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Nonlinear Lattices and Random Matrices

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Abstract

In this thesis, we study problems related to statistical properties of integrable and non-integrable Hamiltonian system, focusing on their relations with random matrix theory.

First, we consider the harmonic chain with short-range interactions. Exploiting the rich theory of circulant and Toeplitz matrices, we are able to explicitly compute the correlation functions for this system. Further, applying the so-called steepest descent method, we compute their long time asymptotic.

In the main part of the thesis, we focus on the interplay between Random Matrix theory and integrable Hamiltonian system. Specifically, we introduce some new tridiagonal random matrix ensembles that we name α ensembles, and we compute their mean density of states. These random matrix models are related to the classical beta ones in the high temperature regime. Moreover, they are also connected to the Toda and the Ablowitz-Ladik lattice, indeed applying our result on the α ensembles, we are able to compute the mean density of states of the Lax matrices of these two lattices.

Next, we focus on the Fermi-Pasta-Ulam-Tsingou (FPUT) system, a non-integrable lattice. We show that the integrals of motion of the Toda lattice are adiabatic invariants, namely statistically almost conserved quantities, for the FPUT system for a time-scale of order $\beta^{1-2\varepsilon}$, here $\varepsilon > 0$, and β is the inverse of the temperature. Moreover, we show that some special linear combinations of the normal modes are adiabatic invariants for the Toda lattice, for all times, and for the FPUT, for times of order $\beta^{1-2\varepsilon}$.

Finally, we consider the classical beta ensembles in the high temperature regime. We compute their mean density of states, making use of the so-called loop equations. Exploiting this formalism, we are able to compute the moments and the linear statistic covariance through recurrence relations. Further, we identify a new α ensemble, which is related to Dyson's study of a disordered chain. Our analysis supplement the results contained in Dyson's work.

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Contents

Introduction

Overview

In this thesis, we study problems related to integrable and almost integrable Hamiltonian systems in connection to Random Matrix Theory (RMT). Specifically, we study densities of states of particular families of random matrices, and we apply our results to integrable and almost integrable lattices with random initial data in the thermodynamic limit. This regime occurs when the number of lattice particles N goes to infinity at a fixed temperature β^{-1}. In such a limit, the energy per particle remains finite, and the total energy of the chain goes to infinity.

The study of the thermodynamic limit arises naturally from the study of interacting particles systems with many degrees of freedom. Despite being so relevant, there are no general tools to study the thermodynamic limit of Hamiltonian systems. Indeed, the analysis is usually different from model to model; therefore, the best we can do to study the thermodynamic limit of a specific system is either adapt techniques exploited in other situations or design new ones tailored for the case at hand. Specifically, we focus on the following systems:

1. harmonic oscillators with short range interactions;
2. lattice systems with nearest neighbour interactions with potential $V_T(r) = e^{-r} + r - 1$ and $V_{FPUT}(r) = \frac{r^2}{2} - \frac{r^3}{6} + \frac{\mathtt{b}r^4}{24}$;
3. lattice systems with interaction potential depending on both the relative elongation r and the momentum.

In case 2. the potential $V_T(r)$ corresponds to the Toda lattice [157], that is an integrable system, while the potential $V_{FPUT}(r)$ corresponds to the Fermi-Pasta-Ulam-Tsingou (FPUT) lattice [50], which is not integrable, and, for low energy, can be considered as a fourth order perturbation of the Toda lattice. The case 3. refers to the Ablowitz-Ladik (AL) lattice [2], that is an integrable discretization of the cubic nonlinear Schrödinger equation. We study each of these systems with periodic boundary conditions and number of particles equal to N. Introducing the Gibbs measure at temperature β^{-1}, we study statistical properties of these systems in the thermodynamic limit.

Regarding case 1., we are able to determine the *correlation functions* and their time scaling in the thermodynamic limit for the chain of harmonic oscillators with short range interactions. Correlation functions are very important physical quantities in statistical mechanics since they encode transport properties of systems. However, for nonlinear systems the rigorous determination of scaling properties of correlation functions still remains an open problem, even in the integrable cases.

Regarding case 2., we show that the integrals of motion of the Toda lattice are *adiabatic invariants*, namely statistically almost conserved quantities, for the FPUT system for a certain time-scale related to the temperature β^{-1}. Therefore, we prove that the FPUT system is not chaotic for the same time-scale. Moreover we computed the *density of states* and the averages of the constants of motion of the Toda lattice in the thermodynamic limit. Regarding case 3., we consider the Ablowitz-Ladik lattice; also this case we determine the *density of states* of the Lax matrix and the averages of the constants of motion of this system in the thermodynamic limit.

The theory of random matrices is essential to our analysis. The novelty of our approach lies in the fact that we are able to connect the study of the thermodynamic limit of Hamiltonian systems with the theory of random matrices, more specifically the Dimitriu-Edelmann β ensembles [42] and the Killip-Nenciu circular β ensemble [93] at *high temperature*, namely when the matrix size N goes to infinity and $\beta = \frac{2\alpha}{N}$, $\alpha > 0$. This allows us to exploit several results and techniques typical of random matrices to obtain new results in the study of Hamiltonian systems in the thermodynamic limit.

The interplay between random matrices and integrable systems has inspired us to define and study a new family of random matrices, which we call α ensembles. These are tridiagonal random matrix ensembles related to the Gaussian, Laguerre, Jacobi, and anti-symmetric Gaussian β ensembles at high temperature. We obtain the mean density of states of the α ensembles and the Gaussian, Laguerre, Jacobi, and anti-symmetric Gaussian β ensembles at high temperature. Further, for the Gaussian, Laguerre, and Jacobi β ensembles at high temperature, we also characterized the monomial linear statistics through recurrence relations.

The structure of the thesis is the following.

In Chapter 1, we analyse the harmonic oscillator chain with short-range interactions. Specifically, we compute the correlation functions for this system and their long time asymptotics in the thermodynamic limit. The result of this chapter are taken from our paper *"Correlation functions for a chain of short range oscillators."* Journal of Statistical Physics (2021), made in collaboration with T.Grava, T. Kriecherbauer, and K. D. T.-R. McLaughlin [75].

In Chapter 2, we introduce three new random matrix ensembles that we called α ensembles. We explicitly compute their *mean density of states*, also called *mean eigenvalue density*. As a corollary of our construction, we obtain the density of states of the Lax matrix of the Toda lattice in thermal equilibrium. The result of this chapter are taken from our paper *"On the mean Density of States of some matrices related to the beta ensembles and an application to the Toda lattice."* arXiv: 2008.04604 (2020) [115].

In Chapter 3, we study the Ablowitz-Ladik lattice. Specifically, we introduce the Generalized Gibbs ensemble for this lattice, and we are able to compute the generalized free energy. We also obtain the density of states for this lattice in thermal equilibrium via a particular solution of the Double Confluent Heun Equation. The results of this chapter are taken from our paper *"Generalized Gibbs ensemble of the Ablowitz-Ladik lattice, circular β-ensemble and double confluent Heun equation."* arXiv: 2107.02303 (2021) [116], made in collaboration with T.Grava.

In Chapter 4, we focus on the Fermi-Pasta-Ulam-Tsingou (FPUT) chain. Specifically, we prove that some integral of motion of the Toda lattice are adiabatic invariants for the FPUT system. Moreover, we prove that some linear combinations of the harmonic energies are also adiabatic invariants for both the FPUT chain, and the Toda lattice. The result of this chapter are taken from our paper *"Adiabatic invariants for the FPUT and Toda chain in the thermodynamic limit."* Communications in Mathematical Physics, 380 (2020) [76], made in collaboration with T.Grava, A. Maspero, and A. Ponno.

In Chapter 5, exploiting the theory of loop equations, we produced a unifying mechanism to characterize the mean density of states for the β ensembles in the high temperature regime. We also characterize the moments and the covariances of monomial linear statistics through recurrence relations. Finally, we define the Gaussian anti-symmetric α ensemble, and we computed its mean density of states and its mean spectral measure. From the explicit formulas, we are able to supplement analytic results obtained by Dyson in the study of the so-called type I disordered chain. The result of this chapter are taken from our paper *"The classical beta ensembles with beta proportional to 1/N: from loop equations to Dyson's disordered chain."* Journal of Mathematical Physics 62, 073505 (2021) [61], made in collaboration with P.J. Forrester.

All chapters are independent from each other, so they can be read separately. We now describe our results in more details, and we put them into context.

Correlation Functions

Correlation functions are an important object in statistical mechanics since they encode transport properties of the system, such as the *thermal conductivity.*

To formally introduce them, we consider the even dimensional phase-space $\mathcal{M} \subseteq \mathbb{R}^{2N}$. We denote by $(\mathbf{p}, \mathbf{q}) \in \mathcal{M}$, where $\mathbf{p}, \mathbf{q} \in \mathbb{R}^N$, the vectors in $\mathcal{M}$. On this phase-space we consider the algebra of smooth real or complex valued functions $\mathcal{C}^\infty(\mathcal{M})$ and a bilinear antisymmetric operation $\{\cdot\,,\cdot\} : \mathcal{C}^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$ that satisfies the Jacobi identity, namely $\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$ for all $f, g, h \in \mathcal{C}^\infty(\mathcal{M})$, and the Leibniz's rule, namely $\{fg, h\} = \{f,h\}g + f\{g,h\}$ for all $f, g, h \in \mathcal{C}^\infty(\mathcal{M})$. Such operation is called Poisson bracket. Given a real function $H(\mathbf{p}, \mathbf{q}) \in \mathcal{C}^\infty(\mathcal{M})$, that we call Hamiltonian, the equations of motion of the system take the form:

$$\dot{q}_j = \{q_j, H\}, \quad \dot{p}_j = \{p_j, H\}, \quad j = 1, \ldots, N\,, \tag{0.1}$$

where $\dot{q}_j := \frac{d}{dt} q_j$, $\dot{p}_j := \frac{d}{dt} p_j$. We denote by $\mathbf{p}(t)$ and $\mathbf{q}(t)$ the evolution of $\mathbf{p}(0)$ and $\mathbf{q}(0)$ along the Hamiltonian flow (0.1). A function F is a *constant of motion* if it commutes with H, namely

$$\dot{F} = \{F, H\} = 0\,.$$

Further, we assume that $\int_\mathcal{M} e^{-\beta H(\mathbf{p},\mathbf{q})} \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}$ is finite so that the classical Gibbs measure $\mathrm{d}\mu_H$ at temperature β^{-1}

$$\mathrm{d}\mu_H = \frac{e^{-\beta H(\mathbf{p},\mathbf{q})} \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}}{\int_\mathcal{M} e^{-\beta H(\mathbf{p},\mathbf{q})} \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}}\,, \tag{0.2}$$

is well-defined. We notice that this measure is invariant under the Hamiltonian dynamics.

We can now define the correlation between two functions $F, G \in L^2(\mathcal{M}, \mathrm{d}\mu_H) \cap C^\infty(\mathcal{M})$, which is the space of infinitely many differentiable functions, such that their modulus squared is integrable with respect to $\mathrm{d}\mu_H$ (0.2).

Definition 0.1. *Let $H(\mathbf{p}, \mathbf{q}) \in \mathcal{C}^\infty(\mathcal{M})$ be a Hamiltonian such that the Gibbs measure $\mathrm{d}\mu_H$ in* (0.2) *is well-defined. Let $F, G \in L^2(\mathcal{M}, \mathrm{d}\mu_H) \cap C^\infty(\mathcal{M})$ be two real or complex valued functions. The correlation function of $F(t) = F(\mathbf{p}(t), \mathbf{q}(t))$ and $G(t) = G(\mathbf{p}(t), \mathbf{q}(t))$ is defined as*

$$Cor(F, G) = \mathbf{E}\left[F(t)G(0)\right] - \mathbf{E}\left[F(0)\right]\mathbf{E}\left[G(0)\right] ,$$

where $\mathbf{E}[\cdot]$ denotes the expected value with respect to the Gibbs measure (0.2).

As an example, we consider a Hamiltonian system with nearest neighbourhood interaction

$$H(\mathbf{p}, \mathbf{q}) = \sum_{j=1}^{N} \left(\frac{p_j^2}{2} + V(q_{j+1} - q_j) \right) ,$$

where $V(q) \in \mathcal{C}^\infty(\mathbb{R})$ is a function bounded from below and with at least polynomial growth at $\pm\infty$. The Poisson bracket is the canonical one, namely

$$\{q_j, q_k\} = \{p_j, p_k\} = 0, \quad \{q_j, p_k\} = \delta_{jk} ,$$

so the Hamiltonian equations take the standard form:

$$\begin{aligned}
\dot{q}_j &= \{q_j, H\} = \frac{\partial H}{\partial p_j} = p_j , \\
\dot{p}_j &= \{p_j, H\} = -\frac{\partial H}{\partial q_j} = V'(q_{j+1} - q_j) - V'(q_j - q_{j-1}) ,
\end{aligned}$$

here with $'$ we denote the derivative with respect to the argument. We assume periodic boundary conditions, i.e. $p_{j+N} = p_j$, $q_{j+N} = q_j$ for all $j \in \mathbb{Z}$ and N positive integer. Since the Hamiltonian is translational invariant, we can consider the phase space $\mathcal{M}$

$$\mathcal{M} := \left\{ (\mathbf{p}, \mathbf{q}) \in \mathbb{R}^{2N} \;\middle|\; \sum_{j=1}^{N} p_j = \sum_{j=1}^{N} q_j = 0 \right\} .$$

On this phase space, we can introduce the Gibbs measure

$$\mathrm{d}\mu_H = \frac{e^{-\beta H(\mathbf{p}, \mathbf{q})} \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}}{\int_{\mathcal{M}} e^{-\beta H(\mathbf{p}, \mathbf{q})} \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}} ,$$

here $\mathrm{d}\mathbf{q} = \prod_{j=1}^{N} \mathrm{d}q_j$, and similarly for $\mathbf{p}$. Notice that our choice of the potential V implies that $\int_{\mathcal{M}} e^{-\beta H(\mathbf{p}, \mathbf{q})} \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}$ is finite. The relevant physical quantities are the *momenta* p_j, the *elongations* $r_j = q_{j+1} - q_j$, and the *local energy* $e_j = \frac{p_j^2}{2} + V(r_j)$, $j = 1, \ldots, N$. These quantities evolve according to local conservation laws, namely

$$\begin{aligned}
\dot{p}_j &= V'(r_j) - V'(r_{j-1}) , \\
\dot{r}_j &= p_{j+1} - p_j , \\
\dot{e}_j &= -V'(r_{j-1}) p_j + V'(r_j) p_{j+1} ,
\end{aligned}$$

where with $'$ we denote the first derivative with respect to the argument. We are interested in computing the following correlation functions

$$\begin{aligned} \operatorname{Cor}(p_j, p_0) &= \mathbf{E}[p_j(t)p_0(0)] - \mathbf{E}[p_j(0)]\,\mathbf{E}[p_0(0)]\ , \\ \operatorname{Cor}(r_j, r_0) &= \mathbf{E}[r_j(t)r_0(0)] - \mathbf{E}[r_j(0)]\,\mathbf{E}[r_0(0)]\ , \\ \operatorname{Cor}(e_j, e_0) &= \mathbf{E}[e_j(t)e_0(0)] - \mathbf{E}[e_j(0)]\,\mathbf{E}[e_0(0)]\ . \end{aligned} \tag{0.3}$$

Correlation functions give some insight on how the actual status of a portion of the system affect the motion of another part. Moreover, the fact that the total momentum and total energy are conserved gives an idea on how the momentum and energy spread in the lattice.

The explicit computation of these correlation functions for general dynamical system is "utterly out of reach" - H. Sphon. Indeed there are no explicit formulas for correlation functions of hamiltonian system, except for the harmonic oscillator case [112]. Based on extensive numerical investigations and some preliminary computations, it is conjectured that, for sufficient long time, the energy-energy correlations should scale as

$$\operatorname{Cor}(e_j, e_0) \simeq \frac{1}{\lambda t^{\gamma}} \mathcal{G}\left(\frac{j - vt}{\lambda t^{\delta}}\right)\ , \tag{0.4}$$

where v is some characteristic speed of the lattice, $\mathcal{G}$ is an analytic function, $\lambda \in \mathbb{R}^+$, and $0 < \gamma$, $\delta \leqslant 1$. The case $\gamma = \frac{1}{2}$ is referred to as *normal diffusion*, and $\gamma > \frac{1}{2}$ is referred as *super diffusion*.

H. Sphon, in a series of papers [119, 148–152], argues that for nearest neighbourhood, non-linear, and non-integrable Hamiltonian systems the exponents in (0.4) are $\gamma = \delta = \frac{2}{3}$ and $\mathcal{G}$ is a universal function, meaning that it is the same for all this class of systems. Specifically $\mathcal{G} = F_{TW}$, where F_{TW} is the Tracy-Widom distribution [158]. For nearest neighbourhood, non-linear, and integrable systems, the decay is expected to be *ballistic*, meaning that $\gamma = \delta = 1$, and the function $\mathcal{G}$ is also universal, specifically $\mathcal{G} = e^{\frac{-x^2}{2}}/\sqrt{2\pi}$, which is just the Gaussian distribution.

Much less in known and conjectured for short range interactions. In chapter 1 we present our work *"Correlation functions for a chain of short range oscillators"*, on Journal of Statistical Physics (2021) made in collaboration with T. Grava, T. Kriecherbauer, and K. D. T.-R. McLaughlin, where we consider systems with short range interactions with quadratic potential, namely the Hamiltonian system:

$$H_{SR}(\mathbf{p}, \mathbf{q}) = \sum_{j=1}^{N} \left(\frac{p_j^2}{2} + \sum_{s=1}^{m} \frac{\kappa_s}{2} (q_j - q_{j+s})^2 \right)\ , \tag{0.5}$$

where $1 \leqslant m \ll N$, $\kappa_1 > 0$, $\kappa_m > 0$, and $\kappa_s \geqslant 0$ for $1 < s < m$. Periodic boundary conditions are assumed, i.e. $q_{j+N} = q_j$, $p_{j+N} = p_j$, for all $j \in \mathbb{Z}$. Our aim is to explicitly compute the correlation functions (0.3) for this system.

One of the main technical difficulties that we encounter is to find the analogous quantities to the elongation, and the local energy, whose definitions are intuitive in the nearest neighbourhood case. We re-write the Hamiltonian system (0.5) in matrix notation as

$$H_{SR}(\mathbf{p}, \mathbf{q}) = \frac{1}{2}(\mathbf{p}, \mathbf{p}) + \frac{1}{2}(\mathbf{q}, A\mathbf{q})\,,$$

where A is a circulant matrix of band size $2m+1$, and $(\cdot,\cdot)$ denotes the standard scalar product in $\mathbb{R}^N$.

We find a *local, and periodic* Toeplitz matrix T of band size $m+1$ such that $A = T^\intercal T$, where $T^\intercal$ is the matrix transpose. We define the *generalized elongation* as

$$\mathbf{r} = T\mathbf{q}\,,$$

so that

$$H_{SR}(\mathbf{p},\mathbf{r}) = \sum_{j=1}^{N} \frac{1}{2}\left(p_j^2 + r_j^2\right)\,,$$

we remark that the previous transformation is *non-canonical.* The generalized elongation r_j extends the concept of elongation in the harmonic oscillators with nearest neighbourhood interaction. With the linear transformation $\mathbf{q} \to \mathbf{r} = T\mathbf{q}$ the local energies take the familiar form $e_j = \dfrac{1}{2}\left(p_j^2 + r_j^2\right)$, $j = 1, \ldots, N$.

The derivation of the matrix T is crucial for our analysis, and it is obtained by exploiting the rich algebraic structure of circulant and Toeplitz matrices [77].

Thanks to this linear transformation, we are able to explicitly compute the correlation functions (0.3). Furthermore, applying the steepest descent method (see e.g. [122]), we analyse the long time asymptotic of the correlation functions in the large N limit, and we obtain the analogue of formula (0.4) for our case. The correlation functions are highly oscillatory. There are two fastest peaks travelling in opposite directions with equal speed. We obtain the asymptotic description for the scaling in time and shape of these peaks according to (0.4). Specifically, we show that the time scaling is ruled by the coefficients $\gamma = \frac{2}{3}$, $\delta = \frac{1}{3}$, and $\mathcal{G} = \mathrm{Ai}^2$, where Ai is the standard Airy function [38, Eq. 9.5.4]. Moreover, we show that the correlation functions may have some non-generic slowly decaying peaks. We proved that their shape is described by the Pearcey integral [38, Eq. 36.2.14].

Density of States of Lax matrices

We consider integrable Hamiltonian lattice systems with $2N$ degree of freedom.

The Hamiltonian system is *Liouville integrable* if it admits N constants of motion independent and in involution. The modern theory of integrable systems was developed by finding powerful tools to detect integrability. One of these tools is the Lax pair formulation. In this case, the integrability is inferred by finding a couple of square matrices L and B such that the equations of motion (0.1) are equivalent to

$$\dot{L} = [L; B]\,,$$

where $\dot{L} = \dfrac{\partial L}{\partial t}$, and $[L;B] = LB - BL$ is the commutator between the matrices. The size of the matrices L and B depends on the specific cases. This formulation implies that the eigenvalues of the Lax matrix L are conserved. The two prototypical examples of discrete integrable systems admitting a Lax formulation are

- the Toda lattice that is an Hamiltonian system with nearest neighbour interactions with Hamiltonian

$$H_T(\mathbf{p},\mathbf{q}) = \sum_{j=1}^{N}\left(\frac{p_j^2}{2} + e^{-(q_{j+1}-q_j)}\right)\,,$$

and canonical Poisson bracket. Here $\mathbf{p} = (p_1, \ldots, p_N)$ and $\mathbf{q} = (q_1, \ldots, q_N)$, $(\mathbf{p}, \mathbf{q}) \in \mathcal{M} \subseteq \mathbb{R}^{2N}$. We consider periodic boundary conditions $q_{j+N} = q_j$ and $p_{j+N} = p_j$ for $j \in \mathbb{Z}$ and N a positive integer. The Toda equations of motions are

$$\begin{aligned} \dot{q}_j &= p_j\,, \\ \dot{p}_j &= e^{q_{j-1}-q_j} - e^{q_j - q_{j+1}}\,. \end{aligned}$$

The corresponding Lax matrix L is a $N \times N$ periodic Jacobi matrix.

- The Ablowitz-Ladik lattice [2, 3] is a discrete spatial version of the cubic nonlinear Schrödinger equation (NLS). There are several discretizations of this equation, and the Ablowtiz-Ladik is among the ones that preserve integrability [137]. For the Ablowitz-Ladik lattice with periodic boundary conditions the dependent variables are the complex quantities $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)$ where $\alpha_j \in \mathbb{D}$ with $\mathbb{D} = \{z \in \mathbb{C}\,||z| < 1\}$. The Hamiltonian takes the form

$$H_{AL}(\alpha_j, \overline{\alpha}_j) = -2\sum_{j=1}^{N} \Re(\alpha_j \overline{\alpha}_{j+1}) - 2\sum_{j=1}^{N} \log\left(1 - |\alpha_j|^2\right). \tag{0.6}$$

The Poisson bracket is defined on the space $\mathcal{C}^\infty(\mathcal{M})$ with $\mathcal{M} = \mathbb{D}^N$ as

$$\{f, g\} = i\sum_{j=1}^{N} \rho_j^2 \left(\frac{\partial f}{\partial \overline{\alpha}_j}\frac{\partial g}{\partial \alpha_j} - \frac{\partial f}{\partial \alpha_j}\frac{\partial g}{\partial \overline{\alpha}_j}\right) \qquad \rho_j = \sqrt{1 - |\alpha_j|^2}.$$

The Ablowitz-Ladik equations of motions are

$$\dot{\alpha}_j = \{\alpha_j, H_{AL}\} = i(\alpha_{j+1} + \alpha_{j-1} - 2\alpha_j) - i|\alpha_j|^2(\alpha_{j-1} + \alpha_{j+1})\,, \tag{0.7}$$

where $j \in \mathbb{Z}$, $\dot{\alpha}_j = \dfrac{d\alpha_j}{dt}$, and $\alpha_{j+N} = \alpha_j$, for all $j \in \mathbb{Z}$. We remark that the quantity $-2\sum_{j=1}^{N} \log\left(1 - |\alpha_j|^2\right)$ is the generator of the shift $\alpha_j(t) \to e^{-2it}\alpha_j(t)$, while $H_1 = -2\sum_{j=1}^{N} \Re(\alpha_j \overline{\alpha}_{j+1})$ generates the flow

$$i\dot{\alpha}_j = -\rho_j^2(\alpha_{j+1} + \alpha_{j-1}), \quad \rho_j = \sqrt{1 - |\alpha_j|^2}, \tag{0.8}$$

which is related to the Schur one [73].

The Lax matrix $\mathcal{E}$ for the flow (0.8) is a $2N \times 2N$ periodic CMV matrix [126, 147] (See chapter 3).

As before, we consider random initial data sampled according to a Gibbs measure. Our novel observation is that for integrable systems this measure endows the entries of the Lax matrix L with a probability distribution. As a consequence L becomes a random matrix. This observation allows us to study the thermodynamic limit of integrable system using the techniques and the rich theory of Random Matrices.

Our first goal is to study the eigenvalue distribution of the Lax matrix L. Such eigenvalues are distributed on the contour $\mathcal{L} \subseteq \mathbb{C}$. For the Toda lattice $\mathcal{L} = \mathbb{R}$, while $\mathcal{L} = S^1 := \{z \in \mathbb{C} \,|\, |z| = 1\}$ for the Ablowitz-Ladik lattice.

Assuming that L is a $N \times N$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_N$, the *empirical spectral distribution* of the Lax matrix L is the random probability measure $\mathrm{d}\nu_N^{(L)}$ as

$$\mathrm{d}\nu_N^{(L)} = \frac{1}{N}\sum_{j=1}^{N} \delta_{\lambda_j} \,,$$

where δ_λ is the Kronecker delta function centred at λ.

We can now define the *mean density of states* $\mathrm{d}\nu^{(L)}$ of the matrix L as the non-random probability measure, such that

$$\int_{\mathcal{L}} f \mathrm{d}\nu^{(L)} = \lim_{N\to\infty} \int_{\mathcal{L}} f \mathrm{d}\nu_N^{(L)} \,,$$

for all bounded, and continuous f defined in $\mathcal{L}$, provided that the previous limit exists.

The mean density of states of the Lax matrix L is used to derive heuristically the behaviour of the correlation functions using the theory of generalized hydrodynamic [39], as it has shown by H. Spohn in [151, 152].

In Chapter 2, we present our work *"On the mean Density of States of some matrices related to the beta ensembles and an application to the Toda lattice"*, arXiv:2008.04604 (2020). We consider the Toda lattice with periodic boundary conditions, and we compute the mean density of states of its Lax matrix. As we have already mentioned, it is a classical result [51, 110] that the equations of motion of the Toda lattice can be rewritten in Lax pair form as

$$\dot{L} = [B; L] \,, \tag{0.9}$$

where L is the periodic Jacobi matrix of the form:

$$L(\mathbf{b}, \mathbf{a}) := \begin{pmatrix} b_1 & a_1 & 0 & \ldots & a_N \\ a_1 & b_2 & a_2 & \ddots & \vdots \\ 0 & a_2 & b_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ a_N & \ldots & 0 & a_{N-1} & b_N \end{pmatrix}, \quad \begin{cases} b_j = p_j \\ a_j = e^{-\frac{r_j}{2}} \end{cases}, \tag{0.10}$$

here $r_j = q_{j+1} - q_j$, and we recall that $a_{j+N} = a_j$, and $b_{j+N} = b_j$. The matrix B in the Lax equation (0.9) is the anti-symmetric matrix

$$B = \frac{1}{2}\left(L_+ - L_+^{\intercal}\right) \,,$$

where for a matrix X we denote by X_+

$$(X_+)_{ij} = \begin{cases} X_{ij}, & j = i + 1 \mod N \\ 0, & \text{otherwise} \end{cases}$$

In this case the phase-space $\mathcal{M}$ takes the form

$$\mathcal{M} := \left\{ (\mathbf{p}, \mathbf{r}) \in \mathbb{R}^N \times \mathbb{R}^N \Big| \quad \sum_{j=1}^N r_j = \sum_{j=1}^N p_j = 0 \right\} , \tag{0.11}$$

we consider the Gibbs measure

$$\mathrm{d}\mu_T^{(N)} := \frac{1}{Z_T(\beta)} \, e^{-\beta H_T(\mathbf{p},\mathbf{r})} \mathrm{d}\mathbf{p} \, \mathrm{d}\mathbf{r} \, ,$$

where $r_j = q_{j+1} - q_j$ are the elongations, β is the inverse of the temperature, and Z_T is the partition function of the system, namely

$$Z_T(\beta) := \int_{\mathcal{M}} e^{-\beta H_T(\mathbf{p},\mathbf{r})} \mathrm{d}\mathbf{p} \, \mathrm{d}\mathbf{r}.$$

This Gibbs measure endows the entries of the matrix L with a probability distribution, thus L becomes a random matrix. We notice that due to the constraints (0.11) the entries of the matrix L are dependent. Our first result is to prove that the density of states of L is equal to the density of states $\mathrm{d}\nu^{(H_\alpha)}$ of H_α

$$H_\alpha \sim \frac{1}{\sqrt{2}} \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & a_{N-1} \\ & & & a_{N-1} & b_N \end{pmatrix} , \quad \begin{cases} b_n \sim \mathcal{N}(0,2) \\ a_n \sim \chi_{2\alpha} \end{cases} , \tag{0.12}$$

here α is a continuous function depending on β, $\mathcal{N}(0,2)$ is the Gaussian distribution with mean zero and variance 2, and $\chi_{2\alpha}$ is the chi-distribution with parameter 2α, where $\alpha = \alpha(\beta) > 0$ (see (4.40)).

Further, we are able to compute the density of states of the matrix H_α connecting it to the Gaussian β-ensemble in the *high temperature regime* or mean field regime, i.e. when the parameter β scales as $1/N$. More specifically, we proved that the mean density of states $\mathrm{d}\nu^{(H_N)}$ of the matrix

$$H_N(\mathbf{a}, \mathbf{b}) = \frac{1}{\sqrt{2}} \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & a_{N-1} \\ & & & a_{N-1} & b_N \end{pmatrix} , \quad \begin{cases} a_n \sim \chi_{2\alpha\left(1-\frac{n}{N}\right)} \\ b_n \sim \mathcal{N}(0,2) \end{cases} ,$$

is related to the density of states $\mathrm{d}\nu^{(H_\alpha)}$ by the formula

$$\partial_\alpha(\alpha \mathrm{d}\nu^{(H_N)}) = \mathrm{d}\nu^{(H_\alpha)} \, .$$

Moreover, since the density of states $\mathrm{d}\nu^{(H_N)}$ was explicitly computed in [7, 44] as

$$\mathrm{d}\nu^{(H_N)}(x) := \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \left| \widehat{f}_\alpha(x) \right|^{-2} \mathrm{d}x \, , \quad \widehat{f}_\alpha(x) := \sqrt{\frac{\alpha}{\Gamma(\alpha)}} \int_0^\infty t^{\alpha-1} e^{-\frac{t^2}{2}} e^{ixt} \mathrm{d}t \, ,$$

we are able to give an explicit expression for the mean density of states $\mathrm{d}\nu^{(H_\alpha)}$.

Furthermore, we generalize this construction introducing three new tridiagonal matrix ensembles, that we name Gaussian, Laguerre, and Jacobi α ensembles. They are related to the classical β ensembles in the so-called *high temperature regime* or mean-field limit, i.e. when $\beta \to 0$ and N, the size of the matrix, approaches infinity in such a way that $\beta N \to 2\alpha$, $\alpha \in \mathbb{R}^+$. For example, the Gaussian α-ensemble is described by the matrix H_α (0.12). Exploiting their connection to the classical beta ensembles in the high temperature regime, we are able to explicitly compute their mean densities of states.

In Chapter 3, we present our work *"Generalized Gibbs ensemble of the Ablowitz-Ladik lattice, circular β-ensemble and double confluent Heun equation"*, arXiv: 2107.02303 (2021), made in collaboration with T. Grava. Here, we focus on a problem similar to the one in chapter 2, namely we consider the *Ablowitz-Ladik* lattice (0.7). We introduce the Generalized Gibbs ensemble for the AL lattice, and we connect it to the Killip-Nenciu [93] matrix circular β-ensemble at high-temperature investigated by Hardy and Lambert [80]. We consider the Ablowitz-Ladik lattice in thermal equilibrium, meaning that we considered the Gibbs measure $\mathrm{d}\mu_{AL}$

$$\mathrm{d}\mu_{AL} = \frac{1}{Z_{AL}(\beta,\eta)} e^{-\frac{\beta}{2}H_{AL} - \frac{\nu}{2}H_1} \mathrm{d}^2\boldsymbol{\alpha}\,,$$

where H_{AL} is in (0.6) and $H_1 = -2\sum_{j=1}^N \Re(\alpha_j \overline{\alpha}_{j+1})$. In this setting, we are able to compute explicitly the mean density of states of the Lax matrix of the Ablowitz-Ladik lattice as a particular solution of the Double Confluent Heun equation (DCH):

$$z^2 v''(z) + \left(-\eta + z(\beta+1) + \eta z^2\right) v'(z) + \eta\beta(z+\lambda)v(z) = 0\,,$$

where $'$ and $''$ denote the first and second derivative with respect to the argument, $\eta = \nu + \beta$, and $\lambda = \lambda(\eta, \beta)$ is a transcendental function related to the Painlevé III equation [53, 104].

Adiabatic Invariants

We consider the phase-space $\mathcal{M} \subseteq \mathbb{R}^{2N}$, and a Hamiltonian $H(\mathbf{p},\mathbf{q})$, here $\mathbf{p} = (p_1, \dots, p_N)$, $\mathbf{q} = (q_1, \dots, q_N)$. Further, we assume that it is possible to define the classical Gibbs measure at temperature β^{-1}.

Let us consider a function on the phase space, namely $F : \mathcal{M} \to \mathbb{C}$ and let $F(t) = F(\mathbf{p}(t), \mathbf{q}(t))$, where $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are the evolution of $\mathbf{p}(0)$ and $\mathbf{q}(0)$ along the flow generated by the Hamiltonian $H(\mathbf{p},\mathbf{q})$. Our goal is to understand whether there exists a class of functions $F(t)$ that are statistically almost conserved on a certain time-scale. Such functions are called *adiabatic invariants.* The presence of adiabatic invariants implies that the system is not chaotic for the specified time-scale. Indeed, having almost conserved quantities implies that the system does not explore all the phase-space.

The definition of adiabatic invariant is the following

Definition 0.2. *Let us consider the phase space $\mathcal{M}$ with Hamiltonian $H(\mathbf{p},\mathbf{q})$ and Gibbs measure $\mathrm{d}\mu_H$ as in (0.2). Let $F : \mathcal{M} \to \mathbb{C}$, such that $F \in L^2(\mathrm{d}\mu_H, \mathcal{M}) \cap L^1(\mathrm{d}\mu_H, \mathcal{M})$. F is an adiabatic invariant for the Hamiltonian $H(\mathbf{p},\mathbf{q})$ if there exist constants $a, b \in \mathbb{R}^+$ and a continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\mathbf{P}\left(|F(t) - F(0)| > \frac{\sigma_{F(0)}}{\beta^a}\right) \leqslant \frac{f(t)}{\beta^b},$$

where the probability is taken with respect to the Gibbs measure $\mathrm{d}\mu_H$ (0.2)*, and* $\sigma_{F(0)}$ *is the variance of* $F(0)$ *with respect to the same measure, that is* $\sigma_{F(0)} = \mathbf{E}\left[F^2(0)\right] - \mathbf{E}\left[F(0)\right]^2$.

We remark that the previous definition is meaningful if $\frac{f(t)}{\beta^b} < 1$, since we are giving an upper bound on a probability. Further, we notice that the concept of adiabatic invariants generalize the one of conserved quantities. For non-integrable systems, the determination of adiabatic invariants is not an immediate task. In Chapter 4, we present our work *"Adiabatic invariants for the FPUT and Toda chain in the thermodynamic limit."* Communications in Mathematical Physics, 380 (2020), made in collaboration with T. Grava, A. Maspero, and A. Ponno. Here, we explicitly compute some adiabatic invariant for the periodic Fermi-Pasta-Ulam-Tsingou (FPUT) lattice. The Hamiltonian of this system reads

$$H_{FPUT}(\mathbf{p}, \mathbf{q}) = \sum_{j=0}^{N-1} \left(\frac{p_j^2}{2} + V(q_{j+1} - q_j)\right), \quad V(r) = \frac{r^2}{2} - \frac{r^3}{6} + \mathtt{b}\frac{r^4}{24},$$

here $\mathtt{b} > 0$, and we consider periodic boundary condition, i.e. $q_{j+N} = q_j$, $p_{j+N} = p_j$ for all $j \in \mathbb{Z}$.

This system, which is not integrable, was introduced by Fermi, Pasta, Ulam, and Tsingou to study the foundations of statistical mechanics. Specifically, they wanted to obtain some numerical evidence of the so-called *ergodic hypothesis*: any non-integrable system would reach an equilibrium state in fairly short-time for any initial data.

Surprisingly, their numerical experiments showed that the FPUT exhibit a recurrent behaviour for a long time-scale, which is a typical feature of integrable systems. This was in contrast with what they expected.

In the last 60 years, several scholars tried to explain the FPUT paradox in different ways. Initially, this phenomenon was interpreted in terms of closeness to some non-linear integrable system, e.g. the Korteweg-de Vries (KdV) [170], the Boussinesq equation [171], and the Toda chain [49, 110]. On larger time-scales the system displays instead an ergodic behaviour and approaches its micro-canonical equilibrium state (in measure), unless the energy is so low to enter a KAM-like regime [16,17,85,89,142]. We also mention the works [16,17], where the authors were able to prove the recurrent behaviour of the FPUT lattice in the regime of specific energy going to zero, by approximating it by the KdV equation and the Toda lattice respectively. However, these analytic results do not hold in the thermodynamic limit, because the energy per particle scales to zero. Adiabatic invariants were introduced in [29] to overcome this problem, since their presence is usually not affected by the size of the system.

In the last few years, lots of efforts were put in constructing adiabatic invariants for the FPUT system see [29, 30, 68, 69, 109]. In all these papers, the authors found some adiabatic invariants by considering the FPUT system as a perturbation of the harmonic oscillator.

In Chapter 4 we exploit a different approach by considering the FPUT system as a fourth order perturbation of the Toda lattice. Using this approach we determine a new family of adiabatic invariants of the FPUT system. Specifically, we proved that the integrals of motion of the Toda lattice $J^{(m)} = Tr\,(L^m)$, where the Lax matrix L is defined in (0.10), are adiabatic invariants for the FPUT system for time of order $\beta^{1-2\varepsilon}$, $\varepsilon > 0$.

The advantage of our approach is that the integrals $J^{(m)}$, also known as Henon integrals, are explicit in physical variables [82], so we avoid the perturbative construction of [67]. On the other hand, we needed to carefully analyse the algebraic structure of the integrals $J^{(m)}$. This is achieved by using ideas coming from Random Matrix Theory. Indeed, we derive a new explicit formula for the integrals $J^{(m)}$ by relating them to the so-called Super Motzkin path [130], a well-known combinatorial object. This observation is fundamental for our analysis.

Loop equations

In the last chapter of the thesis, we present our work *"The classical beta ensembles with beta proportional to 1/N: from loop equations to Dyson's disordered chain."* Journal of Mathematical Physics 62, 073505 (2021), made in collaboration with P.J. Forrester. Here, we focus on a purely random matrix theory problem, namely the explicit computation of the density of states for the classical beta ensembles in the high temperature regime [7,8,44,159,160]. Moreover, we characterize their moments and the linear statistics covariance though recurrence relations. Specifically, we consider the following probability distribution:

$$\mathrm{d}\mathbb{P}_{\mathrm{G}} = \frac{1}{Z_{\mathrm{G}}} \prod_{i \neq j = 1}^{N} (\lambda_i - \lambda_j)^{\frac{2\alpha}{N}} e^{-\sum_{j=1}^{N} \lambda_j^2}\,, \tag{0.13}$$

$$\mathrm{d}\mathbb{P}_{\mathrm{L}} = \frac{1}{Z_{\mathrm{L}}} \prod_{i \neq j = 1}^{N} (\lambda_i - \lambda_j)^{\frac{2\alpha}{N}} \prod_{j=1}^{N} \lambda_j^{\alpha_1} e^{-\lambda_j}\,, \tag{0.14}$$

$$\mathrm{d}\mathbb{P}_{\mathrm{J}} = \frac{1}{Z_{\mathrm{J}}} \prod_{i \neq j = 1}^{N} (\lambda_i - \lambda_j)^{\frac{2\alpha}{N}} \prod_{j=1}^{N} \lambda_j^{\alpha_1} (1 - \lambda_j)^{\alpha_2} e^{-\lambda_j}\,, \tag{0.15}$$

here, $\alpha > -1$, $\alpha_1, \alpha_2 > -1$, and $Z_{\mathrm{G}}, Z_{\mathrm{L}}$, and Z_{J} are the norming constants of the systems, or partition functions.

The joint probability distributions (0.13)-(0.14)-(0.15) are referred to as Gaussian, Laguerre, and Jacobi beta ensembles in the high temperature regime. Each of these ensembles admits a matrix representation, meaning that there exist three random matrices such that their eigenvalues are distributed according to (0.13)-(0.14)-(0.15) respectively. For example, the eigenvalues of the matrix H_N

$$H_N(\mathbf{a}, \mathbf{b}) = \frac{1}{\sqrt{2}} \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & a_{N-1} \\ & & & a_{N-1} & b_N \end{pmatrix}, \quad \begin{cases} a_n \sim \chi_{2\alpha\left(1 - \frac{n}{N}\right)} \\ b_n \sim \mathcal{N}(0, 2) \end{cases},$$

are distributed according to (0.13).

Our first aim is to compute the density of states for these ensembles. We obtain this result by making use of the knowledge of the so-called *loop equations* [23, 25, 63, 123, 166] for the classical beta ensembles. The loop equation formalism allows determining the coefficients of the large N expansion of the rescaled resolvent

$$\frac{1}{N}\left\langle \sum_{j=1}^{N} \frac{1}{x-\lambda_i} \right\rangle^{\mathrm{G}} = W_1^{0,\mathrm{G}}(x) + \frac{1}{N} W_1^{1,\mathrm{G}}(x) + \cdots ,$$

here we used the superscript G to refer to the Gaussian beta ensemble, and λ_j are distributed according to (0.13). The expected value is taken with respect to (0.13), and

$$W_1^{0,\mathrm{G}}(x) = \int_{-\infty}^{\infty} \frac{\rho_{(1),0}^{\mathrm{G}}(\lambda;\alpha)}{x-\lambda}\, \mathrm{d}\lambda ,$$

here $\rho_{(1),0}^{\mathrm{G}}(\lambda;\alpha)$ is the mean density of states of the Gaussian Beta ensemble in the high temperature regime. We notice that $W_1^{0,\mathrm{G}}(x)$ is exactly the Stieltjes transform of the mean density of states. By knowing $W_1^{0,\mathrm{G}}(x)$ it is possible to recover $\rho_{(1),0}^{\mathrm{G}}(\lambda;\alpha)$ using the inversion formula [13]:

$$\rho_{(1),0}^{\mathrm{G}}(x;\alpha) = \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \mathrm{Im}\, W_1^{0,\mathrm{G}}(x-i\varepsilon).$$

There are analogous structures for the Laguerre and Jacobi case.

Studying the loop equations for these ensembles, we are able to recover $W_1^{0,\mathrm{G}}(x)$, and the corresponding quantities for the Laguerre, and Jacobi cases. We also characterize the moments of the density of states and the covariance of monomial linear statistics via recurrence relations.

The explicit expression for the mean density of states for these ensembles were already known by different methods [7, 8, 44, 159, 160]. The novelty of our approach is the fact that we produce a unifying mechanism to characterize the mean density of states for all β ensembles in the high temperature regime, making use of loop equations.

Moreover, following the procedure in [115], we identify a further example of α ensemble, specified by the following random anti-symmetric tridiagonal matrix

$$A_N^{\alpha} = \begin{pmatrix} 0 & a_1 & & & \\ -a_1 & 0 & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_{N-2} & 0 & a_{N-1} \\ & & & -a_{N-1} & 0 \end{pmatrix}, \quad a_n \sim \tilde{\chi}_{\alpha},$$

with $\tilde{\chi}_k$ denoting the square root of the gamma distribution $\Gamma[k/2, 1]$.

Also in this case, we are able to explicitly compute its mean density of states. It is worth to mention that the same random matrix ensemble appears in Dyson's [45] study of a disordered chain of harmonic oscillators. Our analytic results supplement those already contained in Dyson's work.

Chapter 1

Correlation Functions for a chain of Short Range Oscillator

In this chapter, we will consider a system of $N = 2M + 1$ particles interacting with a short range harmonic potential with Hamiltonian of the form

$$H = \sum_{j=0}^{N-1} \frac{p_j^2}{2} + \sum_{s=1}^{m} \frac{\kappa_s}{2} \sum_{j=0}^{N-1} (q_j - q_{j+s})^2 \,, \tag{1.1}$$

where $1 \leqslant m \ll N$, $\kappa_1 > 0$, $\kappa_m > 0$, and $\kappa_s \geqslant 0$ for $1 < s < m$. Throughout this chapter, we consider periodic boundary conditions, i.e. $q_{N+j} = q_j$, $p_{N+j} = p_j$ for all j.

The Hamiltonian (1.1) can be rewritten in the form

$$H(\mathbf{p}, \mathbf{q}) := \frac{1}{2}(\mathbf{p}, \mathbf{p}) + \frac{1}{2}(\mathbf{q}, A\mathbf{q}), \tag{1.2}$$

where $\mathbf{p} = (p_0, \ldots, p_{N-1})$, $\mathbf{q} = (q_0, \ldots, q_{N-1})$, $(\,\cdot\,,\,\cdot\,)$ denotes the standard scalar product in $\mathbb{R}^N$ and where $A \in \mathrm{Mat}(N, \mathbb{R})$ is a positive semidefinite symmetric circulant matrix generated by the vector $\mathbf{a} = (a_0, \ldots, a_{N-1})$ namely $A_{kj} = a_{(j-k)\bmod N}$ or

$$A = \begin{bmatrix} a_0 & a_1 & \ldots & a_{N-2} & a_{N-1} \\ a_{N-1} & a_0 & a_1 & & a_{N-2} \\ \vdots & a_{N-1} & a_0 & \ddots & \vdots \\ a_2 & & \ddots & \ddots & a_1 \\ a_1 & a_2 & \ldots & a_{N-1} & a_0 \end{bmatrix}, \tag{1.3}$$

where

$$a_0 = 2\sum_{s=1}^{m} \kappa_s, \quad a_s = a_{N-s} = -\kappa_s, \quad \text{for } s = 1, \ldots, m \text{ and } a_s = 0 \text{ otherwise.} \tag{1.4}$$

Due to the condition $\kappa_1 > 0$ we have $(\mathbf{q}, A\mathbf{q}) = 0$ iff all spacings $q_{j+1} - q_j$ vanish. Therefore, the kernel of A is one-dimensional, with the constant vector $(1, \ldots, 1)^\intercal$ providing a basis. This also implies that the lattice at rest has zero spacings everywhere. Observe, however, that one may introduce an arbitrary spacing Δ for the lattice at rest by the canonical transformation $Q_j = q_j + j\Delta$, $P_j = p_j$ which does not change the dynamics. The periodicity condition for the positions Q_j then reads $Q_{N+j} = Q_j + L$ with $L = N\Delta$ (see e.g. [148, Sec. 2]).

The harmonic oscillator with only nearest neighbour interactions is recovered by choosing

$$a_0 = 2\kappa_1, \quad a_1 = a_{N-1} = -\kappa_1,$$

and the remaining coefficients are set to zero.

The equations of motion for the Hamiltonian H take the form

$$\frac{d^2}{dt^2} q_j = \sum_{s=1}^{m} \kappa_s (q_{j+s} - 2q_j + q_{j-s}), \quad j = 0, \dots, N-1.$$

The integration is obtained by studying the dynamics in Fourier space (see e.g. [107]). We will study correlations between momentum, position and local versions of energy. Following the standard procedure in the case of nearest neighbour interactions we replace the vector of position $\mathbf{q}$ by a new variable $\mathbf{r}$ so that the Hamiltonian takes the form

$$H = \frac{1}{2}(\mathbf{p}, \mathbf{p}) + \frac{1}{2}(\mathbf{r}, \mathbf{r}).$$

Such a change of variables may be achieved by any linear transformation

$$\mathbf{r} = T\mathbf{q}, \tag{1.5}$$

with an $N \times N$ matrix T that satisfies

$$A = T^{\intercal} T, \tag{1.6}$$

where $T^{\intercal}$ denotes the transpose of T. In the case of nearest neighbour interactions one may choose $r_j = \sqrt{\kappa_1}(q_{j+1} - q_j)$ corresponding to a circulant matrix T generated by the vector $\boldsymbol{\tau} = \sqrt{\kappa_1}(-1, 1, 0, \dots, 0)$. We show in Proposition 1.2 below that short range interactions given by matrices A of the form (1.3), (1.4) also admit such a *localized square root*. More precisely, there exists a circulant $N \times N$ matrix T of the form

$$T = \begin{bmatrix} \tau_0 & \tau_1 & \dots & \tau_m & 0 & \dots & 0 \\ 0 & \tau_0 & \tau_1 & \dots & \tau_m & 0 & \\ & \ddots & \ddots & \ddots & \ddots & & \\ \tau_m & 0 & \ddots & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \tau_2 & \dots & \tau_m & 0 & \dots & \tau_0 & \tau_1 \\ \tau_1 & \tau_2 & \dots & \tau_m & 0 & 0 & \tau_0 \end{bmatrix}. \tag{1.7}$$

that satisfies (1.6). The crucial point here is that T is not the standard (symmetric) square root of the positive semidefinite matrix A, but a localized version generated by some vector $\boldsymbol{\tau}$ with zero entries everywhere, except possibly in the first $m+1$ components. Hence, the j^{th} component of the *generalized elongation* $\mathbf{r}$ defined through (1.5) depends only on the components q_s with $s = j, j+1, \dots, j+m$. It is worth noting that $\mathbf{1} = (1, \dots, 1)^{\intercal}$ satisfies $T\mathbf{1} = 0$ since $(\mathbf{1}, A\mathbf{1}) = 0$. This implies

$$\sum_{s=0}^{m} \tau_s = 0\,, \quad r_j = \sum_{s=1}^{m} \tau_s (q_{j+s} - q_j) \quad \text{and} \quad \sum_{j=0}^{N-1} r_j = (1, \dots, 1) T\mathbf{q} = 0.$$

The local energy e_j takes the form

$$e_j = \frac{1}{2}p_j^2 + \frac{1}{2}r_j^2 \,.$$

Our goal is to study the behaviour of the correlation functions for the momentum p_j, the generalized elongation r_j and the local energy e_j when $N \to \infty$ and $t \to \infty$. Due to the spatial translation invariance of the Hamiltonian $H(\mathbf{p}, \mathbf{q}) = H(\mathbf{p}, \mathbf{q} + \lambda\mathbf{1})$, $\lambda \in \mathbb{R}$, that corresponds to the conservation of total momentum, we reduce the Hamiltonian system by one degree of freedom to obtain a normalizable Gibbs measure. This leads to the reduced phase space

$$\mathcal{M} := \left\{ (\mathbf{p}, \mathbf{q}) \in \mathbb{R}^N \times \mathbb{R}^N \;:\; \sum_{k=0}^{N-1} p_k = 0\,;\; \sum_{k=0}^{N-1} q_k = 0 \right\}. \tag{1.8}$$

We endow $\mathcal{M}$ with the Gibbs measure at temperature β^{-1}, namely:

$$\mathrm{d}\mu = Z_N(\beta)^{-1}\delta_0\left(\sum_{k=0}^{N-1} p_k\right)\delta_0\left(\sum_{k=0}^{N-1} q_k\right)e^{-\beta H(\mathbf{p},\mathbf{q})}\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q} \tag{1.9}$$

where $Z_N(\beta)$ is the norming constant and $\delta_0(x)$ is the delta function centred at 0. For convenience, we introduce the vector

$$\boldsymbol{u}(j,t) = (r_j(t), p_j(t), e_j(t)).$$

We consider the correlation functions

$$S^N_{\alpha\alpha'}(j,t) = \langle u_\alpha(j,t)u_{\alpha'}(0,0)\rangle - \langle u_\alpha(j,t)\rangle\langle u_{\alpha'}(0,0)\rangle\,, \quad \alpha, \alpha' = 1,2,3, \tag{1.10}$$

where the symbol $\langle\,\cdot\,\rangle$ refers to averages with respect to $d\mu$. We calculate the limits

$$\lim_{N\to\infty} S^N_{\alpha\alpha'}(j,t) = S_{\alpha\alpha'}(j,t)\,.$$

For the harmonic oscillator with nearest neighbor interactions such limits have been calculated in [112].

In an interesting series of papers, (see e.g. [149], and also the collection [102]) several researchers have considered the evolution of space-time correlation functions, for "anharmonic chains", which are nonlinear nearest-neighbor Hamiltonian systems of oscillators. The authors consider the deterministic evolution from random initial data sampled from a Gibbs ensemble, with a large number of particles and study the correlation functions $S^N_{\alpha\alpha'}$.

In addition to intensive computational simulations [98], [120], Spohn and collaborators also propose and study a nonlinear stochastic conservation law model [148], [149]. Using deep physical intuition, it has been proposed that the long-time behaviour of space-time correlation functions of the deterministic Hamiltonian evolution from random initial data is equivalent to the behaviour of correlation functions of an analogous nonlinear stochastic system of PDEs. Studying this stochastic model, Spohn eventually arrives at an asymptotic description of the "sound peaks" of the correlation functions in normal modes coordinates which are related to $S_{\alpha\alpha'}$ by orthogonal transformation:

$$\tilde{S}_{\alpha\alpha} \cong (\lambda_s t)^{-2/3} f_{\mathrm{KPZ}}\left((\lambda_s t)^{-2/3}(x - \alpha c t)\right)\,,$$

using the notation of [Formula (3.1)] [148]. Here f_{KPZ} is a universal function that first emerges in the Kardar-Parisi-Zhang equation and it is related to the Tracy-Widom distribution, [158], (for a review see [32] and also [94]). A common element to the above cited papers is the observation that such formulae should hold for non-integrable dynamics, while the correlation functions of integrable lattices of oscillators will exhibit *ballistic scaling*, which means the correlation functions decay as $\frac{1}{t}$ for t large. For example, in [98] the authors present the results of simulations of the Toda lattice in 3 different asymptotic regimes (the harmonic oscillator limit, the hard-particle limit, and the full nonlinear system). They present plots of the quantity $tS(x,t)$ as a function of the scaled spatial variable x/t (here $S(x,t)$ represents any of the correlation functions). The numerical results support the ballistic scaling conjecture in some asymptotic scaling regimes. Further analysis in [151] gives a derivation of the ballistic scaling for the Toda lattice. The decay of equilibrium correlation functions show similar features as anomalous heat transport in one-dimensional systems [37], [101] [36] which leads to conjecture that the two phenomena are related [102].

In [119] the authors also pursue a different connection to random matrices, and in particular to the Tracy-Widom distribution. Over the last 15 years, there has emerged a story originating in the proof that for the totally asymmetric exclusion process on a 1-D lattice (TASEP), the fluctuations of the height function are governed (in a suitable limit) by the Tracy-Widom distribution. Separately, a partial differential equations model for these fluctuations emerged, which takes the form of stochastic Burgers equation:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} - \lambda u \frac{\partial u}{\partial x} + \frac{\partial \zeta}{\partial x} \ ,$$

where ζ is a stationary spatio-temporal white noise process. (The mean behaviour of TASEP is actually described by the simpler Euler equation $\frac{\partial u}{\partial t} = -\lambda u \frac{\partial u}{\partial x}$.). From these origins there have now emerged proofs, for a small collection of initial conditions, that the fluctuations of the solution to (1.12) are indeed connected to the Tracy-Widom distribution (see [32] and the references contained therein). In [119], the authors considered continuum limits of anharmonic lattices with random initial data, in which there are underlying conservation laws describing the mean behaviour that are the analogue of the Euler equation associated to (1.12). By analogy with the connection between TASEP and (1.12), they proposed that the time-integrated currents are the analogue of the height function, and should exhibit fluctuations about their mean described by the Tracy-Widom distribution, again based on the use of the nonlinear stochastic pde system as a model for the deterministic evolution from random initial data. As one example, they consider the quantity

$$\Phi(x,t) = \int_0^t \mathfrak{j}(x,t')dt' - \int_0^x \mathfrak{u}(x',0)dx' \ ,$$

where $\mathfrak{u}(x,t)$ arises as a sort of continuum limit of a particle system obeying a discrete analogue of a system of conservation laws taking the form $\partial_t \mathfrak{u}(x,t) + \partial_x \mathfrak{j}(x,t) = 0$, in which $\mathfrak{j}(x,t)$ is a local current density for $\mathfrak{u}(x,t)$. The authors suggest a dual interpretation of $\Phi(x,t)$ as the height function from a KPZ equation, and thus arrive at the proposal that

$$\Phi(x,t) \simeq a_0 t + (\Gamma t)^{1/3} \xi_{TW} \ ,$$

where a_0 and Γ are model-dependent parameters, and ξ_{TW} is a random amplitude with Tracy-Widom distribution.

Our main result is the analogue of the relations (1.11) for the harmonic oscillator with short range interactions and (1.14) for the harmonic oscillator. For stating our result, we first calculate the dispersion relation $|\omega(k)|$ for the harmonic oscillator with short range interaction in the limit $N \to \infty$ obtaining

$$f(k) = |\omega(k)| = \sqrt{2\sum_{s=1}^{m} \kappa_s \left(1 - \cos(2\pi k s)\right)}\,, \tag{1.15}$$

see (1.28). The points $k = 0, 1$ contribute to the fastest moving peaks of the correlation functions that have a velocity $\pm v_0$ where $v_0 = \sqrt{\sum_{s=1}^{m} s^2\kappa_s} = f'(0)/(2\pi)$. If $f''(k) < 0$ for all $0 < k \leqslant 1/2$ then as $t \to \infty$ the following holds uniformly in $j \in \mathbb{Z}$ (cf. Theorem 1.6 and Figure 1.1):

$$\begin{aligned} S_{\alpha\alpha'}(j,t) &= \frac{1}{2\beta\lambda_0 t^{1/3}}\left[(-1)^{\alpha+\alpha'}\mathrm{Ai}\left(\frac{j - v_0 t}{\lambda_0 t^{1/3}}\right) + \mathrm{Ai}\left(-\frac{j + v_0 t}{\lambda_0 t^{1/3}}\right)\right] + \mathcal{O}\left(t^{-1/2}\right), \quad \alpha, \alpha' = 1, 2 \\ S_{33}(j,t) &= \frac{1}{2\beta^2\lambda_0^2 t^{2/3}}\left[\mathrm{Ai}^2\left(\frac{j - v_0 t}{\lambda_0 t^{1/3}}\right) + \mathrm{Ai}^2\left(-\frac{j + v_0 t}{\lambda_0 t^{1/3}}\right)\right] + \mathcal{O}\left(t^{-5/6}\right)\,, \end{aligned} \tag{1.16}$$

where $\mathrm{Ai}(w) = \frac{1}{\pi}\int_0^\infty \cos(y^3/3 + wy)dy$, $w \in \mathbb{R}$, is the Airy function, and $\lambda_0 := \frac{1}{2}\left(\frac{1}{v_0}\sum_{s=1}^{m} s^4\kappa_s\right)^{1/3}$. The above formula is the linear analogue of the Tracy-Widom distribution in (1.11).

Furthermore, we can tune the spring intensities κ_s, $s = 1, \dots, m$ in (1.15) so that we can find an $(m-1)$-parameter family of potentials such that for $j \sim \pm v^* t$, with $0 \leqslant v^* < v_0$, one has

$$S_{\alpha\alpha'}(j,t) = \mathcal{O}\left(\frac{1}{t^{\frac{1}{4}}}\right), \quad \alpha, \alpha' = 1, 2\,, \quad S_{33}(j,t) = \mathcal{O}\left(\frac{1}{t^{\frac{1}{2}}}\right), \qquad \text{as } t \to \infty\,.$$

In this case the local behaviour of the correlation functions is described by the Pearcey integral (see Theorem 1.7 and Figures 1.2, 1.3 below).

For example a potential with such behaviour is given by a spring interaction of the form $\kappa_s = \dfrac{1}{s^2}$ for $s = 1, \dots, m$ and m even (see Example 1.8 below).

In Section 1.3 we study numerically small nonlinear perturbations of the harmonic oscillator with short range interactions and our results suggest that the behaviour of the fastest peak has a transition from the Airy asymptotic (1.16) to the Tracy-Widom asymptotic (1.11), depending on the strength of the nonlinearity. Namely the asymptotic behaviour in (1.11) that has been conjectured for nearest neighbour interactions seems to persist also for sufficiently strong nonlinear perturbations of the harmonic oscillator with short range interactions. Remarkably, our numerical simulations indicate that the non generic decay in time of other peaks in the correlation functions persists under small nonlinear perturbations with the same power law $t^{-1/4}$ as in the linear case, see e.g. Figures 1.4 and 1.6.

So as not to overlook a large body of related work, we observe that the quantities we consider here are somewhat different than those considered in the study of thermal transport, though there is of course overlap. (We refer to the Lecture Notes [101] for an overview of this research area and also the seminal paper [140].) As mentioned above, we study the dynamical evolution of space-time correlation functions and the statistical description of random height functions, where the only randomness comes from the initial data. By comparison, in the consideration of heat conduction and transport in low dimensions, anharmonic chains are often connected at their ends to heat reservoirs of different temperatures, and randomness is present primarily in the dynamical laws, not only in fluctuations of initial data.

This chapter is organized as follows. In Section 1.1 we study the harmonic oscillator with short range interactions, and we introduce the necessary notation and the change of coordinates $\mathbf{q} \to \mathbf{r}$ that enables us to study correlation functions. We then study the time decay of the correlation functions via steepest descent analysis, and we show that the two fastest peaks travelling in opposite directions originate from the points $k = 0$ and $k = 1$ in the spectrum. Such peaks have a decay described by the Airy scaling. We then show the existence of potentials such that the correlation functions have a slower time decaying with respect to "Airy peaks". In Section 1.2 we show that the harmonic oscillator with short range interactions has a complete set of *local* integrals of motion in involution and the correlation functions of such integrals have the same structure as the energy-energy correlation function. Further, we show that the evolution equations for the generalized position, momentum can be written in the form of conservation laws which have a potential function. For the case of the harmonic oscillator with nearest neighbour interaction, we show that this function is a Gaussian random variable and determine the leading order behaviour of its variance as $t \to \infty$. This may be viewed as the analogue of formula (1.14) for the linear case. Finally, in Section 1.3 we study numerically the evolution of the correlation functions after adding nonlinear perturbations to our model.

1.1 The harmonic oscillator with short range interactions

As it was previously explained, we rewrite the Hamiltonian for the harmonic oscillator with short range interactions

$$H(\mathbf{p},\mathbf{q}) = \sum_{j=0}^{N-1} \frac{p_j^2}{2} + \sum_{s=1}^{m} \frac{\kappa_s}{2} \sum_{j=0}^{N-1} (q_j - q_{j+s})^2 = \sum_{j=0}^{N-1} \left(\frac{p_j^2}{2} + \frac{1}{2} \Big(\sum_{s=1}^{m} \tau_s (q_{j+s} - q_j) \Big)^2 \right)$$

so that we may define a Hamiltonian density

$$e_j = \frac{p_j^2}{2} + \frac{1}{2} \Big(\sum_{s=1}^{m} \tau_s (q_{j+s} - q_j) \Big)^2,$$

which is local in the variables $(\mathbf{p},\mathbf{q})$ for fixed m. Namely, if we let $N \to \infty$, the quantity e_j involves a finite number of physical variables $(\mathbf{p},\mathbf{q})$. Recall that the coefficients τ_s are the entries of the circulant localized square root T of the matrix A by which we mean a solution of the equation (1.6) of the form (1.7). The matrix T will also play a role in constructing a complete set of integrals that have a local density in the sense that we just described for the energy.

In order to state our result we have to introduce some notation. First of all, a matrix A of the form (1.3) with $\mathbf{a} \in \mathbb{R}^N$ is called a circulant matrix generated by the vector $\mathbf{a}$.

Definition 1.1 (m-physical vector and half-m-physical vector)**.** *Fix* $m \in \mathbb{N}$*. For any odd* $N > 2m$*, a vector* $\tilde{\mathbf{x}} \in \mathbb{R}^N$ *is said to be* m-physical *generated by* $\mathbf{x} = (x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$ *if* $x_0 = -2\sum_{s=1}^m x_s$ *and*

$$\begin{aligned}\tilde{x}_0 =& x_0\,,\\ \tilde{x}_1 =& \tilde{x}_{N-1} = x_1 < 0, \;\; \tilde{x}_m = \tilde{x}_{N-m} = x_m < 0,\\ \tilde{x}_k =& \tilde{x}_{N-k} = x_k \leqslant 0, \;\; \textit{for } 1 < k < m,\\ \tilde{x}_k =& 0, \;\; \textit{otherwise,}\end{aligned}$$

while the vector $\tilde{\mathbf{x}} \in \mathbb{R}^N$ *is called* half-m-physical *generated by* $\mathbf{y} \in \mathbb{R}^{m+1}$ *if* $y_0 = -\sum_{s=1}^m y_s$ *and*

$$\begin{aligned}\tilde{x}_k =& y_k, \;\; \textit{for } 0 \leqslant k \leqslant m\\ \tilde{x}_k =& 0, \;\; \textit{for } m < k \leqslant N-1.\end{aligned}$$

Following the proof of a classic lemma by Fejér and Riesz, see e.g. [141, pg. 117 f], one can show that a circulant symmetric matrix A of the form (1.2) generated by a m-physical vector $\mathbf{a}$ always has a circulant localized square root T that is generated by a half-m physical vector $\boldsymbol{\tau}$.

Proposition 1.2. *Fix* $m \in \mathbb{N}$*. Let the circulant matrix* A *be generated by an* m*-physical vector* $\mathbf{a}$*, then there exist a circulant matrix* T *generated by an half-*m*-physical vector* $\boldsymbol{\tau}$ *such that:*

$$A = T^\intercal T\,.$$

Moreover, we can choose $\boldsymbol{\tau}$ *such that* $\sum_{s=1}^m s\tau_s > 0$*. Then one has* $\sum_{s=1}^m s\tau_s = \sqrt{\sum_{s=1}^m s^2\kappa_s}$*.*

Proof. In view of the notation introduced in (1.3), (1.4), and (1.7), we have just to show that there exist $\tau_0, \ldots, \tau_m \in \mathbb{R}$ satisfying $\sum_{s=0}^m \tau_s = 0$ such that

$$Q(z^{-1})\,Q(z) = \ell(z) \qquad \text{for all } z \in \mathbb{C}\backslash\{0\}\,, \tag{1.17}$$

where we have defined

$$\begin{aligned}Q(z) &= \tau_0 + \tau_1 z + \ldots + \tau_m z^m\,,\\ \ell(z) &= -\kappa_m z^{-m} - \ldots - \kappa_1 z^{-1} + a_0 - \kappa_1 z - \ldots - \kappa_m z^m\,.\end{aligned} \tag{1.18}$$

The existence of the τ_j's is a consequence of the Fejér-Riesz lemma. For the convenience, we present a proof following the presentation in [141, pg. 117 f]. Denote by P the polynomial of degree $2m$ given by $P(z) := z^m \ell(z)$. Observe that for all $x \in \mathbb{R}$ we have

$$\ell(e^{ix}) = a_0 - 2\sum_{j=1}^m \kappa_j \cos(jx) \;\geqslant a_0 - 2\sum_{j=1}^m \kappa_j \;=\; 0.$$

By the positivity of κ_1 equality holds in the inequality above iff $\cos(x) = 1$. This implies that P has no zeros on the unit circle $|z| = 1$ except for $z = 1$. We denote by η_k, $1 \leqslant k \leqslant r_<$, the zeros of P that lie within the unit disc $|\eta_k| < 1$ and by ξ_k, $1 \leqslant k \leqslant r_>$, the zeros of P with $|\xi_k| > 1$, recorded repeatedly according to their multiplicities, so that

$$P(z) = -\kappa_m (z-1)^{r_0} \prod_{k=1}^{r_<} (z-\eta_k) \prod_{k=1}^{r_>} (z-\xi_k) \,. \tag{1.19}$$

Using the uniqueness of such a factorization for any polynomial together with the relation $z^{2m}P(z^{-1}) = P(z)$ one obtains that $r_< = r_>$ and that the zeros can be listed in such a way that $\eta_k = \xi_k^{-1}$ for all $1 \leqslant k \leqslant r_<$. Moreover, we learn that r_0 is even with $1 \leqslant \varrho := r_0/2 = m - r_<$. Now it follows from formula (1.19) that

$$l(z) = z^{-m}P(z) = c(z^{-1}-1)^{\varrho}(z-1)^{\varrho} \prod_{k=1}^{r_<}(z^{-1}-\xi_k) \prod_{k=1}^{r_<}(z-\xi_k) \,,$$

here

$$c := -\kappa_m (-1)^{\varrho} \prod_{k=1}^{r_<} (-\xi_k^{-1}) \neq 0 \,.$$

Choosing $d \in \mathbb{C}$ with $d^2 = c$ we see that $Q(z) := d(z-1)^{\varrho} \prod_{k=1}^{r_<}(z-\xi_k)$ satisfies (1.17). Next, we show that the coefficients of the polynomial Q are real. To this end, we observe that P has real coefficients and therefore all non-real zeros of P come in complex conjugate pairs with equal multiplicities. Therefore, the polynomial $d^{-1}Q(z) = \sum_{j=0}^{m} s_j z^j$ has only real coefficients s_j. Relation (1.17) implies $a_0 = d^2 \sum_{j=0}^{m} s_j^2$. Consequently, d^2 is the quotient of two positive numbers and d must be real. Thus, we have $\tau_j = ds_j \in \mathbb{R}$ for all $0 \leqslant j \leqslant m$. We complete the proof by arguing that $\sum_{s=0}^{m} \tau_s = 0$ and $(\sum_{s=1}^{m} s\tau_s)^2 = \sum_{s=1}^{m} s^2\kappa_s$ hold true. This can be deduced from (1.17) via $Q(1)^2 = \ell(1) = 0$ and $-2Q'(1)^2 = \ell''(1) = -\sum_{s=1}^{m} 2s^2\kappa_s$. □

For example, if we consider $m = 1$, and $a_0 = 2\kappa_1$ and $a_1 = a_{N-1} = -\kappa_1$. The matrix T is generated by the vector $\boldsymbol{\tau} = (\tau_0, \tau_1)$ with $\tau_0 = -\sqrt{\kappa_1}$ and $\tau_1 = \sqrt{\kappa_1}$. When $m = 2$ and $a_0 = 2\kappa_1 + 2\kappa_2$, $a_1 = a_{N-1} = -\kappa_1$, $a_2 = a_{N-2} = -\kappa_2$. The matrix T is generated by the vector $\boldsymbol{\tau} = (\tau_0, \tau_1, \tau_2)$ with

$$\begin{aligned} \tau_0 &= -\frac{\sqrt{\kappa_1}}{2} - \frac{1}{2}\sqrt{\kappa_1 + 4\kappa_2}, \quad \tau_1 = \sqrt{\kappa_1}, \\ \tau_2 &= -\frac{\sqrt{\kappa_1}}{2} + \frac{1}{2}\sqrt{\kappa_1 + 4\kappa_2}, \end{aligned}$$

so that the quantities r_j are defined as

$$r_j = \tau_1(q_{j+1} - q_j) + \tau_2(q_{j+2} - q_j) \,, \quad j = 0, \dots N-1 \,.$$

Next we integrate the equation of motions. The Hamiltonian $H(\mathbf{p}, \mathbf{q})$ represents clearly an integrable system that can be integrated passing through Fourier transform. Let $\mathcal{F}$ be the discrete Fourier transform with entries $\mathcal{F}_{j,k} := \frac{1}{\sqrt{N}} e^{-2i\pi jk/N}$ with $j, k = 0, \dots, N-1$. It is immediate to verify that

$$\mathcal{F}^{-1} = \bar{\mathcal{F}} \qquad \mathcal{F}^{\intercal} = \mathcal{F}.$$

Thanks to the above properties, the transformation defined by

$$(\widehat{p}, \widehat{q}) = (\bar{\mathcal{F}}p, \mathcal{F}q)$$

is canonical. Furthermore $\overline{\widehat{p}_j} = \widehat{p}_{N-j}$ and $\overline{\widehat{q}_j} = \widehat{q}_{N-j}$, for $j = 1, \dots, N-1$, while $\widehat{p}_0$ and $\widehat{q}_0$ are real variables. The matrices T and A are circulant matrices and so they are reduced to diagonal form by $\mathcal{F}$:

$$\mathcal{F}A\mathcal{F}^{-1} = \mathcal{F}T^{\intercal}T\mathcal{F}^{-1} = \overline{(\mathcal{F}T\mathcal{F}^{-1})}^{\intercal}(\mathcal{F}T\mathcal{F}^{-1}) .$$

Let ω_j denote the eigenvalues of the matrix T ordered so that $\mathcal{F}T\mathcal{F}^{-1} = \mathrm{diag}(\omega_j)$. Then $|\omega_j|^2$ are the (non-negative) eigenvalues of the matrix A and

$$|\omega_j|^2 = \sqrt{N}(\overline{\mathcal{F}}\tilde{\mathbf{a}})_j, \quad \omega_j = \sqrt{N}(\overline{\mathcal{F}}\tilde{\boldsymbol{\tau}})_j, \quad j = 0, \dots, N-1, \tag{1.20}$$

where $\tilde{\mathbf{a}}$ is the m-physical vector generated by $\mathbf{a}$ and $\tilde{\boldsymbol{\tau}}$ is the half m-physical vector generated by $\boldsymbol{\tau}$ according to Definition 1.1. It follows that

$$\omega_0 = 0, \quad \omega_j = \overline{\omega}_{N-j}\, , \quad j = 1, \dots, N-1,$$

which implies $|\omega_j|^2 = |\omega_{N-j}|^2$, $j = 1, \dots, N-1$. The Hamiltonian H, can be written as the sum of $N-1$ oscillators

$$H(\widehat{\mathbf{p}}, \widehat{\mathbf{q}}) = \frac{1}{2}\left(\sum_{j=1}^{N-1} |\widehat{p}_j|^2 + |\omega_j|^2|\widehat{q}_j|^2\right) = \sum_{j=1}^{\frac{N-1}{2}} |\widehat{p}_j|^2 + |\omega_j|^2|\widehat{q}_j|^2 .$$

There are no terms involving $\widehat{p}_0, \widehat{q}_0$ since the conditions defining $\mathcal{M}$ (1.8) imply that $\widehat{p}_0 = 0$ and $\widehat{q}_0 = 0$. The Hamilton equations are

$$\begin{cases} \dfrac{d}{dt}\widehat{q}_j = \overline{\widehat{p}_j} \\ \dfrac{d}{dt}\overline{\widehat{p}}_j = -|\omega_j|^2\widehat{q}_j . \end{cases}$$

Thus the general solution reads:

$$\begin{aligned} \widehat{q}_j(t) &= \widehat{q}_j(0)\cos(|\omega_j|t) + \frac{\overline{\widehat{p}_j(0)}}{|\omega_j|}\sin(|\omega_j|t)\, , \\ \overline{\widehat{p}_j(t)} &= \overline{\widehat{p}_j(0)}\cos(|\omega_j|t) - |\omega_j|\widehat{q}_j(0)\sin(|\omega_j|t)\, , \quad j = 1, \dots, N-1, \end{aligned} \tag{1.21}$$

and $\widehat{q}_0(t) = 0$ and $\widehat{p}_0(t) = 0$. Inverting the Fourier transform, we recover the variables $\mathbf{q} = \mathcal{F}^{-1}\widehat{\mathbf{q}}$, $\mathbf{p} = \mathcal{F}\widehat{\mathbf{p}}$ and $\mathbf{r} = \mathcal{F}^{-1}\widehat{\mathbf{r}}$ where

$$\widehat{r}_j = \omega_j\widehat{q}_j,\ j = 0, \dots, N-1 . \tag{1.22}$$

Correlation Decay

We now study the decay of correlation functions for Hamiltonian systems of the form (1.2). We recall the definition (1.9) of the Gibbs measure at temperature β^{-1} on the reduced phase space $\mathcal{M}$, namely:

$$\mathrm{d}\mu = Z_N(\beta)^{-1}\delta_0\left(\sum_{k=0}^{N-1} p_k\right)\delta_0\left(\sum_{k=0}^{N-1} q_k\right)e^{-\beta H(\mathbf{p},\mathbf{q})}\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}$$

where $Z_N(\beta)$ is the norming constant of the probability measure. For a function $f = f(\mathbf{p}, \mathbf{q})$ we define its average as

$$\langle f \rangle := \int_{\mathbb{R}^{2N}} f(\mathbf{p}, \mathbf{q}) \, \mathrm{d}\mu.$$

We first compute all correlation functions (1.10), then we will evaluate the limit $N \to \infty$. We first observe that (1.9) in the variables $(\widehat{\mathbf{p}}, \widehat{\mathbf{q}}) := (\bar{\mathcal{F}}\mathbf{p}, \mathcal{F}\mathbf{q})$ becomes

$$\mathrm{d}\mu = Z_N(\beta)^{-1} \prod_{j=1}^{\frac{N-1}{2}} e^{-\beta(|\widehat{p}_j|^2 + |\omega_j|^2 |\widehat{q}_j|^2)} \mathrm{d}\widehat{p}_j \mathrm{d}\widehat{q}_j \tag{1.23}$$

where $\mathrm{d}\widehat{p}_j \mathrm{d}\widehat{q}_j = \mathrm{d}\Re\widehat{p}_j \mathrm{d}\Im\widehat{p}_j \mathrm{d}\Re\widehat{q}_j \mathrm{d}\Im\widehat{q}_j$ and we recall that $\widehat{p}_j = \overline{\widehat{p}}_{N-j}$, $\widehat{q}_j = \overline{\widehat{q}}_{N-j}$, $\widehat{r}_j = \omega_j \widehat{q}_j$, for $j = 1, \dots, N-1$.

