Octonions and the Triple Articulation

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Resumo

Num artigo anterior descrevemos a nossa abordagem para modelar as relações que um ecossistema de negócios pode sustentar para as demandas multi-facetadas dos seus clientes. Esta abordagem distinguia dois tipos de tempo, chronos e kairos, e era triplamente articulada, descrevendo uma empresa como uma realização de possíveis composições de 'capacidades tecnológicas', 'modelos sociais de orquestração e sincronização' e as antecipações de satisfações ‘da diferenciação das organizações dos clientes individuais'. Este artigo descreve os trabalhos subsequentes que necessitaram abandonar os Números Complexos como sua base matemática em favor dos Quaterniones e finalmente adoptar os Octoniones que fornecem um modelo da trialidade, necessário para abstrair as relações entre as articulações. Identificam-se uma série de questões de pesquisa derivadas da abordagem que fornece um meio para relacionar a agilidade necessária para apoiar uma relação dinâmica entre a situação de um cliente individual e a abordagem adoptada para instituir a empresa como um todo.

Palavras-chave: trialidade, octonions, modelagem de ecossistema, chronos, kairo, articulação tripla

Abstract

A previous paper[4] described our approach to modeling the relations that a business ecosystem can sustain to the multi-sided

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[4] De Freitas, Cohen and Boxer, 2015.

demands of its clients. This approach distinguished two kinds of time, *chronos* and *kairos,* and was *triply articulated*, describing an enterprise as a realization of possible compositions of 'technological capabilities', 'social models of orchestration and synchronization' and 'the differing organizations of individual clients' anticipations of satisfaction'. This paper describes subsequent work that necessitated abandoning the Complex Numbers as its mathematical basis in favour of the Quaternions and finally adopting the Octonions which provide a model of triality necessary for abstracting the relations between the articulations. It identifies a number of research questions derived from the approach which provides a means of relating the agility needed to support a dynamic relation to an individual client's situation with the approach taken to instituting the enterprise as a whole.

Key Words: triality, octonions, ecosystem modeling, chronos, Kairos, triple articulation

Introduction

This document describes the current state of our collaboration and the open questions that we are working on.

An enterprise is a social organization instituted to provide products and services. It may be public or private, commercial or non-profit, civil or military, individual or collective. Every enterprise is sensitive to changes:

- in the supply of raw materials, technologies, etc.;
- in the variety of the demands to which it must respond in its environment, including other enterprises, be they clients, competitors or collaborators; and

- in the capabilities of its own processes, people and management.

It must learn from and adapt to these changes if it is to remain viable and sustainable.

Our approach to the analysis of an enterprise as a social organization considers it as being like a 'living system'[5] able to sustain a stable organization of its relation to its environmental niches. Changes in these niches would demand of the enterprise different ways of sustaining its organization, failures to adapt to and learn from such changes incurring risks to its continuing viability. Our approach to the alignment of the 'inside' of an enterprise with its environmental 'outside' niches involved developing a six-layer stratification[6], on the basis of which misalignments between the layers could be analyzed as defining different kinds of risk[7].

In one example of our approach[8], we were asked to evaluate the risks facing an enterprise under changing demand conditions: given the software upgrades to an operational capability that the enterprise was currently committed to making, we needed to identify the risks that its operation would face 10 years out, given the likely changes to its mission environment.

The method we used was to model three kinds of relational knowledge: the way the technology worked (forming an 'existential' model), the way the enterprise organized its uses of the technology (forming a 'deontic' model), and the different kinds of mission

[5] Rosen, 1991.

[6] Boxer, 1998.

[7] Boxer and Cohen, 2000.

[8] Anderson, Boxer et al., 2006.

environment in which operational missions would have to produce their effects (forming a 'referential' model). A fourth model of the relation of the enterprise to its environments was then formed. This fourth model composed these three kinds of model to produce the six-layer stratification on the basis of which risks could be evaluated[9].

The subsequent challenge was to abstract away from the particular modeling to consider the impact of variations in the underlying relational models and their composition. While an approach was developed that could abstract any one kind of relational model in terms of the Quaternions10, the approach could not abstract their composition in the formation of a stratification. It was this difficulty that brought us to consider the Octonions.

Projective Analysis modeling

The E!2187 AgentWorks project was a Eureka Project[11] aimed to develop an integrated environment for Multi-Agent Based Systems design and implementation. Ægis was the French Company with overall leadership of this project.

The project aimed to produce an application development environment that used a multi-agent architecture. In principle this project was to enable the development of tools and processes that would be of value to all knowledge-driven industries and enterprises. The part of Boxer Research Ltd (BRL) in this project was to concern itself with the means of supporting the formulation of the models and

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[9] Boxer and Cohen, 2010.

[10] De Freitas, Cohen et al., 2015.

[11] Eureka is an international network established in 1985 as an agreement between 18 countries to foster European competitiveness and integration and to encourage R&D cooperation.

logics that would govern the way these tools and processes could be used; and to develop the means of supporting these forms of use within enterprises.

The project undertaken in the UK by BRL was thus to develop a Composition Agent, which was a key element of the AgentWorks development. This Composition Agent provided a software tool and supporting practices based on a theoretically sound and technically well-founded approach to the modelling and analysis of Enterprise Architectures where there was a requirement that those architectures be 'agile' i.e. dynamically responsive to changes in the form of demand. DTI funding for this project was to enable BRL to overcome a gap in market provision for such a tool, resulting from market failure.

This Eureka research project aimed at developing triply articulated methods of modeling[12]. The result was the development of particular methods of projective analysis[13] and formal methods for analyzing patterns in the models. These patterns were used to define a stratification of the ways in which a supply-side and demand-side organization were aligned with each other, which could then be analyzed for structural 'gaps' that emerged within and between its different layers[14].

There have been a number of client assignments concerned with developing the use of these methods: with British Telecom, examining the root causes of errors in their relationship to their customers arising from the impact of digitization; with the UK's Ministry of Defence, examining the ways in which the procurement of defense capabilities were 'blind' to network enablement between

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[12] Cohen and Boxer, 2004.

[13] Boxer and Cohen, 1998.

[14] Boxer and Cohen, 2000.

capabilities within the larger defense ecosystem[15]; and with the UK's National Health Service, examining the ways in which the orthotics service could be better enabled to meet the chronic and therefore indirect demands of its patients[16]. In each of these cases, implementation was limited by the client enterprise because of the levels of change they identified as being needed in order to address the structural 'gaps'. This was not so much an issue of how to work with these different forms of analysis as an issue of how to address the strategic challenges identified by them[17]. Working within the context of these strategic challenges, it was the anxieties and resistances that then had to be worked through as an explicit part of an overall process of change[18].

Later projects undertaken while at the Carnegie Mellon University's Software Engineering Institute took the analytic issues further with respect to the pragmatics of demand[19], enterprise architecture[20], organizational agility[21], collaborative systems of systems[22], eGovernment[23], the impact of governance approaches[24] and analyzing the architectures of software-intensive ecosystems[25].

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[15] Whittall and Boxer, 2008, 2009.

[16] Flynn and Boxer, 2004.

[17] Boxer and Veryard, 2006.

[18] Boxer, P. J., 2004.

[19] Boxer, P. J., E. Morris, W. Anderson and B. Cohen, 2008.

[20] Boxer, P. J. and S. Garcia, 2009.

[21] Boxer, P. J., 2009.

