



Ideal, non-extended formulations for disjunctive constraints admitting a network representation

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Received: 17 March 2020 / Accepted: 1 April 2021 / Published online: 16 April 2021
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Abstract

In this paper we reconsider a known technique for constructing strong MIP formulations for disjunctive constraints of the form $x \in \bigcup_{i=1}^m P_i$, where the P_i are polytopes. The formulation is based on the Cayley Embedding of the union of polytopes, namely, $Q := \text{conv}(\bigcup_{i=1}^m P_i \times \{\epsilon^i\})$, where ϵ^i is the i th unit vector in \mathbb{R}^m . Our main contribution is a full characterization of the facets of Q , provided it has a certain network representation. In the second half of the paper, we work-out a number of applications from the literature, e.g., special ordered sets of type 2, logical constraints, the cardinality indicating polytope, union of simplices, etc., along with a more complex recent example. Furthermore, we describe a new formulation for piecewise linear functions defined on a grid triangulation of a rectangular region $D \subset \mathbb{R}^d$ using a logarithmic number of auxiliary variables in the number of gridpoints in D for any fixed d . The series of applications demonstrates the richness of the class of disjunctive constraints for which our method can be applied.

Keywords Disjunctive programming · Mixed-integer linear programming formulations · Facets

Mathematics Subject Classification 90C11

To the memory of Egon Balas.

This work has been supported by the National Research, Development and Innovation Office—NKFIH, Grant No. SNN 129178, and ED_18-2-2018-0006.

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1 Introduction

Disjunctive programming was introduced by Egon Balas [2,3] in the 1970s as an extension of linear programming with disjunctive constraints. Disjunctive constraints can express logical conditions that the feasible solutions must satisfy. A disjunctive constraint can be represented as

$$x \in \bigcup_{i=1}^m P_i, \quad (1)$$

where each P_i is a polyhedron. Suppose $P_i = \{x \in \mathbb{R}^n : A^{(i)}x \leq b^{(i)}\}$, where $A^{(i)}$ is a rational matrix and $b^{(i)}$ is a rational vector of matching dimension. Balas [4], and Jeroslow and Lowe [32] proved that if the P_i are bounded and nonempty, then \bar{x} satisfies (1) if and only if the following set of constraints admits a solution:

$$\begin{aligned} \sum_{i=1}^m x^{(i)} &= \bar{x}, \\ A^{(i)}x^{(i)} &\leq b^{(i)}\lambda_i, \quad i = 1, \dots, m \\ \sum_{i=1}^m \lambda_i &= 1, \\ \lambda &\in \{0, 1\}^m. \end{aligned} \quad (2)$$

More generally, a *MIP formulation* for (1) is a set of constraints

$$Ax + By + Cz \leq b, \quad z \in \mathbb{Z}^k \quad (3)$$

such that \bar{x} satisfies (1) if and only if (3) admits a feasible solution when $x = \bar{x}$ (see e.g. [26,32,44–46]). The *Linear Programming (LP) relaxation* of (3) is the polyhedron Q determined by $Ax + By + Cz \leq b$. For simplicity we assume that Q has at least one extreme point. The formulation is *ideal*, if z is integral in all extreme points of Q , and *non-extended* if it contains no y variables, and *extended* otherwise. Balas' formulation (2) is extended and ideal [32,45]. A lot of work has been done for devising (ideal) formulations for disjunctive constraints that do only involve the original x variables (in particular, no copies of them) as well as some additional integral z -variables, see Sect. 2 for a brief overview, and it is the main topic of this paper.

For illustrating the benefits of such formulations in terms of the size of the formulation, consider the n -dimensional cross-polytope

$$P = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n |x_j| \leq 1 \right\}.$$

In fact, P equals the convex hull of the union of polytopes $P_i = \{x \in \mathbb{R}^n : -1 \leq x_i \leq 1, x_j = 0 \forall j \neq i\}, i = 1, \dots, n$. A minimal linear representation of P consists of 2^n linear inequalities (see e.g. Balas [9], page 146), that is,

$$\begin{aligned} -x_1 - \dots - x_n &\leq 1 \\ x_1 - \dots - x_n &\leq 1 \\ &\vdots \\ x_1 + \dots + x_n &\leq 1. \end{aligned}$$

Now, an ideal, non-extended formulation for $x \in \bigcup_{i=1}^n P_i$ is

$$\begin{aligned} -\lambda_i \leq x_i \leq \lambda_i, \quad i = 1, \dots, n \\ \sum_{i=1}^n \lambda_i = 1, \quad \lambda \in \mathbb{Z}_{\geq 0}^n. \end{aligned}$$

This system has $2n$ continuous variables, and $3n + 1$ linear inequalities (including the nonnegativity of the variables).

A further advantage of ideal, non-extended formulations for (1) is that they may be more efficient computationally than (2), since they do not contain copies of the original problem variables and constraints. This claim is supported by a recent study of Anderson et al. [1], who compared ideal non-extended, and extended formulations for expressing the maximum of affine functions on polyhedral domains, and showed that the former formulations result much shorter computation times.

To derive ideal, non-extended formulations for (1), we focus on a particular technique, pioneered by Vielma [45], which is based on the Cayley Embedding of a finite union of polyhedra. The Cayley Embedding of polyhedra was proposed by Huber et al. [25] for studying the Minkowski sum of point configurations in \mathbb{R}^n , see also [34,49], and for the union of polyhedra in (1) it would be of the form $\bigcup_{i=1}^m P_i \times \{\epsilon^i\}$, where ϵ^i is the i th unit vector in \mathbb{R}^m . Vielma generalized the Cayley Embedding by replacing the unit vectors with distinct binary vectors $h^i \in \{0, 1\}^d$ for some $d \geq \lceil \log_2 m \rceil$. Under the conditions that all the P_i are non-empty, and pointed, Vielma proved that a non-extended, ideal MIP formulation for (1) is given by

$$(x, \lambda) \in Q(\mathcal{P}, \mathcal{H}) := \text{conv} \left(\bigcup_{i=1}^m (P_i \times \{h^i\}) \right), \quad \lambda \in \{0, 1\}^d, \tag{4}$$

where $\mathcal{P} = (P_i)_{i=1}^m$, and $\mathcal{H} = (h^i)_{i=1}^m$ is a family of distinct binary vectors in $\{0, 1\}^d$. This formulation is implicit in the sense that it does not provide a description of $Q(\mathcal{P}, \mathcal{H})$ in terms of linear inequalities. Nevertheless, Vielma determined the non-trivial facets¹ of the convex hull $Q(\mathcal{P}, \mathcal{H})$ for the SOS2 constraints of [13], and also for piecewise linear functions of two variables on grid triangulations of a square. In

¹ Those beyond the non-negativity of the variables.

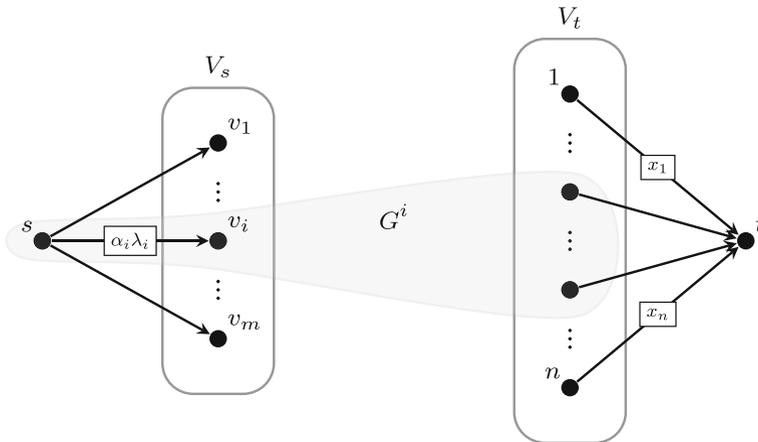


Fig. 1 Network N , and subgraph G^i

both cases, the number of binary variables and also that of the non-trivial facets were bounded by $O(\log_2 n)$.

These ideas were generalized to *combinatorial disjunctive constraints* by Huchette and Vielma [27], which take the form

$$x \in \bigcup_{i=1}^m P(S_i) \tag{5}$$

where the S_i are subsets of $V := \{1, \dots, n\}$ such that $\bigcup_{i=1}^m S_i = V$, $P(S) := \{x \in \Delta^V : x_j = 0 \forall j \in V \setminus S\}$, and $\Delta^V := \{x \in \mathbb{R}_{\geq 0}^n : \sum_{j=1}^n x_j = 1\}$ is the n -dimensional standard simplex. Huchette and Vielma gave an explicit description of $Q(\mathcal{P}, \mathcal{H})$ by linear inequalities, where $\mathcal{P} := (P(S_i))_{i=1}^m$, and \mathcal{H} is a set of m distinct binary vectors, under some mild technical assumptions. However, the construction of the inequalities may be computationally heavy in general, and the authors left open the characterization of those inequalities which induce facets of $Q(\mathcal{P}, \mathcal{H})$.

In this paper we propose a new systematic way of constructing ideal, non-extended formulations for disjunctive constraints (1), when the P_i are polytopes, and a certain network representation exists. We will use the MIP formulation

$$(x, \lambda) \in P^{emb} := \text{conv} \left(\bigcup_{i=1}^m P_i \times \{\epsilon^i\} \right), \lambda \in \{0, 1\}^m \tag{6}$$

of Vielma [45] restricted to the unit vectors ϵ^i . Nevertheless, if P^{emb} admits a network representation of a given structure, to be defined next, then we can characterize the facets of P^{emb} . Our method works not only for combinatorial disjunctions, but for a much wider class of polytopes, which will be illustrated in Sect. 5.

A *network representation* for P^{emb} consists of a network $N = (V, A, c)$ from the following family. The set of nodes V comprises a source node s , and a sink node t ,

the nodes $V_s := \{v_1, \dots, v_m\}$ representing m alternatives, the nodes $V_t := \{1, \dots, n\}$ corresponding to the variables x_1, \dots, x_n in (6), and possibly some other nodes. We assume that $V_s \cap V_t = \emptyset$. The set of arcs A contains, among others, the arcs (s, v_i) for $i \in \{1, \dots, m\}$, and also the arcs (j, t) for $j \in \{1, \dots, n\}$, and neither s , nor t has other nodes adjacent to them. It is required that $N \setminus \{t\}$ decomposes into m directed trees rooted at s , G^1, \dots, G^m , where v_i is a node of G^i , the leaves of each G^i lie in V_t , and $V(G^i) \cap V(G^k) \subseteq V_t \cup \{s\}$ for $1 \leq i < k \leq m$ (see Fig. 1). For each i and arc a of G^i , the capacity of a is $c(a) := k_a \lambda_i$ for some rational number $k_a > 0$. Moreover, $c(j, t) = x_j$ for each $j \in V_t$.

We say that N represents P^{emb} , if for each i , when setting $\lambda_i = 1$, and the other coordinates of λ to 0, and for any $x \geq 0$, $(x, \lambda) \in P^{emb}$ if and only if the network N parametrized by x and λ as above, admits a feasible $s - t$ flow of value $\sum_{j=1}^n x_j$, and (x, λ) satisfies the valid equations for P^{emb} . Deciding whether a particular P^{emb} admits a network representation from the above family is out of scope of this paper. However, in several practical applications, the construction of N is easy, see Sect. 5.

In the sequel we concentrate on two special cases only. Either the only valid equation for P^{emb} is $\sum_{i=1}^m \lambda_i = 1$, or all $(x, \lambda) \in P^{emb}$ also satisfy $\sum_{i=1}^m \alpha_i \lambda_i = \sum_{j=1}^n x_j$, where $\alpha_i \lambda_i = c(s, v_i)$. We distinguish these two cases by writing P_{\leq}^{emb} , and $P_{=}^{emb}$, respectively, for P^{emb} . To simplify the presentation, let P_*^{emb} denote any of P_{\leq}^{emb} and $P_{=}^{emb}$. We assume that each $j \in V_t$ is reachable from s by a directed path. This implies that $x_j = 0$ is not a valid equation for P_*^{emb} . The following assumption is without loss of generality.

Assumption 1 For each $i \in \{1, \dots, m\}$, P_*^{emb} contains a point (x^i, ϵ^i) .

Main results of the paper. In the first part of the paper, we fully characterize the non-trivial facets of P_*^{emb} for any choice of the P_i , provided it has a network representation N . At first, we prove that any $s - t$ cut of the network N yields a valid inequality for P^{emb} , and moreover, along with the non-negativity of the variables, and the valid equation(s), they describe P^{emb} . Then, we fully characterize those $s - t$ cuts of N , that induce facets of P^{emb} . Briefly, some $s - t$ cut (S, \bar{S}) of N induces a facet of P_{\leq}^{emb} if and only if the subgraphs of N spanned by $\bar{S} \setminus \{t\}$, and by S , respectively, admit some connectivity properties. In case of $P_{=}^{emb}$, slightly stronger conditions must hold. These results, along with (6) provide an explicit, ideal, non-extended MIP formulation for (1).

In the second part of the paper, we demonstrate the richness of the class of problems for which our machinery can be applied by deriving well-known formulations for problems in the literature, and working out some new ones. The list of examples include reformulation of SOS2 and SOS k sets, the Cardinality Indicating Polytope, the $EVEN^n$ polytope, the polytope of small cliques, the union of simplices, and some more complicated ones. In addition, based on the ideas of [28], we obtain a new formulation for piecewise linear functions defined on a grid-triangulation of a rectangular domain in \mathbb{R}^d using $O(d \log_2 n + 2^d d!)$ auxiliary variables.

Our results complement those of Vielma [45] and Huchette and Vielma [26–28]. On the one hand, Vielma [45] was able to derive the facets of the generalized Cayley embedding of SOS2 sets for any choice of the (distinct) binary vectors h^i , but our

method works for a much wider class of disjunctive sets than SOS2, and we have a full characterization of the non-trivial facets. On the other hand, in [26] a linear representation is obtained for the convex hull $Q(\mathcal{P}, \mathcal{H})$ of the MIP formulation for combinatorial disjunctive constraints (5) and any choice of distinct binary vectors $\mathcal{H} = (h^i)_{i=1}^m$, while our characterization of facets is valid for any choice of the P_i provided that P^{emb} in (6) admits a network representation.

In Sect. 2 we review some previous work. After some preliminaries in Sect. 3, we state and prove our main results in Sect. 4. In Sect. 5 we describe a series of applications, and conclude the paper in Sect. 6.

1.1 Terminology

Let $P = \{x \in \mathbb{R}^n \mid Bx \leq b\}$ be any convex polyhedron. A *face* F of P is a subset of P obtained by setting some inequalities to equations in the system of inequalities $Bx \leq b$ defining P . In particular, P is a face of itself, and also the empty set is a face of P . A *facet* F is a face of P such that $F \neq P$, and there exists no face F' of P such that $F \subset F' \subset P$ (the containments are proper). A *vertex* of P is a face which consists of exactly one point of P .

The dimension of P , $\dim(P)$, is the maximum number of affinely independent points in P minus one. Equivalently, $\dim(P) = n - m_{=}$, where $m_{=}$ is the maximum number of linearly independent equations which are satisfied by all the points of P .

Let $\epsilon^i \in \mathbb{R}^m$ be the i -th unit vector in \mathbb{R}^m , i.e., $\epsilon_i^i = 1$ and $\epsilon_k^i = 0$ for all $k \in \{1, \dots, m\} \setminus \{i\}$.

A network $N = (V, A, c)$ is a directed graph with vertex set V , arc set A , and arc capacities c . Let V_1, V_2 be disjoint subsets of the set of vertices V of the network N . Then $[V_1, V_2] \subseteq A$ is the set of those arcs $(u, v) \in A$ such that $u \in V_1$ and $v \in V_2$. Let s and t be distinct vertices of N . An $s - t$ *cut* (S, \bar{S}) is a partitioning of V into subsets S and $\bar{S} = V \setminus S$ such that $s \in S$ and $t \in \bar{S}$. The associated *cut-set* is $[S, \bar{S}]$. For any node u , let $\delta_u^{out} := [\{u\}, V \setminus \{u\}]$, and $\Gamma_u^{out} = \{v \in V : (u, v) \in \delta_u^{out}\}$.

Recall the general network N defined above. For fixed x and λ , let $N_{x,\lambda}$ denote the network with arc capacities determined by x and λ .

2 Previous work

The ultimate reference on disjunctive programming is the recent book of Balas [9]. The first results were summarized in a research report which got published only some 24 years later [6], but the first papers appeared in print already in the 1970's, see [2] and [3]. In these early works, extended formulations were obtained for general disjunctive programs, the concept of duality was developed and a sequential convexification procedure was devised. Questions related to facets of general disjunctive programs were raised and partially answered in [11].