From the evolution of $\widehat{p}_j$ and $\widehat{q}_j$ in (1.21) and (1.22), we arrive at the relations

$$\left\langle \widehat{p}_j(t)\overline{\widehat{p}_k(0)} \right\rangle = \left\langle \overline{\widehat{p}_k(0)} \left(\widehat{p}_j(0)\cos(|\omega_j|t) - |\omega_j|\overline{\widehat{q}_j(0)} \sin(|\omega_j|t) \right) \right\rangle = \delta_{j,k}\frac{1}{\beta}\cos(|\omega_j|t),$$

$$\langle \widehat{p}_j(t)\widehat{r}_k(0) \rangle = \left\langle \omega_k \widehat{q}_k(0) \left(\widehat{p}_j(0)\cos(|\omega_j|t) - |\omega_j|\overline{\widehat{q}_j(0)} \sin(|\omega_j|t) \right) \right\rangle = -\delta_{j,k}\frac{\omega_j}{|\omega_j|\beta}\sin(|\omega_j|t),$$

$$\langle \widehat{r}_j(t)\widehat{p}_k(0) \rangle = \left\langle \omega_j \widehat{p}_k(0) \left(\widehat{q}_j(0)\cos(|\omega_j|t) + \frac{\overline{\widehat{p}_j(0)}}{|\omega_j|}\sin(|\omega_j|t) \right) \right\rangle = \delta_{j,k}\frac{\omega_j}{|\omega_j|\beta}\sin(|\omega_j|t)$$

$$\left\langle \widehat{r}_j(t)\overline{\widehat{r}_k(0)} \right\rangle = \left\langle \overline{\omega}_k \omega_j \overline{\widehat{q}_k(0)} \left(\widehat{q}_j(0)\cos(|\omega_j|t) + \frac{\overline{\widehat{p}_j(0)}}{|\omega_j|}\sin(|\omega_j|t) \right) \right\rangle = \delta_{j,k}\frac{1}{\beta}\cos(|\omega_j|t).$$

Now we are ready to compute explicitly the correlation functions in the physical variables. We show the computation just for the case $S_{11}^N(j,t)$, since all the other cases are analogous:

$$\begin{aligned} S_{11}^N(j,t) = \langle r_j(t) r_0(0) \rangle &= \frac{1}{N}\left\langle \sum_{k,l=1}^{N-1} \widehat{r}_k(t)\widehat{r}_l(0) e^{2\pi\imath\frac{jk}{N}} \right\rangle \\ &= \frac{1}{N\beta}\sum_{l=1}^{N-1} \cos(|\omega_l|t)\cos\left(2\pi\frac{lj}{N}\right) = S_{22}^N(j,t)\,. \end{aligned} \tag{1.24}$$

In the same way we have that:

$$
\begin{aligned}
S_{12}^N(j,t) &= \frac{1}{N\beta}\sum_{l=1}^{N-1}\sin(|\omega_l|t)\cos\left(2\pi\frac{lj}{N}+\arg(\omega_l)\right)\\
S_{21}^N(j,t) &= -\frac{1}{N\beta}\sum_{l=1}^{N-1}\sin(|\omega_l|t)\cos\left(2\pi\frac{lj}{N}-\arg(\omega_l)\right) && (1.25)\\
S_{31}^N(j,t) &= S_{32}^N(j,t)=S_{13}^N(j,t)=S_{23}^N(j,t)=0 && (1.26)\\
S_{33}^N(j,t) &= \frac{1}{2}((S_{11}^N)^2+(S_{22}^N)^2+(S_{12}^N)^2+(S_{21}^N)^2)+\frac{3(N-1)}{2N^2\beta^2}\,. && (1.27)
\end{aligned}
$$

The dispersion relation given by (1.20) takes the form

$$
\begin{aligned}
\omega_\ell &= -\sum_{s=1}^{m}\tau_s\left(1-\cos\left(2\pi\frac{s\ell}{N}\right)\right)+\mathrm{i}\sum_{s=1}^{m}\tau_s\sin\left(2\pi\frac{s\ell}{N}\right)\\
|\omega_\ell|^2 &= \sum_{s=0}^{N-1}a_s e^{-2\pi\mathrm{i}\frac{s\ell}{N}} = 2\sum_{s=1}^{m}\kappa_s\left(1-\cos\left(2\pi\frac{s\ell}{N}\right)\right)\,,
\end{aligned}
$$

where we substitute for the a_s their values as in (1.4). We are interested in obtaining the continuum limit of the above correlation functions. We first define $\omega(k)$ to provide continuum limits of ω_ℓ and $|\omega_\ell|^2$, namely

$$
\begin{aligned}
\omega(k) &:= -\sum_{s=1}^{m}\tau_s\left(1-\cos\left(2\pi sk\right)\right)+\mathrm{i}\sum_{s=1}^{m}\tau_s\sin\left(2\pi sk\right)\\
|\omega(k)|^2 &= 2\sum_{s=1}^{m}\kappa_s\left(1-\cos(2\pi ks)\right)\,,
\end{aligned}
\tag{1.28}
$$

where the variable ℓ/N has been approximated with $k\in[0,1]$. One may use equation (1.17) to check the consistency of the two equations of (1.28). To this end observe that $\omega(k)=Q(e^{-2\pi ik})$, $\overline{\omega(k)}=Q(e^{2\pi ik})$, and $|\omega(k)|^2=\ell(e^{2\pi ik})$, we recall that $Q(z),\ell(z)$ are defined in (1.18).

Lemma 1.3. *Let $\omega(k)$ be defined as in (1.28), set $f(k):=|\omega(k)|$, and denote $\theta(k):=arg(\omega(k))$ for $0\leqslant k\leqslant 1$, where the ambiguity in the definition of θ is settled by requiring θ to be continuous with $\theta(0)\in(-\pi,\pi]$. Then, for all $k\in[0,1]$ we have*

$$
\begin{aligned}
\omega(1-k) &= \overline{\omega(k)},\\
f(1-k) &= f(k), && (1.29)\\
\theta(1-k) &\equiv -\theta(k) \quad (mod\ 2\pi). && (1.30)
\end{aligned}
$$

Furthemore, the functions f and $\theta-\frac{\pi}{2}$ are C^∞ on $[0,1]$ and they both possess odd C^∞-extensions at $k=0$ which implies in particular $\theta(0)=\frac{\pi}{2}$.

Proof. The symmetries follow directly from the definition of ω in (1.28). From (1.28) we also learn that $|\omega(k)|^2\geqslant 2\kappa_1(1-\cos(2\pi k))>0$ for $k\in(0,1)$. Thus the smoothness of f and θ only needs to be investigated for $k\in\{0,1\}$. By symmetry we only need to study the case $k=0$. The smoothness of the function θ may be obtained from the expansion near $k=0$

$$
\cot(\theta(k)) = -k\pi\frac{\sum_{s=1}^m s^2\tau_s}{\sum_{s=1}^m s\tau_s}+\mathcal{O}(k^3)
$$

together with $\sum_{s=1}^{m} s\tau_s > 0$ (see Proposition 1.2). Since $\cot(\theta(0)) = 0$ and $\Im\omega(k) > 0$ for small positive values of k we conclude that $\theta(0) = \frac{\pi}{2}$ from the requirement $\theta(0) \in (-\pi, \pi]$. This also implies the existence of a smooth odd extension of $\theta - \frac{\pi}{2}$ at $k = 0$ because $\cot(\theta(k))$ has such an extension. For the function f the claims follow from the representation

$$f(k) = 2\pi k \left(\sum_{s=1}^{m} s^2 \kappa_s \operatorname{sinc}^2(\pi s k) \right)^{1/2}$$

near $k = 0$ where $\operatorname{sinc}(x) = \frac{\sin(x)}{x}$ denotes the smooth and even sinus cardinalis function. □

Lemma 1.4. *In the limit $N \to \infty$ the correlation functions have the following expansion*

$$S^N_{\alpha\alpha'}(j,t) + \frac{\delta_{\alpha\alpha'}}{N\beta} = S_{\alpha\alpha'}(j,t) + \mathcal{O}\left(N^{-\infty}\right), \quad \alpha, \alpha' = 1, 2,$$
$$S^N_{33}(j,t) = S_{33}(j,t) + \mathcal{O}\left(N^{-1}\right),$$

where $\delta_{\alpha\alpha'}$ denotes the Kronecker delta,

$$S_{11}(j,t) = S_{22}(j,t) = \frac{1}{\beta} \int_0^1 \cos\left(|\omega(k)|t\right) \cos\left(2\pi k j\right) \mathrm{d}k \tag{1.31}$$

$$S_{12}(j,t) = \frac{1}{\beta} \int_0^1 \sin\left(|\omega(k)|t\right) \cos\left(2\pi k j + \theta(k)\right) \mathrm{d}k,$$

$$S_{21}(j,t) = -\frac{1}{\beta} \int_0^1 \sin\left(|\omega(k)|t\right) \cos\left(2\pi k j - \theta(k)\right) \mathrm{d}k, \tag{1.32}$$

$$S_{33}(j,t) = \frac{1}{2}(S_{11}^2 + S_{22}^2 + S_{12}^2 + S_{21}^2), \tag{1.33}$$

and $\theta(k) = \arg\omega(k)$ with $\omega(k)$ as in (1.28)*.*

Proof. For any periodic C^∞-function g on the real line with period 1, $g(k) = \sum_{n\in\mathbb{Z}} \hat{g}_n e^{2\pi i k n}$, one has

$$\frac{1}{N} \sum_{\ell=0}^{N-1} g\left(\frac{\ell}{N}\right) = \sum_{m\in\mathbb{Z}} \hat{g}_{mN} = \int_0^1 g(k)\mathrm{d}k + \mathcal{O}\left(N^{-\infty}\right).$$

It follows from Lemma 1.3 that the integrands in (1.31)-(1.32) can be extended to 1-periodic smooth functions because we have for small positive values of k that

$$\begin{aligned}
\cos\left(f(-k)t\right)\cos\left(-2\pi k j\right) &= \cos\left(f(k)t\right)\cos\left(-2\pi k j\right) = \cos\left(f(1-k)t\right)\cos\left(2\pi(1-k)j\right), \\
\sin\left(f(-k)t\right)\cos\left(-2\pi k j \pm \theta(-k)\right) &= -\sin\left(f(k)t\right)\cos\left(-2\pi k j \pm (\pi - \theta(k))\right) \\
&= \sin\left(f(1-k)t\right)\cos\left(2\pi(1-k)j \pm \theta(1-k)\right).
\end{aligned}$$

Observing in addition that the summands corresponding to $\ell = 0$ are missing in (1.24)-(1.25) the first claim is proved. Together with (1.27) this also implies the second claim. □

Next we analyse the leading order behaviour (as $t \to \infty$) of the limiting correlation functions $S_{\alpha\alpha'}(j,t)$ using the method of steepest descent. In order to explain the phenomena that may occur we start by discussing S_{11}. Denote

$$\xi := \frac{j}{t} \quad \text{and} \quad \phi_\pm(k,\xi) := f(k) \pm 2\pi\xi k\,. \tag{1.34}$$

With these definitions and using the symmetry (1.29) we may write

$$S_{11}(j,t) = \frac{1}{2\beta}\Re\int_0^1 \left(e^{it(f(k)+2\pi\xi k)} + e^{it(f(k)-2\pi\xi k)}\right) dk = \frac{1}{\beta}\Re\int_0^1 e^{it\phi_-(k,\xi)}dk\,. \tag{1.35}$$

The leading order behaviour ($t \to \infty$) of such an integral is determined by the stationary phase points $k_0 \in [0,1]$, i.e. by the solutions of the equation $\frac{\partial}{\partial k}\phi_-(k_0,\xi) = 0$ which depend on the value of ξ.

Such stationary phase points do not need to exist. In fact, as we see in Lemma 1.5 b) below, the range of f' is given by some interval $[-2\pi v_0, 2\pi v_0]$ so that there are no stationary phase points for $|\xi| > v_0$. As in the proof of Lemma 1.4 one can argue that the integrand $\Re e^{it\phi_-(k,j/t)}$ can be extended to a periodic smooth function of k on the real line with period 1. It then follows from integration by parts that $S_{11}(j,t)$ decays rapidly in time. More precisely, for every fixed $\delta > 0$ we have

$$S_{11}(j,t) = \mathcal{O}\left(t^{-\infty}\right) \quad \text{as } t \to \infty, \text{ uniformly for } |j| \geqslant (v_0+\delta)t. \tag{1.36}$$

This justifies the name of sound speed for the quantity v_0.

In the case $|\xi| \leqslant v_0$ there always exists at least one stationary phase point $k_0 = k_0(\xi) \in [0,1]$. Each stationary phase point may provide an additive contribution to the leading order behaviour of $\int_0^1 e^{it\phi_-(k,j/t)}dk$ for j near ξt. However, the order of the contribution depends on the multiplicity of the stationary phase point. For example, let k_0 be a stationary phase point of $\phi_-(\cdot,\xi)$, i.e. $\frac{\partial}{\partial k}\phi_-(k_0,\xi) = 0$. Denote by ℓ the smallest integer bigger than 1 for which $\frac{\partial^\ell}{\partial k^\ell}\phi_-(k_0,\xi) \neq 0$. Then k_0 contributes a term of order $t^{1/\ell}$ to the t-asymptotics of $\int_0^1 e^{it\phi_-(k,j/t)}dk$ for j in a suitable neighbourhood of ξt.

Before treating the general situation let us recall the case of nearest neighbour interactions. There we have

$$f(k) = f_1(k) = \sqrt{2\kappa_1(1-\cos(2\pi k))} = 2\sqrt{\kappa_1}\sin(\pi k)\,, \quad k \in [0,1]\,.$$

The range of f_1' equals $[-2\pi v_0, 2\pi v_0]$ with $v_0 = \sqrt{\kappa_1}$. For every $|\xi| \leqslant v_0$ there exists exactly one stationary phase point $k_0(\xi) \in [0,1]$ of $\phi_-(\cdot,\xi)$ that is determined by the relation $\cos(\pi k_0(\xi)) = \xi/v_0$. A straight forward calculation gives

$$\frac{\partial^2}{\partial k^2}\phi_-(k_0(\xi),\xi) = f_1''(k_0(\xi)) = -2\pi^2\sqrt{v_0^2-\xi^2} = 0 \;\Leftrightarrow\; \xi = \pm v_0\,.$$

Moreover, we have $k_0(v_0) = 0$ and $k_0(-v_0) = 1$ and therefore $\frac{\partial^3}{\partial k^3}\phi_-(k_0(\pm v_0),\pm v_0) = \mp 2\pi^3 v_0 \neq 0$. This implies that in addition to (1.36) we have $S_{11}(j,t) = \mathcal{O}(t^{-1/2})$, except for j near $\pm v_0 t$ where $S_{11}(j,t) = \mathcal{O}(t^{-1/3})$. In order to determine the behaviour near the least decaying peaks that travel at speeds $\pm v_0$ we expand f_1 near the stationary phase points. Let us first consider $\xi = v_0$ with $k_0 = 0$. Introducing $\lambda_0 = \frac{1}{2\pi}|f_1'''(0)/2|^{1/3} = \frac{1}{2}v_0^{1/3}$ we obtain

$$f_1(k) = 2\pi v_0 k - \frac{1}{3}(2\pi\lambda_0 k)^3 + \mathcal{O}(k^5)\,, \quad \text{as } k \to 0.$$

Substituting $y = 2\pi\lambda_0 t^{1/3}k$ leads for k close to 0 to the asymptotic expression

$$t\phi_-(k,\, j/t) = \frac{v_0 t - j}{\lambda_0 t^{1/3}}y - \frac{1}{3}y^3 + \mathcal{O}(t^{-2/3})\,, \quad \text{as } t \to \infty.$$

Using the well-known representation $\mathrm{Ai}(w) = \frac{1}{\pi}\int_0^\infty \cos(y^3/3 + wy)dy$, $w \in \mathbb{R}$, of the Airy function and performing a similar analysis around the stationary phase point $k_0 = -1$ for $\xi = -v_0$ one obtains an asymptotic formula for the region not covered by (1.36)

$$S_{11}(j,t) = \frac{1}{2\beta\lambda_0 t^{1/3}}\left[\mathrm{Ai}\left(\frac{j - v_0 t}{\lambda_0 t^{1/3}}\right) + \mathrm{Ai}\left(-\frac{j + v_0 t}{\lambda_0 t^{1/3}}\right)\right] + \mathcal{O}\big(t^{-1/2}\big)\,,\ t \to \infty, \quad (1.37)$$

uniformly for $|j| < (v_0+\delta)t$, for $\delta > 0$ (see e.g. [122]). Observe that due to the decay of $\mathrm{Ai}(w)$ for $w \to \pm\infty$, the Airy term is dominant roughly in the regions described by $v_0 t - o(t) < |j| < v_0 t + o((\ln t)^{2/3})$.

From the arguments just presented it is not difficult to see that the derivation of (1.37) only uses the following properties of $f = f_1$:

$$f''(k) < 0 \quad \text{for all} \quad 0 < k \leqslant \frac{1}{2}\,, \qquad (1.38)$$

together with

$$f''(0) = 0\,, \quad f'''(0) < 0\,, \quad \text{and} \quad f(1-k) = f(k) \quad \text{for all} \quad 0 \leqslant k < \frac{1}{2}\,. \qquad (1.39)$$

Conditions (1.38) and (1.39) imply that statements (1.36) and (1.37) hold with $v_0 = \frac{f'(0)}{2\pi} > 0$ and $\lambda_0 = \frac{1}{2\pi}|f'''(0)/2|^{1/3}$.

It follows from equation (1.29) and from statement a) of Lemma 1.5 below that the conditions of (1.39) are always satisfied in our model. Condition (1.38), however, might fail. Indeed, it is not hard to see that there exist open regions in the $\boldsymbol{\kappa}$-space $\mathbb{R}_+^m$ where there always exist stationary phase points $k_0 \in (0,1)$ of higher multiplicity, i.e. with $f''(k_0) = 0$. In this situation the value of $v := \frac{f'(k_0)}{2\pi}$ lies in the open interval $(-v_0, v_0)$ (cf. Lemma 1.5 b). Then the decay rate of $S_{11}(j,t)$ for j near vt is at most of order $t^{-1/3}$. The decay is even slower (at least of order $t^{-1/4}$) if $f'''(k_0) = 0$ holds in addition. We show in Theorem 1.7 that this may happen for $\boldsymbol{\kappa}$ in some submanifold of $\mathbb{R}_+^m$ of codimension 1 (see also Examples 1.8 and 1.9). Nevertheless, if $\kappa_2, \ldots, \kappa_m$ are sufficiently small in comparison to κ_1 then condition (1.38) is always satisfied as we show in Theorem 1.6 c).

Before stating our main results of this section, Theorems 1.6 and 1.7, we first summarize some more properties of the function f.

Lemma 1.5. *Given $(\kappa_1, \ldots, \kappa_m)$ with $\kappa_1 > 0$, $\kappa_m > 0$, and $\kappa_j \geqslant 0$ for $1 < j < m$. Denote $f(k) = |\omega(k)|$ for $0 \leqslant k \leqslant 1$ as introduced in Lemma 1.3 and define $v_0 := (\sum_{s=1}^m s^2\kappa_s)^{\frac{1}{2}}$. Then the following holds:*

a) $f(0) = f''(0) = 0\,,\ f'(0) = 2\pi v_0\,,$ *and* $f'''(0) = -\frac{2\pi^3}{v_0}\sum_{s=1}^m s^4\kappa_s\,.$

b) $f'([0,1]) = [-2\pi v_0, 2\pi v_0]$. *$f'$ attains its maximum only at $k = 0$ and its minimum only at $k = 1$.*

c) Fix $\kappa_1 > 0$. Then the map f can be extended as a C^∞-function of the variables $(k, \kappa_2, \dots, \kappa_m)$ on the set $[0,1] \times [0,\infty)^{m-1}$.

Proof. Statement a) follows directly from the last formula in the proof of Lemma 1.3 and from the expansion $\operatorname{sinc}^2(x) = 1 - \frac{x^2}{3} + \mathcal{O}(x^4)$ for small values of x:

$$f(k) = 2\pi k\left(\sum_{s=1}^m s^2\kappa_s\,\operatorname{sinc}^2(\pi s k)\right)^{1/2} = 2\pi v_0 k - \frac{\pi^3}{3v_0}\left(\sum_{s=1}^m s^4\kappa_s\right)k^3 + \mathcal{O}(k^5)\,.$$

This representation also settles statement c). As we know already $f'(0) = 2\pi v_0 = -f'(1)$ we may establish statement b) by verifying that $|f'(k)| < 2\pi v_0$ holds for all $k \in (0,1)$. To this end we write $f = (\sum_{s=1}^m h_s^2)^{1/2}$ with $h_s(k) = 2\sqrt{\kappa_s}\sin(\pi s k)$. Using the Cauchy-Schwarz inequality we obtain for $0 < k < 1$ that

$$|f'(k)| = \frac{|\sum_{s=1}^m h_s(k)h_s'(k)|}{\left(\sum_{s=1}^m h_s^2(k)\right)^{1/2}} \leqslant \left(\sum_{s=1}^m (h_s')^2(k)\right)^{1/2} = 2\pi\left(\sum_{s=1}^m s^2\kappa_s\cos^2(\pi s k)\right)^{1/2} < 2\pi v_0\,,$$

where the last inequality follows from $|\cos(\pi k)| < 1$ and $\kappa_1 > 0$. □

We are now ready to state our first main result in this section.

Theorem 1.6. *Let $m \in \mathbb{N}$, fix $\delta > 0$, denote $f(k) = |\omega(k)|$ as introduced in Lemma 1.3, and set*

$$v_0 := \sqrt{\sum_{s=1}^m s^2\kappa_s}, \quad \lambda_0 := \frac{1}{2}\left(\frac{1}{v_0}\sum_{s=1}^m s^4\kappa_s\right)^{1/3}. \tag{1.40}$$

a) For all α, $\alpha' = 1,2,3$ we have rapid decay as $t \to \infty$, uniformly for $|j| > (v_0 + \delta)t$, i.e.

$$S_{\alpha\alpha'}(j,t) = \mathcal{O}\left(t^{-\infty}\right).$$

b) If $f''(k) < 0$ for all $0 < k \leqslant 1/2$ then as $t \to \infty$ the following holds uniformly for $|j| < (v_0 + \delta)t$:

$$S_{11}(j,t) = \frac{1}{2\beta\lambda_0 t^{1/3}}\left[Ai\left(\frac{j - v_0 t}{\lambda_0 t^{1/3}}\right) + Ai\left(-\frac{j + v_0 t}{\lambda_0 t^{1/3}}\right)\right] + \mathcal{O}\left(t^{-1/2}\right) = S_{22}(j,t)\,, \tag{1.41}$$

$$S_{12}(j,t) = \frac{1}{2\lambda_0 t^{1/3}\beta}\left(Ai\left(-\frac{j + v_0 t}{\lambda_0 t^{1/3}}\right) - Ai\left(\frac{j - v_0 t}{\lambda_0 t^{1/3}}\right)\right) + \mathcal{O}(t^{-\frac{1}{2}}) = S_{21}(j,t)\,,$$

$$S_{33}(j,t) = \frac{1}{2\beta^2\lambda_0^2 t^{2/3}}\left[Ai^2\left(\frac{j - v_0 t}{\lambda_0 t^{1/3}}\right) + Ai^2\left(-\frac{j + v_0 t}{\lambda_0 t^{1/3}}\right)\right] + \mathcal{O}\left(t^{-5/6}\right).$$

c) For every $\kappa_1 > 0$ there exists $\varepsilon = \varepsilon(\kappa_1) > 0$ such that for all $(\kappa_2, \dots, \kappa_m) \in [0,\varepsilon)^{m-1}$ we have $f''(k) < 0$ for all $0 < k \leqslant 1/2$.

Proof. The rapid decay claimed in statement a) can be argued in the same way as (1.36) for $S_{11} = S_{22}$. Due to relations (1.26) and (1.33) one only needs to consider S_{12} and S_{21}. Indeed, using Lemma 1.3 one may show that the imaginary parts of the integrands used in the representation of S_{12} and S_{21} in (1.42) below have smooth extensions to all $k \in \mathbb{R}$ that are 1-periodic. This is all that is needed because $|\frac{\partial}{\partial k}\phi_\pm(k, j/t)| > 2\pi\delta$ by Lemma 1.5 b) uniformly for $k \in [0, 1/2]$ and $|j| > (v_0 + \delta)t$.

We have already argued above that conditions (1.38), (1.39) suffice to derive the first claim of statement b) with $v_0 = \frac{f'(0)}{2\pi} > 0$ and $\lambda_0 = \frac{1}{2\pi}|f'''(0)/2|^{1/3}$. The expressions for $f'(0)$ and $f'''(0)$ stated in Lemma 1.5 a) justify the definitions of (1.40).

Using the symmetry relations (1.29) and (1.30) we derive a representation for S_{12} and S_{21} that is suitable for a steepest descent analysis

$$\begin{aligned} S_{12}(j,t) =& \frac{1}{\beta}\int_0^{1/2}\Big(\sin(f(k)t - 2\pi kj - \theta(k)) + \sin(f(k)t + 2\pi kj + \theta(k))\Big)\mathrm{d}k \\ =& \frac{1}{\beta}\Im\int_0^{1/2}\Big(e^{it\phi_-(k,j/t)}e^{-i\theta(k)} + e^{it\phi_+(k,j/t)}e^{i\theta(k)}\Big)dk \\ S_{21}(j,k) =& -\frac{1}{\beta}\Im\int_0^{1/2}\Big(e^{it\phi_-(k,j/t)}e^{i\theta(k)} + e^{it\phi_+(k,j/t)}e^{-i\theta(k)}\Big)dk \end{aligned} \tag{1.42}$$

where $\phi_\pm(k,\xi) = f(k) \pm 2\pi\xi k$ as in (1.34) above. Expanding for k close to zero one obtains $\phi_\pm(k, j/t) = 2\pi v_0 k - \frac{1}{3}(2\pi)^3\lambda_0^3 k^3 \pm 2\pi k\frac{j}{t} + \mathcal{O}(k^5)$. Substituting $y = 2\pi\lambda_0 t^{1/3}k$ leads to the asymptotic expression

$$t\phi_\pm(k, j/t) = \frac{v_0 t \pm j}{\lambda_0 t^{\frac{1}{3}}}y - \frac{1}{3}y^3 + \mathcal{O}(t^{-\frac{2}{3}}) \quad \text{as } t \to \infty.$$

Keeping in mind that $\theta(0) = \frac{\pi}{2}$ we obtain

$$S_{12}(j,t) \;=\; \frac{1}{2\lambda_0 t^{1/3}\beta}\left(\mathrm{Ai}\left(-\frac{j+v_0t}{\lambda_0 t^{1/3}}\right) - \mathrm{Ai}\left(\frac{j-v_0t}{\lambda_0 t^{1/3}}\right)\right) + \mathcal{O}(t^{-\frac{1}{2}}) = S_{21}(j,t)\,.$$

Regarding the expansion for $t \to \infty$ of $S_{33}(j,t)$ it follows immediately from the expression (1.33) and the expansions of $S_{\alpha\alpha'}(j,t)$ with $\alpha, \alpha' = 1, 2$.

Statement c) follows from the continuous dependence of the derivatives f'' and f''' on the parameters $(\kappa_2, \ldots, \kappa_m)$ (see Lemma 1.5 c) and from simple facts for the case of nearest neighbour interactions $f_1(k) = 2\sqrt{\kappa_1}\sin(\pi k)$ discussed above. Indeed, from $f''(0) = 0$ and from $f_1'''(0) < 0$ it follows that there exists such an $\varepsilon > 0$ such that $f'''(k) < 0$ and hence also $f''(k) < 0$ for k in some region $(0, \delta)$ uniformly in $(\kappa_2, \ldots, \kappa_m) \in [0, \varepsilon)^{m-1}$. As $f_1''(k) < -2\pi^2\sqrt{\kappa_1}\sin(\pi\delta)$ for all $k \in [\delta, 1/2]$ we may prove the claim in this region by reducing the value of ε if necessary. □

Theorem 1.6 provides the leading order asymptotics of the limiting correlations $S_{\alpha\alpha'}(j,t)$ for $t \to \infty$ in the simple situation that the second derivative of the dispersion relation is strictly negative on the open interval $(0, 1)$ (cf. condition (1.38)). Moreover, statement c) shows that there is a set of positive measure in parameter space $\boldsymbol{\kappa} \in \mathbb{R}_+^m$ where this happens. For general values of $\boldsymbol{\kappa}$, however, different phenomena may appear. In particular, there might exist stationary phase points of higher order leading to slower time-decay of the correlations (see discussion before the statement of Lemma 1.5). By a naive count of variables and equations one might expect that decay rates $t^{-1/(3+p)}$ occur on submanifolds of parameter space of dimension $m-p$. Theorem 1.7 shows that this is indeed the case for $p=1$. Moreover, we present in this situation a formula for the leading order contribution of the corresponding stationary phase points to the asymptotics of $S_{\alpha\alpha'}(j,t)$. Despite being non-generic in parameter space it is interesting to note that decay rates $t^{-1/4}$ can be observed numerically (see Figures 1.2 and 1.3). There is also a second issue that may arise if condition (1.38) fails. Namely, for $v \in (-v_0, v_0)$ there can be several values of $k \in (0, \frac{1}{2}]$ satisfying $f'(k) \pm 2\pi v = 0$ so that the contributions from all these stationary points need to be added to describe the leading order behaviour for j near vt.

Theorem 1.7. *Recall from* (1.28) *the formula for the dispersion relation*

$$f(k) = |\omega(k)| = \sqrt{2\sum_{s=1}^{m} \kappa_s \left(1-\cos(2\pi k s)\right)}\,.$$

a) For $m \geqslant 3$ there is an $(m-1)$-parameter family of potentials for which there exist $k^* = k^*(\boldsymbol{\kappa}) \in (0, \frac{1}{2})$ with

$$f''(k^*) = 0, \quad f'''(k^*) = 0, \quad f^{(iv)}(k^*) \neq 0, \quad \textit{and} \quad 0 < v^* := \frac{f'(k^*)}{2\pi} < v_0, \tag{1.43}$$

with v_0 as in (1.40)*. Set $\lambda^* := \frac{1}{2\pi}(|f^{(iv)}(k^*)|/4!)^{\frac{1}{4}} > 0$. Then for $j \to \infty$ and $t \to \infty$ in such a way that*

$$\frac{v^*t - j}{\lambda^* t^{\frac{1}{4}}}$$

is bounded, the contribution of the stationary phase point k^* to the correlation functions is given by:

$$S_{11}(j,t), S_{22}(j,t): \ \frac{1}{2\beta\pi\lambda^* t^{\frac{1}{4}}} \Re\left(e^{it\phi_-(k^*,j/t)} \mathcal{P}_\pm\left(\frac{v^*t-j}{\lambda^* t^{\frac{1}{4}}}\right)\right) + \mathcal{O}(t^{-\frac{1}{2}}), \tag{1.44}$$

$$S_{12}(j,t): \ \frac{1}{2\beta\pi\lambda^* t^{\frac{1}{4}}} \Im\left(e^{it\phi_-(k^*,j/t) - i\theta(k^*)} \mathcal{P}_\pm\left(\frac{v^*t-j}{\lambda^* t^{\frac{1}{4}}}\right)\right) + \mathcal{O}(t^{-\frac{1}{2}}), \tag{1.45}$$

$$S_{21}(j,t): \ -\frac{1}{2\beta\pi\lambda^* t^{\frac{1}{4}}} \Im\left(e^{it\phi_-(k^*,j/t) + i\theta(k^*)} \mathcal{P}_\pm\left(\frac{v^*t-j}{\lambda^* t^{\frac{1}{4}}}\right)\right) + \mathcal{O}(t^{-\frac{1}{2}}), \tag{1.46}$$

where $\phi_\pm(k,\xi) = f(k) \pm 2\pi\xi k$, $\theta(k) = \arg\omega(k)$ as defined in Lemma 1.3, $\mathcal{P}_\pm(a)$ denote the Pearcey integrals, [38],

$$\mathcal{P}_\pm(a) = \int_{-\infty}^{\infty} e^{i(\pm y^4 + ay)} dy, \qquad a \in \mathbb{R}, \tag{1.47}$$

and $\mathcal{P}_\pm$ has to be chosen according to the sign of $f^{(iv)}(k^*)$. If $j \to -\infty$ with bounded $(v^*t + j)/(\lambda^* t^{1/4})$ the contributions of the stationary point k^* can be obtained from the ones presented in (1.44)-(1.46) *by replacing ϕ_- by ϕ_+, θ by $-\theta$, and j in the argument of $\mathcal{P}_\pm$ by $-j$.*
b) When $k^* = \frac{1}{2}$ one has $f'(1/2) = 0$ and $f'''(1/2) = 0$ by the symmetry (1.29)*. For each $m \geqslant 2$ there is an $(m-1)$-parameter family of potentials so that $f''(1/2) = 0$*

and $f^{(iv)}(1/2) \neq 0$ holds in addition. In this case the contribution of the stationary phase point $k^* = 1/2$ to the correlation functions in the asymptotic regime $t \to \infty$ with bounded $j/t^{\frac{1}{4}}$ is given by (λ^* defined as in statement a) with $k^* = \frac{1}{2}$)

$$\begin{aligned} S_{12}(j,t), S_{21}(j,t) \ &: - \, sgn \, (\sum_{s \, odd} \tau_s) \frac{(-1)^j}{2\beta\pi\lambda^* t^{\frac{1}{4}}} \Im \left(e^{itf(\frac{1}{2})} \mathcal{P}_{\pm} \left(\frac{j}{\lambda^* t^{\frac{1}{4}}} \right) \right) + \mathcal{O}(t^{-\frac{1}{2}}) \\ S_{11}(j,t), S_{22}(j,t) \ &: \frac{(-1)^j}{2\beta\pi\lambda^* t^{\frac{1}{4}}} \Re \left(e^{itf(\frac{1}{2})} \mathcal{P}_{\pm} \left(\frac{j}{\lambda^* t^{\frac{1}{4}}} \right) \right) + \mathcal{O}(t^{-\frac{1}{2}}) \\ S_{33}(j,t) \ &: \frac{1}{4\beta^2\pi^2(\lambda^*)^2 t^{\frac{1}{2}}} \left| \mathcal{P}_{\pm} \left(\frac{j}{\lambda^* t^{\frac{1}{4}}} \right) \right|^2 + \mathcal{O}(t^{-\frac{3}{4}}) \, . \end{aligned} \tag{1.48}$$

Proof. We begin by proving formula (1.44) for the momentum or position correlations $S_{22}(j,t) = S_{11}(j,t)$ under the assumption that we have found a $k^* \in (0, 1/2)$ for which all the relations of (1.43) are satisfied. From (1.35) and Lemma 1.3 we obtain

$$S_{11}(j,t) = S_{22}(j,t) = \frac{1}{\beta} \Re \int_0^{\frac{1}{2}} \left(e^{it(f(k) + 2\pi k \frac{j}{t})} + e^{it(f(k) - 2\pi k \frac{j}{t})} \right) dk. \tag{1.49}$$

In order to compute the contribution of the stationary phase point k^* to the large t asymptotics of the integral in (1.49) we expand

$$f(k) = f(k^*) + 2\pi v^*(k - k^*) + f^{(iv)}(k^*)(k - k^*)^4/4! + O((k - k^*)^5) \, .$$

Introducing the change of variables

$$y = 2\pi\lambda^*(k - k^*)t^{\frac{1}{4}}, \quad \lambda^* = \frac{1}{2\pi}(|f^{(iv)}(k^*)|/4!)^{\frac{1}{4}}$$

one obtains

$$tf(k) - 2\pi jk = tf(k^*) - 2\pi jk^* + y\frac{v^*t - j}{\lambda^* t^{\frac{1}{4}}} \pm y^4 + \mathcal{O}(t^{-\frac{1}{4}})$$

where the $\pm$ sign is determined by the sign of $f^{(iv)}(k^*)$. Then using the Pearcey integral (1.47), the expansion (1.44) can be derived in a straightforward way from (1.49). In a similar way the expansions (1.45) and (1.46) are obtained by applying the above analysis to the expression (1.42).

In the situation $k^* = 1/2$ of statement b) one uses in addition that $t\phi_{\pm}(1/2, j/t) = tf(1/2) \pm j\pi$, $\omega(1/2) = -\sum_{s=1}^m \tau_s(1 - \cos(\pi s)) = -2\sum_{s \, odd} \tau_s$, see (1.28), and consequently $e^{\pm i\theta(1/2)} = - \, \text{sgn}(\sum_{s \, odd} \tau_s)$. The leading order contribution of the stationary phase point $k^* = 1/2$ to the integral representation of, say, S_{12} in (1.42) is then given by

$$-\text{sgn} \left(\sum_{s \, odd} \tau_s \right) \frac{(-1)^j}{2\beta\pi\lambda^* t^{\frac{1}{4}}} \Im \left(e^{itf(\frac{1}{2})} \left(\int_{-\infty}^0 e^{i(\pm y^4 - wy)} dy + \int_{-\infty}^0 e^{i(\pm y^4 + wy)} dy \right) \right)$$

with $w = \frac{j}{\lambda^* t^{\frac{1}{4}}}$. In this way and with the help of (1.33) all relations of (1.48) can be deduced.

We now show the existence of a codimension 1 manifold in parameter space that exhibits such higher order stationary phase points in the situation of b) where $k^* = 1/2$. As we have $f'''(1/2) = 0$ by symmetry (1.29) we only need to solve

$$f''\left(\frac{1}{2}\right) = 0 \quad \text{which is equivalent to} \quad \sum_{s=1}^{m} s^2(-1)^{s+1}\kappa_s = 0\,. \tag{1.50}$$

The solution of the above equation is

$$\kappa_m = \frac{(-1)^m}{m^2}\sum_{s=1}^{m-1} s^2(-1)^{s+1}\kappa_s. \tag{1.51}$$

It is clear from the above relation that for m even, choosing κ_1 sufficiently big one has $\kappa_m > 0$ while for m odd, it is sufficient to choose $\kappa_{s+1} > \frac{s^2}{(s+1)^2}\kappa_s > 0$, s odd and $1 \leqslant s \leqslant m-2$. Note that in the situation of (1.51) $f^{(iv)}(\frac{1}{2}) \neq 0$ holds iff $\sum_{s=1}^{m} \kappa_s s^4(-1)^{s+1} \neq 0$. This condition simply removes an $(m-2)$-dimensional plane from our manifold (1.51) which defines a hyperplane in the positive cone of the m-dimensional parameter space. Therefore we have found an $(m-1)$-parameter family of potentials such that the correlation functions decay as in (1.48).

Finally, we show for $m \geqslant 4$ our claim about the solution set of (1.43). The case $m = 3$ is treated in Example 1.9. Our strategy is to first show that there exists a $\boldsymbol{\kappa}^*$ that satisfies $f''(1/4, \boldsymbol{\kappa}^*) = 0$, $f'''(1/4, \boldsymbol{\kappa}^*) = 0$, $f'(1/4, \boldsymbol{\kappa}^*) > 0$, and $f^{(iv)}(1/4, \boldsymbol{\kappa}^*) \neq 0$. We then invoke the Implicit Function Theorem to show the existence of the $(m-1)$-dimensional solution manifold, where the stationary phase point $k^* \sim 1/4$ may and will depend on the parameters. The conditions $f''(\frac{1}{4}, \boldsymbol{\kappa}) = 0$ and $f'''(\frac{1}{4}, \boldsymbol{\kappa}) = 0$ imply

$$f'''\left(\frac{1}{4}\right) = 0 \rightarrow \sum_{s\,odd}(-1)^{\frac{s-1}{2}} s^3\kappa_s = 0, \tag{1.52}$$

$$f''\left(\frac{1}{4}\right) = 0 \rightarrow \left(2\sum_{s\,odd}\kappa_s + 2\sum_{s\,even}\kappa_s(1-(-1)^{\frac{s}{2}})\right)\sum_{s\,even} s^2\kappa_s(-1)^{\frac{s}{2}} - \left(\sum_{s\,odd} s\kappa_s(-1)^{\frac{s-1}{2}}\right)^2 = 0\,. \tag{1.53}$$

One needs to treat the case m odd and even separately. Here we consider only the case m even. The odd case can be treated in a similar way. Equation (1.52) gives

$$\kappa_{m-1} = \frac{(-1)^{\frac{m}{2}}}{(m-1)^3}\sum_{s\,odd, s=1}^{m-3}(-1)^{\frac{s-1}{2}} s^3\kappa_s.$$

If $m = 2\ell$ with ℓ even, a positive solution κ_{m-1} exists, provided that κ_1 is sufficiently big. If $m = 2\ell$ with ℓ odd then one needs to require $0 < \kappa_s < \frac{(s+2)^3}{s^3}\kappa_{s+2}$ for $s = 1, 5, 9, \ldots, m-5$.
The equation (1.53) is a linear equation in κ_4 and we solve it for κ_4 obtaining

$$\kappa_4 = \frac{1}{32}\frac{\left(\sum_{s\,odd,s=1}^{m-3}\kappa_s(-1)^{\frac{s-1}{2}}s(1-\frac{s^2}{(m-1)^2})\right)^2}{\sum_{s\,odd,s=1}^{m-3}\kappa_s(1+\frac{s^3(-1)^{\frac{m+s-1}{2}}}{(m-1)^3}) + \sum_{s\,even,s=2}^{m}\kappa_s(1-(-1)^{\frac{s}{2}})} + \frac{1}{16}\sum_{s\,even,s\neq 4,s=2}^{m} s^2\kappa_s(-1)^{\frac{s-2}{2}}\,.$$

We observe that the first term in the above expression is always positive, while the second term is positive if we require that $\kappa_s > \frac{(s+2)^2}{s^2}\kappa_{s+2} > 0$ for $s = 6, 10, 14, \ldots, m-2$. The remaining two conditions $f'(1/4) > 0$ and $f^{(iv)}(1/4) \neq 0$ are easy to satisfy: The sign of $f'(1/4)$ agrees with the sign of $\sum_{s\,odd} s\kappa_s(-1)^{\frac{s-1}{2}}$ and can be made positive by choosing κ_1 sufficiently large. In the situation where (1.52) and (1.53) hold the fourth derivative $f^{(iv)}(1/4)$ does not vanish iff $\sum_{s\,even} s^4\kappa_s(-1)^{\frac{s}{2}} \neq 0$. This can be achieved by adjusting, for example, the value of κ_2. We have now shown that there exists $\boldsymbol{\kappa}^* \in \mathbb{R}_+^m$ such that the first four derivatives of f have all desired properties at $k = 1/4$. In order to obtain the $(m-1)$-dimensional solution manifold in parameter space, we apply the Implicit Function Theorem to $F(k, \boldsymbol{\kappa}) := (f''(k, \boldsymbol{\kappa}), f'''(k, \boldsymbol{\kappa}))$. By a straight forward computation on sees that

$$\det\left[\frac{\partial F}{\partial(k, \kappa_4)}(1/4, \boldsymbol{\kappa}^*)\right] = -f^{(iv)}(1/4, \boldsymbol{\kappa}^*)\,\frac{\partial f''}{\partial \kappa_4}(1/4, \boldsymbol{\kappa}^*) \neq 0\,.$$

We can therefore solve $F(k, \boldsymbol{\kappa}) = 0$ near $(1/4, \boldsymbol{\kappa}^*)$ by choosing (k, κ_4) as functions of the remaining parameters κ_j with $j \neq 4$.

□

Example 1.8. m **even**. *Choosing* $\kappa_s = \frac{1}{s^2}$ *for* $s = 1, \ldots, m$ *one has that conditions* (1.50) *are satisfied and* $f^{(iv)}\left(\frac{1}{2}\right) < 0$.

For $\kappa_s = \frac{1}{s^\alpha}$, $s = 1, \ldots, m-1$, $2 < \alpha < 3$, *and* κ_m *given by* (1.51), *there is* $\alpha = \alpha(m)$ *such that* $\kappa_m < \kappa_{m-1}$.
m **odd**. *Choosing* $\kappa_s = \frac{1}{s}$, *for* $s = 1, \ldots m-1$, *one has from* (1.51) $\kappa_m = \frac{m-1}{2m^2} < \kappa_{m-1}$ *and* $f^{(iv)}(\frac{1}{2}) > 0$.
In all these examples the correlation functions $S_{\alpha\alpha'}(j, t)$, $\alpha, \alpha' = 1, 2$ *decrease as* $t^{-\frac{1}{4}}$ *near* $j = 0$.

Example 1.9. *We consider the case* $m = 3$ *and we want to get a potential that satisfies* (1.43) *with* $v^* > 0$. *We chose as a critical point of* $f(k)$ *the point* $k^* = \frac{1}{3}$ *thus obtaining the equations*

$$\kappa_2 = \frac{1}{8}\kappa_1, \quad \kappa_3 = \frac{7}{72}\kappa_1\,.$$

The speed of the peak is $v^* = \dfrac{\sqrt{2\kappa_1}}{4}$ *and* $f^{(iv)}(\frac{1}{3}) = -\frac{68\sqrt{6}}{6}\pi^4\sqrt{\kappa_1}$.

The correlation functions $S_{\alpha\alpha'}(j, t)$, $\alpha, \alpha' = 1, 2$ *decrease as* $t^{-\frac{1}{4}}$ *and* $S_{33}(j, t)$ *decreases like* $t^{-\frac{1}{2}}$ *as* $t \to \infty$ *and* $j \sim v^* t$, *see Figure 1.3. Note that one may obtain a 2-parameter family of solutions of* (1.43) *by picking, for example, the particular solution related to* $\kappa_1 = 1$ *and by showing that the system of equations* $(f'', f''')(k, \boldsymbol{\kappa}) = 0$ *can be solved near (1/3, 1, 1/8, 7/72) by choosing* k *and* κ_3 *as functions of* κ_1 *and* κ_2 *using the Implicit Function Theorem in the same way as at the end of the proof of Theorem 1.7.*

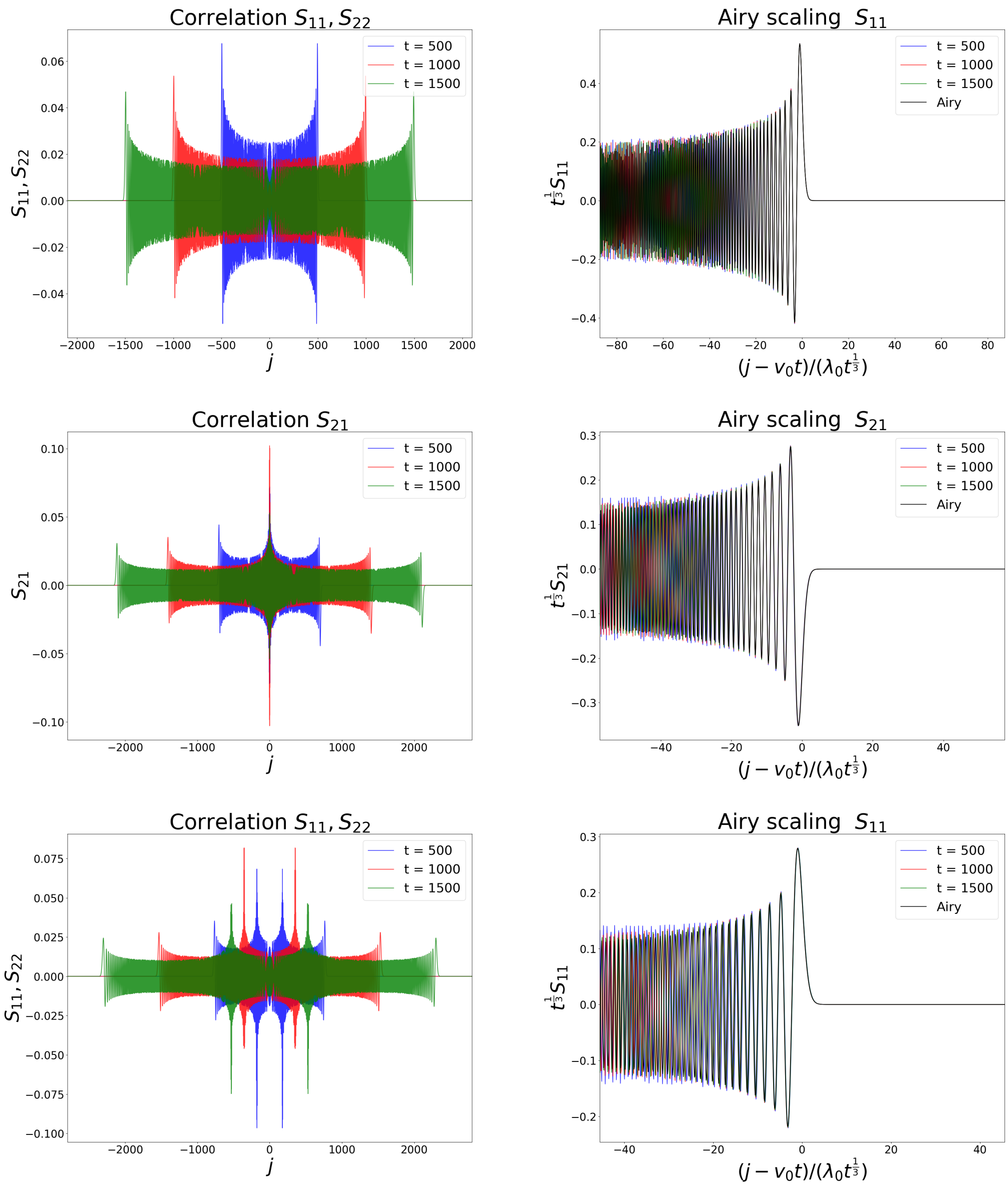


Figure 1.1: Correlation functions $S_{\alpha\alpha'}$ for the harmonic oscillator with nearest neighbour interaction with $\kappa_1 = 1$ (top left) and the harmonic potential with $\kappa_s = \frac{1}{s^2}$ for $s = 1, 2$ in Example1.8 (center left) and the potential of Example 1.9 in the bottom left. In the second column the Airy scaling (1.41) of the corresponding fastest moving peaks. The Airy asymptotic is perfectly matching the fastest peak and capturing several oscillations.

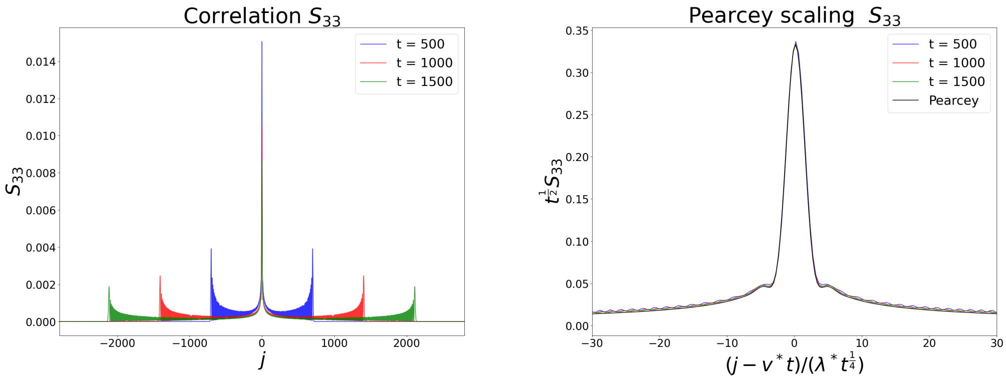


Figure 1.2: Correlation function $S_{33}(j,t)$ for the potential $\kappa_s = 1/s^2$ for $m = 2$ in Example 1.8 for several values of time on the left. On the right one sees that the Pearcey scaling provided in (1.48) matches perfectly for the central peak of $S_{33}(j,t)$.

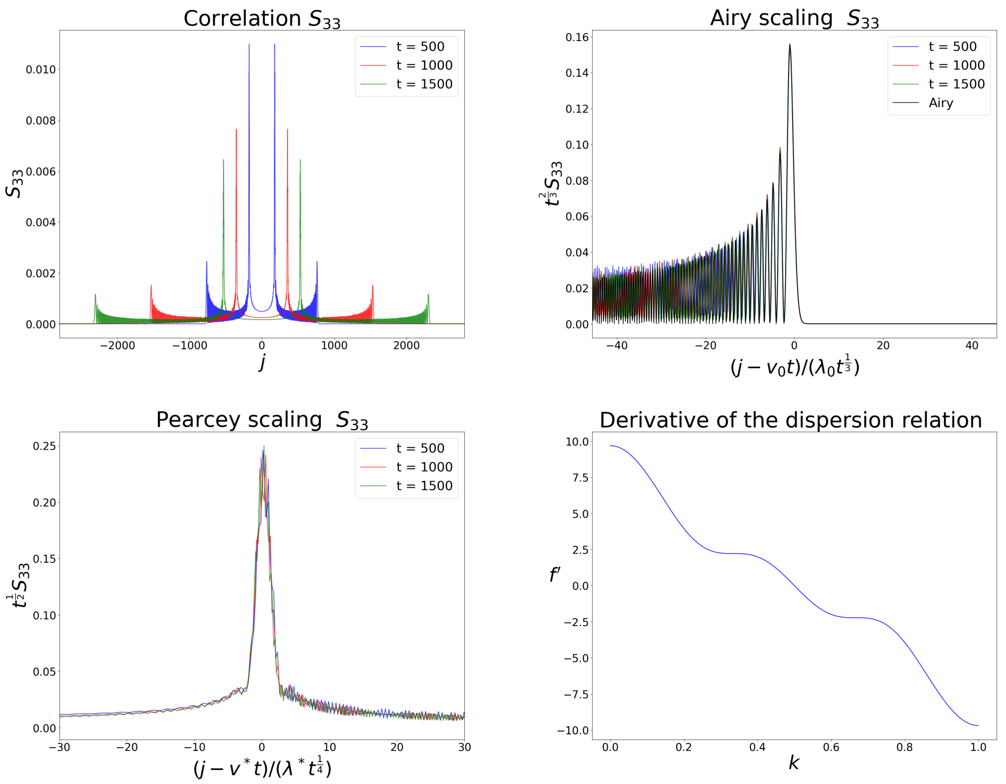


Figure 1.3: Potential of Example 1.9. The top left figure displays $S_{33}(j,t)$ for several values of t. The scaling of S_{33} according to the Airy function in Theorem 1.6 for the fastest moving peak and the scaling of the slower moving peak according to the Pearcey integral are shown top right and bottom left, respectively. The corresponding critical points of the derivative of the dispersion function can be seen in the bottom right figure.

1.2 Complete set of integrals with local densities, currents and potentials

1.2.1 Circulant hierarchy of integrals

In this section we construct a complete set of conserved quantities that have local densities. The harmonic oscillator with short range interaction is clearly an integrable system. A set of integrals of motion is given by the harmonic oscillators in each of the Fourier variables: $\widehat{H}_j = \frac{1}{2}\left(|\widehat{p}_j|^2 + |\omega_j|^2|\widehat{q}_j|^2\right)$, $j = 0, \ldots \frac{N-1}{2}$. However, when written in the physical variables $\mathbf{p}$ and $\mathbf{q}$, the quantities

$$\widehat{H}_j = \frac{1}{2}\sum_{k,l=0}^{N-1} \mathcal{F}_{j,k}\overline{\mathcal{F}_{j,l}}(p_k p_l + |\omega_j|^2 q_k q_l)$$

depend on all components of the physical variables. We now construct integrals of motion each having a density that involves only a limited number of components of the physical variables and this number only depends on the range m of interaction.

For this purpose, we denote by $\{\mathbf{e}_k\}_{k=0}^{N-1}$ the canonical basis in $\mathbb{R}^N$.

Theorem 1.10. *Let us consider the Hamiltonian*

$$H(\mathbf{p},\mathbf{q}) = \frac{1}{2}(\mathbf{p},\mathbf{p}) + \frac{1}{2}(\mathbf{q}, A\mathbf{q}),$$

with the symmetric circulant matrix A as in (1.2), (1.3). *Define the matrices $\{G_k\}_{k=1}^{M}$ to be the symmetric circulant matrix generated by the vector $\frac{1}{2}(\mathbf{e}_k + \mathbf{e}_{N-k})$ and $\{S_k\}_{k=1}^{M}$ to be the antisymmetric circulant matrix generated by the vector $\frac{1}{2}(\mathbf{e}_k - \mathbf{e}_{N-k})$. Then the family of Hamiltonians defined as*

$$H_k(\mathbf{p},\mathbf{q}) = \frac{1}{2}\mathbf{p}^\intercal G_k \mathbf{p} + \frac{1}{2}\mathbf{q}^\intercal T^\intercal G_k T\mathbf{q} = \frac{1}{2}\sum_{j=0}^{N-1}\left[p_j p_{j+k} + r_j r_{j+k}\right], \tag{1.54}$$

$$H_{k+\frac{N-1}{2}}(\mathbf{p},\mathbf{q}) = \mathbf{p}^\intercal T^\intercal S_k T\mathbf{q} = \frac{1}{2}\sum_{j=0}^{N-1}\left[\left(\sum_{\ell=0}^{m}\tau_\ell p_{j+\ell}\right)(r_{j+k} - r_{j-k})\right], \quad k = 1, \ldots, \frac{N-1}{2}$$

together with $H_0 := H$ forms a complete family $(H_j)_{0\leqslant j\leqslant N-1}$ of integrals of motion that, moreover, is in involution.

Proof. Observe first that the Hamiltonian $H_0 = H$ is included in the description of formula (1.54) as G_0 equals the identity matrix. Using the symmetries $G_k^\intercal = G_k$, $0 \leqslant k \leqslant (N-1)/2$, the Poisson bracket $\{F, G\} = (\nabla_{\mathbf{q}}F, \nabla_{\mathbf{p}}G) - (\nabla_{\mathbf{q}}G, \nabla_{\mathbf{p}}F)$ may be evaluated in the form

$$\begin{aligned}
\{H_k, H_\ell\} &= \mathbf{q}^\intercal\big(T^\intercal G_k T G_\ell - T^\intercal G_\ell T G_k\big)\mathbf{p}, && \text{for } 0 \leqslant k,\ell \leqslant \tfrac{N-1}{2},\\
\{H_k, H_\ell\} &= \mathbf{p}^\intercal\big(T^\intercal S_k T T^\intercal S_\ell T - T^\intercal S_\ell T T^\intercal S_k T\big)\mathbf{q}, && \text{for } \tfrac{N+1}{2} \leqslant k,\ell \leqslant N-1,\\
\{H_k, H_\ell\} &= \mathbf{q}^\intercal T^\intercal G_k T T^\intercal S_\ell T\mathbf{q} - \mathbf{p}^\intercal T^\intercal S_\ell T G_k \mathbf{p}, && \text{for } 0 \leqslant k \leqslant \tfrac{N-1}{2},\ \tfrac{N+1}{2} \leqslant \ell \leqslant N-1.
\end{aligned}$$

All these expressions vanish. To see this, it suffices to observe that multiplication is commutative for circulant matrices and, for the bottom line, that S_ℓ is skew symmetric: $S_\ell^\intercal = -S_\ell$.

□

Now we introduce the local densities corresponding to the just defined integrals of motion

$$e_j^{(k)} = \begin{cases} \frac{1}{2}\left(p_j p_{j+k} + r_j r_{j+k}\right) \text{, for } k = 1, \dots, \frac{N-1}{2} \\ \left(\sum_{l=0}^{m} \tau_l p_{j+l}\right)\left(r_{j+k} - r_{j-k}\right), \quad \text{for } k = \frac{N+1}{2}, \dots, N\,. \end{cases}$$

together with their correlation functions

$$S^{(N)}_{(k+3,n+3)}(j,t) := \left\langle e_j^{(k)}(t) e_0^{(n)}(0)\right\rangle - \left\langle e_j^{(k)}(t)\right\rangle\left\langle e_0^{(n)}(0)\right\rangle .$$

and limits

$$S_{k,n}(j,t) = \lim_{N\to\infty} S^{(N)}_{k,n}(j,t).$$

Following the same ideas as before, it is straightforward to compute those limits explicitly as

$$\begin{aligned} S_{k+3,n+3}(j,t) =& \frac{1}{2\beta^2}\int_0^1\int_0^1 \cos\left(f(x)t\right)\cos\left(f(y)t\right)\cos\left(2\pi x(j-n)\right)\cos\left(2\pi y(j+k)\right) \\ &+ \cos\left(f(x)t\right)\cos\left(f(y)t\right)\cos\left(2\pi xj\right)\cos\left(2\pi y(j+k-n)\right) \\ &+ \sin\left(f(x)t\right)\sin\left(f(y)t\right)\cos\left(2\pi x(j-n)\right)\cos\left(2\pi y(j+k)\right)\cos(\theta(x))\cos(\theta(y)) \\ &+ \sin\left(f(x)t\right)\sin\left(f(y)t\right)\sin\left(2\pi x(j-n)\right)\sin\left(2\pi y(j+k)\right)\sin(\theta(x))\sin(\theta(y))\mathrm{d}x\mathrm{d}y\,, \end{aligned}$$

for $k,n \leqslant \frac{N-1}{2}$,

$$\begin{aligned} S_{n+3,k+3}(j,t) =& \frac{1}{2\beta^2}\int_0^1\int_0^1 f(x)f(y)\sin\left(f(x)t\right)\sin\left(f(y)t\right)\sin\left(2\pi xj\right)\sin\left(2\pi yj\right)\sin\left(2\pi xn\right)\sin\left(2\pi yk\right) \\ &+ f^2(x)\cos(f(x)t)\cos(f(y)t)\cos\left(2\pi xj\right)\cos\left(2\pi yj\right)\sin\left(2\pi yn\right)\sin\left(2\pi yk\right)\mathrm{d}x\mathrm{d}y\,, \end{aligned}$$

for $k,n > \frac{N-1}{2}$ and

$$\begin{aligned} S_{n+3,k+3}(j,t) =& \frac{1}{2\beta^2}\int_0^1\int_0^1 \cos\left(2\pi xj - \theta(x)\right)\cos\left(2\pi yj\right)\sin\left(2\pi yk\right)\sin\left(2\pi yn\right)\sin((f(x)+f(y))t) \\ &+ \cos\left(2\pi xj - \theta(x)\right)\sin\left(2\pi yj\right)\sin\left(2\pi yk\right)\cos\left(2\pi yn\right)\sin((f(x)-f(y))t)\mathrm{d}x\mathrm{d}y\,, \end{aligned}$$

for $k > \frac{N-1}{2}, n \leqslant \frac{N-1}{2}$.

From these explicit formulas, one can deduce that they have the same scaling behaviour as the energy-energy correlation function S_{33} when $t \to \infty$.

1.2.2 Currents and potentials

In this subsection we write the evolution with respect to time of r_j, p_j and e_j in the form of a (discrete) conservation law by introducing the currents. Each conservation law has a potential function that is a Gaussian random variable. In the final part of this subsection we determine the leading order behaviour of the variance of this Gaussian random variable as $t \to \infty$ in the case of nearest neighbour interactions.

For introducing the currents we recall that $\mathbf{r} = T\mathbf{q}$ with T as in (1.7). Then one has

$$\begin{aligned} \dot{r}_j &= \sum_\ell T_{j\,\ell} p_\ell = \sum_{\ell=1}^m \tau_\ell (p_{j+\ell} - p_j), \quad r_{j+N} = r_j \\ \dot{p}_j &= -\sum_\ell T_{\ell\, j} r_\ell = \sum_{\ell=1}^m \tau_\ell (r_j - r_{j-\ell}), \quad p_{j+N} = p_j, \;\; j = 0, \dots, N-1. \end{aligned} \tag{1.55}$$

To write the above equation in the form of a discrete conservation law we introduce the local currents

$$\mathcal{J}_j^{(r)} := \sum_{s=0}^{m-1} p_{j+1+s} \sum_{\ell=s+1}^m \tau_\ell \, , \quad \mathcal{J}_j^{(p)} := \sum_{s=1}^m r_{j+1-s} \sum_{\ell=s}^m \tau_\ell .$$

Then the equations of motion (1.55) can be written in the form

$$\begin{aligned} \dot{r}_j &= \mathcal{J}_j^{(r)} - \mathcal{J}_{j-1}^{(r)} \\ \dot{p}_j &= \mathcal{J}_j^{(p)} - \mathcal{J}_{j-1}^{(p)}, \quad j = 0, \dots, N-1. \end{aligned}$$

From the above equations it is clear that the momentum p_j and the generalized elongation r_j are locally conserved. The evolution of the energy $e_j := \frac{1}{2}p_j^2 + \frac{1}{2}r_j^2$ at position j takes the form

$$\dot{e}_j = \mathcal{J}_j^{(e)} - \mathcal{J}_{j-1}^{(e)} \, , \quad \mathcal{J}_j^{(e)} = \sum_{s=1}^m \tau_s \sum_{\ell=0}^{s-1} r_{j+1-s+\ell} p_{j+1+\ell} .$$

We remark that all the currents $\mathcal{J}_j^{(r)}$, $\mathcal{J}_j^{(p)}$ and $\mathcal{J}_j^{(e)}$ are local quantities in the variables $\mathbf{q}$ and $\mathbf{p}$. We recall the notation of the introduction

$$\boldsymbol{u}(j,t) = (r_j(t), p_j(t), e_j(t)),$$

and we introduce the vector of currents $\boldsymbol{J}(j,t) = (\mathcal{J}_j^{(r)}(t), \mathcal{J}_j^{(p)}(t), \mathcal{J}_j^{(e)}(t))$. The equations of motion take the compact form

$$\frac{d}{dt}\boldsymbol{u}(j,t) = \boldsymbol{J}(j,t) - \boldsymbol{J}(j-1,t).$$

We define a potential function for the above conservation law

$$\boldsymbol{\Phi}(j,t) := \int_0^t \boldsymbol{J}(j,t')dt' + \sum_{\ell=0}^j \mathbf{u}(\ell, 0).$$

Then it is straightforward to verify that $\boldsymbol{\Phi}_t(j,t) = \boldsymbol{J}(j,t)$ and $\boldsymbol{\Phi}(j,t) - \boldsymbol{\Phi}(j-1,t) = \mathbf{u}(j,t)$. The quantities $\Phi_1(j,t)$ and $\Phi_2(j,t)$ can be expressed as sums of independent centered Gaussian random variables and are therefore also Gaussian random variables with zero mean and variance $\langle(\Phi_1(j,t))^2\rangle$ and $\langle(\Phi_2(j,t))^2\rangle$, where all the averages are taken with respect to the distribution (1.9), see also (1.23). We calculate the variance for the case of the harmonic oscillator with nearest neighbour interactions. In this particular case

$$\begin{aligned}\Phi_1(j,t) &= \sqrt{\kappa_1}\int_0^t p_{j+1}(t')dt' + \sum_{\ell=0}^{j} r_\ell(0) = \sqrt{\kappa_1}(q_{j+1}(t) - q_0(0))\\ \Phi_2(j,t) &= \sqrt{\kappa_1}\int_0^t r_j(t')dt' + \sum_{\ell=0}^{j} p_\ell(0)\ .\end{aligned} \tag{1.56}$$

After some lengthy calculations one obtains:

$$\lim_{N\to\infty}\langle(\Phi_1(j,t))^2\rangle = \frac{2\kappa_1}{\beta}\int_0^1 |\omega(k)|^{-2}\left[1-\cos\left(|\omega(k)|t\right)\cos\left(2\pi(j+1)k\right)\right]dk \tag{1.57}$$

$$\lim_{N\to\infty}\langle(\Phi_2(j,t))^2\rangle = \frac{2\kappa_1}{\beta}\int_0^1 |\omega(k)|^{-2}(1-\cos\left(|\omega(k)|t\right))\cos\left(2\pi(j+1)k\right)dk + \frac{j+1}{\beta}\ . \tag{1.58}$$

Evaluating the r.h.s. of the above expressions in the limit $t\to\infty$ we arrive to the following theorem.

Theorem 1.11. *In the limit $N\to\infty$ and $t\to\infty$ the quantities $\Phi_1(j,t)$ and $\Phi_2(j,t)$ defined in* (1.56) *are Gaussian random variables that have the following large t behaviour:*

$$\lim_{N\to\infty}\Phi_1(j,t) = \mathcal{N}(0,\sigma_1^2) \qquad \text{and} \qquad \lim_{N\to\infty}\Phi_2(j,t) = \mathcal{N}(0,\sigma_2^2)\,.$$

The leading order behaviour of the variances σ_1^2 and σ_2^2 agrees. In the physically interesting region $\frac{|j|}{t}\leqslant\sqrt{\kappa_1}$ it is given by

$$\sigma_1^2 = \frac{t\sqrt{\kappa_1}}{\beta} + \mathcal{O}\big(t^{\frac{1}{3}}\big) = \sigma_2^2\,.$$

The proof of the above theorem relies on steepest descent analysis of the oscillatory integrals in (1.58). But because the integrand is actually quite large ($\sim Ct^2$) near $k=0$, we consider the following Cauchy-type integral instead,

$$F_0(z) = \frac{1}{2\pi^2\beta}\int_{-1/2}^{1/2}\frac{1-\cos\left(|\omega(k)|t\right)}{(k-z)^2}\cos\left(2\pi(j+1)k\right)dk\ ,$$

which gives the leading order asymptotic behaviour of the integrals appearing in (1.58), since

$$\frac{2\kappa_1}{\beta}\int_{-1/2}^{1/2}|\omega(k)|^{-2}(1-\cos\left(|\omega(k)|t\right))\cos\left(2\pi(j+1)k\right)dk - F_0(0) \to\ 0 \quad \text{as } t,j\to\infty\ .$$

For $\frac{|j|}{t} < (1-\epsilon)\sqrt{\kappa_1}$, $\epsilon>0$, the analysis of $F_0(z)$ is quite straightforward - a standard stationary phase calculation combined with a contour deformation to permit the evaluation at $z=0$. For t and j growing to ∞ such that $\frac{|j|}{t}\approx\sqrt{\kappa_1}$, the analysis is more complicated because the point of stationary phase is encroaching upon the origin, where the integrand itself is actually large as $t\to\infty$. For this case, one must construct a local parametrix, following quite closely the analysis presented in [95], and we omit the details of this analysis. In order to analyse Φ_1 observe that the difference of the integrals in relations (1.57) and (1.58) is given by $\int_0^1|\omega(k)|^{-2}\left[1-\cos\left(2\pi(j+1)k\right)\right]dk$ which can also be treated by a stationary phase calculation combined with a contour deformation.

1.3 Nonlinear Regime

In this section, we consider a nonlinear perturbation of the harmonic oscillators with short range interactions of the form

$$H(\mathbf{p},\mathbf{q}) = \sum_{j=0}^{N-1}\frac{p_j^2}{2}+\sum_{s=1}^{m}\kappa_s\left(\frac{1}{2}\sum_{j=0}^{N-1}(q_j-q_{j+s})^2+\frac{\chi}{3}\sum_{j=0}^{N-1}(q_j-q_{j+s})^3+\frac{\gamma}{4}\sum_{j=0}^{N-1}(q_j-q_{j+s})^4\right). \tag{1.61}$$

We consider Example 1.8 and Example 1.9 with different strengths of nonlinearity namely

$$m=2,\ \kappa_1=1,\ \kappa_2=\tfrac{1}{4},\quad \begin{cases}\chi=0.01 \text{ and } \gamma=0.001\\ \chi=0.1 \text{ and } \gamma=0.01\end{cases}$$

$$m=3,\ \kappa_1=1,\ \kappa_2=\tfrac{1}{8},\ \kappa_2=\tfrac{7}{72},\quad \begin{cases}\chi=0.01 \text{ and } \gamma=0.001\\ \chi=0.1 \text{ and } \gamma=0.01\end{cases}.$$

We numerically compute and study the correlatios functions for these systems sampling the initial conditions according to the Gibbs measures of just their harmonic part at temperature $\beta^{-1}=1$.

