Functional renormalization and the $\overline{\mathrm{MS}}$ scheme

Alessio Baldazzi,[1,*] Roberto Percacci,[1,†] and Luca Zambelli[2,‡]

[1] *International School for Advanced Studies, via Bonomea 265, I-34136 Trieste, Italy and INFN, Sezione di Trieste, Italy*

[2] *Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena, D-07743 Jena, Germany*

(Received 16 February 2021; accepted 23 March 2021; published 20 April 2021)

Working with scalar field theories, we discuss choices of regulator that, inserted in the functional renormalization group equation, reproduce the results of dimensional regularization at one and two loops. The resulting flow equations can be seen as nonperturbative extensions of the $\overline{\mathrm{MS}}$ scheme. We support this claim by recovering all the multicritical models in two dimensions. We discuss a possible generalization to any dimension. Finally, we show that the method also preserves nonlinearly realized symmetries, which is a definite advantage with respect to other regulators.

DOI: 10.1103/PhysRevD.103.076012

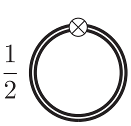
I. INTRODUCTION

Dimensional regularization (dimreg), together with modified minimal subtraction ($\overline{\mathrm{MS}}$),[1] is the most widely used regularization and renormalization method in particle physics. It owes its popularity mainly to its simplicity and to the fact that it respects gauge invariance, one of the cornerstones of particle physics models. It is also remarkably selective: in the language of momentum cutoffs, it extracts only the logarithmic divergences, which for most applications turn out to contain the important information (in particular, the beta functions of the marginal couplings). However, in its standard implementation, dimreg, it is a purely perturbative device, and it works only in even dimensions.

On the other hand, the functional renormalization group (FRG) equation (FRGE) is a convenient way of implementing Wilson's idea of integrating out modes one momentum shell at the time. At its core lies a choice of a "regulator" function R_k that suppresses the contribution of low momentum modes to the path integral.[2] The regulator depends on a scale parameter k with dimension of mass, and the derivative with respect to k gives the contribution to the effective action of an infinitesimal momentum shell. The contribution to the functional integral of a momentum shell of thickness Δk can be written as a loop expansion. The ℓ-loop term is of order $(\Delta k/k)^\ell$, so that the continuous FRGE ($\Delta k/k \to 0$) looks like a one-loop equation [1]. In the 1PI formulation, the FRGE reads [2–5]

$$\frac{\mathrm{d}\Gamma_k}{\mathrm{d}t} = \frac{\hbar}{2}\mathrm{Tr}\left(\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k\right)^{-1}\frac{\mathrm{d}R_k}{\mathrm{d}t}, \tag{1.1}$$

where the functional Γ_k is a scale-dependent version of the effective action (EA), generally called the effective average action (EAA) and $t = \log(k/k_0)$. We refer to [6–10] for reviews of this equation and its applications. The one-loop nature of the FRGE is manifest in the presence of a single trace (momentum integration). In fact, the FRGE can be represented graphically as

$$\partial_t\Gamma_k = \frac{1}{2}$$  , (1.2)

*abaldazz@sissa.it
†percacci@sissa.it
‡luca.zambelli@uni-jena.de

[1] In the FRG one follows the flow of renormalized quantities. Therefore, for a meaningful comparison, we have to supplement dimreg by a renormalization prescription.

[2] The "regulator" only cuts off the IR end of the propagator and it does not remove UV divergences from the functional integral. However, when one computes the derivative of the effective action with respect to k, one is taking the difference of two functional integrals that only differ in their low energy parts, and the UV divergences cancel. In practice, the trace in (1.1) is made UV finite by the presence of the term $\frac{dR_k}{dt}$.

where the double line represents the full propagators and the crossed circle represents the insertion of the regulator $\partial_t R_k$. The counterpart of this simplicity is that the equation is only exact if one takes into account *all* possible terms in the action. Since it is practically impossible to solve the exact equation, its effectiveness hinges crucially on a good choice of approximation. There are three main systematic expansion schemes. We briefly recall their definition, and then discuss the relation among them, and to standard perturbation theory.

A. Loop expansion

This is an expansion in powers of $\hbar$. We write for the EAA:

$$\Gamma_k[\phi] = S_\Lambda[\phi] + \sum_{L=1}^{n} \hbar^L \Gamma_{L,k}[\phi]. \tag{1.3}$$

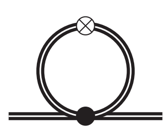
Inserting (1.3) in the flow equation (1.1) one can reproduce the usual beta functions of perturbation theory. First, introducing S in the right-hand side (rhs) of (1.1), one calculates the one-loop beta functional $\partial_t \Gamma_{1,k}$. Integrating over k from Λ to k' gives the one-loop EAA $\Gamma_{1,k'}$, and using this in the rhs of (1.1) one calculates the two-loop beta functional $\partial_t \Gamma_{2,k}$. The procedure can be iterated. Since in many cases the loop expansion coincides with the expansion in the marginal coupling constant, this approximation scheme is close to standard weak-coupling perturbation theory.

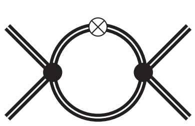
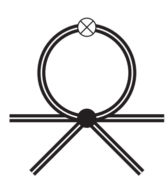
B. Vertex expansion

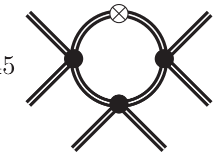
The EAA can be Taylor expanded in powers of the field:

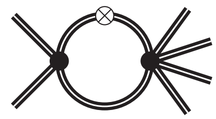
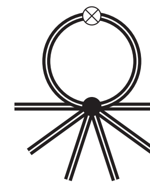
$$\Gamma_k[\phi] = \sum_n \int_{p_1} \dots \int_{p_n} \Gamma_k^{(n)}(p_1, \dots, p_n)\phi(p_1)\dots\phi(p_n), \tag{1.4}$$

where p_n are the external momenta. By functionally differentiating Eq. (1.1) one obtains an infinite sequence of flow equations for the n-point functions $\Gamma^{(n)}$. The vertex expansion consists in truncating this sequence at some finite order. The first three equations of the sequence for a $\mathbb{Z}_2$-invariant scalar theory can be represented graphically as follows:

$\partial_t \Gamma_k^{(2)} = -\frac{1}{2}$  , (1.5a)

$\partial_t \Gamma_k^{(4)} = 3$  $-\frac{1}{2}$  , (1.5b)

$\partial_t \Gamma_k^{(6)} = -45$ 

$+\,15$  $-\frac{1}{2}$  . (1.5c)

Here the black dots represent full vertices. The vertex expansion is clearly a good approximation in weak field situations, and is widely used in particle physics, where one generally deals with just a few quanta of the field. In this approximation one retains the full momentum dependence.

C. Derivative expansion

When one is interested in low energy phenomena, one can expand the action in powers of derivatives. This is close to many applications of the effective field theory approach. For a single scalar field the expansion starts with

$$\Gamma_k[\phi] = \int \mathrm{d}x \left(V_k(\phi) + \frac{1}{2} Z_k(\phi)(\partial\phi)^2 + O(\partial^4) \right) \tag{1.6}$$

where V_k and Z_k are arbitrary functions of the field. Inserting in Eq. (1.1) one obtains flow equations for V_k, Z_k etc. This is complementary to the vertex expansion, because one retains the full field dependence, but only the lowest powers of momentum.

These expansions give rise to different forms of perturbation theory, where different parameters are assumed to be small, and a statement that is perturbative in one expansion is generally nonperturbative in the others. Thus for example, the leading order of the derivative expansion, which is called the local potential approximation (LPA), consists in retaining in (1.6) only the running potential V_k and to put $Z_k = 1$. The beta function of the potential that can be obtained in this way from the FRGE contains information about infinitely many orders of the vertex expansion, and to all loop orders. If furthermore the potential is assumed to be a finite polynomial, then one is working simultaneously in the derivative and vertex expansion. Similarly, truncating the vertex expansion to a finite order gives n-point functions that include all orders of the derivative expansion and of the loop expansion, and the EA calculated at a given order of the loop expansion contains information that includes all orders of the derivative and vertex expansion.

Regardless of the choice of approximation scheme for the FRG equations, each one is able to reproduce standard epsilon-expansions for any regulator choice, if a high-enough order of approximation is considered. This has been shown since the early years of the FRG [1,11,12]. Yet again, in comparison to standard FRG schemes, dimreg/$\overline{\mathrm{MS}}$ occupies a somewhat privileged position for this kind of calculations, as they can be performed more simply in the latter scheme. For a nice direct comparison of the two kinds of computations, FRG versus dimreg/$\overline{\mathrm{MS}}$, see for instance Ref. [13].

In practice, in applications of perturbative quantum field theory to particle physics, one generally considers two-, three- and four-point functions at a finite order of the loop expansion, and therefore one is working simultaneously in the vertex and in the loop expansion. This is what we shall refer to as "standard perturbation theory." A different implementation of the weak coupling expansion is possible in presence of background fields, since perturbation theory can then account for the full dependence of the vertices on the latter variables. This is what might be called "functional perturbation theory." The application of dimreg to such functional methods [14–16] has been recently revived in the study of conformal field theories [17]. In these respects, our study could be interpreted as an attempt to extend these methods to the nonperturbative domain.

Focusing again on a scalar field and starting from a bare action that contains only a quartic interaction $\lambda\phi^4$, one can compute the two-, three- and four-point functions at some finite loop order L, by means of a functional integral, Feynman diagrams etc. How does one obtain such higher-loop information from the FRGE, which is a one-loop equation? One has to recall that the FRGE is only exact when one uses the full propagator and the full vertices. The propagators and vertices that appear in the vertex expansion can be expanded in loops, and this gives rise to the higher-loop effects in the beta function. Integrating the flow from a bare action S at some UV cutoff scale Λ down to $k = 0$ gives the desired terms in the EA. We shall discuss this in some detail in Sec. VI.

In summary, one of the most interesting features of the FRGE is the availability of various approximation schemes that sometimes allow us to follow the flow of infinitely many couplings in a single stroke and to go beyond standard perturbation theory. On the other hand, the arbitrariness in the choice of the regulator means that much of the information contained in the flow is unphysical. One has to learn to extract physical information from it.

Since the strengths and weaknesses of the FRGE and of dimreg/$\overline{\mathrm{MS}}$ are quite complementary, it would be useful to transfer some of the strengths of one method to the other, or at least to use them in a complementary way, so as to overcome the respective weaknesses. This paper is a first attempt in this direction. The main question that we shall address is the following: is there a choice of regulator that reproduces the beta functions of the $\overline{\mathrm{MS}}$ scheme in the standard perturbative domain? We provide here a positive answer to this question: we show that by bending the standard rules and procedures of the FRG it is possible to reproduce the results of dimreg/$\overline{\mathrm{MS}}$, at least up to two loops. In this way, we will make manifest the kind of unphysical features that one has to impose on a regulator so as to reproduce the results of dimreg. For this reason we will talk about a "pseudoregulator" that, upon use in the FRG equation, reproduces the beta functions of $\overline{\mathrm{MS}}$.

More importantly, having shown that the $\overline{\mathrm{MS}}$ pseudoregulator reproduces the results of dimreg in the perturbative domain, we have a tool that potentially provides a nonperturbative extension of dimreg/$\overline{\mathrm{MS}}$. We will indeed show that with the pseudoregulator one can find and study all multicritical fixed points in two dimensions, as well as the critical Sine-Gordon theory. It is remarkable that in this way one can even write the potentials of these models in closed form. Furthermore, the use of this tool is not limited to even dimensions, as we shall show by considering the Wilson-Fisher fixed point in three dimensions.

The use of dimreg is really of great advantage when one deals with gauge theories. We will not attempt here to use the $\overline{\mathrm{MS}}$ pseudoregulator in the FRGE for gauge theories, but we will show that it has definite advantages in the treatment of nonlinearly realized symmetries.

The paper is organized as follows. In Sec. II we state the problem in a precise way, in the most straightforward and simplified setting: the case of a linear scalar field theory in the LPA. The solution of the problem and our pseudoregulator are given in Sec. III. We also explore some of the intrinsic freedom in the construction of the pseudoregulator, and we exhibit a one-parameter family of regulators that continuously connects the results of standard FRG regulators with those of the $\overline{\mathrm{MS}}$ pseudoregulator.

In Sec. IV, we account for the inclusion of the field's anomalous dimension. This transition only requires minor generalizations of the pseudoregulator, allowing for some more free parameters, which come along with corresponding forms of "RG improvement" in the one-loop flow equations. Section V further shows that the same pseudoregulator is appropriate for the $O(\partial^2)$ of the derivative expansion. This discussion offers us the chance to address two exploratory applications of the $\overline{\mathrm{MS}}$ functional RG equations. The first is the description of nonperturbative critical phenomena, namely two-dimensional multicritical scalar theories. We perform this study with the main goal to test the physical content of the "RG improvement," which is the imprint of the FRG origin of our $\overline{\mathrm{MS}}$ equations. The second application is provided by nonlinear $O(N)$ models in two dimensions, whose interest in this context lies in the interplay between nonlinearly realized symmetries and the FRG equations.

An even more general truncation is needed to reproduce the two-loop $\overline{\mathrm{MS}}$ beta functions in massive four-dimensional ϕ^4 theory [the perturbatively renormalizable linear $O(N)$ model]. This is discussed in Sec. VI. This exercise serves as a proof that by means of the FRG and our pseudoregulator one can, by considering large-enough truncations, obtain $\overline{\rm MS}$ flow equations which are beyond a one-loop form.

Finally, in Sec. VII we explore the role of dimensionality in our construction. In fact, while dimreg/$\overline{\rm MS}$ is usually at work in an even number of dimensions d, the FRG equations can be obtained and applied for continuous d. We show that the latter feature can be preserved while taking the limit from the FRG to $\overline{\rm MS}$.

Section VIII contains some concluding remarks and an outlook on possible future developments. Several Appendixes account for the details and the subtleties of the computations presented.

II. STATEMENT OF THE PROBLEM

In order to make our idea more precise, let us begin by stating the conditions that are generally imposed on a regulator for the FRG equation. A regulator is an additive modification of the inverse two-point function, and is therefore a function of a single momentum q, or rather its modulus $z = q^2$, depending on an scale k. The regulator, which is denoted $R_k(z)$, is typically assumed to satisfy the following conditions:

(1) To be positive (must suppress modes).
(2) To be monotonically increasing with k, for all z.
(3) To be monotonically decreasing with z, for all k.
(4) $\lim_{k\to 0} R_k(z) = 0$ for all z.
(5) For $z > k^2$, R_k goes to zero sufficiently fast, e.g., as an exponential.
(6) $R_k(0) = k^2$.

The first three conditions are obvious properties of a cutoff. The fourth guarantees that the path integral reproduces the standard partition function for $k = 0$. The fifth condition ensures that high momentum modes are integrated out unsuppressed and guarantees the UV convergence of the rhs of the flow equation. The sixth and last condition provides a sort of normalization. For certain purposes, one may sometimes forgo the last two conditions and consider cutoffs that either do not decrease very fast for large momenta or even diverge when $z \to 0$. These six conditions are useful in that they provide a clear physical interpretation for the coarse graining implemented by the regulator, and they ensure control on the UV and IR end points of the momentum integrals. However, they are not needed in the derivation of the FRG equation, which would keep its exact one-loop form for any regulator choice.

Both z and the function $R_k(z)$ have dimension of mass squared, so we can write

$$R_k(z) = k^2 r(y), \qquad y = z/k^2, \tag{2.1}$$

where r is a dimensionless "cutoff profile." The following are typical choices:

$$r(y) = \frac{y}{e^y - 1}, \tag{2.2}$$

$$r(y) = \frac{y^2}{e^{y^2} - 1}, \tag{2.3}$$

$$r(y) = (1 - y)\theta(1 - y). \tag{2.4}$$

The third choice has been argued to provide "optimized" results, in a certain class of models and truncations [18,19]. For certain purposes its nondifferentiability is an issue, but it has the great advantage of allowing an analytic evaluation of momentum integrals. Note that k plays the role of an *infrared* cutoff: its effect is to give a mass of order k to the modes with $\sqrt{z} < k$, and no mass to the modes with $\sqrt{z} > k$.

Introducing the cutoff in the functional integral and then performing the Legendre transform leads to the FRGE (1.1). We note that the trace on the rhs is IR and UV finite, and that the equation contains no reference to a bare action or UV physics.

In order to extract useful information from the exact equation one has to approximate it in some way. For definiteness, let us focus on a single scalar field in the LPA

$$\Gamma_k(\phi) = \int \mathrm{d}^d x \left(\frac{1}{2} (\partial_\mu \phi)^2 + V_k(\phi) \right). \tag{2.5}$$

Inserting in the FRGE we obtain the "beta functional"

$$\partial_t V_k = \frac{1}{2(4\pi)^{d/2}} Q_{d/2} \left[\frac{\partial_t R_k}{P_k + V_k''} \right], \tag{2.6}$$

where

$$Q_n[W] = \frac{1}{\Gamma(n)} \int_0^\infty \mathrm{d}z z^{n-1} W(z) \tag{2.7}$$

is the momentum integral. Assuming $\mathbb{Z}_2$ symmetry and Taylor expanding the potential

$$V_k(\phi) = \sum_n \frac{\lambda_{2n}(k)}{(2n)!} \phi^{2n}, \tag{2.8}$$

we can derive infinitely many beta functions $\beta_{2n} = k \frac{\partial \lambda_{2n}}{\partial k}$. These are obtained by expanding both sides of (2.6) in powers of the field and equating the coefficients. For arbitrary regulator, and in any dimension, for the first few couplings this leads to

$$\beta_2 = -\frac{1}{2(4\pi)^{d/2}}\lambda_4 Q_{d/2}\left[\frac{\partial_t R_k}{(P_k+\lambda_2)^2}\right], \tag{2.9a}$$

$$\beta_4 = \frac{1}{2(4\pi)^{d/2}}\left(6\lambda_4^2 Q_{d/2}\left[\frac{\partial_t R_k}{(P_k+\lambda_2)^3}\right] - \lambda_6 Q_{d/2}\left[\frac{\partial_t R_k}{(P_k+\lambda_2)^2}\right]\right), \tag{2.9b}$$

$$\beta_6 = \frac{1}{2(4\pi)^{d/2}}\left(-90\lambda_4^3 Q_{d/2}\left[\frac{\partial_t R_k}{(P_k+\lambda_2)^4}\right] + +30\lambda_4\lambda_6 Q_{d/2}\left[\frac{\partial_t R_k}{(P_k+\lambda_2)^3}\right] - \lambda_8 Q_{d/2}\left[\frac{\partial_t R_k}{(P_k+\lambda_2)^2}\right]\right). \tag{2.9c}$$

We note that these are one-loop beta functions, since no resummation is involved. They coincide with the first three equations of the vertex expansion, namely Eqs. (1.5), when the n-point functions are evaluated at zero momentum. In order to have more explicit formulas, we can use the optimized regulator (2.4) that gives

$$Q_n\left[\frac{\partial_t R_k}{(P_k+\lambda_2)^\ell}\right] = \frac{2}{\Gamma(n+1)}\frac{k^{2(n+1)}}{(k^2+\lambda_2)^\ell}. \tag{2.10}$$

Then, the first beta functions are

$$\beta_2 = -\frac{k^{d+2}}{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1)}\frac{\lambda_4}{(k^2+\lambda_2)^2}, \tag{2.11a}$$

$$\beta_4 = \frac{k^{d+2}}{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1)}\left(\frac{6\lambda_4^2}{(k^2+\lambda_2)^3} - \frac{\lambda_6}{(k^2+\lambda_2)^2}\right), \tag{2.11b}$$

$$\beta_6 = \frac{k^{d+2}}{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1)}\left(-90\frac{\lambda_4^3}{(k^2+\lambda_2)^4} + 30\frac{\lambda_4\lambda_6}{(k^2+\lambda_2)^3} - \frac{\lambda_8}{(k^2+\lambda_2)^2}\right). \tag{2.11c}$$

One can also calculate the beta functions of this theory at one loop using dimreg/$\overline{\mathrm{MS}}$. The corresponding expressions read

$$\beta_2 = \frac{(-1)^{d/2}}{\Gamma(\frac{d}{2})(4\pi)^{d/2}}\lambda_4\lambda_2^{d/2-1}, \tag{2.12a}$$

$$\beta_4 = \frac{(-1)^{d/2}}{(4\pi)^{d/2}}\left(3\lambda_4^2\frac{\lambda_2^{d/2-2}}{\Gamma(\frac{d}{2}-1)} + \lambda_6\frac{\lambda_2^{d/2-1}}{\Gamma(\frac{d}{2})}\right), \tag{2.12b}$$

$$\beta_6 = \frac{(-1)^{d/2}}{(4\pi)^{d/2}}\left(15\lambda_4^3\frac{\lambda_2^{d/2-3}}{\Gamma(\frac{d}{2}-2)} + 15\lambda_4\lambda_6\frac{\lambda_2^{d/2-2}}{\Gamma(\frac{d}{2}-1)} + \lambda_8\frac{\lambda_2^{d/2-1}}{\Gamma(\frac{d}{2})}\right). \tag{2.12c}$$

In fact, one can even derive a functional perturbative beta function for V, analogous to (2.6) [17]. (We shall discuss this in Sec. III A.)

The beta functions obtained by the two procedures are strikingly different. In the beta functions derived from the FRG, the dimension is carried by k, and there are denominators that automatically produce decoupling when one crosses the mass threshold $k^2 = \lambda_2$. In the beta functions of dimreg the dimension is always carried by powers of λ_2, and threshold effects are not accounted for. In fact such beta functions are only valid at energies much higher that λ_2.

The difference persists also in the massless limit. In the beta functions obtained from the FRGE, it is enough to put $\lambda_2 = 0$. In the dimreg calculation, the massless limit has to be taken after fixing the dimension. Then, many terms are absent from the start. For example, in $d = 4$ the first term in (2.12c) is absent because of the Gamma in the denominator. Then taking $\lambda_2 \to 0$ kills the last term. Altogether in $d = 4$ and in the massless limit the beta function of λ_{2n} is proportional to $\lambda_4\lambda_{2n}$, for all $n = 1, 2, 3\ldots$

In spite of these differences, there is a close relationship between these two sets of beta functions. To see this, note that, for a generic regulator, the Q functional with $\ell = n+1$ and $\lambda_2 = 0$ (which is dimensionless) is universal, i.e.,

$$Q_n\left[\frac{\partial_t R_k}{P_k^{n+1}}\right] = \frac{2}{\Gamma(n+1)}, \tag{2.13}$$

independently of the shape of the regulator. The reason for this is that in this case the integrand is a total derivative:

$$\int_0^\infty \mathrm{d}z z^{n-1}\frac{\partial_t R_k}{P_k^{n+1}} = \int_0^\infty \mathrm{d}y y^{n-1} 2\frac{r(y) - y r'(y)}{(y + r(y))^{n+1}}, = \int_0^\infty \mathrm{d}y \frac{2}{n}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{y}{y+r(y)}\right)^n. \tag{2.14}$$

The universal result will hold even if the regulator does not satisfy all the requirements that are listed in the beginning of Sec. II: it is enough that $r(\infty) = 0$ and $r(0) > 0$.

In the presence of a mass λ_2, we can expand the Q functional for $k^2 > \lambda_2$:

$$Q_n\left[\frac{\partial_t R_k}{(P_k+\lambda_2)^\ell}\right] = \sum_{j=0}^\infty \frac{(-1)^j\Gamma(\ell+j)}{\Gamma(\ell)\Gamma(j+1)}\lambda_2^j Q_n\left[\frac{\partial_t R_k}{P_k^{j+\ell}}\right]. \tag{2.15}$$

We see that the term $j = n - \ell + 1$ in the sum is universal and equal to

$$\frac{2(-1)^{n-\ell+1}}{\Gamma(\ell)\Gamma(n-\ell+2)}\lambda_2^{n-\ell+1}. \tag{2.16}$$

The beta functions of dimensional regularization consist exactly of all these universal terms, all the remaining ones being set simply to zero.

This exposes very clearly both the strength and the weakness of dimreg. The universal terms in the beta functions are the ones that may be more easily linked to physically observable quantities, and dimreg very efficiently extracts only these terms. On the other hand, threshold effects do not have a universal form, but are still physical, and dimreg does not see them.