Balas [2,3,6] investigated the convex hull of feasible points of a disjunctive program by giving two distinct linear programming formulations: one involving only the original problem variables, and another using additional continuous variables. We focus

on the latter one. Let

$$P := \text{cl conv} \left(\bigcup_{i=1}^m P_i \right),$$

where $P_i := \{x \in \mathbb{R}^n \mid A^{(i)}x \leq b^{(i)}\}$ for some matrix $A^{(i)}$ and vector $b^{(i)}$ of matching dimensions, for $i = 1, \dots, m$, and $\text{cl conv}(\cdot)$ is the closed convex hull operator. Balas has shown that $\bar{x} \in P$ if and only if the LP-relaxation of (2) admits a feasible solution when $x = \bar{x}$, provided that the P_i are nonempty and bounded, or some additional technical assumptions hold.

Jeroslow and Lowe [32] introduced the concept of MIP-representable sets, namely, $S \subseteq \mathbb{Q}^n$ is *MIP-representable* if there are rational matrices A, B, C , and a vector b with the property that: $x \in S$ if and only if, for some $y, z \geq 0$ with z integer, $Ax + By + Cz \leq b$. They also pointed out the connection to disjunctive programming, and in particular they showed that (2) is ideal. Basu et al. [12] gave an alternative definition, and the concept was generalized to convex sets by Lubin et al. [37]. Vielma [44] provided a broad overview of MIP formulations in mixed-integer programming, including MIP representability of sets and functions.

Conforti and Wolsey [19] generalized the idea of Balas [6] for expressing the union of polyhedra in a higher dimensional space by first lifting each polyhedron in some space where it admits an easy extended formulation. They applied this results to mixed integer sets, like the continuous mixing set with upper bound, a mixing set with two divisible capacities, and a divisible capacity single node flow model.

In the formulation (2), the original x variables are copied m times, which is computationally unattractive. However, Conforti et al. [18] has recently shown that Balas' extended formulation for $\text{conv}(P_1 \cup P_2)$ is optimal in the following sense. For any polynomial σ , there exist polytopes P_1 and P_2 of size² f_1 and f_2 , respectively, such that any formulation of $\text{conv}(P_1 \cup P_2)$ of size bounded by $\sigma(f_1 + f_2)$ must have $\Omega(n)$ additional variables.

In the best case, neither variable copies, nor new λ_i variables are needed to get an explicit linear description of the convex hull of the union of some polyhedra. For instance, Balas et al. [10] considered upper monotone polytopes in $[0, 1]^n$. A polytope $P = \{x \in [0, 1]^n \mid Ax \geq 1\}$ is *upper monotone* if A is a non-negative matrix. Among other results, Balas and his co-authors derived the convex hull of the union of two upper monotone polytopes $P_1 \subseteq [0, 1]^{n_1} \times \{0_{n_2}\}$ and $P_2 \subseteq \{0_{n_1}\} \times [0, 1]^{n_2}$ in disjoint spaces, and in partially overlapping spaces as well without using any extra variables. They also obtained the facets of $\text{conv}(P_1 \cup P_2)$, and applied their characterizations to the union of matroid polytopes, and to logical inference. The latter result extends that of [50], see also [7]. These results were generalized to nonlinear and possibly unbounded orthogonal disjunctive sets in [42].

The second best option is that no variable copies are used in a formulation, only some new (binary) variables. Jeroslow [31] studied a special class of disjunctive constraints, where $P_i := \{x \in \mathbb{R}^n \mid Ax \leq b^{(i)}\}$ for $i \in \{1, \dots, m\}$, that is, only the right-hand-sides change in the definitions of the P_i . Let $P := \text{cl conv}(\bigcup_{i=1}^m P_i)$ and Q the set of

² Size of a (linear) formulation is measured by the number of inequalities.

those $x \in \mathbb{R}^n$ that have an extension $(x, \lambda) \in \mathbb{R}^{n+m}$ satisfying

$$\begin{aligned} Ax - \sum_{i=1}^m b^{(i)} \lambda_i &\leq 0 \\ \sum_{i=1}^m \lambda_i &= 1, \lambda \geq 0. \end{aligned} \quad (7)$$

While $P \subseteq Q$ always holds, the converse inclusion is not true in general. Jeroslow gave two sufficient conditions for $Q = P$. Blair [14] devised new necessary and sufficient conditions for $Q = P$, and proved that deciding whether equality holds is an NP-hard decision problem. Blair's condition was later generalized by Vielma [46] (Corollary 2). Balas [5] gave yet another sufficient condition for $Q = P$, and applied it to multiple network polyhedra.

Vielma [45] developed ideal, non-extended MIP formulations for modeling disjunctive constraints based on the concept of Cayley Embedding, and applied it to SOS2 constraints, and to piecewise linear functions on the grid. In [27], Huchette and Vielma provided a very general ideal formulation for combinatorial disjunctive constraints based on the generalized Cayley Embedding of [45], and applied it to univariate and bivariate piecewise linear functions. Lee and Wilson [36] studied the modeling of piecewise linear functions on arbitrary triangulations of their domains. Their model can be reinterpreted as the Cayley Embedding of a union of polyhedra (one polytope for each triangle) using the vectors ϵ^i . Vielma [46] generalized the Cayley Trick for polyhedra to the union of convex sets. He obtained ideal, non-extended formulations for $\bigcup_{i=1}^m C_i$ using the corresponding gauge functions, and applied them to a wide range of disjunctive constraints. Vielma also characterized the boundary structure of the Cayley Embedding of the union of convex sets, which in the polyhedral case is as follows. Each face of P^{emb} is of the form $\text{conv}(\bigcup_{i=1}^m (F_i \times \{\epsilon^i\}))$, where the F_i are faces of the P_i with intersecting normal vectors, see also [51].

Branching schemes constitute another approach for deriving ideal, non-extended formulations for (combinatorial) disjunctive constraints. Tomlin [43] proposed a new modeling of SOS2 sets with binary variables, while Martin et al. [38] introduced SOS k constraints and a branching scheme for modeling two-variable piecewise linear functions. Vielma and Nemhauser introduced the concept of independent branching schemes for modeling a constraint very similar to the combinatorial disjunctive constraint (5) of [26], the only difference being that $\Delta^V = \{x \in \mathbb{R}_{\geq 0}^V : \sum_{j \in V} x_j \leq 1\}$ is the $|V|$ -dimensional simplex in the definition of the $Q(S_i)$, $i = 1, \dots, m$. An *independent branching scheme of depth d* for (5) is given by pairs of disjoint sets $L_k, R_k \subset V$ for $k = 1, \dots, d$, such that $\bigcup_{i=1}^m Q(S_i) = \bigcap_{k=1}^d (Q(L_k) \cup Q(R_k))$. This yields the formulation

$$x \in \Delta^V, \sum_{j \notin L_k} x_j \leq z_k, \sum_{j \notin R_k} x_j \leq 1 - z_k, z_k \in \{0, 1\}, k = 1, \dots, d. \quad (8)$$

This formulation is ideal. Of course, the existence of a branching scheme of depth d depends on the sets S_i . For SOS1 and SOS2 constraints, the authors obtained formu-

lations with $\lceil \log_2 n \rceil$ binary variables and twice as many additional constraints. They also devised MIP formulations for 2-variable piecewise linear functions on a grid triangulation of a square using a logarithmic number of new variables and constraints in the number of triangles, see also [47]. When Δ^V is replaced by $[0, 1]^V$ in the above definitions, they obtained an independent branching scheme of depth $d = m$. In [26], Huchette and Vielma gave a necessary and sufficient condition for the existence of an independent branching scheme of depth d for a combinatorial disjunctive constraint. The authors described a number of techniques for constructing independent branching schemes of depth $\lceil \log_2 m \rceil$, and applied them to several well-known disjunctive constraints, including SOS2, SOS k , and Grid Triangulations. In [28], new methods are described for constructing independent branching schemes for univariate and bivariate piecewise linear functions (on grid triangulations), and the computational merits of the various approaches are evaluated.

Ceria and Soares [16] generalized the formulation (2) to closed convex sets C_i using the perspective mappings of the convex functions defining the C_i . The authors also described a primal procedure which converges to the optimum under mild conditions. On/off constraints can be used to model disjunctive constraints and the description of the set of feasible solutions in linear and nonlinear mathematical programs is the topic of e.g., [15,22–24,42,46]. In particular, Hijazi et al. [24] considered a linear program with a disjunctive constraint $x \in P_0 \cup P_1$, where P_0 is given by lower and upper bounds for the variables, while P_1 is specified by one linear inequality and another set of lower and upper bounds on the variables. The authors provided a complete description of $\text{conv}(P_0 \cup P_1)$ using one new binary variable only. Note that their description needs exponentially many new constraints in the number of the variables, but the authors left open which inequalities induce facets.

For more results on extended formulations in combinatorial optimization see the review paper [17], and also [35,40].

3 Preliminaries

In this section we derive a (not necessarily minimal) linear representation of P_*^{emb} . Here and below, we always assume that P_*^{emb} admits a network representation $N = (V, A, c)$. As we will see, the linear representation is closely related to the $s - t$ cuts of N . We will define a dominance relation between $s - t$ cuts, and prove that dominated $s - t$ cuts are redundant in the linear representation.

To start with, we describe some reduction rules for the arc capacities of N without affecting any $s - t$ flow of value $\sum_{j=1}^n x_j$. Recall that for any i the arc capacities of the subnetwork G^i of N are specified by $c(a) = k_a \lambda_i$, for $a \in E(G^i)$. Let us define $k_v^{out} := \sum_{a \in \delta_v^{out}} k_a$ for any node $v \in V(G^i) \setminus V_t$. Let $v \in V(G^i) \setminus \{s\}$. The capacity $c(a) = k_a \lambda_i$ of the unique arc $a = (u, v) \in E(G^i)$ can be reduced without affecting the set of feasible $s - t$ flows of value $\sum_{j=1}^n x_j$ in $N_{x,\lambda}$ for any $(x, \lambda) \in P_*^{emb}$ if one of the following two transformations can be applied to it:

- If $v \notin V_t$ and $k_a > k_v^{out}$, then k_a can be reduced to k_v^{out} , thus $c(a)$ becomes $k_v^{out} \lambda_i$.
- If $v \neq v_i$, and $k_a > k_{a'}$ for the unique arc $a' := (u', u) \in E(G^i)$, then k_a can be reduced to $k_{a'}$, thus $c(a)$ becomes $k_{a'} \lambda_i$.

Throughout the paper we assume that N is *reduced*, i.e., the above transformations cannot be applied to it. Now, we express the capacity of the $s - t$ cuts of $N_{x,\lambda}$ in terms of x and λ .

Observation 1 *Let (S, \bar{S}) be an $s - t$ cut of $N_{x,\lambda}$. Then $c_{x,\lambda}([S, \bar{S}]) = \sum_{j \in V_t \cap S} x_j + \sum_{i=1}^m k_i \lambda_i$ for some rational numbers $k_i \geq 0$.*

In order to describe P_{\leq}^{emb} by linear inequalities, we define the polytope

$$Q_{\leq} := \left\{ (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \mid \sum_{i=1}^m \lambda_i = 1, x \geq 0, \lambda \geq 0, \right. \\ \left. c_{x,\lambda}([S, \bar{S}]) \geq \sum_{j=1}^n x_j, \forall s - t \text{ cut } (S, \bar{S}) \right\}, \tag{9}$$

while for $P_{=}^{emb}$ consider

$$Q_{=} := \left\{ (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \mid \sum_{i=1}^m \lambda_i = 1, x \geq 0, \lambda \geq 0, \right. \\ \left. c_{x,\lambda}([S, \bar{S}]) \geq \sum_{j=1}^n x_j, \forall s - t \text{ cut } (S, \bar{S}) \right. \\ \left. \sum_{i=1}^m \alpha_i \lambda_i = \sum_{j=1}^n x_j \right\}. \tag{10}$$

Let Q_* denote one of Q_{\leq} and $Q_{=}$.

Proposition 1 $P_*^{emb} \subseteq Q_*$.

Proof By definition, for any $i \in [m]$, $(x, \epsilon^i) \in P_*^{emb}$ if and only if $N_{(x, \epsilon^i)}$ admits an $s - t$ flow of value $\sum_{j=1}^n x_j$, $x \geq 0$, and (x, ϵ^i) satisfies the valid equations for P_*^{emb} . By the MAX-FLOW MIN-CUT Theorem of Ford and Fulkerson [20], $N_{(x, \epsilon^i)}$ admits an $s - t$ flow of value $\sum_{j=1}^n x_j$ if and only if all the $s - t$ cuts have a capacity of at least $\sum_{j=1}^n x_j$. Now, observe that these are precisely the inequalities defining Q_* , whence $(x, \epsilon^i) \in Q_*$. To finish the proof, take any point $(x, \lambda) \in P_*^{emb}$, and notice that it is a convex combination of some points with λ integral. Since Q_* is a convex polytope, the statement follows. \square

Our next goal is to prove the converse inclusion.

Proposition 2 Q_* has vertices, and in each vertex (x, λ) of Q_* , the λ is a 0/1 vector.

Proof By Assumption 1, and since $P_*^{emb} \subseteq Q_*$, Q_* is not empty. Since all variables are non-negative in the definition of Q_* , it resides in the non-negative orthant of the $(n + m)$ -dimensional real vector space, and thus it has vertices.

Let $H_i := \{(x, \lambda) \in Q_* : \lambda_i = 1\}$ for each $i \in \{1, \dots, m\}$. By Assumption 1, $H_i \neq \emptyset$ for each i . Clearly, H_i is a face of Q_* . We claim that $Q_* = \text{conv}(\cup_{i=1}^m H_i)$. On the one hand, $\text{conv}(\cup_{i=1}^m H_i) \subseteq Q_*$, since $H_i \subseteq Q_*$ for each i , and Q_* is convex. Conversely, consider any point $(x, \lambda) \in Q_*$, and consider any $s-t$ flow of $N_{x,\lambda}$ of value $\sum_{j=1}^n x_j$. Clearly, such a flow must exist, since the minimum capacity of an $s-t$ cut of $N_{x,\lambda}$ is $\sum_{j=1}^n x_j$ by the definition of Q_* . Let f_e^i be the flow on arc e of G^i . Then we have $x_j = \sum_{i=1}^m \sum_{e \in \delta_{G^i}^{\text{in}}(j)} f_e^i$, since a flow of value $\sum_{j=1}^n x_j$ must saturate all the (j, t) arcs of N . If $\lambda_j = 0$, then we pick the vector x^i provided by Assumption 1, and we have $(x^i, \epsilon^i) \in H_i$. Otherwise, we rescale the f_e^i for each arc e of G^i , i.e., let $g_e^i = f_e^i / \lambda_i$. Let $x_j^i = \sum_{e \in \delta_{G^i}^{\text{in}}(j)} g_e^i$ for each $j \in V_t \cap V(G^i)$, and 0 otherwise. Then $(x^i, \epsilon^i) \in H_i$, since (g^i, x^i) is a feasible flow of value $\sum_{j=1}^n x_j^i$ in N_{x^i, ϵ^i} , where the arc (s, v_i) and all the arcs (j, t) are saturated. Hence, $(x, \lambda) = \sum_{i=1}^m \lambda_i (x^i, \epsilon^i) \in \text{conv}(\cup_{i=1}^m H_i)$. \square

Corollary 1 $P_*^{emb} = Q_*$ and $P_{\leq}^{emb} = Q_{\leq}$.

Definition 1 For any $s-t$ cut (S, \bar{S}) of N , the induced face of P_*^{emb} is $F_S := \{(x, \lambda) \in P_*^{emb} \mid c_{x,\lambda}([S, \bar{S}]) = \sum_{j=1}^n x_j\}$.

Observation 2 The vertices of F_S are those vertices of P_*^{emb} that are in F_S .

Consequently, in all the vertices $(\hat{x}, \hat{\lambda})$ of F_S , $\hat{\lambda} = \epsilon^i$ for some i .

Proposition 3 No facet F_S of P_*^{emb} induced by an $s-t$ cut (S, \bar{S}) of N equals the face induced by $x_j = 0$ for any j .

Proof We distinguish two cases. First suppose $j \in V_t \cap S$. Since $F_S = \{(x, \lambda) \in P_*^{emb} : \sum_{i=1}^m k_i \lambda_i = \sum_{j' \in V_t \setminus S} x_{j'}\}$, the equation defining F_S is linearly independent from $x_j = 0$, so F_S cannot be induced by $x_j = 0$.

