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A note on the stochastic version of the Gronwall lemma

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ABSTRACT

We prove a stochastic version of the Gronwall lemma assuming that the underlying martingale has a terminal random value in L_p, where $1 \leq p < \infty$. The proof of the present result is mainly based on a sharp martingale inequality of the Doob-type.

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1. Introduction

In [1], Scheutzow proved a stochastic version of the Gronwall lemma. The proof in [1] and also in the present author's paper [2] are essentially based on the Burkholder martingale inequality [3]. Recently, various authors (see [4–7],etc) have established different versions of the stochastic Gronwall lemma. The aim of the present paper is to prove some new and related results. Throughout the paper, we shall assume that the underlying martingale has a terminal random value in L_p, where $1 \leq p < \infty$. Under this restriction, our proof employs a sharp martingale inequality of the Doob-type ([8,9]) proved independently by Gilat [10] and Jacka [11]. The present results are quite useful in establishing estimates for moments of solutions for a certain class of stochastic differential equations. For estimates of such type or their variants, see for instance Lapeyre [12], where these estimates are established by entirely using Fernique's inequality [13].

Let M be a martingale defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. Assume that M terminates at a random variable X in L_p with $1 \leq p < \infty$. In what follows, $||X||$ denotes the L_p-norm of the random variable X and M^* is the maximal function of the modulus of M. For $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, it is proved in [10] and [11] that

$$\mathbf{E}[M^*] \leq \left(\Gamma(q+1)\right)^{1/q} ||X||_p, \tag{1}$$

where $\Gamma(q+1)$ is a gamma function, and is the best possible. In the special case when $p = q = 2$, see Dubins and Schwarz [14].

In the next section, we shall state and prove a stochastic version of the Gronwall lemma. The proof is a consequence of the martingale inequality (1).

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2. Main result

The following theorem is the main result of this paper. The result supplements Theorem 4 in [1].

Theorem 2.1. *Assume that Z and H are non-negative, adapted processes with continuous paths. Let ψ be non-negative, progressively measurable and M be a continuous martingale terminating at a random variable X in L_p for $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that*

$$Z(t) \leq H(t) + \int_0^t \left(\int_0^s \psi(u) Z(u)\ du \right)\ ds + M(t) \tag{2}$$

for all $0 \leq s \leq t$.
Then, for $0 < \theta < 1$ and $\nu_1, \nu_2 > 1$ such that $\frac{1}{\nu_1} + \frac{1}{\nu_2} = 1$ and $\theta \nu_i < 1$ with i = 1, 2,

$$\mathbf{E} \sup_{0 \leq s \leq t} Z^\theta(s) \leq \left(1 + \mathbf{E}(N^*(t))^{\theta \nu_1}\right)^{1/\nu_1} \left(\mathbf{E}(H^*(t))^{\theta \nu_2} + \left(\Gamma(q+1)\right)^{\frac{\theta \nu_2}{q}} \left(||X||_p \right)^{\theta \nu_2} \right)^{1/\nu_2}, \tag{3}$$

where $\Gamma(q+1)$ is a gamma function, $N^*(t) = \sup_{0 \leq s \leq t} \frac{(1-\beta(s))^{1/q}}{1-(1-\beta(s))^{1/q}}$ and

$$\beta(t) = \exp\left(- \frac{1}{(p+1)^{q/p}} \int_0^t \psi^q(s) s^{q + \frac{q}{p}}\ ds \right). \tag{4}$$

Proof. Using integration by parts in (2), then

$$Z(t) \leq H(t) + \int_0^t (t-s)\psi(s) Z(s)\ ds + M(t). \tag{5}$$

By the Hölder inequality, it follows from (5) that

$$\begin{aligned} Z(t) &\leq H(t) + |M(t)| + \left(\int_0^t (t-s)^p\ ds \right)^{1/p} \left(\int_0^t \psi^q(s) Z^q(s)\ ds \right)^{1/q} \\ &= H(t) + |M(t)| + \frac{t^{1+\frac{1}{p}}}{(p+1)^{1/p}} \left(\int_0^t \psi^q(s) Z^q(s)\ ds \right)^{1/q}. \end{aligned} \tag{6}$$

