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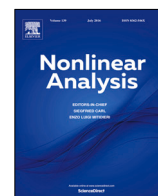
Traveling wave solutions for two species competitive chemotaxis systems

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A B S T R A C T

In this paper, we consider two species chemotaxis systems with Lotka–Volterra competition reaction terms. Under appropriate conditions on the parameters in such a system, we establish the existence of traveling wave solutions of the system connecting two spatially homogeneous equilibrium solutions with wave speed greater than some critical number c^*. We also show the non-existence of such traveling waves with speed less than some critical number c_0^*, which is independent of the chemotaxis. Moreover, under suitable hypotheses on the coefficients of the reaction terms, we obtain explicit range for the chemotaxis sensitivity coefficients ensuring $c^* = c_0^*$, which implies that the minimum wave speed exists and is not affected by the chemoattractant.

1. Introduction

Chemotaxis refers to the tendency of cells, bacteria, or organisms to orient the direction of their movements toward the increasing or decreasing concentration of a signaling chemical substance. It has a crucial role in a wide range of biological phenomena such as immune system response, embryo development, and tumor growth (see [1,2]). Recent studies describe also macroscopic process such as population dynamics or gravitational collapse, in terms of chemotaxis (see [3]). Because of its crucial role in the population dynamics, it is important to investigate the effects of certain chemical substances on the dynamics of populations of species.

There are many existing works on the dynamics of two competitive populations of biological species which are attracted by a chemical stimulus on a bounded environment (see [4–11], etc.). For example, in [10], Tello

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and Winkler considered the following two species competitive chemotaxis system,

$$
\begin{cases}
u_{1,t} = d_1 \Delta u_1 - \chi_1 \nabla \cdot (u_1 \nabla v) + r_1 u_1 (1 - u_1 - a u_2), & x \in \Omega \\
u_{2,t} = d_2 \Delta u_2 - \chi_2 \nabla \cdot (u_2 \nabla v) + r_2 u_2 (1 - b u_1 - u_2), & x \in \Omega \\
0 = \Delta v - \lambda v + \mu_1 u_1 + \mu_2 u_2, & x \in \Omega \\
\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial \Omega,
\end{cases}
\tag{1.1}
$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded smooth domain; $d_1, d_2, \chi_1, \chi_2, a, b, r_1, r_2, \lambda, \mu_1, \mu_2 > 0$ are positive constants; $u_i(t,x)$, $i = 1, 2$ denote the density functions of two mobile species living together in the same habitat and competing for some limited resources available in their environment. These two competing species also produce some chemical substance which affects their reproduction dynamics in the sense that each mobile species has tendency to move toward its higher concentration. The density function of the chemical substance is denoted by $v(t,x)$ and is being produced at the rates μ_i by the species i, for each $i = 1, 2$. The positive constant χ_i measures the sensitivity rate by the species $i \in \{1, 2\}$ of the chemical substance. The chemical substance has a self degradation rate given by the positive constant λ. The positive constants a and b measure the interspecific competition between the mobile species. Assume $0 < a < 1$ and $0 < b < 1$. Among others, they proved that the spatially homogeneous positive equilibrium $(u_1^*, u_2^*) = (\frac{1-a}{1-ab}, \frac{1-b}{1-ab})$ is globally asymptotically stable for any $d_1, d_2, \chi_1, \chi_2 > 0$, $\mu_1 = \mu_2 = 1$, and r_1, r_2 in certain nonempty range.

The current work is concerned with two species competitive chemotaxis systems on the whole space. In particular, we consider the traveling wave solutions of the following two species competition chemotaxis system on $\mathbb{R}$,

$$
\begin{cases}
u_{1,t} = (u_{1,x} - \chi_1 u_1 v_x)_x + u_1 (1 - u_1 - a u_2), & x \in \mathbb{R} \\
u_{2,t} = (d u_{2,x} - \chi_2 u_2 v_x)_x + r u_2 (1 - b u_1 - u_2), & x \in \mathbb{R} \\
0 = v_{xx} - \lambda v + \mu_1 u_1 + \mu_2 u_2, & x \in \mathbb{R},
\end{cases}
\tag{1.2}
$$

where $a, b, d, r, \lambda > 0$ and $\mu_i, \chi_i > 0$ $(i = 1, 2)$ are positive constants. We assume that the first species diffuses at a rate equal to one while the second species diffuses at a rate $d > 0$. The positive constant r is the intrinsic growth rate of the second mobile species.

When $\chi_1 = \chi_2 = 0$, the dynamics of the chemotaxis model (1.2) is governed by the following classical Lotka–Volterra diffusive competition system,

$$
\begin{cases}
u_{1,t} = u_{1,xx} + u_1 (1 - u_1 - a u_2), & x \in \mathbb{R} \\
u_{2,t} = d u_{2,xx} + r u_2 (1 - b u_1 - u_2), & x \in \mathbb{R}.
\end{cases}
\tag{1.3}
$$

The asymptotic dynamics of (1.3) is of significant research interests and considerable results have been established in the literature (see [12–16] and the references therein). It is well known that the large time behavior of solutions to (1.3) is delicately related to the positive constants a and b. For example, consider the kinetic system of ODEs associated with (1.3), that is,

$$
\begin{cases}
u_{1,t} = u_1 (1 - u_1 - a u_2) \\
u_{2,t} = r u_2 (1 - b u_1 - u_2).
\end{cases}
\tag{1.4}
$$

It is easily seen that (1.4) has a trivial equilibrium solution $\mathbf{e}_0 = (0, 0)$, and two semi-trivial equilibrium solutions $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. When $0 < a, b < 1$, or $a, b > 1$, (1.4) has a positive equilibrium given by $\mathbf{e}_* = (\frac{1-a}{1-ab}, \frac{1-b}{1-ab})$. The asymptotic dynamics of (1.4) depends on the strength of the competition coefficients a and b. We say that *the competition on the first species u_1 (resp. on the second species u_2) is strong* if $a > 1$ (resp. $b > 1$). We say that *the competition on the first species u_1 (resp. on the second species u_2) is weak* if $0 < a < 1$ (resp. $0 < b < 1$). Based on the magnitudes of a and b, the following four important cases arise.

(**A1**) $0 < b < 1 < a$, which is referred to as the *strong–weak competition case.*
(**A2**) $0 < a < 1 < b$, which is referred to as the *weak–strong competition case.*
(**A3**) $0 < a, b < 1$, which is referred to as the *weak–weak competition case.*
(**A4**) $a, b > 1$, which is referred to as the *strong–strong competition case*

Due to biological applications, we are only interested in non-negative solutions of (1.2), (1.3), and (1.4).
Let $(u_1(t), u_2(t))$ be a solution of (1.4) with $u_1(0) > 0$ and $u_2(0) > 0$. The following results are well known
(see [17–19] and the references therein). In the case (**A1**), $(u_1(t), u_2(t)) \to \mathbf{e}_2$ as $t \to \infty$ and hence the
second population outcompetes the first. In the case (**A2**), $(u_1(t), u_2(t)) \to \mathbf{e}_1$ as $t \to \infty$ and hence the
first population outcompetes the second. In the case (**A3**), $(u_1(t), u_2(t)) \to \mathbf{e}_*$ as $t \to \infty$ and hence both
species coexist for all the time. In the case (**A4**), the limit of $(u_1(t), u_2(t))$ depends on the choice of the
initial condition $(u_1(0), u_2(0))$.

Let

$$C_{\mathrm{unif}}^b(\mathbb{R}) = \{u \in C(\mathbb{R}) \mid u \text{ is uniformly continuous and bounded on } \mathbb{R}\}$$

with norm $\|u\|_\infty = \sup_{x \in \mathbb{R}} |u(x)|$. The above results also hold for the solutions $(u_1(t, x), u_2(t, x))$ of (1.3)
with initial functions $u_i(0, \cdot) \in C_{\mathrm{unif}}^b(\mathbb{R})$, $\inf_{x \in \mathbb{R}} u_i(0, x) > 0$ $(i = 1, 2)$. For example, in the case (**A2**), if
$(u_1(t, x), u_2(t, x))$ is a classical solution of (1.3) with $u_i(0, \cdot) \in C_{\mathrm{unif}}^b(\mathbb{R})$, $\inf_{x \in \mathbb{R}} u_i(0, x) > 0$ $(i = 1, 2)$, then

$$\lim_{t \to \infty} (u_1(t, x), u_2(t, x)) = \mathbf{e}_1$$

uniformly in $x \in \mathbb{R}$.

Consider (1.3). It is also interesting to know the asymptotic behavior of the solutions $(u_1(t, x), u_2(t, x))$
with front like initial functions $(u_1(0, x), u_2(0, x))$, that is, initial functions connecting two equilibrium
solutions of (1.4). This is strongly related to the so called traveling wave solutions. A *traveling wave solution
of* (1.3) is a classical solution of the form $(u_1(t, x), u_2(t, x)) = (U_1(x - ct), U_2(x - ct))$ for some constant $c \in \mathbb{R}$,
which is called the *speed of the traveling wave*. A traveling wave solution is said to *connect an equilibrium
solution* $\mathbf{e}_+$ *of* (1.4) *at the right end* if

$$\lim_{x \to \infty} (U_1(x), U_2(x)) = \mathbf{e}_+. \tag{1.5}$$

It is said to *connect an equilibrium solution* $\mathbf{e}_-$ *of* (1.4) *at the left end* if

$$\lim_{x \to -\infty} (U_1(x), U_2(x)) = \mathbf{e}_-. \tag{1.6}$$

The existence of traveling wave solutions of (1.3) has been extensively studied (see [12–16,20–25], etc.). For
example, assume $0 < a < 1$. It is well known that there is a minimum wave speed $c_{\min} \geq c_0^* := 2\sqrt{1 - a}$ such
that (1.3) has a monotone traveling wave solution with speed c connecting the equilibrium solutions $\mathbf{e}$ and
$\mathbf{e}_2$ of (1.4) (at the left end and right end, respectively) if and only if $c \geq c_{\min}$, where $\mathbf{e} = \mathbf{e}_1$ in the case (**A2**)
and $\mathbf{e} = \mathbf{e}_*$ in the case (**A3**). There is no explicit formula available for $c_{\min}$ in the literature and it is known
(see [26]) that it is possible to have the strict inequality $c_{\min} > c_0^*(= 2\sqrt{1 - a})$. When $c_{\min} = c_0^*$, it is said
that the minimum wave speed is *linearly determinate*. The works [15,27,28] provide sufficient conditions on
the parameters to ensure that $c_{\min}$ is linearly determinate. For example, it is proved in [15, Theorem 2.1]
that $c_{\min} = c_0^*$ provided the following (**A5**) holds.

(**A5**) $0 < d \leq 2$ and $(ab - 1)r \leq (1 - a)(2 - d)$.

From application point of view, it is important to know whether the presence of the chemotactic substance
have some effect on the invasion process of a weaker competitor by a stronger one. In particular, it is
interesting to know if movement toward some chemoattractant by both species can slow down or speed
up the invasion speed of the weaker competitor by the stronger competitor. This question is closely related
to whether minimum wave speed, if exists, of the chemotaxis system (1.2) is greater or smaller that the

minimum wave speed of the classical Lotka–Volterra system (1.3). Hence the objective of the current work is two folds. First, we would like to investigate how far the traveling wave theory for (1.3) can be extended to the two species chemotaxis system (1.2). Hence we hope to fill a gap in the literature concerning the traveling wave theory for two species driven by a chemotaxis. Second, we would also like to understand how the chemotaxis affects the minimum wave speed. It is clear that the space-independent solutions of the chemotaxis system (1.2) are the solutions of the ODE system (1.4). Note that case (**A1**) and case (**A2**) can be handled similarly. In the following, we focus on case (**A2**) and case (**A3**), and investigate the existence of traveling wave solutions of (1.2) connecting the unstable equilibrium $\mathbf{e}_2$ of (1.4) at the right end and the stable equilibrium $\mathbf{e}$ of (1.4) at the left end, where $\mathbf{e} = \mathbf{e}_1$ in the case (**A2**) and $\mathbf{e} = \mathbf{e}_*$ in the case (**A3**) (see the following subsection for the definition of traveling wave solutions of (1.2)).

1.1. Definition of traveling wave solutions of (1.2)

Similarly to (1.3), a *traveling wave solution of* (1.2) is a classical solution of the form $(u_1(t,x), u_2(t,x),$ $v(t,x)) = (U_1(x-ct), U_2(x-ct), V(x-ct))$ for some constant $c \in \mathbb{R}$, which is called the *speed of the traveling wave*. A traveling wave solution $(u_1(t,x), u_2(t,x), v(t,x)) = (U_1(x-ct), U_2(x-ct), V(x-ct))$ of (1.2) is said to *connect* $\mathbf{e}_2$ *at the right end* if

$$\lim_{x \to \infty} (U_1(x), U_2(x)) = \mathbf{e}_2, \tag{1.7}$$

and to *connect* $\mathbf{e}$ *at the left end* if

$$\lim_{x \to -\infty} (U_1(x), U_2(x)) = \mathbf{e}. \tag{1.8}$$

A traveling wave solution $(u_1(t,x), u_2(t,x), v(t,x)) = (U_1(x-ct), U_2(x-ct), V(x-ct))$ of (1.2) connecting $\mathbf{e}_2$ at the right end is *nontrivial* if $U_1(x) > 0$ for $x \in \mathbb{R}$.

As far as the chemotaxis model (1.2) is concerned, very little is known about the existence of traveling wave solutions. There are some recent works on the existence and non-existence of traveling wave solutions and spreading speeds of the single species chemotaxis model. In this regard, we refer the reader to the works in [29–34] and references therein.

We note that $(u_1(t,x), u_2(t,x), v(t,x)) = (U_1(x-ct), U_2(x-ct), V(x-ct))$ is a traveling wave solution of (1.2) connecting $\mathbf{e_2}$ at the right end and $\mathbf{e}$ at the left end if and only if $(U_1(x), U_2(x), V(x))$ is a steady state solution of the following system

$$\begin{cases} u_{1,t} = u_{1,xx} + (c - \chi_1 v_x)u_{1,x} + u_1(1 - \lambda\chi_1 v - (1 - \chi_1\mu_1)u_1 - (a - \chi_1\mu_2)u_2) \\ u_{2,t} = du_{2,xx} + (c - \chi_2 v_x)u_{2,x} + ru_2(1 - \frac{\lambda\chi_2}{r}v - (1 - \frac{\chi_2\mu_2}{r})u_2 - (b - \frac{\chi_2\mu_1}{r})u_1) \\ 0 = v_{xx} - \lambda v + \mu_1 u_1 + \mu_2 u_2 \end{cases} \tag{1.9}$$

complemented with the boundary conditions

$$(u_1(-\infty), u_2(-\infty)) = \mathbf{e} \quad \text{and} \quad (u_1(\infty), u_2(\infty)) = \mathbf{e_2}. \tag{1.10}$$

Considering the densely defined closed and linear operator induced by the Laplace operator on $C^b_{\text{unif}}(\mathbb{R})$, that is $\Delta : C^{2,b}_{\text{unif}}(\mathbb{R}) \ni u \mapsto \Delta u \in C^b_{\text{unif}}(\mathbb{R})$, it is well known that its spectrum is contained in the left half plane $\{z \in \mathbb{C} : \text{Re}(z) \leq 0\}$. Hence the operator $\lambda I - \Delta : C^{2,b}_{\text{unif}}(\mathbb{R}) \to C^b_{\text{unif}}(\mathbb{R})$ is invertible for every $\lambda > 0$. Hence for any solution $(u_1(t,x), u_2(t,x), v(t,x))$ (respectively steady state $(U_1(x), U_2(x), V(x))$) of (1.2) (respectively (1.9)), the third component $v(t,x)$ (respectively $V(x)$) is uniquely determined by the first two components. Hence for the sake of simplicity in the notations, we write $\mathbf{u} = (u_1, u_2)$ and $\mathbf{U} = (U_1, U_2)$ for vectors and denote by $\mathbf{u}(t,x) = \mathbf{U}^c(x-ct)$ traveling wave solutions of (1.2) with speed $c \in \mathbb{R}$. In the following, we always assume that $u_i(0, \cdot) \in C^b_{\text{unif}}(\mathbb{R})$ with $\inf_{x \in \mathbb{R}} u_i(0, x) \geq 0$ for $i = 1, 2$.