[22] Boxer, P. J. and N. J. Whittall, 2009.

[23] Boxer, P. J. and H. Sassenburg, 2010b.

[24] Boxer, P. J. and H. Sassenburg, 2010a.

[25] Boxer, P. J., Boxer, P. J. and R. Kazman, 2017.

Time and the Triple Articulation

We are using a Leibnizian approach to the concept of a universe whose origin is not absolute but is brought into being in accordance with the lack,[26] to which that universe's instituting is a response[27], the relation to lack represented by Φ.

So far, we had been dealing with an atemporal theory. The need to introduce time led us to seek an extension of this theory.

We considered two different kinds of time, chronologic and kairotic[28]. Chronologic time was that which was experienced as a logic of succession. This logic was present in the Existential articulation, which described the coordinations of behavior (C^1) forming a 'zero-level' directed acyclic graph (ZDAG) representing the causal relations of succession between behaviors; and another directed acyclic graph (SDAG) that subtended C^1 and representing possible coordinations of these coordinations of behavior (C^2). Contrasting with this was the Deontic articulation, which described synchronizations of traces-of-behavior *qua* data (S^1) forming a ZDAG representing the observed relations between traces; and subtending this S^1 an SDAG representing formal synchronizations of synchronizations of traces-of-behavior (S^2). , A logic of succession emerges from the composition of these two articulations through a bundle of possible existential trajectories being made subject to the sovereignty of a particular form

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[26] 'Lack' is understood as an experiencing of the presence of an absence, leading to an anticipation of satisfaction by subsequent behaviors (*qua* responses to the anticipation of satisfaction) that are at the same time not expected to be fully satisfying, i.e., there is no 'escaping' from a being-in-relation-to the presence of an absence *qua* lack.

[27] Evangelides, B., 2018.

[28] καιρός - that when said about time means the right point of time, the proper time or season. It is equivalent to Latin *opportunitas*.

of synchronization[29]. This might or might not be subject to inter-subjective agreement between observers. Insofar as it was, then it was capable of being indexed against the periodicity of an atomic clock.

Kairotic time[30] was the time as realized by an enterprise, but also as experienced by observer-clients in relation to their embodiment, expressed by the way profundity in the Referential articulation of each observer-client was organized.[31] The Referential articulation represented an observer-client's most immediate articulation of situations in which s/he experienced wanting expressed in terms of their drivers (A^1), forming a ZDAG at a 'surface' level. Further situations then represented anticipations of satisfaction at progressively more profound levels, the effect of satisfying any one of which would engender effects on situations closer to the surface and ultimately on the surface situations themselves, forming an SDAG of anticipations of anticipations of satisfaction (A^2). The anticipation of satisfaction of each of these more profound situations was expressed

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[29] Boxer, P. J. and B. Cohen, 2000.

[30] Chronos and kairos are two different realizations of time. A mathematical model that can represent those two different realizations of time is the one developed by Luxemburg, W. A. J. (1967). A New Approach To The Theory of Monads. Technical Report. Pasadena, California Insititute of Technology. and Machover, M. and J. Q. P. Hirschfeld (1969). Lectures on Non-Standard Analysis. Berlin, Springer-Verlag., where a formalization of the concept of monad is made to give a model for the hyper-real numbers, where for each real number there is a lifting formed by hyper-reals, that lifting forms a monad, we can say that a monad is a real number, representing chronos plus all possible hyper-reals, representing kairos.

[31] This leads to two echelons (following Bourbaki) at which behaviors in response to anticipations of satisfaction may be defined: the echelon of an individual observer-client within the context of an ecosystem offering possibilities of realization; and the echelon of the ecosystem itself as a realization of an instituting relation to lack Φ. The ethical question facing an enterprise is: to what extent should the realization of the ecosystem create possibilities of realization for its observer-clients, i.e., possibilities experienced as under-determining?

in terms of its direct effect on some subset of the 'surface' drivers. In this sense each profound situation had a Euclidian relation to the 'surface' situations on which it had effects.

The approach thus associated a universe with its instituting, while also understanding the universe itself to be a universe of monads.[32] The relation between 'height' in the Existential and Deontic articulations and 'profundity'[33] in the Referential articulation depended on the way the referential articulations of the observer-clients could be brought into relation with the other two articulations through the exercise of deontic sovereignty over the existential. This meant that for there to be satisfaction of some aspect of the 'monadic' property of an observer-client, that aspect had both to be made consistent with the logic of succession that emerged from an existential subject to deontic sovereignty, but that some degree of that deontic sovereignty had to be surrendered to the observer-client in order for the resultant behaviors to be aligned to and be made to cohere around the observer-client's situation *qua* pragmatics-of-use[34].

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[32] The 'monads' refer to irreducible elements, each one of which is understood to be constitutive of the presence of an absence. What distinguishes an enterprise and the observer-clients as monads within this universe is the possibility that there can be realizations forming sources of (partial) support to anticipations of satisfaction. This renders these realizations as extimate symptoms of being lacking.

[33] *Profundis* in Latin reffers to height or to depth, that is the sense we are talking about profundity. The heigth in the existencial articulation and the depth in the referential are related each other by a reflexion, that is a 'twist', also it is a quadric form that is linked to the triality of the triple articulated model.

[34] A 'market' thus became a way of aggregating across observer-clients' pragmatics-of-use, uncoupling a supplier from their contexts-of-use. Adding the referential articulations of observer-clients enabled the approach to consider the structural implications of approaching observer-clients one-by-one.

Note that without kairotic time, all we had was a logic of succession – a syntactic formation within the 'mother universe' of the supplier that was expressible from the outside as a denotational semantics[35]. So, a requirement was expressed at best as a constraint on a behavioral semantics that was the semantics of the supplier – a realization of the supplier's relation to lack within a kairos of the supplier. With kairotic time, we introduced a semantics which, from the environment of the observer-client, was expressible as a pragmatics-of-use within the observer-client's context-of-use. Note that kairos was now relevant both to an enterprise and to the referential articulations of the observer-clients i.e. more than one relation to kairos is implicit to the analysis[36].

This kairotic time of the observer-clients allowed us to understand the significance of the third, referential, articulation: without it, any requirement identified by a user could be expressed in any number of different architectures, the behavioral semantics of any one of which could be evaluated in terms of whether it satisfied the requirement, but which would also allow other behaviors which might or might not be of use to the user (viz: the static vs. dynamic parts of product or service customization).

With the third articulation, we got particular 'bundles' of existential supplier behaviors that could be made to satisfy a client's pragmatics-of-use through the appropriate deontic exercise of sovereignty through which those behaviors could be orchestrated and synchronized. The 6-layer stratification emerged as a necessary way of representing the alignment of these existential behaviors and

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[35] This logic of succession is a series-B chronology. See McTaggart, J. E., 1908. "The Unreality of Time." Mind New Series 17(68 (October)), pp. 457-474.

[36] De Freitas, A., 2016. "Hesiod's Cosmos and the Beginning of Greek Philosophical Speculation." Coleção CLE 71.

deontic synchronizations to a third referential articulation of anticipations of satisfaction, which we called a *realization* of the triple articulation. This 6-layer stratification was a representation of the way existential bundles of trajectories could be deontically orchestrated and synchronized in relation to a client's pragmatics-of-use in the referential, where a trajectory represented a possible behavior of a supplier that could be aligned in support of the anticipations of satisfaction of observer-clients.