In the weakly nonlinear case, the fastest peaks of the correlation functions scale numerically according to the Airy parametrices (cf. Theorem 1.6) as can be deduced from the top pictures in Figures 1.4, 1.5 while for stronger nonlinearity the fastest peaks seem to scale like $t^{\frac{2}{3}}$ in equation (1.11), see bottom figures in Figures 1.4, 1.5. The non-generic peaks that are present in the linear cases and scale like $t^{1/4}$ have a fast decay in the case of strong nonlinearity. However, for weak non-linearities, the central peak in the top left Figure 1.4, still scales in time like $t^{-\frac{1}{4}}$. Indeed, performing a regression analysis of the log-log plot one can see a scaling like $t^{-0.267}$ that is slightly faster than $t^{-\frac{1}{4}}$ (see Figure 1.6).

The numerical computations have been implemented with `Python` software, all codes are available on GitHub [114]. Fig. 1.1–1.3 are the result of the numerical evaluation via the standard routine `numpy.trapz` of the integrals in (1.31)–(1.33) for various values of j and t and then we just added the Airy function (1.16) and the Pearcey integral (1.48).

To obtain Fig. 1.4 we proceed in the following way. First, we sampled a random initial data according to the Gibbs measure defined by the corresponding harmonic part of (1.61), namely the Hamiltonian of Example 1.8 with $m=2$. We let these data evolve according to the Hamilton equations of (1.61) and compute the values of the correlations function. Then we repeated this procedure 4×10^6 times, and we averaged the values of the correlations functions. On the left panel we plot the correlation functions, instead on the right one we focus on the extreme peak, and we guess a proper scaling depending on the size of the perturbation. Fig. 1.5 is made similarly, where now the nonlinear potential has the same harmonic part as Example 1.9.

In Fig. 1.6 we focus our attention on the central peak of the chain with potential as is Fig. 1.4. We follow the same procedure as before and plot in logarithmic scale the average scaling of the highest peak in the center of the chain. We decide to

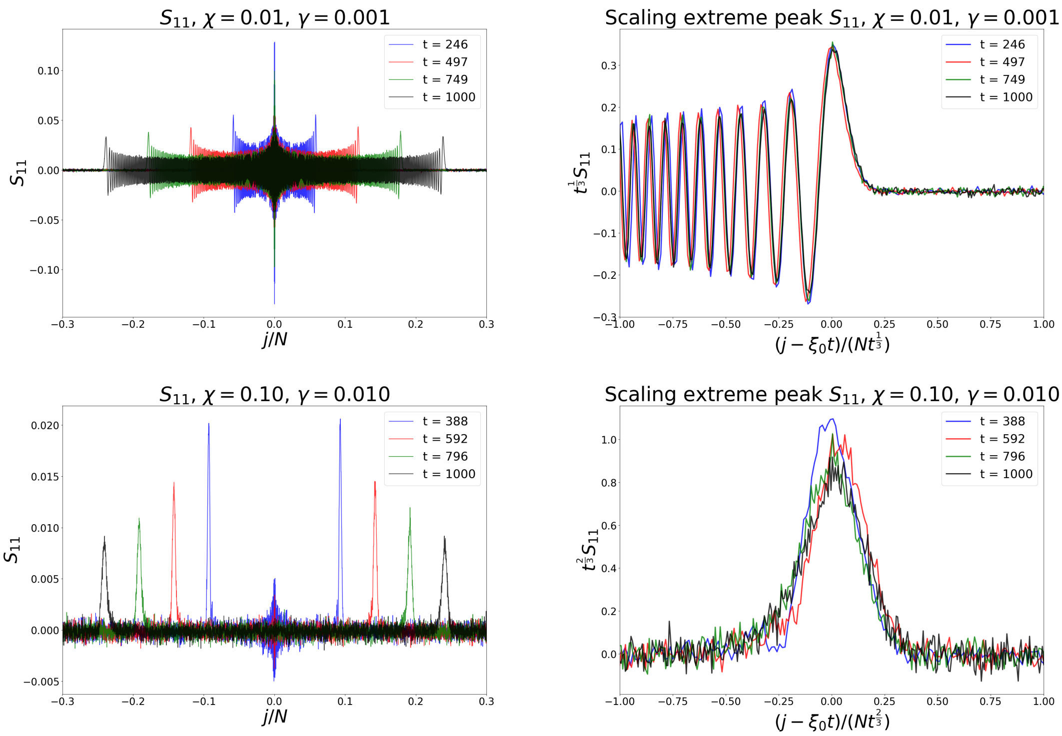


Figure 1.4: Correlation function $S_{11}^{(N)}(j,t)$ for several values of times and for the Hamiltonian (1.61) with κ_s as in Example 1.8, $\chi = 0.01$ and $\gamma = 0.001$ in the top figure and $\chi = 0.1$ and $\gamma = 0.01$ in the lower figure. On the right top figure, the scaling of the fastest peak according to Airy parametrix (see Theorem 1.6 and Figure 1.1) and according to $t^{-2/3}$ in the lower figure. The speed ξ_0 of the fastest peak is determined numerically. One can see that the central peak has a low decay in the top left figure, while in the left bottom figure it is destroyed by the relatively stronger nonlinearity.

plot the average height of this peak since it is highly oscillatory, and it is difficult to precisely track the oscillations.

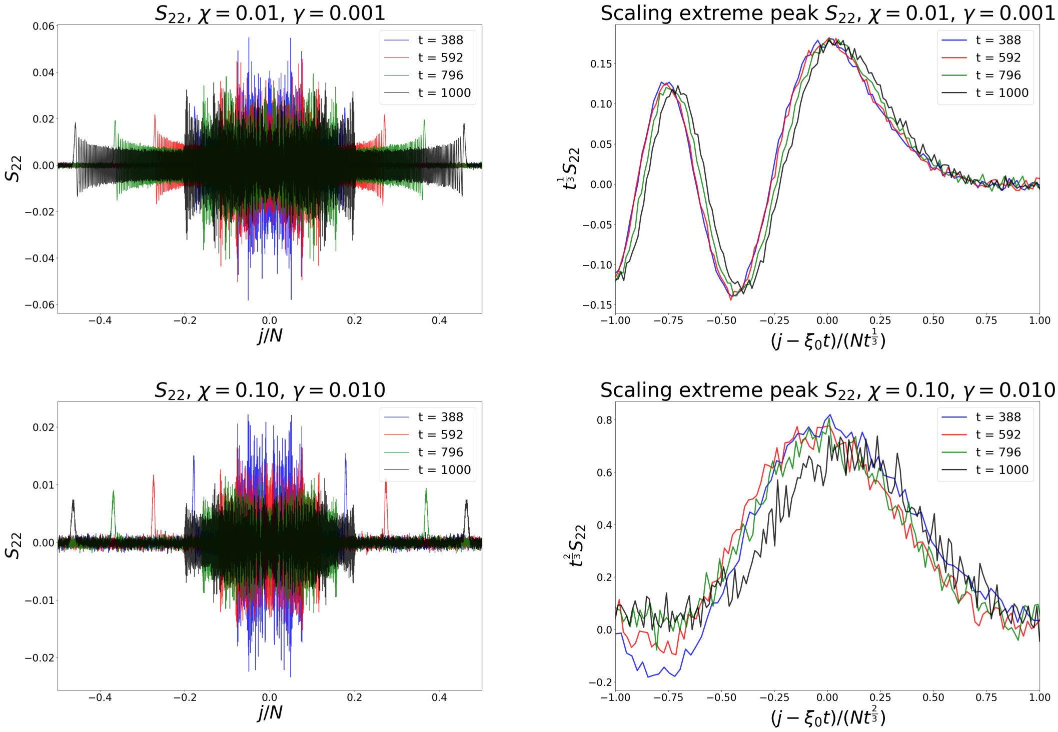
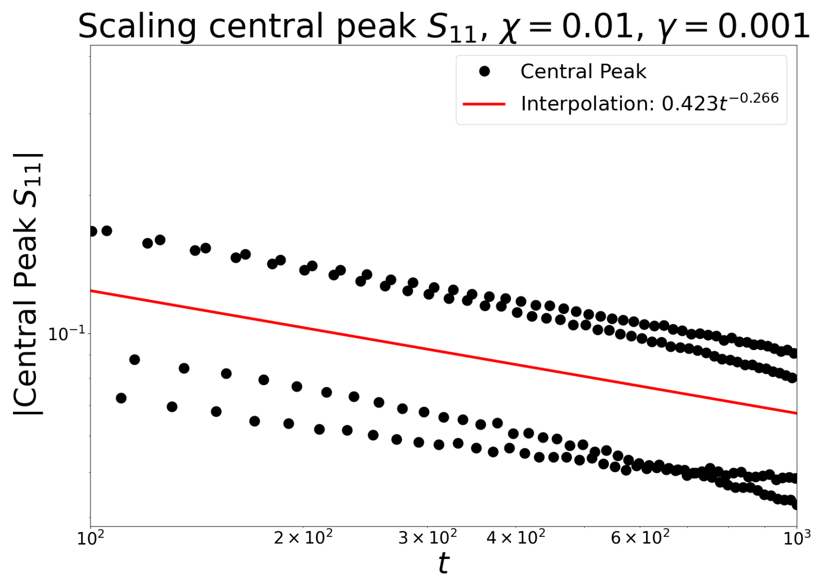


Figure 1.5: Correlation function $S_{22}^{(N)}(j,t)$ for several values of times and for the Hamiltonian (1.61) with κ_s as in Example1.9, $\chi = 0.01$ and $\gamma = 0.001$ in the top figure and $\chi = 0.1$ and $\gamma = 0.01$ in the lower figure. The right top figure shows the scaling of the fastest peak compatible with the Airy parametrix and according to $t^{-2/3}$ in the lower figure. The speed ξ_0 of the fastest peak is determined numerically. The decay rate of the slower moving peaks that are scaling like $t^{-1/4}$ in the linear case (see Figure 1.1), is not very clear due to their highly oscillatory behaviour.



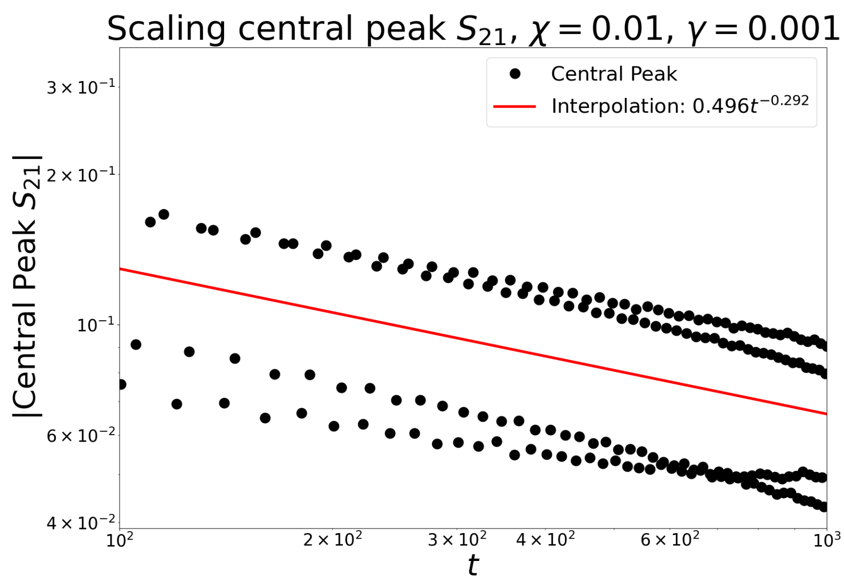


Figure 1.6: Logarithmic plot of the central peak of the example in Figure 1.4 for $S_{11}(j,t)$ and $S_{21}(j,t)$ and several values of times. The peak is highly oscillatory and the oscillations are interpolated by the red line that suggests a scaling of the correlation function $S_{11}(j,t)$ and $S_{21}(j,t)$ near $j \sim 0$ compatible with $t^{-\frac{1}{4}}$.

Chapter 2

Alpha Ensembles, and the Toda lattice

In this chapter, we consider some tridiagonal random matrix models related to the classical β-ensembles [42, 46, 93]. More specifically, we study the mean density of states of the random matrices in Table 2.1 where the quantity $\mathcal{N}(0,\sigma^2)$ is the real Gaussian random variable with density $\frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$ supported on all $\mathbb{R}$, the quantity $\chi_{2\alpha}$ is the chi-distribution with density $\frac{x^{2\alpha-1}e^{-\frac{x^2}{2}}}{2^{\alpha-1}\Gamma(\alpha)}$ supported on $\mathbb{R}^+$, here $\Gamma(\alpha)$ is the gamma function, and $\mathrm{Beta}(a,b)$ is the Beta random variable with density $\frac{\Gamma(a+b)x^{a-1}(1-x)^{b-1}}{\Gamma(a)\Gamma(b)}$ supported on $(0,1)$.

Let us explain some terminology first and then state our result.

A random Jacobi matrix is a symmetric tridiagonal $N\times N$ matrix of the form

$$T_N := \begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{N-1} \\ & & & b_{N-1} & a_N \end{pmatrix} \tag{2.1}$$

where $\{a_i\}_{i=1}^N$ are i.i.d. real random variables and $\{b_i\}_{i=1}^{N-1}$ are i.i.d. positive random variables independent from the a_i. This matrix has the property of having N-distinct eigenvalues [35]. The empirical spectral distribution of T_N is the random probability distribution on $\mathbb{R}$ defined as

$$\mathrm{d}\nu_T^{(N)} := \frac{1}{N}\sum_{j=1}^N \delta_{\lambda_j^{(N)}}, \tag{2.2}$$

where $\lambda_1^{(N)} > \ldots > \lambda_N^{(N)}$ are the eigenvalues of T_N and $\delta_{(\cdot)}$ is the delta function. The mean *Density of States* $\overline{\mathrm{d}\nu_T}$ is the non-random probability distribution, provided it exists, defined as

$$\int f \mathrm{d}\nu_T := \lim_{N\to\infty} \mathbf{E}\left[\int f \mathrm{d}\nu_T^{(N)}\right],$$

for all continuous and bounded functions f, here $\mathbf{E}[\cdot]$ stands for the expectation with respect to the given probability distribution on the matrix entries.

Gaussian α Ensemble	$H_\alpha \sim \frac{1}{\sqrt{2}} \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \\ & & b_{N-1} & a_N \end{pmatrix}$, $H_\alpha \in \mathrm{Mat}(N \times N, \mathbb{R})$, $b_n \sim \chi_{2\alpha} \quad n = 1, \ldots, N-1$, $a_n \sim \mathcal{N}(0, 2) \quad n = 1, \ldots, N$,
Laguerre α Ensemble	$L_{\alpha,\gamma} = B_{\alpha,\gamma} B^{\intercal}_{\alpha,\gamma}, \quad B_{\alpha,\gamma} = \frac{1}{\sqrt{2}} \begin{pmatrix} x_1 & & & \\ y_1 & x_2 & & \\ & \ddots & \ddots & \\ & & y_{N-1} & x_N \end{pmatrix}$, $B_{\alpha,\gamma} \in \mathrm{Mat}(N \times M, \mathbb{R})$, $M \geqslant N$, $x_n \sim \chi_{\frac{2\alpha}{\gamma}} \quad n = 1, \ldots, N$, $y_n \sim \chi_{2\alpha} \quad n = 1, \ldots, N-1$,
Jacobi α Ensemble	$J_\alpha = D_\alpha D^{\intercal}_\alpha, \quad D_\alpha = \begin{pmatrix} s_1 & & & \\ t_1 & s_2 & & \\ & \ddots & \ddots & \\ & & t_{N-1} & s_N \end{pmatrix}$, $D_\alpha \in \mathrm{Mat}(N \times N, \mathbb{R})$, $t_n = \sqrt{q_n(1-p_n)}$, $s_n = \sqrt{p_n(1-q_{n-1})}$, $q_n \sim \mathrm{Beta}\,(\alpha, \alpha + a + b + 2) \quad (q_0 = 0)$, $p_n \sim \mathrm{Beta}\,(\alpha + a + 1, \alpha + b + 1)$.

Table 2.1: The Gaussian, Laguerre and Jacobi α-ensembles.

The main result of this chapter is the following Theorem, which gives the explicit formula for the mean density of states of the Gaussian, Laguerre and Jacobi α-ensembles introduced in Table 2.1.

Theorem 2.1. *Consider the matrices $H_\alpha, L_{\alpha,\gamma}$, and J_α in Table 2.1 with $\alpha \geqslant 0$, $\gamma \in (0,1)$, $a + \alpha > 0$, $b + \alpha > 0$ and $a \notin \mathbb{N}$. Then their empirical spectral distributions $\mathrm{d}\nu_H^{(N)}, \mathrm{d}\nu_L^{(N)}$, and $\mathrm{d}\nu_J^{(N)}$ converge almost surely, in the large N limit, to their corresponding mean density of states, whose formula are given explicitly by:*

$$\overline{\mathrm{d}\nu_H}(x) = \partial_\alpha(\alpha\mu_\alpha(x))\mathrm{d}x\,, \tag{2.3}$$

$$\overline{\mathrm{d}\nu_L}(x) = \partial_\alpha\left(\alpha\mu_{\alpha,\gamma}(x)\right)\mathrm{d}x\,, \quad x \geqslant 0, \tag{2.4}$$

$$\overline{\mathrm{d}\nu_J}(x) = \partial_\alpha\left(\alpha\mu_{\alpha,a,b}(x)\right)\mathrm{d}x\,, \quad 0 \leqslant x \leqslant 1\,. \tag{2.5}$$

Here ∂_α is the derivative with respect to α and

$$\mu_\alpha(x) := \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\left|\widehat{f}_\alpha(x)\right|^{-2}, \quad \widehat{f}_\alpha(x) := \sqrt{\frac{\alpha}{\Gamma(\alpha)}}\int_0^\infty t^{\alpha-1}e^{-\frac{t^2}{2}}e^{ixt}\mathrm{d}t\,, \tag{2.6}$$

$$\mu_{\alpha,\gamma}(x) := \frac{1}{\Gamma(\alpha+1)\Gamma\left(1+\frac{\alpha}{\gamma}+\alpha\right)}\frac{x^{\frac{\alpha}{\gamma}}e^{-x}}{\left|\psi\left(\alpha, -\frac{\alpha}{\gamma}; xe^{-i\pi}\right)\right|^2} \quad x \geqslant 0\,, \tag{2.7}$$

with $\Gamma(z)$ *the gamma-function and* $\psi(v,w;z)$ *is the Tricomi's confluent hypergeometric function, for the definition see [38, §13], and*

$$\mu_{\alpha,a,b}(x) := \frac{\Gamma(\alpha+1)\Gamma(\alpha+a+b+2)}{\Gamma(\alpha+a+1)\Gamma(\alpha+b+1)} \frac{x^a(1-x)^b}{\left|U(x)+e^{i\pi b}V(x)\right|^2} \qquad 0 \leqslant x \leqslant 1, \tag{2.8}$$

where

$$\begin{aligned} U(x) &:= \frac{\Gamma(\alpha+1)\Gamma(a+1)}{\Gamma(1+\alpha+a)} {}_2F_1(\alpha, -\alpha-a-b-1, -a; x)\,, \\ V(x) &:= \frac{-\pi\alpha\Gamma(\alpha+a+b+2)}{\sin(\pi a)\Gamma(1+\alpha+b)\Gamma(a+2)} (1-x)^{b+1} x^{a+1} {}_2F_1(1-\alpha, \alpha+a+b+2, 2+a; x)\,, \end{aligned}$$

here ${}_2F_1(a,b,c;z)$ *is the Hypergeometric function:*

$${}_2F_1(a,b,c;z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}\,, \qquad (a)_n := a(a+1)\cdots(a+n-1)\,.$$

Moreover, for any non-trivial polynomial $P(x)$ *the following limits hold:*

$$\sqrt{N}\left(\int P(x)\mathrm{d}\nu_H^{(N)} - \int P(x)\overline{\mathrm{d}\nu_H}\right) \xrightarrow{d} \mathcal{N}(0,\sigma_P^2) \quad \text{as } N\to\infty, \tag{2.9}$$

$$\sqrt{N}\left(\int P(x)\mathrm{d}\nu_L^{(N)} - \int P(x)\overline{\mathrm{d}\nu_L}\right) \xrightarrow{d} \mathcal{N}(0,\overline{\sigma}_P^2) \quad \text{as } N\to\infty,$$

$$\sqrt{N}\left(\int P(x)\mathrm{d}\nu_J^{(N)} - \int P(x)\overline{\mathrm{d}\nu_J}\right) \xrightarrow{d} \mathcal{N}(0,\widetilde{\sigma}_P^2) \quad \text{as } N\to\infty, \tag{2.10}$$

for some constants $\sigma_P^2, \overline{\sigma}_P^2, \widetilde{\sigma}_P^2 \geqslant 0$*, here* $\xrightarrow{d}$ *is the convergence in distribution.*

In figures 2.1–2.3 we plot the empirical spectral distribution of the α-ensembles for different values of the parameters, the solid black line is the numericaly estimated density. All plots are made using `Python` code available at [113], we made extensive use of the libraries `Seaborn` [163] and `matplotlib` [86].

The measures with density μ_α, $\mu_{\alpha,\gamma}$ and $\mu_{\alpha,a,b}$ have already appeared in the literature as the orthogonality measures of the associated Hermite, Laguerre and Jacobi polynomials [11, 87, 160]. Such measures have also appeared in the study of the classical β-ensembles [42] (see Table 2.2) in the *high temperature regime*, namely in the limit when $N\to\infty$, with $\beta N \to 2\alpha$, $\alpha>0$, [7, 8, 18, 44, 159, 160]. In order to summarize the results of those papers we recall that for the Jacobi matrix T_N in (2.1) the *spectral measure* $\mathrm{d}\mu_T^{(N)}$ is the probability measure supported on its eigenvalues $\lambda_1^{(N)},\dots,\lambda_N^{(N)}$ with weights $q_1^2,\dots,q_N^2$ where $q_j = |\langle v_j^{(N)}, e_1\rangle|$ and $v_1^{(N)},\dots,v_N^{(N)}$ are the orthonormal eigenvectors:

$$\mathrm{d}\mu_T^{(N)} := \sum_{j=1}^{N} q_j^2 \delta_{\lambda_j^{(N)}}. \tag{2.11}$$

As the eigenvectors form an orthonormal basis, and $||e_1|| = 1$ we get that $\sum_{j=1}^N q_j^2 = 1$. Moreover, the set of finite Jacobi matrix of size N is in one to one correspondence with the set of probability measure supported on N real points [35].

For the β-ensembles the quantities $\{q_i\}_{i=1}^N$ are independent of the eigenvalues and are distributed as $(\chi_\beta, \ldots, \chi_\beta)$ normalized to unit length [42, 46, 93]. It follows that $E[q_j^2] = \frac{1}{N}$. Consequently, the mean of the empirical measure (2.2) coincides with the mean of the spectral measure (2.11), namely

$$\mathrm{d}\bar{\nu}_{H_\beta}^{(N)} = \mathrm{d}\bar{\mu}_{H_\beta}^{(N)}, \quad \mathrm{d}\bar{\nu}_{L_\beta}^{(N)} = \mathrm{d}\bar{\mu}_{L_\beta}^{(N)}, \quad \mathrm{d}\bar{\nu}_{J_\beta}^{(N)} = \mathrm{d}\bar{\mu}_{J_\beta}^{(N)},$$

where H_β, L_β and J_β refer to the Hermite, Laguerre and Jacobi β-ensembles, see Table 2.2. It is shown in [44] (see also [7, 18]) that the measures $\mathrm{d}\bar{\nu}_{H_\beta}^{(N)} = \mathrm{d}\bar{\mu}_{H_\beta}^{(N)}$ converge weakly, in the limit $N \to \infty$, with $\beta N = 2\alpha$, to the non-random probability measure with density μ_α defined in (2.6). It is shown in [8,159,160] that the measures $\mathrm{d}\bar{\nu}_{L_\beta}^{(N)} = \mathrm{d}\bar{\mu}_{L_\beta}^{(N)}$ and $\mathrm{d}\bar{\nu}_{J_\beta}^{(N)} = \mathrm{d}\bar{\mu}_{J_\beta}^{(N)}$, under some mild assumptions on the parameters, converge weakly in the limit $N \to \infty$, with $\beta N \to 2\alpha$ and $N/M \to \gamma \in (0, 1)$ to the non-random probability measures with density $\mu_{\alpha,\gamma}$ and $\mu_{\alpha,a,b}$ defined in (2.7) and (2.8) respectively. In [44, 159, 160] it is showed that these measures coincide with the mean spectral measures of the random matrices H_α, L_α and J_α, see Table 2.1.

The problem of convergence of the empirical spectral distribution of the Gaussian, Laguerre and Jacobi α-ensembles has remained unsolved. In this chapter we address such problem in Theorem 2.1 by determining the mean Density of States of such random matrices and their fluctuation. Our strategy to prove the result is the application of the moment method and an astute counting of the super-Motzkin paths [130] to calculate the moments of the the Gaussian, Laguerre and Jacobi α and β-ensembles.

For completeness, we mention also the result in [136] where a different generalization of the Gaussian β ensemble is studied. Indeed, in [136] the author examined the mean spectral measure of a random Jacobi matrix T_N such that there exists a sequence of real number $\{m_k\}_{k\geqslant 0}$ and $m_0 = 1$ such that $\mathbf{E}\left[(b_1/N^\sigma)^k\right] \to m_k$ as $N \to \infty$ for all fixed $k \in \mathbb{N}$, which is a generalization of the classical case where $b_1 \sim \chi_{\beta(N-1)}/\sqrt{2}$ and $\sigma = 1/2$.

Finally we relate the Gibbs ensemble of the classical Toda chain to the Gaussian α-ensemble. In particular, we obtain, as a corollary of Theorem 2.1, the mean density of states of the Toda Lax matrix with periodic boundary conditions when the matrix entries are distributed accordingly to the Gibbs ensemble and when the number of particles goes to infinity. This result is instrumental to study the Toda lattice in the thermodynamic limit. We remark that the mean density of states of the Toda Lax matrix has already appeared in the physics literature [151]. Here we present an alternative proof of this result.

2.1 Preliminary results

In this section we summarize some known results and techniques that we will use along the proof of the main theorem.

The moments of a measure $\mathrm{d}\sigma$, when they exist, are defined as:

Gaussian β Ensemble	$H_\beta \sim \frac{1}{\sqrt{2}}\begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{N-1} \\ & & & b_{N-1} & a_N \end{pmatrix}$ $H_\beta \in \text{Mat}(N\times N, \mathbb{R})$, $b_n \sim \chi_{\beta(N-n)} \quad n = 1,\dots,N-1$, $a_n \sim \mathcal{N}(0,2) \quad n = 1,\dots,N$,
Laguerre β Ensemble	$L_{\beta,\gamma} = B_{\beta,\gamma}B_{\beta,\gamma}^\intercal, \quad B_{\beta,\gamma} = \frac{1}{\sqrt{2}}\begin{pmatrix} x_1 & & & \\ y_1 & x_2 & & \\ & \ddots & \ddots & \\ & & y_{N-1} & x_N \end{pmatrix}$, $B_{\beta,\gamma} \in \text{Mat}(N\times M, \mathbb{R}),\ M \geqslant N$, $x_n \sim \chi_{\beta(M-n+1)} \quad n = 1,\dots,N$, $y_n \sim \chi_{\beta(N-n)} \quad n = 1,\dots,N-1$,
Jacobi β Enseble	$J_\beta = D_\beta D_\beta^\intercal, \quad D_\beta = \begin{pmatrix} s_1 & & & \\ t_1 & s_2 & & \\ & \ddots & \ddots & \\ & & t_{N-1} & s_N \end{pmatrix}$, $D_\beta \in \text{Mat}(N\times N, \mathbb{R})$, $t_n = \sqrt{q_n(1-p_n)},\ s_n = \sqrt{p_n(1-q_{n-1})}$, $q_n \sim \text{Beta}\left(\frac{\beta(N-n)}{2}, \frac{\beta(N-n)}{2}+a+b+2\right) \quad (q_0 = 0)$, $p_n \sim \text{Beta}\left(\frac{\beta(N-n)}{2}+a+1, \frac{\beta(N-n)}{2}+b+1\right)$.

Table 2.2: The Gaussian, Laguerre and Jacobi β-ensembles

$$u^{(\ell)} := \int x^\ell \mathrm{d}\sigma \quad \ell \in \mathbb{N}\,.$$

Under some mild assumptions, they totally define the measure itself, indeed the following Lemma, whose proof can be found in [13, Lemma B.2], holds:

Lemma 2.2. *(cf. [13, Lemma B.2]) Let* $\{u^{(\ell)}\}_{\ell\geqslant 0}$ *be the sequence of moments of a measure* $\mathrm{d}\sigma$*. If*

$$\liminf_{l\to\infty} \frac{(u^{(2\ell)})^{\frac{1}{2\ell}}}{\ell} < \infty\,, \tag{2.12}$$

then $\mathrm{d}\sigma$ *is uniquely determined by the moment sequence* $\{u^{(\ell)}\}_{\ell\geqslant 0}$.

This implies that if two measures have the same moment sequence and (2.12) holds then the two measure are the same. We will exploit this property, indeed we will show that the moments of the random matrices H_α, L_α and J_α coincide, in the large N limit, with the moments of the measure $\overline{\mathrm{d}\nu_H}(x), \overline{\mathrm{d}\nu_L}(x)$ and $\overline{\mathrm{d}\nu_J}(x)$ in (2.3)–(2.5) and we will prove that (2.12) holds for all of them. This technique undergoes the name of *moment method*.

To apply this idea, we need to compute explicitly the moments of the mean density of states for the α and β-ensembles. We will use the following identity for the moments of the mean density of states:

$$\int x^\ell \overline{\mathrm{d}\nu_T} = \lim_{N\to\infty} \frac{1}{N}\mathbf{E}\left[Tr\left(T_N^\ell\right)\right] \quad ,$$

where

$$Tr\left(T_N^\ell\right) := \sum_{j=1}^N T_N^\ell(j,j),$$

and $T_N^\ell(j,i)$ is the entry (j,i) of the matrix T_N^ℓ and the average is made according to the distribution of the matrix entries. From now on we will write $\mathbf{E}\left[f(\mathbf{a},\mathbf{b})\right]_T$ as the mean value of $f(\mathbf{a},\mathbf{b})$ made according to the distributions of the matrix T's entries, here $\mathbf{a}$ is a vector of components $a_1, \ldots, a_N$.

To conclude the computation of the moments, we need an explicit expression for the terms $T_N^\ell(j,j)$. The following lemma proved in [76], and also in Chapter 4, provides us their general expressions:

Theorem 2.3. *(cf. [76, Theorem 3.1]) For any* $1 \leqslant \ell < N$*, consider the tridiagonal matrix* T_N (2.1)*, then one has*

$$Tr\left(T_N^\ell\right) = \sum_{j=1}^N h_j^{(\ell)} \,,$$

where $h_j^{(\ell)} := T_N^\ell(j,j)$ *is given explicitly for* $\lfloor \ell/2 \rfloor < j < N - \lfloor \ell/2 \rfloor$ *by*

$$h_j^{(\ell)}(\mathbf{b},\mathbf{a}) = \sum_{(\mathbf{n},\mathbf{k})\in\mathcal{A}^{(\ell)}} \rho^{(\ell)}(\mathbf{n},\mathbf{k}) \prod_{i=-\lfloor \ell/2 \rfloor}^{\lfloor \ell/2 \rfloor -1} b_{j+i}^{2n_i} \prod_{i=-\lfloor \ell/2 \rfloor+1}^{\lfloor \ell/2 \rfloor -1} a_{j+i}^{k_i} \,. \tag{2.13}$$

Here $\mathcal{A}^{(m)}$ *is the set*

$$\begin{aligned} \mathcal{A}^{(\ell)} := \Big\{ (\mathbf{n},\mathbf{k}) \in \mathbb{N}_0^{\mathbb{Z}} \times \mathbb{N}_0^{\mathbb{Z}} \ : \quad & \sum_{i=-\lfloor \ell/2 \rfloor}^{\lfloor \ell/2 \rfloor -1} (2n_i + k_i) = \ell, \\ & \forall i \geqslant 0, \quad n_i = 0 \Rightarrow n_{i+1} = k_{i+1} = 0, \\ & \forall i < 0, \quad n_{i+1} = 0 \Rightarrow n_i = k_i = 0 \Big\}. \end{aligned} \tag{2.14}$$

The quantity $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and* $\rho^{(\ell)}(\mathbf{n},\mathbf{k}) \in \mathbb{N}$ *is given by*

$$\rho^{(\ell)}(\mathbf{n},\mathbf{k}) := \binom{n_{-1}+n_0+k_0}{k_0}\binom{n_{-1}+n_0}{n_0} \prod_{\substack{i=-\lfloor \ell/2 \rfloor \\ i\neq -1}}^{\lfloor \ell/2 \rfloor -1} \binom{n_i+n_{i+1}+k_{i+1}-1}{k_{i+1}}\binom{n_i+n_{i+1}-1}{n_{i+1}}.$$

Remark 2.4. *Formula* (2.13) *holds for* $\lfloor \ell/2 \rfloor < j < N - \lfloor \ell/2 \rfloor$*, for the other values of* j *the formula is slightly different. This is because for* $j \leqslant \lfloor \ell/2 \rfloor$ *or* $j \geqslant N - \lfloor \ell/2 \rfloor$ *the polynomial* $h_j^{(\ell)}$ *is related to a constrained Super Motzkin path, [111], instead for* $\lfloor \ell/2 \rfloor < j < N - \lfloor \ell/2 \rfloor$ *it is related to a classical Super Motzkin path. In any case the polynomial* $h_j^{(\ell)}$ *is independent of* N *for all* j*.*

We remark that both $|\mathcal{A}_\ell|$ and $\rho^{(\ell)}(\mathbf{n},\mathbf{k})$ do not depend on N and j. Moreover, from the condition $\sum_{i=-\lfloor \ell/2\rfloor}^{\lfloor \ell/2\rfloor -1}(2n_i+k_i)=\ell$ in (2.14) one gets that

$$\begin{aligned} \ell \text{ even} \quad &\Longrightarrow \quad h_j^{(\ell)} \text{ contains only even polynomials in } \mathbf{a},\\ \ell \text{ odd} \quad &\Longrightarrow \quad h_j^{(\ell)} \text{ contains only odd polynomials in } \mathbf{a}. \end{aligned} \tag{2.15}$$

To prove the almost sure convergence of the empirical spectral distributions $\mathrm{d}\nu_H^{(N)},\mathrm{d}\nu_L^{(N)}$ and $\mathrm{d}\nu_J^{(N)}$ to their corresponding mean density of states, we will use two general results. The first one is the following Theorem proved in [125]:

Theorem 2.5. *(cf. [125, Theorem 2.2]) Consider a random Jacobi matrix T_N (2.1) and assume that $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^{N-1}$ have all finite moments. Then for any non-trivial polynomial $P(x)$:*

$$\begin{aligned} &\int P(x)\mathrm{d}\nu_T^{(N)} \overset{a.s.}{\to} \int P(x)\overline{\mathrm{d}\nu_T} \qquad as\ \ N\to\infty\\ &\sqrt{N}\left(\int P(x)\mathrm{d}\nu_T^{(N)} - \int P(x)\overline{\mathrm{d}\nu_T}\right) \overset{d}{\to} \mathcal{N}(0,\sigma_P^2) \qquad as\ \ N\to\infty, \end{aligned} \tag{2.16}$$

for some constant $\sigma_P^2 \geqslant 0$. Here $\overset{a.s.}{\to}$ is the almost sure convergence and $\overset{d}{\to}$ is the convergence in distribution.

We observe that Theorem 2.5 is not stated in the present form in [125] but this formulation is more convenient for our analysis. The second result is the following classical Lemma, whose proof can be found in [9, 44]:

Lemma 2.6. *(cf. [44, Lemma 2.2]) Consider a sequence of random probability measures $\{\mathrm{d}\mu_n\}_{n=1}^\infty$ and $\mathrm{d}\mu$ a probability measure determined by its moments according to Lemma 2.2. Assume that any moment of $\mathrm{d}\mu_n$ converges almost surely to the one of $\mathrm{d}\mu$. Then as $n\to\infty$ the sequence of measures $\{\mathrm{d}\mu_n\}_{n=1}^\infty$ converges weakly, almost surely, to $\mathrm{d}\mu$, namely for all bounded and continuous functions f:*

$$\int f\mathrm{d}\mu_n \to \int f\mathrm{d}\mu \quad a.s\ as\ \ n\to\infty\,.$$

The convergences still holds for a continuous function f of polynomials growth.

Finally, before moving to the actual proof of our main theorem, we summarize the main results of [7, 8, 44, 159, 160] in the following theorem.

Theorem 2.7. *As $N\to\infty, \beta N\to 2\alpha\in(0,\infty), \frac{N}{M}\to\gamma\in(0,1)$, $a+\alpha>0$, $b+\alpha>0$ and $a\notin\mathbb{N}$, the mean spectral measure and the mean density of state of the Gaussian, Laguerre and Jacobi β-ensembles weakly converge to the non-random measures with density $\mu_\alpha(x),\mu_{\alpha,\gamma}(x)$ and $\mu_{\alpha,a,b}(x)$ defined in (2.6), (2.7) and (2.8) respectively. Moreover, (2.12) holds for their moments sequences.*

2.2 Proof of the main result

We are now in position to prove our main result. First, we remark that the density $\partial_\alpha(\alpha\mu_\alpha(x))$, $\partial_\alpha(\alpha\mu_{\alpha,\gamma}(x))$ and $\partial_\alpha(\alpha\mu_{\alpha,a,b}(x))$ define a probability measure since the densities $\mu_\alpha(x)$, $\mu_{\alpha,\gamma}(x)$ and $\mu_{\alpha,a,b}(x)$ define a probability measure. Then, since we want to apply the moment method, we have to compute the moments of the α-ensembles explicitly. To conclude the proof we will need also an explicit expression of the moments of the mean density of states of the β-ensembles. The following lemma lays the ground to conclude both computations.

Lemma 2.8. *Fix $\alpha \in \mathbb{R}^+\backslash\{0\}$, $\gamma \in (0,1), a, b > -1$ and $N/2 > \ell \in \mathbb{N}$. Consider the α and β-ensembles in Table 2.1-2.2, there exist polynomials $w_\ell(x), g_\ell(x)$, and rational and continuous functions $r_\ell(x)$ such that, for N large enough and $\beta N = 2\alpha, \frac{N}{M} = \gamma$, the following holds, for $\lfloor \ell/2 \rfloor < j < N - \lfloor \ell/2 \rfloor$:*

$$\mathbf{E}\left[h_j^{(\ell)}\right]_{H_\beta} = \begin{cases} w_\ell\left(\alpha\left(1-\frac{j}{N}\right)\right) + O\left(N^{-1}\right) & \ell \text{ even} \\ 0 & \ell \text{ odd} \end{cases}, \tag{2.17}$$

$$\mathbf{E}\left[h_j^{(\ell)}\right]_{H_\alpha} = \begin{cases} w_\ell(\alpha) & \ell \text{ even} \\ 0 & \ell \text{ odd} \end{cases}, \tag{2.18}$$

$$\mathbf{E}\left[h_j^{(\ell)}\right]_{L_\beta} = g_\ell\left(\alpha\left(1-\frac{j}{N}\right)\right) + O\left(N^{-1}\right),$$

$$\mathbf{E}\left[h_j^{(\ell)}\right]_{L_\alpha} = g_\ell(\alpha),$$

$$\mathbf{E}\left[h_j^{(\ell)}\right]_{J_\beta} = r_\ell\left(\alpha\left(1-\frac{j}{N}\right)\right) + O\left(N^{-1}\right),$$

$$\mathbf{E}\left[h_j^{(\ell)}\right]_{J_\alpha} = r_\ell(\alpha).$$

Proof of Lemma 2.8. We will just prove (2.17)-(2.18) since the proof of the other cases is similar. Indeed, the only difference in the proofs is that for the Gaussian and Laguerre α and β-ensembles we use the fact that the expected value of any even monomial with respect to a χ_ξ-distribution is a monomial in ξ. While for the Jacobi α and β-ensembles we use the fact that the expected values of any monomial with respect to a Beta(a,b)-distribution is a rational function of the parameters.

First, since $\mathbf{a} = (a_1, \ldots, a_N)$ are normal distributed for both ensembles and thanks to (2.15) we get that

$$\mathbf{E}\left[h_j^{(\ell)}\right]_{H_\alpha} = \mathbf{E}\left[h_j^{(\ell)}\right]_{H_\beta} = 0, \qquad \ell \text{ odd}.$$

For the Gaussian α ensemble we have that, for $\lfloor \ell/2 \rfloor < j < N - \lfloor \ell/2 \rfloor$

$$\mathbf{E}\left[h_j^{(\ell)}\right]_{H_\alpha} = \mathbf{E}\left[\sum_{(\mathbf{n},\mathbf{k})\in\mathcal{A}^{(\ell)}} \rho^{(\ell)}(\mathbf{n},\mathbf{k}) \prod_{i=-\lfloor \ell/2 \rfloor}^{\lfloor \ell/2 \rfloor -1} b_{j+i}^{2n_i} \prod_{i=-\lfloor \ell/2 \rfloor+1}^{\lfloor \ell/2 \rfloor -1} a_{j+i}^{k_i}\right]_{H_\alpha}$$

does not depend on j since $b_{j+i} \sim \chi_{2\alpha}, a_i \sim \mathcal{N}(0,2)$, $i = -\lfloor \ell/2 \rfloor, \ldots, \lfloor \ell/2 \rfloor$, and the coefficients $\rho^{(\ell)}(\mathbf{n},\mathbf{k})$ and the set $\mathcal{A}^{(\ell)}$ are independent of j and N by Theorem 2.3. Moreover, as already pointed out, the expected values of any even monomial with respect to a χ_ξ-distribution is a monomial in ξ. Thus, we have that for fixed $l \in \mathbb{N}$, there exists a polynomial $w_\ell(\alpha)$ such that (2.18) holds.

We can apply a similar reasoning for the Gaussian β ensemble, indeed we notice that if we approximate the distribution of $b_{j+i} \sim \chi_{2\alpha\left(1-\frac{j+i}{N}\right)}$, $i = -\lfloor \ell/2 \rfloor, \ldots, \lfloor \ell/2 \rfloor$

with the one of $b_j \sim \chi_{2\alpha\left(1-\frac{j}{N}\right)}$ we get an error of order N^{-1} when we evaluate the expected value. So we can compute

$$\begin{aligned}\mathbf{E}\left[h_j^{(\ell)}\right]_{H_\beta} &= \mathbf{E}\left[\sum_{(\mathbf{n},\mathbf{k})\in\mathcal{A}^{(\ell)}} \rho^{(\ell)}(\mathbf{n},\mathbf{k}) \prod_{i=-\lfloor \ell/2\rfloor}^{\lfloor \ell/2\rfloor-1} b_{j+i}^{2n_i} \prod_{i=-\lfloor \ell/2\rfloor+1}^{\lfloor \ell/2\rfloor-1} a_{j+i}^{k_i}\right]_{H_\beta} \\ &= w_\ell\left(\alpha\left(1-\frac{j}{N}\right)\right) + O\left(N^{-1}\right),\end{aligned}$$

where the only difference from the previous case is that the parameter of the χ-distribution is $2\alpha\left(1-\frac{j}{N}\right)$ instead of 2α.

□

Using the above lemma we can conclude the computation of the moments for the α and β-ensembles:

Corollary 2.9. *Fix $\ell \in \mathbb{N}$, $\alpha \in \mathbb{R}^+\backslash\{0\}$, $a, b > -1$ and $\gamma \in (0,1)$ then in the large N limit, with $N\beta \to 2\alpha$ and $\frac{N}{M} \to \gamma$, the following holds:*

$$u_\alpha^{(\ell)} := \lim_{N\to\infty} \mathbf{E}\left[\frac{1}{N} Tr\left((H_\beta^\ell)\right)\right]_{H_\beta} = \begin{cases} \int_0^1 w_\ell(\alpha x)\,\mathrm{d}x & \ell \text{ even} \\ 0 & \ell \text{ odd}\end{cases}, \tag{2.19}$$

$$v_\alpha^{(\ell)} := \lim_{N\to\infty} \mathbf{E}\left[\frac{1}{N} Tr\left(H_\alpha^\ell\right)\right]_{H_\alpha} = \begin{cases} w_\ell(\alpha) & \ell \text{ even} \\ 0 & \ell \text{ odd}\end{cases}, \tag{2.20}$$

$$u_{\alpha,\gamma}^{(\ell)} := \lim_{N\to\infty} \mathbf{E}\left[\frac{1}{N} Tr\left(L_\beta^\ell\right)\right]_{L_\beta} = \int_0^1 g_\ell(\alpha x)\mathrm{d}x\,,$$

$$v_{\alpha,\gamma}^{(\ell)} := \lim_{N\to\infty} \mathbf{E}\left[\frac{1}{N} Tr\left(L_\alpha^\ell\right)\right]_{L_\alpha} = g_\ell(\alpha)\,,$$

$$u_{\alpha,a,b}^{(\ell)} := \lim_{N\to\infty} \mathbf{E}\left[\frac{1}{N} Tr\left(J_\beta^\ell\right)\right]_{J_\beta} = \int_0^1 r_\ell(\alpha x)\mathrm{d}x\,,$$

$$v_{\alpha,a,b}^{(\ell)} := \lim_{N\to\infty} \mathbf{E}\left[\frac{1}{N} Tr\left(J_\alpha^\ell\right)\right]_{J_\alpha} = r_\ell(\alpha)\,.$$

Proof. We will just prove (2.19)-(2.20) since the proof of the other cases is analogous.

From Lemma 2.8 and Theorem 2.3 one gets that:

$$v_\alpha^{(\ell)} = \lim_{N\to\infty}\left[\frac{1}{N}\left(\sum_{j=\lfloor \ell/2\rfloor+1}^{N-\lfloor \ell/2\rfloor-1} w_\ell(\alpha) + O(1)\right)\right] = w_\ell(\alpha)\,.$$

Indeed neglecting the terms $h_j^{(\ell)}$ $j = 1,\ldots,\lfloor \ell/2\rfloor, N-\lfloor \ell/2\rfloor,\ldots,N$ in the average of $Tr\left(H_\alpha^\ell\right)$ we get an error of order $O(1)$ since ℓ is fixed, so in the summations we are neglecting a finite number of terms of order $O(1)$, see Remark 2.4.

For the same reason one gets that:

$$u_\alpha^{(\ell)} = \lim_{N\to\infty}\left[\frac{1}{N}\sum_{j=\lfloor \ell/2\rfloor+1}^{N-\lfloor \ell/2\rfloor-1} w_\ell\left(\alpha\left(1-\frac{j}{N}\right)\right) + O(N^{-1})\right].$$

Thus taking the limit for N going to infinity one gets the integral in (2.19). □

Remark 2.10. *We stress that* $u_\alpha^{(\ell)}, u_{\alpha,\gamma}^{(\ell)}$ *and* $u_{\alpha,a,b}^{(\ell)}$ *are respectively the* ℓ^{th} *moments of the Gaussian, Laguerre and Jacobi* β*-ensembles in the high temperature regime. Analogously, the quantities* $v_\alpha^{(\ell)}, v_{\alpha,\gamma}^{(\ell)}$ *and* $v_{\alpha,a,b}^{(\ell)}$ *are the* ℓ^{th} *moments of the Gaussian, Laguerre and Jacobi* α*-ensembles respectively.*

We can now finish the proof of Thereon 2.1

Proof of Theorem 2.1. From Corollary 2.9 one concludes that for all fixed $l \in \mathbb{N}$:

$$v_\alpha^{(\ell)} = \partial_\alpha(\alpha u_\alpha^{(\ell)}) \,, \tag{2.21}$$
$$v_{\alpha,\gamma}^{(\ell)} = \partial_\alpha(\alpha u_{\alpha,\gamma}^{(\ell)}) \,, \tag{2.22}$$
$$v_{\alpha,a,b}^{(\ell)} = \partial_\alpha(\alpha u_{\alpha,a,b}^{(\ell)}) \,.$$

By Theorem 2.7, Corollary 2.9 and Remark 2.10, the quantities $u_\alpha^{(\ell)}$, $u_{\alpha,\gamma}^{(\ell)}$ and $u_{\alpha,a,b}^{(\ell)}$ are the moments of the measures with densities μ_α, $\mu_{\alpha,\gamma}$ and $\mu_{\alpha,a,b}$ defined in (2.3), (2.4) and (2.5). Moreover, by formula (2.12) such moments uniquely determine the corresponding measures.

It follows from relation (2.21) and Lemma 2.2 that the mean density of states $\overline{\mathrm{d}\nu_H}$ of the Gaussian α-ensemble coincides with $\partial_\alpha(\alpha\mu_\alpha)$ with μ_α as (2.3). In a similar way, by (2.22), the measure $\partial_\alpha(\alpha u_{\alpha,\gamma})$ in (2.4) is the mean density of states $\overline{\mathrm{d}\nu_L}$ of the Laguerre α-ensembles and $\partial_\alpha(\alpha u_{\alpha,a,b})$ in (2.5) is the mean density of states $\overline{\mathrm{d}\nu_H}$ of the Jacobi α-ensembles.

Since for the α-ensembles all $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^{N-1}$ have all finite moments, one can apply Theorem 2.5 getting that the moments of the empirical spectral distributions of the α- ensembles $\mathrm{d}\nu_H^{(N)}, \mathrm{d}\nu_L^{(N)}$ and $\mathrm{d}\nu_J^{(N)}$ converge almost surely to the ones of the corresponding mean density of states $\overline{\mathrm{d}\nu_H}, \overline{\mathrm{d}\nu_L}$ and $\overline{\mathrm{d}\nu_J}$ in (2.3), (2.4) and (2.5) respectively. Furthermore applying Lemma 2.6 one obtains that the spectral distributions of the α-ensembles $\mathrm{d}\nu_H^{(N)}, \mathrm{d}\nu_L^{(N)}$ and $\mathrm{d}\nu_J^{(N)}$ converge almost surely to $\overline{\mathrm{d}\nu_H}, \overline{\mathrm{d}\nu_L}$ and $\overline{\mathrm{d}\nu_J}$ in (2.3)-(2.4) and (2.5) respectively.

Finally from (2.16) one gets that formula (2.9)–(2.10) hold, namely that the global fluctuations are Gaussian. □

2.2.1 Parameters limit

In this section we study the behavior of α-ensembles when the parameter α goes to infinity. For this purpose we consider the rescaled version of α-ensembles, i.e. the matrices defined as $\frac{1}{\sqrt{\alpha}}H_\alpha$, $\frac{\gamma}{\alpha}L_{\alpha,\gamma}$ and J_α. The corresponding mean density of states is rescaled to $\mathrm{d}\overline{\nu}_H(\sqrt{\alpha}x), \mathrm{d}\overline{\nu}_L\left(\frac{\alpha x}{\gamma}\right)$ and $\mathrm{d}\overline{\nu}_J(x)$ (see (2.3)–(2.5)).

Now we have to compute the limits of these measures when $\alpha \to \infty$. We will compute these limits using the matrix representations of the normalized α-ensemble and exploit the following weak limits:

$$\lim_{\alpha\to\infty} \frac{\mathcal{N}(0,2)}{\sqrt{\alpha}} \xrightarrow{d} 0 \qquad \lim_{\alpha\to\infty} \frac{\chi_\alpha}{\sqrt{\alpha}} \xrightarrow{d} 1 \qquad \lim_{\alpha\to\infty} \mathrm{Beta}(\alpha,\alpha) \xrightarrow{d} \frac{1}{2}.$$

The above relations imply that the mean density of states of the three normalized α-ensembles weakly converges to the mean density of states of the following matrices:

$$H_\infty = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{pmatrix}, \qquad L_\infty = \begin{pmatrix} 1 & \sqrt{\gamma} & & & \\ \sqrt{\gamma} & 1+\gamma & \sqrt{\gamma} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \sqrt{\gamma} \\ & & & \sqrt{\gamma} & 1+\gamma \end{pmatrix},$$

$$J_\infty = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & & & \\ \frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{4} & & \\ & \frac{1}{4} & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{1}{4} \\ & & & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

The eigenvalues distributions of the above matrices in the large N limit are given by

$$\lim_{\alpha\to\infty} \mathrm{d}\overline{\nu}_H(\sqrt{\alpha}x) = \frac{\mathbb{1}_{(-2,2)}}{\pi\sqrt{4-x^2}}\mathrm{d}x\,, \tag{2.23}$$

$$\lim_{\alpha\to\infty} \mathrm{d}\overline{\nu}_L\left(\frac{\alpha x}{\gamma}\right) = \frac{\mathbb{1}_{((1-\sqrt{\gamma})^2,(1+\sqrt{\gamma})^2)}}{\pi\sqrt{4\gamma-(x-1-\gamma)^2}}\mathrm{d}x\,,$$

$$\lim_{\alpha\to\infty} \mathrm{d}\overline{\nu}_J(x) = \frac{2\mathbb{1}_{(0,1)}}{\pi\sqrt{1-(2x-1)^2}}\mathrm{d}x\,,$$

where $\mathbb{1}_{(a,b)}$ is the indicator function of the interval (a,b).

We observe that for all the three α-ensembles in the large α limit, the corresponding mean density of states is an arcsine distribution. It would be interesting to study the behavior of the fluctuations of the max/min eigenvalue of the α-ensembles in the limit of large α.

2.3 An application to the Toda chain

In this section we will apply Theorem 2.1 to find the mean density of states of the classical Toda chain [157] with periodic boundary conditions. As we already mentioned this is an alternative proof of the result in [151].

2.3.1 Integrable Structure

The classical Toda chain is the dynamical system described by the following Hamiltonian:

$$H_T(\mathbf{p},\mathbf{q}) := \frac{1}{2}\sum_{j=1}^N p_j^2 + \sum_{j=1}^N V_T(q_{j+1}-q_j)\ , \quad V_T(x) = e^{-x}+x-1\,, \tag{2.24}$$

with periodic boundary conditions $p_{j+N} = p_j$, $q_{j+N} = q_j$, $\forall\, j \in \mathbb{Z}$. Its equations of motion take the form

$$\dot{q}_j = \frac{\partial H_T}{\partial p_j} = p_j, \quad \dot{p}_j = -\frac{\partial H_T}{\partial q_j} = V_T'(q_{j+1} - q_j) - V_T'(q_j - q_{j-1}), \;\; j = 1, \dots, N\,. \tag{2.25}$$

It is well known that the Toda chain is an integrable system [82, 157], one way to prove it is to put the Toda equations in Lax pair form. This was introduced by Flaschka [51] and Manakov [110] through the following *non-canonical* change of coordinates:

$$a_j := -p_j\,, \qquad b_j := e^{\frac{1}{2}(q_j - q_{j+1})} \equiv e^{-\frac{1}{2}r_j}, \qquad 1 \leqslant j \leqslant N\,, \tag{2.26}$$

where $r_j = q_{j+1} - q_j$ is the relative distance. The periodic boundary conditions imply

$$\sum_{j=1}^{N} r_j = 0\,.$$

Then, defining the Lax operator L as the periodic Jacobi matrix [162]

$$L(\mathbf{b}, \mathbf{a}) := \begin{pmatrix} a_1 & b_1 & 0 & \dots & b_N \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{N-1} \\ b_N & \dots & 0 & b_{N-1} & a_N \end{pmatrix}, \tag{2.27}$$

and the anti-symmetric matrix B

$$B(\mathbf{b}) := \frac{1}{2} \begin{pmatrix} 0 & b_1 & 0 & \dots & -b_N \\ -b_1 & 0 & b_2 & \ddots & \vdots \\ 0 & -b_2 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{N-1} \\ b_N & \dots & 0 & -b_{N-1} & 0 \end{pmatrix},$$

a straightforward calculation shows that the equations of motions (2.25) are equivalent to

$$\frac{dL}{dt} = [B; L]\ ,$$

so the eigenvalues of L are a set of integrals of motion.

2.3.2 Gibbs ensemble and the density of states for the periodic Toda chain

We consider the evolution of the Toda chain on the subspace:

$$\mathcal{M} := \left\{ (\mathbf{p}, \mathbf{r}) \in \mathbb{R}^N \times \mathbb{R}^N \;:\quad \sum_{j=1}^{N} r_j = \sum_{j=1}^{N} p_j = 0 \right\}, \tag{2.28}$$

which is invariant for the dynamics. Indeed the condition $\sum_{j=1}^{N} r_j = 0$ follows from the periodic boundary conditions and the condition $\sum_{j=1}^{N} p_j = 0$ follows from the fact that the system is translational invariant and therefore the total momentum is conserved. We endow the phase space $\mathcal{M}$ (2.28) with the Gibbs measure for the Toda lattice at temperature β^{-1} as

$$\mathrm{d}\nu_{Toda} := \frac{1}{Z_{Toda}(\beta)} \; e^{-\beta H_T(\mathbf{p},\mathbf{r})} \delta_{\sum_{j=1}^{N} p_j} \delta_{\sum_{j=1}^{N} r_j} \mathrm{d}\mathbf{p}\, \mathrm{d}\mathbf{r}\,, \tag{2.29}$$

here $Z_{Toda}(\beta)$ is the partition function which normalize the measure, and δ_x is the Kronecker delta function centred at x.

We notice that this ensemble makes L (2.27) into a random matrix, thus it makes sense to study its mean density of states. However, the matrix entries of L are not independent random variables because of the constraints (2.28). For this reason, we introduce the *approximate* measure $\mathrm{d}\widetilde{\nu}_{Toda}$ on $\mathbb{R}^N \times \mathbb{R}^N$ as

$$\mathrm{d}\widetilde{\nu}_{Toda} := \frac{1}{\widetilde{Z}_{Toda}(\beta)} \; e^{-\beta H_T(\mathbf{p},\mathbf{r}) - \theta \sum_j r_j} \mathrm{d}\mathbf{p}\, \mathrm{d}\mathbf{r}\,,$$

where $\widetilde{Z}_{Toda}(\beta)$ is the partition function which normalizes the measure and $\theta > 0$ is chosen in such a way that:

$$\int r_j \mathrm{d}\widetilde{\nu}_{Toda} = 0\,.$$

The value of $\theta > 0$ is unique for all $\beta > 0$ since

$$\int r_j \mathrm{d}\widetilde{\nu}_{Toda} = \log(\beta) - \frac{\Gamma'(\beta+\theta)}{\Gamma(\beta+\theta)}\,,$$

which has just one positive solution.

From now on we will write L and $\widetilde{L}$ as the random matrices whose entries are distributed according to the probability measure $\mathrm{d}\nu_{Toda}$ and $\mathrm{d}\widetilde{\nu}_{Toda}$ respectively. In particular applying the change of coordinates (2.26) one gets that

$$\widetilde{L} \sim \frac{1}{\sqrt{2\beta}} \begin{pmatrix} a_1 & b_1 & 0 & \dots & b_N \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{N-1} \\ b_N & \dots & 0 & b_{N-1} & a_N \end{pmatrix}, \quad b_j \sim \chi_{2(\beta+\theta)}\,, a_j \sim \mathcal{N}(0,2) \quad j = 1, \dots, N.$$

To obtain the mean density of states of the Toda lattice with periodic boundary conditions we need the following lemma, whose proof can be found in [76]:

Lemma 2.11. *(cf. [76, Lemma 4.1]) Fix $\widetilde{\beta} > 0$ and let $f\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ depends on just K variables and finite second order moment with respect to $\mathrm{d}\widetilde{\nu}_{Toda}$, uniformly for all $\beta > \widetilde{\beta}$. Then there exist positive constants C, N_0 and β_0 such that for all $N > N_0$, $\beta > max\{\beta_0, \widetilde{\beta}\}$ one has*

$$\left| \int f \mathrm{d}\nu_{Toda} - \int f \mathrm{d}\widetilde{\nu}_{Toda} \right| \leqslant C \frac{K}{N} \sqrt{\int f^2 \mathrm{d}\widetilde{\nu}_{Toda} - \left(\int f \mathrm{d}\widetilde{\nu}_{Toda} \right)^2}\,.$$

Applying this Lemma we can conclude that the matrices L and $\widetilde{L}$ have the same moment sequence in the large N limit. Furthermore, $\widetilde{L}$ is a rank one perturbation of the matrix $\frac{1}{\sqrt{\beta}}H_{\theta+\beta}$ in table 2.1. So we can use the following theorem, whose proof can be found in [13], to show that the mean density of states of the matrices $\widetilde{L}$ and $\frac{1}{\sqrt{\beta}}H_{\theta+\beta}$ in the large N limit are the same.

Theorem 2.12. *(cf. [13, Theorem A.43]) Let A, B be two $N \times N$ Hermitian matrices and F^A, F^B their empirical spectral density defined as:*

$$F^A(x) := \frac{1}{N}\#\{j \leqslant N \,:\, \lambda_j \leqslant x\}\,,$$

where λ_j are the eigenvalues of A. Then

$$||F^A - F^B|| \leqslant \frac{1}{N}\ \mathit{Rank}(A-B)\,,$$

where $||f|| = \sup_x |f(x)|$.

This implies also that the moment sequence of $\widetilde{L}$ and $\frac{1}{\sqrt{\beta}}H_{\theta+\beta}$ are the same in the large N limit, which means that also the moment sequence of $L, \frac{1}{\sqrt{\beta}}H_{\theta+\beta}$ in the large N limit are equal. So applying Lemma 2.2 and Theorem 2.1 one gets the following

Lemma 2.13. *Consider the classical Toda chain* (2.24) *and endow the phase space $\mathcal{M}$* (2.28) *with the Gibbs measure* $\mathrm{d}\nu_{Toda}$ *in* (2.29)*, then there exists a constant $\beta_0 > 0$ such that, for all $\beta > \beta_0$ the mean density of states of the Lax matrix L* (2.27) *in the limit $N \to \infty$ is explicitly given by:*

$$\overline{\mathrm{d}\xi_\ell}(x) = \sqrt{\beta}\partial_\alpha(\alpha\mu_\alpha(\sqrt{\beta}x))_{|\alpha=\beta+\theta}\mathrm{d}x\,,$$

where $\mu_\alpha(x)$ is given in (2.6)*.*

To conclude, we also remark that if we let the inverse temperature β approach infinity, in view of (2.23), we obtain that the mean density of states of the classical Toda chain in this regime is exactly the arcsine law (2.3). From the physical point of view, the system is at rest.

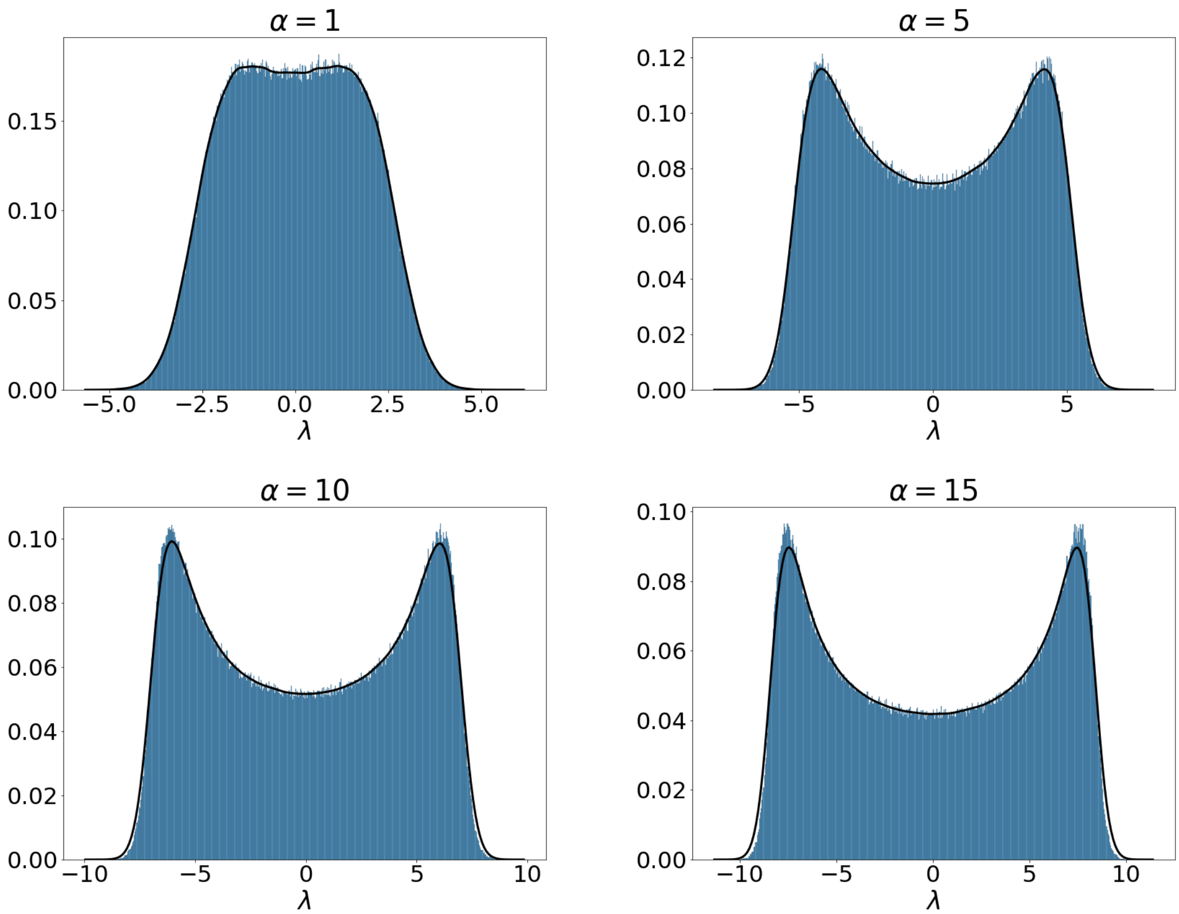


Figure 2.1: Gaussian α ensemble empirical spectral density for different values of the parameters, $N = 500$, trials: 5000. The solid black line is the estimated density, not the actual one.

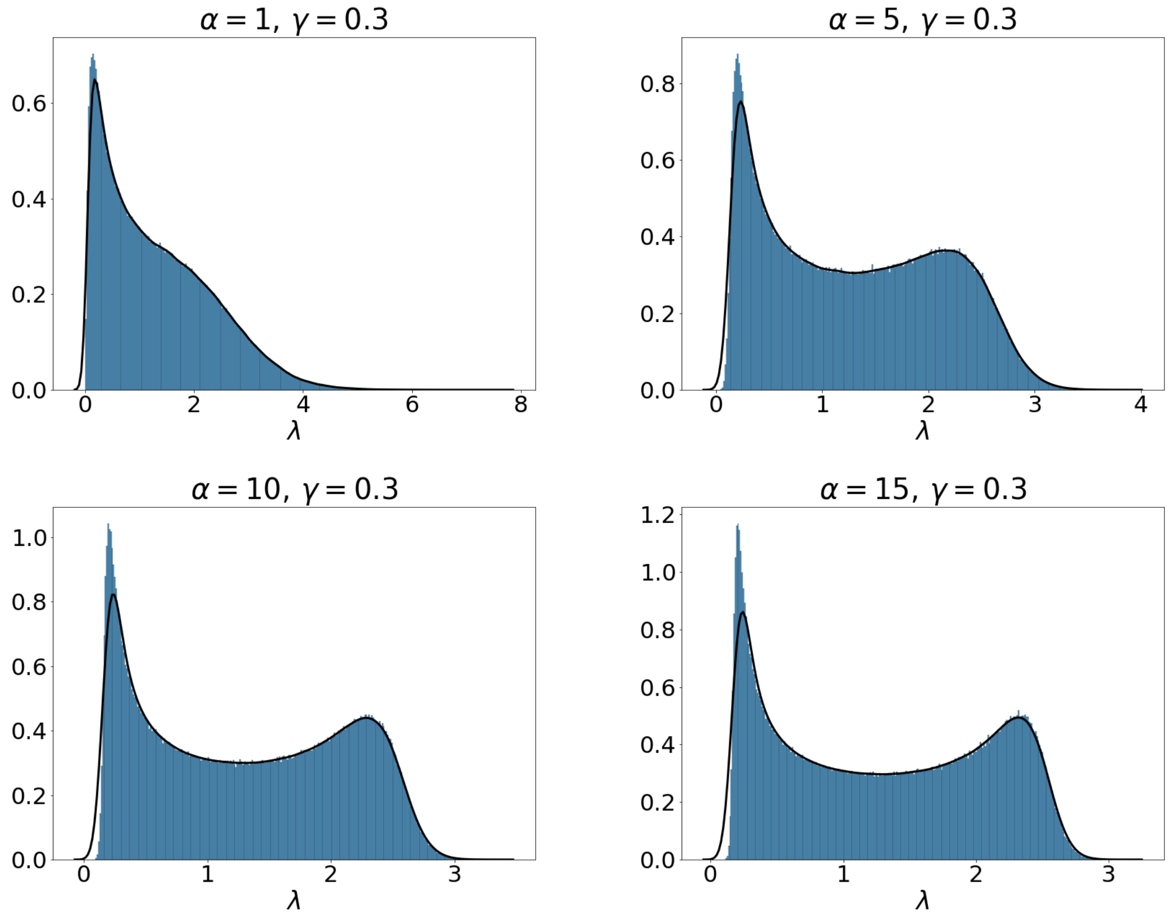


Figure 2.2: Laguerre α ensemble empirical spectral density for different values of the parameters, $N = 500$, trials: 5000. The solid black line is the estimated density, not the actual one.

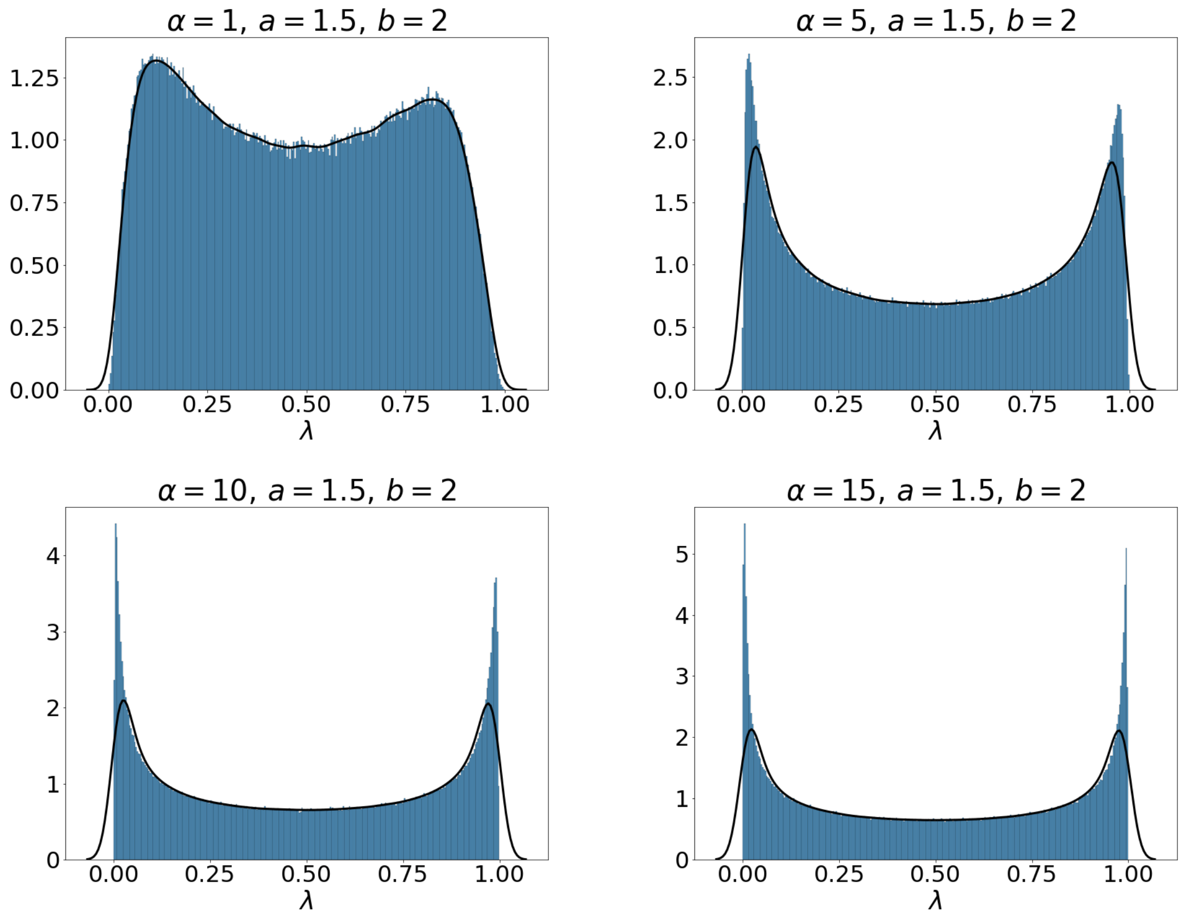


Figure 2.3: Jacobi α ensemble empirical spectral density for different values of the parameters, $N = 500$, trials: 5000. The solid black line is the estimated density, not the actual one.

Chapter 3

Integrable Discrete Non-Linear Schrödinger Equation

In this chapter we study the defocusing Ablowitz-Ladik (AL) lattice for the complex functions $\alpha_j(t)$, $j \in \mathbb{Z}$ and $t \in \mathbb{R}$, which is the system of nonlinear equations

$$i\dot{\alpha}_j = -(\alpha_{j+1} + \alpha_{j-1} - 2\alpha_j) + |\alpha_j|^2(\alpha_{j-1} + \alpha_{j+1}), \tag{3.1}$$

where $\dot{\alpha}_j = \frac{d\alpha_j}{dt}$. We assume N-periodic boundary conditions $\alpha_{j+N} = \alpha_j$, for all $j \in \mathbb{Z}$. The AL lattice was introduced by Ablowitz and Ladik [2, 3] as the spatial discretization of the cubic nonlinear Schrödinger Equation (NLS) for the complex function $\psi(x,t)$, $x \in S^1$ and $t \in \mathbb{R}$:

$$i\partial_t\psi(x,t) = -\frac{1}{2}\partial_x^2\psi(x,t) + |\psi(x,t)|^2\psi(x,t).$$

The cubic NLS equation is an infinite-dimensional integrable system [172]. There are several discretization of the cubic NLS equation and the AL lattice (3.1) is among the several ones that preserve integrability [137]. For applications of the AL lattice see the book [4].