The main question we wish to address is whether the beta functions of dimreg can be obtained *directly* from the FRGE. This will be the case provided R_k is such that

$$Q_n\left[\frac{\partial_t R_k}{(P_k+\lambda_2)^\ell}\right] = \frac{2(-1)^{n-\ell+1}}{\Gamma(\ell)\Gamma(n-\ell+2)}\lambda_2^{n-\ell+1}. \tag{2.17}$$

Thus the question becomes one about the existence of a regulator that gives (2.17). It is immediately clear that any standard regulator, satisfying the criteria given at the beginning of this section, cannot fulfill this requirement. To understand why, it is sufficient to consider the case $\lambda_2 = 0$, in which case the requirement (2.17) becomes

$$Q_n\left[\frac{\partial_t R_k}{P_k^\ell}\right] = \frac{2}{\Gamma(n+1)}\delta_{\ell,n+1}. \tag{2.18}$$

This implies that

$$\frac{1}{\Gamma(n)}\int_0^\infty \mathrm{d}y y^{n-1}\frac{r-yr'}{(y+r)^\ell} = \frac{1}{\Gamma(n+1)}\delta_{\ell,n+1}. \tag{2.19}$$

Using integration by parts and the standard properties of regulators, we obtain

$$\left(1-\frac{\ell-1}{n}\right)Q_n\left[\frac{R_k}{P_k^\ell}\right] = \left(\frac{\ell}{n+1}-\frac{\ell-1}{n}\right)\frac{\delta_{\ell,n+1}}{\Gamma(n+1)} \tag{2.20}$$

that, for $\ell \neq n+1$, gives $Q_n[\frac{R_k}{P_k^\ell}] = 0$. Since the integrand in this Q functional is positive, this implies that $R_k = 0$. While R_k cannot be identically vanishing, it appears possible to reproduce $\overline{\mathrm{MS}}$ beta functions by giving up some of the requirements that are usually made of regulators and taking the $R_k \to 0$ limit in a suitable way, as we shall discuss in the next section.

III. THE $\overline{\mathrm{MS}}$ PSEUDOREGULATOR

The desired "pseudoregulator" depends, in addition to the scale k, also on a dimensionless parameter ϵ and a mass μ, which play a similar role as the ϵ and μ parameters of dimreg:

$$R_k(z) = \lim_{\epsilon\to 0} z\left[\left(\frac{zk^2}{\mu^4}\right)^\epsilon - 1\right], \tag{3.1}$$

or equivalently

$$r(y) = \lim_{\epsilon\to 0} y\left[\left(\frac{y}{\tilde{\mu}^4}\right)^\epsilon - 1\right], \tag{3.2}$$

where $\tilde{\mu} = \mu/k$. A derivation and an explaination of this ansatz are given in Appendix A. Calculations have to be performed with a finite positive ϵ and the limit $\epsilon \to 0$ must be taken at the end of all calculations. Note that expanding for small ϵ

$$R_k(z) = \epsilon z \log\left(\frac{k^2 z}{\mu^4}\right) + O(\epsilon^2). \tag{3.3}$$

The function (3.1) grossly violates the defining properties of a regulator, as spelled out in the beginning of Sec. II. Aside from the fact that it vanishes in the limit $\epsilon \to 0$, it is a growing function of z and goes to zero for $z \to 0$. Nevertheless, it does what we asked for. Calculating the Q functional, we obtain

$$Q_n\left[\frac{\partial_t R_k}{(P_k+m^2)^\ell}\right] = k^{2(n-\ell+1)}\frac{2}{\Gamma(n)}\int_0^\infty \mathrm{d}y y^{n-1}\frac{r - y\partial_y r - \tilde{\mu}/2\partial_{\tilde{\mu}} r}{(y+r+\tilde{m}^2)^\ell},$$
$$= \lim_{\epsilon\to 0}\frac{2\epsilon}{1+\epsilon}m^{2(n-\ell+1)}\left(\frac{\mu^2}{km}\right)^{\frac{2n\epsilon}{1+\epsilon}}\frac{\Gamma(1+\frac{n}{1+\epsilon})\Gamma(\ell-1-n+\frac{n\epsilon}{1+\epsilon})}{\Gamma(n)\Gamma(\ell)}.$$

Here we introduced the more conventional notation m^2 for the mass parameter λ_2, and defined $\tilde{m} = m/k$. The integral in this Q functional is convergent for $\ell > (n+1+\epsilon)/(1+\epsilon)$ and is defined elsewhere by analytic continuation. In the limit $\epsilon \to 0$ it goes to zero except at the points where the second Γ function in the numerator has a pole, namely when $\ell - n - 1$ is zero or a negative integer. In this way we recover (2.17). We note that for $n \leq \ell - 2$ the final result (2.17) is identically zero because of the presence of the gamma function on the denominator. Since ℓ is an integer, n must be integer in order to have a nonzero result: since, for the beta functions of the LPA, $n = d/2$, this implies that only in even dimensions we get a nonzero result. This agrees with the standard lore that dimreg only works in even dimensions.

Sometimes one needs the Q functionals for $n \leq 0$. One can obtain them by observing that

$$Q_n\left[\frac{\partial_t R_k}{(P_k + m^2)^\ell}\right] = \frac{(-1)^j}{\Gamma(n+j)}\int_0^\infty \mathrm{d}z z^{n+j-1}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^j \frac{\partial_t R_k(z)}{(P_k(z)+m^2)^\ell}, \quad (3.4)$$

where j is an integer such that $n + j > 0$. Evaluating this expression for the pseudoregulator, we get

$$Q_n\left[\frac{\partial_t R_k}{(P_k + m^2)^\ell}\right] = 2\delta_{-n,0}\delta_{\ell,1}. \quad (3.5)$$

This agrees with the analytic continuation of (2.17).

A. The effective potential in the LPA

We complete the discussion of the LPA approximation of a scalar theory by giving the functional equation for the potential:

$$\partial_t \tilde{V}_k = -d\tilde{V}_k + \left(\frac{d}{2} - 1\right)\tilde{\phi}\tilde{V}'_k + c_d(-\tilde{V}''_k)^{\frac{d}{2}}, \quad (3.6)$$

where $\tilde{V}_k = k^{-d}V_k$, $\tilde{\phi} = k^{1-\frac{d}{2}}\phi$ and $c_d = \frac{1}{(4\pi)^{d/2}\Gamma[\frac{d}{2}+1]}$. This agrees with the beta functional in $d = 4$ discussed in [17]. For comparison, the optimized regulator leads to the form

$$\partial_t \tilde{V}_k = -d\tilde{V}_k + \left(\frac{d}{2} - 1\right)\tilde{\phi}\tilde{V}'_k + c_d \frac{1}{1 + \tilde{V}''_k}, \quad (3.7)$$

where $c_d = \frac{1}{(4\pi)^{d/2}\Gamma[\frac{d}{2}+1]}$. We observe that (3.6) picks exactly the terms of the expansion of (3.7) with the right power of $\tilde{V}''$ to give a dimension-d operator. Equations (3.6) and (3.7) are one-loop results, and in this sense can be said to be perturbative, but they contain infinitely many terms of the vertex expansion and thus are not perturbative in the standard sense.

Equation (3.6) can be applied only to even dimensions, so it does not admit the Wilson-Fisher fixed point as a solution in $d = 3$. This was to be expected, since dimreg only works in even dimensions. We anticipate however that generalizations to continuous d (including also odd integers) are possible, and will be discussed in Sec. VII.

Equation (3.6) has been used in [20–24] to obtain several new results on statistical models. In $d = 2$ the corresponding fixed-point equation has the critical Sine-Gordon solution

$$V_* = -\frac{m^2}{8\pi}\cos(\sqrt{8\pi}\phi), \quad (3.8)$$

where m is an arbitrary mass. This result holds independently of the shape of the regulator [25].

A related question is whether this pseudoregulator can reproduce some of the (multi)critical theories in $d = 2$. It turns out that the answer is positive, as we shall discuss in greater detail in Sec. VA, where we consider a larger truncation.

B. An external field problem

As a somewhat different application, let us consider a free scalar field with mass m^2 in an external metric $g_{\mu\nu}$. Let $\Delta = -\nabla_\mu\nabla^\mu$ be the covariant Laplacian. In this case we can take over previous formulas for the pseudoregulator, simply reinterpreting $z = \Delta$. We refer to [26] for several examples of this type, both in Lorentzian and Euclidean signature. The Euclidean beta functional is

$$\partial_t\Gamma_E^k = \frac{1}{2(4\pi)^{d/2}}\sum_{j=0}^\infty Q_{\frac{d}{2}-j}\left[\frac{\partial_t R_k}{P_k + \lambda_2}\right]B_{2j}(\Delta) = \frac{(-1)^{d/2}\lambda_2^{d/2}}{(4\pi)^{d/2}}\sum_{j=0}^{d/2}\frac{(-1)^j\lambda_2^{-j}}{\Gamma(\frac{d}{2} - j + 1)}B_{2j}(\Delta), \quad (3.9)$$

where $B_{2j}(\Delta)$ are the heat kernel coefficients of the operator Δ and consist of integrals of powers of the curvature tensor and its covariant derivatives. Note that the sum terminates at $j = d/2$, because of the poles in the Gamma function in the denominator. We get

$$\partial_t\Gamma_E^k = -\frac{1}{4\pi}(\lambda_2 B_0(\Delta) - B_2(\Delta))$$

for $d = 2$, and

$$\partial_t\Gamma_E^k = \frac{1}{(4\pi)^2}\left(\frac{\lambda_2^2}{2}B_0(\Delta) - \lambda_2 B_2(\Delta) + B_4(\Delta)\right)$$

for $d = 4$. These formulas give the cutoff dependence of the effective action for the external metric, generated by the scalar field. By integrating these formulae from some ultraviolet scale Λ down to $k = 0$, one obtains the effective action for the metric. See [27,28] for some calculations of this type.

C. A first generalization

In the definition (3.1) we have used an external, arbitrary mass scale μ. One could use instead a dimensionful coupling of the theory. In particular, in a massive theory, one could use m instead of μ. In the discussion of the two-loop beta functions, it will be convenient to actually use a mixture of the two. Therefore, let us generalize the pseudoregulator to

$$R_k(z) = \lim_{\epsilon\to 0} z\left[\left(\frac{zk^2}{m^{2b}\mu^{4-2b}}\right)^{\epsilon} - 1\right]. \tag{3.10}$$

Note that m is a running parameter, so when we evaluate the Q functional (2.17) it gives rise to an additional term depending on the beta function of the mass $\beta_{m^2} = \partial_t m^2$:

$$Q_n\left[\frac{\partial_t R_k}{(P_k+m^2)^\ell}\right] = \frac{2(-1)^{n-\ell+1}}{\Gamma(\ell)\Gamma(n-\ell+2)} \times\left(1-\frac{b}{2}\frac{\beta_{m^2}}{m^2}\right)m^{2(n-\ell+1)}. \tag{3.11}$$

The term with the beta function of the mass is a higher-loop effect, so at one loop this pseudoregulator still reproduces the result of dimreg.

We note that the above discussion could be generalized replacing m by any combination of couplings with the dimension of mass. This would give rise to additional beta functions in the rhs of (3.11) and may be useful in higher-loop calculations.

In the massless case ($m = 0$) one has to set $b = 0$ and introduce by hand an IR regulator in the Q functionals:

$$Q_n\left[\frac{\partial_t R_k}{P_k^\ell}\right] \mapsto Q_n\left[\frac{\partial_t R_k}{(P_k+\mu^2)^\ell}\right]. \tag{3.12}$$

The limit $\mu \to 0$ has to be taken in the very end. Note that this IR regulator mass is not necessarily equal to the dimreg parameter μ, but we will not need this degree of generality, so the same mass will be used in both rôles. Then we obtain

$$Q_n\left[\frac{\partial_t R_k}{(P_k+\mu^2)^\ell}\right] = \frac{2(-1)^{n-\ell+1}}{\Gamma(\ell)\Gamma(n-\ell+2)}\mu^{2(n-\ell+1)}. \tag{3.13}$$

As we already said above, this formula gives zero for $n < l-1$. Taking the limit for $\mu \to 0$ we get zero for $n > l+1$. So the result is

$$Q_n\left[\frac{\partial_t R_k}{P_k^\ell}\right] = \lim_{\mu\to 0} Q_n\left[\frac{\partial_t R_k}{(P_k+\mu^2)^\ell}\right] = \frac{2\delta_{\ell,n+1}}{\Gamma(n+1)}. \tag{3.14}$$

We note that only one combination of ℓ and n gives a nonvanishing result, which corresponds to the universal result of Eq. (2.13).

D. Interpolation with the optimized regulator

The Q functionals for the optimized regulator have been given in (2.10). Now let us consider the following one-parameter family of regulators:

$$r_a(y) = a(1-y)\theta(1-y). \tag{3.15}$$

For $a \neq 1$ they violate the normalization condition, but otherwise they are acceptable regulators. In fact, the parameter a is used to optimize the results [29–31].

The corresponding Q functionals are given by

$$Q_n\left[\frac{\partial_t R_k}{(P_k+m^2)^\ell}\right] = \frac{2ak^{2(n-l+1)}}{(a+\tilde{m}^2)^\ell}\frac{1}{\Gamma(n+1)} \times {}_2F_1\left(\ell,n,n+1,-\frac{1-a}{a+\tilde{m}^2}\right). \tag{3.16}$$

If $\ell < n+1$ and $\tilde{m} > 0$, then these are monotonically increasing functions of a, which are equal to (2.10) for $a = 1$ and decrease monotonically to zero when $a \to 0$. If $\ell > n+1$ and $\tilde{m} > 0$, they grow as functions of a and they go to zero when $a \to 0$. For $\tilde{m} = 0$ they are monotonic functions on the interval $0 < a < 1$, with either a zero or a pole for $a \to 0$ depending whether $\ell < n+1$ or $\ell > n+1$. Remarkably, the Q functionals $\ell = n+1$, $\tilde{m} = 0$ are independent of a and equal to (2.13). Thus, the universality of these Q functionals is not spoiled by the regulator not being normalized.

On the other hand, if we set $a = 0$ the regulator vanishes identically and so do all the beta functions, including the universal ones. This means that the limit $a \to 0$ is not continuous. We would like to find a way to obtain at least the universal beta functions also for $a = 0$. One can achieve this by introducing an additional parameter ϵ. Consider the following interpolating regulator $R_k = k^2 r(y, \tilde{m}^2, \epsilon, a)$, with

$$r(y,\tilde{m}^2,\epsilon,a) = (a+(1-a)\tilde{\mu}^{-2(2-b)\epsilon}\tilde{m}^{-2b\epsilon}y^{1+\epsilon} - y)\theta\left(1-\frac{a}{a+\epsilon}y\right). \tag{3.17}$$

For $\epsilon \to 0$ it reduces to (3.15) and for $a \to 0$ it reduces to (3.10). Thus we can go continuously from the optimized regulator to the pseudoregulator reproducing dimreg by following a curve of the form shown in Fig. 1. In this way the limit $a \to 0$ can be made continuous. The price one pays is that for $\epsilon \neq 0$ one does not have a good regulator in the sense of Sec. II. In any case we obtain the desired result that all the nonuniversal beta functions go continuously to zero, while the universal ones remain constant.

Let us see how the evaluation of the Q-functionals proceeds. We have

$$Q_n\left[\frac{\partial_t R_k}{(P_k+m^2)^\ell}\right] = \frac{k^{2(n-\ell+1)}}{\Gamma(n)}\int_0^\infty \mathrm{d}y y^{n-1}\frac{2r-2y\partial_y r-2\tilde{\mu}^2\partial_{\tilde{\mu}^2}r-\tilde{m}^2(2-\frac{\beta_{m^2}}{m^2})\partial_{\tilde{m}^2}r}{(y+r+\tilde{m}^2)^\ell}. \tag{3.18}$$

Using (3.17), the fraction in the integral can be written as the sum of three pieces:

$$\begin{aligned}
1) &= \frac{2a}{((1-a)\tilde{m}_b^{-4\epsilon}y^{1+\epsilon}+a+\tilde{m}^2)^\ell}\theta\left(1-\frac{ay}{a+\epsilon}\right),\\
2) &= \frac{2\epsilon(1-a)(1-\frac{b}{2}\frac{\beta_{m^2}}{m^2})\tilde{m}_b^{-4\epsilon}y^{1+\epsilon}}{((1-a)\tilde{m}_b^{-4\epsilon}y^{1+\epsilon}+a+\tilde{m}^2)^\ell}\theta\left(1-\frac{ay}{a+\epsilon}\right),\\
3) &= \frac{2(a+(1-a)\tilde{m}_b^{-4\epsilon}y^{1+\epsilon}-y)}{((1-a)\tilde{m}_b^{-4\epsilon}y^{1+\epsilon}+a+\tilde{m}^2)^\ell}\frac{a}{a+\epsilon}y\delta\left(1-\frac{ay}{a+\epsilon}\right),
\end{aligned} \tag{3.19}$$

where $\tilde{m}_b^2=\tilde{\mu}^{2-b}\tilde{m}^b$. Performing the integral, the Q functional is the sum of three pieces

$$\begin{aligned}
1) &= 2a\frac{(\frac{a+\epsilon}{a})^n}{(a+\tilde{m}^2)^\ell}\frac{1}{\Gamma(n+1)}\,{}_2F_1\left[\ell,\frac{n}{1+\epsilon},1+\frac{n}{1+\epsilon};\frac{(a-1)\tilde{m}_b^{-4\epsilon}(\frac{a+\epsilon}{a})^{1+\epsilon}}{a+\tilde{m}^2}\right],\\
2) &= \frac{2\epsilon(1-a)\tilde{m}_b^{-4\epsilon}}{\Gamma(n)(n+1+\epsilon)}\left(1-\frac{b}{2}\frac{\beta_{m^2}}{m^2}\right)\frac{(\frac{a+\epsilon}{a})^{n+1+\epsilon}}{(a+\tilde{m}^2)^\ell}\,{}_2F_1\left[\ell,1+\frac{n}{1+\epsilon},2+\frac{n}{1+\epsilon};\frac{(a-1)\tilde{m}_b^{-4\epsilon}(\frac{a+\epsilon}{a})^{1+\epsilon}}{a+\tilde{m}^2}\right],\\
3) &= \frac{2}{\Gamma(n)}\frac{(a+(1-a)\tilde{m}_b^{-4\epsilon}(\frac{a+\epsilon}{a})^{1+\epsilon}-\frac{a+\epsilon}{a})}{((1-a)\tilde{m}_b^{-4\epsilon}(\frac{a+\epsilon}{a})^{1+\epsilon}+a+\tilde{m}^2)^\ell}\left(\frac{a+\epsilon}{a}\right)^{n-1}.
\end{aligned}$$

Taking the limit $\epsilon\to 0$

$$\begin{aligned}
1) &\to \frac{2a}{(a+\tilde{m}^2)^\ell}\frac{1}{\Gamma(n+1)}\,{}_2F_1\left[\ell,n,1+n;\frac{a-1}{a+\tilde{m}^2}\right],\\
2) &\to 0,\\
3) &\to 0.
\end{aligned}$$

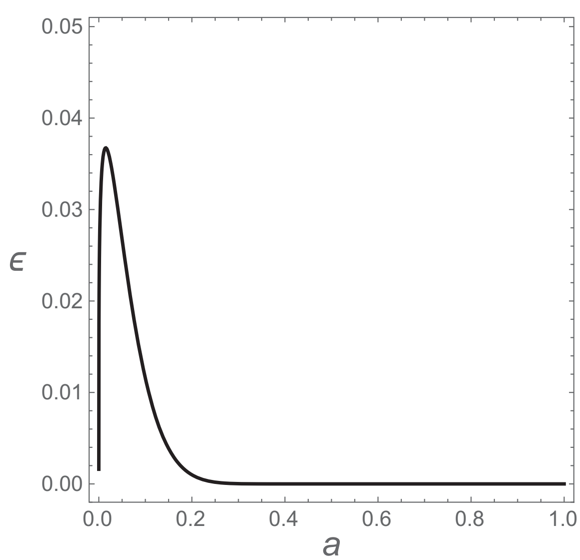


FIG. 1. A path in the $a-\epsilon$ plane interpolating smoothly between the optimized cutoff and the pseudocutoff of dimreg.

Then sending $a\to 1$ we get (2.10). If $a\neq 0$ we can also take the two limits in the reverse order and obtain the same result. For $a\to 0$ the previous expressions are not well defined, so in this case we have to take $\epsilon\to 0$ after the a limit. For $a\to 0$

$$\begin{aligned}
1) &\to 0,\\
2) &\to 2\epsilon\left(1-\frac{b}{2}\frac{\beta_{m^2}}{m^2}\right)\frac{\Gamma(1+\frac{n}{1+\epsilon})\Gamma(\ell-1-\frac{n}{1+\epsilon})}{(1+\epsilon)\Gamma(\ell)\Gamma(n)}\\
&\quad\times\left(\frac{\tilde{m}_b}{\tilde{m}}\right)^{\frac{4n\epsilon}{1+\epsilon}}\tilde{m}^{2(n-\ell+1+\frac{\epsilon n}{1+\epsilon})},\\
3) &\to 0.
\end{aligned}$$

Then taking the limit $\epsilon\to 0$ we get (3.11).

IV. BEYOND THE LPA

In any quantum field theory application, and in the FRG framework as well, the choice of a regularization scheme should be tailored to a specific model and computation. In the process of relaxing the approximations used to solve the exact FRG equations, it is thus inevitable to reconsider the regulator choice. In this section we discuss the adjustment of the $\overline{\mathrm{MS}}$ pseudoregulator to the transition from the LPA to the inclusion of the running wave function renormalization. In the following, after the construction of a more general family of pseudoregulators, we discuss its application to scalar field theory. We show how these pseudoregulators are appropriate for investigations within the LPA′ approximation, which differs from the LPA only for the inclusion of a field- and momentum-independent wave function renormalization factor. The next layer of complexity, namely the $O(\partial^2)$ derivative expansion including the field dependence of the wave function renormalization, will be addressed in Sec. V.

From the point of view of standard perturbation theory, the step from the LPA to the LPA′ already involves the resummation of an infinite class of Feynman diagrams—those self-energy-like one-particle-reducible corrections to the internal propagator lines which are accounted for by a nontrivial field's anomalous dimension—and therefore goes beyond finite-order perturbative calculations.

A. Rôle of the wave function renormalization

If the kinetic term in the action contains a nontrivial wave function renormalization factor $Z_k \neq 1$, one usually includes this global factor inside R_k

$$R_k(z) \mapsto Z_k R_k(z). \tag{4.1}$$

There are several reasons in favor of this choice. First of all, it allows to take over the regulators already working in the LPA, as the relevant regularized kinetic term is then in the functional form $z + R_k(z)$. Furthermore, it is motivated by the desired invariance under rigid rescalings of the fields, also called reparameterizations. In other words, it allows to remove Z_k from the flow equations by simply rescaling the fields according to their quantum dimension

$$d_\phi = \frac{d-2+\eta_k}{2}, \tag{4.2}$$

where

$$\eta_k = -\partial_t \log Z_k \tag{4.3}$$

is the field anomalous dimension. While the former motivation is just a matter of convenience, the latter is deeper and less arbitrary. In fact, this choice is the one that minimizes the spurious breaking of reparameterization invariance due to the truncation of the exact FRG equation [6,32–34].

Following the choice of Eq. (4.1). the flow equations receive further RG resummations encoded in the appearance of η_k on the rhs as

$$\partial_t R_k(z) \mapsto Z_k(\partial_t R_k(z) - \eta_k R_k(z)). \tag{4.4}$$

While the first term on the rhs gives rise to the Q functionals already discussed in Sec. II, the second term leads to the following new integrals

$$Q_n\left[\frac{R_k}{(P_k+m^2)^\ell}\right] = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\ell+j)}{\Gamma(\ell)\Gamma(j+1)} m^{2j} \times Q_n\left[\frac{R_k}{P_k^{j+\ell}}\right]. \tag{4.5}$$

Also for these new Q functionals we see that the term $j = n - \ell + 1$ has no explicit k dependence, but it is not universal. For instance, the exponential regulator of Eq. (2.2) would give

$$Q_n\left[\frac{R_k}{P_k^{n+1}}\right] = \begin{cases} \log(2) & n=1 \\ \log(\frac{4}{3}) & n=2 \\ \frac{1}{2}\log(\frac{32}{27}) & n=3 \end{cases}, \tag{4.6}$$

while the optimized regulator (2.4) leads to

$$Q_n\left[\frac{R_k}{P_k^{n+1}}\right] = \frac{1}{\Gamma(n+2)}. \tag{4.7}$$

This exemplifies the arbitrariness in the construction of an $\overline{\mathrm{MS}}$ pseudoregulator for calculations beyond the LPA.

If we straightforwardly apply the recipe Eq. (4.1), we obtain a divergent result:

$$Q_n\left[\frac{R_k}{(P_k+m^2)^\ell}\right] = \lim_{\epsilon\to 0}\left[\frac{(-m^2)^{n-\ell+1}}{\Gamma(\ell)\Gamma(n-\ell+2)} \times \left(1 - \frac{n}{n+1}\right)\frac{1}{\epsilon}\right]. \tag{4.8}$$

Therefore, including the wave function renormalization in the pseudoregulator requires some additional work. In the following we explore a family of pseudoregulators which achieve the goal of reproducing one-loop $\overline{\mathrm{MS}}$ results, plus RG resummations, in the $\epsilon \to 0$ limit.