We understood a relational enterprise to be a one that, in being triply articulated, could align bundles of trajectories dynamically to its observer-clients' embodied relation to lack. Each existential bundle of trajectories could be represented as a hyperbolic cone (as in the 'light cone' of Einstein-Minkowski space-time). If a bundle could provide no support for the satisfaction of a client's pragmatics-of-use, the cone became just a point (i.e. no behaviors were satisfying); while if any pragmatics-of-use could be supported, the cones became a horizontal line to infinity (i.e. any behaviors were satisfying).

Given the availability of a relevant bundle of trajectories, then the behaviors could be understood as taking place on the surface of an ellipsoid surface, the foci of which represented the sovereign forms of synchronization. With no surrender of sovereignty to the observer-client's pragmatics-of-use, this ellipsoid surface became a sphere. Either way, the emergent logic of succession was a pathway across the ellipsoid surface.

The observer-client's relation to lack in a 'surface' situation produced an inversion in the knowledge of the supplier-as-other, as the observer-client's anticipation of satisfaction was what constituted the particular nature of that observer-client's kairotic time. Below c-level[37], the kairotic time of the observer-client could be ignored while, above c-level, the 'lower' the situations (in the observer-client's kairotic profundity), the more likely they were to be situations shared with other observer-clients. In this sense, kairotic profundity (above c-level) was necessarily *paradeictic*[38]. In contrast, the emergent logic of succession, insofar as it could be defined independently of clients' pragmatics-of-use (i.e. below c-level), became chronological and *apodeictic*[39] in nature.

Instituting the observer-client

The observer-client, too, could be understood as being triply articulated. For there to be behaviors that could support an observer-client's anticipations of satisfaction, there needed to be logics of succession that were consistent with the observer-client's monadic indexing of succession. Equally, the ability of the observer-client to manage their use of these logics in support of their satisfaction involved privileging their paradeictic relation to lack. This involved the profundity of experienced satisfactions being consistent with implied profundity with respect to height in the composition of the other two articulations. The synchronization of experienced trajectories represented a particular identification by an observer-client with his or her experience of embodiment (i.e., a 'world line').

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[37] 'c-level' is that level of profundity beyond which the relation to situation *qua* context-of-use (i.e. embodied context) becomes irrelevant – because the demand at that level of profundity can be expressed in a way that is context-independent.

[38] *Paradeictic* – meaning can only be established by reference to the speaker *qua* observer-client.

[39] *Apodeictic* – meaning can be established inter-subjectively independently of the speaker.

For two trajectories to overlap in hyperbolic space through their synchronization was for them to have some common kairotic time that conserved some aspect of the observer-client's relation to lack. A larger bundle of trajectories thus articulated greater synchronization choices in how the observer-client's anticipations of satisfaction could be supported. Increasing the 'size' of the bundle was thus about elaborating a wider range of possibilities for living (trajectories).

The composition of observer-clients' referential articulations with the other two articulations made possible by the triple articulation of an enterprise induces a particular relation between profundity and height while the composition of the existential and deontic imposes a particular synchronization of succession. These constraints on the elaboration of an observer-client's bundle arises from the way an enterprise exercises sovereignty. A prime interest in our work is thus to be able to model the relation between the instituting of an enterprise and the extent to which the sovereignty constraining the bundles of trajectories can be surrendered through their alignment to observer-clients' anticipations of satisfaction.

The challenge presented by triple articulation

Our original need was to extend our theory with two more coordinates representing time in its two 'variations' i.e. succession and kairos. Just as the first two coordinates 'indexed' a vertex and its 'height' in the existential and deontic articulations, we hypothesized that the second two could be used to index 'succession' and 'profundity'.

As our theory was foundational, any change to it had to be done in the set of atoms.

The unique natural extension to the set of atoms we chose following the earlier formulation in terms of the Complex numbers (**C**)[40] was Hamilton's Quaternions (**H**)[41]. We used the term 'monad'[42], rather than 'atom', when referring to a Quaternion. All information was contained in our monads.

Quaternions provided us with monads formed by four coordinates, the first two representing our previous atoms (based on the Complex numbers) i.e., the name and height of a vertex, and the two new coordinates identified with time in a bi-dimensional sense i.e., the succession and kairos of a vertex.

Quaternions provide infinitely many solutions to polynomial equations. For example, the solutions of the equation $z^2 + 1 = 0$ are $z = bi + cj + dk$, with $b^2 + c^2 + d^2 = 1$. As our third and fourth coordinates were temporal coordinates, this implied that, in each particular relation to embodiment there were infinitely many futures and pasts. In fact, the existence of infinitely many futures implied infinitely many pasts, as they were related by the involution.

Involution, the conjugacy operation on the Quaternions, continued to act as it did over the field of the complex numbers in the atemporal model and would, in addition, 'send' both new coordinates to their additive inverse.

In order to preserve our development in terms of C*-algebra and involution, we had to use a matrix representation of Quaternions. Fortunately, Quaternions could be represented in matrix form and,

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40 De Freitas, A., B. Cohen and P. J. Boxer, 2015.

41 See the Appendix for definitions and examples of the mathematical structures used in our models, including Quaternions, Octonions and bilinear forms.

42 See for example De Risi, 2007.

since involution continued to be an involution, the Bratteli-like constructions worked in this extended theory.

The extension of this work to the Octonions (**O**) arose because of the need to define the distinct nature of each of the three articulations, each with its own geometry, as well as their composition with respect to the instituting of a relation to lack in the form of an enterprise with its realized behaviors.

As a single vertex was denoted by a Quaternion, an ordered pair of vertices was denoted by an Octonion. Given the vertices $x = a + bi + cj + dk$ and $y = p + qi + rj + sk$, the directed arc between them, i.e. the ordered pair (x, y), was denoted by the Octonion

$$e_0a + e_1b + e_2c + e_3d + e_4p + e_5q + e_6r + e_7s$$

The angle between two directed arcs was given by the angle between their respective Octonions. Given a triangle of directed arcs, this meant that we could calculate the sum of its internal angles. The geometry of the surface in which they lay was Riemannian (i.e. elliptic), Euclidian (i.e. flat) or Lobatchewskian (i.e. hyperbolic)[43] if that sum is less than, equal to, or greater than zero, respectively. This gave us a criterion by which we could evaluate abstractions of the different nature of each articulation in terms of the sum of the internal angles of triangles of directed arcs within each kind of articulation.

The distinct nature of each articulation appears to exhibit properties that parallel the triality of the Octonions. This triality attributes different geometries to each articulation. Thus, the geometry of the referential articulation is Euclidean, that of the

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[43] A good introduction to non-Euclidean geometries is the classical book of Coxeter, H. S. M., 1942. Non-Euclidean Geometry (1st Edition), University of Toronto.

existential articulation is Hyperbolic, and that of the deontic articulation is Ellipsoid.[44]

The Fano Plane[45] then appears to provide a means of representing the way compositions derived from the three articulations institute realized behaviors as an expression of an instituting relation to lack (Φ).