Now suppose $j \in V_t \setminus S$. Then $x_j = 0$ is satisfied by all points $(x, \lambda) \in F_S$ if and only if the source s is not connected to $j \in V_t$ by a directed path, which we excluded. \square

In order to characterize those $s-t$ cuts (S, \bar{S}) of N that induce facets of P_*^{emb} , as we will see, the single most important parameter is the intersection of S with V_t .

Definition 2 Let $U \subseteq V_t$ be fixed. $\mathcal{C}(U)$ consists of all the $s-t$ cuts of N such that $S \cap V_t = U$.

A cut $(S, \bar{S}) \in \mathcal{C}(U)$ induces an inequality of the following general form (cf. Observation 1):

$$\sum_{i=1}^m k_i \lambda_i \geq \sum_{j \in V_t \setminus U} x_j \tag{11}$$

We are only interested in the minimum capacity $s - t$ cuts in $\mathcal{C}(U)$. We can identify these cuts without fixing λ and x as follows. Since $N \setminus \{t\}$ is the union of the directed graphs G^i , where the G^i share only the nodes $\{s\} \cup V_t$, we can determine separately for each G^i which of its nodes belong to S in order to minimize the coefficients k_i .

Observation 3 Let $U \subseteq V_t$ be fixed. There exists an $(S, \bar{S}) \in \mathcal{C}(U)$ which minimizes simultaneously all the coefficients k_i in $c_{x,\lambda}([S, \bar{S}]) = \sum_{j \in U} x_j + \sum_{i=1}^m k_i \lambda_i$.

This observation motivates the following definition.

Definition 3 Let $U \subseteq V_t$ be fixed. Let $\mathcal{C}_{\min}(U)$ consist of all the $s - t$ cuts $(S, \bar{S}) \in \mathcal{C}(U)$ which minimize simultaneously all the coefficients k_i in $c_{x,\lambda}([S, \bar{S}]) = \sum_{j \in U} x_j + \sum_{i=1}^m k_i \lambda_i$. The members of $\mathcal{C}_{\min}(U)$ are the *minimum capacity $s - t$ cuts of N with respect to U* .

Remark 1 For any $(S, \bar{S}), (Z, \bar{Z}) \in \mathcal{C}_{\min}(U)$, $c_{x,\lambda}([S, \bar{S}]) = c_{x,\lambda}([Z, \bar{Z}])$.

Proposition 4 Let $U \subseteq V_t$ be fixed. There exists a unique $(S, \bar{S}) \in \mathcal{C}_{\min}(U)$ such that $Z \subseteq S$ for all the $(Z, \bar{Z}) \in \mathcal{C}_{\min}(U)$.

Proof We claim that if both of $(Z_1, \bar{Z}_1), (Z_2, \bar{Z}_2) \in \mathcal{C}_{\min}(U)$, then $(Z_1 \cup Z_2, \bar{Z}_1 \cup \bar{Z}_2) \in \mathcal{C}_{\min}(U)$, which proves the statement. Recall that the set function which assigns to S the capacity of the $s - t$ cut (S, \bar{S}) is submodular, see e.g., Frank [21]. Using this, we can derive

$$\begin{aligned} c_{x,\lambda}([Z_1, \bar{Z}_1]) + c_{x,\lambda}([Z_2, \bar{Z}_2]) &\geq c_{x,\lambda}([Z_1 \cap Z_2, \overline{Z_1 \cap Z_2}]) + c_{x,\lambda}([Z_1 \cup Z_2, \overline{Z_1 \cup Z_2}]) \\ &\geq c_{x,\lambda}([Z_1, \bar{Z}_1]) + c_{x,\lambda}([Z_2, \bar{Z}_2]) \end{aligned}$$

where the first inequality follows from the submodularity of the cut capacity function, and the second from the assumption that both of $(Z_1, \bar{Z}_1), (Z_2, \bar{Z}_2) \in \mathcal{C}_{\min}(U)$, which implies that $(Z_1 \cap Z_2, \overline{Z_1 \cap Z_2}), (Z_1 \cup Z_2, \overline{Z_1 \cup Z_2}) \in \mathcal{C}(U)$. Consequently, equality holds throughout, and $(Z_1 \cup Z_2, \overline{Z_1 \cup Z_2}) \in \mathcal{C}_{\min}(U)$. \square

Definition 4 Let $U \subseteq V_t$. We call $(S, \bar{S}) \in \mathcal{C}_{\min}(U)$ *dominating* if $Z \subseteq S$ for all $(Z, \bar{Z}) \in \mathcal{C}_{\min}(U)$. If $(S, \bar{S}) \in \mathcal{C}(U) \setminus \mathcal{C}_{\min}(U)$, then we say that (S, \bar{S}) is a *dominated $s - t$ cut of N w.r.t. U* , and any member of $\mathcal{C}_{\min}(U)$ is a *non-dominated $s - t$ cut of N w.r.t. U* .

Remark 1 implies that in the following definition, the face F_S does not depend on which non-dominated $s - t$ cut in $\mathcal{C}_{\min}(U)$ is chosen.

Definition 5 The *face induced by $U \subseteq V_t$* is F_S , where (S, \bar{S}) is any non-dominated $s - t$ cut in $\mathcal{C}_{\min}(U)$.

Observation 4 If F_S is a facet of P_*^{emb} induced by the $s - t$ cut (S, \bar{S}) , then $V_t \setminus S$ is not empty.

Proposition 5 Consider any $U \subseteq V_t$. For any non-dominated $s - t$ cut $(S, \bar{S}) \in \mathcal{C}_{\min}(U)$, the face F_S of P_*^{emb} contains a point (x', ϵ^i) for any $i \in \{1, \dots, m\}$.

Proof Fix S and i , and we set $\lambda = \epsilon^i$. Thus, the arcs of G^i have some positive capacities, while all the arcs in the other $G^k, k \neq i$, subgraphs of N have zero capacity. We construct a vector x' such that $(x', \epsilon^i) \in P_*^{emb}$, and (x', ϵ^i) satisfies (11) at equality.

First we consider P_{\leq}^{emb} . We determine x' by constructing an $s - t$ flow f^i saturating all the arcs in $E(G^i) \cap [S, \bar{S}]$, and which has zero flow on any arc $(u, v) \in E(G^i)$ such that $u \in V(G^i) \setminus S$, and $v \in V(G^i) \cap S$. The total capacity of the arcs in $E(G^i) \cap [S, \bar{S}]$ is k_i , when $\lambda = \epsilon^i$. Since $(S, \bar{S}) \in \mathcal{C}_{\min}(U)$, we can send k_i amount of flow in G^i from s to the nodes $V_i \setminus U$. If $k_i > 0$, then this flow necessarily saturates the arcs $E(G^i) \cap [S, \bar{S}]$. Then, we just set $x'_j = f^i_{(j,t)}$ for all j , and it is easy to verify that (11) is satisfied by (x', ϵ^i) at equality.

As for $P_{=}^{emb}$, if the flow f^i constructed above saturates the arc (s, v_i) (which is of capacity α_i), then we are done, since then $\sum_{j=1}^n x'_j = f^i_{(s,v_i)} = \alpha_i$, by flow conservation at the nodes. Otherwise, we have to augment f^i with additional flow restricted to the subgraph $G^i(S)$ of G^i spanned by the nodes $S \cap V(G^i)$. Consider any $s - j$ path in $G^i(S)$ for some $j \in U$ which consist of non-saturated arcs only. We increase the flow along this path until some arc becomes saturated. We repeat this until no more flow can be sent from s to some node in U in $G^i(S)$. Let \hat{f}^i be the resulting flow. If the arc (s, v_i) is saturated by \hat{f}^i , then we are done, since we can set $x'_j = \hat{f}^i_{(j,t)}$ for all j , and $(x', \epsilon^i) \in F_S$. Otherwise, a subset of the arcs saturated by \hat{f}^i constitutes a cut-set separating v_i from all the nodes in V_i having capacity smaller than α_i . But this contradicts Assumption 1. \square

Proposition 6 For any $U \subseteq V_t$, dominated $s - t$ cuts in $\mathcal{C}(U)$ cannot induce facets of P_*^{emb} .

Proof Let (S, \bar{S}) and (Z, \bar{Z}) be a dominating and a dominated $s - t$ cut in $\mathcal{C}(U)$, respectively. We claim that $F_Z \subset F_S$, and $F_Z \neq F_S$. Let $c_{x,\lambda}([S, \bar{S}]) = \sum_{j \in U} x_j + \sum_{i=1}^m k_i \lambda_i$, and $c_{x,\lambda}([Z, \bar{Z}]) = \sum_{j \in U} x_j + \sum_{i=1}^m k'_i \lambda_i$. Since $(Z, \bar{Z}) \in \mathcal{C}(U)$ is dominated, $k_i \leq k'_i$ for all i , and at least one of these inequalities is strict. Hence, any point $(x, \lambda) \in P_*^{emb}$ satisfies

$$\sum_{j \in U} x_j + \sum_{i=1}^m k'_i \lambda_i \geq \sum_{j \in U} x_j + \sum_{i=1}^m k_i \lambda_i \geq \sum_{j=1}^n x_j.$$

Consequently, if $\sum_{j \in U} x_j + \sum_{i=1}^m k'_i \lambda_i = \sum_{j=1}^n x_j$, then also $\sum_{j \in U} x_j + \sum_{i=1}^m k_i \lambda_i = \sum_{j=1}^n x_j$, i.e., $F_Z \subseteq F_S$. Now let i^* be an index such that $k'_{i^*} > k_{i^*}$. Then by Proposition 5, there is an x such that $(x, \epsilon^{i^*}) \in F_S$. Substituting (x, ϵ^{i^*}) into the cut capacity functions we obtain

$$\sum_{j \in U} x_j + \sum_{i=1}^m k'_i \epsilon^{i^*}_i = \sum_{j \in U} x_j + k'_{i^*} > \sum_{j \in U} x_j + k_{i^*} = \sum_{j \in U} x_j + \sum_{i=1}^m k_i \epsilon^{i^*}_i = \sum_{j=1}^n x_j.$$

Hence $(x, \epsilon^{i^*}) \in F_S \setminus F_Z$. \square

4 Main results

Theorem 1 *The set $U \subset V_t$ induces a facet of P_{\leq}^{emb} if and only if*

- (i) *for the dominating $s - t$ cut $(S, \bar{S}) \in \mathcal{C}_{\min}(U)$, the subgraph $N \setminus (S \cup \{t\})$ is connected, and*
- (ii) *for each $j \in U$, there exists some $v_i \in S$ such that there is a directed path from v_i to j with all nodes in S , and $\alpha_i > k_i$, where $c(s, v_i) = \alpha_i \lambda_i$, and $k_i \lambda_i$ is the total capacity of those arcs of G^i in $[S, \bar{S}]$.*

Proof Necessity : suppose that U induces a facet F of P_{\leq}^{emb} , but the dominating $s - t$ cut $(S, \bar{S}) \in \mathcal{C}_{\min}(U)$ is such that $N \setminus (S \cup \{t\})$ is not connected. Let C_1, \dots, C_q ($q \geq 2$) be the connected components of $N \setminus (\{t\} \cup S)$. Consider the face F_Z induced by the $s - t$ cut (Z, \bar{Z}) of N , where $Z := S \cup \left(\bigcup_{g=2}^q V(C_g)\right)$. We claim that $F \subset F_Z$ and $F \neq F_Z$, and thus F cannot be a facet, which contradicts our initial assumption.

Let $(x, \lambda) \in F$, and suppose that the capacity of (S, \bar{S}) is $\sum_{j \in U} x_j + \sum_{i=1}^m k_i \lambda_i$. We can decompose the second term of this sum along the components of $N \setminus (S \cup \{t\})$ as follows:

$$\sum_{i=1}^m k_i \lambda_i = \sum_{g=1}^q \sum_{i=1}^m k_i^g \lambda_i,$$

where $\sum_{i=1}^m k_i^g \lambda_i$ is the capacity of those arcs of $[S, \bar{S}]$ that lead from S to the component C_g of $N \setminus (S \cup \{t\})$. But then (x, λ) satisfies the equation

$$\sum_{i=1}^m k_i^1 \lambda_i = \sum_{j \in V(C_1) \cap V_t} x_j.$$

However, this equation is equivalent to

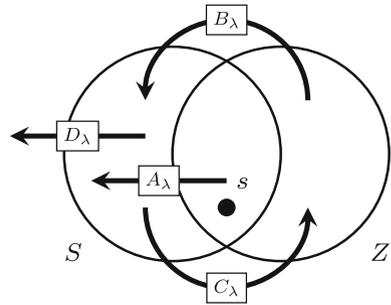
$$\sum_{j \in V_t \setminus V(C_1)} x_j + \sum_{i=1}^m k_i^1 \lambda_i = \sum_{j=1}^n x_j,$$

which is the capacity of the $s - t$ cut (Z, \bar{Z}) of N . This shows that $F \subseteq F_Z$. It remains to construct a point in $F_Z \setminus F$. Since F is a facet, for each $j \in V_t \cap (\bigcup_{g=2}^q V(C_g))$, there exist some x and i such that $(x, \epsilon^i) \in F$, and $x_j > 0$, otherwise all points in F satisfy $x_j = 0$, and thus F cannot be a facet by Proposition 3. For each such j , we define a vector \tilde{x} as follows: $\tilde{x}_j := 0$, and $\tilde{x}_\ell := x_\ell$ for all $\ell \neq j$. Then $(\tilde{x}, \epsilon^i) \in F_Z \setminus F$.

As for (ii), suppose the condition does not hold for some $j \in U$. Then $x_j = 0$ holds for all $(x, \lambda) \in F$, which leads to a contradiction by Proposition 3.

Sufficiency : Assume that conditions (i) and (ii) of the Theorem hold. Let $F := F_S$ be the face of P_{\leq}^{emb} induced by the dominating $s - t$ cut $(S, \bar{S}) \in \mathcal{C}_{\min}(U)$. It suffices to prove that P_{\leq}^{emb} does not admit any facet F' such that F is a proper face of F' . By

Fig. 2 The capacity of the arcs leaving S and Z



condition (ii), for each j there exists i and x such that $(x, \epsilon^i) \in F$ and $x_j > 0$, and therefore, F cannot be a proper face of a facet induced by $x_j = 0$. Furthermore, by Proposition 5, F cannot be a proper face of a facet induced by $\lambda_i = 0$ for any i .

Consider any $s - t$ cut (Z, \bar{Z}) of N which induces a facet $F_Z \neq F$ of P_{\leq}^{emb} . By Proposition 6, we may also assume that (Z, \bar{Z}) is a dominating member of $\mathcal{C}_{\min}(Z \cap V_t)$. We distinguish two cases: $U \setminus Z \neq \emptyset$, and $U \subset Z$.

First suppose $U \setminus Z \neq \emptyset$, and we construct a point in $F \setminus F_Z$ to show that $F \not\subseteq F_Z$. Let $\sum_{i=1}^m k_i^Z \lambda_i$ be the total capacity of those arcs directed from Z to some nodes of $N \setminus (Z \cup \{t\})$. Clearly, all points $(x, \lambda) \in F_Z$ satisfy $\sum_{j' \in V_t \setminus Z} x_{j'} = \sum_{i=1}^m k_i^Z \lambda_i$. Let R_i be the set of those nodes in $V_t \setminus Z$ that are reachable from v_i along a directed path in N . Then choose index i^* such that $R_{i^*} \cap U \neq \emptyset$, $v_{i^*} \in S$, $\alpha_i > k_i$, and there is a directed path with all nodes in S from v_{i^*} to some $j \in R_{i^*} \cap U$. If none of the sets R_i satisfies these conditions, then S violates condition (ii) of the theorem.

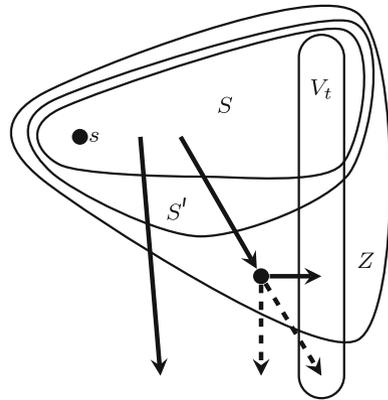
Consider some $j \in R_{i^*} \cap U$ such that there is a directed path from s to $j \in V_t$ in the subnetwork $G^i(S)$ of G^i (such a node exists by the choice of R_{i^*}). Then there exists a point $(\bar{x}, \epsilon^{i^*}) \in F$ such that $\bar{x}_j > 0$, since $\alpha_i > k_i$. Consider the set of points $X_j = \{x \mid 0 \leq x_j \leq \bar{x}_j, x_{j'} = \bar{x}_{j'} \text{ for all } j' \neq j\}$. On the one hand, $(x, \epsilon^{i^*}) \in F$ for all $x \in X_j$. On the other hand, $\sum_{j' \in R_{i^*} \cap V_t} x_{j'}$ is not a constant on X_j . Hence, there exists $x \in X_j$ such that $(x, \epsilon^{i^*}) \notin F_Z$.