1.2. Standing assumptions and notations

To state the main results of this paper, we introduce some standing assumptions and notations in this subsection. It is well known that the solutions $(u_1(t,x), u_2(t,x))$ of (1.3) with initials $u_i(0,\cdot) \in C^b_{\text{unif}}(\mathbb{R})$ and $0 \leq u_i(0,\cdot)$ $(i = 1,2)$ exist for all $t \geq 0$ and are bounded. The first standing assumption is on the global existence and boundedness of classical solutions of (1.2).

(H1) $1 > \chi_1\mu_1$, $r > \chi_2\mu_2$, $a \geq \chi_1\mu_2$, $br \geq \chi_2\mu_1$.

Note that, when $\chi_1 = 0$ and $\chi_2 = 0$, **(H1)** always holds. Note also that (1.2) can be written as

$$\begin{cases} u_{1,t} = u_{1,xx} - \chi_1 v_x u_{1,x} + u_1(1 - \lambda\chi_1 v - (1 - \chi_1\mu_1)u_1 - (a - \chi_1\mu_2)u_2) \\ u_{2,t} = du_{2,xx} - \chi_2 v_x u_{2,x} + ru_2(1 - \frac{\lambda\chi_2}{r}v - (1 - \frac{\chi_2\mu_2}{r})u_2 - (b - \frac{\chi_2\mu_1}{r})u_1) \\ 0 = v_{xx} - \lambda v + \mu_1 u_1 + \mu_2 u_2. \end{cases} \quad (1.11)$$

The assumption **(H1)** implies that the solutions $(u_1(t,x), u_2(t,x))$ of (1.2) with initials $u_i(0,\cdot) \in C^b_{\text{unif}}(\mathbb{R})$ and $0 \leq u_i(0,\cdot)$ $(i = 1,2)$ exist for all $t > 0$ and, moreover, if $0 \leq u_i(0,\cdot) \leq M_i$ $(i = 1,2)$, then $0 \leq u_i(t,\cdot) \leq M_i$ $(i = 1,2)$ for all $t \geq 0$, where

$$M_1 = \frac{1}{1 - \chi_1\mu_1} \quad \text{and} \quad M_2 := \frac{r}{r - \chi_2\mu_2}. \quad (1.12)$$

We point out that the existence of bounded classical solutions of (1.11) for arbitrary choice of positive parameters of the PDE system still remains an interesting problem. Hypothesis **(H1)**, which might not be a necessary condition, is sufficient to ensure that all positive solutions of (1.11) are uniformly bounded in t as well as for the existence of an explicit invariant rectangle. Throughout the rest of this paper, we assume that **(H1)** holds, and that M_1, M_2 are the constants defined in (1.12), and

$$c_0^* = 2\sqrt{1 - a}. \quad (1.13)$$

As mentioned in the above, in the case **(A2)**, that is, $0 < a < 1 < b$, $\mathbf{e_1}$ is a stable equilibrium solution of (1.3). The second standing assumption is on the stability of $\mathbf{e_1}$ for (1.2).

(H2) $1 > \chi_1\mu_1 M_1 + aM_2$ and $b \geq 1$.

Note that **(H2)** implies that $a < 1$ and $b \geq 1$, and when $\chi_1 = 0$ and $\chi_2 = 0$, **(H2)** reduces to $a < 1$ and $b \geq 1$. We will prove that **(H2)** implies that $\mathbf{e_1}$ is a stable solution of (1.2) (see Theorem 1.1(*i*)).

It is known that, in the case **(A3)**, that is, $0 < a, b < 1$, $\mathbf{e_*}$ is a stable equilibrium solution of (1.3). The third standing assumption is on the stability of $\mathbf{e_*}$ for (1.2).

(H3) $1 > \chi_1\mu_1 M_1 + aM_2$ and $r > brM_1 + \chi_2\mu_2 M_2$.

Note that **(H3)** implies that $a < 1$ and $b < 1$, and when $\chi_1 = 0$ and $\chi_2 = 0$, **(H3)** reduces to $a < 1$ and $b < 1$. We will prove that **(H3)** implies that $\mathbf{e_*}$ is a stable solution of (1.2) (see Theorem 1.1(*ii*)).

Consider (1.3). In the case **(A2)**, it has traveling wave solutions connecting $\mathbf{e_2}$ at the right end and connecting $\mathbf{e_1}$ at the left end. In the case **(A3)**, it has traveling wave solutions connecting $\mathbf{e_2}$ at the right end and connecting $\mathbf{e_*}$ at the left end. The next standing assumption is on the existence of traveling wave solutions of (1.2).

(H4) $(1 - a) > r(M_1 a(b - \frac{\chi_2\mu_1}{r}) - \frac{1}{M_2})_+ + \chi_2(\mu_1 M_1 + \mu_2 M_2)$.

Note that, when $\chi_1 = 0$ and $\chi_2 = 0$, **(H4)** becomes $(1 - a) > r(ab - 1)_+$. We will prove that **(H2)**+**(H4)** (resp. **(H3)**+**(H4)**) implies the existence of traveling wave solutions of (1.2) connecting $\mathbf{e_2}$ at the right end and connecting $\mathbf{e_1}$ at the left end (resp. connecting $\mathbf{e_2}$ at the right end and connecting $\mathbf{e_*}$ at the left end) for speed c greater than some number $c^*(\chi_1, \chi_2)(\geq c_0^*)$ (see Theorem 1.2).

As it is mentioned in the above, when **(A5)** holds, the minimal wave speed $c_{\min}$ of (1.3) is linearly determinate, that is, $c_{\min} = c_0^*$. The last standing assumption is on the existence and linear determinacy of the minimal wave speed of (1.2).

(H5)
$$
\begin{cases}
(1-a)(1+\chi_1\mu_1 M_1) < \lambda \\
(1-a)(1-(d-1)_+) \geq r\left(M_1\left(a + \frac{(1-a)\left(\mu_2 M_2 + a\mu_1 M_1 + (1-a)\mu_2/\sqrt{\lambda}\right)}{\lambda-(1-a)(1+\chi_1\mu_1 M_1)}\right)(b - \frac{\chi_2\mu_1}{r}) - \frac{1}{M_2}\right)_+ \\
\quad + \chi_2\left(\frac{\lambda}{\lambda-(1-a)} + \frac{(\sqrt{1-a})}{\sqrt{\lambda}}\right)(\mu_1 M_1 + \mu_2 M_2).
\end{cases}
$$

It is clear that **(H5)** implies **(H4)**. When $\chi_1 = \chi_2 = 0$, the first two equations in (1.2) are independent of λ (hence λ can be chosen large enough such that $1 - a < \lambda$) and **(H5)** reduces to

$$
r(ab-1)_+ \leq (1-a)(1-(d-1)_+), \tag{1.14}
$$

which holds trivially if $d < 2$ and $ab \leq 1$. It will be shown that **(H2)**+ **(H5)** (resp. **(H3)**+**(H5)**) implies the existence and linear determinacy of the minimal wave speed of (1.2) connecting $\mathbf{e}_2$ at the right end and $\mathbf{e}_1$ at the left end (resp. connecting $\mathbf{e}_2$ at the right end and $\mathbf{e}_*$ at the left end) (see Theorem 1.4).

In the following, we introduce some standing notations. For every $0 < \kappa < \kappa_{\max} := \min\{\sqrt{1-a}, \sqrt{\lambda}\}$, let

$$
c_\kappa = \frac{\kappa^2 + 1 - a}{\kappa} \tag{1.15}
$$

and

$$
B_{\lambda,\kappa} = \int_{\mathbb{R}} e^{-\sqrt{\lambda}|z|-\kappa z} dz = \frac{1}{\sqrt{\lambda} - \kappa} + \frac{1}{\sqrt{\lambda} + \kappa} = \frac{2\sqrt{\lambda}}{\lambda - \kappa^2}. \tag{1.16}
$$

It is clear that the maps $(0, \sqrt{\lambda}) \ni \kappa \mapsto B_{\lambda,\kappa}$ and $(0, \sqrt{\lambda}) \ni \kappa \mapsto \kappa B_{\lambda,\kappa}$ are monotone increasing. We define $\kappa_1^*(\chi_1, \chi_2)$ by

$$
\kappa_1^*(\chi_1, \chi_2) := \sup\left\{\kappa \in (0, \kappa_{\max}) \ : \ \kappa B_{\lambda,\kappa} < \frac{2}{\chi_1\mu_1 M_1}\right\}. \tag{1.17}
$$

For every $\kappa \in (0, \kappa_1^*(\chi_1, \chi_2))$ we let $f(\kappa, \chi_1, \chi_2)$ denote the positive solution of the algebraic equation

$$
\frac{\kappa B_{\lambda,\kappa}}{2}\left(\mu_2 M_2 + \mu_1 M_1(a + \chi_1 f)\right) + \frac{\mu_2 \kappa^2 B_{\lambda,\kappa}}{2\sqrt{\lambda}} = f \tag{1.18}
$$

and define the function F by

$$
F(\kappa, \chi_1, \chi_2) = r\left(M_1(a + \chi_1 f(\kappa, \chi_1, \chi_2))(b - \frac{\chi_2\mu_1}{r}) - \frac{1}{M_2}\right)_+
$$
$$
+ \frac{\chi_2(\lambda B_{\lambda,\kappa} + 2\kappa)(\mu_1 M_1 + \mu_2 M_2)}{2\sqrt{\lambda}} - (1-d)\kappa^2. \tag{1.19}
$$

Let

$$
\kappa_{\chi_1,\chi_2}^* := \sup\left\{\tilde{\kappa} \in (0, \kappa_1^*(\chi_1, \chi_2)) \ : \ 1 - a \geq F(\kappa, \chi_1, \chi_2), \ \forall 0 < \kappa < \tilde{\kappa}\right\} \tag{1.20}
$$

and

$$
c^* = \frac{(\kappa_{\chi_1,\chi_2}^*)^2 + 1 - a}{\kappa_{\chi_1,\chi_2}^*}. \tag{1.21}
$$

Observe that

$$
F(0, \chi_1, \chi_2) = r\left(M_1 a(b - \frac{\chi_2\mu_1}{r}) - \frac{1}{M_2}\right)_+ + \chi_2(\mu_1 M_1 + \mu_2 M_2).
$$

It is clear that $\kappa_1^*(\chi_1, \chi_2) > 0$ and is well defined. If hypothesis **(H4)** holds, then κ_{χ_1,χ_2}^* and c^* are well defined as well.

1.3. Statements of the main results

In this subsection, we state the main results of this paper. The first main result is on the stability of spatially homogeneous equilibrium solutions of (1.2) and is stated in the following theorem.

Theorem 1.1. *For given $c \in \mathbb{R}$, let $\mathbf{u}(t, x; c)$ be a classical solution of (1.9) satisfying $\inf_{x \in \mathbb{R}} u_1(0, x; c) > 0$. Then the following hold.*

*(i) If (**H2**) holds, then*

$$\lim_{t \to \infty} \|\mathbf{u}(t, \cdot; c) - \mathbf{e}_1\|_\infty = 0.$$

*(ii) If (**H3**) holds, and $\inf_{x \in \mathbb{R}} u_2(0, x; c) > 0$, then*

$$\lim_{t \to \infty} \|\mathbf{u}(t, \cdot; c) - \mathbf{e}_*\|_\infty = 0.$$

We have the following two theorems on the existence and nonexistence of traveling wave solutions of (1.2).

Theorem 1.2. *Suppose that (**H4**) holds. Then for every $c > c^*$, (1.2) has a nontrivial traveling solution $\mathbf{u}(t, x) = \mathbf{U}^c(x - ct) = (U_1^c(x - ct), U_2^c(x - ct))$ with speed c connecting $\mathbf{e}_2$ at right end and satisfying that*

$$\lim_{x \to \infty} \left| \frac{U_1^c(x)}{e^{-\kappa x}} - 1 \right| = 0 \quad and \quad \lim_{x \to \infty} \left| \frac{U_2^c(x) - 1}{e^{-\kappa x}} - \frac{(\chi_2 \mu_2 - rb)}{(1 - a) + \frac{r}{M_2} - (d - 1)\kappa^2 - \frac{\chi_2 \mu_2 \sqrt{\lambda}}{2} B_{\lambda, \kappa}} \right| = 0, \quad (1.22)$$

where $\kappa \in (0, \kappa^*_{\chi_1, \chi_2})$ satisfies $c = c_\kappa$. Moreover, the following hold.

  *(i) If (**H2**) holds, then the traveling wave solution $\mathbf{u}(t, x) = \mathbf{U}^c(x - ct) = (U_1^c(x - ct), U_2^c(x - ct))$ of (1.2) also connects $\mathbf{e}_1$ at the left end.*
  *(ii) If (**H3**) holds, then the traveling wave solution $\mathbf{u}(t, x) = \mathbf{U}^c(x - ct) = (U_1^c(x - ct), U_2^c(x - ct))$ of (1.2) also connects $\mathbf{e}_*$ at the left end.*

Theorem 1.3. *For any choice of the positive parameters χ_i and μ_i, $i = 1, 2$, there is no nontrivial traveling wave solution $\mathbf{u}(t, x) = \mathbf{U}^c(x - ct)$ of (1.2) with speed $c < c_0^*$ and connecting $\mathbf{e_2}$ at the right end.*

Observe that Theorem 1.3 provides a lower bound $c_0^*(= 2\sqrt{1 - a})$ for the speeds of traveling wave solutions of (1.2). This lower bound is independent of the chemotaxis sensitivity coefficients χ_1 and χ_2. The following theorem shows that this is the greatest lower bound for the speeds of traveling wave solutions of (1.2).

Theorem 1.4.

  *(i) If (**H2**) and (**H5**) hold, then for every $c > c_0^*$, there is a traveling wave solution of (1.2) with speed c connecting $\mathbf{e}_1$ and $\mathbf{e}_2$. If, in addition, $r > 2\chi_2 \mu_2$ then there is a traveling wave solution of (1.2) with speed $c = c_0^*$ connecting $\mathbf{e}_1$ and $\mathbf{e}_2$.*
  *(ii) If (**H3**) and (**H5**) hold, then for any $c \geq c_0^*$, there is a traveling wave solution of (1.2) with speed c connecting $\mathbf{e}_*$ and $\mathbf{e}_2$.*

We remark that under the conditions of Theorem 1.4, the minimum wave speed of (1.2) exists, is linearly determinate, and is not affected by the chemotactic effect. In general, it is not known whether (1.2) has a minimal wave speed, and if so, whether both systems (1.2) and (1.3) have the same minimum wave speed. This question is related to whether the presence of the chemical substance slows down or speeds up the minimum wave speed. Note that Theorem 1.3 shows that the presence of the chemical substance does not

slow down the minimum wave speed of (1.3) whenever it is linearly determinate. Another interesting question that is left open by the current work is whether the traveling wave solutions of (1.2) constructed are stable and unique (up to space translation). This question also remains open for the traveling solutions of the single species chemotaxis model. Recently Shen and Xue [35] present numerical simulations which suggest that the single species traveling waves are unique and stable. But a rigorous theoretical proof is still not yet available to support their work. We plan to continue working on this problem in our future works. We see from (1.14) that Theorem 1.4 recovers [15, Theorem 2.1], which guarantees that the minimum wave speed of (1.3) is linearly determinate under hypothesis (1.14).