Working with the triality of the Octonions

In order to establish the possibility of instituting a realization of the three articulations being in relation to each other, three kinds of mapping between the articulations become necessary:

- The ontic mapping of the referential to the existential, establishing behaviors capable of supporting deontic responses to the demands of observer-clients.
- The epistemic mapping of the existential to the deontic, identifying what is known about how to manage this existential in order to establish ways of orchestrating and synchronizing behaviors, creating the ability to secure particular kinds of direct and indirect outcome in response to the demands of any given observer-client.
- The relational mapping of the deontic to the referential, establishing ways of dynamically aligning direct and indirect outcomes to the situated contexts-of-use identified with observer-clients in the referential.

[44] In the case of the deontic articulation with no possibility of being in relation to observer-clients' contexts-of-use, it has an ellipsoid geometry for which its two foci are identical.

[45] Hirschfeld, J. Q. P., 1979.

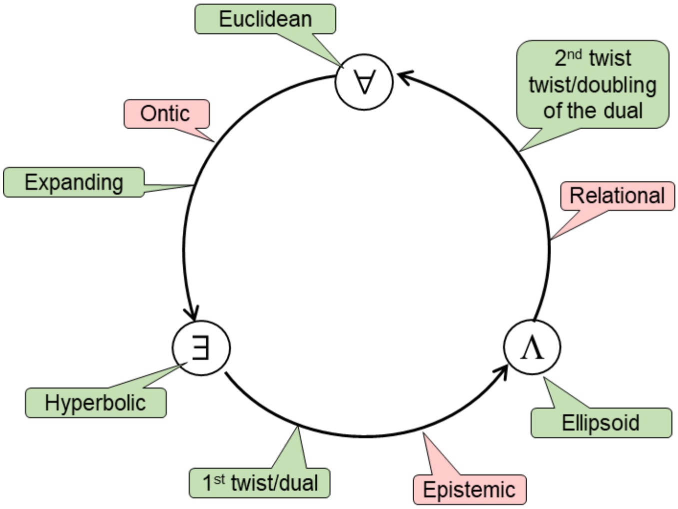

Figure 1: Triality of the articulations

These mappings involve 'pruning' each articulation with respect to each other articulation. There are six different pruning strategies, depending on which articulation is privileged.[46]

Here, the referential is privileged, the ontic mapping to the existential establishing what behaviors are relevant to the referential anticipations of satisfaction; the epistemic mapping establishing what is 'known' deontically about how to manage this existential; and the relational mapping establishing how the dynamic alignment of existential-and-deontic to the referential may be sustained.

The six different pruning strategies create the means of instituting a 'fourth ring' which represents the 'being' of the instituted relation to lack, while the resultant pruned articulations establish a basis from which to realize behaviors that give expression to that

[46] See Table 1 in De Freitas, A., B. Cohen and P. J. Boxer, 2015.

relation (which in turn imposes constraints on the anticipations of satisfaction of observer-clients that can be supported).

The mappings require two kinds of 'twist' to be defined within the Octonions (or the bi-quaternions, a simpler substructure of the Octonions, which are sufficient for our purpose):

- the first 'twist', constructed by taking the involution of the existential articulation, which permits us to define the relation between the dual of the existential (expressing zero-level events) and the traces of the deontic, and between the dual of the deontic (expressing zero-level compositions) and the processes of the existential.
- the second 'twist', constructed by taking the involution of the first, which permits us to define the inverted relation between profundity in the referential and height in the existential and the deontic.

With respect to the geometry, let us say, informally, that if E, L and R represent models for the Euclidean, Hyperbolic and Elliptic geometries, respectively, then we want the diagram $0 \longrightarrow E \longrightarrow L \longrightarrow R \longrightarrow E \longrightarrow 0$ to be an exact sequence.

For an informal and sufficient definition, we say that a geometry is Euclidean if the shortest distance between two points is defined by a line. If it is a hyperbolic curve then the geometry is hyperbolic or Lobatchewskian; and if it is an elliptical curve, then the geometry is elliptic or Riemannian. In more formal terms the geometry of a manifold is defined by its Gaussian curvature: elliptic when positive, hyperbolic when negative and flat when zero.

Using the terminology of the triple articulation we can define the following operations that are, more formally, functors over categories. Let us call 'points' a structure with a Euclidean geometry, 'hyperboloid' a structure with a hyperbolic geometry and 'ellipsoid' a structure with an elliptic geometry. In a way of linking those functors to the Fano model, we call 'solid 1' and 'solid 2' the 'hyperbolic' and 'elliptic' structures with their respective geometries.

In the terminology of the triple articulation, the application of the 'dual' functor models the 'expanding operation', the 'first twist' and the 'second twist'.

$points \xrightarrow{\text{dual}} hyperboloid(solid_1)$ (Expanding operation)

$hyperboloid(solid_1) \xrightarrow{dual} ellipsoid(solid_2)$ (First Twist)

$ellipsoid(solid_2) \xrightarrow{\text{dual}} points$ (Second Twist)

Let us define the pair $(\mathbf{H}, h)$ to be the quaternions with the bilinear form h that is a hyperbolic object in the projective plane, i.e., a quadratic negative definite form. Let us define $(\mathbf{H}, e)$ to be the quaternions with e an ellipsoidal object in the projective plane, i.e., a quadratic form positive definite then, we affirm, without proof, that

$$0 \longrightarrow \Re \longrightarrow (\mathbf{H}, e) \longrightarrow (\mathbf{H}, h) \longrightarrow \Re \longrightarrow 0$$

is an exact sequence, where $\Re$ is the real numbers with a degenerated bilinear form in the projective plane.

$(\mathbf{H}, h)$ is the dual space that associates to each point $x \in \Re$ a quadratic negative-definite form and $(\mathbf{H}, e)$ is the dual space that associates to each point $x \in \Re$ a quadratic positive-definite form. The map $(\mathbf{H}, e) \longrightarrow (\mathbf{H}, h)$ is defined by the involution.

Let $Q(x) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{44}x_4^2$ be a quadratic positive definite form, then $Q^*(x) = a_{11}x_1^2 - a_{22}x_2^2 - a_{33}x_3^2 - a_{44}x_4^2$ that is a quadratic negative definite form and $(\mathbf{H}, e)^* \cong (\mathbf{H}, h)$, because $(\mathbf{H}, e)^* = (\mathbf{H}^*, e^*) \cong (\mathbf{H}, h)$.

The following triplet is a triality over $\mathbf{H}$

$$\Big(\big(\Re, (\mathbf{H}, e)\big), \big((\mathbf{H}, e), (\mathbf{H}, h)\big), \big((\mathbf{H}, h), \Re\big)\Big)$$

That triality over $\mathbf{H}$ induces a triality over $\mathbf{H} \oplus \mathbf{H}$, the bi-quaternions, in the following way

$$\Big(\big(\mathbf{C}, (\mathbf{H}, e) \oplus (\mathbf{H}, e)\big), \big((\mathbf{H}, e) \oplus (\mathbf{H}, e), (\mathbf{H}, h) \oplus (\mathbf{H}, h)\big), \big((\mathbf{H}, h) \oplus (\mathbf{H}, h), \mathbf{C}\big)\Big)$$

where **C** is the complex plane.

