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Mathematical Analysis, Modelling, and Applications

Topological methods for the search of solutions of nonlinear equations

From planar systems to ordinary and partial differential equations

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Contents

Introduction

The present thesis collects some results which I have obtained in the past three years for different *nonlinear problems* by means of topological methods. We start with planar systems of equations, then we pass to planar systems of differential equations with periodic boundary conditions. The so found results will be suitably generalized firstly to an infinite-dimensional setting and then to the case of a finite-dimensional system of partial differential equations. We will next focus our attention to nearly integrable Hamiltonian systems again in the infinite-dimensional case. The thesis is concluded by some related problems arisen independently of the main lines of our research.

The first chapter of the thesis focuses on topological degree and its applications to some generalizations of the Poincaré–Bohl theorem for planar maps defined over a bounded domain Ω in $\mathbb{R}^2$. A central role in the study of these kind of problems is played by the *normal cone*. When Ω is convex and $\bar{x} \in \partial\Omega$, the normal cone is defined as

$$\mathcal{N}_\Omega(\bar{x}) = \left\{ v \in \mathbb{R}^2 : \langle v, x - \bar{x} \rangle \leq 0\,, \text{ for every } x \in \Omega \right\}\,.$$

Here, as usual, $\langle \cdot\,, \cdot \rangle$ denotes the euclidean scalar product in $\mathbb{R}^2$, with associated norm $\|\cdot\|$. It is well known that if f satisfies an *avoiding cones condition*, namely

$$f(x) \notin \mathcal{N}_\Omega(x)\,, \qquad \text{for every } x \in \partial\Omega\,, \tag{1}$$

then the equation $f(x) = 0$ has a solution $x \in \Omega$. Its proof can be found for example in [42, 65]. Our research work focuses on the case when Ω is *not convex*. In this case, we have to adopt a more general definition of normal cone like the one provided in e.g., [91]). Since it could well happen that the normal cone reduces to a point for some $\bar{x} \in \partial\Omega$, the avoiding cones condition at those points $\bar{x}$ gives *no restriction* for $f(\bar{x})$. Nonetheless we will show that, if the avoiding cones condition (1) holds, the topological degree remains a positive number, provided that $\partial\Omega$ is sufficiently regular.

There are many other possible definitions of normal cone in the nonconvex case (see [91, page 232] for a clarifying survey), and several theorems on the existence of equilibria are available (see, e.g., the review paper [65]). The main novelty of our approach is allowing the normal cones to vanish at certain points, still recovering the existence result at least in the planar case. More precisely our purpose is to consider the case where Ω is an open and bounded planar set, whose boundary $\partial\Omega$ is a Jordan curve, $f : \overline{\Omega} \to \mathbb{R}^2$ is a continuous function such that $0 \notin f(\partial\Omega)$ and provide some conditions on the behavior of the function f at the boundary which guarantee that the Brouwer topological degree $\deg(f, \Omega)$ is a positive number. It is well known that, in such a case, there will be some $x \in \Omega$ such that $f(x) = 0$.

Let we start with a definition

Definition. *Let $\mathcal{S} \subseteq \mathbb{R}$ we can define the iterated derived sets of $\mathcal{S}$ as*

$$\mathcal{S}^{(1)} = \mathcal{S}' , \qquad \mathcal{S}^{(n+1)} = [\mathcal{S}^{(n)}]' .$$

We call $\mathcal{S}$ a vanishing set *if, for some positive integer N, the iterated derived set $\mathcal{S}^{(N)}$ is empty.*

Here is the precise statement of our result.

Theorem 1. *Assume $\partial\Omega$ to be a Jordan curve, piecewise graph of a continuous function. Let $\gamma : [0, 1] \to \mathbb{R}^2$ be a continuous parametrization of $\partial\Omega$, with the property that there are a countable number of non-overlapping intervals $[a_j, b_j]$, contained in $[0, 1]$, on the interior of which γ is of class $\mathcal{C}^1$, and $\mathcal{S} = [0, 1] \setminus \bigcup_j \,]a_j, b_j[$ is a vanishing set. If*

$$f(x) \notin \mathcal{N}_\Omega(x) , \quad \text{for every } x \in \partial\Omega .$$

Then, $\deg(f, \Omega) \geq 1$.

This result is achieved by steps. In the first part we prove that the theorem is valid when $\partial\Omega$ is a "curved polygon", namely a piecewise $\mathcal{C}^1$ curve, and in this setting we are also able to give an upper bound to the degree of f with respect to Ω ; while in the second part we extend the result to the case when $\partial\Omega$ is a piecewise regular Jordan curve in which case we lose the upper bound on the degree. The proof is quite peculiar since it mixes geometrical tools with analytical ones. Entering into details the proof in the first case relies on Hopf's theorem, the so called Umlaufsatz that gives the rotation index of the tangent vector to a piecewise regular $\mathcal{C}^1$ curve. (For the original statement and proof made by Hopf in 1935 see e.g.[61].) The second part is based firstly on a generalization of Hopf's Theorem to normal cones defined by Dini derivatives and a generalization of Darboux's theorem to Dini derivatives for continuous functions. These two results are of independent interest and in particular the second one finds an application in proving existence results for planar systems of ordinary differential equations with the methods of the topological degree (e.g. [47]) as we will see in Chapter 4.

Darboux's theorem states that the derivative of a $\mathcal{C}^1$ function has the intermediate value property; we will show that the same holds true for Dini derivatives of a continuous function.

Theorem 2. *Let $F : [a, b] \to \mathbb{R}$ be a continuous function such that, for some $\mu \in \mathbb{R}$,*

$$D^+F(a) > \mu > D_-F(b) .$$

Then, there is a $\xi \in]a, b[$ such that

$$D_-F(\xi) \geq \mu \geq D^+F(\xi) .$$

As an immediate consequence of the above theorem we obtain the following corollaries describing how the value of a Dini derivative at a point depends on the values of Dini derivatives in a neighborhood of that point.

Corollary. *Let $F : I \to \mathbb{R}$ be a continuous function, defined on some interval I, and let τ_0 be a point of I. Consider the set*

$$E = \{\tau \in I : D^+F(\tau) \leq D_-F(\tau)\} .$$

If τ_0 is a cluster point for E from the left, then

$$D_-F(\tau_0) \geq \liminf_{\substack{\tau\to\tau_0^-\\ \tau\in E}} D^+F(\tau) .$$

Similarly, if τ_0 is a cluster point for E from the right, then

$$D^+F(\tau_0) \leq \limsup_{\substack{\tau\to\tau_0^+\\ \tau\in E}} D_-F(\tau) .$$

The next step consists in proving that Theorem 1 remains true if $\mathcal{S}'$, the *derived set* of $\mathcal{S}$, is finite. The same argument holds assuming that $\mathcal{S}'$ is an infinite set, with a finite number of cluster points; iterating the reasoning an arbitrary finite number of times the proof is thus completed.

In Chapter 2 we consider the following periodic problem

$$(\mathfrak{P}) \qquad \begin{cases} x' = f(t, x, y) , \quad y' = g(t, x, y) , \\ x(0) = x(T) , \quad y(0) = y(T) , \end{cases}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ are continuous functions, T-periodic in their first variable. In this chapter we further develop the theory of lower-upper solutions, concentrating on the periodic problem, by the use of topological degree methods. Our purpose is to give a general definition of a lower and an upper solution with the aim of obtaining the existence of a solution to problem $(\mathfrak{P})$. The use of lower and upper solutions in boundary value problems goes back to the papers of Peano [86] in 1885, Picard [87] in 1893, Scorza Dragoni [96] in 1931 and Nagumo [77] in 1937. The first results for the periodic problem were obtained by Knobloch [64] in 1963. At present a huge literature on this subject has developed, dealing with different types of boundary conditions for ordinary and partial differential equations of elliptic or parabolic type (see, e.g., [29, 31] and the references therein).

Let us recall the classical definition of lower/upper solutions for the periodic problem

$$(P) \qquad \begin{cases} x'' = f(t,x)\,, \\ x(0) = x(T)\,, \quad x'(0) = x'(T)\,. \end{cases}$$

In the scalar case when $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous, the $\mathcal{C}^2$-functions $\alpha, \beta : [0,T] \to \mathbb{R}$ are said to be lower/upper solutions of problem (P), respectively, if

$$\alpha''(t) \geq f(t,\alpha(t))\,, \qquad \beta''(t) \leq f(t,\beta(t))\,.$$

for every $t \in [0,T]$, and

$$\alpha(0) = \alpha(T)\,, \quad \beta(0) = \beta(T)\,, \quad \alpha'(0) \geq \alpha'(T)\,, \quad \beta'(0) \leq \beta'(T)\,.$$

We say that (α,β) is a well-ordered pair of lower/upper solutions if $\alpha \leq \beta$. It is well known that, when such a pair exists, problem (P) has a solution x such that $\alpha \leq x \leq \beta$.

When the inequality $\alpha \leq \beta$ does not hold, we say that the lower and upper solutions are non-well-ordered. In this case, further conditions have to be added in order to avoid resonance with the positive eigenvalues of the differential operator $-x''$ with T-periodic conditions (recall that 0 is an eigenvalue, and all the other eigenvalues are positive) thus recovering existence results. See [4, 30, 51, 56, 59, 83] for results in this direction.

Let us first recall the definition of lower solution given in [52]. A continuously differentiable function $\alpha : \mathbb{R} \to \mathbb{R}$ is said to be a *lower solution* for problem $(\mathfrak{P})$ if it is T-periodic and the following properties hold:

(i) there exists a unique function $y_\alpha : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} y < y_\alpha(t) & \Rightarrow \quad f(t,\alpha(t),y) < \alpha'(t)\,, \\ y > y_\alpha(t) & \Rightarrow \quad f(t,\alpha(t),y) > \alpha'(t)\,; \end{cases}$$

(ii) y_α is continuously differentiable, and

$$y'_\alpha(t) \geq g(t, \alpha(t), y_\alpha(t)) \,, \qquad \text{for every } t \in \mathbb{R} \,;$$

(iii) there are two positive constants δ, m such that, when $|y - y_\alpha(t)| \leq \delta$,

$$\begin{cases} y < y_\alpha(t) - m|x - \alpha(t)| & \Rightarrow \quad f(t,x,y) < \alpha'(t) \,, \\ y > y_\alpha(t) + m|x - \alpha(t)| & \Rightarrow \quad f(t,x,y) > \alpha'(t) \,. \end{cases}$$

An analogous definition is provided for an *upper solution* $\beta : \mathbb{R} \to \mathbb{R}$, and an existence result is proved for problem ($\mathfrak{P}$) assuming $\alpha \leq \beta$, the so called *well-ordered case*.

We will generalize the above definition in two directions. First of all, condition (iii) will be removed. Moreover, the function α will not need to be differentiable on all its domain, and the function y_α will be allowed to have some discontinuity points. In proving this result we make use of Theorem 2 and its corollary. At the same time a deeper analysis of the results in [29] leads to a thorough insight in the study of Dini derivatives of real functions that will be the main argument of the second appendix of the thesis. Moreover, after having proved the existence of a solution of problem ($\mathfrak{P}$) in the well-ordered case, we will then be able to prove an existence result also in the *non-well-ordered case* (namely $\alpha \not\leq \beta$) under some growth conditions on f and g in order to avoid resonance.

A natural application of our results is provided by the periodic problem associated with the scalar equation

$$(\phi(x'))' = h(t, x, x') \,, \tag{2}$$

which can be written in the form of problem ($\mathfrak{P}$), with $f(t,x,y) = \phi^{-1}(y)$ and $g(t,x,y) = h(t,x,\phi^{-1}(y))$. Here, $\phi : I \to J$ is an increasing homeomorphism between two intervals I and J containing 0, and $\phi(0) = 0$. Typical examples in the applications involve the choice $\phi(v) = |v|^{p-2}v$, leading to the so-called "scalar p-Laplacian" operator (cf. [22]), or $\phi(v) = v/\sqrt{1+v^2}$, providing a "mean curvature" operator (cf. [80]), or $\phi(v) = v/\sqrt{1-v^2}$, providing a "relativistic" operator (cf. [15]). (See [52] for a detailed discussion in this direction.) A lower solution for the periodic problem associated with (2) is usually defined as a continuously differentiable function $\alpha : [0,T] \to \mathbb{R}$ such that $\alpha'(t) \in I$ for every t, with $\alpha(0) = \alpha(T)$, $\alpha'(0) \geq \alpha'(T)$ and

$$(\phi(\alpha'))'(t) \geq h(t, \alpha(t), \alpha'(t)) \,, \qquad \text{for every } t \in [0,T] \,.$$

We will see that our definition extends also this one, with the natural choice $y_\alpha(t) = \phi(\alpha'(t))$. Similarly for what concerns an upper solution. We remark moreover for our problem $(\mathfrak{P})$ no monotonicity assumptions on $f(t,x,y)$ are made. Indeed, even in the simpler case $f(t,x,y) = f(y)$, the inequalities in (i) resemble some sign condition, which may be satisfied also if f is not an increasing function.

The existence result in the well-ordered case $\alpha \leq \beta$ is based on assuming (like in [52]) the existence of some bounding curves, in order to control the solutions in the phase plane. The construction of these curves can be easily carried out in concrete examples, assuming a Nagumo-type condition (see again [52]).

The non-well-ordered case requires to add an extra technical condition on the lower and upper solutions, namely we will assume the existence of a whole family of bounding curves; existence that is again verified under some type of Nagumo conditions.

The third and the fourth chapter of the thesis is focused on several existence results for periodic solutions of ODEs both in finite-dimensional and in infinite-dimensional settings by means of perturbation theory and degree theory, and for PDEs at least in the finite-dimensional case.

Chapter 3 deals with the methods of lower–upper solutions for both finite and infinite-dimensional second order ODE.

Bebernes and Schmitt [10] generalized the scalar well-ordered case to a system of type (P), in $\mathbb{R}^N$; we slightly generalize this result at the beginning of our analysis. The first advancement we made in this direction is to prove an existence result for a system in $\mathbb{R}^N$ when the components of the lower/upper solutions can be both well-ordered and non-well-ordered. In order to avoid resonance with the higher part of the spectrum, for simplicity we ask the function f to be globally bounded in the non-well-ordered components (other even more general choices can be done.

We conclude the chapter with the generalization of the previous result (under suitable hypotheses) to the case of a system in an infinite-dimensional Hilbert space. The problem has been analyzed by Schmitt and Thompson [95] in 1975 for boundary value problems of Dirichlet type. However, when facing the periodic problem, they needed to assume the space to be finite-dimensional. In our work the lack of compactness is recovered by assuming the lower and upper solutions to take their values in a Hilbert cube. Moreover, we ask the function f to be globally bounded and completely continuous in the non-well-ordered components. It can be seen that these assumptions resemble that of an infinite-dimensional version of the Poincaré–Miranda Theorem as given in [68].

The study of periodic solutions for infinite-dimensional Hamiltonian systems has been already faced by several authors, see, e.g., [9, 18, 36, 45, 49]. Our approach does not need a Hamiltonian structure, and could be applied also to systems with nonlinearity depending on the derivative of x, provided some Nagumo-type condition is assumed. Such kind of systems were studied, e.g., in [95].

The proof of our result in the finite-dimensional case is based on constructing an auxiliary problem that coincides with the given one on a sufficiently big closed set and interpolate the given problem with a suitable one outside it. To prove the existence of a solution of the problem, topological degree theory is applied; the solution so found is a solution of the original problem since it fulfills a suitable a priori bound that guarantees that its trajectory is confined in the region where the original problem has not been modified. In the infinite-dimensional case one more step is needed: we have to approximate the infinite system with a convenient finite-dimensional one, applying the theorem for the finite-dimensional case and prove the convergence of the so found solutions to a solution of the original problem.

In Chapter 4 we further extent the results of the previous chapter to the PDE counterpart in the finite-dimensional case with different boundary value problems. In particular we deal with a boundary value problem for a system of the type

$$\begin{cases} \mathcal{L}u_n = F_n(x, u_1, \dots, u_M) & \text{in } \Omega\,, \\ \mathcal{B}u_n = 0 & \text{on } \partial\Omega\,, \end{cases} \qquad n = 1, \dots, M\,.$$

In this setting, Ω is a regular bounded domain in $\mathbb{R}^N$, and the differential operator $\mathcal{L} : W^{2,r}(\Omega) \to L^r(\Omega)$ is of elliptic type:

$$(\mathcal{L}w)(x) = -\sum_{l,m=1}^{N} a_{lm}(x)\partial^2_{x_l x_m} w(x) + \sum_{i=1}^{N} a_i(x)\partial_{x_i} w(x) + a_0(x)w(x)\,,$$

with $a_i \in L^\infty(\Omega)$, for $i = 0, \dots, N$ and $a_{lm} \in C(\overline{\Omega})$, $a_{lm} = a_{ml}$, for $l, m = 1, \dots, N$, with the assumption that there exists $\bar{a} > 0$ such that

$$\sum_{l,m=1}^{N} a_{lm}(x)\xi_l\xi_m \geq \bar{a}\|\xi\|^2\,, \quad \text{for every } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^N.$$

We may assume without loss of generality that $a_0 \geq 0$. We take $r > N$, so that $W^{2,r}(\Omega)$ is compactly imbedded into $C^1(\overline{\Omega})$. The function $F : \Omega \times \mathbb{R}^M \to \mathbb{R}^M$ is assumed to be L^r - Carathéodory; moreover, with regard to the boundary operator $\mathcal{B} : C^1(\overline{\Omega}) \to C(\partial\Omega)$, we assume that $\partial\Omega$ is the disjoint union of two closed sets Γ_1 and Γ_2 (the cases $\Gamma_1 = \varnothing$ or $\Gamma_2 = \varnothing$ are admitted), and take

$$\mathcal{B}w := \begin{cases} w & \text{on } \Gamma_1\,, \\ \displaystyle\sum_{i=1}^{N} b_i(x)\partial_{x_i} w + b_0(x)w & \text{on } \Gamma_2\,. \end{cases}$$

Here $b_i \in C^1(\partial\Omega)$, for $i = 0, \ldots, N$, and there exists $\bar{b} > 0$ such that

$$b_0(x) \geq 0 \quad \text{and} \quad \sum_{i=1}^{N} b_i(x)\nu_i(x) \geq \bar{b}\,, \quad \text{for every } x \in \partial\Omega\,.$$

The vector $\nu(x) = (\nu_1(x), \ldots \nu_N(x))$ is the unit outer normal to Ω at $x \in \partial\Omega$. The boundary condition on Γ_1 is the (homogeneous) Dirichlet condition and the one on Γ_2 is the (non-homogeneous) regular oblique derivative condition.

We want to ensure the existence of a solution of problem (P), i.e., a function $u \in W^{2,r}(\Omega)$ satisfying the differential equation almost everywhere in Ω and the boundary condition pointwise. A function with these properties is usually called "strong solution" in the literature. This will be done by introducing the concepts of lower and upper solutions in this setting.

For sake of simplicity we do not deal with nonlinearities depending on ∇u; nonetheless our results can be adapted to such a situation, by adding a Nagumo type assumption. Moreover, our choice of taking the same differential operator and boundary conditions for all components has mainly intended to simplifying the exposition, even if our arguments are also suited to a more general setting.

We will follow a semi-abstract approach like the one in [50], with the purpose of highlighting the main features needed in order to obtain the existence result. In this way, slight modifications lead to similar results for different problems. For example, differential operators of parabolic type may also be considered. Entering now for a moment into details, in this hypotheses we can deal with the problem

$$\begin{cases} \mathcal{L}u_n = F_n(x, t, u_1, \ldots, u_M) & \text{in } Q\,, \\ \mathcal{B}u_n = 0 & \text{on } \partial Q\,, \end{cases} \qquad n = 1, \ldots, M\,.$$

where $\mathcal{L} : W_r^{2,1}(Q) \to L^r(Q)$ is defined as follows:

$$\mathcal{L}w = \partial_t w - \sum_{l,m=1}^{N} a_{lm}(x,t)\partial^2_{x_l x_m} w + \sum_{i=1}^{N} a_i(x,t)\partial_{x_i} w + a_0(x,t)w\,.$$

and the boundary boundary operator $\mathcal{B} : C^{1,0}(\overline{Q}) \to C(\partial Q)$ defined as

$$\mathcal{B}w := \begin{cases} w & \text{on } \Gamma_1 \times [0,T]\,, \\ \displaystyle\sum_{i=1}^{N} b_i(x,t)\partial_{x_i} w + b_0(x,t)w & \text{on } \Gamma_2 \times [0,T]\,, \\ w - \tau_T w & \text{in } \Omega \times \{0\}\,, \\ \tau_{(-T)} w - w & \text{in } \Omega \times \{T\}\,. \end{cases}$$

where

$$(\tau_s w)(x,t) = w(x,t+s)\,,$$

and thus having Dirichlet-periodic conditions on Γ_1, and Robin-periodic on Γ_2.

Concerning the problem of non-well-ordered lower and upper solutions, we refer to [4, 30, 57, 59, 83]. An abstract approach to the theory of lower and upper solutions has also been proposed in [1, 2]. Fewer results are known for systems. We refer to [84, Chapter 8] for systems of elliptic or parabolic equations, where some type of monotonicity is assumed in order to get the existence results.

In Chapter 5 we establish the existence of periodic solutions bifurcating from an infinite-dimensional invariant torus for a nearly integrable Hamiltonian system. The finite-dimensional case was treated in [5, 14, 24, 39, 40] by assuming the existence of an invariant torus made of periodic solutions all sharing the same period, under some non-degeneracy conditions. More precisely if we denote by $H(I,\varphi) = \mathcal{K}(I)$ the Hamiltonian of a completely integrable system in $\mathbb{R}^{2N}$ in action–angle variables, we can write the corresponding system

$$\begin{cases} \dot\varphi = \nabla\mathcal{K}(I) \\ -\dot I = 0\,. \end{cases}$$

Assume that there is a $I^0 \in \mathbb{R}^N$ such that

$$\det \mathcal{K}''(I^0) \neq 0 \tag{3}$$

and consider now the perturbed system

$$\begin{cases} \dot\varphi = \nabla\mathcal{K}(I) + \varepsilon\nabla_I P(t,\varphi,I) \\ -\dot I = \varepsilon\nabla_\varphi P(t,\varphi,I)\,, \end{cases}$$

where $P(\cdot,\varphi,I)$ is T-periodic, and $P(t,\cdot,I)$ is τ_k-periodic in φ_k, for every $k = 1,\dots,N$. Assume that there exist some integers $m_1,\dots,m_N$ for which

$$T\nabla\mathcal{K}(I^0) = (m_1\tau_1,\dots,m_N\tau_N)\,. \tag{4}$$

Then, for $|\varepsilon|$ small enough, there are at least $N+1$ solutions $(\varphi(t), I(t))$ satisfying

$$\varphi(t+T) = \varphi(t) + T\,\nabla\mathcal{K}(I^0)\,, \quad I(t+T) = I(t)\,, \quad \text{for every } t \in \mathbb{R}\,, \tag{5}$$

and these solutions are near to some solutions of the unperturbed problem, i.e., briefly,

$$\varphi(t) \approx \varphi(0) + t\nabla\mathcal{K}(I^0)\,, \quad I(t) \approx I^0.$$

Since $P(\cdot, \varphi, I)$ also mT-periodic for every positive integer m, one could search the so-called "subharmonic solutions" namely "periodic solutions" having period mT, as well. We refer to [40] for a complete description of the problem, and for a more general statement.

The above result was recently extended in [41] for systems of the type

$$\begin{cases} \dot\varphi = \nabla\mathcal{K}(I) + \varepsilon\nabla_I P(t, \varphi, I, z) \\ -\dot I = \varepsilon\nabla_\varphi P(t, \varphi, I, z) \\ J\dot z = \mathcal{A}z + \varepsilon\nabla_z P(t, \varphi, I, z)\,, \end{cases}$$

where $J = \begin{pmatrix} 0 & -I_M \\ I_M & 0 \end{pmatrix}$ denotes the standard $2M \times 2M$ symplectic matrix and $\mathcal{A}$ is a symmetric non-resonant matrix. Assuming (3), (4) and that ∇P, the gradient of P with respect to (φ, I, z), is uniformly bounded, the existence of at least $N+1$ solutions $(\varphi(t), I(t), z(t))$ satisfying (5) and $z(t+T) = z(t)$ was proved, when $|\varepsilon|$ is small enough.

In this thesis we will show how to extend the above results to an infinite-dimensional setting. Let X and Z be the separable Hilbert spaces replacing $\mathbb{R}^N$ and $\mathbb{R}^{2M}$, respectively. The spaces X and Z may be infinite-dimensional, finite-dimensional, or even reduced to $\{0\}$. If X is finite-dimensional, the cases $Z = \{0\}$ and Z finite-dimensional correspond to the settings in [40] and [41], respectively. In the case when X or Z are infinite-dimensional, we will be able to prove the bifurcation of *at least one* periodic orbit from an invariant torus, which can also be infinite-dimensional. The multiplicity problem remains open.

In proving our existence result in infinite-dimensions, we suppose all the functions to be Lipschitz continuous on bounded sets, and the perturbing term ∇P to be uniformly bounded. Moreover, we need a special structure of the autonomous Hamiltonian function; roughly speaking, the functions involved must be decomposable in a sequence of finite-dimensional blocks.

We conclude this chapter showing how the previous results apply to systems where second order systems are coupled with linear ones (focusing in particular on the case when the second order system is given by a relativistic or mean curvature operator) and to "superintegrable systems". We will consider the following kind of coupling

$$\begin{cases} \dfrac{d}{dt}\left(\nabla\Phi \circ \dot{x}\right) = \varepsilon\, \nabla_x F(t,x,z) \\ J\dot{z} = \mathcal{A}z + \varepsilon \nabla_z F(t,x,z)\,. \end{cases} \tag{6}$$

Denoting by Φ_j^* the Legendre–Fenchel transform of Φ_j system (6) can be written as a Hamiltonian system

$$\begin{cases} \dot{x} = \nabla\Phi^*(y) \\ \dot{y} = \varepsilon\, \nabla_x F(t,x,z) \\ J\dot{z} = \mathcal{A}z + \varepsilon \nabla_z F(t,x,z)\,. \end{cases}$$

The choice

$$\Phi(y) = \sum_{j=1}^{\infty} \left(1 - \sqrt{1 - \|\vec{y}_j\|^2}\,\right),$$

and its dual, namely

$$\Phi(y) = \sum_{j=1}^{\infty} \left(\sqrt{1 + \|\vec{y}_j\|^2} - 1\,\right).$$

transforms the first equation of system (6) respectively in

$$\frac{d}{dt}\,\frac{\dot{\vec{x}}_j}{\sqrt{1 - \|\dot{\vec{x}}_j\|^2}} = \varepsilon\, \nabla_{\vec{x}_j} F(t,x,z)\,, \qquad j = 1,2,\ldots$$

or

$$\frac{d}{dt}\,\frac{\dot{\vec{x}}_j}{\sqrt{1 + \|\dot{\vec{x}}_j\|^2}} = \varepsilon\, \nabla_{\vec{x}_j} F(t,x,z)\,, \qquad j = 1,2,\ldots$$

The study of systems of the form

$$\begin{cases} \dot{\varphi} = \nabla\mathcal{K}(I) + \eta^2 \nabla_I P(t,\varphi,I,z) \\ -\dot{I} = \eta^2 \nabla_\varphi P(t,\varphi,I,z) \\ J\dot{z} = \eta\mathcal{A}z + \eta^2 \nabla_z P(t,\varphi,I,z)\,, \end{cases}$$

with associated Hamiltonian function

$$H(t,\varphi,I,z) = \mathcal{K}(I) + \frac{\eta}{2}\,\langle \mathcal{A}z\,,\, z\rangle + \eta^2 P(t,\varphi,I,z)\,,$$

extends to an infinite-dimensional setting [41, Theorem 4.1], which was motivated by the study of perturbations of superintegrable systems, cf. [75].

The thesis is concluded by two appendices in which we deal with some spinoffs of our research. Appendix A gives a characterization of the property of having equal p-norms when p varies in some infinite set $P \subseteq [1, +\infty)$. As a result we will prove that the above property is equivalent to the following condition:

$$\mu(\{x \in E : |f(x)| > \alpha\}) = \mu(\{x \in E : |g(x)| > \alpha\}) \quad \text{for all } \alpha \geq 0 .$$

The resulting condition resembles in some way a reverse Cavalieri's principle that links the volume of two solids with the areas of the corresponding parallel sections. Recently this result has been applied to the inverse problem of recovering the non-linearity for the one dimensional variable exponent $p(x)$-Laplace equation from the Dirichlet-to-Neumann map (see e.g. [19]).

Appendix B investigates some properties of Dini derivatives of arbitrary real functions. We will show that for a continuous function f, the following theorem holds.

Theorem 3. *If $f : I \to \mathbb{R}$ is upper well behaved, namely for every compact interval $J \subseteq I$ there exists x_j such that $f(x_j) = \max f(J)$, then the set*

$$V_f := \{x \in I : D_- f(x) < D^+ f(x)\} .$$

is totally disconnected.

The hypothesis is verified if f is continuous although other weaker assumptions preserve this property. Quite surprisingly a function with this property did not have to be nowhere continuous and we will construct a function f whose set V_f coincides with the entire domain, and nevertheless f is continuous on an infinite set possibly having infinitely many cluster points. The study of this kind of functions raises interest not only in and of itself but because it can give some ideas and also tools to further investigate and maybe extend Theorem 1. The proof of Theorem 3 relies deeply on the theory of continued fractions building up a bridge with Appendix A since this is the main tool used by Stieltjes in 1894 (as can be seen in [98]) to build up one of the central examples.

Chapter 1

The Poincaré–Bohl theorem and the avoiding cones condition

1.1 An historical overview

In this chapter we deal with the Poincaré-Bohl Theorem and we explore some possible extensions and variations in the planar case. This theorem is strictly correlated to topological degree theory, differential geometry and topology. Its origins dates back to the pioneering work of Poincaré in 1883 when, searching for particular solutions to the three–body problem, he discovered that their existence depends on the solvability of a nonlinear system of n equations in n unknowns. To tackle this problem he generalized the Bolzano's Theorem in the following way:

Theorem. *Let $f_1, \ldots, f_n$ be continuous functions of n real variables $x_1, \ldots, x_n$, with*

$$x_i \in [-a_i, a_i] \qquad \text{for all } \; i = 1, \ldots, n\,,$$

and that for all i, f_i is always positive for $x_i = a_i$ and always negative if $x_i = -a_i$. Then there exists $\xi_1, \ldots, \xi_n$ such that

$$f_i(\xi_1, \ldots, \xi_n) = 0 \qquad \text{for all } \; i = 1, \ldots, n.$$

This result was announced in 1883 in a note in Comptes-Rendus de l'Académie des sciences de Paris [88] and developed in 1884 in another article in the Bulletin Astronomique [89] although with an incomplete proof. In 1904, Piers Bohl published on the "Journal de Crelle" [16] an article on the asymptotic behavior of a mechanical system in the neighborhood of an equilibrium point and he proved several theorem on the existence of solution of systems of n equations in n unknowns defined over $K = \prod_{i=1}^{n}[-a_i, a_i]$. As a corollary he obtained the following result:

Theorem. *If $g : K \to \mathbb{R}^N$ is continuous and doesn't vanish on the domain there exist $u \in \partial K$ and $\mu < 0$ such that $g(u) = \mu u$.*

He also proved the analogous result for the n- dimensional ball. Another step in this direction is the Poincaré-Miranda Theorem. The theorem stated in 1883 by Jules Henri Poincaré (nowadays known as the Poincaré-Miranda Theorem) was rediscovered in 1940 by Silvio Cinquini who gave an incomplete proof [25]. One year later Carlo Miranda proved the equivalence of this theorem with Brouwer fixed point theorem [74]. For a very complete and detailed historical account on this topic see for example [69]. Recently Alessandro Fonda and Paolo Gidoni in [42] have proved a variant of the Poincaré-Bohl theorem assuming an "avoiding cones" hypothesis obtaining a generalization of the Poincaré-Bohl theorem for convex domains or more generally to domains that are diffeomorphic to a convex set in $\mathbb{R}^N$.

In this chapter we focus on the planar case dropping the convexity hypothesis. To achieve this goal we have to adopt a more general definition of normal cone like the one in e.g., [91]:

$$\mathcal{N}_\Omega(\bar{x}) = \left\{ v \in \mathbb{R}^N : \limsup_{\substack{x \to \bar{x} \\ x \in \Omega}} \frac{\langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\}.$$

This is the polar of the Bouligand cone (also named contingent cone). It has been called *regular normal cone* in [91, def. 6.3]. Since it could well happen that $\mathcal{N}_\Omega(\bar{x}) = \{0\}$ for some $\bar{x} \in \partial\Omega$, the avoiding cones condition at those points $\bar{x}$ gives *no restriction* for $f(\bar{x})$. As a first step in Section 1.2 we treat the case when the boundary of Ω is a "curved polygon" namely a piecewise $\mathcal{C}^1$ curve. One of the most important tools in Section 1.3 is the Hopf's Theorem (the so called Umlaufsatz). This result is highly important, linking the notion of curvature and Euler characteristic, and it was proved in 1935 for curved polygons (see e.g. [61]). In the same section we obtain a generalization of the Darboux Theorem and the Umlaufsatz to Dini derivatives. [1] We conclude giving a generalization of the Poincaré-Bohl theorem in that framework completing the analysis made in [42].

Let us explain our main results, first introducing some notation. Since $\partial\Omega$ is a Jordan curve, there is a continuous function $\gamma : [0,1] \to \mathbb{R}^2$, whose restriction to $[0,1[$ is injective, with $\gamma(0) = \gamma(1)$ and $\gamma([0,1]) = \partial\Omega$. Let us start assuming that $\partial\Omega$ is a *piecewise regular* Jordan curve. By this we mean that there are

$$0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1\,,$$

[1]These two simple results seem not to be present in literature.

such that, for every $j = 1, 2, \dots, n$, if we look at the function $\gamma_j : [a_{j-1}, a_j] \to \mathbb{R}^2$, restriction of γ to the closed interval $[a_{j-1}, a_j]$, this function is of class $\mathcal{C}^1$, and $\gamma_j'(s) \neq 0$ for every $s \in [a_{j-1}, a_j]$. Then, writing

$$\gamma_-'(a_j) = \gamma_j'(a_j)\,, \qquad \gamma_+'(a_j) = \gamma_{j+1}'(a_j)\,,$$

it may be that $\gamma_-'(a_j) \neq \gamma_+'(a_j)$. Among these, there could be inward and outward corner points (see Section 1.2 for a precise definition). Let us denote by N_ι *the number of inward corner points (or cusps).*

We will first prove the following result.

Theorem 1.1.1. *Assume $\partial\Omega$ to be a piecewise regular Jordan curve, and that*

$$f(x) \notin \mathcal{N}_\Omega(x)\,, \quad \text{for every } x \in \partial\Omega\,. \tag{1.1}$$

Then, $1 \leq \deg(f, \Omega) \leq N_\iota + 1$.

As we already said, at certain points a_j it may happen that $\mathcal{N}_\Omega(a_j) = \{0\}$, in which case $f(a_j)$ has no cone to avoid. Let us illustrate this with an example. Using complex notation, we consider the function $f : \mathbb{C} \to \mathbb{C}$ defined as $f(z) = z^2$. As for the set Ω, if we took the disk centered at the origin with radius 1, condition (1.1) would be violated at the point $(1, 0)$. So, we modify the disk in a small neighborhood of that point, by creating an inner corner, as in Figure 1. Now condition (1.1) is satisfied, and Theorem 1.1.1 tells us that $1 \leq \deg(f, \Omega) \leq 2$ (of course, we all know that $\deg(f, \Omega) = 2$ in this case).

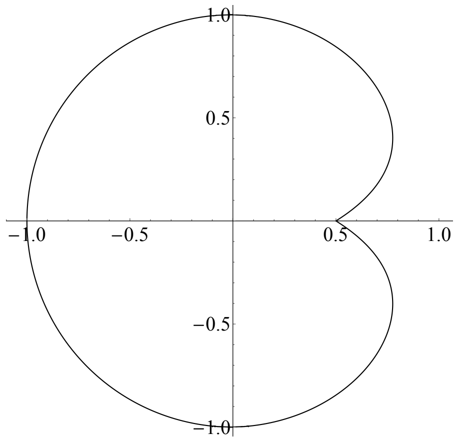


Figure 1.1: *Local deformation of the boundary.*

The proof of Theorem 1.1.1 is provided in the next section. An important tool will be Hopf's Theorem (the so-called *Umlaufsatz*), adapted to our situation.

The extension of Theorem 1.1.1 to sets having an infinite number of corners is discussed in Section 1.3, where we focus our attention on sets whose boundary is piecewise the graph of a continuous function. This difficult task is not fully achieved here, since we eventually need to assume some additional regularity of the boundary. However, in view of some striking examples of sets whose boundary is locally the graph of nowhere differentiable functions (see, e.g., the one in [34]), we expect that further generalizations would require a much deeper insight in the theory of continuous functions (we will return on this argument in Appendix B). As expected, in this framework we lose the upper estimate on the degree, and finally only prove that $\deg(f, \Omega) \geq 1$.

Nevertheless, with the aim of extending Theorem 1.1.1, we will provide in Section 1.3.1 a generalization of Hopf's Theorem to some cases where the curve bounding the set Ω is not regular, and in Section 1.3.3 an extension of Darboux Theorem involving the Dini derivatives. These results could also have an independent interest.

The existence of equilibria of functions defined on sets in abstract spaces with very irregular boundaries has been investigated in [13, 27, 28], typically in situations when the associated topologically degree is equal to 1.

Let us end this introduction by saying that Theorem 1.1.1 and its extension in Section 1.3 could be generalized assuming the vector field $f(x)$ to avoid some more general upper semicontinuous multivalued map having closed convex values.

1.2 Proof of Theorem 1.1.1

Following the usual habit, we assume that $\gamma : [0,1] \to \mathbb{R}^2$ parametrizes $\partial\Omega$ in the *counter-clockwise* direction. Also, without loss of generality, we may ask that $\gamma(0) = \gamma(1)$ is a regular point, i.e., that $\gamma'_+(0) = \gamma'_-(1)$, and that $\gamma'_-(a_j) \neq \gamma'_+(a_j)$, for $j = 1, 2, \dots, n-1$. Moreover, for simplicity we may also assume that γ is an arc-length parametrization.

1.2.1 The angular function

Denoting by $\mathcal{P}(\mathbb{R})$ the collection of all subsets of $\mathbb{R}$, we define a multivalued function $\omega : [0,1] \to \mathcal{P}(\mathbb{R})$, the so-called *angular function*, as follows.

In the open intervals $]a_{j-1}, a_j[$, the function will be single-valued, hence we can write

$$\gamma'(s) = e^{i\omega(s)}, \quad \text{when } s \in]a_{j-1}, a_j[, \tag{1.2}$$

(recall that $\|\gamma'(s)\| = 1$) while at the points a_j, corresponding to corners or cusps, $\omega(a_j)$ will be a closed interval $[\alpha_j, \beta_j]$. Moreover, the multivalued function $\omega : [0,1] \to \mathcal{P}(\mathbb{R})$ will be upper semicontinuous (cf. [7, page 41]). We now enter into details.

Since we have assumed that $\gamma(0)$ is a regular point, we define $\omega(0)$ to be single-valued, such that $e^{i\omega(0)} = \gamma'_+(0)$ and $\omega(0) \in [0, 2\pi[$. Then, the function $\omega(s)$ is uniquely defined on $[0, a_1[$, by continuity, asking that (1.2) holds, and it is single-valued there.

Let us explain how $\omega(s)$ is defined on $[a_1, a_2[$. Since $\gamma'_-(a_1) \neq \gamma'_+(a_1)$, it is easily seen that we have the following alternative: either

(*i*) there is an $\varepsilon > 0$ such that $\gamma(a_1) + \lambda\gamma'_-(a_1) \notin \Omega$, for every $\lambda \in]0, \varepsilon[$,

in which case we say that $\gamma'_-(a_1)$ "points outward", so that $\gamma(a_1)$ is an "outward corner point", or

(*ii*) there is an $\varepsilon > 0$ such that $\gamma(a_1) + \lambda\gamma'_-(a_1) \in \Omega$, for every $\lambda \in]0, \varepsilon[$,

in which case we say that $\gamma'_-(a_1)$ "points inward", so that $\gamma(a_1)$ is an "inward corner point".

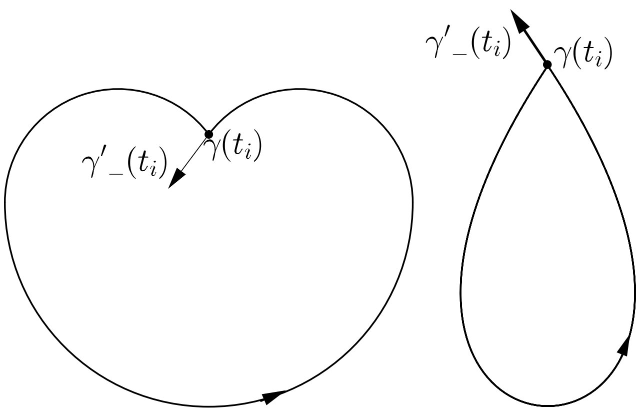


Figure 1.2: *Inward and outward corner points for a curved polygon.*

In case $\gamma'_-(a_1)$ points *outward*, let

$$\alpha_1 = \lim_{s \to a_1^-} \omega(s) \,. \tag{1.3}$$

Such a limit exists and is finite, since $\gamma(s) = \gamma_1(s)$ on $[0, a_1]$ and $\gamma_1 : [0, a_1] \to \mathbb{R}^2$ is of class $\mathcal{C}^1$, with $\|\gamma_1'(s)\| = 1$ for every $s \in [0, a_1]$. Moreover, $e^{i\alpha_1} = \gamma'_-(a_1)$. Let $\beta_1 \in]\alpha_1, \alpha_1 + \pi]$ be such that $e^{i\beta_1} = \gamma'_+(a_1)$, and define $\omega(a_1) = [\alpha_1, \beta_1]$. Now there is a unique way to define $\omega(s)$ on $]a_1, a_2[$, in such a way that (1.2) holds, preserving the upper semicontinuity of the multivalued function ω on the whole interval $[0, a_2[$. Notice that it has to be

$$\beta_1 = \lim_{s \to a_1^+} \omega(s) . \tag{1.4}$$

In case $\gamma'_-(a_1)$ points *inward*, let instead

$$\beta_1 = \lim_{s \to a_1^-} \omega(s) , \tag{1.5}$$

so that $e^{i\beta_1} = \gamma'_-(a_1)$, and let $\alpha_1 \in [\beta_1 - \pi, \beta_1[$ be such that $e^{i\alpha_1} = \gamma'_+(a_1)$. Define $\omega(a_1) = [\alpha_1, \beta_1]$, and extend $\omega(s)$ on $]a_1, a_2[$, in such a way that (1.2) holds, preserving the upper semicontinuity on the whole interval $[0, a_2[$. In this case, it has to be

$$\alpha_1 = \lim_{s \to a_1^+} \omega(s) . \tag{1.6}$$

The definition of $\omega(a_2)$ is analogous to that of $\omega(a_1)$, and we can continue recursively, thus defining $\omega(s)$ on $[a_{j-1}, a_j[$, for every $j = 1, 2, \ldots, n$. When we arrive at the last interval, we define $\omega(1)$ just by continuity: $\omega(1) = \lim_{s \to 1^-} \omega(s)$.

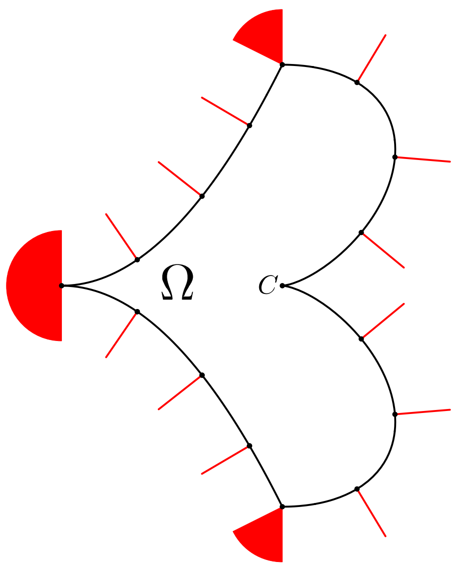


Figure 1.3: *The avoiding cones condition for the set Ω. Notice that in C the normal cone reduces to $\{0\}$ so there is no restriction for f in this point.*

The following lemma will be crucial in the proof of Theorem 1.1.1.

Lemma 1.2.1. *One has that*

$$\omega(1) = \omega(0) + 2\pi\,.$$

Proof The function $\omega : [0,1] \to \mathcal{P}(\mathbb{R})$ defined above is upper semicontinuous and, since $\gamma(0) = \gamma(1)$ is a regular point, there must exist an integer N for which $\omega(1) = \omega(0) + 2\pi N$. If there are no singular points, i.e. if $n = 1$, we can apply Hopf's Theorem [61], stating that for any simple closed $\mathcal{C}^1$-curve γ in the plane it has to be $N = 1$.

Let us now assume $n \geq 2$. We will approximate the curve γ with a $\mathcal{C}^1$-curve $\tilde{\gamma} : [0,1] \to \mathbb{R}^2$, by *smoothing the angles*. We will thus correspondingly obtain an approximation of the multivalued function ω by a continuous single-valued function $\widetilde{\omega} : [0,1] \to \mathbb{R}$.

Let us explain how $\tilde{\gamma}$ is defined, assuming for simplicity $n = 2$, i.e., that a_1 is the only point of discontinuity of γ'. Recalling that ω is upper semicontinuous and $\omega(a_1) = [\alpha_1, \beta_1]$, for any $\varepsilon \in \left]0, \frac{\pi}{2}\right[$ there is a $\delta > 0$ such that

$$s \in [a_1 - \delta, a_1 + \delta] \quad \Rightarrow \quad \mathrm{dist}(\omega(s), [\alpha_1, \beta_1]) \leq \varepsilon\,.$$

(Here and in the following, $\mathrm{dist}(\mathcal{A}, \mathcal{B}) = \inf\{\|x - y\| : x \in \mathcal{A}, y \in \mathcal{B}\}$.) Take δ small enough, and consider the rectangle $I_1 = [a_1 - \delta, a_1 + \delta] \times [\alpha_1 - \varepsilon, \beta_1 + \varepsilon]$. We want the function $\widetilde{\omega}$ to coincide with ω on $[0, a_1 - \delta] \cup [a_1 + \delta, 1]$, while in the interval $[a_1 - \delta, a_1 + \delta]$ we will construct a $\mathcal{C}^1$-function whose graph is contained in I_1 and smoothly glues the endpoints $(a_1 - \delta, \omega(a_1 - \delta))$ and $(a_1 + \delta, \omega(a_1 + \delta))$.

Let $B(\gamma(a_1), r)$ be the open planar disk centered at $\gamma(a_1)$ with a small radius $r > 0$, so small that its boundary is crossed only twice by the curve γ. This choice is possible since there surely are $\bar{r} > 0$ and $\bar{\delta} > 0$ such that, if $r \in]0, \bar{r}]$ and $\mathrm{dist}(\gamma(s), \gamma(a_1)) = r$ for some $s \in]a_1 - \bar{\delta}, a_1 + \bar{\delta}[$, then $\gamma'(s)$ is transversal to $\partial B(\gamma(a_1), r)$. Moreover, there is a $\bar{\varepsilon} > 0$ such that, if $|s - a_1| \geq \bar{\delta}$, then $\mathrm{dist}(\gamma(s), \gamma(a_1)) \geq \bar{\varepsilon}$. It will then be sufficient to choose $r \leq \min\{\bar{r}, \bar{\varepsilon}\}$. With this choice of $r > 0$, there will be an "entrance point" $A = \gamma(a)$ and an "exit point" $B = \gamma(b)$. Notice that $a < a_1 < b$, and $b - a$ can be made arbitrarily small, by reducing the radius r.

Consider the segment AB joining A and B, and take the straight line $\mathcal{L}$, parallel to AB, at a small distance $\hat{\varepsilon} > 0$ from it, lying between the segment itself and the center of the ball $\gamma(a_1)$. Let A' and B' be the intersections of $\mathcal{L}$ with the lines

$$\mathcal{L}_A = \{\gamma(a) + t\gamma'(a) : t \in \mathbb{R}\} \quad \text{and} \quad \mathcal{L}_B = \{\gamma(b) + t\gamma'(b) : t \in \mathbb{R}\}\,,$$

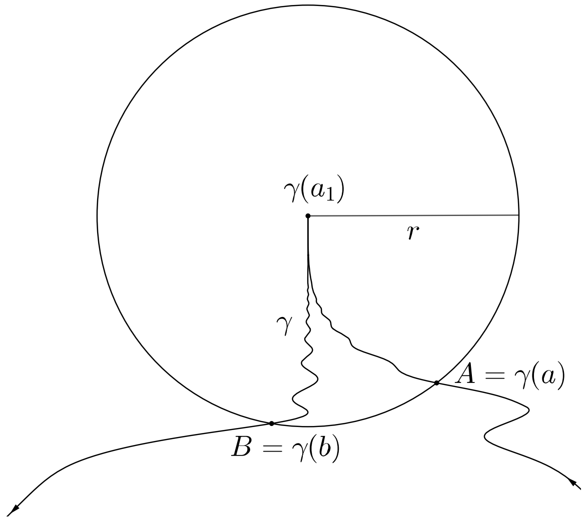


Figure 1.4: *The case of a cusp.*

respectively. Let A'' and B'' be the points on the segment $A'B'$ such that AA' and $A'A''$ have the same length, as well as for for BB' and $B'B''$. Taking $\hat{\varepsilon}$ small enough, the vector from A'' to B'' will have the same direction of the vector from A to B. Consider the circular arc $\mathcal{C}_{AA''}$, starting at A, arriving at A'', and tangent to both $\mathcal{L}$ and $\mathcal{L}_A$. Similarly, consider the circular arc $\mathcal{C}_{BB''}$, starting at B, arriving at B'', and tangent to both $\mathcal{L}$ and $\mathcal{L}_B$. The curve $\tilde{\gamma}$ will be defined as follows (see Figure 2): $\tilde{\gamma}(s)$ coincides with $\gamma(s)$ for $s < a$, i.e., until it reaches the point A; then, it follows the circular arc $\mathcal{C}_{AA''}$ until A''; at this point, it goes straight to B'', thus remaining on the line $\mathcal{L}$; then, it follows the circular arc $\mathcal{C}_{BB''}$ until B, where it rejoins the curve γ. (Notice that, since we must be careful to parametrize $\tilde{\gamma}$ in such a way that $\tilde{\gamma}(b) = B$, this curve will be regular but not necessarily parametrized by arc-length any more.) Finally, $\tilde{\gamma}(s)$ coincides with $\gamma(s)$ for $s > b$.

In the above construction, the constants r ε, δ and $\hat{\varepsilon}$ can be chosen to be arbitrarily small. Moreover, the angle function $\widetilde{\omega} : [0,1] \to \mathbb{R}$, defined by

$$\frac{\tilde{\gamma}'(s)}{\|\tilde{\gamma}'(s)\|} = e^{i\,\widetilde{\omega}(s)}\,,$$

with $\widetilde{\omega}(0) = \omega(0)$, is monotone as s varies in $[a,b]$, and continuous. These facts guarantee that

$$\mathrm{dist}(\widetilde{\omega}(s), \omega(s)) \leq \pi + 2\varepsilon < 2\pi\,, \qquad \text{for every } s \in [0,1]\,.$$

By Hopf's Theorem, $\widetilde{\omega}(1) = \widetilde{\omega}(0) + 2\pi$, hence also $\omega(1) = \omega(0) + 2\pi$, thus finishing the proof. □

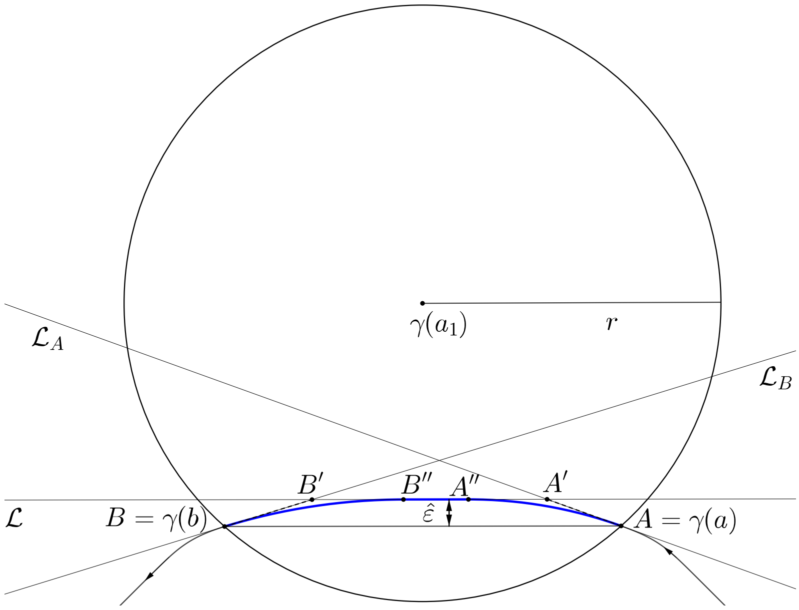


Figure 1.5: *An example of angle-smoothing*

1.2.2 The avoiding cones condition

We consider the restriction of our function $f : \overline{\Omega} \to \mathbb{R}^2$ to the boundary of Ω. More precisely, let us define the new function

$$g = f \circ \gamma : [0,1] \to \mathbb{R}^2 \setminus \{0\}\,.$$

Passing to polar coordinates, in complex notation, we can write

$$g(s) = \rho(s) e^{i\varphi(s)}\,,$$

for some continuous functions $\rho : \mathbb{R} \to]0, +\infty[$ and $\varphi : \mathbb{R} \to \mathbb{R}$. Since $\gamma(0) = \gamma(1)$, the number $\varphi(1)$ differs from $\varphi(0)$ by an integer multiple of 2π, and

$$\deg(f, \Omega) = \frac{\varphi(1) - \varphi(0)}{2\pi}\,.$$

It will be useful to consider the multivalued function $\Theta : [0,1] \to \mathcal{P}(\mathbb{R})$ defined as

$$\Theta(s) = \begin{cases} \emptyset\,, & \text{if } s = a_j \text{ and } \gamma'_-(a_j) \text{ points inward}\,, \\ \omega(s) - \frac{1}{2}\pi + 2\pi\mathbb{Z}\,, & \text{otherwise}\,. \end{cases}$$

We can thus introduce an auxiliary cone $\mathcal{N}^*_\Omega(\gamma(s))$, made of the origin and the union of all the half-lines starting from the origin determined by the angles in

$\Theta(s)$. Precisely,

$$\mathcal{N}^*_\Omega(\gamma(s)) = \begin{cases} \{0\}\,, & \text{if } s = a_j \text{ and } \gamma'_-(a_j) \text{ points inward,} \\ \{\alpha e^{i\theta} : \alpha \geq 0,\, \theta \in \Theta(s)\}\,, & \text{otherwise.} \end{cases} \tag{1.7}$$

In the sequel we will often use without further mention the elementary properties of this kind of cones, like e.g. being closed sets, translation invariant and rotation equivariant. Notice also that neither $\mathcal{N}_\Omega(\gamma(s))$ nor $\mathcal{N}^*_\Omega(\gamma(s))$ can be larger than a half-plane.