B. An extended family of pseudoregulators

The first requirement on a new pseudoregulator which is appropriate for the LPA′, is that it reduces to the pseudoregulator we have adopted for the LPA in the $Z_k \to 1$ limit.[3] Hence we consider a generalization of Eq. (3.1) which amounts to the introduction of two new parameters $Z_0 > 0$ and σ:

$$R_k(z) = Z_0 Z_k^{\sigma\epsilon}\left[\left(\frac{k^2}{\mu^4}\right)^\epsilon z^{1+\epsilon} - z\right]. \tag{4.9}$$

While the most common choice, as in Eq. (4.1), would be $Z_0 = 1$ and $\sigma = 1/\epsilon$, we prefer to keep the two variables arbitrary for the time being. We define

[3] For simplicity we set $b = 0$.

$$G_k(q^2) = (Z_k q^2 + V_k'' + R_k(q^2))^{-1} \tag{4.10}$$

as the regularized propagator. With our pseudoregulator this reads

$$G_k = \frac{1}{Z_0(\frac{Z_k^\sigma k^2}{\mu^4})^\epsilon z^{1+\epsilon} - (Z_0 Z_k^{\sigma\epsilon} - Z_k)z + V_k''}, \tag{4.11}$$

from which it is manifest that having a vanishing $(Z_0 Z_k^{\sigma\epsilon} - Z_k)$ would tremendously simplify the task of evaluating the loop integrals. Though we restrain from this simplifying assumption, we still assume that this difference is small. We calculate all loop integrals by means of their Taylor series in this difference around zero.

Then the generic Q functional becomes

$$\begin{aligned}
Q_n[G_k^\ell \partial_t R_k] &= \frac{\epsilon Z_0 Z_k^{\sigma\epsilon}}{\Gamma(n)\Gamma(\ell)} \sum_{p=0}^{\infty} (Z_0 Z_k^{\sigma\epsilon} - Z_k)^p \frac{\Gamma(\ell+p)}{\Gamma(p+1)} \int_0^\infty \mathrm{d}z z^{n+p} \frac{(2-\sigma\eta_k)(\frac{k^2}{\mu^4})^\epsilon z^\epsilon + \sigma\eta_k}{(Z_0(\frac{Z_k^\sigma k^2}{\mu^4})^\epsilon z^{1+\epsilon} + V_k'')^{\ell+p}} \\
&= \frac{\epsilon Z_0 Z_k^{\sigma\epsilon}}{(1+\epsilon)\Gamma(n)\Gamma(\ell)} \sum_{p=0}^{\infty} \frac{(Z_0 Z_k^{\sigma\epsilon} - Z_k)^p}{\Gamma(p+1)} \left(Z_0 \left(\frac{Z_k^\sigma k^2}{\mu^4} \right)^\epsilon \right)^{-\frac{n+p}{1+\epsilon}} \\
&\quad \times \left[(2-\sigma\eta_k)\Gamma\left(\frac{n+p}{1+\epsilon}+1\right)\Gamma\left(\ell+p-\frac{n+p}{1+\epsilon}-1\right)\left(Z_0\left(\frac{Z_k^\sigma k^2}{\mu^4}\right)^\epsilon\right)^{-1}(V_k'')^{\frac{n+p}{1+\epsilon}-\ell-p+1} \right. \\
&\quad \left. + \sigma\eta_k \Gamma\left(\frac{n+p+1}{1+\epsilon}\right)\Gamma\left(\ell+p-\frac{n+p+1}{1+\epsilon}\right)\left(Z_0\left(\frac{Z_k^\sigma k^2}{\mu^4}\right)^\epsilon\right)^{-\frac{1}{1+\epsilon}}(V_k'')^{\frac{n+p+1}{1+\epsilon}-\ell-p} \right].
\end{aligned} \tag{4.12}$$

From this expression it can be clearly seen that σ cannot diverge for vanishing ϵ [as a comparison of (4.1) and (4.9) would suggest] or both terms would also diverge. On the other hand choosing a vanishing σ in this limit would remove any Z_k and η_k dependence, thus reproducing the same results of the LPA pseudoregulator. Finally, choosing σ to stay constant in the $\epsilon \to 0$ limit leads to

$$\begin{aligned}
Q_n[G_k^\ell \partial_t R_k] &= \frac{Z_0^{-n}(-V_k'')^{n-\ell+1}}{\Gamma(n)\Gamma(\ell)\Gamma(n-\ell+2)} \sum_{p=0}^{\infty} \frac{\Gamma(n+p)}{\Gamma(p+1)} \left(1-\frac{Z_k}{Z_0}\right)^p \left[(2-\sigma\eta_k) + \frac{\sigma\eta_k(n+p)}{n+p+1}\right] \\
&= \frac{Z_0^{-n}(-V_k'')^{n-\ell+1}}{\Gamma(\ell)\Gamma(n-\ell+2)} \left[(2-\sigma\eta_k)\frac{Z_0^n}{Z_k^n} + \sigma\eta_k \frac{n}{n+1} {}_2F_1\left(1+n, 1+n, 2+n; 1-\frac{Z_k}{Z_0}\right)\right].
\end{aligned} \tag{4.13}$$

Summarizing we have

$$\frac{Q_n[G_k^\ell \partial_t R_k]}{(2-\sigma\eta_k(1+H_0))} = \frac{Z_k^{-n}(-V_k'')^{n-\ell+1}}{\Gamma(\ell)\Gamma(n-\ell+2)}, \tag{4.14a}$$

$$\frac{Q_n[G_k^\ell G_k' \partial_t R_k]}{(2-\sigma\eta_k(1+H_1))} = -\frac{n}{n-1} \frac{Z_k^{1-n}(-V_k'')^{n-\ell-1}}{\Gamma(\ell+2)\Gamma(n-\ell)}, \tag{4.14b}$$

$$\frac{Q_n[G_k^\ell G_k'' \partial_t R_k]}{(2-\sigma\eta_k(1+H_2))} = \frac{2n}{n-2} \frac{Z_k^{2-n}(-V_k'')^{n-\ell-2}}{\Gamma(\ell+3)\Gamma(n-\ell-1)}, \tag{4.14c}$$

where primes denote differentiation with respect to z, and we introduced the following notations:

$$H_0(n, Z_k, Z_0) = -\frac{n}{n+1}\left(\frac{Z_k}{Z_0}\right)^n \times {}_2F_1\left(n+1, n+1, n+2; 1-\frac{Z_k}{Z_0}\right), \tag{4.15a}$$

$$H_1(n, Z_k, Z_0) = -\frac{n-1}{n}\frac{Z_0}{Z_k} + \frac{n-1}{n+1}\left(1-\frac{Z_k}{Z_0}\right) \times \left(\frac{Z_k}{Z_0}\right)^{n-1} {}_2F_1\left(n+1, n+1, n+2; 1-\frac{Z_k}{Z_0}\right), \tag{4.15b}$$

$$H_2(n, Z_k, Z_0) = \frac{n-2}{n}\left(1-\frac{2n-1}{n-1}\frac{Z_k}{Z_0}\right)\left(\frac{Z_0}{Z_k}\right)^2 - \frac{n-2}{n+1}\left(1-\frac{Z_k}{Z_0}\right)^2\left(\frac{Z_k}{Z_0}\right)^{n-2} {}_2F_1\left(n+1, n+1, n+2; 1-\frac{Z_k}{Z_0}\right). \tag{4.15c}$$

The dependence of the H_i functions on Z_k/Z_0 signals the expected breaking of reparameterization invariance, which translates in nonautonomous flow equations for the dimensionless renormalized couplings. An autonomous flow can be recovered in special cases: besides the $\sigma \to 0$ limit, other interesting choices are

$$\lim_{Z_0\to 0} H_i(n) = 0, \tag{4.16}$$

$$\lim_{Z_0\to \infty} H_i(n) = -1. \tag{4.17}$$

From (4.14) we see that the second case suppresses the "RG improvement" terms, in the same way as setting $\sigma = 0$, and therefore gives back the LPA result. Thus, in summary, the proper way to use the pseudoregulator (4.9) is to first evaluate the integrals, then take the limit $\epsilon \to 0$ and finally the limit $Z_0 \to 0$.

Finally, it is worth stressing that the previous identities are not restricted to the LPA′ truncation. For truncations where Z_k depends on fields and/or momentum, the relevant wave function renormalization factor appearing inside the pseudoregulator is to be identified with Z_k evaluated at preferred values of momentum and fields, for instance minimizing the potential and the inverse propagator. In the simplest cases the latter are vanishing values. Then the simple propagator G_k of Eq. (4.10), and the loop integrals given in the previous equations, would arise after a polynomial expansion of Z_k to obtain derivative vertices which are local in field space and in spacetime.

C. Scalar field with a generic potential and its anomalous dimension

As a first example of application of the above pseudoregulator, let us turn to a simple scalar field theory within the LPA′ truncation. We consider a most generic effective potential, which can be parametrized as follows

$$V_k = \sum_{n=0} Z_k^{n/2} \frac{\lambda_n}{n!} \phi^n. \tag{4.18}$$

The anomalous dimension η_k is computed by extracting from the exact FRG equation the contributions to the quadratic part of the two-point function

$$\begin{aligned}\eta_k &= -\frac{1}{Z_k} \lim_{p^2\to 0} \frac{\partial}{\partial p^2} \partial_t \frac{\delta^2 \Gamma_k}{\delta\phi_p \phi_{-p}}\bigg|_{\phi_0} \\ &= -\frac{(V_k^{(3)}(\phi_0))^2}{Z_k} \lim_{p^2\to 0} \frac{\partial}{\partial p^2} \int \frac{\mathrm{d}^d q}{(2\pi)^d} G_k(q^2)^2 G_k((q+p)^2) \partial_t R_k(q^2)\bigg|_{\phi_0}. \end{aligned} \tag{4.19}$$

Here ϕ_0 is the minimum of the potential. After taking the derivative and the limit, we obtain

$$\eta_k = -\frac{(V_k^{(3)}(\phi_0))^2}{Z_k (4\pi)^{d/2}} \big(Q_{\frac{d}{2}}[G_k^2 G_k' \partial_t R_k] + Q_{\frac{d}{2}+1}[G_k^2 G_k'' \partial_t R_k]\big). \tag{4.20}$$

For $\phi_0 = 0$ and in the $Z_0 \to 0$ limit, this boils down to

$$\eta_k = -\frac{(-1)^{d/2}}{6\Gamma(\frac{d}{2}-2)} \frac{\lambda_3^2}{(4\pi)^{d/2}} \lambda_2^{\frac{d}{2}-3} \left(1 - \sigma\frac{\eta_k}{2}\right). \tag{4.21}$$

Within the LPA′ truncation for this pseudoregulator we obtain what is essentially a one-loop equation with RG improvement. Therefore in $d = 2$, 4 we find $\eta = 0$.[4] To reproduce a nonvanishing η with this pseudoregulator we need to consider larger truncations, as detailed in Sec. VI.

Now we consider the equation for the potential and we calculate the beta functions

$$\partial_t V_k = \frac{1}{2(4\pi)^d} Q_{\frac{d}{2}}[G_k \partial_t (Z_k R_k)], \tag{4.22}$$

$$\beta_n = \frac{n}{2}\eta\lambda_n + Z_k^{-n/2} \frac{\partial^n \partial_t V_k}{(\partial\phi)^n}\bigg|_{\phi=0}. \tag{4.23}$$

We list the initial beta functions in the $Z_0 \to 0$ limit:

[4] Note that we drop the k subscript in η_k from this point on.

$$\beta_1 = \frac{1}{2}\eta\lambda_1 + \frac{(-1)^{d/2}}{\Gamma(\frac{d}{2})(4\pi)^{d/2}}\left(1-\sigma\frac{\eta}{2}\right)\lambda_3\lambda_2^{d/2-1}, \tag{4.24a}$$

$$\beta_2 = \eta\lambda_2 + \frac{(-1)^{d/2}}{(4\pi)^{d/2}}\left(1-\sigma\frac{\eta}{2}\right)\left(\lambda_3^2\frac{\lambda_2^{d/2-2}}{\Gamma(\frac{d}{2}-1)} + \lambda_4\frac{\lambda_2^{d/2-1}}{\Gamma(\frac{d}{2})}\right), \tag{4.24b}$$

$$\beta_3 = \frac{3}{2}\eta\lambda_3 + \frac{(-1)^{d/2}}{(4\pi)^{d/2}}\left(1-\sigma\frac{\eta}{2}\right)\left(\lambda_3^3\frac{\lambda_2^{d/2-3}}{\Gamma(\frac{d}{2}-2)} + 3\lambda_3\lambda_4\frac{\lambda_2^{d/2-2}}{\Gamma(\frac{d}{2}-1)} + \lambda_5\frac{\lambda_2^{d/2-1}}{\Gamma(\frac{d}{2})}\right), \tag{4.24c}$$

$$\beta_4 = 2\eta\lambda_4 + \frac{(-1)^{d/2}}{(4\pi)^{d/2}}\left(1-\sigma\frac{\eta}{2}\right)\left(\lambda_3^4\frac{\lambda_2^{d/2-4}}{\Gamma(\frac{d}{2}-3)} + 6\lambda_3^2\lambda_4\frac{\lambda_2^{d/2-3}}{\Gamma(\frac{d}{2}-2)} + (4\lambda_3\lambda_5+3\lambda_4^2)\frac{\lambda_2^{d/2-2}}{\Gamma(\frac{d}{2}-1)} + \lambda_6\frac{\lambda_2^{d/2-1}}{\Gamma(\frac{d}{2})}\right). \tag{4.24d}$$

In $d=6$, if $\lambda_2 \to 0$ and $\lambda_{n\geq 4} = 0$, we get the universal one-loop result, plus RG resummations

$$\eta = \frac{\lambda_3^2}{6(4\pi)^3}\left(1-\sigma\frac{\eta}{2}\right), \tag{4.25}$$

$$\beta_3 = -\frac{3\lambda_3^3}{4(4\pi)^3}\left(1-\sigma\frac{\eta}{2}\right). \tag{4.26}$$

Solving these equations and expanding in λ_3

$$\eta = \frac{\lambda_3^2}{6(4\pi)^3 + \frac{\sigma}{2}\lambda_3^2} = \frac{\lambda_3^2}{6(4\pi)^3} - \frac{\sigma\lambda_3^4}{72(4\pi)^6} + O(\lambda_3^6), \tag{4.27}$$

it can be checked that it is possible to adjust σ to reproduce the correct two-loop result [35] for either η ($\sigma = -26/3$) or β_3 ($\sigma = -250/9$) but not both simultaneously.[5] In fact, recovering the full two-loop RG equations requires larger truncations, as we discuss in Sec. VI.

V. THE $O(\partial^2)$ DERIVATIVE EXPANSION

The pseudoregulators introduced in the previous section are also apt for application to a larger class of truncations which accounts for a possible field dependence of the wave function renormalization, the $O(\partial^2)$ of the derivative expansion. While this kind of more elaborate approximation is often an optional for many models, it is in some cases a necessity already as a zeroth order approach, such as for instance in the applications to nonlinear sigma models or for conformal field theories in two dimensions. For this reason, in this section we address these two examples. They allow us to account for a trivial generalization of the LPA′ formulas given in the previous section, and also to discuss more subtle points about the scope of an $\overline{\text{MS}}$ pseudoregulator, such as its applicability to strongly interacting field theories and to models with nonlinear symmetries.

[5]The situation does not change if we insert also the RG improvement proportional to β_2/λ_2, which is finite in the limit $\lambda_2 \to 0$.

A. Multicritical models

We consider the following truncation of Γ_k

$$\Gamma_k[\phi] = \int \mathrm{d}^d x \left(V_k(\phi) + \frac{1}{2}Z_k(\phi)\partial_\mu\phi\partial^\mu\phi\right). \tag{5.1}$$

This kind of ansatz is general enough to capture the emergence of a tower of multicritical ϕ^{2p} scalar field theories below the fractional upper critical dimensions $d_p = 2p/(p-1)$, and to provide good estimates of their properties in $d=2$ [36–38]. As these conclusions apply to conventional FRG regulator choices, it is interesting to check whether these nice results can be obtained even with an $\overline{\text{MS}}$ pseudoregulator.

The flow equations of the functions V_k and Z_k for the pseudoregulator (4.9) can be obtained from those presented in Appendix B for the more general case of $O(N)$ models. More specifically, they correspond to Eqs. (B4) and (B5), for $d=2$ and $N=1$.[6] Rescaling the field

$$\phi = Z_k(0)^{-1/2}\tilde{\phi}, \qquad \eta = -\partial_t \log Z_k(0) \tag{5.2}$$

and introducing the dimensionless renormalized functions

$$v_k(\tilde{\phi}) = k^{-2}V_k(\phi), \qquad \zeta_k(\tilde{\phi}) = Z_k(0)^{-1}Z_k(\phi), \tag{5.3}$$

these flow equations read

[6]Here and in the following sections we identify the wave function renormalization of the single-field theory of Eq. (5.1) with the one of the radial mode in the linear $O(N)$ model, which is $\tilde{Z}_k(\phi)$ in Eq. (B5). Notice that this choice is not the conventional one for FRG studies, which usually associate the single-field Z_k to the $N \to 1$ limit of the Goldstone-modes wave function renormalization.

$$\partial_t v_k = -2v_k + \frac{\eta}{2}\tilde{\phi}v_k' - \frac{1}{4\pi}\left(1-\sigma\frac{\eta}{2}\right)\zeta_k^{-1}v_k'', \tag{5.4a}$$

$$\partial_t \zeta_k = \eta\zeta_k + \frac{\eta}{2}\tilde{\phi}\zeta_k' + \frac{1}{8\pi}\left(1-\sigma\frac{\eta}{2}\right) \times \left[-2\frac{\zeta_k''}{\zeta_k} + 3\left(\frac{\zeta_k'}{\zeta_k}\right)^2\right]. \tag{5.4b}$$

Now we search for the scaling solutions for this system of equations. Setting the $\mathbb{Z}_2$ parity and normalizations conditions

$$v'(0) = 0, \qquad v''(0) = \zeta_0\tilde{m}^2, \tag{5.5a}$$

$$\zeta(0) = \zeta_0, \qquad \zeta'(0) = 0, \tag{5.5b}$$

the previous system of equations has the following family of fixed points

$$v_* = -\frac{2-\sigma\eta}{16\pi}\tilde{m}^2\cos\left(\frac{2}{\sqrt{\eta}}\arctan\sqrt{\frac{\Phi^2}{1-\Phi^2}}\right), \tag{5.6a}$$

$$\zeta_* = \zeta_0(1-\Phi^2)^{-1}, \tag{5.6b}$$

$$\Phi = \sqrt{\frac{4\pi\eta\zeta_0}{2-\sigma\eta}}\tilde{\phi}. \tag{5.6c}$$

It is remarkable that with the $\overline{\mathrm{MS}}$ pseudoregulator the scaling solutions can be written in closed form: usually in the FRGE they are only known numerically. These solutions manifestly preserve reparameterization invariance: the normalization factor ζ_0 cannot influence observable quantities as it can be eliminated from the action (5.1) by rescaling $\tilde{\phi}$. Depending on the sign of m^2, the fixed-point potential can have a maximum or a minimum for zero field.

Note that ζ_* diverges for $\Phi^2 = 1$. In order to have a potential v_* which is smooth at this point and is bounded from below we impose

$$v_*^{(n)}|_{\Phi^2\to 1} = \text{finite}, \qquad \forall\, n \tag{5.7a}$$

$$\lim_{\tilde{\phi}\to\infty} v_* = +\infty. \tag{5.7b}$$

From (5.7a) we get the quantization rule:

$$\sin\frac{\pi}{\sqrt{\eta}} = 0, \tag{5.8}$$

such that

$$\eta = \frac{1}{p^2}, \qquad p = 1, 2, 3\ldots, \tag{5.9}$$

while (5.7b) can be fulfilled by adjusting the sign of $\tilde{m}^2$ (while the modulus remains free):

$$\tilde{m}^2 \lessgtr 0 \quad \text{if } (-1)^{1/\sqrt{\eta}} = \pm 1. \tag{5.10}$$

In this way v_* acquires the typical shape of a $(p-1)$-critical potential.

To compute the critical exponents associated to these fixed points, we linearize the RG flow around them and look for eigenperturbations. In other words, we insert $v_k \to v_* + e^{\theta t}\delta v$, $\zeta_k \to \zeta_* + e^{\theta t}\delta\zeta$ and $\eta = \eta_* + \delta\eta$ into Eq. (5.4), and expand them to the first order in the perturbations δv, $\delta\zeta$ and $\delta\eta$. For $\delta\eta \neq 0$ the corresponding $\delta\zeta$ is complex and furthermore singular at $\Phi^2 = 1$. We therefore impose $\delta\eta = 0$. In this simplified case the linearized equations read

$$\theta\delta v = -2\delta v + \frac{\eta}{2}\tilde{\phi}\delta v' - \frac{1}{4\pi}\left(1-\sigma\frac{\eta}{2}\right) \times \zeta_*^{-1}\left[\delta v'' - v_*''\frac{\delta\zeta}{\zeta_*}\right], \tag{5.11a}$$

$$\theta\delta\zeta = \eta\delta\zeta + \frac{\eta}{2}\tilde{\phi}\delta\zeta' + \frac{1}{4\pi}\left(1-\sigma\frac{\eta}{2}\right) \times \left[-\frac{\delta\zeta''}{\zeta_*} + 3\frac{\zeta_*'\delta\zeta'}{\zeta_*^2} + \left(\frac{\zeta_*''}{\zeta_*} - 3\left(\frac{\zeta_*'}{\zeta_*}\right)^2\right)\frac{\delta\zeta}{\zeta_*}\right]. \tag{5.11b}$$

The condition of fixed η results in LPA-like perturbations with vanishing $\delta\zeta$.

Besides the trivial solutions

$$\theta = -2, \qquad \delta v = 1, \tag{5.12a}$$

$$\theta = -2 + \frac{\eta}{2}, \qquad \delta v = \tilde{\phi}, \tag{5.12b}$$

$$\theta = 0, \qquad \delta v = v_*, \tag{5.12c}$$

we find the even eigenperturbations

$$\theta = -2 + 2\eta n^2, \tag{5.13a}$$

$$\delta v = \cos\left(\sqrt{\frac{4+2\theta}{\eta}}\arctan\sqrt{\frac{\Phi^2}{1-\Phi^2}}\right), \tag{5.13b}$$

and the odd eigenperturbations

$$\theta = -2 + 2\eta\left(n+\frac{1}{2}\right)^2, \tag{5.14a}$$

$$\delta v = \sqrt{\frac{2-\sigma\eta}{8\pi(2+\theta)}}\sin\left(\sqrt{\frac{4+2\theta}{\eta}}\arctan\sqrt{\frac{\Phi^2}{1-\Phi^2}}\right), \tag{5.14b}$$

where η assumes its fixed-point value (5.9). Enforcing regularity of δv at the pole of ζ requires $n = 1, 2, 3\ldots$. From the largest even parity eigenvalue ($n = 1$), excluding the unit operator, we get the critical exponent ν

$$\nu = \frac{1}{2 - 2\eta}. \tag{5.15}$$

In Table I we summarize these results for the critical exponents η and ν and compare them to FRG estimates obtained by means of the optimized regulator and with the homogeneous regulator, as well as with the exact values. Comparing Eq. (5.9) with the exact result

$$\eta = \frac{3}{(p+1)(p+2)}, \tag{5.16}$$

we see that for large p our result is off by a factor 3, whereas ν correctly tends to $1/2$.

B. The nonlinear σ model

Addressing the nonlinear σ model with the $\overline{\mathrm{MS}}$ pseudoregulator requires only a simple generalization of the truncation we just studied, to account for a multiplet of fields, rather than a single one. We therefore start from the following truncation of Γ_k for a $O(N)$-invariant multiplet of scalars:

TABLE I. Estimates of the critical exponents η_p and ν_p for the two-dimensional ϕ^{2p} multicritical scalar models. The first three columns present FRG estimates: the first obtained with the $\overline{\mathrm{MS}}$ pseudoregulator, the second with the optimized regulator of Eq. (2.4), the third with a homogeneous regulator. Finally, the last column shows the exact results, from conformal field theory methods.

