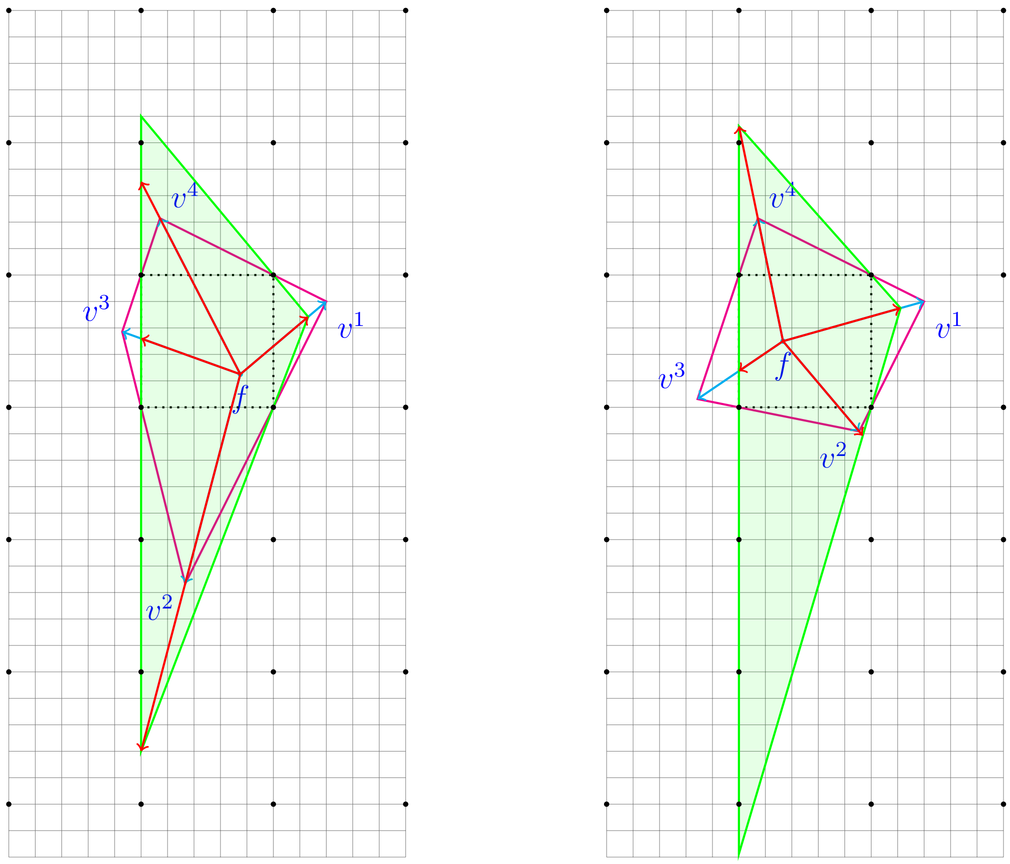

Figure A.5: (left) τ_1^{bp2}-Ray-Sliding Triangle for Ray 1 where $\tau_1^{bp2} = \frac{15}{19}$ and parameters $a = 4$, $b = 2$, $c = 3$, $d = \frac{1}{2}$, $g = \frac{3}{4}$, and $h = \frac{1}{4}$ and (right) τ_1^{bp1}-Ray-Sliding Triangle for Ray 1 for $\tau_1^{bp1} = \frac{5}{6}$ and parameters $a = \frac{1}{5}$, $b = 2$, $c = 3$, $d = \frac{1}{2}$, $g = \frac{1}{3}$, and $h = \frac{1}{2}$.

For all $\tau \in [\tau_1^{bp1}, \tau_1^{max}]$, the intersection cut for ray-sliding triangle $RS_1(\tau)$ is

$$\left(\frac{1}{\tau}\right) s_1 + \left(1 - \frac{a(b+d)(1-\tau)}{(a+b)\tau}\right) s_2 + \left(1 + \frac{1}{(a+c)g}\right) s_3 + \left(1 - \frac{c(b+d)(1-\tau)}{(c+d)\tau}\right) s_4 \geq 1.$$

Note that when $\tau = \tau_1^{bp2}$, two of the above cut formulas apply. A straightforward substitution shows that the two formulas agree on this boundary point and the corresponding intersection cut for ray-sliding triangle $RS_1(\tau_1^{bp2})$ is

$$\left(1 + \frac{b}{a(b+d)g}\right) s_1 + \left(1 - \frac{b}{(a+b)g}\right) s_2 + \left(1 + \frac{1}{(a+c)g}\right) s_3 + \left(1 - \frac{d}{(c+d)g}\right) s_4 \geq 1.$$

Note that when $\tau = \tau_1^{bp1}$, two of the above cut formulas apply. A straightforward substitution shows that the two formulas agree on this boundary point and the corresponding intersection cut for ray-sliding triangle $RS_1(\tau_1^{bp1})$ is the same as in Case 2.

Checking Consistency for $g = v_x^4$ and $g = v_x^2$: Note that when $g = v_x^4$, both Case 1 and Case 2 apply. The formulas are consistent because $\tau_1^{min} = \tau_1^{bp1}$. Note that when $g = v_x^2$, both Case 2 and Case 3 apply. The formulas are consistent because $\tau_1^{min} = \tau_1^{bp2}$.

A.4 τ-Ray-Sliding Triangle for Ray 2

Let $F = \{(x, y) \in \mathbb{R}^2 : y = 1\}$. Fix $\tau \in \mathbb{R}_+$ such that the y-coordinate of $f + \tau r^2$ is at most 0. Let $L_{(0,0)}(\tau)$ be the line through $f + \tau r^2$ and $(0, 0)^T$. Let $L_{(1,0)}(\tau)$ be the line through $f + \tau r^2$ and $(1, 0)^T$. Suppose $F, L_{(0,0)}(\tau)$, and $L_{(1,0)}(\tau)$ bound a triangle T with points $(0, 0)^T, (0, 1)^T, (1, 0)^T$, and $(1, 1)^T$ on its boundary. Then T is $RS_2(\tau)$ as defined in Section 2.4. There are three, four, or five standard breakpoints; the number of breakpoints depends on the directions of r^1 and r^3, which in turn depend on the y-coordinate of f, relative to those of v^1 and v^3. In all cases, the following three breakpoints apply.

1. If $h > 0$, the smallest τ value for which $RS_2(\tau)$ is defined is $\tau_2^{min} := 1 - \frac{ab}{(a+b)h+ab}$. Note that this corresponds to a horizontal split. If $h \leq 0$, triangle $RS_2(\tau)$ is defined for all $\tau \in (0, 1]$. We define $\tau_2^{min} = 0$ in this case for convenience, but note that $f \notin \operatorname{int} RS_2(\tau)$ for $\tau = 0$.
2. The value of τ corresponding to not sliding along ray 2 is $\tau_2^{fixed} := 1$. The corresponding intersection cut for fixed triangle F_2 is
$$s_1 + s_2 + s_3 + \left(1 + \frac{cd}{(c+d)(1-h)}\right) s_4 \geq 1.$$
3. The largest τ value for which $RS_2(\tau)$ is defined is either determined by the intersection of ray 2 with $x = 0$ or the intersection of ray 2 with $x = 1$, depending on the direction r^2. Accordingly we define:
$$\tau_2^{max} := \begin{cases} 1 + \frac{b}{(a+b)g-b} & \text{if } g > v_x^2 \\ 1 + \frac{a}{b-(a+b)g} & \text{if } g < v_x^2 \\ \infty & \text{otherwise} \end{cases}.$$

Note that $RS_2(\tau)$ generates a valid cut for $R(f; \Gamma)$ whenever $f \in \operatorname{int} RS_2(\tau)$.

1. For $g \in [0, 1]$ and $h \in [0, 1)$, $RS_2(\tau)$ generates a valid cut for all $\tau \in [\tau_2^{min}, \tau_2^{max}]$.
2. For $h < 0$, $RS_2(\tau)$ generates a valid cut for all $\tau \in (0, \tau_2^{max}]$.

3. For $h \geq 1$, $RS_2(\tau)$ never generates a valid cut.
4. For $g > 1$, $RS_2(\tau)$ generates a valid cut whenever $\tau \leq 1$ (and for some $\tau \geq 1$, though these values aren't required for our purposes).
5. For $g < 0$, $RS_2(\tau)$ generates a valid cut whenever $\tau \leq 1$ (and again for some $\tau \geq 1$).

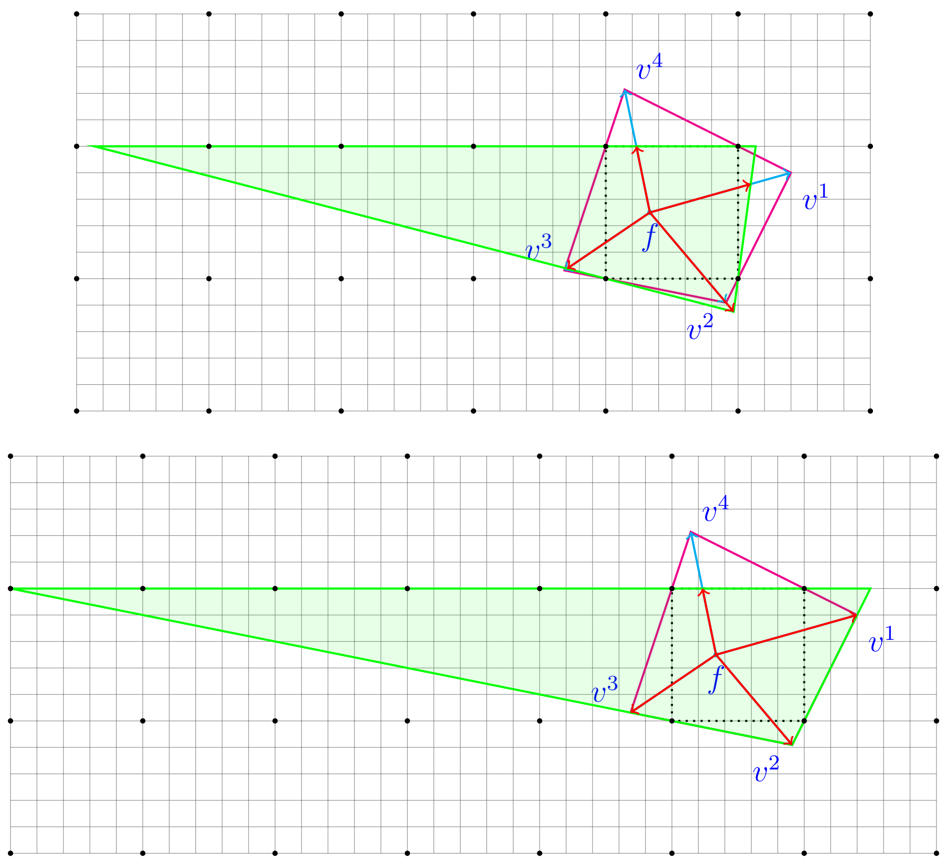

Figure A.6: (top) τ-Ray-Sliding Triangle for Ray 2 $RS_2(\tau)$ for $\tau = \frac{11}{10}$ and (bottom) Fixed Triangle F_2 for parameters $a = \frac{1}{5}$, $b = 2$, $c = 3$, $d = \frac{1}{2}$, $g = \frac{1}{3}$, and $h = \frac{1}{2}$.

Case 1: $h \leq v_y^3$: There are five standard breakpoints: $\tau_2^{min} \leq \tau_2^{bp2} \leq \tau_2^{bp1} \leq \tau_2^{fixed} \leq \tau_2^{max}$ where

$$\tau_2^{bp2} := 1 - \frac{bc}{(a+b)(1-h)+bc}, \text{ and } \tau_2^{bp1} := 1 - \frac{ad}{(a+b)(1-h)+ad}.$$

For all $\tau \in [\tau_2^{min}, \tau_2^{bp2}]$, the intersection cut for ray-sliding triangle $RS_2(\tau)$ is

$$\left(1 - \frac{d}{(b+d)(1-h)}\right) s_1 + \left(\frac{1}{\tau}\right) s_2 + \left(1 - \frac{c}{(a+c)(1-h)}\right) s_3 + \left(1 + \frac{cd}{(c+d)(1-h)}\right) s_4 \geq 1.$$

For all $\tau \in [\tau_2^{bp2}, \tau_2^{bp1}]$, the intersection cut for ray-sliding triangle $RS_2(\tau)$ is

$$\left(1 - \frac{d}{(b+d)(1-h)}\right) s_1 + \left(\frac{1}{\tau}\right) s_2 + \left(1 - \frac{(a+b)(1-\tau)}{b(a+c)\tau}\right) s_3 + \left(1 + \frac{cd}{(c+d)(1-h)}\right) s_4 \geq 1.$$

For all $\tau \in [\tau_2^{bp1}, \tau_2^{max}]$, the intersection cut for ray-sliding triangle $RS_2(\tau)$ is

$$\left(1 - \frac{(a+b)(1-\tau)}{a(b+d)\tau}\right) s_1 + \left(\frac{1}{\tau}\right) s_2 + \left(1 - \frac{(a+b)(1-\tau)}{b(a+c)\tau}\right) s_3 + \left(1 + \frac{cd}{(c+d)(1-h)}\right) s_4 \geq 1.$$

Note that when $\tau = \tau_2^{bp2}$, two of the above cut formulas apply. A straightforward substitution shows that the two formulas agree on this boundary point and that the corresponding intersection cut for ray-sliding triangle $RS_2(\tau_2^{bp2})$ is

$$\left(1 - \tfrac{d}{(b+d)(1-h)}\right) s_1 + \left(1 + \tfrac{bc}{(a+b)(1-h)}\right) s_2 + \left(1 - \tfrac{c}{(a+c)(1-h)}\right) s_3 + \left(1 + \tfrac{cd}{(c+d)(1-h)}\right) s_4 \geq 1.$$

Note that when $\tau = \tau_2^{bp1}$, two of the above cut formulas apply. A straightforward substitution shows that the two formulas agree on this boundary point and that the corresponding intersection cut for ray-sliding triangle $RS_2(\tau_2^{bp1})$ is

$$\left(1 - \tfrac{d}{(b+d)(1-h)}\right) s_1 + \left(1 + \tfrac{ad}{(a+b)(1-h)}\right) s_2 + \left(1 - \tfrac{ad}{b(a+c)(1-h)}\right) s_3 + \left(1 + \tfrac{cd}{(c+d)(1-h)}\right) s_4 \geq 1.$$

Case 2: $v_y^3 \leq h \leq v_y^1$: There are four standard breakpoints: $\tau_2^{min} \leq \tau_2^{bp1} \leq \tau_2^{fixed} \leq \tau_2^{max}$ where

$$\tau_2^{bp1} := 1 - \frac{ad}{(a+b)(1-h)+ad}.$$

For all $\tau \in [\tau_2^{min}, \tau_2^{bp1}]$, the intersection cut for ray-sliding triangle $RS_2(\tau)$ is

$$\left(1 - \frac{d}{(b+d)(1-h)}\right) s_1 + \left(\frac{1}{\tau}\right) s_2 + \left(1 - \frac{(a+b)(1-\tau)}{b(a+c)\tau}\right) s_3 + \left(1 + \frac{cd}{(c+d)(1-h)}\right) s_4 \geq 1.$$

For all $\tau \in [\tau_2^{bp1}, \tau_2^{max}]$, the intersection cut for ray-sliding triangle $RS_2(\tau)$ is

$$\left(1 - \frac{(a+b)(1-\tau)}{a(b+d)\tau}\right) s_1 + \left(\frac{1}{\tau}\right) s_2 + \left(1 - \frac{(a+b)(1-\tau)}{b(a+c)\tau}\right) s_3 + \left(1 + \frac{cd}{(c+d)(1-h)}\right) s_4 \geq 1.$$

Note that when $\tau = \tau_2^{bp1}$, two of the above cut formulas apply. A straightforward substitution shows that the two formulas agree on this boundary point and that the corresponding intersection cut for ray-sliding triangle $RS_2(\tau_2^{bp1})$ is the same as Case 1.