Lemma 1.2.2. *The cones $\mathcal{N}_\Omega(\gamma(s))$ and $\mathcal{N}^*_\Omega(\gamma(s))$ coincide. Therefore, the avoiding cones condition* (1.1) *is equivalent to*

$$\varphi(s) \notin \Theta(s)\,, \qquad \textit{for every } s \in [0,1]\,.$$

Proof We analyze several different situations.

If $s \neq a_j$ for every $j = 1, 2, \ldots, n-1$, the boundary of Ω is smooth at $\gamma(s)$, hence $\mathcal{N}_\Omega(\gamma(s))$ is just a single half-line, orthogonal to $\gamma'(s)$, with angle $\omega(s) - \frac{1}{2}\pi$. It thus coincides with $\mathcal{N}^*_\Omega(\gamma(s))$.

Assume that $s = a_j$ and that $\gamma'_-(a_j)$ points inward, so that $\Theta(a_j) = \varnothing$ and $\mathcal{N}^*_\Omega(\gamma(s)) = \{0\}$. We want to prove that $\mathcal{N}_\Omega(\gamma(a_j)) = \{0\}$, as well. Let us translate $\gamma(a_j)$ to the origin and rotate the reference system of axes in such a way that the two straight lines passing through it determined by $\gamma'_-(a_j)$ and $\gamma'_+(a_j)$ are symmetric with respect to the vertical axis and, roughly speaking, the set Ω locally stays below its boundary. More precisely, if these two lines coincide, in which case we have an inner cusp, they will be equal to $\{(x_1, x_2) : x_1 = 0\}$; otherwise, the first one will have a positive slope m, and the second one a negative slope $-m$. We may also assume, in both cases, that there are two constants $\bar{r} > 0$ and $\mu > 0$ such that

$$\{(x_1, x_2) : x_2 < \mu|x_1|\} \cap B(0, r) \subseteq \Omega\,, \qquad \text{for every } r \in\,]0, \bar{r}]\,.$$

Let $v = (v_1, v_2)$ be a vector with $\|v\| = 1$. We distinguish three cases.

Case 1: $v_2 \leq \mu|v_1|$. Then, choosing $x = \frac{r}{2}v$, we have that

$$\frac{\langle v, x\rangle}{\|x\|} = 1\,.$$

Case 2: $v_2 > \mu|v_1|$ and $v_1 \geq 0$. Here we choose $x = (\epsilon, \mu\epsilon)$, with $\epsilon > 0$ small enough, and we have that

$$\frac{\langle v, x\rangle}{\|x\|} \geq \frac{\mu}{\sqrt{1+\mu^2}}\, v_2\,. \tag{1.8}$$

Case 3: $v_2 > \mu|v_1|$ and $v_1 < 0$. We then take $x = (-\epsilon, \mu\epsilon)$, with $\epsilon > 0$ small enough, and we have (1.8) again.

We have thus shown, in all the three cases, that $v \notin \mathcal{N}_\Omega(0)$. Since it cannot contain any unitary vector v, the cone $\mathcal{N}_\Omega(0)$ is reduced to $\{0\}$.

Assume now that $s = a_j$ and that $\gamma'_-(a_j)$ points outward. In this case, $\omega(a_j) = [\alpha_j, \beta_j]$, so that $\Theta(a_j) = [\alpha_j - \frac{1}{2}\pi, \beta_j - \frac{1}{2}\pi] + 2\pi\mathbb{Z}$. As above, we translate $\gamma(a_j)$ to the origin and take a reference system of axes so that the two straight lines passing through the origin determined by $\gamma'_-(a_j)$ and $\gamma'_+(a_j)$ are symmetric with respect to the vertical axis. If they coincide (in which case $\alpha_j = \pi/2$ and $\beta_j = 3\pi/2 \mod 2\pi$), we have an outer cusp, and they will be equal to $\{(x_1, x_2) : x_1 = 0\}$; otherwise, the first one will have a negative slope $-m$, and the second one a positive slope m (in this case, $\alpha_j = \pi - \arctan(m)$ and $\beta_j = \pi + \arctan(m) \mod 2\pi$). We want to prove that, in the first case, $\mathcal{N}_\Omega(0) = \{(x_1, x_2) : x_2 \geq 0\}$ while, in the second case, $\mathcal{N}_\Omega(0) = \{(x_1, x_2) : x_2 \geq \frac{1}{m}|x_1|\}$. This will imply that $\mathcal{N}_\Omega(0) = \mathcal{N}^*_\Omega(0)$.

Let us consider the case of an outer cusp. We first prove the following inclusion $\{(x_1, x_2) : x_2 > 0\} \subseteq \mathcal{N}_\Omega(0)$. Let v be a vector in $\{(x_1, x_2) : x_2 > 0\}$, and let $m_v > 0$ be such that $v \in \{(x_1, x_2) : x_2 \geq m_v|x_1|\}$. There is a $\bar{r} > 0$ such that

$$\Omega \cap B(0, r) \subseteq \left\{(x_1, x_2) : x_2 < -\frac{2}{m_v}\,|x_1|\right\}, \quad \text{for every } r \in]0, \bar{r}]\,.$$

Therefore, for any $r \in]0, \bar{r}]$ and every $x \in \Omega \cap B(0, r) \setminus \{0\}$, one has that $\langle v, x\rangle < 0$, showing that $v \in \mathcal{N}_\Omega(0)$. Since $\mathcal{N}_\Omega(0)$ is closed (cf. [91, Proposition 6.5]), we conclude that $\{(x_1, x_2) : x_2 \geq 0\} \subseteq \mathcal{N}_\Omega(0)$.

To prove the opposite inclusion, let $v = (v_1, v_2)$ be such that $v_2 < 0$. There exist $c_v > 0$ and $\tilde{\mu}_v > 0$ such that, for every nonzero vector $x = (x_1, x_2)$ with $x_2 \leq -\tilde{\mu}_v|x_1|$, one has

$$\frac{\langle v, x\rangle}{\|x\|} \geq c_v\,. \tag{1.9}$$

Now, there is a $\bar{r}_v > 0$ such that

$$\Omega \cap B(0, r) \subseteq \{(x_1, x_2) : x_2 < -\tilde{\mu}_v\,|x_1|\}\,, \quad \text{for every } r \in]0, \bar{r}_v]\,.$$

Therefore, for any $r \in]0, \bar{r}_v]$ and every $x \in \Omega \cap B(0, r) \setminus \{0\}$, one has that (1.9) holds, showing that $v \notin \mathcal{N}_\Omega(0)$.

Assume now that $\gamma'_-(a_j)$ points outward, but is not a cusp. Let us first prove the inclusion $\{(x_1, x_2) : x_2 > \frac{1}{m}|x_1|\} \subseteq \mathcal{N}_\Omega(0)$. Let v be a vector in $\{(x_1, x_2) : x_2 > \frac{1}{m}|x_1|\}$, and let $m'_v \in]0, m[$ be such that $v \in \{(x_1, x_2) : x_2 \geq \frac{1}{m'_v}|x_1|\}$. There is a $\bar{r} > 0$ such that

$$\Omega \cap B(0, r) \subseteq \{(x_1, x_2) : x_2 < -m'_v\,|x_1|\}\,, \qquad \text{for every } r \in]0, \bar{r}]\,.$$

Therefore, for any $r \in]0, \bar{r}]$ and every $x \in \Omega \cap B(0,r) \setminus \{0\}$, one has that $\langle v, x\rangle < 0$, showing that $v \in \mathcal{N}_\Omega(0)$. Since $\mathcal{N}_\Omega(0)$ is a closed cone, we conclude that $\{(x_1,x_2) : x_2 \geq \frac{1}{m}|x_1|\} \subseteq \mathcal{N}_\Omega(0)$.

Let us now prove the opposite inclusion. Let $v = (v_1, v_2) \notin \{(x_1,x_2) : x_2 \geq \frac{1}{m}|x_1|\}$, and let $\mu_v > m$ be such that $v \notin \{(x_1,x_2) : x_2 \geq \frac{1}{\mu_v}|x_1|\}$. There is a $\bar{r} > 0$ such that

$$\{(x_1,x_2) : x_2 < -\mu_v\,|x_1|\} \cap B(0,r) \subseteq \Omega\,, \qquad \text{for every } r \in]0,\bar{r}]\,.$$

Assume $v_1 \geq 0$, and hence $v_2 < \frac{1}{\mu_v} v_1$. Then, taking $x = (\delta, -\mu_v\delta)$, for any sufficiently small $\delta > 0$ we have that $x \in \Omega$, and

$$\frac{\langle v, x\rangle}{\|x\|} = \frac{1}{\sqrt{1+\mu_v^2}}(v_1 - v_2\mu_v) > 0\,,$$

showing that $v \notin \mathcal{N}_\Omega(0)$. The case $v_1 \leq 0$ is analogous.

The proof of the lemma is thus completed. □

1.2.3 Conclusion of the proof

Recalling that $\gamma(0)$ is a regular point and that, by Lemma 1.2.2,

$$\varphi(0) \notin \omega(0) - \tfrac{1}{2}\pi + 2\pi\mathbb{Z}\,,$$

there is a $K \in \mathbb{Z}$ such that

$$\omega(0) - \tfrac{1}{2}\pi + 2\pi K < \varphi(0) < \omega(0) - \tfrac{1}{2}\pi + 2\pi(K+1)\,. \tag{1.10}$$

Then, by continuity and Lemma 1.2.2, it has to be that

$$\varphi(s) > \omega(s) - \tfrac{1}{2}\pi + 2\pi K\,, \qquad \text{for every } s \in [0, a_1[\,. \tag{1.11}$$

(Notice that $\omega(s)$ is single-valued in $[0, a_1[$, and in each interval $]a_{j-1}, a_j[$.) When we arrive at $s = a_1$, we have two possibilities: either $\gamma'_-(a_1)$ points outward, or it points inward. If it points outward, then

$$\varphi(a_1) \notin \Theta(a_1) = \omega(a_1) - \tfrac{1}{2}\pi + 2\pi\mathbb{Z} = [\alpha_1, \beta_1] - \tfrac{1}{2}\pi + 2\pi\mathbb{Z}\,. \tag{1.12}$$

By (1.11) and (1.3), we know that

$$\varphi(a_1) = \lim_{s\to a_1^-} \varphi(s) \geq \lim_{s\to a_1^-} \omega(s) - \tfrac{1}{2}\pi + 2\pi K = \alpha_1 - \tfrac{1}{2}\pi + 2\pi K\,,$$

hence, by (1.12) and (1.4), it has to be

$$\varphi(a_1) > \beta_1 - \tfrac{1}{2}\pi + 2\pi K = \lim_{s\to a_1^+} \omega(s) - \tfrac{1}{2}\pi + 2\pi K\,.$$

Consequently, if $s > a_1$ and s is sufficiently near a_1, then $\varphi(s) > \omega(s) - \frac{1}{2}\pi + 2\pi K$. This inequality will persist, by continuity and Lemma 1.2.2, for every $s \in]a_1, a_2[$.

On the other hand, if $\gamma'_-(a_1)$ points inward, there is no cone to avoid. However, by (1.11), (1.5) and (1.6),

$$\begin{aligned}\varphi(a_1) = \lim_{s\to a_1^-} \varphi(s) \geq \lim_{s\to a_1^-} \omega(s) - \tfrac{1}{2}\pi + 2\pi K = \beta_1 - \tfrac{1}{2}\pi + 2\pi K > \\ > \alpha_1 - \tfrac{1}{2}\pi + 2\pi K = \lim_{s\to a_1^+} \omega(s) - \tfrac{1}{2}\pi + 2\pi K\,.\end{aligned}$$

Hence, by the same argument as above, we will have that $\varphi(s) > \omega(s) - \frac{1}{2}\pi + 2\pi K$, for every $s \in\,]a_1, a_2[\,$.

Iterating this process, we have that

$$\varphi(s) > \omega(s) - \tfrac{1}{2}\pi + 2\pi K\,, \qquad \text{for every } s \in \bigcup_{j=1}^{n}\,]a_{j-1}, a_j[\,,$$

and finally, by continuity, Lemma 1.2.1 and (1.10),

$$\varphi(1) \geq \omega(1) - \tfrac{1}{2}\pi + 2\pi K = \omega(0) - \tfrac{1}{2}\pi + 2\pi(K+1) > \varphi(0)\,.$$

Since $\varphi(1) - \varphi(0)$ is an integer multiple of 2π, we then deduce that

$$\varphi(1) - \varphi(0) \geq 2\pi\,,$$

i.e., that $\deg(f, \Omega) \geq 1$.

In order to show that $\deg(f, \Omega) \leq N_\iota + 1$, let us go back to $[0, a_1[\,$. Arguing as above, by (1.10) we have that

$$\varphi(s) < \omega(s) + \tfrac{3}{2}\pi + 2\pi K\,, \qquad \text{for every } s \in [0, a_1[\,.$$

If $\gamma'_-(a_1)$ points outward,

$$\varphi(a_1) < \alpha_1 + \tfrac{3}{2}\pi + 2\pi K\,, \tag{1.13}$$

and we see that, if $s > a_1$ and s is sufficiently near a_1, then $\varphi(s) < \omega(s) + \frac{3}{2}\pi + 2\pi K$, and this inequality will persist for every $s \in\,]a_1, a_2[\,$.

Now, if $\gamma'_-(a_j)$ points outward for every j, we would have

$$\varphi(s) < \omega(s) + \tfrac{3}{2}\pi + 2\pi K\,, \qquad \text{for every } s \in \bigcup_{j=1}^{n}\,]a_{j-1}, a_j[\,,$$

and, by Lemma 1.2.1 and (1.10),

$$\varphi(1) \leq \omega(1) + \tfrac{3}{2}\pi + 2\pi K = \omega(0) + \tfrac{3}{2}\pi + 2\pi(K+1) < \varphi(0) + 4\pi\,.$$

Then, $\varphi(1) - \varphi(0) \leq 2\pi$, so that $\deg(f, \Omega) \leq 1$.

On the other hand, if $\gamma'_-(a_1)$ points inward, there is no control like (1.13), and it could be as well that

$$\alpha_1 + \tfrac{3}{2}\pi + 2\pi K < \varphi(a_1) < \beta_1 + \tfrac{3}{2}\pi + 2\pi K \,,$$

giving an increase of 1 in the final computation of the degree. Clearly, the same could happen for any of the N_ι inward corner points.

The proof of Theorem 1.1.1 is thus completed. □

1.3 An extension of Theorem 1.1.1

The aim of this section is to extend Theorem 1.1.1 to the case when $\partial\Omega$ is piecewise the graph of a continuous function. However, this difficult task will not be completely achieved, and we will eventually need to assume some additional regularity on that set. Moreover, as may be expected, in this framework we will lose the upper estimate on the degree, and finally only prove that $\deg(f,\Omega) \geq 1$.

Let us start by giving a precise definition of what we mean by "piecewise graph of a continuous function". As usual, $\partial\Omega$ is a Jordan curve parametrized by a continuous function $\gamma : [0,1] \to \mathbb{R}^2$, in counter-clockwise direction.

Definition 1.3.1. *We say that $\partial\Omega$ is* piecewise the graph of a continuous function *if there are*

$$0 = \hat{a}_0 < \hat{a}_1 < \cdots < \hat{a}_{m-1} < \hat{a}_m = 1 \,,$$

such that, writing $p_k = \gamma(\hat{a}_k)$,

the closed polygonal curve $\Gamma = p_0 p_1 \cdots p_m$ has no self-intersections ;

moreover, denoting by ν_k the outer normal to the segment $\overline{p_{k-1}p_k}$ joining the two points p_{k-1} and p_k, for every $k = 1, 2, \ldots, m$ there are $h_k > 0$ and a continuous function $g_k : \overline{p_{k-1}p_k} \to [-h_k, h_k]$ such that, defining the rectangles

$$R_k = \overline{p_{k-1}p_k} + [-h_k, h_k]\nu_k \,,$$

we have that

$$\Omega \cap R_k = \{p + y\nu_k : p \in \overline{p_{k-1}p_k}\,, y \in [-h_k, g_k(p)[\,\} \,,$$
$$\partial\Omega \cap R_k = \{p + y\nu_k : p \in \overline{p_{k-1}p_k}\,, y = g_k(p)\} \,.$$

Notice that the polygonal curve Γ, being a piecewise regular Jordan curve, can be parametrized by a piecewise regular function $\gamma_\Gamma : [0,1] \to \mathbb{R}^2$ such that $\gamma_\Gamma(\hat{a}_k) = \gamma(\hat{a}_k)$, for every $k = 1, 2, \ldots, m$. Then, there is an associated *angular function* $\omega_\Gamma : [0,1] \to \mathcal{P}(\mathbb{R})$, defined precisely as in Section 1.2 (to simplify the exposition, we may assume that $\gamma_\Gamma(0)$ is a regular point for Γ, i.e., that $\gamma'_\Gamma(0) = \gamma'_\Gamma(1)$). Notice that there are no cusps for Γ, and that $\omega_\Gamma(1) = \omega_\Gamma(0) + 2\pi$, by Lemma 1.2.1.

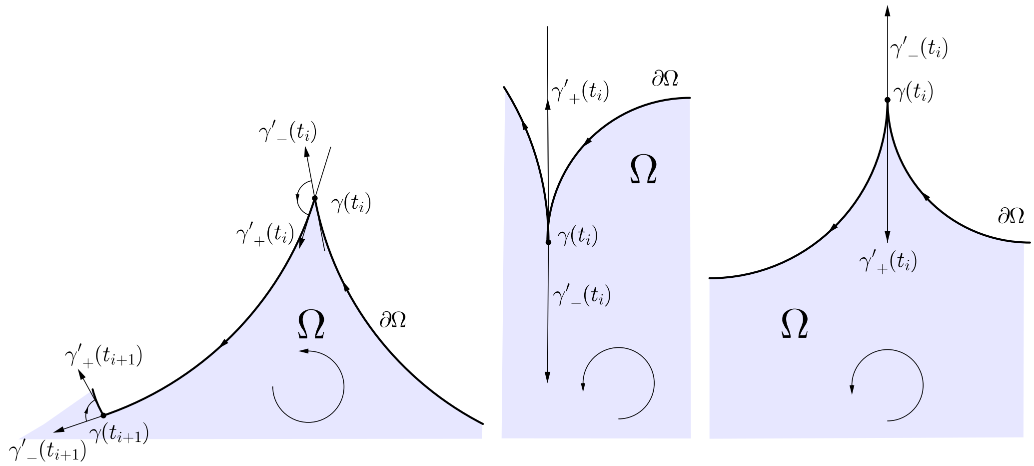


Figure 1.6: *The parametrization of the boundary of* Ω *in the case of a piecewise regular Jordan curve.*

Let us now introduce the concept of "vanishing set". Given a set $\mathcal{S}$, we denote by $\mathcal{S}'$ the *derived set* of $\mathcal{S}$, i.e., the set of cluster points of $\mathcal{S}$.

Definition 1.3.2. *Looking at the iterated derived sets*

$$\mathcal{S}^{(1)} = \mathcal{S}' , \qquad \mathcal{S}^{(n+1)} = [\mathcal{S}^{(n)}]' ,$$

we call $\mathcal{S}$ *a* vanishing set *if, for some positive integer* N*, the iterated derived set* $S^{(N)}$ *is empty.*

We will prove the following extension of Theorem 1.1.1.

Theorem 1.3.3. *Assume* $\partial\Omega$ *to be a Jordan curve, piecewise graph of a continuous function. Let* $\gamma : [0,1] \to \mathbb{R}^2$ *be a continuous parametrization of* $\partial\Omega$*, with the property that there are a countable number of non-overlapping intervals* $[a_j, b_j]$*, contained in* $[0,1]$*, on the interior of which* γ *is of class* $\mathcal{C}^1$*, and* $\mathcal{S} = [0,1] \setminus \bigcup_j \,]a_j, b_j[$ *is a vanishing set. If the avoiding cones condition* (1.1) *holds, then* $\deg(f, \Omega) \geq 1$.

The proof will be carried out in the next four subsections. We will first need to extend Hopf's Theorem in this new setting, and to characterize the normal cones with the new angular function, similarly as in Lemma 1.2.2. We will then make a small detour to provide us with some useful properties of the Dini derivatives (which could also have some independent interest). The proof of Theorem 1.3.3 will then be given first assuming the number of intervals $[a_j, b_j]$ to be finite, and finally in its general form.

1.3.1 An extension of Hopf's Theorem

We need to define the *angular function* $\omega : [0,1] \to \mathcal{P}(\mathbb{R})$ in the case when $\partial\Omega$ is piecewise the graph of a continuous function. This will eventually lead us to an extension of Hopf's Theorem.

So, take some $x \in \partial\Omega$, and assume first that $x = \gamma(s)$ for some $s \in]\hat{a}_{k-1}, \hat{a}_k[$. After a roto-translation $\mathcal{S}_k$, in which the segment $\overline{p_{k-1}p_k}$ becomes horizontal, of the type $[c_k, d_k] \times \{0\}$, we have a corresponding continuous function $F_k : [c_k, d_k] \to \mathbb{R}$, whose graph is the transformation of the graph of g_k by $\mathcal{S}_k$, and $\mathcal{S}_k(\Omega)$ locally "stays below" this graph. More precisely, we can write $\mathcal{S}_k = \mathcal{T}_k \circ \mathcal{R}_k$, where $\mathcal{T}_k$ is a translation and $\mathcal{R}_k$ is the rotation around the origin with angle

$$\hat{\theta}_\Gamma^k = \pi - \omega_\Gamma\left(\frac{\hat{a}_{k-1} + \hat{a}_k}{2}\right).$$

(Notice that ω_Γ is constant on $]\hat{a}_{k-1}, \hat{a}_k[$.) The interval $[c_k, d_k]$ has the same length as the segment $\overline{p_{k-1}p_k}$, and we will have that

$$\mathcal{S}_k(\gamma(s)) = (t(s), F_k(t(s))),$$

with $t(s) \in]c_k, d_k[$ continuously determined by $s \in]\hat{a}_{k-1}, \hat{a}_k[$ through the formula

$$t(s) = c_k + \frac{d_k - c_k}{\hat{a}_k - \hat{a}_{k-1}}(\hat{a}_k - s).$$

Moreover, $t(\hat{a}_{k-1}) = d_k$, $t(\hat{a}_k) = c_k$, and

$$F_k(t(s)) = [\mathcal{S}_k \circ g_k \circ \mathcal{S}_k^{-1}](t(s), 0).$$

To simplify the notation, we will now write F instead of F_k, and t instead of $t(s)$. We consider the four Dini derivatives

$$D_+f(x) = \liminf_{h\to 0^+} \frac{f(x+h) - f(x)}{h}, \qquad D^+f(x) = \limsup_{h\to 0^+} \frac{f(x+h) - f(x)}{h},$$
$$D_-f(x) = \liminf_{h\to 0^-} \frac{f(x+h) - f(x)}{h}, \qquad D^-f(x) = \limsup_{h\to 0^-} \frac{f(x+h) - f(x)}{h}.$$

Let

$$\mathcal{L}_-(t) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \le 0,\ x_2 = D_-F(t)\, x_1\},$$
$$\mathcal{L}^+(t) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \le 0,\ x_2 = D^+F(t)\, x_1\},$$

where it is implicitly assumed that

$$\begin{aligned} D_-F(t) = -\infty \quad &\Rightarrow \quad \mathcal{L}_-(t) = \{0\} \times [0, +\infty[\,, \\ D_-F(t) = +\infty \quad &\Rightarrow \quad \mathcal{L}_-(t) = \{0\} \times]-\infty, 0]\,, \\ D^+F(t) = -\infty \quad &\Rightarrow \quad \mathcal{L}^+(t) = \{0\} \times [0, +\infty[\,, \\ D^+F(t) = +\infty \quad &\Rightarrow \quad \mathcal{L}^+(t) = \{0\} \times]-\infty, 0]\,. \end{aligned}$$

Let $\theta_-(t)$, $\theta^+(t)$ be the two real numbers in $[\frac{\pi}{2}, \frac{3\pi}{2}]$ such that, in complex notation,

$$\mathcal{L}_-(t) = \{\alpha e^{i\theta_-(t)} : \alpha \geq 0\}\,, \qquad \mathcal{L}^+(t) = \{\alpha e^{i\theta^+(t)} : \alpha \geq 0\}\,.$$

(Notice that, whenever the right and left derivatives exist and are finite, the case $\theta_-(t) < \theta^+(t)$ corresponds to an inward corner point, while the case $\theta_-(t) > \theta^+(t)$ corresponds to an outward corner point.) We thus define

$$\omega(s) = [\alpha(s), \beta(s)]\,,$$

where

$$\alpha(s) = \theta^+(t(s)) - \hat{\theta}^k_\Gamma\,, \qquad \beta(s) = \theta_-(t(s)) - \hat{\theta}^k_\Gamma\,, \tag{1.14}$$

with the convention that $[a, b] = [b, a]$ when $b < a$.

Now we look at the cases when $s = \hat{a}_k$, for some $k = 1, 2, \dots, m$. At these points, the limits from the left have to be made with one reference function, while those from the right concern a different one. For example, looking at $s = \hat{a}_k$, the angle $\theta^+(t(\hat{a}_k))$ must be defined through the function $F_k : [c_k, d_k] \to \mathbb{R}$, with $t(\hat{a}_k) = c_k$, while $\theta_-(t(\hat{a}_k))$ is defined using $F_{k+1} : [c_{k+1}, d_{k+1}] \to \mathbb{R}$, with $t(\hat{a}_k) = d_{k+1}$. Once this is done, the definition of $\omega(\hat{a}_k)$ is

$$\omega(\hat{a}_k) = [\alpha(\hat{a}_k), \beta(\hat{a}_k)]\,,$$

where

$$\alpha(\hat{a}_k) = \theta^+(t(\hat{a}_k)) - \hat{\theta}^k_\Gamma\,, \qquad \beta(\hat{a}_k) = \theta_-(t(\hat{a}_k)) - \hat{\theta}^{k+1}_\Gamma\,, \tag{1.15}$$

with the usual convention for $[a, b]$ when $b < a$.

Having defined the multivalued function $\omega : [0, 1] \to \mathcal{P}(\mathbb{R})$, we can now state an analogue of Hopf's Theorem.

Theorem 1.3.4. *Assume that $\partial\Omega$ is piecewise the graph of a continuous function. Then,*

$$\omega(1) = \omega(0) + 2\pi\,.$$

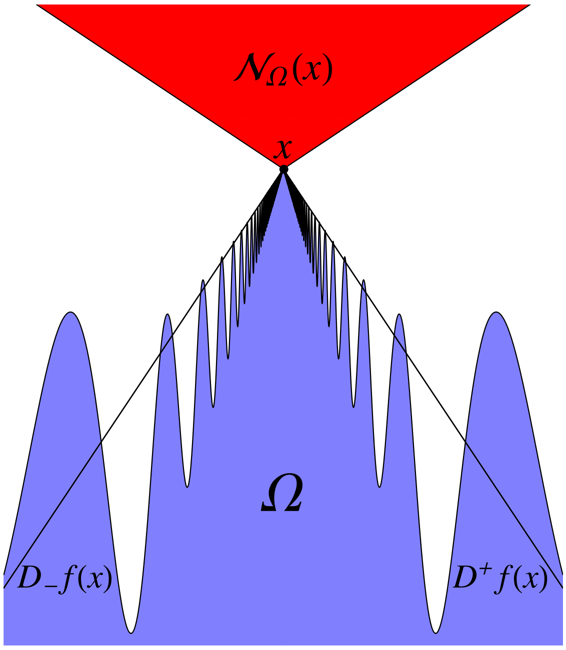


Figure 1.7: *The avoiding cones condition for the set* Ω *in a point* x *of the boundary of* Ω.

<u>Proof</u> We know that $\omega_\Gamma(1) = \omega_\Gamma(0) + 2\pi$ and, for every $s \in [0, 1]$,

$$\alpha', \beta' \in \omega_\Gamma(s) \quad \Rightarrow \quad |\alpha' - \beta'| < \pi \,.$$

Moreover, recalling the assumption that $\partial\Omega$ is piecewise the graph of a continuous function,

$$s \in \,]\hat{a}_{k-1}, \hat{a}_k[\quad \Rightarrow \quad \mathrm{dist}(\omega_\Gamma(s), \omega(s)) \leq \frac{\pi}{2} \,.$$

The conclusion easily follows. □

1.3.2 A characterization of normal cones

We now give a characterization of normal cones, similarly as in Section 1.2, when $\partial\Omega$ is piecewise the graph of a continuous function. It will be useful to consider the following multivalued function $\Theta : [0, 1] \to \mathcal{P}(\mathbb{R})$. Recalling how we have defined $\omega(s) = [\alpha(s), \beta(s)]$, we set

$$\Theta(s) = \begin{cases} \varnothing\,, & \text{if } \alpha(s) > \beta(s)\,, \\ \omega(s) - \frac{1}{2}\pi + 2\pi\mathbb{Z}\,, & \text{if } \alpha(s) \leq \beta(s)\,. \end{cases}$$

We can thus introduce an auxiliary cone $\mathcal{N}^\star_\Omega(\gamma(s))$, made of the origin and the union of all the half-lines starting from the origin determined by the angles in $\Theta(s)$, as in (1.7).

Lemma 1.3.5. *The cones $\mathcal{N}_\Omega(\gamma(s))$ and $\mathcal{N}^\star_\Omega(\gamma(s))$ coincide. Therefore, the avoiding cones condition* (1.1) *is equivalent to*

$$\varphi(s) \notin \Theta(s) , \quad \textit{for every } s \in [0,1] .$$

<u>Proof</u> We fix $s \in [0,1]$ and assume first that $s \in]\hat{a}_{k-1}, \hat{a}_k[$, for some k. After operating the roto-translation $\mathcal{S}_k$, we can assume that the segment $\overline{p_{k-1}p_k}$ coincides with $[c_k, d_k] \times \{0\}$. Moreover, without loss of generality, we can assume that $c_k < 0 < d_k$ and that $\mathcal{S}_k(\gamma(s))$ coincides with the origin.

Let $\alpha(s) > \beta(s)$, so that $\Theta(s) = \varnothing$ and $\mathcal{N}^\star_\Omega(\gamma(s)) = \{0\}$. We want to prove that $\mathcal{N}_\Omega(\gamma(s)) = \{0\}$, as well. In this case, there are two real constants $\bar{\mu} > \bar{\nu}$ such that, for every $\mu \le \bar{\mu}$ and every $\nu \ge \bar{\nu}$, the half-lines

$$\ell^+_\mu = \{(x_1, x_2) : x_1 \ge 0,\ x_2 = \mu x_1\} , \qquad \ell^-_\nu = \{(x_1, x_2) : x_1 \le 0,\ x_2 = \nu x_1\}$$

intersect the set Ω infinitely many times in every small neighborhood of the origin. Hence, for every $v \in \mathbb{R}^2 \setminus \{0\}$, it is possible to find a vector x with $\|x\| = 1$ on one of such half-lines for which $\langle v, x\rangle = \delta > 0$. Hence, there is a sequence of points $(x_n)_n$ of $\Omega \setminus \{0\}$ on this half-line such that $x_n \to 0$ and $\langle v, x_n\rangle = \delta\|x_n\|$. Therefore, if $v \neq 0$, then $v \notin \mathcal{N}_\Omega(\gamma(s))$.

Assume now that $\alpha(s) = \beta(s)$, so that $\omega(s)$ is single-valued and $\mathcal{N}^\star_\Omega(\gamma(s))$ is a half-line. For every $\varepsilon > 0$ there are two sectors $S^\varepsilon_1 \subseteq S^\varepsilon_2$, with the following properties. First of all, both sectors are symmetrical with respect to $\mathcal{N}^\star_\Omega(\gamma(s))$. The sector S^ε_2 has angular amplitude equal to $\pi + 2\varepsilon$, and there is a $\bar{r} > 0$ such that $S^\varepsilon_2 \cap B(0, r)$ contains $\Omega \cap B(0, r)$, for every $r \in]0, \bar{r}[$. The sector S^ε_1 has angular amplitude equal to $\pi - 2\varepsilon$, and every half-line of this sector intersects the set Ω infinitely many times in every small neighborhood of the origin.

Let $v \neq 0$ be a vector not belonging to the half-line $\mathcal{N}^\star_\Omega(\gamma(s))$. Then, taking $\varepsilon > 0$ small enough, it is possible to find a half-line in S^ε_1 and a point x on it, with with $\|x\| = 1$, for which $\langle v, x\rangle = \delta > 0$. Then, there is a sequence of points $(x_n)_n$ of $\Omega \setminus \{0\}$ on this half-line such that $x_n \to 0$ and $\langle v, x_n\rangle = \delta\|x_n\|$, showing that $v \notin \mathcal{N}_\Omega(\gamma(s))$. We have thus proved that $\mathcal{N}_\Omega(\gamma(s)) \subseteq \mathcal{N}^\star_\Omega(\gamma(s))$.

On the other hand, let $v \in \mathcal{N}^\star_\Omega(\gamma(s))$ be a vector with norm $\|v\| = 1$. For every $\varepsilon > 0$, there is a $\bar{r} > 0$ such that, for every $x \in \Omega \cap B(0, \bar{r})$, being $x \in S^\varepsilon_2$, one has

$$\frac{\langle v, x\rangle}{\|x\|} \le \cos\left(\frac{\pi}{2} - \varepsilon\right) . \tag{1.16}$$

Since ε is arbitrary, this shows that $v \in \mathcal{N}_\Omega(\gamma(s))$, and since $\mathcal{N}_\Omega(\gamma(s))$ is a cone, we have proved that $\mathcal{N}^\star_\Omega(\gamma(s)) \subseteq \mathcal{N}_\Omega(\gamma(s))$.

Finally, let $\alpha(s) < \beta(s)$. In this case, $\Theta(s) = [\alpha(s) - \frac{1}{2}\pi, \beta(s) - \frac{1}{2}\pi] + 2\pi\mathbb{Z}$, and $\mathcal{N}^\star_\Omega(\gamma(s))$ is a cone whose angular amplitude is $\iota(s) = \beta(s) - \alpha(s)$. We distinguish two subcases.

Case 1: $\iota(s) < \pi$. For every $\varepsilon \in]0, \frac{1}{2}(\pi - \iota(s))[$ there are two sectors $S^\varepsilon_1 \subseteq S^\varepsilon_2$, symmetrical with respect to $\mathcal{N}^\star_\Omega(\gamma(s))$. The sector S^ε_2 has angular amplitude equal to $\pi - \iota(s) + 2\varepsilon$, and there is a $\bar{r} > 0$ such that $S^\varepsilon_2 \cap B(0,r)$ contains $\Omega \cap B(0,r)$, for every $r \in]0, \bar{r}[$. The sector S^ε_1 has angular amplitude equal to $\pi - \iota(s) - 2\varepsilon$, and every half-line of this sector intersects the set Ω infinitely many times in every small neighborhood of the origin. The proof now is the same as the one seen above in the case $\alpha(s) = \beta(s)$.

Case 2: $\iota(s) = \pi$. In this case, $\mathcal{N}^\star_\Omega(\gamma(s))$ is the half-plane $\{(x_1, x_2) : x_2 \geq 0\}$. For every $\varepsilon > 0$ there is a sector S^ε, symmetrical with respect to the vertical axis, having angular amplitude equal to 2ε, and there is a $\bar{r} > 0$ such that $S^\varepsilon \cap B(0,r)$ contains $\Omega \cap B(0,r)$, for every $r \in]0, \bar{r}[$. Let $v = (v_1, v_2)$ be a vector with $\|v\| = 1$ and $v_2 > 0$. Then, for every sufficiently small $\varepsilon > 0$, taking $r \in]0, \bar{r}[$, we see that, for every $x \in \Omega \cap B(0,r)$, being $x \in S^\varepsilon$, the inequality (1.16) holds true. Since ε is arbitrary, this shows that $v \in \mathcal{N}_\Omega(\gamma(s))$. We have thus proved that $\mathcal{N}_\Omega(\gamma(s))$ contains the open set $\{(x_1, x_2) : x_2 > 0\}$. Being a closed cone, it contains $\{(x_1, x_2) : x_2 \geq 0\}$, hence $\mathcal{N}^\star_\Omega(\gamma(s)) \subseteq \mathcal{N}_\Omega(\gamma(s))$. Then, equality must hold, since $\mathcal{N}_\Omega(\gamma(s))$ cannot be larger than a half-plane.

In the case when $s = \hat{a}_k$ for some $k \in \{0, 1, \ldots, m\}$, the proof is essentially the same, in view of (1.15), taking care of distinguishing the behaviour to the left from the one to the right. We avoid the details, for briefness. □

1.3.3 A generalized version of Darboux's Theorem

In the following theorem and related corollary, we provide some important properties of the Dini derivatives, in the spirit of Darboux's Theorem.

Theorem 1.3.6. *Let $F : [a,b] \to \mathbb{R}$ be a continuous function such that, for some $\mu \in \mathbb{R}$,*

$$D^+F(a) > \mu > D_-F(b)\,. \tag{1.17}$$

Then, there is a $\xi \in]a,b[$ such that

$$D_-F(\xi) \geq \mu \geq D^+F(\xi)\,.$$

<u>Proof</u> By Weierstrass Theorem, the function $\widetilde{F}(t) = F(t) - \mu t$ has a maximum in $[a,b]$. By (1.17), a maximum point ξ must be in $]a,b[$. Then, $D_-\widetilde{F}(\xi) \geq 0 \geq D^+\widetilde{F}(\xi)$, and since

$$D_-\widetilde{F}(\xi) = D_-F(\xi) - \mu\,, \qquad D^+\widetilde{F}(\xi) = D^+F(\xi) - \mu\,,$$

the result follows. □

The following corollary will play an important role in the proof of Theorem 1.3.3.

Corollary 1.3.7. *Let $F : I \to \mathbb{R}$ be a continuous function, defined on some interval I, and let τ_0 be a point of I. Consider the set*

$$E = \{\tau \in I : D^+F(\tau) \le D_-F(\tau)\} .$$

If τ_0 is a cluster point for E from the left, then

$$D_-F(\tau_0) \ge \liminf_{\substack{\tau\to\tau_0^- \\ \tau\in E}} D^+F(\tau) . \tag{1.18}$$

Similarly, if τ_0 is a cluster point for E from the right, then

$$D^+F(\tau_0) \le \limsup_{\substack{\tau\to\tau_0^+ \\ \tau\in E}} D_-F(\tau) . \tag{1.19}$$

Proof Let us prove (1.18). Assume by contradiction that the opposite inequality holds. Then, we can find a $\delta > 0$ and a real number μ such that $[\tau_0 - \delta, \tau_0] \subseteq I$ and

$$D^+F(\tau) > \mu > D_-F(\tau_0) , \quad \text{for every } \tau \in [\tau_0 - \delta, \tau_0[\cap E . \tag{1.20}$$

Fix $\bar\tau \in [\tau_0 - \delta, \tau_0[\cap E$. By Theorem 1.3.6, there is a $\xi \in]\bar\tau, \tau_0[$ such that

$$D_-F(\xi) \ge \mu \ge D^+F(\xi) .$$

Then, we see that $\xi \in E$ and, by (1.20), it should be $D^+F(\xi) > \mu$, a contradiction. The proof of (1.19) is analogous. □

1.3.4 The proof of Theorem 1.3.3

The proof will be divided in three steps.

Step 1. First, we assume that the number of intervals $[a_j, b_j]$ is finite. Hence, besides assuming that $\partial\Omega$ is piecewise the graph of a continuous function, we also ask that there are

$$0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1 ,$$

such that, for every $j = 1, 2, \dots, n$, the restriction of γ to the *open* interval $]a_{j-1}, a_j[$ is of class $\mathcal{C}^1$, and $\gamma_j'(s) \neq 0$ for every $s \in]a_{j-1}, a_j[$. Notice that, in this setting, the limits $\lim_{s\to a_j^\pm} \gamma'(s)$ do not have to exist.

In the following, for simplicity, we will ask that $\gamma(0) = \gamma(1)$ is a regular point, i.e., that $\gamma'_+(0) = \gamma'_-(1)$. Let us start by assuming that each point a_j is contained in the interior of some $]\hat{a}_{k-1}, \hat{a}_k[$.

We consider the function $g = f \circ \gamma : [0,1] \to \mathbb{R}^2 \setminus \{0\}$ and, extending it by 1-periodicity, we write

$$g(s) = \rho(s)e^{i\varphi(s)}\,,$$

for some continuous functions $\rho : \mathbb{R} \to]0, +\infty[$ and $\varphi : \mathbb{R} \to \mathbb{R}$.

Being $\varphi(0) \notin \Theta(0)$, let $K \in \mathbb{Z}$ be such that

$$\beta(0) + 2\pi K < \varphi(0) + \tfrac{1}{2}\pi < \alpha(0) + 2\pi(K+1)\,. \tag{1.21}$$

(Here, since $\omega(0)$ is single-valued, $\alpha(0) = \beta(0)$.) By continuity and Lemma 1.3.5, it has to be that

$$\varphi(s) + \tfrac{1}{2}\pi > \beta(s) + 2\pi K\,, \qquad \text{for every } s \in [0, a_1[\,.$$

We know that $a_1 \in]\hat{a}_{k-1}, \hat{a}_k[$, for some $k \in \{1, 2, \dots, m\}$. We consider the corresponding function $t :]\hat{a}_{k-1}, \hat{a}_k[\to]c_k, d_k[$, and set $\tau_0 = t(a_1)$. Then, recalling (1.14), there exists some $\delta > 0$ for which

$$\varphi(t^{-1}(\tau)) + \tfrac{1}{2}\pi > \theta_-(\tau) - \hat{\theta}^k_\Gamma + 2\pi K\,, \qquad \text{for every } \tau \in]\tau_0, \tau_0 + \delta[\,.$$

Then, by (1.19), recalling (1.14) again,

$$\begin{aligned} \varphi(a_1) + \tfrac{1}{2}\pi &= \limsup_{\tau \to \tau_0^+} \varphi(t^{-1}(\tau)) + \tfrac{1}{2}\pi \\ &\geq \limsup_{\tau \to \tau_0^+} \theta_-(\tau) - \hat{\theta}^k_\Gamma + 2\pi K \\ &\geq \theta^+(\tau_0) - \hat{\theta}^k_\Gamma + 2\pi K = \alpha(a_1) + 2\pi K\,. \end{aligned} \tag{1.22}$$

(Here the set E plays no role.) We have two possibilities.

Case 1: $D_-F_k(\tau_0) \geq D^+F_k(\tau_0)$. Then, by Lemma 1.3.5 and (1.22), it has to be that

$$\varphi(a_1) + \tfrac{1}{2}\pi > \beta(a_1) + 2\pi K\,. \tag{1.23}$$

Case 2: $D_-F_k(\tau_0) < D^+F_k(\tau_0)$. Then, $\alpha(a_1) > \beta(a_1)$, and from (1.22) we get (1.23) again.

On the other hand, by (1.14) and (1.18),

$$\beta(a_1) = \theta_-(\tau_0) - \hat{\theta}^k_\Gamma \geq \liminf_{\tau \to \tau_0^-} \theta^+(\tau) - \hat{\theta}^k_\Gamma\,.$$

(Even here the set E plays no role.) So, by (1.23), there are a sufficiently small $\varepsilon > 0$ and a strictly increasing sequence $(\tau_n)_n$ such that $\lim_n \tau_n = \tau_0$ and, setting $s_n = t^{-1}(\tau_n)$, by (1.14),

$$\varphi(a_1) + \tfrac{1}{2}\pi - \varepsilon > \theta^+(\tau_n) - \hat{\theta}^k_\Gamma + 2\pi K = \alpha(s_n) + 2\pi K = \beta(s_n) + 2\pi K\,.$$

(Here $\alpha(s_n) = \beta(s_n)$, being γ of class $\mathcal{C}^1$ on $]a_1, a_2[$.) Since $s_n \to a_1$, by continuity, for n large enough,

$$\varphi(s_n) + \tfrac{1}{2}\pi > \beta(s_n) + 2\pi K\,.$$

Hence, by Lemma 1.3.5 and the continuity of φ and β on $]a_1, a_2[$,

$$\varphi(s) + \tfrac{1}{2}\pi > \beta(s) + 2\pi K\,, \qquad \text{for every } s \in]a_1, a_2[\,.$$

Iterating this procedure on each interval $]a_{j-1}, a_j[$, we thus prove that

$$\varphi(s) + \tfrac{1}{2}\pi > \beta(s) + 2\pi K\,, \qquad \text{for every } s \in]a_{j-1}, a_j[\,.$$

By continuity and Theorem 1.3.4, recalling that $\omega(1)$ is single-valued and using also (1.21),

$$\varphi(1) + \tfrac{1}{2}\pi \geq \beta(1) + 2\pi K = \alpha(1) + 2\pi K = \alpha(0) + 2\pi(K+1) > \varphi(0) + \tfrac{1}{2}\pi\,.$$

Since $\varphi(1) - \varphi(0)$ is an integer multiple of 2π, we then deduce that

$$\varphi(1) - \varphi(0) \geq 2\pi\,,$$

and the proof is completed. In the case when some a_j coincides with some $\hat{a}_k$ the proof is easily adapted, in view of the definition given in (1.15), taking care of the different functions involved when approaching a_j from the left and from the right.

Step 2. As a second step, we now assume that there are a countable number of non-overlapping intervals $[a_j, b_j]$, contained in $[0, 1]$, on the interior of which γ is of class $\mathcal{C}^1$, and that the *singular set*

$$\mathcal{S} = [0,1] \setminus \bigcup_{j=0}^{\infty}]a_j, b_j[$$

has a finite number of cluster points $a'_1 < a'_2 < \cdots < a'_N$.

Keeping the same notations as above, for simplicity we ask that $\gamma(0) = \gamma(1)$ be a regular point, and we first assume that each point $a'_1, a'_2, \ldots, a'_N$ is contained in the interior of some $]\hat{a}_{k-1}, \hat{a}_k[$. Let $K \in \mathbb{Z}$ be such that (1.21) holds. Then, by induction, using the result proved in Step 1, we see that

$$\varphi(s) + \tfrac{1}{2}\pi > \beta(s) + 2\pi K\,, \qquad \text{for every } s \in [0, a'_1[\,.$$

We know that $a'_1 \in]\hat{a}_{k-1}, \hat{a}_k[$, for some $k \in \{1, 2, \dots, m\}$. We consider the corresponding function $t :]\hat{a}_{k-1}, \hat{a}_k[\to]c_k, d_k[$, and set $\tau'_0 = t(a'_1)$. Using (1.19), we see like in (1.22) that $\varphi(a'_1) + \frac{1}{2}\pi \geq \alpha(a'_1) + 2\pi K$ (here the set E plays no role). We now have two possibilities.

Case 1: $D_-F_k(\tau'_0) \geq D^+F_k(\tau'_0)$. Then, by Lemma 1.3.5, it has to be that

$$\varphi(a'_1) + \tfrac{1}{2}\pi > \beta(a'_1) + 2\pi K\,. \tag{1.24}$$

Case 2: $D_-F_k(\tau'_0) < D^+F_k(\tau'_0)$. Then, $\alpha(a'_1) > \beta(a'_1)$, and we get (1.24) again.

Now, using (1.18), there is a strictly decreasing sequence $(s_n)_n$ such that

$$\lim_n s_n = a'_1\,, \quad \alpha(s_n) \leq \beta(s_n)\,, \quad \lim_n \alpha(s_n) \leq \beta(a'_1)\,.$$

(In this case, the set E plays a crucial role.) By (1.24), taking $\varepsilon > 0$ small enough,

$$\varphi(a'_1) + \tfrac{1}{2}\pi - \varepsilon > \alpha(s_n) + 2\pi K\,,$$

for every sufficiently large n. Being $\Theta(s_n) = [\alpha(s_n), \beta(s_n)] - \frac{1}{2}\pi + 2\pi\mathbb{Z}$, with $\alpha(s_n) \leq \beta(s_n)$, by Lemma 1.3.5 we have that

$$\varphi(a'_1) + \tfrac{1}{2}\pi - \varepsilon > \beta(s_n) + 2\pi K\,,$$

so that, by continuity, for n large enough,

$$\varphi(s_n) + \tfrac{1}{2}\pi > \beta(s_n) + 2\pi K\,.$$

We can now use the argument at the end of Step 1 to show that

$$\varphi(s) + \tfrac{1}{2}\pi > \beta(s) + 2\pi K\,, \qquad \text{for every } s \in]a'_1, a'_2[\,.$$

Iterating this procedure, we easily conclude the proof. The case when some a'_j coincides with some $\hat{a}_k$ is treated similarly, as already observed above.

Step 3. We have thus shown that the topological degree is a positive number if $\mathcal{S}'$, the derived set of $\mathcal{S}$, is finite. We can now repeat the argument in Step 2 assuming that $\mathcal{S}'$ is an infinite set, with a finite number of cluster points. And this procedure can be carried on an arbitrary finite number of times. Since we have assumed that $\mathcal{S}$ is a vanishing set, we will eventually reach an iterated derived set having only a finite number of points. The proof is then completed using once again the argument in Step 2. □

Chapter 2

Well-ordered and non-well-ordered lower and upper solutions for periodic planar systems

2.1 Introduction

The method of lower and upper solutions for scalar second order differential equations of the type

$$x'' = g(t, x, x')$$

can be dated back to the pioneering papers by Picard [87], Scorza Dragoni [96] and Nagumo [77], dealing with separated boundary conditions. Its full extension to the periodic problem is due to Knobloch [64]. Further extensions to partial differential equations of elliptic or parabolic type have also been proposed, and there is nowadays a huge literature on this subject. For a rather complete historical and bibliographical account, we refer to the book [29].

Recently Fonda and Toader [52], extended the main idea in the definition of lower and upper solutions to planar systems of ordinary differential equations, with the aim of finding bounded solutions through the method of Ważewski [103]. As a by-product, the theorem of Massera [66] provided also the existence of periodic solutions. It is the aim of this chapter to further develop this theory, concentrating on the periodic problem, by the use of topological degree methods.

We consider the periodic problem

$$(P) \qquad \begin{cases} x' = f(t, x, y)\,, \quad y' = g(t, x, y)\,, \\ x(0) = x(T)\,, \quad y(0) = y(T)\,, \end{cases}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ are continuous functions, T-periodic in their first variable. Our purpose is to give a general definition of a lower and an upper solution with the aim of obtaining the existence of a solution to problem (P).

In the first part we introduce our main definitions and provide some remarks and preliminaries needed in the sequel. In the second part we prove an existence result in the well-ordered case $\alpha \leq \beta$, assuming (like in [52]) the existence of some bounding curves, in order to control the solutions in the phase plane. The construction of these curves can be easily carried out in concrete examples, assuming a Nagumo-type condition (see [52] or Lemma 2.4.2 below).

The next step is to extend the result dealing with the non-well-ordered case. Here we need to ask an extra technical condition on the lower and upper solutions; it remains an open question if it could possibly be avoided. Moreover, we assume the existence of a whole family of bounding curves. This assumption is again verified under some type of Nagumo conditions.

We conclude presenting some variants of our main theorems and discuss on the possibility of further extending the theory to higher dimensional systems.