	This work	opt. [37, 38]	hom. [36]	exact [39]
η_2	0.25	0.2132	0.309	0.25
ν_2	0.666667	…	0.863	1
η_3	0.111111	0.1310	0.200	0.15
ν_3	0.5625	…	0.566	0.556
η_4	0.0625	0.0910	0.131	0.1
ν_4	0.533333	…	0.545	0.536
η_5	0.04	0.0679	0.0920	0.0714
ν_5	0.520833	…	0.531	0.525
η_6	0.0277778	0.0522	0.0679	0.0535714
ν_6	0.514286	…	0.523	0.519
η_7	0.0204082	…	0.0521	0.0416667
ν_7	0.510417	…	0.517	0.514
η_8	0.015625	…	0.0412	0.0333333
ν_8	0.507937	…	0.514	0.511
η_9	0.0123457	…	0.0334	0.0272727
ν_9	0.50625	…	0.511	0.509
η_{10}	0.01	…	0.0277	0.0227273
ν_{10}	0.505051	…	0.509	0.508
η_{11}	0.00826446	…	0.0233	0.0192308
ν_{11}	0.504167	…	0.508	0.506

$$\Gamma_k[\phi] = \int \mathrm{d}^d x \left(U_k(\rho) + \frac{1}{2} Z_k(\rho)\partial_\mu\phi_a\partial^\mu\phi^a + \frac{1}{4} Y_k(\rho)\partial_\mu\rho\partial^\mu\rho \right), \tag{5.17}$$

where the N fields ϕ^a are in the fundamental representation of $O(N)$, and $\rho = \phi^a\phi^a/2$ is the corresponding local invariant. We further define the radial wave function renormalization

$$\tilde{Z}_k(\rho) = Z_k(\rho) + \rho Y_k(\rho). \tag{5.18}$$

In Appendix B we show the flow equations of this model in the present truncation, for general d and adopting the pseudoregulator of Eq. (4.9) in the $Z_0 \to 0$ limit. For the especially interesting case $d = 2$, we obtain

$$\partial_t U_k = -\frac{1}{4\pi}\left(\frac{U_k' + 2\rho U_k''}{\tilde{Z}_k} + (N-1)\frac{U_k'}{Z_k} \right), \tag{5.19a}$$

$$\partial_t \tilde{Z}_k = -\frac{(\tilde{Z}_k' + 2\rho\tilde{Z}_k'')}{4\pi\tilde{Z}_k} - (N-1)\frac{(Z_k' + \rho Y_k')}{4\pi Z_k} + \frac{3\rho(\tilde{Z}_k')^2}{4\pi\tilde{Z}_k^2} + (N-1)\frac{\rho Z_k'(Y_k - Z_k')}{2\pi Z_k^2}. \tag{5.19b}$$

Here we suppressed the RG improvement by setting $\sigma = 0$; the effect of a nonvanishing σ will be addressed in a moment.

As it stands, this action could still describe a linear model. If we make the assumptions

$$Z_k(\rho) = \frac{Z_k}{g_k^2}, \tag{5.20a}$$

$$\tilde{Z}_k(\rho) = \frac{1}{g_k^2}\left(\frac{1}{Z_k} - 2\rho\right)^{-1}, \tag{5.20b}$$

$$U_k = -h_k\sqrt{\frac{1}{Z_k} - 2\rho}, \tag{5.20c}$$

the EAA becomes

$$\Gamma_k[\phi] = \int \mathrm{d}^d x \left[\frac{Z_k}{2g_k^2}\left(\delta_{ab} + \frac{\phi^a\phi^b}{\frac{1}{Z_k} - 2\rho}\right)\partial_\mu\phi_a\partial^\mu\phi^b - h_k\sqrt{\frac{1}{Z_k} - 2\rho} \right], \tag{5.21}$$

which describes a nonlinear σ model with values in a sphere S^N of radius $Z_k^{-1/2}$ and coupled to an external source h_k [40]. In this case the symmetry group is extended to $O(N+1)$. Inserting this ansatz in the flow equations (5.19) one deduces the correct one-loop beta functions

$$\partial_t g_k = -\frac{N-1}{4\pi} g_k^3, \tag{5.22a}$$

$$\eta = -\partial_t \log Z_k = \frac{N}{2\pi} g_k^2, \tag{5.22b}$$

$$\partial_t h_k = 0. \tag{5.22c}$$

Thus, the flow equations (5.19) maintain the form of the ansatz in Eq. (5.20), that is to say, they preserve the nonlinearly realized $O(N+1)/O(N)$ symmetry. This might appear trivial as we are applying one-loop RG equations, but it is not so for two reasons. First, this compatibility extends beyond one-loop order as we observe in the following by the inclusion of the RG improvement. Second, because this conclusion does not hold for finite ϵ, i.e., within the realm of ordinary FRG computations. In fact, it is well known that the FRG regulator, being a deformation of the two-point function of the N fields, explicitly breaks the nonlinear part of the $O(N+1)$ symmetry. For this reason, most FRG applications to nonlinear sigma models have adopted different formulations based on the background field method [27,41–48].

Let us then turn to the RG improvement, which leads us beyond the one-loop approximation. To this end we should note that the pseudoregulator in Eq. (4.9) has a factor $Z_k^{\sigma\epsilon}$, but here Z_k should be replaced by $Z_k g_k^{-2}$ to be compatible with the ansatz of Eq. (5.20). Then, with this little adjustment of the pseudoregulator, for a generic σ we get

$$\partial_t g_k = -\frac{(N-1)g_k^3}{4\pi + \sigma g_k^2}, \tag{5.23a}$$

$$\eta = \frac{2N g_k^2}{4\pi + \sigma g_k^2}, \tag{5.23b}$$

$$\partial_t h_k = 0. \tag{5.23c}$$

Even though the previous flow equations hold in $d = 2$, it is possible to apply them in $d = 2 + \varepsilon$ by simply augmenting them with their ε-dependent canonical dimensional part. In so doing, one can recover the ε-expansion description of the nontrivial fixed point which exists for $\varepsilon > 0$. We defer this discussion to the end of Sec. VII.

VI. THE TWO-LOOP BETA FUNCTIONS

In the previous sections we have shown that the $\epsilon \to 0$ limit of the FRG beta functions for the $\overline{\text{MS}}$ pseudoregulator reduces them to well-known $\overline{\text{MS}}$ one-loop RG equations, possibly up to a resummation. We have shown this in the LPA, in the LPA′ and in the $O(\partial^2)$ derivative expansion. In this section we show how to reproduce the two-loop result in four dimensions, by considering larger truncations and by taking the $\epsilon \to 0$ limit in a suitable way.

Although the computation of the beta function of the quartic coupling was discussed by several authors already, see Refs. [49–56], part of the arguments adopted in those works do not apply to the $\overline{\text{MS}}$ pseudoregulator, which is not an IR regulator. Furthermore, we crucially rely on analytic continuation of divergent integrals, such that parametric limits are allowed to not commute, whereas standard FRG regulators render all integrals convergent. In addition, we also compute the two-loop running of the mass.

We closely follow the notations and the arguments of the first FRG work addressing this task, namely Ref. [49]. We therefore focus on the linear $O(N)$ models with bare action

$$S[\phi] = \int \mathrm{d}^d x \left\{ \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a + U_\Lambda(\rho) \right\}, \tag{6.1a}$$

$$U_\Lambda(\rho) = \bar{m}^2 \rho + \frac{\bar{\lambda}}{2} \rho^2. \tag{6.1b}$$

Note that compared to Eq. (2.8), we have changed the notation to $\lambda_2 = \bar{m}^2$ and $\lambda_4 = 3\bar{\lambda}$, and the bars denote bare couplings. In a massless scheme such as $\overline{\text{MS}}$, the two-loop beta function of the quartic coupling is universal and mass independent, such that it is usually possible to assume $\bar{m}^2 = 0$ right from the start. We instead focus on a massive theory in the symmetric regime for technical reasons. In fact, we are going to adopt an FRG pseudoregulator which does not regulate IR divergences. This does not prevent us from analysing the massless theory though, as we are allowed to take the $\bar{m}^2 \to 0$ limit of any IR safe quantity after the loop integrals are computed.

In our regularization scheme, it is furthermore essential to account for the k dependence of the renormalized mass parameter m^2, or else the correct two-loop beta function would not be reproduced. In fact, as the latter contributes to the running of λ in any mass-dependent scheme, it does so also in our computations at nonvanishing ϵ. Interestingly, this contribution will survive the $\epsilon \to 0$ limit, if the latter is taken carefully enough. This computation will thus serve as an example of a more general mechanism, according to which the super-renormalizable and the nonrenormalizable sectors of a theory, which show a nontrivial running in any mass-dependent scheme, do feed back into the running of renormalizable operators even in a massless scheme such as $\overline{\text{MS}}$, provided the mass-thresholds effects are correctly accounted for.[7]

As discussed in the introduction, a two-loop result involves arbitrarily high orders of the derivative expansion. We therefore cannot use an ansatz such as (1.6), or its multifield generalization. Instead, we must make the ansatz

[7] This mechanism has been observed also in Ref. [57].

$$\Gamma_k[\phi] = \int \mathrm{d}^d x \left\{ U_k(\rho) + \frac{1}{2}\partial_\mu\phi^a Z_k(\rho, -\partial^2)\partial^\mu\phi_a + \frac{1}{4}\partial_\mu\rho Y_k(\rho, -\partial^2)\partial^\mu\rho \right\}. \tag{6.2}$$

Equivalently, expressing the field in the angular (Goldstone) modes, and in the radial (massive), mode $\sqrt{\rho}$, one finds that their wave function renormalizations are given by the function Z_k and

$$\tilde{Z}_k(\rho, q^2) = Z_k(\rho, q^2) + \rho Y_k(\rho, q^2), \tag{6.3}$$

respectively. In order to compute the beta function of the quartic coupling it may seem sufficient to stop at the fourth order of the vertex expansion, and hence assume that U_k is quadratic in ρ and Z_k is linear. However, in the vertex expansion, the beta function of $\Gamma^{(n)}$ involves also $\Gamma^{(n+1)}$ and $\Gamma^{(n+2)}$, so we need U_k up to order ρ^3 and Z_k up to order ρ^2. In general we shall use the following terminology for the expansion of these functions:

$$U_k(\rho) = \sum_{n=1}^{\infty} \frac{u_n}{n!} Z_k^n k^{d-n(d-2)}(\rho - \rho_0)^n, \tag{6.4a}$$

$$Z_k(\rho, q^2) = \sum_{n=0}^{\infty} \frac{z_n(q^2/k^2)}{n!} Z_k^{n+1} k^{-n(d-2)}(\rho - \rho_0)^n, \tag{6.4b}$$

$$\tilde{Z}_k(\rho, q^2) = \sum_{n=0}^{\infty} \frac{\tilde{z}_n(q^2/k^2)}{n!} Z_k^{n+1} k^{-n(d-2)}(\rho - \rho_0)^n, \tag{6.4c}$$

where $Z_k = Z_k(\rho_0, 0)$ is the wave function renormalization and ρ_0 is the minimum of the potential. In any scheme, the effective potential is already renormalized at one loop. The functions $Z_k(\rho, q^2)$ and $\tilde{Z}_k(\rho, q^2)$ also receive one-loop radiative corrections in mass-dependent schemes. The one-loop contributions to these functions are however field dependent corrections, and therefore correspond to radiatively generated momentum-dependent vertices. In any scheme the field-independent part of the wave function renormalizations, i.e., the field's anomalous dimensions, receive corrections from the two-loop order on. These well-known perturbative facts are recovered from the FRG equations, straightforwardly in mass-dependent schemes, and with a little care also for mass-independent schemes, as we show in this section.

Although the FRGE looks like a one-loop equation, this is only true as long as one uses the full propagators and vertices [double lines and black dots in Eqs. (1.5)]. The full propagators and vertices can be expanded in loops, as briefly explained in the introduction, giving rise to infinite series that can be represented in terms of standard Feynman diagrams. This introduces resummations of perturbative diagrams of two kinds. The first is the so-called "spectral adjustment" of the regulator, i.e., the possible dependence of the regulator on the couplings of the theory, most commonly the wave function renormalization Z_k (as already discussed in Sec. IVA). This produces terms depending on the field's anomalous dimensions. The second source of resummations is provided by the mass thresholds, which in a functional setup may also depend on the point of expansion in the space of field amplitudes,

$$\kappa = k^{2-d} Z_k \rho_0, \tag{6.5}$$

at which we define local couplings. These include the perturbatively renormalizable ones

$$m^2 \equiv k^2 u_1 = Z_k^{-1} U'(\rho_0), \tag{6.6a}$$

$$\lambda \equiv u_2 = k^{d-4} Z_k^{-2} U_k''(\rho_0), \tag{6.6b}$$

as well as the nonrenormalizable ones

$$u_n = k^{n(d-2)-d} Z_k^{-n} U_k^{(n)}(\rho_0), \quad n \geq 3, \tag{6.6c}$$

$$z_n(y) = k^{n(d-2)} Z_k^{-(n+1)} Z_k^{(n)}(\rho_0, k^2 y), \quad n \geq 1, \tag{6.6d}$$

$$\tilde{z}_n(y) = k^{n(d-2)} Z_k^{-(n+1)} \tilde{Z}_k^{(n)}(\rho_0, k^2 y), \qquad n \geq 1. \tag{6.6e}$$

If one were to suppress both these portals towards higher order corrections, the FRGE would boil down to a pure one-loop result. However, thanks to these two contributions, higher loops are generated by solving the RG equations and constructing the RG trajectory, i.e., in the process of renormalizing the theory.

Let us now address the task of integrating the $d = 4$ flow equations from the UV initial condition $\Gamma_{k=\Lambda} = S$ down to $k < \Lambda$.[8] We recall that for the bare theory of Eq. (6.1) the loop expansion corresponds to the expansion in the coupling λ. From now on it is more convenient to think in this way. To reproduce perturbation theory, we need to compute the RG vector field in vicinity of the Gaussian fixed point up to next-to-leading order in λ. This is tantamount to integrating the flow order by order in a Taylor expansion for small λ. We first input the initial condition Γ_Λ on the right-hand sides of the RG equations. This produces a one-loop beta function for the renormalizable couplings m^2 and λ. On the other hand, the RG equations radiatively generate further couplings, namely those whose t derivative at this initial point is nonvanishing. By considering the Feynman diagrams mentioned above,

[8]Here the limit $\Lambda \to \infty$ is allowed as part of the regularization choice, and should not be confused with the possibility to remove a UV cutoff, thus defining a UV complete theory. The latter question is instead emerging when trying to take such a limit at for a fixed IR action $\Gamma_{k=0}$.

we deduce a power counting for the radiatively generated couplings in terms of λ.

The first few couplings to be generated, and the corresponding orders of magnitude in terms of the initial quartic coupling are

$$\eta = O(\lambda^2), \qquad z_1 = O(\lambda^2), \qquad \tilde{z}_1 = O(\lambda^2), \tag{6.7a}$$

$$u_3 = O(\lambda^3), \qquad z_2 = O(\lambda^3), \qquad \tilde{z}_2 = O(\lambda^3), \tag{6.7b}$$

$$u_4 = O(\lambda^4), \qquad z_1^2 = O(\lambda^4), \qquad \tilde{z}_1^2 = O(\lambda^4). \tag{6.7c}$$

Similar relations hold for u_n and z_n, with progressively higher powers of λ for higher n. Thus after an infinitesimal RG step from $k = \Lambda$ to $k = \Lambda - \delta k$, the effective average action changes and the perturbative expansion of the FRG vector field correspondingly adjusts. To compute the most general form of β_λ along such a flow, which is exact at the order λ^3 (still within a local expansion around vanishing fields), we can use the power counting of Eq. (6.7) to eliminate the higher order terms. This results in

$$\beta_{m^2} = -\frac{k^2}{16\pi^2}[(N-1)\lambda l^4_{1,0}(0) + 3\lambda l^4_{0,1}(2\lambda\kappa) + (N-1)\langle z_1\rangle^6_{1,0}(0) + \langle \tilde{z}_1\rangle^6_{0,1}(2\lambda\kappa)] + \eta m^2 + k^2(2\kappa + \partial_t\kappa)\lambda + O(\lambda^3), \tag{6.8}$$

$$\beta_\lambda = \frac{N-1}{16\pi^2} l^4_{2,0}(0)\lambda^2 + \frac{9}{16\pi^2} l^4_{0,2}(2\lambda\kappa)\lambda^2 - \frac{N-1}{16\pi^2} l^4_{1,0}(0)u_3 - \frac{5}{16\pi^2} l^4_{0,1}(2\lambda\kappa)u_3 + (2\kappa + \partial_t\kappa)u_3 + \frac{N-1}{8\pi^2}\lambda\langle z_1\rangle^6_{2,0}(0) + \frac{3}{8\pi^2}\lambda\langle\tilde{z}_1\rangle^6_{0,2}(2\lambda\kappa) - \frac{N-1}{16\pi^2}\langle z_2\rangle^6_{1,0}(0) - \frac{1}{16\pi^2}\langle\tilde{z}_2\rangle^6_{0,1}(2\lambda\kappa) + 2\eta\lambda + O(\lambda^4). \tag{6.9}$$

In these equations the threshold functions $l^4_{n,m}$ and the averages $\langle z_n\rangle^6_{n,m}$ denote one-loop integrals over virtual momenta, with momentum-independent and -dependent vertices, respectively. The precise definitions can be found in Eq. (D5). The λ dependence of mass thresholds should also be expanded, for instance:

$$l^d_{0,n}(2\lambda\kappa) = l^d_{n,0}(0) - 2n\lambda\kappa l^d_{n+1,0}(0) + O(\lambda^2). \tag{6.10}$$

However this would bring corrections only for nonvanishing κ, which is not generated at the two-loop order. We can thus set $\kappa = 0$ in Eqs. (6.8) and (6.9). While the contribution of the nonrenormalizable couplings u_3, $z_{1,2}$ and $\tilde{z}_{1,2}$ is obvious in any mass-dependent scheme, one might expect that it would not be present in $\overline{\text{MS}}$, since all dimensionful integrals, in absence of mass thresholds, need to vanish in the $\epsilon \to 0$ limit. This expectation is however incorrect, because when the computation is performed at nonvanishing mass, and the $m \to 0$ limit is taken after the $\epsilon \to 0$ limit, the contribution of the beta functions of the mass and of the nonrenormalizable couplings inside β_λ attains a finite nonvanishing value.

To illustrate the details of this mechanism, we should first choose a specific form of the ϵ-dependent pseudoregulator which is suitable for the present computation. We adopt the following function[9]

$$R_k(z) = \left[\left(\frac{k^2}{\mu^{4-2b}M^{2b}}\right)^{\epsilon}(z + M^2)^{1+\epsilon} - (z + m^2)\right]. \tag{6.11}$$

Here μ is a k-independent momentum scale and M plays the role of a regularized mass, which is assumed to be k dependent. M should be an analytic function of m, such that the $m \to 0$ limit smoothly removes also M.[10] The precise form of M making all relevant integrals finite and ensuring the k independence of the beta functions in the $\epsilon \to 0$ limit is derived in Appendix C. Imposing the analyticity requirement on M fixes $b = 1$.

We now turn to the beta function of Eq. (6.9), where on the rhs we organized different kinds of contributions on different lines. The first line provides the one-loop expression, as well as also a first type of higher order contribution, due to the RG improvement of the pseudoregulator. More

[9]As η is vanishing at one loop in this model, we discard the precise form of the Z_0 and Z_k dependence of the pseudoregulator: in fact, the η dependence appearing on the rhs of the flow equations through the regularization only contributes to the rhs of the flow from three loops on, and so we can safely replace $Z_k \to 1$.

[10]If M depended also on other couplings aside from m, their contributions would only appear from three loops.

specifically, the threshold functions are responsible for the appearance of β_{m^2} on the rhs of β_λ, as

$$l^4_{0,2}(2\lambda\kappa) = l^4_{2,0}(0) + O(\lambda^2) = 1 - \frac{\beta_{m^2}}{2m^2} + O(\lambda^2),$$
$$= 1 - \frac{N+2}{32\pi^2}\lambda + O(\lambda^2). \tag{6.12}$$

As already anticipated, the ratio β_{m^2}/m^2 attains a finite mass-independent value. In fact, the one-loop $\overline{\mathrm{MS}}$ result for β_{m^2} is recovered also with the present pseudoregulator. Even the two-loop $\overline{\mathrm{MS}}$ coefficient for β_{m^2} can be correctly reproduced, although this requires a careful choice of the function M^2, which is described in Appendix D 5.

The second line of Eq. (6.9) encodes the effect of u_3, which is generated by the flow equation itself, as detailed in Appendix D. It is a general feature of the FRG equations that solving the flow equation for u_3 as a function of λ, at leading order in λ, is equivalent to setting u_3 at its λ-dependent fixed point value. With the present pseudoregulator this value of the sextic coupling reads

$$u_3^{(1)} = \frac{N+26}{32\pi^2}\lambda^3\frac{k^2}{m^2}. \tag{6.13}$$

This illustrates a second mechanism that generates two-loop terms, even with the $\overline{\mathrm{MS}}$ pseudoregulator. In fact, despite all momentum integrals appearing in the beta function of u_3, and any other nonrenormalizable couplings, being dimensionful and thus vanishing in the $\epsilon\to 0$ limit, some of the integrals appearing in the solution of the flow and fixed-point equations for these couplings are dimensionless and therefore survive in $\overline{\mathrm{MS}}$. In other words, the flow equations should be solved before the $\epsilon\to 0$ limit is taken. Then, replacing Eq. (6.13) in the second line of Eq. (6.9) produces further λ^3 terms in the beta function.

A similar fate applies to the third line of Eq. (6.9), although the computational details this time are somewhat more intricate. This is due to the momentum dependence of the nonrenormalizable couplings appearing inside Z and $\tilde{Z}$. In the process of solving the flow equations for these couplings at leading order in λ, and plugging the solution in Eq. (6.9), the following momentum averages are generated

$$\langle z_1\rangle^6_{2,0}(0) = -8(16\pi^2)\lambda^2 A, \tag{6.14a}$$

$$\langle \tilde{z}_1\rangle^6_{0,2}(0) = -4(16\pi^2)(N+8)\lambda^2 A, \tag{6.14b}$$

$$\langle z_2\rangle^6_{1,0}(0) = 32(16\pi^2)\lambda^3 B, \tag{6.14c}$$

$$\langle \tilde{z}_2\rangle^6_{0,1}(0) = 8(16\pi^2)(N+26)\lambda^3 B. \tag{6.14d}$$

Here A and B are dimensionless double momentum integrals whose precise form is given in Appendix D 3. Although these are two-loop integrals, they involve only one copy of $\partial_t R_k$, because one of them disappears in the process of solving the flow equations for the nonrenormalizable couplings. As a consequence, the $1/\epsilon^2$ pole of the integrals is not balanced by the ϵ factor coming from the single $\partial_t R_k$. Thus both A and B exhibit a $1/\epsilon$ pole.[11] Despite this divergence, the flow equation itself is finite, at least at order λ^3, as in fact the only appearance of A and B on the rhs of Eq. (6.9) is through the combination $A+B$, in which the $1/\epsilon$ poles cancel. The final result of this process is therefore

$$A + B = \frac{1}{2(16\pi^2)^2}. \tag{6.15}$$

Also for these terms, taking the $\epsilon\to 0$ limit too early, i.e., before the flow for Z and $\tilde{Z}$ is solved and fed back inside β_λ, would fail to unveil higher order corrections.

Putting all these contributions together, the truncated beta function of Eq. (6.9) in the $\epsilon\to 0$ limit reduces to

$$\beta_\lambda = \frac{N+8}{16\pi^2}\lambda^2 - \frac{2(5N+22)}{(16\pi^2)^2}\lambda^3 + 2\eta\lambda. \tag{6.16}$$

We next turn to the computation of the anomalous dimension. Following Ref. [49], we split η in the sum

$$\eta = \eta^{(1)} + \eta^{(2)}, \tag{6.17}$$

the two terms on the rhs being the contributions of the momentum-independent and -dependent parts of the wave function renormalizations, respectively. Notice that both contributions would vanish in a truncation neglecting the field dependence of the wave function renormalizations, as the vacuum expectation value κ vanishes at this order. Thus both terms are entirely due to the four-point function $\Gamma^{(4)}_k$. In the first part, $\eta^{(1)}$ is proportional to the derivative couplings at zero momenta

$$\eta^{(1)} = \frac{1}{16\pi^2} l^4_{1,0}(0)[(N-1)z_1(0) + \tilde{z}_1(0)]. \tag{6.18}$$

The $O(\lambda^2)$ solution of the flow equation gives

$$z_1(0) = \frac{1}{3(16\pi^2)}\frac{k^2}{m^2}\lambda^2, \tag{6.19a}$$

$$\tilde{z}_1(0) = \frac{(N+8)}{6(16\pi^2)}\frac{k^2}{m^2}\lambda^2, \tag{6.19b}$$

[11] Incidentally, neither A nor B would be divergent within a strict derivative expansion where the RG-generated momentum dependence of Z and $\tilde{Z}$ is truncated to its power series expansion around $p^2 = 0$, because in the latter case the two-loop integrals would exhibit only a $1/\epsilon$ pole. However this truncation would not reproduce the full two-loop beta function, but just part of the $O(\lambda^3)$ contributions.

such that the first contribution to the anomalous dimension reads

$$\eta^{(1)} = -\frac{(N+2)}{2(16\pi^2)}\lambda^2. \tag{6.20}$$

The second part of the anomalous dimension is instead proportional to the nontrivial momentum dependence of $\Gamma_k^{(4)}$. Again taking the limit for $\epsilon \to 0$ at the end of the nesting process, we find

$$\eta^{(2)} = \frac{(N+2)}{(16\pi^2)^2}\lambda^2. \tag{6.21}$$

Thus the whole two-loop anomalous dimension is recovered

$$\eta = \eta^{(1)} + \eta^{(2)} = \frac{(N+2)}{2(16\pi^2)^2}\lambda^2. \tag{6.22}$$

Inserting in Eq. (6.16) we finally arrive at

$$\beta_\lambda = \frac{N+8}{16\pi^2}\lambda^2 - \frac{9N+42}{(16\pi^2)^2}\lambda^3. \tag{6.23}$$

This is the universal part of the beta function at two loops. With different mass-dependent regulators one would obtain additional nonuniversal terms depending on the mass. The contributions to the beta function from three loops up is known not to be universal. In our approach this regulator dependence arises at least from two sources: the freedom of inserting other couplings in the pseudoregulator, as discussed in Sec. III C and footnote 10, and the contributions coming from Z_k, as mentioned in Sec. IV A and footnote 9.