The phase shift $\alpha_j(t) \to e^{-2it}\alpha_j(t)$ transforms the AL lattice into

$$i\dot{\alpha}_j = -\rho_j^2(\alpha_{j+1} + \alpha_{j-1}), \quad \rho_j = \sqrt{1 - |\alpha_j|^2}, \tag{3.2}$$

which is related to the Schur flow [73]. It is straightforward to verify that if $|\alpha_j(0)| < 1$, then $|\alpha_j(t)| < 1$ for all $t > 0$, see [73]. We chose the N-dimensional disc $\mathbb{D}^N$ as the phase space of the AL system, here $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$. On $\mathbb{D}^N$ we introduce the symplectic form [47, 66]

$$\omega = i\sum_{j=1}^N \frac{1}{\rho_j^2} \mathrm{d}\alpha_j \wedge \mathrm{d}\overline{\alpha}_j\,, \quad \rho_j = \sqrt{1 - |\alpha_j|^2}. \tag{3.3}$$

The corresponding Poisson bracket is defined for functions $f, g \in \mathcal{C}^\infty(\mathbb{D}^N)$ as

$$\{f, g\} = i\sum_{j=1}^N \rho_j^2 \left(\frac{\partial f}{\partial \overline{\alpha}_j}\frac{\partial g}{\partial \alpha_j} - \frac{\partial f}{\partial \alpha_j}\frac{\partial g}{\partial \overline{\alpha}_j}\right).$$

The AL equation (3.1) have the Hamiltonian structure

$$\dot{\alpha}_j = \{\alpha_j, H_{AL}\}, \quad H_{AL}(\alpha_j, \overline{\alpha}_j) = -2\ln(K^{(0)}) + K^{(1)} + \overline{K^{(1)}}, \tag{3.4}$$

with $\overline{K^{(1)}}$ complex conjugate of $K^{(1)}$ and the conserved quantities $K^{(0)}$ and $K^{(1)}$ are given by

$$K^{(0)} := \prod_{j=1}^{N} \left(1 - |\alpha_j|^2\right), \quad K^{(1)} := -\sum_{j=1}^{N} \alpha_j \overline{\alpha}_{j+1}. \tag{3.5}$$

We remark that quantity $-2\ln(K^{(0)})$ is the generator of the shift $\alpha_j(t) \to e^{-2it}\alpha_j(t)$, while $H_1 = K^{(1)} + \overline{K^{(1)}}$ generates the flow (3.2).

Integrability. The integrability of the AL lattice was proved by Ablowitz and Ladik by discretizing the 2×2 Zakharov-Shabat Lax pair [1,2], for a comprehensive review see [4]. The integrability of the Ablowitz Ladik system has also been proved by constructing a bi-Hamiltonian structure [10, 47].

More recently different authors (see [5, 126, 147]) worked on the link between orthogonal and biorthogonal polynomials on the unit circle and the Ablowitz–Ladik hierarchy. This is the analogue of the celebrated link between the Toda hierarchy and orthogonal polynomials on the real line. This link was also generalizes to the non-commutative case [27]. The connection between orthogonal polynomials on the unit circle and AL lattice leads to the construction of the so-called "big Lax" matrix that turns out to be a five-diagonal band matrix. Generalization of this construction to other integrable equations has been considered in [128, 137]. Following [126, 147] we double the size of the chain according to the periodic boundary condition, thus we consider a chain of $2N$ particles $\alpha_1, \ldots, \alpha_{2N}$ such that $\alpha_j = \alpha_{j+N}$ for $j = 1, \ldots, N$. Define the 2×2 unitary matrix Ξ_j

$$\Xi_j = \begin{pmatrix} \overline{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix}, \quad j = 1, \ldots, 2N,$$

and the $2N \times 2N$ matrices

$$\mathcal{M} = \begin{pmatrix} -\alpha_{2N} & & & & & \rho_{2N} \\ & \Xi_2 & & & & \\ & & \Xi_4 & & & \\ & & & \ddots & & \\ & & & & \Xi_{2N-2} & \\ \rho_{2N} & & & & & \overline{\alpha}_{2N} \end{pmatrix}, \qquad \mathcal{L} = \begin{pmatrix} \Xi_1 & & & \\ & \Xi_3 & & \\ & & \ddots & \\ & & & \Xi_{2N-1} \end{pmatrix}.$$

Now let us define the Lax matrix

$$\mathcal{E} = \mathcal{L}\mathcal{M}, \tag{3.6}$$

that has the structure

$$\begin{pmatrix} * & * & * & & & & & & & * \\ * & * & * & & & & & & & * \\ & * & * & * & * & & & & & \\ & * & * & * & * & & & & & \\ & & & & & \ddots & \ddots & & & \\ & & & & & & * & * & * & * \\ & & & & & & * & * & * & * \\ * & & & & & & & * & * & * \\ * & & & & & & & * & * & * \end{pmatrix}.$$

The N-periodic AL equation (3.2) is equivalent to the following Lax equation for the matrix $\mathcal{E}$:

$$\dot{\mathcal{E}} = i\left[\mathcal{E}, \mathcal{E}^+ + (\mathcal{E}^+)^\dagger\right], \tag{3.7}$$

where † stands for hermitian conjugate and

$$\mathcal{E}^+_{j,k} = \begin{cases} \frac{1}{2}\mathcal{E}_{j,j} & j = k \\ \mathcal{E}_{j,k} & k = j+1 \bmod N \,\text{or}\, k = j+2 \bmod N \\ 0 & \text{otherwise.} \end{cases}$$

The formulation (3.7) implies that the quantities

$$K^{(\ell)} = \frac{\operatorname{Tr}\left(\mathcal{E}^\ell\right)}{2}, \quad \ell = 1, \dots, N-1, \tag{3.8}$$

are constants of motion for the defocusing AL system. For example

$$K^{(1)} = -\sum_{j=1}^{N} \alpha_j \overline{\alpha}_{j+1}, \quad K^{(2)} = \sum_{j=1}^{N} \left[(\alpha_j \overline{\alpha}_{j+1})^2 - 2\alpha_j \overline{\alpha}_{j+2} \rho^2_{j+1}\right].$$

Furthermore, $K^{(0)}, K^{(1)}, \dots, K^{(N-1)}$ are functionally independent and in involution, showing that the N-periodic AL system is integrable [1–4, 126].

Generalized Gibbs Ensemble for the Ablowitz-Ladik Lattice. The symplectic form ω in (3.3) induces on $\mathbb{D}^N$ the volume form $\mathrm{dvol} = \dfrac{1}{K^{(0)}}\mathrm{d}^2\boldsymbol{\alpha}$, with $\mathrm{d}^2\boldsymbol{\alpha} = \prod_{j=1}^N (i\mathrm{d}\alpha_j \wedge \mathrm{d}\overline{\alpha}_j)$. We observe that $\int_{\mathbb{D}^N} \mathrm{dvol} = \infty$, however we can define the Gibbs measure with respect to the Hamiltonian H_{AL} in (3.4):

$$\frac{1}{Z_\beta} e^{-\frac{\beta}{2}H_{AL}} \mathrm{dvol} = \frac{1}{Z_\beta} e^{\beta \Re(K^{(1)})} \prod_{j=1}^{N} (1 - |\alpha_j|^2)^{\beta-1} \mathrm{d}^2\boldsymbol{\alpha}, \quad \beta > 0, \tag{3.9}$$

where $Z_\beta = \int_{\mathbb{D}^N} e^{\beta\Re(K^{(1)})} \prod_{j=1}^N (1-|\alpha_j|^2)^{\beta-1} \mathrm{d}^2\boldsymbol{\alpha} < \infty$ is the normalizing constant. The above probability measure is clearly invariant under the Hamiltonian flow $\alpha_j(0) \to \alpha_j(t)$ associated to the Ablowitz-Ladik equation (3.1).

Since the Ablowitz-Ladik lattice posses several conserved quantities (3.5)-(3.8), one can introduce a Generalized Gibbs Ensemble on the phase space $\mathbb{D}^N$ in the following way. Fix $\mathbb{N} \ni \kappa < N-1$ and let us define

$$V(z) = \sum_{m=1}^{\kappa} \eta_m \Re(z^m), \tag{3.10}$$

where $\eta_m \in \mathbb{R}$ are called chemical potentials. Then

$$\mathrm{Tr}\left(V\left(\mathcal{E}\right)\right) = \sum_{m=1}^{\kappa} \eta_m (K^{(m)} + \overline{K^{(m)}}),$$

where $K^{(m)}$ are the AL conserved quantities (3.8). The finite volume Generalized Gibbs measure can be written as:

$$\mathrm{d}\mathbb{P}_{AL}(\alpha_1, \dots, \alpha_N) = \frac{1}{Z_N^{AL}(V,\beta)} \prod_{j=1}^{N} \left(1 - |\alpha_j|^2\right)^{\beta-1} \exp\left(-\mathrm{Tr}\left(V\left(\mathcal{E}\right)\right)\right) \mathrm{d}^2\boldsymbol{\alpha}\,, \quad (3.11)$$

where $Z_N^{AL}(V,\beta)$ is the partition function of the system:

$$Z_N^{AL}(V,\beta) = \int_{\mathbb{D}^N} \prod_{j=1}^{N} \left(1 - |\alpha_j|^2\right)^{\beta-1} \exp\left(-\mathrm{Tr}\left(V\left(\mathcal{E}\right)\right)\right) \mathrm{d}^2\boldsymbol{\alpha}\,. \quad (3.12)$$

Choosing the initial data of the Ablowitz-Ladik lattice according to the Generalized Gibbs measure, the Lax matrix $\mathcal{E}$ turns into a random matrix. In [119] Mendl and Spohn study the dynamic of the Ablowitz-Ladik lattice at non-zero temperature. They study numerically correlation functions and in particular, introducing the density $\delta_j = \Re(\alpha_{j+1}\overline{\alpha}_j)$, they study the density-density correlation function

$$\mathbf{E}\left[\delta_j(t)\delta_1(0)\right] - \mathbf{E}\left[\delta_j(t)\right]\mathbf{E}\left[\delta_1(0)\right]$$

where $\mathbf{E}\left[\cdot\right]$ is the expectation with respect to Gibbs measure (3.9). They showed numerically that density-density time correlations in thermal equilibrium have symmetrically located peaks, which travel in opposite directions at constant speed, broaden ballistically and decay as $1/t^{\gamma}$ when $t \to \infty$, where the scaling exponent γ is approximately equal to one. This is behaviour is believed to be typical of integrable nonlinear systems.

More quantitative results have been obtained for linear (integrable) systems and for the Toda lattice, which is a nonlinear integrable system. It was shown in [75] that the fastest peaks of the correlation functions of harmonic oscillators with short range interactions have a Airy type scaling. Namely, the fastest peak of the momentum-momentum and energy-energy correlation functions scales as $t^{-\frac{1}{3}}$, and $t^{-\frac{2}{3}}$ respectively and the shape of the peak is described by the Airy function and its square respectively. Regarding nonlinear integrable systems in [**?**] Spohn was able to connect the Gibbs ensemble of the Toda lattice to the Dumitriu-Edelman β-ensemble [42]. In this way, the generalized Gibbs free energy of the Toda chain turns out to be related to the β-ensembles of random matrix theory in the mean-field regime [7, 44]. The behaviour of the correlation functions of the Toda chains has been derived by applying the theory of generalized hydrodynamic [39, 153]. We mention also the recent work [79], where the authors derive a large deviation principle for the mean density of states for the Generalized Gibbs measure of the Toda lattice.

In this chapter we present the results in [116], and we connect the generalized Gibbs ensemble of the Ablowitz-Ladik lattice to the Killip-Nenciu [93] matrix Circular β-ensemble at high-temperature investigated by Hardy and Lambert [80]. We determine the free energy

$$F_{AL}(V,\beta) = \lim_{N\to\infty} \frac{1}{N} \log Z_N^{AL}(V,\beta),$$

(see Proposition 3.8) and the density of states μ_{AL}^{β} (see Theorem 3.6) of the random Lax matrix $\mathcal{E}$ sampled according to (3.11) in the thermodynamic limit $N \to \infty$. The density of states μ_{AL}^{β} is obtained as follows. Consider the functional

$$\begin{aligned}\mathcal{F}^{(V,\beta)}(\mu) = 2\int_{\mathbb{T}} V(\theta)\mu(\theta)\mathrm{d}\theta - \beta \int\int_{\mathbb{T}\times\mathbb{T}} \ln\sin\left(\frac{|\theta-\phi|}{2}\right)\mu(\theta)\mu(\phi)\mathrm{d}\theta\mathrm{d}\phi \\ + \int_{\mathbb{T}} \ln\left(\mu(\theta)\right)\mu(\theta)\mathrm{d}\theta + \ln(2\pi)\,,\end{aligned} \tag{3.13}$$

where $\mu \in \mathcal{M}(\mathbb{T})$ with $\mathcal{M}(\mathbb{T})$ the space of probability measures on the torus $\mathbb{T} = [-\pi, \pi]$. Such functional has a unique minimizer $\mu_{HT}^{\beta}(\mathrm{d}\theta) = \mu_{HT}^{\beta}(\theta)\mathrm{d}\theta$, [145], that describes the density of states of the β-ensembles at high temperature [80]. For finite β and smooth potentials $V(\theta)$, it has been shown by Hardy and Lambert in [80] that the minimizer $\mu_{HT}^{\beta}(\mathrm{d}\theta)$ has a smooth density and its support is the whole torus $\mathbb{T}$. The minimizer $\mu_{HT}^{\beta}(\mathrm{d}\theta)$ of (3.13) and its minimum value

$$F_{HT}(V,\beta) := \mathcal{F}^{(V,\beta)}(\mu_{HT}^{\beta}),$$

are related to the density of states $\mu_{AL}^{\beta}(\theta)$ (Theorem 3.6) and the free energy $F_{AL}(V,\beta)$ (Proposition 3.8) of the AL lattice by the relations

$$\mu_{AL}^{\beta}(\theta) = \partial_\beta\left(\beta\mu_{HT}^{\beta}(\theta)\right), \quad F_{AL}(V,\beta) = \partial_\beta\left(\beta F_{HT}(V,\beta)\right) + \ln(2).$$

The particular case $V(\theta) = \eta\cos\theta$ corresponds to the free energy associated to the Schur flow (3.2), and we show that the minimizer of the functional (3.13) is obtained via a particular solution of the Double Confluent Heun (DCH) equation:

$$z^2v''(z) + \left(-\eta + z(\beta+1) + \eta z^2\right)v'(z) + \eta\beta(z+\lambda)v(z) = 0\,,$$

where $'$ and $''$ denote the first and second derivative with respect to the argument. The density of states $\mu_{HT}^{\beta}(\theta)$ is recovered from the unique smooth solution (up to a multiplicative non-zero constant) of the DCH equation (see Theorem 3.9) by the relation

$$\mu_{HT}^{\beta}(\theta) = \frac{1}{2\pi} + \frac{1}{\pi\beta}\Re\left(\frac{e^{i\theta}v'(e^{i\theta})}{v(e^{i\theta})}\right)\,. \tag{3.14}$$

The parameter $\lambda = \lambda(\eta,\beta)$ in (3.36) is a transcendental function that is related to the Painlevé III equation [53]. For the case $V = 0$ it was shown in [80] that the minimizer is the uniform measure on the circle, while for the case $V(\theta) \to \beta V(\theta)$ and $\beta \to \infty$ the minimizer of (3.13) was considered in [117] and the particular case $\beta V(\theta) = \beta\eta\cos\theta$ has first been considered by Gross Witten [78] and Baik-Deift-Johansson [14]. Therefore the measure (3.14) generalizes the result by Gross and Witten [78] and Baik-Deift-Johansson [14] to the high-temperature regime.

This chapter is organized as follows. In section 3.1 we introduce the Circular β ensemble and its high-temperature limit. In section 3.2 we state and prove our main results, namely Theorem 3.6 and Theorem 3.9. Finally, the most technical part of our arguments is deferred to section 3.3.

3.1 Circular beta Ensemble at high-temperature

The Circular Ensemble at temperature $\tilde{\beta}^{-1}$ is a system of N identical particles on the one-dimensional torus $\mathbb{T} = [-\pi, \pi]$ with distribution

$$\mathrm{d}\mathbb{P}_{\tilde{\beta}}(\theta_1, \ldots, \theta_N) = \frac{1}{Z_N^{C\tilde{\beta}E}} \prod_{j<\ell} |e^{i\theta_\ell} - e^{i\theta_j}|^{\tilde{\beta}} \mathrm{d}\boldsymbol{\theta}, \quad \mathrm{d}\boldsymbol{\theta} = \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_N \,, \tag{3.15}$$

where $Z_N^{C\tilde{\beta}E} > 0$ is the norming constant, or partition function of the system. For $\tilde{\beta} = 1, 2, 4$ Dyson observed that the above measure corresponds to the eigenvalue distribution of unitary symmetric, unitary and unitary symplectic random matrix ensembles (see e.g. [54, 118]). For general $\tilde{\beta} > 0$, Killip and Nenciu proved that the Circular beta Ensemble can be associated to the eigenvalue distribution of a random sparse matrix, the so called CMV matrix, after Cantero, Moral, Velázquez [28]. To state their result, we need the following definition.

Definition 3.1. *A complex random variable X with values on the unit disk $\mathbb{D}$ is Θ_ν-distributed ($\nu > 1$) if*

$$\mathbf{E}\left[f(X)\right] = \frac{\nu - 1}{2\pi} \int_{\mathbb{D}} f(z)(1 - |z|^2)^{\frac{\nu-3}{2}} \mathrm{d}^2 z \,.$$

In the case $\nu = 1$, let Θ_1 be the uniform distribution on the unit circle S^1.

We recall that for integer $\nu \geqslant 2$, such measure has the following geometrical interpretation: if u is chosen at random according to the surface measure on the unit sphere S^ν in $\mathbb{R}^{\nu+1}$, then $u_1 + iu_2$ is Θ_ν−distributed. We can now state the result of Killip-Nenciu.

Theorem 3.2 (cf. [126] Theorem 1)**.** *Consider the block diagonal $N \times N$ matrices*

$$M = diag\left(\Xi_1, \Xi_3, \Xi_5 \ldots,\right) \qquad and \qquad L = diag\left(\Xi_0, \Xi_2, \Xi_4, \ldots\right) \,,$$

where the block Ξ_j, $j = 1, \ldots, N-1$, takes the form

$$\Xi_j = \begin{pmatrix} \overline{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix} \,, \quad \rho_j = \sqrt{1 - |\alpha_j|^2},$$

while $\Xi_0 = (1)$ and $\Xi_N = (\overline{\alpha}_N)$ are 1×1 matrices. Define the $N \times N$ sparse matrix

$$E = LM,$$

and suppose that the entries α_j are independent complex random variables with $\alpha_j \sim \Theta_{\tilde{\beta}(N-j)+1}$ for $1 \leqslant j \leqslant N-1$ and α_N is uniformly distributed on the unit circle. Then the eigenvalues of E are distributed according to the Circular Ensemble (3.15) *at temperature $\tilde{\beta}$.*

We observe that each of the matrices Ξ_j is unitary, and so are the matrices L and M. As a result, the eigenvalues of E clearly lie on the unit circle. The matrix E is a 5-diagonal unitary matrix that takes the form

$$E = \begin{pmatrix} \bar{\alpha}_1 & \rho_1\bar{\alpha}_2 & \rho_1\rho_2 & & & & & & \\ \rho_1 & -\alpha_1\bar{\alpha}_2 & -\alpha_1\rho_2 & & & & & & \\ & \rho_2\bar{\alpha}_3 & -\alpha_2\bar{\alpha}_3 & \rho_3\bar{\alpha}_4 & \rho_3\rho_4 & & & & \\ & \rho_2\rho_3 & -\alpha_2\rho_3 & -\alpha_3\bar{\alpha}_4 & -\alpha_3\rho_4 & & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & & \rho_{N-3}\bar{\alpha}_{N-2} & -\alpha_{N-3}\bar{\alpha}_{N-2} & \rho_{N-2}\bar{\alpha}_{N-1} & \rho_{N-2}\rho_{N-1} \\ & & & & & \rho_{N-3}\rho_{N-2} & -\alpha_{N-3}\rho_{N-2} & -\alpha_{N-2}\bar{\alpha}_{N-1} & -\alpha_{N-2}\rho_{N-1} \\ & & & & & & & \bar{\alpha}_N\rho_{N-1} & -\alpha_{N-1}\bar{\alpha}_N \end{pmatrix} . \tag{3.16}$$

We are interested in the probability distribution (3.15) when

- we add an external field, namely $d\theta_i \to e^{-2V(\theta_i)}d\theta_i$ with $V:\mathbb{T}\to\mathbb{R}$ a differentiable potential;
- we consider the limit $\widetilde{\beta}\to 0$ and $N\to\infty$ in such a way that $\widetilde{\beta}N = 2\beta$, $\beta>0$. Since $\widetilde{\beta}$ is interpreted as the inverse of the temperature, such limit is called *high-temperature regime.*

With the above changes, we arrive at the probability distribution of the Circular ensemble at high-temperature, and with an external potential:

$$\mathrm{d}\mathbb{P}^V_\beta(\theta_1,\dots,\theta_N) = \frac{1}{\mathcal{Z}_N^{HT}(V,\beta)}\prod_{j<\ell}|e^{i\theta_\ell}-e^{i\theta_j}|^{\frac{2\beta}{N}}e^{-2\sum_{j=1}^N V(\theta_j)}\mathrm{d}\boldsymbol{\theta}\,, \tag{3.17}$$

where $\mathcal{Z}_N^{HT}(V,\beta)$ is the normalizing constant or the partition function of the system. Also in this case, we can associate to the above probability distribution a random CMV matrix. The lemma below has probably already appeared in the literature, but for completeness we provide the proof.

Lemma 3.3. *Let E be the CMV matrix* (3.16)*. Consider the block $2N\times 2N$ matrix*

$$\widetilde{E} = diag(E,E)\,, \tag{3.18}$$

whose entries are distributed according to

$$\mathrm{d}\mathbb{P}(\alpha_1,\dots,\alpha_N) = \frac{1}{Z_N^{HT}(V,\beta)}\prod_{j=1}^{N-1}\left(1-|\alpha_j|^2\right)^{\beta\left(1-\frac{j}{N}\right)-1}e^{-Tr(V(\widetilde{E}))}\prod_{j=1}^{N-1}\mathrm{d}^2\alpha_j\frac{\mathrm{d}\alpha_N}{i\alpha_N}\,. \tag{3.19}$$

Then the eigenvalues of $\widetilde{E}$ are all double, they lie on the unit circle and are distributed according to (3.17)*.*
Moreover

$$Z_N^{HT}(V,\beta) = 2^{1-N}\frac{\Gamma\left(\frac{\beta}{N}\right)^N}{\Gamma(\beta)}\mathcal{Z}_N^{HT}(V,\beta)\,, \tag{3.20}$$

where $\mathcal{Z}_N^{HT}(V,\beta)$ is the norming constant of the probability distribution (3.19)*.*

Proof. First, we notice that the eigenvalues of $\widetilde{E}$ are all double, since it is a block diagonal matrix with two identical blocks.

We consider the change of variables $\alpha_N\to e^{i\varphi}$, thus (3.19) becomes:

$$\mathrm{d}\mathbb{P}(\alpha_1,\dots,\alpha_{N-1},\varphi) = \frac{1}{Z_N^{HT}(V,\beta)}\prod_{j=1}^{N-1}\left(1-|\alpha_j|^2\right)^{\beta\left(1-\frac{j}{N}\right)-1}e^{-\mathrm{Tr}(V(\widetilde{E}))}\prod_{j=1}^{N-1}\mathrm{d}^2\alpha_j\mathrm{d}\varphi\,. \tag{3.21}$$

Now, let $e^{i\theta_1},\dots,e^{i\theta_N}$ be the eigenvalues of the CMV matrix E endowed with probability (3.19), and let $q_1,\dots,q_N$ be the entries of the first row of the unitary matrix Q such that $Q^\dagger\Theta Q = E$ where $\Theta = \mathrm{Diag}(e^{i\theta_1},\dots,e^{i\theta_N})$ and $\sum_{k=1}^N|q_k|^2=1$. Introduce the variable $\gamma_j = |q_j|^2$ for $j=1,\dots,N$, then From [93, Lemma 4.1, and Proposition 4.2 relation (4.14)] we have that

$$|\Delta(e^{i\boldsymbol{\theta}})|^2 \prod_{j=1}^{N} \gamma_j = \prod_{j=1}^{N-1} \left(1-|\alpha_j|^2\right)^{(N-j)} \,, \tag{3.22}$$

$$\left|\frac{\partial\left(\alpha_1,\ldots,\alpha_{N-1},\varphi\right)}{\partial(\boldsymbol{\theta},\boldsymbol{\gamma})}\right| = 2^{1-N}\frac{\prod_{j=1}^{N-1}\left(1-|\alpha_j|^2\right)}{\prod_{j=1}^{N}\gamma_j} \,, \tag{3.23}$$

here $\boldsymbol{\gamma} = (\gamma_1,\ldots,\gamma_{N-1})$, and $\Delta(e^{i\boldsymbol{\theta}}) = \prod_{j<\ell}\left(e^{i\theta_j}-e^{i\theta_\ell}\right)$. Applying the previous equalities to (3.21) we get that:

$$\begin{aligned}
\mathrm{d}\mathbb{P}(\alpha_1,\ldots,\alpha_{N-1},\varphi)) &= \frac{e^{-\mathrm{Tr}(V(\widetilde{E}))}}{Z_N^{HT}(V,\beta)}\mathrm{d}\varphi \prod_{j=1}^{N-1}\left(1-|\alpha_j|^2\right)^{\beta\left(1-\frac{j}{N}\right)-1}\mathrm{d}\alpha_j\mathrm{d}\overline{\alpha}_j \\
&\overset{(3.23)}{=} \frac{1}{Z_N^{HT}(V,\beta)}\frac{2^{1-N}}{\prod_{j=1}^{N}\gamma_j}\prod_{j=1}^{N-1}\left(1-|\alpha_j|^2\right)^{\beta\left(1-\frac{j}{N}\right)} e^{-2\sum_{j=1}^{N}V(e^{i\theta_j})}\mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{\gamma} \\
&\overset{(3.22)}{=} \frac{1}{Z_N^{HT}(V,\beta)}2^{1-N}|\Delta(e^{i\boldsymbol{\theta}})|^{\frac{2\beta}{N}}\prod_{j=1}^{N}\gamma_j^{\frac{\beta}{N}-1}e^{-2\sum_j V(e^{i\theta_j})}\mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{\gamma}\,.
\end{aligned}$$

Thus, we get the relation

$$Z_N^{HT}(V,\beta) = 2^{1-N}\mathcal{Z}_N^{HT}(V,\beta)\int_\Delta \prod_{j=1}^{N}\gamma_j^{\frac{\beta}{N}-1}\mathrm{d}\gamma_1\ldots\mathrm{d}\gamma_{N-1}\,,$$

here Δ is the simplex $\sum_{j=1}^{N}\gamma_j = 1$.

The above integral is a well-known Dirichlet integral that can be computed explicitly (see [93, Lemma 4.4])

$$\int_\Delta \prod_{j=1}^{N}\gamma^{\frac{\beta}{N}-1}\mathrm{d}\gamma_1\ldots\mathrm{d}\gamma_{N-1} = \frac{\Gamma\left(\frac{\beta}{N}\right)^N}{\Gamma(\beta)}\,,$$

proving (3.20).

□

Let $e^{i\theta_1},\ldots,e^{i\theta_N}$ be the double eigenvalues of the CMV Matrix $\tilde{E}$ in (3.18), whose entries are distrbuted according to (3.21). The *empirical measure* is the random probability measure

$$\mu_N = \frac{1}{N}\sum_{j=1}^{N}\delta_{e^{i\theta_j}}\,. \tag{3.24}$$

The mean density of state μ_{HT}^{β} is defined as the non-random probability measure such that

$$\int_{\mathbb{T}} f(\theta)\mu_{HT}^{\beta}(\mathrm{d}\theta) = \lim_{N\to\infty}\mathbf{E}\left[\int_{\mathbb{T}} f(\theta)\mu_N(\mathrm{d}\theta)\right]\,, \tag{3.25}$$

for all continuous function f on the torus $\mathbb{T}$, and the expected value is taken with respect to (3.19). In order to discuss the large N limit of μ_N we have to introduce several quantities. Let $\mathcal{M}(\mathbb{T})$ be the set of probability measures on the one-dimensional torus $\mathbb{T}$ and for $\mu \in \mathcal{M}(\mathbb{T})$ we consider the logarithmic energy [145]

$$E(\mu) := \iint_{\mathbb{T}\times\mathbb{T}} \ln \left| \sin\left(\frac{\theta-\phi}{2}\right)\right|^{-1} \mu(\mathrm{d}\theta)\mu(\mathrm{d}\phi)\,.$$

We define the relative entropy with respect to $\mu_0(\mathrm{d}\theta) = \dfrac{\mathrm{d}\theta}{2\pi}$ as

$$K(\mu|\mu_0) := \int_{\mathbb{T}} \log\left(\frac{\mu}{\mu_0}\right)\mu(\mathrm{d}\theta) \quad \in [0, +\infty]\,,$$

when μ is absolutely continuous with respect to μ_0 and otherwise $K(\mu|\mu_0) := +\infty$. The relevant functional is

$$\mathcal{F}^{(V,\beta)}(\mu) := \beta E(\mu) + K(\mu|\mu_0) + 2\int_{\mathbb{T}} V(\theta)\mu(\mathrm{d}\theta).$$

When $\mathcal{F}^{(V,\beta)}(\mu)$ is finite, it follows that μ is absolutely continuous with respect to the Lebesgue measure μ_0 and we can write $\mu(\mathrm{d}\theta) = \mu(\theta)\mathrm{d}\theta$. We denote by $C^{n,1}(\mathbb{T})$ with $n = 0, 1, 2, \dots$ the space of n-times differentiable functions whose n-derivative is also Lipschitz continuous.

The following result describes the limiting measure μ_{HT}^{β} in (3.25).

Theorem 3.4. *(cf. [80, Proposition 2.1 and 2.5]) Let $\mathcal{M}(\mathbb{T})$ be the set of probability measures on the one-dimensional torus and $V : \mathbb{T} \to \mathbb{R}$ be a measurable and bounded function. For any $\beta > 0$ consider the functional $\mathcal{F}^{(V,\beta)} : \mathcal{M}(\mathbb{T}) \to [0, \infty]$*

$$\mathcal{F}^{(V,\beta)}(\mu) = 2\int_{\mathbb{T}} V(\theta)\mu(\theta)\mathrm{d}\theta + \beta E(\mu) + \int_{\mathbb{T}} \ln\left(\mu(\theta)\right)\mu(\theta)\mathrm{d}\theta + \ln(2\pi)\,. \tag{3.26}$$

Then

(a) the functional $\mathcal{F}^{(V,\beta)}(\mu)$ has a unique minimizer $\mu_{HT}^{\beta}(\mathrm{d}\theta) = \mu_{HT}^{\beta}(\theta)\mathrm{d}\theta$ in $\mathcal{M}(\mathbb{T})$;

(b) μ_{HT}^{β} is absolutely continuous with respect to the Lebesgue measure and there is $0 < \delta < 1$ such that

$$\delta \leqslant \frac{\mu_{HT}^{\beta}(\theta)}{2\pi} \leqslant \delta^{-1}, \quad a.e.\,;$$

(c) if $V = 0$, then $\mu_{HT}^{\beta}(\mathrm{d}\theta) = \frac{1}{2\pi}\mathrm{d}\theta$;

(d) if $V \in C^{m,1}(\mathbb{T})$, then $\mu_{HT}^{\beta} \in C^{m,1}(\mathbb{T})$;

(d) the empirical measure μ_N in (3.24) converges weakly and almost surely to the measure μ_{HT}^{β} as $N \to \infty$.

From the above theorem when the potential V is at least $C^{2,1}(\mathbb{T})$ the minimizer of the function $\mathcal{F}^{(V,\beta)}$ is characterized by the Euler-Lagrange equations

$$\frac{\delta \mathcal{F}^{(V,\beta)}}{\delta \mu} = 2V(\theta) - 2\beta \int_{\mathbb{T}} \ln \sin\left(\frac{|\theta-\phi|}{2}\right)\mu(\phi)\mathrm{d}\phi + \ln \mu(\theta) + 1 = C(V,\beta) \quad (3.27)$$

where $C(V, \beta)$ is a constant in θ.

For convenience, we define $F_{HT}(V, \beta)$ as the value of the functional at the minimizer, namely

$$F_{HT}(V,\beta) := \mathcal{F}^{(V,\beta)}(\mu^{\beta}_{HT}).$$

The quantity $F_{HT}(V, \beta)$ is referred to as free energy of the Circular beta ensemble at high temperature. It is a standard result that (see e.g. [65])

$$F_{HT}(V,\beta) = -\lim_{N\to\infty} \frac{1}{N} \log \mathcal{Z}_N^{HT}(V,\beta) \quad (3.28)$$

where the partition function $\mathcal{Z}_N^{HT}(V, \beta)$ of the Circular beta ensemble at high-temperature is defined in (3.17).

Remark 3.5. *We notice that from* (3.20) *and* (3.28) *we can also obtain the free energy $F_{HT}(V, \beta)$ from the partition function $Z_N^{HT}(V, \beta)$ of the CMV matrix ensemble* (3.17)*, namely:*

$$F_{HT}(V,\beta) = -\lim_{N\to\infty} \frac{\ln(Z_N^{HT}(V,\beta))}{N} - \ln(2)\,.$$

For completeness, we mention that the literature related to the high-temperature regime for the classical beta ensemble is wide. In [7,8,44,61,80,159,160] the authors explicitly computed the mean density of states for the classical Gaussian, Laguerre, Jacobi, and Circular beta ensemble at high-temperature. In [7,8,61,80] the densities of states are computed as a solution of some particular ordinary differential equation. On the other hand, in [44, 159, 160] the authors reconstruct the densities from the moment generating functions. Several authors [18,99,124,125,131,159] investigated the local fluctuations of the eigenvalues, they observed that in this regime they are described by a Poisson process. In particular, in [99] Lambert studied the local fluctuations for general Gibbs ensembles on N-dimensional manifolds, moreover he also studied the asymptotic behaviour of the maximum eigenvalue for the classical beta ensemble at high-temperature. In [57, 61] the loop equations for the classical beta ensemble at high-temperature are studied, in particular in [57] a special kind of duality between high and low temperature is underlined. There are also some results for higher dimensional Coulomb gases [6, 144].

3.2 Statement of the Results

The generalized Gibbs ensemble of the Ablowitz Ladik lattice in (3.11) is very close to the probability distribution (3.19) of the Circular beta ensemble at high-temperature with an external source. Indeed, the only difference between the two ensembles is the exponent of the terms $(1 - |\alpha_j|)$. Our main results are contained in Theorem 3.6 below which relates the mean density of states of the Ablowitz-Ladik lattice to the mean density of states of the Circular beta ensemble at high-temperature, and Theorem 3.9 which derives the mean density of states for the potential $V(z) = \eta\Re(z)$ via a particular solution of the double confluent Heun equation.

Theorem 3.6. *Consider $\beta > 0$ and a potential $V(z)$ as in* (3.10) *smooth and absolutely bounded on the unit circle $\mathbb{T}$. The mean density of states μ_{AL}^{β} of the Ablowitz Ladik Lax matrix $\mathcal{E}$ in* (3.6) *endowed with the probability* (3.11) *is absolutely continuous with respect to the Lebesgue measure and takes the form*

$$\mu_{AL}^{\beta}(\mathrm{d}\theta) = \mu_{AL}^{\beta}(\theta)\mathrm{d}\theta\,, \quad \mu_{AL}^{\beta}(\theta) = \partial_\beta\left(\beta\mu_{HT}^{\beta}(\theta)\right),$$

where μ_{HT}^{β} is the unique minimizer of the functional (3.26) *and the derivative is made in weak sense.*

To prove the result, we will use the *moment matching technique*. We will prove that the derivative with respect to β of μ_{HT}^{β} is well-defined in $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$, meaning that there exists a unique function $\partial_\beta(\mu_{HT}^{\beta}) \in L^2(\mathbb{T})$ such that

$$\partial_\beta\left(\int_{\mathbb{T}} f\mu_{HT}^{\beta}\mathrm{d}\theta\right) = \int_{\mathbb{T}} f\partial_\beta(\mu_{HT}^{\beta})\mathrm{d}\theta\,,$$

for all bounded and continuous f, and that the moment sequence of the measure with density $\partial_\beta\left(\beta\mu_{HT}^{\beta}(\theta)\right)$ coincides with the one of the mean density of states of the Ablowitz-Ladik lattice $\mu_{AL}^{\beta}(\theta)$. Then, we will use the following Lemma to prove that the two measures coincide.

Lemma 3.7. *([13, Lemma B.1 - B.2]) Let $\mathrm{d}\sigma, \mathrm{d}\sigma'$ be two measures with the same moment sequence $\{u^{(l)}\}_{l\geqslant 0}$. If*

$$\liminf_{l\to\infty} \frac{(u^{(2l)})^{\frac{1}{2l}}}{l} < \infty\,,$$

then $\mathrm{d}\sigma = \mathrm{d}\sigma'$.

Next, we define the *free energy* for the Ablowitz-Ladik lattice as:

$$F_{AL}(V,\beta) = -\lim_{N\to\infty}\frac{1}{N}\ln Z_N^{AL}(V,\beta)\,, \tag{3.29}$$

where the partition function $Z_N^{AL}(V,\beta)$ is defined in (3.12). The next proposition shows that the free energy $F_{AL}(V,\beta)$ of the Generalized Gibbs ensemble of the Ablowitz-Ladik lattice and the free energy $F_{HT}(V,\beta)$ in (3.28) of the Circular beta ensemble at high-temperature are related and this fact allows us to calculate the moments of the mean density of states of the CMV matrix E in (3.16) and of the Lax matrix $\mathcal{E}$ in (3.6).

Proposition 3.8. *The free energy $F_{AL}(V,\beta)$ in* (3.29) *of the AL lattice and the free energy $F_{HT}(V,\beta)$ in* (3.28) *of the Circular beta ensemble at high-temperature are differentiable with respect to β, and are related by*

$$\partial_\beta\left(\beta F_{HT}(V,\beta)\right) + \ln(2) = F_{AL}(V,\beta). \tag{3.30}$$

The moments of the density of states μ_{AL}^{β} of the Lax matrix $\mathcal{E}$ in (3.6) *endowed with the probability measure* (3.11) *and the moments of the density of states μ_{HT} of the*

Circular beta ensemble in the high-temperature regime (3.19) *are related to the free energies* $F_{AL}(V,\beta)$ *and* $F_{HT}(V,\beta)$ *by*

$$\begin{aligned}\int_{\mathbb{T}} e^{i\theta m}\mu_{AL}^{\beta}(\mathrm{d}\theta) &= \partial_t F_{AL}\left(V+\frac{t}{2}\Re(z^m),\beta\right)_{|t=0}\,,\\ \int_{\mathbb{T}} e^{i\theta m}\mu_{HT}^{\beta}(\mathrm{d}\theta) &= \partial_t F_{HT}\left(V+\frac{t}{2}\Re(z^m),\beta\right)_{|t=0}\,.\end{aligned} \tag{3.31}$$

Since the proof of this proposition is rather technical, we postpone it to section 3.3.1. We are now ready to prove the first main Theorem 3.6.

Proof of Theorem 3.6. First, we will prove that the derivative with respect to β of the density μ_{HT}^{β} is well-defined in $L^2(\mathbb{T})$. We notice that, since we are considering just smooth potential $V(z)$ as (3.10), from Theorem 3.4 point (*d*) $\mu_{HT}^{\beta}(\theta)\in C^{\infty}(\mathbb{T})$, so we can expand it in Fourier series as

$$\mu_{HT}^{\beta}(\theta)=\sum_{n\in\mathbb{Z}} d_n e^{i\theta n}\,, \tag{3.32}$$

where $d_n=\int e^{i\theta n}\mu_{HT}^{\beta}(\mathrm{d}\theta)$ are the moments of $\mu_{HT}^{\beta}(\mathrm{d}\theta)$ and they decay faster than $n^{-\ell}$ for any positive integer ℓ and any $\beta>0$. Further, since $\mu_{HT}^{\beta}(\theta)$ is a probability density, then $d_0=1$ and the remaining moments satisfy the symmetry

$$\overline{d_n}=d_{-n},\quad n\geqslant 1.$$

By Proposition 3.8, we can differentiate the moments d_n with respect to β and therefore we can formally compute the derivative with respect to β of the density $\mu_{HT}^{\beta}(\theta)$ by differentiating its Fourier expansion term by term

$$\partial_\beta\mu_{HT}^{\beta}(\theta)=\sum_{n\in\mathbb{Z}}\partial_\beta(d_n)e^{i\theta n}\,.$$

In order to prove that $\partial_\beta\mu_{HT}^{\beta}(\theta)$ is well-defined, we will show that its Fourier expansion defines a function in $L^2(\mathbb{T})$.

For this purpose, we define the moments of $\mu_{AL}^{\beta}(\mathrm{d}\theta)$ as $c_n=\int e^{i\theta n}\mu_{AL}^{\beta}(\mathrm{d}\theta)$ for all $\mathbb{N}\ni n\geqslant 1$. From the definition of mean density of states (3.25), we have that:

$$c_n=\lim_{N\to\infty}\frac{\mathbf{E}\left[\mathrm{Tr}(\mathcal{E}^n)\right]}{2N}\,,$$

where the expectation is taken with respect to the probability distribution (3.11). Since the eigenvalues of $\mathcal{E}$ lie on the unit circle, we get the following chain of inequalities:

$$\left|\mathbf{E}\left[\mathrm{Tr}(\mathcal{E}^n)\right]\right|\leqslant\mathbf{E}\left[\left|\mathrm{Tr}(\mathcal{E}^n)\right|\right]\leqslant 2N\,,$$

which implies that:

$$|c_n|\leqslant 1\,.$$

Thus, from Lemma 3.7, we obtain that the measure $\mu_{AL}^{\beta}(\mathrm{d}\theta)$ is uniquely characterized by its moments.

Further, as a corollary of Proposition 3.8 we have the following equality:

$$\partial_\beta d_n = \frac{c_n - d_n}{\beta}\,,$$

thus $|\partial_\beta d_n| \leqslant 2\beta^{-1}$.

Differentiating the Euler-Lagrange equation (3.27) at the minimizer $\mu_{HT}^{\beta}(\theta)$ with respect to θ we obtain the following integral equation (see [80, Proposition 2.5]):

$$\partial_\theta \mu_{HT}^{\beta}(\theta) + \mu_{HT}^{\beta}(\theta)[2\partial_\theta(V(\theta)) + \beta H \mu_{HT}^{\beta}(\theta)] = 0\,, \tag{3.33}$$

where H is the Hilbert transform defined on $L^2(\mathbb{T})$ as

$$H\mu_{HT}^{\beta}(\theta) = -\text{p.v.}\int_{\mathbb{T}} \cot\left(\frac{\theta-\phi}{2}\right)\mu_{HT}^{\beta}(\phi)\mathrm{d}\phi$$

and p.v. is the Cauchy principal value, that is the limit as $\varepsilon \to 0$ of the integral on the torus $\mathbb{T}$ restricted to the domain $|e^{i\theta} - e^{i\phi}| > \varepsilon$. We notice that the Hilbert transform H is diagonal on the bases of exponential $\{e^{in\theta}\}_{n\in\mathbb{Z}}$, meaning that

$$He^{in\theta} = 2\pi i \operatorname{sgn}(n) e^{in\theta}\,,$$

where $\operatorname{sgn}(\cdot)$ is the sign function with the convention that $\operatorname{sgn}(0) = 0$.

Therefore, substituting the Fourier expansion (3.32) of μ_{HT}^{β} into (3.33) we get the following equation

$$\sum_{n\in\mathbb{Z}} 2\sum_{\ell=1}^{\kappa} \eta_\ell d_n\left(z^{n+\ell} - z^{n-\ell}\right) + \sum_{n\in\mathbb{Z}} n d_n z^n + 2\pi\beta\left(\sum_{n\geqslant 1}\sum_{s=1}^{n} d_s d_{n-s} z^n - \sum_{n\geqslant 1}\sum_{s=1}^{n} d_{-s} d_{s-n} z^{-n}\right) = 0\,, \tag{3.34}$$

where $e^{in\theta} = z^n$ and $V(z) = \sum_{\ell=1}^{\kappa}\eta_\ell\left(z^\ell + z^{-\ell}\right)$. Equating terms of the same order in (3.34) we obtain the following recurrence relation for the moments d_n:

$$\begin{cases} \sum_{\ell=1}^{\kappa}\eta_\ell \Im(d_\ell) = 0\,, & n = 0\,,\\ 2\sum_{\ell=1}^{\kappa}\eta_\ell(d_{n-\ell} - d_{n+\ell}) + n d_n + 2\pi\beta\sum_{\ell=1}^{n} d_{n-\ell}d_\ell = 0\,, & n > 0\,,\\ 2\sum_{\ell=1}^{\kappa}\eta_\ell(d_{n-\ell} - d_{n+\ell}) + n d_n - 2\pi\beta\sum_{\ell=1}^{|n|} d_{n+\ell}d_{-\ell} = 0\,, & n < 0\,. \end{cases}$$

From the above recurrence relation, we notice that the moments d_n, $n > \kappa$, are uniquely defined by the moments $d_1,\ldots,d_\kappa$, while the negative moments are obtained by symmetry $d_{-n} = \overline{d_n}$. By differentiating the above recurrence relation with respect to β we get a new recurrence for the sequence $\{\partial_\beta d_n\}_{n\in\mathbb{Z}}$:

$$\begin{cases} \sum_{\ell=1}^{\kappa}\eta_\ell(\partial_\beta d_{-\ell} - \partial_\beta d_\ell) = 0\,, \quad n = 0\\ 2\sum_{\ell=1}^{\kappa}\eta_\ell(\partial_\beta d_{n-\ell} - \partial_\beta d_{n+\ell}) + n\partial_\beta d_n \\ \qquad + 2\pi\sum_{\ell=1}^{n} d_{n-\ell}d_\ell + 2\pi\beta\left(\sum_{\ell=1}^{n}\partial_\beta d_{n-\ell}d_\ell + \sum_{\ell=1}^{n} d_{n-\ell}\partial_\beta d_\ell\right) = 0\,, \quad n > 0\\ 2\sum_{\ell=1}^{\kappa}\eta_\ell(\partial_\beta d_{n-\ell} - \partial_\beta d_{n+\ell}) + n\partial_\beta d_n - 2\pi\sum_{\ell=1}^{|n|} d_{n+\ell}d_{-\ell} \\ \qquad - 2\pi\beta\left(\sum_{\ell=1}^{|n|}\partial_\beta d_{n+\ell}d_{-\ell} + \sum_{\ell=1}^{|n|} d_{n+\ell}\partial_\beta d_{-\ell}\right) = 0\,, \quad n < 0 \end{cases}$$

Recalling that definitely $|d_n| \leqslant n^{-\ell}$, for any $\ell \in \mathbb{N}$, and $|\partial_\beta d_n| \leqslant 2\beta^{-1}$, we deduce from the above recurrence expressions that, for n large enough, there exists a constant $\widetilde{C} > 0$ independent of n such that

$$|\partial_\beta d_n| \leqslant \frac{\widetilde{C}}{n}\,.$$

This is equivalent to $\{\partial_\beta d_n\}_{n\in\mathbb{Z}} \in \ell^2$, thus there exists a unique function in $L^2(\mathbb{T})$ with Fourier coefficients equal to $\{\partial_\beta d_n\}_{n\in\mathbb{Z}}$ (see for example [164, Chapter 12]). We conclude that

$$\partial_\beta \mu^\beta_{HT}(\theta) \in L^2(\mathbb{T}), \tag{3.35}$$

for any $\beta > 0$. To conclude the proof of the main Theorem 3.6, we observe that from Proposition 3.8 we obtain the relation

$$c_n = \partial_\beta\left(\beta d_n\right)$$

between the moments of the measures $\mu^\beta_{AL}(\theta)$ and $\mu^\beta_{HT}(\theta)$ respectively. This, together with Lemma 3.7 and (3.35) implies that

$$\mu^\beta_{AL}(\theta) = \partial_\beta\left(\beta\mu^\beta_{HT}(\theta)\right)\,.$$

□

Our next main result provides an explicit expression of the mean density of states $\mu_{HT}(\theta)$ for the potential $V(z) = \eta\Re(z)$. This generalizes the result by Gross and Witten [78] and Baik-Deift-Johansson [14] obtained for finite temperature to the high-temperature regime.

Theorem 3.9. *Fix $\beta > 0$ and let $V(z) = \eta\Re(z)$, where η is a real parameter. There exists $\varepsilon > 0$ such that for all $\eta \in (-\varepsilon, \varepsilon)$, the minimizer $\mu^\beta_{HT}(\mathrm{d}\theta) = \mu^\beta_{HT}(\theta)\mathrm{d}\theta$ of the functional* (3.4) *takes the form*

$$\mu^\beta_{HT}(\theta) = \frac{1}{2\pi} + \frac{1}{\pi\beta}\Re\left(\frac{e^{i\theta}v'(e^{i\theta})}{v(e^{i\theta})}\right),$$

where $v(z)$ is the unique solution (up to a multiplicative non-zero constant) of Double Confluent Heun (DCH) equation

$$z^2v''(z) + \left(-\eta + z(\beta+1) + \eta z^2\right)v'(z) + \eta\beta(z+\lambda)v(z) = 0 \tag{3.36}$$

analytic for $|z| \leqslant r$ with $r \geqslant 1$. Such solution is differentiable in the parameter η and β. The parameter $\lambda = \lambda(\eta,\beta)$ in (3.36) *is determined for $\eta \in (-\varepsilon,\varepsilon)$ by the solution of the equation*

$$\lambda(R_1)_{11} + \frac{\eta}{\beta+1}(R_1)_{21} = 0,$$

with the condition $\lambda(\eta = 0, \beta) = 0$. In the above expression $(R_1)_{jk}$ is the jk entry of the matrix R_1 which is defined by the infinite product

$$R_1 = M_1M_2\dots M_k\dots, \quad M_k = \begin{pmatrix} 1 + \frac{\lambda\beta\eta}{k(k+\beta)} & \frac{\eta^2}{k(k+\beta+1)} \\ 1 & 0\end{pmatrix}.$$

We remark that the solution of the double confluent Heun equation has generically an essential singularity at $z = 0$ and $z = \infty$, and one needs to tune the parameter λ to obtain an analytic solution, for a review see [143]. The parameter λ is called *accessory parameter*, and it is related to the Painlevé III equation [53], [104]. The proof of Theorem 3.9 is contained in the next section and consists of mainly two parts: we first derive from the variational equations with respect to the functional $\mathcal{F}^{(V,\beta)}$, the double confluent Heun equation (3.36). Then we show that such equation admits an analytic solution in any compact sets of the complex plane containing the origin.

3.2.1 Proof of Theorem 3.9

From Theorem 3.4 we know that the density μ_{HT}^{β} is characterized as the unique minimizer of the functional (3.26). We follow the ideas developed in [7, 8, 34, 61] to find this minimizer explicitly. We consider the Euler-Lagrange equation of the functional (3.26), namely

$$\frac{\delta \mathcal{F}^{(V,\beta)}}{\delta \mu} = 2V(\theta) - 2\beta \int_{\mathbb{T}} \ln\sin\left(\frac{|\theta-\phi|}{2}\right)\mu(\phi)\mathrm{d}\phi + \ln\mu(\theta) + 1 = C(V,\beta)\,, \quad \textit{a.e.}$$

where the equation holds almost everywhere and where $C(V,\beta)$ is a constant depending on the potential and β, but not on the variable θ. As we did in the previous proof, by differentiating with respect to θ the Euler-Lagrange equation we obtain

$$\partial_\theta \mu(\theta) + \mu(\theta)[2\partial_\theta(V(\theta)) + \beta H\mu(\theta)] = 0\,, \tag{3.37}$$

we recall that H is the Hilbert transform defined on $L^2(\mathbb{T})$ as

$$H\mu(\theta) = -\text{p.v.}\int_{\mathbb{T}} \cot\left(\frac{\theta-\phi}{2}\right)\mu(\phi)\mathrm{d}\phi\,,$$

and p.v. is the Cauchy principal value. Setting $e^{i\theta} = z$ and $e^{i\phi} = w$, we recognize the Riesz - Herglotz kernel $\dfrac{z+w}{z-w}$ expressed as

$$\frac{z+w}{z-w} = -i\cot\left(\frac{\theta-\phi}{2}\right)\,.$$

Therefore

$$\int_{\mathbb{T}} \cot\left(\frac{\theta-\phi}{2}\right)\mu(\phi)\mathrm{d}\phi = i + 2\int_{S^1}\frac{\mu(w)dw}{z-w},$$

where S^1 is the anticlockwise oriented circle, and we used the normalization condition $\int_{\mathbb{T}} \mu(\phi)\mathrm{d}\phi = 1$. We can recast (3.37) in the form

$$z\partial_z\mu(z) + \mu(z)\left[2z\partial_z V(z) - \beta + 2i\beta\text{p.v.}\int_{S^1}\mu(w)\frac{\mathrm{d}w}{z-w}\right] = 0\,.$$

For $z \in \mathbb{C}\backslash S^1$ let us define

$$G(z) := \int_{S^1}\mu(w)\frac{\mathrm{d}w}{w-z} = \frac{i}{2} - \frac{1}{2}\int_{\mathbb{T}}\cot\left(\frac{\theta-\phi}{2}\right)\mu(\phi)\mathrm{d}\phi\,,$$

and for $z \in S^1$ let $G_\pm(z) = \lim_{\tilde{z}\to z} G(\tilde{z})$ for $\tilde{z}$ inside and outside the unit circle respectively. Then

$$\begin{aligned} G_\pm(z) &= \pm\pi i\mu(z) + \text{p.v.}\int_{S^1}\mu(w)\frac{\mathrm{d}w}{w-z} \\ &= \pm\pi i\mu(z) + \frac{i}{2} - \frac{2iz\partial_z V(z)}{2\beta} - \frac{iz\partial_z\mu(z)}{2\beta\mu(z)}\,. \end{aligned}$$

This implies that for $z \in S^1$ one has

$$\begin{aligned} &G_+(z) + G_-(z) = i - \frac{2iz\partial_z V(z)}{\beta} - \frac{iz\partial_z\mu(z)}{\beta\mu(z)}\,, \\ &G_+(z) - G_-(z) = 2\pi i\mu(z)\,. \end{aligned}$$

Multiplying the two previous expressions one gets:

$$G_+(z)^2 - G_-(z)^2 = 2\pi i\mu(z)\left(i - \frac{2iz\partial_z V(z)}{\beta} - \frac{zi\partial_z\mu(z)}{\beta\mu(z)}\right)\,.$$

In order to proceed we have to specify our potential $V(z)$, in our case we will consider $V(z) = \frac{\eta}{2}\left(z + \frac{1}{z}\right)$. Applying the Sokhtoski-Plemelj formula to the above boundary value problem one obtains

$$G^2(z) = i\int_{S^1}\frac{\mu(w)}{w-z}\mathrm{d}w - \frac{i\eta}{\beta}\int_{S^1}\frac{(w-\overline{w})\,\mu(w)}{w-z}\mathrm{d}w - \frac{i}{\beta}\int_{S^1}\frac{w\partial_w\mu(w)}{w-z}\mathrm{d}w\,. \tag{3.38}$$

The second term in the r.h.s. of the above expression gives

$$\begin{aligned} \int_{S^1}\frac{(w-\overline{w})\mu(w)}{w-z}\mathrm{d}w &= \int_{S^1}\frac{w\pm z}{w-z}\mu(w)\mathrm{d}w + \frac{1}{z}\int_{S^1}\mu(w)\left(-\frac{1}{w-z}+\frac{1}{w}\right)\mathrm{d}w \\ &= \left(zG(z) + i\lambda - \frac{G(z)}{z} + \frac{i}{z}\right)\,, \end{aligned}$$

where we have defined

$$\lambda := -i\int_{S^1}\mu(w)\mathrm{d}w, \quad \lambda \in \mathbb{R}. \tag{3.39}$$

The third term in the r.h.s. of (3.38) gives

$$\begin{aligned} \int_{S^1}\frac{w\partial_w\mu(w)}{w-z}\mathrm{d}w &= \int_{S^1}\partial_w\mu(w)\mathrm{d}w + z\int_{S^1}\frac{\partial_w\mu(w)}{w-z}\mathrm{d}w \\ &= z\int_{S^1}\frac{\mu(w)}{(w-z)^2}\mathrm{d}w = z\partial_z G(z), \end{aligned}$$

where in these last relations we use the results of Theorem 3.4 about the regularity of μ. Now we can rewrite (3.38) as

$$G^2(z) = iG(z) - \frac{i\eta}{\beta}\left(zG(z) + i\lambda - \frac{G(z)}{z} + \frac{i}{z}\right) - \frac{iz\partial_z G(z)}{\beta}\,. \tag{3.40}$$

Remark 3.10. *In the above ODE, the parameter $\lambda = \lambda(\eta,\beta)$ depends via (3.39) implicitly on the function $G(z)$. Our strategy to solve the above equation is to consider λ as a free parameter that is uniquely fixed by the analytic properties of the function $G(z)$.*

We can now turn the non-linear first order ODE (3.40) into a linear second order ODE through the substitution

$$G(z) = i + \frac{izv'(z)}{\beta v(z)}, \tag{3.41}$$

getting:

$$z^2 v''(z) + \left(-\eta + z(\beta+1) + \eta z^2\right) v'(z) + \eta\beta(z+\lambda)v(z) = 0\,, \tag{3.42}$$

which is the DCH equation in (3.36). The solutions to this equation have generically essential singularities at $z = 0$ and $z = \infty$ and the local description near the singularities depends on the parameter η and β. The quantity λ is usually referred to as *accessory parameter* since it does not change the singular behaviour of the solution. Since $G(z)$ is analytic in the unit disc and continuous up to the boundary and $G(0) = i$, we seek for a solution $v(z)$ of the DCH equation that is analytic in the unit disk and such that $v(z) \xrightarrow[z\to 0]{} v_0$, where v_0 is a nonzero constant.

Construction of the analytic solution. We look for a solution of (3.42) in the form

$$v(z) = \sum_{k=0}^{\infty} a_k z^k, \tag{3.43}$$

which implies the following recurrence relations for the coefficients $\{a_k\}_{k\in\mathbb{N}}$

$$\eta(a_0\lambda\beta - a_1) = 0\,, \tag{3.44}$$

$$a_k(k^2 + k\beta + \lambda\beta\eta) + \eta(k-1+\beta)a_{k-1} - \eta(k+1)a_{k+1} = 0\,, \quad k > 0\,, \tag{3.45}$$

where we have the freedom to chose λ and a_0. Generically, the above recurrence relation for the coefficients $\{a_k\}_{k\in\mathbb{N}}$ gives a divergent series in (3.43). To obtain a convergent series, we follow the ideas in [26, 155].

We start by considering the 2×2 matrices $R_k^{(s)}$ defined as

$$R_k^{(s)} = M_k M_{k+1}\dots M_s,\ s \geqslant k,\ \ M_k = \begin{pmatrix} 1 + \frac{\lambda\beta\eta}{k(k+\beta)} & \frac{\eta^2}{k(k+\beta+1)} \\ 1 & 0 \end{pmatrix}, \tag{3.46}$$

which satisfy the recurrence relation $R_k^{(s)} = R_k^{(s-1)} M_s$. The next lemma shows that the limit of $R_k^{(s)}$ as $s \to \infty$ exists.

Lemma 3.11. *Let $R_k^{(s)}$ be the matrix defined in* (3.46)*. Then the limit of $R_k^{(s)}$ as $s \to \infty$ exists and*

$$R_k := \lim_{s\to\infty} R_k^{(s)}. \tag{3.47}$$

The matrices R_k, $k \geqslant 1$ satisfy the descending recurrence relation:

$$R_k = M_k R_{k+1} \quad k \geqslant 1\,. \tag{3.48}$$

Furthermore each entry of the matrix $R_k = R_k(\beta, \eta, \lambda)$ is differentiable with respect to the parameters β, η, and λ.

Since the proof of this lemma is rather technical, we defer it to section 3.3.2, where we gather the most technical results.

Finally, let us define the following function:

$$\xi(\eta,\beta,\lambda) := \begin{pmatrix}\lambda & \frac{\eta}{\beta+1}\end{pmatrix} R_1 \begin{pmatrix}1\\0\end{pmatrix}. \tag{3.49}$$

We are now ready to prove the following result that will give us a necessary condition to fix the value of λ.

Proposition 3.12. *For the values of λ such that*

$$\xi(\eta,\beta,\lambda) = 0, \tag{3.50}$$

where $\xi(\eta,\beta,\lambda)$ is defined in (3.49)*, the Double Confluent Heun equation* (3.42) *admits a non-zero solution $v = v(z,\eta,\beta)$ defined by the series* (3.43) *that is uniformly convergent in $|z| \leqslant r$ with $r \geqslant 1$. The corresponding coefficients $\{a_k\}_{k\in\mathbb{N}}$ of the Taylor expansion* (3.43) *are given by the relation*

$$a_0 = \frac{1}{\beta}\begin{pmatrix}1 & 0\end{pmatrix} R_1 \begin{pmatrix}1\\0\end{pmatrix}, \tag{3.51}$$

$$a_k = (-1)^k \frac{\eta^k}{k!(k+\beta)}\begin{pmatrix}0 & 1\end{pmatrix} R_k \begin{pmatrix}1\\0\end{pmatrix}, \quad k \geqslant 1, \tag{3.52}$$

where the matrices R_k are defined in (3.47)*. For each λ satisfying* (3.50)*, the solution $v(z)$ of the DCH equation* (3.42)*, analytic at zero is unique up to a multiplicative factor.*

Proof. First, we show that choosing a_k according to (3.51)-(3.52) we obtain a solution of the recurrence (3.45). We notice that due to the recurrence relation for the matrices R_k (3.48), we have that:

$$\begin{pmatrix}0 & 1\end{pmatrix} R_k \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 & 0\end{pmatrix} R_{k+1} \begin{pmatrix}1\\0\end{pmatrix}.$$

Thus, applying the previous equation and (3.51)-(3.52), we can recast (3.45) as:

$$\begin{aligned}
&\Bigg[(-1)^{k-1}\frac{\eta^k}{(k-1)!}\begin{pmatrix}1 & 0\end{pmatrix} R_k + (-\eta)^k\,\frac{k(k+\beta)+\eta\lambda\beta}{k!(k+\beta)}\begin{pmatrix}1 & 0\end{pmatrix} R_{k+1} \\
&+(-1)^k\frac{\eta^{k+2}}{k!(k+1+\beta)}\begin{pmatrix}0 & 1\end{pmatrix} R_{k+1}\Bigg]\begin{pmatrix}1\\0\end{pmatrix} \\
&= \frac{(-\eta)^k}{(k-1)!}\left[-\begin{pmatrix}1 & 0\end{pmatrix} R_k + \begin{pmatrix}1+\frac{\lambda\beta\eta}{k(k+\beta)} & \frac{\eta^2}{k(k+1+\beta)}\end{pmatrix} R_{k+1}\right]\begin{pmatrix}1\\0\end{pmatrix} = 0,
\end{aligned}$$

where in the last equality we have enforced (3.48). Next we can rewrite (3.44) in terms of the matrix R_1 exploiting (3.51)-(3.52), namely

$$0 = \begin{pmatrix}\lambda & \frac{\eta}{\beta+1}\end{pmatrix} R_1 \begin{pmatrix}1\\0\end{pmatrix} = \xi(\eta,\beta,\lambda),$$

which is exactly (3.49). Since the entries of the matrices R_k are uniformly bounded, the solution $v(z) = \sum_{k\geqslant 0} a_k z^k$ with a_k as in (3.52), defines a uniformly convergent Taylor series in $|z| < r$ for any $r \geqslant 0$ and in particular for any $r > 1$.

To show that the solution analytic at $z = 0$ is unique up to a constant, we consider the Wronkstian $W(v, \tilde{v})(z)$ of two independent solution v and $\tilde{v}$ of the Double Confluent Heun equation (3.42), namely

$$W(z) = e^{-\eta(z+\frac{1}{z})} z^{-(\beta+1)} (v'(z)\tilde{v}(z) - v(z)\tilde{v}'(z)).$$

Since $W'(z) = 0$, it follows that $W(z) = C$ a constant. If by contradiction we suppose that there are two analytic solutions at $z = 0$, then from the above relation we obtain

$$e^{-\eta z}(v'(z)\tilde{v}(z) - v(z)\tilde{v}'(z)) = Ce^{\frac{\eta}{z}} z^{\beta+1} .$$

If $\eta \neq 0$ the left-hand side of the above equation is analytic and the right-hand side is not, that is clearly a contradiction. If $\eta = 0$ then (3.42) becomes:

$$z^2 v''(z) + z(\beta + 1)v'(z) = 0 .$$

The above equation has two independent solutions, one is the constant solution, which is analytic, the other one is $v(z) = Cz^{-\beta}$ which is not analytic since $\beta > 0$. □

Remark 3.13. *We observe that the equation* (3.50)*, does not uniquely determined* λ*. Indeed as it is shown in Figure 3.1 the function* $\xi(\eta, \beta, \lambda)$ *may have several zeros for given* η *and* β*.*

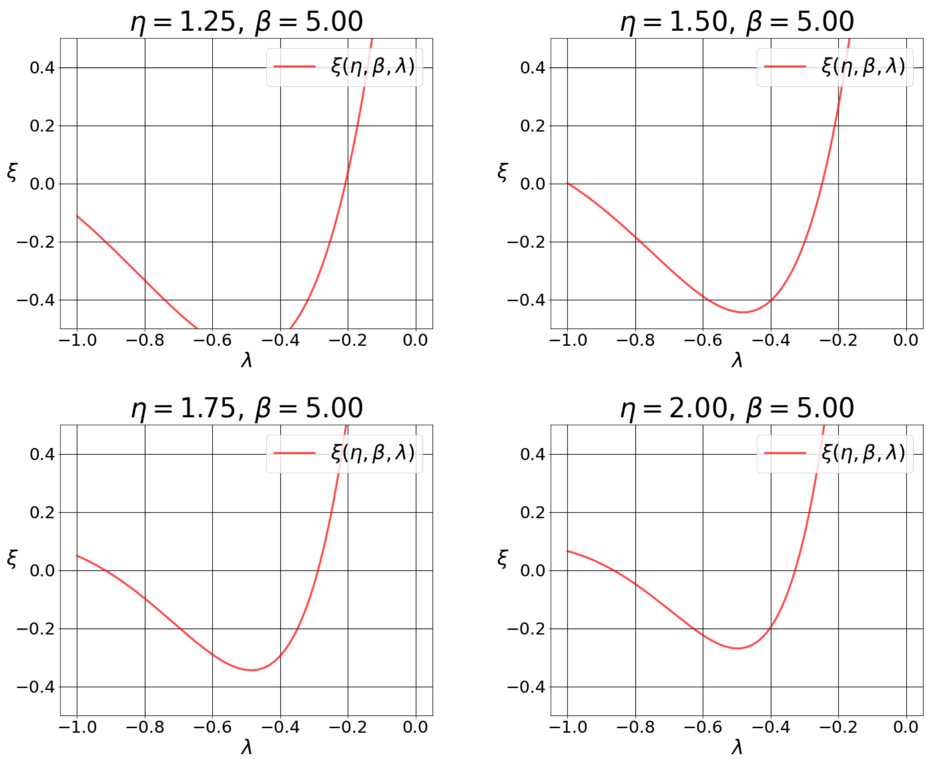


Figure 3.1: Plots of $\xi(\eta, \beta, \lambda)$ for various values of η, β

Choice of the parameter λ. We will now prove that the parameter λ is uniquely determined in a neighbourhood of $\eta = 0$ by requiring that the solution $v = v(z, \eta, \beta)$ depends continuously on the parameter η.

Lemma 3.14. *There exists an $\varepsilon > 0$ such that for all $\eta \in (-\varepsilon, \varepsilon)$ and $\beta > 0$ there is a unique $\lambda = \lambda(\eta, \beta)$ such that $\xi(\eta, \beta, \lambda(\eta, \beta)) = 0$.*

Proof. When $\eta = 0$ the matrix $R_j = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ so that the only solution of the equation (3.50) $\xi(\eta = 0, \beta, \lambda) = 0$ is $\lambda = 0$. To show the existence of the solution (3.50) for $\lambda = \lambda(\eta, \beta)$ near $\eta = 0$, we use the implicit function theorem. We have to show that $\partial_\lambda \xi(\eta, \beta, \lambda)_{|(0,\beta,0)} \neq 0$. For the purpose, we need to evaluate

$$\partial_\lambda \left(M_k\right)_{(\eta=0,\lambda=0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where M_k is defined in (3.47). This equation implies that

$$\partial_\lambda(\xi(\eta, \beta, \lambda))_{|(0,\beta,0)} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1.$$

Thus we can apply the implicit function theorem, and we get the claim. □

We conclude the proof of Theorem 3.9. When $\eta = 0$ the only analytic solution of DCH equation is $v(z) = c$, $c \in \mathbb{C}$. In this case in principle λ is undetermined. However, from Theorem 3.4 the minimizer μ_{HT}^{β} of (3.26) is the uniform measure on the circle and therefore from equation 3.39 one has $\lambda = 0$. From Lemma 3.14 when $\eta \in (-\varepsilon, \varepsilon)$, there exists a unique $\lambda(\eta, \beta)$ that satisfies equation (3.50) and such that $\lambda(\eta = 0, \beta) = 0$ and therefore by Proposition 3.12 we obtain for $\eta \in (-\varepsilon, \varepsilon)$, the unique solution $v(z, \eta, \beta)$ of the DCH equation analytic in any compact set $|z| \leqslant r$, with $r > 0$ and in particular when $r = 1$. Because of lemma 3.11 the solution $v(z, \eta, \beta)$ is differentiable with respect to the parameters η and β.

We remark that $v(z) \neq 0$ on the unit disc $\overline{\mathbb{D}}$ because of the relation (3.41) between the analytic function $G(z)$ and $v(z)$.

To complete our proof we recover the explicit expression of $\mu_{HT}^{\beta}(\theta)$ from $G(z)$ and $v(z)$ using the *Poisson representation formula* (see for example [147, Chapter 1]):

$$\mu_{HT}^{\beta}(\theta) = -\frac{1}{2\pi} - \frac{\Re(iG(e^{i\theta}))}{\pi\beta} = \frac{1}{2\pi} + \frac{1}{\pi\beta}\Re\left(\frac{e^{i\theta}v'(e^{i\theta})}{v(e^{i\theta})}\right).$$

□

In Figure 3.2 we plotted the density of states of the Circular beta ensemble in the high-temperature regime with potential $V(z) = \eta\Re(z)$. To produce this picture and Figure 3.1, we used extensively the `NumPy` [81], and `matplotlib` [86] libraries.

3.3 Technical results

In this section we collect the most technical parts of this chapter.

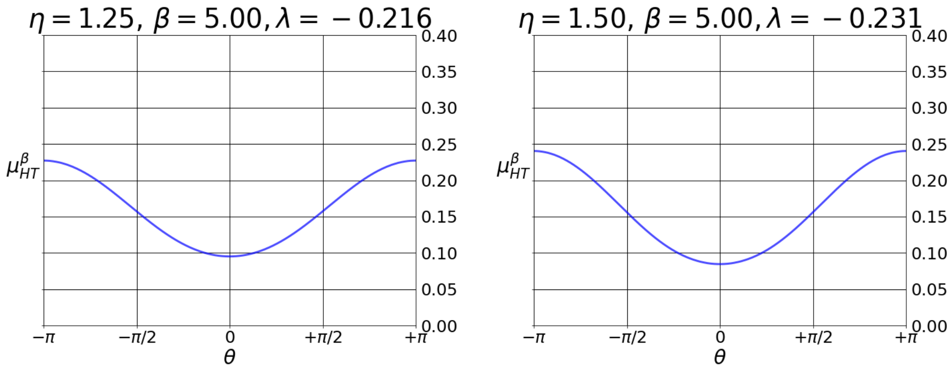


Figure 3.2: The mean density of states μ_{HT}^{β} for different parameters.

3.3.1 Proof of Proposition 3.8

First, we prove the relation between the free energies (3.30):

$$\partial_\beta(\beta F_{HT}(V,\beta)) + \ln(2) = F_{AL}(V,\beta)\,, \tag{3.53}$$

doing that we will also prove that they are differentiable with respect to β.

We notice that the previous expression is equivalent to:

$$\partial_\beta\left(\beta \lim_{N\to\infty} \frac{\ln(Z_N^{HT}(V,\beta))}{N}\right) = \lim_{N\to\infty} \frac{\ln(Z_N^{AL}(V,\beta))}{N}\,.$$

To prove the previous relation, we will use the so-called transfer operator technique [92,96,134]. We are considering a potential of the form $\mathrm{Tr}(V(\mathcal{E}))$ as in (3.10) which is of finite range K, meaning that it can be expressed as a sum of local quantities, i.e. depending on a finite number $2K$ of variables, with K independent of N [126]. For example, if $V(z) = \Re(z)$, then $\mathrm{Tr}(\mathcal{E}) = -2\sum_{j=1}^N \Re(\alpha_j\overline{\alpha}_{j+1})$ and in this case the range is $K = 1$. Let $N = KM + L$ with $M, L \in \mathbb{N}$ and $L < K$. We split the coordinates $(\alpha_1, \ldots, \alpha_N)$ into M blocks of length K and a reminder of length L, and we define $\widetilde{\boldsymbol{\alpha}}_j = (\alpha_{K(j-1)+1}, \alpha_{K(j-1)+2}, \ldots, \alpha_{Kj})$. In this notation,

$$(\alpha_1, \ldots, \alpha_N) = (\overbrace{\widetilde{\boldsymbol{\alpha}}_1, \ldots, \widetilde{\boldsymbol{\alpha}}_M}^{KM}, \overbrace{\alpha_{KM+1}, \ldots, \alpha_N}^{L}),$$

and we can rewrite the potential as

$$\mathrm{Tr}(V(\mathcal{E})) = \sum_{\ell=1}^{M-1} W(\widetilde{\boldsymbol{\alpha}}_\ell, \widetilde{\boldsymbol{\alpha}}_{\ell+1}) + W(\widetilde{\boldsymbol{\alpha}}_M, \alpha_{KM+1}, \ldots, \alpha_N, \alpha_1, \ldots, \alpha_{K-L})\,,$$

where W is a continuous function $W : \mathbb{D}^K \times \mathbb{D}^K \to \mathbb{R}$, and $W(\widetilde{\boldsymbol{\alpha}}_1, \widetilde{\boldsymbol{\alpha}}_2) = W(\widetilde{\boldsymbol{\alpha}}_2, \widetilde{\boldsymbol{\alpha}}_1)$. The last term in the above expression is different from the others since we may have an off-set of length L, due to periodicity. For convenience, we define

$$\widetilde{\boldsymbol{\alpha}}_{M+1} = (\alpha_{KM+1}, \ldots, \alpha_N, \alpha_1, \ldots, \alpha_{K-L}).$$

In the case $V(z) = \Re(z)$, then $W(\alpha_1, \alpha_2) = -2\Re(\alpha_1\overline{\alpha}_2)$ and there is no off-set.

We can now rewrite $Z_N^{AL}(V,\beta)$ in (3.12) as

$$Z_N^{AL}(V,\beta) = \int_{\mathbb{D}^N} \prod_{j=1}^{N} \left(1-|\alpha_j|^2\right)^{\beta-1} \exp\left(-\sum_{\ell=1}^{M} W(\widetilde{\boldsymbol{\alpha}}_\ell, \widetilde{\boldsymbol{\alpha}}_{\ell+1})\right) \mathrm{d}^2\boldsymbol{\alpha}\,.$$

We are now in position to apply the transfer operator technique to compute this partition function. On $L^2(\mathbb{D}^K)$ we introduce the scalar product

$$(f,g) = \int_{\mathbb{D}^K} f(\mathbf{z})\overline{g(\mathbf{z})}\mathrm{d}\mathbf{z}\,, \tag{3.54}$$

where $\mathbf{z} = (z_1,\ldots,z_K)$. This scalar product induces a norm on $L^2(\mathbb{D})$ and also a norm on the operators $T : L^2(\mathbb{D}^K) \to L^2(\mathbb{D}^K)$ as

$$||T|| := \sup_{f\,:\,||f||_2=1} ||Tf||_2\,,$$

where $||f||_2$ is the standard L^2 norm.