Finally, assume $U \subset Z$.

Claim $S \subset Z$.

Proof Suppose $S \setminus Z$ is not empty. We define the quantities $A_\lambda, B_\lambda, C_\lambda$, and D_λ as follows. Let A_λ be the total capacity of those arcs (u, v) such that $u \in S \cap Z$ and $v \in S \setminus Z$. We have $A_\lambda := c_{x,\lambda}([S \cap Z, S \setminus Z]) = \sum_{i=1}^m k_i^A \lambda_i$ for some non-negative rationals k_i^A . Likewise, $B_\lambda := c_{x,\lambda}([Z \setminus S, S \setminus Z])$, $C_\lambda := c_{x,\lambda}([S \setminus Z, Z \setminus S])$, and $D_\lambda := c_{x,\lambda}([S \setminus Z, \overline{S \cup S \setminus \{t\}}])$, see Fig. 2. Since $S \cap Z \subseteq S$, and $S \cap Z \cap V_t = U$ by assumption, $(S \cap Z, \overline{S \cap Z}) \in \mathcal{C}(U)$, and therefore its capacity is at least that of (S, \bar{S}) . Consequently, $A_\lambda \geq C_\lambda + D_\lambda$ for all $(x, \lambda) \in P$ (where we compare the coefficients of the λ_i). On the other hand, (Z, \bar{Z}) is a dominating member of $\mathcal{C}_{\min}(Z \cap V_t)$, whence $A_\lambda + B_\lambda < D_\lambda$ for some $(x, \lambda) \in P$. Combining the two inequalities, we obtain $B_\lambda + C_\lambda + D_\lambda < D_\lambda$ for some $(x, \lambda) \in P$, which is impossible, since $B_\lambda, C_\lambda \geq 0$ always hold. \square

Fig. 3 Sets $S \subseteq S' \subset Z$. Dashed arcs do not exist



Indirectly, suppose $F \subseteq F_Z$, and let $c_{x,\lambda}([S, \bar{S}]) = \sum_{j \in U} x_j + \sum_{i=1}^m k_i \lambda_i$ for some rationals $k_i \geq 0$, and $c_{x,\lambda}([Z, \bar{Z}]) = \sum_{j \in V_t \cap Z} x_j + \sum_{i=1}^m k_i^Z \lambda_i$ for some rationals $k_i^Z \geq 0$. Then, for all $(x, \lambda) \in F$, $\sum_{i=1}^m k_i \lambda_i = \sum_{j \in V_t \setminus U} x_j$, and $\sum_{i=1}^m k_i^Z \lambda_i = \sum_{j \in V_t \setminus Z} x_j$. By condition (i), there is some i , and x such that $(x, \epsilon^i) \in F$, $\sum_{j \in V_t \cap Z \setminus U} x_j > 0$, $\sum_{j \in V_t \setminus Z} x_j > 0$, and $G^i \setminus S$ has a connected component with non-empty intersection with $V_t \cap Z \setminus U$ and also with $V_t \setminus Z$. However, $\sum_{j \in V_t \setminus Z} x_j = k_i^Z (\lambda$ is fixed to $\epsilon^i)$. Hence, $k_i > k_i^Z > 0$. Therefore, for any x such that $(x, \epsilon^i) \in F$, we cannot send less flow than k_i^Z through the edges of $[Z, \bar{Z} \setminus \{t\}]$ in a feasible $s - t$ flow of value $\sum_{j=1}^n x_j$ in N_{x,ϵ^i} . But this is only possible, if the maximum amount of flow that can be sent in G^i from S to $Z \cap V_t \setminus U$ is at most $(k_i - k_i^Z) \lambda_i$.

Claim There exists $S' \subset Z$ such that $S \subseteq S'$, $S' \setminus S \subset V(G^i) \setminus V_t$, and (a) $c_{x,\lambda}([S' \cap V(G^i), Z \cap V(G^i) \setminus S']) = (k_i - k_i^Z) \lambda_i$, and (b) there is no arc (u, v) of G^i such that $u \in Z \cap V(G^i) \setminus S'$ and $v \in \bar{Z}$.

Proof Suppose indirectly that any $S' \subset Z$ such that $S \subseteq S'$ and $S' \setminus S \subset V(G^i) \setminus V_t$, violates condition (a), see Fig. 3. If $c_{x,\lambda}([S' \cap V(G^i), Z \cap V(G^i) \setminus S']) > (k_i - k_i^Z) \lambda_i$ for all S' such that $S \subseteq S' \subset Z$, then the total flow that can be sent from S to those nodes $j \in V_t \cap V(G^i) \cap Z \setminus U$ can be more than $(k_i - k_i^Z) \lambda_i$. Therefore, there exists $(x, \epsilon^i) \in F$ such $\sum_{j \in V_t \cap Z \setminus U} x_j > (k_i - k_i^Z) \lambda_i$, while $\sum_{j \in V_t \setminus Z} x_j < k_i^Z \lambda_i$, which implies $(x, \epsilon^i) \notin F_Z$, a contradiction.

Let S' be a subset of nodes that satisfies condition (a). Indirectly, suppose it violates (b). Then the cut-set $[V(G^i) \cap Z, V(G^i) \cap \bar{Z}]$ contains those edges (u, v) of G^i such that $u \in Z \cap V(G^i) \setminus S'$ and $v \in \bar{Z}$. Then, in order to saturate all arcs of $[V(G^i) \cap Z, V(G^i) \cap \bar{Z}]$, the flow through the arcs of $[S' \cap V(G^i), Z \cap V(G^i) \setminus S']$ must be split, and less than $(k_i - k_i^Z) \lambda_i$ of it reaches the nodes $j \in Z \cap V_t \setminus U$ for any $(x, \epsilon^i) \in F$. But then $\sum_{j \in V_t \cap Z \setminus U} x_j < k_i - k_i^Z \lambda_i$, again a contradiction. \square

Using this claim, we can replace S by $S_2 := S' \cup (Z \cap V(G^i) \setminus S')$. Then $F_{S_2} = F$. Therefore, if $S_2 \neq S$, then S is not dominating, a contradiction. Otherwise, $G^i \setminus S$ has no connected component which contains some nodes $j \in V(G^i) \cap V_t \cap Z \setminus U$ and $j' \in V(G^i) \cap V_t \setminus Z$ simultaneously, again a contradiction. \square

Now we turn to P_{\leq}^{emb} . By definition, $\sum_{i=1}^m \lambda_i = 1$, and $\sum_{i=1}^m \alpha_i \lambda_i = \sum_{j=1}^n x_j$ are valid for P_{\leq}^{emb} . However, adding the latter equation to Q_{\leq} may render some of the inequalities induced by $s - t$ cuts to implicit equations for P_{\leq}^{emb} (for the same problem data).

Example 1 Suppose $P_1 = \{x \in \mathbb{R}_{\geq 0}^3 : x_1 \leq \frac{1}{3}, x_2 \leq \frac{2}{3}, x_3 = 0\}$, and $P_2 = \{x \in \mathbb{R}_{\geq 0}^3 : x_1 = 0, x_2 \leq 1, x_3 \leq 1\}$. Then

$$P_{\leq}^{emb} = Q_{\leq} = \left\{ (x, \lambda) \in \mathbb{R}_{\geq 0}^3 \times \mathbb{R}_{\geq 0}^2 : \begin{array}{l} x_1 \leq \frac{1}{3}\lambda_1, \quad x_2 \leq \frac{2}{3}\lambda_1 + \lambda_2, \quad x_3 \leq \lambda_2 \\ \lambda_1 + \lambda_2 = 1 \end{array} \right\}.$$

Now we add $x_1 + x_2 + x_3 = 1$ to Q_{\leq} to obtain $Q_{=}$. But then $x_1 = \frac{1}{3}\lambda_1$ is a valid equation for $Q_{=}$ and it is linearly independent of $x_1 + x_2 + x_3 = 1$ and $\lambda_1 + \lambda_2 = 1$. Thus, a minimal linear representation of $Q_{=}$ is

$$P_{=}^{emb} = Q_{=} = \left\{ (x, \lambda) \in \mathbb{R}_{\geq 0}^3 \times \mathbb{R}_{\geq 0}^2 : \begin{array}{l} x_1 = \frac{1}{3}\lambda_1, \quad x_2 \leq \frac{2}{3}\lambda_1 + \lambda_2, \quad x_3 \leq \lambda_2 \\ \lambda_1 + \lambda_2 = 1, \quad x_1 + x_2 + x_3 = 1 \end{array} \right\}.$$

After these preliminaries, we determine the maximum number of linearly independent valid equations for $P_{=}^{emb}$.

Proposition 7 Let E_1, \dots, E_{κ} be a maximum number of disjoint subsets of V_t such that $\bigcup_{\ell=1}^{\kappa} E_{\ell} = V_t$, and for arbitrary $(S_{\ell}, \bar{S}_{\ell}) \in C_{\min}(E_{\ell})$, $\ell = 1, \dots, \kappa$,

$$\sum_{i=1}^m \beta_{i\ell} \lambda_i = \sum_{j \in E_{\ell}} x_j, \tag{12}$$

are all valid equations for $P_{=}^{emb}$, where $c_{x,\lambda}([S_{\ell}, \bar{S}_{\ell}]) = \sum_{i=1}^m k_{i\ell} \lambda_i + \sum_{j \in E_{\ell}} x_j$, and $\beta_{i\ell} = \alpha_i - k_{i\ell}$. Then $\sum_{i=1}^m \lambda_i = 1$, and (12) are linearly independent, and imply all valid equations for $P_{=}^{emb}$. Moreover, $\sum_{\ell=1}^{\kappa} \beta_{i\ell} = \alpha_i$ for all $i \in [m]$, and the partitioning is unique.

Proof First notice that (12) is obtained from $c_{x,\lambda}([S_{\ell}, \bar{S}_{\ell}]) = \sum_{j=1}^n x_j$ by subtracting it from the equation $\sum_{i=1}^m \alpha_i \lambda_i = \sum_{j=1}^n x_j$ valid for $P_{=}^{emb}$. Since $\sum_{i=1}^m \alpha_i \lambda_i = \sum_{j=1}^n x_j$ is a defining equation for $P_{=}^{emb}$, the claimed partitioning exists and $\kappa \geq 1$. The linear independence of the given equations is straightforward. Now we turn to the second part of the statement. Since a complete linear description of $P_{=}^{emb}$ is known, any valid equation must correspond to some $s - t$ cut (S, \bar{S}) of N . Using the submodularity of the cut capacity function $c_{x,\lambda}$, we derive

$$\begin{aligned} 2 \sum_{j=1}^n x_j &= c_{x,\lambda}([S_{\ell}, \bar{S}_{\ell}]) + c_{x,\lambda}([S, \bar{S}]) \\ &\geq c_{x,\lambda}([S_{\ell} \cap S, \overline{S_{\ell} \cap S}]) + c_{x,\lambda}([S_{\ell} \cup S, \overline{S_{\ell} \cup S}]) \geq 2 \sum_{j=1}^n x_j. \end{aligned}$$

Hence, equality holds throughout, and in particular $(S_\ell \cap S, \overline{S_\ell \cap S})$ induces a valid equation for P_{\leq}^{emb} . Now, if $V_t \cap S_\ell \cap S \neq \emptyset$, and $V_t \cap S_\ell \setminus (V_t \cap S) \neq \emptyset$, then E_ℓ can be further split, whence κ is not maximal, a contradiction. On the other hand, if $S \cap V_t$ is the union of some of the E_ℓ , then the equation induced by (S, \bar{S}) is implied by (12). The same argument shows that the above partitioning is unique. Finally, by substituting ϵ_i to λ_i , we immediately derive $\sum_{\ell=1}^\kappa \beta_{i\ell} = \alpha_i$. \square

Corollary 2 *Let κ be defined as in Proposition 7. Then $\dim(P_{\leq}^{emb}) = n + m - 1 - \kappa$.*

To facilitate the presentation, we assume that the rooted trees G^i that constitute the network N have the following additional structure. For each $i \in \llbracket m \rrbracket$, let $A_i \subseteq \llbracket \kappa \rrbracket$ be the subset of those indices such that $\beta_{i\ell} > 0$. If $|A_i| \geq 2$, then G^i has exactly $|A_i|$ outgoing arcs at node v_i , i.e., $\delta_{v_i}^{out} = \{(v_i, w_{i\ell}) : \ell \in A_i\}$, and the capacity of $(v_i, w_{i\ell})$ is $\beta_{i\ell}\lambda_i$. Moreover, the subtree of G^i rooted at $w_{i\ell}$ has all leaves in E_ℓ . If $A_i = \{\ell\}$ for some $\ell \in \llbracket \kappa \rrbracket$, then all leafs of G^i are in E_ℓ , and no further assumptions are needed. Let N^ℓ be the subnetwork of N spanned by the nodes $\{s, t\} \cup E_\ell$ along with those nodes of G^i , $i \in \llbracket \kappa \rrbracket$, which are on a directed path from v_i to E_ℓ .

Due to the valid equations, the facets of P_{\leq}^{emb} may have several equivalent forms. We argue that there is a normal form, which is easy to construct.

Proposition 8 *Any facet F_S of P_{\leq}^{emb} is induced by the dominating $s - t$ cut $(S, \bar{S}) \in \mathcal{C}_{\min}(U)$ for some $U \subset V_t$ such that $U \subset E_\ell$ for some $\ell \in \llbracket \kappa \rrbracket$.*

Proof The facet inducing inequality induced by (S, \bar{S}) is $\sum_{i=1}^m k_i \lambda_i \geq \sum_{j \in V_t \setminus U} x_j$. If U is the union of some of the E_ℓ , then F_S is not a proper face of P_{\leq}^{emb} , a contradiction. Hence, there exists some E_ℓ such that $E_\ell \setminus U \neq \emptyset$ and $U \cap E_\ell \neq \emptyset$. If $U \cap E_\mu = \emptyset$ for all $\mu \neq \ell$, then $U \subset E_\ell$ and we are done. Otherwise, there are two cases. First suppose there exists $E_\mu \subset U$. If we add the valid equation (12) for E_ℓ to the inequality induced by (S, \bar{S}) we obtain

$$\sum_{i=1}^m (k_i + \beta_{i\ell}) \lambda_i \geq \sum_{j \in (V_t \setminus U) \cup E_\ell} x_j.$$

This inequality corresponds to the dominating $s - t$ cut $(S_1, \bar{S}_1) \in \mathcal{C}_{\min}(U \setminus E_\ell)$, and $F_S = F_{S_1}$. Finally, suppose $U \cap E_\mu \neq \emptyset$ for some $E_\mu \neq E_\ell$. Since $E_\ell \cap E_\mu = \emptyset$ by definition, it means that a proper subset of E_μ is contained in U . We will show that then F_S is not a facet of P_{\leq}^{emb} , a contradiction. For any $(x, \lambda) \in F_S$, consider a feasible flow of value $\sum_{j=1}^n x_j$. This flow saturates the arcs of G^i in the cut-set $[S, \bar{S}]$ for any $i \in \llbracket m \rrbracket$, and it is split between the nodes $E_\mu \setminus U$ and $V_t \setminus (U \cup E_\mu)$. Hence, there exists $\beta_{i\mu}^1 \in [0, \beta_{i\mu}]$, independent of (x, λ) , such that the amount of flow toward $E_\mu \setminus U$ is $\beta_{i\mu}^1 \lambda_i$, since the total flow toward the nodes of E_μ is $\beta_{i\mu}$ for any $(x, \lambda) \in P_{\leq}^{emb}$. Therefore, the dominating $s - t$ cut $(Z, \bar{Z}) \in \mathcal{C}_{\min}(U \cup E_\mu)$ is such that $F_S \subseteq F_Z$. Moreover, $F_S \neq F_Z$, otherwise E_μ could be further split, and $F_Z \neq P_{\leq}^{emb}$, since $Z \cap E_\ell = S \cap E_\ell \neq \emptyset$, and $V_t \cap Z \setminus E_\ell = V_t \cap S \setminus E_\ell \neq \emptyset$. Then, F_S is not a facet of P_{\leq}^{emb} . \square

Theorem 2 For any $\ell \in \llbracket \kappa \rrbracket$, the set $U \subset E_\ell$ induces a facet of P_{\leq}^{emb} if and only if

- (i) for the dominating $s - t$ cut $(S, \bar{S}) \in \mathcal{C}_{\min}(U)$, the subgraph $N^\ell \setminus (S \cup \{t\})$ is connected, and
- (ii) if $|U| \geq 2$, then for any partitioning of U into non-empty subsets U_1 and U_2 , there exist $i \in \llbracket m \rrbracket$, and a connected component C of $(V(G^i) \cap S, E(V(G^i) \cap S))$ which contains v_i , and at least one point from each of U_1 and U_2 , such that for $\lambda = \epsilon^i$, the minimum capacity of a cut in C separating v_i from $V(G^i) \cap U$ is larger than $\alpha_i - k_i$, where k_i is the total capacity of those arcs of G^i which are in $[S, \bar{S}]$, and $\alpha_i - k_i > 0$.