For example, the first element of the triplet is a space that sends each complex number into the bi-quaternions with a quadratic form that is the multiplication of two quadratic definite positive forms, that it is itself a quadratic definite positive form over the bi-quaternions. Indeed, if we consider $Q_1(x_1) = x_1^T A_1 x_1$ and $Q_2(x_2) = x_2^T A_2 x_2$ are quadratic form positive definite over the quaternions, $Q(x) = x^T A x$ is a quadratic form over the bi-quaternions, where $A = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}$ is a Hermitian matrix in the Jordan form, $x = (x_1, x_2)$, in matricial form $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $x^T = [x_1 x_2]$, then

$$Q(x) = [x_1 x_2] \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 x_2] \begin{bmatrix} A_1 x_1 \\ A_2 x_2 \end{bmatrix} = x_1 A_1 x_1 + x_2 A_2 x_2$$
$$= A_1 x_1^2 + A_2 x_2^2$$

is a quadratic form in the bi-quaternions that 'inherited' the property of definiteness of Q_1, Q_2.

Some relevant observations:

1. A quadratic form can be written in matricial form as $Q(x) = x^T A x$, where x is a n-vector, considered as a $n \times 1$ matrix, and there transpose x^T is a $1 \times n$ matrix, where at least one coordinate is non-zero, and A is a $n \times n$ symmetric matrix, that means that $A = A^T$.
2. If A is a diagonal matrix, then $Q(x) = a_{11} x_1^2 + a_{22} x_2^2 + \cdots + a_{nn} x_n^2$. In other cases $Q(x)$ has 'crossed' terms, that indicates a rigid transformation has been applied to the canonical quadratic form.

3. A Hermitian matrix (i.e., a complex matrix equal to its conjugate transpose) M is positive-definite if the real number z^*Mz is positive for every non-zero column complex vector z.

4. A Hermitian matrix M is positive semi-definite if the real number z^*Mz is positive or zero for every non-zero column complex vector z.

5. Analogously for negative-definite and negative semi-definite.

Implications

A triple articulation is needed to describe not only the way individuals *use* the doubly articulated medium of language when speaking[47], but also the way individuals *use* organizations in support of realizing their desires[48]. Triple articulation can thus provide a means of considering how the architecture of an enterprise within an ecosystem, which is realized by an instituting relation to lack, constrains the possibilities for realization available to its individual clients[49]. For example, it makes it possible to relate the architecture of the Facebook business ecosystem to the possibilities for realization available to individual communities of interest. We anticipate that the approach described in this paper may be a first application of Octonion algebra to a psycho-social analysis of business ecosystems beyond its usual domain of fundamental physics[50].

At the echelon of the 'vertical' interests of an enterprise, we can identify the observer-client's realization at the level of their

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[47] Nack, F. and L. Hardman, 2001, Lacan, J., 2002[1996].

[48] Boxer, P. J., 1979.

[49] Boxer, P. J., 2021b.

[50] Wolchover, N., 2018.

embodiment with the ways in which the enterprise over-determines possibilities-for-realization.

At the echelon of the 'horizontal' interests of individual clients' possibilities-for-realization, we can identify the instituting of a way of constraining possible realizations of relations between the articulations with bundles of behavioral trajectories (in a Minkowski geometry) mediated by the deontics of orchestration and synchronization (in an Ellipsoid geometry), in which the size of the bundle reflects the extent of under-determination of individuals' possibilities.

A vertically dominant form of governance views these possibilities to be a side-effect of the architecture of the enterprise. The means of relating the two echelons makes possible horizontally dominant forms of governance designed to privilege the interests of the client without jeopardizing the sustainability of the enterprise[51]. This is in contrast with the vertically dominant form that over-determines what may constitute a 'good' for the individual client, placing the interests of the enterprise before those of its clients. This ability to articulate a structural relation between horizontal and vertical axes of governance will be a new development within institution theory.

Research Questions

The extension to the Octonions appears to offer the possibility that structural accounts can be given of eight characteristics of our work

1. The relation of the triality of the Octonions to the triple

[51] Boxer, P. J., 2021a.

articulation.

2. The formulation of these relations in terms of the two 'twists'.
3. The necessity of at-least-6 layers of a stratification that brings the existential into relation with the referential via the deontic.
4. The formulation of a fourth 'ring' from the original three articulations generated by the derivation of complex objects using relational operators[52], this fourth 'ring' having the same characteristics as a 4-ring borromean knotting.
5. The parallel between the structural characteristics of the four further articulations generated by these complex objects and the Fano plane.
6. The parallel between the relation of the instituting relation of lack Φ to the triple articulation and the relation of the Fano plane to an origin Φ, expressed in terms of two echelons.
7. The possibilities for supporting the anticipations of satisfaction of an individual observer-client within an ecosystem taking the form of a bundle of trajectories, analogous to a 'sheaf' in category-theoretic terms, which can be aligned to and made to cohere around the observer-client's situation.
8. The relation between the echelons of enterprise and of observer-clients expressible in terms of the way an ecosystem over-determines the *variety* of bundles of trajectories that can be aligned and made to cohere around the contexts-of-use of observer-clients while at the same time under-determining the 'size' of each bundle in order to support the requisite agility within each situation.

[52] Boxer, P. J. and Cohen, B., 2010.

Appendix: Mathematical Structures

Bilinear forms and the dual space

A non-degenerate bilinear form is a form that is linear in each component and for two different non-zero vectors the image is non-zero.

Each non-degenerate bilinear form on a finite-dimensional vector space, V, induces an isomorphism $V \longrightarrow V^{*}: v \mapsto v^{*}, v^{*}(w) := \langle v, w \rangle \forall w \in V$, where the bilinear form is denoted by $\langle . , . \rangle$ in V. If the space considered is Euclidean, then the bilinear form is the scalar product.

The inverse isomorphism is

$V^{*} \longrightarrow V: v^{*} \mapsto v$ where v is the unique element of V such that $\langle v, w \rangle = v^{*}(w), \forall w \in V$. The vector $v^{*} \in V^{*}$ is the dual vector of v and V^{*} is the dual space of V.

If V is a vectorial space that has a basis $e_1, \ldots, e_n$ that is not necessarily orthogonal, then the dual space V^{*} has a basis $\widetilde{\omega}^{i}(e_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$, where that uses the Dirac delta, $w^{i}(e_j) = \delta_{ij}$.

A linear functional $\tilde{u}$ belonging to the dual space V^{*} can be expressed as the linear combination of basis functionals with components u_i, has the form $\tilde{u} = \sum_{i=1}^{n} u_i \, \widetilde{\omega}^{i}$. Applying the functional $\tilde{u}$ to the basis, we have that $\tilde{u}(e_j) = u_j$.

Given two vector spaces over a field F with a non-degenerate bilinear form, then

$V_1 \times V_2 \longrightarrow F$ is a linear functional. That is, for each vector in each of the vector spaces, the pairing is a non-zero linear functional of the other.

For example, if we represent the elements of $\Re^n$ as column vectors

$x = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$, then for each vector $a = [a_1 \dots a_n]$, we define the scalar product as

$$f_a(x) = a \cdot x = [a_1 \dots a_n] \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}.$$

This is a linear functional, and each linear functional can be defined in that way, equivalently $f_a(x) = a_1 x_1 + \cdots + a_n x_n$.

Duality and triality

A *duality* between two vector spaces over a field F is a non-degenerate bilinear form $V_1 \times V_2 \longrightarrow F$.

A *triality* among three vector spaces over a field F is a non-degenerate trilinear form $V_1 \times V_2 \times V_3 \longrightarrow F$, meaning that each non-zero vector in one of the three vector spaces induces a duality between the other two.

In order to continue with our definition, let us define the evaluation functional $ev_c f = f(c)$, where $ev_c : P_n \longrightarrow \Re$, with P_n is the real-valued polynomial functions of degree $\leq n$ defined on an interval $[a, b]$, with $c \in [a, b]$.