We also remark that there are also several interesting works on the dynamics of solutions to (1.2) when considered on bounded domains, see [6–8,10,36–41] and the references therein. For example, the works in [6,10,36] studied the stability of the equilibria of (1.2) on bounded domains with Neumann boundary conditions, while the works [7,37] considered (1.2) on bounded domains with some non-local term in the reaction terms.

The rest of the paper is organized as follows. In Section 2 we present the proof of Theorem 1.1. In Section 3 we construct some super and sub-solutions to be used in the proof of Theorem 1.2. Section 4 is devoted to the proof of Theorem 1.2. In Section 5 we present the proof of Theorem 1.3. The proof of Theorem 1.4 is presented in Section 6.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Throughout this section $c \in \mathbb{R}$ is an arbitrary fixed number. We first prove two lemmas, which are fundamental for the proofs of most of our results in the subsequent sections.

Lemma 2.1. *Suppose that* (**H2**) *holds and let* $\mathbf{u}(t, x; c)$ *be a bounded classical solution of* (1.9) *defined for every* $x \in \mathbb{R}$ *and* $t \in \mathbb{R}$. *If* $\inf_{x,t \in \mathbb{R}} u_1(t, x; c) > 0$, *then* $\mathbf{u}(t, x; c) \equiv \mathbf{e}_1$.

Proof. Following the idea in the proof of [42, Theorem 1.8], we introduce

$$l_1 = \inf_{t,x \in \mathbb{R}} u_1(t, x; c), \quad L_1 = \sup_{t,x \in \mathbb{R}} u_1(t, x; c), \quad l_2 = \inf_{t,x \in \mathbb{R}} u_2(t, x; c) \quad \text{and} \quad L_2 = \sup_{t,x \in \mathbb{R}} u_2(t, x; c), \quad (2.1)$$

and prove that $l_1 = L_1 = 1$ and $l_2 = L_2 = 0$. Some new arguments are needed in the proof due to the presence of a second species. For the completeness, we provide a proof in the following.

Let M_1 and M_2 be as in (1.12). Observe from the comparison principle for elliptic equations that $\mu_1 l_1 + \mu_2 l_2 \le \lambda v(x; \mathbf{u}(\cdot, \cdot; c)) \le \mu_1 L_1 + \mu_2 L_2$. Hence

$$u_{1,t} = u_{1,xx} + (c - \chi_1 v_x(\cdot; \mathbf{u}))u_{1,x} + (1 - (1 - \chi_1\mu_1)u_1 - (a - \chi_1\mu_2)u_2 - \chi_1\lambda v(\cdot; \mathbf{u}))u_1$$
$$\le u_{1,xx} + (c - \chi_1 v_x(\cdot; \mathbf{u}))u_{1,x} + (1 - \chi_1\mu_1 l_1 - (1 - \chi_1\mu_1)u_1)u_1.$$

Then by the comparison principle for parabolic equations, we have

$$u_1(t, x; c) \le M_1(1 - \chi_1\mu_1 l_1)_+, \quad \forall\, t, x \in \mathbb{R}.$$

This implies that $1 > \chi_1\mu_1 l_1$ and that

$$L_1 \le M_1(1 - \chi_1\mu_1 l_1). \quad (2.2)$$

Similarly, observe that

$$u_{1,t} \ge u_{1,xx} + (c - \chi_1 v_x(\cdot; \mathbf{u}))u_{1,x} + (1 - \chi_1\mu_1 L_1 - aL_2 - (1 - \chi_1\mu_1)u_1)u_1.$$

Thus, as in the above, we conclude from the comparison principle for parabolic equations that

$$l_1 \geq M_1(1 - \chi_1\mu_1L_1 - aL_2). \tag{2.3}$$

Observe also that

$$u_{2,t} = du_{2,xx} + (c - \chi_2 v_x(\cdot; \mathbf{u}))u_{2,x} + (r - \frac{r}{M_2}u_2 - (br - \chi_2\mu_1)u_1 - \chi_2\lambda v(\cdot; u))u_2$$

$$\leq du_{2,xx} + (c - \chi_2 v_x(\cdot; \mathbf{u}))u_{2,x} + r(1 - bl_1 - \frac{1}{M_2}u_2)u_2.$$

We conclude from the comparison principle for parabolic equations that

$$L_2 \leq M_2(1 - bl_1)_+. \tag{2.4}$$

From this point, we distinguish two cases and show that $L_1 = l_1 = 1$ and $L_2 = 0$.

Case 1. $(1 - bl_1)_+ = 0$. In this case, by (2.4), $L_2 = 0$. Thus, taking the difference of (2.2) and (2.3) side-by-side yields

$$(1 - \chi_1\mu_1 M_1)(L_1 - l_1) \leq 0.$$

Since $1 > \chi_1\mu_1 M_1$ (see **(H2)**) and $l_1 \leq L_1$, we obtain that $L_1 = l_1$, which combined with (2.2) and (2.3) and the fact $L_2 = 0$ yield $l_1 = L_1 = 1$.

Case 2. $1 - bl_1 > 0$. In this case, it follows from (2.2)–(2.4) that

$$l_1 \geq M_1 \left(1 - \chi_1\mu_1 M_1(1 - \chi_1\mu_1 l_1) - aM_2(1 - bl_1)\right) = M_1(1 - \chi_1\mu_1 M_1 - aM_2) + M_1((\chi_1\mu_1)^2 M_1 + abM_2)l_1.$$

That is

$$\left(\frac{1}{M_1} - (\chi_1\mu_1)^2 M_1 - abM_2\right) l_1 \geq 1 - \chi_1\mu_1 M_1 - aM_2.$$

Using the fact that $\chi_1\mu_1 + \frac{1}{M_1} = 1$, a simple computation shows that $\frac{1}{M_1} - (\chi_1\mu_1)^2 M_1 = 1 - \chi_1\mu_1 M_1$, and hence

$$(1 - \chi_1\mu_1 M_1 - abM_2)l_1 \geq 1 - \chi_1\mu_1 M_1 - aM_2.$$

This implies that $l_1 \geq 1$ since $b \geq 1$ (since **(H2)** holds). Thus, we get from (2.4) that $L_2 = 0$, so we can proceed as in **case 1** to show that $l_1 = L_1 = 1$ as well.

From both cases and the definition of $l_1, L_1,$ and L_2, we obtain $\mathbf{u}(t, x; c) \equiv (1, 0)$, which completes the proof of the lemma. $\quad\square$

Lemma 2.2. *Suppose that $0 < b < 1$ and let $\mathbf{u}(t, x; c)$ be a bounded classical solution of (1.9) defined for every $x \in \mathbb{R}$ and $t \in \mathbb{R}$ such that $\min\{\inf_{x,t\in\mathbb{R}} u_2(t, x; c), \inf_{x,t\in\mathbb{R}} u_2(t, x)\} > 0$. If there holds*

$$(1 - \chi_1\mu_1 M_1)_+(r - \chi_2\mu_2 M_2)_+ > abr M_1 M_2, \tag{2.5}$$

*then $\mathbf{u}(t, x; c) \equiv \mathbf{e}_*$. In particular if **(H3)** holds then $\mathbf{u}(t, x; c) \equiv \mathbf{e}_*$.*

Proof. Similarly, following the idea in the proof of [42, Theorem 1.8], we introduce l_i and L_i, $i = 1, 2$, as in (2.1) and prove that $l_1 = L_1 = \frac{1-a}{1-ab}$ and $l_2 = L_2 = \frac{1-b}{1-ab}$. Again, for the completeness, we provide a proof in the following.

First, note that **(H3)** implies that

$$1 - \chi_1\mu_1 M_1 > aM_2 \quad \text{and} \quad r - \chi_2\mu_2 M_2 > br M_1,$$

which implies (2.5). It then suffices to prove $\mathbf{u}(t, x; c) \equiv \mathbf{e}_*$ provided that (2.5) holds.

Second, note that $\mu_1 l_1 + \mu_2 l_2 \le \lambda v(x; u) \le \mu_1 L_1 + \mu_2 L_2$ for every $x \in \mathbb{R}$. We can proceed as in the proof of Lemma 2.1 by using comparison principle for parabolic equations and the fact that $\min\{l_1, l_2, L_1, L_2\} \ge 0$ to obtain the following inequalities:

$$L_1 \le M_1(1 - \chi_1 \mu_1 l_1 - al_2), \tag{2.6}$$

$$l_1 \ge M_1(1 - \chi_1 \mu_1 L_1 - aL_2), \tag{2.7}$$

$$rL_2 \le M_2(r - \chi_2 \mu_2 l_2 - brl_1), \tag{2.8}$$

$$rl_2 \ge M_2(r - \chi_2 \mu_2 L_2 - brL_1). \tag{2.9}$$

Taking difference side-by-sides of inequalities (2.6) and (2.7) yields

$$(1 - \chi_1 \mu_1 M_1)(L_1 - l_1) \le aM_1(L_2 - l_2). \tag{2.10}$$

Similarly, it follows from inequalities (2.8) and (2.9) that

$$(r - \chi_2 \mu_2 M_2)(L_2 - l_2) \le rbM_2(L_1 - l_1). \tag{2.11}$$

The last two inequalities imply that

$$(1 - \chi_1 \mu_1 M_1)(r - \chi_2 \mu_2 M_2)(L_1 - l_1)(L_2 - l_2) \le abr M_1 M_2(L_1 - l_1)(L_2 - l_2)$$

since (2.5) implies that $1 - \chi_1 \mu_1 M_1 > 0$ and $r - \chi_2 \mu_2 M_2 > 0$. Since (2.5) holds, we must have from the above inequality that $(L_1 - l_1)(L_2 - l_2) = 0$, which combined with (2.10) and (2.11) yields $l_1 = L_1$ and $l_2 = L_2$. Therefore, recalling the identities $(1 - \chi_1 \mu_1)M_1 = 1$ and $(r - \chi_2 \mu_2)M_2 = r$, it follows from (2.6)–(2.9) that

$$\begin{cases} L_1 = 1 - aL_2 \\ L_2 = 1 - bL_1. \end{cases}$$

It then follows that $l_1 = L_1 = \frac{1-a}{1-ab}$ and $l_2 = L_2 = \frac{1-b}{1-ab}$. That is $\mathbf{u}(t, x; c) \equiv e_*$, which completes the proof of the lemma. $\square$

Now, we prove Theorem 1.1.

Proof of Theorem 1.1. Let $c \in \mathbb{R}$ be given and $\mathbf{u}(t, x; c)$ be a classical solution of (1.9) satisfying $\inf_{x \in \mathbb{R}} u_1(0, x; c) > 0$. It is easy to see that

$$\limsup_{t \to \infty} \|u_i(t, \cdot; c)\|_\infty \le M_i \quad \forall i = 1, 2.$$

Hence without loss of generality we may suppose that $\|u_i(t, \cdot; c)\|_\infty \le M_i$ for every $t \ge 0$ and $i = 1, 2$.

(*i*) Suppose that hypothesis (**H2**) holds. Observe that $u_1(t, x; c)$ satisfies

$$u_{1,t} \ge u_{1,xx} + (c - \chi_1 v_x)u_{1,x} + u_1(1 - (\chi_1 \mu_1 M_1 + aM_2) - (1 - \chi_1 \mu_1)u_1), \quad \forall\, t > 0,\ x \in \mathbb{R}.$$

Hence, since (**H2**) holds, we may employ the comparison principle for parabolic equations to conclude that

$$u_1(t, x; c) \ge \underline{m}_1 := \min\left\{ \inf_{x \in \mathbb{R}} u_1(0, x; c), \frac{1 - (\chi_1 \mu_1 M_1 + aM_2)}{1 - \chi_1 \mu_1} \right\} > 0, \quad \forall\, t \ge 0,\ x \in \mathbb{R}. \tag{2.12}$$

Now, if we suppose by contradiction that the statement of Theorem 1.1 (*i*) is false, then there exist a sequence $t_n \to \infty$ and x_n such that

$$\inf_{n \ge 1} |\mathbf{u}(t_n, x_n; c) - \mathbf{e}_1| > 0. \tag{2.13}$$

By a priori estimates for parabolic equations, without loss of generality, we may suppose that there is some $\mathbf{u}^* \in C^{1,2}(\mathbb{R} \times \mathbb{R})$ such that $\mathbf{u}(t + t_n, x + x_n; c) \to \mathbf{u}^*(t, x)$ as $n \to \infty$ in $C^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{R})$. Note that $\mathbf{u}^*(t, x)$ is an entire solution of (1.9) and it follows from (2.12) that

$$u_1(t, x) \geq \underline{m}_1, \quad \forall\, t, x \in \mathbb{R}.$$

Thus, by Lemma 2.1, we conclude that $\mathbf{u}^*(t, x) \equiv \mathbf{e}_1$. This contradicts with (2.13) since $\mathbf{u}^*(0, 0) = \lim_{n\to\infty} \mathbf{u}(t_n, x_n; c)$. Hence $\lim_{t\to\infty} \|\mathbf{u}(t, \cdot; c) - \mathbf{e}_1\|_\infty = 0$, which completes the proof of (i).

(ii) Suppose that hypothesis (**H3**) holds. The proof follows similar arguments as in (i). Indeed, note that in addition to (2.12), it also holds that

$$u_{2,t} \geq d u_{2,xx} + (c - \chi_2 v_x) u_{2,x} + u_2(r - (\chi_2 \mu_2 M_2 + b r M_1) - (r - \chi_2 \mu_2) u_2), \quad \forall\, t > 0, \ x \in \mathbb{R}.$$

Hence, since (**H3**) holds, we may employ the comparison principle for parabolic equations to conclude that

$$u_2(t, x; c) \geq \underline{m}_2 := \min\left\{\inf_{x\in\mathbb{R}} u_2(0, x; c), \frac{r - (\chi_2 \mu_2 M_2 + b r M_1)}{r - \chi_2 \mu_2}\right\} > 0, \quad \forall\, t \geq 0, \ x \in \mathbb{R}. \tag{2.14}$$

Therefore, similar arguments used to prove (i) together with Lemma 2.2 yield that $\lim_{t\to\infty} \|\mathbf{u}(t, \cdot; c) - \mathbf{e}_*\|_\infty = 0$. This completes the proof of the theorem. $\quad\square$

3. Super- and sub-solutions

In this section, we construct some super- and sub-solutions to some elliptic equations related to (1.9). These super- and sub-solutions will be used in the proof of Theorem 1.2 in the next section.

We first introduce some notations. For fixed $0 < \kappa < \kappa_{\max} = \min\{\sqrt{1-a}, \sqrt{\lambda}\}$, $D_1, D_2, \tilde{D}_2 > 0$ and $0 < \varepsilon_1 \ll 1$, we define

$$\overline{u}_1^\kappa(x) = \min\{M_1, M_1 D_2 e^{-\kappa x}\}, \quad \underline{u}_1^\kappa(x) = M_1 D_2 \left(1 - D_1 e^{-\varepsilon_1 x}\right)_+ e^{-\kappa x},$$

$$\overline{u}_2^\kappa(x) = \min\{M_2, 1 + M_2 \tilde{D}_2 e^{-\kappa x}\} \quad \text{and} \quad \underline{u}_2^\kappa(x) = \left(1 - \tilde{D}_2 e^{-\kappa x}\right)_+,$$

where $(m)_+ = \max\{m, 0\}$ for every real number $m \in \mathbb{R}$, and M_1 and M_2 are as in (1.12), that is, $M_1 = \frac{1}{1 - \chi_1 \mu_1}$ and $M_2 = \frac{r}{r - \chi_2 \mu_2}$. We shall provide more information on how to choose the positive constants $D_1, D_2, \tilde{D}_2$ and ε_1 in Lemma 3.3 below. Define the convex set

$$\mathcal{E}(\kappa) := \{\mathbf{u} \in C^b_{\text{unif}}(\mathbb{R}) \times C^b_{\text{unif}}(\mathbb{R}) \mid \underline{u}_i^\kappa(x) \leq u_i(x) \leq \overline{u}_i^\kappa(x) \ \forall\, x \in \mathbb{R}, \ i = 1, 2\}. \tag{3.1}$$

Observe that for every $\mathbf{u} \in C^b_{\text{unif}}(\mathbb{R}) \times C^b_{\text{unif}}(\mathbb{R})$, the scalar valued function

$$v(x; \mathbf{u}) = \frac{1}{2\sqrt{\lambda}} \int_\mathbb{R} e^{-\sqrt{\lambda}|x-y|} (\mu_1 u_1(y) + \mu_2 u_2(y)) dy, \quad x \in \mathbb{R}, \tag{3.2}$$

is twice continuously differentiable and solves the elliptic equation

$$0 = v_{xx} - \lambda v_x + \mu_1 u_1 + \mu_2 u_2, \quad x \in \mathbb{R}.$$

Next, we present some lemmas.