Proof Necessity : Suppose that the dominating $s - t$ cut $(S, \bar{S}) \in \mathcal{C}_{\min}(U)$ induces a facet of P_{\leq}^{emb} . Observe that condition (i) coincides with Theorem 1(i). Since $P_{\leq}^{emb} \subset P_{\leq}^{emb}$ (for the same data), it is enough to verify that U satisfies condition (ii). Indirectly, suppose this is not the case, and there exists a partitioning $U_1 \cup U_2$ of U into nonempty subsets along with non-negative numbers β_1^i, β_2^i for all $i \in \{1, \dots, m\}$ such that $\sum_{i=1}^m \beta_k^i \lambda_i = \sum_{j \in U_k} x_j$ is satisfied by all $(x, \epsilon^i) \in F_S$ for $k = 1, 2$. Therefore, F_S cannot be a facet of P_{\leq}^{emb} .

Sufficiency: Suppose conditions (i) and (ii) are satisfied by U , and we have to prove that $F := F_S$ is a facet of P_{\leq}^{emb} for the dominating $s - t$ cut $(S, \bar{S}) \in \mathcal{C}_{\min}(U)$. As in the proof of Theorem 1, we may assume that F is not contained in the faces of P_{\leq}^{emb} induced by $x_j = 0$ for any j , or by $\lambda_i = 0$ for any i . So, suppose (Z, \bar{Z}) induces a facet F_Z of P_{\leq}^{emb} and $F \subset F_Z$. Since F_Z is a facet, by Proposition 6, (Z, \bar{Z}) is a dominating $s - t$ cut in $\mathcal{C}_{\min}(Z \cap V_t)$. We distinguish two cases: $U \setminus Z \neq \emptyset$, and $U \subset Z$.

First suppose $U \setminus Z \neq \emptyset$, and $U \cap Z \neq \emptyset$. We construct a point (x, λ) in $F \setminus F_Z$ to show that $F \not\subseteq F_Z$. By condition (ii), there exist i and a connected component C of $(V(G^i) \cap S, A(V(G^i) \cap S))$ such that $v_i \in C$, C has non-empty intersection with $U \setminus Z$ and also with $U \cap Z$, and for $\lambda = \epsilon^i$, the minimum capacity of a cut separating v_i from $V(G^i) \cap U$ is larger than $\alpha_i - k_i$, where $\alpha_i - k_i$ is the remaining quantity to be covered by a flow from v_i . Since this minimum capacity is larger than the amount of flow to be sent from v_i to the nodes $V(G^i) \cap U$, the flows on the arcs of component C are not fixed. Therefore, there exist points $(x^1, \epsilon^i), (x^2, \epsilon^i) \in F$ such that $\sum_{j \in U \setminus Z} x_j^1 \neq \sum_{j \in U \setminus Z} x_j^2$, while $\sum_{j \in V_t \setminus (U \cup Z)} x_j^1 = \sum_{j \in V_t \setminus (U \cup Z)} x_j^2$. But then $\sum_{j \in V_t \setminus Z} x_j^1 \neq \sum_{j \in V_t \setminus Z} x_j^2$, whence it is impossible that both of (x^1, ϵ^i) and (x^2, ϵ^i) are in F_Z .

Now suppose $U \cap Z = \emptyset$. Let k_i^Z be the coefficient of λ_i in the linear expression for the capacity of (Z, \bar{Z}) as in Observation 1. Consider the partitioning $V_t = U \cup (V_t \cap Z) \cup (V_t \setminus (U \cup Z))$. We argue that for any i , and any $(x, \epsilon^i) \in F$, the sum of the x_j in the three subsets equals $\alpha_i - k_i, \alpha_i - k_i^Z$, and $k_i + k_i^Z - \alpha_i$, respectively. For any $(x, \epsilon^i) \in F$, $\sum_{j \in V_t \setminus U} x_j = k_i$ by definition, and thus $\sum_{j \in U} x_j = \alpha_i - k_i$. Since $F \subset F_Z$, x also satisfies $\sum_{j \in V_t \cap Z} x_j = \alpha_i - k_i^Z$. Since $\sum_{j=1}^n x_j = \alpha_i$, we get that $\sum_{j \in (V_t \setminus (U \cup Z))} x_j = \alpha_i - (\alpha_i - k_i) - (\alpha_i - k_i^Z) = k_i + k_i^Z - \alpha_i$. Therefore, if $U \cup (V_t \cap Z) = V_t$, then $k_i + k_i^Z = \alpha_i$, and thus $F_Z \subseteq F$, a contradiction. Now suppose $U \cup (V_t \cap Z) \subset V_t$. Then, by condition (i), there exists $v_i \in V_s \setminus S$, such that v_i is connected to some $j_1 \in V_t \cap Z$, and also to some node $j_2 \in V_t \setminus (U \cup Z)$. Moreover,

the capacity of the cut (in G^i) separating v_i from these two sets must be greater than k_i , since (S, \bar{S}) is a dominating $s - t$ cut. Clearly, there must exist $(x, \epsilon^i) \in F$ such that $x_{j_2} > 0$, say. Let $\delta > 0$ be small enough such that increasing x_{j_1} by δ , while decreasing x_{j_2} by δ yields a vector x' such that $(x', \epsilon^i) \in F$. But x' does not satisfy the equation $\sum_{j \in V_i \cap Z} x'_j = \alpha_i - k_i^Z$, hence $(x', \epsilon^i) \notin F_Z$, a contradiction.

Finally, if $U \subset Z$, then we can proceed as in the proof of Theorem 1. □

5 Applications

In this section we present a number of applications where we apply Theorem 1 or Theorem 2 in order to derive the non-trivial facets (different to the non-negativity of the variables) of the corresponding formulations. For each application we will present a MIP formulation, then a network representation, and finally derive the dominating and facet defining inequalities, but we abandon straightforward technical details such as the network correctly represents the feasible solutions of the MIP formulation in a higher dimensional space. If not stated otherwise, the formulations contain no implicit equations.

5.1 Special ordered sets of type 2

A special ordered set of type 2 (SOS2) consists of vectors $x \in \mathbb{R}_{\geq 0}^n$ such that $\sum_{j=1}^n x_j = 1$, and x has either one nonzero coordinate, or two consecutive nonzero coordinates, see [13].

Let $2 \leq n \in \mathbb{Z}$ and consider the polytope $(P_n^{SOS2})^{emb} \subseteq \mathbb{R}_{\geq 0}^{2n-1}$ defined as the convex hull of points

$$\left\{ (x, \lambda) \in \mathbb{R}^n \times \{0, 1\}^{n-1} : \sum_{j=1}^n x_j = 1, \sum_{i=1}^{n-1} \lambda_i = 1, \right. \\ \left. x_1 \leq \lambda_1, x_j \leq \lambda_{j-1} + \lambda_j \text{ for all } j = 2, \dots, n-1, x_n \leq \lambda_{n-1} \right\}. \quad (13)$$

Observe that in this formulation, if $\lambda_j = 1$, then x_j and x_{j+1} may be positive, but all other coordinates of x must be 0. Hence, if λ is integral, then $\text{proj}_x(P_n^{SOS2})^{emb}$ is indeed a SOS2. However, relaxing the integrality of the λ_i variables in (13), we obtain a polytope \tilde{P}_n^{SOS2} which has a number of vertices with fractional λ_i variables. For instance, for $n = 5$, $x = (1/2, 1/2, 0, 0, 0)$, and $\lambda = (1/2, 0, 1/2, 0)$, (x, λ) is a vertex of \tilde{P}_5^{SOS2} .

We create a network $N_n^{SOS2} = (V, A, c)$ with $V = \{s\} \cup V_s \cup V_t \cup \{t\}$, where $V_s = \{v_1, \dots, v_{n-1}\}$ and $V_t = \{1, \dots, n\}$. For all $i = 1, \dots, n-1$ we add arcs (s, v_i) , (v_i, i) and $(v_i, i + 1)$ to the network of capacity λ_i each, and the arcs (j, t) of capacity x_j for $j \in V_t$, see Fig. 4.

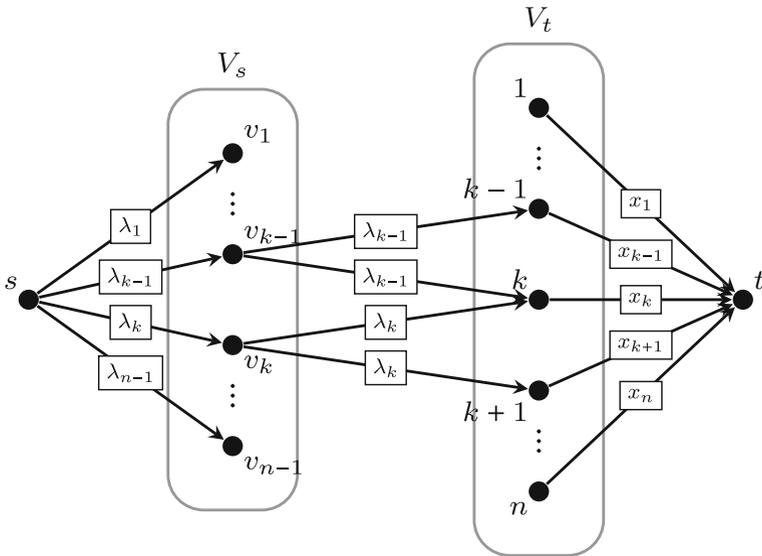


Fig. 4 Construction of network N_n^{SOS2} for special ordered sets of type 2

By construction, we have

$$(P_n^{SOS2})^{emb} = \left\{ (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^{n-1} : \sum_{i=1}^{n-1} \lambda_i = 1, \sum_{j=1}^n x_j = 1, x \geq 0, \lambda \geq 0, \right. \\ \left. c_{x,\lambda}([S, \bar{S}]) \geq \sum_{j=1}^n x_j \text{ for all } s-t \text{ cuts } (S, \bar{S}) \right\}.$$

Proposition 9 *The non-trivial facets of $(P_n^{SOS2})^{emb}$ are*

$$\sum_{j=\ell+1}^n x_j \leq \sum_{i=\ell}^{n-1} \lambda_i, \quad \ell = 1, \dots, n-1 \tag{14}$$

$$\sum_{i=\ell}^{n-1} \lambda_i \leq \sum_{j=\ell}^n x_j, \quad \ell = 2, \dots, n. \tag{15}$$

Proof Consider a nonempty set $U \subset V_t$ and the corresponding unique dominating $s-t$ cut (S, \bar{S}) w.r.t. U . By definition, $v_k \in S$ if and only if $S \cap \{k, k+1\} \neq \emptyset$. Clearly, if $i, k \notin U$ and $j \in U$ for some $1 \leq i < j < k \leq n$, then $N_n^{SOS2} \setminus (S \cup \{t\})$ is disconnected, and thus the set U does not induce a facet of $(P_n^{SOS2})^{emb}$. We distinguish three cases.

First, suppose $U = \{1, \dots, \ell\}$ for some $1 \leq \ell \leq n-1$, then $S = \{s\} \cup \{v_1, \dots, v_\ell\} \cup U$ which clearly satisfies the conditions of Theorem 2, and thus yields the inequality (14).

Second, suppose $U = \{\ell, \dots, n\}$ for some $2 \leq \ell \leq n$, then $S = \{s\} \cup \{v_{\ell-1}, \dots, v_{n-1}\} \cup U$ which clearly satisfies the conditions of Theorem 2, and thus yields the inequality

$$\sum_{j=1}^{\ell-1} x_j \leq \sum_{k=1}^{\ell-1} \lambda_k$$

which can be reformulated as (15).

Third, suppose $V_t \setminus U = \{k, \dots, \ell\}$ for some $2 \leq k \leq \ell \leq n-1$. Then $S = \{s\} \cup \{v_i \in V_s \mid 1 \leq i \leq k-1 \text{ or } \ell \leq i \leq n-1\} \cup U$. However, (S, \bar{S}) violates condition (ii) of Theorem 2, since $(S \setminus \{s\}, A(S \setminus \{s\}))$ is not connected. \square

Observe that (14) and (15) along with the non-negativity of the variables and the two valid equations, is precisely the reformulation of Padberg [39] for SOS2 (obtained by completely different means), see also [26,28,45].

5.2 Special ordered sets of type k

A special ordered set of type k (SOSk) consists of vectors $x \in \mathbb{R}_{\geq 0}^n$ such that $\sum_{j=1}^n x_j = 1$, and x has at most k consecutive non-zero components, see [26,38]. Clearly, it generalizes SOS2 discussed in the previous section. Let $(P_n^{\text{SOS}k})^{emb}$ be the convex hull of those (x, λ) that satisfy the constraints

$$\begin{aligned} x_j - \sum_{i=\max\{j-k+1, 1\}}^j \lambda_i &\leq 0, \quad j = 1, \dots, n \\ \sum_{j=1}^n x_j &= 1, \\ \sum_{i=1}^{n-k+1} \lambda_i &= 1, \\ x &\geq 0, \lambda \in \{0, 1\}^n. \end{aligned}$$

The corresponding network $N = (V, A, c)$ has a set of nodes V comprising a source s , a sink t , the nodes $V_s = \{v_1, \dots, v_{n-k+1}\}$, and also $V_t := \{1, \dots, n\}$. s is connected to each of the v_i by a directed arc of capacity λ_i , and each $j \in V_t$ to t by a directed arc of capacity x_j . Finally, v_i is connected to the nodes $i, \dots, i+k-1$ in V_t by directed arcs of capacity λ_i each. By applying Theorem 2 to the network N , one can prove analogously to the SOS2 special case the following:

Proposition 10 *The non-trivial facets of $(P_n^{SOSk})^{emb}$ are*

$$\sum_{j=h}^n x_j \leq \sum_{i=h-k+1}^{n-k+1} \lambda_i, \quad h = k, \dots, n, \text{ and}$$

$$\sum_{j=h}^n x_j \geq \sum_{i=h}^{n-k+1} \lambda_i, \quad h = 2, \dots, n - k + 1.$$

5.3 Logical constraints

Consider a set of binary variables x_j indexed by $j \in J = \{1, \dots, n\}$. For each $i = 1, \dots, m$, let $\mathcal{E}_i \subseteq \{H \subseteq J : H \neq \emptyset\}$ be a set of pairwise disjoint subsets of J , that is, $E_{ik} \cap E_{ik'} = \emptyset$ for distinct sets $E_{ik}, E_{ik'} \in \mathcal{E}_i$. Note however, that for distinct i and i' , there may exist $E \in \mathcal{E}_i$ and $E' \in \mathcal{E}_{i'}$ such that $E \cap E' \neq \emptyset$. We also assume that $V = \bigcup_{i=1}^m \bigcup_{k=1}^{|\mathcal{E}_i|} E_{ik}$. Let P_{\leq}^{LOG} be the convex hull of those binary vectors $x \in \{0, 1\}^J$ that satisfy the following disjunctive constraints:

$$\bigvee_{i=1}^m \left(\bigwedge_{k=1}^{|\mathcal{E}_i|} \left(\sum_{j \in E_{ik}} x_j \leq p_{ik} \right) \right) \tag{16}$$

In order to obtain a MIP formulation for P_{\leq}^{LOG} , we introduce a binary variable λ_i for $i = 1, \dots, m$. Let $(P_{\leq}^{LOG})^{emb}$ be the convex hull of points $(x, \lambda) \in \{0, 1\}^n \times \{0, 1\}^m$ that satisfy the constraints

$$\sum_{i=1}^m \lambda_i = 1, \tag{17}$$

$$(|E_{ik}| - p_{ik}) \lambda_i + \sum_{j \in E_{ik}} x_j \leq |E_{ik}|, \quad i = 1, \dots, m, k = 1, \dots, |\mathcal{E}_i|. \tag{18}$$

Define the network $N = (V, A, c)$, where $V = \{s, t\} \cup V_s \cup W \cup V_t$, where $V_s := \{v_1, \dots, v_m\}$, $V_t = J$, and $W = \{w_{ik} \mid i = 1, \dots, m, k = 1, \dots, |\mathcal{E}_i|\}$. The set of arcs A comprises the (s, v_i) of capacity $(n - \sum_{k=1}^{|\mathcal{E}_i|} (|E_{ik}| - p_{ik})) \lambda_i$ for each $i = 1, \dots, m$; the (v_i, w_{ik}) of capacity $p_{ik} \lambda_i$ for $i = 1, \dots, m, k = 1, \dots, |\mathcal{E}_i|$; the arcs (v_i, j) of capacity λ_i for $i = 1, \dots, m$, and $j \in V_t \setminus \cup (E_{ik} \mid k = 1, \dots, |\mathcal{E}_i|)$; and the (w_{ik}, j) of capacity λ_i for $i = 1, \dots, m, k = 1, \dots, |\mathcal{E}_i|, j \in E_{ik}$.