Let $\boldsymbol{\zeta} = (\zeta_1,\ldots,\ldots\zeta_{2K})$ with $\zeta_{K+j} = \zeta_j > 0$ for $j = 1,\ldots,K$, we define the continuous family of transfer operators $\mathcal{T}_{\boldsymbol{\zeta}} : L^2(\mathbb{D}^K) \to L^2(\mathbb{D}^K)$ as

$$(\mathcal{T}_{\boldsymbol{\zeta}} f)(\widetilde{\boldsymbol{\alpha}}_2) = \int_{\mathbb{D}^K} f(\widetilde{\boldsymbol{\alpha}}_1) \prod_{j=1}^{2K} \left(1-|\alpha_j|^2\right)^{\frac{\zeta_j-1}{2}} \exp\left(-W(\widetilde{\boldsymbol{\alpha}}_1,\widetilde{\boldsymbol{\alpha}}_2)\right) \mathrm{d}^2\widetilde{\boldsymbol{\alpha}}_1\,. \tag{3.55}$$

We notice that $\mathcal{T}_{\boldsymbol{\zeta}}$ is symmetric with respect to the scalar product (3.54), indeed $(f,\mathcal{T}g) = (\mathcal{T}f,g)$. Furthermore, $\mathcal{T}_{\boldsymbol{\zeta}}$ is an integral operator whose kernel $\prod_{j=1}^{2K} (1-|\alpha_j|^2)^{\frac{\zeta_j-1}{2}} \exp\left(-W(\widetilde{\boldsymbol{\alpha}}_1,\widetilde{\boldsymbol{\alpha}}_2)\right)$ belongs to $L^2(\mathbb{D}^K\times\mathbb{D}^K)$, and therefore $\mathcal{T}_{\boldsymbol{\zeta}}$ is an Hilbert-Schimdt operator. We conclude that there exists a complete set of normalized eigenfunctions $\{\psi_j\}_{j\geqslant 1}$ with real eigenvalues $\{\lambda_j\}_{j\geqslant 1}$ in descending order, differentiable functions of the parameters $\boldsymbol{\zeta} = (\zeta_1,\ldots,\zeta_{2K})$ [91], such that:

$$(\mathcal{T}_{\boldsymbol{\zeta}}\psi_j)(\mathbf{z},V,\boldsymbol{\zeta}) = \lambda_j(V,\boldsymbol{\zeta})\psi_j(\mathbf{z}',V,\boldsymbol{\zeta}) \tag{3.56}$$

$$\sum_{n=1}^{\infty} \overline{\psi}_n(\mathbf{z},V,\boldsymbol{\zeta})\psi_n(\mathbf{z}',V,\boldsymbol{\zeta}) = \delta_{\mathbf{z}}(\mathbf{z}') \tag{3.57}$$

where $\delta_{\mathbf{z}}(\cdot)$ is the Dirac delta function at $\mathbf{z}\in\mathbb{D}^K$. Moreover $\sum_{j=1}^{\infty} |\lambda_j(V,\boldsymbol{\zeta})|^2 < \infty$.

We artificially rewrite Z_N^{AL} as

$$\begin{aligned} Z_N^{AL}(V,\beta) = \int_{\mathbb{D}^{N+K}} & \delta_{\widetilde{\boldsymbol{\alpha}}_1}(\boldsymbol{\gamma}) \prod_{\ell=1}^{K} \left(1-|\gamma_\ell|^2\right)^{\frac{\beta-1}{2}} \prod_{\ell=1}^{K} \left(1-|\alpha_j|^2\right)^{\frac{\beta-1}{2}} \prod_{\ell=K+1}^{N} \left(1-|\alpha_j|^2\right)^{\beta-1} \\ & \times \exp\left(-\sum_{\ell=1}^{M-1} W(\widetilde{\boldsymbol{\alpha}}_\ell,\widetilde{\boldsymbol{\alpha}}_{\ell+1}) - W(\widetilde{\boldsymbol{\alpha}}_M, \alpha_{KM+1},\ldots,\alpha_N,\gamma_1,\ldots,\gamma_{K-L})\right) \mathrm{d}^2\boldsymbol{\alpha}\mathrm{d}^2\boldsymbol{\gamma}\,, \end{aligned}$$

where $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_K)$ and $\boldsymbol{\gamma} \in \mathbb{D}^K$. We can use (3.57) with $\boldsymbol{\zeta} = \boldsymbol{\beta} = \overbrace{(\beta, \ldots, \beta)}^{2K}$ to rewrite the previous equation as:

$$
\begin{aligned}
Z_N^{AL}(V, \beta) = & \sum_{n=1}^{\infty} \int_{\mathbb{D}^{N+K}} \overline{\psi}_n(\boldsymbol{\gamma}, V, \boldsymbol{\beta}) \psi_n(\widetilde{\boldsymbol{\alpha}}_1, V, \boldsymbol{\beta}) \prod_{\ell=1}^{K} \left(1 - |\gamma_\ell|^2\right)^{\frac{\beta-1}{2}} \\
& \times \prod_{\ell=1}^{K} \left(1 - |\alpha_j|^2\right)^{\frac{\beta-1}{2}} \prod_{\ell=K+1}^{N} \left(1 - |\alpha_j|^2\right)^{\beta-1} \\
& \times \exp\left(- \sum_{\ell=1}^{M-1} W(\widetilde{\boldsymbol{\alpha}}_\ell, \widetilde{\boldsymbol{\alpha}}_{\ell+1}) - W(\widetilde{\boldsymbol{\alpha}}_M, \alpha_{KM+1}, \ldots, \alpha_N, \gamma_1, \ldots, \gamma_{K-L}) \right) \mathrm{d}^2\boldsymbol{\alpha} \mathrm{d}^2\boldsymbol{\gamma} \, .
\end{aligned}
$$

In the above integral, we can identify the integral operator $\mathcal{T}_{\boldsymbol{\beta}}$ where $\boldsymbol{\beta} = \overbrace{(\beta, \ldots, \beta)}^{2K}$. We repeatedly apply (3.56) $M-1$ times to the above integral, to obtain:

$$
Z_N^{AL}(V, \beta) = \sum_{n=1}^{\infty} (\lambda_n(V, \boldsymbol{\beta}))^{M-1} R_n, \tag{3.58}
$$

$$
\begin{aligned}
R_n = & \int_{\mathbb{D}^{2K+L}} \overline{\psi}_n(\boldsymbol{\gamma}, V, \boldsymbol{\beta}) \psi_n(\widetilde{\boldsymbol{\alpha}}_M, V, \boldsymbol{\beta}) \prod_{\ell=1}^{K} \left(1 - |\gamma_\ell|^2\right)^{\frac{\beta-1}{2}} \prod_{\ell=(M-1)K+1}^{MK} \left(1 - |\alpha_j|^2\right)^{\frac{\beta-1}{2}} \\
\times & \prod_{\ell=MK+1}^{N} \left(1 - |\alpha_j|^2\right)^{\beta-1} \exp\left(-W(\widetilde{\boldsymbol{\alpha}}_M, \alpha_{KM+1}, \ldots, \alpha_N, \gamma_1, \ldots, \gamma_{K-L})\right) \prod_{\ell=(M-1)K+1}^{N} \mathrm{d}^2\alpha_j \prod_{\ell=1}^{K} \mathrm{d}^2\gamma_\ell \, .
\end{aligned} \tag{3.59}
$$

The modulus of the reminder $|R_n|$ in (3.59) can be bounded from above and below by two constants $C_1, C_2 > 0$ independent of N, therefore we conclude from (3.58) that

$$
F_{AL}(V, \beta) = - \lim_{N \to \infty} \frac{1}{N} \ln\left(Z_N^{AL}(V, \beta)\right) = -\frac{1}{K} \ln\left(\lambda_1(V, \boldsymbol{\beta})\right) \, .
$$

As a consequence of the previous relation and [169, Theorem 137.4], we get that $F_{AL}(V, \beta)$ is differentiable with respect to β, since $\lambda_1(V, \beta)$ is differentiable in β and strictly positive.

We can apply the same procedure to the partition function $Z_N^{HT}(V, \beta)$ in (3.19). Also in this case the potential $\mathrm{Tr}(V(\widetilde{E}))$ with V as in (3.10) and the matrix $\widetilde{E}$ as in (3.18) is of finite range K, meaning that it can be expressed as a sum of local quantities [126]. More precisely, assuming $N = KM + L$ with $L < K$ and $M, N, L \in \mathbb{N}$ we have

$$
\mathrm{Tr}(V(\widetilde{E})) = \sum_{\ell=1}^{M-1} W(\widetilde{\boldsymbol{\alpha}}_\ell, \widetilde{\boldsymbol{\alpha}}_{\ell+1}) + W(\overbrace{0, \ldots, 0}^{K-1}, -1, \widetilde{\boldsymbol{\alpha}}_1) + W(\widetilde{\boldsymbol{\alpha}}_M, \alpha_{KM+1}, \ldots, \alpha_N, \overbrace{0, \ldots, 0}^{K-L}) \, ,
$$

For example for $V(z) = z^2 + \bar{z}^2$ one has $K = 2$ and $N = 2M + L$ where $L = 0, 1$. The vector $\widetilde{\boldsymbol{\alpha}}_\ell$ takes the form $\widetilde{\boldsymbol{\alpha}}_\ell = (\alpha_{2\ell-1}, \alpha_{2\ell})$ for $\ell = 1, \ldots, M$. In this notation, we can rewrite the potential as

$$
\mathrm{Tr}(V(\widetilde{E})) = \sum_{\ell=1}^{M-1} W(\widetilde{\boldsymbol{\alpha}}_\ell, \widetilde{\boldsymbol{\alpha}}_{\ell+1}) + \delta_{L,0} W(\widetilde{\boldsymbol{\alpha}}_M, \alpha_N, 0) + \underbrace{2\Re(\alpha_1^2 + 2\bar{\alpha}_2\rho_1^2)}_{=W(0,-1,\alpha_1,\alpha_2)}
$$

where in this case $W(\tilde{\boldsymbol{\alpha}}_\ell, \tilde{\boldsymbol{\alpha}}_{\ell+1}) = 2\Re\left[\sum_{s=0}^{1}(\alpha_{2\ell-1+s}\bar{\alpha}_{2\ell+s})^2 - 2\alpha_{2\ell-1+s}\bar{\alpha}_{2\ell+1+s}\rho^2_{2\ell+s}\right]$.
In this way, the partition function can be written in the form

$$Z_N^{HT}(V,\beta) = \int\limits_{\mathbb{D}^{N-1}\times S^1} \prod_{j=1}^{N-1} \mathrm{d}^2\alpha_j \frac{\mathrm{d}\alpha_N}{i\alpha_N} \prod_{j=1}^{N-1} \left(1-|\alpha_j|^2\right)^{\beta\left(1-\frac{j}{N}\right)-1} \tag{3.60}$$
$$\times \exp\left(-\sum_{\ell=1}^{M-1} W(\tilde{\boldsymbol{\alpha}}_\ell, \tilde{\boldsymbol{\alpha}}_{\ell+1}) - W(\tilde{\boldsymbol{\alpha}}_M, \alpha_{KM+1}, \dots, \alpha_N, \overbrace{0,\dots,0}^{K-L}) - W(\overbrace{0,\dots,0}^{K-1}, -1, \tilde{\boldsymbol{\alpha}}_1)\right).$$

We want to apply the same technique as in the previous case, but we have to pay attention to one important detail: in this situation the eigenvalues and the eigenfunctions of the transfer operators will be dependent on the block number. Indeed, in this case the exponents of $(1-|\alpha_j|^2)$ are not identical, but they depend on the index j as in (3.60).
For this reason, we define the vector $\boldsymbol{\zeta}^{(1)} \in \mathbb{R}^{2K}$ as

$$\begin{aligned}\boldsymbol{\zeta}^{(1)} =&\Big(\beta\left(1-\frac{1}{N}\right), \beta\left(1-\frac{2}{N}\right), \dots, \beta\left(1-\frac{K}{N}\right)\\ &, \beta\left(1-\frac{1}{N}\right), \beta\left(1-\frac{2}{N}\right), \dots, \beta\left(1-\frac{K}{N}\right)\Big),\end{aligned}$$

and

$$\boldsymbol{\zeta}^{(j)} = \boldsymbol{\zeta}^{(1)} - \beta\frac{j-1}{N}\boldsymbol{K}, \quad j = 1, \dots, M-1$$

where $\boldsymbol{K}_j = K$ for $j = 1, \dots, 2K$. For K integer and $K < N$ we introduce the multiplication operator $M_K : L^2(\mathbb{D}^K) \to L^2(\mathbb{D}^K)$ defined as

$$(M_K f)(\boldsymbol{\alpha}) = \prod_{j=1}^{K} \left(1-|\alpha_j|^2\right)^{-\frac{K}{2N}} f(\boldsymbol{\alpha}).$$

We observe that $M_{-K} = (M_K)^{-1}$ and the operators $\mathcal{T}_{\boldsymbol{\zeta}^{(j)}} : L^2(\mathbb{D}^K) \to L^2(\mathbb{D}^K)$ defined in (3.55) satisfy the relation

$$\mathcal{T}_{\boldsymbol{\zeta}^{(j+1)}} = M_K \mathcal{T}_{\boldsymbol{\zeta}^{(j)}} M_K, \quad j = 1, \dots, M-1.$$

We notice that the operators $\mathcal{T}_{\boldsymbol{\zeta}^{(j)}}$ are compact and symmetric.
Let us define the operator

$$\widetilde{\mathcal{T}} = M_K \mathcal{T}_{\boldsymbol{\zeta}^{(M-1)}} M_K \mathcal{T}_{\boldsymbol{\zeta}^{(M-2)}} \cdots \mathcal{T}_{\boldsymbol{\zeta}^{(1)}}, \tag{3.61}$$

we notice that it is a compact operator, since all $\mathcal{T}_{\boldsymbol{\zeta}^{(j)}}$ are Hilbert-Schimdt and the multiplication operator M_K is bounded. We will now prove the following technical proposition:

Proposition 3.15. *Let $\widetilde{\mathcal{T}}$ as in (3.61) and Z_N^{HT} as in (3.60) then:*

$$\lim_{N\to\infty} \frac{1}{N} \ln\left(\frac{Z_N^{HT}}{Tr(\widetilde{\mathcal{T}})}\right) = 0 \tag{3.62}$$

Proof. We will estimate both Z_N^{HT}, and $\mathrm{Tr}(\widetilde{\mathcal{T}})$ from above and below, then combining these estimates we will obtain (3.62). We start with Z_N^{HT}.

$$
\begin{aligned}
Z_N^{HT} = & \int\limits_{\mathbb{D}^{N-1}\times S^1} \prod_{j=1}^{N-1} \mathrm{d}^2\alpha_j \frac{\mathrm{d}\alpha_N}{i\alpha_N} \prod_{j=1}^{N-1} \left(1-|\alpha_j|^2\right)^{\beta\left(1-\frac{j}{N}\right)-1} \\
& \times \exp\left(-\sum_{\ell=1}^{M-1} W(\widetilde{\boldsymbol{\alpha}}_\ell, \widetilde{\boldsymbol{\alpha}}_{\ell+1}) - W(\widetilde{\boldsymbol{\alpha}}_M, \alpha_{KM+1}, \dots, \alpha_N, \overbrace{0,\dots,0}^{K-L}) - W(\overbrace{0,\dots,0}^{K-1}, -1, \widetilde{\boldsymbol{\alpha}}_1)\right) \\
& \leqslant C_1(V,\beta) \int\limits_{\mathbb{D}^{N-1}\times S^1} \prod_{j=1}^{N-1} \mathrm{d}^2\alpha_j \frac{\mathrm{d}\alpha_N}{i\alpha_N} \prod_{j=1}^{N-1} \left(1-|\alpha_j|^2\right)^{\beta\left(1-\frac{j}{N}\right)-1} \exp\left(-\sum_{\ell=2}^{M-3} W(\widetilde{\boldsymbol{\alpha}}_\ell, \widetilde{\boldsymbol{\alpha}}_{\ell+1})\right),
\end{aligned}
$$

here $C_1(V,\beta)$ is a constant depending on V, β, but not on N. We can explicitly integrate in α_j for $j=1,\dots,K$ and $j=(M-2)K+1,\dots,N$ using the formula

$$
\int_{\mathbb{D}} \left(1-|z|^2\right)^{t-1} \mathrm{d}^2 z = \pi t^{-1},
$$

obtaining that there exists a constant $C(V,\beta)$ depending on V, β such that :

$$
\begin{aligned}
Z_N^{HT} \leqslant & C(V,\beta) N^{2K+L-1} \int\limits_{\mathbb{D}^{(M-3)K}} \prod_{j=1}^{(M-2)K} \mathrm{d}^2\alpha_j \prod_{j=K+1}^{(M-2)K} \left(1-|\alpha_j|^2\right)^{\beta\left(1-\frac{j}{N}\right)-1} \\
& \times \exp\left(-\sum_{\ell=2}^{M-3} W(\widetilde{\boldsymbol{\alpha}}_\ell, \widetilde{\boldsymbol{\alpha}}_{\ell+1})\right).
\end{aligned}
\tag{3.63}
$$

With analogous computation, we can obtain a lower bound for Z_N^{HT}

$$
\begin{aligned}
Z_N^{HT} \geqslant & c(V,\beta) N^{2K+L-1} \int\limits_{\mathbb{D}^{(M-3)K}} \prod_{j=1}^{(M-2)K} \mathrm{d}^2\alpha_j \prod_{j=K+1}^{(M-2)K} \left(1-|\alpha_j|^2\right)^{\beta\left(1-\frac{j}{N}\right)-1} \\
& \times \exp\left(-\sum_{\ell=2}^{M-3} W(\widetilde{\boldsymbol{\alpha}}_\ell, \widetilde{\boldsymbol{\alpha}}_{\ell+1})\right),
\end{aligned}
\tag{3.64}
$$

here $c(V,\beta)$ is a constant depending on V, β, but not on N.

We can proceed analogously to estimate the trace of $\widetilde{\mathcal{T}}$:

$$
\begin{aligned}
\mathrm{Tr}(\widetilde{\mathcal{T}}) = & \int\limits_{\mathbb{D}^{(M-2)K}} \prod_{j=1}^{K} \left(1-|\alpha_j|^2\right)^{\frac{\beta}{2}\left(1-\frac{j}{N}\right)-\frac{1}{2}} \prod_{j=1}^{K} \left(1-|\alpha_j|^2\right)^{\frac{\beta}{2}\left(1-\frac{(M-2)K+j}{N}\right)-\frac{1}{2}} \prod_{j=K+1}^{(M-2)K} \left(1-|\alpha_j|^2\right)^{\beta\left(1-\frac{j}{N}\right)-1} \\
& \times \exp\left(-\sum_{j=1}^{M-3} W(\widetilde{\boldsymbol{\alpha}}_j, \widetilde{\boldsymbol{\alpha}}_{j+1}) - W(\widetilde{\boldsymbol{\alpha}}_{M-2}, \widetilde{\boldsymbol{\alpha}}_1)\right) \prod_{j=1}^{M-2} \mathrm{d}^2 \widetilde{\boldsymbol{\alpha}}_j \\
& \leqslant \widetilde{C}_1(V,\beta) \int\limits_{\mathbb{D}^{(M-2)K}} \prod_{j=1}^{K} \left(1-|\alpha_j|^2\right)^{\frac{\beta}{2}\left(1-\frac{j}{N}\right)-\frac{1}{2}} \\
& \times \prod_{j=1}^{K} \left(1-|\alpha_j|^2\right)^{\frac{\beta}{2}\left(1-\frac{(M-2)K+j}{N}\right)-\frac{1}{2}} \prod_{j=K+1}^{(M-2)K} \left(1-|\alpha_j|^2\right)^{\beta\left(1-\frac{j}{N}\right)-1} \\
& \times \exp\left(-\sum_{j=2}^{M-3} W(\widetilde{\boldsymbol{\alpha}}_j, \widetilde{\boldsymbol{\alpha}}_{j+1})\right) \prod_{j=1}^{M-2} \mathrm{d}^2 \widetilde{\boldsymbol{\alpha}}_j \\
& \leqslant \widetilde{C}(V,\beta) \int\limits_{\mathbb{D}^{(M-3)K}} \prod_{j=1}^{(M-2)K} \mathrm{d}^2 \alpha_j \prod_{j=K+1}^{(M-2)K} \left(1-|\alpha_j|^2\right)^{\beta\left(1-\frac{j}{N}\right)-1} \exp\left(-\sum_{\ell=2}^{M-3} W(\widetilde{\boldsymbol{\alpha}}_\ell, \widetilde{\boldsymbol{\alpha}}_{\ell+1})\right),
\end{aligned}
\tag{3.65}
$$

here $\widetilde{C}_1(V,\beta)\,, \widetilde{C}(V,\beta)$ is a constant depending on V, β, but not on N.
With the same kind of computation, one gets that:

$$
\mathrm{Tr}(\mathcal{T}) \geqslant \widetilde{c}(V,\beta) \int\limits_{\mathbb{D}^{(M-3)K}} \prod_{j=1}^{(M-2)K} \mathrm{d}^2 \alpha_j \prod_{j=K+1}^{(M-2)K} \left(1-|\alpha_j|^2\right)^{\beta\left(1-\frac{j}{N}\right)-1} \exp\left(-\sum_{\ell=2}^{M-3} W(\widetilde{\boldsymbol{\alpha}}_\ell, \widetilde{\boldsymbol{\alpha}}_{\ell+1})\right),
\tag{3.66}
$$

here $\widetilde{c}(V,\beta)$ is a constant depending on V, β, but not on N. From (3.63)-(3.64)-(3.65)-(3.66) we deduce (3.62). $\square$

Applying the previous proposition, we can express the Free energy of the Circular beta ensemble in the high-temperature regime in terms of $\mathrm{Tr}(\widetilde{\mathcal{T}})$:

$$
\begin{aligned}
F_{HT}(V,\beta) &= -\lim_{N\to\infty} \frac{1}{N} \ln\left(Z_N^{HT}\right) = -\lim_{N\to\infty} \frac{1}{N}\left(\ln\left(\frac{Z_N^{HT}}{\mathrm{Tr}(\mathcal{T})}\right) + \ln(\mathrm{Tr}(\mathcal{T}))\right) \\
&= -\lim_{N\to\infty} \frac{\ln(\mathrm{Tr}(\mathcal{T}))}{N},
\end{aligned}
\tag{3.67}
$$

where in the last equality we used Proposition 3.15.

Thus, we have to understand the behaviour of $\mathrm{Tr}(\widetilde{\mathcal{T}})$, to do that we have to carefully analyse the compact operators $\mathcal{T}_{\boldsymbol{\zeta}^{(j)}}$.

Let us define the functions $\psi_n(\mathbf{z}, V, \boldsymbol{\zeta}^{(j)})$ to be the eigenfunctions of $\mathcal{T}_{\boldsymbol{\zeta}^{(j)}}$ with corresponding eigenvalues $\lambda_n(V, \boldsymbol{\zeta}^{(j)})$ in descending order. From a generalized version of Jentzsch Theorem (see [169, Theorem 137.4]), we get that $\lambda_n(V, \boldsymbol{\zeta}^{(j)}) < \lambda_1(V, \boldsymbol{\zeta}^{(j)})$ for all $n \geqslant 2$. Moreover, the largest eigenvalues of each $\mathcal{T}_{\boldsymbol{\zeta}^{(j)}}$ is a differentiable function of the parameters $\boldsymbol{\zeta}^{(j)}$, so we can conclude that:

$$\lambda_1(V,\boldsymbol{\zeta}^{(j+1)}) = \lambda_1(V,\boldsymbol{\zeta}^{(j)})\left(1+O\left(\frac{1}{N}\right)\right).$$

Furthermore, we claim that

$$\left(\psi_1(\mathbf{z},V,\boldsymbol{\zeta}^{(j+1)}), M_K\psi_n(\mathbf{z},V,\boldsymbol{\zeta}^{(j)})\right) = \delta_{1,n} + O\left(\frac{1}{N}\right). \tag{3.68}$$

Indeed let us consider the integral

$$\begin{aligned}&\left(\psi_1(\mathbf{z},V,\boldsymbol{\zeta}^{(j+1)}), T_{\boldsymbol{\zeta}^{(j+1)}}M_K\psi_n(\mathbf{z},V,\boldsymbol{\zeta}^{(j)})\right)\\ &= \left(\psi_1(\mathbf{z},V,\boldsymbol{\zeta}^{(j+1)}), M_kT_{\boldsymbol{\zeta}_j}M_KM_K\psi_n(\mathbf{z},V,\boldsymbol{\zeta}^{(j)})\right)\end{aligned}$$

$$\begin{aligned}&\lambda_1(V,\boldsymbol{\zeta}^{(j+1)})\left(\psi_1(\mathbf{z},V,\boldsymbol{\zeta}^{(j+1)}), M_K\psi_n(\mathbf{z},V,\boldsymbol{\zeta}^{(j)})\right)\\ &\qquad= \lambda_n(V,\boldsymbol{\zeta}^{(j)})\left(\psi_1(\mathbf{z},V,\boldsymbol{\zeta}^{(j+1)}), M_K\psi_n(\mathbf{z},V,\boldsymbol{\zeta}^{(j)})\right)\left(1+O\left(\frac{1}{N}\right)\right),\end{aligned}$$

where in the right-hand side we have expanded in power series of $1/N$ the operator M_K. From the above relation we would obtain $\lambda_1(V,\boldsymbol{\zeta}^{(j+1)}) = \lambda_n(V,\boldsymbol{\zeta}^{(j)})\left(1+O\left(\frac{1}{N}\right)\right)$ for every n which is a contradiction unless (3.68) holds. We also conclude that

$$T_{\boldsymbol{\zeta}^{(j+1)}}M_K\psi_1(\mathbf{z},V,\boldsymbol{\zeta}^{(j)}) = \lambda_1(V,\boldsymbol{\zeta}^{(j+1)})\psi_1(\mathbf{z},V,\boldsymbol{\zeta}^{(j+1)})\left(1+O\left(\frac{1}{N}\right)\right).$$

We are now in position to prove the following proposition

Proposition 3.16. *Let $\psi_n(\mathbf{z},V,\boldsymbol{\zeta}^{(j)})$ be the eigenfunctions of $\mathcal{T}_{\boldsymbol{\zeta}^{(j)}}$* (3.55) *with corresponding eigenvalues $\lambda_n(V,\boldsymbol{\zeta}^{(j)})$ in decreasing order. Consider the operator $\widetilde{\mathcal{T}}$ in* (3.61)*, then the following holds:*

$$\lim_{N\to\infty}\frac{1}{N}\ln\left(\frac{\left(\psi_1(\mathbf{z},V,\boldsymbol{\zeta}^{(1)}),\widetilde{\mathcal{T}}\psi_1(\mathbf{z},V,\boldsymbol{\zeta}^{(1)})\right)}{\prod_{j=1}^{M-1}\lambda_1(V,\boldsymbol{\zeta}^{(j)})}\right) = 0, \tag{3.69}$$

$$\lim_{N\to\infty}\frac{\sum_{\ell\geqslant 2}\left(\psi_\ell(\mathbf{z},V,\boldsymbol{\zeta}^{(1)}),\widetilde{\mathcal{T}}\psi_\ell(\mathbf{z},V,\boldsymbol{\zeta}^{(1)})\right)}{\prod_{j=1}^{M-1}\lambda_1(V,\boldsymbol{\zeta}^{(j)})} = 0. \tag{3.70}$$

Proof. To simplify the notation, we will drop the V dependence of the eigenvalues $\lambda_n(V,\boldsymbol{\zeta}^{(j)})$, and of the eigenfunctions $\psi_n(\mathbf{z},V,\boldsymbol{\zeta}^{(j)})$. We will prove by induction on M that there exists $a_1,\dots,a_{M-1}$ constants independent of N, and so on M, such that

$$\left(\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)}),\widetilde{\mathcal{T}}\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right) = \prod_{j=1}^{M-1}\lambda_1(\boldsymbol{\zeta}^{(j)})\left(1+\frac{a_j}{N}\right),$$

if this expression holds, then (3.69) follows.

For $M=2$, we have that $\widetilde{\mathcal{T}} = M_K\mathcal{T}_{\boldsymbol{\zeta}^{(1)}}$, so we have to compute:

$$\left(\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)}), M_K\mathcal{T}_{\boldsymbol{\zeta}^{(1)}}\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right) = \lambda_1(\boldsymbol{\zeta}^{(1)})\left(\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)}), M_K\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right) .$$

For N big enough, we have that there exists a constant a_1 independent of N such that:

$$\left(\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)}), M_K\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right) = 1 + \frac{a_1}{N} ,$$

so the first inductive step is proved.
For general M, we have to compute:

$$\begin{aligned}&\left(\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)}), M_K\mathcal{T}_{\boldsymbol{\zeta}^{(M-1)}}\cdots M_K\mathcal{T}_{\boldsymbol{\zeta}^{(1)}}\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right) = \lambda_1(\boldsymbol{\zeta}^{(1)})\left(\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)}), M_K\mathcal{T}_{\boldsymbol{\zeta}^{(M-1)}}\cdots M_K\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right)\\ &= \lambda_1(\boldsymbol{\zeta}^{(1)})\sum_{\ell\geqslant 1}\left(\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)}),\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(2)})\right)\left(\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(2)}), M_K\mathcal{T}_{\boldsymbol{\zeta}^{(M-1)}}\cdots\mathcal{T}_{\boldsymbol{\zeta}^{(2)}}\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(2)})\right)\\ &\times\left(\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(2)}), M_K\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right) .\end{aligned} \tag{3.71}$$

We notice that for N big enough, we get that there exists a constant $\xi_{1,\ell}$ independent of N such that:

$$\left(\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(2)}), M_K\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right) = \left(\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(2)}),\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right)\left(1+\frac{\xi_{1,\ell}}{N}\right) .$$

Defining $c_{1,\ell} = \left(\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(2)}),\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right)$, we can recast (3.71) as:

$$\begin{aligned}\left(\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)}), M_K\mathcal{T}_{\boldsymbol{\zeta}^{(M-1)}}\cdots M_K\mathcal{T}_{\boldsymbol{\zeta}^{(1)}}\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right) &= \lambda_1(\boldsymbol{\zeta}^{(1)})\sum_{\ell\geqslant 1}|c_{1,\ell}|^2\left(1+\frac{\xi_{1,\ell}}{N}\right)\\ &\times\left(\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(2)}), M_K\mathcal{T}_{\boldsymbol{\zeta}^{(M-1)}}\cdots\mathcal{T}_{\boldsymbol{\zeta}^{(2)}}\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(2)})\right) .\end{aligned} \tag{3.72}$$

Since $\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(2)})$, $\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(1)})$ are complete orthonormal bases of $L^2(\mathbb{D}^K)$, we can conclude that $\sum_{\ell\geqslant 1}|c_{1,\ell}|^2 = ||\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)})||_2 = 1$. Moreover, from (3.68) we get that there exists a constant χ_1 independent of N such that $|c_{1,1}|^2 = 1-\chi_1/N$. Thus, we deduce that

$$\sum_{\ell\geqslant 2}|c_{1,\ell}|^2 = \frac{\chi_1}{N} .$$

We can rewrite (3.72) in a more convenient way as:

$$\begin{aligned}&\left(\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)}), M_K\mathcal{T}_{\boldsymbol{\zeta}^{(M-1)}}\cdots M_K\mathcal{T}_{\boldsymbol{\zeta}^{(1)}}\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right)\\ &= \lambda_1(\boldsymbol{\zeta}^{(1)})\left(1-\frac{\chi_1}{N}\right)\left(1+\frac{\xi_{1,1}}{N}\right)\left(\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(2)}), M_K\mathcal{T}_{\boldsymbol{\zeta}^{(M-1)}}\cdots\mathcal{T}_{\boldsymbol{\zeta}^{(2)}}\psi_1(\mathbf{z},\boldsymbol{\zeta}^{(2)})\right)\\ &+ \lambda_1(\boldsymbol{\zeta}^{(1)})\sum_{\ell\geqslant 2}|c_{1,\ell}|^2\left(1+\frac{\xi_{1,\ell}}{N}\right)\left(\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(2)}), M_K\mathcal{T}_{\boldsymbol{\zeta}^{(M-1)}}\cdots\mathcal{T}_{\boldsymbol{\zeta}^{(2)}}\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(2)})\right) .\end{aligned}$$

We can apply the inductive hypothesis to the first term of the previous expression, thus to complete the proof we have just to show that there exists a constant C such that

$$\frac{\left|\sum_{\ell\geqslant 2}|c_{1,\ell}|^2\left(1+\frac{\xi_{1,\ell}}{N}\right)\left(\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(2)}),M_K\mathcal{T}_{\boldsymbol{\zeta}^{(M-1)}}\cdots\mathcal{T}_{\boldsymbol{\zeta}^{(2)}}\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(2)})\right)\right|}{\prod_{j=1}^{M-1}\lambda_1(\boldsymbol{\zeta}^{(j)})}\leqslant\frac{C}{N}\,. \tag{3.73}$$

We notice that there exists a constant $\widetilde{C}$ independent of N such that

$$\left|\left(\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(2)}),M_K\mathcal{T}_{\boldsymbol{\zeta}^{(M-1)}}\cdots\mathcal{T}_{\boldsymbol{\zeta}^{(2)}}\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(2)})\right)\right|\leqslant\widetilde{C}\prod_{j=2}^{M-1}\lambda_1(\boldsymbol{\zeta}^{(j)})\,, \tag{3.74}$$

so, from the previous proof, we get that there exists a constant C independent of N such that:

$$\left|\sum_{\ell\geqslant 2}|c_{1,\ell}|^2\left(1+\frac{\xi_{1,\ell}}{N}\right)\left(\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(2)}),M_K\mathcal{T}_{\boldsymbol{\zeta}^{(M-1)}}\cdots\mathcal{T}_{\boldsymbol{\zeta}^{(2)}}\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(2)})\right)\right|\leqslant\frac{C}{N}\prod_{j=1}^{M-1}\lambda_1(\boldsymbol{\zeta}^{(j)})\,,$$

which leads to (3.73), thus we proved (3.69).

To prove (3.70), we rewrite the numerator as:

$$\begin{aligned}\sum_{\ell\geqslant 2}\left(\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(1)}),\widetilde{\mathcal{T}}\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right)&=\sum_{\ell\geqslant 2}\lambda_\ell(\boldsymbol{\zeta}^{(1)})\left(\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(1)}),M_K\mathcal{T}_{\boldsymbol{\zeta}^{(M-1)}}\cdots\mathcal{T}_{\boldsymbol{\zeta}^{(2)}}M_K\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right)\\&=\sum_{\ell\geqslant 2}\lambda_\ell(\boldsymbol{\zeta}^{(1)})\sum_{n\geqslant 1}\left(\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(1)}),\psi_n(\mathbf{z},\boldsymbol{\zeta}^{(2)})\right)\left(\psi_n(\mathbf{z},\boldsymbol{\zeta}^{(2)}),M_K\mathcal{T}_{\boldsymbol{\zeta}^{(M-1)}}\cdots\mathcal{T}_{\boldsymbol{\zeta}^{(2)}}\psi_n(\mathbf{z},\boldsymbol{\zeta}^{(2)})\right)\\&\times\left(\psi_n(\mathbf{z},\boldsymbol{\zeta}^{(2)}),M_K\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right)\,.\end{aligned} \tag{3.75}$$

As in the previous case, we notice that for N big enough there exist constants $\xi_{n,\ell}$ independent of N such that:

$$\left(\psi_n(\mathbf{z},\boldsymbol{\zeta}^{(2)}),M_K\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right)=\left(\psi_n(\mathbf{z},\boldsymbol{\zeta}^{(2)}),\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right)\left(1+\frac{\xi_{n,\ell}}{N}\right)\,.$$

Defining $c_{n,\ell}=\left(\psi_n(\mathbf{z},\boldsymbol{\zeta}^{(2)}),\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right)$, we can recast (3.75) as:

$$\begin{aligned}&\sum_{\ell\geqslant 2}\left(\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(1)}),\widetilde{\mathcal{T}}\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right)\\&\quad=\sum_{\ell\geqslant 2}\lambda_\ell(\boldsymbol{\zeta}^{(1)})\sum_{n\geqslant 1}|c_{n,\ell}|^2\left(1+\frac{\xi_{n,\ell}}{N}\right)\left(\psi_n(\mathbf{z},\boldsymbol{\zeta}^{(2)}),M_K\mathcal{T}_{\boldsymbol{\zeta}^{(M-1)}}\cdots\mathcal{T}_{\boldsymbol{\zeta}^{(2)}}\psi_n(\mathbf{z},\boldsymbol{\zeta}^{(2)})\right)\,.\end{aligned} \tag{3.76}$$

As before, we have that there exist a constant χ_1 independent of N such that $\sum_{\ell\geqslant 2}|c_{n,1}|^2=\chi_1/N$, so, applying (3.74), we get the following estimate for (3.76)

$$
\begin{aligned}
&\left|\sum_{\ell\geqslant 2}\left(\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(1)}),\widetilde{\mathcal{T}}\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right)\right| \\
&\leqslant \frac{C_1}{N}|\lambda_2(\boldsymbol{\zeta}^{(1)})|\prod_{j=2}^{M-1}\lambda_1(\boldsymbol{\zeta}^{(j)}) + \left(1+\frac{C_2}{N}\right)|\lambda_2(\boldsymbol{\zeta}^{(1)})|\left|\sum_{n\geqslant 2}\left(\psi_n(\mathbf{z},\boldsymbol{\zeta}^{(2)}), M_K\mathcal{T}_{\boldsymbol{\zeta}^{(M-1)}}\cdots\mathcal{T}_{\boldsymbol{\zeta}^{(2)}}\psi_n(\mathbf{z},\boldsymbol{\zeta}^{(2)})\right)\right|,
\end{aligned}
$$

where we used that $|c_{n,\ell}|^2 \leqslant 1$ and $\xi_{n,\ell} \leqslant C_2$, here C_2 is a constant independent of N.

Inductively, we get that there exists a constant C independent of N such that:

$$
\left|\sum_{\ell\geqslant 2}\left(\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(1)}),\widetilde{\mathcal{T}}\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right)\right| \leqslant \frac{C}{N}\sum_{j=1}^{M-1}\left(\prod_{\ell=1}^{j}\lambda_2(\boldsymbol{\zeta}^{(j)})\prod_{m=j}^{M-1}\lambda_1(\boldsymbol{\zeta}^{m})\right).
$$

Thus, we get that

$$
\lim_{N\to\infty}\frac{\left|\sum_{\ell\geqslant 2}\left(\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(1)}),\widetilde{\mathcal{T}}\psi_\ell(\mathbf{z},\boldsymbol{\zeta}^{(1)})\right)\right|}{\prod_{j=1}^{M-1}\lambda_1(\boldsymbol{\zeta}^{(j)})} \leqslant \lim_{N\to\infty}\frac{C}{N}\sum_{j=1}^{M-1}\prod_{\ell=1}^{j}\frac{|\lambda_2(\boldsymbol{\zeta}^{(j)})|}{\lambda_1(\boldsymbol{\zeta}^{(j)})} = 0\,,
$$

here in the last equality we used that $|\lambda_2(\boldsymbol{\zeta}^{(\ell)})| < \lambda_1(\boldsymbol{\zeta}^{(\ell)})$ for all ℓ, which implies that the previous sum is not divergent in N. Thus, we got the claim. □

Applying Proposition 3.16 to (3.67) we get that:

$$
\begin{aligned}
F_{HT}(V,\beta) &= -\lim_{N\to\infty}\frac{1}{N}\ln\left(\mathrm{Tr}(\widetilde{\mathcal{T}})\right) = -\lim_{N\to\infty}\frac{1}{N}\ln\left(\sum_{n\geqslant 1}\left(\psi_n(\mathbf{z},V,\boldsymbol{\zeta}^{(1)}),\widetilde{\mathcal{T}}\psi_n(\mathbf{z},V,\boldsymbol{\zeta}^{(1)})\right)\right) \\
&= -\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{M-1}\ln(\lambda_1(V,\boldsymbol{\zeta}^{(j)}))\,.
\end{aligned}
\tag{3.77}
$$

Since the maximum eigenvalue of each $\mathcal{T}_{\boldsymbol{\zeta}^{(j)}}$ is positive and a continuous function of the parameters (see [91, 139]), we get that $\lambda_1(V,\boldsymbol{\zeta}^{(j)}) = \lambda_1\left(V,\boldsymbol{\beta}\left(1-\frac{jK}{N}\right)\right) + O(N^{-1})$. Therefore, we can rewrite (3.77) as

$$
F_{HT}(V,\beta) = -\lim_{N\to\infty}\sum_{j=1}^{M-1}\ln\left(\lambda_1\left(V,\boldsymbol{\beta}\left(1-\frac{jK}{N}\right)\right)\right) = -\frac{1}{K}\int_0^1\ln\left(\lambda_1\left(V,\boldsymbol{\beta}x\right)\right)\mathrm{d}x\,.
$$

This leads to (3.53), moreover, as a consequence of the previous relation, we get that $F_{HT}(V,\beta)$ is differentiable with respect to β.

We notice that the proof is heavily based on the assumption that the potential that we are considering is of finite range, otherwise our approach would not work.

We now prove the moments relations (3.31). Thanks to the symmetries of the measures (3.19)-(3.11) and the definition of mean density of states (3.25) we get that the imaginary part of the moments of μ_{HT}^{β} and μ_{AL}^{β} is equal to zero, meaning that:

$$\int_{\mathbb{T}} e^{i\theta m}\mu_{HT}^{\beta}(\mathrm{d}\theta) = \lim_{N\to\infty}\frac{\mathbf{E}\left[\mathrm{Tr}(E^m)\right]}{N} = \lim_{N\to\infty}\frac{\mathbf{E}\left[\Re\left(\mathrm{Tr}(E^m)\right)\right]}{N} = \int_{\mathbb{T}}\cos(\theta m)\mu_{HT}^{\beta}(\mathrm{d}\theta)\,,$$

$$\int_{\mathbb{T}} e^{i\theta m}\mu_{AL}^{\beta}(\mathrm{d}\theta) = \lim_{N\to\infty}\frac{\mathbf{E}\left[\mathrm{Tr}(\mathcal{E}^m)\right]}{2N} = \lim_{N\to\infty}\frac{\mathbf{E}\left[\Re\left(\mathrm{Tr}(\mathcal{E}^m)\right)\right]}{2N} = \int_{\mathbb{T}}\cos(\theta m)\mu_{AL}^{\beta}(\mathrm{d}\theta)\,,$$

where in the first equation the expected values are taken with respect to (3.19) and to (3.11) in the second one. Therefore, we have just to prove that

$$\int_{\mathbb{T}}\cos(\theta m)\mu_{HT}^{\beta}(\mathrm{d}\theta) = \partial_t F_{HT}\left(V+\frac{t}{2}\Re(z^m),\beta\right)_{|t=0}\,, \tag{3.78}$$

$$\int_{\mathbb{T}}\cos(\theta m)\mu_{AL}^{\beta}(\mathrm{d}\theta) = \partial_t F_{AL}\left(V+\frac{t}{2}\Re(z^m),\beta\right)_{|t=0}\,. \tag{3.79}$$

We focus on (3.78). From Remark 3.5, we have that $F_{HT}(V,\beta) = \mathcal{F}^{(V;\beta)}(\mu_{HT}^{\beta}(\theta))$, we recall that $\mu_{HT}^{\beta}(\theta)$ is the density of the mean density of states of Circular beta ensemble at high-temperature and the functional $\mathcal{F}^{(V,\beta)}$ is defined in (3.26). We write the Euler-Lagrange equation for this functional, getting that $\mu_{HT}^{\beta}(\theta)$ satisfies:

$$2V(\theta) - 2\beta\int_{\mathbb{T}}\ln\left(\sin\left(\frac{|\theta-\gamma|}{2}\right)\right)\mu_{HT}^{\beta}(\gamma)\mathrm{d}\gamma + \ln(\mu_{HT}^{\beta}(\theta)) + C(V,\beta) = 0\,, \tag{3.80}$$

where $C(V,\beta)$ is a constant not depending on θ.

Consider the same functional as before, but with potential $\widetilde{V}(\theta) = V(\theta) + \frac{t}{2}\cos(m\theta)$:

$$\begin{aligned}\mathcal{F}^{\left(V(\theta)+\frac{t}{2}\cos(m\theta),\beta\right)}(\mu) = {}& 2\int_{\mathbb{T}} V(\theta)\mu(\theta)\mathrm{d}\theta + t\int_{\mathbb{T}}\cos(m\theta)\mu(\theta)\mathrm{d}\theta \\ & -\beta\int\int_{\mathbb{T}\times\mathbb{T}}\ln\sin\left(\frac{|\theta-\gamma|}{2}\right)\mu(\theta)\mu(\gamma)\mathrm{d}\theta\mathrm{d}\gamma + \int_{\mathbb{T}}\ln\left(\mu(\theta)\right)\mu(\theta)\mathrm{d}\theta + \ln(2\pi)\,.\end{aligned}$$

Also this functional has a unique minimizer $\mu^{(t)}(\theta)$, and we notice that $\mu^{(0)}(\theta) = \mu_{HT}^{\beta}(\theta)$. Evaluating the previous functional at $\mu^{(t)}(\theta)$, and computing its derivative at $t=0$, we get that:

$$
\begin{aligned}
\partial_t \mathcal{F}^{\left(V(\theta)+\frac{t}{2}\cos(m\theta),\beta\right)}(\mu)_{|t=0} &= 2\int_{\mathbb{T}} V(\theta)\partial_t\mu^{(t)}(\theta)_{|t=0}\mathrm{d}\theta \\
&+ \int_{\mathbb{T}} \cos(m\theta)\mu^{\beta}_{HT}(\theta)\mathrm{d}\theta - \beta\int\int_{\mathbb{T}\times\mathbb{T}} \ln\sin\left(\frac{|\theta-\gamma|}{2}\right)\mu^{\beta}_{HT}(\gamma)\partial_t\mu^{(t)}(\theta)_{|t=0}\mathrm{d}\theta\mathrm{d}\gamma \\
&+ \int_{\mathbb{T}} \ln\left(\mu^{\beta}_{HT}(\theta)\right)\partial_t\mu^{(t)}(\theta)_{|t=0}\mathrm{d}\theta\,.
\end{aligned}
\tag{3.81}
$$

Testing (3.80) against $\partial_t\mu^{(t)}(\theta)_{|t=0}$ we obtain

$$
\begin{aligned}
2\int_{\mathbb{T}} V(\theta)\partial_t\mu^{(t)}(\theta)_{|t=0}\mathrm{d}\theta - \beta\int\int_{\mathbb{T}\times\mathbb{T}} \ln\sin\left(\frac{|\theta-\gamma|}{2}\right)\mu^{\beta}_{HT}(\gamma)\partial_t\mu^{(t)}(\theta)_{|t=0}\mathrm{d}\theta\mathrm{d}\gamma \\
+ \int_{\mathbb{T}} \ln\left(\mu^{\beta}_{HT}(\theta)\right)\partial_t\mu^{(t)}(\theta)_{|t=0}\mathrm{d}\theta = 0\,,
\end{aligned}
$$

where we used that $\int_{\mathbb{T}}\partial_t\mu^{(t)}(\theta)\mathrm{d}\theta = 0$. Thus, we can simplify (3.81) as :

$$
\partial_t \mathcal{F}^{\left(V(\theta)+\frac{t}{2}\cos(m\theta),\beta\right)}(\mu)_{|t=0} = \int_{\mathbb{T}} \cos(m\theta)\mu^{\beta}_{HT}(\theta)\mathrm{d}\theta\,,
$$

which is equivalent to (3.78).

To complete the proof of Proposition 3.8 we have to show that (3.79) holds. From the definition of mean density of states (3.25) we have that:

$$
\int_{\mathbb{T}} \cos(\theta m)\mu^{\beta}_{AL}(\mathrm{d}\theta) = \lim_{N\to\infty}\frac{\mathbf{E}\left[\Re\left(\mathrm{Tr}(\mathcal{E}^m)\right)\right]}{2N} = -\lim_{N\to\infty}\frac{\partial_t\left(Z_N^{(AL)}\left(V+\frac{t}{2}\Re(z^m),\beta\right)\right)_{|t=0}}{NZ_N^{(AL)}(V,\beta)}\,,
$$

where the expected value is taken with respect to the generalized Gibbs ensemble of the Ablowitz-Ladik lattice.

Exploiting (3.58) we can rewrite the previous expression as:

$$
\begin{aligned}
-\lim_{N\to\infty}\frac{\partial_t\left(Z_N^{(AL)}\left(V+\frac{t}{2}\Re(z^m),\beta\right)\right)_{|t=0}}{NZ_N^{(AL)}(V,\beta)} &= -\lim_{N\to\infty}\frac{\partial_t\left(\sum_{\ell=1}^{\infty}\lambda_\ell^M\left(V+\frac{t}{2}\Re(z^m),\beta\right)\right)_{|t=0}}{N\sum_{\ell=1}^{\infty}\lambda_\ell^M(V,\beta)} \\
&= -\frac{\partial_t\lambda_{\max}\left(\left(V+\frac{t}{2}\Re(z^m),\beta\right)\right)_{|t=0}}{K\lambda_{\max}(V,\beta)} \\
&= \partial_t\left(F_{AL}\left(V+\frac{t}{2}\Re(z^m),\beta\right)\right)_{|t=0}\,.
\end{aligned}
$$

Thus, we have completed the proof of Proposition 3.8. □

3.3.2 Proof of lemma 3.14

We prove (3.47) for $k=1$ and the cases $k>1$ easily follow.

Let us define $R_1^{(s)} = \begin{pmatrix} f_s & h_s \\ p_s & q_s \end{pmatrix}$ where $s \geqslant 1$. If follows from (3.46) that

$$\begin{pmatrix} f_s & h_s \\ p_s & q_s \end{pmatrix} = \begin{pmatrix} f_{s-1} & h_{s-1} \\ p_{s-1} & q_{s-1} \end{pmatrix} \begin{pmatrix} 1 + \frac{\lambda\beta\eta}{s(s+\beta)} & \frac{\eta^2}{s(s+\beta+1)} \\ 1 & 0 \end{pmatrix}, \quad s > 1,$$

where $\beta > 0$ and $\lambda, \eta \in \mathbb{R}\backslash\{0\}$. Note that in the case $\eta = 0$ the lemma is trivially satisfied. We will show that all the sequences $\{f_s, h_s, p_s, q_s\}_{s\geqslant 1}$ converge as $s \to \infty$, moreover $h_s, q_s \xrightarrow{s\to\infty} 0$. First of all, we notice that $h_s = \frac{\eta^2 f_{s-1}}{s(s+\beta+1)}$ and $q_s = \frac{\eta^2 p_{s-1}}{s(s+\beta+1)}$, thus the convergence to zero of these two sequences follows from the convergence of p_s and f_s as $s \to \infty$. Moreover, the terms of these last two sequences obey to the 3-terms recurrence:

$$f_s = \left(1 + \frac{\lambda\beta\eta}{s(s+\beta)}\right) f_{s-1} + \frac{\eta^2}{(s-1)(s+\beta)} f_{s-2},$$

and the same holds for p_s in place of f_s. Thus, we have just to prove that the sequence $\{f_s\}_{s\geqslant 1}$ converges. For this purpose we bound $|f_s|$ from above as:

$$|f_s| \leqslant \left(1 + \frac{2\eta^2 + |\lambda\beta\eta|}{s(s+\beta)}\right) \max\left(|f_{s-1}|, |f_{s-2}|\right),$$

inductively we get that there exists a constant C depending just on the initial condition such that:

$$|f_s| \leqslant C \prod_{\ell=1}^{s} \left(1 + \frac{2\eta^2 + |\lambda\beta\eta|}{\ell(\ell+\beta)}\right) \leqslant C \prod_{\ell=1}^{\infty} \left(1 + \frac{2\eta^2 + |\lambda\beta\eta|}{\ell(\ell+\beta)}\right). \tag{3.82}$$

Since the infinite product on the right-hand side of (3.82) is convergent by a classical result, see for example [100, Chapter XIII, Lemma 1], this implies that the sequence $\{f_s\}_{s\geqslant 1}$ is uniformly bounded. Moreover, we have that:

$$|f_{s+1} - f_s| \leqslant \frac{|f_s\lambda\beta\eta|}{(s+1)(s+1+\beta)} + \frac{\eta^2|f_{s-1}|}{s(s-1+\beta)} \leqslant \tilde{C}\frac{\eta^2 + |\lambda\beta\eta|}{s(s-1+\beta)},$$

for some constant $\tilde{C} > 0$. This last equation implies that the sequence $\{f_s\}_{s\geqslant 1}$ is a Cauchy sequence, thus it is convergent. So we get the claim (3.46). The claim (3.48) easily follows from (3.46).

Regarding the differentiability of the matrix R_k with respect to the parameters, we consider only the η-dependence and the other cases can be treated in the same way. We observe that

$$\lim_{h\to 0} \frac{R_k(\eta+h) - R_k(\eta)}{h} = \sum_{s=k}^{\infty} R_k^{(s-1)} \begin{pmatrix} \frac{\lambda\beta}{s(s+\beta)} & \frac{2\eta}{s(s+\beta+1)} \\ 0 & 0 \end{pmatrix} R_s.$$

The r.h.s. of the above expression is a convergent sum since each entry of the matrix R_s and $R_k^{(s-1)}$ is uniformly bounded because of (3.82). □

Chapter 4

Adiabatic Invariant for the FPUT chain

The FPUT chain with N particles is the system with Hamiltonian

$$H_F(\mathbf{p},\mathbf{q}) = \sum_{j=0}^{N-1} \frac{p_j^2}{2} + \sum_{j=0}^{N-1} V_F(q_{j+1}-q_j) \ , \quad V_F(x) = \frac{x^2}{2} - \frac{x^3}{6} + \mathtt{b}\frac{x^4}{24} \ , \tag{4.1}$$

which we consider with periodic boundary conditions $q_{j+N} = q_j$, $p_{j+N} = p_j$ and $\mathtt{b} > 0$. We observe that any generic nearest neighbourhood quartic potential can be set in the form of $V_F(x)$ through a canonical change of coordinates.

Over the last 60 years, the FPUT system has been the object of intense numerical and analytical research. Nowadays, it is well understood that the system displays, on a relatively short time-scale, an integrable-like behaviour, first uncovered by Fermi, Pasta, Ulam and Tsingou [50] and later interpreted in terms of closeness to a nonlinear integrable system by some authors, e.g. the Korteweg-de Vries (KdV) equation by Zabusky and Kruskal [170], the Boussinesq equation by Zakharov [171], and the Toda chain by Manakov first [110], and then by Ferguson, Flaschka and McLauglin [49]. On larger time-scales the system displays instead an ergodic behavior and approaches its micro-canonical equilibrium state (i.e. measure), unless the energy is so low to enter a KAM-like regime [85, 89, 142].

In this chapter, we give a quantitative result of the integrable behaviour of the FPUT system that hold in the thermodynamic limit. Namely, we show that a family of first integrals of the Toda system are adiabatic invariants (namely almost constant quantities) for the FPUT system. We bound their variation for times of order $\beta^{1-2\varepsilon}$, $\varepsilon > 0$, where β is the inverse of the temperature of the chain. Such estimates hold for a large set of initial data with respect to the Gibbs measure of the chain and they are uniform in the number of particles, thus they persist in the thermodynamic limit. In this way we show that the FPUT chain has, in measure, an integrable-like behaviour on time scales of order $\beta^{1-2\varepsilon}$, thus we give an insight of the so-called FPUT paradox.

In the last few years, there has been a lot of activity in the problem of constructing adiabatic invariants of nonlinear chain systems in the thermodynamic limit, see [29, 30, 68, 69, 108, 109]. In particular, adiabatic invariants in measure for the FPUT chain have been recently introduced by Maiocchi, Bambusi, Carati [108] by considering the FPUT chain a perturbation of the linear harmonic chain. Our approach is based on the remark [49,110] that the FPUT chain (4.1) can be regarded as a perturbation of the (nonlinear) Toda chain [156]

$$H_T(\mathbf{p},\mathbf{q}) := \frac{1}{2}\sum_{j=0}^{N-1} p_j^2 + \sum_{j=0}^{N-1} V_T(q_{j+1}-q_j)\ , \quad V_T(x) = e^{-x} + x - 1\,, \tag{4.2}$$

which we consider again with periodic boundary conditions $q_{j+N} = q_j$, $p_{j+N} = p_j$. The equations of motion of (4.1) and (4.2) take the form

$$\dot{q}_j = \frac{\partial H}{\partial p_j} = p_j, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} = V'(q_{j+1}-q_j) - V'(q_j - q_{j-1}), \ \ j = 0,\ldots,N-1, \tag{4.3}$$

where H stands for H_F or H_T and V for V_F and V_T respectively.
According to the values of $\mathtt{b}$ in (4.1), the Toda chain is either an approximation of the FPUT chain of third order (for $\mathtt{b} \neq 1$), or fourth order (for $\mathtt{b} = 1$). We remark that the Toda chain is the only nonlinear integrable FPUT-like chain [41,146].

The Toda chain admits several families of N integrals of motion in involution (e.g. [52, 84, 162]). Among the various families of integrals of motion, the ones constructed by Henon [82] and Flaschka [51] are explicit and easy to compute, being the trace of the powers of the Lax matrix associated to the Toda chain. In the following, we refer to them simply as *Toda integrals* and denote them by $J^{(k)}$, $1 \leqslant k \leqslant N$ (see (4.13)).

As the $J^{(k)}$'s are conserved along the Toda flow, and the FPUT chain is a perturbation of the Toda one, the Toda integrals are good candidates to be adiabatic invariants when computed along the FPUT flow. This intuition is supported by several numerical simulations, the first by Ferguson-Flaschka-McLaughlin [49] and more recently by other authors [19,21,31,72,135]. Such simulations show that the variation of the Toda integrals along the FPUT flow is very small on long times for initial data of small specific energy. In particular, the numerical results in [19,21,72] suggest that such phenomenon should persist in the thermodynamic limit and for "generic" initial conditions.

Our first result is a quantitative, analytical proof of this phenomenon. More precisely, we fix an arbitrary $m \in \mathbb{N}$ and provided N and β sufficiently large, we bound the variations of the first m Toda integrals computed along the flow of FPUT, for times of order

$$\frac{\beta^{1-2\varepsilon}}{((\mathtt{b}-1)^2 + C_1\beta^{-1})^{\frac{1}{2}}}, \tag{4.4}$$

where $\varepsilon > 0$ is arbitrary small and C_1 is a positive constant, independent of β, N. Such a bound holds for initial data in a set of large Gibbs measure. Note that the bound (4.4) improves to $\beta^{\frac{3}{2}-2\varepsilon}$ when $\mathtt{b} = 1$, namely when the Toda chain becomes a fourth order approximation of the FPUT chain. Such analytical time-scales are compatible with (namely smaller than) the numerical ones determined in [19–21].

An interesting question is whether the Toda integrals $J^{(k)}$'s control the normal modes of FPUT, namely the action of the linearized chain. It turns out that this is indeed the case: we prove that the quadratic parts $J_2^{(2k)}$ (namely the Taylor polynomials of order 2) of the integral of motions $J^{(2k)}$, are linear combinations of the normal modes. Namely, one has

$$J^{(2k)} = \sum_{j=0}^{N-1} \widehat{c}_j^{(k)}\, E_j + O((\widehat{\mathbf{p}},\widehat{\mathbf{q}})^3),$$

where E_j is the j^{th} normal mode (see (4.18) for its formula), $(\widehat{\mathbf{p}}, \widehat{\mathbf{q}})$ are the discrete Hartley transform of $(\mathbf{p}, \mathbf{q})$ (see definition below in (4.16)) and $\widehat{\mathbf{c}}^{(k)}$ are real coefficients.

So we consider linear combinations of the normal modes of the form

$$\sum_{j=0}^{N-1} \widehat{g}_j E_j \tag{4.5}$$

where $(\widehat{g}_j)_j$ is the discrete Hartley transform of a vector $\mathbf{g} \in \mathbb{R}^N$ which has only $2\lfloor \frac{m}{2} \rfloor + 2$ non-zero entries with m independent of N, here $\lfloor \frac{m}{2} \rfloor$ is the integer part of $\frac{m}{2}$. Our second result shows that linear combinations of the form (4.5), when computed along the FPUT flow, are adiabatic invariants for the same time-scale as in (4.4).

Actually, exploiting the fact that the Toda integrals are invariant for the Toda dynamics, we deduce also that the linear combinations in (4.5), when computed along the flow of Toda chain, are adiabatic invariants *for all times*. This is our third result.

Examples of linear combinations (4.5) that we control are

$$\sum_{j=1}^{N} \sin^{2\ell}\left(\frac{j\pi}{N}\right) E_j, \qquad \sum_{j=1}^{N} \cos^{2\ell}\left(\frac{j\pi}{N}\right) E_j, \qquad \forall \ell = 0, \ldots, \left\lfloor \frac{m}{2} \right\rfloor. \tag{4.6}$$

These linear combinations weight in different ways low and high energy modes.

Our results are mainly based on two ingredients. The first one is a detailed study of the algebraic properties of the Toda integrals. The second ingredient comes from adapting to our case, methods of statistical mechanics developed by Carati [29] and Carati-Maiocchi [30], and also in [68, 69, 108, 109].

4.1 Statement of results

4.1.1 Toda integrals as adiabatic invariants for FPUT

We come to precise statements of the main results of the present chapter. We consider the FPUT chain (4.1) and the Toda chain (4.2) in the subspace

$$\mathcal{M} := \left\{ (\mathbf{p}, \mathbf{q}) \in \mathbb{R}^N \times \mathbb{R}^N : \quad \sum_{j=0}^{N-1} q_j = \mathcal{L} \ , \sum_{j=0}^{N-1} p_j = 0 \right\} , \tag{4.7}$$

which is invariant for the dynamics. Here $\mathcal{L}$ is a positive constant.

Since both H_F and H_T depend just on the relative distance between q_{j+1} and q_j, it is natural to introduce on $\mathcal{M}$ the *elongations* r_j's as

$$r_j := q_{j+1} - q_j, \qquad 0 \leqslant j \leqslant N-1 \, , \tag{4.8}$$

which are naturally constrained to

$$\sum_{j=0}^{N-1} r_j = 0 \, , \tag{4.9}$$

due to the periodic boundary condition $q_N = q_0$. We observe that the change of coordinates (4.8) together with the condition (4.9) is well-defined on the phase space $\mathcal{M}$, but not on the whole phase space $\mathbb{R}^N \times \mathbb{R}^N$. In these variables the phase space $\mathcal{M}$ reads

$$\mathcal{M} := \left\{ (\mathbf{p}, \mathbf{r}) \in \mathbb{R}^N \times \mathbb{R}^N : \quad \sum_{j=0}^{N-1} r_j = \sum_{j=0}^{N-1} p_j = 0 \right\} . \tag{4.10}$$

We endow $\mathcal{M}$ by the Gibbs measure of H_F at temperature β^{-1}, namely we put

$$\mathrm{d}\mu_F := \frac{1}{Z_F(\beta)} \; e^{-\beta H_F(\mathbf{p},\mathbf{r})} \; \delta_0 \left(\sum_{j=0}^{N-1} r_j \right) \; \delta_0 \left(\sum_{j=0}^{N-1} p_j \right) \; \mathrm{d}\mathbf{p} \, \mathrm{d}\mathbf{r}, \tag{4.11}$$

where as usual $Z_F(\beta)$ is the partition function which normalize the measure, namely

$$Z_F(\beta) := \int_{\mathbb{R}^N \times \mathbb{R}^N} e^{-\beta H_F(\mathbf{p},\mathbf{r})} \; \delta_0 \left(\sum_{j=0}^{N-1} r_j \right) \; \delta_0 \left(\sum_{j=0}^{N-1} p_j \right) \; \mathrm{d}\mathbf{p} \, \mathrm{d}\mathbf{r}.$$

Given a function $f \colon \mathcal{M} \to \mathbb{C}$, we will use the probability (4.11) to compute its average $\langle f \rangle$, its L^2 norm $\|f\|$, its variance σ_f^2 defined as

$$\begin{aligned} \langle f \rangle &:= \mathbf{E}\,[f] \equiv \int_{\mathbb{R}^{2N}} f(\mathbf{p}, \mathbf{r}) \; \mathrm{d}\mu_F, \\ \|f\|^2 &:= \mathbf{E}\left[|f|^2\right] \equiv \int_{\mathbb{R}^{2N}} |f(\mathbf{p}, \mathbf{r})|^2 \, \mathrm{d}\mu_F, \\ \sigma_f^2 &:= \|f - \langle f \rangle\|^2. \end{aligned}$$

In order to state our first theorem, we must introduce the Toda integrals of motion. It is well known that the Toda chain is an integrable system [82, 156]. The standard way to prove its integrability is to put it in a Lax-pair form. The Lax form was introduced by Flaschka in [51] and Manakov [110] and it is obtained through the change of coordinates

$$b_j := -p_j \, , \qquad a_j := e^{\frac{1}{2}(q_j - q_{j+1})} \equiv e^{-\frac{1}{2} r_j}, \qquad 0 \leqslant j \leqslant N-1 \, .$$

By the geometric constraint (4.9) and the momentum conservation $\sum_{j=0}^{N-1} p_j = 0$ (see (4.7)), such variables are constrained by the conditions

$$\sum_{j=0}^{N-1} b_j = 0, \qquad \prod_{j=0}^{N-1} a_j = 1 \, .$$

The Lax operator for the Toda chain is the periodic Jacobi matrix [162]

$$L(b, a) := \begin{pmatrix} b_0 & a_0 & 0 & \dots & a_{N-1} \\ a_0 & b_1 & a_1 & \ddots & \vdots \\ 0 & a_1 & b_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-2} \\ a_{N-1} & \dots & 0 & a_{N-2} & b_{N-1} \end{pmatrix} . \tag{4.12}$$

We introduce the matrix $B = L_+ - L_+^\intercal$ where for a square matrix X we call X_+ the upper triangular part of X

$$(X_+)_{ij} = \begin{cases} X_{ij}, & i = j+1 \mod N \\ 0, & \text{otherwise} \end{cases}$$

A straightforward calculation shows that the Toda equations of motions (4.3) are equivalent to

$$\frac{dL}{dt} = [B, L].$$

It follows that the eigenvalues of L are integrals of motion in involutions.

In particular, the trace of powers of L,

$$J^{(m)} := \frac{1}{m}\mathrm{Tr}\left(L^m\right), \qquad \forall 1 \leqslant m \leqslant N \tag{4.13}$$

are N independent, commuting, integrals of motions in involution. Such integrals were first introduced by Henon [82] (with a different method), and we refer to them as *Toda integrals*. We give the first few of them explicitly, written in the variables $(\mathbf{p}, \mathbf{r})$:

$$J^{(1)}(\mathbf{p}) := -\sum_{i=0}^{N-1} p_i, \qquad J^{(2)}(\mathbf{p},\mathbf{r}) := \sum_{i=0}^{N-1}\left[\frac{p_i^2}{2} + e^{-r_i}\right],$$

$$J^{(3)}(\mathbf{p},\mathbf{r}) := -\sum_{i=0}^{N-1}\left[\frac{1}{3}p_i^3 + (p_i + p_{i+1})e^{-r_i}\right],$$

$$J^{(4)}(\mathbf{p},\mathbf{r}) := \sum_{i=0}^{N-1}\left[\frac{1}{4}p_i^4 + (p_i^2 + p_i p_{i+1} + p_{i+1}^2)e^{-r_i} + \frac{1}{2}e^{-2r_i} + e^{-r_i - r_{i+1}}\right].$$

Note that $J^{(2)}$ coincides with the Toda Hamiltonian H_T.

Our first result shows that the Toda integral $J^{(m)}$, computed along the Hamiltonian flow $\phi^t_{H_F}$ of the FPUT chain, is an adiabatic invariant for long times and for a set of initial data in a set of large Gibbs measure. Here is the precise statement:

Theorem 4.1. *Fix $m \in \mathbb{N}$. There exist $N_0, \beta_0, C_0, C_1 > 0$ (depending on m), such that for any $N > N_0$, $\beta > \beta_0$, $0 < \varepsilon < \frac{1}{4}$, one has*

$$\mathbf{P}\left(\left|J^{(m)} \circ \phi^t_{H_F} - J^{(m)}\right| > \frac{\sigma_{J^{(m)}}}{\beta^\varepsilon}\right) \leqslant \frac{C_0}{\beta^{2\varepsilon}}, \tag{4.14}$$

for every time t fulfilling

$$|t| \leqslant \frac{\beta^{1-2\varepsilon}}{\left((\mathtt{b}-1)^2 + C_1\beta^{-1}\right)^{1/2}}. \tag{4.15}$$

In (4.14) $\mathbf{P}$ *stands for the probability with respect to the Gibbs measure* (4.11).

We observe that the time-scale (4.15) increases to $\beta^{\frac{3}{2}-2\varepsilon}$ for $\mathtt{b} = 1$, namely if the Toda chain is a fifth order approximation of the FPUT chain.

Remark 4.2. *We observe that our estimates in* (4.14) *and* (4.15) *are independent of the number of particles N. Therefore, we can claim that the result of theorem 4.1 holds true in the thermodynamic limit, i.e. when $\lim_{N\to\infty}\frac{\langle H_F\rangle}{N}=e>0$ where $\langle H_F\rangle$ is the average over the Gibbs measure* (4.11) *of the FPUT Hamiltonian H_F. The same observation applies to theorem 4.4 and theorem 4.5 below.*

Our Theorem 4.1 gives a quantitative, analytical proof of the adiabatic invariance of the Toda integrals, at least for a set of initial data of large measure. It is an interesting question whether other integrals of motion of the Toda chain are adiabatic invariants for the FPUT chain. Natural candidates are the actions and spectral gaps.

Action-angle coordinates and the related Birkhoff coordinates (a cartesian version of action-angle variables) were constructed analytically by Henrici and Kappeler [83,84] for any finite N, and by Bambusi and one of the author [16] uniformly in N, but in a regime of specific energy going to 0 when N goes to infinity (thus not the thermodynamic limit).
The difficulty in dealing with these other sets of integrals is that they are not explicit in the physical variables $(\mathbf{p},\mathbf{r})$. As a consequence, it appears very difficult to compute their averages with respect to the Gibbs measure of the system.

Despite these analytical challenges, recent numerical simulations by Goldfriend and Kurchan [72] suggest that the spectral gaps of the Toda chain are adiabatic invariants for the FPUT chain for long times also in the thermodynamic limit.

4.1.2 Packets of normal modes

Our second result concerns adiabatic invariance of some special linear combination of normal modes. To state the result, we first introduce the normal modes through the discrete Hartley transform. Such transformation, which we denote by $\mathcal{H}$, is defined as

$$\widehat{\mathbf{p}}:=\mathcal{H}\mathbf{p},\quad \mathcal{H}_{j,k}:=\frac{1}{\sqrt{N}}\left(\cos\left(2\pi\frac{jk}{N}\right)+\sin\left(2\pi\frac{jk}{N}\right)\right),\qquad j,k=0,\ldots,N-1 \tag{4.16}$$

and one easily verifies that it fulfils

$$\mathcal{H}^2=\mathbb{1},\qquad \mathcal{H}^\intercal=\mathcal{H}.$$

The Hartley transform is closely related to the classical Fourier transform $\mathcal{F}$, whose matrix elements are $\mathcal{F}_{j,k}:=\frac{1}{\sqrt{N}}e^{-\mathrm{i}2\pi jk/N}$, as one has $\mathcal{H}=\Re\mathcal{F}-\Im\mathcal{F}$. The advantage of the Hartley transform is that it maps real variables into real variables, a fact which will be useful when calculating averages of quadratic Hamiltonians (see Section 4.4.2).

A consequence of (4.16) is that the change of coordinates

$$\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N\times\mathbb{R}^N,\quad (\mathbf{p},\mathbf{q})\mapsto(\widehat{\mathbf{p}},\widehat{\mathbf{q}}):=(\mathcal{H}\mathbf{p},\mathcal{H}\mathbf{q})$$

is a canonical one. Due to $\sum_j p_j=0$, $\sum_j q_j=\mathcal{L}$, one has also $\widehat{p}_0=0$, $\widehat{q}_0=\mathcal{L}/\sqrt{N}$. In these variables the quadratic part H_2 of the Toda Hamiltonian (4.1), i.e. its Taylor expansion of order two nearby the origin, takes the form

$$H_2(\widehat{\mathbf{p}},\widehat{\mathbf{q}}):=\sum_{j=1}^{N-1}\frac{\widehat{p}_j^2+\omega_j^2\widehat{q}_j^2}{2},\qquad \omega_j:=2\sin\left(\pi\frac{j}{N}\right). \tag{4.17}$$

We observe that (4.17) is exactly the Hamiltonian of the Harmonic Oscillator chain. We define

$$E_j := \frac{\widehat{p}_j^2 + \omega_j^2 \widehat{q}_j^2}{2}, \qquad j = 1, \ldots, N-1, \tag{4.18}$$

the j^{th} *normal mode.*

To state our second result, we need the following definition:

Definition 4.3 (m-admissible vector)**.** *Fix $m \in \mathbb{N}$ and $\widetilde{m} := \lfloor \frac{m}{2} \rfloor$, where $\lfloor \frac{m}{2} \rfloor$ is the integer part of $\frac{m}{2}$. For any $N > m$, a vector $\mathbf{x} \in \mathbb{R}^N$ is said to be m-*admissible *if there exists a non-zero vector $\mathbf{y} = (y_0, y_1, \ldots, y_{\widetilde{m}}) \in \mathbb{R}^{\widetilde{m}+1}$ with $K^{-1} \leqslant \sum_j |y_j| \leqslant K$, K independent of N, such that*

$$x_k = x_{N-k} = y_k, \text{ for } 0 \leqslant k \leqslant \widetilde{m} \text{ and } x_k = 0 \text{ otherwise.}$$

We are ready to state our second result, which shows that special linear combinations of normal modes are adiabatic invariants for the FPUT dynamics for long times. Here is the precise statement:

Theorem 4.4. *Fix $m \in \mathbb{N}$ and let $\boldsymbol{g} = (g_0, \ldots, g_{N-1}) \in \mathbb{R}^N$ be an m-admissible vector (according to Definition 4.3). Define*

$$\Phi := \sum_{j=0}^{N-1} \widehat{g}_j E_j, \tag{4.19}$$

where $\widehat{\mathbf{g}}$ is the discrete Hartley transform (4.16) *of $\mathbf{g}$, and E_j is the harmonic energy* (4.18)*. Then there exist $N_0, \beta_0, C_0, C_1 > 0$ (depending on m), such that for any $N > N_0$, $\beta > \beta_0$, $0 < \varepsilon < \frac{1}{4}$, one has*

$$\mathbf{P}\left(\left| \Phi \circ \phi^t_{H_F} - \Phi \right| > \frac{\sigma_\Phi}{\beta^\varepsilon} \right) \leqslant \frac{C_0}{\beta^{2\varepsilon}},$$

for every time t fulfilling (4.15).