Case 3: $v_y^1 \leq h < 1$: There are three standard breakpoints: $\tau_2^{min} \leq \tau_2^{fixed} \leq \tau_2^{max}$. For all $\tau \in [\tau_2^{min}, \tau_2^{max}]$, the intersection cut for ray-sliding triangle $RS_2(\tau)$ is

$$\left(1 - \frac{(a+b)(1-\tau)}{a(b+d)\tau}\right) s_1 + \left(\frac{1}{\tau}\right) s_2 + \left(1 - \frac{(a+b)(1-\tau)}{b(a+c)\tau}\right) s_3 + \left(1 + \frac{cd}{(c+d)(1-h)}\right) s_4 \geq 1.$$

Checking Consistency for $h = v_y^3$ and $h = v_y^1$: Note that when $h = v_y^3$, both Case 1 and Case 2 apply. The formulas are consistent because $\tau_2^{min} = \tau_2^{bp2}$. Note that when $g = v_y^1$, both Case 2 and Case 3 apply. The formulas are consistent because $\tau_2^{min} = \tau_2^{bp1}$.

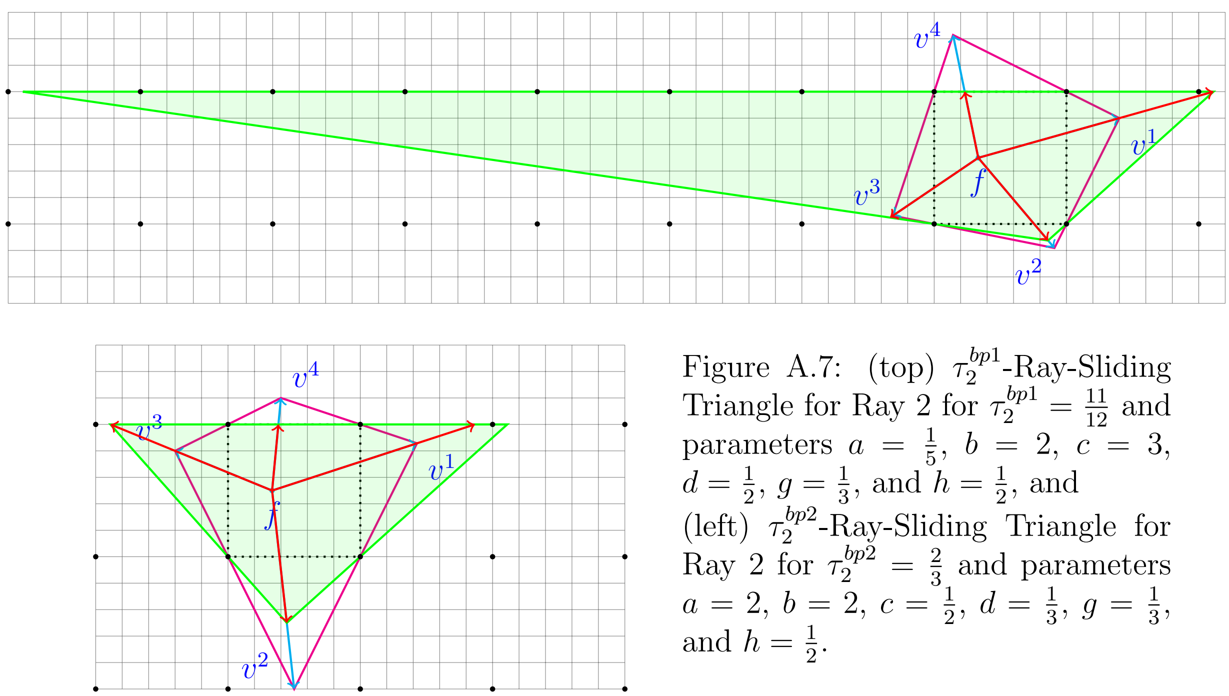

Figure A.7: (top) τ_2^{bp1}-Ray-Sliding Triangle for Ray 2 for $\tau_2^{bp1} = \frac{11}{12}$ and parameters $a = \frac{1}{5}$, $b = 2$, $c = 3$, $d = \frac{1}{2}$, $g = \frac{1}{3}$, and $h = \frac{1}{2}$, and (left) τ_2^{bp2}-Ray-Sliding Triangle for Ray 2 for $\tau_2^{bp2} = \frac{2}{3}$ and parameters $a = 2$, $b = 2$, $c = \frac{1}{2}$, $d = \frac{1}{3}$, $g = \frac{1}{3}$, and $h = \frac{1}{2}$.

A.5 τ-Ray-Sliding Triangle for Ray 3

Let $F = \{(x, y) \in \mathbb{R}^2 : x = 1\}$. Fix $\tau \in \mathbb{R}_+$ such that the x-coordinate of $f + \tau r^3$ is at most 0. Let $L_{(0,0)}(\tau)$ be the line through $f + \tau r^3$ and $(0, 0)^T$. Let $L_{(0,1)}(\tau)$ be the line through $f + \tau r^3$ and $(0, 1)^T$. Suppose $F, L_{(0,0)}(\tau)$, and $L_{(0,1)}(\tau)$ bound a triangle T with points $(0, 0)^T, (0, 1)^T, (1, 0)^T$, and $(1, 1)^T$ on its boundary. Then T is $RS_3(\tau)$ as defined in Section 2.4. There are three, four, or five standard breakpoints; the number of breakpoints depends on the directions of r^2 and r^4, which in turn depend on the x-coordinate of f, relative to those of v^2 and v^4. In all cases, the following three breakpoints apply.

1. If $g > 0$, the smallest τ value for which $RS_3(\tau)$ is defined is $\tau_3^{min} := 1 - \frac{1}{(a+c)g+1}$. Note that this corresponds to a vertical split. If $g \leq 0$, triangle $RS_3(\tau)$ is defined for all $\tau \in (0,1]$. We define $\tau_3^{min} = 0$ in this case for convenience, but note that $f \notin \operatorname{int} RS_3(\tau)$ for $\tau = 0$.
2. The value of τ corresponding to not sliding along ray 3 is $\tau_3^{fixed} := 1$. The corresponding intersection cut for fixed triangle F_3 is
$$\left(1 + \frac{1}{(b+d)(1-g)}\right) s_1 + s_2 + s_3 + s_4 \geq 1.$$
3. The largest τ value for which $RS_3(\tau)$ is defined is either determined by the intersection of ray 3 with $y = 0$ or the intersection of ray 3 with $y = 1$, depending on the direction r^3. Accordingly we define:
$$\tau_3^{max} := \begin{cases} 1 + \frac{a}{(a+c)h-a} & \text{if } h > v_y^3 \\ 1 + \frac{c}{(a+c)(1-h)-c} & \text{if } h < v_y^3 \\ \infty & \text{otherwise} \end{cases}.$$

Note that $RS_3(\tau)$ generates a valid cut for $R(f;\Gamma)$ whenever $f \in \operatorname{int} RS_3(\tau)$.

1. For $g \in [0,1)$ and $h \in [0,1]$, $RS_3(\tau)$ generates a valid cut for all $\tau \in [\tau_3^{min}, \tau_3^{max}]$.
2. For $g < 0$, $RS_3(\tau)$ generates a valid cut for all $\tau \in (0, \tau_3^{max}]$.
3. For $g \geq 1$, $RS_3(\tau)$ never generates a valid cut.
4. For $h > 1$, $RS_3(\tau)$ generates a valid cut whenever $\tau \leq 1$ (and for some $\tau \geq 1$, though these values aren't required for our purposes).
5. For $h < 0$, $RS_3(\tau)$ generates a valid cut whenever $\tau \leq 1$ (and again for some $\tau \geq 1$).

<u>Case 1: $g \leq v_x^4$:</u> There are five standard breakpoints: $\tau_3^{min} \leq \tau_3^{bp2} \leq \tau_3^{bp1} \leq \tau_3^{fixed} \leq \tau_3^{max}$ where
$$\tau_3^{bp2} := 1 - \frac{c}{d(a+c)(1-g)+c} \quad \text{and} \quad \tau_3^{bp1} := 1 - \frac{a}{b(a+c)(1-g)+a}.$$

For all $\tau \in [\tau_3^{min}, \tau_3^{bp2}]$ the intersection cut for ray-sliding triangle $RS_3(\tau)$ is
$$\left(1 + \frac{1}{(b+d)(1-g)}\right) s_1 + \left(1 - \frac{a}{(a+b)(1-g)}\right) s_2 + \left(\frac{1}{\tau}\right) s_3 + \left(1 - \frac{c}{(c+d)(1-g)}\right) s_4 \geq 1.$$

For all $\tau \in [\tau_3^{bp2}, \tau_3^{bp1}]$ the intersection cut for ray-sliding triangle $RS_3(\tau)$ is
$$\left(1 + \frac{1}{(b+d)(1-g)}\right) s_1 + \left(1 - \frac{a}{(a+b)(1-g)}\right) s_2 + \left(\frac{1}{\tau}\right) s_3 + \left(1 - \frac{(a+c)d(1-\tau)}{(c+d)\tau}\right) s_4 \geq 1.$$

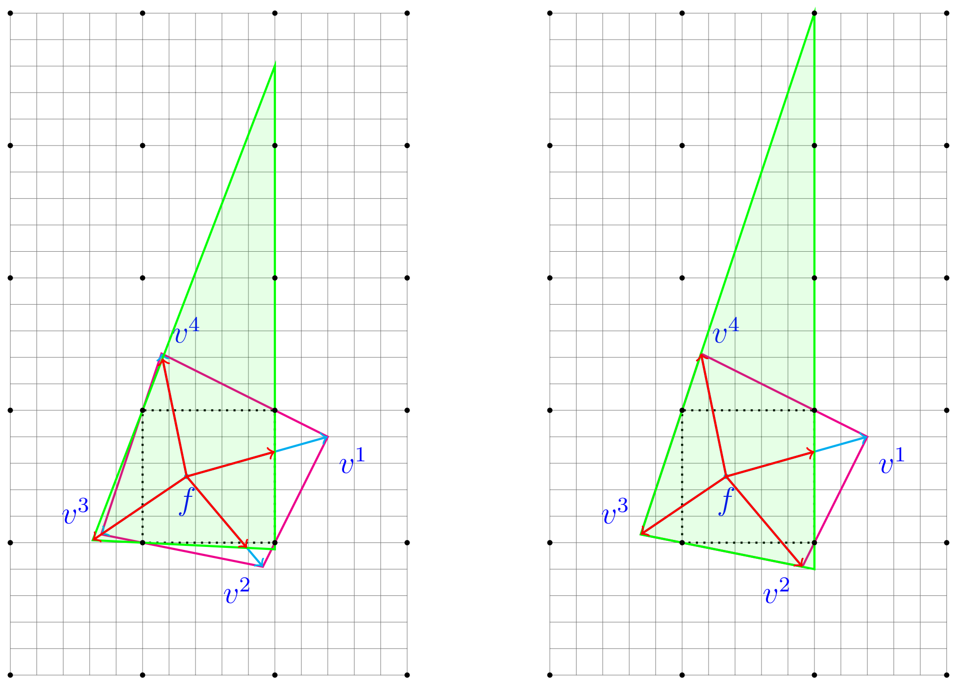

Figure A.8: (left) τ-Ray-Sliding Triangle for Ray 3 $RS_3(\tau)$ for $\tau = \frac{11}{10}$ and (right) Fixed Triangle F_3 for parameters $a = \frac{1}{5}$, $b = 2$, $c = 3$, $d = \frac{1}{2}$, $g = \frac{1}{3}$, and $h = \frac{1}{2}$.

For all $\tau \in [\tau_3^{bp1}, \tau_3^{max}]$ the intersection cut for ray-sliding triangle $RS_3(\tau)$ is

$$\left(1 + \frac{1}{(b+d)(1-g)}\right) s_1 + \left(1 - \frac{b(a+c)(1-\tau)}{(a+b)\tau}\right) s_2 + \left(\frac{1}{\tau}\right) s_3 + \left(1 - \frac{(a+c)d(1-\tau)}{(c+d)\tau}\right) s_4 \geq 1.$$

Note that when $\tau = \tau_3^{bp2}$, two of the above cut formulas apply. A straightforward substitution shows that the two formulas agree on this boundary point and the corresponding intersection cut for ray-sliding triangle $RS_3(\tau_3^{bp2})$ is

$$\left(1 + \tfrac{1}{(b+d)(1-g)}\right) s_1 + \left(1 - \tfrac{a}{(a+b)(1-g)}\right) s_2 + \left(1 + \tfrac{c}{(a+c)d(1-g)}\right) s_3 + \left(1 - \tfrac{c}{(c+d)(1-g)}\right) s_4 \geq 1.$$

Note that when $\tau = \tau_3^{bp1}$, two of the above cut formulas apply. A straightforward substitution shows that the two formulas agree on this boundary point and the corresponding intersection cut for ray-sliding triangle $RS_3(\tau_3^{bp1})$ is

$$\left(1 + \tfrac{1}{(b+d)(1-g)}\right) s_1 + \left(1 - \tfrac{a}{(a+b)(1-g)}\right) s_2 + \left(1 + \tfrac{a}{b(a+c)(1-g)}\right) s_3 + \left(1 - \tfrac{ad}{b(c+d)(1-g)}\right) s_4 \geq 1.$$

Case 2: $v_x^4 \leq g \leq v_x^2$: There are four standard breakpoints: $\tau_3^{min} \leq \tau_3^{bp1} \leq \tau_3^{fixed} \leq \tau_3^{max}$

where

$$\tau_3^{bp1} := 1 - \frac{a}{b(a+c)(1-g)+a}.$$

For all $\tau \in [\tau_3^{min}, \tau_3^{bp1}]$ the intersection cut for ray-sliding triangle $RS_3(\tau)$ is

$$\left(1 + \frac{1}{(b+d)(1-g)}\right) s_1 + \left(1 - \frac{a}{(a+b)(1-g)}\right) s_2 + \left(\frac{1}{\tau}\right) s_3 + \left(1 - \frac{(a+c)d(1-\tau)}{(c+d)\tau}\right) s_4 \geq 1.$$

For all $\tau \in [\tau_3^{bp1}, \tau_3^{max}]$ the intersection cut for ray-sliding triangle $RS_3(\tau)$ is

$$\left(1 + \frac{1}{(b+d)(1-g)}\right) s_1 + \left(1 - \frac{b(a+c)(1-\tau)}{(a+b)\tau}\right) s_2 + \left(\frac{1}{\tau}\right) s_3 + \left(1 - \frac{(a+c)d(1-\tau)}{(c+d)\tau}\right) s_4 \geq 1.$$

Note that when $\tau = \tau_3^{bp1}$, two of the above cut formulas apply. A straightforward substitution shows that the two formulas agree on this boundary point and the corresponding intersection cut for ray-sliding triangle $RS_3(\tau_3^{bp1})$ is the same as in Case 1.

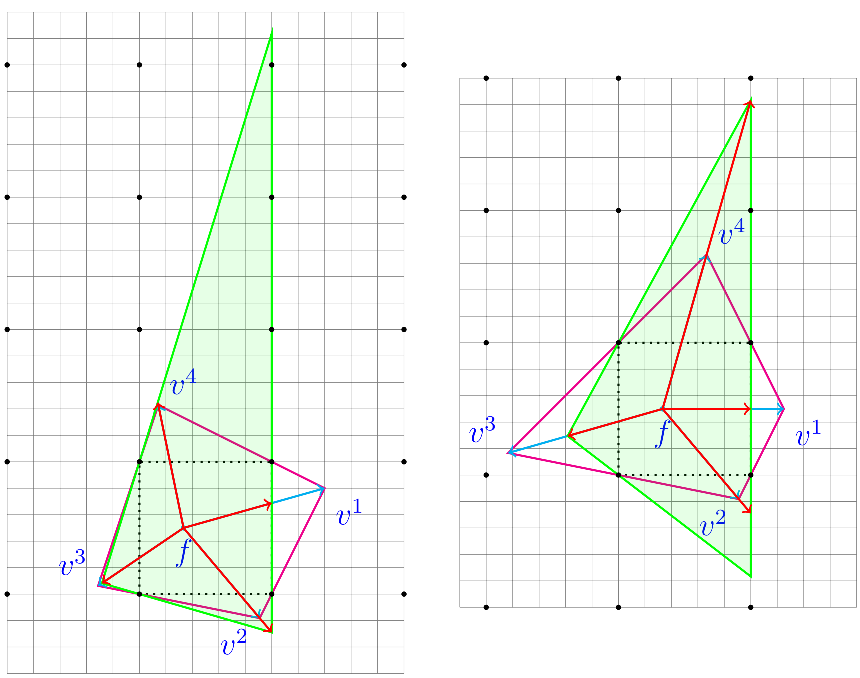

Figure A.9: (left) τ_3^{bp1}-Ray-Sliding Triangle for Ray 3 for $\tau_3^{bp1} = \frac{64}{67}$ and parameters $a = \frac{1}{5}$, $b = 2$, $c = 3$, $d = \frac{1}{2}$, $g = \frac{1}{3}$, and $h = \frac{1}{2}$ and (right) τ_3^{bp2}-Ray-Sliding Triangle for Ray 3 for $\tau_3^{bp2} = \frac{8}{13}$ and parameters $a = \frac{1}{5}$, $b = 2$, $c = 1$, $d = 2$, $g = \frac{1}{3}$, and $h = \frac{1}{2}$.