Observe that $f \mapsto f(c)$ is linear since $(f + g)(c) = f(c) + g(c)$ and $(\alpha f)(c) = \alpha f(c)$. If $x_0, \dots, x_n$ are $n + 1$distinct points in $[a, b]$, then the set of functionals ev_{x_i} , $i = 0,1, \dots, n$ form the basis for the dual space.

By choosing vectors e_i in each V_i on which the trilinear form evaluates to 1, we find that the three vector spaces are all isomorphic to each other, and their duals. Denoting this common vector space by V, the triality can be expressed as a bilinear multiplication $V \times V \longrightarrow V$, where each e_i corresponds to the identity element in V. The non-degeneracy condition implies that V is a composition algebra, because any non-degenerate quadratic form N satisfies $N(xy) = N(x)N(y)$. As a composition algebra, V has an involution called the conjugation, $x \mapsto x^*$. The quadratic form is $N(x) = xx^*$, which is the norm. Hurwitz's theorem states that the dimensions of V are 1,2,4 or 8. If $F = \Re$ and the form is defined positive, then V is isomorphic to $\Re$, $\mathbb{C}$, **H**, or **O**.

If we have a composition algebra, and we take each V_i equal to the algebra, pairing each space with its dual, contracting the multiplication with the inner product on the algebra, we create a trilinear form.

The Quaternions

While the complex numbers are obtained by adding to the real numbers the element *i* which satisfies i^2=-1, the Quaternions **H** are obtained by adding the elements *i*, *j* and *k* to the real numbers, which satisfy the relations: $i^2 = j^2 = k^2 = ijk = -1$.

If the multiplication is assumed to be associative (as indeed it is), the following relations follow directly:

$ij = k$	$jk = i$	$ki = j$
$ji = -k$	$kj = -i$	$ik = -j$

As a real vector space, every Quaternion is a real linear combination of the basis Quaternions, {1, *i*, *j*, *k*}, i.e. every Quaternion is uniquely expressible in the form *a*+*bi*+*cj*+*dj* where *a*, *b*, *c*, *d* .

In order to help us to understand the geometry, we can think of a Quaternion as formed by a real number, called the real or scalar part, plus a vector, called the imaginary part. The scalar part of the Quaternion is a while the remainder is the vector part. Thus a vector in the context of Quaternions has zero for scalar part.

The Quaternions are an example of a division ring, an algebraic structure similar to a field except for commutativity of multiplication. In particular, multiplication is still associative and every non-zero element has a unique inverse.

Quaternions form a 4-dimensional associative algebra over the reals (in fact a division algebra) and contain the complex numbers, but they do not form an associative algebra over the complex numbers.

The Quaternions, along with the complex and real numbers, are the only finite-dimensional associative division algebras over the field of real numbers.

The non-commutativity of multiplication has some unexpected consequences, among them that polynomial equations over the Quaternions can have more distinct solutions than the degree of the polynomial.

The set of Quaternions whose square is -1 is the set of unit vectors (of absolute value 1), that is $\mathbf{H} = \{q | q^2 = -1\} =$
$\{q | q^* = -1 \wedge qq^* = 1\}$

From this set equality one can view **H** as the union of complex planes sharing the same real line and taking an imaginary unit from the set. Furthermore, the unit sphere in **H**, the 3-sphere, is formed by the collection of unit circles in these complex planes.

As the general point on a circle is $e^{i\varphi}$ (known as Euler's formula), the general point on the 3-sphere is e^{ar} where r is a unit vector of **H**.

The multiplicative group of non-zero Quaternions acts by conjugation on the copy of $\Re^3$ consisting of Quaternions with real part equal to zero.

Let $z = a + u$, where u is a 3-dimensional real vector, be a non-zero Quaternion.

Then $|z| = \sqrt{a^2 + \|u\|^2}$ and $z^* = a - u$.

Now consider the function $f(v) = z\ v\ z^{-1}$ where z^{-1} is the multiplicative inverse of z and v is a vector, considered as a Quaternion with zero real part.

The function f is known as **conjugation by** z. Note that the real part of $f(\boldsymbol{v})$ is zero, because in general zw and wz have the same real part for any Quaternions z and w.

Furthermore, f is $\Re$-linear and we have $f(\boldsymbol{v}) = \boldsymbol{v}$ if and only if $\boldsymbol{v}$ and the imaginary part $\boldsymbol{u}$ of z are collinear (because $f(\boldsymbol{v}) = \boldsymbol{v}$ means $\boldsymbol{v}\,z = z\,\boldsymbol{v}$).

Hence conjugation by $z = a + \boldsymbol{u}$ is a rotation whose axis passes through the origin and is given by the vector $\boldsymbol{u}$. Note that conjugation by z is the same as conjugation by rz for any real number r, so that conjugation by z and by $-z$ represent the same rotation.

We can therefore restrict our attention to the Quaternions of absolute value 1, the so-called **unit Quaternions** (over the 4-dimensional sphere). The inverse of a unit Quaternion is its conjugate: if $|z| = 1$, then $z^{-1} = z^*$.

Conjugation by a unit Quaternion, $z = a + \boldsymbol{v}$, is a rotation by an angle θ, where $a = cos(\theta/2)$.

A counterclockwise rotation through an angle θ about an axis u can be represented via conjugation by the unit Quaternion $z = \big(cos(\theta/2) + sin(\theta/2)\big)\hat{u}$, where $\hat{u} = u/\|u\|$ is the *normalizer vector*.

The composition of two rotations corresponds to Quaternion multiplication: if the rotation *f* is represented by conjugation with the Quaternion *z* and the rotation *g* is represented by conjugation with *w*, then the composition $f°g$ is represented by conjugation with *zw*.

If one wishes to rotate about an axis that does not pass through the origin, then one first translates the vectors into the origin, conjugates, and translates back.

The angle between two Quaternions should not be confused with the angle of rotation involved in the rotation between the orientations corresponding to these Quaternions: the former is half of the latter (or 180° minus half the latter).

Some examples of operations with Quaternions

The *conjugate* of the Quaternion $z = a + bi + cj + dk$ is defined as

$z^* = a - bi - cj - dk$ and

the *absolute value* of *z* is the non-negative real number defined by

$|z| = \sqrt{(zz^*)} = \sqrt{(a^2 + b^2 + c^2 + d^2)}$

Note that $(wz)^* = z^*w^*$, which is not in general equal to w^*z^*.

The multiplicative inverse of the non-zero Quaternion *z* is

$$z^{-1} = z^*/(a^2 + b^2 + c^2 + d^2)$$

By using the distance function $d(z, w) = |z - w|$, the Quaternions form a metric space (isometric to the usual Euclidean metric on $\mathfrak{R}^4$) and the arithmetic operations are continuous. We also have $|zw| = |z|\,|w|$ for all Quaternions *z* and *w*. Using the absolute value as norm, the Quaternions form a real Banach Algebra.

A pair of Quaternions allows for compact representations of rotations in four-dimensional space because the four-dimensional rotation group SO(4) may be written as a semi-direct product of the three-dimensional rotation group SO(3). The Quaternions are, in turn closely related to the double covering of SO(3) by SU(2). Also closely related is the Lorenz group.