Firstly, we identify the dominating $s - t$ cut (S, \bar{S}) w.r.t. some $U \subset V_t$. Clearly, $S = \{s\} \cup V_s \cup U \cup \{w_{ik} \in W : |E_{ik} \setminus U| \leq p_{ik}\}$, because the capacity of the arc (v_i, w_{ik}) is $p_{ik} \lambda_i$, while the total capacity of those arcs from w_{ik} to $V_t \setminus U$ is $|E_{ik} \setminus U| \lambda_i$.

We can apply Theorem 1 to determine those dominating $s - t$ cuts of N with respect to some $U \subset V_t$ that induce facets of $(P_{\leq}^{LOG})^{emb}$, however, the condition we get is not much more specific than that of the theorem. All we can say is that the dominating $s - t$ cut w.r.t. any $U \subset V_t$ satisfies condition (ii) of Theorem 1.

As a special case we reconsider the logical constraints of [7], which we summarized in Sect. 2. By splitting J into two subsets J_1 and J_2 , and after complementing all the variables, i.e., $\bar{x}_j = 1 - x_j$ for all j , we can express P_1 and P_2 as

$$P^1 = \left\{ \left(x^1, \dots, x^r; y^1, \dots, y^s \right) \in [0, 1]^{|M_1|+\dots+|M_r|+|N_1|+\dots+|N_r|} : \right. \\ \left. \bar{x}^i(M_i) \leq |M_i| - p_i, i = 1, \dots, r \right\},$$

and

$$P^2 = \left\{ \left(x^1, \dots, x^r; y^1, \dots, y^s \right) \in [0, 1]^{|M_1|+\dots+|M_r|+|N_1|+\dots+|N_r|} : \right. \\ \left. \bar{y}^j(N_j) \leq |N_j| - \ell_j, j = 1, \dots, s \right\}.$$

Since P_1 and P_2 are in disjoint subspaces, the characterization of Theorem 1 becomes more apparent. Let $E_{1i} = M_i$, and $E_{2j} = N_j$.

Proposition 11 *The subset $U \subset V_t$ induces a facet of $\text{conv}(P_1 \cup P_2)$ if $|V_t \setminus U| = 1$, or there exists w_{1i} or w_{2j} such that for the dominating $s - t$ cut $(S, \bar{S}) \in \mathcal{C}_{\min}(U)$, $w_{1i} \in \bar{S}$ and $M_i \setminus U = V_t \setminus U$, or $w_{2j} \in \bar{S}$ and $N_j \setminus U = V_t \setminus U$.*

Proof By Theorem 1, $N \setminus (S \cup \{t\})$ must be connected. However, the M_i and the N_j are disjoint by assumption, so, condition (i) holds only if the conditions of this statement are satisfied. On the other hand, $V_s \subset S$, and thus condition (ii) of the same theorem is met as well. □

5.4 Cardinality indicating polytope

The cardinality indicating polytope $P_n^{CARD} \subseteq \mathbb{R}_{\geq 0}^{2n+1}$ is defined as the convex hull of the points

$$\left\{ (x, \lambda) \in \{0, 1\}^n \times \{0, 1\}^{n+1} : \lambda_k = 1 \text{ and } \lambda_j = 0 \text{ for } j \neq k, \text{ where } k = \sum_{i=1}^n x_i \right\}.$$

Clearly, $\sum_{k=0}^n \lambda_k = 1$ holds for each $(x, \lambda) \in P_n^{CARD}$.

First, we create a network $N_n^{CARD} = (V, A, c)$ with $V = \{s\} \cup \{t\} \cup V_s \cup V_t$, where $V_s = \{v_0, v_1, \dots, v_n\}$, and $V_t = \{1, \dots, n\}$, and A comprising the following arcs. For $k = 0, \dots, n$, the arcs $(s, v_k) \in A$ of capacity $k\lambda_k$. For $k = 0, \dots, n$ and $j = 1, \dots, n$, the arcs $(v_k, j) \in A$ of capacity λ_k . For $j = 1, \dots, n$, the arcs $(j, t) \in A$ of capacity x_j (Fig. 5).

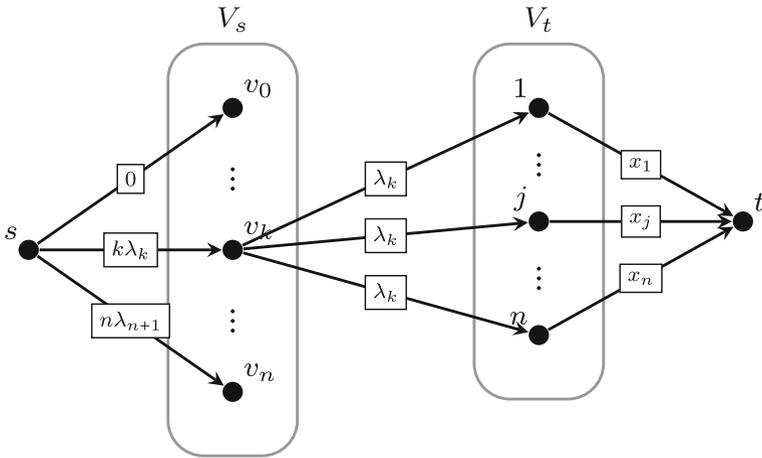


Fig. 5 Network for the cardinality indicating polytope

By construction, we have

$$P_n^{CARD} = \left\{ (x, \lambda) \in \mathbb{R}^{n+1} \times \mathbb{R}^n : \sum_{k=0}^n \lambda_k = 1, \sum_{k=0}^n k\lambda_k = \sum_{j=1}^n x_j, x \geq 0, \lambda \geq 0, \right. \\ \left. c_{x,\lambda}([S, \bar{S}]) \geq \sum_{j=1}^n x_j \text{ for all } s-t \text{ cut } (S, \bar{S}) \right\}.$$

Proposition 12 *The non-trivial facets of P_n^{CARD} are*

$$\sum_{j \in I} x_j \leq \sum_{k=1}^{|I|} k\lambda_k + |I| \sum_{k=|I|+1}^n \lambda_k, \quad \emptyset \neq I \subset \{1, \dots, n\}. \tag{19}$$

Proof By Proposition 6 it suffices to consider dominating $s-t$ cuts to identify the facets of P_n^{CARD} . For any nonempty subset $I \subseteq \{1, \dots, n\}$, let $U := V_t \setminus I$, and (S, \bar{S}) the unique dominating $s-t$ cut w.r.t. U . Since (S, \bar{S}) is of minimum capacity w.r.t. U , $v_k \in S$ for all $k > |I|$, and $v_k \in \bar{S}$ for all $k < |I|$. Since, (S, \bar{S}) is dominating w.r.t. U , $v_{|I|} \in S$. Clearly, (S, \bar{S}) satisfies the conditions of Theorem 2, thus the corresponding face F_S is a facet of P_n^{CARD} , and takes the form (19). \square

Notice that Theorem 10 of [35] gives precisely the same description of P_n^{CARD} .

5.5 All even subsets

The parity polytope P_n^{EVEN} is defined as the convex hull of points

$$\{x \in \{0, 1\}^n : x \text{ has even an number of 1's}\}.$$

The minimal description of P_n^{EVEN} in terms of linear inequalities in the space of original variables is attributed to Jeroslow [30]. We apply our technique to get one in a higher dimensional space. Let $(P_n^{EVEN})^{emb}$ be the convex hull of the set of points

$$\left\{ (x, \lambda) \in \{0, 1\}^n \times \{0, 1\}^{\lfloor n/2 \rfloor + 1} : \exists k \text{ such that } \lambda = \epsilon^k, \text{ and } 2k = \sum_{i=1}^n x_i \right\}.$$

Similarly to Sect. 5.4, we create a network $N_n^{EVEN} = (V, A, c)$ with $V = \{s\} \cup V_s \cup V_t \cup \{t\}$, where $V_s := \{v_0, \dots, v_{\lfloor n/2 \rfloor}\}$ and $V_t := \{1, \dots, n\}$, and the set of arcs is defined analogously to that of N_n^{card} . The arc capacities are $c(s, v_k) = 2k\lambda_k$, for $k = 0, \dots, \lfloor n/2 \rfloor$, $c(v_k, j) = \lambda_k$ for all $j \in V_t$, and $c(j, t) = x_j$ for $j \in V_t$. By construction we have

$$(P_n^{EVEN})^{emb} = \left\{ (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^{n+1} : \sum_{k=0}^n \lambda_k = 1, \sum_{k=0}^{\lfloor n/2 \rfloor} 2k\lambda_k = \sum_{j=1}^n x_j, x \geq 0, \lambda \geq 0, \right. \\ \left. c_{x,\lambda}([S, \bar{S}]) \geq \sum_{j=1}^n x_j \text{ for all } s - t \text{ cut } (S, \bar{S}) \right\}.$$

Proposition 13 *The non-trivial facets of $(P_n^{EVEN})^{emb}$ are*

$$\sum_{j \in I} x_j \leq \sum_{k=1}^{\lfloor |I|/2 \rfloor} 2k\lambda_k + |I| \sum_{k=\lfloor |I|/2 \rfloor + 1}^n \lambda_k, \quad I \in \mathcal{I}_n, \tag{20}$$

where $\mathcal{I}_n := \{I \subseteq \{1, \dots, n\} : |I| \neq 2, \exists k \in \mathbb{Z} \text{ such that } n > 2k > |I|\}$.

Proof By Proposition 6 only dominating $s - t$ cuts of N_n^{EVEN} can define facets. For any nonempty subset $I \subseteq \{1, \dots, n\}$, let $U := V_t \setminus I$, and (S, \bar{S}) the unique dominating $s - t$ cut w.r.t. U . Since (S, \bar{S}) is of minimum capacity w.r.t. U , $v_k \in S$ for all $2k \geq |I|$, and $v_k \in \bar{S}$ for all $2k < |I|$. In contrast to the cardinality indicating polytope, not all I determine facets of $(P_n^{EVEN})^{emb}$, namely, $|I|$ induces a facet if and only if $|I| \neq 2$, and there exists an integer k such that $n > 2k$ and $2k > |I|$. If this condition fails, then the $s - t$ cut (S, \bar{S}) corresponding to I does not satisfy condition (i) or (ii) of Theorem 2. The statement follows. \square

5.6 Polytopes of small cliques

Given a simple graph $G = (\mathcal{V}, \mathcal{E})$, and a constant c , let

$$P_{\leq c}^{CLQ} := \text{conv} \left\{ x(C) \in [0, 1]^{\mathcal{V}} \mid C \subseteq \mathcal{V} \text{ clique of size } \leq c \right\}$$

be the polytope whose vertices correspond to cliques of G of size at most c , and

$$P_{=c}^{CLQ} := \text{conv} \left\{ x(C) \in [0, 1]^{\mathcal{V}} \mid C \subseteq \mathcal{V} \text{ clique of size } c \right\}$$

Let C_1, \dots, C_m be the node sets of the cliques of size c of G , and $n = |\mathcal{V}|$. Clearly, $m \leq \binom{n}{c}$. The corresponding MIP formulations can be derived using a network $N = (V, A, c)$, where $V = \{s, t\} \cup V_s \cup V_t$ such that V_s comprises a distinct node v_i for each C_i , and $V_t = \mathcal{V}$ (the set of nodes of G). The source s is connected to each v_i , each j is connected to the sink t , and each node v_i is connected to every $j \in C_i$. $c(s, v_i) = c\lambda_i$, $c(v_i, j) = \lambda_i$ for $j \in C_i$, and $c(j, t) = x_j$ for all $j \in V_t$. First we consider $(P_{=c}^{CLQ})^{emb}$. It is easy to see that

$$\begin{aligned} (P_{=c}^{CLQ})^{emb} = & \left\{ (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \mid \sum_{j \in \mathcal{V}} x_j = c, \sum_{i=1}^m \lambda_i = 1, x \geq 0, \lambda \geq 0, \right. \\ & \left. c_{x,\lambda}([S, \bar{S}]) \geq \sum_{j=1}^n x_j, \forall s-t \text{ cut } (S, \bar{S}) \right\}. \end{aligned}$$

In fact, this polytope has a very simple structure.

Proposition 14

$$(P_{=c}^{CLQ})^{emb} = \left\{ (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m : x_j = \sum_{i:j \in C_i} \lambda_i, \sum_{i=1}^m \lambda_i = 1, x \geq 0, \lambda \geq 0 \right\}.$$

Proof The validity of the equations is easy to verify. They correspond to the dominating $s - t$ cuts for $U \subset V_t$ such that $|U| = n - 1$, but they hold at equality for all $(x, \lambda) \in (P_{=c}^{CLQ})^{emb}$ and together imply $\sum_{j \in \mathcal{V}} x_j = c$. Moreover, the polyhedron on the right is integral, which can be verified by ad-hoc methods. \square

When $c = 2$, we can easily project out the λ_i variables, by considering the stable sets of G . That is, using a result of [8], we apply the extreme rays of the polyhedron $yB \geq 0$, where $B\lambda = x$ is the subsystem defining the connection between the x and λ in the definition of $(P_{=c}^{CLQ})^{emb}$. For any stable set I of G , let $y_j = -1$ for $j \in I$, and $y_j = 1$ for $j \in \Gamma(I)$, the neighbors of I in G . These extreme rays y yield the valid

inequalities

$$x(I) - x(\Gamma(I)) \leq 0, \quad \text{for all stable set } I \text{ of } G.$$

These inequalities were derived in [33] by starting out from a different extended formulation, and Kaibel and Loos also show that along with $x_j \geq 0, j \in \mathcal{V}$, and $\sum_{j \in \mathcal{V}} x_j = 2$, they suffice to describe $P_{=c}^{CLQ}$, and they give conditions for these inequalities to define facets.

Now let us turn to $(P_{\leq c}^{CLQ})^{emb}$. Clearly,

$$\left(P_{\leq c}^{CLQ} \right)^{emb} = \left\{ (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \mid \sum_{i=1}^m \lambda_i = 1, x \geq 0, \lambda \geq 0, \right. \\ \left. c_{x,\lambda}([S, \bar{S}]) \geq \sum_{j=1}^n x_j, \forall S - t \text{ cut } (S, \bar{S}) \right\}.$$

Proposition 15 *The non-trivial facets of $(P_{\leq c}^{CLQ})^{emb}$ are*

$$\sum_{i:j \in C_i} \lambda_i \geq x_j, \quad \forall j \in \{1, \dots, n\}.$$

Proof By Theorem 1, the only sets $U \subset V_t$ that induce facets of $(P_{\leq c}^{CLQ})^{emb}$ are those, where U contains all but one of the nodes $j \in V_t$, and the statement follows. \square

The special case with $c = 2$ is extensively studied in [29] and [33], where all facets of $P_{\leq c}^{CLQ}$ are determined in the space of the original problem variables.