Case 3: $v_x^2 \le g < 1$: There are three standard breakpoints: $\tau_3^{min} \le \tau_3^{fixed} \le \tau_3^{max}$. For all $\tau \in [\tau_3^{min}, \tau_3^{max}]$ the intersection cut for ray-sliding triangle $RS_3(\tau)$ is

$$\left(1 + \frac{1}{(b+d)(1-g)}\right) s_1 + \left(1 - \frac{b(a+c)(1-\tau)}{(a+b)\tau}\right) s_2 + \left(\frac{1}{\tau}\right) s_3 + \left(1 - \frac{(a+c)d(1-\tau)}{(c+d)\tau}\right) s_4 \ge 1.$$

Checking Consistency for $g = v_x^4$ and $g = v_x^2$: Note that when $g = v_x^4$, both Case 1 and Case 2 apply. The formulas are consistent because $\tau_3^{min} = \tau_3^{bp2}$. Note that when $g = v_x^2$, both Case 2 and Case 3 apply. The formulas are consistent because $\tau_3^{min} = \tau_3^{bp1}$.

A.6 τ-Ray Sliding Triangle for Ray 4

Let $F = \{(x,y) \in \mathbb{R}^2 : y = 0\}$. Fix $\tau \in \mathbb{R}_+$ such that the y-coordinate of $f + \tau r^4$ is at least 1. Let $L_{(0,1)}(\tau)$ be the line through $f + \tau r^4$ and $(0,1)^T$. Let $L_{(1,1)}(\tau)$ be the line through $f + \tau r^4$ and $(1,1)^T$. Suppose $F, L_{(0,1)}(\tau)$, and $L_{(1,1)}(\tau)$ bound a triangle T with points $(0,0)^T, (0,1)^T, (1,0)^T$, and $(1,1)^T$ on its boundary. Then T is $RS_4(\tau)$ as defined in Section 2.4. There are three, four, or five standard breakpoints; the number of breakpoints depends on the directions of r^1 and r^3, which in turn depend on the y-coordinate of f, relative to those of v^1 and v^3. In all cases, the following three breakpoints apply.

1. If $h < 1$, the smallest τ value for which $RS_4(\tau)$ is defined is $\tau_4^{min} := 1 - \frac{cd}{(c+d)(1-h)+cd}$. Note that this corresponds to a horizontal split. If $h \ge 1$, triangle $RS_4(\tau)$ is defined for all $\tau \in (0,1]$. We define $\tau_4^{min} = 0$ in this case for convenience, but note that $f \notin \operatorname{int} RS_4(\tau)$ for $\tau = 0$.
2. The value of τ corresponding to not sliding along ray 4 is $\tau_4^{fixed} := 1$. The corresponding intersection cut for fixed triangle F_4 is

$$s_1 + \left(1 + \frac{ab}{(a+b)h}\right) s_2 + s_3 + s_4 \ge 1.$$

3. The largest τ value for which $RS_4(\tau)$ is defined is either determined by the intersection of ray 4 with $x = 0$ or the intersection of ray 4 with $x = 1$. Accordingly we define:

$$\tau_4^{max} := \begin{cases} 1 + \frac{d}{g(c+d)-d} & \text{if } g > v_x^4 \\ 1 + \frac{c}{(1-g)(c+d)-c} & \text{if } g < v_x^4 \\ \infty & \text{otherwise} \end{cases} .$$

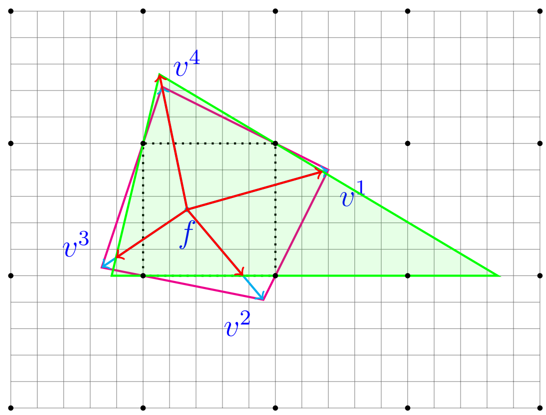

Figure A.10: (left) τ-Ray-Sliding Triangle for Ray 4 $RS_4(\tau)$ for $\tau = \frac{11}{10}$ and (right) Fixed Triangle F_4 for parameters $a = \frac{1}{5}$, $b = 2$, $c = 3$, $d = \frac{1}{2}$, $g = \frac{1}{3}$, and $h = \frac{1}{2}$.

Note that $RS_4(\tau)$ generates a valid cut for $R(f;\Gamma)$ whenever $f \in \operatorname{int} RS_4(\tau)$.

1. For $g \in [0,1]$ and $h \in (0,1]$, $RS_4(\tau)$ generates a valid cut for all $\tau \in [\tau_4^{min}, \tau_4^{max}]$.
2. For $h > 1$, $RS_4(\tau)$ generates a valid cut for all $\tau \in (0, \tau_4^{max}]$.
3. For $h \leq 0$, $RS_4(\tau)$ never generates a valid cut.
4. For $g > 1$, $RS_4(\tau)$ generates a valid cut whenever $\tau \leq 1$ (and for some $\tau \geq 1$, though these values aren't required for our purposes).
5. For $g < 0$, $RS_4(\tau)$ generates a valid cut whenever $\tau \leq 1$ (and again for some $\tau \geq 1$).

Case 1: $0 < h \leq v_y^3$: There are three standard breakpoints: $\tau_4^{min} \leq \tau_4^{fixed} \leq \tau_4^{max}$. For all $\tau \in [\tau_4^{min}, \tau_4^{max}]$, the intersection cut for ray-sliding triangle $RS_4(\tau)$ is

$$\left(1 - \frac{(c+d)(1-\tau)}{c(b+d)\tau}\right) s_1 + \left(1 + \frac{ab}{(a+b)h}\right) s_2 + \left(1 - \frac{(c+d)(1-\tau)}{(a+c)d\tau}\right) s_3 + \left(\frac{1}{\tau}\right) s_4 \geq 1.$$

Case 2: $v_y^3 \leq h \leq v_y^1$: There are four standard breakpoints: $\tau_4^{min} \leq \tau_4^{bp1} \leq \tau_4^{fixed} \leq \tau_4^{max}$ where

$$\tau_4^{bp1} := 1 - \frac{ad}{(c+d)h + ad}.$$

For all $\tau \in [\tau_4^{min}, \tau_4^{bp1}]$, the intersection cut for ray-sliding triangle $RS_4(\tau)$ is

$$\left(1 - \frac{(c+d)(1-\tau)}{c(b+d)\tau}\right) s_1 + \left(1 + \frac{ab}{(a+b)h}\right) s_2 + \left(1 - \frac{a}{(a+c)h}\right) s_3 + \left(\frac{1}{\tau}\right) s_4 \geq 1.$$

For all $\tau \in [\tau_4^{bp1}, \tau_4^{max}]$, the intersection cut for ray-sliding triangle $RS_4(\tau)$ is

$$\left(1 - \frac{(c+d)(1-\tau)}{c(b+d)\tau}\right) s_1 + \left(1 + \frac{ab}{(a+b)h}\right) s_2 + \left(1 - \frac{(c+d)(1-\tau)}{(a+c)d\tau}\right) s_3 + \left(\frac{1}{\tau}\right) s_4 \geq 1.$$

Note that when $\tau = \tau_4^{bp1}$, two of the above cut formulas apply. A straightforward substitution shows that the two formulas agree on this boundary point and that the corresponding intersection cut for ray-sliding triangle $RS_4(\tau_4^{bp1})$ is

$$\left(1 - \frac{ad}{c(b+d)h}\right) s_1 + \left(1 + \frac{ab}{(a+b)h}\right) s_2 + \left(1 - \frac{a}{(a+c)h}\right) s_3 + \left(1 + \frac{ad}{(c+d)h}\right) s_4 \geq 1.$$

Case 3: $v_y^1 \leq h$: There are five standard breakpoints: $\tau_4^{min} \leq \tau_4^{bp2} \leq \tau_4^{bp1} \leq \tau_4^{fixed} \leq \tau_4^{max}$ where

$$\tau_4^{bp2} := 1 - \frac{bc}{(c+d)h + bc}, \quad \text{and} \quad \tau_4^{bp1} := 1 - \frac{ad}{(c+d)h + ad}.$$

For all $\tau \in [\tau_4^{min}, \tau_4^{bp2}]$, the intersection cut for ray-sliding triangle $RS_4(\tau)$ is

$$\left(1 - \frac{b}{(b+d)h}\right) s_1 + \left(1 + \frac{ab}{(a+b)h}\right) s_2 + \left(1 - \frac{a}{(a+c)h}\right) s_3 + \left(\frac{1}{\tau}\right) s_4 \geq 1.$$

For all $\tau \in [\tau_4^{bp2}, \tau_4^{bp1}]$, the intersection cut for ray-sliding triangle $RS_4(\tau)$ is

$$\left(1 - \frac{(c+d)(1-\tau)}{c(b+d)\tau}\right) s_1 + \left(1 + \frac{ab}{(a+b)h}\right) s_2 + \left(1 - \frac{a}{(a+c)h}\right) s_3 + \left(\frac{1}{\tau}\right) s_4 \geq 1.$$

For all $\tau \in [\tau_4^{bp1}, \tau_4^{max}]$, the intersection cut for ray-sliding triangle $RS_4(\tau)$ is

$$\left(1 - \frac{(c+d)(1-\tau)}{c(b+d)\tau}\right) s_1 + \left(1 + \frac{ab}{(a+b)h}\right) s_2 + \left(1 - \frac{(c+d)(1-\tau)}{(a+c)d\tau}\right) s_3 + \left(\frac{1}{\tau}\right) s_4 \geq 1.$$

Note that when $\tau = \tau_4^{bp2}$, two of the above cut formulas apply. A straightforward substitution shows that the two formulas agree on this boundary point and that the corresponding intersection cut for ray-sliding triangle $RS_4(\tau_4^{bp2})$ is

$$\left(1 - \frac{b}{(b+d)h}\right) s_1 + \left(1 + \frac{ab}{(a+b)h}\right) s_2 + \left(1 - \frac{a}{(a+c)h}\right) s_3 + \left(1 + \frac{bc}{(c+d)h}\right) s_4 \geq 1.$$

Note that when $\tau = \tau_4^{bp1}$, two of the above cut formulas apply. A straightforward substitution shows that the two formulas agree on this boundary point and that the corresponding intersection cut for ray-sliding triangle $RS_4(\tau_4^{bp1})$ is the same as Case 2.

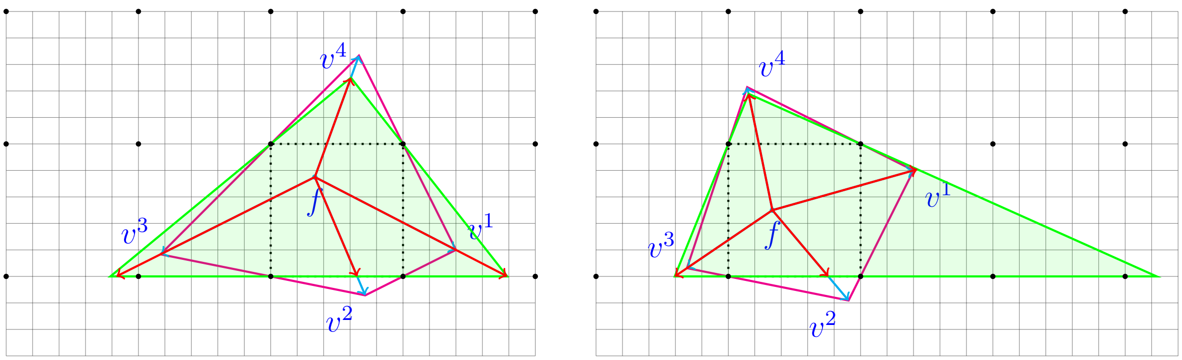

Figure A.11: (left) τ_4^{bp2}-Ray-Sliding Triangle for Ray 4 for $\tau_4^{bp2} = \frac{9}{11}$ and parameters $a = \frac{1}{5}$, $b = \frac{1}{2}$, $c = 1$, $d = 2$, $g = \frac{1}{3}$, and $h = \frac{3}{4}$ and (right) τ_4^{bp1}-Ray-Sliding Triangle for Ray 4 for $\tau_4^{bp1} = \frac{35}{37}$ and parameters $a = \frac{1}{5}$, $b = 2$, $c = 3$, $d = \frac{1}{2}$, $g = \frac{1}{3}$, and $h = \frac{1}{2}$.

Checking Consistency for $h = v_y^3$ and $h = v_y^1$: Note that when $h = v_y^3$, both Case 1 and Case 2 apply. The formulas are consistent because $\tau_4^{min} = \tau_4^{bp1}$. Note that when $g = v_y^1$, both Case 2 and Case 3 apply. The formulas are consistent because $\tau_4^{min} = \tau_4^{bp2}$.

Appendix B

Proofs of Proposition 3.4.2 and Proposition 3.5.2

B.1 Proof of Proposition 3.4.2.

We prove Proposition 3.4.2, the statement of which is reproduced below for convenience.

Proposition 3.4.2. *The optimal value of* (P') *in Lemma 3.4.1 is exactly* $LB_{SW}(\alpha, \beta, g, h)$.

Proof. For fixed $\alpha, \beta > 0$ such that $\alpha \leq \beta$ and g, h such that $(g, h)^T \in R_{SW}(\alpha, \beta)$, the optimization problem (P') is given by

$$\begin{aligned}
\min \quad & s_1 + s_2 + s_3 + s_4 \\
\text{s. t.} \quad & \begin{bmatrix} \left(1 - \frac{\alpha}{(\alpha+\beta)(1-h)}\right) & \left(1 + \frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right) & \left(1 - \frac{\alpha}{(\alpha+\beta)(1-h)}\right) & \left(1 + \frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right) \\ \left(1 + \frac{1}{(\alpha+\beta)(1-g)}\right) & \left(1 - \frac{\alpha}{(\alpha+\beta)(1-g)}\right) & \left(1 + \frac{1}{(\alpha+\beta)(1-g)}\right) & \left(1 - \frac{\alpha}{(\alpha+\beta)(1-g)}\right) \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \geq \mathbb{1} \\
& s_1, s_2, s_3, s_4 \geq 0.
\end{aligned}$$

By Lemma 3.4.1, $LB_{SW}(\alpha, \beta, g, h)$ is a lower bound on the optimal value of (P'). Here we find a feasible solution to (P') of objective function value $LB_{SW}(\alpha, \beta, g, h)$ and conclude the optimal value is exactly $LB_{SW}(\alpha, \beta, g, h)$ as claimed. Our choice for feasible solution to (P') depends on the sign of g and h. There are three cases: (1) $g \geq 0, h \geq 0$, (2) $g < 0, h > 0$ and (3) $g > 0, h < 0$.