Again, when $\mathtt{b} = 1$ the time-scale improves by a factor $\beta^{\frac{1}{2}}$.

Finally, we consider the Toda dynamics generated by the Hamiltonian H_T in (4.2). In this case we endow $\mathcal{M}$ in (4.10) by the Gibbs measure of H_T at temperature β^{-1}, namely we put

$$\mathrm{d}\mu_T := \frac{1}{Z_T(\beta)} e^{-\beta H_T(\mathbf{p},\mathbf{r})} \, \delta\left(\sum_j r_j = 0 \right) \delta\left(\sum_j p_j = 0 \right) \mathrm{d}\mathbf{p} \, \mathrm{d}\mathbf{r}, \tag{4.20}$$

where as usual $Z_T(\beta)$ is the partition function which normalize the measure, namely

$$Z_T(\beta) := \int_{\mathbb{R}^N \times \mathbb{R}^N} e^{-\beta H_T(\mathbf{p},\mathbf{r})} \, \delta\left(\sum_j r_j = 0 \right) \delta\left(\sum_j p_j = 0 \right) \mathrm{d}\mathbf{p} \, \mathrm{d}\mathbf{r}.$$

We prove that the quantity (4.19), computed along the Hamiltonian flow $\phi^t_{H_T}$ of the Toda chain, is an adiabatic invariant *for all times* and for a large set of initial data:

Theorem 4.5. *Fix $m \in \mathbb{N}$; let $\mathbf{g} \in \mathbb{R}^N$ be an m-admissible vector and define Φ as in* (4.19)*. Then there exist $N_0, \beta_0, C > 0$ such that for any $N > N_0$, $\beta > \beta_0$, $0 < \varepsilon < \frac{1}{2}$, one has*

$$\mathbf{P}\left(\left|\Phi \circ \phi^t_{H_T} - \Phi\right| > \frac{\sigma_\Phi}{\beta^\varepsilon}\right) \leqslant \frac{C}{\beta^{1-2\varepsilon}}\,, \tag{4.21}$$

for all times.

Remark 4.6. *It is easy to verify that the functions Φ in* (4.19) *are linear combinations of*

$$\sum_{j=0}^{N-1} \cos\left(\frac{2\ell j\pi}{N}\right) E_j, \qquad \ell = 0, \ldots, \left\lfloor \frac{m}{2} \right\rfloor$$

(choose $g_\ell = g_{N-\ell} = 1$, $g_j = 0$ otherwise). Then, using the multi-angle trigonometric formula

$$\cos(2nx) = (-1)^n T_{2n}(\sin x), \qquad \cos(2nx) = T_{2n}(\cos x),$$

where the T_n's are the Chebyshev polynomial of the first kind, it follows that we can control (4.6)*.*

Let us comment about the significance of Theorem 4.4 and Theorem 4.5. The study of the dynamics of the normal modes of FPUT goes back to the pioneering numerical simulations of Fermi, Pasta, Ulam and Tsingou [50]. They observed that, corresponding to initial data with only the first normal mode excited, namely initial data with $E_1 \neq 0$ and $E_j = 0 \ \forall j \neq 1$, the dynamics of the normal modes develops a recurrent behavior, whereas their time averages $\frac{1}{t}\int_0^t E_j \circ \phi^\tau_{H_F} \mathrm{d}\tau$ quickly relaxed to a sequence exponentially localized in j. This is what is known under the name of FPUT packet of modes.

Subsequent numerical simulations have investigated the persistence of the phenomenon for large N and in different regimes of specific energies [19, 21, 22, 64, 105, 129] (see also [15] for a survey of results about the FPUT dynamics).

Analytical results controlling packets of normal modes along the FPUT system are proven in [16, 17]. All these results deal with specific energies going to zero as the number of particles go to infinity, thus they do not hold in the thermodynamic limit. Our result controls linear combination of normal modes and holds in the thermodynamic limit.

4.1.3 Ideas of the proof

The starting point of our analysis is to estimate the probability that the time evolution of an observable $\Phi(t)$, computed along the Hamiltonian flow of H, slightly deviates from its initial value. In our application Φ is either the Toda integral of motion or a special linear combination of the harmonic energies and H is either the FPUT or Toda Hamiltonian. Quantitatively, Chebyshev inequality gives

$$\mathbf{P}\Big(\,|\Phi(t) - \Phi(0)| > \lambda\sigma_{\Phi(0)}\Big) \leqslant \frac{1}{\lambda^2}\frac{\sigma^2_{\Phi(t)-\Phi(0)}}{\sigma^2_{\Phi(0)}}\,, \qquad \forall \lambda > 0. \tag{4.22}$$

So our first task is to give an upper bound on the variance $\sigma_{\Phi(t)-\Phi(0)}$ and a lower bound on the variance $\sigma_{\Phi(0)}$. Regarding the former bound, we exploit the Carati-Maiocchi inequality [30]

$$\sigma^2_{\Phi(t)-\Phi(0)} \leqslant \left\langle \{\Phi, H\}^2 \right\rangle t^2, \qquad \forall t \in \mathbb{R}, \tag{4.23}$$

where $\{\Phi, H\}$, denotes the canonical Poisson bracket

$$\{\Phi, H\} := (\partial_{\mathbf{q}}\Phi)^\intercal \partial_{\mathbf{p}} H - (\partial_{\mathbf{p}}\Phi)^\intercal \partial_{\mathbf{q}} H \equiv \sum_{i=0}^{N-1} \partial_{q_i}\Phi\, \partial_{p_i} H - \partial_{p_i}\Phi\, \partial_{q_i} H.$$

Next we fix $m \in \mathbb{N}$, consider the m^{th} Toda integral $J^{(m)}$, and prove that the quotient

$$\frac{\left\langle \{J^{(m)}, H_F\}^2 \right\rangle}{\sigma^2_{J^{(m)}}} \tag{4.24}$$

scales appropriately in β (as $\beta \to \infty$) and it is bounded uniformly in N (provided N is large enough). It is quite delicate to prove that the quotient in (4.24) is bounded uniformly in N and for the purpose we exploit the rich structure of the Toda integral of motions.

This chapter is organized as follows. In section 3 we study the structure of the Toda integrals. In particular, we prove that for any $m \in \mathbb{N}$ fixed, and N sufficiently large, the m^{th} Toda integral $J^{(m)}$ can be written as a sum $\frac{1}{m}\sum_{j=1}^N h_j^{(m)}$ where each term depends only on at most m consecutive variables, moreover $h_j^{(m)}$ and $h_k^{(m)}$ have disjoint supports if the distance between j and k is larger than m. Then we make the crucial observation that the quadratic part of the Toda integrals $J^{(m)}$ are quadratic forms in $\mathbf{p}$ and $\mathbf{q}$ generated by symmetric circulant matrices. In section 3 we approximate the Gibbs measure with the measure were all the variable are independent random variables. and we calculate the error of our approximation. In section 4 we obtain a bound on the variance of $J^{(m)}(t) - J^{(m)}(0)$ with respect to the FPUT flow and a bound of linear combination of harmonic energies with respect to the FPUT flow and the Toda flow. Finally in section 5 we prove our main results, namely Theorem 4.1, Theorem 4.4 and Theorem 4.5. In section 4.6 we describe the more technical results.

4.2 Structure of the Toda integrals of motion

In this section we study the algebraic and the analytic properties of the Toda integrals defined in (4.13). First we write them explicitly:

Theorem 4.7. *For any $1 \leqslant m \leqslant N-1$, one has*

$$J^{(m)} = \frac{1}{m}\sum_{j=1}^{N} h_j^{(m)}\,, \tag{4.25}$$

where $h_j^{(m)} := [L^m]_{jj}$ is given explicitly by

$$h_j^{(m)}(\mathbf{p}, \mathbf{r}) = \sum_{(\mathbf{n},\mathbf{k})\in\mathcal{A}^{(m)}} (-1)^{|\mathbf{k}|}\, \rho^{(m)}(\mathbf{n}, \mathbf{k}) \prod_{i=-\tilde{m}}^{\tilde{m}-1} e^{-n_i r_{j+i}} \prod_{i=-\tilde{m}+1}^{\tilde{m}-1} p_{j+i}^{k_i}\,, \tag{4.26}$$

where it is understood $r_j \equiv r_{j \bmod N}$, $p_j \equiv p_{j \bmod N}$ *and* $\mathcal{A}^{(m)}$ *is the set*

$$\begin{aligned}\mathcal{A}^{(m)} := \Big\{(\mathbf{n},\mathbf{k}) \in \mathbb{N}_0^{\mathbb{Z}} \times \mathbb{N}_0^{\mathbb{Z}} \ : \quad & \sum_{i=-\widetilde{m}}^{\widetilde{m}-1} (2n_i + k_i) = m, \\ & \forall i \geqslant 0, \quad n_i = 0 \Rightarrow n_{i+1} = k_{i+1} = 0, \\ & \forall i < 0, \quad n_{i+1} = 0 \Rightarrow n_i = k_i = 0\Big\}.\end{aligned} \tag{4.27}$$

The quantity $\widetilde{m} := \lfloor m/2 \rfloor$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and* $\rho^{(m)}(\mathbf{n},\mathbf{m}) \in \mathbb{N}$ *is given by*

$$\rho^{(m)}(\mathbf{n},\mathbf{k}) := \binom{n_{-1}+n_0+k_0}{k_0}\binom{n_{-1}+n_0}{n_0}\prod_{\substack{i=-\widetilde{m}\\ i\neq -1}}^{\widetilde{m}-1}\binom{n_i+n_{i+1}+k_{i+1}-1}{k_{i+1}}\binom{n_i+n_{i+1}-1}{n_{i+1}}. \tag{4.28}$$

We give the proof of this theorem in section 4.6.1.

Remark 4.8. *The structure of* $J^{(N)}$ *is slightly different, but we will not use it here.*

We now describe some properties of the Toda integrals, which we will use several times. The Hamiltonian density $h_j^{(m)}(\mathbf{p},\mathbf{r})$ depends on the set $\mathcal{A}^{(m)}$ and the coefficient $\rho^{(m)}(\mathbf{n},\mathbf{k})$ which are independent of the index j. This implies that $h_j^{(m)}$ is obtained by $h_1^{(m)}$ just by shifting $1 \to j$; we formalize this property below with the notion of cyclic functions.

A second immediate property, as one sees inspecting the formulas (4.27) and (4.28), is that there exists $C^{(m)} > 0$ (depending only on m) such that

$$|\mathcal{A}^{(m)}| \leqslant C^{(m)}, \quad \rho^{(m)}(\mathbf{n},\mathbf{k}) \leqslant C^{(m)},$$

namely the cardinality of the set $\mathcal{A}^{(m)}$ and the values of the coefficients $\rho^{(m)}(\mathbf{n},\mathbf{k})$ are *independent of* N.

The last elementary property, which follows from the condition $2|\mathbf{n}| + |\mathbf{k}| = m$ in (4.27), is that

$$\begin{aligned} m \text{ even} \quad &\Longrightarrow \quad h_j^{(m)} \text{ contains only even polynomials in } p, \\ m \text{ odd} \quad &\Longrightarrow \quad h_j^{(m)} \text{ contains only odd polynomials in } p. \end{aligned}$$

Now we describe three other important properties of the Toda integrals, which are less trivial and require some preparation. Such properties are

(*i*) *cyclicity*;

(*ii*) *uniformly bounded support*;

(*iii*) the quadratic parts of the Toda integrals are represented by *circulant matrices*.

We first define each of these properties rigorously, and then we show that the Toda integrals enjoy them.

Cyclicity. Cyclic functions are characterized by being invariant under left and right cyclic shift. For any $\ell \in \mathbb{Z}$, and $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$ we define the *cyclic shift of order* ℓ as the map

$$S_\ell \colon \mathbb{R}^N \to \mathbb{R}^N, \qquad (S_\ell x)_j := x_{(j+\ell) \bmod N}. \tag{4.29}$$

For example S_1 and S_{-1} are the left respectively right shifts:

$$S_1(x_1, x_2, \ldots, x_N) := (x_2, \ldots, x_N, x_1), \qquad S_{-1}(x_1, x_2, \ldots, x_N) := (x_N, x_1, \ldots, x_{N-1}).$$

It is immediate to check that for any $\ell, \ell' \in \mathbb{Z}$, cyclic shifts fulfils:

$$S_\ell \circ S_{\ell'} = S_{\ell+\ell'}, \qquad S_\ell^{-1} = S_{-\ell}, \qquad S_0 = \mathbb{1}, \qquad S_{\ell+N} = S_\ell.$$

Consider now a a function $H \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{C}$; we shall denote $S_\ell H \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{C}$ as the operator

$$(S_\ell H)(\mathbf{p}, \mathbf{r}) := H(S_\ell \mathbf{p}, S_\ell \mathbf{r}), \qquad \forall (\mathbf{p}, \mathbf{r}) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Clearly S_ℓ is a linear operator. We can now define cyclic functions:

Definition 4.9 (Cyclic functions)**.** *A function $H \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{C}$ is called* cyclic *if $S_1 H = H$.*

It is clear from the definition that a cyclic function fulfils $S_\ell H = H \;\; \forall \ell \in \mathbb{Z}$. It is easy to construct cyclic functions as follows: given a function $h \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{C}$ we define the new function H by

$$H(\mathbf{p}, \mathbf{r}) := \sum_{\ell=0}^{N-1} (S_\ell h)(\mathbf{p}, \mathbf{r}).$$

H is clearly cyclic, and we say that H is *generated* by h.

Support. Given a differentiable function $F \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{C}$, we define its *support* as the set

$$\operatorname{supp} F := \left\{ \ell \in \{0, \ldots, N-1\} \colon \quad \frac{\partial F}{\partial p_\ell} \not\equiv 0 \quad \text{or} \quad \frac{\partial F}{\partial r_\ell} \not\equiv 0 \right\}$$

and its *diameter* as

$$\operatorname{diam}(\operatorname{supp} F) := \sup_{i,j \in \operatorname{supp} F} \mathtt{d}(i,j) + 1,$$

where $\mathtt{d}$ is the *periodic distance*

$$\mathtt{d}(i,j) := \min\left(|i-j|, \; N - |i-j|\right).$$

Note that $0 \leqslant \mathtt{d}(i,j) \leqslant \lfloor N/2 \rfloor$.

We often use the following property: if f is a function with diameter $K \in \mathbb{N}$, and $K \ll N$, then

$$\mathtt{d}(i,j) > K \quad \Longrightarrow \quad \operatorname{supp} S_j f \cap \operatorname{supp} S_i f = \varnothing, \tag{4.30}$$

where S_j is the shift operator (4.29). With the above notation and definition, we arrive to the following elementary result.

Lemma 4.10. *Consider the Toda integral* $J^{(m)} = \frac{1}{m}\sum_{j=1}^{N} h_j^{(m)}$, $1 \leqslant m \leqslant N$ *in* (4.25). *Then* $J^{(m)}$ *is a cyclic function generated by* $\frac{1}{m}h_1^{(m)}$, *namely*

$$J^{(m)}(\mathbf{p},\mathbf{r}) = \frac{1}{m}\sum_{j=1}^{N} S_{j-1}h_1^{(m)}(\mathbf{p},\mathbf{r}). \tag{4.31}$$

Further, each term $h_j^{(m)}$ *has diameter at most* m. *In particular,* $h_j^{(m)}$ *and* $h_k^{(m)}$ *have disjoint supports provided* $\mathtt{d}(j,k) > m$.

Circulant symmetric matrices. We begin recalling the definition of circulant matrices (see e.g. [77, Chap. 3]).

Definition 4.11 (Circulant matrix)**.** *An* $N \times N$ *matrix* A *is said to be* circulant *if there exists a vector* $\mathbf{a} = (a_j)_{j=0}^{N-1} \in \mathbb{R}^N$ *such that*

$$A_{j,k} = a_{(j-k) \bmod N}.$$

We will say that A *is* represented by the vector $\mathbf{a}$.

In particular, circulant matrices have all the form

$$A = \begin{bmatrix} a_0 & a_{N-1} & \dots & a_2 & a_1 \\ a_1 & a_0 & a_{N-1} & & a_2 \\ \vdots & a_1 & a_0 & \ddots & \vdots \\ a_{N-2} & & \ddots & \ddots & a_{N-1} \\ a_{N-1} & a_{N-2} & \dots & a_1 & a_0 \end{bmatrix}$$

where each row is the right shift of the row above.
Moreover, A is circulant symmetric if and only if its representing vector $\mathbf{a}$ is even, i.e. one has

$$a_k = a_{N-k}\ , \quad \forall k. \tag{4.32}$$

One of the most remarkable property of circulant matrices is that they are all diagonalized by the discrete Fourier transform (see e.g. [77, Chap. 3]). We show now that circulant symmetric matrices are diagonalized by the Hartley transform:

Lemma 4.12. *Let* A *be a circulant symmetric matrix represented by the vector* $\mathbf{a} \in \mathbb{R}^N$. *Then*

$$\mathcal{H}A\mathcal{H}^{-1} = \sqrt{N}\,\mathrm{diag}\{\widehat{a}_j :\ 0 \leqslant j \leqslant N-1\}, \tag{4.33}$$

where $\widehat{\mathbf{a}} = \mathcal{H}\mathbf{a}$.

Proof. First remark that a circulant matrix acts on a vector $\mathbf{x} \in \mathbb{R}^N$ as a periodic discrete convolution,

$$A\mathbf{x} = \mathbf{a} \star \mathbf{x}, \qquad (\mathbf{a} \star \mathbf{x})_j := \sum_{k=0}^{N-1} a_{j-k}\, x_k, \qquad 0 \leqslant j \leqslant N-1,$$

where it is understood $a_\ell \equiv a_{\ell \bmod N}$. As the Hartley transform of a discrete convolution is given by

$$[\mathcal{H}(\mathbf{a} \star \mathbf{x})]_k = \frac{\sqrt{N}}{2}\Big((\widehat{a}_k + \widehat{a}_{N-k})\widehat{x}_k + (\widehat{a}_k - \widehat{a}_{N-k})\widehat{x}_{N-k}\Big),$$

we obtain (4.33), using that the Hartley transform maps even vectors (see (4.32)) to even vectors. □

Our interest in circulant matrices comes from the following fact: *quadratic cyclic functions are represented by circulant matrices*. More precisely, consider a quadratic function of the form

$$Q(\mathbf{p},\mathbf{r}) = \frac{1}{2}\mathbf{p}^\intercal A\mathbf{p} + \frac{1}{2}\mathbf{r}^\intercal B\mathbf{r} + \mathbf{p}^\intercal C\mathbf{r},$$

where A, B, C are $N \times N$ matrices. Then one has

$$Q \text{ is cyclic} \quad \Longleftrightarrow \quad A, B, C \text{ are circulant .} \tag{4.34}$$

This result, which is well known (see e.g. [77]), follows from the fact that Q cyclic is equivalent to A, B, C commuting with the left cyclic shift S_1, and that the set of matrices which commute with S_1 coincides with the set of circulant matrices.

We conclude this section collecting some properties of Toda integrals. Denote by $J_2^{(m)}$ the Taylor polynomial of order 2 of $J^{(m)}$ at zero; being a quadratic, symmetric, cyclic function, it is represented by circulant symmetric matrices. We have the following lemma.

Lemma 4.13. *Let us consider the Toda integral*

$$J^{(m)}(\mathbf{p},\mathbf{r}) = \frac{1}{m}\sum_{j=1}^{N} S_{j-1}h_1^{(m)}(\mathbf{p},\mathbf{r}).$$

Then $h_1^{(m)}(\mathbf{p},\mathbf{q})$ has the following Taylor expansion at $\mathbf{p}=\mathbf{r}=0$:

$$h_1^{(m)}(\mathbf{p},\mathbf{r}) = \varphi_0^{(m)} + \varphi_1^{(m)}(\mathbf{p},\mathbf{r}) + \varphi_2^{(m)}(\mathbf{p},\mathbf{r}) + \varphi_{\geqslant 3}^{(m)}(\mathbf{p},\mathbf{r}) \tag{4.35}$$

where each $\varphi_k^{(m)}(\mathbf{p},\mathbf{r})$ is a homogeneous polynomial of degree $k=0,1,2$ in $\mathbf{p}$ and $\mathbf{r}$ of diameter m and coefficients independent from N. The reminder $\varphi_{\geqslant 3}^{(m)}(\mathbf{p},\mathbf{r})$ takes the form

$$\varphi_{\geqslant 3}^{(m)}(\mathbf{p},\mathbf{r}) := \sum_{\substack{(\mathbf{k},\mathbf{n})\in\mathcal{A}^{(m)}\\ |k|\geqslant 3}} (-1)^{|\mathbf{k}|}\rho^{(m)}(\mathbf{n},\mathbf{k})\,\mathbf{p}^{\mathbf{k}}\left(1-\mathbf{n}^\intercal\mathbf{r}+\frac{1}{2}(\mathbf{n}^\intercal\mathbf{r})^2+\frac{(\mathbf{n}^\intercal\mathbf{r})^3}{2}\int_0^1 e^{-s\mathbf{n}^\intercal r}\,(1-s)^2\,\mathrm{d}s\right), \tag{4.36}$$

with $\mathcal{A}^{(m)}$ and $\rho^{(m)}$ defined in (4.27) *and* (4.28) *respectively. Moreover the Taylor expansion of $J^{(m)}(\mathbf{p},\mathbf{r})$ at $\mathbf{p}=\mathbf{r}=0$ takes the form*

$$J^{(m)}(\mathbf{p},\mathbf{r}) = J_0^{(m)} + J_2^{(m)}(\mathbf{p},\mathbf{r}) + J_{\geqslant 3}^{(m)}(\mathbf{p},\mathbf{r}),$$

where

- $J_0^{(m)} = \begin{cases} c\in\mathbb{R}, & m \text{ even} \\ 0\,, & m \text{ odd .}\end{cases}$

- $J_2^{(m)}(\mathbf{p},\mathbf{r})$ *is a cyclic function of the form*

$$J_2^{(m)}(\mathbf{p},\mathbf{r}) = \begin{cases} \mathbf{p}^\intercal A^{(m)}\mathbf{p} + \mathbf{r}^\intercal A^{(m)}\mathbf{r}, & m \text{ even} \\ \mathbf{p}^\intercal B^{(m)}\mathbf{r}, & m \text{ odd}\end{cases} \tag{4.37}$$

  with $A^{(m)}, B^{(m)}$ circulant, symmetric $N\times N$ matrices; their representing vectors $\mathbf{a}^{(m)}$, $\mathbf{b}^{(m)}$ are m-admissible (according to Definition 4.3) and

$$a_k^{(m)} = a_{N-k}^{(m)} > 0, \qquad b_k^{(m)} = b_{N-k}^{(m)} > 0, \qquad \forall 0 \leqslant k \leqslant \widetilde{m} := \left\lfloor \frac{m}{2} \right\rfloor. \tag{4.38}$$

- *The reminder* $J^{(m)}_{\geqslant 3}$ *is a cyclic function generated by* $\frac{\varphi^{(m)}_{\geqslant 3}}{m}$.

The proof is postponed to section 4.6.2. We conclude this section giving the definition of *m-admissible* functions, and we prove a lemma that characterize them in terms of $\{J_2^{(l)}\}_{l=1}^N$:

Definition 4.14. *$G_1, G_2 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ are called m-*admissible *functions of the first and second kind respectively if there exists a m-admissible vector $\boldsymbol{g} \in \mathbb{R}^N$ such that*

$$G_1 := \sum_{j,l=0}^{N-1} g_l p_j r_{j+l}\,, \qquad G_2 := \sum_{j,l=0}^{N-1} g_l\left(p_j p_{j+l} + r_j r_{j+l}\right)\,.$$

Remark 4.15. *From definition 4.14 and* (4.34) *one can deduce that both G_1 and G_2 can be represented with circulant and symmetric matrices. Indeed we have that $G_1 = \mathbf{p}^\intercal \mathcal{G}_1 \mathbf{r}$ where $(\mathcal{G}_1)_{jk} = g_{(j-k) \bmod N}$ and similarly for G_2.*

An immediate, but very useful, corollary of Lemma 4.13 , is the fact that the quadratic parts of Toda integrals are a basis of the vector space of m-admissible functions.

Lemma 4.16. *Fix $m \in \mathbb{N}$ and let G_1 and G_2 be m-admissible functions of the first and second kind defined by a m-admissible vector $\boldsymbol{g} \in \mathbb{R}^N$. Then there are two unique sequences $\{c_j\}_{j=0}^{\tilde{m}}$, $\{d_j\}_{j=0}^{\tilde{m}}$, with $\max_j |c_j|$, $\max_j |d_j|$ independent from N, such that:*

$$G_1 = \sum_{l=0}^{\tilde{m}} c_l J_2^{(2l+1)}, \quad G_2 = \sum_{l=0}^{\tilde{m}} d_l J_2^{(2l+2)},$$

where $J_2^{(m)}$ is the quadratic part (4.37) *of the Toda integrals $J^{(m)}$ in* (4.25).

Proof. We will prove the statement just for functions of the first kind. The proof for functions of the second kind can be obtained in a similar way. Let $J_2^{(2l+1)} = \mathbf{p}^\intercal B^{(2l+1)} \mathbf{r}$ where the circulant matrix $B^{(2l+1)}$ is represented by the vector $\mathbf{b}^{(2l+1)}$ and let $G_1 = \mathbf{p}^\intercal \mathcal{G}_1 \mathbf{r}$ where $(\mathcal{G}_1)_{jk} = g_{(j-k) \bmod N}$. Then

$$\mathcal{G}_1 = \sum_{l=0}^{\tilde{m}} c_l B^{(2l+1)} \Longrightarrow g_k = \sum_{l=0}^{\tilde{m}} b_k^{(2l+1)} c_l\,.$$

From Lemma 4.13 the matrix $\mathfrak{B} = [b_k^{(2l+1)}]_{k,l=0}^{\tilde{m}}$ is upper triangular and the diagonal elements are always different from 0 (see in particular formula (4.38)). This implies that the above linear system is uniquely solvable for $(c_0, \ldots, c_{\tilde{m}})$. □

4.3 Averaging and covariance

In this section, we collect some properties of the Gibbs measure $\mathrm{d}\mu_F$ in (4.11). The first property if the invariance with respect to the shift operator. Namely, for a function $f\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$; we have that

$$\langle S_j f \rangle = \langle f \rangle\,, \qquad \forall j = 0, \ldots, N-1\,,$$

which follows from the fact that $(S_j)_* \mathrm{d}\mu_F = \mathrm{d}\mu_F$.

It is in general not possible to compute exactly the average of a function with respect to the Gibbs measure $\mathrm{d}\mu_F$ in (4.11). This is mostly due to the fact that the variables $p_0, \ldots, p_{N-1}$ and $r_0, \ldots, r_{N-1}$ are not independent with respect to the measure $\mathrm{d}\mu_F$, being constrained by the conditions $\sum_i r_i = \sum_i p_i = 0$.

We will therefore proceed as in [108], by considering a new measure $\mathrm{d}\mu_{F,\theta}$ on the extended phase space according to which all variables are independent. We will be able to compute averages and correlations with respect to this measure, and estimate the error derived by this approximation.

For any $\theta \in \mathbb{R}$, we define the measure $\mathrm{d}\mu_{F,\theta}$ on the extended space $\mathbb{R}^N \times \mathbb{R}^N$ by

$$\mathrm{d}\mu_{F,\theta} := \frac{1}{Z_{F,\theta}(\beta)}\, e^{-\beta H_F(\mathbf{p},\mathbf{r})}\, e^{-\theta \sum_{j=0}^{N-1} r_j}\, \mathrm{d}\mathbf{p}\, \mathrm{d}\mathbf{r}, \tag{4.39}$$

where we define $Z_{F,\theta}(\beta)$ as the normalizing constant of $\mathrm{d}\mu_{F,\theta}$. We denote the expectation of a function f with respect to $\mathrm{d}\mu_{F,\theta}$ by $\langle f \rangle_\theta$. We also denote by

$$\|f\|_\theta^2 := \int_{\mathbb{R}^{2N}} |f(\mathbf{p},\mathbf{r})|^2\, \mathrm{d}\mu_{F,\theta}.$$

If $\|f\|_\theta < \infty$ we say that $f \in L^2(\mathrm{d}\mu_{F,\theta})$.

The measure $\mathrm{d}\mu_{F,\theta}$ depends on the parameter $\theta \in \mathbb{R}$ and we fix it in such a way that

$$\int_{\mathbb{R}} r\, e^{-\theta r - \beta V_F(r)}\, \mathrm{d}r = 0. \tag{4.40}$$

Following [108], it is not difficult to prove that there exists $\beta_0 > 0$ and a compact set $\mathcal{I} \subset \mathbb{R}$ such that for any $\beta > \beta_0$, there exists $\theta = \theta(\beta) \in \mathcal{I}$ for which (4.40) holds true. We remark that (4.40) is equivalent to require that $\langle r_j \rangle_\theta = 0$ for $j = 0, \ldots, N-1$ and as a consequence $\left\langle \sum_{j=0}^{N-1} r_j \right\rangle_\theta = 0$. We observe that $\left\langle \sum_{j=0}^{N-1} r_j \right\rangle = 0$ with respect to the measure $\mathrm{d}\mu_F$.

The main reason for introducing the measure $\mathrm{d}\mu_{F,\theta}$ is that it approximates averages with respect to $\mathrm{d}\mu_F$ as the following result shows.

Lemma 4.17. *Fix $\widetilde{\beta} > 0$ and let $f\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ have support of size K (according to Definition 4.2) and finite second order moment with respect to $\mathrm{d}\mu_{F,\theta}$, uniformly for all $\beta > \widetilde{\beta}$. Then there exist positive constants C, N_0 and β_0 such that for all $N > N_0$, $\beta > max\{\beta_0, \widetilde{\beta}\}$ one has*

$$|\langle f \rangle - \langle f \rangle_\theta| \leqslant C \frac{K}{N} \sqrt{\langle f^2 \rangle_\theta - \langle f \rangle_\theta^2}\,.$$

The above lemma is an extension to the periodic case of a result from [108], and we shall prove it in section 4.6.3. As an example of applications of Lemma 4.17, we give a bound to correlations functions.

Lemma 4.18. *Fix $K \in \mathbb{N}$. Let $f, g\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{C}$ such that :*

1. *f, g and $fg \in L^2(\mathrm{d}\mu_{F,\theta})$,*

2. *the supports of f and g have size at most $K \in \mathbb{N}$.*

Then there exist $C, N_0, \beta_0 > 0$ such that for all $N > N_0$, $\beta > \beta_0$

$$|\langle fg\rangle - \langle f\rangle\langle g\rangle| \leqslant 2\|f\|_\theta \|g\|_\theta + \frac{CK}{N}\Big(\|f\|_\theta \|g\|_\theta + \|fg\|_\theta\Big). \tag{4.41}$$

Moreover, if f and g have disjoint supports, then

$$|\langle fg\rangle - \langle f\rangle\langle g\rangle| \leqslant \frac{CK}{N}\Big(\|f\|_\theta \|g\|_\theta + \|fg\|_\theta\Big). \tag{4.42}$$

Proof. We substitute the measure $d\mu_F$ with $d\mu_{F,\theta}$ and then we control the error by using Lemma 4.17. With this idea, we write

$$\begin{aligned}\langle fg\rangle - \langle f\rangle\langle g\rangle =& \langle fg\rangle - \langle fg\rangle_\theta && (4.43)\\ &+ \langle fg\rangle_\theta - \langle f\rangle_\theta \langle g\rangle_\theta && (4.44)\\ &+ \langle f\rangle_\theta \langle g\rangle_\theta - \langle f\rangle\langle g\rangle\,, && (4.45)\end{aligned}$$

and estimate the different terms. We will often use the inequality

$$|\langle f\rangle_\theta| \leqslant \|f\|_\theta\,, \tag{4.46}$$

valid for any function $f \in L^2(d\mu_{F,\theta})$.
Estimate of (4.43): By Lemma 4.17, and the assumption that fg depends on at most $2K$ variables,

$$|\langle fg\rangle - \langle fg\rangle_\theta| \leqslant C\frac{2K}{N}\sqrt{\langle (fg)^2\rangle_\theta - \langle fg\rangle_\theta^2} \leqslant \frac{C'K}{N}\|fg\|_\theta\,.$$

Estimate of (4.44): By Cauchy-Schwartz and (4.46) we have

$$|\langle fg\rangle_\theta - \langle f\rangle_\theta \langle g\rangle_\theta| \leqslant 2\|f\|_\theta \|g\|_\theta\,.$$

Estimate of (4.45): We decompose further

$$\langle f\rangle_\theta \langle g\rangle_\theta - \langle f\rangle\langle g\rangle = \langle g\rangle_\theta (\langle f\rangle_\theta - \langle f\rangle) + (\langle g\rangle_\theta - \langle g\rangle)\langle f\rangle_\theta + (\langle g\rangle_\theta - \langle g\rangle)(\langle f\rangle - \langle f\rangle_\theta)\,,$$

again by Lemma 4.17 and (4.46) we obtain

$$|\langle f\rangle_\theta \langle g\rangle_\theta - \langle f\rangle\langle g\rangle| \leqslant C\frac{K}{N}\|g\|_\theta \|f\|_\theta\,.$$

Combining the three bounds above and redefining $C = \max\{C, C'\}$ one obtains (4.41). To prove (4.42) it is sufficient to observe that if f and g have disjoint supports, then $\langle fg\rangle_\theta = \langle f\rangle_\theta \langle g\rangle_\theta$ and consequently (4.44) is equal to zero. □

In order to make Lemma 4.18 effective, we need to show how to compute averages according to the measure (4.39).

Lemma 4.19. *There exists $\beta_0 > 0$ such that for any $\beta > \beta_0$, the following holds true. For any fixed multi-index $\mathbf{k}, \mathbf{l}, \mathbf{n}, \mathbf{s} \in \mathbb{N}_0^N$ and $d, d' \in \{0, 1, 2\}$, there are two constants $C^{(1)}_{\mathbf{k},\mathbf{l}} \in \mathbb{R}$ and $C^{(2)}_{\mathbf{k},\mathbf{l}} > 0$ such that*

$$\frac{C^{(1)}_{\mathbf{k},\mathbf{l}}}{\beta^{\frac{|\mathbf{k}|+|\mathbf{l}|}{2}}} \leqslant \left\langle \mathbf{p}^{\mathbf{k}}\,\mathbf{r}^{\mathbf{l}} \left(\int_0^1 e^{-\xi \mathbf{n}^\intercal \mathbf{r}}(1-\xi)^2 d\xi\right)^d \left(\int_0^1 e^{-\xi \mathbf{s}^\intercal \mathbf{r}}(1-\xi)^3 d\xi\right)^{d'} \right\rangle_\theta \leqslant \frac{C^{(2)}_{\mathbf{k},\mathbf{l}}}{\beta^{\frac{|\mathbf{k}|+|\mathbf{l}|}{2}}}$$

where $\mathbf{p}^{\mathbf{k}} = \prod_{j=1}^N p_j^{k_j}$ and $\mathbf{r}^{\mathbf{l}} = \prod_{j=1}^N r_j^{l_j}$. Moreover:

(i) if k_i is odd for some i then $C^{(1)}_{\mathbf{k},\mathbf{l}} = C^{(2)}_{\mathbf{k},\mathbf{l}} = 0$*;*

(ii) if k_i, l_i are even for all i then $C^{(1)}_{\mathbf{k},\mathbf{l}} > 0$*.*

The lemma is proved in section 4.6.4.

Remark 4.20. *Actually all the results of this section hold true (with different constants) also when we endow $\mathcal{M}$ with the Gibbs measure of the Toda chain in* (4.20) *and we use as approximating measure*

$$\mathrm{d}\mu_{T,\theta} := \frac{1}{Z_{T,\theta}(\beta)}\, e^{-\beta H_T(\mathbf{p},\mathbf{r})}\, e^{-\theta\sum_{j=0}^{N-1} r_j}\, \mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{r};$$

here θ is selected in such a way that

$$\int_{\mathbb{R}} r\, e^{-\theta r - \beta V_T(r)}\,\mathrm{d}r = 0. \tag{4.47}$$

We show in section 4.6.4 that it is always possible to choose θ to fulfil (4.47) *(see Lemma 4.35) and we also prove Lemma 4.19 for Toda. In section 4.6.3 we prove Lemma 4.17 for the Toda chain.*

4.4 Bounds on the variance

In this section, we prove upper and lower bounds on the variance of the quantities relevant to prove our main theorems.

4.4.1 Upper bounds on the variance of $J^{(m)}$ along the flow of FPUT

In this subsection, we only consider the case $\mathcal{M}$ endowed by the FPUT Gibbs measure. We denote by $J^{(m)}(t) := J^{(m)} \circ \phi^t_{H_F}$ the Toda integral computed along the Hamiltonian flow $\phi^t_{H_F}$ of the FPUT Hamiltonian. The aim is to prove the following result:

Proposition 4.21. *Fix $m \in \mathbb{N}$. There exist $N_0, \beta_0, C_0, C_1 > 0$ such that for any $N > N_0$, $\beta > \beta_0$, one has*

$$\sigma^2_{J^{(m)}(t)-J^{(m)}(0)} \leqslant C_0 N \left(\frac{(\mathtt{b}-1)^2}{\beta^4} + \frac{C_1}{\beta^5}\right) t^2, \qquad \forall t \in \mathbb{R}.$$

Proof. As explained in the introduction, applying formula (4.23) we get

$$\sigma^2_{J^{(m)}(t)-J^{(m)}(0)} \leqslant \left\langle \{J^{(m)}, H_F\}^2 \right\rangle t^2, \qquad \forall t \in \mathbb{R}.$$

Therefore we need to bound $\left\langle \{J^{(m)}, H_F\}^2 \right\rangle$. For the purpose, we rewrite this term in a more convenient form. Since $\langle\cdot\rangle$ is an invariant measure with respect to the Hamiltonian flow of H_F, one has

$$\left\langle \{J^{(m)}, H_F\} \right\rangle = 0. \tag{4.48}$$

Furthermore, since $J^{(m)}$ is an integral of motion of the Toda Hamiltonian H_T, we have

$$\left\{J^{(m)}, H_T\right\} = 0. \tag{4.49}$$

We apply identities (4.48) and (4.49) to write

$$\left\langle \left\{J^{(m)}, H_F\right\}^2 \right\rangle = \left\langle \left\{J^{(m)}, H_F - H_T\right\}^2 \right\rangle - \left\langle \left\{J^{(m)}, H_F - H_T\right\}\right\rangle^2 . \tag{4.50}$$

The above expression enables us to exploit the fact that the FPUT system is a fourth order perturbation of the Toda chain. To proceed with the proof we need the following technical result.

Lemma 4.22. *One has*

$$\{J^{(m)}, H_F - H_T\} = \sum_{j=1}^{N} H_j^{(m)} , \tag{4.51}$$

where the functions $H_j^{(m)}$ fulfil

(i) $H_j^{(m)} = S_{j-1} H_1^{(m)} \;\; \forall j$*, moreover the diameter of the support of H_j is at most m;*

(ii) there exist $N_0, \beta_0, C, C' > 0$ such that for any $N > N_0$, $\beta > \beta_0$, any $i, j = 1, \dots, N$, the following estimates hold true:

$$\|H_j^{(m)}\|_\theta \leqslant C \left(\frac{(\mathtt{b}-1)^2}{\beta^4} + \frac{C'}{\beta^5} \right)^{1/2}, \qquad \|H_i^{(m)} H_j^{(m)}\|_\theta \leqslant C \left(\frac{(\mathtt{b}-1)^4}{\beta^8} + \frac{C'}{\beta^{10}} \right)^{1/2}. \tag{4.52}$$

The proof of the lemma is postponed at the end of the subsection.
We are now ready to finish the proof of Proposition 4.21. Substituting (4.51) in (4.50) we obtain

$$\left\langle \left\{J^{(m)}, H_F\right\}^2 \right\rangle = \sum_{j,i=1}^{N} \left[\left\langle H_i^{(m)} H_j^{(m)} \right\rangle - \left\langle H_i^{(m)} \right\rangle \left\langle H_j^{(m)} \right\rangle \right]. \tag{4.53}$$

Therefore estimating $\left\langle \left\{J^{(m)}, H_F\right\}^2 \right\rangle$ is equivalent to estimate the correlations between $H_i^{(m)}$ and $H_j^{(m)}$. Exploiting Lemma 4.18 and observing that if $\mathtt{d}(i,j) > m$ then $H_i^{(m)}$ and $H_j^{(m)}$ have disjoint supports (see Lemma 4.22 (*i*) and (4.30)), we get that there are positive constants that for convenience we still call C and C', such that $\forall N, \beta$ large enough

$$\left| \left\langle H_i^{(m)} H_j^{(m)} \right\rangle - \left\langle H_i^{(m)} \right\rangle \left\langle H_j^{(m)} \right\rangle \right| \leqslant C \left(\frac{(\mathtt{b}-1)^2}{\beta^4} + \frac{C'}{\beta^5} \right), \qquad \forall i, j, \tag{4.54}$$

$$\left| \left\langle H_i^{(m)} H_j^{(m)} \right\rangle - \left\langle H_i^{(m)} \right\rangle \left\langle H_j^{(m)} \right\rangle \right| \leqslant \frac{C}{N} \left(\frac{(\mathtt{b}-1)^2}{\beta^4} + \frac{C'}{\beta^5} \right), \qquad \forall i, j \colon \mathtt{d}(i,j) > m\,. \tag{4.55}$$

From (4.53) we split the sum in two terms:

$$\left\langle \left\{ J^{(m)}, H_F \right\}^2 \right\rangle = \sum_{\mathtt{d}(i,j)\leqslant m} \left[\left\langle H_i^{(m)} H_j^{(m)} \right\rangle - \left\langle H_i^{(m)} \right\rangle \left\langle H_j^{(m)} \right\rangle \right] + \sum_{\mathtt{d}(i,j)> m} \left[\left\langle H_i^{(m)} H_j^{(m)} \right\rangle - \left\langle H_i^{(m)} \right\rangle \left\langle H_j^{(m)} \right\rangle \right] .$$

We now apply estimates (4.54), (4.55) to get

$$\begin{aligned} \left\langle \left\{ J^{(m)}, H_F \right\}^2 \right\rangle &\leqslant NC \left(\frac{(\mathtt{b}-1)^2}{\beta^4} + \frac{C'}{\beta^5} \right) + N^2 \frac{\widetilde{C}}{N} \left(\frac{(\mathtt{b}-1)^2}{\beta^4} + \frac{C'}{\beta^5} \right) \\ &\leqslant NC_1 \left(\frac{(\mathtt{b}-1)^2}{\beta^4} + \frac{C_2}{\beta^5} \right) \end{aligned}$$

for some positive constants C_1 and C_2. □

Proof of Lemma 4.22

We start by writing the Poisson bracket $\{J^{(m)}, H_F - H_T\}$ in an explicit form. First, we observe that for any $1 \leqslant m < N$ one has from (4.13)

$$\frac{\partial J^{(m)}}{\partial p_{j-1}} = \frac{1}{m} \frac{\partial \mathrm{Tr}\,(L^m)}{\partial p_{j-1}} = \mathrm{Tr} \left(L^{m-1} \frac{\partial L}{\partial p_{j-1}} \right) = -[L^{m-1}]_{j,j} = -h_j^{(m-1)},$$

for all $j = 1, \dots, N$. In the above relation $h_j^{(m-1)}$ is the generating function of the $m-1$ Toda integral defined in (4.26).

Next we observe that

$$H_F(\mathbf{p},\mathbf{q}) - H_T(\mathbf{p},\mathbf{q}) = \sum_{j=0}^{N-1} R(q_{j+1}-q_j), \qquad R(x) := \frac{x^2}{2} - \frac{x^3}{6} + \mathtt{b}\frac{x^4}{24} - (e^{-x} - 1 + x).$$

This implies also that

$$\begin{aligned} \left\{ J^{(m)}, H_F - H_T \right\} &= \sum_{j=1}^{N} h_j^{(m-1)} \; (R'(r_{j-2}) - R'(r_{j-1})) \\ &= \sum_{j=1}^{N} (h_j^{(m-1)} - h_j^{(m-1)}(\mathbf{0},\mathbf{0})) \; (R'(r_{j-2}) - R'(r_{j-1})) \end{aligned}$$

where, to obtain the second identity, we are using that $h_j^{(m-1)}(\mathbf{0},\mathbf{0})$ is by (4.31) and (4.35) a constant independent of j and the second term in the last relation is a telescopic sum. Define

$$H_j^{(m)} := \left(h_j^{(m-1)}(\mathbf{p},\mathbf{r}) - h_j^{(m-1)}(\mathbf{0},\mathbf{0}) \right) (R'(r_{j-2}) - R'(r_{j-1})) \,, \quad j = 1, \dots, N; \tag{4.56}$$

then item (i) of Lemma 4.22 follows because clearly $H_j^{(m)} = S_{j-1} H_1^{(m)}$. Furthermore, since $h_j^{(m-1)}$ has diameter bounded by $m-1$, the same property applies to $H_j^{(m)}$.

To prove item (ii) we start by expanding $R'(r_{j-1}) - R'(r_j)$ in Taylor series with integral remainder. Since

$$R'(x) = \frac{(\mathtt{b}-1)}{6}x^3 + \frac{x^4}{6}\int_0^1 e^{-\xi x}(1-\xi)^3 \mathrm{d}\xi\,,$$

we get that

$$R'(r_{j-2}) - R'(r_{j-1}) = \frac{(\mathtt{b}-1)}{6}S_{j-1}\psi_3(\mathbf{r}) + \frac{1}{6}S_{j-1}\psi_4(\mathbf{r})\,, \tag{4.57}$$

where explicitly

$$\begin{aligned} \psi_3(\mathbf{r}) &:= r_{N-1}^3 - r_0^3\,, \\ \psi_4(\mathbf{r}) &:= r_{N-1}^4 \int_0^1 e^{-\xi r_{N-1}}(1-\xi)^3\mathrm{d}\xi - r_0^4\int_0^1 e^{-\xi r_0}(1-\xi)^3\mathrm{d}\xi\,. \end{aligned} \tag{4.58}$$

Combining (4.35) with (4.57) we rewrite $H_j^{(m)}$ in (4.56) in the form

$$H_j^{(m)} = \frac{S_{j-1}}{6}\Big((\varphi_1^{(m)} + \varphi_2^{(m)} + \varphi_{\geqslant 3}^{(m)})\Big((\mathtt{b}-1)\psi_3 + \psi_4\Big)\Big)\,,$$

where $\varphi_j^{(m)}$, $j = 0,1,2$, are defined in (4.35). Thus the squared L^2 norm of H_j is given by (we suppress the superscript to simplify the notation)

$$\|H_j\|_\theta^2 = \frac{1}{36}(\mathtt{b}-1)^2\Big(\sum_{\ell,\ell'=1}^{2}\left\langle\psi_3^2\,\varphi_\ell\,\varphi_{\ell'}\right\rangle_\theta + \left\langle\psi_3^2\varphi_{\geqslant 3}\,(\varphi_{\geqslant 3}+2\varphi_1+2\varphi_2)\right\rangle_\theta\Big) \tag{4.59}$$

$$+\frac{\mathtt{b}-1}{18}\Big(\sum_{\ell,\ell'=1}^{2}\langle\psi_3\psi_4\,\varphi_\ell\,\varphi_{\ell'}\rangle_\theta + \langle\psi_3\psi_4\varphi_{\geqslant 3}\,(\varphi_{\geqslant 3}+2\varphi_1+2\varphi_2)\rangle_\theta\Big) \tag{4.60}$$

$$+\frac{1}{36}\sum_{\ell,\ell'=1}^{2}\left\langle\psi_4^2\,\varphi_\ell\,\varphi_{\ell'}\right\rangle_\theta + \frac{1}{36}\left\langle\psi_4^2\,\varphi_{\geqslant 3}\,(\varphi_{\geqslant 3}+2\varphi_1+2\varphi_2)\right\rangle_\theta\,. \tag{4.61}$$

Consider now the terms in (4.59); by (4.37),(4.36) and (4.58), we know that each element is a linear combination of functions of the form

$$\mathbf{p^k}\,\mathbf{r^l}\left(\int_0^1 e^{-\xi\mathbf{n}^\intercal\mathbf{r}}(1-\xi)^2\mathrm{d}\xi\right)^d\left(\int_0^1 e^{-\xi\mathbf{s}^\intercal\mathbf{r}}(1-\xi)^3\mathrm{d}\xi\right)^{d'}\,, \tag{4.62}$$

with $|\mathbf{k}| + |\mathbf{l}| \geqslant 6 + \ell + \ell' \geqslant 8$, $d, d' \in \{0,1,2\}$. The number of these functions and their coefficients are independent of N (see Lemma 4.13). By Lemma 4.19 it follows that there exists a constant $C > 0$, depending only on m, such that

$$|\text{r.h.s. of (4.59)}| \leqslant C\,(\mathtt{b}-1)^2\,\beta^{-4}. \tag{4.63}$$

Analogously, line (4.60) is a linear combination of functions of the form (4.62) with $|\mathbf{k}| + |\mathbf{l}| \geqslant 9$, $d, d' \in \{0,1,2\}$. Applying Lemma 4.19 we get the estimate

$$|(4.60)| \leqslant C'\,|\mathtt{b}-1|\,\beta^{-9/2} \tag{4.64}$$

for some constant $C' > 0$. Similarly, the expression (4.61) is a linear combination of functions of the form (4.62) with $|\mathbf{k}| + |\mathbf{l}| \geqslant 10$, $d, d' \in \{0,1,2\}$. Applying Lemma 4.19 we get the estimate

$$|(4.61)| \leqslant C''\,\beta^{-5}\,, \tag{4.65}$$

for some constant $C'' > 0$. Combining (4.63),(4.64) and (4.65) we obtain estimate (4.52) for $\|H_j\|_\theta$. The estimate for $\|H_i^{(m)}H_j^{(m)}\|_\theta$ can be proved in an analogous way. □

4.4.2 Lower bounds on the variance of m-admissible functions

From now on we consider $\mathcal{M}$ endowed with either the FPUT or the Toda Gibbs measure; the following result holds in both cases.

Proposition 4.23. *Fix $m \in \mathbb{N}$, let G be an m-admissible function of the first or second kind (see Definition 4.14). There exist $N_0, \beta_0, C > 0$ such that for any $N > N_0$, $\beta > \beta_0$, one has*

$$\sigma_G^2 = \left\langle G^2 \right\rangle - \left\langle G \right\rangle^2 \geqslant C\frac{N}{\beta^2}. \tag{4.66}$$

Proof. We first prove (4.66) when $G = G_1 = \mathbf{p}^\intercal \mathcal{G}_1 \mathbf{r}$ where $\mathcal{G}_1$ is a circulant, symmetric matrix represented by the m-admissible vector $\mathbf{a} \in \mathbb{R}^N$. We now make the change of coordinates $(\mathbf{p}, \mathbf{r}) = (\mathcal{H}\widehat{\mathbf{p}}, \mathcal{H}\widehat{\mathbf{r}})$ which diagonalizes the matrix $\mathcal{G}_1$ (see (4.33)), getting

$$G_1(\widehat{\mathbf{p}}, \widehat{\mathbf{r}}) = \sqrt{N} \sum_{j=0}^{N-1} \widehat{g}_j \widehat{p}_j \widehat{r}_j .$$

So we have just to compute

$$\begin{aligned}
\sigma_{G_1}^2 &= N\left\langle \sum_{i,j=0}^{N-1} \widehat{g}_j \widehat{g}_i \widehat{p}_j \widehat{p}_i \widehat{r}_i \widehat{r}_j \right\rangle - N\left(\left\langle \sum_{j=0}^{N-1} \widehat{g}_j \widehat{p}_j \widehat{r}_j \right\rangle \right)^2 \\
&= N\sum_{i,j=0}^{N-1} \widehat{g}_j \widehat{g}_i \left\langle \widehat{p}_j \widehat{p}_i \right\rangle \left\langle \widehat{r}_i \widehat{r}_j \right\rangle - N\left(\sum_{j=0}^{N-1} \widehat{g}_j \left\langle \widehat{p}_j \right\rangle \left\langle \widehat{r}_j \right\rangle \right)^2 ,
\end{aligned} \tag{4.67}$$

where we used that $\widehat{p}_k, \widehat{r}_j$ are random variables independent of each other.

We notice that $\widehat{p}_1, \widehat{p}_2, \dots, \widehat{p}_{N-1}$ are i.i.d. Gaussian random variable with variance β^{-1}, $\widehat{p}_0 = 0$ (see (4.7)), so that we have $\left\langle \widehat{p}_j \right\rangle = 0$ and $\left\langle \widehat{p}_j \widehat{p}_i \right\rangle = \frac{\delta_{i,j}}{\beta}$ $i, j = 1, \dots, N-1$ (remark that this holds true both for the FPUT and Toda's potentials as the p-variables have the same distributions).
As a consequence, (4.67) becomes:

$$\sigma_{G_1}^2 = \frac{N}{\beta} \sum_{j=1}^{N-1} \widehat{g}_j^2 \left\langle \widehat{r}_j^2 \right\rangle = \frac{1}{\beta} \left\langle \widehat{\mathbf{r}}^\intercal \mathcal{H} \mathcal{G}_1^2 \mathcal{H} \widehat{\mathbf{r}} \right\rangle = \frac{1}{\beta} \left\langle \mathbf{r}^\intercal \mathcal{G}_1^2 \mathbf{r} \right\rangle . \tag{4.68}$$

Since $\mathcal{G}_1$ is circulant symmetric matrix so is $\mathcal{G}_1^2$ and its representing vector is $\mathbf{d} := \mathbf{g} \star \mathbf{g}$.
Next we remark that the identity $\left\langle \left(\sum_{j=0}^{N-1} r_j \right)^2 \right\rangle = 0$ implies

$$\left\langle r_j r_i \right\rangle = -\frac{1}{N-1} \left\langle r_0^2 \right\rangle, \quad \forall i \neq j .$$

Applying this property to (4.68) we get

$$\begin{aligned}
\sigma_{G_1}^2 &= \frac{1}{\beta} \sum_{j,l=0}^{N-1} d_l \left\langle r_j r_{j+l} \right\rangle = \frac{N}{\beta} \left\langle r_0^2 \right\rangle d_0 + \frac{1}{\beta} \sum_{\substack{j,l \\ l\neq 0}}^{N-1} d_l \left\langle r_j r_{j+l} \right\rangle \\
&= \frac{1}{\beta} \left\langle r_0^2 \right\rangle \left(N d_0 - \frac{N}{N-1} \sum_{l\neq 0}^{N} d_l \right) .
\end{aligned} \tag{4.69}$$

By Lemmas 4.17 and 4.19 we have that, for N sufficiently large, $\langle r_0^2 \rangle \geqslant c\beta^{-1}$. Finally, since the vectors $\mathtt{g}, \mathtt{d}$ are m-admissible and $2m$-admissible respectively, we have that

$$d_0 = (\mathbf{g} \star \mathbf{g})_0 = \sum_{j=0}^{\tilde{m}} g_j^2 \geqslant c_m, \qquad \sum_{l \neq 0}^{N-1} d_l = \sum_{l \neq 0}^{2\tilde{m}} d_l \leqslant C_m, \tag{4.70}$$

for some constants $c_m > 0$ and $C_m > 0$. Plugging (4.70) into (4.69) we obtain (4.66) for the case of m-admissible functions of the first kind.

For the case of admissible functions of the second kind, one has $G_2 = p^\intercal \mathcal{G}_2 p + r^\intercal \mathcal{G}_2 r$ with $\mathcal{G}_2$ circulant, symmetric and represented by an m-admissible vector. Since p and r are independent random variables one gets

$$\sigma_{G_2} = \sigma_{\mathbf{p}^\intercal \mathcal{G}_2 \mathbf{p} + \mathbf{r}^\intercal \mathcal{G}_2 \mathbf{r}} = \sigma_{\mathbf{p}^\intercal \mathcal{G}_2 \mathbf{p}} + \sigma_{\mathbf{r}^\intercal \mathcal{G}_2 \mathbf{r}} \geqslant \sigma_{\mathbf{p}^\intercal \mathcal{G}_2 \mathbf{p}}.$$

Then arguing as in the previous case one gets (4.66). □

By applying Proposition 4.23 to the quantity $J_2^{(m)}$ that is an m-admissible function of the first or second kind, depending on the parity of m, we obtain the following result.

Corollary 4.24. *The quadratic part $J_2^{(m)}$ of the Taylor expansion of the Toda integral $J^{(m)}$ near $(\mathbf{p}, \mathbf{r}) = (0, 0)$ satisfies*

$$\sigma^2_{J_2^{(m)}} \geqslant C \frac{N}{\beta^2} \,,$$

for some constant $C > 0$.

Similarly, we obtain a lower bound on the reminder $J_{\geqslant 3}^{(m)}$ of the Taylor expansion of the Toda integral $J^{(m)}$ near $\mathbf{p} = 0$ and $\mathbf{r} = 0$.

Lemma 4.25. *Fix $m \in \mathbb{N}$. There exist $N_0, \beta_0, C > 0$ such that for any $N > N_0$, $\beta > \beta_0$, one has*

$$\sigma^2_{J_{\geqslant 3}^{(m)}} \leqslant C \frac{N}{\beta^3}. \tag{4.71}$$

Proof. Recall from Lemma 4.13 that $J_{\geqslant 3}^{(m)}$ is a cyclic function generated by $\widetilde{h}_1^{(m)} := \frac{1}{m} \varphi_{\geqslant 3}^{(m)}$. Thus, denoting $h_j^{(m)} := S_{j-1} \widetilde{h}_1^{(m)}$, we have $J_{\geqslant 3}^{(m)} = \sum_{j=1}^N \widetilde{h}_j^{(m)}$ and its variance is given by

$$\sigma^2_{J_m^{\geqslant 3}} = \sum_{i,j=1}^N \left\langle \widetilde{h}_i^{(m)} \widetilde{h}_j^{(m)} \right\rangle - \left\langle \widetilde{h}_i^{(m)} \right\rangle \left\langle \widetilde{h}_j^{(m)} \right\rangle. \tag{4.72}$$

We can bound the correlations in (4.72) exploiting Lemma 4.18, provide we estimate first the $L^2(\mathrm{d}\mu_{F,\theta})$ and $L^2(\mathrm{d}\mu_{T,\theta})$ norms of $\widetilde{h}_i^{(m)}$ and $\widetilde{h}_i^{(m)} \widetilde{h}_j^{(m)}$. Proceeding with the same arguments as in Lemma 4.22, one proves that there exists $\tilde{C} > 0$ such that for any $N > N_0$, $\beta > \beta_0$,

$$\| \widetilde{h}_i^{(m)} \|_\theta \leqslant \tilde{C} \beta^{-3/2}, \qquad \| \widetilde{h}_i^{(m)} \, \widetilde{h}_j^{(m)} \|_\theta \leqslant \tilde{C} \beta^{-3}.$$

By Lemma 4.13, the function $\widetilde{h}_1^{(m)}$ has diameter at most m, so in particular if $\mathtt{d}(i,j) > m$, the functions $\widetilde{h}_i^{(m)}$ and $\widetilde{h}_j^{(m)}$ have disjoint supports (recall (4.30)).
We are now in position to apply Lemma 4.18 and obtain

$$\left|\left\langle \widetilde{h}_i^{(m)}\widetilde{h}_j^{(m)}\right\rangle - \left\langle \widetilde{h}_i^{(m)}\right\rangle\left\langle \widetilde{h}_j^{(m)}\right\rangle\right| \leqslant \frac{C'}{\beta^3}, \qquad \forall i,j \tag{4.73}$$

$$\left|\left\langle \widetilde{h}_i^{(m)}\widetilde{h}_j^{(m)}\right\rangle - \left\langle \widetilde{h}_i^{(m)}\right\rangle\left\langle \widetilde{h}_j^{(m)}\right\rangle\right| \leqslant \frac{C'}{N\beta^3}, \qquad \forall i,j\colon \mathtt{d}(i,j) > m, \tag{4.74}$$

for some constant $C' > 0$. Thus we split the variance in (4.72) in two parts

$$\sigma^2_{J_{\geqslant 3}^{(m)}} = \sum_{\mathtt{d}(i,j)\leqslant m} \left\langle \widetilde{h}_i^{(m)}\widetilde{h}_j^{(m)}\right\rangle - \left\langle \widetilde{h}_i^{(m)}\right\rangle\left\langle \widetilde{h}_j^{(m)}\right\rangle + \sum_{\mathtt{d}(i,j)> m} \left\langle \widetilde{h}_i^{(m)}\widetilde{h}_j^{(m)}\right\rangle - \left\langle \widetilde{h}_i^{(m)}\right\rangle\left\langle \widetilde{h}_j^{(m)}\right\rangle$$

and apply estimates (4.73), (4.74) to get (4.71). □

Combining Corollary 4.24 and Lemma 4.25 we arrive to the following crucial proposition.

Proposition 4.26. *Fix $m \in \mathbb{N}$. There exist $N_0, \beta_0, C > 0$ such that for any $N > N_0$, $\beta > \beta_0$, one has*

$$\sigma^2_{J^{(m)}} \geqslant C\frac{N}{\beta^2}. \tag{4.75}$$

Proof. By Lemma 4.13, we write $J^{(m)} = J_0^{(m)} + J_2^{(m)} + J_{\geqslant 3}^{(m)}$ with $J_0^{(m)}$ constant. By Corollary 4.24 and Lemma 4.25 we deduce that for N and β large enough,

$$\sigma_{J^{(m)}} = \sigma_{J_2^{(m)}+J_{\geqslant 3}^{(m)}} \geqslant \sigma_{J_2^{(m)}} - \sigma_{J_{\geqslant 3}^{(m)}} \geqslant \frac{\sqrt{N}}{\beta}\left(\sqrt{C'} - \sqrt{\frac{C}{\beta}}\right),$$

which leads immediately to the claimed estimate (4.75). □

4.5 Proof of the main results

In this section, we give the proofs of the main theorems of our paper.

4.5.1 Proof of Theorem 4.1

The proof is a straightforward application of Proposition 4.21 and 4.26. Having fixed $m \in \mathbb{N}$, we apply (4.22) with $\Phi = J^{(m)}$ and $\lambda = \beta^{-\varepsilon}$ to get

$$\mathbf{P}\left(\left|J^{(m)}(t) - J^{(m)}(0)\right| \geqslant \frac{\sigma_{J^{(m)}(0)}}{\beta^\varepsilon}\right) \leqslant \frac{C_1}{C}\left(\frac{|\mathtt{b}-1|^2}{\beta^2} + \frac{C_2}{\beta^3}\right)\beta^{2\varepsilon}t^2$$

from which one deduces the statement of Theorem 4.1.

4.5.2 Proof of Theorem 4.4 and Theorem 4.5

The proofs of Theorem 4.4 and Theorem 4.5 are quite similar and we develop them at the same time. As in the proof of Theorem 4.1, the first step is to use Chebyshev inequality to bound

$$\mathbf{P}\left(|\Phi(t)-\Phi| > \lambda\sigma_{\Phi}\right) \leqslant \frac{1}{\lambda^2}\frac{\sigma^2_{\Phi(t)-\Phi}}{\sigma^2_{\Phi}}\,, \tag{4.76}$$

where the time evolution is intended with respect to the FPUT flow ϕ_F^t or the Toda flow ϕ_T^t. Accordingly, the probability is calculated with respect to the FPUT Gibbs measure (4.11) or the Toda Gibbs measure (4.20).

Next we observe that the quantity $\Phi := \sum_{j=1}^{N-1}\widehat{g}_j E_j$ defined in (4.19) can be written in the form

$$\Phi(\mathbf{p},\mathbf{r}) = \sum_{j=1}^{N-1}\widehat{g}_j E_j = \frac{1}{2\sqrt{N}}\sum_{j,l=0}^{N-1} g_l\left(p_j p_{j+l} + r_j r_{j+l}\right) = \frac{1}{2\sqrt{N}}G_2(\mathbf{p},\mathbf{r}),$$

where $\mathbf{g} \in \mathbb{R}^N$ is a m-admissible vector and $G_2(\mathbf{p},\mathbf{r})$ is a m-admissible function of the second kind, as in Definition 4.14. As the inequality (4.22) is scaling invariant, prove (4.76) is equivalent to obtain that

$$\mathbf{P}\left(|G_2(t)-G_2| > \lambda\sigma_{G_2}\right) \leqslant \frac{1}{\lambda^2}\frac{\sigma^2_{G_2(t)-G_2}}{\sigma^2_{G_2}}.$$

Applying Proposition 4.23 we can estimate $\sigma^2_{G_2}$. We are then left to give an upper bound to $\sigma^2_{G_2(t)-G_2}$. By Lemma 4.16, there exists a unique sequence $\{c_j\}_{j=0}^{\widetilde{m}-1}$, with $\max_j |c_j|$ independent from N, such that $G_2(p,r) = \sum_{l=0}^{\widetilde{m}-1} c_l J_2^{(2l+2)}$, where $J_2^{(2l+2)}$ are defined in (4.37). Hence we bound

$$\sigma_{G_2(t)-G_2(0)} \leqslant \sum_{l=0}^{\widetilde{m}-1} |c_l|\,\sigma_{J_2^{(2l+2)}(t)-J_2^{(2l+2)}(0)}.$$

Next we interpolate $J_2^{(2l)}$ with the integrals $J^{(2l)}$ and exploit the fact that they are adiabatic invariants for the FPUT flow and integrals of motion for the Toda flow. More precisely

$$\begin{aligned}\sigma_{J_2^{(2l)}(t)-J_2^{(2l)}(0)} \leqslant{}& \sigma_{J_2^{(2l)}(t)-J^{(2l)}(t)} + \sigma_{J^{(2l)}(0)-J_2^{(2l)}(0)}\\ &+ \sigma_{J^{(2l)}(t)-J^{(2l)}(0)}.\end{aligned} \tag{4.77}$$

By the invariance of the two measures with respect to their corresponding flow and Lemma 4.25, we get both for FPUT and Toda the estimate

$$\sigma_{J_2^{(2l)}(t)-J^{(2l)}(t)} = \sigma_{J_2^{(2l)}(0)-J^{(2l)}(0)} = \sigma_{J_{\geqslant 3}^{(2l)}} \leqslant \sqrt{\frac{\tilde{C}_1 N}{\beta^3}},$$

for some constant $\tilde{C}_1 > 0$ and for $\beta > \beta_0$ and $N > N_0$. As (4.77) is zero for the Toda flow (being $J^{(2l)}(t)$ constant along the flow), we get

$$\sigma^2_{G_2\circ\phi_T^t - G_2} \leqslant \frac{C_1 N}{\beta^3}, \tag{4.78}$$

for some constant $C_1 > 0$ and for $\beta > \beta_0$ and $N > N_0$. Combing Proposition 4.23 with (4.78) we conclude that

$$\mathbf{P}\left(\left|G_2 \circ \phi_T^t - G_2\right| > \lambda \sigma_{G_2}\right) \leqslant \frac{C_1}{\lambda^2 \beta}$$

and by choosing $\lambda = \beta^{-\varepsilon}$ with $0 < \varepsilon < \frac{1}{2}$ we arrive to the expression (4.21), namely we have concluded the proof of Theorem 4.5.

We are left to estimate (4.77) for FPUT, but this is exactly the quantity bounded in Proposition 4.21. We conclude that

$$\sigma^2_{G_2 \circ \phi_F^t - G_2} \leqslant \frac{C_1 N}{\beta^3} + C_3 N \left(\frac{|\mathtt{b} - 1|^2}{\beta^4} + \frac{C_2}{\beta^5}\right) t^2, \tag{4.79}$$

for some constant $C_j > 0$, $j = 1, 2, 3$ and for $\beta > \beta_0$ and $N > N_0$.

Combing Proposition 4.23 with (4.79) we obtain

$$\mathbf{P}\left(\left|G_2 \circ \phi_F^t - G_2\right| > \lambda \sigma_{G_2}\right) \leqslant \frac{C_1}{\lambda^2 \beta} + \frac{C_3}{\lambda^2}\left(\frac{|\mathtt{b} - 1|^2}{\beta^2} + \frac{C_2}{\beta^3}\right) t^2. \tag{4.80}$$

Choosing $\lambda = \beta^{-\varepsilon}$ with $0 < \varepsilon < \frac{1}{4}$, (4.80) is equivalent to

$$\mathbf{P}\left(\left|G_2 \circ \phi_F^t - G_2\right| > \frac{\sigma_{G_2}}{\beta^{\varepsilon}}\right) \leqslant \frac{C_1}{\beta^{2\varepsilon}},$$

for some redefine constant $C_1 > 0$ and for every time t fulfilling

$$|t| \leqslant \frac{\beta^{1-2\varepsilon}}{\left((\mathtt{b} - 1)^2 + C_2 \beta^{-1}\right)^{1/2}}.$$

We have thus concluded the proof of Theorem 4.4.

4.6 Technical Results

4.6.1 Proof of Theorem 4.7

In this subsection we prove Theorem 4.7. From the structure of the matrix Lax matrix L in (4.12), we immediately get

$$[L^m]_{jj}(\mathbf{a}, \mathbf{b}) = S_{j-1}\left([L^m]_{11}(\mathbf{a}, \mathbf{b})\right),$$

where S_j is the shift defined in (4.29), thus we have to prove formula (4.26) just for the case $j = 1$.

To accomplish this result we need to introduce the notion of super Motzkin path and super Motzkin polynomial, that generalize the notion of Motzkin path and Motzkin polynomial [130, 154].

Definition 4.27. *A super Motzkin path p of size m is a path in the integer plane $\mathbb{N}_0 \times \mathbb{Z}$ from $(0,0)$ to $(m,0)$ where the permitted steps from $(0,0)$ are: the step up $(1,1)$, the step down $(1,-1)$ and the horizontal step $(1,0)$. A similar definition applies to all other vertices of the path.*

The set of all super Motzkin paths of size m will be denoted by $s\mathcal{M}_m$.

In order to introduce the super Motzkin polynomial associated to these paths we have to define their *weight*. This is done in the following way: to each up step that occurs at height k, i.e. it joins the points (l,k) and $(l+1,k+1)$, we associate the weight a_k, to a down step that joins the points (l,k) and $(l+1,k-1)$ we associate the weight a_{k-1}, to each horizontal step from (l,k) to $(l+1,k)$ we associate the weight b_k. Since $k \in \mathbb{Z}$, the index of a_k and b_k are understood modulus N.

At this point we can define the total weight $w(p)$ of a super Motzkin path p to be the product of weights of its individual steps. So it is a monomial in the commuting variables $(\mathbf{b},\mathbf{a}) = (b_{-\widetilde{m}},\dots,b_{\widetilde{m}},a_{-\widetilde{m}},\dots,a_{\widetilde{m}})$, where $\widetilde{m} = \lfloor m/2 \rfloor$. We remark that the total weight do not characterize uniquely the path. We are now ready to give the definition of Motzkin polynomial:

Definition 4.28. *The super Motzkin polynomial $sP_m(\mathbf{a},\mathbf{b})$ is the sum of all weight of the elements of $s\mathcal{M}_m$:*

$$sP_m(\mathbf{a},\mathbf{b}) = \sum_{p\in s\mathcal{M}_m} w(p)\,. \tag{4.81}$$

We are now ready to relate the Toda integrals to the super Motzkin polynomial $sP_m(\mathbf{a},\mathbf{b})$.

Proposition 4.29. *Given the Lax matrix L in* (4.12) *then:*

$$\left[L^m\right]_{1,1}(\mathbf{a},\mathbf{b}) = sP_m(\mathbf{a},\mathbf{b})\,.$$

where the super Motzkin polynomial $sP_m(\mathbf{a},\mathbf{b})$ is defined in (4.81) *and $a_j \equiv a_{j \bmod N}$, $b_j \equiv b_{j \bmod N}$.*

Proof. In general we have that:

$$\left[L^m\right]_{1,1} = \sum_{\mathbf{j}\in\mathbb{N}^{m-1}} L_{1,j_1}L_{j_1,j_2}\dots L_{j_{m-1},1}$$

To every element of the sum we associate the path with vertices

$$(0,0),\ (1,\widetilde{j}_1-1,),\ (2,\widetilde{j}_2-1),\dots,(\ell,\widetilde{j}_\ell-1),\dots,(m-1,\widetilde{j}_{m-1}-1),\ (m,0)$$

where

$$\widetilde{j}_k = \begin{cases} j_k & \text{if } j_k < \widetilde{m} \\ j_k - N & \text{if } j_k \geqslant \widetilde{m} \end{cases}$$

This is a super Motzkin path $p_{\mathbf{j}}$ and we can associate the weight $w(p_{\mathbf{j}})$ as in the description above therefore we have

$$L_{1,j_1}L_{j_1,j_2}\dots L_{j_{m-1},1} = w(p_{\mathbf{j}})$$

This is clearly a bijection. The sum of the weights of all possible super Motzkin paths, is defined to be the super Motzkin polynomial $sP_m(\mathbf{a},\mathbf{b})$ and thus we get the claim. □

Proceeding as in [130, Proposition 1], we are able to prove the following result, which together with Proposition 4.29 proves Theorem 4.7:

Proposition 4.30. *The super Motzkin polynomial of size m is given explicitly as*

$$sP_m(\mathbf{a},\mathbf{b}) = \sum_{(\mathbf{n},\mathbf{k})\in\mathcal{A}^{(m)}} \rho(\mathbf{n},\mathbf{k}) \prod_{i=-\widetilde{m}}^{\widetilde{m}} a_i^{2n_i} b_i^{k_i}$$

where $\mathcal{A}^{(m)}$ is the set

$$\mathcal{A}^{(m)} := \Big\{(\mathbf{n},\mathbf{k}) \in \mathbb{N}_0^m \times \mathbb{N}_0^m \ : \quad \begin{aligned} &\sum_{i=-\widetilde{m}}^{\widetilde{m}} (2n_i + k_i) = m, \\ &\forall i \geqslant 0, \quad n_i = 0 \Rightarrow n_{i+1} = k_{i+1} = 0, \\ &\forall i < 0, \quad n_{i+1} = 0 \Rightarrow n_i = k_i = 0 \Big\} \end{aligned}, \tag{4.82}$$

where $\widetilde{m} = \lfloor m/2 \rfloor$ and $\rho^{(m)}(\mathbf{n},\mathbf{m}) \in \mathbb{N}$ is given by

$$\rho^{(m)}(\mathbf{n},\mathbf{k}) := \binom{n_{-1}+n_0+k_0}{k_0}\binom{n_{-1}+n_0}{n_0} \prod_{\substack{i=-\widetilde{m} \\ i\neq -1}}^{\widetilde{m}} \binom{n_i+n_{i+1}+k_{i+1}-1}{k_{i+1}}\binom{n_i+n_{i+1}-1}{n_{i+1}}.$$

Proof. For a give super Motzkin path p starting at $(0,0)$ and finishing at $(0,m)$ let k_i be the number of horizontal steps at height i and let n_i be the number of step up from height i to $i+1$. We remark the number n_i of step up from height i to $i+1$ is equal to the number of step down from $i+1$ to i. We define the vectors $\mathbf{k} = (k_{-\widetilde{m}}, k_{-\widetilde{m}+1}, \dots, k_{\widetilde{m}})$ and $\mathbf{n} = (n_{-\widetilde{m}}, n_{-\widetilde{m}+1}, \dots, n_{\widetilde{m}})$ and we associate the product

$$\prod_{i=-\widetilde{m}}^{\widetilde{m}} a_i^{2n_i} b_i^{k_i}.$$

Next we need to sum over all possible super Motzkin path p of length m connecting $(0,0)$ to $(0,m)$. Since the number of steps up is equal to the number of steps down, one necessarily have

$$\sum_{i=-\widetilde{m}}^{\widetilde{m}} (2n_i + k_i) = m\,.$$

Furthermore since the path is connected it follows that it is not possible to have a vertex at height $i+1$ without have a vertex at height $i > 0$ and the other way round if $i < 0$. Therefore one has

$$\begin{aligned} &\forall i \geqslant 0, \quad n_i = 0 \Rightarrow n_{i+1} = k_{i+1} = 0, \\ &\forall i < 0, \quad n_{i+1} = 0 \Rightarrow n_i = k_i = 0\,. \end{aligned}$$

This proves the definition of the set $\mathcal{A}^{(m)}$ in (4.82). The final step of the proof is to count the number of paths associated to the vectors $\mathbf{k} = (k_{-\widetilde{m}}, k_{-\widetilde{m}+1}, \dots, k_{\widetilde{m}})$ and $\mathbf{n} = (n_{-\widetilde{m}}, n_{-\widetilde{m}+1}, \dots, n_{\widetilde{m}})$. We want to show that this number is equal to $\rho^{(m)}(\mathbf{n},\mathbf{k})$.