5.7 Union of simplicies

This application is from Jeroslow [31]. Let $P^\Delta := \text{conv}(\cup_{i=1}^m P_i^\Delta)$, where

$$P_i^\Delta := \left\{ x \in \mathbb{R}^n \mid \sum_{j=1}^n x_j \leq UB^i, x_j \geq LB_j^i, \quad j = 1, \dots, n \right\}.$$

We assume that $\sum_{j=1}^n LB_j^i \leq UB^i$ for all i , and all bounds are finite. Let $(P^\Delta)^{emb} = \text{conv}(\cup_{i=1}^m (P_i^\Delta \times \epsilon^i))$. Clearly, $(P^\Delta)^{emb}$ is the convex hull of those (x, λ) that satisfy

the constraints

$$\begin{aligned}
 \sum_{j=1}^n x_j &\leq UB^i \lambda_i, \quad i = 1, \dots, m \\
 x_j &\geq LB_j^i \lambda_i, \quad i = 1, \dots, m, \quad j = 1, \dots, n \\
 \sum_{i=1}^m \lambda_i &= 1 \\
 \lambda &\in \{0, 1\}^m.
 \end{aligned}
 \tag{21}$$

In order to describe $(P^\Delta)^{emb}$, we form a network $N = (V, A, c, lb)$, where V comprises a source s , sink t , the nodes $V_s = \{v_1, \dots, v_m\}$ as well as $V_t = \{1, \dots, n\}$. The set of arcs A comprises (s, v_i) for all $v_i \in V_s$, (v_i, j) for all $v_i \in V_s$, and $j \in V_t$, and (j, t) for all $j \in V_t$. There are both capacities c , and lower bounds lb on the arcs. Let $c(s, v_i) = UB^i \lambda_i$, $lb(s, v_i) = \lambda_i \sum_{j=1}^n LB_j^i$, $c(v_i, j) = (UB^i - \sum_{k \neq j} LB_k^i) \lambda_i$, $lb(v_i, j) = LB_j^i \lambda_i$, and $c(j, t) = x_j$, and $lb(j, t) = 0$ for all i and j . The lower bounds on the arcs (s, v_i) , and (v_i, j) can be eliminated by the following transformation. We derive a new network $N' = (V, A, c')$ from N by modifying the upper bounds as follows. $c'(s, v_i) := (UB^i - \sum_{k=1}^n LB_k^i) \lambda_i$, $c'(v_i, j) := (UB^i - \sum_{k=1}^n LB_k^i) \lambda_i$, and $c'(j, t) := x'_j$, where $x'_j = x_j - \sum_{i=1}^m LB_j^i \lambda_i$. We have the following relation between the feasible flows of N and N' .

Proposition 16 *Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ be vectors such that $\lambda \geq 0$, and $\sum_{i=1}^n \lambda_i = 1$. We have the equivalences*

- i) $(x, \lambda) \in (P^\Delta)^{emb}$ if and only if $N_{x,\lambda}$ admits a feasible $s - t$ flow of value $\sum_{j=1}^n x_j$.
- ii) $N_{x,\lambda}$ admits a feasible $s - t$ flow of value $\sum_{j=1}^n x_j$ if and only if $N'_{x',\lambda}$ admits a feasible $s - t$ flow of value $\sum_{j=1}^n x'_j$, where $x'_j = x_j - \sum_{i=1}^m LB_j^i \lambda_i$.

Proof Part i) is obvious. As for ii), first suppose $N_{x,\lambda}$ admits a feasible flow f of value $\sum_{j=1}^n x_j$. Let $f'_{(s,v_i)} = f_{(s,v_i)} - \lambda_i \sum_{j=1}^n LB_j^i$, $f'_{(v_i,j)} = f_{(v_i,j)} - \lambda_i LB_j^i$, and $f'_{(j,t)} = f_{(j,t)} - \sum_{i=1}^m \lambda_i LB_j^i$. Clearly, f' is a feasible $s - t$ flow in $N'_{x',\lambda}$ of value $\sum_{j=1}^n x'_j$, as claimed. The opposite direction can be proved similarly. \square

Let Q' be convex hull of those (x', λ) such that $x', \lambda \geq 0$, and $N'_{x',\lambda}$ admits a feasible $s - t$ flow of value $\sum_{j=1}^n x'_j$. Now we use Theorem 1 to identify those facets of Q' that corresponds to some $s - t$ cuts of N' and then we lift the description of Q' to that of $(P^\Delta)^{emb}$.

Observation 5 Q' is of dimension $n + m - 1$ if there exists $i \in \{1, \dots, m\}$ such that $\sum_{j=1}^n LB_j^i < UB^i$, otherwise it has dimension $m - 1$.

Proposition 17 Any proper subset $U \subset V_t$ does not induce a facet of Q' . If $\dim Q' = n + m - 1$, then the set $U = V_t$ induces the facet

$$\sum_{j=1}^n x'_j \leq \sum_{i=1}^m \left(UB^i - \sum_{j=1}^n LB_j^i \right) \lambda_i. \tag{22}$$

Proof Let $U \subseteq V_t$. If $|U| \leq n - 2$, then any $s - t$ cut (S, \bar{S}) in $\mathcal{C}_{\min}(U)$ has the following structure: $S = \{s\} \cup U$. But then condition (ii) of Theorem 1 is not satisfied by (S, \bar{S}) . Now suppose $|U| = n - 1$. Then the dominating $s - t$ cut (S, \bar{S}) in $\mathcal{C}_{\min}(U)$ has the following structure: $S = \{s\} \cup V_t \cup U$. But then condition (i) of Theorem 1 is not satisfied by (S, \bar{S}) , since for each v_i , a single arc of capacity $\lambda_i(UB^i - \sum_{k=1}^n LB_k^i)$ leaves S , but this matches the capacity of (s, v_i) . Finally, if $U = V_t$, then the conditions of Theorem 1 are satisfied (the first one is void), and thus we get the inequality in the statement. \square

So, we have the following description for Q' :

$$\begin{aligned} \sum_{j=1}^n x'_j &\leq \sum_{i=1}^m \lambda_i \left(UB^i - \sum_{j=1}^n LB_j^i \right) \\ \sum_{i=1}^m \lambda_i &= 1 \\ x, \lambda &\geq 0 \end{aligned}$$

Notice that this description is valid even if $UB^i = \sum_{j=1}^n LB_j^i$ for all i . Now we transform this description to one for $(P^\Delta)^{emb}$. But this is easy. By Proposition 16 we know that $(x, \lambda) \in (P^\Delta)^{emb}$ if and only if $x_j \geq \sum_{i=1}^m LB_j^i \lambda_i$ for each j , and $(x', \lambda) \in Q'$, where $x'_j = x_j - \sum_{i=1}^m LB_j^i \lambda_i$ for all j . Consequently, we have the following:

Proposition 18 A minimal linear description of $(P^\Delta)^{emb}$ is

$$\begin{aligned} \sum_{j=1}^n x_j &\leq \sum_{i=1}^m UB^i \lambda_i, \\ x_j &\geq \sum_{i=1}^m LB_j^i \lambda_i, \quad j = 1, \dots, m \\ \sum_{i=1}^m \lambda_i &= 1. \\ \lambda &\geq 0. \end{aligned}$$

This is precisely the description of Jeroslow obtained by completely different means.

5.8 Multi-variate piecewise linear functions

This application is from [26], see also [28]. The objective is to model the graph of a continuous piecewise linear function $f : D \rightarrow \mathbb{R}$ on a bounded domain $D \subseteq \mathbb{R}^d$, where D admits a polyhedral partition $\bigcup_{i=1}^m C_i = D$ ($\text{relint}(C_i) \cap \text{relint}(C_j) = \emptyset$ for $i \neq j$) such that $x \in C_i$ implies $f(x) = a_i \cdot x + b_i$, for some $a_i \in \mathbb{R}^d$ and $b_i \in \mathbb{R}$. The graph of f is $\text{gr}(f) := \{(x, f(x)) : x \in D\}$. Let $W := \bigcup_{i=1}^m \text{vert}(C_i)$. In fact, we have

$$\text{gr}(f) = \left\{ \sum_{w \in W} x_w(w, f(w)) : x \in \bigcup_{i=1}^m P_i \right\},$$

where $P_i := \{x \in \mathbb{R}_{\geq 0}^W : \sum_{w \in W} x_w = 1, x_w = 0 \ \forall w \in W \setminus \text{vert}(C_i)\}$. Here, $x \in \bigcup_{i=1}^m P_i$ is the combinatorial disjunctive constraint (5).

Let $P^{pwl} = \text{conv}(\bigcup_{i=1}^m P_i)$. Suppose $W = \{w_1, \dots, w_n\}$. Then $(P^{pwl})^{emb}$ is the convex hull of those (x, λ) which satisfy

$$\begin{aligned} \sum_{j=1}^n x_j &= 1, \\ x_j &\leq \sum_{i \in [m] : w_j \in \text{vert}(C_i)} \lambda_i, \quad j = 1, \dots, n \\ \sum_{i=1}^m \lambda_i &= 1, \\ \lambda &\in \{0, 1\}^m. \end{aligned} \tag{23}$$

It is easy to see that $(P^{pwl})^{emb}$ admits a network representation $N = (V, A, c)$, where $V = \{s, t\} \cup V_s \cup V_t$ such that $V_s = \{v_1, \dots, v_m\}$, $V_t = \{1, \dots, n\}$, and $(v_i, j) \in A$ if and only if $w_j \in C_i$. Furthermore, A contains all the arcs (s, v_i) for $i = 1, \dots, m$, and (j, t) for $j = 1, \dots, n$. As usual, $c(j, t) = x_j$ for $j \in V_t$, while $c(s, v_i) = \lambda_i$, and $c(v_i, j) = \lambda_i$ for all $(v_i, j) \in A$. The facets of $(P^{pwl})^{emb}$ have a particularly nice geometrical characterization. Let $\kappa(T)$ denote the number of connected components³ of some $T \subset \mathbb{R}^d$. Clearly, the dimension of $(P^{pwl})^{emb}$ is $n + m - 1 - \kappa(D)$.

Proposition 19 *The non-trivial facets of $(P^{pwl})^{emb}$ take the form*

$$\sum_{i \in M} \lambda_i \leq \sum_{j \in \bigcup_{i \in M} \Gamma_{v_i}^{out}} x_j, \tag{24}$$

where $M \subset \{1, \dots, m\}$ is such that $\kappa(D \setminus (\bigcup_{i \in M} C_i)) + \kappa(\bigcup_{i \in M} C_i) = \kappa(D) + 1$.

Proof We construct a dominating $s - t$ cut (S, \bar{S}) from M and show that it determines an inequality equivalent to (24). Let $U := \bigcup_{i \in M} \Gamma_{v_i}^{out}$, and $S := \{s\} \cup \{v_i \in V_s : \dots$

³ Path-connected in topological sense.

$|\Gamma_{v_i}^{out} \setminus U| \leq 1\} \cup U$. Observe that $v_i \notin S$ if and only if $|\Gamma_{v_i}^{out} \setminus U| \geq 2$ and $i \in M$ if and only if $\Gamma_{v_i}^{out} \subseteq U$. Therefore, the corresponding inequality for (S, \bar{S}) is

$$\sum_{v_i \in V_s : |\Gamma_{v_i}^{out} \setminus U| \geq 2} \lambda_i + \sum_{v_i \in V_s : |\Gamma_{v_i}^{out} \setminus U| = 1} \lambda_i + \sum_{j \in U} x_j \geq \sum_{j=1}^n x_j.$$

Since $\sum_{i=1}^m \lambda_i = \sum_{j=1}^n x_j = 1$ by assumption, we can rewrite this as

$$1 - \sum_{i \in M} \lambda_i + \sum_{j \in U} x_j \geq 1.$$

Rearranging terms yields (24).

We still have to verify that S meets the conditions of Theorem 2. In fact, if either of the conditions does not hold, then either the nodes in U are disconnected in the subnetwork spanned by $S \setminus \{s\}$, or those in $V_t \setminus U$ are disconnected in $N \setminus (S \cup \{t\})$, which implies $\kappa(D \setminus (\bigcup_{i \in M} C_i)) + \kappa(\bigcup_{i \in M} C_i) > \kappa(D) + 1$. \square

In fact, this characterization is the same as that of Lee and Wilson [36].

The 2-dimensional special case, where $D = [0, S_1] \times [0, S_2]$, and the $\{C_i\}_{i=1}^m$ constitute a grid triangulation of D with $m = 2S_1S_2$ triangles, is extensively studied in [26–28,45,48], and ideal formulations with $O(\log_2 m)$ new (binary) variables and constraints are provided. Different triangulation of the grid may define different piecewise linear functions, even if they agree on the grid points, for an example we refer the reader to [28].

The naive modelling (23) of the grid triangulations leads to m auxiliary λ_i variables, which is far more than $O(\log_2 m)$ needed by the methods of [26–28,45,48]. On the other hand, the techniques of the above papers are difficult to generalize for 3 or more dimensions. Nevertheless, Huchette and Vielma [28] proved that intersecting ideal formulations for combinatorial disjunctions yields an ideal formulation:

Theorem 3 [28] *Fix $s \in \mathbb{Z}_{\geq 1}$, and consider s distinct combinatorial disjunctive constraints $B^t = \bigcup_{i=1}^{m_t} Q(T^{i,t})$ on the same set V , where $\bigcup_{i=1}^{m_t} T^{i,t} = V$, for $t \in \llbracket s \rrbracket$. Let $\Pi^t \subseteq \mathbb{R}^V \times \mathbb{R}^{r_t}$ be such that $\{(x, \lambda^t) \in \Pi^t : \lambda^t \in \mathbb{Z}^{r_t}\}$ is an ideal formulation for B^t , for each $t \in \llbracket s \rrbracket$. Then, an ideal formulation for $\bigcap_{t=1}^s B^t$ is*

$$\left\{ (x, \lambda^1, \dots, \lambda^s) : (x, \lambda^t) \in \Pi^t, \lambda^t \in \mathbb{Z}^{r_t} \forall t \in \llbracket s \rrbracket \right\}. \tag{25}$$

Huchette and Vielma suggested that this result permits to go beyond the 2-variable special case. Below we describe a possible realization of their idea. In d dimensions, the unit grid for $D := \times_{k=1}^d [0, S_k]$ consists of $\prod_{t=1}^d S_t$ unit cubes. Let $n = \prod_{t=1}^d (S_t + 1)$ be the number of grid points in $W := D \cap \mathbb{Z}^d$. For each unit cube in the grid, a partitioning into $d!$ simplices of $[0, 1]^d$ is specified. Let \mathcal{C} be the set of all the simplices given in all the unit cubes of the grid W . We can use d distinct combinatorial disjunctions for

Fig. 6 Covering the grid $[0, 4]^2 \cap \mathbb{Z}^2$ by 4 sets of node disjoint unit squares. The squares with the same number constitute one set of the partitioning

1	2	1	2
3	4	3	4
1	2	1	2
3	4	3	4

selecting a cube, and then some additional combinatorial disjunction(s) to select a single simplex in the selected cube (this is how the compact formulation of [28] works in 2 dimensions). So, for $t \in \llbracket d \rrbracket$, let $m_t = S_t$, and $T^{i,t} = \{w \in W : w_t \in \{i - 1, i\}\}$ for $i \in \llbracket m_t \rrbracket$, namely, in $T^{i,t}$ we select all the grid points $w \in D$ such that $w_t \in \{i - 1, i\}$. Let $B^t = \bigcup_{i=1}^{m_t} Q(T^{i,t})$, and observe that this is the SOS2 constraint along the t^{th} axis. Let $\Pi^t \subset \mathbb{R}^n \times \mathbb{R}^{r_t}$ be any ideal formulation for B^t with $r_t = O(\log S_t)$ variables and $O(\log S_t)$ constraints, see e.g., [28], and Sect. 2. By Theorem 3, joining the Π^t , $t \in \llbracket d \rrbracket$, yields an ideal formulation for $\bigcap_{t=1}^d B^t$, and in fact it selects a single cube from the grid (by setting the λ^t to unit vectors). However, we still have to select a simplex. All we need is one more combinatorial disjunction, which covers the vertices of all the $d! \prod_{t=1}^d S_t$ simplices such that each term in this disjunction covers simplices of node disjoint cubes. In fact, $2^d d!$ terms suffice, as the following statement shows.

Proposition 20 *There is a combinatorial disjunction $B^{d+1} = \bigcup_{i=1}^{2^d d!} Q(T^{i,d+1})$ such that*

- $\bigcup_{i=1}^{2^d d!} T^{i,d+1} = D \cap \mathbb{Z}^d$,
- for any unit cube in the grid $D \cap \mathbb{Z}^d$, and any simplex $C \in \mathcal{C}$ from the cube, there is a unique $T^{i,d+1}$ containing $\text{vert}(C)$, and
- Any $T^{i,d+1}$ consists of the vertices of simplices from vertex disjoint unit cubes of the grid.