Lemma 3.1. *Let $\mathbf{u} \in \mathcal{E}(\kappa)$ and $v(x; \mathbf{u})$ be given by (3.2). Then for any $x \in \mathbb{R}$,*

$$v(x; \mathbf{u}) \leq \mu_1 \min\left\{\frac{M_1}{\lambda}, \frac{M_1 D_2 B_{\lambda,\kappa}}{2\sqrt{\lambda}} e^{-\kappa x}\right\} + \mu_2 \min\left\{\frac{M_2}{\lambda}, \frac{1}{\lambda} + \frac{M_2 \tilde{D}_2 B_{\lambda,\kappa}}{2\sqrt{\lambda}} e^{-\kappa x}\right\} \tag{3.3}$$

and

$$v(x; \mathbf{u}) \geq \frac{\mu_1 M_1 D_2 e^{-\kappa x}}{2\sqrt{\lambda}} \left(B_{\lambda,\kappa} - D_1 B_{\lambda,\kappa+\varepsilon_1} e^{-\varepsilon_1 x}\right)_+ + \mu_2 \left(\frac{1}{\lambda} - \frac{\tilde{D}_2 B_{\lambda,\kappa}}{2\sqrt{\lambda}} e^{-\kappa x}\right)_+, \tag{3.4}$$

where $B_{\lambda,\kappa}$ is as in (1.16).

Lemma 3.2. *Let* $\mathbf{u} \in \mathcal{E}(\kappa)$ *and* $v(x; \mathbf{u})$ *be given by* (3.2). *Then*

$$|\frac{d}{dx}v(x; \mathbf{u})| \leq \mu_1 \min\{\frac{M_1 D_2 B_{\lambda,\kappa}}{2}e^{-\kappa x}, \frac{M_1}{\sqrt{\lambda}}\} + \mu_2 \min\{\frac{\tilde{D}_2 M_2 B_{\lambda,\kappa}}{2}e^{-\kappa x}, \frac{M_2}{\sqrt{\lambda}}\}, \quad \forall\, x \in \mathbb{R}, \tag{3.5}$$

where $B_{\lambda,\kappa}$ *is introduced in* (1.16).

We delay the proofs of Lemmas 3.1 and 3.2 to the Appendix. For every $\mathbf{u} \in \mathcal{E}(\kappa)$, we associate the differential operator $\mathcal{F}^{\mathbf{u}}(\mathbf{U}) = (\mathcal{F}_1^{\mathbf{u}}(U_1), \mathcal{F}_2^{\mathbf{u}}(U_2))$ defined by

$$\mathcal{F}_1^{\mathbf{u}}(U_1) = U_{1,xx} + (c_\kappa - \chi_1 v_x(\cdot; \mathbf{u}))U_{1,x} + (1 - (a - \chi_1\mu_2)u_2 - \lambda\chi_1 v(\cdot; \mathbf{u}) - \frac{(U_1)_+}{M_1})U_1 + R(u_1 - U_1), \tag{3.6}$$

$$\mathcal{F}_2^{\mathbf{u}}(U_2) = dU_{2,xx} + (c_\kappa - \chi_2 v_x(\cdot, \mathbf{u}))U_{2,x} + (r - (br - \chi_2\mu_1)u_1 - \lambda\chi_2 v(\cdot; \mathbf{u}) - \frac{r(U_2)_+}{M_2})U_2 + R(u_2 - U_2) \tag{3.7}$$

with $(U_i)_+ = \max\{U_i, 0\}$, $\mathbf{U} = (U_1, U_2)$, where $c_\kappa = \frac{\kappa^2+1-a}{\kappa}$ and $R \gg 1$ is a positive constant satisfying

$$R > \max\{1 - (a - \chi_1\mu_2)M_2 - \chi_1(\mu_1 M_1 + \mu_2 M_2), r - (br - \chi_2\mu_1)M_1 - \chi_2(\mu_1 M_1 + \mu_2 M_2)\}. \tag{3.8}$$

In the rest of this section, we assume that **(H4)** holds. Let $\kappa^*_{\chi_1,\chi_2}$ be defined as in (1.20). By the definition of $\kappa^*_{\chi_1,\chi_1}$, for every $0 < \kappa < \kappa^*_{\chi_1,\chi_2}$, it holds that

$$\frac{\kappa\chi_1\mu_1 M_1 B_{\lambda,\kappa}}{2} < 1 \tag{3.9}$$

and

$$1 - a \geq F(\kappa, \chi_1, \chi_2), \tag{3.10}$$

where $F(\kappa, \chi_1, \chi_2)$ is as in (1.19). Our aim is to prove Theorem 1.2 in the next section by showing that there is $\mathbf{u}^* \in \mathcal{E}(\kappa)$ such that $\mathcal{F}^{\mathbf{u}^*}(\mathbf{u}^*) = 0$ for $0 < \kappa < \kappa^* := \kappa^*_{\chi_1,\chi_2}$.

Observe that $\bar{u}_i^k$ and $\underline{u}_i^k$ are analogues of $U_\mu^+(\cdot)$ and $U_\mu^-(\cdot)$ in [33] (see (2.4) and (2.6) in [33]), and the following lemma is the analog of [33, Theporem 2.1]. Some new arguments are needed due to the presence of a second species. For the completeness, we provide a proof of the following lemma.

Lemma 3.3. *Let* $0 < \kappa < \kappa^*$, $0 < \varepsilon_1 < \min\{\kappa, c_\kappa - 2\kappa\}$, $D_2 = \frac{1}{M_1}$, *and* $\tilde{D}_2 = D_2/(a + \chi_1 f(\kappa, \chi_1, \chi_2))$, *where* $f(\kappa, \chi_1, \chi_2)$ *is the positive solution of the algebraic equation* (1.18). *Then there is* $D_1 \gg 1$ *such that for every* $\mathbf{u} \in \mathcal{E}(\kappa)$ *and* $i \in \{1, 2\}$ *the following hold.*

(i) $\mathcal{F}_i^{\mathbf{u}}(M_i) \leq 0$ *and* $\mathcal{F}_i^{\mathbf{u}}(0) \geq 0$ *for every* $x \in \mathbb{R}$.
(ii) $\mathcal{F}_i^{\mathbf{u}}(\bar{u}_i^\kappa) \leq 0$ *on the open interval* $\{\bar{u}_i^\kappa < M_i\}$.
(iii) $\mathcal{F}_i^{\mathbf{u}}(\underline{u}_i^\kappa) \geq 0$ *on the open interval* $\{\underline{u}_i^\kappa > 0\}$.

Proof. Let $\mathbf{u} \in \mathcal{E}(\kappa)$.
(i) Note that $\mathcal{F}_i^{\mathbf{u}}(0) = Ru_i(x) \geq 0$ for every $x \in \mathbb{R}$ and

$$\mathcal{F}_1^{\mathbf{u}}(M_1) = -M_1((a - \chi_1\mu_2)u_2(x) + \chi_1\lambda v(x; \mathbf{u})) - R(M_1 - u_1(x)) < 0, \quad \forall\, x \in \mathbb{R}$$

since $v(\cdot; \mathbf{u}) > 0$, $\|u_1\|_\infty \leq M_1$ and $u_2(\cdot) \geq 0$. Similarly

$$\mathcal{F}_2^{\mathbf{u}}(M_2) = -M_2((br - \chi_2\mu_1)u_1(x) + \chi_2\lambda v(x; \mathbf{u})) - R(M_2 - u_2(x)) < 0, \quad \forall\, x \in \mathbb{R}$$

since $v(\cdot; \mathbf{u}) \geq 0$, $\|u_2\|_\infty \leq M_2$ and $u_1(\cdot) \geq 0$.

(ii) Recalling inequalities (3.4) and (3.5), with $\overline{u}_1^\kappa = M_1 D_2 e^{-\kappa x}$, we get

$$
\begin{aligned}
\mathcal{F}_1^{\mathbf{u}}(\overline{u}_1^\kappa) &= \left((a + \kappa\chi_1 v_x(;\mathbf{u}) - D_2 e^{-\kappa x} - (a - \chi_1\mu_2)u_2 - \chi_1\lambda v(\cdot;\mathbf{u})) \right)\overline{u}_1^\kappa - R(\overline{u}_1^\kappa - u_1) \\
&\leq \left(a - \chi_1\mu_2 + \kappa\chi_1 v_x(;\mathbf{u}) - \tilde{D}_2\Big(\frac{D_2}{\tilde{D}_2} - \frac{D_2\chi_1\mu_2\sqrt{\lambda}B_{\lambda,\kappa}}{2}\Big)e^{-\kappa x} - (a - \chi_1\mu_2)(1 - \tilde{D}_2 e^{-\kappa x}) \right)\overline{u}_1^\kappa \\
&= \left(\kappa\chi_1 v_x(;\mathbf{u}) - \tilde{D}_2\Big(\frac{D_2}{\tilde{D}_2} - \frac{\chi_1\mu_2\sqrt{\lambda}B_{\lambda,\kappa}}{2} - (a - \chi_1\mu_2)\Big)e^{-\kappa x} \right)\overline{u}_1^\kappa \\
&\leq \tilde{D}_2\left(\frac{\chi_1\kappa B_{\lambda,\kappa}}{2}\Big(\frac{\mu_1 M_1 D_2}{\tilde{D}_2} + \mu_2 M_2\Big) - \Big(\frac{D_2}{\tilde{D}_2} - \frac{\chi_1\mu_2\sqrt{\lambda}B_{\lambda,\kappa}}{2} - (a - \chi_1\mu_2)\Big) \right)\overline{u}_1^\kappa e^{-\kappa x} \\
&= \chi_1\tilde{D}_2\Big(\frac{\kappa B_{\lambda,\kappa}}{2}\big(\mu_1 M_1(a + \chi_1 f) + \mu_2 M_2\big) + \frac{\mu_2\kappa^2 B_{\lambda,\kappa}}{2\sqrt{\lambda}} - f \Big)\overline{u}_1^\kappa e^{-\kappa x} = 0.
\end{aligned}
$$

Note that we have used (1.18). Similarly, for $\overline{u}_2^\kappa(x) = 1 + M_2 D_2 e^{-\kappa x} \leq M_2$, by Lemmas 3.1 and 3.2,

$$
\begin{aligned}
\mathcal{F}_2^{\mathbf{u}}(\overline{u}_2^\kappa) &= M_2(d\kappa^2 - \kappa(c_\kappa - \chi_2 v_x(\cdot;\mathbf{u})))\frac{\tilde{D}_2}{e^{\kappa x}} \\
&\quad + (r - \frac{r\overline{u}_2^\kappa}{M_2} - (br - \chi_2\mu_1)u_1 - \chi_2\lambda v(\cdot;\mathbf{u}))\overline{u}_2^\kappa + R(u_2 - \overline{u}_2^\kappa) \\
&\leq M_2(d\kappa^2 - \kappa(c_\kappa - \chi_2 v_x(\cdot;\mathbf{u})))\tilde{D}_2 e^{-\kappa x} \\
&\quad + \left(r - \frac{r}{M_2}(1 + M_2\tilde{D}_2 e^{-\kappa x}) - \chi_2\mu_2\Big(1 - \frac{\tilde{D}_2\sqrt{\lambda}B_{\lambda,\kappa}}{2e^{\kappa x}}\Big) \right)\overline{u}_2^\kappa \\
&= \tilde{D}_2\left((d\kappa^2 - \kappa(c_\kappa - \chi_2 v_x(\cdot;\mathbf{u})))M_2 - \Big(r - \frac{\chi_2\mu_2\sqrt{\lambda}B_{\lambda,\kappa}}{2}\Big)\overline{u}_2^\kappa \right)e^{-\kappa x} \\
&\leq \tilde{D}_2\left((d\kappa^2 - \kappa c_\kappa + \frac{\kappa\chi_2}{\sqrt{\lambda}}(\mu_1 M_1 + \mu_2 M_2))M_2 - \Big(r - \frac{\chi_2\mu_2\sqrt{\lambda}B_{\lambda,\kappa}}{2}\Big)\overline{u}_2^\kappa \right)e^{-\kappa x} \\
&\leq \frac{M_2\tilde{D}_2}{e^{\kappa x}}\left((d-1)\kappa^2 - (1-a) + \frac{\kappa\chi_2}{\sqrt{\lambda}}(\mu_1 M_1 + \mu_2 M_2) + \frac{\chi_2\mu_2\sqrt{\lambda}B_{\lambda,\kappa}}{2} - \frac{r}{M_2} \right) \leq 0,
\end{aligned}
$$

whenever (3.10) holds. This completes the proof of (ii).