A similar treatment of Eq. (6.8) leads to

$$\beta_{m^2} = m^2\left[\frac{(N+2)}{16\pi^2}\lambda - \frac{(N+2)}{4(16\pi^2)^2}\lambda^2 \times ((1+2f_1)(N+2) - 8\sqrt{3}\pi + 70)\right]. \tag{6.24}$$

Some more details are reported in Appendix D 5. We observe that the two-loop term is not universal, and that the $\overline{\text{MS}}$ result can be reproduced by suitably fixing the parameter f_1, which enters the pseudoregulator (6.11) through the choice of M.

VII. GENERALIZATION TO CONTINUOUS DIMENSIONS

Despite the fact that $\overline{\text{MS}}$ is limited to applications in an even number of dimensions, the pseudoregulator we discussed lends itself to generalizations to any continuous d, thanks to the intimate relation that exists between dispersion relations and the dimensionality of spacetime. Consider the following pseudoregulator

$$R_k(z) = Z_0 Z_k^{\sigma\epsilon}\left[\mu^{2(1-\alpha)}\left(\frac{k^2}{\mu^4}\right)^{\epsilon} z^{\alpha+\epsilon} - z\right], \tag{7.1}$$

which generalizes (4.9) in that the regularized propagator is now an homogeneous function of momentum with power $\alpha+\epsilon$, rather than $1+\epsilon$. This allows to correspondingly generalize the formulae (4.14) for the Q functionals, whenever the dimension of the momentum integrals, after having factored out all μ dependence, is a nonnegative integer. In fact, in this case the $\epsilon \to 0$ and the $Z_0 \to 0$ limits give

$$Q_n(G_k^\ell \partial_t R_k) = 2\mu^{2n(1-\frac{1}{\alpha})}\frac{\Gamma(\frac{n}{\alpha})Z_k^{-\frac{n}{\alpha}}(-V_k'')^{\frac{n}{\alpha}-\ell+1}}{\Gamma(n)\Gamma(\ell)}\frac{(1-\frac{\sigma\eta}{2})}{\Gamma(\frac{n}{\alpha}-\ell+2)}, \tag{7.2a}$$

$$Q_n(G_k^\ell G_k' \partial_t R_k) = -\frac{\alpha^2\Gamma(\frac{n-1}{\alpha}+2)\mu^{2(n-1)(1-\frac{1}{\alpha})}}{(n-1)\Gamma(n)}\frac{2Z_k^{1-\frac{n}{\alpha}}(-V_k'')^{\frac{n-1}{\alpha}-\ell}}{\Gamma(\ell+2)}\frac{(1-\frac{\sigma\eta}{2})}{\Gamma(\frac{n-1}{\alpha}-\ell+1)}, \tag{7.2b}$$

$$Q_n(G_k^\ell G_k'' \partial_t R_k) = \frac{\alpha^2(2\alpha - \frac{(\alpha-1)(\ell+2)}{\frac{n-2}{\alpha}+2})\Gamma(\frac{n-2}{\alpha}+3)\mu^{2(n-2)(1-\frac{1}{\alpha})}}{(n-2)\Gamma(n)}\frac{2Z_k^{2-\frac{n}{\alpha}}(-V_k'')^{\frac{n-2}{\alpha}-\ell}}{\Gamma(\ell+3)}\frac{(1-\frac{\sigma\eta}{2})}{\Gamma(\frac{n-2}{\alpha}-\ell+1)}, \tag{7.2c}$$

where the α-dependent arguments of the Gamma functions in the denominators are positive integers. Recall that for a scalar field theory and in the derivative expansion the index n takes the values $d/2+l$ with $l = 0, 1, 2, \ldots$. Hence, if α is a continuous power, these formulas are applicable to continuous d.

We illustrate the use of this generalized pseudoregulator, by addressing the description of the Wilson-Fisher fixed point for $2 < d < 4$, for the linear $O(N)$ models. We focus on the flow equations we presented in Appendix B within the derivative expansion, namely Eqs. (B2) and (B3). As we expect the effective potential to play a dominant role in the description of the Wilson-Fisher fixed point, we demand that the corresponding quantum contributions be nonvanishing in our regularization scheme. Specifically, the first kind of Q functional, given in Eq. (7.2), is nontrivial in $2 < d < 4$ only if $d/2\alpha$ is a positive integer. Under this assumption the flow equations of the derivative expansion become

$$\partial_t u = -du + (d-2+\eta)\tilde{\rho}u' + \frac{\alpha\tilde{\mu}^{d(1-\frac{1}{\alpha})}}{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1)}[\tilde{\zeta}^{-\frac{d}{2\alpha}}(-u'-2\tilde{\rho}u'')^{\frac{d}{2\alpha}} + (N-1)\zeta^{-\frac{d}{2\alpha}}(-u')^{\frac{d}{2\alpha}}], \tag{7.3a}$$

$$\begin{aligned}\partial_t\tilde{\zeta} = {} & \eta\tilde{\zeta} + (d-2+\eta)\tilde{\rho}\tilde{\zeta}' - \frac{\tilde{\mu}^{d(1-\frac{1}{\alpha})}}{(4\pi)^{d/2}\Gamma(\frac{d}{2})}[(\tilde{\zeta}'+2\tilde{\rho}\tilde{\zeta}'')\tilde{\zeta}^{-\frac{d}{2\alpha}}(-u'-2\tilde{\rho}u'')^{\frac{d}{2\alpha}-1} + (N-1)\tilde{\zeta}'\zeta^{-\frac{d}{2\alpha}}(-u')^{\frac{d}{2\alpha}-1}] \\ & - \frac{(d+2(\alpha-6))(d-2\alpha)\tilde{\mu}^{d(1-\frac{1}{\alpha})}}{6\alpha(4\pi)^{d/2}\Gamma(\frac{d}{2})}\tilde{\rho}(3u''+2\tilde{\rho}u''')\tilde{\zeta}'\tilde{\zeta}^{-\frac{d}{2\alpha}}(-u'-2\tilde{\rho}u'')^{\frac{d}{2\alpha}-2} \\ & - (N-1)\frac{d(d+2)(d^2-4\alpha^2)\tilde{\mu}^{d(1-\frac{1}{\alpha})}}{24\alpha(4\pi)^{d/2}\Gamma(\frac{d}{2}+2)}\tilde{\rho}u''\zeta'\zeta^{-\frac{d}{2\alpha}}(-u')^{\frac{d}{2\alpha}-2} \\ & + (N-1)\frac{(d-2\alpha)\tilde{\mu}^{d(1-\frac{1}{\alpha})}}{\alpha(4\pi)^{d/2}\Gamma(\frac{d}{2})}u''(\tilde{\zeta}-\zeta)\zeta^{-\frac{d}{2\alpha}}(-u')^{\frac{d}{2\alpha}-2},\end{aligned} \tag{7.3b}$$

where u, ζ and $\tilde{\zeta}$ are the dimensionless renormalized counterparts of the U_k, Z_k and $\tilde{Z}_k$ of Eqs. (5.17) and (5.18), defined in analogy to Eq. (5.3). Here for notational simplicity we dropped the RG improvement, by setting $\sigma = 0$; the full equations contain the factor $(1-\sigma\eta/2)$ in front of every quantum contribution. The $\tilde{\mu}$ dependence of these flow equations can be canceled by a further rescaling of all dimensionful quantities with respect to $\tilde{\mu}$, which casts the RG equations in a genuine $\overline{\mathrm{MS}}$ form.

We look for fixed points of the previous flow equations by means of a small-fields polynomial expansion

$$u = \sum_{i=0} \frac{\lambda_{2i}}{i!}\tilde{\rho}^i, \tag{7.4a}$$

$$\tilde{\zeta} = 1 + \sum_{i=1} \frac{\tilde{z}_{2i}}{i!}\tilde{\rho}^i, \qquad \zeta = 1 + \sum_{k=i} \frac{z_{2i}}{i!}\tilde{\rho}^i. \tag{7.4b}$$

We find the Gaussian fixed point for every value of α and a nontrivial fixed point only for $\alpha = d/4$, located at

$$\lambda_4^* = \frac{(4\pi)^{d/2}\Gamma(\frac{d}{2})(4-d)}{8+N}\tilde{\mu}^{4-d}, \tag{7.5}$$

with $\eta = 0$, and all others couplings being vanishing. We note that with standard regulators the dimensionless potential of the WF fixed point has a nontrivial minimum, but the dimensionful mass (deduced from the limit $k \to 0$) is zero, in accordance with the fact that the theory is scale invariant at quantum level. With the $\overline{\mathrm{MS}}$ pseudoregulator the "dimensionless mass" λ_2 is zero even for finite k. The same phenomenon happens also in the functional perturbative approach [17].

For $\alpha = d/4$ the power of $\tilde{\mu}$ appearing in the Eqs. (7.3) is $(d-4)$. Therefore, the rescaling which maps these equations into those of $\overline{\mathrm{MS}}$, is effectively declaring the dimensionality of the couplings to be the one expected in four dimensions, the rescaling factors differing only by powers of $(k/\tilde{\mu})^{4-d}$. Furthermore, the value $\alpha = d/4$ is precisely the one that makes the quartic interaction marginal for continuous d. In fact, the effective kinetic term of the regularized theory has a dispersion relation z^α, which changes the dimensionality of the scalar field from $(d-2)/2$ to $(d-2\alpha)/2$. In other words, within the present truncation, our one-loop-like equations for ϕ^4 theory are able to detect the Wilson-Fisher fixed point in continuous d only when the pseudoregulator turns d into an "effective upper critical dimension." This interpretation is also consistent with the apparent absence of the multicritical models with ρ^p interactions, for $p > 2$. In fact, the effective upper critical dimensions for these models would be at $d = 2p\alpha/(p-1)$, which is not compatible with our simplifying assumption of an integer $d/2\alpha$.

The stability matrix at the fixed point of Eq. (7.5) is triangular and the eigenvalues are

$$\theta_i = -d + i(d-2) + i\left[1 + \frac{6(i-2)}{N+8}\right](4-d), \tag{7.6a}$$

$$\omega_i = i(d-2) + \left[i + 1 + \frac{3(2i^2-2i-3)}{N+8}\right](4-d) + \frac{3i}{2(N+8)}(4-d)^2. \tag{7.6b}$$

By setting $d = 4-\varepsilon$ we recognize that this prediction agrees with the usual first order of the ε expansion. For instance, for $N = 1$ we get

$$\theta = \left(-2 + \frac{\varepsilon}{3}, \varepsilon, 2 + 3\varepsilon, 4 + \frac{19}{3}\varepsilon, 6 + 11\varepsilon \ldots\right), \tag{7.7a}$$

$$\omega = \left(2, 4 + \frac{4}{3}\varepsilon, 6 + 4\varepsilon, 8 + 8\varepsilon \ldots\right). \tag{7.7b}$$

Notice that ε here should not be confused with the ϵ of Eq. (7.1), the latter having been removed by the limit $\epsilon \to 0$. Also, the one-loop predictions of Eq. (7.6) become exact in the $N \to \infty$ limit.

Order ε^2 corrections affect the estimate of ω_i in Eq. (7.6b) but are missing in Eq. (7.6a). This is related to the fact that the fixed point value of $\eta = 0$ is vanishing in this truncation, such that the RG resummations triggered by the dependence of R_k on Z_k are ineffective at the fixed point. In fact, improvements on the estimate of θs can be obtained by allowing for the feedback of other couplings in the pseudoregulator. For instance, it is natural to allow for the replacement of the mass parameter μ^2 with the running λ_2 through a tunable parameter b,[12] and write

$$R_k(z) = Z_0 Z_k^{\sigma\epsilon}\left[\mu^{2(1-\alpha)}\left(\frac{k^2}{\mu^{2(2-b)}\lambda_2^b}\right)^{\epsilon} z^{\alpha+\epsilon} - z\right]. \tag{7.8}$$

This would result in a different RG improvement of Eqs. (7.3), where each quantum term is now multiplied by the factor $(1 - b\beta_2/(2\lambda_2) - \sigma\eta/2)$, which leads to the following b-dependent quartic coupling evaluated at the fixed point and critical exponents

$$\lambda_4^* = \frac{(4\pi)^{d/2}\Gamma(\frac{d}{2})(4-d)}{N + 8 - \frac{b}{2}(N+2)(4-d)}\tilde{\mu}^{4-d}, \tag{7.9a}$$

$$\theta_i = -d + i(d-2) + i\left[1 + \frac{6(i-2)}{N+8}\right](4-d) + \delta_{i,4}\frac{b}{2}\left(\frac{N+2}{N+8}\right)(4-d)^2, \tag{7.9b}$$

$$\omega_i = i(d-2) + \left[i + 1 + \frac{3(2i^2 - 2i - 3)}{N+8}\right](4-d) + \frac{3i}{2(N+8)}(4-d)^2. \tag{7.9c}$$

Getting better estimates of the critical η and the correlation-length exponent ν does instead require a larger truncation, accounting for at least part of the two-loop contributions, as discussed in Sec. VI. This kind of more elaborate analysis of the Wilson-Fisher fixed point by means of the $\overline{\text{MS}}$ pseudoregulator is left for future studies.

We reiterate that the flow equations (7.3) have been obtained under the assumptions that $2 < d < 4$ and that $d/2\alpha$ is a positive integer. In even dimensions the equations have additional terms. In fact, taking the $d \to 2$ limit in (7.3) would miss relevant contributions which are present in the flow equations studied in Sec. V A. The latter can indeed be reproduced by the α-generalized $\overline{\text{MS}}$ pseudoregulator, by applying Eq. (7.2) directly in $d = 2$ and taking the $\alpha \to 1$ limit.

Finally, let us comment on the extension of the nonlinear sigma model of Sec. V B to dimension $d > 2$. Instead of just using the ε expansion, it is possible to use directly the generalized $\overline{\text{MS}}$ pseudoregulator (7.1). As a result, in $d = 2 + \varepsilon$ we recover the well-known nontrivial fixed point

$$g_*^2 = \frac{2\pi\varepsilon}{N-1} + \frac{\pi\sigma\varepsilon^2}{(N-1)^2} + O(\varepsilon^3), \tag{7.10a}$$

$$\nu^{-1} = \varepsilon - \frac{\sigma\varepsilon^2}{2(N-1)} + O(\varepsilon^3), \tag{7.10b}$$

$$\eta = \frac{\varepsilon}{N-1} + O(\varepsilon^3). \tag{7.10c}$$

Here it is possible to adjust σ ($\sigma = -2$) to get the full two-loop result for ν, but not for η.[13] The latter correction would arise by considering a truncation where Z_k depends also on the momenta, as is discussed in Sec. VI.

VIII. CONCLUSIONS

Mass-dependent Wilsonian RG schemes, such as for instance momentum subtraction with a sharp UV cutoff, simultaneously achieve the two goals of regularizing a field theory and of defining the heavy modes to be integrated out while constructing an effective description of the system. In these schemes, information about mass thresholds is essential and built in the effective theory at all scales. Mass-independent schemes instead, e.g., dimreg/$\overline{\text{MS}}$, remove the latter piece of information by taking the limit of infinite separation between the physical scales of applicability of the effective field theory and the heavy masses of the underlying microscopic description. It is therefore natural that, in the construction of a

[12] We cannot replace every occurrence of μ^2 with λ_2, otherwise singularities of the form β_2/ϵ would arise in the beta functions. The insertion of λ_2 in the pseudoregulator must preserve the cancellation of such poles.

[13] This situation is similar to the one observed Sec. IV C for ϕ^3 theory in six dimensions, although there we could tune σ to get the right two-loop result for either β_3 or η.

mass-independent scheme out of a Wilsonian one, the infinite-separation-of-scales limit also becomes a regularization-removal process.

This is precisely what has been observed in the present work. More specifically, we have focused on the functional renormalization group (FRG) equations, a prototypical Wilsonian representation of field theory based on shell-by-shell integration of modes according to a coarse-graining-defining function $R_k(q^2)$, which acts as a smooth infrared cutoff on modes with momentum $q^2 \ll k^2$. As a matter of fact, we have found that it is indeed possible to achieve a continuous transition from this exact mass-dependent scheme to functional RG equations within $\overline{\mathrm{MS}}$, at the price of taking a parametric limit $\epsilon \to 0$ that in even dimensions also results in the removal of the cutoff: $R_k \to 0$.[14]

The dependence of R_k on continuous parameters, such as our ϵ, is allowed and welcome in the FRG setup. In fact, it is often used in FRG applications as a diagnostic tool (weak dependence on such parameters is taken as a sign of a good truncation) or even as a selection criterion for the "best" regulator (e.g., through the principle of minimum sensitivity [30,31]). However we find that, while taking the $\epsilon \to 0$ limit, the regulator R_k at some point must leave the domain of acceptable IR Wilsonian cutoffs and violate some of the conditions that define physical coarse grainings. This is quite to be expected, as dimreg is by no means a physical IR cutoff. As such, also the pseudocutoff form which should be attained by R_k for asymptotically small ϵ, our Eq. (3.1), defies every interpretation as a conventional regulator, and is well suited for its goal only when augmented by analytic continuation of the momentum integrals in ϵ.

Quite interestingly, there is a certain degree of freedom in how to attain the $\overline{\mathrm{MS}}$ limit, and it is even possible to achieve it starting from the most popular functional forms of R_k, as is explained in Sec. III D. This exercise renews interest on several nontrivial aspects of vanishing-R_k limits of FRG equations, to which we will devote a subsequent paper [58]. There, we will also show that this limit can help to weed out certain unphysical features that are introduced in the FRGE by some choices of regulator, and which are intimately related to the preservation of nonlinear symmetries.

Indeed, one of the most tantalizing aspects of this research direction, is the possibility to look at the $\overline{\mathrm{MS}}$ limit of FRG equations as a novel way to approach the challenging problem of gauge and nonlinear symmetries. Here, we have limited ourselves to explore these aspects in Sec. V B, where we have observed that the $\epsilon \to 0$ limit of the RG equations of a linear $O(N)$ model have the pleasant property of preserving also a nonlinearly realized $O(N+1)$ symmetry. Further systematic studies of this problem are in order to assess whether taking the $\overline{\mathrm{MS}}$ limit might ease the task of fulfilling Ward-Takahashi identities and master equations (actually R_k-deformed versions of the latter).

Although it might be possible to study the problem of reproducing $\overline{\mathrm{MS}}$ at the level of the exact FRG equation (1.1), we have only addressed this goal within specific approximation schemes, mainly the first orders of the derivative expansion and up to two loops. One major conceptual problem which might be raised against our efforts is whether it makes any sense to try to join the FRG framework, whose strength is in the exact and nonperturbative nature of the formalism, with dimreg/$\overline{\mathrm{MS}}$, which is widely believed to be applicable only within perturbation theory. Another way of formulating this question, would be to ask for a way of performing numerical FRG computations and still make sense of an $\overline{\mathrm{MS}}$ limit.

Keeping this aspect in mind, after the construction of an FRG pseudoregulator which successfully reproduces the one-loop $\overline{\mathrm{MS}}$ beta functions for vanishing ϵ, see Sec. III, we have addressed the question as to whether this pseudoregulator choice and the $\epsilon \to 0$ limit spoil the nonperturbative nature of the exact FRG equation. We have provided reasons to argue for a negative answer. In Secs. IV and V we have first illustrated the physical content of the RG resummations contained in the RG improvement of one-loop beta functions, showing that they account for higher-order perturbative contributions and can even fairly describe some nonperturbative critical phenomena in two dimensions, see for instance Table I.

Furthermore, we have found strong hints suggesting that the FRG equations remain nonperturbative, as long as the limit of vanishing ϵ is taken at the end of the relevant computations (indeed, analytic continuation of integrals and parametric limits are processes which can be performed also numerically). To substantiate this conclusion, and to illustrate how the limit of vanishing pseudoregulator should be dealt with, in Sec. VI we have derived the two-loop beta functions for massive ϕ^4 theory in four dimensions. The details of the computation, presented in Appendix D, show how the typical higher-orders FRG contributions, which are tied to dimensionful momentum integrals and stay rightly nonvanishing for $\epsilon \neq 0$, survive even for $\epsilon \to 0$ as long as the RG equations are solved before taking the latter limit.

Other interesting aspects we have just touched upon, and that certainly deserve further attention, are the use of the $\overline{\mathrm{MS}}$ pseudoregulator in theories featuring background fields (Sec. III B) and in continuous dimensions (Sec. VII). Finally, as in this work we have confined our

[14]The generalization to continuous dimensions discussed in Sec. VII is an exception, as in this case the deformation of the dispersion relation operated by the pseudoregulator survives the $\epsilon \to 0$ limit.

attention to scalar field theories, future works will need to explore the generalization to fermions, gauge theories and gravity.

ACKNOWLEDGMENTS

We are grateful to O. Zanusso for seminal discussions on this topic, and to J. M. Pawlowski for his challenging questions. L. Z. acknowledges support by the DFG under Grants No. GRK1523/2, No. Gi328/9-1 and No. 406116891 within the Research Training Group RTG 2522/1, for some of the duration of this project.

APPENDIX A: DERIVATION OF THE PSEUDOREGULATOR

In this section we present a derivation of the functional form of the $\overline{\mathrm{MS}}$ pseudoregulator in Eq. (3.10). We want to obtain the result (2.17) from the FRGE. Inspired by the deformation of integrands which takes place in dimensional regularization, we consider the following family of regulators

$$R_k(z) = \mu^{-2\epsilon} F(k,\mu,m,\epsilon) z^{1+\epsilon} - z. \tag{A1}$$

Here the first term containing the momentum power $(1+\epsilon)$ is needed to reproduce the $1/\epsilon$ pole of the analytically continued dimensionless integral and can be obtained for instance by starting with a $(d+\epsilon)$ dimensional integral, and by integrating out the $(+\epsilon)$ dimensions [59]. In it, F is an arbitrary dimensionless function and μ is a classical arbitrary mass parameter. The second term in Eq. (A1) is there to cancel the original inverse propagator of the bare theory [we recall that the regulator R_k acts additively, and that the regulated inverse propagator is the combination $z + R_k(z)$].

Note that $\partial_t R_k(z)$ has two contributions: one coming from the explicit dependence of F on k and another one proportional to $\beta_{m^2} = \partial_t m^2$. Assuming that $\partial_t F \propto \epsilon$ and using the following identity

$$\Gamma[-n+\epsilon] = \frac{(-1)^n}{\Gamma[n+1]}\frac{1}{\epsilon} + O(\epsilon^0), \tag{A2}$$

the Mellin transform of the first term inside $\partial_t R_k$ is

$$\begin{aligned} &\frac{\partial_t F}{F^{1+\frac{n}{1+\epsilon}}}\left(\frac{\mu}{m}\right)^{\frac{2n\epsilon}{1+\epsilon}} \frac{\Gamma(1+\frac{n}{1+\epsilon})\Gamma(l-n-1+\frac{n\epsilon}{1+\epsilon})}{(1+\epsilon)\Gamma(n)\Gamma(l)(m^2)^{-(n-l+1)}} \\ &= \frac{\partial_t F}{\epsilon F^{1+n}} \frac{(-1)^{n-l+1}}{\Gamma(l)\Gamma(n-l+2)}(m^2)^{n-l+1} + O(\epsilon). \end{aligned} \tag{A3}$$

So taking the limit for $\epsilon \to 0$, we find (2.17) if

$$\partial_t F(k,\mu,m,\epsilon) = 2\epsilon F(k,\mu,m,\epsilon)^{1+n}, \tag{A4}$$

that is

$$F = 1 + \epsilon \log\left(\frac{k^2}{\mu^{2-2b} m^{2b}}\right) + O(\epsilon^2) \approx \left(\frac{k^2}{\mu^{2-2b} m^{2b}}\right)^{\epsilon}. \tag{A5}$$

With this F, the second piece of $\partial_t R_k$ proportional to β_{m^2} can be calculated in the same way. This agrees with Eq. (3.10), of which Eq. (3.1) is a special case, corresponding to $b = 0$. As described in the main text, different choices of b only affect higher-order corrections.