2.2 Main definitions and preliminaries

For any function $\nu : \mathbb{R} \to \mathbb{R}$ we use the notation

$$\nu(\tau^-) = \lim_{t \to \tau^-} \nu(t)\,, \qquad \nu(\tau^+) = \lim_{t \to \tau^+} \nu(t)\,.$$

Definition 2.2.1. *A continuous function $\alpha : \mathbb{R} \to \mathbb{R}$ is said to be a* lower solution *for problem (P) if it is T-periodic and there exist a T-periodic function $y_\alpha : \mathbb{R} \to \mathbb{R}$ and a finite number of points $0 = \tau_0 < \tau_1 < \cdots < \tau_n = T$ such that the following properties hold:*

1. *the restriction of α [resp. y_α] to each open interval $]\tau_{k-1}, \tau_k[$, with $k \in \{1, \ldots, n\}$, is continuously differentiable [resp. differentiable];*
2. *$\alpha'(\tau_k^\pm)$ and $y_\alpha(\tau_k^\pm)$ exist in $\mathbb{R}$ for every $k \in \{1, \ldots, n\}$, with*
$$\alpha'(\tau_k^-) \leq \alpha'(\tau_k^+) \quad \text{and} \quad y_\alpha(\tau_k^-) \leq y_\alpha(\tau_k^+)\,; \tag{2.1}$$
3. *for every $t \in \cup_{k=1}^n \,]\tau_{k-1}, \tau_k[$,*
$$\begin{cases} y < y_\alpha(t) & \Rightarrow \quad f(t, \alpha(t), y) < \alpha'(t)\,, \\ y > y_\alpha(t) & \Rightarrow \quad f(t, \alpha(t), y) > \alpha'(t)\,, \end{cases} \tag{2.2}$$

and

$$y'_\alpha(t) \geq g(t, \alpha(t), y_\alpha(t))\,. \tag{2.3}$$

Definition 2.2.2. *A continuous function $\beta : \mathbb{R} \to \mathbb{R}$ is said to be an* upper solution *for problem (P) if it is T-periodic and there exist a T-periodic function $y_\beta : \mathbb{R} \to \mathbb{R}$ and a finite number of points $0 = \tau_0 < \tau_1 < \cdots < \tau_n = T$ such that the following properties hold:*

1. *the restriction of β [resp. y_β] to each open interval $]\tau_{k-1}, \tau_k[$, with $k \in \{1, \dots, n\}$, is continuously differentiable [resp. differentiable];*
2. *$\beta'(\tau_k^\pm)$ and $y_\beta(\tau_k^\pm)$ exist in $\mathbb{R}$ for every $k \in \{1, \dots, n\}$, with*
$$\beta'(\tau_k^-) \geq \beta'(\tau_k^+) \quad \textit{and} \quad y_\beta(\tau_k^-) \geq y_\beta(\tau_k^+)\,; \tag{2.4}$$
3. *for every $t \in \cup_{k=1}^n]\tau_{k-1}, \tau_k[$,*
$$\begin{cases} y < y_\beta(t) & \Rightarrow \quad f(t, \beta(t), y) < \beta'(t)\,, \\ y > y_\beta(t) & \Rightarrow \quad f(t, \beta(t), y) > \beta'(t)\,, \end{cases} \tag{2.5}$$
and
$$y'_\beta(t) \leq g(t, \beta(t), y_\beta(t))\,. \tag{2.6}$$

In what follows, when dealing with a couple (α, β) of a lower and an upper solution, we will assume, without loss of generality, that the points $\{\tau_0, \tau_1, \dots, \tau_n\}$ provided in the previous definitions are the same, both for α and β. Moreover, since we are dealing with T-periodic functions, it is worth defining the sets

$$\mathcal{J} := \{t = \tau_k + \iota T \mid k \in \{1, \dots, n\}\,,\ \iota \in \mathbb{Z}\}\,, \qquad \mathcal{I} := \mathbb{R} \setminus \mathcal{J}\,.$$

Therefore, (2.1), (2.4) hold with τ_k replaced by any $\tau \in \mathcal{J}$, and (2.2), (2.3), (2.5), (2.6) hold for every $t \in \mathcal{I}$.

Remark 2.2.3. *When dealing with the periodic problem associated with the scalar equation*

$$(\phi(x'))' = h(t, x, x')\,, \tag{2.7}$$

the usual definitions of lower/upper solutions are contained in the above ones, taking $f(t, x, y) = \phi^{-1}(y)$, $g(t, x, y) = h(t, x, \phi^{-1}(y))$, and defining $y_\alpha(t) = \phi(\alpha'(t))$, $y_\beta(t) = \phi(\beta'(t))$. Indeed, the conditions $\alpha(0) = \alpha(T)$, $\beta(0) = \beta(T)$ permit to continuously extend the functions $\alpha, \beta : [0, T] \to \mathbb{R}$ to the whole real line $\mathbb{R}$, and the conditions $\alpha'(0) \geq \alpha'(T)$, $\beta'(0) \leq \beta'(T)$ are included in (2.1), (2.4)*. The possibility of having some discontinuity points τ_k can be useful in the applications, e.g., when taking as a lower solution the maximum of two or more smooth lower solutions, and as an upper solution the minimum of two or more smooth upper solutions.*

From (2.2) we have that

$$\alpha'(t) = f(t, \alpha(t), y_\alpha(t)), \quad \text{for every } t \in \mathcal{I}, \tag{2.8}$$

and $y_\alpha(t)$ is the only value for which this identity holds. Similarly, from (2.5) we have

$$\beta'(t) = f(t, \beta(t), y_\beta(t)), \quad \text{for every } t \in \mathcal{I}, \tag{2.9}$$

and $y_\beta(t)$ is uniquely defined on $\mathcal{I}$ by this identity.

It is well known in the case of scalar second order equations that if a function is at the same time a lower and an upper solution, then it is a solution. Let us write the analogous statement in our situation.

Proposition 2.2.4. *Let $x : \mathbb{R} \to \mathbb{R}$ be at the same time a lower and an upper solution for problem (P). Then, there exists a function $y : \mathbb{R} \to \mathbb{R}$ such that (x, y) is a solution of problem (P).*

Proof. Denote by y_α and y_β the functions provided by Definitions 2.2.1 and 2.2.2 taking $x = \alpha$ and $x = \beta$, respectively. From (2.8) and (2.9) we deduce that

$$x'(t) = f(t, x(t), y_\alpha(t)) \quad \text{and} \quad y_\alpha(t) = y_\beta(t), \quad \text{for every } t \in \mathcal{I}.$$

Then, from (2.1) and (2.4) we first see that $x'(\tau_k^-) = x'(\tau_k^+)$, thus implying that $x : \mathbb{R} \to \mathbb{R}$ is continuously differentiable; moreover, on one hand we have $y_\alpha(\tau_k^-) \leq y_\alpha(\tau_k^+)$, and on the other hand

$$y_\alpha(\tau_k^-) = y_\beta(\tau_k^-) \geq y_\beta(\tau_k^+) = y_\alpha(\tau_k^+),$$

showing that $y_\alpha(\tau_k^\pm) = y_\beta(\tau_k^\pm)$ for every k. We can thus define

$$y(t) = \begin{cases} y_\alpha(t), & \text{if } t \in \mathcal{I}, \\ y_\alpha(t^\pm), & \text{if } t \in \mathcal{J}, \end{cases}$$

a continuous function.

Since $x : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $y : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^3 \to \mathbb{R}$ are continuous, from (2.8) we deduce that $x'(t) = f(t, x(t), y(t))$ for every $t \in \mathbb{R}$. Moreover, by (2.3) and (2.6) we get $y'(t) = g(t, x(t), y(t))$ for every $t \in \mathcal{I}$; since $y : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ are continuous, we first see that $y : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, and then also that $y'(t) = g(t, x(t), y(t))$ for every $t \in \mathbb{R}$, thus completing the proof. $\square$

We will need the following estimates involving our lower and upper solutions, where we adopt the usual definition of the Dini derivatives:

$$D_\pm F(t_0) = \liminf_{t \to t_0^\pm} \frac{F(t) - F(t_0)}{t - t_0}, \qquad D^\pm F(t_0) = \limsup_{t \to t_0^\pm} \frac{F(t) - F(t_0)}{t - t_0}.$$

Proposition 2.2.5. *If α is a lower solution for problem (P), then*

$$D_{\pm}y_{\alpha}(\tau) \geq g(\tau, \alpha(\tau), y_{\alpha}(\tau^{\pm}))\,, \quad \textit{for every } \tau \in \mathcal{J}\,.$$

If β is an upper solution for problem (P), then

$$D^{\pm}y_{\beta}(\tau) \leq g(\tau, \beta(\tau), y_{\beta}(\tau^{\pm}))\,, \quad \textit{for every } \tau \in \mathcal{J}\,.$$

Proof. Let us fix k and consider the restrictions of the functions y_α and y_β to the interval $[\tau_k, \tau_{k+1}]$, redefining the two functions at the extremes in such a way to make them continuous. Then, since both y_α and y_β are differentiable in the interval $]\tau_k, \tau_{k+1}[$, by [43, Corollary 3.7] we have

$$\begin{aligned} D_{-}y_{\alpha}(\tau_{k+1}) &\geq \liminf_{t \to \tau_{k+1}^{-}} D^{+}y_{\alpha}(t) = \liminf_{t \to \tau_{k+1}^{-}} y_{\alpha}'(t) \\ &\geq \liminf_{t \to \tau_{k+1}^{-}} g(t, \alpha(t), y_{\alpha}(t)) = g(\tau_{k+1}, \alpha(\tau_{k+1}), y_{\alpha}(\tau_{k+1}^{-}))\,, \end{aligned}$$

and

$$\begin{aligned} D^{+}y_{\beta}(\tau_{k}) &\leq \limsup_{t \to \tau_{k}^{+}} D_{-}y_{\beta}(t) = \limsup_{t \to \tau_{k}^{+}} y_{\beta}'(t) \\ &\leq \limsup_{t \to \tau_{k}^{+}} g(t, \beta(t), y_{\beta}(t)) = g(\tau_{k}, \beta(\tau_{k}), y_{\beta}(\tau_{k}^{+}))\,. \end{aligned}$$

Similarly, we have

$$\begin{aligned} D_{+}y_{\alpha}(\tau_{k}) &\geq \liminf_{t \to \tau_{k}^{+}} D^{-}y_{\alpha}(t) = \liminf_{t \to \tau_{k}^{+}} y_{\alpha}'(t) \\ &\geq \liminf_{t \to \tau_{k}^{+}} g(t, \alpha(t), y_{\alpha}(t)) = g(\tau_{k}, \alpha(\tau_{k}), y_{\alpha}(\tau_{k}^{+}))\,, \end{aligned}$$

and

$$\begin{aligned} D^{-}y_{\beta}(\tau_{k+1}) &\leq \limsup_{t \to \tau_{k+1}^{-}} D_{+}y_{\beta}(t) = \limsup_{t \to \tau_{k+1}^{-}} y_{\beta}'(t) \\ &\leq \limsup_{t \to \tau_{k+1}^{-}} g(t, \beta(t), y_{\beta}(t)) = g(\tau_{k+1}, \beta(\tau_{k+1}), y_{\beta}(\tau_{k+1}^{-}))\,, \end{aligned}$$

thus ending the proof. □

2.3 Well-ordered lower and upper solutions

We will say that (α, β) is a well-ordered couple of lower/upper solutions of problem (P) if α and β are respectively a lower and an upper solution of problem (P), and $\alpha(t) \leq \beta(t)$ for every $t \in \mathbb{R}$. The following result generalizes that part of [52, Theorem 2.5] concerning the existence of periodic solutions.

Theorem 2.3.1. *Assume the existence of a well-ordered couple* (α, β) *of lower/upper solutions of problem* (P). *Set* $A = \min \alpha$ *and* $B = \max \beta$, *with* $A < B$. *Let there exist two continuously differentiable functions* $\gamma_\pm : [A, B] \to \mathbb{R}$ *such that, for every* $t \in \mathbb{R}$ *and* $x \in [\alpha(t), \beta(t)]$,

$$\gamma_-(x) < \min\{y_\alpha(t^-), y_\beta(t^+)\} \leq \max\{y_\alpha(t^+), y_\beta(t^-)\} < \gamma_+(x)\,,$$

and

$$g(t, x, \gamma_-(x)) < f(t, x, \gamma_-(x))\gamma'_-(x) \tag{2.10}$$

$$g(t, x, \gamma_+(x)) > f(t, x, \gamma_+(x))\gamma'_+(x)\,. \tag{2.11}$$

Then there exists at least one solution of problem (P) *such that*

$$\alpha(t) \leq x(t) \leq \beta(t) \quad \textit{and} \quad \gamma_-(x(t)) < y(t) < \gamma_+(x(t))\,,$$

for every $t \in \mathbb{R}$.

Some remarks are in order.

1) We will discuss in Section 2.5 on the possibility of reversing the inequalities in (2.10) and (2.11).

2) We will provide in Lemma 2.4.2 some Nagumo-type conditions which guarantee the existence of the curves $\gamma_\pm$.

3) The assumption $A < B$ is inessential, since if $A = B$ we have that $\alpha = \beta$, hence by Proposition 2.2.4 we immediately get a solution.

2.3.1 Proof of Theorem 2.3.1

2.3.1.1 An auxiliary problem

Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^2$ be defined as

$$\Phi(t, x, y) = (f(t, x, y), g(t, x, y))\,.$$

Fix $D > 0$ such that

$$-D < \gamma_-(x) < \gamma_+(x) < D\,, \quad \text{for every } x \in [A, B]\,.$$

Define

$$\|\alpha'\|_\infty = \max_{t\in[0,T]} |\alpha'(t^\pm)|\,, \qquad \|\beta'\|_\infty = \max_{t\in[0,T]} |\beta'(t^\pm)|\,, \tag{2.12}$$

$$\mu_1 = \max_{t\in[0,T]} |f(t, \alpha(t), \gamma_\pm(\alpha(t)))|\,, \qquad \mu_2 = \max_{t\in[0,T]} |f(t, \beta(t), \gamma_\pm(\beta(t)))|\,,$$

choose

$$M_X > \max\{\mu_1, \mu_2, \|\alpha'\|_\infty, \|\beta'\|_\infty\}\,, \tag{2.13}$$

and

$$M_Y > \|\gamma'_\pm\|_\infty M_X\,. \tag{2.14}$$

We interpolate the vector field $\Phi(t,x,y)$ on $\{A \le x \le B\,,\, \gamma_-(x) \le y \le \gamma_+(x)\}$ with a constant vector field on $\{A \le x \le B\,,\, |y| \ge D\}$. Precisely, we define $\widehat{\Phi} : \mathbb{R} \times [A,B] \times \mathbb{R} \to \mathbb{R}^2$ as

$$\widehat{\Phi}(t,x,y) = \begin{cases} (M_X, M_Y)\,, & \text{if } y \ge D\,, \\ \Phi(t,x,\gamma_+(x)) + \dfrac{y-\gamma_+(x)}{D-\gamma_+(x)}\Big((M_X,M_Y) - \Phi(t,x,\gamma_+(x))\Big), & \\ & \text{if } \gamma_+(x) \le y \le D\,, \\ \Phi(t,x,y)\,, & \text{if } \gamma_-(x) \le y \le \gamma_+(x)\,, \\ \Phi(t,x,\gamma_-(x)) - \dfrac{y-\gamma_-(x)}{D+\gamma_-(x)}\Big((-M_X,-M_Y) - \Phi(t,x,\gamma_-(x))\Big), & \\ & \text{if } -D \le y \le \gamma_-(x)\,, \\ (-M_X, -M_Y)\,, & \text{if } y \le -D\,. \end{cases}$$

We will write $\widehat{\Phi}(t,x,y) = \Big(\hat f(t,x,y), \hat g(t,x,y)\Big)$.

By the use of the auxiliary functions

$$\zeta(s;\mu,\nu) = \begin{cases} \mu\,, & \text{if } s < \mu\,, \\ s\,, & \text{if } \mu \le s \le \nu\,, \\ \nu\,, & \text{if } s > \nu\,, \end{cases}$$

and

$$e(s;\mu,\nu) = s - \zeta(s;\mu,\nu) = \begin{cases} s-\mu\,, & \text{if } s < \mu\,, \\ 0\,, & \text{if } \mu \le s \le \nu\,, \\ s-\nu\,, & \text{if } s > \nu\,, \end{cases}$$

we define, for every $(t,x,y) \in \mathbb{R}^3$,

$$\begin{aligned} \tilde f(t,x,y) =& \hat f\Big(t, \zeta\big(x;\alpha(t),\beta(t)\big)\,,\, \zeta\big(y;-D,D\big)\Big) + e\big(y;-D,D\big)\,, \\ \tilde g(t,x,y) =& \hat g\Big(t\,,\, \zeta\big(x;\alpha(t),\beta(t)\big)\,,\, \zeta\big(y;-D,D\big)\Big) + e\big(x;\alpha(t),\beta(t)\big)\,, \end{aligned}$$

so to introduce the modified problem

$$(\widetilde{P}) \qquad \begin{cases} x' = \tilde{f}(t,x,y)\,, \quad y' = \tilde{g}(t,x,y)\,, \\ x(0) = x(T)\,, \quad y(0) = y(T)\,. \end{cases}$$

We will write $\widetilde{\Phi}(t,x,y) = \big(\tilde{f}(t,x,y), \tilde{g}(t,x,y)\big)$. In the space

$$\mathcal{C}^0_T = \big\{ v \in \mathcal{C}^0([0,T],\mathbb{R}^2) : v(0) = v(T) \big\}$$

we introduce the open set

$$\mathcal{V} = \{ u \in \mathcal{C}^0_T \mid (t,u(t)) \in V \text{ for every } t \in [0,T] \}\,, \tag{2.15}$$

where, see Figure 2.1,

$$V = \{ (t,x,y) \in \mathbb{R}^3 \mid \alpha(t) < x < \beta(t)\,,\ \gamma_-(x) < y < \gamma_+(x) \}\,.$$

Our aim is to prove that there exists a solution $u = (x,y)$ of problem $(\widetilde{P})$ belonging to $\overline{\mathcal{V}}$. Since $\tilde{f} = f$ and $\tilde{g} = g$ on the set $\overline{V}$, then u will solve also (P).

2.3.1.2 No solutions of $(\widetilde{P})$ outside $\overline{\mathcal{V}}$

We show that all the solutions $u = (x,y)$ of system $(\widetilde{P})$ are such that $(t,u(t)) \in \overline{V}$, for every $t \in \mathbb{R}$.

Let us start proving a preliminary lemma.

Lemma 2.3.2. *For every $t \in \mathcal{I}$, the following inequalities hold:*

$$\begin{cases} \tilde{f}(t,x,y) < \alpha'(t)\,, & \text{if } x \le \alpha(t) \text{ and } y < y_\alpha(t)\,, \\ \tilde{f}(t,x,y) > \alpha'(t)\,, & \text{if } x \le \alpha(t) \text{ and } y > y_\alpha(t)\,; \end{cases} \tag{2.16}$$

$$\begin{cases} \tilde{f}(t,x,y) < \beta'(t)\,, & \text{if } x \ge \beta(t) \text{ and } y < y_\beta(t)\,, \\ \tilde{f}(t,x,y) > \beta'(t)\,, & \text{if } x \ge \beta(t) \text{ and } y > y_\beta(t)\,; \end{cases} \tag{2.17}$$

$$\begin{cases} \tilde{g}(t,x,y_\alpha(t)) < y_\alpha'(t)\,, & \text{if } x < \alpha(t)\,, \\ \tilde{g}(t,x,y_\beta(t)) > y_\beta'(t)\,, & \text{if } x > \beta(t)\,. \end{cases} \tag{2.18}$$

Moreover, for every $\tau \in \mathcal{J}$,

$$\begin{cases} \tilde{g}(\tau,x,y_\alpha(\tau^\pm)) < D_\pm y_\alpha(\tau)\,, & \text{if } x < \alpha(\tau)\,, \\ \tilde{g}(\tau,x,y_\beta(\tau^\pm)) > D^\pm y_\beta(\tau)\,, & \text{if } x > \beta(\tau)\,. \end{cases} \tag{2.19}$$

Proof. Let us prove the first inequality in (2.16). Suppose $t \in \mathcal{I}$, $x \leq \alpha(t)$ and $y < y_\alpha(t)$. We have that

$$\begin{aligned}\tilde{f}(t,x,y) &= \hat{f}\Big(t, \zeta\big(x;\alpha(t),\beta(t)\big)\,,\,\zeta\big(y;-D,D\big)\Big) + e(y;-D,D)\\ &= \hat{f}\Big(t,\alpha(t)\,,\,\zeta\big(y;-D,D\big)\Big) + e(y;-D,D)\,.\end{aligned}$$

We need to consider three different cases.

Case 1. If $\gamma_-(\alpha(t)) \leq y < y_\alpha(t)$, then

$$\tilde{f}(t,x,y) = \hat{f}(t,\alpha(t),y) = f(t,\alpha(t),y) < \alpha'(t)\,.$$

Case 2. If $-D \leq y < \gamma_-(\alpha(t))$, then

$$\begin{aligned}&\tilde{f}(t,x,y) = \hat{f}(t,\alpha(t),y)\\ &= f(t,\alpha(t),\gamma_-(\alpha(t))) - \frac{y-\gamma_-(\alpha(t))}{D+\gamma_-(\alpha(t))}\Big[-M_X - f(t,\alpha(t),\gamma_-(\alpha(t)))\Big]\\ &\leq f(t,\alpha(t),\gamma_-(\alpha(t))) < \alpha'(t)\,.\end{aligned}$$

Case 3. If $y < -D$ then, by (2.13),

$$\tilde{f}(t,x,y) = \hat{f}(t,\alpha(t),-D) + y + D = -M_X + y + D < -M_X < \alpha'(t)\,.$$

Hence, the first inequality in (2.16) is proved. The second one can be proved analogously, as well as the inequalities in (2.17).

We now prove the first inequality of (2.18). Let $x < \alpha(t)$. Since $-D < \gamma_-(\alpha(t)) \leq y_\alpha(t) \leq \gamma_+(\alpha(t)) < D$, we have

$$\begin{aligned}\tilde{g}(t,x,y_\alpha(t)) &= \hat{g}\Big(t\,,\,\zeta\big(x;\alpha(t),\beta(t)\big)\,,\,\zeta\big(y_\alpha(t);-D,D\big)\Big) + e\big(x;\alpha(t),\beta(t)\big)\\ &= \hat{g}(t,\alpha(t),y_\alpha(t)) + x - \alpha(t)\\ &< \hat{g}(t,\alpha(t),y_\alpha(t))\\ &= g(t,\alpha(t),y_\alpha(t)) \leq y_\alpha'(t)\,.\end{aligned}$$

The second inequality in (2.18) follows analogously, and a similar computation proves the ones in (2.19). □

Let us define the sets

$$\begin{aligned}A_{NW} &= \{(t,x,y) \in \mathbb{R}^3 \mid x < \alpha(t)\,,\; y > y_\alpha(t^+)\}\,,\\ A_{SW} &= \{(t,x,y) \in \mathbb{R}^3 \mid x < \alpha(t)\,,\; y < y_\alpha(t^-)\}\,,\\ A_{NE} &= \{(t,x,y) \in \mathbb{R}^3 \mid x > \beta(t)\,,\; y > y_\beta(t^-)\}\,,\\ A_{SE} &= \{(t,x,y) \in \mathbb{R}^3 \mid x > \beta(t)\,,\; y < y_\beta(t^+)\}\end{aligned}$$

(see Figure 2.1).

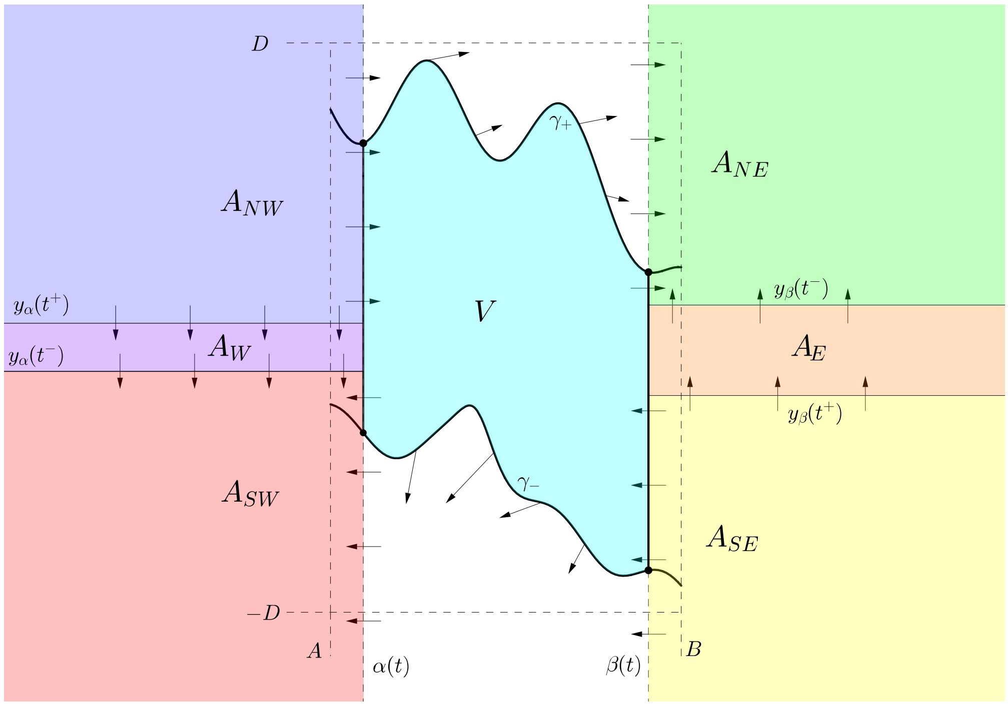


Figure 2.1: A sketch of the section at a fixed time t of the regions where to study the dynamics of $u' = \widetilde{\Phi}(t,u)$. Notice that the vertical lines $x = \alpha$, $x = \beta$ and the horizontal lines $y = y_\alpha$, $y = y_\beta$ move in time, while the curves $\gamma_\pm$ are fixed.

Lemma 2.3.3. *For every solution $u = (x,y)$ of*

$$x' = \tilde{f}(t,x,y)\,, \qquad y' = \tilde{g}(t,x,y)\,, \tag{2.20}$$

the following assertions hold true:

$$\begin{aligned}(t_0,u(t_0)) \in A_{NW} &\Rightarrow (t,u(t)) \in A_{NW} \text{ for every } t < t_0\,,\\ (t_0,u(t_0)) \in A_{SE} &\Rightarrow (t,u(t)) \in A_{SE} \text{ for every } t < t_0\,,\\ (t_0,u(t_0)) \in A_{NE} &\Rightarrow (t,u(t)) \in A_{NE} \text{ for every } t > t_0\,,\\ (t_0,u(t_0)) \in A_{SW} &\Rightarrow (t,u(t)) \in A_{SW} \text{ for every } t > t_0\,.\end{aligned}$$

Proof. We will prove only the validity of the first assertion, since the others follow similarly. We argue by contradiction and assume the existence of $t_1 < t_0$ and of a solution $u = (x,y)$ of (2.20) such that $(t,u(t)) = (t,x(t),y(t)) \in A_{NW}$ for every $t \in]t_1,t_0]$ and $(t_1,u(t_1)) = (t_1,x(t_1),y(t_1)) \in \partial A_{NW}$, where (see Figure 2.2)

$$\begin{aligned}\partial A_{NW} =&\{(t,x,y) \in \mathbb{R}^3 \mid x = \alpha(t)\,,\, y \geq y_\alpha(t^+)\}\\ &\cup \{(t,x,y) \in \mathbb{R}^3 \mid x \leq \alpha(t)\,,\, y_\alpha(t^-) \leq y(t) \leq y_\alpha(t^+)\}\,.\end{aligned} \tag{2.21}$$

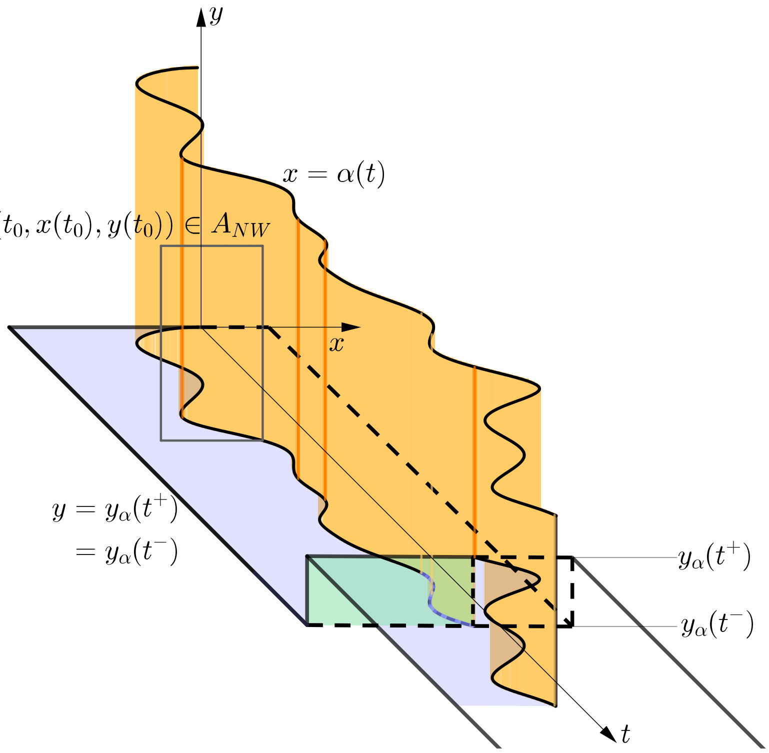


Figure 2.2: A sketch of the boundary of the set A_{NW}. It consists of a *wall* $x = \alpha(t)$, a *floor* $y = y_\alpha(t^+)$ and a possible *step* $y_\alpha(t^-) \leq y < y_\alpha(t^+)$. For simplicity, the function y_α is drawn as being piecewise constant.

Without loss of generality we can assume the existence of $\delta > 0$ such that $]t_1, t_1+\delta] \subseteq \mathcal{I}$. We define $G(t) = x(t) - \alpha(t)$, for every $t \in [t_1, t_1+\delta]$. We have $G(t_1+\delta) < 0$ and, from (2.16),

$$G'(t) = x'(t) - \alpha'(t) = \tilde{f}(t, x(t), y(t)) - \alpha'(t) > 0\,,$$

for every $t \in]t_1, t_1 + \delta]$. Hence, $G(t_1) < 0$. We conclude that $x(t) < \alpha(t)$ for every $t \in [t_1, t_0]$. So, being $x(t_1) < \alpha(t_1)$, recalling (2.21), we necessarily have $y_\alpha(t_1^-) \leq y(t_1) \leq y_\alpha(t_1^+)$.

If $y(t_1) = y_\alpha(t_1^+)$, then the function $H(t) = y(t) - y_\alpha(t^+)$ is continuous in the interval $[t_1, t_0]$ with $H(t_1) = 0$ and $H(t) > 0$ for all $t \in]t_1, t_0]$. Recalling that $x(t) < \alpha(t)$ for all $t \in [t_1, t_0]$, by (2.18) or (2.19) we have

$$D^+H(t_1) = y'(t_1) - D_+y_\alpha(t_1) = \tilde{g}(t_1, x(t_1), y_\alpha(t_1^+)) - D_+y_\alpha(t_1) < 0\,,$$

leading again to a contradiction.

The case $y_\alpha(t_1^-) \le y(t_1) < y_\alpha(t_1^+)$ could arise only if $t_1 \in \mathcal{J}$. However, such a situation is not possible, indeed we would have the existence of $\delta > 0$ such that $H(t) < 0$ for every $t \in (t_1, t_1 + \delta)$ which gives a contradiction, since we have assumed $(t, u(t)) \in A_{NW}$ for every $t \in]t_1, t_0[$. $\square$

We have thus proved that the sets A_{NW}, A_{SE} are *invariant in the past*, while the sets A_{NE}, A_{SW} are *invariant in the future*. We also define the sets

$$\begin{aligned} A_W &= \{(t,x,y) \in \mathbb{R}^3 \mid x < \alpha(t)\,,\ y_\alpha(t^-) \le y \le y_\alpha(t^+)\}\,, \\ A_E &= \{(t,x,y) \in \mathbb{R}^3 \mid x > \beta(t)\,,\ y_\beta(t^+) \le y \le y_\beta(t^-)\}\,, \end{aligned}$$

(see Figure 2.1).

Lemma 2.3.4. *If $u = (x,y)$ is a solution of* (2.20) *such that $(t_0, u(t_0)) \in A_W$, then there exists $\delta > 0$ such that*

$$\begin{aligned} t \in]t_0 - \delta, t_0[\quad &\Rightarrow \quad (t, u(t)) \in A_{NW}\,, \\ t \in]t_0, t_0 + \delta[\quad &\Rightarrow \quad (t, u(t)) \in A_{SW}\,. \end{aligned}$$

Similarly, if $u = (x,y)$ is a solution of (2.20) *such that $(t_0, u(t_0)) \in A_E$, then there exists $\delta > 0$ such that*

$$\begin{aligned} t \in]t_0 - \delta, t_0[\quad &\Rightarrow \quad (t, u(t)) \in A_{SE}\,, \\ t \in]t_0, t_0 + \delta[\quad &\Rightarrow \quad (t, u(t)) \in A_{NE}\,. \end{aligned}$$

Proof. We give the proof of the first part of the statement, the second one being similar. Let $u = (x,y)$ be a solution of (2.20) such that $(t_0, u(t_0)) \in A_W$. If $y(t_0) = y_\alpha(t_0^+)$ then, defining as above the function $H(t) = y(t) - y_\alpha(t^+)$,

$$D^+H(t_0) = y'(t_0) - D_+y_\alpha(t_0) = \tilde{g}(t_0, x(t_0), y_\alpha(t_0^+)) - D_+y_\alpha(t_0) < 0\,,$$

using (2.18) or (2.19). So, there exists $\delta > 0$ such that $y(t) < y_\alpha(t^+) = y_\alpha(t^-)$ and $x(t) < \alpha(t)$ for every $t \in]t_0, t_0 + \delta[$.

On the other hand, if $y_\alpha(t_0^-) \le y(t_0) < y_\alpha(t_0^+)$, then $t_0 \in \mathcal{J}$ and the strict inequalities $y(t_0) < y_\alpha(t_0^+)$ and $x(t_0) < \alpha(t_0)$ provide the same conclusion as before by a continuity argument.

We now give the proof for $t < t_0$. If $y(t_0) = y_\alpha(t_0^-)$ then

$$D^-H(t_0) = y'(t_0) - D_-y_\alpha(t_0) = \tilde{g}(t_0, x(t_0), y_\alpha(t_0^-)) - D_-y_\alpha(t_0) < 0\,,$$

and we get the existence of $\delta > 0$ such that $y(t) > y_\alpha(t^-) = y_\alpha(t^+)$ and $x(t) < \alpha(t)$, for every $t \in]t_0 - \delta, t_0[$. On the other hand, if $y_\alpha(t_0^-) < y(t_0) \le y_\alpha(t_0^+)$, we reach the same conclusion, by continuity. $\square$

Lemma 2.3.5. *If $u = (x, y)$ is a solution of $(\widetilde{P})$, then*

$$\alpha(t) \leq x(t) \leq \beta(t)\,, \quad \textit{for every } t \in \mathbb{R}\,. \tag{2.22}$$

Proof. Suppose that there exists a solution $u = (x, y)$ of $(\widetilde{P})$ such that $x(t_0) < \alpha(t_0)$ for a certain $t_0 \in [0, T]$. If $(t_0, u(t_0)) \in A_{NW}$, then, from Lemma 2.3.3, we have that $(t, u(t)) \in A_{NW}$ for every $t \in \mathbb{R}$. Moreover, from (2.16) we get

$$t \in \mathcal{I} \quad \Rightarrow \quad (x - \alpha)'(t) = \tilde{f}(t, x(t), y(t)) - \alpha'(t) > 0\,,$$

a contradiction, since $x - \alpha$ is a periodic function.

The same reasoning can be adopted if $(t_0, u(t_0)) \in A_{SW}$. Finally, if $(t_0, u(t_0))$ belongs to A_W, Lemma 2.3.4 brings us to the previous contradicting situations.

A similar argument can be adopted in order to show that there are no solutions $u = (x, y)$ of $(\widetilde{P})$ such that $\max_{[0,T]}(x - \beta) > 0$. □

Lemma 2.3.6. *If $u = (x, y)$ is a solution of $(\widetilde{P})$, then*

$$\gamma_-(x(t)) < y(t) < \gamma_+(x(t))\,, \quad \textit{for every } t \in \mathbb{R}\,. \tag{2.23}$$

Proof. We already know from Lemma 2.3.5 that any solution of $(\widetilde{P})$ is such that $\alpha(t) \leq x(t) \leq \beta(t)$ for every $t \in [0, T]$. We claim that $|y(t)| < D$, for every $t \in [0, T]$. Indeed, if the function y has minimum at $t = t_m$ such that $y(t_m) < -D$, then we would have

$$y'(t_m) = \tilde{g}(t_m, x(t_m), y(t_m)) = -M_Y < 0\,,$$

a contradiction. Similarly, $\max_{[0,T]} y < D$ must hold.

We now define the periodic function $F_-(t) = y(t) - \gamma_-(x(t))$. Let $s_m \in [0, T]$ such that $F_-(s_m) = \min_{[0,T]} F_-$. If $F_-(s_m) \leq 0$, we get the following contradiction:

$$\begin{aligned}
F_-'(s_m) &= y'(s_m) - \gamma_-'(x(s_m))x'(s_m) \\
&= \tilde{g}(s_m, x(s_m), y(s_m)) - \gamma_-'(x(s_m))\tilde{f}(t, x(s_m), y(s_m)) \\
&= \hat{g}(s_m, x(s_m), y(s_m)) - \gamma_-'(x(s_m))\hat{f}(t, x(s_m), y(s_m)) \\
&= \langle \hat{\Phi}(s_m, x(s_m), y(s_m)), (-\gamma_-'(x(s_m)), 1) \rangle \\
&= \left(1 - \frac{\gamma_-(x(s_m)) - y(s_m)}{D + \gamma_-(x(s_m))}\right) \Big\langle \Phi(s_m, x(s_m), \gamma_-(x(s_m)))\,,\, (-\gamma_-'(x(s_m)), 1) \Big\rangle \\
&\quad - \frac{\gamma_-(x(s_m)) - y(s_m)}{D + \gamma_-(x(s_m))} \Big\langle (M_X, M_Y), (-\gamma_-'(x(s_m)), 1) \Big\rangle < 0\,,
\end{aligned}$$

where we have used both (2.10) and (2.14). So, $\min_{[0,T]} F_- > 0$. Similarly we can prove that $\max_{[0,T]} F_+ < 0$, where $F_+(t) = y(t) - \gamma_+(x(t))$, thus concluding the proof. □

2.3.1.3 A topological degree argument

We define the operators

$$\mathcal{L} : \mathcal{C}^1_T \to \mathcal{C}^0_T\,, \qquad \mathcal{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix},$$

where $\mathcal{C}^1_T = \{v \in \mathcal{C}^1([0,T],\mathbb{R}^2) : v(0) = v(T)\}$ and

$$\widetilde{\mathcal{N}} : \mathcal{C}^0_T \to \mathcal{C}^0_T\,, \qquad \widetilde{\mathcal{N}}\begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} \tilde{f}(t,x(t),y(t)) \\ \tilde{g}(t,x(t),y(t)) \end{pmatrix}. \tag{2.24}$$

So, a solution $u(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ of problem $(\widetilde{P})$ corresponds to a solution of

$$\mathcal{L}u - \widetilde{\mathcal{N}}u = 0\,. \tag{2.25}$$

In the previous section we have found the a priori bound $\overline{\mathcal{V}}$ for all the possible solutions of problem $(\widetilde{P})$. In order to apply the degree theory we need to consider an open ball $\mathcal{B}_R$ containing $\overline{\mathcal{V}}$. By the above arguments, we can deduce that if u solves (2.25), then $u \notin \partial\mathcal{B}_R$, so that the coincidence degree $d_{\mathcal{L}}(\mathcal{L}-\widetilde{\mathcal{N}},\mathcal{B}_R)$ is well defined. We refer to [70] for more details on this topic.

Since (2.22) and (2.23) hold, we can rewrite system (2.20) as

$$x' = y + p(t,x,y)\,, \qquad y' = x + q(t,x,y)\,,$$

where

$$\begin{aligned} p(t,x,y) &= \hat{f}\Big(t, \zeta\big(x;\alpha(t),\beta(t)\big)\,,\,\zeta\big(y;-D,D\big)\Big) - \zeta\big(y;-D,D\big)\,,\\ q(t,x,y) &= \hat{g}\Big(t\,,\,\zeta\big(x;\alpha(t),\beta(t)\big)\,,\,\zeta\big(y;-D,D\big)\Big) - \zeta\big(x;\alpha(t),\beta(t)\big)\,, \end{aligned}$$

are bounded functions. We now introduce the functions

$$\mathcal{F}_\lambda(t,u) = \mathcal{F}_\lambda(t,x,y) = \Big(y + \lambda p(t,x,y)\,,\,x + \lambda q(t,x,y)\Big),$$

and the problems

$$(Q_\lambda) \qquad \begin{cases} u' = \mathcal{F}_\lambda(t,u)\,,\\ u(0) = u(T)\,. \end{cases}$$

We define the Nemytskii operator related to the family of problem (Q_λ) as

$$(\mathcal{M}_\lambda u)(t) = \mathcal{F}_\lambda(t,u(t))\,,$$

Since the function $(p,q) : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ is bounded, by a classical argument we can find a sufficiently large $R > 0$, such that, for every $\lambda \in [0,1]$, all the periodic solutions of (Q_λ) satisfy

$$\|u\|_\infty^2 = \sup_{t\in[0,T]} [x^2(t) + y^2(t)] < R^2 .$$

Since for $\lambda = 0$ we have an autonomous linear problem ruled by the function $\mathcal{G}(u) = \mathcal{G}(x,y) = \big(y,x\big)$, by [23, Lemma 1] we can conclude that

$$d_{\mathcal{L}}(\mathcal{L} - \widetilde{\mathcal{N}}, \mathcal{B}_R) = d_{\mathcal{L}}(\mathcal{L} - \mathcal{M}_1, \mathcal{B}_R) = d_{\mathcal{L}}(\mathcal{L} - \mathcal{M}_0, \mathcal{B}_R) = \deg(\mathcal{G}, B_R) = -1 ,$$

where $\deg(\mathcal{G}, B_R)$ denotes the Brouwer degree of the function $\mathcal{G}$ on the ball $B_R = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < R^2\}$ and $\mathcal{B}_R$ is the set of continuous functions having image in B_R. We have so found a solution of problem $(\widetilde{P})$ belonging to the set $\mathcal{B}_R$. However, such a solution belongs indeed to the a priori bound $\overline{\mathcal{V}}$, and so it is also a solution of problem (P), thus concluding the proof of Theorem 2.3.1.

2.3.2 An important consequence of the proof

We first recall the definition (2.15) of the open set

$$\mathcal{V} = \{u \in \mathcal{C}^0_T \mid (t, u(t)) \in V \text{ for every } t \in [0,T]\} , \tag{2.26}$$

where

$$V = \{(t,x,y) \in \mathbb{R}^3 \mid \alpha(t) < x < \beta(t) \, , \, \gamma_-(x) < y < \gamma_+(x)\} .$$

Let us introduce the Nemytskii operator related to problem (P) as

$$\mathcal{N} : \mathcal{C}^0_T \to \mathcal{C}^0_T \qquad \mathcal{N}\begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} f(t,x(t),y(t)) \\ g(t,x(t),y(t)) \end{pmatrix} .$$

Corollary 2.3.7. *Under the assumptions of Theorem 2.3.1, if there are no solutions of (P) in $\partial\mathcal{V}$, then*

$$d_{\mathcal{L}}(\mathcal{L} - \mathcal{N}, \mathcal{V}) = -1 .$$

Proof. Since $\Phi = \widetilde{\Phi}$ on $\overline{V}$, and so $\mathcal{N} = \widetilde{\mathcal{N}}$ on $\overline{\mathcal{V}}$, the additional assumption permits us to evaluate the coincidence degree also on the set $\mathcal{V}$. Recalling that all the solutions of problem $(\widetilde{P})$ satisfy the a priori bounds (2.22) and (2.23), by the excision property we have

$$-1 = d_{\mathcal{L}}(\mathcal{L} - \widetilde{\mathcal{N}}, \mathcal{B}_R) = d_{\mathcal{L}}(\mathcal{L} - \widetilde{\mathcal{N}}, \mathcal{V}) = d_{\mathcal{L}}(\mathcal{L} - \mathcal{N}, \mathcal{V}) ,$$

and the proof is completed. □

Remark 2.3.8. *The set $\mathcal{V}$ introduced in* (2.26) *depends on the well-ordered couple (α,β) of lower/upper solutions of problem (P) and the functions $\gamma_\pm$ given in the assumptions of Theorem 2.3.1. In the following section, we will denote this set by $\mathcal{V}(\alpha,\beta,\gamma_\pm)$ when we need to underline such a dependence.*

2.4 Non-well-ordered lower and upper solutions

We still consider the periodic problem

$$(P) \qquad \begin{cases} x' = f(t,x,y)\,, \quad y' = g(t,x,y)\,, \\ x(0) = x(T)\,, \quad y(0) = y(T)\,, \end{cases}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ are continuous functions, T-periodic in their first variable.

We will say that (α, β) is a non-well-ordered couple of lower/upper solutions of problem (P) if α and β are respectively a lower and an upper solution of problem (P), such that there exists $\hat{t}_0 \in [0,T]$ satisfying

$$\alpha(\hat{t}_0) > \beta(\hat{t}_0)\,. \tag{2.27}$$

Let us set

$$a(t) := \min\{\alpha(t), \beta(t)\}\,, \qquad b(t) := \max\{\alpha(t), \beta(t)\}\,,$$

$$A := \min a\,, \qquad B := \max b\,.$$

Notice that $A < B$, by (2.27).

Let us introduce our assumptions.

(H1) There is a continuous function $\chi : \mathbb{R} \to [0, +\infty[$ and a constant $M > 0$ such that

$$|f(t,x,y)| \le \chi(y)(1 + |x|)\,, \qquad \text{for every } (t,x,y) \in \mathbb{R}^3\,, \tag{2.28}$$

$$|g(t,x,y)| \le M(1 + |y|)\,, \qquad \text{for every } (t,x,y) \in \mathbb{R}^3\,. \tag{2.29}$$

(H2) There exist two continuous functions $\gamma_\pm : [A,B] \times [1, +\infty[\to \mathbb{R}$, continuously differentiable with respect to the first variable, such that

$$\lim_{\lambda \to +\infty} \gamma_\pm(x;\lambda) = \pm\infty\,, \quad \text{uniformly with respect to } x \in [A,B]\,,$$

and

$$g\big(t,x,\gamma_-(x;\lambda)\big) < f\big(t,x,\gamma_-(x;\lambda)\big)\gamma_-'(x;\lambda)\,, \tag{2.30}$$

$$g\big(t,x,\gamma_+(x;\lambda)\big) > f\big(t,x,\gamma_+(x;\lambda)\big)\gamma_+'(x;\lambda)\,, \tag{2.31}$$

for every $t \in \mathbb{R}$, $x \in [a(t), b(t)]$ and $\lambda \in [1, +\infty[$. (Here we denote by $\gamma_\pm'$ the derivative with respect to the first variable.)

Theorem 2.4.1. *Assume the existence of a non-well-ordered couple (α, β) of lower/upper solutions of problem (P) with the additional property that there exists a constant $\hat{c} > 0$ such that, for every $k \in \{1, \dots, n\}$,*

$$\begin{cases} y \le -\hat{c} & \Rightarrow \quad f(\tau_k, \alpha(\tau_k^-), y) < \alpha'(\tau_k^-)\,, \\ y \ge \hat{c} & \Rightarrow \quad f(\tau_k, \alpha(\tau_k^+), y) > \alpha'(\tau_k^+)\,, \end{cases} \tag{2.32}$$

$$\begin{cases} y \le -\hat{c} & \Rightarrow \quad f(\tau_k, \beta(\tau_k^+), y) < \beta'(\tau_k^+)\,, \\ y \ge \hat{c} & \Rightarrow \quad f(\tau_k, \beta(\tau_k^-), y) > \beta'(\tau_k^-)\,. \end{cases} \tag{2.33}$$

If (H1) and (H2) hold, there exists at least one solution of problem (P) such that, for some $t_1, t_2 \in [0, T]$, one has $x(t_1) \le \alpha(t_1)$ and $x(t_2) \ge \beta(t_2)$.

This theorem extends some classical results for scalar second order differential equations of the type (2.7). We will show below two examples of applications. Conditions (H1) and (H2) will be necessary in order to avoid resonance phenomena, and to obtain a priori bounds. Notice that (2.2) and (2.5) imply a weaker form of (2.32) and (2.33), i.e., with only weak inequalities. It remains an open problem if these additional assumptions can be omitted.

We will discuss in Section 2.5 on the possibility of reversing the inequalities in (2.30) and (2.31). Concerning the existence of the functions $\gamma_\pm$, let us prove the following lemma.

Lemma 2.4.2. *Let the following assumptions hold:*

(G1) *there are a constant $d > 0$ and two continuous functions $f_+ : [d, +\infty[\to \mathbb{R}$ and $f_- :]-\infty, -d] \to \mathbb{R}$ such that*

$$\begin{cases} y \ge d & \Rightarrow \quad f(t, x, y) \ge f_+(y) > 0\,, \\ y \le -d & \Rightarrow \quad f(t, x, y) \le f_-(y) < 0\,, \end{cases}$$

for every $(t, x) \in [0, T] \times [A, B]$;

(G2) *there is a positive continuous function $\varphi : [0, +\infty[\to \mathbb{R}$ such that*

$$|g(t, x, y)| \le \varphi(|y|)\,, \qquad \text{for every } (t, x, y) \in [0, T] \times [A, B] \times \mathbb{R}\,;$$

(G3) *the above functions are such that*

$$\int_d^{+\infty} \frac{f_+(s)}{\varphi(s)}\, ds = +\infty\,, \qquad \int_{-\infty}^{-d} \frac{f_-(s)}{\varphi(|s|)}\, ds = -\infty\,. \tag{2.34}$$

Then, there exist four continuous functions $\gamma_{\pm,1}$, $\gamma_{\pm,2} : [A, B] \times [1, +\infty[\to \mathbb{R}$, *continuously differentiable with respect to the first variable, such that*

$$\lim_{\lambda\to+\infty} \gamma_{\pm,1}(x;\lambda) = \pm\infty \quad \textit{and} \quad \lim_{\lambda\to+\infty} \gamma_{\pm,2}(x;\lambda) = \pm\infty\,, \qquad \textit{uniformly with respect to } x \in [A,B]\,, \tag{2.35}$$

and

$$g\big(t, x, \gamma_{+,1}(x;\lambda)\big) > f\big(t, x, \gamma_{+,1}(x;\lambda)\big)\gamma'_{+,1}(x;\lambda)\,, \tag{2.36}$$
$$g\big(t, x, \gamma_{+,2}(x;\lambda)\big) < f\big(t, x, \gamma_{+,2}(x;\lambda)\big)\gamma'_{+,2}(x;\lambda)\,, \tag{2.37}$$
$$g\big(t, x, \gamma_{-,1}(x;\lambda)\big) < f\big(t, x, \gamma_{-,1}(x;\lambda)\big)\gamma'_{-,1}(x;\lambda)\,, \tag{2.38}$$
$$g\big(t, x, \gamma_{-,2}(x;\lambda)\big) > f\big(t, x, \gamma_{-,2}(x;\lambda)\big)\gamma'_{-,2}(x;\lambda)\,, \tag{2.39}$$

for every $t \in [0, T]$, $x \in [A, B]$ *and* $\lambda \in [1, +\infty[$.

Proof. For every $y_0 \geq d$, we introduce the continuous strictly increasing function $\mathcal{F}_{y_0} : [d, +\infty[\to \mathbb{R}$ defined as

$$\mathcal{F}_{y_0}(\xi) = \int_{y_0}^{\xi} \frac{f_+(s)}{\varphi(s)}\, ds\,.$$

We can easily verify that $\mathcal{F}_{y_0}(y_0) = 0$ and, from (2.34),

$$\lim_{\xi\to+\infty} \mathcal{F}_{y_0}(\xi) = +\infty\,.$$

Construction of $\gamma_{+,1}$. For every $y_0 \geq d$ and for every $x \in [A, B]$ there exists a unique $\xi \geq y_0$ such that $\mathcal{F}_{y_0}(\xi) = 2(B - x)$. Hence, we can define $\gamma_{+,1}(x; \lambda)$, for $\lambda \geq 1$, as the unique solution of equation

$$\mathcal{F}_{\lambda-1+d}\big(\gamma_{+,1}(x;\lambda)\big) = 2(B - x)\,. \tag{2.40}$$

In particular, since $\mathcal{F}_{\lambda-1+d}\big(\gamma_{+,1}(B;\lambda)\big) = 0$, we get

$$\gamma_{+,1}(x;\lambda) \geq \gamma_{+,1}(B;\lambda) = \lambda - 1 + d\,,$$

which provides the validity of (2.35) for the function $\gamma_{+,1}$. Differentiating in (2.40) we see that $\gamma'_{+,1}(x;\lambda) < 0$ for every $x \in [A, B]$, and

$$\begin{aligned} f(t, x, \gamma_{+,1}(x;\lambda))\gamma'_{+,1}(x;\lambda) &\leq f_+(\gamma_{+,1}(x;\lambda))\gamma'_{+,1}(x;\lambda) \\ &= -2\varphi(\gamma_{+,1}(x;\lambda)) < -\varphi(\gamma_{+,1}(x;\lambda)) \\ &< g(t, x, \gamma_{+,1}(x;\lambda))\,, \end{aligned}$$

thus proving (2.36).

Construction of $\gamma_{+,2}$. Arguing similarly as above, for every $y_0 \geq d$ and for every $x \in [A, B]$ there exists a unique $\xi \geq y_0$ such that $\mathcal{F}_{y_0}(\xi) = 2(x - A)$. Hence we can define $\gamma_{+,2}(x; \lambda)$ by

$$\mathcal{F}_{\lambda-1+d}\big(\gamma_{+,2}(x;\lambda)\big) = 2(x - A)\,. \tag{2.41}$$

In particular, since $\mathcal{F}_{\lambda-1+d}\big(\gamma_{+,2}(A;\lambda)\big) = 0$, we get

$$\gamma_{+,2}(x;\lambda) \geq \gamma_{+,2}(A;\lambda) = \lambda - 1 + d\,,$$

so that (2.35) holds for the function $\gamma_{+,2}$. Differentiating in (2.41),

$$\begin{aligned} f(t, x, \gamma_{+,2}(x;\lambda))\gamma'_{+,2}(x;\lambda) &\geq f_+(\gamma_{+,2}(x;\lambda))\gamma'_{+,2}(x;\lambda) \\ &= 2\varphi(\gamma_{+,2}(x;\lambda)) > \varphi(\gamma_{+,2}(x;\lambda)) \\ &> g(t, x, \gamma_{+,2}(x;\lambda))\,, \end{aligned}$$

thus proving (2.37).