(iii) For $x \in \{\underline{u}_2^\kappa > 0\}$ and recalling Lemmas 3.1 and 3.2, we obtain

$$
\begin{aligned}
&\mathcal{F}_2^{\mathbf{u}}(\underline{u}_2^\kappa) + R(\underline{u}_2^\kappa - u_2) \\
&= \left(r - \frac{r}{M_2}(1 - \tilde{D}_2 e^{-\kappa x}) - (br - \chi_2\mu_1)u_1 - \chi_2\lambda v(\cdot;\mathbf{u}) \right) \\
&\quad \times \underline{u}_2^\kappa + (1 - a + (1-d)\kappa^2 - \kappa\chi_2 v_x(\cdot;\mathbf{u}))\tilde{D}_2 e^{-\kappa x} \\
&\geq \left(r - \frac{r}{M_2}(1 - \tilde{D}_2 e^{-\kappa x}) - (br - \chi_2\mu_1)u_1 - \chi_2\lambda v(\cdot;\mathbf{u}) \right) \\
&\quad \times \underline{u}_2^\kappa + \left(1 - a + (1-d)\kappa^2 - \frac{\kappa\chi_2(\mu_1 M_1 + \mu_2 M_2)}{\sqrt{\lambda}} \right)\frac{\tilde{D}_2}{e^{\kappa x}} \\
&\geq \left(r - \frac{r(1 - \tilde{D}_2 e^{-\kappa x})}{M_2} - \Big(\frac{M_1 D_2}{\tilde{D}_2}(br - \chi_2\mu_1) + \frac{\chi_2\sqrt{\lambda}B_{\lambda,\kappa}(\mu_1 M_1 + \mu_2 M_2)}{2}\Big)\frac{\tilde{D}_2}{e^{\kappa x}} - \chi_2\mu_2 \right)\underline{u}_2^\kappa \\
&\quad + \left(1 - a + (1-d)\kappa^2 - \frac{\kappa\chi_2(\mu_1 M_1 + \mu_2 M_2)}{\sqrt{\lambda}} \right)\tilde{D}_2 e^{-\kappa x} \\
&\geq \left(1 - a - (d-1)\kappa^2 - r\left(M_1\big(a + \chi_1 f(\kappa,\chi_1,\chi_2)\big)\big(b - \frac{\chi_2\mu_1}{r}\big) \right. \right. \\
&\quad \left. \left. - \frac{1}{M_2} \right)_+ - \frac{\chi_2(\lambda B_{\lambda,\kappa} + 2\kappa)(\mu_1 M_1 + \mu_2 M_2)}{2\sqrt{\lambda}} \right)\frac{\tilde{D}_2}{e^{\kappa x}} \\
&= ((1 - a) - F(\kappa,\chi_1,\chi_2))\tilde{D}_2 e^{-\kappa x} \geq 0
\end{aligned}
$$

since (3.10) holds. On the other hand, for $x \in \{\underline{u}_1^\kappa > 0\}$, it holds

$$
\begin{aligned}
\mathcal{F}_1^{\mathbf{u}}(\underline{u}_1^\kappa) =& (a-1)\underline{u}_1^\kappa + D_2 M_1 e^{-\kappa x} \left(D_1 \varepsilon_1 (c_\kappa - \varepsilon_1 - 2\kappa)e^{-\varepsilon_1 x} + \chi_1(\kappa - D_1(\varepsilon_1 + \kappa)e^{-\varepsilon_1 x})v_x(\cdot;\mathbf{u}) \right) \\
& + (1 - \frac{1}{M_1}\underline{u}_1^\kappa - (a - \chi_1\mu_2)u_2 - \chi_1\lambda v(\cdot;\mathbf{u}))\underline{u}_1^\kappa + R(u_1 - \underline{u}_1^\kappa) \\
\geq& (a-1)\underline{u}_1^\kappa + D_2 M_1 e^{-\kappa x}\left(D_1 \varepsilon_1(c_\kappa - \varepsilon_1 - 2\kappa)e^{-\varepsilon_1 x} + \chi_1(\kappa - D_1(\varepsilon_1 + \kappa)e^{-\varepsilon_1 x})v_x(\cdot;\mathbf{u}) \right) \\
& + \left(1 - \frac{1}{M_1}\underline{u}_1^\kappa - (a - \chi_1\mu_2)(1 + M_2\tilde{D}_2 e^{-\kappa x}) - \chi_1\mu_2 - \frac{D_2\chi_1}{2}(\mu_2 M_2 \frac{\tilde{D}_2}{D_2} + \mu_1 M_1)\sqrt{\lambda}B_{\lambda,\kappa}e^{-\kappa x} \right)\underline{u}_1^\kappa \\
=& D_2 M_1 e^{-\kappa x}\left(D_1\varepsilon_1(c_\kappa - \varepsilon_1 - 2\kappa)e^{-\varepsilon_1 x} + \chi_1(\kappa - D_1(\varepsilon_1 + \kappa)e^{-\varepsilon_1 x})v_x(\cdot;\mathbf{u}) \right) \\
& - \left(\frac{1}{M_1}\underline{u}_1^\kappa + D_2\left(\frac{(a - \chi_1\mu_2)M_2\tilde{D}_2}{D_2} + \frac{\chi_1}{2}(\frac{\mu_2 M_2\tilde{D}_2}{D_2} + \mu_1 M_1)\sqrt{\lambda}B_{\lambda,\kappa} \right)e^{-\kappa x} \right)\underline{u}_1^\kappa \\
\geq& D_2 M_1 e^{-\kappa x}\left(D_1\varepsilon_1(c_\kappa - \varepsilon_1 - 2\kappa)e^{-\varepsilon_1 x} + \chi_1(\kappa - D_1(\varepsilon_1 + \kappa)e^{-\varepsilon_1 x})v_x(\cdot;\mathbf{u}) \right) \\
& - D_2\left(1 + \frac{(a - \chi_1\mu_2)M_2\tilde{D}_2}{D_2} + \frac{\chi_1}{2}(\frac{\mu_2 M_2\tilde{D}_2}{D_2} + \mu_1 M_1)\sqrt{\lambda}B_{\lambda,\kappa} \right)M_1 D_2 e^{-2\kappa x},
\end{aligned}
$$

where we have used $\underline{u}_1^\kappa(x) \leq D_2 M_1 e^{-\kappa x}$ and Lemma 3.1. Using Lemma 3.2 and the fact that $D_1 e^{-\varepsilon_1 x} \leq 1$ and $e^{-\varepsilon_1 x} > e^{-\kappa x}$ for $x \in \{\underline{u}_1^\kappa > 0\}$ and $D_1 > 1$, the last inequality is improved to

$$
\begin{aligned}
\mathcal{F}_1^{\mathbf{u}}(\underline{u}_1^\kappa) \geq& D_2 M_1 e^{-2\kappa x}\left(D_1\varepsilon_1(c_\kappa - \varepsilon_1 - 2\kappa) - \frac{D_2\chi_1(2\kappa + \varepsilon_1)}{2}(\mu_1 M_1 + \frac{\mu_2 M_2\tilde{D}_2}{D_2})B_{\lambda,\kappa} \right) \\
& - D_2\left(1 + \frac{(a - \chi_1\mu_2)M_2\tilde{D}_2}{D_2} + \frac{\chi_1}{2}(\frac{\mu_2 M_2\tilde{D}_2}{D_2} + \mu_1 M_1)\sqrt{\lambda}B_{\lambda,\kappa} \right)D_2 M_1 e^{-2\kappa x}. \quad (3.11)
\end{aligned}
$$

Hence with the choice of $\frac{D_1}{D_2} = D_1 M_1 \gg 1$ satisfying

$$
\frac{D_1}{D_2}\varepsilon_1(c_\kappa - \varepsilon_1 - 2\kappa) \geq 1 + \frac{(a - \chi_1\mu_2)M_2\tilde{D}_2}{D_2} + \frac{\chi_1(\sqrt{\lambda} + 2\kappa + \varepsilon_1)}{2}(\frac{\mu_2 M_2\tilde{D}_2}{D_2} + \mu_1 M_1)B_{\lambda,\kappa} \quad (3.12)
$$

we conclude from (3.11) that

$$
\mathcal{F}_1^{\mathbf{u}}(\underline{u}_1^\kappa)(x) \geq 0 \quad \forall\, x \in \{\underline{u}_1^\kappa > 0\},
$$

which completes the proof of (iii). $\square$

4. Existence of traveling wave solutions for $c > c^*$

In this section, we investigate the existence of traveling wave solutions of (1.2) and prove Theorem 1.2.

Throughout this section, we assume that **(H4)** holds and that the constants $\varepsilon_1, D_2, \tilde{D}_2$ and D_1 are fixed and satisfy the hypotheses of Lemma 3.3. Recall that $c^* = \frac{(k^*)^2 + 1 - a}{k^*}$, where $\kappa^* = \kappa^*_{\chi_1,\chi_2}$ is as in (1.20). Our idea to prove Theorem 1.2 is to prove that, for any $0 < \kappa < k^*$, there is $\mathbf{u}^* \in \mathcal{E}(\kappa)$ such that $\mathcal{F}^{\mathbf{u}^*}(\mathbf{u}^*) = 0$.

To this end, for every $y > 0$ and $0 < \kappa < \kappa^*$, and $\mathbf{u} \in \mathcal{E}(\kappa)$, consider the following elliptic system

$$
\begin{cases} 0 = \mathcal{F}_i^{\mathbf{u}}(U_i^{\mathbf{u},y}), & |x| < y,\ i = 1, 2 \\ U_i^{\mathbf{u},y}(x) = \overline{u}_i^\kappa(x), & x = \pm y,\ i = 1, 2. \end{cases} \quad (4.1)
$$

Lemma 4.1. *For every $y > 0$ and $0 < \kappa < \kappa^*$, and $\mathbf{u} \in \mathcal{E}(\kappa)$ there exists a unique $\mathbf{U}^{\mathbf{u},y}$ satisfying (4.1).*

Proof. Let $\mathbf{u} \in \mathcal{E}(\kappa)$ and $y > 0$ be given. We first show the uniqueness. Observe that system (4.1) is decoupled, hence the theory of elliptic scalar equations applies for each equation. Since the equations of (4.1) are of the same type for both $i = 1$ and $i = 2$, we shall only provide the arguments for the proof of $U_1^{\mathbf{u},y}$.

Observe from the choice of the positive constant R, see (3.8), and Lemma 3.1 that

$$
1 - (a - \chi_1\mu_1)u_2(x) - \lambda\chi_1 v(x;\mathbf{u}) < R, \quad \forall\, x \in \mathbb{R}.
$$

Whence for each $x \in \mathbb{R}$ fixed, the function

$$\mathbb{R} \ni U_1 \mapsto \mathcal{A}_1^{\mathbf{u}}(x, U_1) := (1 - (a - \chi_1\mu_2)u_2(x) - \lambda\chi_1 v(x, \mathbf{u}) - \frac{(U_1)_+}{M_1})U_1 + R(u_1(x) - U_1) \quad (4.2)$$

is monotone decreasing. Thus by [43, Theorem 10.2, page 264] we deduce that a solution $U^{\mathbf{u},y}(x)$, if exists, is unique.

Now, we show the existence of solution to (4.1). Again, we note from the choice of R (see (3.8)) and Lemma 3.1 that

$$\mathcal{A}_1^{\mathbf{u}}(x, M_1) \leq 0 \quad \text{and} \quad \mathcal{A}_1^{\mathbf{u}}(x, -M_1) \geq 0$$

for every $x \in \mathbb{R}$. We also note, by Lemma 3.2,

$$|c_\kappa - \chi_1 v_x(x; \mathbf{u})| \leq c_\kappa + \frac{\chi_1}{\sqrt{\lambda}}(\mu_1 M_1 + \mu_2 M_2), \quad \forall\, x \in \mathbb{R}.$$

Since $\|\overline{u}_1^\kappa\|_\infty \leq M_1$, it follows from [44, Theorem 5.1, Corollary 5.2, page 433] that there is at least one classical solution to the elliptic equation

$$\begin{cases} 0 = \mathcal{F}_1^{\mathbf{u},y}(U_1) & |x| < y \\ U_1(x) = \overline{u}_1^\kappa(x) & x = \pm y. \end{cases} \qquad \square$$

For later reference, we introduce the function

$$\mathcal{A}_2^{\mathbf{u}}(x, U_2) = (r - (br - \chi_2\mu_1)u_1 - \lambda\chi_2 v(x, \mathbf{u}) - \frac{r}{M_2}(U_2)_+)U_2 + R(u_2(x) - U_2). \quad (4.3)$$

Then (4.1) can be rewritten as

$$\begin{cases} 0 = d_i U_{i,xx}^{\mathbf{u},y} + (c_\kappa - \chi_i v_x(\cdot; \mathbf{u}))U_{i,xx}^{\mathbf{u},y} + \mathcal{A}_i^{\mathbf{u}}(U_i^{\mathbf{u},y}) & |x| < y \\ U_i^{\mathbf{u},y}(x) = \overline{u}_i^\kappa(x) & x = \pm y, \end{cases} \quad (4.4)$$

with $d_1 = 1$ and $d_2 = d$. For every $y > 0$, $0 < \kappa < \kappa^*$, and $\mathbf{u} \in \mathcal{E}(\kappa)$ we define $\mathbf{U}^{\mathbf{u},y}(x) = (\overline{u}_1^\kappa(x), \overline{u}_2^\kappa(x))$ for every $|x| > y$. With this extension, we have the following lemma. For convenience we let $y_0 > 1$ such that $\overline{u}_i^\kappa(-y) = M_i$ for each $i \in \{1, 2\}$ and $y \geq y_0$.

Lemma 4.2. *Let $0 < \kappa < \kappa^*$, $\mathbf{u} \in \mathcal{E}(\kappa)$, and $y \geq y_0$ be given. The following hold for every $i \in \{1, 2\}$.*

(i) *For every $x \in \mathbb{R}$, $\underline{u}_i^\kappa(x) \leq U_i^{\mathbf{u},y}(x) \leq \overline{u}_i^\kappa(x)$*

(ii) *If (**H2**) holds and $U_1^{\mathbf{u},y}(x) \equiv u_1(x)$, then there exist $0 < m_1^* \ll 1$ and $x_1 > 0$ such that $U_1^{\mathbf{u},y}(x) \geq m_1^*$ for every $-y \leq x \leq x_1$ whenever $y > x_1$.*

(iii) *If (**H3**) holds, and $U_2^{\mathbf{u},y}(x) \equiv u_2(x)$, then there exist $0 < m_2 \ll 1$ and $x_2 > 0$ such that $U_2^{\mathbf{u},y}(x) \geq m_2$ for every $-y \leq x \leq x_2$ whenever $y > x_2$.*

Proof. (i) Let $i \in \{1, 2\}$. Since for every $x \in \mathbb{R}$ fixed, the function $U_i \mapsto \mathcal{A}_i^{\mathbf{u}}(U_i)$ is monotone decreasing, then it follows from Lemma 3.3 and the comparison principle for elliptic equations (see [43, Theorem 10.1, page 263]) that

$$\underline{u}_i^\kappa(x) \leq U_i^{\mathbf{u},y}(x) \leq \overline{u}_i^\kappa(x), \quad \forall\, -y < x < y,$$

which together with the fact that $\mathbf{U}^{\mathbf{u},y}(x) = \overline{u}_1^\kappa(x)$ for every $|x| \geq y$, completes the proof of (i).

(ii) Suppose that (**H2**) holds and that $U_1^{\mathbf{u},y}(x) = u_1(x)$ for every $|x| \leq y$. Hence $u_1(x)$ satisfies

$$\begin{cases} 0 = u_{1,xx} + (c_\kappa - \chi_1 v_x(\cdot; \mathbf{u}))u_{1,x} + (1 - (1 - \chi_1\mu_1)u_1 - (a - \chi_1\mu_2)u_2 - \chi_1\lambda v(\cdot; \mathbf{u}))u_1 & |x| < y \\ u_1(x) = \overline{u}_1^\kappa(x) & x = \pm y. \end{cases} \quad (4.5)$$

Since $u_1(x) \geq \underline{u}_1^\kappa(x) \geq 0$ and $u_1(\pm y) > 0$, then the Harnack's inequality for elliptic equations implies that $u_1(x) > 0$ for every $|x| \leq y$. Observe that with $x_1 := \frac{1}{\varepsilon_1} \ln\left(\frac{D_1(\varepsilon_1 + \kappa)}{\kappa}\right)$, it holds that

$$0 < m_1 := \max_{x \in \mathbb{R}} \underline{u}_1^\kappa(x) = \underline{u}_1^\kappa(x_1) = \frac{D_2 M_1 \varepsilon_1 \kappa^{\frac{\kappa}{\varepsilon_1}}}{(\kappa + \varepsilon_1)^{\frac{\kappa}{\varepsilon}+1}} D_1^{-\frac{\kappa}{\varepsilon_1}} = \frac{\varepsilon_1 \kappa^{\frac{\kappa}{\varepsilon_1}}}{(\kappa + \varepsilon_1)^{\frac{\kappa}{\varepsilon}+1}} D_1^{-\frac{\kappa}{\varepsilon_1}} \leq u_1(x_1)$$

and that

$$1 > \eta_1 := 1 - (a - \chi_1\mu_2)M_2 - \chi_1(\mu_1 M_1 - \mu_2 M_2) = 1 - aM_2 - \chi_1\mu_1 M_1 > 0.$$

Now we take $m_1^* := \min\{\eta_1 M_1, m_1\}$. We claim that for every $y > \max\{y_0, x_1\}$ it holds that $m_1^* \leq u_1(x)$ for every $x \in [-y, x_1]$. Indeed, let $y > \max\{x_1, y_0\}$ and suppose that $u_1(x)$ attains its minimum at some point, say $\tilde{x}_1 \in [-y, x_1]$. If $\tilde{x}_1$ is a boundary point, then the claim easily follows. On the other hand, if $\tilde{x}_1$ is an interior point then $u_{1,x}(\tilde{x}_1) = 0$ and $u_{1,xx}(\tilde{x}_1) \geq 0$. This along with (4.5) and the fact that $0 \leq u_i \leq M_i$ for each $i = 1, 2$ imply that

$$0 \geq (1 - (1 - \chi_1\mu_1)u_1(\tilde{x}_1) - (a - \chi_1\mu_2)u_2(\tilde{x}_1) - \chi_1\lambda v((\tilde{x}_1); \mathbf{u}))u_1(\tilde{x}_1) \geq (\eta_1 - (1 - \chi_1\mu_1)u_1(\tilde{x}_1))u_1(\tilde{x}_1).$$

This clearly implies that $u_1(\tilde{x}_1) \geq \frac{\eta_1}{1 - \chi_1\mu_1} = \eta_1 M_1 \geq m_1^*$ since we have shown in the above that $u_1(\tilde{x}_1) = \min_{|x| \leq y} u_1(x) > 0$. So, the claim holds and the result follows.