APPENDIX B: FLOW EQUATIONS IN THE $O(\partial^2)$ DERIVATIVE EXPANSION

In this section we present part of the flow equations of the linear $O(N)$ model in the $O(\partial^2)$ of the derivative expansion, which can be found for instance in Ref. [60]. These descend from the exact FRG equation upon specifying the truncation of Eq. (5.17). We introduce the following notations

$$G_0 = (Z_k q^2 + R_k(q^2) + U'_k)^{-1}, \tag{B1a}$$

$$G_1 = (\tilde{Z}_k q^2 + R_k(q^2) + U'_k + 2\rho U''_k)^{-1}, \tag{B1b}$$

for the Goldstone-bosons and radial-mode propagators. The flow equations for U_k reads

$$\partial_t U_k = \frac{Q_{\frac{d}{2}}[G_1 \partial_t R_k] + (N-1) Q_{\frac{d}{2}}[G_0 \partial_t R_k]}{2(4\pi)^{d/2}}. \tag{B2}$$

The beta functional for Z_k is instead presented with slightly different notations in Appendix D, more precisely in Eq. (D37). Finally, the flow equation for $\tilde{Z}_k$, which is defined in Eq. (5.18), reads

$$
\begin{aligned}
\partial_t \tilde{Z}_k = & -\frac{(\tilde{Z}_k' + 2\rho \tilde{Z}_k'')}{2(4\pi)^{d/2}} Q_{\frac{d}{2}}[G_1^2 \partial_t R_k] - (N-1)\frac{(Z_k' + \rho Y_k')}{2(4\pi)^{d/2}} Q_{\frac{d}{2}}[G_0^2 \partial_t R_k] \\
& + \frac{2\rho(\tilde{Z}_k')^2}{(4\pi)^{d/2}}\left[\frac{2d+1}{2} Q_{\frac{d}{2}+1}[G_1^3\partial_t R_k] + \frac{(d+2)(d+4)}{4}\left(Q_{\frac{d}{2}+2}[G_1^2 G_1' \partial_t R_k] + Q_{\frac{d}{2}+3}[G_1^2 G_1'' \partial_t R_k]\right)\right] \\
& + \frac{2\rho(3U_k'' + 2\rho U_k''')^2}{(4\pi)^{d/2}}\left(Q_{\frac{d}{2}}[G_1^2 G_1' \partial_t R_k] + Q_{\frac{d}{2}+1}[G_1^2 G_1'' \partial_t R_k]\right) \\
& + \frac{2\rho \tilde{Z}_k'(3U_k'' + 2\rho U_k''')}{(4\pi)^{d/2}}\left[(d+2)\left(Q_{\frac{d}{2}+1}[G_1^2 G_1' \partial_t R_k] + Q_{\frac{d}{2}+2}[G_1^2 G_1'' \partial_t R_k]\right) + 2Q_{\frac{d}{2}}[G_1^3 \partial_t R_k]\right] \\
& + (N-1)\frac{\rho Y_k}{(4\pi)^{d/2}}(2U_k'' Q_{\frac{d}{2}}[G_0^3 \partial_t R_k] + dZ_k' Q_{\frac{d}{2}+1}[G_0^3 \partial_t R_k]) \\
& + (N-1)\frac{2\rho(Z_k')^2}{(4\pi)^{d/2}}\left[\frac{(d+2)(d+4)}{4}\left(Q_{\frac{d}{2}+2}[G_0^2 G_0' \partial_t R_k] + Q_{\frac{d}{2}+3}[G_0^2 G_0'' \partial_t R_k]\right) + \frac{1}{2} Q_{\frac{d}{2}+1}[G_0^3 \partial_t R_k]\right] \\
& + (N-1)\frac{2\rho(U_k'')^2}{(4\pi)^{d/2}}(Q_{\frac{d}{2}}[G_0^2 G_0' \partial_t R_k] + Q_{\frac{d}{2}+1}[G_0^2 G_0'' \partial_t R_k]) \\
& + (N-1)\frac{2\rho Z_k' U_k''}{(4\pi)^{d/2}}(d+2)(Q_{\frac{d}{2}+1}[G_0^2 G_0' \partial_t R_k] + Q_{\frac{d}{2}+2}[G_0^2 G_0'' \partial_t R_k]).
\end{aligned} \tag{B3}
$$

Using the pseudoregulator (4.9), in the $Z_0 \to 0$ limit and suppressing the RG improvement by setting $\sigma = 0$, we find

$$
\partial_t U_k = \frac{(-U_k' - 2\rho U_k'')^{\frac{d}{2}}}{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1)\tilde{Z}_k^{\frac{d}{2}}} + \frac{(N-1)(-U_k')^{\frac{d}{2}}}{(4\pi)^{d/2}\Gamma(\frac{d}{2}+1) Z_k^{\frac{d}{2}}}, \tag{B4}
$$

$$
\begin{aligned}
\partial_t \tilde{Z}_k = & -\frac{(\tilde{Z}_k' + 2\rho\tilde{Z}_k'')}{(4\pi)^{d/2}\Gamma(\frac{d}{2})}\tilde{Z}_k^{-\frac{d}{2}}(-U_k' - 2\rho U_k'')^{\frac{d}{2}-1} - (N-1)\frac{(Z_k' + \rho Y_k')}{(4\pi)^{d/2}\Gamma(\frac{d}{2})} Z_k^{-\frac{d}{2}}(-U_k')^{\frac{d}{2}-1} \\
& + \frac{(4+18d-d^2)}{12(4\pi)^{d/2}\Gamma(\frac{d}{2})}\rho(\tilde{Z}_k')^2 \tilde{Z}_k^{-\frac{d}{2}-1}(-U_k' - 2\rho U_k'')^{\frac{d}{2}-1} \\
& + \frac{(10-d)}{3(4\pi)^{d/2}\Gamma(\frac{d}{2}-1)}\rho\tilde{Z}_k'(3U_k'' + 2\rho U_k''')\tilde{Z}_k^{-\frac{d}{2}}(-U_k' - 2\rho U_k'')^{\frac{d}{2}-2} \\
& - \frac{\rho(3U_k'' + 2\rho U_k''')^2}{3(4\pi)^{d/2}\Gamma(\frac{d}{2}-2)}\tilde{Z}_k^{-\frac{d}{2}+1}(-U_k' - 2\rho U_k'')^{\frac{d}{2}-3} \\
& + (N-1)\frac{(4-6d-d^2)}{6(4\pi)^{d/2}\Gamma(\frac{d}{2})}\rho(Z_k')^2 Z_k^{-\frac{d}{2}-1}(-U_k')^{\frac{d}{2}-1} \\
& - (N-1)\frac{(d+2)}{3(4\pi)^{d/2}\Gamma(\frac{d}{2}-1)}\rho Z_k' U_k'' Z_k^{-\frac{d}{2}}(-U_k')^{\frac{d}{2}-2} \\
& - (N-1)\frac{\rho(U_k'')^2}{3(4\pi)^{d/2}\Gamma(\frac{d}{2}-2)} Z_k^{-\frac{d}{2}+1}(-U_k')^{\frac{d}{2}-3} \\
& + (N-1)\frac{2\rho Y_k}{(4\pi)^{d/2}\Gamma(\frac{d}{2}-1)}\left(U_k'' - \frac{d}{d-2} Z_k' Z_k^{-1} U_k'\right) Z_k^{-\frac{d}{2}}(-U_k')^{\frac{d}{2}-2}.
\end{aligned} \tag{B5}
$$

For general σ, the right-hand sides have to be simply multiplied by $(1 - \sigma\eta/2)$.

APPENDIX C: THRESHOLD FUNCTIONS FOR A MASS-DEPENDENT PSEUDOREGULATOR

In this Appendix we detail the computation of the following threshold functions

$$l^d_{n,0}(0) = \frac{nZ_k^n}{2} k^{2n-d} \int_0^\infty \mathrm{d}z z^{\frac{d}{2}-1} \frac{\partial_t R_k(z)}{P_k(z)^{n+1}}, \quad \text{(C1)}$$

where

$$P_k(z) = Z_k z + R_k(z), \quad \text{(C2)}$$

by means of the mass-dependent pseudoregulator of Eq. (6.11). As we need the result for the computation of the two-loop beta function in four-dimensional $\lambda\phi^4$ theory, we content ourselves of the first orders in a perturbative expansion in λ. In particular, we neglect the η dependence appearing on the rhs of the flow equations through the regularization, as it would lead to higher orders in λ. Our pseudoregulator choice results in simple propagators but a somewhat more convoluted contribution of the differentiated pseudoregulator:

$$P_k(z) = Z_k \left(\frac{k^2}{\mu^{4-2b} M^{2b}}\right)^{\epsilon} (z + M^2)^{1+\epsilon}, \quad \text{(C3a)}$$

$$\partial_t R_k(z) = 2\epsilon \left(1 - \frac{b\partial_t M^2}{2M^2}\right) P_k(z) + (1+\epsilon)\partial_t M^2 P_k(z) - \beta_{m^2}. \quad \text{(C3b)}$$

The loop integral can then be split into three different kinds of contributions, corresponding to the three pieces of $\partial_t R_k$

$$\begin{aligned}
\frac{2 l^d_{n,0}}{n k^{2n-d}} &= \frac{2\epsilon}{\Gamma(\frac{d}{2})}\left(1 - \frac{b\partial_t M^2}{2M^2}\right)\left(\frac{k^2}{\mu^{4-2b}M^{2b}}\right)^{-n\epsilon} \int_0^\infty \mathrm{d}z \frac{z^{\frac{d}{2}-1}}{(z+M^2)^{n(1+\epsilon)}} \\
&\quad + \frac{(1+\epsilon)}{\Gamma(\frac{d}{2})}\partial_t M^2 \left(\frac{k^2}{\mu^{4-2b}M^{2b}}\right)^{-n\epsilon} \int_0^\infty \mathrm{d}z \frac{z^{\frac{d}{2}-1}}{(z+M^2)^{1+n(1+\epsilon)}} \\
&\quad - \frac{\beta_{m^2}}{\Gamma(\frac{d}{2})}\left(\frac{k^2}{\mu^{4-2b}M^{2b}}\right)^{-(n+1)\epsilon} \int_0^\infty \mathrm{d}z \frac{z^{\frac{d}{2}-1}}{(z+M^2)^{(1+\epsilon)(n+1)}}, \\
&= 2\epsilon\left(1 - \frac{b\partial_t M^2}{2M^2}\right)\left(\frac{k^2}{\mu^{4-2b}M^{2b}}\right)^{-n\epsilon} \frac{\Gamma(n+n\epsilon-\frac{d}{2})}{\Gamma(n+n\epsilon)} (M^2)^{\frac{d}{2}-n-n\epsilon} \\
&\quad + (1+\epsilon)\partial_t M^2 \left(\frac{k^2}{\mu^{4-2b}M^{2b}}\right)^{-n\epsilon} \frac{\Gamma(n+1+n\epsilon-\frac{d}{2})}{\Gamma(n+1+n\epsilon)} (M^2)^{\frac{d}{2}-n-1-n\epsilon} \\
&\quad - \beta_{m^2}\left(\frac{k^2}{\mu^{4-2b}M^{2b}}\right)^{-(n+1)\epsilon} \frac{\Gamma(n+1+(n+1)\epsilon-\frac{d}{2})}{\Gamma(n+1+(n+1)\epsilon)} (M^2)^{\frac{d}{2}-(1+\epsilon)(n+1)}. \quad \text{(C4)}
\end{aligned}$$

To extract the $\epsilon \to 0$ asymptotics we make use of the standard expansion

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{\Gamma(n+1)}\left[\frac{1}{\epsilon} - \gamma + h(n)\right] + O(\epsilon), \quad \text{(C5)}$$

where $h(n) = \sum_{i=1}^n \frac{1}{i}$. Furthermore, we need to parametrize the possible dependence of M^2 on ϵ. Recalling that for vanishing ϵ also R_k needs to vanish, i.e., M^2 should reduce to m^2, we can write

$$M^2 = m^2 + \epsilon m_1^2(k, m, \mu) + O(\epsilon^2), \quad \text{(C6)}$$

$$\partial_t M^2 = \beta_{m^2}(f_0 + \epsilon F_1(k, m, \mu)) + O(\epsilon^2). \quad \text{(C7)}$$

Here m_1^2 and F_1 are two independent functions and f_0 is a proportionality factor. Thus, we allow for the possibility that $\lim_{\epsilon\to 0} \partial_t M^2 \neq \partial_t \lim_{\epsilon\to 0} M^2$, which can be achieved e.g., by means of the choice

$$M^2 = \left(1 + (f_0 - 1)\int_{\epsilon^2}^{\epsilon^2\frac{m^2}{\mu^2}} \mathrm{d}s\Gamma(s)\right) m^2 + \epsilon m_1^2. \quad \text{(C8)}$$

The need for this behavior of M^2 can be appreciated by inspecting the integrals

$$\left.\frac{2l^d_{n,0}}{nk^{2n-d}}\right|_{n\le\frac{d}{2}-1} = \frac{\beta_{m^2}((n+1)f_0-n)(-m^2)^{\frac{d}{2}-n-1}}{n\Gamma(n+2)\Gamma(\frac{d}{2}-n)\epsilon} + \frac{2(1-\frac{bf_0\beta_{m^2}}{2m^2})(-m^2)^{\frac{d}{2}-n}}{\Gamma(n+1)\Gamma(\frac{d}{2}-n+1)}$$
$$+\frac{\beta_{m^2}(-m^2)^{\frac{d}{2}-n-1}\{n(1-f_0)[\log(\frac{k^2}{\mu^{4-2b}m^{2b-2}})+h(n)-h(\frac{d}{2}-n-1)]+f_0+F_1(k)\}}{n\Gamma(n+1)\Gamma(\frac{d}{2}-n)}$$
$$-\frac{\beta_{m^2}m_1^2(-m^2)^{\frac{d}{2}-n-2}((n+1)f_0-n)}{n\Gamma(n+2)\Gamma(\frac{d}{2}-n-1)}+O(\epsilon). \tag{C9}$$

These exhibit a $1/\epsilon$ pole which can be eliminated by tuning $f_0 \neq 1$. To fulfill this, as well as the condition of removing the renormalization scale k from the beta functions, we set

$$f_0 = \frac{n}{n+1}, \tag{C10}$$

$$F_1 = f_1 + \frac{n}{n+1}\left(h\left(\frac{d}{2}-n-1\right) - h(n) - 1 - \log\left(\frac{k^2}{\mu^{4-2b}m^{2b-2}}\right)\right), \tag{C11}$$

$$m_1^2 = \left[f_1 + \frac{n}{n+1}\left(h\left(\frac{d}{2}-n-1\right) - h(n) - 1 - \log\left(\frac{k}{\mu}\right)\right)\right]\beta_{m^2}\log\frac{k}{\mu}$$
$$+(b-1)\frac{n}{2(n+1)}m^2\left(\log\frac{m^2}{\mu^2}\right)^2 + O(\lambda^2). \tag{C12}$$

As a result we have

$$\left.\frac{2l^d_{n,0}}{nk^{2n-d}}\right|_{n\le\frac{d}{2}-1} = \frac{2(1-\frac{bn\beta_{m^2}}{2(n+1)m^2})(-m^2)^{\frac{d}{2}-n}}{\Gamma(n+1)\Gamma(\frac{d}{2}-n+1)} + f_1\frac{\beta_{m^2}(-m^2)^{\frac{d}{2}-n-1}}{n\Gamma(n+1)\Gamma(\frac{d}{2}-n)} + O(\epsilon)$$
$$= \frac{2(-m^2)^{\frac{d}{2}-n}}{\Gamma(n+1)\Gamma(\frac{d}{2}-n+1)} - \frac{\beta_{m^2}}{m^2}\frac{(-m^2)^{\frac{d}{2}-n}}{\Gamma(n+2)\Gamma(\frac{d}{2}-n)}\left[\frac{bn}{\frac{d}{2}-n} + f_1\frac{n+1}{n}\right] + O(\epsilon). \tag{C13}$$

Recall that M^2 must be analytic around $m^2 = 0$. From Eq. (C12) we see that this can be achieved only if $b = 1$.

On the other hand, the remaining loop integrals are harmless, as they read

$$\frac{2l^d_{n,0}}{nk^{2n-d}} = (f_0-1)\frac{\beta_{m^2}}{m^2}\frac{\Gamma(n-\frac{d}{2}+1)}{\Gamma(n+1)}(m^2)^{\frac{d}{2}-n},$$
$$+O(\epsilon) \quad \text{for } n \ge \frac{d}{2}-1, \tag{C14}$$

$$\frac{2l^d_{n,0}}{nk^{2n-d}} = \frac{2}{\Gamma(\frac{d}{2}+1)}\left[1-((b-1)f_0+1)\frac{\beta_{m^2}}{2m^2}\right]$$
$$+O(\epsilon) \quad \text{for } n = \frac{d}{2}. \tag{C15}$$

For completeness we list some of these integrals in the lowest even numbers of dimensions. If $d = 2$ there is no divergent l function, and in particular

$$l^2_{1,0}(0) = 1 - \frac{1}{2}\frac{\beta_{m^2}}{m^2}, \tag{C16a}$$

$$l^2_{n>1,0}(0) = \frac{\beta_{m^2}}{m^2}\frac{(f_0-1)}{2}\left(\frac{k^2}{m^2}\right)^{n-1}. \tag{C16b}$$

If $d = 4$, the function $l^4_{1,0}$ which enters in the determination of β_{m^2} has a pole unless we choose $f_0 = 1/2$ according to Eq. (C10). This results in

$$l^4_{1,0}(0) = -\frac{m^2}{k^2} + (1+2f_1)\frac{\beta_{m^2}}{4k^2}, \tag{C17a}$$

$$l^4_{2,0}(0) = 1 - \frac{\beta_{m^2}}{2m^2}, \tag{C17b}$$

$$l^4_{n>2,0}(0) = -\frac{1}{4(n-1)}\left(\frac{k^2}{m^2}\right)^{n-2}\frac{\beta_{m^2}}{m^2}. \tag{C17c}$$

These equations are easily interpreted by applying them e.g., to a $\lambda\phi^4$ theory within the LPA. To zeroth order in λ, i.e., neglecting β_{m^2} on the rhs, we recover the standard result that integrals with negative mass dimension do not contribute to the one-loop beta functions. Moreover the positive dimensional integral leads to the usual one-loop RG equation for the mass:

$$\beta_{m^2} = \frac{N+2}{16\pi^2}\lambda m^2 + O(\lambda^2). \tag{C18}$$

Further details of the pseudoregulator choice, such as the coefficient f_1, would affect higher perturbative orders. In fact, in Appendix D 5 we show that the latter coefficient is fixed by requiring that β_{m^2} agrees with the $\overline{\mathrm{MS}}$ result also at two loops.

APPENDIX D: TWO-LOOP COMPUTATION

In this Appendix we detail the computation of the universal part of the two-loop beta function in ϕ^4 theory in four dimensions.

According to our priority, i.e., the computation of β_λ at order λ^3, we first focus on the flow equation for the effective potential,

$$\partial_t U_k = \int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{\partial_t R_k(q)}{2}\left[\frac{N-1}{M_0(\rho,q^2)} + \frac{1}{M_1(\rho,q^2)}\right], \tag{D1a}$$

where

$$M_0(\rho,q^2) = Z_k(\rho,q^2)q^2 + R_k(q) + U_k'(\rho), \tag{D1b}$$

$$M_1(\rho,q^2) = \tilde{Z}_k(\rho,q^2)q^2 + R_k(q) + U_k'(\rho) + 2\rho U_k''(\rho). \tag{D1c}$$

From this functional equation, the beta functions of the mass and of the quartic coupling can be derived by differentiation with respect to ρ. Defining ρ_0 as the field expansion point and

$$w_0 = 2\rho_0 U_k''(\rho_0), \tag{D2a}$$

$$P = Z_k(\rho_0,z)z + R_k(z) + U_k'(\rho_0), \tag{D2b}$$

$$\tilde{P} = \tilde{Z}_k(\rho_0,z)z + R_k(z) + U_k'(\rho_0), \tag{D2c}$$

the flow equations for the two renormalizable couplings read

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}U_k'(\rho_0) &= \partial_t U_k'(\rho_0) + U_k''(\rho_0)\frac{\mathrm{d}}{\mathrm{d}t}\rho_0\\ &= -2v_d(N-1)k^{d-2}Z_k^{-1}U_k''(\rho_0)l_{1,0}^d(0) - 2v_d(N-1)k^d Z_k^{-1}\langle Z_k'(\rho_0)\rangle_{1,0}^{d+2}(0),\\ &\quad - 2v_d k^{d-2}Z_k^{-1}(3U_k''(\rho_0)+2\rho_0 U_k'''(\rho_0))l_{0,1}^d(w_0) - 2v_d k^d Z_k^{-1}\langle \tilde{Z}_k'(\rho_0)\rangle_{0,1}^{d+2}(w_0) + U_k''(\rho_0)\frac{\mathrm{d}}{\mathrm{d}t}\rho_0,\end{aligned} \tag{D3}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}U_k''(\rho_0) &= \partial_t U_k''(\rho_0) + U_k'''(\rho_0)\frac{\mathrm{d}}{\mathrm{d}t}\rho_0\\ &= 2v_d(N-1)k^{d-4}Z_k^{-2}(U_k''(\rho_0))^2 l_{2,0}^d(0) + 2v_d k^{d-4}Z_k^{-2}(3U_k''(\rho_0)+2\rho_0U_k'''(\rho_0))^2 l_{0,2}^d(w_0)\\ &\quad + 4v_d(N-1)k^{d-2}Z_k^{-2}U_k''(\rho_0)\langle Z_k'(\rho_0)\rangle_{2,0}^{d+2}(0)\\ &\quad + 4v_d k^{d-2}Z_k^{-2}(3U_k''(\rho_0)+2\rho_0U_k'''(\rho_0))\langle\tilde{Z}_k'(\rho_0)\rangle_{0,2}^{d+2}(w_0)\\ &\quad + 2v_d(N-1)k^d Z_k^{-2}\langle Z_k'(\rho_0)^2\rangle_{2,0}^{d+4}(0) + 2v_d k^d Z_k^{-2}\langle\tilde{Z}_k'(\rho_0)^2\rangle_{0,2}^{d+4}(w_0)\\ &\quad - 2v_d(N-1)k^{d-2}Z_k^{-1}U_k'''(\rho_0)l_{1,0}^d(0) - 2v_d k^{d-2}Z_k^{-1}(5U_k'''(\rho_0)+2\rho_0U_k^{(4)}(\rho_0))l_{0,1}^d(w_0)\\ &\quad - 2v_d(N-1)k^d Z_k^{-1}\langle Z_k''(\rho_0)\rangle_{1,0}^{d+2}(0) - 2v_d k^d Z_k^{-1}\langle\tilde{Z}_k''(\rho_0)\rangle_{0,1}^{d+2}(w_0) + U_k'''(\rho_0)\frac{\mathrm{d}}{\mathrm{d}t}\rho_0.\end{aligned} \tag{D4}$$

Here we adopted standard notations for the loop integrals

$$l_{n_1,n_2}^d(w) = -\frac{Z_k^{n_1+n_2}}{2}k^{2(n_1+n_2)-d}\int_0^\infty \mathrm{d}z z^{\frac{d}{2}-1}\partial_t\{P(z)^{-n_1}(\tilde{P}(z)+w)^{-n_2}\}, \tag{D5a}$$

$$\langle D_k(\rho_0)\rangle_{n_1,n_2}^d(w) = -\frac{Z_k^{n_1+n_2}}{2}k^{2(n_1+n_2)-d}\int_0^\infty \mathrm{d}z z^{\frac{d}{2}-1}D_k(\rho_0,z)\partial_t\{P(z)^{-n_1}(\tilde{P}(z)+w)^{-n_2}\}, \tag{D5b}$$

and $v_d^{-1} = 2(4\pi)^{d/2}\Gamma(d/2)$. Notice however that our convention for the inverse propagators P_k and $\tilde{P}_k$ slightly departs from the most common choice [49], in that we include the mass parameter $\bar{m}^2$ therein. Furthermore, while ρ_0 is usually chosen as the running minimum of the potential, such that $\rho_0 > 0$ corresponds to a regime of spontaneous symmetry breaking, we instead assume that $U_k'(\rho_0) > 0$. We can safely choose $\rho_0 = 0$ for our goals, as no dynamical symmetry breaking is within reach of a two-loop computation in the present model.