A horizontal step at height i can occur just after a step up to height i, another horizontal step at height i, or a step down to height i. This leaves a total of $n_i + n_{i+1}$

different positions at which a horizontal step at height i can occur. Since we have k_i of horizontal steps, the number of different configurations with these step counts is the number of ways to choose k_i elements from a set of cardinality $n_i + n_{i+1}$ with repetitions allowed, i.e. $\binom{n_i+n_{i+1}+k_i-1}{k_i}$.

The number of different configurations with n_i steps at height i and n_{i+1} at height $i+1$ is given by the number of multi-sets of cardinality n_{i+1} taken from a set of cardinality n_i and this number is equal to $\binom{n_i+n_{i+1}-1}{n_{i+1}}$.

For the horizontal steps at height 0, they can also occur at the beginning of the path, this increase the number of possible positions by 1, so the number of these configurations with these steps counts is $\binom{n_0+n_{-1}+k_0}{k_0}$. In this way we have obtained the coefficient $\rho^{(m)}(\mathbf{n},\mathbf{k})$.

□

4.6.2 Proof of Lemma 4.13

In order to prove Lemma 4.13 we describe more specifically the Toda integrals and characterize their quadratic parts. Equation (4.31) follows by the explicit expression of $h_j^{(m)}$ in (4.26), as the coefficients $\rho^{(m)}(\mathbf{n},\mathbf{k})$ do not depend on the index j. We recall that $h_1^{(m)}$ takes the form

$$h_1^{(m)}(\mathbf{p},\mathbf{r}) = \sum_{(\mathbf{k},\mathbf{n})\in\mathcal{A}^{(m)}} (-1)^{|\mathbf{k}|}\rho^{(m)}(\mathbf{n},\mathbf{k})\,\mathbf{p}^{\mathbf{k}}e^{-\mathbf{n}^\intercal\mathbf{r}}\,,$$

with

$$\operatorname{supp}\mathbf{k},\quad \operatorname{supp}\mathbf{n} \subseteq B^{\mathtt{d}}_{\widetilde{m}}(0) := \{j\colon \mathtt{d}(0,j)\leqslant \widetilde{m}\}, \qquad |\mathbf{k}|+2|\mathbf{n}| = m.$$

In particular it is clear that $h_1^{(m)}$ has diameter $2\widetilde{m}\leqslant m$.

Now we Taylor expand around $\mathbf{r}=0$ the exponential with integral remainder:

$$e^{-\mathbf{n}^\intercal\mathbf{r}} = 1-\mathbf{n}^\intercal\mathbf{r}+\frac{1}{2}(\mathbf{n}^\intercal\mathbf{r})^2+\frac{(\mathbf{n}^\intercal\mathbf{r})^3}{2}\int_0^1 e^{-s\mathbf{n}^\intercal\mathbf{r}}\,(1-s)^2\,\mathrm{d}s$$

and we substitute it in $h_1^{(m)}$, obtaining an expansion of the form:

$$h_1^{(m)}(\mathbf{p},\mathbf{r}) = \sum_{(\mathbf{k},\mathbf{n})\in\mathcal{A}^{(m)}} (-1)^{|\mathbf{k}|}\rho^{(m)}(\mathbf{n},\mathbf{k})\,\mathbf{p}^{\mathbf{k}}\left(1-\mathbf{n}^\intercal\mathbf{r}+\frac{1}{2}(\mathbf{n}^\intercal\mathbf{r})^2+\frac{(\mathbf{n}^\intercal\mathbf{r})^3}{2}\int_0^1 e^{-s\mathbf{n}^\intercal\mathbf{r}}\,(1-s)^2\,\mathrm{d}s\right).$$

We can rewrite the above expression in the form

$$h_1^{(m)}(\mathbf{p},\mathbf{r}) = \varphi_0^{(m)}+\varphi_1^{(m)}(\mathbf{p},\mathbf{r})+\varphi_2^{(m)}(\mathbf{p},\mathbf{r})+\varphi_{\geqslant 3}^{(m)}(\mathbf{p},\mathbf{r})\,,$$

where $\varphi_\ell^{(m)}$, $\ell=0,1,2$, are the Taylor polynomials at $(\mathbf{p},\mathbf{r})=(\mathbf{0},\mathbf{0})$. Their explicit expressions read

$$\varphi_0^{(m)} = \sum_{(\mathbf{0},\mathbf{n})\in\mathcal{A}^{(m)}}\rho^{(m)}(\mathbf{n},\mathbf{0})\,,\quad \varphi_1^{(m)} = -\sum_{(\mathbf{0},\mathbf{n})\in\mathcal{A}^{(m)}}\rho^{(m)}(\mathbf{n},\mathbf{0})\mathbf{n}^\intercal\mathbf{r} - \sum_{\substack{(\mathbf{k},\mathbf{n})\in\mathcal{A}^{(m)}\\ |k|=1}}\rho^{(m)}(\mathbf{n},\mathbf{k})\mathbf{p}^{\mathbf{k}},$$

$$\varphi_2^{(m)} = \sum_{(\mathbf{0},\mathbf{n})\in\mathcal{A}^{(m)}}\rho^{(m)}(\mathbf{n},\mathbf{0})\frac{(\mathbf{n}^\intercal\mathbf{r})^2}{2} + \sum_{\substack{(\mathbf{k},\mathbf{n})\in\mathcal{A}^{(m)}\\ |k|=1}}\rho^{(m)}(\mathbf{n},\mathbf{k})\mathbf{p}^{\mathbf{k}}\mathbf{n}^\intercal\mathbf{r} + \sum_{\substack{(\mathbf{k},\mathbf{n})\in\mathcal{A}^{(m)}\\ |k|=2}}\rho^{(m)}(\mathbf{n},\mathbf{k})\mathbf{p}^{\mathbf{k}}.$$

We deduce from these explicit formulas that if m is odd then $\varphi_0^{(m)} \equiv 0$ as well as the first sum defining $\varphi_1^{(m)}$ and the first and last one defining $\varphi_2^{(m)}$. Indeed the sums are carried on an empty set. If m is even the second sum defining $\varphi_1^{(m)}$ and the second one defining $\varphi_2^{(m)}$ are zero for the same reason. Concerning $\varphi_{\geqslant 3}^{(m)}$, it has a zero of order greater than 3 in the variables $(\mathbf{p}, \mathbf{r})$, and it has the form

$$\varphi_{\geqslant 3}^{(m)}(\mathbf{p},\mathbf{r}) := \sum_{\substack{(\mathbf{k},\mathbf{n})\in\mathcal{A}^{(m)} \\ |k|\geqslant 3}} (-1)^{|\mathbf{k}|}\rho^{(m)}(\mathbf{n},\mathbf{k})\,\mathbf{p}^{\mathbf{k}}\left(1-\mathbf{n}^{\intercal}\mathbf{r}+\frac{1}{2}(\mathbf{n}^{\intercal}\mathbf{r})^2+\frac{(\mathbf{n}^{\intercal}\mathbf{r})^3}{2}\int_0^1 e^{-s\mathbf{n}^{\intercal}r}\,(1-s)^2\,\mathrm{d}s\right).$$

These, together with the explicit formula of $\rho^{(m)}(\mathbf{n},\mathbf{k})$, prove (4.35).

It is easy to see that defining

$$J_0^{(m)} := \frac{1}{m}\sum_{j=0}^{N-1} S_j\varphi_0^{(m)}, \qquad J_1^{(m)} := \frac{1}{m}\sum_{j=0}^{N-1} S_j\varphi_1^{(m)},$$
$$J_2^{(m)} := \frac{1}{m}\sum_{j=0}^{N-1} S_j\varphi_2^{(m)}, \qquad J_{\geqslant 3}^{(m)} := \frac{1}{m}\sum_{j=0}^{N-1} S_j\varphi_{\geqslant 3}^{(m)},$$

we immediately get that

$$J^{(m)} = J_0^{(m)} + J_1^{(m)} + J_2^{(m)} + J_{\geqslant 3}^{(m)}.$$

Clearly $J_0^{(m)}$ it is a constant that is zero for m odd; moreover thanks to the boundary condition (4.10) and the linearity of $J_1^{(m)}$ we have that $J_1^{(m)} = 0$. Further, $J_{\geqslant 3}^{(m)}$ is clearly a cyclic function. In order to get (4.37) and (4.38) for $J_2^{(m)}$ we have to split the proof in two different cases.

Case m odd. In this case thanks to the property of $\varphi_2^{(m)}$, the definition of $J_2^{(m)}$ and (4.34) we get that there exists a cyclic and symmetric matrix $B^{(m)}$ such that:

$$J_2^{(m)} = \mathbf{p}^{\intercal}B^{(m)}\mathbf{r}.$$

Moreover since the diam$(\mathbf{k})$, diam$(\mathbf{n})$ defining $\varphi_2^{(m)}$ are at most $\widetilde{m}$ (see Remark 4.10) we have that the vector $\mathtt{b}^{(m)}$ representing the matrix $B^{(m)}$ is m-admissible and from (4.28) we have that $\mathtt{b}_j^{(m)} = \mathtt{b}_{N-j}^{(m)}$ are positive integers for all $j = 0, \dots, \widetilde{m}$.

Case m even. As before there exist two matrices $A^{(m)}, D^{(m)}$ represented by m-admissible vectors such that:

$$J_2^{(m)} = \mathbf{p}^{\intercal}A^{(m)}\mathbf{p} + \mathbf{r}^{\intercal}D^{(m)}\mathbf{r}\,, \quad \mathtt{a}_k^{(m)} = \mathtt{a}_{N-k}^{(m)} \in \mathbb{N}\,, \quad \mathtt{d}_k^{(m)} = \mathtt{d}_{N-k}^{(m)} \in \mathbb{N}\,, \quad 0 \leqslant k \leqslant \widetilde{m}.$$

We have just to prove that the two matrices are equal; to do this we exploit the involution property of the Toda integrals. Indeed we know that $\left\{J^{(j)}, J^{(k)}\right\} = 0$, for any j, k. It follows easily that also their quadratic parts must commute:

$$\left\{J_2^{(k)}, J_2^{(j)}\right\} = 0, \qquad \forall\, k, j. \tag{4.83}$$

To compute explicitly the Poisson bracket we change coordinates via the Hartley transform (4.16) getting that:

$$
\begin{aligned}
J_2^{(m)} &= \sqrt{N}\sum_{j=1}^{N-1} \widehat{a}_j\widehat{p}_j^2 + \widehat{d}_j\widehat{r}_j^2 = \sqrt{N}\sum_{j=1}^{N-1} \widehat{a}_j\widehat{p}_j^2 + \widehat{d}_j\omega_j^2\widehat{q}_j^2,\\
J_2^{(2)} &= \frac{1}{2}\sum_j \widehat{p}_j^2 + \omega_j^2\widehat{q}_j^2,
\end{aligned}
$$

where $\omega_j = 2\sin\left(\pi\frac{j}{N}\right)$. As the Hartley transform is a symplectic map, by (4.83) we get

$$
0 = \left\{J_2^{(2)}, J_2^{(m)}\right\} = \sqrt{N}\sum_{j=1}^{N-1}\omega_j^2\left(\widehat{a}_j - \widehat{d}_j\right)\widehat{p}_j\widehat{q}_j,
$$

which implies that $\widehat{a}_j = \widehat{d}_j$ for all $j \neq 0$. To prove that also $\widehat{a}_0 = \widehat{d}_0$ we come back to the original variables getting that:

$$
\left.
\begin{aligned}
\mathtt{a}_j^{(m)} &= \frac{1}{\sqrt{N}}\widehat{a}_0 + \frac{1}{\sqrt{N}}\sum_{k=1}^{N-1}\widehat{a}_j\left(\cos\left(2\pi\frac{jk}{N}\right) + \sin\left(2\pi\frac{jk}{N}\right)\right),\\
\mathtt{d}_j^{(m)} &= \frac{1}{\sqrt{N}}\widehat{d}_0 + \frac{1}{\sqrt{N}}\sum_{k=1}^{N-1}\widehat{a}_j\left(\cos\left(2\pi\frac{jk}{N}\right) + \sin\left(2\pi\frac{jk}{N}\right)\right),
\end{aligned}
\right\}\ \forall\, j.
$$

This means that $\mathtt{a}_j^{(m)} - \mathtt{d}_j^{(m)} = \frac{\widehat{a}_0 - \widehat{d}_0}{\sqrt{N}}$ for all $j = 0,\dots,N-1$. Since $\mathtt{a}^{(m)}, \mathtt{d}^{(m)}$ are m-admissible it follows that $\mathtt{a}_{\widetilde{m}+1}^{(m)} = \mathtt{d}_{\widetilde{m}+1}^{(m)} = 0$ so that

$$
\frac{\widehat{a}_0 - \widehat{d}_0}{\sqrt{N}} = \mathtt{a}_{\widetilde{m}+1}^{(m)} - \mathtt{d}_{\widetilde{m}+1}^{(m)} = 0,
$$

which proves the statement.

4.6.3 Measure approximation

In this section we show how to approximate the measure $\mathrm{d}\mu$, in which the variables are constrained, with the measure $\mathrm{d}\mu_\theta$, where all variables are independent. The proof follows the construction of [108] (where it is done for Dirichlet boundary conditions) which applies both to the Gibbs measure of FPUT (4.11) and Toda (4.20). To simplify the construction we consider a general potential $V\colon \mathbb{R}\to\mathbb{R}$ and make the following assumptions:

(V1) There exist $\beta_0 > 0$ and a compact interval $\mathcal{I} \subset \mathbb{R}$ such that for any $\beta > \beta_0$, there exists $\theta \equiv \theta(\beta) \in \mathcal{I}$ such that

$$
\int_{\mathbb{R}} r\, e^{-\theta r - \beta V(r)}\,\mathrm{d}r = 0. \tag{4.84}
$$

(V2) There exist $\beta_0, \mathtt{C}_1, \mathtt{C}_2 > 0$ such that for any $\beta > \beta_0$, with $\theta = \theta(\beta)$ of (V1), one has

$$
\frac{\mathtt{C}_1}{\beta^{k/2}} < \int_{\mathbb{R}} |r|^k e^{-\theta r - \beta V(r)}\,\mathrm{d}r < \frac{\mathtt{C}_2}{\beta^{k/2}}, \qquad k = 0,\dots,4. \tag{4.85}
$$

In particular the moments up to order 4 are finite.

(V3) There exists $\beta_0 > 0$ such that $\forall \beta > \beta_0$, with $\theta = \theta(\beta)$ of (V1), one has

$$\inf_{r\in\mathbb{R}} |\theta r + \beta V(r)| > -\infty,$$

namely the function $r \mapsto \theta r + \beta V(r)$ is bounded from below.

Both the FPUT potential $V_F(x)$ and the Toda potential $V_T(x)$ satisfy the assumptions (V1)–(V3) by the results of section 4.6.4.

We define the constraint measure $\mathrm{d}\mu^V$ on the restricted phase space $\mathcal{M}$ as

$$\mathrm{d}\mu^V := \frac{1}{Z_V(\beta)}\, e^{-\beta\sum_{j=1}^N \frac{p_j^2}{2}}\, e^{-\beta\sum_{j=1}^N V(r_j)}\, \delta\left(\sum_j r_j = 0\right)\, \delta\left(\sum_j p_j = 0\right)\, \mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{r},$$

and the unconstrained measure $\mathrm{d}\mu_\theta^V$ on the extended phase space $\mathbb{R}^N \times \mathbb{R}^N$ as

$$\mathrm{d}\mu_\theta^V := \frac{1}{Z_{V,\theta}(\beta)}\, e^{-\beta\sum_{j=1}^N p_j^2/2}\, e^{-\beta\sum_{j=1}^N V(r_j)}\, e^{-\theta\sum_{j=1}^N r_j}\, \mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{r};$$

as usual $Z_V(\beta)$ and $Z_{V,\theta}(\beta)$ are the normalizing constants of $\mathrm{d}\mu^V$, $\mathrm{d}\mu_\theta^V$ respectively . We denote the expectation of f with respect to the measure $\mathrm{d}\mu^V$ as $\langle f\rangle_V$, and with respect to the measure $\mathrm{d}\mu_\theta^V$ as $\langle f\rangle_{V,\theta}$.
We also denote by $\|f\|_{V,\theta} := \left\langle f^2\right\rangle_{V,\theta}^{1/2}$ the L^2 norm of f with respect to the measure $\mathrm{d}\mu_\theta^V$.

The main result is the following one:

Theorem 4.31. *Assume that (V1)–(V3) hold true. Fix $K \in \mathbb{N}$ and assume that $f\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ have support of size K (according to definition 4.2) and finite second order moment with respect to $\mathrm{d}\mu_\theta^V$. Then there exist C, N_0 and β_0 such that for all $N > N_0$, $\beta > \beta_0$ one has*

$$\left|\langle f\rangle_V - \langle f\rangle_{V,\theta}\right| \leqslant C\frac{K}{N}\sqrt{\langle f^2\rangle_{V,\theta} - \langle f\rangle_{V,\theta}^2}. \tag{4.86}$$

Proof of Theorem 4.31

Introduce the structure function

$$\Omega_N(x) := \int_{x_1+\ldots+x_N=x} e^{-\beta\sum_{j=1}^N V(x_j)}\,\mathrm{d}x_1\ldots\mathrm{d}x_N, \qquad \forall x \in \mathbb{R}.$$

The important remark is that $\Omega_N(x)$ is N-times the convolution of the function $e^{-\beta V(x)}$ with itself, thus it is the density function of the sum of N iid random variables distributed as $e^{-\beta V(x)}$.

Next, for $\theta \in \mathbb{R}$, we define the conjugate distribution

$$U_N^{(\theta)}(x) := \frac{1}{\left(z_\theta(\beta)\right)^N}\, e^{-\theta x}\, \Omega_N(x), \qquad z_\theta(\beta) := \int_{\mathbb{R}} e^{-\beta V(x)-\theta x}\,\mathrm{d}x, \tag{4.87}$$

As before, we remark that $U_N^{(\theta)}(x)$ it is N-times the convolution of the function $e^{-\beta V(x)-\theta x}$ with itself thus it is the density function of the sum of N iid random variables $\{Y_n^{(\theta)}(\beta)\}_{1\leqslant n\leqslant N}$ distribute as

$$Y_n^{(\theta)}(\beta) \sim Y^{(\theta)} := \frac{1}{z_\theta(\beta)} e^{-\beta V(x)-\theta x}\,\mathrm{d}x,$$

moreover thanks to (4.84) we know that $\langle Y^{(\theta)} \rangle = 0$.

The central limit theorem says that the rescaled random variable $\frac{1}{\sigma\sqrt{N}} \sum_{n=1}^{N} Y_n^{(\theta)}(\beta)$ converges in distribution to a normal $\mathcal{N}(0,1)$. We want to apply a more refined version of this result, called *local central limit theorem*, which describes the asymptotic of this convergence.

In particular we will use a local central theorem whose proof can be found in [133, Theorem VII.15]; to state it, we first define the functions

$$\mathtt{q}_\nu(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum_{\mathcal{B}(\nu)} \mathtt{H}_{j+2s}(x) \prod_{d=1}^{\nu} \frac{1}{k_d!} \left(\frac{\gamma_{d+2}}{(d+2)!\, \sigma^{d+2}} \right)^{k_d} \tag{4.88}$$

where $\mathtt{H}_j$ is the j-th Hermite polynomial, γ_d is the d-th cumulant[1] of $Y_n^{(\theta)}(\beta)$, and $\mathcal{B}(\nu)$ is the set of all non-negative integer solutions $k_1, \ldots, k_\nu$ of the equalities $k_1 + 2k_2 + \cdots + \nu k_\nu = \nu$, and $s = k_1 + k_2 + \cdots + k_\nu$.

Theorem 4.32 (Local central limit)**.** *Let $\{X_n\}$ be a sequence of iid variables such that*

(i) For any $1 \leqslant n \leqslant N$, one has $\mathbf{E}[X_n] = 0$.

(ii) There exists $k \geqslant 3$ such that $\mathbf{E}\left[|X_n|^k\right] < +\infty$ for all n. Moreover $\sigma^2 := \mathbf{E}[X_n^2] > 0$.

(iii) The random variable $\frac{1}{\sigma\sqrt{N}} \sum_{n=1}^{N} X_n$ has a bounded density $\mathtt{p}_N(x)$.

Then there exists $C > 0$ such that

$$\sup_x \left| \mathtt{p}_N(x) - \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + \sum_{\nu=1}^{k-2} \frac{\mathtt{q}_\nu(x)}{N^{\nu/2}} \right| \leqslant \frac{C}{N^{(k-2)/2}},$$

where the $\mathtt{q}_\nu$'s are defined in (4.88).

Applying this theorem in case $X_n = Y_n^{(\theta)}(\beta)$, one gets the following result:

Corollary 4.33. *Assume (V1)–(V3). There exist $N_0, \beta_0, C > 0$ such that for all $N \geqslant N_0$, $\beta > \beta_0$ one has*

$$\left| U_N^{(\theta)}(x) - \frac{1}{\sqrt{2\pi\sigma^2 N}} \exp\left(-\frac{x^2}{2\sigma^2 N} \right) + \sum_{\nu=1}^{2} \frac{\mathtt{q}_\nu(x/\sigma\sqrt{N})}{N^{(\nu+1)/2}\sigma} \right| \leqslant \frac{C}{N^{3/2}\sigma}. \tag{4.89}$$

Proof. We verify that the assumptions of Theorem 4.32 are met in case $X_n = Y_n^{(\theta)}(\beta)$.
Item (i) and (ii) hold true thanks to assumptions (V1) and (V2), in particular (ii) is true with $k = 4$. To verify (iii), we note that $\frac{1}{\sigma\sqrt{N}} \sum_{n=1}^{N} Y_n^{(\theta)}(\beta)$ has density given by $\sigma\sqrt{N}\, U_N^{(\theta)}(\sigma\sqrt{N}x)$. This last function is N-times the convolution

[1] We recall that $\gamma_d = \sum_{\mathcal{C}(d)} d!(-1)^{m_1+\ldots+m_d-1}(m_1+\ldots+m_d-1)! \prod_{l=1}^{d} \frac{\alpha_l^{m_l}}{m_l!(l!)^{m_l}}$ where α_l is the l^{th} moment of the random variable and $\mathcal{C}(d)$ is the set of all non-negative integer solution of $\sum_l l m_l = d$.

of $g_\theta(r) := e^{-\theta r - \beta V(r)}$. By assumption (V3), $g_\theta \in L^\infty(\mathbb{R})$ and by (V2) it belongs also to $L^1(\mathbb{R})$. So Young's convolution inequality implies that $\sigma\sqrt{N}\, U_N^{(\theta)}(\sigma\sqrt{N}x)$ is bounded uniformly in x, hence (iii) of Theorem 4.32 is verified.
We apply Theorem 4.32 with $\mathtt{p}_N(x) = \sigma\sqrt{N}\, U_N^{(\theta)}(\sigma\sqrt{N}x)$, then rescale the variable x to get (4.89). □

We study also the structure function

$$\widetilde{\Omega}_N(\xi) := \int\limits_{\xi_1+\ldots+\xi_N=\xi} e^{-\frac{\beta}{2}\sum_{j=1}^N \xi_j^2}\, \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_N.$$

and the normalized distribution

$$\widetilde{U}_N(\xi) := \frac{1}{\left(\widetilde{z}_\theta(\beta)\right)^N}\ \widetilde{\Omega}_N(\xi), \qquad \widetilde{z}_\theta(\beta) := \int_{\mathbb{R}} e^{-\frac{\beta}{2}\xi^2}\, \mathrm{d}\xi. \tag{4.90}$$

We have the following result:

Lemma 4.34. *For any $N \geqslant 1$, any $\beta > 0$, one has*

$$\widetilde{U}_N(\xi) = \sqrt{\frac{\beta}{2\pi N}}\, \exp\left(-\frac{\beta\xi^2}{2N}\right). \tag{4.91}$$

Proof. The function $\widetilde{U}_N$ is the N-times convolution of Gaussian functions of the form $g(\xi) := \sqrt{\frac{\beta}{2\pi}}\, e^{-\frac{\beta}{2}\xi^2}$. Since convolution of Gaussians is a Gaussian whose variance is the sum of the variances, (4.90) follows. □

We can finally prove Theorem 4.31:

Proof of Theorem 4.31. The proof follows closely [108]. We assume that f is supported on $1, \ldots, K$, the other cases being analogous. Using that

$$Z_V(\beta) = \Omega_N(0)\, \widetilde{\Omega}_N(0),$$

and denoting $\widetilde{\mathbf{p}} := (p_1, \ldots, p_K)$ and $\widetilde{\mathbf{r}} := (r_1, \ldots, r_K)$, we write

$$\langle f(\widetilde{\mathbf{p}}, \widetilde{\mathbf{r}})\rangle_V = \int\limits_{\mathbb{R}^K \times \mathbb{R}^K} f(\widetilde{\mathbf{p}}, \widetilde{\mathbf{r}})\ \frac{\Omega_{N-K}\left(-\sum_{j=1}^K r_j\right)}{\Omega_N(0)}\, \frac{\widetilde{\Omega}_{N-K}\left(-\sum_{j=1}^K p_k\right)}{\widetilde{\Omega}_N(0)}\, \mathrm{d}\widetilde{\mu}$$

where $\mathrm{d}\widetilde{\mu} := \exp\left(-\beta\sum_{j=1}^K \frac{p_j^2}{2} - \beta\sum_{j=1}^K V(r_j)\right) \mathrm{d}\widetilde{p}\mathrm{d}\widetilde{r}$. As, by (4.87) and (4.90),

$$\frac{\Omega_{N-K}(x)}{\Omega_N(0)} = \frac{U_{N-K}^{(\theta)}(x)}{U_N^{(\theta)}(0)}\, \frac{e^{\theta x}}{(z_\theta(\beta))^K}, \qquad \frac{\widetilde{\Omega}_{N-K}(\xi)}{\widetilde{\Omega}_N(0)} = \frac{\widetilde{U}_{N-K}^{(\theta)}(\xi)}{\widetilde{U}_N^{(\theta)}(0)}\, \frac{1}{(\widetilde{z}_\theta(\beta))^K},$$

we write the difference $\langle f\rangle_V - \langle f\rangle_{V,\theta}$ as

$$\langle f\rangle_V - \langle f\rangle_{V,\theta} = \int\limits_{\mathbb{R}^K \times \mathbb{R}^K} f(\widetilde{\mathbf{p}}, \widetilde{\mathbf{r}})\, \frac{e^{-\theta\sum_{j=1}^K r_j}}{(z_\theta(\beta))^K\, (\widetilde{z}_\theta(\beta))^K}\ \mathtt{U}^{(\theta)}(\widetilde{\mathbf{p}}, \widetilde{\mathbf{r}})\, \mathrm{d}\widetilde{\mu}$$

where

$$\mathtt{U}^{(\theta)}(\widetilde{\mathbf{p}}, \widetilde{\mathbf{r}}) := \frac{U^{(\theta)}_{N-K}\left(-\sum_{j=1}^{K} r_j\right)}{U^{(\theta)}_N(0)} \frac{\widetilde{U}^{(\theta)}_{N-K}\left(-\sum_{j=1}^{K} p_j\right)}{\widetilde{U}^{(\theta)}_N(0)} - 1.$$

Now we use that

$$\int_{\mathbb{R}^K \times \mathbb{R}^K} \frac{e^{-\theta \sum_{j=1}^{K} r_j}}{(z_\theta(\beta))^K \, (\widetilde{z}_\theta(\beta))^K} \, \mathtt{U}^{(\theta)}(\widetilde{\mathbf{p}}, \widetilde{\mathbf{r}}) \, \mathrm{d}\widetilde{\mu} = \langle 1 \rangle_V - \langle 1 \rangle_{V,\theta} = 0$$

so that we can write the difference $\langle f \rangle_V - \langle f \rangle_{V,\theta}$ as

$$\langle f \rangle_V - \langle f \rangle_{V,\theta} = \int\limits_{\mathbb{R}^K \times \mathbb{R}^K} \Big(f(\widetilde{\mathbf{p}}, \widetilde{\mathbf{r}}) - \langle f \rangle_{V,\theta} \Big) \, \frac{e^{-\theta \sum_{j=1}^{K} r_j}}{(z_\theta(\beta))^K \, (\widetilde{z}_\theta(\beta))^K} \, \mathtt{U}^{(\theta)}(\widetilde{\mathbf{p}}, \widetilde{\mathbf{r}}) \, \mathrm{d}\widetilde{\mu}$$

Using Cauchy-Schwartz we obtain that

$$\Big| \langle f \rangle_V - \langle f \rangle_{V,\theta} \Big| \leqslant \| f - \langle f \rangle_{V,\theta} \|_{V,\theta} \, \| \mathtt{U}^{(\theta)} \|_{V,\theta},$$

so in order to prove (4.86) we are left to show that uniformly in N and β one has

$$\| \mathtt{U}^{(\theta)} \|_{V,\theta} \leqslant C \frac{K}{N}. \tag{4.92}$$

Using (4.89) and (4.91), we have that

$$\left| \frac{U^{(\theta)}_{N-K}(x)}{U^{(\theta)}_N(0)} \frac{\widetilde{U}^{(\theta)}_{N-K}(\xi)}{\widetilde{U}^{(\theta)}_N(0)} - 1 \right| \leqslant C \Bigg(\left| e^{-\frac{x^2}{2\sigma^2 (N-K)} - \frac{\beta \xi^2}{2(N-K)}} - 1 \right| + \frac{N}{(N-K)^{3/2}} \mathtt{q}_1 \times \left(\frac{x}{\sigma \sqrt{N-K}} \right) + \frac{K}{N-K} \Bigg)$$

Next we use that $|e^{-a^2-b^2} - 1| \leqslant a^2 + b^2$, the explicit expression

$$\mathtt{q}_1(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} (x^3 - 3x) \frac{\gamma_3}{6\,\sigma^3},$$

the estimate $\frac{\gamma_3}{6\,\sigma^3} \leqslant \mathtt{C}$ for some $\mathtt{C}$ independent of β (which follows by (4.85) as in our case $\gamma_3 \leqslant C\beta^{-3/2}$), to obtain that there exists $C > 0$ such that $\forall N \geqslant N_0$, $\forall \beta \geqslant \beta_0$,

$$\left| \frac{U^{(\theta)}_{N-K}(x)}{U^{(\theta)}_N(0)} \frac{\widetilde{U}^{(\theta)}_{N-K}(\xi)}{\widetilde{U}^{(\theta)}_N(0)} - 1 \right| \leqslant \frac{C}{N} \left(K + \beta \xi^2 + \frac{x}{\sigma} + \frac{x^2}{\sigma^2} + \frac{x^3}{\sigma^3 N} \right).$$

Substituting $x \equiv -\sum_{j=1}^{K} r_j$, $\xi \equiv -\sum_{j=1}^{K} p_j$, and computing the L^2 norm (with respect to $\mathrm{d}\mu^V_\theta$) of the terms in the r.h.s. of the last formula give the claimed estimate (4.92). □

4.6.4 Proof of Lemma 4.19

We prove the lemma for both the FPUT and Toda measure.

First, we observe that for $d, v = 2, 3$:

$$\frac{1}{4^d} \prod_{j \in \operatorname{Supp} \mathbf{n}} \min\left(e^{-dn_j r_j}, 1\right) \leqslant \left(\int_0^1 e^{-\xi \mathbf{n}^\intercal \mathbf{r}} (1-\xi)^v \mathrm{d}\xi\right)^d \leqslant \frac{1}{3^d} \prod_{j \in \operatorname{Supp} \mathbf{n}} \max\left(e^{-dn_j r_j}, 1\right) .$$

This means that we have actually to prove that for any fixed multi-index $\mathbf{k}, \mathbf{l}, \mathbf{n} \in \mathbb{N}_0^N$ there exist two constants $C^{(1)}_{\mathbf{k},\mathbf{l}} \in \mathbb{R}$ and $C^{(2)}_{\mathbf{k},\mathbf{l}} > 0$ such that:

$$\begin{aligned} \left\langle \mathbf{p}^{\mathbf{k}} \mathbf{r}^{\mathbf{l}} \prod_{j \in \operatorname{Supp} \mathbf{n}} \min\left(e^{-n_j r_j}, 1\right) \right\rangle_\theta &\geqslant C^{(1)}_{\mathbf{k},\mathbf{l}} \beta^{-\frac{|\mathbf{k}|+|\mathbf{l}|}{2}} , \\ \left\langle \mathbf{p}^{\mathbf{k}} \mathbf{r}^{\mathbf{l}} \prod_{j \in \operatorname{Supp} \mathbf{n}} \max\left(e^{-n_j r_j}, 1\right) \right\rangle_\theta &\leqslant C^{(2)}_{\mathbf{k},\mathbf{l}} \beta^{-\frac{|\mathbf{k}|+|\mathbf{l}|}{2}} . \end{aligned}$$

Moreover since for the two measures $\mathrm{d}\mu_{F,\theta}, \mathrm{d}\mu_{T,\theta}$ all $\mathbf{p}$ and $\mathbf{r}$ are independent random variables and moreover the p_j are independent and normally distributed according to $\mathcal{N}(0, \beta^{-1})$, it follows

$$\begin{aligned} \left\langle \mathbf{p}^{\mathbf{k}} \mathbf{r}^{\mathbf{l}} \prod_{j \in \operatorname{Supp} \mathbf{n}} \min\left(e^{-n_j r_j}, 1\right) \right\rangle_\theta &= \left\langle \mathbf{p}^{\mathbf{k}} \right\rangle_\theta \left\langle \mathbf{r}^{l} \prod_{j \in \operatorname{Supp} \mathbf{n}} \min\left(e^{-n_j r_j}, 1\right) \right\rangle_\theta \\ \left\langle \mathbf{p}^{\mathbf{k}} \mathbf{r}^{\mathbf{l}} \prod_{j \in \operatorname{Supp} \mathbf{n}} \max\left(e^{-n_j r_j}, 1\right) \right\rangle_\theta &= \left\langle \mathbf{p}^{\mathbf{k}} \right\rangle_\theta \left\langle \mathbf{r}^{\mathbf{l}} \prod_{j \in \operatorname{Supp} \mathbf{n}} \max\left(e^{-n_j r_j}, 1\right) \right\rangle_\theta \end{aligned}$$

where

$$\left\langle \mathbf{p}^{\mathbf{k}} \right\rangle_\theta = \left\langle \prod_i p_i^{k_i} \right\rangle_\theta = \begin{cases} \prod_i \frac{(k_i - 1)!!}{\beta^{\frac{k_i}{2}}}, & k_i \text{ all even} \\ 0, & \text{some } k_i \text{ odd} \end{cases}$$

Here $k!!$ denotes the double factorial. Instead the distribution of the r_j is different for the two measures, so we need to calculate it separately for the FPUT and Toda chain.

FPUT chain. Let's start considering $\left\langle r^l \min\left(e^{-nr}, 1\right) \right\rangle_\theta$:

$$\begin{aligned} \left\langle r^l \min\left(e^{-nr}, 1\right) \right\rangle_\theta &= \frac{\int_{\mathbb{R}^-} r^l e^{-\theta r - \beta\left(\frac{r^2}{2} + \frac{r^3}{3} + \frac{r^4}{4}\right)} \mathrm{d}r + \int_{\mathbb{R}^+} r^l e^{-nr} e^{-\theta r - \beta\left(\frac{r^2}{2} + \frac{r^3}{3} + \frac{r^4}{4}\right)} \mathrm{d}r}{\int_{\mathbb{R}} e^{-\theta r - \beta\left(\frac{r^2}{2} + \frac{r^3}{3} + \frac{r^4}{4}\right)} \mathrm{d}r} \\ &= \beta^{-\frac{l}{2}} \frac{\int_{\mathbb{R}^-} r^l e^{-\frac{\theta}{\sqrt{\beta}} r - \left(\frac{r^2}{2} + \frac{r^3}{3\sqrt{\beta}} + \frac{r^4}{4\beta}\right)} \mathrm{d}r + \int_{\mathbb{R}^+} r^l e^{-\frac{n}{\sqrt{\beta}} r} e^{-\frac{\theta}{\sqrt{\beta}} r - \left(\frac{r^2}{2} + \frac{r^3}{3\sqrt{\beta}} + \frac{r^4}{4\beta}\right)} \mathrm{d}r}{\int_{\mathbb{R}} e^{-\frac{\theta}{\sqrt{\beta}} r - \left(\frac{r^2}{2} + \frac{r^3}{3\sqrt{\beta}} + \frac{r^4}{4\beta}\right)} \mathrm{d}r} \\ &\geqslant \beta^{-\frac{l}{2}} \frac{\int_{\mathbb{R}^-} r^l e^{-\frac{\theta}{\sqrt{\beta}} r - \left(\frac{r^2}{2} + \frac{r^3}{3\sqrt{\beta}} + \frac{r^4}{4\beta}\right)} \mathrm{d}r}{\int_{\mathbb{R}} e^{-\frac{\theta}{\sqrt{\beta}} r - \left(\frac{r^2}{2} + \frac{r^3}{3\sqrt{\beta}} + \frac{r^4}{4\beta}\right)} \mathrm{d}r} . \end{aligned}$$

Since for β large enough $\theta(\beta)$ is uniformly bounded, it follows that there is a positive constant C_l such that:

$$\left\langle r^l \min\left(e^{-nr}, 1\right)\right\rangle_\theta \geqslant (-1)^l \frac{C_l}{\beta^{\frac{l}{2}}} . \tag{4.93}$$

We notice that if l is even then the right end side of (4.93) is positive. The proof for $\left\langle r^l \max\left(e^{-nr}, 1\right)\right\rangle_\theta$ follows in the same way so we get the claim for the FPUT chain.

□

Toda chain. For the Toda chain the computation is a little bit more involved, so we prefer to split it in different parts.

Lemma 4.35. *Consider the measure 4.39, then there exists a $\beta_0 > 0$ such that for all $\beta > \beta_0$ there exists $\theta \equiv \theta(\beta) \in [1/3, 2]$ such that*

$$\left\langle r_j^k \right\rangle_\theta = \begin{cases} 0 & k = 1 \\ \mathcal{O}\left(\frac{1}{\beta^{\frac{k}{2}}}\right) & k \neq 1 \end{cases} .$$

Proof. First we prove that, for any β large enough, we can chose $\theta(\beta)$ in a compact interval $\mathcal{I}$ such that $\langle r_j \rangle_\theta = 0$. We notice that:

$$\left\langle r^k \right\rangle_\theta = (-1)^k \frac{\partial_\theta^k \int_{\mathbb{R}} e^{-(\theta+\beta)r - \beta e^{-r}} \mathrm{d}r}{\int_{\mathbb{R}} e^{-(\theta+\beta)r - \beta e^{-r}} \mathrm{d}r} \overset{(e^{-r}=x)}{=} (-1)^k \frac{\partial_\theta^k \int_{\mathbb{R}^+} x^{\theta+\beta-1} e^{-\beta x} \mathrm{d}x}{\int_{\mathbb{R}^+} x^{\theta+\beta-1} e^{-\beta x} \mathrm{d}x} = (-1)^k \frac{\partial_\theta^k \frac{\Gamma(\beta+\theta)}{\beta^\theta}}{\frac{\Gamma(\beta+\theta)}{\beta^\theta}} , \tag{4.94}$$

where $\Gamma(z)$ is the usual Gamma function and we used the following equality:

$$\int_0^\infty t^{z-1} e^{-xt} \mathrm{d}t = \frac{\Gamma(z)}{x^z} .$$

In the case $k = 1$ one obtains

$$\langle r \rangle_\theta = \log(\beta) - \frac{\Gamma'(\theta+\beta)}{\Gamma(\theta+\beta)} . \tag{4.95}$$

Introducing the digamma function $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ [106] and using the inequality

$$\log x - \frac{1}{x} \leqslant \psi(x) \leqslant \log x - \frac{1}{2x}, \qquad \forall x > 0,$$

it is easy to show that there exists $\beta_0 > 0$ such that $\forall \beta > \beta_0$ one has

$$\psi\left(\frac{1}{3} + \beta\right) \leqslant \log\left(\frac{1}{3} + \beta\right) - \frac{1}{2(1/3 + \beta)} \leqslant \log \beta$$

and

$$\psi(2 + \beta) \geqslant \log(2 + \beta) - \frac{1}{2 + \beta} \geqslant \log \beta.$$

Since $x \mapsto \psi(x)$ is continuous on $(1, +\infty)$, by the intermediate value theorem there exists $\theta(\beta) \in [1/3, 2]$ fulfilling $\psi(\theta + \beta) = \log \beta$ which implies by (4.95) that

$$\langle r_j \rangle_\theta = \log(\beta) - \frac{\Gamma'(\theta+\beta)}{\Gamma(\theta+\beta)} = 0. \tag{4.96}$$

We will prove the remaining part of the claim by induction; (4.94) leads in the case $k = 2$ to:

$$\begin{aligned}
\left\langle r^2 \right\rangle_\theta &= \frac{\beta^\theta}{\Gamma(\theta+\beta)} \partial_\theta \left(\frac{\Gamma'(\theta+\beta) - \ln(\beta)\Gamma(\theta+\beta)}{\beta^\theta} \right) \\
&= \frac{\beta^\theta}{\Gamma(\theta+\beta)} \partial_\theta \left(\frac{\beta^\theta}{\Gamma(\theta+\beta)} \left(\psi(\theta+\beta) - \ln(\beta)\right) \right) \\
&= \langle r_j \rangle_\theta \left(\psi(\theta+\beta) - \ln(\beta)\right) + \psi^{(1)}(\theta+\beta) \\
&= \psi^{(1)}(\theta+\beta),
\end{aligned}$$

where $\psi^{(s)}$ is the s^{th} polygamma function defined as $\psi^{(s)}(z) := \frac{\partial^s \psi(z)}{\partial z^s}$. For $x \in \mathbb{R}$ it has the following expansion as $x \to +\infty$:

$$\psi^{(s)}(x) \sim (-1)^{s+1} \sum_{k=0}^{\infty} \frac{(k+s-1)!}{k!} \frac{B_k}{x^{k+s}}, \qquad s \geqslant 1\,, \tag{4.97}$$

where B_k are the Bernoulli number of the second kind. Therefore

$$\left\langle r^2 \right\rangle_\theta = \psi^{(1)}(\theta+\beta) \overset{\beta > \beta_0}{=} \mathcal{O}\left(\frac{1}{\beta}\right).$$

So the first inductive step is proved. Next suppose the statement true for k and let us prove it for $k+1$.

$$\begin{aligned}
\left\langle r^{k+1} \right\rangle_\theta &= (-1)^{k+1} \frac{\beta^\theta}{\Gamma(\theta+\beta)} \partial_\theta^k \left(\frac{\Gamma'(\theta+\beta) - \ln(\beta)\Gamma(\theta+\beta)}{\beta^\theta} \right) \\
&= (-1)^{k+1} \frac{\beta^\theta}{\Gamma(\theta+\beta)} \partial_\theta^k \left(\frac{\beta^\theta}{\Gamma(\theta+\beta)} \left(\psi(\theta+\beta) - \ln(\beta)\right) \right) \\
&= (-1)^{k+1} \frac{\beta^\theta}{\Gamma(\theta+\beta)} \partial_\theta^k \left(\frac{\beta^\theta}{\Gamma(\theta+\beta)} \right) \left(\psi(\theta+\beta) - \ln(\beta)\right) \\
&+ (-1)^{k+1} \frac{\beta^\theta}{\Gamma(\theta+\beta)} \sum_{n=1}^{k} \binom{k}{n} \partial_\theta^{k-n} \left(\frac{\beta^\theta}{\Gamma(\theta+\beta)} \right) \partial_\theta^n \psi(\theta+\beta) \\
&= 0 + \sum_{n=1}^{k} \binom{k}{n} (-1)^{n+1} \left\langle r^{k-n} \right\rangle_\theta \partial_\theta^n \psi(\theta+\beta) = \mathcal{O}\left(\frac{1}{\beta^{\frac{k}{2}}}\right),
\end{aligned}$$

where we used (4.96) and (4.97). □

We are now ready to prove the last part of Lemma 4.19 for the Toda chain:

$$\begin{aligned}
\left\langle r^l \max(1, e^{-nr}) \right\rangle_\theta &= \frac{\int_{\mathbb{R}^+} r^l e^{-(\theta+\beta)r - \beta e^{-r}} \mathrm{d}r + \int_{\mathbb{R}^-} r^l e^{-(\theta+\beta-n)r - \beta e^{-r}} \mathrm{d}r}{\int_{\mathbb{R}} e^{-\theta r - \beta e^{-r}} \mathrm{d}r} \\
&\leqslant \frac{\int_{\mathbb{R}^+} r^l e^{-(\theta+\beta)r - \beta e^{-r}} \mathrm{d}r}{\int_{\mathbb{R}} e^{-(\theta+\beta)r - \beta e^{-r}} \mathrm{d}r}\,.
\end{aligned}$$

The last integral can be estimated in the same way as in the previous lemma, moreover the lower bound follows in the same way, so we get the claim also for the Toda chain. □

Chapter 5

Loop equation for the classical Beta ensembles in the high-temperature regime, and the Dyson disordered chain

The Gaussian β-ensemble refers to the eigenvalue probability density function (PDF) proportional to

$$\prod_{l=1}^{N} e^{-\lambda_l^2} \prod_{1 \leqslant j < k \leqslant N} |\lambda_k - \lambda_j|^{\beta}. \tag{5.1}$$

Upon the scaling of the eigenvalues by setting

$$\lambda_l = \sqrt{\beta N} x_l, \tag{5.2}$$

it is a well known fact that the eigenvalue density $\overline{\rho}_{(1)}(x)$, normalized to integrate to unity, has the limiting form of the Wigner semi-circle law (see e.g. [54, 1.4.2])

$$\lim_{N \to \infty} \overline{\rho}_{(1)}(x) = \frac{2}{\pi}(1 - x^2)^{1/2} \chi_{|x|<1}, \tag{5.3}$$

where $\chi_A = 1$ for A true and $\chi_A = 0$ otherwise. The use of the scaling variables (5.2) — often referred to as corresponding to the global regime; see e.g. [55] — also leads to many other consequences. For example, introduce the linear statistic $A = \sum_{j=1}^{N} a(x_j)$ for $a(x)$ smooth and bounded. The average with respect to (5.1) then permits the $1/N$ expansion [90]

$$\mathbf{E}\left[\sum_{j=1}^{N} a(x_j)\right] = N \int_{-\infty}^{\infty} a(x) \overline{\rho}_{(1),0}(x)\, \mathrm{d}x + \int_{-\infty}^{\infty} a(x) \overline{\rho}_{(1),1}(x)\, \mathrm{d}x + \mathrm{O}\Big(\frac{1}{N}\Big), \tag{5.4}$$

where

$$\overline{\rho}_{(1),1}(x) = \Big(\frac{1}{\beta} - \frac{1}{2}\Big)\Big(\frac{1}{2}(\delta(x-1) + \delta(x+1)) - \frac{1}{\pi\sqrt{1-x^2}}\Big). \tag{5.5}$$

Equivalently, the smoothed eigenvalue density (i.e. effective eigenvalue density upon integrating over a smooth test function), $\overline{\rho^{\mathrm{s}}}_{(1)}(x)$ say, admits an expansion in $1/N$ powers,

$$\overline{\rho^{\mathrm{s}}}_{(1)}(x) = \sum_{j=0}^{\infty} \overline{\rho}_{(1),j}(x) N^{-j},$$

where the first two terms are given by (5.3) and (5.5) respectively.

Furthermore, (see e.g. [132])

$$\lim_{N\to\infty} \operatorname{Var} A = \int_{-1}^{1} \mathrm{d}x \int_{-1}^{1} \mathrm{d}y \left(\frac{a(x)-a(y)}{x-y}\right)^2 r_{(2),0}(x,y), \tag{5.6}$$

where

$$r_{(2),0}(x,y) = \frac{1}{4\pi^2}\frac{1-xy}{\sqrt{1-x^2}\sqrt{1-y^2}}.$$

The global regime is characterized by the spacing between the eigenvalues tending to zero, at such a rate that the statistical properties like those reviewed in the above paragraph have a well-defined limit. This latter property is also shared by another choice of limit, corresponding to a scaled high temperature regime, as specified by setting

$$\beta = 2\alpha/N, \qquad \alpha > -1 \text{ fixed}, \tag{5.7}$$

before taking the large N limit. The study of this limit was introduced in the context of the Gaussian β-ensemble in [7]. Later was considered for the Laguerre and Jacobi variants of (5.1) [8, 159, 160], i.e. the primary examples of the classical ensembles in random matrix theory. Related to the β-ensembles with the scaling (5.7) are certain classes of random tridiagonal matrices, with i.i.d. entries along the diagonal, and (separately) along the leading diagonal, now referred to as specifying α-ensembles [115].

After making the N, β-independent change of scale $\lambda_j = x_j/\sqrt{2}$ in (5.1), upon the limit (5.7) the density $\rho_{(1),0}(x) = \rho_{(1),0}(x;\alpha)$ is specified by the functional form [7, 44, 115]

$$\rho_{(1),0}(x;\alpha) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}|\hat{f}_\alpha(x)|^{-2}, \qquad \hat{f}_\alpha(x) = \sqrt{\frac{\alpha}{\Gamma(\alpha)}}\int_0^\infty t^{\alpha-1}e^{-t^2/2}e^{ixt}\,dt. \tag{5.8}$$

While it is to be anticipated that a $1/N$ expansion of the form (5.4) will again hold — and thus with the first term known by way of (5.8) the task remaining is to characterize the analogue for $\overline{\rho}_{(1),1}(x)$ — results from [124, 161] (see also [80]) tell us that in relation to the variance

$$\frac{1}{N}\operatorname{Var} A \tag{5.9}$$

has a well-defined limit. Note that the factor of $1/N$ is absent on the LHS of (5.6). However, it remains to obtain explicit formulas in relation to A.

In this chapter, we introduce a new approach — making use of knowledge of the loop equations for the classical β-ensembles [23, 25, 63, 123, 166] — to systematically study the high temperature scaling (5.7). Choosing the Gaussian β-ensemble for definiteness in this Introduction, the loop equation formalism allows for a systematic quantification of the coefficients in the large N-expansion of the resolvent

$$\frac{1}{N}\mathbf{E}\left[\sum_{j=1}^{N}\frac{1}{x-\lambda_i}\right]^{\mathrm{G}}\Bigg|_{\beta=2\alpha/N} = W_1^{0,\mathrm{G}}(x) + \frac{1}{N}W_1^{1,\mathrm{G}}(x) + \cdots,$$

where

$$W_1^{0,\mathrm{G}}(x) = \int_{-\infty}^{\infty}\frac{\rho^{\mathrm{G}}_{(1),0}(\lambda;\alpha)}{x-\lambda}\,d\lambda, \qquad W_1^{1,\mathrm{G}}(x) = \int_{-\infty}^{\infty}\frac{\rho^{\mathrm{G}}_{(1),1}(\lambda;\alpha)}{x-\lambda}\,d\lambda, \tag{5.10}$$

as well as in the large N-expansion

$$\frac{1}{N}\mathrm{Cov}\left(\sum_{i=1}^{N}\frac{1}{x-\lambda_i},\sum_{i=1}^{N}\frac{1}{y-\lambda_i}\right)^{\mathrm{G}}\Big|_{\beta=2\alpha/N} := W_2^{0,\mathrm{G}}(x,y)+\frac{1}{N}W_2^{1,\mathrm{G}}(x,y)+\cdots,$$

where $\mathrm{Cov}(\cdot,\cdot)$ denotes the covariance of the respective linear statistics, and we use the superscript "G" to indicate the Gaussian ensemble in the scaling limit with $\beta = 2\alpha/N$. Note that from knowledge of $W_1^{0,\mathrm{G}}(x)$ as specified in (5.10), which is the Stieltjes transform of $\rho^{\mathrm{G}}_{(1),0}(\lambda;\alpha)$, the corresponding inversion formula gives

$$\rho^{\mathrm{G}}_{(1),0}(x;\alpha) = \lim_{\epsilon\to 0^+}\frac{1}{\pi}\mathrm{Im}\, W_1^{0,\mathrm{G}}(x-i\epsilon). \tag{5.11}$$

For the classical ensembles generally, we will show that the loop equation formalism implies W_1^0 can be computed as the solution of a differential equation. This fact is already known in the Gaussian and Laguerre cases, but not for the Jacobi ensemble. This then allows for the computation of the leading order scaled density via (5.11). The differential equation characterization also allows for the corresponding moments of the spectral density to be determined via a recurrence. Again specializing to the Gaussian ensemble for definiteness, we see from performing an appropriate geometric series expansion in the first expression of (5.10) that

$$W_1^{0,\mathrm{G}}(x) = \frac{1}{x}\sum_{p=0}^{\infty}\frac{m^{\mathrm{G}}_{p,0}}{x^p}, \qquad m^{\mathrm{G}}_{p,0} = \int_{-\infty}^{\infty} x^p \rho^{\mathrm{G}}_{(1),0}(x;\alpha)\,dx, \tag{5.12}$$

where the formula for $m^{\mathrm{G}}_{p,0}$ in terms of $\rho^{\mathrm{G}}_{(1),0}(x;\alpha)$ tells us that $\{m^{\mathrm{G}}_{p,0}\}$ are the moments of the limiting eigenvalue density.

Proposition 5.1. *(Duy and Shirai [44, Prop. 3.1].) The moments* $\{m^{\mathrm{G}}_{p,0}\}_{p\,\mathrm{even}}$ *satisfy the recurrence*

$$m^{\mathrm{G}}_{p+2,0} = (p+1)m^{\mathrm{G}}_{p,0}+\alpha\sum_{s=0}^{p/2} m^{\mathrm{G}}_{p-2s,0}m^{\mathrm{G}}_{2s,0}, \qquad m^{\mathrm{G}}_{0,0}=1, (p=0,2,4,\dots\mathrm{even}), \tag{5.13}$$

while the odd moments all vanish by symmetry.

The loop equation formalism shows that $W^{\mathrm{G}}_{2,0}(x_1,x_2)$ satisfies a partial differential equation involving $W^{\mathrm{G}}_{1,0}(x_i)$ $(i=1,2)$. No closed form solution is to be expected, but analogous to (5.12) if we expand about infinity by noting

$$W_2^{0,\mathrm{G}}(x_1,x_2) = \frac{1}{x_1x_2}\sum_{p,q=0}^{\infty}\frac{\mu^{\mathrm{G}}_{(p,q),0}}{x_1^p x_2^q}, \qquad \mu^{\mathrm{G}}_{(p,q),0} = \lim_{\substack{N\to\infty\\ \beta=2\alpha/N}} \mathrm{Cov}\left(\sum_{i=1}^{N}x_i^p,\sum_{i=1}^{N}x_i^q\right)^{\mathrm{G}}, \tag{5.14}$$

then the partial differential equation allows $\{\mu^{\mathrm{G}}_{(p,q),0}\}$ to be determined by a coupled recurrence involving $\{m^{\mathrm{G}}_{p,0}\}$, already determined by (5.13).

Proposition 5.2. *(Equivalent to Spohn [152, Eqns. (5.14), (5.15)].) For* $p,q\geqslant 1$ *of the same parity, meaning that they are either both even or odd, we have*

$$\mu^{\mathrm{G}}_{(p,q),0} = (p-1)\mu^{\mathrm{G}}_{(p-2,q),0}+q m^{\mathrm{G}}_{p+q-2,0}+2\alpha\sum_{s=0}^{\lfloor p/2-1\rfloor} m^{\mathrm{G}}_{2s,0}\mu^{\mathrm{G}}_{(p-2-2s,q),0}. \tag{5.15}$$

If $p=0$*, or* $q=0$*, or* p,q *have the opposite parity,* $\mu^{\mathrm{G}}_{(p,q),0}=0$*.*

Our study of $W_1^{1,\mathrm{G}}(x)$ proceeds analogously. Introducing the expansions

$$W_1^{1,\mathrm{G}}(x) = \frac{1}{x}\sum_{p=1}^{\infty}\frac{m_{2p,1}^{\mathrm{G}}}{x^{2p}}, \tag{5.16}$$

$$W_2^{0,\mathrm{G}}(x,x) = \frac{1}{x^2}\sum_{p=1}^{\infty}\frac{\tilde{\mu}_{2p,0}^{\mathrm{G}}}{x^{2p}}, \qquad \tilde{\mu}_{2p,0}^{\mathrm{G}} = \sum_{p_1+q_1=2p}\mu_{(p_1,q_1),0}^{\mathrm{G}}, \tag{5.17}$$

from the loop equations, we can determine that $\{m_{2p,1}^{\mathrm{G}}\}$ satisfies a coupled recurrence with $\{\tilde{\mu}_{2p,0}^{\mathrm{G}}\}$ and $\{m_{2p,0}^{\mathrm{G}}\}$.

Proposition 5.3. *We have*

$$m_{2(p+1),1}^{\mathrm{G}} = -\alpha(2p+1)m_{2p,0}^{\mathrm{G}}+(2p+1)m_{2p,1}^{\mathrm{G}}+\alpha\tilde{\mu}_{2p,0}^{\mathrm{G}}+2\alpha\sum_{s=0}^{p-1}m_{2s,0}^{\mathrm{G}}m_{2(p-s),1}^{\mathrm{G}}, \quad m_{0,1}^{\mathrm{G}} = 0, \tag{5.18}$$

where $\{m_{2p,0}^{\mathrm{G}}\}$ *are determined by the recurrence* (5.13)*, and* $\{\tilde{\mu}_{2p,0}^{\mathrm{G}}\}$ *by the recurrence* (5.47) *below.*

After revising relevant results relating to the loop equation formalism for the classical ensembles in Section 5.1, we proceed in Sections 5.2, 5.3, 5.4 respectively to derive Propositions 5.1–5.3 and their analogues for the Gaussian, Laguerre and Jacobi β-ensembles with high temperature scaling (5.7). Our strategy also gives a unifying method to derive the functional form of the limiting density, given by (5.8) in the Gaussian case; for the Jacobi ensemble this is new. Thus, the loop equations give a particular Riccati equation for the Stieltjes transform of the limiting density, which implies a linear second order differential equation when the latter is written as a logarithmic derivative. In the Jacobi case, the linear second order differential equation is a hypergeometric differential equation, which leads to a functional form for the limiting density in terms of a linear combination of Gauss hypergeometric, in agreement with a recent result of Trinh and Trinh [160].

In the recent work, [115] the classical β-ensembles in the high temperature limit have been used to construct a family of tridiagonal matrices referred to as α-ensembles. Moreover, an application was given to the study of generalized Gibbs ensembles associated with the classical Toda lattice [152]. In section 5.5, beginning with the anti-symmetric Gaussian β-ensemble we identify a further example of an α-ensemble, specified as a random anti-symmetric tridiagonal matrix, with i.i.d. gamma distributed random variables. Knowledge of the limiting spectral density for the Laguerre β-ensemble in the scaled high temperature limit can be used to determine the limit spectral density of this particular α-ensemble. It is pointed out that the same random matrix ensemble appears in Dyson's [45] study of a disordered chain of harmonic oscillators. Our analytic results supplement those already contained in Dyson's work.

5.1 Preliminaries

5.1.1 Quantities of interest in the loop equation formalism

Introduce the notation $\mathrm{ME}_{\beta,N}[w]$ to denote a matrix ensemble with eigenvalue PDF proportional to

$$\prod_{l=1}^{N} w(\lambda_l) \prod_{1 \leqslant j<k \leqslant N} |\lambda_k - \lambda_j|^{\beta}, \tag{5.19}$$

where $w(\lambda)$ is referred to as the weight function. Collectively, the terminology β-ensemble is used in relation to (5.19), and the name associated with the weight is specified as an adjective. Thus, for example, $\mathrm{ME}_{\beta,N}[e^{-\lambda^2}]$ is referred to as the Gaussian β-ensemble, in agreement with the terminology used in relation to (5.1). Let $\rho_{(1)}(\lambda)$ denote the corresponding eigenvalue density, specified by the requirement that $\int_a^b \rho_{(1)}(\lambda)\,\mathrm{d}\lambda$ be equal to the expected number of eigenvalues in a general interval $[a,b]$. Its Stieltjes transform is given by

$$\overline{U}_1(x) = \int_{-\infty}^{\infty} \frac{\rho_{(1)}(\lambda)}{x-\lambda}\,\mathrm{d}\lambda.$$

Note that

$$\overline{U}_1(x) = \mathbf{E}\left[\sum_{j=1}^{N} \frac{1}{x-\lambda_j}\right]_{\mathrm{ME}_{\beta,N}[w]} = \mathbf{E}\left[\mathrm{Tr}\,(x\mathbb{I}_N - H)^{-1}\right]_{\mathrm{ME}_{\beta,N}[w]}, \tag{5.20}$$

where in the second average $H = \mathrm{diag}\,(\lambda_1,\dots,\lambda_N)$. In matrix theory $(x\mathbb{I}_N - H)^{-1}$ is referred to as the resolvent. It is thus by abuse of terminology that $\overline{U}_1(x)$ itself is often referred to as the resolvent. The average in the first equality in (5.20) is an example of a one-point correlator. Its generalization to an n-point correlator is

$$\overline{W}_n(x_1,\dots,x_n) = \mathbf{E}\left[\sum_{j_1,\dots,j_n=1}^{N} \frac{1}{(x_1-\lambda_{j_1})\cdots(x_n-\lambda_{j_n})}\right]_{\mathrm{ME}_{\beta,N}[w]}. \tag{5.21}$$

A feature of (5.20) is that for a large class of weights w, there is a scale $x = c_N s$ such that in the variable s and as $N \to \infty$ the eigenvalue support is a finite interval, and moreover $\overline{W}_1(c_N s)$ can be expanded as a series in $1/N$ [24]

$$c_N \overline{W}_1(c_N s) = N \sum_{\ell=0}^{\infty} \frac{\overline{W}_1^{\ell}(s)}{(N\sqrt{\kappa})^{\ell}}, \qquad \kappa = \beta/2, \tag{5.22}$$

where $\{\overline{W}_1^{l}(s)\}$ are independent of N. For example, from (5.2), in the case of the Gaussian β-ensemble $c_N = \sqrt{\beta N}$. An analogous expansion holds true in relation to the n-point statistic (5.21), but only after forming appropriate linear combinations of $\overline{W}_n$. These are the connected components of $\overline{U}_n$, specified by

$$\begin{aligned}
\overline{W}_1(x) &= \overline{U}_1(x) \\
\overline{W}_2(x_1,x_2) &= \overline{U}_2(x_1,x_2) - \overline{U}_1(x_1)\overline{U}_1(x_2) \\
\overline{W}_3(x_1,x_2,x_3) &= \overline{U}_3(x_1,x_2,x_3) - \overline{U}_2(x_1,x_2)\overline{U}_1(x_3) - \overline{U}_2(x_1,x_2)\overline{U}_1(x_2) \\
&\quad - \overline{U}_2(x_2,x_3)\overline{U}_1(x_1) + 2\overline{U}_1(x_1)\overline{U}_1(x_2)\overline{U}_1(x_3),
\end{aligned} \tag{5.23}$$

with the general case $\overline{W}_k$ being formed by an analogous inclusion/ exclusion construction. Going in the reverse direction, and thus specifying $\{\overline{U}_j\}$ in terms of $\{\overline{W}_j\}$, the inductive relation

$$\overline{U}_n(x_1, J_n) = \overline{W}_n(x_1, J_n) + \sum_{\varnothing \neq J \subseteq J_n} \overline{W}_{n-|J|}(x_1, J_n \backslash J) \overline{U}_{|J|}(J), \tag{5.24}$$

where

$$J_n = (x_2, \dots, x_n), \quad J_1 = \varnothing,$$

holds true (see e.g. [167, pp. 8-9]). The utility of the connected components $\overline{W}_n$ is that (5.22) admits the generalization [24],

$$c_N^n \overline{W}_n(c_N s_1, \dots, c_N s_n) = N^{2-n} \kappa^{1-n} \sum_{l=0}^{\infty} \frac{W_n^l(s_1, \dots, s_n)}{(N\sqrt{\kappa})^l}, \tag{5.25}$$

where again $\kappa = \beta/2$. Thus as n increases by one, the large N form decreases by a factor of $1/N$, with all lower order terms given by a series in $1/N$.

5.1.2 Explicit form of the loop equations for the classical ensembles

Consider first the Gaussian β-ensemble $\mathrm{ME}_{\beta,N}[e^{-\lambda^2/2}]$ (here the rescaling of the eigenvalues $\lambda \mapsto \lambda/\sqrt{2}$ is for convenience; recall the text above (5.7)). With J_n as in (5.24) the n^{th} loop equation is [23, 25, 123, 166]

$$\begin{aligned} 0 = &\left[(\kappa - 1) \frac{\partial}{\partial x_1} - x_1 \right] \overline{W}_n(x_1, J_n) + N \chi_{n=1} \\ &+ \chi_{n \neq 1} \sum_{k=2}^{n} \frac{\partial}{\partial x_k} \left\{ \frac{\overline{W}_{n-1}(x_1, \dots, \hat{x}_k, \dots, x_n) - \overline{W}_{n-1}(J_n)}{x_1 - x_k} \right\} \\ &+ \kappa \left[\overline{W}_{n+1}(x_1, x_1, J_n) + \sum_{J \subseteq J_n} \overline{W}_{|J|+1}(x_1, J) \overline{W}_{n-|J|}(x_1, J_n \backslash J) \right]. \end{aligned} \tag{5.26}$$

Here the notation $\hat{x}_k$ indicates that the variable x_k is not present in the argument, and thus $\overline{W}_{n-1}(x_1, \dots, \hat{x}_k, \dots, x_n) = \overline{W}_{n-1}(\{x_j\}_{j=1}^n \backslash \{x_k\})$.

Consider next the Laguerre β-ensemble $\mathrm{ME}_{\beta,N}[x^{\alpha_1} e^{-x} \chi_{x>0}]$. The n^{th} loop equation is [63, Eq. (3.9)]

$$\begin{aligned} 0 = &\left[(\kappa - 1) \frac{\partial}{\partial x_1} + \left(\frac{\alpha_1}{x_1} - 1 \right) \right] \overline{W}_n(x_1, J_n) + \chi_{n=1} \frac{N}{x_1} \\ &+ \chi_{n \neq 1} \sum_{k=2}^{n} \frac{\partial}{\partial x_k} \left\{ \frac{\overline{W}_{n-1}(x_1, \dots, \hat{x}_k, \dots, x_n) - \overline{W}_{n-1}(J_n)}{x_1 - x_k} + \frac{1}{x_1} \overline{W}_{n-1}(J_n) \right\} \\ &+ \kappa \left[\overline{W}_{n+1}(x_1, x_1, J_n) + \sum_{J \subseteq J_n} \overline{W}_{|J|+1}(x_1, J) \overline{W}_{n-|J|}(x_1, J_n \backslash J) \right]. \end{aligned} \tag{5.27}$$

Finally, consider the Jacobi β-ensemble $\mathrm{ME}_{\beta,N}[x^{\alpha_1}(1-x)^{\alpha_2}\chi_{0<x<1}]$. The n^{th} loop equation is [63, Eq. (4.6)]

$$\begin{aligned}
0 = &\left((\kappa-1)\frac{\partial}{\partial x_1} + \left(\frac{\alpha_1}{x_1} - \frac{\alpha_2}{1-x_1}\right)\right)\overline{W}_n(x_1, J_n) - \frac{n-1}{x_1(1-x_1)}\overline{W}_{n-1}(J_n) \\
&+ \frac{\chi_{n=1}}{x_1(1-x_1)}\left[(\alpha_1+\alpha_2+1)N + \kappa N(N-1)\right] - \frac{\chi_{n\neq 1}}{x_1(1-x_1)}\sum_{k=2}^{n} x_k \frac{\partial}{\partial x_k}\overline{W}_{n-1}(J_n) \\
&+ \chi_{n\neq 1}\sum_{k=2}^{n}\frac{\partial}{\partial x_k}\left\{\frac{\overline{W}_{n-1}(x_1,\dots,\hat{x}_k,\dots,x_n) - \overline{W}_{n-1}(J_n)}{x_1 - x_k} + \frac{1}{x_1}\overline{W}_{n-1}(J_n)\right\} \\
&+ \kappa\left[\overline{W}_{n+1}(x_1,x_1,J_n) + \sum_{J\subseteq J_n}\overline{W}_{|J|+1}(x_1,J)\overline{W}_{n-|J|}(x_1,J_n\backslash J)\right].
\end{aligned} \tag{5.28}$$

5.2 Solving the loop equations at low order with $\beta = 2\alpha/N$ — the Gaussian β-ensemble

Our interest is in the scaling of β proportional to the reciprocal of N, as specified by (5.7). As a modification of (5.25), we make the ansatz for the large N expansion of $\overline{W}_n$ to have the form

$$\overline{W}_n(x_1,\dots,x_n) = N\sum_{l=0}^{\infty}\frac{W_n^l(x_1,\dots,x_n)}{N^l}. \tag{5.29}$$

Note that this N dependence is consistent with both (5.4) and the factor of $1/N$ in (5.9). We will consider each of the three ensembles separately, beginning with the Gaussian β-ensemble.

Consider (5.26) with $n=1$. Upon the substitution (5.29), by equating terms $\mathrm{O}(N)$ we read off the equation for $W_1^0 = W_1^{0,\mathrm{G}}$

$$\left(-\frac{d}{dx} - x\right)W_1^{0,\mathrm{G}}(x) + 1 + \alpha(W_1^{0,\mathrm{G}}(x))^2 = 0. \tag{5.30}$$

Equating terms $\mathrm{O}(1)$ gives an equation relating $W_1^{0,\mathrm{G}}(x), W_1^{1,\mathrm{G}}(x), W_2^{0,\mathrm{G}}(x,x)$,

$$\alpha\frac{d}{dx}W_1^{0,\mathrm{G}}(x) + \left(-\frac{d}{dx} - x\right)W_1^{1,\mathrm{G}}(x) + 2\alpha W_1^{0,\mathrm{G}}(x)W_1^{1,\mathrm{G}}(x) + \alpha W_2^{0,\mathrm{G}}(x,x) = 0. \tag{5.31}$$

For the appearance of Riccati equations specifying the Stieltjes transform W_1^0 of other random matrix models in the context of loop equations, see [48].

We next consider (5.26) with $n=2$. Equating terms $\mathrm{O}(N)$ gives an equation relating $W_2^{0,\mathrm{G}}(x_1,x_2)$ to $W_i^{0,\mathrm{G}}(x_i)$ $(i=1,2)$. Thus

$$\left(-\frac{\partial}{\partial x_1} - x_1\right)W_2^{0,\mathrm{G}}(x_1,x_2) + \frac{\partial}{\partial x_2}\left\{\frac{W_1^{0,\mathrm{G}}(x_1) - W_1^{0,\mathrm{G}}(x_2)}{x_1 - x_2}\right\} + 2\alpha W_1^{0,\mathrm{G}}(x_1)W_2^{0,\mathrm{G}}(x_1,x_2) = 0. \tag{5.32}$$

Note that with $W_1^{0,\mathrm{G}}(x)$ specified by (5.30), (5.32) then allows us to specify $W_2^{0,\mathrm{G}}(x_1,x_2)$. In relation to $W_2^{0,\mathrm{G}}(x,x)$ appearing in (5.31), we can first take the limit $x_1 \to x_2 = x$ in (5.32) to deduce

$$\left(-\frac{1}{2}\frac{d}{dx} - x\right)W_2^{0,\mathrm{G}}(x,x) + \frac{1}{2}\frac{d^2}{dx^2}W_1^{0,\mathrm{G}}(x) + 2\alpha W_1^{0,\mathrm{G}}(x)W_2^{0,\mathrm{G}}(x,x) = 0,$$

where in the derivation use has been made of the symmetry $W_2^{0,\mathrm{G}}(x_1,x_2) = W_2^{0,\mathrm{G}}(x_2,x_1)$. With $W_2^{0,\mathrm{G}}(x,x)$ so now specified (albeit in terms of $W_1^{0,\mathrm{G}}(x)$), substituting in (5.31) then allows for $W_1^{1,\mathrm{G}}(x)$ to be specified.