Case 1: $g \geq 0, h \geq 0$
Note that since $(g,h)^T \in R_{SW}(\alpha,\beta)$ we have $g \in [0\ ,\ \frac{\alpha}{\alpha+\beta}]$ and $h \in [0\ ,\ \frac{\alpha}{\alpha+\beta}]$. We will show that

$$\hat{s} := \begin{bmatrix} \frac{\alpha(\alpha+\beta)[\beta(1-g)+(1-h)+(\alpha-\beta)g]}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)} \\ \frac{(\beta^2-\alpha^2)h}{\beta[\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)]} \\ \frac{\alpha(\beta^2-\alpha^2)g}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)} \\ \frac{(\alpha+\beta)[\alpha\beta(1-g)+\beta(1-h)+(\alpha-\beta)h]}{\beta[\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)]} \end{bmatrix}$$

is a feasible solution of (P') with objective function value $LB_{SW}(\alpha,\beta,g,h)$. The solution $\hat{s}$ can be constructed by adding the constraints for fixed triangles F_1 and F_4 to (P') and then solving the linear system obtained by setting all four constraints tight. First we check the corresponding objective function value by calculating

$$\begin{aligned}
\mathbb{1}^T\hat{s} &= \frac{\alpha(\alpha+\beta)[\beta(1-g)+(1-h)+(\alpha-\beta)g] + \frac{(\beta^2-\alpha^2)h}{\beta} + \alpha(\beta^2-\alpha^2)g + \frac{(\alpha+\beta)[\alpha\beta(1-g)+\beta(1-h)+(\alpha-\beta)h]}{\beta}}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)} \\
&= \frac{\alpha(\alpha+\beta)[\beta(1-g)+(1-h)] + \frac{(\alpha+\beta)[\alpha\beta(1-g)+\beta(1-h)]}{\beta}}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)} \\
&= \frac{\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)} \\
&= 1 - \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}.
\end{aligned}$$

To check $\hat{s}$ is feasible for (P'), first we show that $\hat{s}$ is non-negative. Each entry of $\hat{s}$ is of the form $C \cdot \frac{1}{K}$ for some constant C where

$$K := \alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h) > 0$$

for any fixed $\alpha,\beta > 0$ with $\alpha \leq \beta$, $g \in \left[0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$, and $h \in \left[0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$. So, to verify $\hat{s} \geq \mathbb{0}$, we check that the corresponding constant C is non-negative for each entry. To verify $\hat{s}_1 \geq 0$, note that

$$\begin{aligned}
\alpha(\alpha+\beta)[\beta(1-g)+(1-h)+(\alpha-\beta)g] &= \alpha(\alpha+\beta)[1+\beta-h+(\alpha-2\beta)g] \\
&\geq \alpha[(1+\beta)(\alpha+\beta)-\alpha+(\alpha-2\beta)\alpha] \\
&= \alpha[\alpha+\beta+\alpha\beta+\beta^2-\alpha+\alpha^2-2\alpha\beta] \\
&= \alpha[\beta+\beta^2+\alpha^2-\alpha\beta] \\
&= \alpha[\beta+\beta(\beta-\alpha)+\alpha^2] \\
&\geq 0
\end{aligned}$$

for any fixed $\alpha, \beta > 0$ with $\alpha \leq \beta$, $g \in \left[0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$, and $h \in \left[0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$. To verify $\hat{s}_2 \geq 0$, note that $\frac{(\beta^2-\alpha^2)h}{\beta} \geq 0$ for any fixed $\alpha, \beta > 0$ with $\alpha \leq \beta$, and $h \in \left[0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$. To verify $\hat{s}_3 \geq 0$, note that $\alpha(\beta^2 - \alpha^2)g \geq 0$ for any fixed $\alpha, \beta > 0$ with $\alpha \leq \beta$, and $g \in \left[0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$. To verify $\hat{s}_4 \geq 0$, note that

$$
\begin{aligned}
\frac{(\alpha+\beta)[\alpha\beta(1-g)+\beta(1-h)+(\alpha-\beta)h]}{\beta} &= \frac{(\alpha+\beta)[\beta+\alpha\beta-\alpha\beta g+(\alpha-2\beta)h]}{\beta} \\
&\geq \frac{(\alpha+\beta)(\beta+\alpha\beta)-\alpha^2\beta+(\alpha-2\beta)\alpha}{\beta} \\
&= \frac{\alpha\beta+\alpha^2\beta+\beta^2+\alpha\beta^2-\alpha^2\beta+(\alpha-2\beta)\alpha}{\beta} \\
&= \frac{\beta^2+\alpha\beta^2+\alpha^2-\alpha\beta}{\beta} \\
&= \frac{\alpha\beta^2+\alpha^2+\beta(\beta-\alpha)}{\beta} \\
&\geq 0
\end{aligned}
$$

for any fixed $\alpha, \beta > 0$ with $\alpha \leq \beta$, $g \in \left[0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$, and $h \in \left[0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$.

Having shown $\hat{s}$ is non-negative, we next verify that $\hat{s}$ satisfies the constraints of (P'). Note that each constraint is of the form $(\mathbb{1}^T + [k_1, k_2, k_3, k_4])s \geq 1$. Since we know $\mathbb{1}^T\hat{s} = 1 - \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}$, we only need to check $k_1\hat{s}_1 + k_2\hat{s}_2 + k_3\hat{s}_3 + k_4\hat{s}_4 \geq \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}$. To verify $\hat{s}$ satisfies the first constraint, we calculate

$$
\begin{aligned}
LHS &= \left(\frac{1}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}\right)\cdot \\
&\quad \left[-\left(\frac{\alpha}{(\alpha+\beta)(1-h)}\right)(\alpha(\alpha+\beta)[\beta(1-g)+(1-h)+(\alpha-\beta)g]) + \left(\frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right)\left(\frac{(\beta^2-\alpha^2)h}{\beta}\right) + \dots \right. \\
&\quad \left. \dots - \left(\frac{\alpha}{(\alpha+\beta)(1-h)}\right)(\alpha(\beta^2-\alpha^2)g) + \left(\frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right)\left(\frac{(\alpha+\beta)[\alpha\beta(1-g)+\beta(1-h)+(\alpha-\beta)h]}{\beta}\right)\right] \\
&= \left(\frac{1}{(1-h)[\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)]}\right)\cdot \\
&\quad [-\alpha^2[\beta(1-g)+(1-h)+(\alpha-\beta)g] + \alpha(\beta-\alpha)h - \alpha^2(\beta-\alpha)g + \alpha[\alpha\beta(1-g)+\beta(1-h)+(\alpha-\beta)h]] \\
&= \left(\frac{-\alpha^2(1-h)+\alpha(\beta-\alpha)h+\alpha[\beta(1-h)+(\alpha-\beta)h]}{(1-h)[\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)]}\right) \\
&= \left(\frac{(1-h)\alpha(\beta-\alpha)}{(1-h)[\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)]}\right) \\
&= \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}
\end{aligned}
$$

and conclude that $\hat{s}$ satisfies the first constraint of (P'); moreover, it satisfies this constraint with equality. To verify $\hat{s}$ satisfies the second constraint, we calculate

$$\begin{aligned}
LHS =& \left(\frac{1}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}\right)\cdot \\
&\left[\left(\frac{1}{(\alpha+\beta)(1-g)}\right)(\alpha(\alpha+\beta)[\beta(1-g)+(1-h)+(\alpha-\beta)g])-\left(\frac{\alpha}{(\alpha+\beta)(1-g)}\right)\left(\frac{(\beta^2-\alpha^2)h}{\beta}\right)+\dots\right. \\
&\left.\quad \dots+\left(\frac{1}{(\alpha+\beta)(1-g)}\right)(\alpha(\beta^2-\alpha^2)g)-\left(\frac{\alpha}{(\alpha+\beta)(1-g)}\right)\left(\frac{(\alpha+\beta)[\alpha\beta(1-g)+\beta(1-h)+(\alpha-\beta)h]}{\beta}\right)\right] \\
=& \left(\frac{1}{(1-g)[\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)]}\right)\cdot \\
&\left[(\alpha[\beta(1-g)+(1-h)+(\alpha-\beta)g])-\left(\frac{\alpha(\beta-\alpha)h}{\beta}\right)+\alpha(\beta-\alpha)g-\left(\frac{\alpha[\alpha\beta(1-g)+\beta(1-h)+(\alpha-\beta)h]}{\beta}\right)\right] \\
=& \left(\frac{\alpha\beta(1-g)-\alpha^2(1-g)}{(1-g)[\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)]}\right) \\
=& \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}
\end{aligned}$$

and conclude that $\hat{s}$ satisfies the second constraint of (P'); moreover, it satisfies this constraint with equality. Thus $\hat{s}$ is a feasible solution of (P') with objective function value $LB_{SW}(\alpha, \beta, g, h)$ as required.

<u>Case 2: $g < 0, h > 0$</u>

Note that since $(g, h)^T \in R_{SW}(\alpha, \beta)$ we have $h \in \left(0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$ and $g \in \left[-\frac{h}{\alpha}\ ,\ 0\right)$. We will show that

$$\hat{s} := \begin{bmatrix} \frac{\alpha(\alpha+\beta)[\beta(1-g)+(1-h)]}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)} \\ \frac{(\beta^2-\alpha^2)h}{\beta[\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)]} \\ 0 \\ \frac{(\alpha+\beta)[\alpha\beta(1-g)+\beta(1-h)+(\alpha-\beta)h]}{\beta[\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)]} \end{bmatrix}$$

is a feasible solution of (P') with objective function value $LB_{SW}(\alpha, \beta, g, h)$. The solution $\hat{s}$ can be constructed by adding the constraint for fixed triangle F_4 to (P') and then solving the linear system obtained by setting all three constraints tight and restricting s_3 to be 0. First we check the corresponding objective function value by calculating

$$\begin{aligned}
\mathbb{1}^T\hat{s} &= \frac{\alpha\beta(\alpha+\beta)[\beta(1-g)+(1-h)]+(\beta^2-\alpha^2)h+(\alpha+\beta)[\alpha\beta(1-g)+\beta(1-h)+(\alpha-\beta)h]}{\beta[\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)]} \\
&= \frac{\beta[\alpha(\alpha+\beta)(1+\beta)(1-g)+(\alpha+\beta)(1+\alpha)(1-h)]}{\beta[\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)]} \\
&= 1-\frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}.
\end{aligned}$$

To check $\hat{s}$ is feasible for (P'), first we show that $\hat{s}$ is non-negative. Each entry of $\hat{s}$ is of the form $C \cdot \frac{1}{K}$ for some constant C where

$$K := \alpha(\beta-\alpha) + \alpha(1+\beta)(\alpha+\beta)(1-g) + (1+\alpha)(\alpha+\beta)(1-h) > 0$$

for any fixed $\alpha, \beta > 0$ with $\alpha \leq \beta$, $h \in \left(0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$, and $g \in \left[-\frac{h}{\alpha}, 0\right)$. So, to verify $\hat{s} \geq \mathbb{0}$, we check that the corresponding constant C is non-negative for each entry. To verify $\hat{s}_1 \geq 0$, note that $\alpha(\alpha+\beta)[\beta(1-g)+(1-h)] \geq 0$ for any fixed $\alpha, \beta > 0$ with $\alpha \leq \beta$, $h \in \left(0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$, and $g \in \left[-\frac{h}{\alpha}, 0\right)$. To verify $\hat{s}_2 \geq 0$, note that $\frac{(\beta^2-\alpha^2)h}{\beta} \geq 0$ for any fixed $\alpha, \beta > 0$ with $\alpha \leq \beta$, and $h \in \left(0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$. Clearly $\hat{s}_3 \geq 0$ because $\hat{s}_3 = 0$. To verify $\hat{s}_4 \geq 0$, note that

$$\begin{aligned}
\frac{(\alpha+\beta)[\alpha\beta(1-g)+\beta(1-h)+(\alpha-\beta)h]}{\beta} &= \frac{(\alpha+\beta)[\beta+\alpha\beta-\alpha\beta g+(\alpha-2\beta)h]}{\beta} \\
&\geq \frac{(\alpha+\beta)(\beta+\alpha\beta)+(\alpha-2\beta)\alpha}{\beta} \\
&= \frac{\alpha\beta+\alpha^2\beta+\beta^2+\alpha\beta^2+\alpha^2-2\alpha\beta}{\beta} \\
&= \frac{\alpha^2\beta+\beta(\beta-\alpha)+\alpha\beta^2+\alpha^2}{\beta} \\
&\geq 0
\end{aligned}$$

for any fixed $\alpha, \beta > 0$ with $\alpha \leq \beta$, $h \in \left(0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$, and $g \in \left[-\frac{h}{\alpha}, 0\right)$.

Having shown $\hat{s}$ is non-negative, we next verify that $\hat{s}$ satisfies the constraints of (P'). Note that each constraint is of the form $(\mathbb{1}^T + [k_1, k_2, k_3, k_4])s \geq 1$. Since we know $\mathbb{1}^T\hat{s} = 1 - \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}$, we only need to check $k_1\hat{s}_1 + k_2\hat{s}_2 + k_3\hat{s}_3 + k_4\hat{s}_4 \geq \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}$. To verify $\hat{s}$ satisfies the first constraint, we calculate

$$\begin{aligned}
LHS = &\left(\frac{1}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}\right) \cdot \\
&\left[-\left(\frac{\alpha}{(\alpha+\beta)(1-h)}\right)(\alpha(\alpha+\beta)[\beta(1-g)+(1-h)]) + \left(\frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right)\left(\frac{(\beta^2-\alpha^2)h}{\beta}\right) + \ldots\right. \\
&\left.\ldots + \left(\frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right)\left(\frac{(\alpha+\beta)[\alpha\beta(1-g)+\beta(1-h)+(\alpha-\beta)h]}{\beta}\right)\right] \\
= &\left(\frac{-\alpha^2[\beta(1-g)+(1-h)]+\alpha(\beta-\alpha)h+\alpha[\alpha\beta(1-g)+\beta(1-h)+(\alpha-\beta)h]}{(1-h)[\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)]}\right)
\end{aligned}$$

$$= \left(\frac{\alpha(\beta-\alpha)(1-h)}{(1-h)[\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)]} \right)$$
$$= \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}$$

and conclude that $\hat{s}$ satisfies the first constraint of (P'); moreover, it satisfies this constraint with equality. To verify $\hat{s}$ satisfies the second constraint, we calculate

$$LHS = \left(\frac{1}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)} \right) \cdot$$
$$\left[\left(\frac{1}{(\alpha+\beta)(1-g)} \right) (\alpha(\alpha+\beta)[\beta(1-g)+(1-h)]) - \left(\frac{\alpha}{(\alpha+\beta)(1-g)} \right) \left(\frac{(\beta^2-\alpha^2)h}{\beta} \right) + \dots \right.$$
$$\left. \dots - \left(\frac{\alpha}{(\alpha+\beta)(1-g)} \right) \left(\frac{(\alpha+\beta)[\alpha\beta(1-g)+\beta(1-h)+(\alpha-\beta)h]}{\beta} \right) \right]$$
$$= \left(\frac{\alpha\beta[\beta(1-g)+(1-h)] - \alpha(\beta-\alpha)h - \alpha[\alpha\beta(1-g)+\beta(1-h)+(\alpha-\beta)h]}{\beta(1-g)[\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)]} \right)$$
$$= \left(\frac{\alpha\beta(\beta-\alpha)(1-g)}{\beta(1-g)[\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)]} \right)$$
$$= \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}$$

and conclude that $\hat{s}$ satisfies the second constraint of (P'); moreover, it satisfies this constraint with equality. Thus $\hat{s}$ is a feasible solution of (P') with objective function value $LB_{SW}(\alpha,\beta,g,h)$ as required.

Case 3: $g > 0, h < 0$

Note that since $(g,h)^T \in R_{SW}(\alpha,\beta)$ we have $g \in \left(0 \ , \ \frac{\alpha}{\alpha+\beta}\right]$ and $h \in [-\alpha g \ , \ 0)$. We will show that

$$\hat{s} := \begin{bmatrix} \frac{\alpha(\alpha+\beta)[\beta(1-g)+(1-h)+(\alpha-\beta)g]}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)} \\ 0 \\ \frac{\alpha(\beta^2-\alpha^2)g}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)} \\ \frac{(\alpha+\beta)[\alpha(1-g)+(1-h)]}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)} \end{bmatrix}$$

is a feasible solution of (P') with objective function value $LB_{SW}(\alpha,\beta,g,h)$. The solution $\hat{s}$ can be constructed by adding the constraint for fixed triangle F_1 to (P') and then solving the linear system obtained by setting all three constraints tight and restricting s_2 to be 0.