The construction of the functions $\gamma_{-,1}$ and $\gamma_{-,2}$ satisfying (2.38) and (2.39) is similar. □

Let us illustrate how our result applies to two classical scalar second order differential equations of the type (2.7), involving a *scalar p-Laplacian* and a *mean curvature* operator, with $\phi(s) = |s|^{p-2}s$ and $\phi(s) = s/\sqrt{1+s^2}$, respectively.

Consider first the problem

$$\begin{cases} (|x'|^{p-2}x')' = h(t, x, x')\,, \\ x(0) = x(T)\,, \quad x'(0) = x'(T)\,, \end{cases} \tag{2.42}$$

with $p > 1$, which is equivalent to problem (P), taking $f(t, x, y) = f(y) = |y|^{q-2}y$, with $(1/p) + (1/q) = 1$, and $g(t, x, y) = h(t, x, |y|^{q-2}y)$.

Corollary 2.4.3. *Assume the existence of a non-well-ordered couple (α, β) of lower/upper solutions of problem* (2.42)*, and of a constant $M > 0$ for which*

$$|h(t, x, z)| \leq M(1 + |z|^{p-1})\,, \qquad \textit{for every } (t, x, z) \in \mathbb{R}^3. \tag{2.43}$$

Then, there exists at least one solution of problem (2.42) *such that, for some $t_1, t_2 \in [0, T]$, one has $x(t_1) \leq \alpha(t_1)$ and $x(t_2) \geq \beta(t_2)$.*

Proof. Notice that (2.43) implies (2.29). We can use Lemma 2.4.2 with $\varphi(s) = M(1 + |y|)$ to construct the curves $\gamma_\pm$. Then, Theorem 2.4.1 applies. □

Consider now the problem

$$\begin{cases} \left(\dfrac{x'}{\sqrt{1+(x')^2}} \right)' = h(t,x,x') \,, \\ x(0) = x(T) \,, \quad x'(0) = x'(T) \,, \end{cases} \tag{2.44}$$

which is equivalent to problem (P), taking $f(t,x,y) = \phi^{-1}(y) = y/\sqrt{1-y^2}$ and $g(t,x,y) = h\big(t,x,y/\sqrt{1-y^2}\,\big)$. Notice that these functions are now only defined on $\mathbb{R} \times \mathbb{R} \times \,]-1,1[\,$.

Corollary 2.4.4. *Assume the existence of a non-well-ordered couple (α, β) of lower/upper solutions of problem* (2.44)*, and of a positive continuous function $\zeta : [0,+\infty[\, \to \mathbb{R}$ such that*

$$|h(t,x,z)| \le \zeta(|z|) \,, \qquad \text{for every } (t,x,z) \in \mathbb{R}^3, \tag{2.45}$$

and

$$\int_0^{+\infty} \frac{ds}{(1+s^2)^{3/2}\,\zeta(s)} > \frac{T}{2} \,. \tag{2.46}$$

Then, there exists at least one solution of problem (2.44) *such that, for some $t_1, t_2 \in [0,T]$, one has $x(t_1) \le \alpha(t_1)$ and $x(t_2) \ge \beta(t_2)$.*

Proof. Recalling that $\phi(s) = s/\sqrt{1+s^2}$, by (2.46) there is a $c \in \,]0,1[$ such that

$$\int_0^{\phi^{-1}(c)} \frac{\phi'(s)}{\zeta(s)}\, ds > \frac{T}{2} \,. \tag{2.47}$$

We define the functions $f_c : \mathbb{R} \to \mathbb{R}$ and $g_c : \mathbb{R}^3 \to \mathbb{R}$ as

$$f_c(y) = \begin{cases} \phi^{-1}(-c) + y + c \,, & \text{if } y < -c \,, \\ \phi^{-1}(y) \,, & \text{if } |y| \le c \,, \\ \phi^{-1}(c) + y - c \,, & \text{if } y > c \,, \end{cases}$$

and

$$g_c(t,x,y) = \begin{cases} g(t,x,-c) \,, & \text{if } y < -c \,, \\ g(t,x,y) \,, & \text{if } |y| \le c \,, \\ g(t,x,c) \,, & \text{if } y > c \,, \end{cases}$$

and we consider the system

$$x' = f_c(y) \,, \qquad y' = g_c(t,x,y) \,. \tag{2.48}$$

Using Lemma 2.4.2, we see that all the assumptions of Theorem 2.4.1 hold, so that problem (2.48) has a T-periodic solution (x,y).

We now show that $|y(t)| \leq c$ for every t, implying that (x,y) is indeed a solution of problem (2.44). By contradiction, assume that $\max y > c$, or $\min y < -c$. Let us treat the first case, the other one being similar. By the periodicity, there exists a $\xi \in [0,T]$ such that $x'(\xi) = 0$ and, correspondingly, $y(\xi) = 0$. Let ξ_1 and ξ_2 be such that $|\xi_2 - \xi_1| \leq \frac{T}{2}$, $y(\xi_1) = 0$, $y(\xi_2) = c$, and $y(t) \in]0,c[$ for every $t \in]\xi_1, \xi_2[$. (When $\xi_1 > \xi_2$, we write $]\xi_1,\xi_2[=]\xi_2,\xi_1[$ and $[\xi_1,\xi_2] = [\xi_2,\xi_1]$.) For every $t \in [\xi_1, \xi_2]$, by (2.45) we have

$$|y'(t)| \leq \zeta(\phi^{-1}(y(t)))\,,$$

so that, by (2.47),

$$|\xi_2 - \xi_1| \geq \left| \int_{\xi_1}^{\xi_2} \frac{y'(t)}{\zeta(\phi^{-1}(y(t)))}\,dt \right| = \int_0^{\phi^{-1}(c)} \frac{\phi'(s)}{\zeta(s)}\,ds > \frac{T}{2}\,,$$

a contradiction. □

The above corollary generalizes [81, Proposition 3.7], where $\zeta(s)$ is a constant function with positive value $K < \frac{2}{T}$.

2.4.1 Proof of Theorem 2.4.1

2.4.1.1 An auxiliary problem

Let us set

$$d_y := \max\{\|y_\alpha\|_\infty\,,\ \|y_\beta\|_\infty\,,\ \|\alpha'\|_\infty\,,\ \|\beta'\|_\infty\,,\ \hat{c}\}\,, \tag{2.49}$$

where, for all these functions, the norm $\|\cdot\|_\infty$ can be defined as in (2.12).

We recall here a classical result, which is a straightforward consequence of the Gronwall Lemma, often mentioned as *elastic property*.

Lemma 2.4.5. *For every constant $\mathcal{K} > 0$ we can define a function $\mathcal{E}_\mathcal{K} : [0,+\infty[\to [0,+\infty[$ with the following property: given a differentiable function $z : \mathbb{R} \to \mathbb{R}$ satisfying*

$$|z'(t)| \leq \mathcal{K}(1 + |z(t)|)\,, \quad \text{for every } t \in \mathbb{R}\,,$$

if $|z(\bar{t})| \leq Z$ for a certain $\bar{t} \in \mathbb{R}$, then $|z(t)| \leq \mathcal{E}_\mathcal{K}(Z)$ for every $t \in [\bar{t} - T, \bar{t} + T]$.

For example, we can take $\mathcal{E}_\mathcal{K}(Z) = (Z + \mathcal{K}T)e^{\mathcal{K}T}$.

Using the notation introduced in the previous lemma, let us now set

$$D := \mathcal{E}_M(d_y)\,,$$

where M and d_y have been introduced respectively in (2.29) and (2.49).

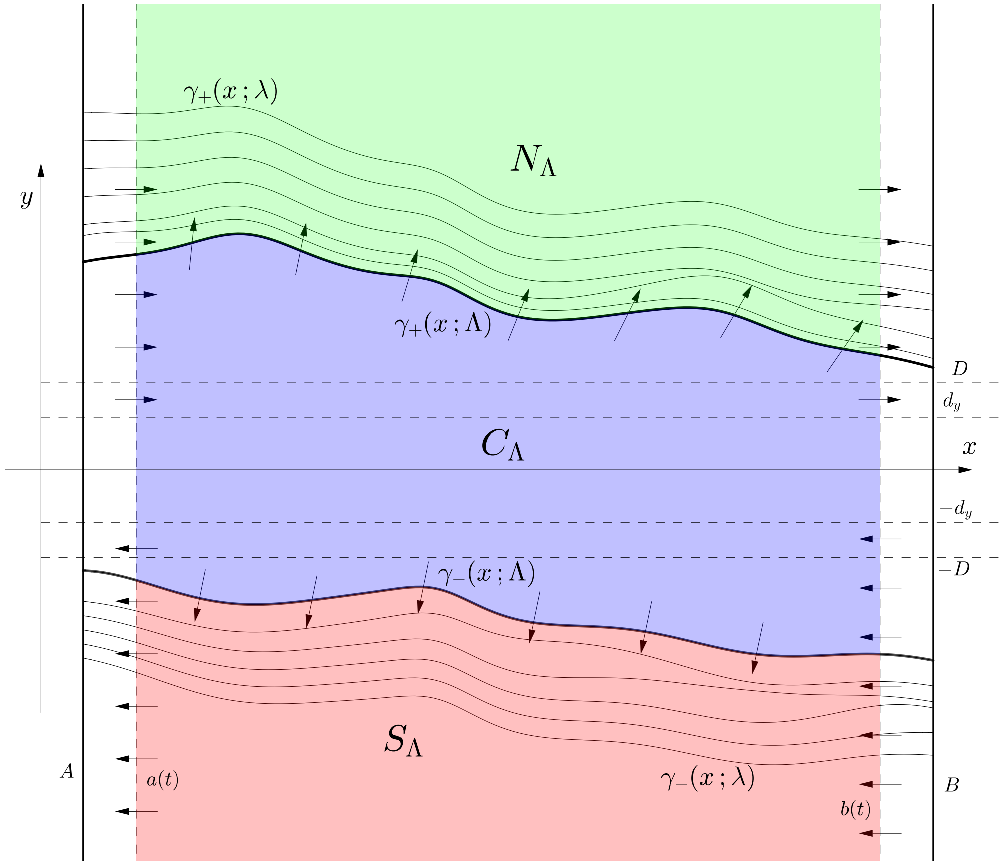


Figure 2.3: A sketch of the section at a fixed time t of the regions N_Λ, C_Λ, and S_Λ. Notice that the vertical lines $x = \alpha$ and $x = \beta$ move in time, while the curves $\gamma_\pm(\cdot, \Lambda)$ are fixed.

By assumption (H2), we can find a sufficiently large constant $\Lambda > 1$ such that

$$|\gamma_\pm(x;\lambda)| > D\,, \quad \text{for every } x \in [A, B] \text{ and } \lambda \geq \Lambda\,. \tag{2.50}$$

Let us introduce the sets

$$\begin{aligned} N_\Lambda &:= \{(t, x, y) \in \mathbb{R}^3 \,:\, a(t) \leq x \leq b(t)\,,\; y > \gamma_+(x;\Lambda)\}\,, \\ C_\Lambda &:= \{(t, x, y) \in \mathbb{R}^3 \,:\, a(t) \leq x \leq b(t)\,,\; \gamma_-(x;\Lambda) \leq y \leq \gamma_+(x;\Lambda)\}\,, \\ S_\Lambda &:= \{(t, x, y) \in \mathbb{R}^3 \,:\, a(t) \leq x \leq b(t)\,,\; y < \gamma_-(x;\Lambda)\}\,, \end{aligned}$$

(see Figure 2.3).

Lemma 2.4.6. *There are two constants ℓ_x and ℓ_y with the following property: if $u = (x, y)$ is a solution of*

$$x' = f(t, x, y)\,, \quad y' = g(t, x, y) \tag{2.51}$$

such that $(t_0, u(t_0)) \in C_\Lambda$ *for a certain* $t_0 \in [0,T]$*, then*

$$|x(t)| \le \ell_x \quad \textit{and} \quad |y(t)| \le \ell_y\,, \quad \textit{for every } t \in [0,T]\,.$$

Proof. Since the set C_Λ is bounded, we can fix two constants $X > 0$ and $Y > 0$ such that

$$C_\Lambda \subseteq [-X, X] \times [-Y, Y]\,.$$

Hence, applying Lemma 2.4.5 in the setting $(z, \mathcal{K}, Z, \bar{t}) = (y, M, Y, t_0)$, we see that every solution $u = (x, y)$ of (2.51) such that $(t_0, u(t_0)) \in C_\Lambda$, for a certain $t_0 \in [0,T]$, satisfies

$$|y(t)| \le \ell_y := \mathcal{E}_M(Y)\,, \quad \text{for every } t \in [0,T]\,.$$

Now, recalling (2.28) and setting $M_\chi = \max_{[-\ell_y, \ell_y]} \chi$, applying Lemma 2.4.5 in the setting $(z, \mathcal{K}, Z, \bar{t}) = (x, M_\chi, X, t_0)$, we see that any such solution also satisfies

$$|x(t)| \le \ell_x := \mathcal{E}_{M_\chi}(X)\,, \quad \text{for every } t \in [0,T]\,.$$

The lemma is thus proved. □

We will now modify the functions f, g by a procedure which resembles the one in Section 2.3.1.1. From assumption (H2) we can find $\Lambda_1 > \Lambda$ such that

$$|\gamma_\pm(x;\lambda)| > \ell_y + 1\,, \qquad \text{for every } x \in [A,B] \text{ and } \lambda \ge \Lambda_1\,. \tag{2.52}$$

We introduce the constants

$$c_\gamma := \max\{|\gamma'_\pm(x;\lambda)| \,:\, x \in [A,B]\,,\, \lambda \in [1,\Lambda_1]\,\}\,, \tag{2.53}$$

$$\mathcal{M}_X := \max\{|f(t,x,y)| \,:\, t \in [0,T]\,, |x| \le \ell_x\,,\, |y| \le \ell_y + 1\}\,, \tag{2.54}$$

and choose

$$\mathcal{M}_Y > c_\gamma\, \mathcal{M}_X\,. \tag{2.55}$$

Setting $\Phi(t,x,y) = \big(f(t,x,y), g(t,x,y)\big)$, we define $\widehat{\Phi} : \mathbb{R} \times [-\ell_x, \ell_x] \times \mathbb{R} \to \mathbb{R}^2$ as

$$\widehat{\Phi}(t,x,y) = \begin{cases} (\mathcal{M}_X, \mathcal{M}_Y)\,, & \text{if } y \ge \ell_y + 1\,, \\ \Phi(t,x,y) + (y - \ell_y)\Big((\mathcal{M}_X, \mathcal{M}_Y) - \Phi(t,x,y)\Big), & \\ & \text{if } \ell_y \le y \le \ell_y + 1\,, \\ \Phi(t,x,y)\,, & \text{if } -\ell_y \le y \le \ell_y\,, \\ \Phi(t,x,y) - (y + \ell_y)\Big((-\mathcal{M}_X, -\mathcal{M}_Y) - \Phi(t,x,y)\Big), & \\ & \text{if } -\ell_y - 1 \le y \le -\ell_y\,, \\ (-\mathcal{M}_X, -\mathcal{M}_Y)\,, & \text{if } y \le -\ell_y - 1\,. \end{cases}$$

We will write $\widehat\Phi(t,x,y) = \big(\hat f(t,x,y), \hat g(t,x,y)\big)$. Finally, we define $\widetilde\Phi : \mathbb{R}^3 \to \mathbb{R}^2$ as

$$\widetilde\Phi(t,x,y) = \begin{cases} (y,1)\,, & \text{if } x \geq \ell_x + 1\,,\\ \widehat\Phi(t,\ell_x,y) + (x-\ell_x)\Big((y,1) - \widehat\Phi(t,\ell_x,y)\Big), & \\ & \text{if } \ell_x \leq x \leq \ell_x + 1\,,\\ \widehat\Phi(t,x,y)\,, & \text{if } -\ell_x \leq x \leq \ell_x\,,\\ \widehat\Phi(t,-\ell_x,y) - (x+\ell_x)\Big((y,-1) - \widehat\Phi(t,-\ell_x,y)\Big), & \\ & \text{if } -\ell_x - 1 \leq y \leq -\ell_x\,,\\ (y,-1)\,, & \text{if } x \leq -\ell_x - 1\,. \end{cases}$$

We will write $\widetilde\Phi(t,x,y) = \big(\tilde f(t,x,y), \tilde g(t,x,y)\big)$.

Remark 2.4.7. *The functions $\tilde f$, $\tilde g$ coincide with f, g on the rectangle $[-\ell_x,\ell_x] \times [-\ell_y,\ell_y]$, and $\tilde f(t,x,y) = y$ when $|x| \geq \ell_x + 1$. Moreover, the function $\tilde g$ is bounded, so we can find a constant $\widetilde M > 0$ such that*

$$|\tilde g(t,x,y)| \leq \widetilde M\,, \quad \textit{for every } (t,x,y) \in \mathbb{R}^3\,.$$

We will consider the modified problem

$$(\widetilde P) \qquad \begin{cases} x' = \tilde f(t,x,y)\,, & y' = \tilde g(t,x,y)\,,\\ x(0) = x(T)\,, & y(0) = y(T)\,. \end{cases}$$

We have the following a priori bound.

Lemma 2.4.8. *If $u = (x,y)$ is a solution of $(\widetilde P)$ such that $(t_0,u(t_0)) \in C_\Lambda$ for a certain $t_0 \in [0,T]$, then*

$$|x(t)| \leq \ell_x \quad \textit{and} \quad |y(t)| \leq \ell_y\,, \quad \textit{for every } t \in \mathbb{R}\,.$$

Hence, u is also a solution of (P).

Proof. As long as the solution u of $(\widetilde P)$ is such that $u(t) \in [-\ell_x,\ell_x] \times [-\ell_y,\ell_y]$, it is a solution of (2.51). Hence Lemma 2.4.6 applies, guaranteeing that indeed $u(t) \in [-\ell_x,\ell_x] \times [-\ell_y,\ell_y]$ for every $t \in [0,T]$. □

Remark 2.4.9. *Since we have assumed the validity of* (2.2)*,* (2.5)*,* (2.32) *and* (2.33)*, thanks to the choice* (2.54) *we have the following assertions.*

If a solution (x,y) of $(\widetilde P)$ is such that $y(t_0) > d_y$ and $x(t_0) = a(t_0)$ [resp. $x(t_0) = b(t_0)$], then we have $x > a$ [resp. $x > b$], in a right neighborhood of t_0.

If a solution (x,y) of $(\widetilde P)$ is such that $y(t_0) < -d_y$ and $x(t_0) = a(t_0)$ [resp. $x(t_0) = b(t_0)$], then we have $x < a$ [resp. $x < b$], in a right neighborhood of t_0.

2.4.1.2 An a priori bound for the desired solutions

Our aim is to show the existence of a solution of $(\widetilde{P})$ belonging to the set

$$\mathcal{S} = \big\{ u = (x,y) \in \mathcal{C}^0_T \,:\, \text{there exist } t_1, t_2 \in [0,T] \text{ such that} \\ x(t_1) \leq \alpha(t_1) \text{ and } x(t_2) \geq \beta(t_2) \big\} . \tag{2.56}$$

In the following lemma we will prove that a solution belonging to $\mathcal{S}$ satisfies the hypotheses of Lemma 2.4.8, permitting us to conclude that it is a solution of the original problem (P).

Lemma 2.4.10. *If $u = (x,y) \in \mathcal{S}$ is a solution of $(\widetilde{P})$, then there exists a $t_0 \in [0,T]$ such that $(t_0, u(t_0)) \in C_\Lambda$, where Λ is given in* (2.50).

Proof. Let us first prove the following preliminary assertion.

Claim. For any solution u of $(\widetilde{P})$, it cannot be that $(t, u(t)) \in N_\Lambda$ for every $t \in [0,T]$.

By contradiction, assume this is true. We distinguish two cases.

If $y(t) \geq \ell_y + 1$ for every $t \in [0,T]$, then, since $N_\Lambda \subseteq [-\ell_x, \ell_x] \times \mathbb{R}$, we get

$$y'(t) = \tilde{g}(t, x(t), y(t)) = \hat{g}(t, x(t), y(t)) = \mathcal{M}_Y > 0\,,$$

which is in contradiction with the periodicity of the function y.

If $y(\bar{t}_0) < \ell_y + 1$ for some $\bar{t}_0 \in [0,T]$, recalling (2.52) and the continuity of the function γ_+ with respect to λ, we can find $\lambda_0 \in [1, \Lambda_1[$ such that $y(\bar{t}_0) = \gamma_+(x(\bar{t}_0); \lambda_0)$. By (2.53) and (2.55), we have

$$\big\langle \big(\mathcal{M}_X, \mathcal{M}_Y\big)\,,\, \big(-\gamma_+'(x;\lambda), 1\big) \big\rangle \geq -c_\gamma \mathcal{M}_X + \mathcal{M}_Y > 0\,, \tag{2.57}$$

for every $x \in [A,B]$ and $\lambda \in [1, \Lambda_1]$. Moreover, we can rewrite (2.31) as

$$\big\langle \Phi\big(t, x, \gamma_+(x;\lambda)\big)\,,\, \big(-\gamma_+'(x;\lambda), 1\big) \big\rangle > 0\,. \tag{2.58}$$

The function $F_+(t;\lambda_0) = y(t) - \gamma_+(x(t);\lambda_0)$ is T-periodic in t, and $F_+(\bar{t}_0;\lambda_0) = 0$. From the above estimates (2.57) and (2.58), since $a(\bar{t}_0) \leq x(\bar{t}_0) \leq b(\bar{t}_0)$,

$$\begin{aligned} F_+'(\bar{t}_0;\lambda_0) &= \tilde{g}\big(\bar{t}_0, x(\bar{t}_0), \gamma_+(x(\bar{t}_0);\lambda_0)\big) \\ &\qquad - \tilde{f}\big(\bar{t}_0, x(\bar{t}_0), \gamma_+(x(\bar{t}_0);\lambda_0)\big)\gamma_+'\big(x(\bar{t}_0);\lambda_0\big) \\ &= \Big\langle \widetilde{\Phi}\big(\bar{t}_0, x(\bar{t}_0), \gamma_+(x(\bar{t}_0);\lambda_0)\big)\,,\, \big(-\gamma_+'(x(\bar{t}_0);\lambda_0), 1\big) \Big\rangle \\ &= \Big\langle \widehat{\Phi}\big(\bar{t}_0, x(\bar{t}_0), \gamma_+(x(\bar{t}_0);\lambda_0)\big)\,,\, \big(-\gamma_+'(x(\bar{t}_0);\lambda_0), 1\big) \Big\rangle > 0\,. \end{aligned} \tag{2.59}$$

So, there exists $\varepsilon \in]0, T/2[$ such that

$$F_+(\bar{t}_0 + \varepsilon; \lambda_0) > 0 > F_+(\bar{t}_0 + T - \varepsilon; \lambda_0)\,,$$

providing the existence of a certain $\bar{t}_1 \in [\bar{t}_0+\varepsilon\,,\ \bar{t}_0+T-\varepsilon]$ such that $F_+(\bar{t}_1; \lambda_0) = 0$ and $F'_+(\bar{t}_1; \lambda_0) \leq 0$. However, similarly as in (2.59), we get the contradiction $F'_+(\bar{t}_1; \lambda_0) > 0$.

The proof of the Claim is thus completed. Similarly one proves that it cannot be that $(t, u(t)) \in S_\Lambda$ for every $t \in [0, T]$.

Now, let $u = (x, y)$ be a solution of $(\widetilde{P})$ belonging to $\mathcal{S}$. Then, there exists a $t_0 \in [0, T]$ such that

$$A \leq a(t_0) \leq x(t_0) \leq b(t_0) \leq B\,.$$

We will prove that $(t_0, u(t_0)) \in C_\Lambda$.

Assume by contradiction that $(t_0, u(t_0)) \in N_\Lambda$. Recalling the Claim, let

$$t_1 := \inf\{t \in [t_0, t_0 + T[:\ (t, u(t)) \notin N_\Lambda\}\,. \tag{2.60}$$

Since $(t_1, u(t_1)) \in \partial N_\Lambda$, we need to treat the following three cases (see Figure 2.3).

Case 1: $y(t_1) \geq \gamma_+(x(t_1); \Lambda)$ and $x(t_1) = b(t_1)$. Let

$$t_2 := \sup\{t \in [t_1, t_0 + T]\,;\ x(s) \geq b(s)\,\forall s \in [t_1, t]\}\,. \tag{2.61}$$

Since $y(t_1) > D > d_y$, from Remark 2.4.9 we have that $t_2 > t_1$. By Lemma 2.4.5, since $y(t_1) > D = \mathcal{E}_M(d_y)$, we get $y(t_2) > d_y$. Again from Remark 2.4.9 we have $x - b > 0$ in a right neighborhood of t_2, in contradiction with its definition in (2.61).

Case 2: $y(t_1) = \gamma_+(x(t_1); \Lambda)$ and $a(t_1) \leq x(t_1) < b(t_1)$. The function $F_+(\cdot; \Lambda)$ is well defined and non-negative in the nontrivial interval $[t_0, t_1]$. Reasoning as in (2.59), we can show that $F'_+(t_1; \Lambda) > 0$, contradicting the definition of t_1 in (2.60).

Case 3: $y(t_1) > \gamma_+(x(t_1); \Lambda)$ and $x(t_1) = a(t_1) < b(t_1)$. This situation is forbidden, by Remark 2.4.9.

Hence, we can conclude that $(t_0, u(t_0)) \notin N_\Lambda$. Similarly one proves that $(t_0, u(t_0)) \notin S_\Lambda$, and the proof is thus completed. □

2.4.1.3 Creating well-ordered couples of lower/upper solutions of $(\widetilde{P})$

Lemma 2.4.11. *Both the constant $\hat{\alpha} \equiv -\ell_x - 2$ and α are lower solutions of problem $(\widetilde{P})$. At the same time, both the constant $\hat{\beta} \equiv \ell_x + 2$ and β are upper solutions of problem $(\widetilde{P})$.*

Proof. We first verify that the constant functions $\hat{\alpha} \equiv -\ell_x - 2$ and $\hat{\beta} \equiv \ell_x + 2$ are respectively a lower solution and an upper solution of $(\widetilde{P})$. Indeed, setting $y_{\hat{\alpha}} \equiv 0$ and $y_{\hat{\beta}} \equiv 0$, since $\tilde{f}(t, -\ell_x - 2, y) = \tilde{f}(t, \ell_x + 2, y) = y$, then (2.2) and (2.5) easily follow. Moreover, (2.3) and (2.6) are an immediate consequence of $\tilde{g}(t, -\ell_x - 2, 0) = -1 < 0$ and $\tilde{g}(t, \ell_x + 2, 0) = 1 > 0$.

In order to check that the functions α and β are respectively a lower solution and an upper solution also for problem $(\widetilde{P})$, we need to verify the validity of (2.2), (2.5), (2.32) and (2.33), where we replace the functions f with $\widetilde{f}$. This fact is guaranteed by the choice (2.54). The validity of both (2.3) and (2.6) with g replaced by $\tilde{g}$ is trivial since $g = \widetilde{g}$ at the points we have to deal with. □

Remark 2.4.12. *The couples $(\hat{\alpha}, \hat{\beta})$, $(\hat{\alpha}, \beta)$, and $(\alpha, \hat{\beta})$ are well-ordered couples of lower/upper solutions of problem $(\widetilde{P})$.*

Lemma 2.4.13. *There exist two continuously differentiable functions $\Gamma_\pm : [\hat{\alpha}, \hat{\beta}] \to \mathbb{R}$, such that*

$$\tilde{g}\big(t, x, \Gamma_-(x)\big) < \tilde{f}\big(t, x, \Gamma_-(x)\big)\Gamma'_-(x)\,,$$
$$\tilde{g}\big(t, x, \Gamma_+(x)\big) > \tilde{f}\big(t, x, \Gamma_+(x)\big)\Gamma'_+(x)\,,$$

for every $t \in \mathbb{R}$ and $x \in [\hat{\alpha}, \hat{\beta}]$.

Proof. From Remark 2.4.7 we deduce the validity of the hypotheses of Lemma 2.4.2 adopting the following choices

$$[A, B] = [\hat{\alpha}, \hat{\beta}]\,, \quad f_+(y) = -f_-(y) \equiv d = \max\{\mathcal{M}_X, \ell_y + 1\}\,, \quad \varphi \equiv \widetilde{M}\,.$$

We take $\Gamma_- = \gamma_{-,1}(\cdot; \lambda)$ and $\Gamma_+ = \gamma_{+,1}(\cdot; \lambda)$, for $\lambda > 0$ sufficiently large. □

2.4.1.4 Degree theory and conclusion of the proof of Theorem 2.4.1

We define the sets

$$U_1 = \big\{(t, x, y) \in \mathbb{R}^3 \,:\, \hat{\alpha} < x < \hat{\beta}\,,\, \Gamma_-(x) < y < \Gamma_+(x)\big\}\,,$$
$$U_2 = \big\{(t, x, y) \in \mathbb{R}^3 \,:\, \hat{\alpha} < x < \beta(t)\,,\, \Gamma_-(x) < y < \Gamma_+(x)\big\}\,,$$
$$U_3 = \big\{(t, x, y) \in \mathbb{R}^3 \,:\, \alpha(t) < x < \hat{\beta}\,,\, \Gamma_-(x) < y < \Gamma_+(x)\big\}\,.$$

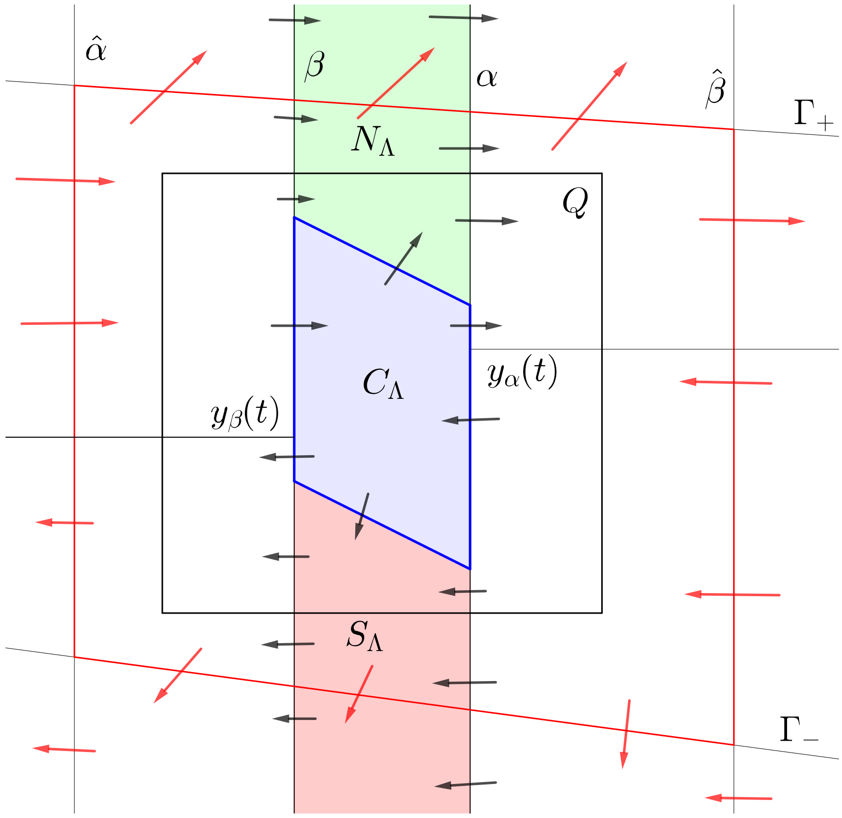


Figure 2.4: The view of the phase plane at a fixed time t in the large. Notice that the horizontal lines $y = y_\alpha$, $y = y_\beta$ move in time, while the curves $\Gamma_\pm$ are fixed.

Notice that $U_2 \cup U_3 \subseteq U_1$. Correspondingly, define the sets

$$\begin{aligned}
\mathcal{U}_1 &= \big\{u = (x, y) \in \mathcal{C}^0_T \,:\, (t, u(t)) \in U_1 \text{ for every } t \in [0, T]\big\}\,,\\
\mathcal{U}_2 &= \big\{u = (x, y) \in \mathcal{C}^0_T \,:\, (t, u(t)) \in U_2 \text{ for every } t \in [0, T]\big\}\,,\\
\mathcal{U}_3 &= \big\{u = (x, y) \in \mathcal{C}^0_T \,:\, (t, u(t)) \in U_3 \text{ for every } t \in [0, T]\big\}\,,\\
\mathcal{U}_4 &= \mathcal{U}_1 \setminus (\overline{\mathcal{U}}_2 \cup \overline{\mathcal{U}}_3)\,.
\end{aligned}$$

The last set can also be written as

$$\begin{aligned}
\mathcal{U}_4 = \big\{u = (x, y) \in \mathcal{C}^0_T \,:\, &(t, u(t)) \in U_1 \text{ for every } t \in [0, T] \text{ and}\\
&\text{there exist } t_1, t_2 \in [0, T] \text{ such that } x(t_1) < \alpha(t_1) \text{ and } x(t_2) > \beta(t_2)\big\}\,.
\end{aligned}$$

So, $\mathcal{U}_4 \subseteq \mathcal{S}$, the set $\mathcal{S}$ being defined in (2.56).

With the notation introduced in Remark 2.3.8, the sets $\mathcal{U}_i$, with $i \in \{1, 2, 3\}$, can be written as

$$\mathcal{U}_1 = \mathcal{V}(\hat{\alpha}, \hat{\beta}, \Gamma_\pm)\,, \quad \mathcal{U}_2 = \mathcal{V}(\hat{\alpha}, \beta, \Gamma_\pm)\,, \quad \mathcal{U}_3 = \mathcal{V}(\alpha, \hat{\beta}, \Gamma_\pm)\,.$$

Remark 2.4.14. *The validity of Lemma 2.4.13 forbids the possibility of finding a solution $u = (x, y)$ of $(\widetilde{P})$ belonging to $\overline{\mathcal{U}}_j$, with $j \in \{1, 2, 3\}$, satisfying $y(t_0) = \Gamma_\pm(x(t_0))$ at a certain time $t_0 \in [0, T]$. Indeed, we would have $(t, u(t)) \notin U_j$ in a right neighborhood of t_0, since $\pm\frac{d}{dt}(y - \Gamma_\pm(x))(t_0) > 0$.*

We now prove that there are no solutions of $(\widetilde{P})$ in $\partial\mathcal{U}_1$, i.e., if $u \in \overline{\mathcal{U}}_1$ solves $(\widetilde{P})$ then $u \in \mathcal{U}_1$. Assume that $x(t) \geq \hat{\alpha}$ for every $t \in [0, T]$ and there exists t_0 such that $x(t_0) = \hat{\alpha}$. Then $\hat{\alpha} \leq x(t) < -\ell_x - 1$ in a neighborhood of t_0, where $x'(t) = \tilde{f}(t, x(t), y(t)) = y(t)$, so that

$$x''(t_0) = y'(t_0) = \tilde{g}(t_0, \hat{\alpha}, y(t_0)) = -1 < 0\,,$$

providing a contradiction. Similarly, the situation when $x(t) \leq \hat{\beta}$ for every $t \in [0, T]$ and there exists t_0 such that $x(t_0) = \hat{\beta}$ cannot arise. Remark 2.4.14 completes the argument.

Since there are no solutions of $(\widetilde{P})$ in $\partial\mathcal{U}_1$, we can apply Corollary 2.3.7 and get

$$d_{\mathcal{L}}(\mathcal{L} - \widetilde{\mathcal{N}}, \mathcal{U}_1) = -1\,, \tag{2.62}$$

where $\widetilde{\mathcal{N}}$ is the Nemytskii operator associated to problem $(\widetilde{P})$, defined as in (2.24).

Assume the existence of a solution belonging to $\partial\mathcal{U}_2$. Then, recalling the above argument and Remark 2.4.14, we have

$$\hat{\alpha} < x(t) \leq \beta(t)\,, \quad \Gamma_-(x) < y(t) < \Gamma_+(x)\,, \quad \text{for every } t \in [0, T]\,,$$

and there exists a $t_0 \in [0, T]$ such that $x(t_0) = \beta(t_0)$. So, such a solution belongs to $\mathcal{S}$, with $t_2 = t_0$ and $t_1 = \hat{t}_0$, where $\hat{t}_0$ was defined in (2.27).

Similarly, if we assume the existence of a solution belonging to $\partial\mathcal{U}_3$, then we have necessarily

$$\alpha(t) \leq x(t) < \hat{\beta}\,, \quad \Gamma_-(x) < y(t) < \Gamma_+(x)\,, \quad \text{for every } t \in [0, T]\,,$$

and there exists a $t_0 \in [0, T]$ such that $x(t_0) = \alpha(t_0)$. So, such a solution belongs to $\mathcal{S}$, with $t_1 = t_0$ and $t_2 = \hat{t}_0$.

If at least one of the previous two situations arises, then we have found the solution we are looking for, and the proof of Theorem 2.4.1 is concluded. Otherwise, we are in the hypotheses of Corollary 2.3.7, which provides

$$d_{\mathcal{L}}(\mathcal{L} - \widetilde{\mathcal{N}}, \mathcal{U}_2) = -1 \quad \text{and} \quad d_{\mathcal{L}}(\mathcal{L} - \widetilde{\mathcal{N}}, \mathcal{U}_3) = -1\,. \tag{2.63}$$

Then, from (2.62) and (2.63), by the excision property,

$$d_{\mathcal{L}}(\mathcal{L} - \widetilde{\mathcal{N}}, \mathcal{U}_4) = d_{\mathcal{L}}(\mathcal{L} - \widetilde{\mathcal{N}}, \mathcal{U}_1) - \Big(d_{\mathcal{L}}(\mathcal{L} - \widetilde{\mathcal{N}}, \mathcal{U}_2) + d_{\mathcal{L}}(\mathcal{L} - \widetilde{\mathcal{N}}, \mathcal{U}_3)\Big) = 1\,,$$

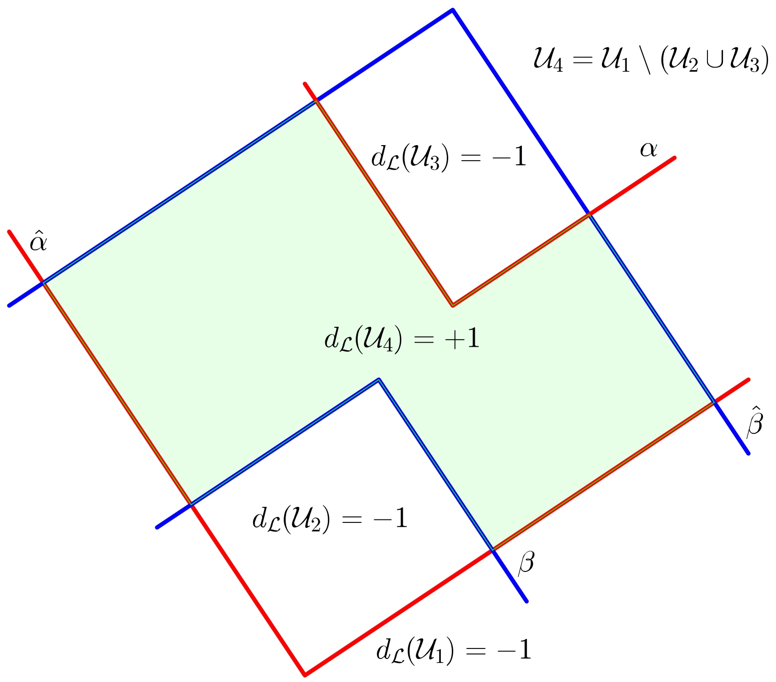


Figure 2.5: A sketch of the calculus of the degree in the proof of Theorem 2.4.1.

and we thus find a solution of $(\widetilde{P})$ belonging to $\mathcal{U}_4 \subseteq \mathcal{S}$. The proof of Theorem 2.4.1 is thus completed, recalling Lemmas 2.4.10 and 2.4.8, in this order.

2.5 Further generalizations and applications

The inequalities in (2.10) and (2.11) can be reversed, and we can restate Theorem 2.3.1 as follows.

Theorem 2.5.1. *Assume the existence of a well-ordered couple (α, β) of lower/upper solutions of problem (P). Set $A = \min \alpha\,, B = \max \beta$, with $A < B$. Let there exist two continuously differentiable functions $\gamma_\pm : [A, B] \to \mathbb{R}$ such that*

$$\gamma_-(x) < \inf_{[0,T]}\{y_\alpha(t^-), y_\beta(t^+)\} \leq \sup_{[0,T]}\{y_\alpha(t^+), y_\beta(t^-)\} < \gamma_+(x)\,,$$

with the following property:

$$\begin{aligned} &\textit{either} \quad g(t, x, \gamma_-(x)) < f(t, x, \gamma_-(x))\gamma'_-(x)\,, \quad \forall t \in \mathbb{R}\,, \forall x \in [\alpha(t), \beta(t)]\,, && (2.64)\\ &\textit{or} \quad g(t, x, \gamma_-(x)) > f(t, x, \gamma_-(x))\gamma'_-(x)\,, \quad \forall t \in \mathbb{R}\,, \forall x \in [\alpha(t), \beta(t)]\,; && (2.65) \end{aligned}$$

and

$$\begin{aligned} \textit{either} \quad & g(t,x,\gamma_+(x)) > f(t,x,\gamma_+(x))\gamma_+'(x)\,, \quad \forall t \in \mathbb{R}\,, \forall x \in [\alpha(t),\beta(t)]\,, && (2.66) \\ \textit{or} \quad & g(t,x,\gamma_+(x)) < f(t,x,\gamma_+(x))\gamma_+'(x)\,, \quad \forall t \in \mathbb{R}\,, \forall x \in [\alpha(t),\beta(t)]\,. && (2.67) \end{aligned}$$

Then there exists at least one solution of problem (P) such that

$$\alpha(t) \le x(t) \le \beta(t) \qquad \textit{and} \qquad \gamma_-(x(t)) < y(t) < \gamma_+(x(t))\,,$$

for every $t \in \mathbb{R}$.

In this statement we allow the additional situations (2.65) and (2.67). Similar conditions were given, e.g., in [8]. The proof of Theorem 2.5.1 needs minor changes with respect to the one of Theorem 2.3.1. For example, if we assume the validity of (2.65) and (2.67) instead of (2.64) and (2.66), in the proof of Theorem 2.3.1 we simply need to modify the function $\widehat{\Phi}$ as follows:

$$\widehat{\Phi}(t,x,y) = \begin{cases} (M_X, -M_Y)\,, & \text{if } y \ge D\,, \\ \Phi(t,x,\gamma_+(x)) + \dfrac{y-\gamma_+(x)}{D-\gamma_+(x)}\Big((M_X,-M_Y) - \Phi(t,x,\gamma_+(x))\Big), & \\ & \text{if } \gamma_+(x) \le y \le D\,, \\ \Phi(t,x,y)\,, & \text{if } \gamma_-(x) \le y \le \gamma_+(x)\,, \\ \Phi(t,x,\gamma_-(x)) - \dfrac{y-\gamma_-(x)}{D+\gamma_-(x)}\Big((-M_X,M_Y) - \Phi(t,x,\gamma_-(x))\Big), & \\ & \text{if } -D \le y \le \gamma_-(x)\,, \\ (-M_X, M_Y)\,, & \text{if } y \le -D\,. \end{cases}$$

In general, the definition of $\widehat{\Phi}$ for $y \ge D$ is related to the choice (2.64) vs. (2.65), while its definition for $y \le -D$ is related to the choice (2.66) vs. (2.67). The proof of Lemma 2.3.6 can be adapted to all the possible settings of Theorem 2.5.1.

Concerning Theorem 2.4.1, hypothesis (H2) can be similarly modified as follows.

(H2′) There exist two continuous functions $\gamma_\pm : [A,B] \times [1,+\infty[\to \mathbb{R}$, continuously differentiable with respect to the first variable, such that

$$\lim_{\lambda \to +\infty} \gamma_\pm(x;\lambda) = \pm\infty\,, \text{uniformly with respect to } x \in [A,B]\,,$$

with the following property

$$\begin{aligned}
\text{either}\quad & g\big(t,x,\gamma_-(x;\lambda)\big) < f\big(t,x,\gamma_-(x;\lambda)\big)\gamma_-'(x;\lambda)\,, \\
& \qquad \forall t\in\mathbb{R}\,,\ \forall x\in[A,B]\,,\ \forall\lambda\in[1,+\infty[\,, \\
\text{or}\quad & g\big(t,x,\gamma_-(x;\lambda)\big) > f\big(t,x,\gamma_-(x;\lambda)\big)\gamma_-'(x;\lambda)\,, \\
& \qquad \forall t\in\mathbb{R}\,,\ \forall x\in[A,B]\,,\ \forall\lambda\in[1,+\infty[\,,
\end{aligned}$$

and

$$\begin{aligned}
\text{either}\quad & g\big(t,x,\gamma_+(x;\lambda)\big) > f\big(t,x,\gamma_+(x;\lambda)\big)\gamma_+'(x;\lambda)\,, \\
& \qquad \forall t\in\mathbb{R}\,,\ \forall x\in[A,B]\,,\ \forall\lambda\in[1,+\infty[\,, \\
\text{or}\quad & g\big(t,x,\gamma_+(x;\lambda)\big) > f\big(t,x,\gamma_+(x;\lambda)\big)\gamma_+'(x;\lambda)\,, \\
& \qquad \forall t\in\mathbb{R}\,,\ \forall x\in[A,B]\,,\ \forall\lambda\in[1,+\infty[\,.
\end{aligned}$$

Assuming (H2′) instead of (H2) in Theorem 2.4.1 we get the same conclusion. However, some steps of the proof need some wise adjustments. In particular, the small changes due in the definition of the function $\widehat{\Phi}$, some lines above, in the setting of Theorem 2.5.1, can be proposed again similarly for the function $\widehat{\Phi}$ introduced in the proof of Theorem 2.4.1. Indeed, in the proof of Lemma 2.4.10, the estimates in (2.59) must provide a different sign. Then, in the second part of the same proof we need to *go back in time*: instead of (2.60), the following definition is in order

$$t_1 := \sup\{t\in\,]t_0-T,t_0[\,:\,(t,u(t))\notin N_\Lambda\}\,.$$

A similar reasoning in the interval $[t_0-T,t_0]$ can be performed. We omit to enter in major details for briefness.

A further extension of our results could lead to systems in $\mathbb{R}^{2N}$ of the type (P), with $f,g:\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$. The case $f(t,x,y)=y$ and $g(t,x,y)=g(t,x)$ has been treated in [46], where also an infinite-dimensional system of the type $x''=g(t,x)$ has been proposed. However, in the non-well-ordered case, the existence of *strict* lower and upper solutions was needed there. We believe that a similar procedure could be undertaken also in the more general framework of system (P). The notion of *strict* lower and upper solutions would probably be the one introduced in [52]; going back to the Introduction, one would need the strict inequality in (ii) and condition (iii) would also be necessary.

Chapter 3

Second order differential equations in Hilbert spaces

3.1 Introduction

In this chapter we consider the periodic problem

$$(P) \qquad \begin{cases} \ddot{x} = f(t,x)\,, \\ x(0) = x(T)\,, \quad \dot{x}(0) = \dot{x}(T)\,. \end{cases}$$

In the scalar case when $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous, the $\mathcal{C}^2$-functions $\alpha, \beta : [0,T] \to \mathbb{R}$ are said to be lower/upper solutions of problem (P), respectively, if

$$\ddot{\alpha}(t) \geq f(t, \alpha(t))\,, \qquad \ddot{\beta}(t) \leq f(t, \beta(t))\,.$$

for every $t \in [0,T]$, and

$$\alpha(0) = \alpha(T)\,, \quad \beta(0) = \beta(T)\,, \quad \dot{\alpha}(0) \geq \dot{\alpha}(T)\,, \quad \dot{\beta}(0) \leq \dot{\beta}(T)\,.$$

We say that (α, β) is a well-ordered pair of lower/upper solutions if $\alpha \leq \beta$. It is well known that, when such a pair exists, problem (P) has a solution x such that $\alpha \leq x \leq \beta$. If we rewrite the problem (P) as a planar problem in the form

$$(P) \qquad \begin{cases} \dot{x} = y \\ \dot{y} = f(t,x) \\ x(0) = x(T)\,, \quad y(0) = y(T)\,. \end{cases}$$

the above definition of lower and upper solutions is a particular case of that given at the beginning of Chapter 2 since it suffices to choose $y_\alpha(t) = \alpha'(t)$ and $y_\beta(t) = \beta'(t)$.

Starting with the paper by Amann, Ambrosetti and Mancini [4] in 1978, there have been several improvements in the existence and localization of the solutions by Omari [83] in 1988, Gossez and Omari [56] in 1994, Habets and Omari [59] in 1996 and De Coster and Henrard [30] in 1998 (see also [50] for an abstract setting of the results).

Here we want to extend those classical existence results for scalar equations to systems, both in a finite-dimensional and in an infinite-dimensional setting. The first step in this direction is represented by the work of Bebernes and Schmitt [10] that generalized the scalar well-ordered case to a system of type (P), with $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$. We report here, in a slightly more general version, their result. We provide an existence result for a system in $\mathbb{R}^N$ when the components of the lower/upper solutions can be both well-ordered and non-well-ordered to generalize it in an infinite-dimensional setting by an approximating process passing to the limit in the dimension by the use of the Ascoli-Arzelà theorem.

3.2 Well-ordered lower and upper solutions for systems

In this section and the next one we consider the problem

$$(P) \qquad \begin{cases} \ddot{x} = f(t,x)\,, \\ x(0) = x(T)\,, \quad \dot{x}(0) = \dot{x}(T)\,, \end{cases}$$

where $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function. We are thus in a finite-dimensional setting. Let us recall a standard procedure to reduce the search of solutions of (P) to a fixed point problem in Banach space. We define the set

$$\mathcal{C}^2_T = \{x \in \mathcal{C}^2([0,T],\mathbb{R}^N) : x(0) = x(T),\ \dot{x}(0) = \dot{x}(T)\}\,,$$

and the linear operator

$$\mathcal{L} : \mathcal{C}^2_T \to \mathcal{C}([0,T],\mathbb{R}^N)\,, \quad \mathcal{L}x = -\ddot{x} + x\,.$$

which is invertible and has a bounded inverse. We consider as well the Nemytskii operator

$$\mathcal{N} : \mathcal{C}([0,T],\mathbb{R}^N) \to \mathcal{C}([0,T],\mathbb{R}^N)\,, \quad (\mathcal{N}x)(t) = x(t) - f(t,x(t))\,.$$

Problem (P) is thus equivalent to the fixed point problem in $\mathcal{C}([0,T],\mathbb{R}^N)$

$$x = \mathcal{L}^{-1}\mathcal{N}x\,.$$

Notice that $\mathcal{L}^{-1}\mathcal{N} : \mathcal{C}([0,T],\mathbb{R}^N) \to \mathcal{C}([0,T],\mathbb{R}^N)$ is completely continuous.

Here, we recall and slightly generalize [10, Theorem 4.1].

Definition 3.2.1. *Given two $\mathcal{C}^2$-functions $\alpha, \beta : [0,T] \to \mathbb{R}^N$, we say that (α, β) is a* well-ordered pair of lower/upper solutions *of problem (P) if, for every $j \in \{1, \ldots, N\}$ and $t \in [0,T]$,*

$$\alpha_j(t) \le \beta_j(t)\,,$$

$$\alpha_j(0) = \alpha_j(T)\,, \quad \beta_j(0) = \beta_j(T)\,, \quad \dot\alpha_j(0) \ge \dot\alpha_j(T)\,, \quad \dot\beta_j(0) \le \dot\beta_j(T)\,,$$

and, for every $x \in \prod_{m=1}^N [\alpha_m(t), \beta_m(t)]$,

$$\begin{aligned} \ddot\alpha_j(t) &\ge f_j(t, x_1, \ldots, x_{j-1}, \alpha_j(t), x_{j+1}, \ldots, x_N)\,,\\ \ddot\beta_j(t) &\le f_j(t, x_1, \ldots, x_{j-1}, \beta_j(t), x_{j+1}, \ldots, x_N)\,. \end{aligned}$$

Theorem 3.2.2 (Bebernes–Schmitt)**.** *If there exists a well-ordered pair of lower/upper solutions (α, β), then problem (P) has a solution $x(t)$ such that*

$$\alpha_j(t) \le x_j(t) \le \beta_j(t)\,, \quad \textit{for every } j \in \{1, \ldots, N\} \textit{ and } t \in [0,T]\,. \tag{3.1}$$

Proof. Step 1. Define the functions $\gamma_j : [0,T] \times \mathbb{R} \to \mathbb{R}$ as

$$\gamma_j(t,s) = \begin{cases} \alpha_j(t) & \text{if } s < \alpha_j(t)\,,\\ s & \text{if } \alpha_j(t) \le s \le \beta_j(t)\,,\\ \beta_j(t) & \text{if } s > \beta_j(t)\,, \end{cases}$$

and the functions $\Gamma, \bar f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ as

$$\Gamma(t,x) = (\gamma_1(t,x_1), \ldots, \gamma_N(t,x_N))\,, \quad \bar f(t,x) = f(t, \Gamma(t,x))\,.$$

Consider the auxiliary problem

$$(P') \qquad \begin{cases} \ddot x = \bar f(t,x) + x - \Gamma(t,x)\,,\\ x(0) = x(T)\,, \quad \dot x(0) = \dot x(T)\,, \end{cases}$$

and the corresponding Nemytskii operator

$$\widetilde{\mathcal{N}} : \mathcal{C}([0,T], \mathbb{R}^N) \to \mathcal{C}([0,T], \mathbb{R}^N)\,, \quad (\widetilde{\mathcal{N}}x)(t) = \Gamma(t, x(t)) - \bar f(t, x(t))\,.$$

Problem (P') is then equivalent to a fixed point problem in $\mathcal{C}([0,T], \mathbb{R}^N)$, namely

$$x = \mathcal{L}^{-1}\widetilde{\mathcal{N}}x\,.$$

By Schauder Theorem, since $\mathcal{L}^{-1}\widetilde{\mathcal{N}} : \mathcal{C}([0,T], \mathbb{R}^N) \to \mathcal{C}([0,T], \mathbb{R}^N)$ is completely continuous and has a bounded image, it has a fixed point, so that (P') has a solution $x(t)$.