Let us now carry through this program, and in particular, quantify to what extent it is possible to specify the quantities of interest. With regards to $W_1^{0,\mathrm{G}}$, the differential equation (5.30) was first obtained in the present context in [7], having appeared much earlier in the orthogonal polynomial literature [11] where it relates to so-called associated Hermite polynomials (for a different line of work in the recent random matrix theory literature relating to associated Hermite polynomials, see [74]). It is an example of a Ricatti nonlinear equation, and as such can be linearised by setting

$$W_1^{0,\mathrm{G}}(x) = -\frac{1}{\alpha}\frac{d}{dx}\log u(x), \qquad u(x) \underset{|x|\to\infty}{\sim} \frac{A_1}{x^\alpha}, \tag{5.33}$$

for some constant A_1. This substitution gives the second order linear equation for $u(x)$,

$$u'' + xu' + \alpha u = 0. \tag{5.34}$$

The solution of (5.34) satisfying the asymptotic condition in (5.33) is [7] (see also [168])

$$u(x) = A_2 e^{-x^2/4} D_{-\alpha}(ix),$$

where $D_{-\alpha}(z)$ is the so-called parabolic cylinder function with integral representation

$$D_{-\alpha}(z) = \frac{e^{-z^2/4}}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1} e^{-zt-t^2/2}\,dt. \tag{5.35}$$

Substituting in (5.33) it follows

$$W_1^{0,\mathrm{G}}(x) = \frac{x}{2\alpha} - \frac{1}{\alpha}\frac{d}{dx}\log D_{-\alpha}(ix). \tag{5.36}$$

According to (5.10) at leading order in N, $W_1^{0,\mathrm{G}}(x)$ is the Stieltjes transform of $\rho^{\mathrm{G}}_{(1),0}(x;c)$. The inversion formula (5.11), with $W_1^{0,\mathrm{G}}(x)$ given by (5.36), implies [7] the explicit form of the density (5.8), or equivalently, upon recalling (5.35)

$$\rho^{\mathrm{G}}_{(1),0}(x;\alpha) = \frac{1}{\sqrt{2\pi}\Gamma(1+\alpha)}\frac{1}{|D_{-\alpha}(ix)|^2}.$$

Remark 5.4. *Suppose in* (5.30) *we scale* $x \mapsto \sqrt{\alpha}y$ *and* $W_1^{0,\mathrm{G}}(x) \mapsto \frac{1}{\sqrt{\alpha}}W_1^{0,\mathrm{G}}(y)$. *Then for large* α (5.30) *reduces to the quadratic equation*

$$-yW_1^{0,\mathrm{G}}(y) + 1 + (W_1^{0,\mathrm{G}}(y))^2 = 0,$$

with solution obeying $W_1^{0,\mathrm{G}}(y) \sim 1/y$ *as* $y \to \infty$

$$W_1^{0,\mathrm{G}}(y) = \frac{y - (y^2-4)^{1/2}}{2}.$$

The inversion formula (5.11) *then implies*

$$\lim_{\alpha\to\infty}\sqrt{\alpha}\rho^{\mathrm{G}}_{(1),0}(\sqrt{\alpha}y;\alpha) = \frac{1}{\pi}(4-y^2)^{1/2}, \qquad |y|<2,$$

which up to scaling is the Wigner semi-circle law (5.3)*); see also [7] for a discussion of this limit.*

Knowledge of the functional form (5.36) is not itself of practical use to specify $W_2^{0,\mathrm{G}}(x_1,x_2)$ from (5.32). Instead we view (5.30) as specifying the coefficients $\{m_{p,0}^{\mathrm{G}}\}$ in the expansion about $x=\infty$ of $W_1^{0,\mathrm{G}}(x)$ (5.12)). Since the density $\rho_{(1),0}^{\mathrm{G}}(x;\alpha)$ is even in x, we see that

$$m_{p,0}^{\mathrm{G}} = 0, \qquad \text{for } p \text{ odd.} \tag{5.37}$$

Substituting in (5.30) gives the recurrence (5.13). An alternative specification of $\{m_{2p,0}^{\mathrm{G}}\}$ follows by substituting the known $x\to\infty$ expansion of $D_{-\alpha}(ix)$ [38, Eq. 12.9] in (5.36). This shows

$$\begin{aligned} W_1^{0,\mathrm{G}}(x) &= \frac{1}{x} - \frac{1}{\alpha}\frac{d}{dx}\log\Big(1+\sum_{s=1}^{\infty}\frac{(\alpha)_{2s}}{s!(2x^2)^s}\Big) \\ &= \frac{1}{x} + \frac{(1+\alpha)}{x^3} + \frac{(3+5\alpha+2\alpha^2)}{x^5} + \frac{(15+32\alpha+22\alpha^2+5\alpha^3)}{x^7} + \cdots \end{aligned} \tag{5.38}$$

and thus (extending (5.38)) to include the term $\mathrm{O}(1/x^9)$)

$$\begin{aligned} m_{2,0}^{\mathrm{G}} &= 1+\alpha \\ m_{4,0}^{\mathrm{G}} &= 3+5\alpha+2\alpha^2 \\ m_{6,0}^{\mathrm{G}} &= 15+32\alpha+22\alpha^2+5\alpha^3 \\ m_{8,0}^{\mathrm{G}} &= 105+260\alpha+234\alpha^2+93\alpha^3+14\alpha^4. \end{aligned} \tag{5.39}$$

Remark 5.5. *1. For even $p \geqslant 2$ the recurrence* (5.13) *can be rewritten*

$$m_{p+2,0}^{\mathrm{G}} = (p+1+2\alpha)m_{p,0}^{\mathrm{G}} + \alpha\sum_{s=1}^{p/2-1} m_{p-2s,0}^{\mathrm{G}} m_{2s,0}^{\mathrm{G}}. \tag{5.40}$$

Indeed for even $p \geqslant 2$ we can rewrite (5.13) *as*

$$\begin{aligned} m_{p+2,0}^{\mathrm{G}} &= (p+1)m_{p,0}^{\mathrm{G}} + \alpha\sum_{s=0}^{p/2} m_{p-2s,0}^{\mathrm{G}} m_{2s,0}^{\mathrm{G}} \\ &= (p+1)m_{p,0}^{\mathrm{G}} + 2\alpha m_{p,0}^{\mathrm{G}} m_{0,0}^{\mathrm{G}} + \alpha\sum_{s=1}^{p/2-1} m_{p-2s,0}^{\mathrm{G}} m_{2s,0}^{\mathrm{G}} \\ &= (p+1+2\alpha)m_{p,0}^{\mathrm{G}} + \alpha\sum_{s=1}^{p/2-1} m_{p-2s,0}^{\mathrm{G}} m_{2s,0}^{\mathrm{G}}, \end{aligned}$$

where in the last equality we used that $m_{0,0}^{\mathrm{G}} = 1$. According to (5.39) *$m_{2,0}^{\mathrm{G}} = (1+\alpha)$, so it follows from* (5.40) *that*

$$m_{2n,0}^{\mathrm{G}} = (1+\alpha)\tilde{m}_{n,0}^{\mathrm{G}}, \qquad n \geqslant 1,$$

where $\tilde{m}_{n,0}^{\mathrm{G}}$ is a polynomial in α of degree $n-1$ satisfying the recurrence

$$\tilde{m}_{n+1,0}^{\mathrm{G}} = (2n+1+2\alpha)\tilde{m}_{n,0}^{\mathrm{G}} + \alpha(1+\alpha)\sum_{s=1}^{n-1}\tilde{m}_{n-s,0}^{\mathrm{G}}\tilde{m}_{s,0}^{\mathrm{G}}, \qquad \tilde{m}_{1,0}^{\mathrm{G}} = 1, \ (n\in\mathbb{N}).$$

For combinatorial interpretations, see [40].

2. As is well known in the theory of the Selberg integral (see [54, Chapter 4.1]) the PDF (5.1) *specifying the Gaussian β-ensemble is well defined for $\beta > -2/N$, implying that the scaling* (5.7) *is well defined for $\alpha > -1$ as stated; see also [7, 8]. In particular this implies all the moments are non-negative for $\alpha > -1$. From point 1. above, we see that exactly at $\alpha = -1$ all the moments $\tilde{m}^{\rm G}_{2p,0}$ vanish for $p \geqslant 1$.*

3. For the ensemble $ME_{\beta,N}(e^{-x^2/2})$ the moments $m^{(G)}_{2k}$ are polynomials in N of degree $(k+1)$. We know from [42, 123, 166] the explicit forms

$$\begin{aligned} m_2^{(G)} &= \kappa\Big(N^2 + N(-1+\kappa^{-1})\Big) \\ m_4^{(G)} &= \kappa^2\Big(2N^3 + 5N^2(-1+\kappa^{-1}) + N(3 - 5\kappa^{-1} + 3\kappa^{-2})\Big) \\ m_6^{(G)} &= \kappa^3\Big(5N^4 + 22N^3(-1+\kappa^{-1}) + N^2(32 - 54\kappa^{-1} + 32\kappa^{-2}) \\ &\qquad + N(-15 + 32\kappa^{-1} - 32\kappa^{-2} + 15\kappa^{-3})\Big), \end{aligned} \tag{5.41}$$

where $\kappa := \beta/2$; in fact [166] gives the explicit form of all moments up to and including $m^{(G)}_{20}$. It follows from (5.41) *that*

$$\begin{aligned} m_2^{(G)}\Big|_{\kappa=\alpha/N} &= (1+\alpha)N - \alpha \\ m_4^{(G)}\Big|_{\kappa=\alpha/N} &= (3+5\alpha+\alpha^2)N - 5\alpha(1+\alpha) + \frac{3\alpha^2}{N} \\ m_6^{(G)}\Big|_{\kappa=\alpha/N} &= (15+32\alpha+22\alpha^2+5\alpha^3)N - 2\alpha(16+27\alpha+11\alpha^2) + \frac{32\alpha^2(1+\alpha)}{N} - \frac{15\alpha^3}{N^2}. \end{aligned} \tag{5.42}$$

We see that the polynomials in α multiplied by N in these expansions agree with the leading moments in the scaling limit with β specified by (5.7) *as displayed in* (5.39).

To see the utility of (5.12) in relation to the equation (5.32) relating $W_2^{0,\rm G}(x_1,x_2)$ to $W_1^{0,\rm G}(x)$, analogous to (5.12) introduce the coefficients $\{\mu^{\rm G}_{(p,q),0}\}$ in the expansion about $x_1, x_2 = \infty$ (5.14). We remark that the reasoning behind the formula in (5.14) expressing $\mu^{\rm G}_{(p,q),0}$ in terms of the covariance is to first note from (5.21) with $n = 2$, and the second equation in (5.23), that

$$\overline{W}_2^{\rm G}(x_1,x_2) = \mathbf{E}\Big[\Big(A(x_1) - \mathbf{E}\left[A(x_1)\right]\Big)\Big(A(x_2) - \mathbf{E}\left[A(x_2)\right]\Big)\Big]^{\rm G} =: \mathrm{Cov}\left(A(x_1), A(x_2)\right)^{\rm G}$$

where $A(x) = \sum_{j=1}^N 1/(x-\lambda_j)$. Expanding about $x_1, x_2 = \infty$ and taking the limit $N \to \infty$ with β specified by (5.7) gives (5.14).

The definition in (5.14) implies the symmetry property

$$\mu^{\rm G}_{(p,q),0} = \mu^{\rm G}_{(q,p),0}. \tag{5.43}$$

It is also immediate that

$$\mu^{\rm G}_{(0,q),0} = \mu^{\rm G}_{(p,0),0} = 0. \tag{5.44}$$

In addition, the symmetry of the PDF (5.1) under the mapping $\lambda_l \mapsto -\lambda_l$ ($l = 1, \dots, N$) implies

$$\mu^{\rm G}_{(p,q),0} = 0, \qquad \text{for } p, q \text{ of different parity}.$$

We substitute both (5.14) and expansion of (5.12) in (5.32). After straightforward manipulation, this shows

$$\frac{1}{x_1^2}\sum_{p,q=0}^{\infty}(p+1)\frac{\mu^{\mathrm{G}}_{(p,q),0}}{x_1^p x_2^q}-\sum_{p,q=0}^{\infty}\frac{\mu^{\mathrm{G}}_{(p,q),0}}{x_1^p x_2^q}+\frac{1}{x_1x_2}\sum_{p=0}^{\infty}m^{\mathrm{G}}_{p,0}\sum_{s=0}^{p}\frac{(s+1)}{x_2^s x_1^{p-s}}$$
$$+\frac{2\alpha}{x_1^2}\sum_{q=0}^{\infty}\frac{1}{x_2^q}\sum_{k=0}^{\infty}\frac{1}{x_1^k}\sum_{s=0}^{k}m^{\mathrm{G}}_{s,0}\,\mu^{\mathrm{G}}_{(k-s,q),0}=0. \quad (5.45)$$

Taking into consideration the vanishing properties (5.37) and (5.44), we can further manipulate (5.45)) to read

$$\sum_{p=3,q=1}^{\infty}(p-1)\frac{\mu^{\mathrm{G}}_{(p-2,q),0}}{x_1^{p-1}x_2^{q-1}}-\sum_{p,q=0}^{\infty}\frac{\mu^{\mathrm{G}}_{(p,q),0}}{x_1^{p-1}x_2^{q-1}}+\sum_{r=0}^{\infty}m^{\mathrm{G}}_{2r,0}\sum_{s=0}^{2r}\frac{(s+1)}{x_2^s x_1^{2r-s}}$$
$$+2\alpha\sum_{q=1}^{\infty}\frac{1}{x_2^{q-1}}\sum_{k=2}^{\infty}\frac{1}{x_1^{k-1}}\sum_{s=0}^{\lfloor k/2-1\rfloor}m^{\mathrm{G}}_{2s,0}\,\mu^{\mathrm{G}}_{(k-2-2s,q),0}=0.$$

Equating coefficients of $(x_1^{-p+1}x_2^{-q+1})$ throughout gives the recurrence (5.15).

Corollary 5.6. *Let* $\{m^{\mathrm{G}}_{p,0}\}$ *be specified by* (5.37) *and* (5.13). *For* $q\in\mathbb{Z}^+$ *we have*

$$\begin{aligned}\mu^{\mathrm{G}}_{(1,q),0}&=qm^{\mathrm{G}}_{q-1,0}\\ \mu^{\mathrm{G}}_{(2,q),0}&=qm^{\mathrm{G}}_{q,0}\\ \mu^{\mathrm{G}}_{(3,q),0}&=2(1+\alpha)qm^{\mathrm{G}}_{q-1,0}+qm^{\mathrm{G}}_{q+1,0}\\ \mu^{\mathrm{G}}_{(4,q),0}&=(3+2\alpha)qm^{\mathrm{G}}_{q,0}+qm^{\mathrm{G}}_{q+2,0}.\end{aligned} \quad (5.46)$$

Remark 5.7. *1. The symmetry* (5.43) *is not apparent in* (5.15), *and thus not in* (5.46) *either. Nonetheless, on a case-by-case basis, the evaluations* (5.48) *can be checked to be consistent with* (5.43). *As an example, for* $\mu^{\mathrm{G}}_{(2,4),0}=\mu^{\mathrm{G}}_{(4,2),0}$, *the equality of the corresponding expressions in* (5.46) *requires* $m^{\mathrm{G}}_{4,0}=(3+2\alpha)m^{\mathrm{G}}_{2,0}$ *which from* (5.39) *is seen to hold true.*
2. The covariances $\{\mu^{\mathrm{G}}_{(p,q),0}\}$ *have been studied in a recent work of Spohn [152], where they were specified by a certain matrix equation with entries permitting a recursive evaluation. In fact the entry* (p,q) $(0\leqslant p\leqslant q)$ *of the matrix equation can be checked to be equivalent to the recurrence* (5.15).

We now turn our attention to the relation (5.32). Setting $x_1=x_2=x$ in (5.14) gives the expansion about $x=\infty$ (5.17). Substituting this and (5.12) in (5.32) gives the recurrence for $\{\tilde{\mu}^{\mathrm{G}}_{2p,0}\}$

$$\tilde{\mu}^{\mathrm{G}}_{2p+2,0}=(p+1)\tilde{\mu}^{\mathrm{G}}_{2p,0}+(2p+1)(p+1)m^{\mathrm{G}}_{2p,0}+2\alpha\sum_{l=1}^{p}\tilde{\mu}^{\mathrm{G}}_{2l,0}m^{\mathrm{G}}_{2(p-l),0}, \qquad \tilde{\mu}^{\mathrm{G}}_{0,0}=0, \quad (5.47)$$

valid for $p=0,1,2,\dots$. Here $\{m^{\mathrm{G}}_{2p}\}$ are input, having been determined by (5.13). The first three non-zero values implied by (5.17) are

$$\begin{aligned}\tilde{\mu}^{\mathrm{G}}_{2,0}&=1\\ \tilde{\mu}^{\mathrm{G}}_{4,0}&=8(\alpha+1)\\ \tilde{\mu}^{\mathrm{G}}_{6,0}&=3(\alpha+1)(23+16\alpha).\end{aligned} \quad (5.48)$$

Each can be checked to be consistent with the relationship between $\{\tilde{\mu}^{\rm G}_{2p,0}\}$ and $\{\mu^{\rm G}_{(p_1,q_1),0}\}$ as specified in (5.17).

With knowledge of both $\{W_1^0(x)\}$ as determined by (5.12), (5.37), (5.13) and $\{W_2^{0,{\rm G}}(x,x)\}$ as determined by (5.17)), (5.47)), introducing the expansion (5.16) the equation (5.31) can be used to deduce a recurrence specifying $\{m^{\rm G}_{2p,0}\}$, which is (5.18) in Proposition 5.3 above. Iterating shows

$$\begin{aligned} m^{\rm G}_{2,1} &= -\alpha \\ m^{\rm G}_{4,1} &= -5\alpha(\alpha+1) \\ m^{\rm G}_{6,1} &= -2\alpha(16+27\alpha+11\alpha^2), \end{aligned}$$

which we see are all in agreement with the term independent of N in the expansions (5.42).

5.3 Solving the loop equations at low order with $\beta = 2\alpha/N$ — the Laguerre β-ensemble

For the Laguerre β-ensemble ${\rm ME}_{\beta,N}(x^{\alpha_1}e^{-x})$, $\alpha_1 > -1$, let the moments of the spectral density be denoted $m_j^{(L)}$. Analogous to (5.41), each $m_j^{(L)}$ is a polynomial of degree j in N and κ (and also in α_1). A listing of $\{\tilde{m}_j^{(L)}\}_{j=1,2,3}$ is given in [63, Prop. 3.11] (see also [121]), where

$$\tilde{m}_j^{(L)} = (N\kappa)^{-j} m_j^{(L)}.$$

We read off that

$$\begin{aligned} \alpha\tilde{m}_1^{(L)}\Big|_{\kappa=\alpha/N} &= (1+\alpha+\alpha_1) - \frac{\alpha}{N} \\ \alpha^2\tilde{m}_2^{(L)}\Big|_{\kappa=\alpha/N} &= \Big((2+3\alpha_1+\alpha_1^2) + \alpha(4+3\alpha_1) + 2\alpha^2\Big) - \frac{\alpha}{N}\Big((4+3\alpha_1)+4\alpha\Big) + {\rm O}\Big(\frac{1}{N^2}\Big) \\ \alpha^3\tilde{m}_3^{(L)}\Big|_{\kappa=\alpha/N} &= \Big((6+11\alpha_1+6\alpha_1^2+\alpha_1^3) + \alpha(17+21\alpha_1+6\alpha_1^2) + \alpha^2(16+10\alpha_1) + 5\alpha^3\Big) \\ &\quad - \frac{\alpha}{N}\Big((17+21\alpha_1+6\alpha_1^2) + \alpha(33+21\alpha_1) + 16\alpha^2\Big) + {\rm O}\Big(\frac{1}{N^2}\Big). \end{aligned} \tag{5.49}$$

Note that here, in distinction to the case of fixed N,β, after taking the scaling limit with $\beta = 2c/N$ setting $\alpha_1 = -1$ is now well defined. This is of importance for our application of the final section.

In view of this expansion we hypothesise that the functions $\overline{W}_n$ again exhibit the N^{-1} expansion (5.29). We begin by enforcing this expansion in the loop equation (5.27) with $n = 1$. Equating terms ${\rm O}(N)$ we read off the equation for $W_1^0 = W_1^{0,{\rm L}}$ (the use of the superscript "L" is to indicate the Laguerre ensemble in the scaling limit with $\beta = 2c/N$),

$$-\frac{d}{dx}W_1^{0,{\rm L}}(x) + \Big(\frac{\alpha_1}{x} - 1\Big) W_1^{0,{\rm L}}(x) + \frac{1}{x} + \alpha\Big(W_1^{0,{\rm L}}(x)\Big)^2 = 0. \tag{5.50}$$

Equating terms ${\rm O}(1)$ gives an equation relating $W_1^{0,{\rm L}}(x), W_1^{1,{\rm L}}(x), W_2^{0,{\rm L}}(x,x)$,

$$\alpha\frac{d}{dx}W_1^{0,{\rm L}}(x) - \frac{d}{dx}W_1^{1,{\rm L}}(x) + \Big(\frac{\alpha_1}{x} - 1\Big) W_1^{1,{\rm L}}(x) + \alpha W_2^{0,{\rm L}}(x,x) + 2\alpha W_1^{0,{\rm L}}(x) W_1^{1,{\rm L}}(x) = 0. \tag{5.51}$$

We next consider the substitution of (5.29)in (5.27) with $n = 2$. Equating terms $\mathrm{O}(N)$ gives an equation relating $W_2^{0,\mathrm{L}}(x_1, x_2)$ to $W_i^{0,\mathrm{L}}(x_i)$ $(i = 1, 2)$. Thus

$$-\frac{\partial}{\partial x_1} W_2^{0,\mathrm{L}}(x_1, x_2) + \left(\frac{\alpha_1}{x_1} - 1\right) W_2^{0,\mathrm{L}}(x_1, x_2) \\ + \frac{\partial}{\partial x_2} \left\{ \frac{W_1^{0,\mathrm{L}}(x_1) - W_1^{0,\mathrm{L}}(x_2)}{x_1 - x_2} + \frac{W_1^{0,\mathrm{L}}(x_2)}{x_1} \right\} + 2\alpha W_2^{0,\mathrm{L}}(x_1, x_2) W_1^{0,\mathrm{L}}(x_1) = 0\,. \tag{5.52}$$

We notice that the differential equation (5.50) — a particular Ricatti equation — can be solved explicitly. This was studied by Allez and collaborators in [8], and also appeared earlier in the orthogonal polynomial literature in the context of associated Laguerre polynomials [103]. Analogous to (5.33), the substitution

$$W_1^{0,\mathrm{L}}(x) = -\frac{1}{\alpha} \frac{d}{dx} \log u(x), \qquad u(x) \underset{|x|\to\infty}{\sim} \frac{B_1}{x^{\alpha}}, \tag{5.53}$$

for some constant B_1, gives rise to the second order differential equation,

$$u'' + \left(1 - \frac{\alpha_1}{x}\right) u' + \frac{\alpha}{x} u = 0.$$

The required solution is given by [8, Eq. (3.41), $\alpha \mapsto -2\alpha_1$, $\zeta = \alpha + \alpha_1/2$, $\mu = (1 + \alpha_1)/2$]

$$u(x) = B_2 e^{-x/2} x^{-\alpha_1/2} W_{-\alpha-\alpha_1/2,(1+\alpha_1)/2}(-x),$$

for some constant B_2, where $W_{\zeta,\mu}(z)$ denotes the Whittaker function. Substituting in (5.53) shows

$$W_1^{0,\mathrm{L}}(x) = \frac{1}{2\alpha} + \frac{\alpha_1}{2\alpha x} - \frac{1}{\alpha} \frac{d}{dx} \log W_{-\alpha-\alpha_1/2,(1+\alpha_1)/2}(-x), \tag{5.54}$$

and this, upon substituting the known large x form of the Whittaker function [38, Eq. 13.19] implies

$$\begin{aligned} W_1^{0,\mathrm{L}}(x) &= \frac{1}{x} - \frac{1}{\alpha} \frac{d}{dx} \log \Big(1 + \sum_{s=1}^{\infty} \frac{(\alpha)_s (1 + \alpha_1 + \alpha)_s}{s!} \frac{1}{x^s}\Big) \\ &= \frac{1}{x} + \frac{(1 + \alpha_1 + \alpha)}{x^2} + \frac{(1 + \alpha_1 + \alpha)(2\alpha + 2 + \alpha_1)}{x^3} \\ &\quad + \frac{(1 + \alpha_1 + \alpha)(6 + 11\alpha + 5\alpha^2 + 5(1 + \alpha)\alpha_1 + \alpha_1^2)}{x^4} + \cdots \end{aligned} \tag{5.55}$$

Analogous to (5.12) $W_1^{0,\mathrm{L}}(x)$ is the moment generating function of the corresponding Laguerre α-ensemble density,

$$W_1^{0,\mathrm{L}}(x) = \frac{1}{x} \sum_{p=0}^{\infty} \frac{m_{p,0}^{\mathrm{L}}}{x^p}, \qquad m_{p,0}^{\mathrm{L}} = \int_0^{\infty} x^p \rho_{(1),0}^{\mathrm{L}}(x; \alpha_1, \alpha)\, dx. \tag{5.56}$$

We thus read off from (5.55) that

$$\begin{aligned} m_{1,0}^{\mathrm{L}} &= (1 + \alpha_1 + \alpha) \\ m_{2,0}^{\mathrm{L}} &= (1 + \alpha_1 + \alpha)(2\alpha + 2 + \alpha_1) \\ m_{3,0}^{\mathrm{L}} &= (1 + \alpha_1 + \alpha)(6 + 11\alpha + 5\alpha^2 + 5(1 + \alpha)\alpha_1 + \alpha_1^2). \end{aligned} \tag{5.57}$$

These are all in agreement with the leading terms (in N) on the RHS of (5.49). Furthermore, by substituting the expansion (5.56) in the differential equation (5.50) we see $\{m^{\rm L}_{p,0}\}$ satisfies the recurrence

$$m^{\rm L}_{p+1,0} = (p+1+\alpha_1+\alpha)m^{\rm L}_{p,0} + \alpha\sum_{s=0}^{p-1} m^{\rm L}_{s,0}m^{\rm L}_{p-s,0}, \qquad m^{\rm L}_{0,0} = 1.$$

An immediate corollary is that $m^{\rm L}_{p,0}$ is a polynomial of degree p in both α_1, α, as seen in the tabulation (5.57) for the low order cases. Moreover, analogous to point 1. of Remark 5.5, writing

$$m^{\rm L}_{p,0} = (1+\alpha_1+\alpha)\tilde{m}^{\rm L}_{p,0}, \qquad p \geqslant 1$$

we see that $\tilde{m}^{\rm L}_{p,0}$ is a polynomial of degree $p-1$ in both α_1, α, satisfying the recurrence

$$\tilde{m}^{\rm L}_{p+1,0} = (p+1+\alpha_1+2\alpha)\tilde{m}^{\rm L}_{p,0} + \alpha(1+\alpha_1+\alpha)\sum_{s=1}^{p-1} \tilde{m}^{\rm L}_{s,0}\tilde{m}^{\rm L}_{p-s,0}, \qquad \tilde{m}^{\rm L}_{1,0} = 1. \tag{5.58}$$

The density $\rho^{\rm L}_{(1),0}(x;\alpha_1,\alpha)$ in (5.56) can be deduced from knowledge of $W_1^{0,{\rm L}}$ as specified by (5.54), together with the analogue of the inversion formula (5.11). One finds [8, Eq. (3.49), $\lambda \mapsto 2x$, $\zeta = \alpha + \alpha_1/2$, $\mu = (1+\alpha_1)/2$]

$$\rho^{\rm L}_{(1),0}(x;\alpha_1,\alpha) = \frac{1}{\Gamma(\alpha+1)\Gamma(\alpha+\alpha_1+1)}\frac{1}{|W_{-\alpha-\alpha_1/2,(1+\alpha_1)/2}(-x)|^2}, \tag{5.59}$$

supported on $x > 0$.

In relation to (5.52), introduce the Laguerre analogue of (5.14)

$$W_2^{0,{\rm L}}(x_1,x_2) = \frac{1}{x_1x_2}\sum_{p,q=1}^{\infty}\frac{\mu^{\rm L}_{(p,q),0}}{x_1^px_2^q}, \qquad \mu^{\rm L}_{(p,q),0} = \lim_{\substack{N\to\infty\\ \beta=2\alpha/N}} {\rm Cov}\left(\sum_{i=1}^N x_i^p, \sum_{i=1}^N x_i^q\right)^{\rm L}.$$

Proceeding as in the derivation of (5.15) shows

$$\begin{aligned}\mu^{\rm L}_{(p+1,q),0} &= (p+1+\alpha_1)\mu^{\rm L}_{(p,q),0} + qm^{\rm L}_{p+q,0} + 2\alpha\sum_{s=0}^{p-1} m^{\rm L}_{s,0}\mu_{(p-s,q),0}\\ &= (p+1+\alpha_1+2\alpha)\mu^{\rm L}_{(p,q),0} + qm^{\rm L}_{p+q,0} + 2\alpha\sum_{s=1}^{p-1} m^{\rm L}_{s,0}\mu_{(p-s,q),0}.\end{aligned} \tag{5.60}$$

Corollary 5.8. *Let $\{m^{\rm L}_{p,0}\}$ be specified by* (5.56) *and* (5.58)*. For $q \in \mathbb{Z}^+$ we have*

$$\begin{aligned}\mu^{\rm L}_{(1,q),0} &= qm^{\rm L}_{q,0}\\ \mu^{\rm L}_{(2,q),0} &= (2+\alpha_1+2\alpha)qm^{\rm L}_{q,0} + qm^{\rm L}_{q+1,0}\\ \mu^{\rm L}_{(3,q),0} &= (3+\alpha_1+2\alpha)\mu^{\rm L}_{(2,q)} + 2\alpha(1+\alpha_1+2\alpha)qm^{\rm L}_{q,0} + qm^{\rm L}_{q+2,0}.\end{aligned} \tag{5.61}$$

Remark 5.9. *As with $\{\mu^{\rm G}_{(p,q),0}\}$, the recurrence* (5.60) *is not symmetric upon the interchange $p \leftrightarrow q$, yet from the definition $\mu^{\rm L}_{(p,q),0}$ has this symmetry. As observed in Remark 5.7 in the Gaussian case, on a case-by-case basis this symmetry can be checked from the explicit forms, in particular those in Corollary 5.8 combined with the tabulation* (5.57)*.*

The equation (5.51) for $W_1^{1,\mathrm{L}}$ requires knowledge of $W_2^{0,\mathrm{L}}(x,x)$. In regards to this quantity, letting $x_1 \to x_2 = x$ in (5.52) shows

$$-\frac{d}{dx}W_2^{0,\mathrm{L}}(x,x) + 2\left(\frac{\alpha_1}{x} - 1\right)W_2^{0,\mathrm{L}}(x,x) + \frac{d^2}{dx^2}W_1^{0,\mathrm{L}}(x) + \frac{2}{x}\frac{d}{dx}W_1^{0,\mathrm{L}}(x) + 4\alpha W_2^{0,\mathrm{L}}(x,x)W_1^{0,\mathrm{L}}(x) = 0. \tag{5.62}$$

Introducing the expansion about $x = \infty$

$$W_2^{0,\mathrm{L}}(x,x) = \frac{1}{x^2}\sum_{p=1}^{\infty}\frac{\tilde{\mu}_{p,0}^{\mathrm{L}}}{x^p}, \qquad \tilde{\mu}_{2p,0}^{\mathrm{L}} = \sum_{p_1+q_1=p}\mu_{(p_1,q_1),0}^{\mathrm{L}} \tag{5.63}$$

(cf. (5.17)), as well as the analogous expansion for $W_1^{0,\mathrm{L}}$ from (5.56), reduces (5.62) to the recurrence

$$\tilde{\mu}_{p+1,0}^{\mathrm{L}} = \frac{1}{2}(p+2+2\alpha_1+4\alpha)\tilde{\mu}_{p,0}^{\mathrm{L}} + \frac{p(p+1)}{2}m_{p,0}^{\mathrm{L}} + 2\alpha\sum_{s=1}^{p-1}\tilde{\mu}_{s,0}^{\mathrm{L}}m_{p-s,0}^{\mathrm{L}}. \tag{5.64}$$

As done in relation to (5.47), the implied evaluations for members of $\{\tilde{\mu}_{p,0}^{\mathrm{L}}\}$ can be checked, for small p at least, to be consistent with the relationship to $\{\mu_{(p_1,q_1),0}^{\mathrm{G}}\}$, and thus the tabulation (5.61), as required by the second equation in (5.63).

With $\{m_{p,0}^{\mathrm{L}}\}$ determined by (5.55) or the recurrence (5.58), and $\{\tilde{\mu}_{p,0}^{\mathrm{L}}\}$ determined by the recurrence (5.64), by introducing the expansion

$$W_1^{1,\mathrm{L}}(x) = \frac{1}{x}\sum_{p=1}^{\infty}\frac{m_{p,1}^{\mathrm{L}}}{x^p}$$

we see from (5.51) that $\{m_{p,1}^{\mathrm{L}}\}$ can be determined by the recurrence

$$m_{p+1,1}^{\mathrm{L}} = -\alpha(p+1)m_{p,0}^{\mathrm{L}} + (p+1+\alpha_1+2\alpha)m_{p,1}^{\mathrm{L}} + \alpha\tilde{\mu}_{p,0}^{\mathrm{L}} + 2\alpha\sum_{l=1}^{p-1}\tilde{\mu}_{l,0}^{\mathrm{L}}m_{p-l,0}^{\mathrm{L}},$$

valid for $p = 1, 2, \dots$ with initial condition $m_{0,1}^{\mathrm{L}} = 0$. In particular, iteration shows

$$\begin{aligned}
m_{1,1}^{\mathrm{L}} &= -\alpha \\
m_{2,1}^{\mathrm{L}} &= -\alpha\Big((4+3\alpha_1)+4\alpha\Big) \\
m_{3,1}^{\mathrm{L}} &= -\alpha\Big((17+21\alpha_1+6\alpha_1^2)+\alpha(33+21\alpha_1)+16\alpha^2\Big),
\end{aligned}$$

which we see are all in agreement with the term independent of N exhibited in the expansions (5.49).

5.4 Solving the loop equations at low order with $\beta = 2\alpha/N$ — the Jacobi β-ensemble

For the Jacobi β-ensemble $\mathrm{ME}_{\beta,N}(x^{\alpha_1}(1-x)^{\alpha_2})$, let the moments of the spectral density be denoted $m_j^{(J)}$. In distinction to the Gaussian and Laguerre β-ensembles, the moments of the Jacobi β-ensemble spectral density are no longer polynomials in N and κ, but rather rational functions. The first two are given explicitly in [121, App. B]. From these we deduce

$$\begin{aligned}
\frac{1}{N}m_1^{(J)} &= \frac{\alpha_1+1+\alpha}{\alpha_1+\alpha_2+2+2\alpha} - \frac{1}{N}\frac{\alpha(\alpha_2-\alpha_1)}{(\alpha_1+\alpha_2+2+2\alpha)^2} + \mathrm{O}\Big(\frac{1}{N^2}\Big) \\
\frac{1}{N}m_2^{(J)} &= \frac{(1+\alpha+\alpha_1)\Big((2+\alpha_1)(2+\alpha_1+\alpha_2)+\alpha(7+3\alpha_1+2\alpha_2)+3\alpha^2\Big)}{(2+2\alpha+\alpha_1+\alpha_2)^2(3+2\alpha+\alpha_1+\alpha_2)} \\
&\quad + \frac{\alpha}{N}\frac{Q_1(\alpha_1,\alpha_2,\alpha)+Q_2(\alpha_1,\alpha_2,\alpha)}{(2+2\alpha+\alpha_1+\alpha_2)^3(3+2\alpha+\alpha_1+\alpha_2)^2} + \mathrm{O}\Big(\frac{1}{N^2}\Big),
\end{aligned} \tag{5.65}$$

where

$$Q_1(\alpha_1,\alpha_2,\alpha) = -(1+\alpha+\alpha_1)(2+2\alpha+\alpha_1+\alpha_2)(3+2\alpha+\alpha_1+\alpha_2)(9+7\alpha+4\alpha_1+2\alpha_2),$$

$$\begin{aligned}
Q_2(\alpha_1,\alpha_2,\alpha) &= \Big((2+\alpha_1)(2+\alpha_1+\alpha_2)+\alpha(7+3\alpha_1+2\alpha_2)+3\alpha^2\Big) \\
&\quad\times\Big(13+21\alpha_1+2\alpha_2+(6\alpha_1-\alpha_2)(\alpha_1+\alpha_2)+\alpha(23+17\alpha_1+3\alpha_2)+10\alpha^2\Big).
\end{aligned}$$

As in the Gaussian and Laguerre cases, to solve the Jacobi β-ensemble loop equations (5.28) in the regime $\beta = 2\alpha/N$ we will make the ansatz (5.29). We see that with $n=1$ the latter is consistent with the form of the expansions (5.65). Equating the terms of order $\mathrm{O}(N)$ gives the particular Riccati type equation for $W_1^{0,\mathrm{J}}$ (here we use the superscript "J" to indicate the Jacobi ensemble in the scaling limit with $\beta = 2c/N$)

$$-\frac{d}{dx}W_1^{0,\mathrm{J}}(x)+\left(\frac{\alpha_1}{x}-\frac{\alpha_2}{1-x}\right)W_1^{0,\mathrm{J}}(x)+\frac{1}{x(1-x)}(1+\alpha_1+\alpha_2+\alpha)+\alpha\left(W_1^{0,\mathrm{J}}(x)\right)^2=0. \tag{5.66}$$

Being a Riccati type equation, it is most natural to proceed in the analysis of (5.30) and (5.50) and perform the change of variables

$$W_1^{0,\mathrm{J}}(x) = -\frac{\psi'(x)}{\alpha\psi(x)}, \qquad \psi(x) \underset{|x|\to\infty}{\sim} \frac{C_1}{x^\alpha} \tag{5.67}$$

for some constant C_1. We see that $\psi(x)$ satisfies the second order linear differential equation

$$x(x-1)\psi''(x)+(\alpha_1-(\alpha_1+\alpha_2)x)\,\psi'(x)-\alpha\,(1+\alpha+\alpha_1+\alpha_2)\,\psi(x)=0,$$

this being a particular hypergeometric differential equation [38, Eq. 15.10]. Due to the condition (5.67) we have for the general solution

$$\psi(x) = C_1 x^{-\alpha}{}_2F_1\left(\alpha,\alpha+\alpha_1+1,2\alpha+\alpha_1+\alpha_2+2;x^{-1}\right),$$

where ${}_2F_1$ denotes the usual Gauss hypergeometric function. After some algebraic manipulations, this implies

$$\begin{aligned}
W_1^{0,\mathrm{J}}(x) &= \frac{1}{x}+\frac{\alpha+\alpha_1+1}{2\alpha+\alpha_1+\alpha_2+2}\,\frac{{}_2F_1\left(\alpha+1,\alpha+\alpha_1+2,2\alpha+\alpha_1+\alpha_2+3;x^{-1}\right)}{x^2{}_2F_1\left(\alpha,\alpha+\alpha_1+1,2\alpha+\alpha_1+\alpha_2+2;x^{-1}\right)} \\
&= -\frac{{}_2F_1(\alpha+1,\alpha+\alpha_1+1,2\alpha+\alpha_1+\alpha_2+2;1/x)}{x\,{}_2F_1(\alpha,\alpha+\alpha_1+1,2\alpha+\alpha_1+\alpha_2+2;1/x)}.
\end{aligned} \tag{5.68}$$

This latter form was given recently by Trinh and Trinh [160], using a different set of ideas stemming from the theory of associated Jacobi polynomials [165], and making no direct use of differential equations.

Since analogous to (5.12) and (5.56)

$$W_1^{0,\mathrm{J}}(x) = \frac{1}{x}\sum_{p=0}^{\infty}\frac{m_{p,0}^{\mathrm{J}}}{x^p}, \qquad m_{p,0}^{\mathrm{J}} = \int_0^1 x^p \rho_{(1),0}^{\mathrm{J}}(x;\alpha_1,\alpha_2,\alpha)\,dx, \tag{5.69}$$

we can use (5.68) (with the help of computer algebra) to compute $\{m_{p,0}^{\mathrm{J}}\}_{p=1}^{\infty}$, at least for small p. Agreement with the leading order (in N) rational functions known from (5.65) is found.

It is furthermore the case that substitution of (5.69) in (5.66) implies a recurrence for $\{m_{p,0}^{\mathrm{J}}\}_{p=1}^{\infty}$. Thus we find

$$m_{p,0}^{\mathrm{J}} = \frac{1}{p+1+\alpha_1+\alpha_2+2\alpha}\Big((1+\alpha_1+\alpha) - \alpha_2\sum_{s=1}^{p-1} m_{s,0}^{\mathrm{J}} - \alpha\sum_{s=1}^{p-1} m_{s,0}^{\mathrm{J}} m_{p-s,0}^{\mathrm{J}}\Big), \tag{5.70}$$

valid for $p = 1, 2, \dots$ and subject to the initial condition $m_{0,0}^{\mathrm{J}} = 1$. We can verify that iterating for small p ($p = 1, 2$) reproduces the leading order terms from (5.65), and is thus in agreement with (5.68).

Remark 5.10. *1. Moving the denominator in the RHS of* (5.70) *to the LHS, replacing p by $p+1$, then subtracting from the form without this latter replacement shows*

$$m_{p+1,0}^{\mathrm{J}} = \frac{1}{p+2+\alpha_1+\alpha_2+2\alpha}\Big((p+1+\alpha_1)m_{p,0}^{\mathrm{J}} - \alpha\sum_{s=1}^{p} m_{s,0}^{\mathrm{J}} m_{p+1-s,0}^{\mathrm{J}} + \alpha\sum_{s=0}^{p} m_{s,0}^{\mathrm{J}} m_{p-s,0}^{\mathrm{J}}\Big).$$

This recurrence was obtained recently in the work [160, Eq. (15)], which as in the derivation of (5.68) *in that work uses a different set of ideas.*
2. Changing variables $x = (X+1)/2$ in (5.66) *shows*

$$-\frac{d}{dX}W_1^{0,\mathrm{J}*}(X) + \left(\frac{\alpha_2}{1-X} - \frac{\alpha_2}{1+X}\right)W_1^{0,\mathrm{J}*}(X) + \frac{1}{1-X^2}(1+\alpha_1+\alpha_2+\alpha) + \alpha\left(W_1^{0,\mathrm{J}*}(X)\right)^2 = 0, \tag{5.71}$$

where

$$W_1^{0,\mathrm{J}*}(X) = \int_{-1}^{1}\frac{\rho_{(1),0}^{\mathrm{J}*}(Y;\alpha_1,\alpha_2,\alpha)}{X-Y}\,dY.$$

Here $\rho_{(1),0}^{\mathrm{J}*}$ denotes the density for the Jacobi β-ensemble with high temperature scaling (5.7) *relating to the weight $(1-X)^{\alpha_1}(1+X)^{\alpha_2}$ supported on $(-1,1)$. In the case $\alpha_1 = \alpha_2 = a$ (symmetric Jacobi weight) the corresponding moments, as for the Gaussian ensemble, must vanish for p odd. It follows from* (5.71) *that the even moments $\{m_{2p,0}^{\mathrm{J}*}\}$ satisfy the recurrence*

$$m_{2p,0}^{\mathrm{J}*} = \frac{1}{2p+2\alpha+2a+1}\Big((1+\alpha) - 2a\sum_{s=1}^{p-1} m_{2s,0}^{\mathrm{J}*} - \alpha\sum_{s=1}^{p-1} m_{2s,0}^{\mathrm{J}*} m_{2(p-s),0}^{\mathrm{J}*}\Big), \tag{5.72}$$

valid for $p = 1, 2, \dots$ *with initial condition* $m^{\mathrm{J}*}_{0,0} = 1$. *Moments for the symmetric Jacobi* β*-ensembles, with* $\beta = 1, 2$ *or 4 have been the subject of the recent work [62]. In fact a number of recent works in random matrix theory have identified recurrences for moments and also distribution functions; see e.g. [12, 33, 58–60, 70, 71, 97, 138].*

Manipulation of (5.72) *as in the derivation of* (5.70) *shows*

$$m^{\mathrm{J}*}_{2(p+1),0} = \frac{1}{2p+2\alpha+2a+3}\Big((2p+1)m^{\mathrm{J}*}_{2p,0} - \alpha\sum_{s=1}^{p} m^{\mathrm{J}*}_{2s,0}m^{\mathrm{J}*}_{2(p+1-s),0} + \alpha\sum_{s=0}^{p} m^{\mathrm{J}*}_{2s,0}m^{\mathrm{J}*}_{2(p-s),0}\Big). \tag{5.73}$$

Scaling $x \mapsto x/\sqrt{2a}$ *in the definition of the symmetric moment* $m^{\mathrm{J}*}_{2k,0}$ *shows*

$$m^{\mathrm{J}*}_{2k,0} = \frac{1}{(2a)^{k+1/2}}\int_{-\sqrt{a}}^{\sqrt{a}} x^{2k}\rho^{\mathrm{J}*}_{(1),0}(x/\sqrt{2a}; a, \alpha)\, dx \sim \frac{1}{(2a)^{k+1/2}} m^{\mathrm{G}}_{2k,0},$$

where the asymptotic relation follows from the elementary limit $(1 - x^2/2a)^a \to e^{-x^2}$ *as* $x \to \infty$. *Using this to equate leading order terms in* (5.73) *reclaims* (5.13).

The work [160, Th. A.4] also contains an explicit formula for the density $\rho^{\mathrm{J}}_{(1),0}$. This is derived not from (5.67) and an inversion formula analogous to (5.11), but rather by using theory relating to the asymptotic of associated Jacobi polynomials [88] and general relations between tridiagonal matrices and orthogonal polynomials [127]. With

$$U(x) = \frac{\Gamma(\alpha+1)\Gamma(\alpha_1+1)}{\Gamma(1+\alpha+\alpha_1)}\, {}_2F_1(\alpha, -\alpha-\alpha_1-\alpha_2-1, -\alpha_1; x),$$
$$V(x) = \frac{-\pi\alpha\Gamma(\alpha+\alpha_1+\alpha_2+2)}{\sin(\pi\alpha_1)\Gamma(1+\alpha+\alpha_2)\Gamma(2+\alpha_1)}(1-x)^{1+\alpha_2}x^{1+\alpha_1}\, {}_2F_1(1-\alpha, 2+\alpha+\alpha_1+\alpha_2, 2+\alpha_1; x),$$

we read off from [160] that

$$\rho^{\mathrm{J}}_{(1),0}(x; \alpha_1, \alpha_2, \alpha) = \frac{\Gamma(\alpha+1)\Gamma(\alpha+\alpha_1+\alpha_2+2)}{\Gamma(\alpha+\alpha_1+1)\Gamma(\alpha+\alpha_2+1)}\frac{x^{\alpha_1}(1-x)^{\alpha_2}}{|U(x)+e^{\pi i\alpha_1}V(x)|^2}, \tag{5.74}$$

supported on $0 < x < 1$.

Knowledge of (5.67) and (5.68), together with the inversion formula

$$\rho^{\mathrm{J}}_{(1),0}(x; \alpha_1, \alpha_2, \alpha) = \lim_{\epsilon\to 0^+}\frac{1}{\pi}\mathrm{Im}\, W^{0,\mathrm{J}}_1(x - i\epsilon), \tag{5.75}$$

can in fact be used to derive (5.74). The starting point is to make use of the connection formula [38, Eq. 15.10(ii)]

$$e^{\pi i\alpha}\psi(x) = U(x) + e^{-\pi i\alpha_1}V(x).$$

Substituting in (5.67), then substituting the result in (5.75) shows

$$\rho^{\mathrm{J}}_{(1),0}(x; \alpha_1, \alpha_2, \alpha) = C_2\frac{u'(x)v(x) - v'(x)u(x)}{|U(x)+e^{-\pi i\alpha_1}V(x)|^2} \tag{5.76}$$

where

$$C_2 = -\frac{1}{(\alpha_1+1)}\frac{\Gamma(\alpha+1)\Gamma(\alpha+\alpha_1+\alpha_2+2)}{\Gamma(1+\alpha+\alpha_1)\Gamma(1+\alpha+\alpha_2)}$$

and

$$u(x) = {}_2F_1(a,b,c;x), \qquad v(x) = x^{1-c}\,{}_2F_1(a-c+1,b-c+1,2-c;x)$$

with

$$a=\alpha, \qquad b=-(\alpha+\alpha_1+\alpha_2+1), \qquad c=-\alpha_1. \tag{5.77}$$

Here $u(x), v(x)$ satisfies the same hypergeometric differential equation. We can use this to show

$$u'(x)v(x)-v'(x)u(x) = (c-1)x^a(1-x)^{c-a-b-1}. \tag{5.78}$$

Substituting (5.78) with parameters given by (5.77) in (5.76) we reclaim (5.74).

Remark 5.11. *From the relationship between the Jacobi and Laguerre weights we must have*

$$\lim_{\alpha_2\to\infty}\frac{1}{\alpha_2}\rho^{\mathrm{J}}_{(1),0}(x/\alpha_2;\alpha_1,\alpha_2,\alpha) = \rho^{\mathrm{L}}_{(1),0}(x;\alpha_1,\alpha).$$

Starting from (5.74)*, and upon making sue of standard asymptotics for the gamma function and the hypergeometric function confluent limit formula*

$$\lim_{b\to\infty}{}_2F_1(a,b,c;x/b) = {}_1F_1(a,c;x)$$

we see that

$$\lim_{\alpha_2\to\infty}\frac{1}{\alpha_2}\rho^{\mathrm{J}}_{(1),0}(x/\alpha_2;\alpha_1,\alpha_2,\alpha) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\alpha_1+1)}\frac{x^{\alpha_1}e^{-x}}{|\tilde U(x)+e^{\pi i\alpha_1}\tilde V(x)|^2}, \tag{5.79}$$

where

$$\begin{aligned}\tilde U(x) &= \frac{\Gamma(\alpha+1)\Gamma(\alpha_1+1)}{\Gamma(1+\alpha+\alpha_1)}\,{}_1F_1(\alpha,-\alpha_1;-x)\\ \tilde V(x) &= -\frac{\pi\alpha}{\sin(\pi\alpha_1)\Gamma(2+\alpha_1)}x^{1+\alpha_1}e^{-x}\,{}_1F_1(1-\alpha,2+\alpha_1;x).\end{aligned}$$

We see that (5.79) *is consistent with* (5.59) *if it is true*

$$\frac{1}{\Gamma(\alpha+1)}|\tilde U(x)+e^{\pi i\alpha_1}\tilde V(x)| = x^{\alpha_1/2}e^{-x/2}|W_{-\alpha-\alpha_1/2,(1+\alpha_1)/2}(-x)|$$

By writing the Whittaker function in terms of the Tricomi hypergeometric function, then writing the latter in terms of the confluent hypergeometric function (see [159, below Lemma 2.1]) this is indeed seen to be valid.

Coming back to the loop equation for the Jacobi ensemble, applying (5.28) with $n=2$ and equating the terms of $\mathrm{O}(N)$ we get a partial differential equation for $W_2^{0,J}$ in terms of $W_1^{0,J}$,

$$\begin{aligned}&-\frac{\partial}{\partial x_1}W_2^{0,\mathrm{J}}(x_1,x_2)+\left(\frac{\alpha_1}{x_1}-\frac{\alpha_2}{1-x_1}\right)W_2^{0,\mathrm{J}}(x_1,x_2)-\frac{1}{x_1(1-x_1)}\left(1+x_2\frac{\partial}{\partial x_2}\right)W_1^{0,\mathrm{J}}(x_2)\\ &+\frac{\partial}{\partial x_2}\left\{\frac{W_1^{0,\mathrm{J}}(x_1)-W_1^{0,\mathrm{J}}(x_2)}{x_1-x_2}+\frac{W_1^{0,\mathrm{J}}(x_2)}{x_1}\right\}+2\alpha W_2^{0,\mathrm{J}}(x_1,x_2)W_1^{0,\mathrm{J}}(x_1)=0\,.\end{aligned} \tag{5.80}$$

Introducing

$$W_2^{0,\mathrm{J}}(x_1,x_2)=\frac{1}{x_1x_2}\sum_{p,q=1}^{\infty}\frac{\mu^{\mathrm{J}}_{(p,q),0}}{x_1^p x_2^q},\qquad \mu^{\mathrm{J}}_{(p,q),0}=\lim_{\substack{N\to\infty\\ \beta=2\alpha/N}}\mathrm{Cov}\left(\sum_{i=1}^N x_i^p,\sum_{i=1}^N x_i^q\right)^{\mathrm{J}}.$$

Proceeding as in the derivation of (5.15) and (5.60) shows

$$\mu^{\mathrm{J}}_{(p,q),0}=\frac{1}{(p+\alpha_1+\alpha_2+2\alpha+1)}\Big(q(m^{\mathrm{J}}_{q,0}-m^{\mathrm{J}}_{p+q,0})-\alpha_2\sum_{s=1}^{p-1}\mu^{\mathrm{J}}_{(s,q),0}-2\alpha\sum_{s=1}^{p-1}m^{\mathrm{J}}_{s,0}\mu^{\mathrm{J}}_{(p-s,q),0}\Big). \tag{5.81}$$

Note that this is consistent with the requirement that $\mu^{\mathrm{J}}_{(0,q),0}=\mu^{\mathrm{J}}_{(p,0),0}=0$. Beyond this, the simplest case is $p=1$ which gives

$$\mu^{\mathrm{J}}_{(1,q),0}=\frac{q}{(\alpha_1+\alpha_2+2\alpha+2)}(m^{\mathrm{J}}_{q,0}-m^{\mathrm{J}}_{p+1,0}).$$

Thus, for example, making use of knowledge of $m^{\mathrm{J}}_{1,0}$ and $m^{\mathrm{J}}_{2,0}$ as implied by (5.70), or as can be read off from (5.65), we have

$$\mu^{\mathrm{J}}_{(1,1),0}=\frac{(1+\alpha+\alpha_1)(1+\alpha+\alpha_2)(2+\alpha+\alpha_1+\alpha_2)}{(2+2\alpha+\alpha_1+\alpha_2)^3(3+2\alpha+\alpha_1+\alpha_2)}.$$

We remark that iterating (5.81) with the help of computer algebra, we can check the required symmetry $\mu^{\mathrm{J}}_{(p,q),0}=\mu^{\mathrm{J}}_{(q,p),0}$ in low order cases.

Finally, we return to the Jacobi ensemble loop equation (5.28) in the case $n=1$. With $\beta=2\alpha/N$ and the ansatz corresponding to (5.29), equating the terms of order O(1) gives the equation relating $W_1^{0,\mathrm{J}},W_1^{1,\mathrm{J}},W_2^{0,\mathrm{J}}$ (the latter at coincident points)

$$\begin{aligned}\alpha\frac{d}{dx}W_1^{0,\mathrm{J}}(x)-\frac{d}{dx}W_1^{1,\mathrm{J}}(x_1)+\left(\frac{\alpha_1}{x}-\frac{\alpha_2}{1-x}\right)W_1^{1,\mathrm{J}}(x)-\frac{\alpha}{x(1-x)}\\ +\alpha W_2^{0,\mathrm{J}}(x,x)+2\alpha W_1^{0,\mathrm{J}}(x)W_1^{1,\mathrm{J}}(x)=0.\end{aligned} \tag{5.82}$$

In relation to $W_2^{0,J}(x,x)$ herein, letting $x_1\to x_2$ and redefining $x_2=x$ in (5.80) shows

$$\begin{aligned}-\frac{1}{2}\frac{d}{dx}W_2^{0,\mathrm{J}}(x,x)+\left(\frac{\alpha_1}{x}-\frac{\alpha_2}{1-x}\right)W_2^{0,\mathrm{J}}(x,x)-\frac{W_1^{0,\mathrm{J}}(x)}{x(1-x)}-\frac{1}{1-x}\frac{d}{dx}W_1^{0,\mathrm{J}}(x)\\ +\frac{1}{2}\frac{d^2}{dx^2}W_1^{0,\mathrm{J}}(x)+\frac{1}{x}\frac{d}{dx}W_1^{0,\mathrm{J}}(x)+2\alpha W_2^{0,\mathrm{J}}(x,x)W_1^{0,\mathrm{J}}(x)=0\,.\end{aligned} \tag{5.83}$$

As with (5.17) and (5.63), introducing the expansion about $x=\infty$

$$W_2^{0,\mathrm{J}}(x,x)=\frac{1}{x^2}\sum_{p=1}^{\infty}\frac{\tilde{\mu}^{\mathrm{J}}_{p,0}}{x^p},\qquad \tilde{\mu}^{\mathrm{J}}_{p,0}=\sum_{p_1+q_1=p}\mu^{\mathrm{J}}_{(p_1,q_1),0}, \tag{5.84}$$

together with the analogous expansion of $W_1^{0,\mathrm{J}}$ from (5.69), we obtain from (5.83) the recurrence

$$\tilde{\mu}^{\mathrm{J}}_{p,0}=\frac{1}{\alpha_1+\alpha_2+2\alpha+1+p/2}\Big(\sum_{s=1}^{p-1}sm^{\mathrm{J}}_{s,0}-\frac{(p-1)p}{2}m^{\mathrm{J}}_{p,0}-\alpha_2\sum_{s=1}^{p-1}\tilde{\mu}^{\mathrm{J}}_{s,0}-2\alpha\sum_{s=1}^{p-1}\tilde{\mu}^{\mathrm{J}}_{s,0}m^{\mathrm{J}}_{p-s,0}\Big). \tag{5.85}$$

With the help of computer algebra, we can check in low order cases that the sequence $\{\tilde{\mu}^{\mathrm{J}}_{p,0}\}_{p=1,2,\dots}$ generated by this recurrence is consistent with its relationship to $\{\mu^{\mathrm{J}}_{(p_1,q_1),0}\}$ as implied by the second equation in (5.84).

In (5.82) we have now have $\{m^{\mathrm{J}}_{p,0}\}$ determined by the recurrence (5.70), and $\{\tilde{\mu}^{\mathrm{J}}_{p,0}\}$ determined by the recurrence (5.85). Now introducing the expansion

$$W_1^{1,\mathrm{J}}(x\varkappa) = \frac{1}{x}\sum_{p=2}^{\infty}\frac{m^{\mathrm{J}}_{p,1}}{x^p}$$

we see that $\{m^{\mathrm{J}}_{p,1}\}$ can be determined by the recurrence

$$m^{\mathrm{J}}_{p,1} = \frac{1}{p+1+\alpha_1+\alpha_2+2\alpha}\left(\alpha(p+1)m^{\mathrm{J}}_{p,0} - \alpha(\tilde{\mu}_{p,0}+1) - 2\alpha\sum_{s=1}^{p-1} m^{\mathrm{J}}_{s,1}m^{\mathrm{J}}_{p-s,0} - \alpha_2\sum_{s=1}^{p-1} m^{\mathrm{J}}_{s,1}\right), \tag{5.86}$$

valid for $p = 1, 2, \dots$ with initial condition $m^{\mathrm{J}}_{0,1} = 0$. By the aid of computer algebra, it can be checked that (5.86) correctly reproduces the values of $m^{\mathrm{J}}_{p,1}$ for $p = 1$ and $p = 2$ as implied by (5.65).

5.5 Application to Dyson's disordered chain

5.5.1 Anti-symmetric Gaussian β-ensemble in the high temperature regime

Starting with the work [42], it has been known how to construct random tridiagonal matrices whose eigenvalue probability density function realises the classical β ensembles and thus have functional form given by (5.19) for appropriate $w(x)$. A systematic discussion in the context of the high temperature regime as specified by the relation (5.7) is given in [115]. Our interest for subsequent application is a particular tridiagonal anti-symmetric matrix that gives rise to a variant of (5.19) involving the Laguerre weight, but with squared variables. This is the anti-symmetric Gaussian β-ensemble introduced in [43]. With $\tilde{\chi}_k$ denoting the square root of the gamma distribution $\Gamma[k/2, 1]$, the latter random tridiagonal matrix is specified by with entries directly above the diagonal being distributed by

$$(\tilde{\chi}_{(N-1)\beta/2}, \tilde{\chi}_{\beta(N-2)/2}, \dots, \tilde{\chi}_{\beta/2}). \tag{5.87}$$

It was shown in [43] that the eigenvalue PDF can be explicitly determined, with the precise functional form depending on the parity of N. Replacing N by $2N+1$ so the size of the matrix is odd, there is one zero eigenvalue, with the remaining eigenvalues coming in pairs $\{\pm i x_j\}_{j=1}^N$, $x_j > 0$. Their squares $x_j^2 =: y_j$ are distributed according to the PDF proportional to

$$\prod_{l=1}^{N} y_l^{3\beta/4-1}e^{-y_l}\prod_{1\leqslant j<k\leqslant N}|y_k-y_j|^{\beta},$$

and is thus an example of the Laguerre β-ensemble with $\alpha_1 = 3\beta/4-1$.

As observed in the recent work [56], it follows from the theory of the Laguerre β-ensemble with high temperature scaling (5.7) that the anti-symmetric Gaussian

β-ensemble too permits a well defined high temperature limit specified by the scaling (5.7) with $\alpha > 0$. Specifically, taking the limit of (5.59) for $\alpha_1 \to -1$, we get that that the limiting density of the squared eigenvalues is given in terms of a particular Whittaker function according to

$$\rho_{(1),0}^{(\text{a-s})^2}(y;\alpha) = \frac{1}{\Gamma(\alpha+1)\Gamma(\alpha)} \frac{1}{|W_{-\alpha+1/2,0}(-y)|^2}, \tag{5.88}$$

supported on $y > 0$. This relates to the density of the eigenvalues themselves (i.e. without squaring) by the simple relation

$$\rho_{(1),0}^{\text{a-s}}(x;\alpha) = 2x\rho_{(1),0}^{(\text{a-s})^2}(x^2;\alpha). \tag{5.89}$$

In particular, combining this with (5.57) shows

$$\int_0^\infty x^2 \rho_{(1),0}^{\text{a-s}}(x;\alpha)\,dx = \int_0^\infty y\rho_{(1),0}^{(\text{a-s})^2}(y;\alpha)\,dy = \alpha.$$

The result (5.88) in the form implied by (5.89) it is illustrated through numerical simulation of the eigenvalue density of the anti-symmetric Gaussian β-ensemble scaled by (5.7) in Figure 5.1. To tabulate (5.88) for $\alpha \notin \mathbb{Z}$, use is made of the connection formula for the Whittaker function

$$W_{k,\mu}(z) = -\frac{W_{-k,\mu}(-z)\Gamma(\frac{1}{2}+\mu+k)}{\Gamma(\frac{1}{2}+\mu-k)} e^{-(\mu+\frac{1}{2})\pi i} + \frac{\Gamma(\frac{1}{2}+\mu+k)e^{k\pi i}}{\Gamma(1+2\mu)} M_{-k,\mu}(-z),$$

where $M_{k,\mu}(z)$ is the second solution of the Whittaker equation [38, Eq. 13.14(i)].

5.5.2 Anti-symmetric Gaussian α-ensemble

We define the anti-symmetric Gaussian α-ensemble, with $\alpha > 0$, in analogy with the other α-ensembles defined in the recent work [115]. This is done by noting that in the high temperature regime the entries in the top left corner of the tridiagonal realization of the classical β-ensembles are to leading order independent of the row and thus i.i.d. The prescription then is to construct a random tridiagonal matrix with these random variables. In the case of the anti-symmetric Gaussian β-ensemble, where the off diagonal distributions before the high temperature scaling (5.7) are given by (5.87), this gives for an element of anti-symmetric Gaussian α-ensemble as the random $N \times N$ tridiagonal matrix

$$A_N^\alpha = \begin{pmatrix} 0 & a_1 & & & \\ -a_1 & 0 & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_{N-2} & 0 & a_{N-1} \\ & & & -a_{N-1} & 0 \end{pmatrix}, \quad a_n \sim \tilde{\chi}_\alpha, \tag{5.90}$$

Following the same idea as in [115], the limiting mean spectral measure and mean density of states can be determined.

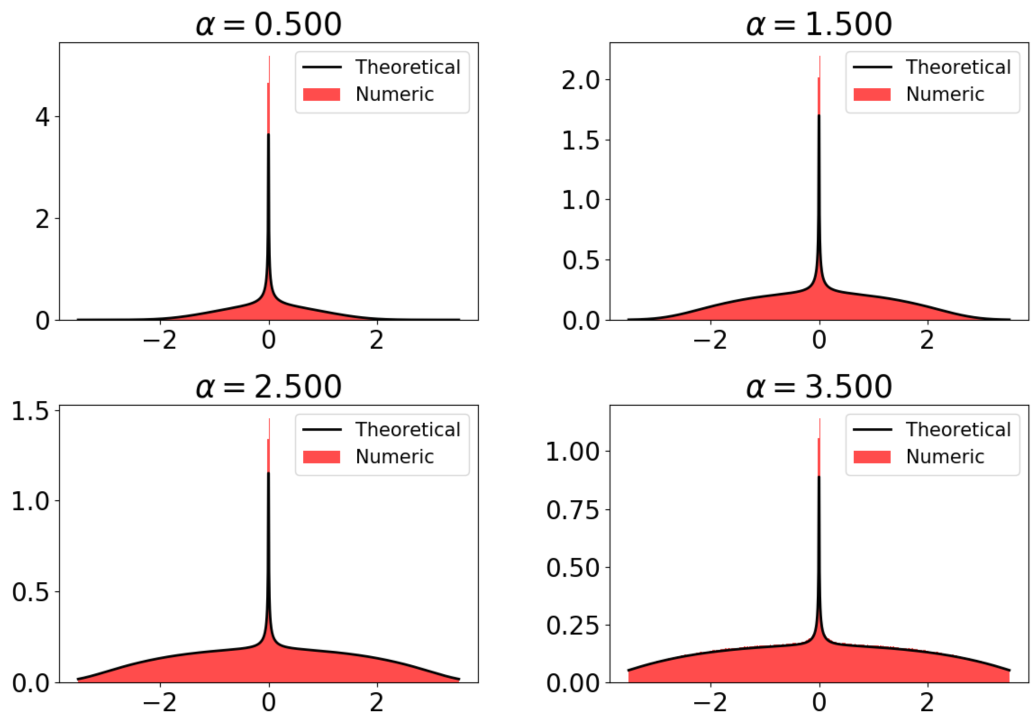


Figure 5.1: Simulation of anti-symmetric Gaussian β-ensemble density of states in the high temperature regime, $n = 5000$, trials $= 500$. The density smoothly goes to zero outside the plotted area.

Theorem 5.12. *Consider the matrix A_N^α in (5.90), $\alpha \in \mathbb{R}^+$, then the mean spectral measure of A_N^α has density $\rho_{(1),0}^{a\text{-}s}(x;\alpha)$ as specified in (5.88) and (5.89), and the mean density of states of A_N^α has density $\mu_\alpha^{a\text{-}s}$ where*

$$\mu_\alpha^{a\text{-}s}(x) = \frac{\partial}{\partial\alpha}(\alpha\rho_{(1),0}^{a\text{-}s}(x;\alpha)).$$

Consequently, with $\mu_\alpha^{(a\text{-}s)^2}(y)$ the density in squared variables, $y = x^2$, we have

$$\mu_\alpha^{(a\text{-}s)^2}(y) = \frac{\partial}{\partial\alpha}\left(\frac{1}{|\Gamma(\alpha)W_{-\alpha+1/2,0}(-y)|^2}\right). \tag{5.91}$$

We give a sketch of the proof, which in fact is a combination of two lemmas.

Lemma 5.13. *Consider the matrix A_N^α in (5.90), $\alpha \in \mathbb{R}^+$, then the mean spectral measure of A_N^α has density $\rho_{(1),0}^{a\text{-}s}(x;\alpha)$ as specified in (5.88) and (5.89).*

Proof. Denote by $B_n(\beta)$ the top $n \times n$ sub-block of the random tridiagonal matrix specified by the distribution of its leading diagonal (5.87). One just has to realize that for any fixed $\kappa \in \mathbb{N}$, $\kappa < n$ the $\kappa \times \kappa$ upper left block of $B_n(2\alpha/n)$ weakly converges to the corresponding one of A_n^α. This implies that the two matrices have the same spectral measure, so applying the result of the previous subsection we get the claim. □

Lemma 5.14. *Consider the matrix A_N^α in (5.90), $\alpha \in \mathbb{R}^+$, let $v_\ell^{a\text{-}s}(\alpha)$ be the ℓ^{th} moment of the mean spectral measure of A_N^α, and $w_\ell^{a\text{-}s}(\alpha)$ the ℓ^{th} moment of the* *mean density of states. We have*

$$w_\ell^{a\text{-}s}(\alpha) = \frac{\partial}{\partial\alpha}(\alpha v_\ell^{a\text{-}s}(\alpha)).$$

Equivalently, with reference to the mean spectral measure, and mean density of states in squared variables

$$w_\ell^{(a\text{-}s)^2}(\alpha) = \frac{\partial}{\partial\alpha}(\alpha v_\ell^{(a\text{-}s)^2}(\alpha)) = \frac{\partial}{\partial\alpha}(\alpha m_{\ell,0}^{\mathrm{L}})\Big|_{\alpha_1=-1}. \tag{5.92}$$

Proof. The argument of [115, Lemma 3.1 – Corollary 3.2] is valid in this case too. □

Proof of Theorem 5.12. The first part of the claim follows immediately from Lemma 5.13. Regarding the second part of the claim, from Lemma 5.14 we have that, in the same notation as before,

$$w_\ell^{\mathrm{a\text{-}s}}(\alpha) = \frac{\partial}{\partial\alpha}(\alpha v_\ell^{\mathrm{a\text{-}s}}(\alpha)).$$

This relation must carry over to relate the densities $\mu_\alpha^{\mathrm{a\text{-}s}}(x)$ of the mean density of states and $\rho_{(1),0}^{\mathrm{a\text{-}s}}(x;\alpha)$ of the mean spectral measure according to

$$\mu_\alpha^{\mathrm{a\text{-}s}}(x) = \frac{\partial}{\partial\alpha}(\alpha\rho_{(1),0}^{\mathrm{a\text{-}s}}(x;\alpha)),$$

and the claim follows. □

Combining (5.92) with (5.57) shows

$$\begin{aligned} w_1^{(\mathrm{a\text{-}s})^2}(\alpha) &= 2\alpha \\ w_2^{(\mathrm{a\text{-}s})^2}(\alpha) &= 2\alpha(1+3\alpha) \\ w_3^{(\mathrm{a\text{-}s})^2}(\alpha) &= 2\alpha(2+9\alpha+10\alpha^2). \end{aligned}$$

5.5.3 Dyson's disordered chain

As a mathematical model of a disordered system, Dyson [45] made a study of the distribution of the squared frequencies for N coupled oscillators along a line, in the circumstance that the spring constants, and/or the masses are random variables (for some example of lattices with random initial data see [75,76] and the references therein). Let K_j denote the spring constant of the j^{th} spring, and let m_j denote the attached mass. With free boundary conditions it was shown in [45] that the allowed frequencies ω of the chain are given by the $(N-1)$ positive eigenvalues of the matrix $i\mathbf{\Lambda}$, where $\mathbf{\Lambda}$ is the $(2N-1)\times(2N-1)$ anti-symmetric tridiagonal matrix specified by having the diagonal above the main diagonal with entries

$$(\lambda_1^{1/2}, \lambda_2^{1/2}, \dots, \lambda_{2N-1}^{1/2}), \qquad \lambda_{2j-1} = K_j/m_j,\ \lambda_{2j} = K_j/m_{j+1}. \tag{5.93}$$

This matrix also has one zero eigenvalue, in keeping with the choice of free boundary conditions.

As observed in [45], the structure (5.93) implies that the simplest type of disorder is to choose $\{\lambda_j\}$ from a common probability distribution, giving rise to what was termed a a Type I disordered chain. Moreover, with the common probability distribution equalling the gamma distribution $\Gamma[\alpha, 1/\kappa]$, Dyson was able to obtain a number of analytic results. Substituting the gamma distribution in (5.93), up to scaling by a factor of $1/\sqrt{\kappa}$, we see that Dyson was in fact studying matrices from the anti-symmetric Gaussian α-ensemble (5.90). One of the analytic results obtained in [45, Eq. (63))] was, in the case $\alpha \in \mathbb{N}$, an explicit functional form for the integrated mean density of states in squared variables. Our Theorem 5.12 generalizes the result of Dyson by giving a special function evaluation of the mean density of states for general $\alpha > 0$.

Two features of Dyson's exact solution have received particular prominence as illustrating universal features, shared by models beyond the solvable case (see the recent review [55] for a discussion and references). One is the functional form of the singularity $x \to 0^+$ [45, consequence of (72)]:

$$\mu_\alpha^{(\text{a-s})^2}(x) \sim \frac{c}{x|\ln(x)|^3},$$

for some constant $c = c_\alpha > 0$, now referred to as the Dyson singularity. For the constant, Dyson's result implies that for $\alpha \in \mathbb{N}$

$$c_\alpha = 2\left(\frac{\pi^2}{6} - \sum_{l=1}^{\alpha-1} \frac{1}{l^2}\right). \tag{5.94}$$

We can also recover this result from the explicit expression (5.91). First, we require knowledge of the asymptotic behaviour of the Whittaker function [38, Eq. (13.14.19)] for $x \to 0^+$,

$$|W_{-\alpha+1/2,0}(-x)| \sim \frac{\sqrt{x}}{\Gamma(\alpha)}|\ln(x) + \psi(\alpha) + 2\gamma + i\pi|,$$

where $\psi(\alpha)$ denotes the digamma function and γ denotes Euler's constant. This substituted in (5.91) show that for $x \to 0^+$

$$\mu_\alpha^{(\text{a-s})^2}(x) \sim \frac{2}{x}\frac{\psi'(\alpha)}{|\ln(x)|^3}. \tag{5.95}$$

From the explicit formula for the trigamma function

$$\psi'(\alpha) = \sum_{n=0}^{\infty} \frac{1}{(\alpha+n)^2}$$

we see that the constant of proportionality in (5.95) reduces to Dyson's result (5.94) for $\alpha \in \mathbb{N}$.

The other prominent feature of Dyson's exact solution solution relates to the (scaled) limit $\alpha \to \infty$, which corresponds to weak disorder; see the discussion of [55, Chapter 3.4] for more details and references. Proceeding analogously to the analysis of Remark 5.4 in the Gaussian case, we see that upon the scaling $x \mapsto \kappa y$ and $W_1^{0,\mathrm{L}}(x) \mapsto \frac{1}{\kappa} W_1^{0,\mathrm{L}}(y)$, for $\kappa, \alpha \to \infty$ with $\kappa/\alpha = \mathrm{O}(1)$ (5.50) reduces to the quadratic equation

$$W_1^{0,\mathrm{L}}(y) + \frac{1}{y} + \frac{\alpha}{\kappa}(W_1^{0,\mathrm{L}}(y))^2 = 0.$$

Subject to the requirement that for large y this behaves as $1/y$, the solution of this quadratic equation is

$$W_1^{0,\mathrm{L}}(y) = \frac{\kappa}{2\alpha}\Big(1 - (1 - 4\alpha/\kappa y)^{1/2}\Big).$$

Consequently, in the same limit,

$$\kappa\rho^{\mathrm{L}}_{(1),0}(y;\alpha_1,\alpha) \to \frac{\kappa}{2\pi\alpha}(4\alpha/\kappa y - 1)^{1/2}, \qquad 0 < y < 4\alpha/\kappa.$$

But $\rho^{\mathrm{L}}_{(1),0}(y;\alpha_1,\alpha)|_{\alpha_1=-1} = \rho^{((\mathrm{a-s})^2}_{(1),0}(y;\alpha)$ and so according to (5.92)

$$\kappa\mu_\alpha^{(\mathrm{a-s})^2)}(\kappa y) \to \frac{\kappa}{2\pi}\frac{\partial}{\partial\alpha}(4\alpha/(\kappa y) - 1)^{1/2} = \frac{1}{\pi}(4\alpha y^{1/2}/\kappa - y)^{-1/2}, \qquad 0 < y < 4\alpha/\kappa.$$

With $\kappa = \alpha$, this is in precise agreement with the limiting result obtained by Dyson [45, Eq. (43)].

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