First we check the corresponding objective function value by calculating

$$
\begin{aligned}
\mathbb{1}^T\hat{s} &= \frac{\alpha(\alpha+\beta)[\beta(1-g)+(1-h)+(\alpha-\beta)g]+\alpha(\beta^2-\alpha^2)g+(\alpha+\beta)[\alpha(1-g)+(1-h)]}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)} \\
&= \frac{\alpha(\alpha+\beta)(1+\beta)(1-g)+(\alpha+\beta)(1+\alpha)(1-h)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)} \\
&= 1-\frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}.
\end{aligned}
$$

To check $\hat{s}$ is feasible for (P'), first we show that $\hat{s}$ is non-negative. Each entry of $\hat{s}$ is of the form $C\cdot\frac{1}{K}$ for some constant C where

$$K := \alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h) > 0$$

for any fixed $\alpha,\beta>0$ with $\alpha\le\beta$, $g\in\left(0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$, and $h\in[-\alpha g\ ,\ 0)$. So, to verify $\hat{s}\ge\mathbb{0}$, we check that the corresponding constant C is non-negative for each entry. To verify $\hat{s}_1\ge 0$, note that

$$
\begin{aligned}
\alpha(\alpha+\beta)[\beta(1-g)+(1-h)+(\alpha-\beta)g] &= \alpha(\alpha+\beta)[1+\beta-h+(\alpha-2\beta)g] \\
&\ge \alpha[(\alpha+\beta)(1+\beta)+(\alpha-2\beta)\alpha] \\
&= \alpha(\alpha+\beta+\beta(\beta-\alpha)+\alpha^2) \\
&\ge 0
\end{aligned}
$$

for any fixed $\alpha,\beta>0$ with $\alpha\le\beta$, $g\in\left(0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$, and $h\in[-\alpha g\ ,\ 0)$. Clearly $\hat{s}_2\ge 0$ because $\hat{s}_2=0$. To verify $\hat{s}_3\ge 0$, note that $\alpha(\beta^2-\alpha^2)g\ge 0$ for any fixed $\alpha,\beta>0$ with $\alpha\le\beta$, and $g\in\left(0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$. To verify $\hat{s}_4\ge 0$, note that $(\alpha+\beta)[\alpha(1-g)+(1-h)]\ge 0$ for any fixed $\alpha,\beta>0$ with $\alpha\le\beta$, $g\in\left(0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$, and $h\in[-\alpha g\ ,\ 0)$.

Having shown $\hat{s}$ is non-negative, we next verify that $\hat{s}$ the constraints of (P'). Note that each constraint is of the form $(\mathbb{1}^T+[k_1,k_2,k_3,k_4])s\ge 1$. Since we know $\mathbb{1}^T\hat{s}=1-\frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}$, we only need to check $k_1\hat{s}_1+k_2\hat{s_2}+k_3\hat{s_3}+k_4\hat{s_4}\ge$

$\frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}$. To verify $\hat{s}$ satisfies the first constraint, we calculate

$$\begin{aligned}
LHS &= \left(\frac{1}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}\right)\cdot \\
&\quad\Bigg[-\left(\frac{\alpha}{(\alpha+\beta)(1-h)}\right)(\alpha(\alpha+\beta)[\beta(1-g)+(1-h)+(\alpha-\beta)g])+\dots \\
&\qquad \dots-\left(\frac{\alpha}{(\alpha+\beta)(1-h)}\right)\left(\alpha(\beta^2-\alpha^2)g\right)+\left(\frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right)((\alpha+\beta)[\alpha(1-g)+(1-h)])\Bigg] \\
&= \left(\frac{-\alpha^2[\beta(1-g)+(1-h)+(\alpha-\beta)g]-\alpha^2(\beta-\alpha)g+\alpha\beta[\alpha(1-g)+(1-h)]}{(1-h)[\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)]}\right) \\
&= \left(\frac{\alpha(\beta-\alpha)(1-h)}{(1-h)[\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)]}\right) \\
&= \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}
\end{aligned}$$

and conclude that $\hat{s}$ satisfies the first constraint of (P'); moreover, it satisfies this constraint with equality. To verify $\hat{s}$ satisfies the second constraint, we calculate

$$\begin{aligned}
LHS &= \left(\frac{1}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}\right)\cdot \\
&\quad\Bigg[\left(\frac{1}{(\alpha+\beta)(1-g)}\right)(\alpha(\alpha+\beta)[\beta(1-g)+(1-h)+(\alpha-\beta)g])+\dots \\
&\qquad \dots+\left(\frac{1}{(\alpha+\beta)(1-g)}\right)\left(\alpha(\beta^2-\alpha^2)g\right)-\left(\frac{\alpha}{(\alpha+\beta)(1-g)}\right)((\alpha+\beta)[\alpha(1-g)+(1-h)])\Bigg] \\
&= \left(\frac{\alpha[\beta(1-g)+(1-h)+(\alpha-\beta)g]+\alpha(\beta-\alpha)g-\alpha[\alpha(1-g)+(1-h)]}{(1-g)[\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)]}\right) \\
&= \left(\frac{\alpha(\beta-\alpha)(1-g)}{(1-g)[\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)]}\right) \\
&= \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha(1+\beta)(\alpha+\beta)(1-g)+(1+\alpha)(\alpha+\beta)(1-h)}
\end{aligned}$$

and conclude that $\hat{s}$ satisfies the second constraint of (P'); moreover, it satisfies this constraint with equality. Thus $\hat{s}$ is a feasible solution of (P') with objective function value $LB_{SW}(\alpha,\beta,g,h)$ as required. $\square$

B.2 Proof of Proposition 3.5.2

We prove Proposition 3.5.2, the statement of which is reproduced below for convenience.

Proposition 3.5.2. *The optimal value of* (P') *in Lemma 3.5.1 is exactly* $LB_S(\alpha, \beta, g, h)$.

Proof. For fixed $\alpha, \beta > 0$ such that $\alpha \leq \beta$ and g, h such that $(g, h)^T \in R_S(\alpha, \beta)$, the optimization problem (P') is given by

$$\begin{array}{ll} \min & s_1 + s_2 + s_3 + s_4 \\ \text{s. t.} & \begin{bmatrix} \left(1 + \frac{\alpha}{\beta(\alpha+\beta)g}\right) & \left(1 - \frac{\alpha^2}{\beta(\alpha+\beta)g}\right) & \left(1 + \frac{1}{(\alpha+\beta)g}\right) & \left(1 - \frac{\alpha}{(\alpha+\beta)g}\right) \\ \left(1 - \frac{\alpha}{(\alpha+\beta)(1-h)}\right) & \left(1 + \frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right) & \left(1 - \frac{\alpha}{(\alpha+\beta)(1-h)}\right) & \left(1 + \frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right) \\ \left(1 + \frac{1}{(\alpha+\beta)(1-g)}\right) & \left(1 - \frac{\alpha}{(\alpha+\beta)(1-g)}\right) & \left(1 + \frac{\alpha}{\beta(\alpha+\beta)(1-g)}\right) & \left(1 - \frac{\alpha^2}{\beta(\alpha+\beta)(1-g)}\right) \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \geq \mathbb{1} \\ & s_1, s_2, s_3, s_4 \geq 0. \end{array}$$

By Lemma 3.5.1, $LB_S(\alpha, \beta, g, h)$ is a lower bound on the optimal value of (P'). Here we find a feasible solution to (P') of objective function value $LB_S(\alpha, \beta, g, h)$ and conclude the optimal value is exactly $LB_S(\alpha, \beta, g, h)$ as claimed. Our choice for feasible solution to (P') depends on the sign of h. There are two cases: (1) $h \geq 0$, and (2) $h < 0$.

<u>Case 1: $h \geq 0$</u>

Note that since $(g, h)^T \in R_S(\alpha, \beta)$ we have $g \in \left[\frac{\alpha}{\alpha+\beta}\ ,\ \frac{\beta}{\alpha+\beta}\right]$ and $h \in \left[0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$. We will show that

$$\hat{s} := \begin{bmatrix} \frac{\alpha[\beta^3 - \beta^2(\alpha+\beta)g + (\beta^2-\alpha^2)h]}{\beta[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]} \\ \frac{(\beta^2-\alpha^2)h}{\beta[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]} \\ \frac{\alpha(\alpha+\beta)[\beta+\beta^2 g+(\alpha-2\beta)h]}{\beta[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]} \\ \frac{\alpha\beta^2+\alpha\beta+\beta^2+(\alpha+\beta)(\alpha-2\beta)h}{\beta[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]} \end{bmatrix}$$

is a feasible solution of (P') with objective function value $LB_S(\alpha, \beta, g, h)$. The solution $\hat{s}$ can be constructed by adding the constraint for fixed triangle F_4 to (P') and then solving the linear system obtained by setting all four constraints tight. First we check the corresponding objective function value by calculating

$$\begin{aligned}
\mathbb{1}^T\hat{s} &= \frac{\begin{array}{c}\alpha[\beta^3-\beta^2(\alpha+\beta)g+(\beta^2-\alpha^2)h]+(\beta^2-\alpha^2)h+\alpha(\alpha+\beta)[\beta+\beta^2 g+(\alpha-2\beta)h]+\dots\\ \dots+\alpha\beta^2+\alpha\beta+\beta^2+(\alpha+\beta)(\alpha-2\beta)h\end{array}}{\beta[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}\\
&= \frac{\alpha[\beta^3+(\beta^2-\alpha^2)h]+(\beta^2-\alpha^2)h+\alpha(\alpha+\beta)[\beta+(\alpha-2\beta)h]+\alpha\beta^2+\alpha\beta+\beta^2+(\alpha+\beta)(\alpha-2\beta)h}{\beta[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}\\
&= \frac{\alpha\beta^3+(1+\alpha)(\beta^2-\alpha^2)h+\alpha\beta(\alpha+\beta)+\alpha\beta^2+\alpha\beta+\beta^2+(1+\alpha)(\alpha+\beta)(\alpha-2\beta)h}{\beta[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}\\
&= \frac{\alpha\beta^3+\alpha^2\beta+\alpha\beta^2+\alpha\beta^2+\alpha\beta+\beta^2-\beta(1+\alpha)(\alpha+\beta)h}{\beta[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}\\
&= \frac{\alpha\beta^3+\alpha\beta^2+\beta(1+\alpha)(\alpha+\beta)(1-h)}{\beta[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}\\
&= 1-\frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}.
\end{aligned}$$

To check $\hat{s}$ is feasible for (P'), first we show that $\hat{s}$ is non-negative. Each entry of $\hat{s}$ is of the form $C\cdot\frac{1}{D}$ for some constant C where

$$D=\beta[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]>0$$

for any fixed $\alpha,\beta>0$ with $\alpha\le\beta$, and $h\in\left[0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$. So, to verify $\hat{s}\ge\mathbb{0}$, we check that the corresponding constant C is non-negative for each entry. To verify $\hat{s}_1\ge 0$, note that $\alpha[\beta^3-\beta^2(\alpha+\beta)g+(\beta^2-\alpha^2)h]\ge\alpha[\beta^3-\beta^3]=0$ for any fixed $\alpha,\beta>0$ with $\alpha\le\beta$, $g\in\left[\frac{\alpha}{\alpha+\beta}\ ,\ \frac{\beta}{\alpha+\beta}\right]$, and $h\in\left[0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$. To verify $\hat{s}_2\ge 0$, note that $(\beta^2-\alpha^2)h\ge 0$ for any fixed $\alpha,\beta>0$ with $\alpha\le\beta$, and $h\in\left[0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$. To verify $\hat{s}_3\ge 0$, note that

$$\begin{aligned}
\alpha(\alpha+\beta)[\beta+\beta^2 g+(\alpha-2\beta)h] &\ge \alpha[\beta(\alpha+\beta)+\alpha\beta^2+(\alpha-2\beta)\alpha]\\
&= \alpha[\alpha\beta^2+\beta(\beta-\alpha)+\alpha^2]\\
&\ge 0
\end{aligned}$$

for any fixed $\alpha,\beta>0$ with $\alpha\le\beta$, $g\in\left[\frac{\alpha}{\alpha+\beta},\frac{\beta}{\alpha+\beta}\right]$, and $h\in\left[0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$. To verify $\hat{s}_4\ge 0$, note that

$$\begin{aligned}
\alpha\beta^2+\alpha\beta+\beta^2+(\alpha+\beta)(\alpha-2\beta)h &\ge \alpha\beta^2+\alpha\beta+\beta^2+(\alpha-2\beta)\alpha\\
&= \alpha\beta^2+\beta(\beta-\alpha)+\alpha^2\\
&\ge 0
\end{aligned}$$

for any fixed $\alpha, \beta > 0$, and $h \in \left[0\ ,\ \frac{\alpha}{\alpha+\beta}\right]$.

Having shown $\hat{s}$ is non-negative, we next verify that $\hat{s}$ satisfies the constraints of (P'). Note that each constraint is of the form $(\mathbb{1}^T + [k_1, k_2, k_3, k_4])s \geq 1$. Since we know $\mathbb{1}^T \hat{s} = 1 - \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}$, we only need to check $k_1\hat{s}_1 + k_2\hat{s_2} + k_3\hat{s_3} + k_4\hat{s_4} \geq \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}$. To verify $\hat{s}$ satisfies the first constraint, we calculate

$$
\begin{aligned}
LHS = &\left(\frac{1}{\beta[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}\right)\cdot \\
&\left[\left(\frac{\alpha}{\beta(\alpha+\beta)g}\right)(\alpha[\beta^3-\beta^2(\alpha+\beta)g+(\beta^2-\alpha^2)h]) - \left(\frac{\alpha^2}{\beta(\alpha+\beta)g}\right)((\beta^2-\alpha^2)h) + \dots\right. \\
&\left.\dots + \left(\frac{1}{(\alpha+\beta)g}\right)(\alpha(\alpha+\beta)[\beta+\beta^2 g+(\alpha-2\beta)h]) - \left(\frac{\alpha}{(\alpha+\beta)g}\right)(\alpha\beta^2+\alpha\beta+\beta^2+(\alpha+\beta)(\alpha-2\beta)h)\right] \\
= &\left(\frac{1}{\beta^2(\alpha+\beta)g[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}\right)\cdot \\
&\left[(\alpha^2[\beta^3-\beta^2(\alpha+\beta)g+(\beta^2-\alpha^2)h]) - \alpha^2(\beta^2-\alpha^2)h + \dots\right. \\
&\left.\dots + \alpha\beta(\alpha+\beta)[\beta+\beta^2 g+(\alpha-2\beta)h] - \alpha\beta[\alpha\beta^2+\alpha\beta+\beta^2+(\alpha+\beta)(\alpha-2\beta)h]\right] \\
= &\left(\frac{-\alpha^2\beta^2(\alpha+\beta)g+\alpha\beta^3(\alpha+\beta)g}{\beta^2(\alpha+\beta)g[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}\right) \\
= &\frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}
\end{aligned}
$$

and conclude that $\hat{s}$ satisfies the first constraint of (P'); moreover, it satisfies this constraint with equality. To verify $\hat{s}$ satisfies the second constraint, we calculate

$$
\begin{aligned}
LHS = &\left(\frac{1}{\beta[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}\right)\cdot \\
&\left[-\left(\frac{\alpha}{(\alpha+\beta)(1-h)}\right)(\alpha[\beta^3-\beta^2(\alpha+\beta)g+(\beta^2-\alpha^2)h]) + \left(\frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right)((\beta^2-\alpha^2)h) + \dots\right. \\
&\left.\dots - \left(\frac{\alpha}{(\alpha+\beta)(1-h)}\right)(\alpha(\alpha+\beta)[\beta+\beta^2 g+(\alpha-2\beta)h]) + \left(\frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right)(\alpha\beta^2+\alpha\beta+\beta^2+(\alpha+\beta)(\alpha-2\beta)h)\right] \\
= &\left(\frac{1}{\beta(\alpha+\beta)(1-h)[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}\right)\cdot \\
&\left[-\alpha^2[\beta^3-\beta^2(\alpha+\beta)g+(\beta^2-\alpha^2)h] + \alpha\beta(\beta^2-\alpha^2)h + \dots\right. \\
&\left.\dots - \alpha^2(\alpha+\beta)[\beta+\beta^2 g+(\alpha-2\beta)h] + \alpha\beta[\alpha\beta^2+\alpha\beta+\beta^2+(\alpha+\beta)(\alpha-2\beta)h]\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{1}{\beta(\alpha+\beta)(1-h)[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}\right)\cdot \\
&\quad \Big[-\alpha^2(\beta^2-\alpha^2)h+\alpha\beta(\beta^2-\alpha^2)h-\alpha^2(\alpha+\beta)[\beta+(\alpha-2\beta)h+\alpha\beta[\alpha\beta+\beta^2+(\alpha+\beta)(\alpha-2\beta)h]\Big] \\
&= \left(\frac{1}{\beta(\alpha+\beta)(1-h)[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}\right)\cdot \\
&\quad \Big[\alpha(\beta-\alpha)(\beta^2-\alpha^2)h-\alpha^2(\alpha+\beta)(\alpha-2\beta)h+\alpha\beta(\alpha+\beta)(\alpha-2\beta)h-\alpha^2\beta(\alpha+\beta)+\alpha\beta^2(\alpha+\beta)\Big] \\
&= \frac{\alpha(\beta-\alpha)^2(\alpha+\beta)h+\alpha(\alpha-2\beta)(\beta-\alpha)(\alpha+\beta)h+\alpha\beta(\beta-\alpha)(\alpha+\beta)}{\beta(\alpha+\beta)(1-h)[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]} \\
&= \frac{\alpha\beta(\beta-\alpha)(\alpha+\beta)(1-h)}{\beta(\alpha+\beta)(1-h)[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]} \\
&= \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}
\end{aligned}
$$