Quaternions' matrix representation

There are at least two ways of doing this in such a way that Quaternion addition and multiplication correspond to matrix addition and matrix multiplication (i.e., Quaternion-matrix homomorphisms). One is to use 2×2 complex matrices, and the other is to use 4×4 real matrices.

In the first way, the Quaternion $a + bi + cj + dk$ is represented as

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}.$$

This representation has several nice properties.

- Complex numbers ($c = d = 0$) correspond to diagonal matrices.
- The square of the absolute value of a Quaternion is the determinant of the corresponding matrix.
- The conjugate of a Quaternion corresponds to the conjugate transpose of the matrix.
- Restricted to unit Quaternions, this representation provides the isomorphism between S3 and SU(2). The latter group is important in quantum mechanics when dealing with spin.

In the second way, the Quaternion $a + bi + cj + dk$ is represented as

$$\begin{pmatrix} a & -b & d & -c \\ b & a & -c & -d \\ -d & c & a & -b \\ c & d & b & a \end{pmatrix}$$

In this representation, the conjugate of a Quaternion corresponds to the transpose of the matrix.

The Octonions

The only possible extension of the Quaternions with mathematical relevance is the Octonions, also known as Cayley numbers. By mathematical relevance, we understand a structure that preserve a well-known property. We saw that the Quaternions lose the commutativity property. The Octonions have an operation that we call multiplication that is neither commutative nor associative. Still the Octonions are a division algebra over the real numbers. As the Quaternions are an 'extension' of the complex numbers, creating a hyper-complex numbers structure, the Octonions are an 'extension' of the Quaternions, creating a hyper-hyper-complex number structure over the real numbers.

The definition goes back to the 19th century, defined by John T. Graves, following the steps that Hamilton used to define the Quaternions.

Octonions' arithmetic

The Octonions, as its name suggest is a set of 8-tuples of real numbers. Thus, an Octonion $x \in O$ is expressed as

$$x = e_0x_0 + e_1x_1 + e_2x_2 + e_3x_3 + e_4x_4 + e_5x_5 + e_6x_6 + e_7x_7$$

where $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ is a linear base for an 8-dimensional space, defined by the following table of multiplication:

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_7	$-e_6$	e_5	$-e_4$
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	$-e_5$	e_4	e_7	$-e_6$
e_4	$-e_7$	$-e_6$	e_5	-1	$-e_3$	e_2	e_1
e_5	e_6	$-e_7$	$-e_4$	e_3	-1	$-e_1$	e_2
e_6	$-e_5$	e_4	$-e_7$	$-e_2$	e_1	-1	e_3
e_7	e_4	e_5	e_6	$-e_1$	$-e_2$	$-e_3$	-1

Octonions' Algebra

Let $x = e_0$ be an Octonion, then we define the *conjugate* as

$$\bar{x} = e_0x_0 - e_1x_1 - e_2x_2 - e_3x_3 - e_4x_4 - e_5x_5 - e_6x_6 - e_7x_7$$

As usual we can define its scalar and vector parts*:*

$$Sc(x) = \frac{1}{2}\left(x + \bar{x}\right)$$

$$Vec(x) = \frac{1}{2}(x - \overline{x})$$

Observe that

$$\overline{(xy)} = \bar{y}\bar{x}$$

$$\overline{\overline{(x)}} = x$$

1. The scalar product of two Octonions is defined by

$$\langle x, y\rangle = \frac{1}{2}(x\overline{y} + y\overline{x}) = \frac{1}{2}(\overline{x}y + \overline{y}x)$$

2. The norm over **O** is defined by $N(x) = x\overline{x} = \overline{x}x = \sum_{\alpha=0}^{7} x_\alpha^2 e_0$
3. The inverse of a non-zero Octonion is $x^{-1} = \frac{\overline{x}}{N(x)}$

from where we deduce $xx^{-1} = x^{-1}x = 1$

The norm of an Octonion is zero, if and only if $x = 0$, otherwise is positive.

The norm has the following property: $N(xy) = N(x)N(y) = N(y)N(x)$

Quaternions and Octonions

It is common to think that $\Re \subset \Re^2 \subset \Re^3$, where $\Re, \Re^2, \Re^3$ are the real numbers field, the 2-dimensional real space over the reals, and the 3-dimensional space over the reals. Strictly speaking there is an isomorphism between $\Re$ and a subset of $\Re^2$, and there is an isomorphism between $\Re^2$ and a subset of $\Re^3$. That isomorphism is defined by the injection:

$$\Re \text{ into } \Re^2 \qquad a \mapsto (a, 0)$$

$$\Re^2 \text{ into } \Re^3 \qquad (a, b) \mapsto (a, b, 0)$$

In the same way we can think that the Quaternions H are a subset of the O.

We can think that a Quaternion is an Octonion with the last four coordinates equal to zero, using the injection:

$$\mathbf{H} \text{ into } \mathbf{O} \qquad (a, b, c, d) \mapsto (a, b, c, d, 0,0,0,0)$$

Other algebraic representations of the Octonions and the Quaternions are as follows:

$$\mathbf{H} \simeq \Re \oplus \Re^3$$

$$\mathbf{O} \simeq \Re \oplus \Re^7$$

Another way to represent the Octonions is to observe that any Quaternion $q = w + ix + jy + kz$ satisfies the quadratic equation

$$q^2 - 2wq + |q|^2 = 0.$$

Note, also that

$$\mathbf{C} \simeq \Re \oplus \Re i$$

$$\mathbf{H} \simeq \mathbf{C} \oplus \mathbf{C}j$$

then we can expect that there is another unit l, such that $l^2 = -1$ satisfy the following equation

$$\mathbf{O} \simeq \mathbf{H} \oplus \mathbf{H}l$$

From $\mathbf{O} \simeq \mathbf{H} \oplus \mathbf{H}l$ we get that any Octonion x has a representation $a = a' + a''l$, where $a', a'' \in \mathbf{H}$.

The addition and multiplication of Octonions, are defined as follows:

$$a + b = (a' + a''l) + (b' + b''l) = (a' + b') + (a'' + b'')l$$

$$ab = (a' + a''l)(b' + b''l) = (a'b' - \overline{b''}a'') + (b''a' + a''\overline{b'})l$$

Let $a, b \in \mathbf{O}$, the following properties hold:

1. $a(ab) = a^2 b$
2. $(ba)a = ba^2, (ab)a = a(ba) := aba$
3. $a^{-1} = \frac{\overline{a}}{|a|^2}$
4. $a^2 - 2(Rea)a + |a|^2 = 0$ and $(Ima)^2 = -|Ima|^2$
5. $|ab| = |a||b|$

A Quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$ has a matrix representation as follows:

$\phi: q = q_0 + q_1 i + q_2 j + q_3 k \in \text{H} \longrightarrow$

$$\phi(a) = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix},$$

where ϕ defines an isomorphism between the Quaternions and a subset of the 4×4 matrices.

We can define an isomorphism ω between the Octonions and a subset of the 8×8 matrices , as follows:

$$\omega(a) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & -a_3 & a_2 & -a_5 & a_4 & a_7 & -a_6 \\ a_2 & a_3 & a_0 & -a_1 & -a_6 & -a_7 & a_4 & a_5 \\ a_3 & -a_2 & a_1 & a_0 & -a_7 & a_6 & -a_5 & a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ a_5 & -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & -a_2 \\ a_6 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & a_0 & a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{bmatrix}$$

In this way, we have a matrix representation of an Octonion, which leads us into the geometry.