(*iii*) Suppose that (**H3**) holds and $U_2^{\mathbf{u},y}(x) = u_2(x)$ for $|x| \leq y$. The proof follows similar arguments as in (*ii*) by observing that

$$\eta_2 := r - (br - \chi_2\mu_1)M_1 - \chi_2(\mu_1 M_1 + \mu_2 M_2) = r - rbM_1 - \chi_2\mu_2 M_2 > 0$$

and with $x_2 = \frac{\ln(\tilde{D}_2 + \varepsilon_2)}{\kappa}$,

$$u_2(x_2) > m_2 = \underline{u}_2^\kappa(x_2) = 1 - \frac{\tilde{D}_2}{\tilde{D}_2 + \varepsilon_2} = \frac{\varepsilon_2}{\tilde{D}_2 + \varepsilon_2} \to 0 \quad \text{as } \varepsilon_2 \to 0^+.$$

Hence we can take $0 < \varepsilon_2 \ll 1$ such that we take $m_2^* = m_2 = \min\{m_2, \eta_2 M_2, M_2\}$ and then proceed as in the proof of (*ii*) to show that $u_2(x) \geq m_2^*$ for every $x \in [-y, x_2]$ with $y > \max\{x_2, y_0\}$. $\square$

Theorem 4.3. *Let $0 < \kappa < \kappa^*$ and $y \geq y_0$ be given. Then $\mathbf{U}^{\mathbf{u},y} \in \mathcal{E}(\kappa)$ for every $\mathbf{u} \in \mathcal{E}(\kappa)$. Moreover, the mapping $\mathcal{E}(\kappa) \ni \mathbf{u} \mapsto \mathbf{U}^{\mathbf{u},y} \in \mathcal{E}(\kappa)$ is continuous and compact with respect to the compact open topology. Therefore, by Schauder's fixed point theorem, it has a fixed point, say $\mathbf{u}^{*,y}$.*

Proof. Let $y > y_0$ be fixed. It follows from Lemma 4.2(i) that $\mathbf{U}^{\mathbf{u},y} \in \mathcal{E}(\kappa)$ for every $\mathbf{u} \in \mathcal{E}(\kappa)$. Since $\|U_i^{\mathbf{u},y}\|_\infty \leq M_i$ for every $i = 1, 2$ and $u \in \mathcal{E}(\kappa)$, by a priori estimates for elliptic equations and the uniqueness of solution to (4.1) guaranteed by Lemma 4.1 and the Arzela–Ascot's theorem, it follows that the mapping $\mathcal{E}(\kappa) \ni \mathbf{u} \mapsto \mathbf{U}^{\mathbf{u},y} \in \mathcal{E}(\kappa)$ is continuous and compact with respect to the compact open topology. Therefore, by Schauder's fixed point theorem, it has a fixed point, say $\mathbf{u}^{*,y}$. $\square$

Fix $0 < \kappa < \kappa^*$. For every $y > y_0$, let $\mathbf{u}^{*,y}$ be a fixed point of the mapping $\mathcal{E}(\kappa) \ni \mathbf{u} \mapsto \mathbf{U}^{\mathbf{u},y} \in \mathcal{E}(\kappa)$. Since $\|u_i^{*,y}\|_\infty \leq M_1 + M_2$, $i = 1, 2$, and by Lemmas 3.1 and 3.2 it holds that

$$\|v(\cdot; \mathbf{u}^{*,y})\|_{C_{\text{unif}}^{2,b}(\mathbb{R})} \leq \lambda\|v(\cdot; \mathbf{u}^{*,y})\| + \mu_1\|u_1^{*,y}\|_\infty + \mu_2\|u_2^{*,y}\|_\infty \leq 2(\mu_1 M_1 + \mu_2 M_2)$$

for every $y > y_0$. By a priori estimates for elliptic equations and Arzela–Ascoli's theorem, there are a sequence $\{y_n\}_{n \geq 1}$ with $y_n \to \infty$ as $n \to \infty$ and a function $\mathbf{u}^* \in C_{\text{unif}}^{2,b}(\mathbb{R}) \times C_{\text{unif}}^{2,b}(\mathbb{R})$ such that $\mathbf{u}^{*,y_n} \to \mathbf{u}^*$ locally uniformly in $C^2(\mathbb{R}) \times C^2(\mathbb{R})$. Moreover, the function $\mathbf{u}^*$ satisfies the elliptic system

$$\begin{cases} 0 = u_{1,xx}^* + (c_\kappa - \chi_1 v_x^*(\cdot))u_{1,x}^* + (1 - (1 - \chi_1\mu_1)u_1^* - (a - \chi_1\mu_2)u_2^* - \chi_1\lambda v^*)u_1^*, & x \in \mathbb{R} \\ 0 = du_{1,xx}^* + (c_\kappa - \chi_2 v_x^*(\cdot))u_{1,x}^* + (r - (r - \chi_2\mu_2)u_2^* - (br - \chi_2\mu_1)u_1^* - \chi_2\lambda v)u_1^*, & x \in \mathbb{R} \\ 0 = v_{xx}^* - \lambda v^* + \mu_1 u_1^* + \mu_2 u_2^*, & x \in \mathbb{R} \end{cases} \tag{4.6}$$

where $v^*(\cdot) = v(\cdot; \mathbf{u}^*)$. Since $\mathbf{u}^* \in \mathcal{E}(\kappa)$, then

$$\lim_{x \to \infty} \frac{u_1^*(x)}{e^{-\kappa x}} = \frac{1}{D_2 M_1} = 1 \quad \text{and} \quad \lim_{x \to \infty} |u_2^*(x) - 1| = 0. \tag{4.7}$$

Whence

$$\lim_{x \to \infty} \mathbf{u}^*(x) = \mathbf{e}_2. \tag{4.8}$$

In fact $\mathbf{u}^*(x)$ satisfies the following.

Lemma 4.4. *It holds that*

$$\lim_{x \to \infty} \left| \frac{u_1^*(x)}{e^{-\kappa x}} - 1 \right| = 0 \quad \text{and} \quad \lim_{x \to \infty} \left| \frac{u_2^*(x) - 1}{e^{-\kappa x}} - \frac{(\chi_2 \mu_2 - rb)}{(1 - a) + \frac{r}{M_2} - (d - 1)\kappa^2 - \frac{\chi_2 \mu_2 \sqrt{\lambda}}{2} B_{\lambda,\kappa}} \right| = 0, \tag{4.9}$$

where $B_{\lambda,\kappa}$ is as in (1.16).

Proof. It is clear that the first limit in (4.9) is established in (4.7). So, it remains to show that the second limit in (4.7) holds. We proceed by contradiction and suppose that there is a sequence $\{x_n\}_{n \geq 1}$ with $x_n \to \infty$ as $n \to \infty$ such that

$$\inf_{n \geq 1} \left| \frac{u_2^*(x_n) - 1}{e^{-\kappa x_n}} - \frac{(\chi_2 \mu_2 - rb)}{(1 - a) + \frac{r}{M_2} - (d - 1)\kappa^2 - \frac{\chi_2 \mu_2 \sqrt{\lambda}}{2} B_{\lambda,\kappa}} \right| > 0. \tag{4.10}$$

Consider the functions

$$w_2^n(x + x_n) = \frac{u_2^*(x_n + x) - 1}{e^{-\kappa(x_n + x)}} \quad \text{and} \quad w_1^n(x) = \frac{u_1^*(x_n + x)}{e^{-\kappa(x_n + x)}}, \quad x \in \mathbb{R}$$

and note that

$$\|w_1^n\|_\infty \leq 1 \quad \text{and} \quad \|w_2^n\|_\infty \leq M_2 \tilde{D}_2 \quad \forall\, n \geq 1,$$

since $\mathbf{u}^* \in \mathcal{E}(\kappa)$. A simple computation shows that $\{(w_1^n, w_2^n)\}_{n \geq 1}$ satisfy

$$0 = d(\kappa^2 w_2^n - 2\kappa w_{2,x}^n + w_{2,xx}^n) + (c_\kappa - \chi_2 v_x(\cdot + x_n; \mu_1 u_1^* + \mu_2 u_2^*))(w_{2,x}^n - \kappa w_2^n)$$
$$- \left(\frac{r w_2^n}{M_2} + (br - \chi_2 \mu_1)w_1^n + \chi_2 \lambda \tilde{v}(\cdot + x_n; \mu_1 w_1^n + \mu_2 w_2^n) \right) u_2^*(\cdot + x_n) \tag{4.11}$$

where the linear and bounded operator $C_{\text{unif}}^b(\mathbb{R}) \ni g \mapsto \tilde{v}(\cdot; g)$ is given by

$$\tilde{v}(x; g) = \frac{1}{2\sqrt{\lambda}} \int_{\mathbb{R}} e^{-\sqrt{\lambda}|z| - \kappa z} g(z + x)dz \quad \forall\, g \in C_{\text{unif}}^b(\mathbb{R}). \tag{4.12}$$

By a priori-estimates for parabolic equations, we may suppose that $(w_1^n(x), w_2^n(x)) \to (w_1^\infty(x), w_2^\infty(x))$ as $n \to \infty$ locally uniformly in $C_{loc}^2(\mathbb{R})$. Furthermore, we note from (4.7) that $\lim_{n \to \infty} v(x + x_n; \mu_1 u_1^* + \mu_2 u_2^*) = \frac{\mu_2}{\lambda}$ and $w_1^\infty(x) = 1$ for every $x \in \mathbb{R}$. Hence $w_2^\infty(\cdot) \in C_{\text{unif}}^{2,b}(\mathbb{R})$ and satisfies

$$0 = d(\kappa^2 w_2^\infty - 2\kappa w_{2,x}^\infty + w_{2,xx}^\infty) + c_\kappa(w_{2,x}^\infty - \kappa w_2^\infty) - \frac{r w_2^\infty}{M_2} - (br - \chi_2 \mu_1) - \chi_2 \lambda \tilde{v}(x; \mu_2 w_2^\infty)$$
$$= dw_{2,xx}^\infty + (c_\kappa - 2\kappa d)w_{2,x}^\infty - \left((1 - a) - (d - 1)\kappa^2 + \frac{r}{M_2}\right) w_2^\infty - \chi_2 \mu_2 \lambda \tilde{v}(x; w_2^\infty) - (br - \chi_2 \mu_1). \tag{4.13}$$

We denote by $W(\cdot; g)$ the solution of the elliptic equation

$$0 = dW_{xx} + (c_\kappa - 2\kappa d)W_x - \left((1 - a) - (d - 1)\kappa^2 + \frac{r}{M_2}\right)W + g(x), \quad x \in \mathbb{R},$$

for every $g \in C^b_{\text{unif}}(\mathbb{R})$. Since $(1-a)-(d-1)\kappa^2+\frac{r}{M_2} > 0$, it follows from standard theory of elliptic operators, that $W(\cdot;g) \in C^{2,b}_{\text{unif}}(\mathbb{R})$ is uniquely determined by g. Moreover, the maximum principle implies that

$$\|W(\cdot;g)\|_\infty \leq \frac{1}{(1-a)-(d-1)\kappa^2+\frac{r}{M_2}}\|g\|_\infty \quad \forall \ g \in C^b_{\text{unif}}(\mathbb{R}). \tag{4.14}$$

Now, observe from (4.13) that

$$w_2^\infty(\cdot) = -\chi_2\mu_2\lambda W(\cdot;\tilde{v}(\cdot;w_2^\infty)) - W(\cdot;(br-\chi_2\mu_1)).$$

Equivalently, we have

$$w_2^\infty(\cdot) + \chi_2\mu_2\lambda W(\cdot;\tilde{v}(\cdot;w_2^\infty)) = -W(\cdot;(br-\chi_2\mu_1)) \tag{4.15}$$

Observe from (4.12) that

$$\|\tilde{v}(\cdot;g)\|_\infty \leq \frac{\|g\|_\infty}{2\sqrt{\lambda}}\int_\mathbb{R} e^{-\sqrt{\lambda}|z|-\kappa z}dz = \frac{B_{\lambda,\kappa}}{2\sqrt{\lambda}}\|g\|_\infty \quad \forall \ g \in C^b_{\text{unif}}(\mathbb{R}), \tag{4.16}$$

where $B_{\lambda,\kappa}$ is as in (1.16). Hence, by (4.14) and (4.16),

$$\|\chi_2\mu_2\lambda W(\cdot;\tilde{v}(\cdot;g))\|_\infty \leq \frac{\chi_2\mu_2\sqrt{\lambda}B_{\lambda,\kappa}}{2\big((1-a)-(d-1)\kappa^2+\frac{r}{M_2}\big)}\|g\|_\infty \quad \forall \ g \in C^b_{\text{unif}}(\mathbb{R}).$$

We remark from (3.10) that $\frac{\chi_2\mu_2\sqrt{\lambda}B_{\lambda,\kappa}}{2\big((1-a)-(d-1)\kappa^2+\frac{r}{M_2}\big)} < 1$. Hence 1 belongs to the resolvent set of the linear bounded operator $C^b_{\text{unif}}(\mathbb{R}) \ni g \mapsto -\chi_2\mu_2\lambda W(\cdot;\tilde{v}(\cdot;g))$. As a result, we obtain that the solution of Eq. (4.15), equivalently solution of (4.13), is uniquely determined in $C^{2,b}_{\text{unif}}(\mathbb{R})$. However, it is easily verified that the constant function

$$w(x) = \frac{(\chi_2\mu_2 - rb)}{(1-a)+\frac{r}{M_2}-(d-1)\kappa^2-\frac{\chi_2\mu_2\sqrt{\lambda}}{2}B_{\lambda,\kappa}} \quad \forall \ x \in \mathbb{R}$$

is a solution of (4.13) in $C^{2,b}_{\text{unif}}(\mathbb{R})$. Thus we conclude that $w_2^\infty(x) \equiv \frac{(\chi_2\mu_2-rb)}{(1-a)+\frac{r}{M_2}-(d-1)\kappa^2-\frac{\chi_2\mu_2\sqrt{\lambda}}{2}B_{\lambda,\kappa}}$, in particular we obtain

$$\frac{(\chi_2\mu_2 - rb)}{(1-a)+\frac{r}{M_2}-(d-1)\kappa^2-\frac{\chi_2\mu_2\sqrt{\lambda}}{2}B_{\lambda,\kappa}} = w_2^\infty(0) = \lim_{n\to\infty} w_2(x_n),$$

which contradicts with (4.10). Therefore (4.9) must hold. $\quad\square$

Thanks to Lemma 4.4, to complete the proof of Theorem 1.2, it remains to study the asymptotic behavior of $\mathbf{u}^*(x)$ and $x \to -\infty$, which we complete in next two lemmas.