Step 2. Let us show that (3.1) holds for every solution of (P'), thus proving the theorem. By contradiction, assume that there is a $j \in \{1, \dots, N\}$ and a $t_j \in [0,T]$ for which $x_j(t_j) \notin [\alpha_j(t_j), \beta_j(t_j)]$. For instance, let $x_j(t_j) < \alpha_j(t_j)$ (the case $x_j(t_j) > \beta_j(t_j)$ being similar). Set $v_j(t) = \alpha_j(t) - x_j(t)$, and let $\hat{t}_j \in [0,T]$ be such that $v_j(\hat{t}_j) = \max\{v_j(t) : t \in [0,T]\}$. We distinguish two cases.

Case 1: $\hat{t}_j \in]0,T[$. In this case, surely $\ddot{v}_j(\hat{t}_j) \leq 0$. On the other hand,

$$\begin{aligned}\ddot{v}_j(\hat{t}_j) &= \ddot{\alpha}_j(\hat{t}_j) - \ddot{x}_j(\hat{t}_j) \\ &= \ddot{\alpha}_j(\hat{t}_j) - \bar{f}_j(\hat{t}_j, x(\hat{t}_j)) - x_j(\hat{t}_j) + \gamma_j(\hat{t}_j, x_j(\hat{t}_j)) \\ &> \ddot{\alpha}_j(\hat{t}_j) - f_j(\hat{t}_j, \gamma_1(\hat{t}_j, x_1(\hat{t}_j)), \dots, \alpha_j(\hat{t}_j), \dots, \gamma_N(\hat{t}_j, x_N(\hat{t}_j))) \geq 0\,,\end{aligned}$$

leading to a contradiction.

Case 2: $\hat{t}_j = 0$ or $\hat{t}_j = T$. Assume for instance that $\hat{t}_j = 0$ (the other situation being similar). Then,

$$0 \geq \dot{v}_j(0) = \dot{\alpha}_j(0) - \dot{x}_j(0) \geq \dot{\alpha}_j(T) - \dot{x}_j(T) = \dot{v}_j(T)\,,$$

so that, being $v_j(T) = v_j(0)$ the maximum value of $v_j(t)$ over $[0,T]$, it has to be that $\dot{v}_j(T) = 0$, hence also $\dot{v}_j(0) = 0$. Now, since $v_j(0) > 0$, there is a small $\delta > 0$ such that $v_j(s) > 0$, for every $s \in [0,\delta]$. Then, if $t \in [0,\delta]$, we have that $x_j(s) < \alpha_j(s)$, for every $s \in [0,t]$, hence

$$\begin{aligned}\dot{v}_j(t) &= \dot{v}_j(0) + \int_0^t \ddot{v}_j(s)\,\mathrm{d}s \\ &= \int_0^t \Big(\ddot{\alpha}_j(s) - \ddot{x}_j(s)\Big)\,\mathrm{d}s \\ &= \int_0^t \Big(\ddot{\alpha}_j(s) - \bar{f}_j(s, x(s)) - x_j(s) + \gamma_j(s, x_j(s))\Big)\,\mathrm{d}s \\ &> \int_0^t \Big(\ddot{\alpha}_j(s) - f_j(s, \gamma_1(s, x_1(x)), \dots, \alpha_j(s), \dots, \gamma_N(s, x_N(x)))\Big)\,\mathrm{d}s \geq 0\,,\end{aligned}$$

a contradiction, since 0 is a maximum point for $v_j(t)$. The proof is thus completed. □

We now provide some illustrative examples.

Example 3.2.3. *Let, for every* $j \in \{1, \dots, N\}$,

$$f_j(t,x) = a_j x_j^3 + h_j(t,x)\,,$$

for some constants $a_j > 0$*, and assume that there is a* $c > 0$ *such that*

$$|h(t,x)| \leq c\,, \quad \textit{for every } (t,x) \in [0,T] \times \mathbb{R}^N. \tag{3.2}$$

Then, taking the constant functions $\alpha_j = -\sqrt[3]{c/a_j}$ *,* $\beta_j = \sqrt[3]{c/a_j}$ *, we see that Theorem 3.2.2 applies, hence* (P) *has a solution.*

Example 3.2.4. *Let us consider, for every* $j \in \{1, \dots, N\}$*,*

$$f_j(t,x) = x_j^2 \sin x_j + h_j(t,x)\,,$$

and assume that there is a $c > 0$ *such that* (3.2) *holds. Then, for every* $\ell \in \mathbb{Z}$ *with* $|\ell|$ *sufficiently large, taking the constant functions* $\alpha_j = -\pi/2 + 2\ell\pi$ *,* $\beta_j = \pi/2 + 2\ell\pi$ *, we see that Theorem 3.2.2 applies, and we conclude that* (P) *admits an infinite number of solutions.*

In order to work with Leray-Schauder degree, we need to introduce the notions of *strict* lower/upper solutions.

Definition 3.2.5. *The well-ordered pair of lower/upper solutions* (α, β) *of problem* (P) *is said to be* strict *if* $\alpha_j(t) < \beta_j(t)$ *for every* $j \in \{1, \dots, N\}$ *and* $t \in [0,T]$*, and the following property holds: if* $x(t)$ *is a solution of* (P) *satisfying* (3.1)*, then*

$$\alpha_j(t) < x_j(t) < \beta_j(t)\,, \quad \textit{for every } j \in \{1, \dots, N\} \textit{ and } t \in [0,T]\,.$$

When we have a well-ordered pair of strict lower/upper solutions, the previous theorem provides some additional information.

Theorem 3.2.6. *If* (α, β) *is a strict well-ordered pair of lower/upper solutions of problem* (P)*, then*

$$d(I - \mathcal{L}^{-1}\mathcal{N}, \Omega) = 1\,,$$

where

$$\begin{aligned}\Omega := \big\{ x \in \mathcal{C}([0,T], \mathbb{R}^N) \;:\; & \alpha_j(t) < x_j(t) < \beta_j(t)\,, \\ & \textit{for every } j \in \{1, \dots, N\} \textit{ and } t \in [0,T] \big\}\,.\end{aligned}$$

Proof. Arguing as in Step 1 of the proof of Theorem 3.2.2, we can introduce the modified problem (P') and we know, by Schauder Theorem, that

$$d(I - \mathcal{L}^{-1}\widetilde{\mathcal{N}}, B_R) = 1\,,$$

where B_R is an open ball in $\mathcal{C}([0,T], \mathbb{R}^N)$ centered at the origin with a sufficiently large radius $R > 0$. In particular, we may assume that $\Omega \subseteq B_R$. By the argument in Step 2 of the same proof and the fact that the pair of lower/upper solutions is strict, we have that all the solutions of (P') belong to Ω. In other words, there are no zeroes of $I - \mathcal{L}^{-1}\widetilde{\mathcal{N}}$ in the set $B_R \setminus \overline{\Omega}$. Then, by the excision property of the degree,

$$d(I - \mathcal{L}^{-1}\widetilde{\mathcal{N}}, \Omega) = 1\,.$$

Finally, since $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ coincide on the set Ω, the conclusion follows. $\square$

3.3 Non-well-ordered lower and upper solutions for systems

In this section we still consider problem (P) in the finite-dimensional space $\mathbb{R}^N$. We will treat the case in which we can find lower and upper solutions which are not well-ordered. To this aim, we need to distinguish the components which are well-ordered from the others.

We will say that the couple $(\mathcal{J}, \mathcal{K})$ is a partition of the set of indices $\{1, \dots, N\}$ if and only if $\mathcal{J} \cap \mathcal{K} = \varnothing$ and $\mathcal{J} \cup \mathcal{K} = \{1, \dots, N\}$. Correspondingly we can decompose a vector

$$x = (x_1, \dots, x_N) = (x_n)_{n \in \{1,\dots,N\}} \in \mathbb{R}^N$$

as $x = (x_{\mathcal{J}}, x_{\mathcal{K}})$ where $x_{\mathcal{J}} = (x_j)_{j \in \mathcal{J}} \in \mathbb{R}^{\#\mathcal{J}}$ and $x_{\mathcal{K}} = (x_k)_{k \in \mathcal{K}} \in \mathbb{R}^{\#\mathcal{K}}$. Here $\#\mathcal{J}$ and $\#\mathcal{K}$ denote respectively the cardinality of the sets $\mathcal{J}$ and $\mathcal{K}$.

Similarly, every function $\mathcal{F} : \mathcal{A} \to \mathbb{R}^N$ can be written as $\mathcal{F}(x) = \big(\mathcal{F}_{\mathcal{J}}(x), \mathcal{F}_{\mathcal{K}}(x)\big)$ where $\mathcal{F}_{\mathcal{J}} : \mathcal{A} \to \mathbb{R}^{\#\mathcal{J}}$ and $\mathcal{F}_{\mathcal{K}} : \mathcal{A} \to \mathbb{R}^{\#\mathcal{K}}$.

Definition 3.3.1. *Given two $\mathcal{C}^2$-functions $\alpha, \beta : [0,T] \to \mathbb{R}^N$ we will say that (α, β) is a* pair of lower/upper solutions of (P) related to the partition $(\mathcal{J}, \mathcal{K})$ of $\{1, \dots, N\}$ *if the following four conditions hold:*

1. *for any $j \in \mathcal{J}$, $\alpha_j(t) \le \beta_j(t)$ for every $t \in [0,T]$;*

2. *for any $k \in \mathcal{K}$, there exists $t_k^0 \in [0,T]$ such that $\alpha_k(t_k^0) > \beta_k(t_k^0)$;*

3. *for any $n \in \{1, \dots, N\}$ we have*

$$\ddot{\alpha}_n(t) \ge f_n(t, x_1, \dots, x_{n-1}, \alpha_n(t), x_{n+1}, \dots, x_N)\,, \tag{3.3}$$
$$\ddot{\beta}_n(t) \le f_n(t, x_1, \dots, x_{n-1}, \beta_n(t), x_{n+1}, \dots, x_N)\,, \tag{3.4}$$

   for every $(t,x) \in \mathcal{E}$, where

$$\mathcal{E} := \Big\{(t,x) \in [0,T] \times \mathbb{R}^N \;:\; x = (x_{\mathcal{J}}, x_{\mathcal{K}})\,, x_{\mathcal{J}} \in \prod_{j \in \mathcal{J}} [\alpha_j(t), \beta_j(t)]\Big\}\,.$$

4. *for any $n \in \{1, \dots, N\}$,*

$$\alpha_n(0) = \alpha_n(T)\,, \qquad \beta_n(0) = \beta_n(T)\,,$$
$$\dot{\alpha}_n(0) \ge \dot{\alpha}_n(T)\,, \qquad \dot{\beta}_n(0) \le \dot{\beta}_n(T)\,.$$

Definition 3.3.2. *The pair* (α, β) *of lower/upper solutions of* (P) *is said to be* strict with respect to the j-th component, *with* $j \in \mathcal{J}$, *if* $\alpha_j(t) < \beta_j(t)$ *for every* $t \in [0, T]$, *and for every solution* x *of* (P) *we have*

$$\big(\forall t \in [0,T]\,,\ \alpha_j(t) \le x_j(t) \le \beta_j(t)\big) \Rightarrow \big(\forall t \in [0,T]\,,\ \alpha_j(t) < x_j(t) < \beta_j(t)\big)\,; \quad (3.5)$$

it is said to be strict with respect to the k-th component, *with* $k \in \mathcal{K}$, *if for every solution* x *of* (P) *we have*

$$\big(\forall t \in [0,T]\,,\ x_k(t) \ge \alpha_k(t)\big) \Rightarrow \big(\forall t \in [0,T]\,,\ x_k(t) > \alpha_k(t)\big)\,, \quad (3.6)$$

$$\big(\forall t \in [0,T]\,,\ x_k(t) \le \beta_k(t)\big) \Rightarrow \big(\forall t \in [0,T]\,,\ x_k(t) < \beta_k(t)\big)\,. \quad (3.7)$$

The following proposition provides a sufficient condition in order to guarantee the *strictness* property of a pair of lower/upper solutions of (P) with respect to a certain component.

Proposition 3.3.3. *Given a pair* (α, β) *of lower/upper solutions of (P),*

1. *if, for any* $n \in \mathcal{J}$, *both* (3.3) *and* (3.4) *hold with strict inequalities, then* (3.5) *holds for* $n = j$;
2. *if, for any* $n \in \mathcal{K}$, (3.3) *holds with strict inequality, then* (3.6) *holds for* $n = k$;
3. *if, for any* $n \in \mathcal{K}$, (3.4) *holds with strict inequality, then* (3.7) *holds for* $n = k$.

The proof can be easily adapted from the corresponding scalar result in [29, Proposition III-1.1] and is omitted.

We are able to prove the existence of a solution of (P) in presence of a pair of lower/upper solutions (α, β) provided that we ask the *strictness* property when the components α_k, β_k are non-well-ordered.

Theorem 3.3.4. *Let* (α, β) *be a pair of lower/upper solutions of* (P) *related to the partition* $(\mathcal{J}, \mathcal{K})$ *of* $\{1, \dots, N\}$, *and assume that it is strict with respect to the* k*-th component, for every* $k \in \mathcal{K}$. *Assume moreover the existence of a constant* $C > 0$ *such that*

$$|f_{\mathcal{K}}(t,x)| \le C\,, \qquad \text{for every } (t,x) \in \mathcal{E}\,.$$

Then, (P) *has a solution* x *with the following property: for any* $(j,k) \in \mathcal{J} \times \mathcal{K}$,

(W_j) $\alpha_j(t) \le x_j(t) \le \beta_j(t)$, *for every* $t \in [0,T]$;

(NW_k) *there exist* $t_k^1, t_k^2 \in [0,T]$ *such that* $x_k(t_k^1) < \alpha_k(t_k^1)$ *and* $x_k(t_k^2) > \beta_k(t_k^2)$.

In Section 3.3.2 we will provide a generalization of the above result, removing the strictness assumption on one of the components $\kappa \in \mathcal{K}$. Let us now present two illustrative examples.

Example 3.3.5. *Assume $\mathcal{J} = \varnothing$ and let, for every $k \in \mathcal{K}$,*

$$f_k(t,x) = -\frac{a_k x_k}{1+|x_k|} + h_k(t,x)\,,$$

for some $a_k > 0$, with

$$\|h_k\|_\infty := \sup\left\{|h_k(t,x)| : (t,x) \in [0,T] \times \mathbb{R}^N\right\} < a_k\,. \tag{3.8}$$

Then, taking the constant functions

$$\alpha_k = \frac{\|h_k\|_\infty}{a_k - \|h_k\|_\infty} + 1\,, \qquad \beta_k = -\frac{\|h_k\|_\infty}{a_k - \|h_k\|_\infty} - 1\,,$$

we see that Theorem 3.3.4 applies. The same would be true if $\mathcal{J} \neq \varnothing$, assuming for $j \in \mathcal{J}$, e.g., a situation like in Examples 3.2.3 and 3.2.4.

Example 3.3.6. *Let*

$$f_n(t,x) = -a_n \sin x_n + h_n(t,x)\,,$$

with $a_n > 0$ and h_n satisfying (3.8) *with $k = n$. For every $n \in \{1,\dots,N\}$ we have constant lower and upper solutions*

$$\alpha_n \in \left\{\frac{\pi}{2} + 2m\pi : m \in \mathbb{Z}\right\}, \qquad \beta_n \in \left\{-\frac{\pi}{2} + 2m\pi : m \in \mathbb{Z}\right\}.$$

Then, for each equation $\ddot{x} = f_n(t,x)$ we have both well-ordered and non-well-ordered pairs of lower/upper solutions. Let us fix, e.g.,

$$\alpha_n = \frac{\pi}{2}\,, \qquad \beta_n^\iota = \frac{\pi}{2} + \iota\pi\,, \quad \text{with } \iota \in \{-1,1\}\,.$$

Choosing $\vec{\iota} = (\iota_1,\dots,\iota_N) \in \{-1,1\}^N$, and defining (α,β) with $\beta_n = \beta_n^{\iota_n}$, by Theorem 3.3.4 we get the existence of at least 2^N solutions $x^{\vec{\iota}}$ of problem (P), whose components are such that

$$\begin{aligned} \iota_n = 1 \Rightarrow &\quad \forall t \in [0,T],\ x_n^{\vec{\iota}}(t) \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \\ \iota_n = -1 \Rightarrow &\quad \exists \bar{t}_n \in [0,T],\ x_n^{\vec{\iota}}(\bar{t}_n) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \end{aligned}$$

We notice that, even if the function $h(t,x_1,\dots,x_n)$ is 2π-periodic in each variable x_n, the solutions we find are indeed geometrically distinct. We thus get a generalization of a result obtained for the scalar equation in [71].

3.3.1 Proof of Theorem 3.3.4

Notice that the case $\mathcal{K} = \varnothing$ reduces to Theorem 3.2.2. We thus assume $\mathcal{K} \neq \varnothing$ and, without loss of generality, we take either $\mathcal{J} = \varnothing$, or $\mathcal{J} = \{1, \dots, M\}$ and $\mathcal{K} = \{M+1, \dots, N\}$ for a certain $M \in \{1, \dots, N\}$. Indeed, mixing the coordinates of $x = (x_1, \dots, x_N)$, we can always reduce to such a situation. We continue the proof in the case $\mathcal{J} \neq \varnothing$. (The case $\mathcal{J} = \varnothing$ can be treated essentially in the same way.)

We need to suitably modify problem (P). For every $r > 0$, we consider the problem

$$(P_r) \qquad \begin{cases} \ddot{x} = g_r(t, x)\,, \\ x(0) = x(T)\,, \quad \dot{x}(0) = \dot{x}(T)\,, \end{cases}$$

where $g_r : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$, with

$$g_r(t, x) = \big(g_{r,1}(t, x), \dots, g_{r,M}(t, x), g_{r,M+1}(t, x), \dots, g_{r,N}(t, x)\big)\,,$$

is defined as follows.

We first introduce the functions $\bar{f} : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$ and $\Gamma : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$ as

$$\begin{aligned} &\bar{f}(t, x) = f(t, \Gamma(t, x))\,, \\ &\Gamma(t, x) = \big(\gamma_1(t, x_1), \dots, \gamma_M(t, x_M), x_{M+1}, \dots, x_N\big)\,. \end{aligned}$$

where, for $j \in \mathcal{J}$,

$$\gamma_j(t, s) = \begin{cases} \alpha_j(t)\,, & \text{if } s < \alpha_j(t)\,, \\ s\,, & \text{if } \alpha_j(t) \le s \le \beta_j(t)\,, \\ \beta_j(t)\,, & \text{if } s > \beta_j(t)\,. \end{cases}$$

Now we define, for every index $j \in \mathcal{J}$,

$$g_{r,j}(t, x) = \bar{f}_j(t, x) + x_j - \gamma_j(t, x_j)\,,$$

and for every index $k \in \mathcal{K}$,

$$g_{r,k}(t, x) = \begin{cases} \bar{f}_k(t, x) & \text{if } |x_k| \le r\,, \\ (|x_k| - r)\, C \dfrac{x_k}{|x_k|} + (1 + r - |x_k|)\bar{f}_k(t, x) & \text{if } r < |x_k| < r + 1\,, \\ C \dfrac{x_k}{|x_k|} & \text{if } |x_k| \ge r + 1\,. \end{cases}$$

Notice that, for the indices $j \in \mathcal{J}$, the value $r > 0$ does not affect the definition of the components $g_{r,j}$.

Proposition 3.3.7. *If x is a solution of (P_r), then $\alpha_j(t) \leq x_j(t) \leq \beta_j(t)$ for every $j \in \mathcal{J}$ and $t \in [0,T]$.*

The proof follows from a classical reasoning and can be easily adapted from Step 2 of the proof of Theorem 3.2.2.

Proposition 3.3.8. *There is a constant $K > 0$ such that, if x is a solution of (P_r), for any $r > 0$, which satisfies (NW_k) for a certain index $k \in \mathcal{K}$, then $\|x_k\|_{C^2} \leq K$.*

Proof. Notice that

$$|g_{r,k}(t,x)| \leq C\,, \qquad \text{for every } (t,x) \in [0,T] \times \mathbb{R}^N, k \in \mathcal{K} \text{ and } r > 0\,. \tag{3.9}$$

Fix any $k \in \mathcal{K}$. If $x(t)$ is a solution of (P_r), multiplying the k-th equation by $\tilde{x}_k$ and integrating, we have that

$$\|\tilde{x}_k\|_2^2 \leq \left(\frac{T}{2\pi}\right)^2 \|\dot{x}_k\|_2^2 \leq \left(\frac{T}{2\pi}\right)^2 C\sqrt{T}\|\tilde{x}_k\|_2\,.$$

Therefore, by a classical reasoning, there exist a constant $C_1 > 0$ such that $\|\tilde{x}_k\|_{H^1} \leq C_1$, and there is a constant $C_0 > 0$ such that $\|\tilde{x}_k\|_\infty \leq C_0$, for every solution x of (P_r). Define

$$u_k(t) = \min\{\alpha_k(t), \beta_k(t)\}\,, \quad \mathcal{U}_k(t) = \max\{\alpha_k(t), \beta_k(t)\}\,. \tag{3.10}$$

Since (NW_k) holds, there is a $\tau_0 \in [0,T]$ such that

$$u_k(\tau_0) \leq x_k(\tau_0) \leq \mathcal{U}_k(\tau_0)\,. \tag{3.11}$$

Then, if x is a solution of (P_r),

$$\begin{aligned}|x_k(t)| = \left|x_k(\tau_0) + \int_{\tau_0}^t \dot{x}_k(s)\,\mathrm{d}s\right| \leq |x_k(\tau_0)| + \int_0^T |\dot{x}_k(s)|\,\mathrm{d}s \leq |x_k(\tau_0)| + \sqrt{T}\|\dot{x}_k\|_2 \\ \leq \max\{\|\alpha\|_\infty\,,\ \|\beta\|_\infty\} + \sqrt{T}C_1 =: K_0\,,\end{aligned}$$

hence $\|x_k\|_\infty \leq K_0$. Moreover, by periodicity, there is a $\tau_1 \in [0,T]$ such that $\dot{x}_k(\tau_1) = 0$, hence by (3.9)

$$\begin{aligned}|\dot{x}_k(t)| &= \left|\dot{x}_k(\tau_1) + \int_{\tau_1}^t \ddot{x}_k(s)\,\mathrm{d}s\right| = \left|\int_{\tau_1}^t g_{r,k}(s,x(s))\,\mathrm{d}s\right| \\ &\leq \int_0^T |g_{r,k}(s,x(s))|\,\mathrm{d}s \leq CT\,,\end{aligned}$$

so that $\|\dot{x}_k\|_\infty \leq CT$. Then,

$$\|x_k\|_{C^2} = \|x_k\|_\infty + \|\dot{x}_k\|_\infty + \|\ddot{x}_k\|_\infty \leq K_0 + CT + C =: K\,,$$

thus proving the proposition. □

From now on, we will fix $r > \max\{K, \|\alpha\|_\infty\,, \|\beta\|_\infty\}$, where K is given by Lemma 3.3.8. Problem (P_r) is equivalent to the fixed point problem

$$x = \mathcal{L}^{-1}\mathcal{N}_r x\,, \qquad x \in \mathcal{C}([0,T],\mathbb{R}^N)\,,$$

where we have introduced the Nemytskii operator

$$\mathcal{N}_r : \mathcal{C}([0,T],\mathbb{R}^N) \to \mathcal{C}([0,T],\mathbb{R}^N)\,, \quad (\mathcal{N}_r x)(t) = x(t) - g_r(t,x(t))\,.$$

Since we are looking for zeros of

$$\mathcal{T}_r x := (I - \mathcal{L}^{-1}\mathcal{N}_r)(x)\,,$$

we are going to compute the Leray-Schauder degree on a family of open sets. Let us define the constant functions

$$\hat{\alpha} = -r-1\,, \qquad \hat{\beta} = r+1\,,$$

as well as the functions

$$\check{\alpha}_j(t) = \alpha_j(t) - 1\,, \quad \text{and} \quad \check{\beta}_j(t) = \beta_j(t) + 1\,,$$

for every $j \in \mathcal{J}$.

We define, for every multi-index $\mu = (\mu_{M+1}, \ldots, \mu_N) \in \{1,2,3,4\}^{N-M}$, the open set

$$\Omega_\mu := \{x \in \mathcal{C}([0,T],\mathbb{R}^N) \,:\, (\mathcal{O}_j^0) \text{ and } (\mathcal{O}_k^{\mu_k}) \text{ hold for every } j \in \mathcal{J} \text{ and } k \in \mathcal{K}\}\,, \tag{3.12}$$

where the conditions $(\mathcal{O}_j^0)$ and $(\mathcal{O}_k^{\mu_k})$ read as

$(\mathcal{O}_j^0)$ $\check{\alpha}_j(t) < x_j(t) < \check{\beta}_j(t)$, for every $t \in [0,T]$,

$(\mathcal{O}_k^1)$ $\hat{\alpha} < x_k(t) < \hat{\beta}$, for every $t \in [0,T]$,

$(\mathcal{O}_k^2)$ $\hat{\alpha} < x_k(t) < \beta_k(t)$, for every $t \in [0,T]$,

$(\mathcal{O}_k^3)$ $\alpha_k(t) < x_k(t) < \hat{\beta}$, for every $t \in [0,T]$,

$(\mathcal{O}_k^4)$ $\hat{\alpha} < x_k(t) < \hat{\beta}$, for every $t \in [0,T]$, and there are $t_k^1, t_k^2 \in [0,T]$ such that $x(t_k^1) < \alpha_k(t_k^1)$ and $x(t_k^2) > \beta_k(t_k^2)$.

Proposition 3.3.9. *The Leray-Schauder degree $d(\mathcal{T}_r, \Omega_\mu)$ is well-defined for every $\mu \in \{1,2,3,4\}^{N-M}$.*

Proof. Assume by contradiction that there is $x \in \partial\Omega_\mu$ such that $\mathcal{T}_r x = 0$, i.e., x is a solution of (P_r). All the several different situations which may arise lead back to the following four cases.

Case A. For some index $j \in \mathcal{J}$, $\check{\alpha}_j(t) \le x_j(t) \le \check{\beta}_j(t)$, for every $t \in [0,T]$, and $\check{\alpha}_j(\tau) = x_j(\tau)$ for a certain $\tau \in [0,T]$ (the case when $x_j(\tau) = \check{\beta}_j(\tau)$ is similar). We can prove that

$$\ddot{\check{\alpha}}_j(t) > g_{r,j}(t, x_1(t), \dots, x_{j-1}(t), \check{\alpha}_j(t), x_{j+1}(t), \dots, x_N(t)), \quad \text{for every } t \in [0,T],$$

so that arguing as in Step 2 of the proof of Theorem 3.2.2 we obtain a contradiction.

Case B. For some index $k \in \mathcal{K}$, $\hat{\alpha} \le x_k(t) \le \hat{\beta}$, for every $t \in [0,T]$, and $\hat{\alpha} = x_k(\tau)$ for a certain $\tau \in [0,T]$ (the case when $x_k(\tau) = \hat{\beta}$ is similar). Since

$$g_{r,k}(t, x_1(t), \dots, x_{k-1}(t), \hat{\alpha}, x_{k+1}(t), \dots, x_N(t)) = -C < 0, \quad \text{for every } t \in [0,T],$$

we easily get a contradiction as before.

Case C. For some index $k \in \mathcal{K}$, $\hat{\alpha} < x_k(t) \le \beta_k(t)$, for every $t \in [0,T]$, and $x_k(\tau) = \beta_k(\tau)$ for a certain $\tau \in [0,T]$. Such a situation cannot arise since (3.7) holds by assumption.

Case D. For some index $k \in \mathcal{K}$, $\alpha_k(t) \le x_k(t) < \hat{\beta}$, for every $t \in [0,T]$, and $x_k(\tau) = \alpha_k(\tau)$ for a certain $\tau \in [0,T]$. Such a situation cannot arise since (3.6) holds by assumption. □

Proposition 3.3.10. *For every multi-index $\mu \in \{1,2,3\}^{N-M}$ we have $d(\mathcal{T}_r, \Omega_\mu) = 1$.*

Proof. In this case, it can be verified by the arguments of the previous proof, that the definition of the set Ω_μ provides us a well-ordered pair of strict lower/upper solutions of problem (P_r). The conclusion is then an immediate consequence of Theorem 3.2.6. □

For any multi-index $\hat{\mu} \in \{1,2,3\}^{N-M-1}$ we can consider, for every $\ell \in \{1,2,3,4\}$, the multi-index

$$(\ell, \hat{\mu}) = (\ell, \mu_{M+2}, \dots, \mu_N) \in \{1,2,3,4\}^{N-M}.$$

We can verify that $\Omega_{(2,\hat{\mu})}, \Omega_{(3,\hat{\mu})}, \Omega_{(4,\hat{\mu})}$ are pairwise disjoint and all contained in $\Omega_{(1,\hat{\mu})}$ so that

$$\Omega_{(4,\hat{\mu})} = \Omega_{(1,\hat{\mu})} \setminus \overline{\Omega_{(2,\hat{\mu})} \cup \Omega_{(3,\hat{\mu})}}. \tag{3.13}$$

Proposition 3.3.11. *For every multi-index $\hat{\mu} \in \{1,2,3\}^{N-M-1}$ we have $d(\mathcal{T}_r, \Omega_{(4,\hat{\mu})}) = -1$.*

Proof. By Proposition 3.3.10 and equation (3.13),

$$\begin{aligned} 1 &= d(\mathcal{T}_r, \Omega_{(1,\hat{\mu})}) \\ &= d(\mathcal{T}_r, \Omega_{(2,\hat{\mu})}) + d(\mathcal{T}_r, \Omega_{(3,\hat{\mu})}) + d(\mathcal{T}_r, \Omega_{(4,\hat{\mu})}) \\ &= 2 + d(\mathcal{T}_r, \Omega_{(4,\hat{\mu})}) \end{aligned}$$

and the conclusion follows. □

Arguing similarly we can prove by induction the following result.

Proposition 3.3.12. *For every $K \in \{1, \dots, N-M\}$ and every multi-index $\mu \in \{4\}^K \times \{1,2,3\}^{N-M-K}$, we have*

$$d(\mathcal{T}_r, \Omega_\mu) = (-1)^K .$$

Proof. We proceed by induction. The validity of the statement for $K = 1$ follows by Proposition 3.3.11. So, we fix $K \geq 2$ and assume that

$$d(\mathcal{T}_r, \Omega_\mu) = (-1)^{K-1}, \text{ for every } \mu \in \{4\}^{K-1} \times \{1,2,3\}^{N-M-K+1} .$$

Consider the multi-index $\mu = (4, \dots, 4, \mu_{M+K}, \mu_{M+K+1}, \dots, \mu_N) \in \{4\}^{K-1} \times \{1,2,3\}^{N-M-K+1}$ and define for every $\ell \in \{1,2,3,4\}$, the multi-index

$$\bar{\mu}^\ell = (4, \dots, 4, \ell, \mu_{M+K+1}, \dots, \mu_N) .$$

We then see that

$$\begin{aligned} (-1)^{K-1} &= d(\mathcal{T}_r, \Omega_{\bar{\mu}^1}) \\ &= d(\mathcal{T}_r, \Omega_{\bar{\mu}^2}) + d(\mathcal{T}_r, \Omega_{\bar{\mu}^3}) + d(\mathcal{T}_r, \Omega_{\bar{\mu}^4}) \\ &= 2 \cdot (-1)^{K-1} + d(\mathcal{T}_r, \Omega_{\bar{\mu}^4}) , \end{aligned}$$

yielding $d(\mathcal{T}_r, \Omega_{\bar{\mu}^4}) = (-1)^K$. The proof is complete. □

By the previous proposition we conclude that

$$d(\mathcal{T}_r, \Omega_{(4,\dots,4)}) = (-1)^{N-M} . \tag{3.14}$$

As a consequence, there is a solution x of problem (P_r) in the set $\Omega_{(4,\dots,4)}$. Recalling the a priori bounds in Propositions 3.3.7 and 3.3.8, we see that the solution x is indeed a solution of problem (P) and satisfies (W_j) and (NW_k), for every $j \in \mathcal{J}$ and $k \in \mathcal{K}$. The proof is thus completed.

3.3.2 An extension of Theorem 3.3.4

The existence of a solution of (P) can be obtained also removing from the assumptions of Theorem 3.3.4 the *strictness* assumption on *one* of the components.

Theorem 3.3.13. *Let (α, β) be a pair of lower/upper solutions of (P) related to the partition $(\mathcal{J}, \mathcal{K})$ of $\{1, \ldots, N\}$. Fix $\kappa \in \mathcal{K}$ and assume that (α, β) is strict with respect to the k-th component, for every $k \in \mathcal{K} \setminus \{\kappa\}$. Assume moreover the existence of a constant $C > 0$ such that*

$$|f_{\mathcal{K}}(t, x)| \leq C\,, \quad \text{for every } (t, x) \in \mathcal{E}\,.$$

Then, (P) has a solution x such that (W_j) and (NW_k) hold for every $(j, k) \in \mathcal{J} \times (K \setminus \{\kappa\})$, and

$(\widetilde{NW}_\kappa)$ *there exist $t^1_\kappa, t^2_\kappa \in [0, T]$ such that $x_\kappa(t^1_\kappa) \leq \alpha_\kappa(t^1_\kappa)$ and $x_\kappa(t^2_\kappa) \geq \beta_\kappa(t^2_\kappa)$.*

Proof. Without loss of generality we can choose $\mathcal{J} = \{1, \ldots, M\}$, $\mathcal{K} = \{M + 1, \ldots, N\}$ and $\kappa = N$. We can follow the proof of Theorem 3.3.4 step by step in the first part, noticing that Proposition 3.3.8 holds with the same constant when we assume $(\widetilde{NW}_N)$. Moreover, since we do not ask the strictness assumption with respect to the N-th component, when we introduce the sets Ω_μ as in (3.12), we can consider only multi-indices with the last component frozen to 1, i.e. $\mu = (\mu_{M+1}, \ldots, \mu_{N-1}, 1) \in \{1, 2, 3, 4\}^{N-M-1} \times \{1\}$. Indeed, with this new choice of the multi-indices we can still guarantee that the Leray-Schauder degree is well-defined.

Then, arguing as in Propositions 3.3.10, 3.3.11 and 3.3.12 we have

- $d(\mathcal{T}_r, \Omega_\mu) = 1$ for every $\mu \in \{1, 2, 3\}^{N-M-1} \times \{1\}$,
- $d(\mathcal{T}_r, \Omega_\mu) = -1$ for every $\mu \in \{4\} \times \{1, 2, 3\}^{N-M-2} \times \{1\}$,
- for every $K \in \{1, \ldots, N-M-1\}$, $d(\mathcal{T}_r, \Omega_\mu) = (-1)^K$ for every multi-index $\mu \in \{4\}^K \times \{1, 2, 3\}^{N-M-K-1} \times \{1\}$.

However, we cannot conclude the proof saying that the Leray-Schauder degree is different from zero in $\Omega_{(4,\ldots,4)}$ as in (3.14), since we cannot ensure that it is well defined in the sets $\Omega_{(4,\ldots,4,\ell)}$ with $\ell = 2, 3, 4$.

Anyhow, at this step of the proof, we can follow the classical reasoning adopted in the scalar case in presence of non-well-ordered lower/upper solutions, cf. [29, Theorem III-3.1]. If there exists $x \in \partial\Omega_{(4,\ldots,4,2)}$ such that $\mathcal{T}_r x = 0$, then we can easily see that x must be a solution of (P_r) such that $x_N(t) \leq \beta_N(t)$ for every $t \in [0, T]$ and $x_N(\tau) = \beta_N(\tau)$ for a certain $\tau \in [0, T]$. Since the components α_N, β_N are non-well-ordered, we have $\alpha_N(t^0_N) > \beta_N(t^0_N) \geq x_N(t^0_N)$ for some $t^N_0 \in [0, T]$. So $(\widetilde{NW}_N)$ holds, thus giving us that x is a solution of (P_r) satisfying all the required assumptions.

We can argue similarly if there exists $x \in \partial\Omega_{(4,\dots,4,3)}$ such that $\mathcal{T}_r x = 0$.

If the previous situations do not occur, we can compute the degree both in $\Omega_{(4,\dots,4,2)}$ and $\Omega_{(4,\dots,4,3)}$. As in (3.13), we have

$$\Omega_{(4,\dots,4,4)} = \Omega_{(4,\dots,4,1)} \setminus \overline{\Omega_{(4,\dots,4,2)} \cup \Omega_{(4,\dots,4,3)}}\,.$$

so that the degree is well defined also for $\Omega_{(4,\dots,4,4)}$. Performing exactly the same computation adopted in Propositions 3.3.11 and 3.3.12 we can conclude that $d(\mathcal{T}_r, \Omega_{(4,\dots,4)}) = (-1)^{N-M}$, thus finding also in this case a solution x with the desired properties. The proof is thus completed. $\square$

3.4 Infinite-dimensional systems

We now focus our attention on a system defined in a separable Hilbert space H with scalar product $\langle\cdot,\cdot\rangle$ and corresponding norm $|\cdot|$. We study the problem

$$(P) \qquad \begin{cases} \ddot{x} = f(t,x)\,, \\ x(0) = x(T)\,, \quad \dot{x}(0) = \dot{x}(T)\,, \end{cases}$$

where $f : [0,T] \times H \to H$ is a continuous function. In what follows, we extend the results of Section 3.3 to an infinite-dimensional setting, trying to maintain similar notations.

Let $\mathbb{N}_+ = \{1,2,3,\dots\}$. Choosing a Hilbert basis $(e_n)_{n\in\mathbb{N}_+}$, every vector $x \in H$ can be written as $x = \sum_{n\in\mathbb{N}_+} x_n e_n$, or $x = (x_n)_{n\in\mathbb{N}_+} = (x_1, x_2, \dots)$. Similarly, for the function f, we will write

$$f(t,x) = (f_1(t,x), f_2(t,x), \dots)\,.$$

We will sometimes identify H with ℓ^2.

As in the finite-dimensional case, we will say that the couple $(\mathcal{J},\mathcal{K})$ is a partition of $\mathbb{N}_+$ if and only if $\mathcal{J} \cap \mathcal{K} = \varnothing$ and $\mathcal{J} \cup \mathcal{K} = \mathbb{N}_+$. Correspondingly, we can decompose the Hilbert space as $H = H_{\mathcal{J}} \times H_{\mathcal{K}}$, where every $x \in H$ can be written as $x = (x_{\mathcal{J}}, x_{\mathcal{K}})$ with $x_{\mathcal{J}} = (x_j)_{j\in\mathcal{J}} \in H_{\mathcal{J}}$ and $x_{\mathcal{K}} = (x_k)_{k\in\mathcal{K}} \in H_{\mathcal{K}}$.

Similarly, every function $\mathcal{F} : \mathcal{A} \to H$ can be written as $\mathcal{F}(x) = \big(\mathcal{F}_{\mathcal{J}}(x), \mathcal{F}_{\mathcal{K}}(x)\big)$ where $\mathcal{F}_{\mathcal{J}} : \mathcal{A} \to H_{\mathcal{J}}$ and $\mathcal{F}_{\mathcal{K}} : \mathcal{A} \to H_{\mathcal{K}}$.

We rewrite Definition 3.3.1 in this context.

Definition 3.4.1. *Given two $\mathcal{C}^2$-functions $\alpha, \beta : [0,T] \to H$ we will say that (α,β) is a* pair of lower/upper solutions of (P) related to the partition $(\mathcal{J},\mathcal{K})$ of $\mathbb{N}_+$ *if the* *four conditions of Definition 3.3.1 hold replacing* $\{1,\dots,N\}$ *by* $\mathbb{N}_+$ *and the inequalities* (3.3), (3.4) *by*

$$\ddot{\alpha}_n(t) \geq f_n(t, x_1, \dots, x_{n-1}, \alpha_n(t), x_{n+1}, \dots)\,,$$
$$\ddot{\beta}_n(t) \leq f_n(t, x_1, \dots, x_{n-1}, \beta_n(t), x_{n+1}, \dots)\,.$$

Moreover, it is said to be strict with respect to the n-th component, *with* $n \in \mathbb{N}_+$, *if the conditions of Definition 3.3.2 hold.*

We recall the definition of the set

$$\mathcal{E} := \Big\{(t,x) \in [0,T] \times \mathbb{R}^N \,:\, x = (x_{\mathcal{J}}, x_{\mathcal{K}})\,, x_{\mathcal{J}} \in \prod_{j \in \mathcal{J}} [\alpha_j(t), \beta_j(t)]\Big\}\,.$$

Here is our result in this infinite-dimensional setting.

Theorem 3.4.2. *Let* (α, β) *be a pair of lower/upper solutions of* (P) *related to the partition* $(\mathcal{J}, \mathcal{K})$ *of* $\mathbb{N}_+$*, and assume the following conditions:*

- *there exists a sequence* $(d_n)_{n\in\mathbb{N}_+} \in \ell^2$ *such that*

  $-d_n \leq \alpha_n(t) \leq d_n$ *and* $-d_n \leq \beta_n(t) \leq d_n$ *, for every* $n \in \mathbb{N}_+$ *and* $t \in [0,T]$*;*

- (α, β) *is strict with respect to the* k*-th component, for every* $k \in \mathcal{K}$*;*

- *there exists a constant* $C > 0$ *such that*

  $$|f_{\mathcal{K}}(t,x)| \leq C\,, \quad \text{for every } (t,x) \in \mathcal{E}\,;$$

- *for every bounded set* $\mathcal{B} \subset \mathcal{E}$*, the set* $f_{\mathcal{K}}(\mathcal{B})$ *is precompact.*

Then, (P) *has a solution* x *with the following property: for any* $(j,k) \in \mathcal{J} \times \mathcal{K}$*,*

(W_j) $\alpha_j(t) \leq x_j(t) \leq \beta_j(t)$*, for every* $t \in [0,T]$*;*

$(\widetilde{NW}_k)$ *there exist* $t_k^1, t_k^2 \in [0,T]$ *such that* $x_k(t_k^1) \leq \alpha_k(t_k^1)$ *and* $x_k(t_k^2) \geq \beta_k(t_k^2)$*.*

The proof of the theorem is carried out in Section 3.4.2.

Remark 3.4.3. *As in Theorem 3.3.13, we can drop the strictness assumption for a certain index* $\kappa \in \mathcal{K}$*.*

As an immediate consequence of Theorem 3.4.2, taking α and β constant functions, we have the following.

Corollary 3.4.4. *Let there exist two sequences $(p_n)_{n\in\mathbb{N}_+}$ and $(q_n)_{n\in\mathbb{N}_+}$ in ℓ^2, with $p_n < q_n$ for every $n \in \mathbb{N}_+$, and a partition $(\mathcal{J},\mathcal{K})$ of $\mathbb{N}_+$, such that, for every $(t,x) \in [0,T] \times \prod_{j\in\mathcal{J}}[p_j,q_j] \times H_{\mathcal{K}}$,*

$$\begin{aligned} j \in \mathcal{J} \Rightarrow f_j(t,x_1,\dots,x_{j-1},p_j,x_{j+1},\dots) &\le 0 \\ &\le f_j(t,x_1,\dots,x_{j-1},q_j,x_{j+1},\dots)\,; \end{aligned} \tag{3.15}$$

$$\begin{aligned} k \in \mathcal{K} \Rightarrow f_k(t,x_1,\dots,x_{k-1},p_k,x_{k+1},\dots) &> 0 \\ &> f_k(t,x_1,\dots,x_{k-1},q_k,x_{k+1},\dots)\,. \end{aligned} \tag{3.16}$$

Furthermore, let there exists a sequence $(C_k)_{k\in\mathcal{K}} \in \ell^2$ such that, for every $k \in \mathcal{K}$,

$$|f_k(t,x)| \le C_k\,, \quad \text{for every } (t,x) \in [0,T] \times \prod_{j\in\mathcal{J}}[p_j,q_j] \times H_{\mathcal{K}}\,.$$

Then, (P) has a solution $x(t)$ such that, for every $j \in \mathcal{J}$, $k \in \mathcal{K}$,

$$\begin{aligned} &\{x_j(t) : t \in [0,T]\} \subseteq [p_j,q_j]\,; \\ &\{x_k(t) : t \in [0,T]\} \cap [p_k,q_k] \ne \varnothing\,. \end{aligned} \tag{3.17}$$

We now give some examples of applications, with $H = \ell^2$, where we implicitly assume all functions to be continuous.

Example 3.4.5. *Let, for every $j \in \mathbb{N}_+$,*

$$f_j(t,x) = x_j^3 + h_j(t,x)\,,$$

and assume that there is a $c > 0$ such that

$$|h_j(t,x)| \le \frac{c}{j^3}\,, \quad \text{for every } (t,x) \in [0,T] \times \ell^2\,. \tag{3.18}$$

Then, $f : [0,T] \times \ell^2 \to \ell^2$ is well-defined and taking $q_j = -p_j = \sqrt[3]{c}/j$, we see that both $(p_j)_j, (q_j)_j$ belong to ℓ^2, and (3.15) *is satisfied, so that Corollary 3.4.4 applies with $\mathcal{K} = \varnothing$.*

Example 3.4.6. *Let us consider, for every $j \in \mathbb{N}_+$,*

$$f_j(t,x) = x_j^2 \sin x_j + h_j(t,x)\,,$$

and assume that there is a $c > 0$ such that (3.18) *holds. Then, $f : [0,T] \times \ell^2 \to \ell^2$ is well-defined. Since $x^2 \sin x \ge \frac{1}{2}x^3$ in the interval $[0,\pi/2]$, taking $q_j = -p_j = \sqrt[3]{2c}/j$, we see that both $(p_j)_j, (q_j)_j$ belong to ℓ^2, and* (3.15) *is satisfied, so that Corollary 3.4.4 applies with $\mathcal{K} = \varnothing$.*

Furthermore, for every $\ell \in \mathbb{Z}$ with $|\ell|$ sufficiently large, we can see that the constants $p^\ell = -\pi/2+2\ell\pi$, $q^\ell = \pi/2+2\ell\pi$ satisfy (3.15)*, for every $j \in \mathbb{N}_+$. Thus, we can replace a finite number of couples (p_j, q_j) with some couples (p^ℓ, q^ℓ). Such a replacement must be performed only for a* finite *number of indices $j \in \mathbb{N}_+$ since we need to guarantee that the new sequences $(p_j)_j$ and $(q_j)_j$ remain in ℓ^2. Recalling that the so found solution of problem (P) must satisfy* (3.17) *then we conclude that (P) admits an infinite number of solutions.*

Example 3.4.7. *Let, for every $k \in \mathbb{N}_+$,*

$$f_k(t,x) = -\frac{x_k}{1+k|x_k|} + h_k(t,x)\,,$$

and assume that there is a $c \in]0,1[$ such that

$$|h_k(t,x)| \le \frac{c}{k}\,, \quad \text{for every } (t,x) \in [0,T] \times \ell^2\,.$$

Then, $f : [0,T] \times \ell^2 \to \ell^2$ is well-defined and taking $q_k = -p_k = \frac{c}{(1-c)k}$, we see that both $(p_k)_k, (q_k)_k$ belong to ℓ^2, and (3.16) *is verified, so that Corollary 3.4.4 applies with $\mathcal{J} = \varnothing$.*

Example 3.4.8. *Let $(a_n)_n$ and $(\sigma_n)_n$ be sequences of positive numbers in ℓ^2 and let, for every $n \in \mathbb{N}_+$,*

$$f_n(t,x) = -a_n \sin\Big(\frac{2\pi x_n}{\sigma_n}\Big) + h_n(t,x)\,.$$

If h_n satisfies

$$\sup\big\{|h_n(t,x)| : (t,x) \in [0,T] \times \ell^2\big\} < a_n\,,$$

we see that, for every $n \in \{1,\dots,N\}$, it is possible to find pairs of constant lower and upper solutions

$$\alpha_n \in \Big\{\frac{\sigma_n}{4} + m\sigma_n : m \in \mathbb{Z}\Big\}\,, \qquad \beta_n \in \Big\{-\frac{\sigma_n}{4} + m\sigma_n : m \in \mathbb{Z}\Big\}\,.$$

Then, for each equation $\ddot{x} = f_n(t,x)$ we have both well-ordered and non-well-ordered pairs of lower/upper solutions. Applying Corollary 3.4.4 we thus get the existence of infinitely many solutions of problem (P). By the same argument in Example 3.3.6 we notice that, even if the function $h(t,x_1,x_2,\dots)$ is σ_n-periodic in each variable x_n, the solutions we find are indeed geometrically distinct.

Remark 3.4.9. *This result should be compared with the ones in [18, 49], where one or two geometrically distinct solutions were found assuming a Hamiltonian structure of the problem, i.e.,*

$$h_n(t,x) = \frac{\partial \mathcal{V}}{\partial x_n}(t,x)\,,$$

for some function $\mathcal{V}(t, x_1, x_2, \dots)$ which is σ_n-periodic in each variable x_n. It was said in the final section of [49] that it remained an open problem to know if the existence of more than two T-periodic solutions could be proved, and in [18] that "it would be natural to conjecture the existence of infinitely many T-periodic solutions". It is interesting to notice that even in [18, 49], in order to recover some compactness, it was assumed that the sequence of the periods $(\sigma_n)_n$ belongs to ℓ^2.

Remark 3.4.10. *For any choice of a partition $(\mathcal{J}, \mathcal{K})$ of $\mathbb{N}_+$, we can consider functions f satisfying the requirements of Examples 3.4.5, 3.4.6 or 3.4.8 for every $j \in \mathcal{J}$ and of Examples 3.4.7 or 3.4.8 for every $k \in \mathcal{K}$. Corollary 3.4.4 applies also in this case.*

In the next section we provide some preliminary lemmas, which will be used in order to prove Theorem 3.4.2.

3.4.1 Some compactness lemmas

For every sequence $\tau = (\tau_n)_{n\in\mathbb{N}_+}$ contained in $[0, T]$ and every function $u \in \mathcal{C}([0, T], H)$, define the function $P_\tau u : [0, T] \to H$ as

$$(P_\tau u)_n(t) = \int_{\tau_n}^{t} u_n(s)\, \mathrm{d}s\,, \quad n \in \mathbb{N}_+\,.$$

We will need the following extension of [49, Lemma 3.2].

Lemma 3.4.11. *Let $E \subseteq \mathcal{C}([0, T], H)$ be such that the set*

$$A = \{u(t) : u \in E,\, t \in [0, T]\}$$

is precompact in H. Then the set

$$\Sigma = \left\{P_\tau u : \tau \in [0, T]^{\mathbb{N}_+}, u \in E\right\}$$

is precompact in $\mathcal{C}([0, T], H)$. As a consequence, the set

$$\Xi = \left\{P_\tau u(t) : \tau \in [0, T]^{\mathbb{N}_+}, u \in E,\, t \in [0, T]\right\}$$

is precompact in H.

Proof. Fix $\varepsilon > 0$. Since A is precompact, there exist $v_1, \dots, v_m$ in H such that

$$A \subseteq \bigcup_{\iota=1}^{m} B(v_\iota, \varepsilon)\,. \tag{3.19}$$

Let $V = \mathrm{Span}(v_1, \dots, v_m)$, and denote by $Q : H \to V$ the corresponding orthogonal projection. We first prove that the set

$$\mathcal{R} = \left\{P_\tau(Qu) : u \in E,\ \tau \in [0, T]^{\mathbb{N}_+}\right\}$$

is precompact in $\mathcal{C}([0, T], V)$.