and conclude that $\hat{s}$ satisfies the second constraint of (P'); moreover, it satisfies this constraint with equality. To verify $\hat{s}$ satisfies the third constraint, we calculate

$$
\begin{aligned}
LHS &= \left(\frac{1}{\beta[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}\right)\cdot \\
&\quad \Bigg[\left(\frac{1}{(\alpha+\beta)(1-g)}\right)(\alpha[\beta^3-\beta^2(\alpha+\beta)g+(\beta^2-\alpha^2)h])-\left(\frac{\alpha}{(\alpha+\beta)(1-g)}\right)((\beta^2-\alpha^2)h)+\dots \\
&\quad \dots+\left(\frac{\alpha}{\beta(\alpha+\beta)(1-g)}\right)(\alpha(\alpha+\beta)[\beta+\beta^2g+(\alpha-2\beta)h])-\left(\frac{\alpha^2}{\beta(\alpha+\beta)(1-g)}\right)(\alpha\beta^2+\alpha\beta+\beta^2+(\alpha+\beta)(\alpha-2\beta)h)\Bigg] \\
&= \left(\frac{1}{\beta^2(\alpha+\beta)(1-g)[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}\right)\cdot \\
&\quad \Big[\alpha\beta[\beta^3-\beta^2(\alpha+\beta)g+(\beta^2-\alpha^2)h]-\alpha\beta(\beta^2-\alpha^2)h+\dots \\
&\quad \dots+\alpha^2(\alpha+\beta)[\beta+\beta^2g+(\alpha-2\beta)h]-\alpha^2[\alpha\beta^2+\alpha\beta+\beta^2+(\alpha+\beta)(\alpha-2\beta)h]\Big] \\
&= \frac{\alpha\beta[\beta^3-\beta^2(\alpha+\beta)g]+\alpha^2(\alpha+\beta)[\beta+\beta^2g]-\alpha^2[\alpha\beta^2+\alpha\beta+\beta^2]}{\beta^2(\alpha+\beta)(1-g)[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]} \\
&= \frac{\alpha\beta^4-\alpha\beta^3(\alpha+\beta)g+\alpha^2\beta^2(\alpha+\beta)g-\alpha^3\beta^2}{\beta^2(\alpha+\beta)(1-g)[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]} \\
&= \frac{\alpha\beta^2(\alpha+\beta)(\beta-\alpha)(1-g)}{\beta^2(\alpha+\beta)(1-g)[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]} \\
&= \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}
\end{aligned}
$$

and conclude that $\hat{s}$ satisfies the third constraint of (P'); moreover, it satisfies this constraint with equality. Thus $\hat{s}$ is a feasible solution of (P') with objective function value $LB_S(\alpha,\beta,g,h)$ as required.

Case 2: $h < 0$

Note that since $(g,h)^T \in R_S(\alpha,\beta)$ we have $g \in \left[\frac{\alpha}{\alpha+\beta}\ ,\ \frac{\beta}{\alpha+\beta}\right]$ and $h \in [-\alpha g, 0)$. We will show that

$$\hat{s} := \begin{bmatrix} \frac{\alpha\beta^2-\alpha\beta(\alpha+\beta)g}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)} \\ 0 \\ \frac{\alpha(\alpha+\beta)[(1-h)+\beta g]}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)} \\ \frac{\alpha\beta+(\alpha+\beta)(1-h)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)} \end{bmatrix}$$

is a feasible solution of (P') with objective function value $LB_S(\alpha,\beta,g,h)$. The solution $\hat{s}$ can be constructed by solving the linear system obtained by setting all three constraints of (P') tight and restricting s_2 to be 0. First we check the corresponding objective function value by calculating

$$\begin{aligned} \mathbb{1}^T\hat{s} &= \frac{\alpha\beta^2-\alpha\beta(\alpha+\beta)g+\alpha(\alpha+\beta)[(1-h)+\beta g]+\alpha\beta+(\alpha+\beta)(1-h)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)} \\ &= \frac{\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)} \\ &= 1-\frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}. \end{aligned}$$

To check $\hat{s}$ is feasible for (P'), first we show that $\hat{s}$ is non-negative. Each entry of $\hat{s}$ is of the form $C \cdot \frac{1}{D}$ for some constant C where

$$D = \alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h) > 0$$

for any fixed $\alpha,\beta > 0$ with $\alpha \leq \beta$, and $h < 0$. So, to verify $\hat{s} \geq \mathbb{0}$, we check that the corresponding constant C is non-negative for each entry. To verify $\hat{s}_1 \geq 0$, note that $\alpha\beta^2-\alpha\beta(\alpha+\beta)g \geq \alpha\beta^2-\alpha\beta^2 = 0$ for any fixed $\alpha,\beta > 0$ with $\alpha \leq \beta$, and $g \in \left[\frac{\alpha}{\alpha+\beta}\ ,\ \frac{\beta}{\alpha+\beta}\right]$. Clearly $\hat{s}_2 \geq 0$ because $\hat{s}_2 = 0$. To verify $\hat{s}_3 \geq 0$, note that $\alpha(\alpha+\beta)[(1-h)+\beta g] \geq 0$ for any fixed $\alpha,\beta > 0$, $g \in \left[\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta}\right]$, and $h \in [-\alpha g\ ,\ 0)$. To verify $\hat{s}_4 \geq 0$, note that $\alpha\beta+(\alpha+\beta)(1-h) \geq 0$ for any fixed $\alpha,\beta > 0$, and $h < 0$.

Having shown $\hat{s}$ is non-negative, we next verify that $\hat{s}$ satisfies the constraints of (P'). Note that each constraint is of the form $(\mathbb{1}^T + [k_1,k_2,k_3,k_4])s \geq 1$. Since we know $\mathbb{1}^T\hat{s} = 1-\frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}$, we only need to check $k_1\hat{s}_1+k_2\hat{s_2}+k_3\hat{s_3}+k_4\hat{s_4} \geq$

$\frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}$. To verify $\hat{s}$ satisfies the first constraint, we calculate

$$\begin{aligned}
LHS &= \left(\frac{1}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}\right)\cdot\left[\left(\frac{\alpha}{\beta(\alpha+\beta)g}\right)\left(\alpha\beta^2-\alpha\beta(\alpha+\beta)g\right)+\dots\right.\\
&\qquad\left.\dots+\left(\frac{1}{(\alpha+\beta)g}\right)(\alpha(\alpha+\beta)[(1-h)+\beta g])-\left(\frac{\alpha}{(\alpha+\beta)g}\right)(\alpha\beta+(\alpha+\beta)(1-h))\right]\\
&= \left(\frac{\alpha^2\beta-\alpha^2(\alpha+\beta)g+\alpha(\alpha+\beta)(1-h)+\alpha\beta(\alpha+\beta)g-\alpha^2\beta-\alpha(\alpha+\beta)(1-h)}{(\alpha+\beta)g[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}\right)\\
&= \left(\frac{\alpha(\beta-\alpha)(\alpha+\beta)g}{(\alpha+\beta)g[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}\right)\\
&= \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}
\end{aligned}$$

and conclude that $\hat{s}$ satisfies the first constraint of (P'); moreover, it satisfies this constraint with equality. To verify $\hat{s}$ satisfies the second constraint, we calculate

$$\begin{aligned}
LHS &= \left(\frac{1}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}\right)\cdot\left[-\left(\frac{\alpha}{(\alpha+\beta)(1-h)}\right)\left(\alpha\beta^2-\alpha\beta(\alpha+\beta)g\right)+\dots\right.\\
&\qquad\left.\dots-\left(\frac{\alpha}{(\alpha+\beta)(1-h)}\right)(\alpha(\alpha+\beta)[(1-h)+\beta g])+\left(\frac{\alpha\beta}{(\alpha+\beta)(1-h)}\right)(\alpha\beta+(\alpha+\beta)(1-h))\right]\\
&= \frac{-\alpha^2\beta^2+\alpha^2\beta(\alpha+\beta)g-\alpha^2(\alpha+\beta)[(1-h)+\beta g]+\alpha^2\beta^2+\alpha\beta(\alpha+\beta)(1-h)}{(\alpha+\beta)(1-h)[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}\\
&= \frac{\alpha(\beta-\alpha)(\alpha+\beta)(1-h)}{(\alpha+\beta)(1-h)[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}\\
&= \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}
\end{aligned}$$

and conclude that $\hat{s}$ satisfies the second constraint of (P'); moreover, it satisfies this constraint with equality. To verify $\hat{s}$ satisfies the third constraint, we calculate

$$LHS = \left(\frac{1}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}\right)\cdot\left[\left(\frac{1}{(\alpha+\beta)(1-g)}\right)\left(\alpha\beta^2-\alpha\beta(\alpha+\beta)g\right)+\dots\right.$$

$$\left.\dots+\left(\frac{\alpha}{\beta(\alpha+\beta)(1-g)}\right)(\alpha(\alpha+\beta)[(1-h)+\beta g])-\left(\frac{\alpha^2}{\beta(\alpha+\beta)(1-g)}\right)(\alpha\beta+(\alpha+\beta)(1-h))\right]$$

$$= \frac{\alpha\beta^3-\alpha\beta^2(\alpha+\beta)g+\alpha^2(\alpha+\beta)[(1-h)+\beta g]-\alpha^3\beta-\alpha^2(\alpha+\beta)(1-h)}{\beta(\alpha+\beta)(1-g)[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}$$

$$= \frac{\alpha\beta^3-\alpha\beta^2(\alpha+\beta)g+\alpha^2\beta(\alpha+\beta)g-\alpha^3\beta}{\beta(\alpha+\beta)(1-g)[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}$$

$$= \frac{\alpha\beta(\alpha+\beta)(\beta-\alpha)(1-g)}{\beta(\alpha+\beta)(1-g)[\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)]}$$

$$= \frac{\alpha(\beta-\alpha)}{\alpha(\beta-\alpha)+\alpha\beta(1+\beta)+(1+\alpha)(\alpha+\beta)(1-h)}$$

and conclude that $\hat{s}$ satisfies the third constraint of (P'); moreover, it satisfies this constraint with equality. Thus $\hat{s}$ is a feasible solution of (P') with objective function value $LB_S(\alpha,\beta,g,h)$ as required. □

Appendix C

Proofs of Lemma 4.1.1, Lemma 4.2.1 and Lemma 4.3.1 Covering $ad = bc$ Case

We prove Lemma 4.1.1, Lemma 4.2.1, and Lemma 4.3.1, the statements of which are reproduced below for convenience.

Lemma 4.1.1. *For fixed $a, b, c, d > 0$ with $ad \leq bc$ and g, h with $(g, h)^T \in R_{Central}(a, b, c, d)$, let (P') be the parameterized linear program obtained from $(P(a, b, c, d, g, h))$ by replacing constraint (4.7) with the constraint $s \geq \mathbb{0}$ and the intersection cuts for ray-sliding triangles $RS_1(\tau_1^{bp1})$, $RS_2(\tau_2^{bp1})$, $RS_3(\tau_3^{bp1})$, and $RS_4(\tau_4^{bp1})$. Then, the optimal value of (P') is equal to*

$$LB_{Central}(a, b, c, d, g, h) := 1 - \frac{bc - ad}{bc - ad + t_1^C + t_2^C + t_3^C + t_4^C}$$

where

$$t_1^C = (b + d)[bc(1 - g) + ch] \quad , \quad t_2^C = \left(\frac{a + b}{a}\right)[ch + acg],$$
$$t_3^C = (a + c)[bcg + b(1 - h)] \quad , \quad \textit{and} \quad t_4^C = \left(\frac{c + d}{d}\right)[b(1 - h) + bd(1 - g)].$$

Proof. Optimization problem (P') is given by

$$\begin{aligned} \min \quad & \mathbb{1}^T s \\ \text{subject to} \quad & \left[\mathbb{1}\mathbb{1}^T + A\right] s \geq \mathbb{1} \\ & s \geq \mathbb{0} \end{aligned}$$

where

$$A := \begin{bmatrix} \frac{1}{1-g} & 0 & 0 & 0 \\ 0 & \frac{a}{h} & 0 & 0 \\ 0 & 0 & \frac{1}{g} & 0 \\ 0 & 0 & 0 & \frac{d}{1-h} \end{bmatrix} \begin{bmatrix} 1 & -1 & \frac{a}{b} & -\frac{a}{b} \\ -\frac{d}{c} & \frac{b}{a} & -1 & 1 \\ \frac{d}{c} & -\frac{d}{c} & 1 & -1 \\ -1 & 1 & -\frac{a}{b} & \frac{c}{d} \end{bmatrix} \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{a}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{d}{c+d} \end{bmatrix}.$$

The dual (D') of (P') is given by

$$\begin{aligned} \max \quad & \mathbb{1}^T v \\ \text{subject to} \quad & \left[\mathbb{1}\mathbb{1}^T + A^T\right] v \leq \mathbb{1} \\ & v \geq \mathbb{0}. \end{aligned}$$

Let

$$\hat{s} := \left(\frac{1}{bc - ad + t_1^C + t_2^C + t_3^C + t_4^C}\right) \begin{bmatrix} t_1^C \\ t_2^C \\ t_3^C \\ t_4^C \end{bmatrix}$$

and

$$\begin{aligned} \hat{v} &:= \left(\frac{1}{bc - ad + t_1^C + t_2^C + t_3^C + t_4^C}\right) \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \\ &:= \left(\frac{1}{bc - ad + t_1^C + t_2^C + t_3^C + t_4^C}\right) \begin{bmatrix} (1-g)[bc(b+d) + b(c+d)] \\ h[c(b+d) + c\left(\frac{a+b}{a}\right)] \\ g[c(a+b) + bc(a+c)] \\ (1-h)[b\left(\frac{c+d}{d}\right) + b(a+c)] \end{bmatrix}. \end{aligned}$$

We claim that $\hat{s}$ and $\hat{v}$ are a pair of primal and dual optimal solutions for (P') and (D').