Lemma 4.5. *Suppose that hypothesis* (**H2**) *holds. Then*

$$\lim_{x\to-\infty} \mathbf{u}^*(x) = \mathbf{e}_1.$$

Proof. We prove this result by contradiction. Suppose that there is a sequence $\{x_n\}$ with $x_n \to -\infty$ such that

$$\inf_{n\geq 1} |\mathbf{u}^*(x_n) - \mathbf{e}_1| > 0. \tag{4.17}$$

Consider the sequence $\mathbf{u}^{*,n}(x) = \mathbf{u}^*(x + x_n)$ for every $x \in \mathbb{R}$ and $n \geq 1$. By a priori estimates for elliptic equations and Arzela–Ascoli's theorem, without loss of generality, we may suppose that there is some $\tilde{\mathbf{u}} \in C^2$

such that $\mathbf{u}^{*,n} \to \tilde{\mathbf{u}}$ as $n \to \infty$ locally uniformly in $C^2(\mathbb{R})$. Moreover, $\tilde{\mathbf{u}}$ also satisfies (4.6). Recalling m_1^* and x_1 given by Lemma 4.2 (ii), we deduce that

$$m_1^* \leq \tilde{u}_1(x) \leq M_1, \quad \forall \, x \in \mathbb{R} \tag{4.18}$$

since $x_n \to -\infty$ as $n \to \infty$. Therefore by Lemma 2.1, we obtain $\tilde{\mathbf{u}}(x) \equiv (1,0)$. In particular, $\tilde{\mathbf{u}}(0) = e_1$, which contradicts with (4.17), since $\tilde{\mathbf{u}}(0) = \lim_{n\to\infty} \mathbf{u}^*(x_n)$. $\quad\square$

Lemma 4.6. *Suppose that hypothesis* (**H3**) *holds. Then*

$$\lim_{x \to -\infty} \mathbf{u}^*(x) = \mathbf{e}_*.$$

Proof. We proceed also by contradictions. The ideas are similar to that of the proof of Lemma 4.5. Suppose that there is a sequence $\{x_n\}$ with $x_n \to -\infty$ such that

$$\inf_{n \geq 1} |\mathbf{u}^*(x_n) - \mathbf{e}_*| > 0. \tag{4.19}$$

Consider the sequence $\mathbf{u}^{*,n}(x) = \mathbf{u}^*(x + x_n)$ for every $x \in \mathbb{R}$ and $n \geq 1$. By a priori estimates for elliptic equations and the Arzela–Ascoli's theorem, without loss of generality, we may suppose that there is some $\tilde{\mathbf{u}} \in C^2$ such that $\mathbf{u}^{*,n} \to \tilde{\mathbf{u}}$ as $n \to \infty$ locally uniformly in $C^2(\mathbb{R})$. Moreover, $\tilde{\mathbf{u}}$ also satisfies (4.6). Recalling the positive constants m_i^* and x_i, $i = 1, 2$, and given by Lemma 4.2 (ii)–(iii), we deduce that

$$m_i^* \leq \tilde{u}_i(x) \leq M_i, \quad \forall \, x \in \mathbb{R}, \; i = 1, 2 \tag{4.20}$$

since $x_n \to -\infty$ as $n \to \infty$. Therefore by Lemma 2.2, we obtain $\tilde{\mathbf{u}}(x) \equiv e_*$. In particular, $\tilde{\mathbf{u}}(0) = \mathbf{e}_*$, which contradicts with (4.19), since $\tilde{\mathbf{u}}(0) = \lim_{n\to\infty} \mathbf{u}^*(x_n)$. $\quad\square$

Now we complete the proof of Theorem 1.2.

Proof of Theorem 1.2. For every $c > c_{\kappa^*}$, there is a unique $0 < \kappa_c < \kappa^*$ satisfying $c = c_{\kappa_c} = \frac{\kappa_c^2 + 1 - a}{\kappa_{\kappa_c}}$. This implies that $\mathbf{u}(t, x) = \mathbf{u}^*(x - ct)$ is a traveling solution of (1.2) with speed c. Moreover, it follows from (4.8) that $\mathbf{u}^*$ connect $\mathbf{e}_2$ at right end. Since $\mathbf{u}^* \in \mathcal{E}(\kappa)$, then $u_1^*(x) > 0$ by comparison principle for elliptic equations. This in turn implies that $\|u_2^* - 1\|_\infty > 0$. Thus $\mathbf{u}^*$ is not a trivial solution of (1.2). Assertion (i) and (ii) of the theorem follows from Lemmas 4.5 and 4.6 respectively. $\quad\square$

5. Proof of Theorem 1.3

In this section, we present the proof of nonexistence of nontrivial traveling wave solutions of (1.2) with speed $c < c_0^*(= 2\sqrt{1-a})$ connecting $\mathbf{e}_2$ at right end. Our first step toward the proof of the non-existence is to show that, for any nontrivial traveling solution $\mathbf{u}(x - ct)$ of (1.2) connecting $\mathbf{u}(\infty) = \mathbf{e}_2$ at the right end, there holds $u_{1,x} \leq 0$ for $x \gg 1$.

Lemma 5.1. *Let* $\mathbf{u}(t, x) = \mathbf{u}(x - ct)$ *be a nontrivial traveling wave solution of* (1.2) *connecting* $\mathbf{e}_2$ *at the right end. Then there is* $X_0 \gg 1$ *such that* $u_{1,x}(x) \leq 0$ *for every* $x > X_0$.

Proof. We proceed by contraction. Suppose that the statement of the lemma is false. Then, since $u_1(\infty) = 0$ and $u_1(x) > 0$ for every $x \in \mathbb{R}$, then there is sequence of local minimum points $\{x_n\}_{n\geq 1}$ of $u_1(x)$ satisfying $x_n \to \infty$ as $n \to \infty$. Since $u_2(\infty) = 1$, then $\lim_{n\to\infty} u_2(x_n) = 1$. From the representation formula

$$v(x; \mathbf{u}) = \frac{1}{2\sqrt{\lambda}} \int_{\mathbb{R}} e^{-\sqrt{\lambda}|z|} (\mu_1 u_1(z + x) + \mu_2 u_2(z + x)) dz,$$

it follows from the dominated convergence theorem that

$$\lim_{n\to\infty} v(x_n; \mathbf{u}) = \frac{\mu_2}{\lambda}.$$

Hence

$$\lim_{n\to\infty} (1 - (1 - \chi_1\mu_1)u_1(x_n) - (a - \chi_1\mu_2)u_2(x_n) - \lambda\chi_1 v(x_n; \mathbf{u})) = 1 - a.$$

Thus there is $n_0 \gg 1$ such that

$$
\begin{aligned}
u_{1,xx}(x_n) + (c - \chi_1 v_x(x_n; \mathbf{u}))u_{1,x}(x_n) &= ((1 - \chi_1\mu_1)u_1(x_n) + (a - \chi_1\mu_2)u_2(x_n) \\
&\quad + \lambda\chi_1 v(x_n; \mathbf{u}) - 1)\, u_1(x_n) \\
&< \frac{-(1-a)u_1(x_n)}{2} < 0, \quad \forall\, n \geq n_0.
\end{aligned}
\tag{5.1}
$$

Since $\{x_n\}$ is a sequence of local minimum points, we have that $u_{1,xx}(x_n) \geq 0$ and $u_{1,x}(x_n) = 0$ for every $n \geq 1$, which clearly contradicts with (5.1). Thus the statement of the lemma must hold. $\square$

Now, we present the proof of Theorem 1.3.

Proof of Theorem 1.3. We prove this result by contradiction. Suppose that (1.2) has a nontrivial traveling wave solution $\mathbf{u}(t, x) = \mathbf{u}(x - ct)$ with speed $c < c_0^*$ connecting $\mathbf{e}_2$ at the right end. Choose $q > 0$ and $0 < \varepsilon \ll 1$ satisfying $\max\{c, 0\} + \varepsilon < q < 2\sqrt{1 - a - \varepsilon}$. By Lemma 5.1, there is $X_0 \gg 1$ such that $u_{1,x}(x) \leq 0$ for every $x > X_0$. Moreover, since $\mathbf{u}(+\infty) = \mathbf{e}_2$, we deduce that

$$\lim_{x\to\infty} (1 - (1 - \chi_1\mu_1)u_1(x) - (a - \chi_1\mu_2)u_2(x) - \chi_1\lambda v(x; \mathbf{u})) = 1 - a \quad \text{and} \quad \lim_{x\to\infty} v_x(x; \mathbf{u}) = 0.$$

Thus, there is $X_1 \gg X_0$, such that

$$|\chi_1 v_x(x; \mathbf{u})| < \varepsilon \quad \text{and} \quad 1 - (1 - \chi_1\mu_1)u_1(x) - (a - \chi_1\mu_2)u_2(x) - \chi_1\lambda v(x; \mathbf{u}) > 1 - a - \varepsilon \quad \forall\, x \geq X_1.$$

Hence, it holds that the function $\overline{u}(t, x) = u_1(x - (c + \varepsilon)t)$ satisfies that

$$
\begin{aligned}
\overline{u}_t &= u_{1,xx} - (\varepsilon + \chi_1 v_x)u_{1,x} + (1 - (1 - \chi_1\mu_1)u_1 - (a - \chi_1\mu_2 u_2) - \lambda\chi_1 v(x - (x + \varepsilon)t; \mathbf{u}))\overline{u} \\
&\geq u_{1,xx} + (1 - a - \varepsilon)\overline{u}, \quad x \geq X_1 + (c + \varepsilon)t, \ t > 0.
\end{aligned}
\tag{5.2}
$$

A simple computation shows that the function

$$\underline{u}(t, x) = \sigma e^{-\frac{q}{2}(x - x_1 - l - qt)} \cos\left(\frac{\beta}{2}(x - x_1 - l - qt)\right), \quad x_1 + qt \leq x \leq l + x_1 + qt$$

where $l = \frac{\pi}{\beta}$, $\beta = \sqrt{4(1 - a - \varepsilon) - q^2}$ and $\sigma = e^{-\frac{lq}{2}} \min_{x_1 \leq x \leq x_1 + L} \overline{u}(0, x)$, satisfies

$$
\begin{cases}
\underline{u}_t = \underline{u}_{xx} + (1 - a - \varepsilon)\underline{u}, & x_1 + qt \leq x \leq l + x_1 + qt, \\
\underline{u}(t, x) = 0 & x = x_1 + qt, x = l + x_1 + qt.
\end{cases}
$$

Thus, since $q > c + \varepsilon$, then $(c + \varepsilon)t < qt$ for every $t > 0$. Moreover the choice of σ guarantees that $\underline{u}(0, x) \leq \overline{u}(0, x)$ for every $x_1 \leq x \leq x_1 + l$ and $\overline{u}(t, x) > 0$ for $x \in \{x_1 + qt, x_1 + l + qt\}$ for every $t > 0$. We now infer to the comparison principle for parabolic equations to conclude that

$$\underline{u}(t, x) \leq \overline{u}(t, x), \quad \forall x_1 + qt < x < x_1 + l + qt, \ t > 0.$$

In particular, taking $x = x_1 + \frac{l}{2} + qt$, we get

$$\sigma e^{\frac{ql}{4}} \cos\left(\frac{\pi}{4}\right) = \underline{u}(t, qt + x_1 + \frac{l}{2}) \leq \overline{u}(t, x_1 + qt + \frac{l}{2}) = u_1\left((q - c - \varepsilon)t + x_1 + \frac{l}{2}\right), \quad \forall\, t > 0.$$

Letting $t \to \infty$ yield $0 < \sigma e^{\frac{ql}{4}} \cos\left(\frac{\pi}{4}\right) \leq u_1(\infty)$, which is impossible since $u_1(\infty) = 0$. Therefore, we conclude that the statement of the theorem must hold. $\square$

6. Proof of Theorem 1.4

In this section, we present the proof of Theorem 1.4. To this end, we first recall some results on the spreading speeds and stability for single species chemotaxis model.

Lemma 6.1 ([31,34]).

Consider the single species chemotaxis model

$$\begin{cases} u_t = \tilde{d}u_{xx} - \chi(uw_x)_x + u(\tilde{a} - \tilde{b}u) & x \in \mathbb{R}, \ t > 0 \\ 0 = w_{xx} - \tilde{\lambda}w + \mu u & x \in \mathbb{R}, t > 0, \end{cases} \tag{6.1}$$

where all the parameters are positive, and let $(u(t, x; u_0), w(t, x; u_0))$ denote the unique nonnegative classical solution of (6.1) for every $u_0 \in C_{\text{unif}}^b(\mathbb{R})$ with $u_0 \geq 0$ defined on a maximal interval of existence $[0, t_{\max, u_0})$. Then the following hold.

(i) *If $\chi\mu < \tilde{b}$, then $t_{\max, u_0} = +\infty$ and $\|u\|_\infty \leq \max\{\|u_0\|, \frac{\tilde{a}}{\tilde{b} - \chi\mu}\}$ for every $t \geq 0$. Moreover, if $\|u_0\|_\infty > 0$ then*

$$\liminf_{t \to \infty} \inf_{|x| \leq (2\sqrt{\tilde{a}\tilde{d}} - \varepsilon)t} u(t, x) > 0 \quad \forall \, 0 < \varepsilon \ll 1.$$

(ii) *If $2\chi\mu < \tilde{b}$ and $\inf_{x \in \mathbb{R}} u_0(x) > 0$ then*

$$\lim_{t \to \infty} \|u(t, \cdot) - \frac{\tilde{a}}{\tilde{b}}\|_\infty = 0.$$

Throughout the rest of this section, we assume that **(H5)** holds. Note that **(H5)** implies **(H4)**. By the definition of the function $F_2(\kappa, \chi_1, \chi_2)$, we have

$$1 - a > (d-1)_+(1-a) + F_2(\kappa, \chi_1, \chi_2) + (1-d)\kappa^2 \geq F_2(\kappa, \chi_1, \chi_2) \quad \forall \kappa \in (0, \sqrt{1-a}),$$

which means that inequality (3.10) also holds for every $\kappa \in (0, \sqrt{1-a})$. Hence $c^* = c_0^*$.

As a result, to complete the proof of Theorem 1.4, it remains to show the existence of a non-trivial traveling wave connecting $\mathbf{e}_2$ at the right end with minimum speed $c_0^* = 2\sqrt{1-a}$.

Proof of Theorem 1.4.

(i) Suppose that hypotheses **(H2)** and **(H5)** hold and $r > 2\chi_2\mu_2$. Choose a decreasing sequence $\{c_n\}_{n \geq 1}$ such that $c_n \to c_0^*$ as $n \to \infty$. For every $n \geq 1$, let $\tilde{\mathbf{u}}^n = \tilde{\mathbf{U}}^{c_n}(x - c_n t)$ be a traveling wave solution of (1.2) connecting $\mathbf{e}_1$ and $\mathbf{e}_2$ given by Theorem 1.2. Let

$$\tilde{x}_n = \min\{x \in \mathbb{R} \ : \ \tilde{U}_1^{c_n}(x) = \frac{1}{2}\} \tag{6.2}$$

and define $\mathbf{U}^{c_n} = \tilde{\mathbf{U}}^{c_n}(x + \tilde{x}_n)$ for every $x \in \mathbb{R}$ and $n \geq 1$. Note that $U_1^{c_n}$ satisfies

$$U_1^{c_n}(x) \begin{cases} \geq \frac{1}{2} & \text{if } x \leq 0 \\ = \frac{1}{2} & \text{if } x = 0 \end{cases} \tag{6.3}$$

for every $n \geq 1$. Recall that $\|U_i^{c_n}\|_\infty < M_i$ for every $n \geq 1$ and $i = 1, 2$. Hence, by a priori estimates for elliptic equations, if possible after passing to a subsequence, we may suppose that there is some $\mathbf{U} \in C^{2,b}(\mathbb{R})$ such that $(\mathbf{U}^{c_n}, V(\cdot; \mathbf{U}^{c_n})) \to (\mathbf{U}, V(\cdot; \mathbf{U}))$ as $n \to \infty$ in $C_{\text{loc}}^{2,b}(\mathbb{R})$. Moreover, $\mathbf{U}(x)$ satisfies

$$\begin{cases} 0 = U_{1,xx} + (c_0^* - \chi_1 V_x(\cdot; \mathbf{U}))U_{1,x} + U_1(1 - (1 - \chi_1\mu_1)U_1 - (a - \chi_1\mu_2)U_2 - \chi_1\lambda V(\cdot; \mathbf{u})) & x \in \mathbb{R} \\ 0 = dU_{2,xx} + (c_0^* - \chi_2 V_x(\cdot; \mathbf{U}))U_{2,x} + U_2(r - (r - \chi_2\mu_2)U_2 - (br - \chi_2\mu_1)U_2 1 - \chi_2\lambda V(\cdot; \mathbf{u})) & x \in \mathbb{R} \\ 0 = V_{xx} - \lambda V + \mu_1 U_1 + \mu_2 U_2 & x \in \mathbb{R}. \end{cases} \tag{6.4}$$

We note that $U_1(\cdot)$ also satisfies properties (6.3). From this point, we complete the proof of the spatial asymptotic behavior of $\mathbf{U}(x)$ in the following six steps.