The set $Q(A)$ is precompact in V and hence bounded; there exists a real constant D such that

$$|Qu(t)| < D\,, \quad \text{for all } u \in E \text{ and } t \in [0,T]\,. \tag{3.20}$$

Moreover, for every $u \in E$, $\tau \in [0,T]^{\mathbb{N}_+}$ and $t \in [0,T]$,

$$|(P_\tau(Qu))_n(t)| = \left|\int_{\tau_n}^t (Qu)_n(s)\,\mathrm{d}s\right| \le \left|\int_{\tau_n}^t |(Qu)_n(s)|\,\mathrm{d}s\right|\,, \quad n \in \mathbb{N}_+$$

and consequently

$$\begin{aligned}|P_\tau(Qu)(t)|^2 &= \sum_{n=1}^{\infty} |(P_\tau(Qu))_n(t)|^2 \\ &\le \sum_{n=1}^{\infty} \left|\int_{\tau_n}^t |(Qu)_n(s)|\,\mathrm{d}s\right|^2 \le \sum_{n=1}^{\infty} \Big(\int_0^T |(Qu)_n(s)|\,\mathrm{d}s\Big)^2;\end{aligned}$$

by the Hölder Inequality and the use of the Monotone Convergence Theorem, recalling (3.20),

$$\begin{aligned}\sum_{n=1}^{\infty} \Big(\int_0^T |(Qu)_n(s)|\,\mathrm{d}s\Big)^2 &\le T\sum_{n=1}^{\infty} \int_0^T |(Qu)_n(s)|^2\,\mathrm{d}s \\ &= T\int_0^T \sum_{n=1}^{\infty} |(Qu)_n(s)|^2\,\mathrm{d}s \\ &= T\int_0^T |Qu(s)|^2\,\mathrm{d}s < T^2D^2,\end{aligned}$$

and then

$$|P_\tau(Qu)(t)| \le TD\,.$$

Since V is finite-dimensional, the set $\mathcal{S} = \{w(t) : w \in \mathcal{R}\} \subseteq V$ is precompact. On the other hand, for every $u \in E$, $\tau \in [0,T]^{\mathbb{N}_+}$ and every $t_1, t_2 \in [0,T]$ with $t_1 < t_2$, we have

$$|P_\tau(Qu)(t_1) - P_\tau(Qu)(t_2)| = \left|\int_{t_1}^{t_2} (Qu)(s)\,\mathrm{d}s\right| \le \int_{t_1}^{t_2} |(Qu)(s)|\,\mathrm{d}s \le D(t_1 - t_2)\,,$$

so that $\mathcal{R}$ is equi-uniformly continuous as a subset of $\mathcal{C}([0,T],V)$. By the Ascoli–Arzelà Theorem, the set $\mathcal{R}$ is precompact in $\mathcal{C}([0,T],V)$.

Consequently, there exist $f_1, \dots, f_\ell$ in $\mathcal{C}([0,T], V)$ such that

$$\mathcal{R} \subseteq \bigcup_{\iota=1}^{\ell} B(f_\iota, \varepsilon)\,. \tag{3.21}$$

Now, for every $u \in E$, $\tau \in [0,T]^{\mathbb{N}_+}$ and $t \in [0,T]$, by (3.19),

$$\begin{aligned}
|P_\tau u(t) - P_\tau(Qu)(t)|^2 &= \sum_{n=1}^{\infty} |(P_\tau u)_n(t) - (P_\tau(Qu))_n(t)|^2 \\
&\le \sum_{n=1}^{\infty} \left| \int_{\tau_n}^{t} |u_n(s) - (Qu)_n(s)|\,\mathrm{d}s \right|^2 \\
&\le \sum_{n=1}^{\infty} T \int_0^T |u_n(s) - (Qu)_n(s)|^2\,\mathrm{d}s \\
&= T \int_0^T \sum_{n=1}^{\infty} |u_n(s) - (Qu)_n(s)|^2\,\mathrm{d}s \\
&= T \int_0^T |u(s) - (Qu)(s)|^2\,\mathrm{d}s \le T^2 \varepsilon^2,
\end{aligned}$$

and so

$$|P_\tau u(t) - P_\tau(Qu)(t)| \le T\varepsilon.$$

On the other hand, since $P_\tau(Qu) \in \mathcal{R}$, by (3.21) there exists $\bar\iota$ such that

$$\|P_\tau(Qu) - f_{\bar\iota}\|_\infty < \varepsilon\,,$$

hence

$$|P_\tau u(t) - f_{\bar\iota}(t)| \le |P_\tau u(t) - P_\tau(Qu)(t)| + |P_\tau(Qu)(t) - f_{\bar\iota}(t)| \le \varepsilon T + \varepsilon = \varepsilon(T+1).$$

We have thus shown that, given $\varepsilon > 0$, there are $f_1, \dots, f_\ell$ in $\mathcal{C}([0,T], H)$ such that

$$\Sigma \subseteq \bigcup_{\iota=1}^{\ell} B(f_\iota, (T+1)\varepsilon)\,,$$

hence proving that Σ is precompact.

The fact that Ξ is precompact in H now follows again from the Ascoli–Arzelà Theorem, recalling that this theorem gives a necessary and sufficient condition for precompactness. □

Let us denote by $\Pi_N : H \to H$ the projection

$$\Pi_N(x) = (x_1, \dots, x_N, 0, 0, \dots)\,. \tag{3.22}$$

Lemma 3.4.12. *Let A be a compact subset of H. Then, for every $\varepsilon > 0$, there is a $M \geq 1$ such that, for every $a = (a_n)_{n\in\mathbb{N}_+}$ in A,*

$$\sum_{n=M}^{\infty} |a_n|^2 \leq \varepsilon^2.$$

In particular $\lim_{N\to\infty}(\Pi_N - \mathrm{Id})x = 0$ uniformly for $x \in A$.

Proof. By contradiction, let there exist an $\varepsilon > 0$ such that, for every $M \geq 1$, there is $a^M = (a_n^M)_{n\in\mathbb{N}_+} \in A$ such that $\sum_{n=M}^{\infty} |a_n^M|^2 > \varepsilon^2$. By compactness, the sequence $(a^M)_{M\in\mathbb{N}_+}$ has a subsequence, for which we keep the same notation, such that $a^M \to a^*$, for some $a^* \in A$. Let M_* be any positive integer. Then, taking $M \geq M_*$ sufficiently large,

$$\begin{aligned}
\Big(\sum_{n=M_*}^{\infty} |a_n^*|^2\Big)^{1/2} &\geq \Big(\sum_{n=M}^{\infty} |a_n^*|^2\Big)^{1/2} \\
&\geq \Big(\sum_{n=M}^{\infty} |a_n^M|^2\Big)^{1/2} - \Big(\sum_{n=M}^{\infty} |a_n^M - a_n^*|^2\Big)^{1/2} \\
&\geq \varepsilon - \|a^M - a^*\|_{\ell^2} \geq \frac{\varepsilon}{2}\,.
\end{aligned}$$

We thus get a contradiction with the fact that $a^* \in H$. □

As an immediate consequence we find the following compactness property.

Lemma 3.4.13. *Let A be a compact subset of H. Then, the set*

$$A^{\mathcal{P}} := \bigcup_{N\in\mathbb{N}_+} \Pi_N A$$

is precompact in H.

Proof. Let us consider a sequence $(x_n)_{n\in\mathbb{N}_+}$ contained in $A^{\mathcal{P}}$.

If there exists $N_0 \in \mathbb{N}_+$ and a subsequence $(x_{n_\ell})_\ell$ such that $x_{n_\ell} \in \Pi_{N_0}A$ for every ℓ, then the conclusion is reached since $\Pi_{N_0}A$ is compact.

If the previous situation does not arise, then we can find a diverging sequence $(N_\ell)_\ell \subset \mathbb{N}_+$ and a subsequence $(x_{n_\ell})_\ell$ such that $x_{n_\ell} \in \Pi_{N_\ell}A$ for every ℓ. So, there is a sequence $(y_{n_\ell})_\ell \subseteq A$ such that $x_{n_\ell} = \Pi_{N_\ell} y_{n_\ell}$. Since A is compact, then, up to a subsequence, we have $y_{n_\ell} \to \bar{y} \in A$. Hence,

$$|x_{n_\ell} - \bar{y}| \leq |x_{n_\ell} - y_{n_\ell}| + |y_{n_\ell} - \bar{y}| \leq |(\Pi_{N_\ell} - \mathrm{Id})y_{n_\ell}| + |y_{n_\ell} - \bar{y}| \to 0\,,$$

where Lemma 3.4.12 has been applied. □

Remark 3.4.14. *The above statements have been formulated for a Hilbert space H. We will apply them also treating the previously introduced Hilbert spaces $H_{\mathcal{K}}$ and $H_{\mathcal{J}}$.*

3.4.2 Proof of Theorem 3.4.2

We consider, for every $N \in \mathbb{N}_+$, the auxiliary system

$$\begin{cases} \ddot{x}_1 = f_1(t, x_1, \dots, x_N, \alpha_{N+1}(t), \alpha_{N+2}(t), \dots) \\ \quad \vdots \\ \ddot{x}_N = f_N(t, x_1, \dots, x_N, \alpha_{N+1}(t), \alpha_{N+2}(t), \dots) \\ \ddot{x}_{N+1} = 0 \\ \ddot{x}_{N+2} = 0 \\ \quad \vdots \end{cases}$$

We recall the projections Π_N, introduced in (3.22), and define the function

$$\begin{aligned} &\widehat{\Pi}_N : \mathcal{C}([0,T], H) \to \mathcal{C}([0,T], H) \\ &\widehat{\Pi}_N x(t) = (x_1(t), \dots, x_N(t), \alpha_{N+1}(t), \alpha_{N+2}(t), \dots) \,. \end{aligned}$$

The auxiliary problem can then be written as

$$(\widehat{P}_N) \qquad \begin{cases} \ddot{x} = \Pi_N f(t, \widehat{\Pi}_N x(t)) \,, \\ x(0) = x(T) \,, \quad \dot{x}(0) = \dot{x}(T) \,. \end{cases}$$

Notice that

$$(t, \widehat{\Pi}_N x^N(t)) \in \mathcal{E} \,, \qquad \text{for every } N \in \mathbb{N}_+ \text{ and } t \in [0,T] \,. \tag{3.23}$$

By Theorem 3.3.4, for every $N \in \mathbb{N}_+$, there is a solution $x^N(t)$ of $(\widehat{P}_N)$ such that (W_j) and (NW_k) hold for every $j \in \mathcal{J} \cap [1, N]$ and $k \in \mathcal{K} \cap [1, N]$. We impose

$$x_n^N(t) = 0 \,, \qquad \text{for every } n > N \text{ and } t \in [0,T] \,.$$

Arguing as in the proof of Proposition 3.3.8, cf. (3.10) and (3.11), we conclude that x^N satisfies

$$\begin{aligned} &\{x_j^N(t) : t \in [0,T]\} \subseteq [-d_j, d_j] \,, \\ &\{x_k^N(t) : t \in [0,T]\} \cap [-d_k, d_k] \neq \varnothing \,, \end{aligned} \tag{3.24}$$

for every $k \in \mathcal{K}$ and $j \in \mathcal{J}$. Concerning the indices $j \in \mathcal{J}$ we thus have

$$x_{\mathcal{J}}^N(t) \in \mathcal{D}_{\mathcal{J}} := \prod_{j \in \mathcal{J}} [-d_j, d_j] \,, \tag{3.25}$$

for every $N \in \mathbb{N}_+$ and $t \in [0,T]$.

Now, we repeat the arguments of Proposition 3.3.8 with a slight modification. Given the solution x^N of $(\widehat{P}_N)$, we can compute

$$\|\tilde{x}_{\mathcal{K}}^N\|_2^2 \leq \left(\frac{T}{2\pi}\right)^2 \|\dot{x}_{\mathcal{K}}^N\|_2^2 \leq \left(\frac{T}{2\pi}\right)^2 C\sqrt{T}\|\tilde{x}_{\mathcal{K}}^N\|_2 \,,$$

so that $\|\tilde{x}_{\mathcal{K}}^N\|_{H^1} \leq C_1$ and $\|\tilde{x}_{\mathcal{K}}^N\|_\infty \leq C_0$ for some constants C_1 and C_0.

Recalling the validity of (3.24), we can find a sequence $\tau_{\mathcal{K}}^N = (\tau_k^N)_{k\in\mathcal{K}} \subset [0,T]$ such that

$$|x_k^N(\tau_k^N)| \leq d_k \,, \qquad \text{for every } k \in \mathcal{K} \,. \tag{3.26}$$

Then, we can prove that the sequence $(x_{\mathcal{K}}^N)_{N\in\mathbb{N}_+}$ is uniformly bounded. Indeed,

$$\begin{aligned}
|x_{\mathcal{K}}^N(t)|^2 = \sum_{k\in\mathcal{K}} |x_k^N(t)|^2 &= \sum_{k\in\mathcal{K}} \left| x_k^N(\tau_k^N) + \int_{\tau_k^N}^t \dot{x}_k^N(s)\,ds \right|^2 \\
&\leq 2\sum_{k\in\mathcal{K}} \left(|x_k^N(\tau_k^N)|^2 + \left| \int_{\tau_k^N}^t \dot{x}_k^N(s)\,ds \right|^2 \right) \\
&\leq 2\sum_{k\in\mathcal{K}} d_k^2 + 2T\|\dot{x}_{\mathcal{K}}^N\|_2^2 \leq 2\sum_{k\in\mathcal{K}} d_k^2 + 2TC_1^2 =: \varrho^2 \,,
\end{aligned}$$

Then, choosing $\mathcal{B} = \{(t,x) \in \mathcal{E} \;:\; |x_{\mathcal{K}}| \leq \varrho\}$ and recalling (3.23) and that $f_{\mathcal{K}}$ is completely continuous in $\mathcal{E}$, we notice that the set $A = \{f_{\mathcal{K}}(t, \widehat{\Pi}_N x^N(t)) \;:\; N \in \mathbb{N}_+ \,, t \in [0,T]\} \subseteq f_{\mathcal{K}}(\mathcal{B})$ is precompact. Then, using Lemma 3.4.13, we deduce that the set $\{\ddot{x}_{\mathcal{K}}^N(t) \;:\; N \in \mathbb{N}_+ \,, t \in [0,T]\}$ is precompact. By periodicity, there exists a sequence $t_{\mathcal{K}}^N = (t_k^N)_{k\in\mathcal{K}}$ such that $\dot{x}_k^N(t_k^N) = 0$ for every $k \in \mathcal{K}$. Writing

$$\dot{x}_k^N(t) = \dot{x}_k^N(t_k^N) + \int_{t_k^N}^t \ddot{x}_k^N(s)\,\mathrm{d}s = \int_{t_k^N}^t \ddot{x}_k^N(s)\,\mathrm{d}s = \left(P_{t_{\mathcal{K}}^N}\ddot{x}_{\mathcal{K}}^N\right)(t) \,,$$

we deduce from Lemma 3.4.11 that the set $\{\dot{x}_{\mathcal{K}}^N(t) : N \in \mathbb{N}_+ \,, t \in [0,T]\}$ is precompact.

Finally we prove that also the set $\{x_{\mathcal{K}}^N(t) \;:\; N \in \mathbb{N}_+ \,, t \in [0,T]\}$ is precompact. Recalling the sequence $\tau_{\mathcal{K}}^N = (\tau_k^N)_{k\in\mathcal{K}}$ in (3.26), we can write using the notation of Section 3.4.1,

$$x_{\mathcal{K}}^N(t) = \xi_{\mathcal{K}}^N + \left(P_{\tau_{\mathcal{K}}^N}\dot{x}_{\mathcal{K}}^N\right)(t) \,, \qquad \text{where } \xi_{\mathcal{K}}^N := (x_k^N(\tau_k^N))_{k\in\mathcal{K}} \,.$$

By construction $\xi_{\mathcal{K}}^N \in \mathcal{D}_{\mathcal{K}} := \prod_{k\in\mathcal{K}}[-d_k, d_k]$, so that, by Lemma 3.4.11, we conclude that both the addenda are in a compact set. Hence there is a compact set $\widehat{\mathcal{D}}_{\mathcal{K}}$ such that

$$x_{\mathcal{K}}^N(t) \in \widehat{\mathcal{D}}_{\mathcal{K}} \,, \qquad \text{for every } N \in \mathbb{N}_+ \text{ and } t \in [0,T] \,. \tag{3.27}$$

We can now prove similar properties for the components of $x^N(t)$, and their derivatives, with indices $j \in \mathcal{J}$. At this step, the continuity of $f_{\mathcal{J}}$ is sufficient. Indeed, from (3.25) and (3.27), the compactness of $\{f_{\mathcal{J}}(t, \widehat{\Pi}_N x^N(t)) : N \in \mathbb{N}_+, t \in [0,T]\}$ follows. Then, arguing as above, we can prove that both $\{\ddot{x}^N_{\mathcal{J}}(t) : N \in \mathbb{N}_+, t \in [0,T]\}$ and $\{\dot{x}^N_{\mathcal{J}}(t) : N \in \mathbb{N}_+, t \in [0,T]\}$ are precompact.

Consider now the sequence $(u^N)_{N\in\mathbb{N}_+}$ of functions $u^N : [0,T] \to H \times H$ defined by

$$u^N(t) = (x^N(t), \dot{x}^N(t)) .$$

By the above arguments, the sequence $(u^N)_{N\in\mathbb{N}_+}$ takes its values in a compact set, and it is equi-uniformly continuous. By the Ascoli–Arzelà Theorem there exists a subsequence, for which we keep the same notation, which uniformly converges to some $u^* : [0,T] \to H \times H$. Writing $u^*(t) = (x^*(t), y^*(t))$, we have that $(x^N, \dot{x}^N)$ uniformly converges to (x^*, y^*). In particular $x^*(0) = x^*(T)$, $y^*(0) = y^*(T)$. Rewriting the differential equation in $(\widehat{P}_N)$ as a planar system, we have

$$(\widehat{Q}_N) \qquad \begin{cases} \dot{x} = y, \\ \dot{y} = \Pi_N f(t, \widehat{\Pi}_N x(t)) , \end{cases}$$

or equivalently

$$\dot{u} = F^N(t,u) ,$$

where $F^N(t,x,y) = (y, \Pi_N f(t, \widehat{\Pi}_N x(t)))$. The corresponding integral formulation is then

$$u(t) = u(0) + \int_0^t F^N(s, u(s))\,\mathrm{d}s . \tag{3.28}$$

System $(\widehat{Q}_N)$ has a solution $u^N = (x^N, \dot{x}^N)$ such that $u^N(0) = u^N(T)$ for every $N \in \mathbb{N}_+$. We want to show that

$$F^N(t, u^N(t)) \to F(t, u^*(t)) , \qquad \text{uniformly in } t \in [0,T] , \tag{3.29}$$

where $F(t,x,y) = (y, f(t,x))$. Fix $\varepsilon > 0$; for N sufficiently large, we have

$$\begin{aligned} &|F^N(t,u^N(t)) - F(t,u^*(t))| \\ &\qquad\qquad \leq |y^N(t) - y^*(t)| + |\Pi_N f(t, \widehat{\Pi}_N x^N(t)) - f(t, x^*(t))| \\ &\qquad\qquad \leq \varepsilon + |\Pi_N f(t, \widehat{\Pi}_N x^N(t)) - f(t, \widehat{\Pi}_N x^N(t))| + \\ &\qquad\qquad + |f(t, \widehat{\Pi}_N x^N(t)) - f(t, x^*(t))| . \end{aligned}$$

Since $\{\widehat{\Pi}_N x^N(t) : N \in \mathbb{N}_+, t \in [0,T]\}$ is precompact, cf. (3.25) and (3.27), then by continuity $\{f(t, \widehat{\Pi}_N x^N(t)) : N \in \mathbb{N}_+, t \in [0,T]\}$ is precompact, too. So, by Lemma 3.4.12, for N sufficiently large,

$$|\Pi_N f(t, \widehat{\Pi}_N x^N(t)) - f(t, \widehat{\Pi}_N x^N(t))| = |(\Pi_N - \mathrm{Id}) f(t, \widehat{\Pi}_N x^N(t))| \leq \varepsilon .$$

Moreover,

$$
\begin{aligned}
|\widehat{\Pi}_N x^N(t) - \Pi_N x^N(t)| &= |(0,\dots,0,\alpha_{N+1}(t),\alpha_{N+2}(t),\dots)| \\
&\le \sum_{n=N}^{\infty} d_n^2 \to 0\,, \qquad \text{as } N\to\infty\,.
\end{aligned}
$$

Then, applying Lemma 3.4.12,

$$
|\widehat{\Pi}_N x^N(t) - x^*| \le |\widehat{\Pi}_N x^N(t) - \Pi_N x^N(t)| + |\Pi_N x^N(t) - x^N(t)| + |x^N(t) - x^*(t)| \to 0\,,
$$

as $N\to\infty$, so that by continuity, for N large enough,

$$
|f(t,\widehat{\Pi}_N x^N(t)) - f(t,x^*(t))| \le \varepsilon\,.
$$

Summing up, if N is large, then

$$
|F^N(t,u^N(t)) - F(t,u^*(t))| \le 3\varepsilon\,, \qquad \text{for every } t\in[0,T]\,,
$$

thus proving (3.29). Passing to the limit in (3.28), we get

$$
u^*(t) = u^*(0) + \int_0^t F(s,u^*(s))\,\mathrm{d}s\,,
$$

and so $x^*(t)$ is a solution of (P). The conditions (W_j) and $(\widetilde{NW}_k)$ are easily seen to be preserved in the limit process. The proof is thus completed. □

3.5 Final remarks

In this final section, we briefly outline some possible extensions of the previous results.

1. The boundedness assumption on the function $f_{\mathcal{K}}(t,x)$ could be replaced by a nonresonance condition with respect to the higher part of the spectrum of the differential operator $-\ddot{x}$ with T-periodic conditions. For instance, denoting by λ_2 the first positive eigenvalue $(2\pi/T)^2$, one could assume that

$$
-f_{\mathcal{K}}(t,x) = \gamma_{\mathcal{K}}(t,x)x + r_{\mathcal{K}}(t,x)\,,
$$

where $\gamma_{\mathcal{K}}(t,x) \le c < \lambda_2$ and $r_{\mathcal{K}}(t,x)$ is bounded. Or, more generally, one could assume an asymmetric behaviour of the type

$$
-f_{\mathcal{K}}(t,x) = \mu_{\mathcal{K}}(t,x)x^+ - \nu_{\mathcal{K}}(t,x)x^- + r_{\mathcal{K}}(t,x)\,,
$$

where $(\mu_{\mathcal{K}}(t,x),\nu_{\mathcal{K}}(t,x))$ lie below the first curve of the Fučík spectrum (here, as usual, $x^+ = \max\{x,0\}$ and $x^- = \max\{-x,0\}$).

2. One could deal with nonlinearities of the type $f(t,x,\dot{x})$, depending also on the derivative of x, assuming some type of Nagumo growth condition (see [29]). Such a situation has already been studied in the infinite-dimensional setting, e.g., in [95], in the well-ordered case.

3. In this Chapter we defined the lower and upper solutions as $\mathcal{C}^2$-functions. However, this regularity could be weakened, and different definitions could be adopted. We do not enter into the details, for briefness, and we refer to the book [29] for further possible developments.

4. The results of this Chapter hold the same for the Neumann problem

$$\begin{cases} \ddot{x} = f(t,x)\,, \\ \dot{x}(0) = 0 = \dot{x}(T)\,, \end{cases}$$

with almost identical proofs. Concerning the Dirichlet problem

$$\begin{cases} \ddot{x} = f(t,x)\,, \\ x(0) = 0 = x(T)\,, \end{cases}$$

some modifications are needed in the non-well-ordered case.

Chapter 4

Lower and upper solutions for PDE

The method of upper and lower solutions and the construction of monotone sequences with the goal to prove the existence of maximal and minimal solutions of elliptic boundary-value problems were developed as early as the 1930s by Nagumo in [77, 78, 79] for both ordinary and partial differential equations. In the early 1970s, Amann [3] and Sattinger [94] formalized the properties of upper and lower solutions and obtained a more systematic approach for the construction of monotone sequences. Amann considered a general elliptic boundary value problems with nonlinear boundary conditions, while Sattinger extended the definition of upper and lower solutions to parabolic boundary value problems. The same idea has been extended by Pao [85] to parabolic problems with nonlinear boundary conditions. At the beginning of this chapter we introduce the abstract setting, and we provide an existence result in the case of well-ordered lower and upper solutions. This is the analogue, in the setting of PDEs, of a result obtained in [10] for periodic systems of ODEs. For similar results concerning elliptic or parabolic problems see, e.g., [11, 32, 33, 60, 84] and the references therein.

We then consider non-well-ordered lower and upper solutions and we state our main theorem, whose proof is provided in Section 4.4. We emphasize that we do not need any monotonicity assumptions on our nonlinearities. This part is then concluded by some illustrative examples of applications, and comment on some possible extensions of our result in different directions; in particular we show how to adapt our main theorem to systems involving differential operators of parabolic type.

4.1 Well-ordered lower and upper solutions

4.1.1 The abstract setting

Let Ω be a bounded domain in $\mathbb{R}^N$, and denote by $W(\Omega)$ a Banach space of real-valued functions which is continuously and compactly imbedded in $C^1(\overline{\Omega})$. Assume that $\mathcal{L} : W(\Omega) \to L^r(\Omega)$ is a linear operator, with $r > 1$, and $\mathcal{B} : C^1(\overline{\Omega}) \to C(\partial\Omega)$ is a linear and continuous operator. We are concerned with the boundary value problem

$$(P) \qquad \begin{cases} \mathcal{L}u_n = F_n(x, u_1, \dots, u_M) & \text{in } \Omega\,, \\ \mathcal{B}u_n = 0 & \text{on } \partial\Omega\,, \end{cases} \qquad n = 1, \dots, M\,.$$

The function $F : \Omega \times \mathbb{R}^M \to \mathbb{R}^M$ is L^r- Carathéodory, i.e.,

(i) $F(\cdot, u)$ is measurable in Ω, for every $u \in \mathbb{R}^M$;

(ii) $F(x, \cdot)$ is continuous in $\mathbb{R}^M$, for almost every $x \in \Omega$;

(iii) for every $\rho > 0$ there is a $h_\rho \in L^r(\Omega)$ such that, if $|u| \le \rho$, then

$$|F(x, u)| \le h_\rho(x) \quad \text{for a.e. } x \in \Omega\,.$$

We now introduce our abstract hypotheses (see [50, Assumptions A1 and A2]).

Assumption 1. *If $w \in W(\Omega)$ is such that*

$$\min_{\overline{\Omega}} w < 0\,, \quad \text{and} \quad \mathcal{B}w \ge 0\,,$$

then there exists a point $x_0 \in \overline{\Omega}$ satisfying the following properties:
a) $w(x_0) < 0$,
b) there is no neighborhood U of x_0 such that $(\mathcal{L}w)(x) > 0$ for a.e. $x \in U \cap \Omega$.

Remark 4.1.1. *Concerning the elliptic operator, taking $W(\Omega) = W^{2,r}(\Omega)$ with $r > N$, Assumption 1 is a consequence of the Strong Maximum Principle (see, e.g., [50, 55, 102]).*

We define

$$C^1_{\mathcal{B}}(\overline{\Omega}) = \{w \in C^1(\overline{\Omega}) : \mathcal{B}w = 0\}\,, \qquad W_{\mathcal{B}}(\Omega) = \{w \in W(\Omega) : \mathcal{B}w = 0\}\,.$$

These are subspaces of $C^1(\overline{\Omega})$ and $W(\Omega)$, with the respective norms. Both $C^1_{\mathcal{B}}(\overline{\Omega})$ and $W_{\mathcal{B}}(\Omega)$ are Banach spaces, since the operator $\mathcal{B}$ is linear and continuous. Let us denote by $\mathcal{L}_{\mathcal{B}} : W_{\mathcal{B}}(\Omega) \to L^r(\Omega)$ the restriction of $\mathcal{L}$ to $W_{\mathcal{B}}(\Omega)$, and by I the identity operator.

Assumption 2. *There is a constant $\sigma < 0$ such that $\mathcal{L}_{\mathcal{B}} - \sigma I : W_{\mathcal{B}}(\Omega) \to L^r(\Omega)$ is invertible, and $(\mathcal{L}_{\mathcal{B}} - \sigma I)^{-1} : L^r(\Omega) \to W_{\mathcal{B}}(\Omega)$ is continuous.*

Remark 4.1.2. *For the elliptic operator, any sufficiently negative constant σ can be taken.*

We will write any $u \in [W_{\mathcal{B}}(\Omega)]^M$ as $u = (u_1, \ldots, u_M)$. Let $L : [W_{\mathcal{B}}(\Omega)]^M \to [L^r(\Omega)]^M$ be defined as

$$(Lu)(x) = \big((\mathcal{L}_{\mathcal{B}} u_1)(x), \ldots, (\mathcal{L}_{\mathcal{B}} u_M)(x)\big)\,,$$

and let us introduce the nonlinear operator $N : [C^1_{\mathcal{B}}(\overline{\Omega})]^M \to [L^r(\Omega)]^M$, defined as

$$(Nu)(x) = F(x, u(x))\,.$$

It is readily seen that N is continuous and maps bounded sets into bounded sets. Our problem (P) can then be rewritten as

$$Lu = Nu\,.$$

A solution of problem (P) will be a function $u \in [W_{\mathcal{B}}(\Omega)]^M$ which satisfies this equality in $[L^r(\Omega)]^M$, hence almost everywhere.

If σ is the constant introduced in Assumption 2, (P) is equivalent to the fixed point problem

$$u = \mathcal{S}u\,,$$

where $\mathcal{S} : [C^1_{\mathcal{B}}(\overline{\Omega})]^M \to [C^1_{\mathcal{B}}(\overline{\Omega})]^M$ is defined by $\mathcal{S}u = (L - \sigma I)^{-1}(Nu - \sigma u)$. Since $(L - \sigma I)^{-1} : [L^r(\Omega)]^M \to [W(\Omega)]^M$ is continuous and $[W(\Omega)]^M$ is compactly imbedded in $[C^1(\overline{\Omega})]^M$, the operator $\mathcal{S}$ is completely continuous, and this will allow us to use Leray–Schauder degree theory.

4.1.2 An existence result

Let us introduce the concept of lower and upper solutions.

Definition 4.1.3. *Given two functions $\alpha, \beta \in [W(\Omega)]^M$, we say that (α, β) is a* well-ordered pair of lower/upper solutions *of (P) if $\alpha \le \beta$ and there exists a negligible set $\mathcal{N} \subseteq \Omega$ such that*

$$\begin{cases} \mathcal{L}\alpha_j(x) \le F_j(x, u_1, \ldots, u_{j-1}, \alpha_j(x), u_{j+1}, \ldots, u_M)\,, \\ \mathcal{L}\beta_j(x) \ge F_j(x, u_1, \ldots, u_{j-1}, \beta_j(x), u_{j+1}, \ldots, u_M)\,, \\ \mathcal{B}\alpha_j \le 0 \le \mathcal{B}\beta_j\,, \end{cases}$$

for every $j \in \{1, \ldots, M\}$ and $(x, u) \in (\Omega \setminus \mathcal{N}) \times \prod_{m=1}^M [\alpha_m(x), \beta_m(x)]$.

Here is our result, in the well-ordered case; it generalizes [10, Theorem 4.1].

Theorem 4.1.4. *Let Assumptions 1 and 2 hold true. If there exists a well-ordered pair of lower/upper solutions (α, β), then problem (P) has a solution u such that $\alpha \le u \le \beta$.*

<u>Proof</u> Let us define the functions

$$\gamma_j(x,s) = \begin{cases} \alpha_j(x) & \text{if } s \le \alpha_j(x)\,, \\ s & \text{if } \alpha_j(x) < s < \beta_j(x)\,, \\ \beta_j(x) & \text{if } u \ge \beta_j(x)\,, \end{cases}$$

and the function

$$\Gamma(x,u) = \big(\gamma_1(x,u_1), \dots, \gamma_M(x,u_M)\big)\,.$$

Consider the auxiliary problem

$$(\bar{P}) \qquad \begin{cases} \mathcal{L}u_j - \sigma u_j = F_j(x, \Gamma(x,u)) - \sigma\gamma_j(x,u_j) & \text{in } \Omega\,, \\ \mathcal{B}u_j = 0 & \text{on } \partial\Omega\,, \end{cases} \qquad j = 1, \dots, M\,.$$

The remaining part of the proof is divided in two steps.

Step 1: Problem ($\bar{P}$) admits a solution.

Let us introduce the operator $\bar{N} : [C^1_{\mathcal{B}}(\overline{\Omega})]^M \to [L^r(\Omega)]^M$ defined by

$$(\bar{N}u)(x) = F(x, \Gamma(x,u(x))) - \sigma\Gamma(x,u(x))\,.$$

One can see that $\bar{N}$ is continuous and has a bounded image. Problem ($\bar{P}$) is equivalent to the fixed point problem

$$u = \overline{\mathcal{S}}u\,,$$

where the operator $\overline{\mathcal{S}} : [C^1_{\mathcal{B}}(\overline{\Omega})]^M \to [C^1_{\mathcal{B}}(\overline{\Omega})]^M$ is defined by

$$\overline{\mathcal{S}}u = (L - \sigma I)^{-1}\bar{N}u\,.$$

We have that $\overline{\mathcal{S}}$ is completely continuous, and its image is bounded. By Schauder Theorem, it has a fixed point, hence problem ($\bar{P}$) has a solution.

Step 2: Every solution u of ($\bar{P}$) is such that $\alpha \le u \le \beta$.

Let us prove that $\alpha \le u$. Set $v = u - \alpha$, and assume by contradiction that $\min v_j < 0$, for some $j \in \{1, \dots, M\}$. Since $\mathcal{B}v_j = \mathcal{B}u_j - \mathcal{B}\alpha_j = -\mathcal{B}\alpha_j \ge 0$, by Assumption 1 there is a point $x_0 \in \overline{\Omega}$ such that $v_j(x_0) < 0$, and there is no neighborhood U of x_0 such that $\mathcal{L}v_j > 0$, almost everywhere on $U \cap \Omega$. On the other hand, as $v_j(x_0) < 0$, there is a neighborhood V of x_0 such that $v_j < 0$ on $V \cap \Omega$, i.e., $u_j < \alpha_j$ on $V \cap \Omega$. Hence,

$$\begin{aligned}
\mathcal{L}v_j &= \mathcal{L}u_j - \mathcal{L}\alpha_j \\
&= F_j(x, \Gamma(x,u)) - \sigma(\gamma_j(x,u_j) - u_j) - \mathcal{L}\alpha_j \\
&= F_j(x, \big(\gamma_1(x,u_1), \dots, \alpha_j(x), \dots, \gamma_M(x,u_M)\big) - \sigma(\alpha_j - u_j) - \mathcal{L}\alpha_j \\
&\geq \sigma v_j > 0\,,
\end{aligned}$$

almost everywhere on $V \cap \Omega$, a contradiction. In a similar way it can be shown that $u \leq \beta$.

Hence, every solution u of $(\bar{P})$ solves (P), and the proof is completed. □

4.1.3 Computation of the degree

In the following, for any two continuous real-valued functions v, w, we write

$$v < w \quad \Leftrightarrow \quad v(x) < w(x)\,, \quad \text{for every } x \in \overline{\Omega}\,.$$

We define

$$C^1_{\mathcal{B}^-}(\overline{\Omega}) = \{w \in C^1(\overline{\Omega}) : \mathcal{B}w \leq 0\}\,, \quad C^1_{\mathcal{B}^+}(\overline{\Omega}) = \{w \in C^1(\overline{\Omega}) : \mathcal{B}w \geq 0\}\,,$$

both endowed with the norm of $C^1(\overline{\Omega})$.

We now introduce a further hypothesis (see [50, Assumption A4]).

Assumption 3. *An order relation $v \ll w$ can be defined in $C^1(\overline{\Omega})$, with the following properties:*

$$\begin{array}{l}
v < w \quad \Rightarrow \quad v \ll w \quad \Rightarrow \quad v \leq w\,, \\
[\, v \leq w \quad \textit{and} \quad w \ll z\,] \quad \Rightarrow \quad v \ll z\,, \\
[\, v \ll w \quad \textit{and} \quad w \leq z\,] \quad \Rightarrow \quad v \ll z\,, \\
v \ll w \quad \Rightarrow \quad v + z \ll w + z\,, \\
[\, c > 0 \quad \textit{and} \quad v \ll w\,] \quad \Rightarrow \quad cv \ll cw\,,
\end{array}$$

for every $v, w, z \in C^1(\overline{\Omega})$ and every real constant c. Moreover, we assume that the set

$$\{w \in C^1_{\mathcal{B}^-}(\overline{\Omega}) : w \ll 0\}$$

is open in $C^1_{\mathcal{B}^-}(\overline{\Omega})$.

As usual, we can write $w \gg v$ instead of $v \ll w$. Consequently, the set $\{w \in C^1_{\mathcal{B}^+}(\overline{\Omega}) : w \gg 0\}$ is open in $C^1_{\mathcal{B}^+}(\overline{\Omega})$, and the sets

$$\{w \in C^1_{\mathcal{B}}(\overline{\Omega}) : w \ll 0\} \quad \text{and} \quad \{w \in C^1_{\mathcal{B}}(\overline{\Omega}) : w \gg 0\}$$

are open in $C^1_{\mathcal{B}}(\overline{\Omega})$. Notice also that the closures of these sets are contained in

$$\{w \in C^1_{\mathcal{B}}(\overline{\Omega}) : w \le 0\} \quad \text{and} \quad \{w \in C^1_{\mathcal{B}}(\overline{\Omega}) : w \ge 0\}\,,$$

respectively.

Remark 4.1.5. *For the system with the elliptic operator, we will write $v \ll w$ if the following two conditions hold:*

a) for every $x \in \Omega$, $v(x) < w(x)$,

b) for every $x \in \partial\Omega$, either $v(x) < w(x)$, or

$$v(x) = w(x) \quad \text{and} \quad \partial_\nu v(x) > \partial_\nu w(x)\,.$$

Here, ν denotes the outer unit normal to $\partial\Omega$ at the point x.

If the two functions v, w have values in $\mathbb{R}^d$, for any dimension d, then we write

$$\begin{cases} v \le w & \Leftrightarrow \quad v_m \le w_m\,, \\ v \ll w & \Leftrightarrow \quad v_m \ll w_m\,, \end{cases} \qquad \text{for every } m \in \{1, \dots, d\}\,.$$

Definition 4.1.6. *A well-ordered pair of lower/upper solutions (α, β) is said to be* strict *if $\alpha \ll \beta$, and any solution u of (P) satisfying $\alpha \le u \le \beta$ is such that*

$$\alpha \ll u \ll \beta\,.$$

If (α, β) is strict, then the set

$$\mathcal{U}_{(\alpha,\beta)} = \{u \in [C^1_{\mathcal{B}}(\overline{\Omega})]^M : \alpha \ll u \ll \beta\}$$

is open in $[C^1_{\mathcal{B}}(\overline{\Omega})]^M$, by Assumption 3. Moreover, if u is a fixed point of $\mathcal{S}$ in $\overline{\mathcal{U}}_{(\alpha,\beta)}$, then $\alpha \le u \le \beta$ and, by the strictness hypothesis, $u \in \mathcal{U}_{(\alpha,\beta)}$. So, there are no fixed points of $\mathcal{S}$ on the boundary of $\mathcal{U}_{(\alpha,\beta)}$, and we can define the Leray–Schauder degree

$$\deg\left(I - \mathcal{S}\,, \mathcal{U}_{(\alpha,\beta)}\right)\,.$$

Theorem 4.1.7. *Let Assumptions 1, 2 and 3 hold true. If there exists a strict well-ordered pair of lower/upper solutions (α, β), then*

$$\deg\left(I - \mathcal{S}\,, \mathcal{U}_{(\alpha,\beta)}\right) = 1\,.$$

Proof Going back to the proof of Theorem 4, any fixed point u of $\overline{\mathcal{S}}$ is such that $\alpha \le u \le \beta$, and it is a fixed point of $\mathcal{S}$. Hence, all fixed points of $\overline{\mathcal{S}}$ belong to $\mathcal{U}_{(\alpha,\beta)}$, and since $\mathcal{S}$ and $\overline{\mathcal{S}}$ coincide on $\mathcal{U}_{(\alpha,\beta)}$, we have

$$\deg\left(I - \mathcal{S}\,, \mathcal{U}_{(\alpha,\beta)}\right) = \deg\left(I - \overline{\mathcal{S}}\,, \mathcal{U}_{(\alpha,\beta)}\right)\,.$$

By Schauder Theorem and the excision property of the degree, taking $R > 0$ large enough, we have

$$\deg\left(I - \overline{\mathcal{S}}\,, \mathcal{U}_{(\alpha,\beta)}\right) = \deg\left(I - \overline{\mathcal{S}}\,, B(0,R)\right) = 1\,,$$

thus ending the proof. $\square$

4.2 Non-well-ordered lower and upper solutions

Here again, in perfect analogy with what we have done in Section 3.3 we will decompose each element of the space separating the well-ordered components from the non-well-ordered ones. For the convenience of the reader we recall the following notation. We will say that the couple $(\mathcal{J}, \mathcal{K})$ is a partition of the set of indices $\{1, \ldots, M\}$ if and only if $\mathcal{J} \cap \mathcal{K} = \varnothing$ and $\mathcal{J} \cup \mathcal{K} = \{1, \ldots, M\}$. Correspondingly we can decompose a vector $u = (u_1, \ldots, u_M) \in \mathbb{R}^M$ as $u = (u_{\mathcal{J}}, u_{\mathcal{K}})$ where $u_{\mathcal{J}} = (u_j)_{j \in \mathcal{J}} \in \mathbb{R}^{\#\mathcal{J}}$ and $u_{\mathcal{K}} = (u_k)_{k \in \mathcal{K}} \in \mathbb{R}^{\#\mathcal{K}}$. Here $\#\mathcal{J}$ and $\#\mathcal{K}$ denote respectively the cardinality of the sets $\mathcal{J}$ and $\mathcal{K}$. Also every function $\mathcal{F} : \mathcal{A} \to \mathbb{R}^M$ can be written as $\mathcal{F}(x) = \big(\mathcal{F}_{\mathcal{J}}(x), \mathcal{F}_{\mathcal{K}}(x)\big)$ where $\mathcal{F}_{\mathcal{J}} : \mathcal{A} \to \mathbb{R}^{\#\mathcal{J}}$ and $\mathcal{F}_{\mathcal{K}} : \mathcal{A} \to \mathbb{R}^{\#\mathcal{K}}$.

Definition 4.2.1. *Given two functions $\alpha, \beta \in [W(\Omega)]^M$, we say that (α, β) is a* pair of lower/upper solutions of (P) related to the partition $(\mathcal{J}, \mathcal{K})$ of $\{1, \ldots, M\}$ *if the following four conditions hold:*

1. $\alpha_{\mathcal{J}} \leq \beta_{\mathcal{J}}$;
2. $\alpha_k \not\leq \beta_k$ *, for every* $k \in \mathcal{K}$;
3. *there is a negligible set* $\mathcal{N} \subseteq \Omega$ *such that*

$$\begin{cases} \mathcal{L}\alpha_n(x) \leq F_n(x, u_1, \ldots, u_{n-1}, \alpha_n(x), u_{n+1}, \ldots, u_M)\,, \\ \mathcal{L}\beta_n(x) \geq F_n(x, u_1, \ldots, u_{n-1}, \beta_n(x), u_{n+1}, \ldots, u_M)\,, \end{cases}$$

   for any $n \in \{1, \ldots, M\}$ *and every* $(x, u) \in \mathcal{E}$, *where*

$$\mathcal{E} := \Big\{ (x, u) \in (\Omega \setminus \mathcal{N}) \times \mathbb{R}^M \;:\; u = (u_{\mathcal{J}}, u_{\mathcal{K}})\,, u_{\mathcal{J}} \in \prod_{j \in \mathcal{J}} [\alpha_j(x), \beta_j(x)] \Big\}\,.$$

4. $\mathcal{B}\alpha_n \leq 0 \leq \mathcal{B}\beta_n$, *for every* $n \in \{1, \ldots, M\}$.

Definition 4.2.2. *The pair (α, β) of lower/upper solutions of (P) is said to be* strict with respect to the $\mathcal{J}$-th component *if $\alpha_{\mathcal{J}} \ll \beta_{\mathcal{J}}$ and, for every solution u of (P) we have*

$$\alpha_{\mathcal{J}} \leq u_{\mathcal{J}} \leq \beta_{\mathcal{J}} \quad \Rightarrow \quad \alpha_{\mathcal{J}} \ll u_{\mathcal{J}} \ll \beta_{\mathcal{J}}\,;$$

it is said to be strict with respect to the k-th component, *with $k \in \mathcal{K}$, if for every solution u of (P) we have*

$$\begin{aligned} u_k \geq \alpha_k \quad &\Rightarrow \quad u_k \gg \alpha_k\,, \\ u_k \leq \beta_k \quad &\Rightarrow \quad u_k \ll \beta_k\,. \end{aligned}$$

We need to introduce some further assumptions.

Assumption 4. *There is a number $\lambda_1 \geq 0$ and a function $\varphi_1 \in W_{\mathcal{B}}(\Omega)$, with $\varphi_1 \gg 0$, such that*

$$\ker(\mathcal{L}_{\mathcal{B}} - \lambda_1 I) = \{c\varphi_1 : c \in \mathbb{R}\}\,.$$

We will assume that $\max\limits_{\Omega} \varphi_1 = 1$.

Remark 4.2.3. *The existence of a "first" eigenvalue λ_1 with the required properties is standard in the elliptic case, where the spectrum is made of isolated eigenvalues $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$, all contained in $[0 + \infty[$, cf. [55, 102].*

Lemma 4.2.4. *Let Assumptions 3 and 4 hold. Given a bounded set $\mathcal{A}$ in $W(\Omega)$, there is a constant $C_{\mathcal{A}} \geq 0$ such that, if $w \in \mathcal{A}$ satisfies $\mathcal{B}w \leq 0$, then $w \leq C_{\mathcal{A}}\varphi_1$, and if $w \in \mathcal{A}$ satisfies $\mathcal{B}w \geq 0$, then $w \geq -C_{\mathcal{A}}\varphi_1$.*

Proof See [50, Lemma 4.1]. □

Definition 4.2.5. *A pair of functions $(\psi, \Psi) \in L^r(\Omega) \times L^r(\Omega)$ is said to be* admissible *if it satisfies $\psi \leq \lambda_1 \leq \Psi$ almost everywhere in Ω and, for every $q \in L^r(\Omega)$, with $\psi \leq q \leq \Psi$ almost everywhere in Ω, if w is a solution of*

$$(P_{lin}) \qquad \begin{cases} \mathcal{L}w = q(z)w & \text{in } \Omega\,, \\ \mathcal{B}w = 0 & \text{on } \partial\Omega\,, \end{cases}$$

then, either $w = 0$, or $w \ll 0$, or $w \gg 0$.

Remark 4.2.6. *For a self-adjoint elliptic problem, the above property of the couple ψ, Ψ is satisfied, e.g., if $\Psi \leq \lambda_2$ (the second eigenvalue), with strict inequality on a subset of positive measure (cf. [59]).*

Lemma 4.2.7. *Let Assumptions 2, 3 and 4 hold. Given an admissible pair of functions (ψ, Ψ), there are two positive constants $c_{\psi,\Psi}$ and $C_{\psi,\Psi}$ such that, for every $q \in L^r(\Omega)$, with $\psi \leq q \leq \Psi$ almost everywhere in Ω, if u is a solution of (P_{lin}), then*

$$c_{\psi,\Psi}\|u\|_{L^\infty}\varphi_1 \leq |u| \leq C_{\psi,\Psi}\|u\|_{L^\infty}\varphi_1\,.$$

Proof See [50, Lemma 4.3]. □

Assumption 5. *There is a function $\varphi_0 \in W(\Omega)$ such that*

$$\mu := \min_{\overline{\Omega}} \varphi_0 > 0\,, \qquad \mathcal{L}\varphi_0 \geq 0 \quad \text{and} \quad \mathcal{B}\varphi_0 \geq 0\,.$$

We will assume that $\max\limits_{\overline{\Omega}} \varphi_0 = 1$.

Remark 4.2.8. *In the applications to the elliptic case the function φ_0 can be taken constantly equal to* 1*.*

Here is the main result of this Chapter.

Theorem 4.2.9. *Let Assumptions 1–5 hold true. Let (α, β) be a pair of lower/upper solutions of (P) related to the partition $(\mathcal{J}, \mathcal{K})$ of $\{1, \ldots, M\}$ which is strict with respect to the k-th component, for every $k \in \mathcal{K}$, except at most one. Assume that there exist two L^r-Carathéodory functions $f, g : \Omega \times \mathbb{R}^M \to \mathbb{R}^M$ with the following property: for every $k \in \mathcal{K}$,*

$$F_k(x, u) = g_k(x, u)u_k + f_k(x, u)\,,$$

and there is an admissible pair (ψ_k, Ψ_k), and a function $h_k \in L^r(\Omega)$ such that

$$\psi_k(x) \leq g_k(x, u) \leq \Psi_k(x) \quad \text{and} \quad |f_k(x, u)| \leq h_k(x)\,,$$

for almost every $x \in \Omega$ and every $u \in \mathbb{R}^M$. Then, problem (P) has a solution u such that

(W_j) $\alpha_{\mathcal{J}} \leq u_{\mathcal{J}} \leq \beta_{\mathcal{J}}$*;*

(NW_k) $\alpha_k \not\lll u_k$ *and* $u_k \not\lll \beta_k$*, for every* $k \in \mathcal{K}$*.*

4.3 Examples and remarks

As an illustrative example, consider the Neumann problem

$$\begin{cases} -\Delta u_1 = |u_1|^\gamma \sin u_1 + w_1(x, u_1, u_2) & \text{in } \Omega\,, \\ -\Delta u_2 = \pm \arctan u_2 + w_2(x, u_1, u_2) & \text{in } \Omega\,, \\ \partial_\nu u_1 = \partial_\nu u_2 = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Here γ is any positive exponent, and $w_1, w_2 : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous and bounded functions, with

$$\|w_2\|_\infty := \sup\left\{|w_2(x, u_1, u_2)| : x \in \overline{\Omega}, u_1, u_2 \in \mathbb{R}\right\} < \frac{\pi}{2}\,.$$

Applying Theorem 4.2.9, we obtain the existence of infinitely many solutions $u = (u_1, u_2)$. Indeed, it is sufficient to choose the constant pairs of lower/upper solutions (α, β), with

$$\alpha = \left(\frac{\pi}{2} + 2m\pi, \pm n\right), \quad \beta = \left(\frac{3\pi}{2} + 2m\pi, \mp n\right),$$

for sufficiently large positive integers m, n. Notice that these will be well-ordered if the minus sign appears in the second differential equation, otherwise non-well-ordered in the second component.

As a second example, we consider the mixed Dirichlet–Neumann problem

$$\begin{cases} -\Delta u_1 = -u_1^3 + f_1(u_2) + w_1(x, u_1, u_2, u_3) & \text{in } \Omega\,, \\ -\Delta u_2 = \pm \arctan u_2 + w_2(x, u_1, u_2, u_3) & \text{in } \Omega\,, \\ -\Delta u_3 = \pm \arctan u_3 + w_3(x, u_1, u_2, u_3) & \text{in } \Omega\,, \\ \quad u_1 = 0\,, \quad \partial_\nu u_2 = \partial_\nu u_3 = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Here $f_1 : \mathbb{R} \to \mathbb{R}$ is any continuous function, and $w_1, w_2, w_3 : \overline{\Omega} \times \mathbb{R}^3 \to \mathbb{R}$ are continuous and bounded, with

$$\|w_2\|_\infty < \frac{\pi}{2}\,, \quad \|w_3\|_\infty < \frac{\pi}{2}\,.$$

Applying Theorem 4.2.9, we obtain the existence of at least one solution, taking the constant pairs of lower/upper solutions (α, β), with

$$\alpha = \Big(-m, \pm n, \pm n \Big)\,, \quad \beta = \Big(m, \mp n, \mp n \Big)\,,$$

for a sufficiently large positive integer n and $m = m(n)$. Indeed, it is sufficient to fix $n > \tan(\max\{\|w_2\|_\infty, \|w_3\|_\infty\})$ and

$$m > \Big(\max\big\{ |f_1(s)| : s \in [-n, n] \big\} + \|w_1\|_\infty \Big)^{1/3}.$$

Remark 4.3.1. *All the results of this Chapter hold if the nonlinearities depend also on the gradient ∇u, provided that a Nagumo-type condition is assumed. See [50] for the details.*

Remark 4.3.2. *Asymmetric nonlinearities can also be considered, as in [59, 30, 50]. We do not enter into details, for briefness.*

Remark 4.3.3. *Concerning a system with a p-Laplacian differential operator, some difficulties may arise. If we consider, e.g., the associated Dirichlet problem, then the inverse function $(L-\sigma I)^{-1}$ transforms any $h \in L^\infty(\Omega)$ into $(L-\sigma I)^{-1}h \in W_0^{1,p}(\Omega) \cap C^{1,\nu}(\overline{\Omega})$, for some $\nu > 0$, and this function might not have regular second order derivatives. In [30], this problem is overcome by defining lower and upper solutions in a weak form, and carrying out the same construction as for the linear case. A similar procedure can also be practiced in our situation, leading to an existence result analogous to Theorem 4.2.9.*

Remark 4.3.4. *The periodic problem for a system of ordinary differential equations has been treated in [44]. Infinite-dimensional systems were also considered there. It is an open problem whether it could be possible to extend the results of the present Chapter to an infinite-dimensional setting.*

4.4 Proof of Theorem 4.2.9

Notice that the case $\mathcal{K} = \varnothing$ reduces to Theorem 4.1.4. We thus assume $\mathcal{K} \neq \varnothing$ and, without loss of generality, we take either $\mathcal{J} = \varnothing$, or $\mathcal{J} = \{1, \dots, J\}$ and $\mathcal{K} = \{J+1, \dots, M\}$ for a certain $J \in \{1, \dots, N\}$. We moreover suppose that the component on which the lower/upper solution is possibly not strict is the last one, i.e., $k = M$. Indeed, mixing the coordinates we can always reduce to such a situation. We continue the proof in the case $\mathcal{J} \neq \varnothing$. (The case $\mathcal{J} = \varnothing$ can be treated essentially in the same way.)

We need to suitably modify problem (P). For $j = 1, \dots, J$ we define

$$G_j(x,u) = F_j(x, \gamma_1(x,u_1), \dots, \gamma_J(x,u_J), u_{J+1}, \dots, u_M) + u_j - \sigma\gamma_j(x,u_j)\,,$$

where the functions γ_j are the ones introduced in the proof of Theorem 4.1.4.