First we show that $\hat{s}$ is feasible for (P'). Note that for all $i \in \{1, 2, 3, 4\}$, we have $t_i^C > 0$ because $a, b, c, d > 0$ and $(g, h)^T \in R_{Central}(a, b, c, d) \subseteq (0, 1)^2$. Thus $\hat{s} \geq \mathbb{0}$ because

$bc - ad \geq 0$. To check $\hat{s}$ satisfies the four intersection cut constraints we calculate

$$\begin{aligned}
\left(\mathbb{1}\mathbb{1}^T + A\right)\hat{s} &= \left(\mathbb{1}\mathbb{1}^T + \begin{bmatrix} \frac{1}{1-g} & 0 & 0 & 0 \\ 0 & \frac{a}{h} & 0 & 0 \\ 0 & 0 & \frac{1}{g} & 0 \\ 0 & 0 & 0 & \frac{d}{1-h} \end{bmatrix} \begin{bmatrix} 1 & -1 & \frac{a}{b} & -\frac{a}{b} \\ -\frac{d}{c} & \frac{b}{a} & -1 & 1 \\ \frac{d}{c} & -\frac{d}{c} & 1 & -1 \\ -1 & 1 & -\frac{a}{b} & \frac{c}{d} \end{bmatrix} \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{a}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{d}{c+d} \end{bmatrix}\right)\hat{s} \\
&= \left(\frac{1}{bc - ad + t_1^C + t_2^C + t_3^C + t_4^C}\right)\left(\hat{s}^1 + \hat{s}^2\right)
\end{aligned}$$

where

$$\begin{aligned}
\hat{s}^1 &:= \mathbb{1}\mathbb{1}^T \begin{bmatrix} t_1^C \\ t_2^C \\ t_3^C \\ t_4^C \end{bmatrix} \\
&= (t_1^C + t_2^C + t_3^C + t_4^C)\mathbb{1}
\end{aligned}$$

and

$$\begin{aligned}
\hat{s}^2 &:= \begin{bmatrix} \frac{1}{1-g} & 0 & 0 & 0 \\ 0 & \frac{a}{h} & 0 & 0 \\ 0 & 0 & \frac{1}{g} & 0 \\ 0 & 0 & 0 & \frac{d}{1-h} \end{bmatrix} \begin{bmatrix} 1 & -1 & \frac{a}{b} & -\frac{a}{b} \\ -\frac{d}{c} & \frac{b}{a} & -1 & 1 \\ \frac{d}{c} & -\frac{d}{c} & 1 & -1 \\ -1 & 1 & -\frac{a}{b} & \frac{c}{d} \end{bmatrix} \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{a}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{d}{c+d} \end{bmatrix} \begin{bmatrix} t_1^C \\ t_2^C \\ t_3^C \\ t_4^C \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{1-g} & 0 & 0 & 0 \\ 0 & \frac{a}{h} & 0 & 0 \\ 0 & 0 & \frac{1}{g} & 0 \\ 0 & 0 & 0 & \frac{d}{1-h} \end{bmatrix} \begin{bmatrix} 1 & -1 & \frac{a}{b} & -\frac{a}{b} \\ -\frac{d}{c} & \frac{b}{a} & -1 & 1 \\ \frac{d}{c} & -\frac{d}{c} & 1 & -1 \\ -1 & 1 & -\frac{a}{b} & \frac{c}{d} \end{bmatrix} \begin{bmatrix} bc(1-g) + ch \\ ch + acg \\ bcg + b(1-h) \\ b(1-h) + bd(1-g) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{1-g} & 0 & 0 & 0 \\ 0 & \frac{a}{h} & 0 & 0 \\ 0 & 0 & \frac{1}{g} & 0 \\ 0 & 0 & 0 & \frac{d}{1-h} \end{bmatrix} \begin{bmatrix} (bc-ad)(1-g) \\ (bc-ad)\left[\frac{h}{a}\right] \\ (bc-ad)g \\ (bc-ad)\left[\frac{(1-h)}{d}\right] \end{bmatrix} \\
&= (bc-ad)\mathbb{1}.
\end{aligned}$$

Thus

$$\begin{aligned}
\left(\mathbb{1}\mathbb{1}^T + A\right)\hat{s} &= \left(\frac{1}{bc - ad + t_1^C + t_2^C + t_3^C + t_4^C}\right)\left(t_1^C + t_2^C + t_3^C + t_4^C + bc - ad\right)\mathbb{1} \\
&= \mathbb{1}
\end{aligned}$$

and so $\hat{s}$ satisfies all the primal constraints. Moreover, it satisfies these constraints with equality. The objective function value of $\hat{s}$ is

$$\mathbb{1}^T \hat{s} = \frac{t_1^C + t_2^C + t_3^C + t_4^C}{bc - ad + t_1^C + t_2^C + t_3^C + t_4^C} = 1 - \frac{bc - ad}{bc - ad + t_1^C + t_2^C + t_3^C + t_4^C}.$$

Next we show that $\hat{v}$ is feasible for (D'). As we noted before $bc - ad \geq 0$ and $t_i^C > 0$ for all $i \in \{1, 2, 3, 4\}$. It is clear that u_1, u_2, u_3, and u_4 are non-negative because $a, b, c, d > 0$ and $(g, h)^T \in R_{Central}(a, b, c, d) \subseteq (0, 1)^2$. Thus $\hat{v} \geq \mathbb{0}$. To check $\hat{v}$ satisfies the four dual constraints we calculate

$$\begin{aligned}
\left(\mathbb{1}\mathbb{1}^T + A^T\right)\hat{v} &= \left(\mathbb{1}\mathbb{1}^T + \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{a}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{d}{c+d} \end{bmatrix} \begin{bmatrix} 1 & -\frac{d}{c} & \frac{d}{c} & -1 \\ -1 & \frac{b}{a} & -\frac{d}{c} & 1 \\ \frac{a}{b} & -1 & 1 & -\frac{a}{b} \\ -\frac{a}{b} & 1 & -1 & \frac{c}{d} \end{bmatrix} \begin{bmatrix} \frac{1}{1-g} & 0 & 0 & 0 \\ 0 & \frac{a}{h} & 0 & 0 \\ 0 & 0 & \frac{1}{g} & 0 \\ 0 & 0 & 0 & \frac{d}{1-h} \end{bmatrix}\right)\hat{v} \\
&= \left(\frac{1}{bc - ad + t_1^C + t_2^C + t_3^C + t_4^C}\right)\left(\hat{v}^1 + \hat{v}^2\right)
\end{aligned}$$

where

$$\begin{aligned}
\hat{v}^1 &= \mathbb{1}\mathbb{1}^T \begin{bmatrix} (1-g)[bc(b+d) + b(c+d)] \\ h[c(b+d) + c\left(\frac{a+b}{a}\right)] \\ g[c(a+b) + bc(a+c)] \\ (1-h)[b\left(\frac{c+d}{d}\right) + b(a+c)] \end{bmatrix} \\
&= (t_1^C + t_2^C + t_3^C + t_4^C)\mathbb{1}
\end{aligned}$$

and

$$\begin{aligned}
\hat{v}^2 &= \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{a}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{d}{c+d} \end{bmatrix} \begin{bmatrix} 1 & -\frac{d}{c} & \frac{d}{c} & -1 \\ -1 & \frac{b}{a} & -\frac{d}{c} & 1 \\ \frac{a}{b} & -1 & 1 & -\frac{a}{b} \\ -\frac{a}{b} & 1 & -1 & \frac{c}{d} \end{bmatrix} \begin{bmatrix} \frac{1}{1-g} & 0 & 0 & 0 \\ 0 & \frac{a}{h} & 0 & 0 \\ 0 & 0 & \frac{1}{g} & 0 \\ 0 & 0 & 0 & \frac{d}{1-h} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{a}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{d}{c+d} \end{bmatrix} \begin{bmatrix} 1 & -\frac{d}{c} & \frac{d}{c} & -1 \\ -1 & \frac{b}{a} & -\frac{d}{c} & 1 \\ \frac{a}{b} & -1 & 1 & -\frac{a}{b} \\ -\frac{a}{b} & 1 & -1 & \frac{c}{d} \end{bmatrix} \begin{bmatrix} bc(b+d) + b(c+d) \\ ac(b+d) + c(a+b) \\ c(a+b) + bc(a+c) \\ b(c+d) + bd(a+c) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{a}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{d}{c+d} \end{bmatrix} \begin{bmatrix} (bc-ad)(b+d) \\ (bc-ad)\left[\frac{a+b}{a}\right] \\ (bc-ad)(a+c) \\ (bc-ad)\left[\frac{c+d}{d}\right] \end{bmatrix} \\
&= (bc - ad)\mathbb{1}.
\end{aligned}$$

Thus

$$\left(\mathbb{1}\mathbb{1}^T + A^T\right)\hat{v} = \left(\frac{1}{bc - ad + t_1^C + t_2^C + t_3^C + t_4^C}\right)\left(t_1^C + t_2^C + t_3^C + t_4^C + bc - ad\right)\mathbb{1} = \mathbb{1}$$

and so $\hat{v}$ satisfies all the dual constraints. Moreover, it satisfies these constraints with equality.

So $\hat{s}$ and $\hat{v}$ are a pair of primal and dual feasible solutions satisfying complementary slackness. Therefore, $\hat{s}$ is an optimal solution for (P'). So the optimal value of (P') is exactly $1 - \frac{bc-ad}{bc-ad+t_1^C+t_2^C+t_3^C+t_4^C} = LB_{Central}(a,b,c,d,g,h)$. □

Lemma 4.2.1. *For fixed $a,b,c,d > 0$ with $ad \leq bc$ and g,h with $(g,h)^T \in R_{SW}(a,b,c,d)$, let (P') be the parameterized linear program obtained from $(P(a,b,c,d,g,h))$ by replacing constraint (4.7) with the constraint $s \geq \mathbb{0}$ and the intersection cuts for ray-sliding triangles $RS_2(\tau_2^{bp2})$, $RS_2(\tau_2^{bp1})$, $RS_3(\tau_3^{bp2})$, and $RS_3(\tau_3^{bp1})$. Then, the optimal value of (P') is equal to*

$$LB_{SW}(a,b,c,d,g,h) := 1 - \frac{bc - ad}{bc - ad + t_1^{SW} + t_2^{SW} + t_3^{SW} + t_4^{SW}}$$

where

$$\begin{aligned} t_1^{SW} &= (b+d)[a(1-h) + bc(1-g)] \quad , \quad t_2^{SW} = (a+b)[(1-h) + d(1-g)], \\ t_3^{SW} &= (a+c)[b(1-h) + bd(1-g)] \quad , \quad \text{and} \quad t_4^{SW} = (c+d)\left[\frac{b(1-h)}{d} + b(1-g)\right]. \end{aligned}$$

Proof. Optimization problem (P') is given by

$$\begin{aligned} \min \quad & \mathbb{1}^T s \\ \text{subject to} \quad & \left[\mathbb{1}\mathbb{1}^T + A\right] s \geq \mathbb{1} \\ & s \geq \mathbb{0} \end{aligned}$$

where

$$A := \begin{bmatrix} \frac{1}{1-h} & 0 & 0 & 0 \\ 0 & \frac{1}{1-h} & 0 & 0 \\ 0 & 0 & \frac{1}{1-g} & 0 \\ 0 & 0 & 0 & \frac{1}{1-g} \end{bmatrix} \begin{bmatrix} -d & bc & -c & cd \\ -d & ad & -\frac{ad}{b} & cd \\ 1 & -a & \frac{c}{d} & -c \\ 1 & -a & \frac{a}{b} & -\frac{ad}{b} \end{bmatrix} \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{1}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{1}{c+d} \end{bmatrix}.$$

The dual (D') of (P') is given by

$$\begin{aligned} \max \quad & \mathbb{1}^T v \\ \text{subject to} \quad & \left[\mathbb{1}\mathbb{1}^T + A^T\right] v \leq \mathbb{1} \\ & v \geq \mathbb{0}. \end{aligned}$$

Let

$$\hat{s} := \left(\frac{1}{bc - ad + t_1^{SW} + t_2^{SW} + t_3^{SW} + t_4^{SW}} \right) \begin{bmatrix} t_1^{SW} \\ t_2^{SW} \\ t_3^{SW} \\ t_4^{SW} \end{bmatrix}$$

and

$$\begin{aligned} \hat{v} &:= \left(\frac{1}{bc - ad + t_1^{SW} + t_2^{SW} + t_3^{SW} + t_4^{SW}} \right) \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \\ &:= \left(\frac{1}{bc - ad + t_1^{SW} + t_2^{SW} + t_3^{SW} + t_4^{SW}} \right) \begin{bmatrix} (1-h)[a(b+d) + (a+b)] \\ (1-h)[b(a+c) + \frac{b}{d}(c+d)] \\ (1-g)[d(a+b) + bd(a+c)] \\ (1-g)[bc(b+d) + b(c+d)] \end{bmatrix}. \end{aligned}$$

We claim that $\hat{s}$ and $\hat{v}$ are a pair of primal and dual optimal solutions for (P') and (D').

First we show that $\hat{s}$ is feasible for (P'). Note that for all $i \in \{1, 2, 3, 4\}$, we have $t_i^{SW} > 0$ for all $a, b, c, d > 0$ and $(g, h)^T \in R_{SW}(a, b, c, d) \subseteq \{(x, y)^T \in \mathbb{R}^2 : x < 1, y < 1\}$. Thus $\hat{s} \geq \mathbb{0}$ because $bc - ad \geq 0$. To check $\hat{s}$ satisfies the four intersection cut constraints we calculate

$$\begin{aligned} \left(\mathbb{1}\mathbb{1}^T + A\right)\hat{s} &= \left(\mathbb{1}\mathbb{1}^T + \begin{bmatrix} \frac{1}{1-h} & 0 & 0 & 0 \\ 0 & \frac{1}{1-h} & 0 & 0 \\ 0 & 0 & \frac{1}{1-g} & 0 \\ 0 & 0 & 0 & \frac{1}{1-g} \end{bmatrix} \begin{bmatrix} -d & bc & -c & cd \\ -d & ad & -\frac{ad}{b} & cd \\ 1 & -a & \frac{c}{d} & -c \\ 1 & -a & \frac{a}{b} & -\frac{ad}{b} \end{bmatrix} \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{1}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{1}{c+d} \end{bmatrix} \right) \hat{s} \\ &= \left(\frac{1}{bc - ad + t_1^{SW} + t_2^{SW} + t_3^{SW} + t_4^{SW}} \right) \left(\hat{s}^1 + \hat{s}^2\right) \end{aligned}$$

where

$$\hat{s}^1 := \mathbb{1}\mathbb{1}^T \begin{bmatrix} t_1^{SW} \\ t_2^{SW} \\ t_3^{SW} \\ t_4^{SW} \end{bmatrix} = (t_1^{SW} + t_2^{SW} + t_3^{SW} + t_4^{SW})\mathbb{1}$$

and

$$\hat{s}^2 = \begin{bmatrix} \frac{1}{1-h} & 0 & 0 & 0 \\ 0 & \frac{1}{1-h} & 0 & 0 \\ 0 & 0 & \frac{1}{1-g} & 0 \\ 0 & 0 & 0 & \frac{1}{1-g} \end{bmatrix} \begin{bmatrix} -d & bc & -c & cd \\ -d & ad & -\frac{ad}{b} & cd \\ 1 & -a & \frac{c}{d} & -c \\ 1 & -a & \frac{a}{b} & -\frac{ad}{b} \end{bmatrix} \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{1}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{1}{c+d} \end{bmatrix} \begin{bmatrix} t_1^{SW} \\ t_2^{SW} \\ t_3^{SW} \\ t_4^{SW} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{1-h} & 0 & 0 & 0 \\ 0 & \frac{1}{1-h} & 0 & 0 \\ 0 & 0 & \frac{1}{1-g} & 0 \\ 0 & 0 & 0 & \frac{1}{1-g} \end{bmatrix} \begin{bmatrix} -d & bc & -c & cd \\ -d & ad & -\frac{ad}{b} & cd \\ 1 & -a & \frac{c}{d} & -c \\ 1 & -a & \frac{a}{b} & -\frac{ad}{b} \end{bmatrix} \begin{bmatrix} a(1-h) + bc(1-g) \\ (1-h) + d(1-g) \\ b(1-h) + bd(1-g) \\ \frac{b(1-h)}{d} + b(1-g) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{1-h} & 0 & 0 & 0 \\ 0 & \frac{1}{1-h} & 0 & 0 \\ 0 & 0 & \frac{1}{1-g} & 0 \\ 0 & 0 & 0 & \frac{1}{1-g} \end{bmatrix} \begin{bmatrix} (bc-ad)(1-h) \\ (bc-ad)(1-h) \\ (bc-ad)(1-g) \\ (bc-ad)(1-g) \end{bmatrix}$$

$$= (bc - ad)\mathbb{1}.$$

Thus

$$\begin{aligned} \left(\mathbb{1}\mathbb{1}^T + A\right)\hat{s} &= \left(\frac{1}{bc - ad + t_1^{SW} + t_2^{SW} + t_3^{SW} + t_4^{SW}}\right)\left(t_1^{SW} + t_2^{SW} + t_3^{SW} + t_4^{SW} + bc - ad\right)\mathbb{1} \\ &= \mathbb{1} \end{aligned}$$

and so $\hat{s}$ satisfies all the primal constraints. Moreover, it satisfies these constraints with equality. The objective function value of $\hat{s}$ is

$$\mathbb{1}^T\hat{s} = \frac{t_1^{SW} + t_2^{SW} + t_3^{SW} + t_4^{SW}}{bc - ad + t_1^{SW} + t_2^{SW} + t_3^{SW} + t_4^{SW}} = 1 - \frac{bc - ad}{bc - ad + t_1^{SW} + t_2^{SW} + t_3^{SW} + t_4^{SW}}.$$