We define the scalar and vector part of an Octonion by the following formulae:

$$Sc(a) = \frac{1}{2}(a + \overline{a})$$

$$Vec(x) = \frac{1}{2}(a - \overline{a})$$

This is equivalent to

$$Sc(a) = a_0 e_0$$

$$Vec(a) = \sum_{i=1}^{7} a_i e_i$$

The polar form of an Octonion a is

$$a = \sqrt{N(a)}\left(\frac{Sc(a)}{\sqrt{N(a)}} + \frac{Vec(a)}{\sqrt{N(a)}}\right) = \sqrt{N(a)}\left(cos\theta + \hat{a}sin\theta\right)$$

If we define $\hat{a} = \frac{Vec(a)}{N(Vec(a))}$, by the De Moivre formula we have

$$a^n = cos(n\theta) + \hat{a}sin(n\theta)$$

where a is an unit Octonion.

We deduce the formula $Sc^2(a) + N(Vec(a)) = N(a)$, that allows us to calculate the angle between two Octonions a and b as $\lambda = cos^{-1}\frac{Sc(a\bar{b})}{\sqrt{N(a)}\sqrt{N(b)}}$

A trivial result is that two Octonions a and b are perpendicular if $Sc(a\bar{b}) = 0$ and they are parallel if $Vec(a\bar{b}) = 0$.

Now we define the function $\phi_x : \Re^8 \to \Re^8, \phi_x(a) = x(ax^{-1})$ that geometrically represents a rotation of the vector part of a about the vector x through an angle 2ϕ.

If $\hat{p}$ is a normalised unit Octonion, then $\hat{p}^{-1} = -\hat{p}$, and the angle of this Octonion is $\frac{\pi}{2}$, in this way $\hat{p}(a\hat{p}^{-1}) = -\hat{p}(a\hat{p})$ describes a rotation of $Vec(a)$ through π about $\hat{p}$.

Similarly, $\hat{p}(a\hat{p})$ describes a reflection in the plane.

The 7-dimensional cross-product and the second twist: an example

If x, y are 7-dimensional vectors with coordinates $x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$

and $y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7)$, then their cross-product is defined by bilinear operator

$$T_x = \begin{bmatrix} 0 & -x_4 & -x_7 & x_2 & -x_6 & x_5 & x_3 \\ x_4 & 0 & -x_5 & -x_1 & x_3 & -x_7 & x_6 \\ x_7 & x_5 & 0 & -x_6 & -x_2 & x_4 & -x_1 \\ -x_2 & x_1 & x_6 & 0 & -x_7 & -x_3 & x_5 \\ x_6 & -x_3 & x_2 & x_7 & 0 & -x_1 & -x_4 \\ -x_5 & x_7 & -x4 & x_3 & x_1 & 0 & -x_2 \\ -x_3 & -x_6 & x_1 & -x_5 & x_4 & x_2 & 0 \end{bmatrix},$$

where $x \times y = T_x y$

Let us remember that the second twist of the triple articulation creates a pseudo-referential when applied to the referential *per se*.

As we defined before the second twist is a rotation by π in respect to itself, for each one of the vertices of the referential articulation. Once we have a matrix representation of Octonions, we can calculate the pseudo-referential articulation by the formula $\phi_x(a) = x(ax^{-1})$ rotating each vector by π.

Let us suppose that the vector that represents 'urgency' is represented by the Octonion $urg = (0,0,2,5,0,0,2,6)$ then the rotation of urg by π is defined by $\phi_x(urg) = x(urgx^{-1})$, where x is a vector that we agree to make rotation of all vectors of the referential.

Now, we normalize the vector $urg = (0,0,2,5,0,0,2,6)$, that has a matrix form

$$urg = \begin{bmatrix} 0 & 0 & -2 & -5 & 0 & 0 & -2 & -6 \\ 0 & 0 & -5 & 2 & 0 & 0 & 6 & -2 \\ 2 & 5 & 0 & 0 & -2 & -6 & 0 & 0 \\ 5 & -2 & 0 & 0 & -6 & 2 & 0 & 0 \\ 0 & 0 & 2 & 6 & 0 & 0 & -2 & -5 \\ 0 & 0 & 6 & -2 & 0 & 0 & 2 & -2 \\ 2 & -6 & 0 & 0 & 0 & -5 & 0 & 0 \\ 6 & 2 & 0 & 0 & 5 & 2 & 0 & 0 \end{bmatrix}$$

For the normalisation of the vector $urg = (0,0,2,5,0,0,2,6)$, we calculate

$$N(urg) = \sqrt{4+25+4+36} = \sqrt{69}$$

then, the normalised vector is $\hat{urg} = \left(0,0,\frac{2}{\sqrt{69}},\frac{5}{\sqrt{69}},0,0,\frac{2}{\sqrt{69}},\frac{6}{\sqrt{69}}\right)$.

Let us suppose that the rotation of urg is made respect to $p = (0,2,3,0,2,0,2,2)$,

then we calculate $N(p) = 5$, then $\hat{p} = \left(0,\frac{2}{5},\frac{3}{5},0,\frac{2}{5},0,\frac{2}{5},\frac{2}{5}\right)$.

We know that $\hat{p}((urg)\hat{p}^{-1}) = -\hat{p}((urg)\hat{p})$ rotates urg by π.

The vector

$-\hat{p}((urg)\hat{p}) =$

$$-\left(0,\frac{2}{5},\frac{3}{5},0,\frac{2}{5},0,\frac{2}{5},\frac{2}{5}\right)\left((0,0,0,2,5,0,0,2,6)\left(0,\frac{2}{5},\frac{3}{5},0,\frac{2}{5},0,\frac{2}{5},\frac{2}{5}\right)\right).$$

Finally we calculate the matrices associated to urg and $\hat{p}$ and multiply them according to the formula.

The resulting vector is the rotation of urg by π through p, and is one of the vectors of the pseudo-referential articulation.

The orthogonal group

The orthogonal group in dimension n, or the n-dimensional general orthogonal group, denoted $O(n)$, is a group of distance-preserving transformations of the n-dimensional Euclidean space that preserves a fixed point. The operation of a group is the composition of transformations.

Another equivalent definition of the orthogonal group of dimension n, it is the group of $n \times n$ orthogonal matrices. Where an orthogonal or orthonormal matrix Q is an $n \times n$ matrix such that $Q^T Q = QQ^T = I_n$, where Q^T is the transpose matrix of Q, from where we conclude that, a matrix Q is orthogonal iff $Q^T = Q^{-1}$.

Exemples of orthogonal matrices:

The group formed by the matrices $\{[1], [-1]\}$, that geometrically represents the identity and the reflexion.

$\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$ represents a rotation

$\begin{bmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{bmatrix}$ represents a reflection

$\begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix}$ represents an inversion with respect to the origin

$\begin{bmatrix} & -1 & \\ 1 & & \\ & & -1 \end{bmatrix}$ represents a roto-inversion through the z-axis

The set of all n-orthogonal matrices is a Lie compact group of dimension $\frac{n(n-1)}{2}$ called the orthogonal group and denoted by $O(n)$.

The special orthogonal group $SO(n)$ is a subgroup of $O(n)$, where all the matrices have determinant 1.

We have that $\frac{O(n)}{SO(n)} \simeq O(1)$.

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