Using Lemma 4.2.4 and the fact that α_k and β_k are bounded, we can find a constant $c > 0$ such that, for $k \in \mathcal{K}$,

$$-c\varphi_1 - c \le \alpha_k \le c\varphi_1\,, \quad -c\varphi_1 \le \beta_k \le c\varphi_1 + c\,.$$

For any $k \in \mathcal{K}$ and $\Lambda > 0$ large enough, to be fixed, we define

$$\tilde g_k(x,u) = \begin{cases} \lambda_1 & \text{if } \ u_k \le -(\Lambda\varphi_1(x) + \frac{2c}{\mu}\varphi_0(x))\,, \\ \dots & \\ g_k(x,u) & \text{if } \ |u_k| \le \Lambda\varphi_1(x) + \frac{c}{\mu}\varphi_0(x)\,, \\ \dots & \\ \lambda_1 & \text{if } \ u_k \ge \Lambda\varphi_1(x) + \frac{2c}{\mu}\varphi_0(x)\,, \end{cases}$$

$$\tilde f_k(x,u) = \begin{cases} \frac{3c\lambda_1+1}{\mu} & \text{if } \ u_k \le -(\Lambda\varphi_1(x) + \frac{2c}{\mu}\varphi_0(x))\,, \\ \dots & \\ f_k(x,u) & \text{if } \ |u_k| \le \Lambda\varphi_1(x) + \frac{c}{\mu}\varphi_0(x)\,, \\ \dots & \\ -\frac{3c\lambda_1+1}{\mu} & \text{if } \ u_k \ge \Lambda\varphi_1(x) + \frac{2c}{\mu}\varphi_0(x)\,, \end{cases}$$

(here, the dots mean "linear interpolation"), and

$$G_k(x,u) = \tilde g_k(x,u)u_k + \tilde f_k(x,u)\,.$$

We consider the problem

$$(\widetilde{P}_\Lambda) \qquad \begin{cases} \mathcal{L}u_n = G_n(x, u_1, \dots, u_M) & \text{in } \Omega\,, \\ \mathcal{B}u_n = 0 & \text{on } \partial\Omega\,, \end{cases} \qquad n = 1, \dots, M\,.$$

Proposition 4.4.1. *If u is a solution of $(\widetilde{P}_\Lambda)$, for any constant $\Lambda > 0$, then $\alpha_{\mathcal{J}} \le u_{\mathcal{J}} \le \beta_{\mathcal{J}}$.*

Proof It is easily adapted from Step 2 of the proof of Theorem 4.1.4. □

We define $\tilde{\alpha}_{\mathcal{K}}$ and $\tilde{\beta}_{\mathcal{K}}$ by setting $\tilde{\alpha}_k = -(\Lambda\varphi_1 + \frac{3c}{\mu}\varphi_0)$ and $\tilde{\beta}_k = \Lambda\varphi_1 + \frac{3c}{\mu}\varphi_0$, for every $k \in \mathcal{K}$. Notice that, taking $\Lambda > c$,

$$\tilde{\alpha}_{\mathcal{K}} \ll \alpha_{\mathcal{K}} \ll \tilde{\beta}_{\mathcal{K}}\,, \qquad \tilde{\alpha}_{\mathcal{K}} \ll \beta_{\mathcal{K}} \ll \tilde{\beta}_{\mathcal{K}}\,.$$

Finally, we choose $\tilde{\alpha} = (\alpha_{\mathcal{J}}, \tilde{\alpha}_{\mathcal{K}})$ and $\tilde{\beta} = (\beta_{\mathcal{J}}, \tilde{\beta}_{\mathcal{K}})$.

Let us prove that $(\tilde{\alpha}, \tilde{\beta})$ is pair of lower/upper solutions of $(\widetilde{P}_\Lambda)$. We have not modified the components of $\tilde{\alpha}_{\mathcal{J}}$ and $\tilde{\beta}_{\mathcal{J}}$, so we just need to check what happens for $\tilde{\alpha}_{\mathcal{K}}$ and $\tilde{\beta}_{\mathcal{K}}$. For every $k \in \mathcal{K}$ we have

$$\mathcal{L}\tilde{\alpha}_k(x) \le -\Lambda\lambda_1\varphi_1(x) = \lambda_1\tilde{\alpha}_k(x) + \frac{3c\lambda_1}{\mu}\varphi_0(x) < \tilde{g}_k(x, \tilde{\alpha}(x))\tilde{\alpha}_k(x) + \tilde{f}_k(x, \tilde{\alpha}(x))\,,$$

and $\mathcal{B}\tilde{\alpha}_k = -\frac{3c}{\mu}\mathcal{B}\varphi_0 \le 0$. Similar computations can be done for $\tilde{\beta}_k$. So, $(\tilde{\alpha}, \tilde{\beta})$ is pair of lower/upper solutions for $(\widetilde{P}_\Lambda)$.

Let us prove that $(\tilde{\alpha}, \tilde{\beta})$ is strict with respect to the k-th component, for every $k \in \mathcal{K}$. Let u be a solution such that $u_k \ge \tilde{\alpha}_k$. We want to show that $u_k > \tilde{\alpha}_k$. By contradiction, let $v_k = u_k - \tilde{\alpha}_k$ be such that $\min v_k = 0$. Let $z_k = v_k - \frac{c}{\mu}\varphi_0$, so that $\min z_k < 0$ and

$$\mathcal{B}z_k = \mathcal{B}u_k - \mathcal{B}\tilde{\alpha}_k - \frac{c}{\mu}\mathcal{B}\varphi_0 = \frac{2c}{\mu}\mathcal{B}\varphi_0 \ge 0\,.$$

By Assumption 1 there is a $x_0 \in \Omega$ such that $z_k(x_0) < 0$ and there is no neighborhood U of x_0 on which $(\mathcal{L}z_k)(x) > 0$, for almost every $x \in U \cap \Omega$. By continuity, there is a neighborhood V of x_0 on which $z_k < 0$. So, on V, we have that $u_k < \tilde{\alpha}_k + \frac{c}{\mu}\varphi_0$, and so

$$\begin{aligned}
\mathcal{L}z_k &= \mathcal{L}u_k - \mathcal{L}\tilde{\alpha}_k - \frac{c}{\mu}\mathcal{L}\varphi_0 \\
&= \Big(\lambda_1 u_k + \frac{3c\lambda_1 + 1}{\mu}\Big) + \Lambda\lambda_1\varphi_1 + \frac{2c}{\mu}\mathcal{L}\varphi_0 \\
&\ge \Big(\lambda_1 \tilde{\alpha}_k + \frac{3c\lambda_1 + 1}{\mu}\Big) + \Lambda\lambda_1\varphi_1 \\
&= -\frac{3c\lambda_1}{\mu}\varphi_0 + \frac{3c\lambda_1 + 1}{\mu} > 0\,,
\end{aligned}$$

a contradiction. Similar estimates can be written for $\tilde{\beta}_k$, so we conclude that the pair of lower/upper solutions is strict with respect to the k-th component.

Let $\mathcal{X}$ be the subset of $[W(\Omega)]^M$ made of those solutions u of $(\widetilde{P}_\Lambda)$, for any $\Lambda > c$, satisfying $\alpha_{\mathcal{K}} \not\leq u_{\mathcal{K}}$ and $u_{\mathcal{K}} \not\leq \beta_{\mathcal{K}}$.

Claim. There exists a constant $C_1 > 0$ (independent of Λ) such that, for any $u \in \mathcal{X}$, one has $|u_k| \leq C_1\varphi_1$, for every $k \in \mathcal{K}$.

Proof of the Claim. We first prove that there is a constant $K > 0$ (independent of Λ) such that, for any $u \in \mathcal{X}$, one has $\|u_k\|_\infty \leq K$, for every $k \in \mathcal{K}$. By contradiction, let $(u^n)^n$ be a sequence in $\mathcal{X}$, such that $\|u_k^n\|_\infty \to \infty$, for some $k \in \mathcal{K}$. Let us now fix such $k \in \mathcal{K}$.

We know that u^n is a solution of $(\widetilde{P}_{\Lambda_n})$, for some $\Lambda_n > c$. Let us denote by $\tilde{g}_k^n$ and $\tilde{f}_k^n$ the corresponding modified functions. Then $w_k^n = u_k^n/\|u_k^n\|_\infty$ satisfies

$$\mathcal{L}w_k^n(x) = \tilde{g}_k^n(x, u^n(x))w_k^n(x) + \frac{1}{\|u_k^n\|_\infty}\tilde{f}_k^n(x, u^n(x)), \quad \mathcal{B}w_k^n = 0.$$

Let us consider the set of functions

$$\mathcal{D}_k = \{p \in L^r(\Omega) : \psi_k(x) \leq p(x) \leq \Psi_k(x), \text{ for a.e. } x \in \Omega\},$$

which is bounded, closed and convex, hence weakly compact. Since the sequence $(\tilde{g}_k^n(\cdot, u^n(\cdot)))_n$ belongs to $\mathcal{D}_k$, up to a subsequence it weakly converges in $L^r(\Omega)$ to some $q(\cdot) \in \mathcal{D}_k$, while

$$\frac{1}{\|u_k^n\|_\infty}\tilde{f}_k^n(x, u^n(x)) \to 0 \quad \text{in } L^r(\Omega).$$

Let $\widetilde{N}_k^n : C^1_{\mathcal{B}}(\overline{\Omega}) \to L^r(\Omega)$ be defined as

$$(\widetilde{N}_k^n w)(x) = \tilde{g}_k^n(x, u^n(x))w(x) + \frac{1}{\|u_k^n\|_\infty}\tilde{f}_k^n(x, u^n(x)).$$

Let $\sigma \in \mathbb{R}$ be the number given by Assumption 2, and let $\widetilde{\mathcal{S}}_k^n : C^1_{\mathcal{B}}(\overline{\Omega}) \to C^1_{\mathcal{B}}(\overline{\Omega})$ be defined as

$$\widetilde{\mathcal{S}}_k^n v = (L - \sigma I)^{-1}(\widetilde{N}_k^n v - \sigma v).$$

Notice that

$$w_k^n = \widetilde{\mathcal{S}}_k^n w_k^n.$$

Since $(\widetilde{N}_k^n w_k^n - \sigma w_k^n)_n$ is bounded in $L^r(\Omega)$ and $(L - \sigma I)^{-1} : L^r(\Omega) \to C^1_{\mathcal{B}}(\overline{\Omega})$ is compact, there is a $w \in C^1_{\mathcal{B}}(\overline{\Omega})$ such that, up to a subsequence,

$$\widetilde{\mathcal{S}}_k^n w_k^n = (L - \sigma I)^{-1}(\tilde{N}_k^n w_k^n - \sigma w_k^n) \to w \quad \text{in } C^1_{\mathcal{B}}(\overline{\Omega}).$$

Hence, $w_k^n \to w$ in $C^1_{\mathcal{B}}(\overline{\Omega})$. Since $\tilde{N}_k^n w_k^n - \sigma w_k^n$ weakly converges to $q(\cdot)w - \sigma w$, we conclude that

$$w = (L - \sigma I)^{-1}(q(\cdot)w - \sigma w),$$

so that $w \in W_{\mathcal{B}}(\Omega)$ and

$$Lw = q(\cdot)w\,,$$

i.e., w satisfies (P_{lin}). Then, either $w = 0$, or $w \ll 0$, or $w \gg 0$. Since $\|w_k^n\|_\infty = 1$, for every n, we know that $w \neq 0$. Assume for instance $w \gg 0$ (the case $w \ll 0$ is similar). By Lemma 4.2.4, there is a constant $\hat{c}_k > 0$ such that $\alpha_k \leq \hat{c}_k\varphi_1$. By Lemma 4.2.7, $w \geq c_{\psi_k,\Psi_k}\varphi_1 \gg (c_{\psi_k,\Psi_k}/2)\varphi_1$, since $\varphi_1 \gg 0$, by Assumption 2. So,

$$w \gg \frac{c_{\psi_k,\Psi_k}}{2}\varphi_1 \geq \frac{c_{\psi_k,\Psi_k}}{2\hat{c}_k}\alpha_k := b_k\alpha_k\,.$$

By Assumption 3, for n large enough, $w_k^n \gg b_k\alpha_k$, and increasing n still more, $u_k^n = \|u_k^n\|_\infty w_k^n \gg \|u_k^n\|_\infty b_k\alpha_k \geq \alpha_k$, a contradiction.

We have thus seen that $\mathcal{X}_{\mathcal{K}}$, the projection of the set $\mathcal{X}$ on the $\mathcal{K}$-th component, is uniformly bounded. Now recall that problem $(\widetilde{P}_\Lambda)$ is equivalent to a fixed point problem

$$u = (L - \sigma I)^{-1}(\widetilde{N}u - \sigma u)\,.$$

By Assumption 2, we deduce that $\mathcal{X}_{\mathcal{K}}$ is indeed bounded in $[W(\Omega)]^{\sharp\mathcal{K}}$. Then, by Lemma 4.2.4, we find a constant $C_1 > 0$ such that $|u_k| \leq C_1\varphi_1$, for every $k \in \mathcal{K}$. The proof of the Claim is thus completed. □

From now on, we fix $\Lambda \geq C_1$. We are going to compute the Leray–Schauder degree of $I - \mathcal{F}$ on a family of open sets, where

$$\mathcal{F} : [C^1_{\mathcal{B}}(\overline{\Omega})]^M \to [C^1_{\mathcal{B}}(\overline{\Omega})]^M\,, \qquad \mathcal{F}(u) = (L - \sigma I)^{-1}(\widetilde{N}u - \sigma u)\,.$$

Let us define the functions

$$\check{\alpha}_j = \alpha_j - \varphi_0\,, \quad \text{and} \quad \check{\beta}_j = \beta_j + \varphi_0\,,$$

for every $j \in \mathcal{J}$.

We need to introduce a multi-index $\vec{\eta} = (\eta_{J+1}, \ldots, \eta_M) \in \{1, 2, 3\}^{M-J}$, in order to define the open sets

$$\Omega_{\vec{\eta}} := \left\{u \in [C^1_{\mathcal{B}}(\overline{\Omega})]^M \,:\, \check{\alpha}_{\mathcal{J}} \ll u_{\mathcal{J}} \ll \check{\beta}_{\mathcal{J}} \text{ and } (\mathcal{O}_k^{\eta_k}) \text{ holds for every } k \in \mathcal{K}\right\},$$

where

$(\mathcal{O}_k^1)$ $\tilde{\alpha}_k \ll u_k \ll \tilde{\beta}_k\,,$

$(\mathcal{O}_k^2)$ $\tilde{\alpha}_k \ll u_k \ll \beta_k\,,$

$(\mathcal{O}_k^3)$ $\alpha_k \ll u_k \ll \tilde{\beta}_k\,.$

We now end the proof of Thoerem 4.2.9 assuming first that the lower/upper solutions are *strict with respect to all the components* $k \in \mathcal{K}$.

Proposition 4.4.2. *For every multi-index* $\vec{\eta}$*, the degree* $d(I - \mathcal{F}, \Omega_{\vec{\eta}})$ *is well-defined, and*

$$d(I - \mathcal{F}, \Omega_{\vec{\eta}}) = 1 .$$

<u>Proof</u> Assume by contradiction that there is $u \in \partial\Omega_{\vec{\eta}}$ such that $(I - \mathcal{F})u = 0$, i.e., u is a solution of $(\widetilde{P}_\Lambda)$. All the several different situations which may arise lead back to the following four cases.

<u>*Case A*</u>. For some index $j \in \mathcal{J}$, $\breve{\alpha}_j \leq u_j \leq \breve{\beta}_j$, and either $\breve{\alpha}_j \not\ll u_j$, or $u_j \not\ll \breve{\beta}_j$. We have seen in the proof of Theorem 4.1.4 that $\alpha_j \leq u_j \leq \beta_j$. Since $\varphi_0 > 0$, by Assumption 5, we then have $\breve{\alpha}_j < u_j < \breve{\beta}_j$, hence $\breve{\alpha}_j \ll u_j \ll \breve{\beta}_j$, a contradiction.

<u>*Case B*</u>. For some index $k \in \mathcal{K}$, $\tilde{\alpha}_k \leq u_k \leq \tilde{\beta}_k$, and either $\tilde{\alpha}_k \not\ll u_k$, or $u_k \not\ll \tilde{\beta}_k$. This is impossible, since $(\tilde{\alpha}, \tilde{\beta})$ is strict with respect to the k-th component, for every $k \in \mathcal{K}$.

<u>*Case C*</u>. For some index $k \in \mathcal{K}$, $\tilde{\alpha}_k \ll u_k \leq \beta_k$, and $u_k \not\ll \beta_k$. Such a situation cannot arise, by assumption.

<u>*Case D*</u>. For some index $k \in \mathcal{K}$, $\alpha_k \leq u_k \ll \tilde{\beta}_k$, and $\alpha_k \not\ll u_k$. Such a situation cannot arise, by assumption.

Since the sets $\Omega_{\vec{\eta}}$ provide us a well-ordered pair of strict lower/upper solutions of problem $(\widetilde{P}_\Lambda)$, the conclusion is a consequence of Theorem 4.1.7. □

We now start an iterative process, defining a series of open sets and computing the corresponding degrees. This process will eventually lead us to the conclusion.

For every $\ell \in \{1, 2, 3\}$ and any $\vec{\eta} = (\eta_{J+2}, \dots, \eta_M) \in \{1, 2, 3\}^{M-(J+1)}$, we now define the open sets

$$\Omega^0_{(\ell,\vec{\eta})} = \Omega_{(\ell,\eta_{J+2},\dots,\eta_M)} .$$

Notice that $\Omega^0_{(2,\vec{\eta})}$ and $\Omega^0_{(3,\vec{\eta})}$ are disjoint subsets of $\Omega^0_{(1,\vec{\eta})}$. We also define the open set

$$\Omega^0_{(4,\vec{\eta})} = \Omega^0_{(1,\vec{\eta})} \setminus \overline{\Omega^0_{(2,\vec{\eta})} \cup \Omega^0_{(3,\vec{\eta})}} .$$

Proposition 4.4.3. *For every multi-index* $\vec{\eta}$*, the degree* $d(I - \mathcal{F}, \Omega^0_{(4,\vec{\eta})})$ *is well-defined, and*

$$d(I - \mathcal{F}, \Omega^0_{(4,\vec{\eta})}) = -1 .$$

Proof Using the fact that the sets $\Omega^0_{(\ell,\vec{\eta})}$ are open, for $\ell = 1,2,3$, we can see that

$$\partial\Omega^0_{(4,\vec{\eta})} \subseteq \partial\Omega^0_{(1,\vec{\eta})} \cup \partial\Omega^0_{(2,\vec{\eta})} \cup \partial\Omega^0_{(3,\vec{\eta})} .$$

Since we already know that there are no solutions of $(\widetilde{P}_\Lambda)$ on $\partial\Omega^0_{(\ell,\vec{\eta})}$, for $\ell = 1,2,3$, we consequently have that there are no solutions of $(\widetilde{P}_\Lambda)$ on $\partial\Omega^0_{(4,\vec{\eta})}$, hence the degree is well-defined. By the additivity property of the degree and Proposition 4.4.2,

$$d(I-\mathcal{F},\Omega^0_{(4,\vec{\eta})}) = d(I-\mathcal{F},\Omega^0_{(1,\vec{\eta})}) - d(I-\mathcal{F},\Omega^0_{(2,\vec{\eta})}) - d(I-\mathcal{F},\Omega^0_{(3,\vec{\eta})}) = -1 ,$$

so that the proof is completed. □

Now, for every $\ell \in \{1,2,3\}$ and any $\vec{\eta} = (\eta_{J+3},\dots,\eta_M) \in \{1,2,3\}^{M-(J+2)}$, we define the open sets

$$\Omega^1_{(\ell,\vec{\eta})} = \Omega_{(4,\ell,\eta_{J+3},\dots,\eta_M)} .$$

Notice that $\Omega^1_{(2,\vec{\eta})}$ and $\Omega^1_{(3,\vec{\eta})}$ are disjoint subsets of $\Omega^1_{(1,\vec{\eta})}$. We also define the open set

$$\Omega^1_{(4,\vec{\eta})} = \Omega^1_{(1,\vec{\eta})} \setminus \overline{\Omega^1_{(2,\vec{\eta})} \cup \Omega^1_{(3,\vec{\eta})}} .$$

Proceeding by induction, for $K \in \{0,1,\dots,M-(J+1)\}$, any $\ell \in \{1,2,3\}$ and any $\vec{\eta} = (\eta_{J+K+2},\dots,\eta_M) \in \{1,2,3\}^{M-(J+K+1)}$ we can define the open sets

$$\Omega^K_{(\ell,\vec{\eta})} = \Omega_{(\underbrace{4,\dots,4}_{K\ times},\ell,\eta_{J+K+2},\dots,\eta_M)} .$$

Notice that $\Omega^K_{(2,\vec{\eta})}$ and $\Omega^K_{(3,\vec{\eta})}$ are disjoint subsets of $\Omega^K_{(1,\vec{\eta})}$. We also define the open set

$$\Omega^K_{(4,\vec{\eta})} = \Omega^K_{(1,\vec{\eta})} \setminus \overline{\Omega^K_{(2,\vec{\eta})} \cup \Omega^K_{(3,\vec{\eta})}} .$$

Proposition 4.4.4. *For every $K \in \{0,1,\dots,M-(J+2)\}$ and every multi-index $\vec{\eta}$, the degree $d(I-\mathcal{F},\Omega^K_{(4,\vec{\eta})})$ is well-defined, and*

$$d(I-\mathcal{F},\Omega^K_{(4,\vec{\eta})}) = (-1)^{K+1} .$$

Proof We proceed by induction. The validity of the statement for $K=0$ follows by Proposition 4.4.3. Assume that it holds for some $K \in \{0,1,\dots,M-(J+3)\}$. The same argument in the proof of Proposition 4.4.3 shows us that the degree is well-defined. Then, for every $\vec{\eta} = (\eta_{J+K+3}, \dots, \eta_M) \in \{1,2,3\}^{M-(J+K+2)}$,

$$\begin{aligned}
& d(I-\mathcal{F}, \Omega^{K+1}_{(4,\vec{\eta})}) = d(I-\mathcal{F}, \Omega^{K+1}_{(1,\vec{\eta})}) - d(I-\mathcal{F}, \Omega^{K+1}_{(2,\vec{\eta})}) - d(I-\mathcal{F}, \Omega^{K+1}_{(3,\vec{\eta})}) \\
&= d(I-\mathcal{F}, \Omega_{(\underbrace{4,\dots,4}_{K+1\,times},1,\eta_{J+K+3},\dots,\eta_M)}) - \\
&\quad - d(I-\mathcal{F}, \Omega_{(\underbrace{4,\dots,4}_{K+1\,times},2,\eta_{J+K+3},\dots,\eta_M)}) - d(I-\mathcal{F}, \Omega_{(\underbrace{4,\dots,4}_{K+1\,times},3,\eta_{J+K+3},\dots,\eta_M)}) \\
&= d(I-\mathcal{F}, \Omega_{(\underbrace{4,\dots,4}_{K\,times},4,1,\eta_{J+K+3},\dots,\eta_M)}) - \\
&\quad - d(I-\mathcal{F}, \Omega_{(\underbrace{4,\dots,4}_{K\,times},4,2,\eta_{J+K+3},\dots,\eta_M)}) - d(I-\mathcal{F}, \Omega_{(\underbrace{4,\dots,4}_{K\,times},4,3,\eta_{J+K+3},\dots,\eta_M)}) \\
&= d(I-\mathcal{F}, \Omega^{K}_{(4,1,\vec{\eta})}) - d(I-\mathcal{F}, \Omega^{K}_{(4,2,\vec{\eta})}) - d(I-\mathcal{F}, \Omega^{K}_{(4,3,\vec{\eta})}) \\
&= (-1)^{K+1} - (-1)^{K+1} - (-1)^{K+1} = (-1)^{K+2},
\end{aligned}$$

yielding the conclusion. □

By the previous proposition, in the special case $K = M-(J+2)$ we have that for every $\ell \in \{1,2,3\}$,

$$d_\ell := d(I-\mathcal{F}, \Omega_{(\underbrace{4,\dots\dots,4}_{M-J-2\,times},4,\ell)}) = (-1)^{M-(J+1)}.$$

We now consider the set

$$\Omega_{(4,\dots,4,4)} = \Omega_{(4,\dots,4,1)} \setminus \overline{\Omega_{(4,\dots,4,2)} \cup \Omega_{(4,\dots,4,3)}}.$$

By the same argument as above,

$$d(I-\mathcal{F}, \Omega_{(4,\dots,4)}) = d_1 - d_2 - d_3 = (-1)^{M-J}.$$

As a consequence, there exists a solution u of problem $(\widetilde{P}_\Lambda)$ in the set $\Omega_{(4,\dots,4)}$. Recalling the above a priori bounds, we see that the solution u is indeed a solution of problem (P) and satisfies (W_j) and (NW_k). The proof is thus completed, in the case when the lower/upper solutions are strict with respect to all the components $k \in \mathcal{K}$.

If the lower/upper solutions are not strict with respect to the M-th component, the previous propositions all continue to hold provided that $\eta_M = 1$, but we cannot ensure that the degree is well-defined if $\eta_M = 2$ or $\eta_M = 3$. We thus have that

$$d(I-\mathcal{F}, \Omega_{(4,\dots,4,1)}) = (-1)^{M-(J+1)},$$

and there are two possibilities: either, there is a solution of problem $(\widetilde{P}_\Lambda)$ on $\partial\Omega_{(4,\dots,4,2)} \cup \partial\Omega_{(4,\dots,4,3)}$, or the degrees $d(I-\mathcal{F}, \Omega_{(4,\dots,4,2)})$ and $d(I-\mathcal{F}, \Omega_{(4,\dots,4,3)})$ are well-defined, and we conclude as above. □

4.5 The parabolic case

In this section we briefly describe how our results can be adapted to the study of systems of parabolic type. For the details, we refer to [50, Section 7].

Let Ω be a regular bounded domain in $\mathbb{R}^N$ and, given $T > 0$, set $Q = \Omega\times\,]0, T[$. Choosing $r > N + 2$, we define $\mathcal{L} : W_r^{2,1}(Q) \to L^r(Q)$ by

$$\mathcal{L}w = \partial_t w - \sum_{l,m=1}^{N} a_{lm}(x,t)\partial^2_{x_l x_m} w + \sum_{i=1}^{N} a_i(x,t)\partial_{x_i} w + a_0(x,t)w\,.$$

Here $a_{lm} \in C(\overline{Q})$, $a_{lm} = a_{ml}$, $a_{lm}(x,0) = a_{lm}(x,T)$ in $\overline{\Omega}$, for $l, m = 1, \ldots, N$, there exists $\bar{a} > 0$ such that

$$\sum_{l,m=1}^{N} a_{lm}(x,t)\xi_i\xi_j \geq \bar{a}\|\xi\|^2\,, \quad \text{for every } (x,t,\xi) \in \overline{Q} \times \mathbb{R}^N\,,$$

and $a_i \in L^\infty(Q)$, for $i = 0, \ldots, N$.

Assume that $\partial\Omega$ is the disjoint union of two closed sets Γ_1 and Γ_2 (the cases $\Gamma_1 = \varnothing$ or $\Gamma_2 = \varnothing$ are admitted). Let τ_s be the operator defined by

$$(\tau_s w)(x,t) = w(x, t+s)\,,$$

and define $\mathcal{B} : C^{1,0}(\overline{Q}) \to C(\partial Q)$ by

$$\mathcal{B}w = \begin{cases} w & \text{on } \Gamma_1 \times [0,T]\,, \\ \displaystyle\sum_{i=1}^{N} b_i(x,t)\partial_{x_i} w + b_0(x,t)w & \text{on } \Gamma_2 \times [0,T]\,, \\ w - \tau_T w & \text{in } \Omega \times \{0\}\,, \\ \tau_{(-T)}w - w & \text{in } \Omega \times \{T\}\,. \end{cases}$$

Here $b_i \in C^1(\partial\Omega \times [0,T])$, $b_i(x,0) = b_i(x,T)$ in $\partial\Omega$, for $i = 0, \ldots, N$, and there exists $\bar{b} > 0$ such that

$$b_0(x,t) \geq 0 \quad \text{and} \quad \sum_{i=1}^{N} b_i(x,t)\nu_i(x) \geq \bar{b}\,, \quad \text{for every } (x,t) \in \partial\Omega\times\,]0,T[\,.$$

We thus have Dirichlet-periodic conditions on Γ_1, and Robin-periodic on Γ_2.

We can deal with the problem

$$\begin{cases} \mathcal{L}u_n = F_n(x,t,u_1,\ldots,u_M) & \text{in } Q\,, \\ \mathcal{B}u_n = 0 & \text{on } \partial Q\,, \end{cases} \qquad n = 1,\ldots,M\,.$$

Also in this setting our choice of taking the same differential operator and boundary conditions for all components has only the aim of simplifying the exposition. A *solution* of problem (P) is a function $u \in W_r^{2,1}(Q)$ which satisfies the differential equation almost everywhere in Q and the boundary conditions pointwise. A function with these properties is usually called "strong solution" in the literature. All the existence results of this Chapter can be adapted to this situation. See [50] for the verification of the corresponding Assumptions 1–5.

As a final example, we consider the system of the mixed Dirichlet-periodic and Neumann-periodic problem

$$\begin{cases} \partial_t u_1 - \Delta u_1 = -u_1^3 + w_1(x,t,u_1,u_2) & \text{in } Q\,, \\ \partial_t u_2 - \Delta u_2 = \pm \arctan u_2 + w_2(x,t,u_1,u_2) & \text{in } Q\,, \\ u_1 = 0\,, \quad \partial_\nu u_2 = 0 & \text{on } \partial\Omega \times [0,T]\,, \\ u_1(x,0) = u_1(x,T)\,, \quad u_2(x,0) = u_2(x,T) & \text{on } \Omega\,. \end{cases}$$

If $w_1, w_2 : \overline{Q} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous and bounded, with $\|w_2\|_\infty < \pi/2$, we obtain the existence of at least one solution, taking the constant pairs of lower/upper solutions (α,β), with $\alpha = (-m, \pm n)$ and $\beta = (m, \mp n)$, for sufficiently large positive integers m and n.

Chapter 5

Periodic solutions of nearly integrable Hamiltonian systems

The search of periodic solutions of Hamiltonian systems and more specifically of *perturbations* of completely integrable Hamiltonian systems naturally arises in the study of mechanical systems. Historically the main inspiration has come from celestial mechanics (see e.g. [88, 89, 90]) since in this framework in the hypotheses when one body is significantly more massive than all the other the system can be studied as a system of two body problems plus a perturbative term. The main goal in this field is to identify under what assumptions the perturbed system maintains one or more solutions having the same period of the perturbation itself. This branch of research goes under the name of *KAM theory* from the initials of Kolmogorov, Arnold and Moser which gave a fundamental contribution in this subject (for an exhaustive introduction to Hamiltonian perturbation see [6, 12], while for an outline of KAM theory see e.g. [37]). For non-planar Hamiltonian systems, a local approach can be found in the work of Bernstein and Katok in [14] then generalized by Chen in [24]. When we look for the existence and multiplicity of periodic solutions and only the global behavior of the nonlinearity is assumed to be known, the approach combines topological and variational methods. In this chapter we want to prove the existence of periodic solutions bifurcating from an infinite-dimensional invariant torus for a nearly integrable Hamiltonian system. More precisely we want to extend the existence result given in [41] for a system of the type

$$\begin{cases} \dot{\varphi} = \nabla\mathcal{K}(I) + \varepsilon\nabla_I P(t,\varphi,I,z) \\ -\dot{I} = \varepsilon\nabla_\varphi P(t,\varphi,I,z) \\ J\dot{z} = \mathcal{A}z + \varepsilon\nabla_z P(t,\varphi,I,z)\,, \end{cases}$$

to its infinite-dimensional analogue.

5.1 The main result

Let X and E be two separable Hilbert spaces, and set $\mathcal{X} = X^2 \times E^2$. We will use the notation $\omega = (\varphi, I, z)$ for the elements of $\mathcal{X}$, with $\varphi, I \in X$ and $z = (x, y) \in E^2$. For simplicity, we will write $Z = E^2$, and we define $J : Z \to Z$ as $J(x, y) = (-y, x)$. (The same notation J will also be used with the same meaning in similar settings.) Let us introduce all the assumptions we need.

The continuous functions $\mathcal{K} : X \to \mathbb{R}$ and $P : \mathbb{R} \times \mathcal{X} \to \mathbb{R}$ are assumed to be continuously differentiable with respect to I and ω, respectively. The function $t \mapsto P(t, \omega)$ is T-periodic, for some $T > 0$. Moreover, we assume the following Lipschitz condition on bounded sets.

(L) For every $R > 0$ there exist two positive constants $L_R, \mathcal{L}_R$ such that

$$\|\nabla\mathcal{K}(I') - \nabla\mathcal{K}(I'')\| \leq L_R \|I' - I''\| ,$$

for every $I', I'' \in X$ with $\|I'\| < R$, $\|I''\| < R$, and

$$\|\nabla_\omega P(t, \omega') - \nabla_\omega P(t, \omega'')\| \leq \mathcal{L}_R \|\omega' - \omega''\| ,$$

for every $t \in [0, T]$ and $\omega', \omega'' \in \mathcal{X}$ with $\|\omega'\| < R$ and $\|\omega''\| < R$.

Introducing some Hilbert bases of X and E, we can identify these spaces either with some $\mathbb{R}^n$, if they are finite-dimensional, or with ℓ^2, the space of real sequences $(\alpha_k)_k$ which satisfy $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$. Each of the vectors φ, I in X and z in Z will then be written in their coordinates, e.g., $\varphi = (\varphi_1, \varphi_2, \dots)$, or $\varphi = (\varphi_k)_k$, with $\varphi_k \in \mathbb{R}$, while $I = (I_k)_k$ and $z = (z_l)_l$, with $z_l = (x_l, y_l) \in \mathbb{R}^2$. Notice that these sequences may be finite.

We also ask P to be periodic in the φ-variables, as follows.

($\mathbf{P}_\tau$) The function $P(t, \varphi, I, z)$ is τ_k-periodic in each φ_k, i.e., for $k = 1, 2, \dots$,

$$P(t, \dots, \varphi_k + \tau_k, \dots, I, z) = P(t, \dots, \varphi_k, \dots, I, z) ,$$

for every $(t, \varphi, I, z) \in [0, T] \times \mathcal{X}$; moreover, if $\dim X = \infty$, then the sequence $(\tau_k)_k$ belongs to ℓ^2.

Concerning $\nabla_\omega P$, we assume it to be bounded and precompact, in the following sense.

(P$_{bd}$) There exist $(\alpha_k^\star)_k$ and $(\alpha_l^\sharp)_l$ such that, for every $k, l = 1, 2, \dots,$

$$\left|\frac{\partial P}{\partial \varphi_k}(t,\omega)\right| + \left|\frac{\partial P}{\partial I_k}(t,\omega)\right| \le \alpha_k^\star, \qquad \left|\frac{\partial P}{\partial x_l}(t,\omega)\right| + \left|\frac{\partial P}{\partial y_l}(t,\omega)\right| \le \alpha_l^\sharp,$$

for every $(t,\omega) \in [0,T] \times \mathcal{X}$. If $\dim X = \infty$ or $\dim Z = \infty$, then $(\alpha_k^\star)_k$ or $(\alpha_l^\sharp)_l$ belong to ℓ^2, respectively.

Notice that the sets $\prod_{k=1}^\infty [-\alpha_k^\star, \alpha_k^\star]$ and $\prod_{l=1}^\infty [-\alpha_l^\sharp, \alpha_l^\sharp]$ are Hilbert cubes, hence compact sets in ℓ^2.

Let $\mathcal{A} : Z \to Z$ be a linear *bounded selfadjoint* operator. We need the following non-resonance assumption.

(NR) Denoting by

$$\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset L^2([0,T], Z) \to L^2([0,T], Z), \quad \mathcal{L}z = J\dot{z},$$

the unbounded selfadjoint operator with domain

$$\mathcal{D}(\mathcal{L}) = \{z \in H^1([0,T], Z) : z(0) = z(T)\},$$

we assume that $0 \notin \sigma(\mathcal{L} - \mathcal{A})$.

In the case when Z is infinite-dimensional, we need to assume a particular structure for the function $\mathcal{A}$.

(Dec1) If $\dim Z = \infty$, there exists a sequence of positive integers $(N_m^\sharp)_{m\ge1}$ and functions $\mathcal{A}_m : \mathbb{R}^{2N_m^\sharp} \to \mathbb{R}^{2N_m^\sharp}$ such that, writing any vector $z \in Z$ as $z = (\vec{z}_1, \dots, \vec{z}_m, \dots)$, with $\vec{z}_m = (\vec{x}_m, \vec{y}_m) \in \mathbb{R}^{2N_m^\sharp}$, we have that

$$\mathcal{A}z = (\mathcal{A}_1\vec{z}_1, \dots, \mathcal{A}_m\vec{z}_m, \dots).$$

Concerning the function $\mathcal{K}$, its gradient will be "guided" by some linear *bounded selfadjoint invertible* operator $\mathcal{B} : X \to X$, with bounded inverse, as we now specify. First of all, similarly as before, in the case when X is infinite-dimensional, we need to assume a particular structure for the functions $\mathcal{B}$ and $\mathcal{K}$.

(Dec2) If $\dim X = \infty$, there exists a sequence of positive integers $(N_j^\star)_{j\ge1}$ and functions $\mathcal{B}_j : \mathbb{R}^{N_j^\star} \to \mathbb{R}^{N_j^\star}$, $\mathcal{K}_j : \mathbb{R}^{N_j^\star} \to \mathbb{R}$ such that, writing any vector $I \in X$ as $I = (\vec{I}_1, \dots, \vec{I}_j, \dots)$, with $\vec{I}_j \in \mathbb{R}^{N_j^\star}$, we have that

$$\mathcal{B}\,I = (\mathcal{B}_1\vec{I}_1, \dots, \mathcal{B}_j\vec{I}_j, \dots), \qquad \mathcal{K}(I) = \sum_{j=1}^\infty \mathcal{K}_j(\vec{I}_j).$$

We now fix $I^0 \in X$, and introduce our *twist condition*.

(Tw) There exist two positive constants $\bar{c}, \bar{\rho}$ such that, for every $j = 1, 2, \dots$,

$$\|\vec{I}_j - \vec{I}_j^0\| \leq \bar{\rho} \quad \Rightarrow \quad \Big\langle \nabla\mathcal{K}_j(\vec{I}_j) - \nabla\mathcal{K}_j(\vec{I}_j^0) \,,\, \mathcal{B}_j(\vec{I}_j - \vec{I}_j^0) \Big\rangle \geq \bar{c}\, \|\vec{I}_j - \vec{I}_j^0\|^2 \,.$$

Finally, we assume a compatibility condition between T and the periods introduced in **(P$_\tau$)**.

(C$_\tau$) There exist some integers $m_1, m_2, \dots$ for which

$$T\nabla\mathcal{K}(I^0) = (m_1\tau_1, m_2\tau_2, \dots) \,.$$

We are now ready to state our main result.

Theorem 5.1.1. *Let the above assumptions hold. Then, for every $\sigma > 0$ there exists $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}$, there is a solution of system*

$$\begin{cases} \dot{\varphi} = \nabla\mathcal{K}(I) + \varepsilon\nabla_I P(t, \varphi, I, z) \\ -\dot{I} = \varepsilon\nabla_\varphi P(t, \varphi, I, z) \\ J\dot{z} = \mathcal{A}z + \varepsilon\nabla_z P(t, \varphi, I, z) \,, \end{cases} \tag{5.1}$$

satisfying

$$\varphi(t+T) = \varphi(t) + T\nabla\mathcal{K}(I^0) \,, \quad I(t+T) = I(t) \,, \quad z(t+T) = z(t) \,, \tag{5.2}$$

and such that

$$\|\varphi(t) - \varphi(0) - t\nabla\mathcal{K}(I^0)\| + \|I(t) - I^0\| + \|z(t)\| < \sigma \,, \quad \textit{for every } t \in \mathbb{R} \,. \tag{5.3}$$

Remark 5.1.2. When X is finite-dimensional, we will see that condition **(Tw)** can be generalized to

(Tw′) There exists a positive constant $\bar{\rho}$ such that

$$\|I - I^0\| \leq \bar{\rho} \quad \Rightarrow \quad \big\langle \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0) \,,\, \mathcal{B}(I - I^0) \big\rangle > 0 \,;$$

a still more general condition, adopted in [40], is the following:

$$0 \in \mathrm{cl}\Big\{\rho \in \,]0, +\infty[\, : \min_{\|I - I^0\| = \rho} \big\langle \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0) \,,\, \mathcal{B}(I - I^0) \big\rangle > 0 \Big\} \,,$$

where $\mathrm{cl}\,\mathcal{S}$ denotes the closure of a set $\mathcal{S}$.

5.2 Preliminaries for the proof

We will carry out the proof of Theorem 5.1.1 in the case $\dim X = \infty$ and $\dim Z = \infty$, with some specific remarks on the finite-dimensional cases. By the change of variables

$$(\xi(t), I(t), z(t)) = (\varphi(t) - t\nabla\mathcal{K}(I^0), I(t), z(t)), \tag{5.4}$$

system (5.1) becomes

$$\begin{cases} \dot{\xi} = \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0) + \varepsilon\nabla_I \widehat{P}(t,\xi,I,z) \\ -\dot{I} = \varepsilon\nabla_\xi \widehat{P}(t,\xi,I,z) \\ J\dot{z} = \mathcal{A}z + \varepsilon\nabla_z \widehat{P}(t,\xi,I,z), \end{cases} \tag{5.5}$$

where

$$\widehat{P}(t,\xi,I,z) = P(t,\xi + t\nabla\mathcal{K}(I^0), I, z).$$

We use the notation $\zeta = (\xi, I, z)$; the Hamiltonian function is thus

$$\widehat{H}(t,\zeta) = \mathcal{K}(I) - \left\langle \nabla\mathcal{K}(I^0), I \right\rangle + \tfrac{1}{2}\left\langle \mathcal{A}z, z\right\rangle + \varepsilon\widehat{P}(t,\zeta).$$

Combining **(P$_\tau$)** with **(C$_\tau$)**, we see that the function $\widehat{P}(\cdot,\xi,I,z)$ is T-periodic, and $\widehat{P}(t,\cdot,I,z)$ is τ_k-periodic in ξ_k, for every $k = 1, 2, \ldots$

Some additional notations are now necessary. By assumption **(Dec2)**, the vectors $\xi, I \in X$ decompose in vectors $\vec{\xi}_j, \vec{I}_j \in \mathbb{R}^{N_j^\star}$. Setting

$$S_0^\star = 0, \quad S_j^\star = \sum_{i=1}^{j} N_i^\star \quad \text{for } j \geq 1,$$

we can explicitly write the components of $\vec{\xi}_j, \vec{I}_j$ as

$$\vec{\xi}_j = (\xi_{S_{j-1}^\star+1}, \xi_{S_{j-1}^\star+2}, \ldots, \xi_{S_j^\star}), \qquad \vec{I}_j = (I_{S_{j-1}^\star+1}, I_{S_{j-1}^\star+2}, \ldots, I_{S_j^\star}).$$

Similarly, by assumption **(Dec1)**, the vector $z \in Z$ decomposes in vectors $\vec{z}_m \in \mathbb{R}^{2N_m^\sharp}$. Setting

$$S_0^\sharp = 0, \quad S_m^\sharp = \sum_{i=1}^{m} N_i^\sharp \quad \text{for } m \geq 1,$$

we can explicitly write the components of $\vec{z}_m$ as

$$\vec{z}_m = (z_{S_{m-1}^\sharp+1}, z_{S_{m-1}^\sharp+2}, \ldots, z_{S_m^\sharp}).$$

We define the sequences $(a^\star_j)_j$, $(a^\sharp_m)_m$ in ℓ^2 by

$$a^\star_j = \Big(\sum_{i=1}^{N^\star_j} (\alpha^\star_{S^\star_{j-1}+i})^2\Big)^{1/2}, \quad a^\sharp_m = \Big(\sum_{i=1}^{N^\sharp_m} (\alpha^\sharp_{S^\sharp_{m-1}+i})^2\Big)^{1/2}.$$

Notice that $\|a^\star\|_{\ell^2} = \|\alpha^\star\|_{\ell^2}$ and $\|a^\sharp\|_{\ell^2} = \|\alpha^\sharp\|_{\ell^2}$.

Remark 5.2.1. When X has a finite dimension d_X, we can define the sequence $(N^\star_j)_j$ taking $N^\star_1 = d_X$ and $N^\star_j = 0$ for $j \geq 2$. Similarly when Z is finite-dimensional.

Without loss of generality, from now on we will assume that $I^0 = 0$, a situation which can be recovered by a simple translation. The strategy of the proof of Theorem 5.1.1 will be to construct a finite-dimensional approximation of system (5.5), and then pass to the limit on the dimension. Precisely, we define the projections $\Pi_{S^\star_{\mathcal{J}}} : X \to X$ and $\Pi_{S^\sharp_{\mathcal{J}}} : Z \to Z$ as

$$\Pi_{S^\star_{\mathcal{J}}} v = (\vec{v}_1, \dots, \vec{v}_{\mathcal{J}}, 0, 0, \dots), \qquad \Pi_{S^\sharp_{\mathcal{J}}} z = (\vec{z}_1, \dots, \vec{z}_{\mathcal{J}}, 0, 0, \dots),$$

and consider the truncated system

$$\begin{cases} \dot\xi = \Pi_{S^\star_{\mathcal{J}}}[\nabla\mathcal{K}(I) - \nabla\mathcal{K}(0) + \varepsilon\nabla_I \widehat{P}(t,\xi,I,z)] \\ -\dot I = \Pi_{S^\star_{\mathcal{J}}}[\varepsilon\nabla_\xi \widehat{P}(t,\xi,I,z)] \\ J\dot z = \Pi_{S^\sharp_{\mathcal{J}}}[\mathcal{A}z + \varepsilon\nabla_z \widehat{P}(t,\xi,I,z)] . \end{cases} \tag{5.6}$$

We thus have the Hamiltonian function

$$\begin{aligned} \widehat{H}_{\mathcal{J}}(t,\zeta) &= \mathcal{K}(\Pi_{S^\star_{\mathcal{J}}} I) - \big\langle \nabla\mathcal{K}(0)\,,\, \Pi_{S^\star_{\mathcal{J}}} I\big\rangle + \tfrac{1}{2}\big\langle \mathcal{A}\Pi_{S^\sharp_{\mathcal{J}}} z\,,\, \Pi_{S^\sharp_{\mathcal{J}}} z\big\rangle \\ &\quad + \varepsilon\widehat{P}(t, \Pi_{S^\star_{\mathcal{J}}}\xi, \Pi_{S^\star_{\mathcal{J}}} I, \Pi_{S^\sharp_{\mathcal{J}}} z) . \end{aligned}$$

Notice that the function

$$\widehat{P}_{\mathcal{J}}(t,\xi,I,z) = \widehat{P}(t, \Pi_{S^\star_{\mathcal{J}}}\xi, \Pi_{S^\star_{\mathcal{J}}} I, \Pi_{S^\sharp_{\mathcal{J}}} z)$$

satisfies both **(L)** and **(P$_\tau$)** with the same constants, for every index $\mathcal{J} \geq 1$, and observe that system (5.6) is equivalent to

$$\begin{cases} \dot{\vec\xi}_j = \nabla\mathcal{K}_j(\vec I_j) - \nabla\mathcal{K}_j(0) + \varepsilon\nabla_{\vec I_j}\widehat{P}_{\mathcal{J}}(t,\xi,I,z) & \\ -\dot{\vec I}_j = \varepsilon\nabla_{\vec\xi_j}\widehat{P}_{\mathcal{J}}(t,\xi,I,z) & j \leq \mathcal{J}\,, \\ J\dot{\vec z}_j = \mathcal{A}_j\vec z_j + \varepsilon\nabla_{\vec z_j}\widehat{P}_{\mathcal{J}}(t,\xi,I,z) & \\ & \\ \dot{\vec\xi}_i = 0 & \\ -\dot{\vec I}_i = 0 & i > \mathcal{J}\,. \\ J\dot{\vec z}_i = 0 & \end{cases} \tag{5.7}$$

It can be viewed as two uncoupled systems, the first one in a finite-dimensional space (the "approximating system"), and the second one, infinite-dimensional, having only constant solutions. From now on, we will take $\vec{\xi}_i(t), \vec{I}_i(t), \vec{z}_i(t)$ identically equal to zero when $i \geq \mathcal{J}$.

Concerning the "approximating system", we will need the following slight modification of [41, Corollary 2.3]. Let us consider the finite-dimensional Hamiltonian system

$$J\dot{\zeta} = \nabla_\zeta H(t,\zeta)\,, \tag{5.8}$$

with $\zeta = (\xi, I, z) \in \mathbb{R}^{N+N+2M}$, where the Hamiltonian function is T-periodic in t. Here we use the notation $\xi = (\vec{\xi}_1, \dots, \vec{\xi}_{\mathcal{J}})$, $I = (\vec{I}_1, \dots, \vec{I}_{\mathcal{J}})$.

Theorem 5.2.2. *Assume that $H(t,\zeta) = \frac{1}{2}\langle \mathbb{A}z, z\rangle + G(t,\zeta)$, where $\mathbb{A}$ is a symmetric $2M \times 2M$ matrix such that $z \equiv 0$ is the unique T-periodic solution of equation $J\dot{z} = \mathbb{A}z$, and there exists a constant c_1 such that*

$$|\nabla_\zeta G(t,\zeta)| \leq c_1\,, \qquad \textit{for every } (t,\zeta) \in \mathbb{R} \times \mathbb{R}^{2(M+N)}\,.$$

Let $G(t,\xi,I,z)$ be periodic in the variables $\xi_1, \dots, \xi_N$. Assume moreover the existence of some positive constants $r'_j < r''_j$ and symmetric invertible matrices $\mathcal{B}_j$, with $j = 1, \dots, \mathcal{J}$, such that, for any solution $\zeta(t) = (\xi(t), I(t), z(t))$ of (5.8)*, if*

$$r'_j \leq \|\vec{I}_j(0) - \vec{I}^0_j\| \leq r''_j \quad \textit{and} \quad \|\vec{I}_i(0) - \vec{I}^0_i\| \leq r''_i \ \textit{ for every } i \neq j\,,$$

then

$$\Big\langle \vec{\xi}_j(T) - \vec{\xi}_j(0)\,,\, \mathcal{B}_j(\vec{I}_j(0) - \vec{I}^0_j)\Big\rangle > 0\,.$$

Then, there exist at least $N+1$ geometrically distinct T-periodic solutions $\zeta(t) = (\xi(t), I(t), z(t))$ of (5.8)*, such that*

$$\|\vec{I}_j(0) - \vec{I}^0_j\| < r'_j\,, \quad \textit{for every } j = 1, \dots, \mathcal{J}\,.$$

5.3 Proof of Theorem 5.1.1

In what follows, we always assume that $|\varepsilon| \leq 1$, and we denote by $\bar{\rho}$ the constant introduced in assumption **(Tw)**. Moreover, as in the previous section, we assume $I^0 = 0$.