Proof To cover a grid in \mathbb{Z}^d by sets of node disjoint unit cubes, we can partition the grid into 2^d subsets, see Fig. 6. Each of the 2^d sets of cubes yields $d!$ disjunctive terms, namely, we identify each simplex in each unit cube uniquely by a number from $1, \dots, d!$, and then we collect all vertices of those simplices having the same id. in a term of the disjunction. More formally, let R^k for $k \in \llbracket 2^d \rrbracket$ be the sets of node disjoint unit cubes from the grid which together cover D . Each R^k gives rise to $d!$ sets $T^{i,d+1}$ for $i \in \{d!(k - 1) + 1, \dots, d!k\}$, where $T^{i,d+1}$ contains $\text{vert}(C)$ for those simplices C from the cubes of R^k having id $(i - d!(k - 1))$. It is easy to see that the sets $T^{i,d+1}$ satisfy the conditions of the statement. □

Corollary 3 *Let $\Delta_C^W = \{x \in \mathbb{R}_{\geq 0}^W : \sum_{w \in W} x_w = 1, x_w = 0 \forall w \notin \text{vert}(C)\}$. Then $\text{conv}(\bigcap_{t=1}^{d+1} B^t) = \text{conv}\{\Delta_C^W : C \in \mathcal{C}\}$. Moreover, joining any ideal formulations Π^t for B^t , $t \in \llbracket d + 1 \rrbracket$, as in Theorem 3, yields an ideal formulation for $\bigcap_{t=1}^{d+1} B^t$.*

Finally, using our techniques, it is straightforward to obtain an ideal formulation for $P^{d+1} := \text{conv}(B^{d+1})$. Let $m = 2^d d!$, and $n = |W|$. The underlying network $N = (V, A, c)$ has nodes $\{s, t\} \cup V_s \cup V_t$, where $V_s = \{v_1, \dots, v_m\}$, and $V_t = \{1, \dots, n\}$. Node s is connected to each $v_i \in V_s$ by an arc of capacity λ_i , node v_i

is connected to each node $j \in V_t$ such that $w_j \in T^{i,d+1}$ by an arc of capacity λ_i , and finally, each $j \in V_t$ is connected to t by an arc of capacity x_j . We also have the equations $\sum_{i=1}^m \lambda_i = \sum_{j=1}^n x_j = 1$. However, there are no implicit equations for $(P_{\leq}^{d+1})^{emb}$, because if a partitioning E_1, \dots, E_κ of W , for some $\kappa \geq 2$, induces linearly independent valid equations for $(P_{\leq}^{d+1})^{emb}$, then each $T^{i,d+1}$ must be a subset of some E_ℓ . It is easy to verify that this is impossible.

Observation 6 $\dim((P_{\leq}^{d+1})^{emb}) = n + m - 2$.

We can apply Theorem 2 to obtain the non-trivial facets of $(P_{\leq}^{d+1})^{emb} := \text{conv}\{\bigcup_{i=1}^{2^d d!} Q(T^{i,d+1}) \times \{\epsilon^i\}\}$. Note that there are only $2^d d!$ auxiliary variables in the resulting formulation, which is just 8 for $d = 2$, 48 for $d = 3$, and 384 for $d = 4$, and it is independent of the size of n . As for the number of facets of $(P_{\leq}^{d+1})^{emb}$, it can be exponential in n even if $d = 2$. However, to our best knowledge, for $d \geq 3$, no ideal formulation is known with a logarithmic number of auxiliary variables and linear inequalities in n .

Proposition 21 *Let $U \subset V_t$, and $(S, \bar{S}) \in C_{\min}(U)$. Then F_S is a facet of $(P_{\leq}^{d+1})^{emb}$ if and only if (a) $N \setminus (S \cup \{t\})$ is connected, and (b) for any partitioning of U into nonempty subsets U_1 and U_2 , there is some i such that $T^{i,d+1} \cap U_\ell \neq \emptyset$ for $\ell = 1, 2$, and $T^{i,d+1} \subseteq U$.*

5.9 Modeling of variable intensity activities in a resource loading problem

This application stems from the paper [41], where a resource loading problem is studied. Here we give a new derivation of the polyhedral description of feasible intensity assignments to an activity.

Suppose there is a time horizon of n consecutive time periods of unit length each, and an activity which has to be scheduled in an interval of time periods uninterruptedly. In those time periods when it is processed, a fraction between given lower and upper bounds, LB and UB , must be assigned such that the sum of fractions is 1, where $0 < LB \leq UB \leq 1$. Let \mathcal{E} consist of all the possible execution intervals for the activity, i.e., $[k, \ell] \in \mathcal{E}$ if and only if $1 \leq k \leq \ell \leq n$, and $(\ell - k + 1)LB \leq 1 \leq (\ell - k + 1)UB$. For ease of notation, $[k, \ell]$ also represents the set of integers $\{t \in \mathbb{Z} : k \leq t \leq \ell\}$. If we select $[k, \ell] \in \mathcal{E}$, then the intensity assignment $x \in \mathbb{R}^n$ of the activity must satisfy the conditions $\sum_{j=k}^\ell x_j = 1$, $LB \leq x_j \leq UB$ for $j \in [k, \ell]$, and $x_j = 0$ otherwise.

For each interval $[k, \ell] \in \mathcal{E}$, we can strengthen the lower and upper bounds as follows: $LB_{k\ell} = \max\{LB, 1 - (\ell - k)UB\}$ and $UB_{k\ell} = \min\{UB, 1 - (\ell - k)LB\}$. To avoid implicit equations, we assume that for each pair of consecutive time periods j , and $j + 1$, there exists some execution interval $[k, \ell] \in \mathcal{E}$ such that $j, j + 1 \in [k, \ell]$, and $(\ell - k + 1)LB_{k\ell} < 1 < (\ell - k + 1)UB_{k\ell}$.

We can model the feasible intensity assignments by means of the following integer program:

$$\sum_{[k,\ell] \in \mathcal{E}: j \in [k,\ell]} \lambda_{k\ell} LB_{k\ell} \leq x_j \leq \sum_{[k,\ell] \in \mathcal{E}: j \in [k,\ell]} \lambda_{k\ell} UB_{k\ell}, \quad \forall j \in \{1, \dots, n\} \tag{26}$$

$$\sum_{[k,\ell] \in \mathcal{E}} \lambda_{k\ell} = 1 \tag{27}$$

$$\sum_{j=1}^n x_j = 1 \tag{28}$$

$$x_j \geq 0, \quad \forall j \in \{1, \dots, n\}$$

$$\lambda \in \{0, 1\}^{\mathcal{E}} \tag{29}$$

Let P^{RLP} be the convex hull of those (x, λ) vectors that satisfy these constraint. In order to get the (non-trivial) facets of P^{RLP} , we model the above integer program by a network $N = (V, A, c, lb)$, where V comprises a source node s , a sink node t , the nodes $V_s := \{v_{k\ell} \mid [k, \ell] \in \mathcal{E}\}$, and the nodes $V_t := \{1, \dots, n\}$. We have one arc from s to each $v_{k\ell}$ with lower bound $lb(s, v_{k\ell}) = (\ell - k + 1)LB_{k\ell}\lambda_{k\ell}$, and capacity $c(s, v_{k\ell}) = \lambda_{k\ell}$. Further on, for each $v_{k\ell} \in V_s$ and $j \in V_t$, we have one arc $(v_{k\ell}, j)$ of lower bound $\lambda_{k\ell}LB_{k\ell}$, and capacity $\lambda_{k\ell}UB_{k\ell}$. Finally, we have one arc from each $j \in V_t$ to t of lower bound 0, and capacity x_j . Observe that the positive lower bounds stem from the left hand side of (26). Let $x'_j = x_j - \sum_{[k,\ell] \in \mathcal{E}: j \in [k,\ell]} \lambda_{k\ell}LB_{k\ell}$, and we derive network $N' = (V, A, c')$ from N as follows. The capacity of the arcs $(s, v_{k\ell})$ reduces to $(1 - (\ell - k + 1)LB_{k\ell})\lambda_{k\ell}$, that of $(v_{k\ell}, j)$ becomes $(UB_{k\ell} - LB_{k\ell})\lambda_{k\ell}$, while on the arcs (j, t) the new capacity is x'_j . Let $m = |\mathcal{E}|$. The following statement is analogous to Proposition 16.

Proposition 22 *Suppose $\sum_{[k,\ell] \in \mathcal{E}} \lambda_{k\ell} = 1, x \geq 0$, and $\lambda \geq 0$. $(x, \lambda) \in P^{RLP}$ if and only if $N_{x,\lambda}$ admits a feasible $s - t$ flow of value 1 if and only if $N'_{x',\lambda}$ admits a feasible $s - t$ flow of value $\sum_{[k,\ell] \in \mathcal{E}} (1 - (\ell - k + 1)LB_{k\ell})\lambda_{k\ell}$.*

Let $P^{RLP'}$ be the convex hull of those (x', λ) such that $x, \lambda \geq 0, \sum_{[k,\ell] \in \mathcal{E}} \lambda_{k\ell} = 1$, and $N'_{x',\lambda}$ admits a feasible $s - t$ flow of value $\sum_{[k,\ell] \in \mathcal{E}} (1 - (\ell - k + 1)LB_{k\ell})\lambda_{k\ell}$. We apply the MAX FLOW-MIN CUT theorem to N' to obtain a linear representation of $P^{RLP'}$:

$$P^{RLP'} = \left\{ (x', \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \mid \sum_{[k,\ell] \in \mathcal{E}} \lambda_{k\ell} = 1, x' \geq 0, \lambda \geq 0, \right.$$

$$\left. \sum_{j=1}^n x'_j = \sum_{[k,\ell] \in \mathcal{E}} (1 - (\ell - k + 1)LB_{k\ell})\lambda_{k\ell}, \right.$$

$$\left. c_{x',\lambda}([S, \bar{S}]) \geq \sum_{j=1}^n x'_j, \forall s - t \text{ cut } (S, \bar{S}) \text{ of } N' \right\}.$$

Let (S, \bar{S}) be any $s - t$ cut of N' , the corresponding inequality is

$$\sum_{v_{k\ell} \in S} \lambda_{k\ell} |[k, \ell] \setminus U| \cdot (UB_{k\ell} - LB_{k\ell}) + \sum_{v_{k\ell} \notin S} \lambda_{k\ell} (1 - (\ell - k + 1) \cdot LB_{k\ell}) \geq \sum_{j \in V_t \setminus U} x'_j. \tag{30}$$

If we increase both sides by $\sum_{[k, \ell] \in \mathcal{E}} |[k, \ell] \setminus U| \cdot LB_{k\ell} \lambda_{k\ell}$, we obtain

$$\sum_{v_{k\ell} \in S} \lambda_{k\ell} |[k, \ell] \setminus U| \cdot UB_{k\ell} + \sum_{v_{k\ell} \notin S} \lambda_{k\ell} (1 - |[k, \ell] \cap U| \cdot LB_{k\ell}) \geq \sum_{j \in V_t \setminus U} x_j, \tag{31}$$

where we used $x_j = x'_j + \sum_{[k, \ell] \in \mathcal{E}: j \in [k, \ell]} LB_{k\ell} \lambda_{k\ell}$. These are precisely the inequalities (27) obtained in [41] by a different argument. The following statement is analogous to Proposition 18.

Proposition 23 *A minimal linear description of P^{RLP} consists $\lambda \geq 0$, the inequalities $x_j \geq \sum_{[k, \ell] \in \mathcal{E}} LB_{k\ell} \lambda_{k\ell}$, (27)–(29), and those inequalities from (31) for which the corresponding inequality from (30) induces a facet of $P^{RLP'}$.*

It remains to characterize those $U \subset V_t$ that induce facets of $P^{RLP'}$. First, for a fixed $U \subset V_t$, we determine the non-dominated $s - t$ cuts of N' w.r.t. U . So we build a set S containing $\{s\} \cup U$, and some of the nodes $v_{k\ell}$. We can decide for each $v_{k\ell}$ whether to include in S or not in order to minimize the capacity of (S, \bar{S}) .

- If $v_{k\ell} \in S$, then the arcs $(v_{k\ell}, j)$ for $j \in [k, \ell] \setminus U$ are adjacent to $v_{k\ell}$ and leave S . The total capacity of these arcs is

$$\lambda_{k\ell} |[k, \ell] \setminus U| \cdot (U_{k\ell} - LB_{k\ell}) = \lambda_{k\ell} |[k, \ell] \setminus U| \cdot U_{k\ell} - \lambda_{k\ell} |[k, \ell] \setminus U| \cdot LB_{k\ell}. \tag{32}$$

- If $v_{k\ell} \notin S$, then the arc $(s, v_{k\ell})$ leaves S and its capacity is

$$\lambda_{k\ell} (1 - (\ell - k + 1) \cdot LB_{k\ell}) = \lambda_{k\ell} (1 - |[k, \ell] \cap U| \cdot LB_{k\ell}) - \lambda_{k\ell} |[k, \ell] \setminus U| \cdot LB_{k\ell}. \tag{33}$$

Comparing (32) and (33), we can derive the following.

Observation 7 $(S, \bar{S}) \in \mathcal{C}_{\min}(U)$ if and only if $v_{k\ell} \in S$ when $|[k, \ell] \setminus U| \cdot UB_{k\ell} < 1 - |[k, \ell] \cap U| \cdot LB_{k\ell}$, and $v_{k\ell} \notin S$ when $|[k, \ell] \setminus U| \cdot UB_{k\ell} > 1 - |[k, \ell] \cap U| \cdot LB_{k\ell}$.

Now we apply Theorem 2 to single out those subsets $U \subset V_t$ that induce facets of $P^{RLP'}$. Let \mathcal{E}^+ consist of those $[k, \ell] \in \mathcal{E}$ for which $(\ell - k + 1)UB_{k\ell} > 1 > (\ell - k + 1)LB_{k\ell}$.

Theorem 4 *Fix $U \subset V_t$, and let (S, \bar{S}) the dominating $s - t$ cut of N' w.r.t. U . It induces a facet of $P^{RLP'}$ if and only if both of the following conditions hold.*

- i) If $|V_t \setminus U| \geq 2$, then for each $j \in V_t \setminus U$ except the last one, there exists $[k, \ell] \in \mathcal{E}^+$ such that $v_{k\ell} \in \bar{S}$, $[k, \ell] \cap [1, j] \cap (V_t \setminus U) \neq \emptyset$ and $[k, \ell] \cap [j+1, n] \cap (V_t \setminus U) \neq \emptyset$.
- ii) If $|U| \geq 2$, then for each $j \in U$ except the last one, there exists $[k, \ell] \in \mathcal{E}^+$ such that $v_{k\ell} \in S$, $[k, \ell] \cap [1, j] \cap U \neq \emptyset$ and $[k, \ell] \cap [j+1, n] \cap U \neq \emptyset$.

Proof We derive the two conditions of the theorem from those of Theorem 2. First note that if $v_{k\ell} \notin \mathcal{E}^+$, then $v_{k\ell} \in S$, as (S, \bar{S}) is the dominating $s - t$ cut of N' w.r.t. U . Consider the first condition of Theorem 2. $N \setminus (S \cup \{t\})$ is not connected if and only if there exists a partitioning of $V_t \setminus U$ into two subsets $\bar{U}_1 \neq \emptyset$ and $\bar{U}_2 \neq \emptyset$, such that each $v_{k\ell} \in \bar{S} \cap V_s$ is adjacent to some nodes of \bar{U}_1 only, or to some nodes of \bar{U}_2 only, that is, $[k, \ell] \setminus U \subseteq \bar{U}_1$ or $[k, \ell] \setminus U \subseteq \bar{U}_2$. Since $[k, \ell]$ is an interval, $[k, \ell] \setminus U$ consists of consecutive members of $V_t \setminus U$, and thus the latter condition is equivalent to the existence of some $j \in V_t \setminus U$ (but not the last one) such that $[k, \ell] \setminus U \subseteq [1, j]$ or $[k, \ell] \setminus U \subseteq [j+1, n]$ for each $v_{k\ell} \in \bar{S}$. But this is precisely the negation of the first condition of this theorem.

The equivalence of the second condition of Theorem 2 to that of the present theorem can be shown analogously. □

The above characterization of the facets of $P^{RLP'}$ is equivalent to that obtained in [41] for the non-trivial facets of P^{RLP} . However, in that paper a more problem specific approach is used.

6 Final remarks

In this paper we have proposed a new systematic way of constructing ideal, non-extended MIP formulations for disjunctive constraints of the form $x \in \bigcup_{i=1}^m P_i$, where the P_i are polytopes. Our method works if the Cayley Embedding of this union of polytopes admits a certain network representation. Then we can characterize the facets of the LP-relaxation of the MIP formulation. We have illustrated the richness of the class of problems that fits this framework by several examples from the literature, and we have also worked out a new one. We emphasize that our characterization of facets is valid for any union of polytopes such that the corresponding Cayley Embedding admits a network representation.

These results may also pave the way for modeling problems using building blocks representing the convex hull of polytopes with the above properties. Then, the solver could generate cuts for such building blocks as needed in the course of solving the problem by branch-and-cut.

Acknowledgements The authors are grateful to an anonymous referee for many constructive comments that helped to improve the paper significantly, and to the editor and another anonymous referee for comments that led to better positioning our results.

Funding Open access funding provided by ELKH Institute for Computer Science and Control.

Declarations

Conflict of interest There are no conflicts of interest.

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