Hence, using Theorem 1 in [15], we now have

$$Z(t) \leq H(t) + |M(t)| + \frac{t^{1+\frac{1}{p}}}{(p+1)^{1/p}} \frac{\left(\int_0^t \psi^q(s)(H(s) + |M(s)|)^q \beta(s)\ ds \right)^{1/q}}{1-(1-\beta(t))^{1/q}}. \tag{7}$$

Define $N(t)$ by

$$N(t) := \frac{t^{1+\frac{1}{p}}}{(p+1)^{1/p}} \frac{\left(\int_0^t \psi^q(s) \beta(s)\ ds \right)^{1/q}}{1-(1-\beta(t))^{1/q}}. \tag{8}$$

Therefore, we have

$$Z(t) \le (1 + N(t))(H^*(t) + M^*(t)) \tag{9}$$

which follows immediately from (7).

Let $\beta(t)$ be a process given in (4). Using Ito's formula, we have

$$d\beta(t) = -\frac{1}{(p+1)^{q/p}} t^{q+\frac{q}{p}} \psi^q(t)\beta(t)dt. \tag{10}$$

This implies that

$$N(t) = \frac{(1-\beta(t))^{1/q}}{1-(1-\beta(t))^{1/q}}. \tag{11}$$

Combining (9, 11) and using the Hölder inequality, we deduce that

$$\begin{aligned} \mathbf{E} \sup_{0 \le s \le t} Z^\theta(s) &\le \mathbf{E}(1 + N^*(t))^\theta (H^*(t) + M^*(t))^\theta \\ &\le \left(\mathbf{E}(1 + N^*(t))^{\theta\nu_1}\right)^{1/\nu_1} \left(\mathbf{E}(H^*(t) + M^*(t))^{\theta\nu_2}\right)^{1/\nu_2} \\ &\le \left(1 + \mathbf{E}(N^*(t))^{\theta\nu_1}\right)^{1/\nu_1} \left(\mathbf{E}(H^*(t))^{\theta\nu_2} + \mathbf{E}(M^*(t))^{\theta\nu_2}\right)^{1/\nu_2}. \end{aligned} \tag{12}$$

Then, applying the Jensen inequality and the martingale inequality in (1) (see [10,11]), it follows that

$$\begin{aligned} \mathbf{E}(M^*(t))^{\theta\nu_2} &\le (\mathbf{E}(M^*(t)))^{\theta\nu_2} \\ &\le (\Gamma(q+1))^{\frac{\theta\nu_2}{q}} \left(||X||_p\right)^{\theta\nu_2}. \end{aligned} \tag{13}$$

The desired result now follows immediately from (12) and (13). This completes the proof. □

Remark 1. It is interesting to compare the upper estimate obtained in Theorem 4 in [1] with that established in our Theorem 2.1. Assuming that the process $Z(u)$ under the integral sign in (2) is replaced by its running maximum process, then one could prove an interesting extension of our present result and Theorem 2.1 in [6]. The details are left to the interested reader.

Remark 2. It should be noted that (5) is a stochastic integral inequality with a convolution kernel $K(t,s) = (t-s)^\lambda$ of order one ($\lambda = 1$). The results and proofs in [7] do not cover the particular case $\lambda = 1$, but deal mainly with the case $-1 < \lambda < 0$, see for instance Theorems 2.2, 2.3 and 2.4 in [7]. We also note that the proofs in [7] employ the Burkholder martingale inequality [3] and the martingale inequality in Bismut and Yor [16]. For related results, see [5], where the proofs are based on the well-known Métivier-Pellaumail inequality for semimartingales.

Remark 3. Finally, it is important to note that the present result and its proof do not immediately extend to the case when the exponent $\theta \ge 1$. For versions of the stochastic Gronwall lemma, assuming that the underlying process is a general Itô process under certain restrictions, these are proved in [4] strictly for the case when $\theta \in [2, \infty)$. The result in Banuelos and Osekowski [17] plays an important role in proving versions of the stochastic Gronwall lemma in [4].

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