Equations (D3) and (D4) can be rewritten as

$$(\beta_{m^2} - \eta m^2)k^{-2} = \lambda((d-2+\eta)\kappa + \partial_t\kappa) - 2v_d(N-1)(\lambda l_{1,0}^d(0) + \langle z_1\rangle_{1,0}^{d+2}(0)) - 2v_d(3\lambda + 2\kappa u_3)l_{0,1}^d(2\lambda\kappa) - 2v_d\langle\tilde{z}_1\rangle_{0,1}^{d+2}(2\lambda\kappa), \tag{D6}$$

$$\begin{aligned}\beta_\lambda = {}&(d-4+2\eta)\lambda + u_3((d-2+\eta)\kappa + \partial_t\kappa) + 2v_d(N-1)\lambda^2 l_{2,0}^d(0) + 2v_d(3\lambda+2\kappa u_3)^2 l_{0,2}^d(2\lambda\kappa)\\ &- 2v_d(N-1)u_3 l_{1,0}^d(0) - 2v_d(5u_3 + 2\kappa u_4)l_{0,1}^d(2\lambda\kappa) + 4v_d(N-1)\lambda\langle z_1\rangle_{2,0}^{d+2}(0)\\ &+ 4v_d(3\lambda+2\kappa u_3)\langle\tilde{z}_1\rangle_{0,2}^{d+2}(2\lambda\kappa) - 2v_d(N-1)\langle z_2\rangle_{1,0}^{d+2}(0) - 2v_d\langle\tilde{z}_2\rangle_{0,1}^{d+2}(2\lambda\kappa)\\ &+ 2v_d(N-1)\langle z_1^2\rangle_{2,0}^{d+4}(0) + 2v_d\langle\tilde{z}_1^2\rangle_{0,2}^{d+4}(2\lambda\kappa),\end{aligned} \tag{D7}$$

where $\eta = -\partial_t \log Z_k$ is the field anomalous dimension. As described in the main text, introducing the power counting of Eq. (6.7), which is generated by the flow equation itself, into Eqs. (D6) and (D7), and truncating them to order λ^3, result in the simplified perturbative Eqs. (6.8) and (6.9) for $d = 4$. In the following we address the $O(\lambda^3)$ contributions arising on the rhs of Eq. (6.9), organizing them line by line, as these also correspond to different kinds of corrections.

$$\begin{aligned}\beta_\lambda = {}&\frac{N+8}{16\pi^2}\lambda^2 - \frac{(N+8)(N+2)}{2(16\pi^2)^2}\lambda^3 + 2\eta\lambda\\ &- \frac{N-1}{16\pi^2}l_{1,0}^4(0)u_3 - \frac{5}{16\pi^2}l_{0,1}^4(2\lambda\kappa)u_3 + 2\kappa u_3\\ &+ \frac{N-1}{8\pi^2}\lambda\langle z_1\rangle_{2,0}^6(0) + \frac{3}{8\pi^2}\lambda\langle\tilde{z}_1\rangle_{0,2}^6(2\lambda\kappa)\\ &- \frac{N-1}{16\pi^2}\langle z_2\rangle_{1,0}^6(0) - \frac{1}{16\pi^2}\langle\tilde{z}_2\rangle_{0,1}^6(2\lambda\kappa).\end{aligned} \tag{D8}$$

1. β_{m^2} contribution

Using the previous pseudoregulator and the one-loop beta function for m^2 the threshold functions can be expanded at leading order in λ, as in Eq. (6.12). Inserting this into the beta function (6.9) we get

2. u_3 contribution

To evaluate the contribution of the sextic coupling generated by the flow equation, it is enough to consider a uniform and field-independent wave function renormalization for all modes, as in the LPA′; that is, it is safe to set $Z_k(\rho, q^2) = \tilde{Z}_k(\rho, q^2) = Z_k$ at order $O(\lambda^3)$. The flow of the sextic coupling can be deduced by taking the third derivative of Eq. (D1)

$$\begin{aligned}\partial_t U_k'''(\rho) = {}&+4v_d(N-1)k^{d-4}Z_k^{-2}U_k''(\rho)U_k'''(\rho)l_{2,0}^d(0) - 4v_d(N-1)k^{d-6}Z_k^{-3}(U_k''(\rho))^3 l_{3,0}^d(0)\\ &+ 4v_d k^{d-4}Z_k^{-2}(3U_k''(\rho) + 2\rho U_k'''(\rho))(5U_k'''(\rho) + 2\rho U_k^{(4)}(\rho))l_{0,2}^d(w)\\ &- 4v_d k^{d-6}Z_k^{-3}(3U_k''(\rho) + 2\rho U_k'''(\rho))^3 l_{0,3}^d(w)\\ &- 2v_d(N-1)k^{d-2}Z_k^{-1}U_k^{(4)}(\rho)l_{1,0}^d(0) + 2v_d(N-1)k^{d-4}Z_k^{-2}U_k'''(\rho)U_k''(\rho)l_{2,0}^d(0)\\ &- 2v_d k^{d-2}Z_k^{-1}(7U_k^{(4)}(\rho) + 2\rho U_k^{(5)}(\rho))l_{0,1}^d(w)\\ &+ 2v_d k^{d-4}Z_k^{-2}(5U_k'''(\rho) + 2\rho U_k^{(4)}(\rho))(3U_k''(\rho) + 2\rho U_k'''(\rho))l_{0,2}^d(w)\end{aligned} \tag{D9}$$

and evaluating it at $\rho = \rho_0$, such that $w \to w_0$. Using the fact that

$$\partial_t u_3 = (2d-6)u_3 + Z_k^{-3}\left[\partial_t U'''(\rho_0) + U^{(4)}(\rho_0)\frac{\mathrm{d}\rho_0}{\mathrm{d}t}\right], \tag{D10}$$

one deduces

$$\partial_t u_3 = (2d-6)u_3 - 4v_d(N+26)\lambda^3 l^d_{3,0}(0) + O(\lambda^4). \quad \text{(D11)}$$

At one loop and for $d = 4$, u_3 is given by the fixed-point solution of the previous equation

$$u_3^{(1)} = \frac{N+26}{16\pi^2}\lambda^3 k^2 Z_k^3 \int_0^\infty \mathrm{d}z \frac{z}{P^3}. \quad \text{(D12)}$$

By evaluating the momentum integral with the previous pseudoregulator we find an expression which is finite in the $\epsilon \to 0$ limit, namely Eq. (6.13) in the main text. Now let us compute κ at one loop, by looking for a scaling solution for it, i.e., by solving $\partial_t \kappa = 0$, which gives

$$\beta_{m^2} k^{-2} = \lambda(d-2)\kappa - 2v_d(N-1) l^d_{1,0}(0)\lambda - 2v_d(3\lambda + 2\kappa u_3) l^d_{0,1}(2\lambda\kappa). \quad \text{(D13)}$$

Specifying $d = 4$ and using the previous identities we get

$$\frac{N+2}{16\pi^2}\lambda m^2 k^{-2} = 2\lambda\kappa + \frac{(N-1)}{16\pi^2} m^2 k^{-2}\lambda + \frac{1}{16\pi^2} 3\lambda m^2 k^{-2} + O(\lambda^2). \quad \text{(D14)}$$

So as anticipated in the main text, $\kappa = O(\lambda)$ and as such would not affect the $O(\lambda^3)$ of β_λ. Inserting this result for u_3 into the beta function (D8) we obtain

$$\beta_\lambda = \frac{N+8}{16\pi^2}\lambda^2 + \frac{2(5N+22)}{(16\pi^2)^2}\lambda^3 + \frac{N-1}{8\pi^2}\lambda\langle z_1\rangle^6_{2,0}(0) + \frac{3}{8\pi^2}\lambda\langle \tilde{z}_1\rangle^6_{0,2}(2\lambda\kappa) - \frac{N-1}{16\pi^2}\langle z_2\rangle^6_{1,0}(0) - \frac{1}{16\pi^2}\langle \tilde{z}_2\rangle^6_{0,1}(2\lambda\kappa) + 2\eta\lambda. \quad \text{(D15)}$$

3. Wave function renormalization contribution

Recalling that κ and Z_k can be neglected in the third line of Eq. (6.9), as they would give higher order corrections, the wave function renormalization contribution is encoded in the following averages

$$\langle z_1\rangle^6_{2,0}(0) = 16\pi^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} p^2 Z'_k(0,p^2) \frac{\partial_t R_k(p^2)}{P(p^2)^3}, \quad \text{(D16a)}$$

$$\langle z_2\rangle^6_{1,0}(0) = 8\pi^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} p^2 Z''_k(0,p^2) \frac{\partial_t R_k(p^2)}{P(p^2)^2}, \quad \text{(D16b)}$$

and similar relations for $\tilde{z}_1$ and $\tilde{z}_2$. Here we should input the momentum dependence of the wave function renormalization as generated at one loop, that is

$$Z'_k(\rho_0, p^2) = -4\lambda^2 \frac{I_k(p^2)}{p^2}, \quad \text{(D17a)}$$

$$\tilde{Z}'_k(\rho_0, p^2) = -2(N+8)\lambda^2 \frac{I_k(p^2)}{p^2}, \quad \text{(D17b)}$$

$$Z''_k(\rho_0, p^2) = 32\lambda^3 \frac{J_k(p^2)}{p^2}, \quad \text{(D17c)}$$

$$\tilde{Z}''_k(\rho_0, p^2) = 8(N+26)\lambda^3 \frac{J_k(p^2)}{p^2}, \quad \text{(D17d)}$$

where I_k and J_k are the following one-loop integrals

$$I_k(p^2) = \frac{1}{2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{P_k(q)} \left(\frac{1}{P_k(q+p)} - \frac{1}{P_k(q)}\right), \quad \text{(D18a)}$$

$$J_k(p^2) = \frac{1}{2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{P_k(q)^2} \left(\frac{1}{P_k(q+p)} - \frac{1}{P_k(q)}\right). \quad \text{(D18b)}$$

Nesting these expressions leads to Eq. (6.11), where the averages of z_1, $\tilde{z}_1$ and z_2, $\tilde{z}_2$ are, respectively, proportional to the dimensionless two-loop integrals

$$A = \frac{1}{2}\int \frac{\mathrm{d}^4 p}{(2\pi)^4} I_k(p^2) \frac{\partial_t R_k(p)}{P_k(p)^3}, \quad \text{(D19a)}$$

$$B = \frac{1}{2}\int \frac{\mathrm{d}^4 p}{(2\pi)^4} J_k(p^2) \frac{\partial_t R_k(p)}{P_k(p)^2}. \quad \text{(D19b)}$$

We first compute $I_k(p^2)$ with the pseudoregulator (6.11):

$$\begin{aligned}
I_k(p^2,\epsilon) &= \frac{1}{2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{P_k(q)} \left(\frac{1}{P_k(q+p)} - \frac{1}{P_k(q)}\right), \\
&= \frac{1}{2}\left(\frac{\mu^2 m^2}{k^2}\right)^{2\epsilon} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2+m^2)^{1+\epsilon}((q+p)^2+m^2)^{1+\epsilon}} - (p\to 0), \\
&= \frac{1}{2}\left(\frac{\mu^2 m^2}{k^2}\right)^{2\epsilon} \frac{\Gamma(2+2\epsilon)}{\Gamma(1+\epsilon)^2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \int_0^1 \mathrm{d}x \frac{x^\epsilon (1-x)^\epsilon}{(xq^2 + (1-x)(q+p)^2 + m^2)^{2+2\epsilon}} - (p\to 0), \\
&= \frac{1}{2}\left(\frac{\mu^2 m^2}{k^2}\right)^{2\epsilon} \frac{\Gamma(2+2\epsilon)}{\Gamma(1+\epsilon)^2} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{x^\epsilon (1-x)^\epsilon}{(q^2 + x(1-x)p^2 + m^2)^{2+2\epsilon}} - (p\to 0), \\
&= \frac{1}{32\pi^2}\left(\frac{\mu^2}{k^2}\right)^{2\epsilon} \frac{\Gamma(2\epsilon)}{\Gamma(1+\epsilon)^2} \int_0^1 \mathrm{d}x x^\epsilon (1-x)^\epsilon \left(1 + x(1-x)\frac{p^2}{m^2}\right)^{-2\epsilon} - (p\to 0). \quad \text{(D20)}
\end{aligned}$$

Taking the limit for $\epsilon \to 0$ results in the following finite expression

$$I_k(p^2) = -\frac{1}{32\pi^2}\int_0^1 \mathrm{d}x \log\left(1 + x(1-x)\frac{p^2}{m^2}\right) = \frac{1}{16\pi^2}\left[1 - \sqrt{\frac{4m^2+p^2}{p^2}}\mathrm{atanh}\left(\sqrt{\frac{p^2}{4m^2+p^2}}\right)\right]. \tag{D21}$$

We can then insert this result in the expression (D19) for the A coefficient

$$\begin{aligned}
A &= \epsilon\left(\frac{\mu^2 m^2}{k^2}\right)^{2\epsilon}\int\frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{I_k(p^2,\epsilon)}{(p^2+m^2)^{2+2\epsilon}},\\
&= \frac{\epsilon}{(16\pi^2)^2}\left(\frac{\mu^2 m^2}{k^2}\right)^{4\epsilon}\frac{\Gamma(2\epsilon)}{\Gamma(1+\epsilon)^2}\int_0^1 \mathrm{d}x x^\epsilon(1-x)^\epsilon\int_0^\infty \mathrm{d}p p^3\frac{(m^2+x(1-x)p^2)^{-2\epsilon}}{(p^2+m^2)^{2+2\epsilon}}\\
&\quad - \frac{\epsilon}{(16\pi^2)^2}\left(\frac{\mu^2 m^2}{k^2}\right)^{4\epsilon}\frac{\Gamma(2\epsilon)}{\Gamma(1+\epsilon)^2}\int_0^1 \mathrm{d}x x^\epsilon(1-x)^\epsilon\int_0^\infty \mathrm{d}p p^3\frac{m^{-4\epsilon}}{(p^2+m^2)^{2+2\epsilon}},\\
&= \frac{\epsilon}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{\Gamma(2\epsilon)}{\Gamma(1+\epsilon)^2}\int_0^1 \mathrm{d}x x^\epsilon(1-x)^\epsilon\frac{1}{4(1-x(1-x))}\\
&\quad\times\left[\frac{\sqrt{\pi}16^\epsilon\Gamma(2\epsilon+\frac{1}{2})((1-x)x)^{2\epsilon}(1+2\epsilon-(1-x)x(1-2\epsilon))}{\sin(2\pi\epsilon)\Gamma(2\epsilon+2)((1-x)x-1)^{4\epsilon}}\right.\\
&\quad\left.-\frac{2(1-\epsilon)x(1-x)+(1+2\epsilon-(1-x)x(1-2\epsilon))\,{}_2F_1(1,2+2\epsilon;3-2\epsilon;\frac{1}{x-x^2})}{(1-x)^2x^2(1-2\epsilon)(1-\epsilon)}\right]\\
&\quad - \frac{\epsilon}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{\Gamma(2\epsilon)}{\Gamma(1+\epsilon)^2}\int_0^1 \mathrm{d}x x^\epsilon(1-x)^\epsilon\frac{1}{4\epsilon(1+2\epsilon)}.
\end{aligned}\tag{D22}$$

If we first expand the integrand around $\epsilon = 0$ and then perform the integral over x we find

$$A = \frac{1}{(16\pi^2)^2}\left[-\frac{1}{16\epsilon} + \frac{3-2\log(\frac{\mu^2}{k^2})}{8} + O(\epsilon)\right]. \tag{D23}$$

Notice that the coefficient of the pole is equal to one fourth of the coefficient in front of $\log(\frac{\mu^2}{k^2})$.

To demonstrate that the $\epsilon \to 0$ limit and the x integration do commute, let us compute the two also in the opposite order. Thus, we first perform the integral over x and then take $\epsilon \to 0$. For notational convenience we split A in four different terms

$$A = a_1 + a_2 + a_3 + a_4, \tag{D24}$$

where we define

$$a_1 = \frac{1}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{\epsilon\Gamma(-1+2\epsilon)}{2\Gamma(\epsilon+1)^2}\int_0^1 \mathrm{d}x\frac{((1-x)x)^{\epsilon-1}}{(1+x(1-x))}, \tag{D25a}$$

$$a_2 = -\frac{1}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{\sqrt{\pi}2^{4\epsilon-3}\Gamma(\frac{1}{2}+2\epsilon)}{\sin(2\pi\epsilon)(2\epsilon+1)\Gamma(\epsilon+1)^2}\int_0^1 \mathrm{d}x\frac{((1-x)x)^{3\epsilon}(1+2\epsilon-x(1-x)(1-2\epsilon))}{(-1+x(1-x))^{1+4\epsilon}}, \tag{D25b}$$

$$\begin{aligned}
a_3 &= -\frac{1}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{\pi\epsilon}{2\sin(2\pi\epsilon)\Gamma(3-2\epsilon)\Gamma(\epsilon+1)^2}\\
&\quad\times\int_0^1 \mathrm{d}x\frac{((1-x)x)^\epsilon(1+2\epsilon-(1-x)x(1-2\epsilon))\,{}_2F_1(1,2(\epsilon+1);3-2\epsilon;\frac{1}{x-x^2})}{x^2(1-x)^2(1-x(1-x))},
\end{aligned}\tag{D25c}$$

$$a_4 = -\frac{1}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{1}{8\epsilon(1+2\epsilon)^2}. \tag{D25d}$$

By performing the integrals over x and then expanding them around $\epsilon = 0$ they become

$$a_1 = \frac{1}{(16\pi^2)^2}\left[-\frac{1}{2\epsilon} - 1 - \frac{\pi\sqrt{3}}{18} - 2\log\left(\frac{\mu^2}{k^2}\right) + O(\epsilon)\right], \tag{D26a}$$

$$a_2 = \frac{1}{(16\pi^2)^2}\left[\frac{1}{16\epsilon} - \frac{1}{8} - \frac{\pi}{36}(\sqrt{3} + 9i) + \frac{1}{4}\log\left(\frac{\mu^2}{k^2}\right) + O(\epsilon)\right], \tag{D26b}$$

$$a_3 = \frac{1}{(16\pi^2)^2}\left[\frac{1}{2\epsilon} + 1 + \frac{\pi}{12}(\sqrt{3} + 3i) + 2\log\left(\frac{\mu^2}{k^2}\right) + O(\epsilon)\right], \tag{D26c}$$

$$a_4 = \frac{1}{(16\pi^2)^2}\left[-\frac{1}{8\epsilon} + \frac{1}{2} - \frac{1}{2}\log\left(\frac{\mu^2}{k^2}\right) + O(\epsilon)\right]. \tag{D26d}$$

Combining these results we recover Eq. (D23).

Then we turn to the computation of $J_k(p^2)$

$$\begin{aligned}
J_k(p^2, \epsilon) &= \frac{1}{2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{P_k(q)^2}\left(\frac{1}{P_k(q+p)} - \frac{1}{P_k(q)}\right), \\
&= \frac{1}{2}\left(\frac{\mu^2 m^2}{k^2}\right)^{3\epsilon}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{(q^2+m^2)^{2+2\epsilon}((q+p)^2+m^2)^{1+\epsilon}} - (p \to 0), \\
&= \frac{1}{2}\left(\frac{\mu^2 m^2}{k^2}\right)^{2\epsilon}\frac{\Gamma(3+3\epsilon)}{\Gamma(1+\epsilon)\Gamma(2+2\epsilon)}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\int_0^1 \mathrm{d}x \frac{x^{2\epsilon+1}(1-x)^\epsilon}{(xq^2 + (1-x)(q+p)^2 + m^2)^{3+3\epsilon}} - (p \to 0), \\
&= \frac{1}{2}\left(\frac{\mu^2 m^2}{k^2}\right)^{3\epsilon}\frac{\Gamma(3+3\epsilon)}{\Gamma(1+\epsilon)\Gamma(2+2\epsilon)}\int_0^1 \mathrm{d}x\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{x^{2\epsilon+1}(1-x)^\epsilon}{(q^2 + x(1-x)p^2 + m^2)^{3+3\epsilon}} - (p \to 0), \\
&= \frac{1}{32\pi^2 m^2}\left(\frac{\mu^2}{k^2}\right)^{3\epsilon}\frac{\Gamma(1+3\epsilon)}{\Gamma(1+\epsilon)\Gamma(1+2\epsilon)}\int_0^1 \mathrm{d}x \frac{x^{2\epsilon+1}(1-x)^\epsilon}{(1 + x(1-x)\frac{p^2}{m^2})^{1+3\epsilon}} - (p \to 0).
\end{aligned} \tag{D27}$$

Taking the limit $\epsilon \to 0$ we again find a finite one-loop result

$$\begin{aligned}
J_k(p^2) &= \frac{1}{32\pi^2}\int_0^1 \mathrm{d}x\left[\frac{x}{x(1-x)p^2 + m^2} - \frac{x}{m^2}\right], \\
&= \frac{1}{64\pi^2 m^2}\left[\frac{2m^2}{\sqrt{p^2(4m^2+p^2)}}\log\left(1 + \frac{(\sqrt{p^2(4m^2+p^2)} + p^2)}{2m^2}\right) - 1\right],
\end{aligned} \tag{D28}$$

which enters the computation of the B coefficient through Eq. (D19b). The latter proceeds along the same lines as for A. Namely, we exchange again the p and the x integrals

$$\begin{aligned}
B &= \epsilon\left(\frac{\mu^2 m^2}{k^2}\right)^{\epsilon}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{J_k(p^2,\epsilon)}{(p^2+m^2)^{1+\epsilon}}, \\
&= \frac{\epsilon}{(16\pi^2)^2}\left(\frac{\mu^2 m^2}{k^2}\right)^{4\epsilon}\frac{\Gamma(1+3\epsilon)}{\Gamma(1+\epsilon)\Gamma(1+2\epsilon)}\int_0^1 \mathrm{d}x x^{2\epsilon+1}(1-x)^\epsilon\int_0^\infty \mathrm{d}p\, p^3\,\frac{(m^2 + x(1-x)p^2)^{-1-3\epsilon}}{(p^2+m^2)^{1+\epsilon}} \\
&\quad - \frac{\epsilon}{(16\pi^2)^2}\left(\frac{\mu^2 m^2}{k^2}\right)^{4\epsilon}\frac{\Gamma(1+3\epsilon)}{\Gamma(1+\epsilon)\Gamma(1+2\epsilon)}\int_0^1 \mathrm{d}x x^{2\epsilon+1}(1-x)^\epsilon\int_0^\infty \mathrm{d}p\, p^3\,\frac{m^{-2(1+3\epsilon)}}{(p^2+m^2)^{1+\epsilon}}, \\
&= \frac{\epsilon}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{\Gamma(1+3\epsilon)}{\Gamma(1+\epsilon)\Gamma(1+2\epsilon)}\int_0^1 \mathrm{d}x x^{2\epsilon+1}(1-x)^\epsilon\frac{1}{6\epsilon(1-3\epsilon)x^2(1-x)^2} \\
&\quad \times\left(-\frac{\pi\epsilon\Gamma(4\epsilon)x(1-x)(1+3x(1-x))(1-\frac{1}{x(1-x)})^{-\epsilon}}{\sin(3\pi\epsilon)\Gamma(1+\epsilon)\Gamma(-1+3\epsilon)(x(1-x)-1)^{1+3\epsilon}}\right. \\
&\quad \left. + \frac{(1-3\epsilon)x(1-x) + \epsilon(1+3x(1-x))\,{}_2F_1[1,1+\epsilon,2-3\epsilon;\frac{1}{x-x^2}]}{1-(1-x)x}\right) \\
&\quad - \frac{\epsilon}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{\Gamma(1+3\epsilon)}{\Gamma(1+\epsilon)\Gamma(1+2\epsilon)}\int_0^1 \mathrm{d}x x^{2\epsilon+1}(1-x)^\epsilon\frac{1}{2(\epsilon-1)\epsilon}.
\end{aligned} \tag{D29}$$

This time however we are not allowed to take the $\epsilon \to 0$ limit before computing the x integral. In fact, this would result in the wrong answer

$$B = \frac{1}{(16\pi^2)^2} \int_0^1 \mathrm{d}x \left[\frac{1}{8(1-x)} + \frac{x}{2}\right]. \tag{D30}$$

In other words, the integral over x does not commute with the $\epsilon \to 0$ limit, and the latter must be taken as the last step of the computation. To perform the integral over x of the ϵ-dependent expressions, we split also B in four different contributions

$$B = b_1 + b_2 + b_3 + b_4, \tag{D31a}$$

$$b_1 = \frac{1}{(16\pi^2)^2} \left(\frac{\mu^2}{k^2}\right)^{4\epsilon} \frac{\pi\Gamma(4\epsilon)}{2\sin(3\pi\epsilon)\Gamma(\epsilon)^2\Gamma(2\epsilon+2)} \int_0^1 \mathrm{d}x \frac{(1+3x(1-x))(1-x)^{2\epsilon-1}x^{3\epsilon}}{(x(1-x)-1)^{1+4\epsilon}}, \tag{D31b}$$

$$b_2 = \frac{1}{(16\pi^2)^2} \left(\frac{\mu^2}{k^2}\right)^{4\epsilon} \frac{\Gamma(3\epsilon)}{2\Gamma(\epsilon)\Gamma(2\epsilon+2)} \int_0^1 \mathrm{d}x \frac{(1-x)^{\epsilon-1}x^{2\epsilon}}{(1-x(1-x))}, \tag{D31c}$$

$$b_3 = \frac{1}{(16\pi^2)^2} \left(\frac{\mu^2}{k^2}\right)^{4\epsilon} \frac{\pi\epsilon}{2\sin(3\pi\epsilon)\Gamma(2-3\epsilon)\Gamma(\epsilon)\Gamma(2\epsilon+2)} \times \int_0^1 \mathrm{d}x \frac{(1+3x(1-x))(1-x)^{-2+\epsilon}x^{2\epsilon-1}{}_2F_1(1,\epsilon+1;2-3\epsilon;\frac{1}{x-x^2})}{(1-x(1-x))}, \tag{D31d}$$

$$b_4 = \frac{1}{(16\pi^2)^2} \left(\frac{\mu^2}{k^2}\right)^{4\epsilon} \frac{(1+2\epsilon)}{2(1-\epsilon)(2+3\epsilon)(1+3\epsilon)}. \tag{D31e}$$