Next we show that $\hat{v}$ is feasible for (D'). As we noted before $bc - ad \geq 0$ and $t_i^{SW} > 0$ for all $i \in \{1, 2, 3, 4\}$. It is clear that u_1, u_2, u_3, and u_4 are non-negative because $a, b, c, d > 0$ and $(g, h)^T \in R_{SW}(a, b, c, d) \subseteq \{(x, y)^T \in \mathbb{R}^2 : x < 1, y < 1\}$. Thus $\hat{v} \geq \mathbb{0}$. To check $\hat{v}$ satisfies the four dual constraints we calculate

$$\begin{aligned} \left(\mathbb{1}\mathbb{1}^T + A^T\right)\hat{v} &= \left(\mathbb{1}\mathbb{1}^T + \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{1}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{1}{c+d} \end{bmatrix} \begin{bmatrix} -d & -d & 1 & 1 \\ bc & ad & -a & -a \\ -c & -\frac{ad}{b} & \frac{c}{d} & \frac{a}{b} \\ cd & cd & -c & -\frac{ad}{b} \end{bmatrix} \begin{bmatrix} \frac{1}{1-h} & 0 & 0 & 0 \\ 0 & \frac{1}{1-h} & 0 & 0 \\ 0 & 0 & \frac{1}{1-g} & 0 \\ 0 & 0 & 0 & \frac{1}{1-g} \end{bmatrix}\right)\hat{v} \\ &= \left(\frac{1}{bc - ad + t_1^{SW} + t_2^{SW} + t_3^{SW} + t_4^{SW}}\right)\left(\hat{v}^1 + \hat{v}^2\right) \end{aligned}$$

where

$$
\begin{aligned}
\hat{v}^1 &= \mathbb{1}\mathbb{1}^T \begin{bmatrix} (1-h)[a(b+d)+(a+b)] \\ (1-h)[b(a+c)+\frac{b}{d}(c+d)] \\ (1-g)[d(a+b)+bd(a+c)] \\ (1-g)[bc(b+d)+b(c+d)] \end{bmatrix} \\
&= (t_1^{SW}+t_2^{SW}+t_3^{SW}+t_4^{SW})\mathbb{1}
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{v}^2 &= \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{1}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{1}{c+d} \end{bmatrix} \begin{bmatrix} -d & -d & 1 & 1 \\ bc & ad & -a & -a \\ -c & -\frac{ad}{b} & \frac{c}{d} & \frac{a}{b} \\ cd & cd & -c & -\frac{ad}{b} \end{bmatrix} \begin{bmatrix} \frac{1}{1-h} & 0 & 0 & 0 \\ 0 & \frac{1}{1-h} & 0 & 0 \\ 0 & 0 & \frac{1}{1-g} & 0 \\ 0 & 0 & 0 & \frac{1}{1-g} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{1}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{1}{c+d} \end{bmatrix} \begin{bmatrix} -d & -d & 1 & 1 \\ bc & ad & -a & -a \\ -c & -\frac{ad}{b} & \frac{c}{d} & \frac{a}{b} \\ cd & cd & -c & -\frac{ad}{b} \end{bmatrix} \begin{bmatrix} a(b+d)+(a+b) \\ b(a+c)+\frac{b}{d}(c+d) \\ d(a+b)+bd(a+c) \\ bc(b+d)+b(c+d) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{1}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{1}{c+d} \end{bmatrix} \begin{bmatrix} (bc-ad)(b+d) \\ (bc-ad)(a+b) \\ (bc-ad)(a+c) \\ (bc-ad)(c+d) \end{bmatrix} \\
&= (bc-ad)\mathbb{1}
\end{aligned}
$$

Thus

$$
(\mathbb{1}\mathbb{1}^T + A^T)\,\hat{v} = \left(\frac{1}{bc-ad+t_1^{SW}+t_2^{SW}+t_3^{SW}+t_4^{SW}}\right)\left(t_1^{SW}+t_2^{SW}+t_3^{SW}+t_4^{SW}+bc-ad\right)\mathbb{1} = \mathbb{1}
$$

and so $\hat{v}$ satisfies all the dual constraints. Moreover, it satisfies these constraints with equality.

So $\hat{s}$ and $\hat{v}$ are a pair of primal and dual feasible solutions satisfying complementary slackness. Therefore, $\hat{s}$ is an optimal solution for (P'). So the optimal value of (P') is exactly $1 - \frac{bc-ad}{bc-ad+t_1^{SW}+t_2^{SW}+t_3^{SW}+t_4^{SW}} = LB_{SW}(a,b,c,d,g,h)$. □

Lemma 4.3.1. *For fixed $a,b,c,d > 0$ with $ad \le bc$ and g,h with $(g,h)^T \in R_S(a,b,c,d)$, let (P') be the parameterized linear program obtained from $(P(a,b,c,d,g,h))$ by replacing constraint (4.7) with the constraint $s \ge \mathbb{0}$ and the intersection cuts for ray-sliding triangles*

$RS_1(\tau_1^{bp1})$, $RS_2(\tau_2^{bp2})$, $RS_2(\tau_2^{bp1})$, and $RS_3(\tau_3^{bp1})$. Then, the optimal value of (P') is equal to

$$LB_S(a,b,c,d,g,h) := 1 - \frac{bc-ad}{bc-ad+t_1^S+t_2^S+t_3^S+t_4^S}$$

where

$$t_1^S = (b+d)[a(1-h)+bc(1-g)] \quad , \quad t_2^S = (a+b)[cg+(1-h)],$$
$$t_3^S = (a+c)[bcg+b(1-h)] \quad , \quad \textit{and} \;\; t_4^S = (c+d)\left[\frac{b}{d}(1-h)+b(1-g)\right].$$

Proof. Optimization problem (P') is given by

$$\begin{aligned} \min \quad & \mathbb{1}^T s \\ \text{subject to} \quad & \left[\mathbb{1}\mathbb{1}^T + A\right] s \geq \mathbb{1} \\ & s \geq \mathbb{0} \end{aligned}$$

where

$$A := \begin{bmatrix} \frac{1}{g} & 0 & 0 & 0 \\ 0 & \frac{1}{1-h} & 0 & 0 \\ 0 & 0 & \frac{1}{1-h} & 0 \\ 0 & 0 & 0 & \frac{1}{1-g} \end{bmatrix} \begin{bmatrix} \frac{d}{c} & -\frac{ad}{c} & 1 & -d \\ -d & bc & -c & cd \\ -d & ad & -\frac{ad}{b} & cd \\ 1 & -a & \frac{a}{b} & -\frac{ad}{b} \end{bmatrix} \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{1}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{1}{c+d} \end{bmatrix}.$$

The dual (D') of (P') is given by

$$\begin{aligned} \max \quad & \mathbb{1}^T v \\ \text{subject to} \quad & \left[\mathbb{1}\mathbb{1}^T + A^T\right] v \leq \mathbb{1} \\ & v \geq \mathbb{0}. \end{aligned}$$

Let

$$\hat{s} := \left(\frac{1}{bc-ad+t_1^S+t_2^S+t_3^S+t_4^S}\right) \begin{bmatrix} t_1^S \\ t_2^S \\ t_3^S \\ t_4^S \end{bmatrix}$$

and

$$\hat{v} := \left(\frac{1}{bc - ad + t_1^S + t_2^S + t_3^S + t_4^S}\right)\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$:= \left(\frac{1}{bc - ad + t_1^S + t_2^S + t_3^S + t_4^S}\right)\begin{bmatrix} g[c(a+b) + bc(a+c)] \\ (1-h)[a(b+d) + (a+b)] \\ (1-h)[b(a+c) + \frac{b}{d}(c+d)] \\ (1-g)[bc(b+d) + b(c+d)] \end{bmatrix}.$$

We claim that $\hat{s}$ and $\hat{v}$ are a pair of primal and dual optimal solutions for (P') and (D'). First we show that $\hat{s}$ is feasible for (P'). Note that for all $i \in \{1,2,3,4\}$, we have $t_i^S > 0$ for all $a,b,c,d > 0$ and $(g,h)^T \in R_S(a,b,c,d) \subseteq \{(x,y)^T \in \mathbb{R}^2 : 0 < x < 1, y < 1\}$. Thus $\hat{s} \geq \mathbb{0}$ because $bc - ad \geq 0$. To check $\hat{s}$ satisfies the four intersection cut constraints we calculate

$$\left(\mathbb{1}\mathbb{1}^T + A\right)\hat{s} = \left(\mathbb{1}\mathbb{1}^T + \begin{bmatrix} \frac{1}{g} & 0 & 0 & 0 \\ 0 & \frac{1}{1-h} & 0 & 0 \\ 0 & 0 & \frac{1}{1-h} & 0 \\ 0 & 0 & 0 & \frac{1}{1-g} \end{bmatrix}\begin{bmatrix} \frac{d}{c} & -\frac{ad}{c} & 1 & -d \\ -d & bc & -c & cd \\ -d & ad & -\frac{ad}{b} & cd \\ 1 & -a & \frac{a}{b} & -\frac{ad}{b} \end{bmatrix}\begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{1}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{1}{c+d} \end{bmatrix}\right)\hat{s}$$

$$= \left(\frac{1}{bc - ad + t_1^S + t_2^S + t_3^S + t_4^S}\right)\left(\hat{s}^1 + \hat{s}^2\right)$$

where

$$\hat{s}^1 := \mathbb{1}\mathbb{1}^T\begin{bmatrix} t_1^S \\ t_2^S \\ t_3^S \\ t_4^S \end{bmatrix} = (t_1^S + t_2^S + t_3^S + t_4^S)\mathbb{1}$$

and

$$\hat{s}^2 = \begin{bmatrix} \frac{1}{g} & 0 & 0 & 0 \\ 0 & \frac{1}{1-h} & 0 & 0 \\ 0 & 0 & \frac{1}{1-h} & 0 \\ 0 & 0 & 0 & \frac{1}{1-g} \end{bmatrix}\begin{bmatrix} \frac{d}{c} & -\frac{ad}{c} & 1 & -d \\ -d & bc & -c & cd \\ -d & ad & -\frac{ad}{b} & cd \\ 1 & -a & \frac{a}{b} & -\frac{ad}{b} \end{bmatrix}\begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{1}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{1}{c+d} \end{bmatrix}\begin{bmatrix} t_1^S \\ t_2^S \\ t_3^S \\ t_4^S \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{g} & 0 & 0 & 0 \\ 0 & \frac{1}{1-h} & 0 & 0 \\ 0 & 0 & \frac{1}{1-h} & 0 \\ 0 & 0 & 0 & \frac{1}{1-g} \end{bmatrix}\begin{bmatrix} \frac{d}{c} & -\frac{ad}{c} & 1 & -d \\ -d & bc & -c & cd \\ -d & ad & -\frac{ad}{b} & cd \\ 1 & -a & \frac{a}{b} & -\frac{ad}{b} \end{bmatrix}\begin{bmatrix} a(1-h) + bc(1-g) \\ cg + (1-h) \\ bcg + b(1-h) \\ \frac{b}{d}(1-h) + b(1-g) \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} \frac{1}{g} & 0 & 0 & 0 \\ 0 & \frac{1}{1-h} & 0 & 0 \\ 0 & 0 & \frac{1}{1-h} & 0 \\ 0 & 0 & 0 & \frac{1}{1-g} \end{bmatrix} \begin{bmatrix} (bc-ad)g \\ (bc-ad)(1-h) \\ (bc-ad)(1-h) \\ (bc-ad)(1-g) \end{bmatrix} \\
&= (bc-ad)\mathbb{1}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left(\mathbb{1}\mathbb{1}^T + A\right)\hat{s} &= \left(\frac{1}{bc-ad+t_1^S+t_2^S+t_3^S+t_4^S}\right)\left(t_1^S+t_2^S+t_3^S+t_4^S+bc-ad\right)\mathbb{1} \\
&= \mathbb{1}
\end{aligned}
$$

and so $\hat{s}$ satisfies all the primal constraints. Moreover, it satisfies these constraints with equality. The objective function value of $\hat{s}$ is

$$
\mathbb{1}^T\hat{s} = \frac{t_1^S+t_2^S+t_3^S+t_4^S}{bc-ad+t_1^S+t_2^S+t_3^S+t_4^S} = 1 - \frac{bc-ad}{bc-ad+t_1^S+t_2^S+t_3^S+t_4^S}.
$$

Next we show that $\hat{v}$ is feasible for (D'). As we noted before $bc - ad \geq 0$ and $t_i^S > 0$ for all $i \in \{1,2,3,4\}$. It is clear that u_1, u_2, u_3, and u_4 are non-negative because $a,b,c,d > 0$ and $(g,h)^T \in R_S(a,b,c,d) \subseteq \{(x,y)^T \in \mathbb{R}^2 : 0 < x < 1, y < 1\}$. Thus $\hat{v} \geq \mathbb{0}$. To check $\hat{v}$ satisfies the four dual constraints we calculate

$$
\begin{aligned}
\left(\mathbb{1}\mathbb{1}^T + A^T\right)\hat{v} &= \left(\mathbb{1}\mathbb{1}^T + \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{1}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{1}{c+d} \end{bmatrix} \begin{bmatrix} \frac{d}{c} & -d & -d & 1 \\ -\frac{ad}{c} & bc & ad & -a \\ 1 & -c & -\frac{ad}{b} & \frac{a}{b} \\ -d & cd & cd & -\frac{ad}{b} \end{bmatrix} \begin{bmatrix} \frac{1}{g} & 0 & 0 & 0 \\ 0 & \frac{1}{1-h} & 0 & 0 \\ 0 & 0 & \frac{1}{1-h} & 0 \\ 0 & 0 & 0 & \frac{1}{1-g} \end{bmatrix}\right)\hat{v} \\
&= \left(\frac{1}{bc-ad+t_1^S+t_2^S+t_3^S+t_4^S}\right)\left(\hat{v}^1 + \hat{v}^2\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{v}^1 &= \mathbb{1}\mathbb{1}^T \begin{bmatrix} g[c(a+b)+bc(a+c)] \\ (1-h)[a(b+d)+(a+b)] \\ (1-h)[b(a+c)+\frac{b}{d}(c+d)] \\ (1-g)[bc(b+d)+b(c+d)] \end{bmatrix} \\
&= (t_1^S+t_2^S+t_3^S+t_4^S)\mathbb{1}
\end{aligned}
$$

and

$$\hat{v}^2 = \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{1}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{1}{c+d} \end{bmatrix} \begin{bmatrix} \frac{d}{c} & -d & -d & 1 \\ -\frac{ad}{c} & bc & ad & -a \\ 1 & -c & -\frac{ad}{b} & \frac{a}{b} \\ -d & cd & cd & -\frac{ad}{b} \end{bmatrix} \begin{bmatrix} \frac{1}{g} & 0 & 0 & 0 \\ 0 & \frac{1}{1-h} & 0 & 0 \\ 0 & 0 & \frac{1}{1-h} & 0 \\ 0 & 0 & 0 & \frac{1}{1-g} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{1}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{1}{c+d} \end{bmatrix} \begin{bmatrix} \frac{d}{c} & -d & -d & 1 \\ -\frac{ad}{c} & bc & ad & -a \\ 1 & -c & -\frac{ad}{b} & \frac{a}{b} \\ -d & cd & cd & -\frac{ad}{b} \end{bmatrix} \begin{bmatrix} c(a+b) + bc(a+c) \\ a(b+d) + (a+b) \\ b(a+c) + \frac{b}{d}(c+d) \\ bc(b+d) + b(c+d) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{b+d} & 0 & 0 & 0 \\ 0 & \frac{1}{a+b} & 0 & 0 \\ 0 & 0 & \frac{1}{a+c} & 0 \\ 0 & 0 & 0 & \frac{1}{c+d} \end{bmatrix} \begin{bmatrix} (bc-ad)(b+d) \\ (bc-ad)(a+b) \\ (bc-ad)(a+c) \\ (bc-ad)(c+d) \end{bmatrix}$$

$$= (bc - ad)\mathbb{1}$$

Thus

$$\left(\mathbb{1}\mathbb{1}^T + A^T\right)\hat{v} = \left(\frac{1}{bc - ad + t_1^S + t_2^S + t_3^S + t_4^S}\right)\left(t_1^S + t_2^S + t_3^S + t_4^S + bc - ad\right)\mathbb{1} = \mathbb{1}$$

and so $\hat{v}$ satisfies all the dual constraints. Moreover, it satisfies these constraints with equality.

So $\hat{s}$ and $\hat{v}$ are a pair of primal and dual feasible solutions satisfying complementary slackness. Therefore, $\hat{s}$ is an optimal solution for (P'). So the optimal value of (P') is exactly $1 - \frac{bc-ad}{bc-ad+t_1^S+t_2^S+t_3^S+t_4^S} = LB_S(a, b, c, d, g, h)$. $\square$