Step 1. In this step, we prove that $U_2(x) > 0$ for every $x \in \mathbb{R}$. Suppose not. Then $u(t,x) = U_1(x - c_* t)$ is a solution of the single species chemotaxis model (6.1) with $(\tilde{a}, \tilde{b}, \mu, \tilde{\lambda}, \tilde{d}) = (1, 1, \mu_1, \lambda, 1)$. Since $u(0,x) = U_1(x) \geq \frac{1}{2}$ for every $x \leq 0$ (by (6.3)), then it follows from Lemma 6.1(i) that

$$\liminf_{x \to \infty} U_1(x) = \liminf_{t \to \infty} u((2 - \varepsilon)t - c_0^* t) > 0, \quad \forall 0 < \varepsilon \ll 1.$$

Hence we conclude that $\inf_{x \in \mathbb{R}} U_1(x) > 0$, which yield that $\inf_{t,x \in \mathbb{R}} u(t,x) > 0$. It then follows from Lemma 2.1 that $U_1(x) \equiv 1$. Clearly, this contradicts with (6.3) since $U_1(0) = \frac{1}{2}$. Thus we must have that $U_2(x) > 0$ for every $x \in \mathbb{R}$.

Step 2. In this step, we prove that $\liminf_{x \to \infty} U_1(x) = 0$. If not, since $U_1(\cdot)$ satisfies (6.3), we would have that $\inf_{x \in \mathbb{R}} U_1(x) > 0$. And hence since (**H2**) holds, it follows from Lemma 2.1 that $\mathbf{U}(x) \equiv \mathbf{e}_1$, so $U_2 \equiv 0$, which contradicts with **Step 1**. Hence $\liminf_{x \in \mathbb{R}} U_1(x) = 0$.

Step 3. In this step, we prove that $\limsup_{x \to \infty} U_1(x) = 0$. Suppose not. According to **Step 2**, there would exist a sequence of minimum points $\{x_n\}_{n \geq 1}$ of the function U_1 satisfying $x_n \to \infty$ and $U_1(x_n) \to 0$ as $n \to \infty$ with $U_{1,x}(x_n) = 0$ and $U_{1,xx}(x_n) \leq 0$ for every $n \geq 0$. Hence, we deduce from (6.4) that

$$0 \geq U_1(x_n)(1 - (1 - \chi_1 \mu_1)U_1(x_n) - (a - \chi_1 \mu_2)U_2(x_n) - \chi_1 \lambda V(x_n; \mathbf{u})) \quad \forall\, n \geq 1.$$

In particular, we obtain that

$$1 \leq (1 - \chi_1 \mu_1)U_1(x_n) + (a - \chi_1 \mu_2)U_2(x_n) + \chi_1 \lambda V(x_n; \mathbf{u}) \quad \forall\, n \geq 1,$$

since $U_1(x_n) > 0$ for $n \geq 1$. Letting $n \to \infty$ in the last inequality and using the facts that $\|U_2\|_\infty \leq M_2$ and $\|V(\cdot, \mathbf{U})\|_\infty \leq \frac{1}{\lambda}(\mu_1 M_1 + \mu_2 M_2)$, we obtain that

$$1 \leq (a - \chi_1 \mu_2)M_2 + \chi_1(\mu_1 M_1 + \mu_2 M_2) = \chi_1 \mu_1 M_1 + a M_2.$$

This clearly contradicts with hypothesis (**H2**). Thus we must have that $\limsup_{x \to \infty} U_1(x) = 0$.

Step 4. In this step, we prove that $\limsup_{x \to \infty} U_2(x) > 0$. If not, then a slight modification of the proof of Theorem 1.3 shows that $c_0^* \geq 2\sqrt{1 - a U_2(\infty)} = 2$, which is absurd. Hence, we must have $\limsup_{x \to \infty} U_2(x) > 0$.

Step 5. In this step, we prove that $\liminf_{x \to \infty} U_2(x) > 0$. Suppose not. In this case, since $U_1(\infty) = 0$ and $\limsup_{x \to \infty} U_2(x) > 0$, we can repeat the arguments used in **Step 3** for the equation satisfied by $U_2(x)$ to end up with the inequality $r \leq \chi_2 \mu_2 M_2$. This clearly contradicts the fact that $r > 2\chi_2 \mu_2$.

Step 6. In this step, we prove that $\lim_{x \to \infty} U_2(x) = 1$. Suppose not. Then there is sequence $\{y_n\}_n$ with $y_n \to \infty$ as $n \to \infty$ such that

$$\inf_{n \geq 1} |U_2(y_n) - 1| > 0. \tag{6.5}$$

By a priori estimates for elliptic equations, without loss of generality, we may suppose that there is some $\mathbf{U}^* \in C^{2,b}_{\text{unif}}(\mathbb{R})$ such that $\mathbf{U}(x + y_n) \to \mathbf{U}^*(x)$ as $n \to \infty$ in $C^{2,b}_{\text{loc}}(\mathbb{R})$. Note by **Step 5** (respectively **Step 3**) that $\inf_{x \in \mathbb{R}} U_2^*(x) > 0$ (respectively $U_1^* \equiv 0$). Hence, by Lemma 6.1 (ii), we conclude that $U_2^*(x) \equiv 1$ since $r > 2\chi_2 \mu_2$. This contradicts with (6.5). Hence $U_2(\infty) = 1$.

Finally, we can employ Lemma 4.5 together with the fact that $U_1(x) \geq \frac{1}{2}$ for every $x \leq 0$ to conclude that $\lim_{x \to -\infty} \mathbf{U}(x) = \mathbf{e}_1$. This completes the proof of (i).

(ii) Suppose that (**H3**) and **H(5)** hold. Note that hypothesis (**H3**) implies that $r > 2\chi_2 \mu_2$. The proof of the minimal wave in this case follows similar arguments as in (i). So, we shall provide general ideas of the proof. As in (i), choose a sequence $\{c_n\}_{n \geq 1}$ such that $c_n \to c_0^*+$ as $n \to \infty$. For every $n \geq 1$, let

$\tilde{\mathbf{u}}^n = \tilde{\mathbf{U}}^{c_n}(x - c_n t)$ be a traveling wave solution of (1.2) connecting $\mathbf{e}_1$ and $\mathbf{e}_2$ given by Theorem 1.2. Next, we let

$$\tilde{x}_n^i = \min\{x \in \mathbb{R} \ : \ \tilde{U}_i^{c_n}(x) = \frac{\min\{1-a, 1-b\}}{2(1-ab)}\}, \ i = 1, 2 \quad \text{and} \quad \tilde{x}_n = \min\{\tilde{x}_n^1, \tilde{x}_n^2\}, \tag{6.6}$$

and define $\mathbf{U}^{c_n}(x) = \tilde{\mathbf{U}}^{c_n}(x + \tilde{x}_n)$ for every $x \in \mathbb{R}$ and $n \geq 1$. Note that $U_1^{c_n}$ satisfies

$$U_i^{c_n}(x) \geq \frac{\min\{1-a, 1-b\}}{2(1-ab)} \quad \forall \, x \leq 0, i = 1, 2, n \geq 1. \tag{6.7}$$

By a priori estimates for elliptic equations, if possible after passing to a subsequence, we may suppose that there is some $\mathbf{U} \in C^{2,b}(\mathbb{R})$ such that $(\mathbf{U}^{c_n}, V(\cdot; \mathbf{U}^{c_n})) \to (\mathbf{U}, V(\cdot; \mathbf{U}))$ as $n \to \infty$ in $C_{\text{loc}}^{2,b}(\mathbb{R})$. Moreover $\mathbf{U}(x)$ satisfies (6.4). It is clear from (6.7) that $U_i(x) \geq \frac{\min\{1-a,1-b\}}{2(1-ab)}$ for every $x \leq 0$ and $j = 1, 2$. Hence we may employ Lemma 4.6 to conclude that $\lim_{x \to -\infty} \mathbf{U}(x) = e_*$. Since $c_0^* = 2\sqrt{1-a} < 2$, we can proceed as in the proof of **Step 5** to conclude that $\limsup_{x \to \infty} U_2(x) > 0$. Now, we can proceed as in the proof of **Step 6** by using the fact that (**H3**) to conclude that $\liminf_{x \to \infty} U_2(x) > 0$. Next, observe from (6.7) that there is some $i_0 \in \{1, 2\}$ such that $U_{i_0}(0) = \frac{\min\{1-a,1-b\}}{2(1-ab)}$. Hence $\mathbf{U}(x) \not\equiv e_*$. Hence, we can proceed as in the proof of **Step 2** using the stability of e_* to conclude that $\liminf_{x \to \infty} U_1(x) = 0$. This in turn, as is **Step 3** yields that $U_1(\infty) = 0$. As result, since $\liminf_{x \to \infty} U_2(x) > 0$ and $r > 2\chi_2 \mu_3$, we may use Lemma 6.1 (ii) to conclude that $\lim_{x \to \infty} U_2(x) = 1$. This completes the proof of (ii). $\quad \square$

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Appendix

In this section we present the proof of Lemmas 3.1 and 3.2.

Proof of Lemma 3.1. Observe that for every $x \in \mathbb{R}$

$$\int_{\mathbb{R}} e^{-\sqrt{\lambda}|x-y|}\underline{u}_2^\kappa(y)dy \geq \int_{\mathbb{R}} e^{-\sqrt{\lambda}|x-y|}\left(1 - \tilde{D}_2 e^{-\kappa y}\right)dy = \frac{2}{\sqrt{\lambda}} - \tilde{D}_2 \int_{\mathbb{R}} e^{-\sqrt{\lambda}y-\kappa y}dy = \frac{2}{\sqrt{\lambda}} - \tilde{D}_2 B_{\lambda,\kappa} e^{-\kappa x}.$$

It is clear that $\int_{\mathbb{R}} e^{-\sqrt{\lambda}|x-y|}\underline{u}_2^\kappa(y)dy > 0$ for all $x \in \mathbb{R}$. Hence

$$\int_{\mathbb{R}} e^{-\sqrt{\lambda}|x-y|}\underline{u}_2^\kappa(y)dy \geq \left(\frac{2}{\sqrt{\lambda}} - \tilde{D}_2 B_{\lambda,\kappa} e^{-\kappa x}\right)_+. \tag{A.1}$$

Similarly for every $x \in \mathbb{R}$

$$\int_{\mathbb{R}} e^{-\sqrt{\lambda}|x-y|}\underline{u}_1^\kappa(y)dy \geq M_1 D_2 \int_{\mathbb{R}} e^{-\sqrt{\lambda}|x-y|}\left(1 - D_1 e^{-\varepsilon_1 y}\right)e^{-\kappa y}dy$$
$$= M_1 D_2 e^{-\kappa x}\left(B_{\lambda,\kappa} - D_1 B_{\lambda,\kappa+\varepsilon_1}e^{-\varepsilon_1 x}\right).$$

Hence, since $\int_{\mathbb{R}} e^{-\sqrt{\lambda}|x-y|}\underline{u}_1^\kappa(y)dy > 0$ for every $x \in \mathbb{R}$, we deduce that

$$\int_{\mathbb{R}} e^{-\sqrt{\lambda}|x-y|}\underline{u}_1^\kappa(y)dy \geq M_1 D_2 e^{-\kappa x}\left(B_{\lambda,\kappa} - D_1 B_{\lambda,\kappa+\varepsilon_1}e^{-\varepsilon_1 x}\right)_+,$$

which together with (A.1) yields (3.4) since

$$v(x; \mathbf{u}) \geq \frac{\mu_1}{2\sqrt{\lambda}} \int_{\mathbb{R}} e^{-\sqrt{\lambda}|x-y|}\underline{u}_1^\kappa(y)dy + \frac{\mu_2}{2\sqrt{\lambda}} \int_{\mathbb{R}} e^{-\sqrt{\lambda}|x-y|}\underline{u}_2^\kappa(y)dy. \quad \square$$

Proof of Lemma 3.2. . For every $x \in \mathbb{R}$, observe from (3.2) that

$$\frac{d}{dx}v(x;\mathbf{u}) = \frac{1}{2}\int_{\mathbb{R}} \text{sign}(y-x)e^{-\sqrt{\lambda}|y-x|}(\mu_1 u_1(y) + \mu_2 u_2(y))dy, \quad \forall\, x \in \mathbb{R}. \tag{A.2}$$

Hence, since $0 \le u_1(x) \le \overline{u}_1^{\kappa}(x) = \min\{M_1, M_1 D_2 e^{-\kappa x}\}$, we obtain

$$\left|\int_{\mathbb{R}} \text{sign}(y-x)e^{-\sqrt{\lambda}|y-x|}u_1(y)dy\right| \le \int_{\mathbb{R}} e^{-\sqrt{\lambda}|y-x|}\overline{u}_1^{\kappa}(y)dy \le \min\left\{\frac{2M_1}{\sqrt{\lambda}}, M_1 D_2 B_{\lambda,\kappa}e^{\kappa x}\right\}, \quad \forall\, x \in \mathbb{R}. \tag{A.3}$$

On the other hand using the fact that $\int_{\mathbb{R}} \text{sign}(z)e^{-\sqrt{\lambda}z}dz = 0$, and that $|u_2(x) - 1| \le M_2 \tilde{D}_2 e^{-\kappa x}$, we obtain

$$\left|\int_{\mathbb{R}} \text{sign}(y-x)e^{-\sqrt{\lambda}|y-x|}u_2(y)dy\right| = \left|\int_{\mathbb{R}} \text{sign}(y-x)e^{-\sqrt{\lambda}|y-x|}(\overline{u}_1^{\kappa}(y) - 1)dy\right|$$
$$\le M_2 \tilde{D}_2 B_{\lambda,\kappa}e^{\kappa x}, \quad \forall\, x \in \mathbb{R},$$

which combined with the fact that

$$\left|\int_{\mathbb{R}} \text{sign}(y-x)e^{-\sqrt{\lambda}|y-x|}u_2(y)dy\right| \le \int_{\mathbb{R}} e^{-\sqrt{\lambda}|y-x|}\|u_2\|_{\infty}dy \le \frac{2M_2}{\sqrt{\lambda}}, \quad \forall\, x \in \mathbb{R}$$

yields

$$\left|\int_{\mathbb{R}} \text{sign}(y-x)e^{-\sqrt{\lambda}|y-x|}u_2(y)dy\right| \le \min\left\{\frac{2M_2}{\sqrt{\lambda}}, M_2 \tilde{D}_2 B_{\lambda,\kappa}e^{\kappa x}\right\} \quad \forall\, x \in \mathbb{R}. \tag{A.4}$$

The statement of Lemma 3.2 follows from (A.2)–(A.4). □

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