Now we compute the integrals over x and then expand around $\epsilon = 0$, obtaining

$$b_1 = \frac{1}{(16\pi^2)^2} \left[-\frac{1}{48\epsilon} + \frac{9 + 18i\pi - 4\sqrt{3}\pi - 18\log(\frac{\mu^2}{k^2})}{216} + O(\epsilon)\right], \tag{D32a}$$

$$b_2 = \frac{1}{(16\pi^2)^2} \left[\frac{1}{6\epsilon} + \frac{-18 + \sqrt{3}\pi + 36\log(\frac{\mu^2}{k^2})}{54} + O(\epsilon)\right], \tag{D32b}$$

$$b_3 = \frac{1}{(16\pi^2)^2} \left[-\frac{1}{12\epsilon} + \frac{18 - 9i\pi - 36\log(\frac{\mu^2}{k^2})}{108} + O(\epsilon)\right], \tag{D32c}$$

$$b_4 = \frac{1}{4(16\pi^2)^2} + O(\epsilon). \tag{D32d}$$

The sum of these terms leads to the result

$$B = \frac{1}{(16\pi^2)^2} \left[\frac{1}{16\epsilon} + \frac{1 + 2\log(\frac{\mu^2}{k^2})}{8} + O(\epsilon)\right]. \tag{D33}$$

Also in this case the coefficient of the pole is equal to one fourth of the coefficient in front of $\log(\frac{\mu^2}{k^2})$. As a consequence, the sum $A + B$ which determines the wave function renormalization contribution to the two-loop beta function is finite, as given in Eq. (6.15), and the third line of Eq. (6.9) evaluates to

$$\frac{N-1}{8\pi^2}\lambda\langle z_1\rangle^6_{2,0}(0)+\frac{3}{8\pi^2}\lambda\langle\tilde{z}_1\rangle^6_{0,2}(2\lambda\kappa)$$
$$-\frac{N-1}{16\pi^2}\langle z_2\rangle^6_{1,0}(0)-\frac{1}{16\pi^2}\langle\tilde{z}_2\rangle^6_{0,1}(2\lambda\kappa)$$
$$=-8(5N+22)(A+B)\lambda^3=-\frac{4(5N+22)}{(16\pi^2)^2}\lambda^3. \quad \text{(D34)}$$

4. Anomalous dimension contribution

Within the truncation accounting for a field dependent wave function renormalization, we define the anomalous dimension as

$$\eta=-\frac{\mathrm{d}}{\mathrm{d}t}\log Z_k(\rho_0),$$
$$=-Z_k^{-1}(\rho_0)\partial_t Z_k(\rho_0)-Z_k(\rho_0)^{-1}Z_k'(\rho_0)\frac{\mathrm{d}}{\mathrm{d}t}\rho_0. \quad \text{(D35)}$$

Possible differences between this definition and a similar one based on $\tilde{Z}_k(\rho_0)$ are beyond the $O(\lambda^2)$ we are after. Also, the second term on the rhs of Eq. (D35) would not contribute at this perturbative order. Hence, the relevant term which can be deduced from the exact flow equation is

$$\partial_t Z_k(\rho_0)=\frac{1}{2}\lim_{Q^2\to 0}\frac{\partial}{\partial Q^2}\frac{\delta^2}{\delta\phi(Q)\delta\phi(-Q)}\mathrm{Tr}\{\partial_t R_k(q)[\Gamma_k^{(2)}(q)+R_k(q)]^{-1}\}|_{\rho_0},$$
$$=\frac{1}{2}\lim_{Q^2\to 0}\frac{\partial}{\partial Q^2}\mathrm{Tr}\{\partial_t R_k(q)(\Gamma_k^{(2)}(q)+R_k(q))^{-1}[-\Gamma_k^{(4)}(Q,-Q,q,-q)(\Gamma_k^{(2)}(q)+R_k(q))^{-1}$$
$$+2\Gamma_k^{(3)}(Q,q,-q-Q)(\Gamma_k^{(2)}(q)+R_k(q))^{-1}\Gamma_k^{(3)}(-Q,q+Q,-q)(\Gamma_k^{(2)}(q)+R_k(q))^{-1}]\}|_{\rho_0}. \quad \text{(D36)}$$

As described in the main text we have that the anomalous dimension is given by the sum of two terms, $\eta^{(1)}$ and $\eta^{(2)}$: the first one is the contribution at zero momentum, while in the second one is the momentum contribution.

The flow equation which encodes $\eta^{(1)}$ is the one within the $O(\partial^2)$ derivative expansion, that is:

$$\partial_t Z_k(\rho)=-2v_d k^{d-2}Z_k^{-1}\{[(N-1)Z_k'(\rho)+Y_k(\rho)]l^d_{1,0}(0)+[Z_k'(\rho)+2\rho Z_k''(\rho)]l^d_{0,1}(w)\}$$
$$+4v_d k^{d-6}\rho(U_k''(\rho))^2Q^{d,0}_{2,1}(w)+4v_d k^{d-4}\rho Y_k(\rho)U_k''(\rho)Q^{d,1}_{2,1}(w)$$
$$+v_d k^{d-2}\rho(Y_k(\rho))^2Q^{d,2}_{2,1}(w)+16v_d k^{d-4}Z_k^{-2}\rho Z_k'(\rho)U_k''(\rho)l^d_{1,1}(w)$$
$$+\frac{8v_d}{d}k^{d-2}Z_k^{-2}\rho(Z_k'(\rho))^2 l^{d+2}_{1,1}(w)+8v_d k^{d-2}Z_k^{-2}\rho Z_k'(\rho)Y_k(\rho)l^{d+2}_{1,1}(w)$$
$$+\frac{16v_d}{d}k^{d-4}\rho Z_k'(\rho)U_k''(\rho)N^d_{2,1}(w)+\frac{8v_d}{d}k^{d-2}\rho Z_k'(\rho)Y_k(\rho)N^{d+2}_{2,1}(w). \quad \text{(D37)}$$

Following Ref. [49] we define the threshold functions

$$N^d_{n_1,n_2}(w)=k^{2(n_1+n_2-1)-d}\int_0^\infty \mathrm{d}z z^{\frac{d}{2}}\partial_t\{\dot{P}P^{-n_1}(\tilde{P}+w)^{-n_2}\}, \quad \text{(D38a)}$$

$$Q^{d,\alpha}_{n_1,n_2}(w)=k^{2(n_1+n_2-\alpha)-d}\int_0^\infty \mathrm{d}z z^{\frac{d}{2}-1+\alpha}\partial_t\left\{\left[\dot{P}+\frac{2z}{d}\ddot{P}-\frac{4z}{d}P^{-1}\dot{P}^2\right]P^{-n_1}(\tilde{P}+w)^{-n_2}\right\}, \quad \text{(D38b)}$$

$$M^d_{n_1,n_2}(w)=k^{2(n_1+n_2-1)-d}\int_0^\infty \mathrm{d}z z^{\frac{d}{2}}\partial_t\{\dot{P}^2P^{-n_1}(\tilde{P}+w)^{-n_2}\}. \quad \text{(D38c)}$$

These quantities are related in the following way

$$Q^{d,\alpha}_{n_1,n_2}(w)=\frac{2n_1-4}{d}M^{d+2\alpha}_{n_1+1,n_2}(w)+\frac{2n_2}{d}M^{d+2\alpha}_{n_1,n_2+1}(w)+\frac{2n_2}{d}\rho Y_k(\rho)N^{d+2\alpha}_{n_1,n_2+1}(w)-\frac{2\alpha}{d}N^{d+2\alpha-2}_{n_1,n_2}(w). \quad \text{(D39)}$$

Taking $\rho\to\rho_0$ and $w\to w_0$ in Eq. (D37) we get the simplified expression

$$\partial_t Z_k(\rho_0)=\frac{8}{d}v_d k^{d-6}\bar{\lambda}^2\rho_0 M^d_{4,0}(0)-2v_d k^{d-2}Z_k^{-1}(NZ_k'(\rho_0,0)+Y_k(\rho_0,0))l^d_{1,0}(0), \quad \text{(D40)}$$

which in $d=4$ can be rewritten

$$\eta^{(1)} = \frac{1}{8\pi^2} m_4^4 \kappa \lambda^2 + \frac{Z_k^{-2} k^2}{16\pi^2} l_{1,0}^4(0)[(N-1)Z_k'(\rho_0,0) + \tilde{Z}_k'(\rho_0,0)] - Z_k^{-1} Z_k'(\rho_0,0)\frac{\mathrm{d}}{\mathrm{d}t}\rho_0, \tag{D41}$$

$$m_n^d = -\frac{Z_k^{n-2}}{2} M_{n,0}^d(0). \tag{D42}$$

As at the present order and with our pseudoregulator both κ and m_4^4 vanish, we are left with Eq. (6.18). On the other hand, the derivative couplings generated at one loop are

$$Z_k'(\rho_0,0) = -4\lambda^2 Z_k^4 \lim_{p^2\to 0} \frac{I_k(p^2)}{p^2} = \frac{1}{3(16\pi^2)} Z_k^2 m^{-2}\lambda^2, \tag{D43a}$$

$$\tilde{Z}_k'(\rho_0,0) = -2(N+8)\lambda^2 Z_k^4 \lim_{p^2\to 0} \frac{I_k(p^2)}{p^2} = \frac{(N+8)}{6(16\pi^2)} Z_k^2 m^{-2}\lambda^2, \tag{D43b}$$

which leads to Eq. (6.19). Nesting the latter in Eq. (6.18) results in the final expression (6.20) for $\eta^{(1)}$.

We then turn to the momentum dependent contribution. As in Ref. [49] we define the latter by subtracting the momentum independent part from the four-point vertex:

$$\begin{aligned}\Delta_k(Q,-Q,q,-q) &= \Gamma_k^{(4)}(Q,-Q,q,-q) - \Gamma_k^{(4)}(0,0,q,-q) - \Gamma_k^{(4)}(Q,-Q,0,0) - \Gamma_k^{(4)}(0,0,0,0),\\ &= -\lambda^2 \mathrm{diag}(2,N+8,\overbrace{2\ldots 2}^{N-2})\frac{1}{2}\int\frac{\mathrm{d}^4 p}{(2\pi)^4} P^{-1}(p)[2P^{-1}(p) + P^{-1}(p-Q-q)\\ &\quad + P^{-1}(p-Q+q) - 2P^{-1}(p+Q) - 2P^{-1}(p+q)].\end{aligned} \tag{D44}$$

For a ϕ^4 theory at one loop

$$\begin{aligned}\lim_{Q^2\to 0}\frac{\partial}{\partial Q^2}\Delta_k(Q,-Q,q,-q) &= -\lambda^2 \mathrm{diag}(2,N+8,\overbrace{2\ldots 2}^{N-2})\\ &\quad\times\frac{1}{2}\lim_{Q^2\to 0}\frac{\partial}{\partial Q^2}\int\frac{\mathrm{d}^4 p}{(2\pi)^4} P^{-1}(p)[P^{-1}(p-Q-q) + P^{-1}(p-Q+q) - 2P^{-1}(p+Q)].\end{aligned} \tag{D45}$$

To evaluate this expression it is convenient to define

$$H(p^2,Q^2) = P^{-1}(p)P^{-1}(p+Q) = \left(\frac{\mu^2 m^2}{k^2}\right)^{2\epsilon}\frac{\Gamma(2+2\epsilon)}{\Gamma(1+\epsilon)^2}\int_0^1 \mathrm{d}x \frac{x^\epsilon(1-x)^\epsilon}{(p^2+x(1-x)Q^2+m^2)^{2+2\epsilon}}. \tag{D46}$$

We then need to expand the following function for small Q

$$\begin{aligned}H(p^2,(Q\pm q)^2) &= H(q^2) + (Q^2 \pm 2Q\cdot q)H'(q^2) + 2(Q\cdot q)^2 H''(q^2) + O(Q^3)\\ &\underset{q}{=} H(q^2) + Q^2 H'(q^2) + \frac{1}{2}Q^2 q^2 H''(q^2) + O(Q^4),\end{aligned} \tag{D47}$$

where the second equal sign denotes equivalence upon integration over $q\in\mathbb{R}^4$, and primes denote derivatives with respect to q^2. The anomalous dimension involves the trace of the four-point vertex, which then reads

$$\mathrm{Tr}\lim_{Q^2\to 0}\frac{\partial}{\partial Q^2}\Delta_k(Q,-Q,q,-q) \underset{q^2}{=} -3(N+2)\lambda^2\int\frac{\mathrm{d}^4 p}{(2\pi)^4}\left[H'(p^2,q^2) + \frac{1}{2}q^2 H''(p^2,q^2) - H'(p^2,0)\right]. \tag{D48}$$

This one-loop expression for the momentum dependence of the four-point vertex is to be nested in the momentum dependent part of Eq. (D36), thus obtaining

$$\eta^{(2)} = \frac{1}{2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{\partial_t R_k(q)}{P(q)^2}\mathrm{Tr}\lim_{Q^2\to 0}\frac{\partial}{\partial Q^2}\Delta_k(Q,-Q,q,-q). \tag{D49}$$

For our pseudoregulator, we can specify all the terms in the integrand according to Eqs. (C3) and (D46). Taking the limit for $\epsilon \to 0$ after all integrals have been performed, we find the result of Eq. (6.21).

5. Two-loop flow of m^2

In this Appendix we show that also the two-loop beta function of the mass can be obtained as the $\epsilon \to 0$ limit of the corresponding FRG equation. We start from Eq. (D6), and neglect higher-loop contributions, e.g., by inserting $\kappa = 0$, thus obtaining the simplified result:

$$\beta_{m^2} - \eta m^2 = -\frac{k^2}{16\pi^2}[(N+2)\lambda l^4_{1,0}(0) + (N-1)\langle z_1\rangle^6_{1,0}(0) + \langle \tilde{z}_1\rangle^6_{0,1}(0)]. \tag{D50}$$

The contribution of the one-loop wave function renormalization is similar to the one discussed in the previous section

$$\langle z_1\rangle^6_{1,0}(0) = 8\pi^2 k^{-2}\int \frac{\mathrm{d}^4 p}{(2\pi)^4} p^2 Z'_k(0,p^2)\frac{\partial_t R_k(p^2)}{P(p^2)^2}, \tag{D51a}$$

$$\langle \tilde{z}_1\rangle^6_{0,1}(0) = 8\pi^2 k^{-2}\int \frac{\mathrm{d}^4 p}{(2\pi)^4} p^2 \tilde{Z}'_k(0,p^2)\frac{\partial_t R_k(p^2)}{P(p^2)^2}, \tag{D51b}$$

where Z'_k and $\tilde{Z}'_k$ are given in Eqs. (D17a) and (D17b). Then the two-loop contributions arise by replacing in Eq. (D50) the following expressions

$$l^4_{1,0}(0) = -\frac{m^2}{k^2} + (1+2f_1)\frac{\beta_{m^2}}{4k^2}, \tag{D52}$$

$$(N-1)\langle z_1\rangle^6_{1,0}(0) + \langle \tilde{z}_1\rangle^6_{0,1}(0) = (N+2)\frac{(9-\sqrt{3}\pi)}{8\pi^2}\frac{m^2}{k^2}\lambda^2, \tag{D53}$$

$$\eta = \frac{(N+2)}{2(16\pi^2)^2}\lambda^2, \tag{D54}$$

where f_1 is a free regularization parameter as described in Appendix C. The combination of these corrections gives Eq. (6.24), from which it is apparent that the unique choice

$$f_1 = -\frac{1}{2} + \frac{4\sqrt{3}\pi - 30}{N+2} \tag{D55}$$

produces the $\overline{\mathrm{MS}}$ two-loop result

$$\partial_t \log m^2 = \frac{(N+2)}{16\pi^2}\lambda - \frac{5(N+2)}{2(16\pi^2)^2}\lambda^2. \tag{D56}$$

[1] F. J. Wegner and A. Houghton, Renormalization group equation for critical phenomena, Phys. Rev. A **8**, 401 (1973).
[2] T. R. Morris, The exact renormalization group and approximate solutions, Int. J. Mod. Phys. A **09**, 2411 (1994).
[3] C. Wetterich, Exact evolution equation for the average potential, Phys. Lett. B **301**, 90 (1993).
[4] M. Bonini, M. D'Attanasio, and G. Marchesini, Perturbative renormalization and infrared finiteness in the Wilson renormalization group: The massless scalar case, Nucl. Phys. **B409**, 441 (1993).
[5] U. Ellwanger, Flow equations for N point functions and bound states, Z. Phys. C **62**, 503 (1994).
[6] C. Bagnuls and C. Bervillier, Exact renormalization group equations. An introductory review, Phys. Rep. **348**, 91 (2001).
[7] J. M. Pawlowski, Aspects of the functional renormalisation group, Ann. Phys. (Amsterdam) **322**, 2831 (2007).
[8] H. Gies, Introduction to the functional RG and applications to gauge theories, Lect. Notes Phys. **852**, 287 (2012).
[9] B. Delamotte, An introduction to the nonperturbative renormalization group, Lect. Notes Phys. **852**, 49 (2012).
[10] N. Dupuis, L. Canet, A. Eichhorn, W. Metzner, J. M. Pawlowski, M. Tissier, and N. Wschebor, The nonperturbative functional renormalization group and its applications, Phys. Rep. https://doi.org/10.1016/j.physrep.2021.01.001 (2021).
[11] K. G. Wilson and J. B. Kogut, The renormalization group and the epsilon expansion, Phys. Rep. **12**, 75 (1974).
[12] J. F. Nicoll, T. S. Chang, and H. E. Stanley, Approximate Renormalization Group Based on the Wegner-Houghton Differential Generator, Phys. Rev. Lett. **33**, 540 (1974).
[13] J. O'Dwyer and H. Osborn, Epsilon expansion for multicritical fixed points and exact renormalisation group equations, Ann. Phys. (Amsterdam) **323**, 1859 (2008).

[14] I. Jack and H. Osborn, Two loop background field calculations for arbitrary background fields, Nucl. Phys. **B207**, 474 (1982).

[15] C. Ford, I. Jack, and D. Jones, The standard model effective potential at two loops, Nucl. Phys. **B387**, 373 (1992).

[16] S. P. Martin, Two loop effective potential for a general renormalizable theory and softly broken supersymmetry, Phys. Rev. D **65**, 116003 (2002).

[17] A. Codello, M. Safari, G. P. Vacca, and O. Zanusso, Functional perturbative RG and CFT data in the ϵ-expansion, Eur. Phys. J. C **78**, 30 (2018).

[18] D. F. Litim, Optimized renormalization group flows, Phys. Rev. D **64**, 105007 (2001).

[19] D. F. Litim, Critical exponents from optimized renormalization group flows, Nucl. Phys. **B631**, 128 (2002).

[20] A. Codello, M. Safari, G. P. Vacca, and O. Zanusso, New universality class in three dimensions: The critical Blume-Capel model, Phys. Rev. D **96**, 081701 (2017).

[21] M. Safari and G. P. Vacca, Multicritical scalar theories with higher-derivative kinetic terms: A perturbative RG approach with the ϵ-expansion, Phys. Rev. D **97**, 041701 (2018).

[22] M. Safari and G. P. Vacca, Uncovering novel phase structures in $\Box^k$ scalar theories with the renormalization group, Eur. Phys. J. C **78**, 251 (2018).

[23] A. Codello, M. Safari, G. P. Vacca, and O. Zanusso, Leading order CFT analysis of multi-scalar theories in $d > 2$, Eur. Phys. J. C **79**, 331 (2019).

[24] A. Codello, M. Safari, G. P. Vacca, and O. Zanusso, Multicritical multi-field models: A CFT approach to the leading order, Universe **5**, 151 (2019).

[25] A. Codello, Scaling solutions in continuous dimension, J. Phys. A **45**, 465006 (2012).

[26] A. Baldazzi, R. Percacci, and V. Skrinjar, Quantum fields without Wick rotation, Symmetry **11**, 373 (2019).

[27] A. Codello, R. Percacci, L. Rachwał, and A. Tonero, Computing the effective action with the functional renormalization group, Eur. Phys. J. C **76**, 226 (2016).

[28] N. Ohta and L. Rachwal, Effective action from the functional renormalization group, Eur. Phys. J. C **80**, 877 (2020).

[29] L. Canet, B. Delamotte, D. Mouhanna, and J. Vidal, Nonperturbative renormalization group approach to the Ising model: A derivative expansion at order partial**4, Phys. Rev. B **68**, 064421 (2003).

[30] I. Balog, H. Chaté, B. Delamotte, M. Marohnic, and N. Wschebor, Convergence of Nonperturbative Approximations to the Renormalization Group, Phys. Rev. Lett. **123**, 240604 (2019).

[31] G. De Polsi, I. Balog, M. Tissier, and N. Wschebor, Precision calculation of critical exponents in the $O(N)$ universality classes with the nonperturbative renormalization group, Phys. Rev. E **101**, 042113 (2020).

[32] T. R. Morris, Derivative expansion of the exact renormalization group, Phys. Lett. B **329**, 241 (1994).

[33] J. Comellas, Polchinski equation, reparameterization invariance and the derivative expansion, Nucl. Phys. **B509**, 662 (1998).

[34] H. Osborn and D. Twigg, Reparameterisation invariance and RG equations: Extension of the local potential approximation, J. Phys. A **42**, 195401 (2009).

[35] J. Gracey, Four loop renormalization of ϕ^3 theory in six dimensions, Phys. Rev. D **92**, 025012 (2015).

[36] T. R. Morris, The renormalization group and two-dimensional multicritical effective scalar field theory, Phys. Lett. B **345**, 139 (1995).

[37] N. Defenu and A. Codello, Scaling solutions in the derivative expansion, Phys. Rev. D **98**, 016013 (2018).

[38] N. Defenu and A. Codello, The fate of $O(N)$ multi-critical universal behaviour, arXiv:2005.10827.

[39] A. B. Zamolodchikov, Conformal symmetry and multicritical points in two-dimensional quantum field theory (In Russian), Sov. J. Nucl. Phys. **44**, 529 (1986).

[40] J. Zinn-Justin, *Phase Transitions and Renormalization Group*, (Oxford University Press, New York, 2007), p. 452.

[41] A. Codello and R. Percacci, Fixed points of nonlinear sigma models in $d > 2$, Phys. Lett. B **672**, 280 (2009).

[42] R. Percacci and O. Zanusso, One loop beta functions and fixed points in higher derivative sigma models, Phys. Rev. D **81**, 065012 (2010).

[43] M. Fabbrichesi, R. Percacci, A. Tonero, and O. Zanusso, Asymptotic safety and the gauged SU(N) nonlinear σ-model, Phys. Rev. D **83**, 025016 (2011).

[44] F. Bazzocchi, M. Fabbrichesi, R. Percacci, A. Tonero, and L. Vecchi, Fermions and Goldstone bosons in an asymptotically safe model, Phys. Lett. B **705**, 388 (2011).

[45] R. Flore, A. Wipf, and O. Zanusso, Functional renormalization group of the non-linear sigma model and the $O(N)$ universality class, Phys. Rev. D **87**, 065019 (2013).

[46] R. Flore, Renormalization of the nonlinear O(3) model with θ-term, Nucl. Phys. **B870**, 444 (2013).

[47] R. Percacci and M. Safari, Functional renormalization of N scalars with O(N) invariance, Phys. Rev. D **88**, 085007 (2013).

[48] M. Safari, One-loop functional renormalization group flow of scalar theories with electroweak symmetry, Phys. Rev. D **90**, 065009 (2014).

[49] T. Papenbrock and C. Wetterich, Two loop results from one loop computations and nonperturbative solutions of exact evolution equations, Z. Phys. C **65**, 519 (1995).

[50] M. Bonini, G. Marchesini, and M. Simionato, Beta function and infrared renormalons in the exact Wilson renormalization group in Yang-Mills theory, Nucl. Phys. **B483**, 475 (1997).

[51] A. Bonanno and D. Zappala, Two loop results from the derivative expansion of the blocked action, Phys. Rev. D **57**, 7383 (1998).

[52] M. Pernici and M. Raciti, Wilsonian flow and mass independent renormalization, Nucl. Phys. **B531**, 560 (1998).

[53] P. Kopietz, Two loop beta function from the exact renormalization group, Nucl. Phys. **B595**, 493 (2001).

[54] D. Zappala, Perturbative and nonperturbative aspects of the proper time renormalization group, Phys. Rev. D **66**, 105020 (2002).

[55] S. Arnone, A. Gatti, T. R. Morris, and O. J. Rosten, Exact scheme independence at two loops, Phys. Rev. D **69**, 065009 (2004).

[56] A. Codello, M. Demmel, and O. Zanusso, Scheme dependence and universality in the functional renormalization group, Phys. Rev. D **90**, 027701 (2014).

[57] H. Gies, R. Sondenheimer, A. Ugolotti, and L. Zambelli, Scheme dependence of asymptotically free solutions, Eur. Phys. J. C **79**, 463 (2019).

[58] A. Baldazzi, R. Percacci, and L. Zambelli, Vanishing regulators (to be published).

[59] H. Georgi, Effective field theory, Annu. Rev. Nucl. Part. Sci. **43**, 209 (1993).

[60] N. Tetradis and C. Wetterich, Critical exponents from effective average action, Nucl. Phys. **B422**, 541 (1994).