Lemma 5.3.1. *There is a constant $C > 0$ with the following property: if $\zeta(t) = (\xi(t), I(t), z(t))$ is a solution of* (5.5) *with $\|\vec{I}_j(0)\| \leq \bar{\rho}$, for some $j \geq 1$, then*

$$\|\vec{\xi}_j(t) - \vec{\xi}_j(0) - t[\nabla\mathcal{K}_j(\vec{I}_j(0)) - \nabla\mathcal{K}_j(0)]\| + \|\vec{I}_j(t) - \vec{I}_j(0)\| \leq C|\varepsilon|a^\star_j$$

for every $t \in [0,T]$. The same property holds for the solutions of (5.7)*, when $j = 1, \dots, \mathcal{J}$.*

Proof. Let us start computing the following estimate, for every $t \in [0, T]$ and every $k \in \{S^\star_{j-1} + 1, \dots, S^\star_{j-1} + N^\star_j = S^\star_j\}$,

$$|I_k(t) - I_k(0)| \le \int_0^t |\dot I_k(s)|\, ds \le |\varepsilon| \int_0^T \Big| \frac{\partial \widehat P}{\partial \xi_k}(s, \zeta(s)) \Big|\, ds \le |\varepsilon|\, T \alpha^\star_k .$$

Then we easily get

$$\|\vec I_j(t) - \vec I_j(0)\| \le |\varepsilon|\, T \Big(\sum_{i=1}^{N^\star_j} (\alpha^\star_{S^\star_{j-1}+i})^2 \Big)^{1/2} = |\varepsilon|\, T a^\star_j .$$

Moreover,

$$\begin{aligned}
&\|\vec\xi_j(t) - \vec\xi_j(0) - t[\nabla \mathcal K_j(\vec I_j(0)) - \nabla \mathcal K_j(0)]\| \\
&\le \int_0^t \|\dot{\vec\xi}_j(s) - [\nabla \mathcal K_j(\vec I_j(0)) - \nabla \mathcal K_j(0)]\|\, ds \\
&\le \int_0^T \|\nabla \mathcal K_j(\vec I_j(s)) - \nabla \mathcal K_j(\vec I_j(0))\|\, ds + |\varepsilon| \int_0^T \|\nabla_{\vec I_j} \widehat P(s, \zeta(s))\|\, ds \\
&\le \int_0^T L \|\vec I_j(s) - \vec I_j(0)\|\, ds + |\varepsilon|\, T a^\star_j \\
&\le |\varepsilon|\, T(1 + LT) a^\star_j ,
\end{aligned}$$

where L is a suitable Lipschitz constant provided by **(L)**. The proof is thus completed. □

Lemma 5.3.2. *There exist $\bar\varepsilon > 0$ and a sequence $(\delta_j)_j$ in ℓ^2, with $\delta_j \in]0, \bar\rho]$, satisfying the following property: if $\zeta(t) = (\xi(t), I(t), z(t))$ is a solution of* (5.5)*, with $|\varepsilon| < \bar\varepsilon$ and $\delta_j \le \|\vec I_j(0)\| \le \bar\rho$, for some $j \ge 1$, then*

$$\Big\langle \vec\xi_j(T) - \vec\xi_j(0)\, ,\, \mathcal B_j\, \vec I_j(0) \Big\rangle > 0 .$$

The same property holds for the solutions of (5.7)*, when $j = 1, \dots, \mathcal J$.*

Proof. If $\|\vec I_j(0)\| \le \bar\rho$ for some $j \ge 1$, then, by Lemma 5.3.1 and **(Tw)**,

$$\begin{aligned}
&\Big\langle \vec\xi_j(T) - \vec\xi_j(0)\, ,\, \mathcal B_j\, \vec I_j(0) \Big\rangle \\
&= \Big\langle \vec\xi_j(T) - \vec\xi_j(0) - T[\nabla \mathcal K_j(\vec I_j(0)) - \nabla \mathcal K_j(0)]\, ,\, \mathcal B_j\, \vec I_j(0) \Big\rangle + \\
&\qquad + T \Big\langle \nabla \mathcal K_j(\vec I_j(0)) - \nabla \mathcal K_j(0)\, ,\, \mathcal B_j\, \vec I_j(0) \Big\rangle \\
&\ge -C|\varepsilon| a^\star_j\, \|\mathcal B_j\|\, \|\vec I_j(0)\| + T\bar c \|\vec I_j(0)\|^2 \\
&= \Big(-C|\varepsilon| a^\star_j\, \|\mathcal B_j\| + T\bar c \|\vec I_j(0)\| \Big) \|\vec I_j(0)\| .
\end{aligned}$$

Setting

$$\delta_j := \min\left\{\bar{\rho}\,,\, \frac{2C}{\bar{c}T}\, a_j^\star\, \|\mathcal{B}_j\|\right\}\,,$$

we easily verify that $(\delta_j)_j \in \ell^2$, since $(\|\mathcal{B}_j\|)_j$ is bounded by $\|\mathcal{B}\|$ and $(a_j^\star)_j \in \ell^2$; in particular, there exists an integer j_0 such that

$$\delta_j = \frac{2C}{\bar{c}T}\, a_j^\star\, \|\mathcal{B}_j\|\,, \qquad \text{for every } j \geq j_0\,.$$

So, we see that, since $|\varepsilon| \leq 1$ and $\|\vec{I}_j(0)\| \geq \delta_j$,

$$-C|\varepsilon|a_j^\star\, \|\mathcal{B}_j\| + T\bar{c}\|\vec{I}_j(0)\| > 0\,,$$

for every $j \geq j_0$. For the remaining finite number of integers $j \in \{1, \ldots, j_0 - 1\}$ we simply need to choose $|\varepsilon|$ sufficiently small, thus finishing the proof. □

Remark 5.3.3. When X is finite-dimensional, the above estimate simplifies, in view of the compactness of the closed balls centered at the origin, so the first condition in **(Tw')** is sufficient in this case. Concerning the second condition in **(Tw')**, we see that it guarantees the existence of a sequence of balls, with smaller and smaller radii, over which the twist condition still holds.

Notice that the set

$$\Xi_I = \prod_{j=1}^{\infty} B^{N_j^\star}[0, \delta_j + Ca_j^\star]\,,$$

where $B^n[0,R]$ denotes the closed ball $\{v \in \mathbb{R}^n : \|v\| \leq R\}$, is compact, being homeomorphic to a Hilbert cube. We now modify the function $\mathcal{K}$ outside Ξ_I, in order that the gradient of the modified function be bounded. Let $R_I > 0$ be such that $\Xi_I \subseteq \{v \in X : \|v\| \leq R_I\}$, and $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth decreasing function such that

$$\psi(s) = 1 \;\text{ if } s \leq R_I\,, \qquad \psi(s) = 0 \;\text{ if } s \geq 2R_I\,.$$

Define $\widetilde{\mathcal{K}} : X \to \mathbb{R}$ as $\widetilde{\mathcal{K}}(I) = \psi(\|I\|)\mathcal{K}(I)$. Then, when $I \neq 0$,

$$\|\nabla\widetilde{\mathcal{K}}(I)\| = \left\|\psi'(\|I\|)\mathcal{K}(I)\frac{I}{\|I\|} + \psi(\|I\|)\nabla\mathcal{K}(I)\right\| \leq c_1|\mathcal{K}(I)| + \|\nabla\mathcal{K}(I)\|\,,$$

for some $c_1 > 0$. By assumption **(L)**, we can find a Lipschitz constant L such that, for every $s \in [0,1]$, if $\|I\| \leq 2R_I$,

$$\|\nabla\mathcal{K}(sI)\| \leq \|\nabla\mathcal{K}(sI) - \nabla\mathcal{K}(0)\| + \|\nabla\mathcal{K}(0)\| \leq L\|I\| + \|\nabla\mathcal{K}(0)\|\,.$$

Moreover,

$$|\mathcal{K}(I)| = \left|\mathcal{K}(0) + \int_0^1 \langle \nabla\mathcal{K}(sI)\,,\, I\rangle\; ds\right| \le |\mathcal{K}(0)| + \sup_{s\in[0,1]} \|\nabla\mathcal{K}(sI)\|\, \|I\|$$
$$\le |\mathcal{K}(0)| + \big(L\|I\| + \|\nabla\mathcal{K}(0)\|\big)\,\|I\|\,.$$

Hence,

$$\|\nabla\widetilde{\mathcal{K}}(I)\| \le c_1|\mathcal{K}(0)| + (2R_I c_1 + 1)(2R_I L + \|\nabla\mathcal{K}(0)\|)\,, \qquad \text{for every } I \in X.$$

We define $\mathbb{A} = \mathrm{diag}(\mathcal{A}_1, \dots, \mathcal{A}_{\mathcal{J}})$ as a *block-diagonal* matrix having a diagonal formed by the matrices $\mathcal{A}_1, \dots, \mathcal{A}_{\mathcal{J}}$ introduced in **(Dec1)**, i.e. such that

$$\mathbb{A}(\vec{z}_1, \dots, \vec{z}_{\mathcal{J}}) = (\mathcal{A}_1\vec{z}_1, \dots, \mathcal{A}_{\mathcal{J}}\vec{z}_{\mathcal{J}})\,.$$

It is easy to verify, using **(NR)**, that $z \equiv 0$ is the unique T-periodic solution of equation $J\dot{z} = \mathbb{A}z$. Then, by Theorem 5.2.2, for every $\mathcal{J}$ there is a T-periodic solution

$$\zeta_{\mathcal{J}}(t) = (\xi_{\mathcal{J}}(t), I_{\mathcal{J}}(t), z_{\mathcal{J}}(t))$$

of (5.7), with

$$\|\vec{I}_{\mathcal{J}_j}(0)\| < \delta_j\,, \qquad \text{for every } j \ge 1\,. \tag{5.9}$$

(Recall that we have chosen the last constant components of the solutions of (5.7) to be equal to zero.) By Lemma 5.3.1, these solutions satisfy

$$\|\vec{I}_{\mathcal{J}_j}(t)\| \le \delta_j + Ca_j^\star\,, \qquad \text{for every } t \in [0,T]\,,$$

i.e.,

$$I_{\mathcal{J}}(t) \in \Xi_I\,, \qquad \text{for every } t \in [0,T]\,. \tag{5.10}$$

Let us now consider the component $\xi_{\mathcal{J}}(t)$ of the solution. By the periodicity assumption **(P$_\tau$)**, we can assume without loss of generality that $\xi_k(0) \in [0, \tau_k]$, for every $k \ge 1$. From Lemma 5.3.1, property **(L)** and (5.9), we have

$$|\xi_k(t) - \xi_k(0)| \le \|\vec{\xi}_j(t) - \vec{\xi}_j(0)\| \le Ca_j^\star + TL\delta_j\,, \qquad \text{for every } t \in [0,T]\,,$$

for a suitable Lipschitz constant L. Setting $b_k := Ca_j^\star + TL\delta_j$, where j is the index such that $S_{j-1}^\star < k \le S_j^\star$, and defining

$$\Xi_\xi = \prod_{k=1}^{\infty}[-b_k, \tau_k + b_k]\,,$$

we have that

$$\xi_{\mathcal{J}}(t) \in \Xi_\xi\,, \qquad \text{for every } t \in [0,T]\,. \tag{5.11}$$

We now need an a priori estimate on $z_{\mathcal{J}}(t)$.

Lemma 5.3.4. *There exists a sequence $(R_j)_j \in \ell^2$ of positive constants such that, for every T-periodic solution $\zeta(t) = (\xi(t), I(t), z(t))$ of* (5.5)*, we have*

$$\|\vec{z}_j\|_{\mathcal{C}([0,T],\mathbb{R}^{2N_j^\sharp})} \leq |\varepsilon| R_j \,,$$

for every $j \geq 1$. The same property holds for every T-periodic solution of (5.7)*, when $j = 1, \dots, \mathcal{J}$.*

Proof. Fix $j \geq 1$ and consider the j-th block of the third equation in (5.5), i.e.

$$\mathcal{L}_j \vec{z}_j = \mathcal{A}_j \vec{z}_j + \varepsilon \nabla_{\vec{z}_j} \widehat{P}(t, \zeta) \,, \tag{5.12}$$

where $\mathcal{L}_j$ denotes the j-th block of the linear operator $\mathcal{L}$ introduced in **(NR)**, i.e.

$$\mathcal{L}_j \vec{z}_j = \mathcal{L}_j(z_{S_{j-1}^\sharp+1}, \dots, z_{S_j^\sharp}) = (J\dot{z}_{S_{j-1}^\sharp+1}, \dots, J\dot{z}_{S_j^\sharp}) \,. \tag{5.13}$$

From hypothesis **(Dec1)**, we have $\sigma(\mathcal{L}_j - \mathcal{A}_j) \subseteq \sigma(\mathcal{L} - \mathcal{A})$. Hence, using **(NR)**, $0 \notin \sigma(\mathcal{L}_j - \mathcal{A}_j)$ and (5.12) is equivalent to

$$\vec{z}_j = \varepsilon(\mathcal{L}_j - \mathcal{A}_j)^{-1} \nabla_{\vec{z}_j} \widehat{P}(t, \zeta) \,.$$

Moreover,

$$\|(\mathcal{L}_j - \mathcal{A}_j)^{-1}\| = \frac{1}{\operatorname{dist}(0, \sigma(\mathcal{L}_j - \mathcal{A}_j))} \leq \frac{1}{\operatorname{dist}(0, \sigma(\mathcal{L} - \mathcal{A}))} = \|(\mathcal{L} - \mathcal{A})^{-1}\| \,,$$

and consequently, setting $r_j := \sqrt{T} a_j^\sharp \|(\mathcal{L} - \mathcal{A})^{-1}\|$, we have that

$$\|\vec{z}_j\|_{L^2([0,T],\mathbb{R}^{2N_j^\sharp})} \leq |\varepsilon| \, \|(\mathcal{L}_j - \mathcal{A}_j)^{-1}\| \cdot \|\nabla_{\vec{z}_j} \widehat{P}\|_{L^2([0,T],\mathbb{R}^{2N_j^\sharp})} \leq |\varepsilon| r_j \,.$$

Since $\vec{z}_j$ solves (5.12), we have that $\dot{\vec{z}}_j \in L^2([0,T], \mathbb{R}^{2N_j^\sharp})$, and

$$\|\dot{\vec{z}}_j\|_{L^2([0,T],\mathbb{R}^{2N_j^\sharp})} \leq \|\mathcal{A}_j\| \|\vec{z}_j\|_{L^2([0,T],\mathbb{R}^{2N_j^\sharp})} + |\varepsilon| \sqrt{T} a_j^\sharp \leq |\varepsilon| \Big(\|\mathcal{A}_j\| r_j + \sqrt{T} a_j^\sharp \Big) \,.$$

So, setting $C_j = (1 + \|\mathcal{A}_j\|) r_j + \sqrt{T} a_j^\sharp$,

$$\|\vec{z}_j\|_{H^1([0,T],\mathbb{R}^{2N_j^\sharp})} \leq |\varepsilon| C_j \,. \tag{5.14}$$

By the continuous immersion of $H^1([0,T], Z)$ in $\mathcal{C}([0,T], Z)$, cf. [108, §23.6], we can find a constant $\chi > 0$ such that

$$\|z\|_{\mathcal{C}([0,T],Z)} \leq \chi \|z\|_{H^1([0,T],Z)} \,,$$

for every $z \in H^1([0,T], Z)$. Since $\mathcal{C}([0,T], \mathbb{R}^{2N_j^\sharp})$ and $H^1([0,T], \mathbb{R}^{2N_j^\sharp})$ can be seen as a subsets of $\mathcal{C}([0,T], Z)$ and $H^1([0,T], Z)$, respectively, simply adding an infinite number of null components, we obtain

$$\|\vec{z}_j\|_{\mathcal{C}([0,T],\mathbb{R}^{2N_j^\sharp})} \le \chi \|\vec{z}_j\|_{H^1([0,T],\mathbb{R}^{2N_j^\sharp})} \le |\varepsilon| \chi C_j \,.$$

The proof is thus completed, taking $R_j = \chi C_j$. $\square$

Defining

$$\Xi_z = \prod_{j=1}^{\infty} B^{2N_j^\sharp}[0, R_j] \,,$$

we have thus proved that

$$z_{\mathcal{J}}(t) \in \Xi_z \,, \qquad \text{for every } t \in [0,T] \,. \tag{5.15}$$

Summing up, by (5.10), (5.11), (5.15), we have that, setting $\Xi = \Xi_\xi \times \Xi_I \times \Xi_z$, the T-periodic solutions we found satisfy

$$\zeta_{\mathcal{J}}(t) = (\xi_{\mathcal{J}}(t), I_{\mathcal{J}}(t), z_{\mathcal{J}}(t)) \in \Xi \,, \qquad \text{for every } t \in [0,T] \,.$$

Notice that Ξ is compact, being the product of three compact sets. We will now prove that there is a subsequence of $(\zeta_{\mathcal{J}})_{\mathcal{J}}$ which uniformly converges to a solution of (5.5).

From (5.14), recalling that $|\varepsilon| \le 1$, we have

$$\|z_{\mathcal{J}}(t_1) - z_{\mathcal{J}}(t_2)\| \le |t_1 - t_2|^{1/2} \left(\int_0^T \|\dot{z}_{\mathcal{J}}(s)\|^2 \, ds \right)^{1/2} \le |t_1 - t_2|^{1/2} \left(\sum_{j=1}^{\infty} C_j^2 \right)^{1/2} .$$

Looking at the variables $I_{\mathcal{J}}(t)$, by **($\mathbf{P}_{bd}$)** we have that

$$\|I_{\mathcal{J}}(t_1) - I_{\mathcal{J}}(t_2)\| \le |t_2 - t_1|^{1/2} \left(\int_0^T \|\dot{I}_{\mathcal{J}}(s)\|^2 \, ds \right)^{1/2} \le |t_2 - t_1|^{1/2} \sqrt{T} \, \|a^\star\|_{\ell^2} \,.$$

Concerning the variables $\xi_{\mathcal{J}}(t)$, we first observe that

$$\begin{aligned} \|\dot{\xi}_{\mathcal{J}}(s)\| &\le \|\nabla \mathcal{K}(I_{\mathcal{J}}(s)) - \nabla \mathcal{K}(0)\| + \|a^\star\|_{\ell^2} \\ &\le L \|I_{\mathcal{J}}(s)\| + \|a^\star\|_{\ell^2} \le L \Big(\sum_{j=1}^{\infty} (\delta_j + C a_j^\star)^2 \Big)^{1/2} + \|a^\star\|_{\ell^2} := \widehat{C} \,, \end{aligned}$$

where L is a suitable Lipschitz constant provided by **(L)**. Then,

$$\|\xi_{\mathcal{J}}(t_1) - \xi_{\mathcal{J}}(t_2)\| \le |t_2 - t_1|^{1/2} \left(\int_0^T \|\dot{\xi}_{\mathcal{J}}(s)\|^2 \, ds \right)^{1/2} \le |t_2 - t_1|^{1/2} \sqrt{T} \, \widehat{C} \,.$$

Hence, the sequence $(\zeta_{\mathcal{J}})_{\mathcal{J}}$ is equi-uniformly continuous on $[0,T]$ and takes its values in a compact subset of $\mathcal{X}$. By the Ascoli–Arzelà Theorem, we find a subsequence, still denoted by $(\zeta_{\mathcal{J}})_{\mathcal{J}}$, which uniformly converges to a certain continuous function $\zeta^\natural : [0,T] \to \mathcal{X}$, such that $\zeta^\natural(t) \in \Xi$ for every $t \in [0,T]$, and $\zeta^\natural(0) = \zeta^\natural(T)$. We are going to prove that $\zeta^\natural$ solves (5.5), following the lines of the proof of [18, Theorem 3].

Let us consider the solution ζ_∞ of system (5.5) such that $\zeta_\infty(0) = \zeta^\natural(0)$ which, by the boundedness of $\nabla\mathcal{K}$ and $\nabla_\zeta \widehat{P}$, is certainly defined on $[0,T]$. We will prove that the sequence $(\zeta_{\mathcal{J}})_{\mathcal{J}}$ converges uniformly to ζ_∞, thus obtaining that $\zeta_\infty = \zeta^\natural$. To this aim, we write the integral formulation of systems (5.5) and (5.6), for $\mathcal{J} \geq 1$:

$$\zeta_\infty(t) = \zeta_\infty(0) - \int_0^t J\nabla_\zeta \widehat{H}(s, \zeta_\infty(s))\, ds\,, \tag{5.16}$$

$$\zeta_{\mathcal{J}}(t) = \zeta_{\mathcal{J}}(0) - \int_0^t J\nabla_\zeta \widehat{H}_{\mathcal{J}}(s, \zeta_{\mathcal{J}}(s))\, ds\,. \tag{5.17}$$

In order to simplify the notations, we introduce the projection

$$\mathscr{P}_{\mathcal{J}}(\zeta) = \mathscr{P}_{\mathcal{J}}(\xi, I, z) = (\Pi_{S^\star_{\mathcal{J}}}\xi, \Pi_{S^\star_{\mathcal{J}}} I, \Pi_{S^\sharp_{\mathcal{J}}} z)\,.$$

Let us write

$$\|\zeta_{\mathcal{J}}(t) - \zeta_\infty(t)\| \leq \|\zeta_{\mathcal{J}}(t) - \mathscr{P}_{\mathcal{J}}\zeta_\infty(t)\| + \|\mathscr{P}_{\mathcal{J}}\zeta_\infty(t) - \zeta_\infty(t)\|\,.$$

By an elementary argument,

$$\|\mathscr{P}_{\mathcal{J}}\zeta_\infty(t) - \zeta_\infty(t)\| \to 0\,, \quad \text{as } \mathcal{J} \to \infty\,, \tag{5.18}$$

uniformly with respect to $t \in [0,T]$. From (5.16) and (5.17), since $\mathscr{P}_{\mathcal{J}} J = J\mathscr{P}_{\mathcal{J}}$, we have

$$\begin{aligned} \|\zeta_{\mathcal{J}}(t) - \mathscr{P}_{\mathcal{J}}\zeta_\infty(t)\| &\leq \|\zeta_{\mathcal{J}}(0) - \mathscr{P}_{\mathcal{J}}\zeta_\infty(0)\| + \\ &\quad + \int_0^t \|J\nabla_\zeta \widehat{H}_{\mathcal{J}}(s, \zeta_{\mathcal{J}}(s)) - J\mathscr{P}_{\mathcal{J}}\nabla_\zeta \widehat{H}(s, \zeta_\infty(s))\|\, ds\,. \end{aligned} \tag{5.19}$$

Notice that

$$\|\zeta_{\mathcal{J}}(0) - \mathscr{P}_{\mathcal{J}}\zeta_\infty(0)\| \leq \|\zeta_{\mathcal{J}}(0) - \zeta_\infty(0)\| = \|\zeta_{\mathcal{J}}(0) - \zeta^\natural(0)\| \to 0\,, \quad \text{as } \mathcal{J} \to \infty\,. \tag{5.20}$$

Since $\nabla_\zeta \widehat{H}_{\mathcal{J}}(s, \zeta_{\mathcal{J}}(s)) = \mathscr{P}_{\mathcal{J}}\nabla_\zeta \widehat{H}(s, \zeta_{\mathcal{J}}(s))$, the integral term in (5.19) satisfies

$$\int_0^t \Big\| J\mathscr{P}_{\mathcal{J}}\Big(\nabla_\zeta \widehat{H}(s, \zeta_{\mathcal{J}}(s)) - \nabla_\zeta \widehat{H}(s, \zeta_\infty(s))\Big)\Big\|\, ds \leq L \int_0^t \|\zeta_{\mathcal{J}}(s) - \zeta_\infty(s)\|\, ds\,,$$

where L is a suitable Lipschitz constant. Summing up, we have

$$\|\zeta_{\mathcal{J}}(t) - \zeta_\infty(t)\| \le c_{\mathcal{J}} + L \int_0^t \|\zeta_{\mathcal{J}}(s) - \zeta_\infty(s)\| \, ds \,,$$

where $(c_{\mathcal{J}})_{\mathcal{J}}$ is a sequence, provided by the limits in (5.18) and (5.20), such that $\lim_{\mathcal{J}} c_{\mathcal{J}} = 0$. Hence, by Gronwall's Lemma,

$$\|\zeta_{\mathcal{J}}(t) - \zeta_\infty(t)\| \le c_{\mathcal{J}} e^{Lt} \,, \qquad \text{for every } t \in [0, T] \,,$$

implying that $\zeta_{\mathcal{J}} \to \zeta_\infty$ uniformly on $[0, T]$. We conclude that $\zeta_\infty = \zeta^\natural$ on $[0, T]$, thus showing that $\zeta_\infty(0) = \zeta_\infty(T)$, so that ζ_∞ is a T-periodic solution of (5.5).

By the inverse change of variables

$$(\varphi(t), I(t), z(t)) = (\xi(t) + t\nabla\mathcal{K}(I^0), I(t), z(t)) \,,$$

cf. (5.4), we have a solution of (5.1), satisfying (5.2). Moreover, condition (5.3) holds true, by Lemmas 5.3.1 and 5.3.4, suitably reducing, if necessary, the value of $\bar{\varepsilon}$. The proof of Theorem 5.1.1 is thus completed. □

5.4 Applications

5.4.1 Coupling second order with linear systems

We first state a simple lemma, which may be useful for the verification of the twist condition.

Lemma 5.4.1. *If there exists $I^0 \in X$ such that $\mathcal{K} : X \to \mathbb{R}$ is twice continuously differentiable at I^0 and $\mathcal{K}''(I^0) : X \to X$ is invertible, with bounded inverse, then there exist two positive constants $\bar{c}$, $\bar{\rho}$ such that*

$$\|I - I^0\| \le \bar{\rho} \quad \Rightarrow \quad \big\langle \nabla\mathcal{K}(y) - \nabla\mathcal{K}(I^0) \,,\, \mathcal{K}''(I^0)(y - I^0) \big\rangle \ge \bar{c}\, \|y - I^0\|^2.$$

Moreover, if $\dim X = \infty$ and, with the usual notation, $\mathcal{K}(I) = \sum_{j=1}^\infty \mathcal{K}_j(\vec{I}_j)$, then condition **(Tw)** *holds.*

Proof. Since $\mathcal{B} := \mathcal{K}''(I^0) : X \to X$ is invertible with bounded inverse, there exists $\gamma > 0$ such that $\|\mathcal{B}\, I\| \ge \gamma \|I\|$ for every $I \in X$. Then,

$$\begin{aligned}
&\big\langle \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0) \,,\, \mathcal{B}(I - I^0) \big\rangle = \\
&\quad = \int_0^1 \big\langle \mathcal{K}''\big(I^0 + s(I - I^0)\big)(I - I^0) \,,\, \mathcal{B}(I - I^0) \big\rangle \, ds \\
&\quad = \|\mathcal{B}(I - I^0)\|^2 + \int_0^1 \big\langle \big[\mathcal{K}''\big(I^0 + s(I - I^0)\big) - \mathcal{B}\big] \,(I - I^0) \,,\, \mathcal{B}(I - I^0) \big\rangle \, ds \\
&\quad \ge \big(\gamma^2 - \|\mathcal{B}\| \cdot \|\mathcal{K}''\big(I^0 + s(I - I^0)\big) - \mathcal{B}\|\big) \, \|I - I^0\|^2 \,.
\end{aligned}$$

Since $\mathcal{K}''$ is continuous at I^0, there exists $\bar{\rho} > 0$ such that, if $I \in X$ satisfies $\|I - I^0\| \leq \bar{\rho}$, then

$$\|\mathcal{K}''(I) - \mathcal{B}\| = \|\mathcal{K}''(I) - \mathcal{K}''(I^0)\| \leq \frac{\gamma^2}{2\|\mathcal{B}\|},$$

so

$$\left\langle \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0)\,,\, \mathcal{B}(I - I^0) \right\rangle \geq \frac{\gamma^2}{2}\|I - I^0\|^2, \tag{5.21}$$

and the first part of the lemma is thus proved.

Assume now that $\mathcal{K}(I) = \sum_{j=1}^{\infty} \mathcal{K}_j(\vec{I}_j)$. We have that

$$\mathcal{B}\, I = (\mathcal{B}_1\vec{I}_1, \dots, \mathcal{B}_j\vec{I}_j, \dots)\,,$$

where $\mathcal{B}_j = \mathcal{K}_j''(\vec{I}_j^{\,0})$. Then, **(Tw)** is verified directly from (5.21) defining, for every $j \in \{1, 2, \dots\}$, the vector I as $\vec{I}_i = \vec{I}_i^{\,0}$ if $i \neq j$, once $\vec{I}_j$ has been chosen. □

We thus have the following.

Corollary 5.4.2. *Assume* **(L)**, **(P$_\tau$)**, **(P$_{bd}$)**, **(NR)**, **(Dec1)**, **(Dec2)** *and* **(C$_\tau$)** *hold. If* $\mathcal{K} : X \to \mathbb{R}$ *is twice continuously differentiable at* I^0 *and* $\mathcal{K}''(I^0) : X \to X$ *is invertible, with bounded inverse, then there exists* $\bar{\varepsilon} > 0$ *such that, if* $|\varepsilon| \leq \bar{\varepsilon}$*, system* (5.1) *has a* T*-periodic solution.*

Let us now consider an equation in an infinite-dimensional space of the type

$$\begin{cases} \dfrac{d}{dt}\,(\nabla\Phi \circ \dot{x}) = \varepsilon\, \nabla_x F(t, x, z) \\ J\dot{z} = \mathcal{A}z + \varepsilon\nabla_z F(t, x, z)\,. \end{cases} \tag{5.22}$$

Let, for definiteness, $\dim X = \infty$ and $\dim Z = \infty$. Concerning the bounded selfadjoint operator $\mathcal{A}$, we require the nonresonance assumption **(NR)** and that it decomposes as in **(Dec1)**. For the differential operator in the first equation, we suppose that there exists a sequence of positive integers $(N_j)_{j\geq 1}$ such that, writing any vector $y \in X$ as $y = (\vec{y}_1, \dots, \vec{y}_j, \dots)$, with $\vec{y}_j \in \mathbb{R}^{N_j}$,

$$\Phi(y) = \sum_{j=1}^{\infty} \Phi_j(\vec{y}_j)\,,$$

where each Φ_j is a continuous real valued strictly convex function defined on a closed ball $\overline{B}(0, a_j)$ in $\mathbb{R}^{N_j}$, continuously differentiable in the open ball $B(0, a_j)$, with $\nabla\Phi_j : B(0, a_j) \to X$ being a homeomorphism, and $\nabla\Phi_j(0) = 0$.

Denoting by Φ_j^* the Legendre–Fenchel transform of Φ_j, we have that $\Phi_j^* : X \to \mathbb{R}$ is strictly convex and coercive, with $\nabla\Phi^* = (\nabla\Phi)^{-1} : X \to B(0,a)$, cf. [72, Chapter 2]. We can define

$$\Phi^*(y) = \sum_{j=1}^{\infty} \Phi_j^*(\vec{y}_j)\,,$$

so that system (5.22) can be written as a Hamiltonian system

$$\begin{cases} \dot{x} = \nabla\Phi^*(y) \\ \dot{y} = \varepsilon\,\nabla_x F(t,x,z) \\ J\dot{z} = \mathcal{A}z + \varepsilon\nabla_z F(t,x,z)\,. \end{cases}$$

So, we are in the situation of system (5.1), taking $\mathcal{K}(I) = \Phi^*(I)$ and $P(t,\varphi,I,z) = F(t,\varphi,z)$.

An example is provided by the choice

$$\Phi(y) = \sum_{j=1}^{\infty} \Big(1 - \sqrt{1 - \|\vec{y}_j\|^2}\,\Big)\,,$$

for which, writing $x = (\vec{x}_1, \dots, \vec{x}_j, \dots)$, system (5.22) becomes

$$\begin{cases} \dfrac{d}{dt}\dfrac{\dot{\vec{x}}_j}{\sqrt{1 - \|\dot{\vec{x}}_j\|^2}} = \varepsilon\,\nabla_{\vec{x}_j} F(t,x,z)\,, \qquad j = 1,2,\dots \\ J\dot{z} = \mathcal{A}z + \varepsilon\nabla_z F(t,x,z)\,, \end{cases} \tag{5.23}$$

so that, in the first equation, we can see a kind of *relativistic operator*. We then have the following.

Corollary 5.4.3. *In the above setting, assume moreover the following conditions:*

(L) *for every $R > 0$ there exists a positive constant L_R such that*

$$\|\nabla_u F(t,u') - \nabla_u F(t,u'')\| \le L_R\|u' - u''\|\,,$$

for every $t \in [0,T]$ and $u' = (x',z'), u'' = (x'',z'') \in X \times Z$ with $\|u'\| < R$ and $\|u''\| < R$;

($\mathbf{F}_\tau$) *the function $F(t,x,z)$ is τ_k-periodic in each x_k, and the sequence $(\tau_k)_k$ belongs to ℓ^2;*

($\mathbf{F}_{bd}$) *there exist* $(\alpha_k^\star)_k$ *and* $(\alpha_l^\sharp)_l$ *in* ℓ^2 *such that, for every* $k, l = 1, 2, \dots,$

$$\left| \frac{\partial F}{\partial x_k}(t, x, z) \right| \le \alpha_k^\star , \qquad \|\nabla_{z_l} F(t, x, z)\| \le \alpha_l^\sharp ,$$

for every $(t, x, z) \in [0, T] \times X \times Z$.

Then, there exists $\bar\varepsilon > 0$ *such that, if* $|\varepsilon| \le \bar\varepsilon$, *system* (5.23) *has a* T*-periodic solution.*

Proof. Taking $I^0 = 0$, we have that $\nabla\Phi^*(0) = 0$ and $(\Phi^*)''(0) = \mathrm{Id}$. So, assumption **($\mathbf{C}_\tau$)** is fulfilled taking $m_1 = m_2 = \cdots = 0$ and, in view of Lemma 5.4.1, we can apply Theorem 5.1.1 to conclude. □

We have thus obtained an extension to infinite-dimensional systems of a result in [67].

Another possible situation where Theorem 5.1.1 applies is provided by the choice

$$\Phi(y) = \sum_{j=1}^{\infty} \left(\sqrt{1 + \|\vec y_j\|^2} - 1 \right).$$

In this case, we find

$$\Phi^*(y) = \sum_{j=1}^{\infty} \Phi_j^*(\vec y_j) = \sum_{j=1}^{\infty} \left(1 - \sqrt{1 - \|\vec y_j\|^2} \right),$$

and the first equation in system (5.22) becomes

$$\frac{d}{dt} \frac{\dot{\vec x}_j}{\sqrt{1 + \|\dot{\vec x}_j\|^2}} = \varepsilon \, \nabla_{\vec x_j} F(t, x, z) , \qquad j = 1, 2, \dots$$

involving a kind of *mean curvature operator*.

Since each $\nabla\Phi_j^*$ is defined only on the open ball $B(0, 1)$, we must first modify and extend the Hamiltonian function outside a ball $B(0, r)$, with $r \in]0, 1[$, and then be careful that the $\vec y_j$ component of the T-periodic solution we find remains in $B(0, r)$. We omit the details, for briefness. Stating the analogue of Corollary 5.4.3, we thus obtain an infinite-dimensional version of some results obtained in [51, 53] (see also [80], where bounded variation solutions are considered).

5.4.2 Perturbations of "superintegrable" systems

In this section we study a slightly different situation with respect to system (5.1). We are going to consider the Hamiltonian system

$$\begin{cases} \dot\varphi = \nabla\mathcal{K}(I) + \eta^2\nabla_I P(t,\varphi,I,z) \\ -\dot I = \eta^2\nabla_\varphi P(t,\varphi,I,z) \\ J\dot z = \eta\mathcal{A}z + \eta^2\nabla_z P(t,\varphi,I,z)\,, \end{cases} \tag{5.24}$$

with Hamiltonian function

$$H(t,\varphi,I,z) = \mathcal{K}(I) + \frac{\eta}{2}\langle \mathcal{A}z\,,\, z\rangle + \eta^2 P(t,\varphi,I,z)\,.$$

The following result extends to an infinite-dimensional setting [41, Theorem 4.1], which was motivated by the study of perturbations of superintegrable systems, cf. [75].

Theorem 5.4.4. *Assume* **(L)**, **(P$_\tau$)**, **(P$_{bd}$)**, **(Dec1)**, **(Dec2)**, **(Tw)** *and* **(C$_\tau$)**. *Moreover let the operator $\mathcal{A}$ be invertible with a bounded inverse. Then, for every $\sigma > 0$ there exists $\bar\eta > 0$ such that, if $|\eta| \le \bar\eta$, system* (5.24) *has a solution satisfying* (5.2) *and* (3.19).

Notice that the nonresonance assumption **(NR)** is not required here.

Proof. Arguing as above we can perform the change of variable (5.4) and set without loss of generality $I^0 = 0$, so to obtain

$$\begin{cases} \dot\xi = \nabla\mathcal{K}(I) - \nabla\mathcal{K}(0) + \eta^2\nabla_I \widehat{P}(t,\xi,I,z) \\ -\dot I = \eta^2\nabla_\xi \widehat{P}(t,\xi,I,z) \\ J\dot z = \eta\mathcal{A}z + \eta^2\nabla_z \widehat{P}(t,\xi,I,z)\,, \end{cases} \tag{5.25}$$

and, for every index $\mathcal{J} \ge 1$, its approximation

$$\begin{cases} \dot\xi = \Pi_{S^\star_{\mathcal{J}}}[\nabla\mathcal{K}(I) - \nabla\mathcal{K}(0) + \eta^2\nabla_I \widehat{P}(t,\xi,I,z)] \\ -\dot I = \Pi_{S^\star_{\mathcal{J}}}[\eta^2\nabla_\xi \widehat{P}(t,\xi,I,z)] \\ J\dot z = \Pi_{S^\sharp_{\mathcal{J}}}[\eta\mathcal{A}z + \eta^2\nabla_z \widehat{P}(t,\xi,I,z)]\,. \end{cases} \tag{5.26}$$

Lemmas 5.3.1 and 5.3.2 holds again, simply replacing $|\varepsilon|$ with η^2 and $\bar\varepsilon$ with $\bar\eta^2$. The statement and the proof of Lemma 5.3.4, however, must be modified as follows.

Lemma 5.4.5. *There exists a sequence $(r_j)_j \in \ell^2$ of positive constants such that, for every T-periodic solution $\zeta(t) = (\xi(t), I(t), z(t))$ of* (5.25) *we have*

$$\|\vec z_j\|_{L^2([0,T],\mathbb{R}^{2N^\sharp_j})} \le |\eta| r_j\,,$$

for every $j \geq 1$. *The same conclusion holds for every solution of* (5.26), *when* $j = 1, \ldots, \mathcal{J}$.

Proof. Fix $j \geq 1$ and consider the j-th block of the third equation in (5.26), i.e.

$$\mathcal{L}_j \vec{z}_j = \eta \mathcal{A}_j \vec{z}_j + \eta^2 \nabla_{\vec{z}_j} \widehat{P}(t, \zeta) \,, \tag{5.27}$$

where $\mathcal{L}_j$ denotes the j-th block of the linear operator $\mathcal{L}$, cf. (5.13). From hypothesis **(Dec1)**, we have that $\sigma(\mathcal{L}_j - \eta\mathcal{A}_j) \subseteq \sigma(\mathcal{L} - \eta\mathcal{A})$. We set $\eta_0 = \min\{1, \frac{\pi}{T\|\mathcal{A}\|}\}$ and, recalling that $0 \notin \sigma(\mathcal{A})$, we choose $\delta \in (0, \frac{\pi}{T})$ such that $\sigma(\mathcal{A}) \cap [-\delta, \delta] = \varnothing$.

Claim. When $|\eta| < \eta_0$, every $\lambda \in \sigma(\mathcal{L} - \eta\mathcal{A})$ satisfies $|\lambda| > \delta|\eta|$.

In order to prove this Claim, notice that, if $\lambda \in \sigma(\mathcal{L} - \eta\mathcal{A})$, there exists a non-trivial T-periodic solution z of $Jz' = (\eta\mathcal{A} - \lambda I)z$, so

$$\sigma\big(J(\eta\mathcal{A} - \lambda I)\big) \cap \frac{2\pi}{T} i\mathbb{Z} \neq \varnothing \,. \tag{5.28}$$

If $|\lambda| \geq \pi/T$, then $|\lambda| > \delta > \delta|\eta|$. So, we can assume $|\lambda| < \pi/T$. In this case, we have

$$\|J(\eta\mathcal{A} - \lambda I)\| \leq |\eta| \, \|\mathcal{A}\| + |\lambda| < \frac{2\pi}{T} \,,$$

so,

$$\mu \in \sigma\big(J(\eta\mathcal{A} - \lambda I)\big) \quad \Rightarrow \quad |\mu| \leq \|J(\eta\mathcal{A} - \lambda I)\| < \frac{2\pi}{T} \,.$$

By (5.28), we have that $0 \in \sigma(J(\eta\mathcal{A} - \lambda I))$ and, since J is invertible, $0 \in \sigma(\eta\mathcal{A} - \lambda I)$. Hence, $\frac{\lambda}{\eta} \in \sigma(\mathcal{A})$ and so $|\frac{\lambda}{\eta}| > \delta$, thus proving the Claim.

From now on we assume $|\eta| < \eta_0$. By the Claim, in particular, $0 \notin \sigma(\mathcal{L} - \eta\mathcal{A})$ and so $\mathcal{L} - \eta\mathcal{A}$ is invertible, as well as $\mathcal{L}_j - \eta\mathcal{A}_j$, with bounded inverses. Hence, (5.27) is equivalent to

$$\vec{z}_j = \eta^2 (\mathcal{L}_j - \eta\mathcal{A}_j)^{-1} \nabla_{\vec{z}_j} \widehat{P}(t, \zeta) \,.$$

Moreover,

$$\|(\mathcal{L}_j - \eta\mathcal{A}_j)^{-1}\| = \frac{1}{\mathrm{dist}(0, \sigma(\mathcal{L}_j - \eta\mathcal{A}_j))} \leq \frac{1}{\mathrm{dist}(0, \sigma(\mathcal{L} - \eta\mathcal{A}))} \leq \frac{1}{\delta|\eta|} \,,$$

and consequently

$$\|\vec{z}_j\|_{L^2([0,T], \mathbb{R}^{2N_j^\sharp})} \leq \eta^2 \, \|(\mathcal{L}_j - \eta\mathcal{A}_j)^{-1}\| \cdot \|\nabla_{\vec{z}_j} \widehat{P}\|_{L^2([0,T], \mathbb{R}^{2N_j^\sharp})} \leq \frac{\eta^2 \sqrt{T} a_j^\sharp}{\delta|\eta|} = |\eta| \frac{\sqrt{T} a_j^\sharp}{\delta} \,,$$

thus concluding the proof of the lemma. □

The proof of Theorem 5.4.4 can now be completed following again the lines of the proof of Theorem 5.1.1. □

Appendix A

Recovering a function from its $p-$norms

In this appendix we will investigate the problem of finding necessary and sufficient conditions for a couple of real valued functions f and g to have coincident p-norms on the same set of p values. The result, besides the intrinsic interest, has a deeply correlation with Probability and in particular with the so called *Stieltjes moment problem*. Let us briefly recall its description.

Suppose μ be a positive Radon measure. If the number

$$\sigma_n = \int_0^{+\infty} x^n \, \mathrm{d}\mu(x)$$

exists and is finite, it is called the n-th moment of the measure μ. If for all $n \geq 0$ the moments exist and are finite, the sequence $(\sigma_n)_n$ is called the moment sequence of μ. The moment problem consists in answering to the following two questions.

1. Given a sequence $(\sigma_n)_n$ of real numbers, find, if possible, a measure μ having it as its moment sequence.
2. Is μ uniquely determined by this sequence?

We will focus our attention on the second question. Stieltjes, in [98], shows that in general the answer to this question is no. (Other examples can be found in [99] while a complete discussion of the problem can be found in [97, 105].)

Our result is then extended to functions defined over $\mathbb{N}$. The problem will be tackled by means of complex analysis techniques and a central role here is played by the *Mellin Transform* named after the Finnish mathematician Hjalmar Mellin who introduce this transformation to study some properties of the gamma functions and of hypergeometric functions (see e.g. [73]). Nowadays Mellin transforms find application in number theory [101], mathematical physics, statistics, the theory of asymptotic expansions, special functions and integral transformations. (For a very detailed exposition of the argument see e.g. [100, 21]).

Before stating our main results, let us recall the standard notation for the norms in $\mathcal{L}^p(E)$:

$$\|f\|_p = \left(\int_E |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \quad \text{if } 1 \le p < +\infty\,, \qquad\qquad \|f\|_\infty = \operatorname*{ess\,sup}_E |f|\,.$$

Moreover, for a measurable function $f : E \to \mathbb{R}$, we write

$$\{f > \alpha\} = \{x \in E : f(x) > \alpha\}\,, \qquad \mu(f > \alpha) = \mu(\{f > \alpha\})\,.$$

Here is our main result.

Theorem A.0.1. *Suppose $f, g \in \mathcal{L}^p(E)$ for all $p \ge 1$, and $P = (p_j)_{j=1}^\infty$ is a sequence of distinct real numbers $p_j \ge 1$. Suppose also that at least one of the following two conditions holds:*

a) P has an accumulation point in $(1, +\infty)$;

b) $f, g \in \mathcal{L}^\infty(E)$ and

$$\sum_{j=1}^\infty \frac{p_j - 1}{(p_j - 1)^2 + 1} = +\infty\,. \tag{A.1}$$

Then the following two statements are equivalent:

i) $\|f\|_p = \|g\|_p$ for all $p \in P$;

ii) $\mu(|f| > \alpha) = \mu(|g| > \alpha)$ for all $\alpha \ge 0$.

Observe that, if $\lim_j p_j = +\infty$, then (A.1) is equivalent to

$$\sum_{j=1}^\infty \frac{1}{p_j} = +\infty\,.$$

Notice moreover that, in this case, if we drop the boundedness hypothesis on f and g, the result is no longer true, as we will show with a counterexample in Section A.3. Theorem A.0.1 applies for example if

$$P = \{j \in \mathbb{N},\ j \ge 10\}\,,\ P = \left\{1 + \frac{1}{\sqrt{j+1}} : j \in \mathbb{N}\right\}, \text{or } P = \left\{2 + \frac{1}{j+1} : j \in \mathbb{N}\right\}.$$

On the contrary, the set $P = \{j^2 : j \in \mathbb{N} \setminus \{0\}\}$ is not admissible, since in this case the series in (A.1) converges.

It can be seen, as a consequence of the Chebyshev inequality, that $ii)$ implies the identity $\|f\|_p = \|g\|_p$, for all $p \in [1, +\infty)$, and for any functions $f, g \in \mathcal{L}^p(E)$. Hence, we will focus on proving that $i)$ implies $ii)$.

After having recalled some results in measure theory and complex analysis we will give a proof of Theorem A.0.1 by the use of the "Full Müntz Theorem in $\mathcal{C}[0,1]$", elementary complex analysis and the Mellin transform. Next we construct the counterexample to the conclusion of Theorem A.0.1 if the boundedness hypothesis on f and g is dropped in $b)$. At last we will provide a generalization of the theorem for sequences and final remarks.

A.1 Some preliminaries for the proof

We will need the following preliminary results.

Lemma A.1.1. *Suppose $P = (p_j)_{j=1}^{\infty}$ is a sequence of distinct real numbers $p_j \geq 1$ satisfying* (A.1) *and having at most 1 as a finite accumulation point. If $\varphi \in \mathcal{L}^1[0,1]$ satisfies*

$$\int_0^1 \varphi(x) x^{p-1}\, \mathrm{d}x = 0 \quad \text{for all } \; p \in P, \tag{A.2}$$

then $\varphi(x) = 0$ for almost every $x \in [0,1]$.

Proof. By the assumptions given on P, one of the following two cases occurs:

($b1$) P has a strictly increasing subsequence $(p_{j_k})_k$ such that $p_{j_k} \to +\infty$ and $\sum_{k=1}^{\infty} \frac{1}{p_{j_k}} = +\infty$;

($b2$) P has a strictly decreasing subsequence $(p_{j_k})_k$ such that $p_{j_k} \to 1$ and $\sum_{k=1}^{\infty} (p_{j_k} - 1) = +\infty$.

With no loss of generality, we may replace P by the subsequence p_{j_k}, in either case.

Suppose that ($b1$) holds; then by (A.2) we have

$$\int_0^1 \varphi(x) x^{p_1 - 1} x^{p - p_1}\, \mathrm{d}x = 0\,.$$

By the "Full Müntz Theorem in $\mathcal{C}[0,1]$" given in [17, Theorem 2.1] the linear span of $\{x^{p-p_1}, p \in P\}$ is a dense subset in $\mathcal{C}^0[0,1]$ because in this case

$$\sum_{p \in P} \frac{p - p_1}{(p - p_1)^2 + 1} = +\infty\,,$$

by (A.1). Hence, for any $h \in \mathcal{C}^0[0,1]$ we have:

$$\int_0^1 \varphi(x)x^{p_1-1}h(x)\,\mathrm{d}x = 0\,.$$

Extend $\varphi \equiv 0$ outside $[0,1]$. Let χ_ε be a family of mollifying kernels on $\mathbb{R}$ (as shown for example in the article [54] or [104, Chapter 9.2]). Then picking $h(x) = \chi_\varepsilon(x-y)$ and setting $k(x) = \varphi(x)x^{p_1-1}$ we obtain:

$$\int_0^1 k(x)h(x)\,\mathrm{d}x = \int_{\mathbb{R}} k(x)\chi_\varepsilon(x-y)\,\mathrm{d}x = (k * \chi_\varepsilon)(y) = 0 \qquad \text{for every } y \in \mathbb{R}.$$

Now, by [104, Chapter 9, Theorem 9.6],

$$k * \chi_\varepsilon \to k \qquad \text{in } \mathcal{L}^1[0,1].$$

Consequently, (A.2) holds if and only if $\varphi = 0$ almost everywhere.

If case $(b2)$ holds then we can define a sequence of functions $(f_j)_j$ by

$$f_j(x) = \varphi(x)x^{p_j-1}\,.$$

The sequence $(f_j)_j$ converges to φ pointwise. Moreover, choosing $\sigma(x) = |\varphi(x)|$, we obtain that $\sigma \in \mathcal{L}^1$ and

$$|f_j(x)| \leq \sigma(x) \qquad \text{for all } x \in [0,1] \text{ and all } j \in \mathbb{N}\,, j \geq 1.$$

By the Dominated Convergence Theorem we obtain that

$$\int_0^1 \varphi(x)\,\mathrm{d}x = \lim_{j\to\infty}\int_0^1 \varphi(x)x^{p_j-1}\,\mathrm{d}x = 0\,.$$

Consequently we can suppose, without losing in generality, that 1 belongs to P and the proof proceeds as before applying the "Full Müntz Theorem in $\mathcal{C}[0,1]$" [17, Theorem 2.1]. $\square$

Remark A.1.2. *If $f \in \mathcal{L}^p(E)$ for all $p \geq 1$, let $\mu_f : (0,+\infty) \to \mathbb{R}$ be defined by*

$$\mu_f(t) = \mu\left(|f| > t\right).$$

It is well known that μ_f is a monotone nonincreasing function, continuous from the right, and that

$$\int_E |f|^p \mathrm{d}\mu = p\int_0^{+\infty} \mu_f(t)t^{p-1}\mathrm{d}t\,,$$

for every $p \in [1,+\infty)$, cf. [104, Theorem 5.51]. Note in particular that $\mu_f \in \mathcal{L}^1(0,+\infty)$.

To deal with the first part of Theorem A.0.1, namely when case $a)$ holds, we have to recall some tools in complex analysis. The Mellin transform of a function $\upsilon(t)$ is defined as

$$\{\mathcal{M}\upsilon\}(z) = F(z) = \int_0^\infty \upsilon(t)t^{z-1}\mathrm{d}t\,, \quad z \in \mathbb{C}\,,$$

whenever the integral exists for at least one value z_0 of z (cf. [100, 105, 107]).

Lemma A.1.3. *Let $\upsilon : [0, +\infty) \to \mathbb{R}$ be a function such that*

$$\upsilon(t)\, t^{z-1} \in \mathcal{L}^1([0, +\infty)) \qquad \text{for all } z \geq 1.$$

Then $\mathcal{M}\upsilon$ is analytic in $S = \{w \in \mathbb{C} : \Re(w) > 1\}$.

Proof. Let $\gamma : [0, 1] \to \mathbb{C}$ be a triangle in S and let $\Phi(s, z) = \upsilon(s)s^{z-1}$. Then,

$$\int_\gamma F = \int_\gamma \left(\int_0^{+\infty} \Phi(s, z)\,\mathrm{d}s\right)\mathrm{d}z = \int_0^1 \left(\int_0^{+\infty} \Phi(s, \gamma(t))\gamma'(t)\,\mathrm{d}s\right)\mathrm{d}t\,,$$

with $\gamma'(t)$ defined for all but three points $t \in [0, 1]$. Observe that

$$\begin{aligned}\int_\gamma |F| &= \int_0^1 \left(\int_0^{+\infty} |\Phi(s, \gamma(t))\gamma'(t)|\,\mathrm{d}s\right)\mathrm{d}t \\ &\leq \int_0^1 \left(\int_0^{+\infty} |\upsilon(s)||s^{\gamma(t)-1}|\,|\gamma'(t)|\,\mathrm{d}s\right)\mathrm{d}t\,.\end{aligned}$$

Being γ a triangle, $|\gamma'(t)|$ is constant on every side and then there exists C_1 such that $|\gamma'(t)| < C_1$ for all $t \in [0, 1]$ where the tangent vector is defined. Let $R > 0$ be such that $\mathrm{Supp}(\gamma) \subseteq B(0, R)$. Then,

$$|s^{\gamma(t)-1}| = s^{\Re[\gamma(t)-1]} \leq s^{R+1},$$

and so

$$\int_0^1 \left(\int_0^{+\infty} |\upsilon(s)||s^{\gamma(t)-1}|\,|\gamma'(t)|\,\mathrm{d}s\right)\mathrm{d}t \leq C_1 \int_0^1 \left(\int_0^{+\infty} |\upsilon(s)s^{R+1}|\,\mathrm{d}s\right)\mathrm{d}t\,.$$

By hypothesis, $\upsilon(s)s^p$ is Lebesgue integrable for all $p \geq 0$, so

$$C_1 \int_0^1 \left(\int_0^{+\infty} |\upsilon(s)s^{R+1}|\,\mathrm{d}s\right)\mathrm{d}t \leq C_1 C_R < +\infty\,.$$

Then, by Fubini-Tonelli Theorem,

$$\begin{aligned}\int_\gamma F &= \int_\gamma \left(\int_0^{+\infty} \Phi(s,z)\,\mathrm{d}s\right)\mathrm{d}z \\ &= \int_0^{+\infty}\left(\int_\gamma \Phi(s,z)\,\mathrm{d}z\right)\mathrm{d}s = \int_0^{+\infty}\left(\int_\gamma \upsilon(s)s^{z-1}\,\mathrm{d}z\right)\mathrm{d}s\,.\end{aligned}$$

But now $\upsilon(s)s^{z-1}$ is a holomorphic function of z, and then by the Cauchy integral theorem

$$\int_\gamma \upsilon(s)s^{z-1}\,\mathrm{d}z = 0\,,$$

and then

$$\int_\gamma F = 0\,,$$

for every triangular path. Consequently, by Morera's theorem for triangles (see for example [26]), $F(s)$ is holomorphic on $\{w \in \mathbb{C} : \Re(w) > 1\}$. □

A.2 Proof of Theorem A.0.1

Being f and g in $\mathcal{L}^p(E)$ for every $p \in [1,+\infty)$, the functions

$$t \to \mu\left(|f| > t^{\frac{1}{p}}\right) \qquad \text{and} \qquad t \to \mu\left(|g| > t^{\frac{1}{p}}\right)$$

are finite almost everywhere, the function

$$\mathcal{I}(p) = \int_E [\,|f|^p - |g|^p\,]\,\mathrm{d}\mu$$

is well defined and finite for every $p \in [1,+\infty)$, and

$$\begin{aligned}\mathcal{I}(p) &= \int_E \big(|f|^p - |g|^p\big)\mathrm{d}\mu = \\ &= \int_0^{+\infty} \big[\mu(|f|^p > t) - \mu(|g|^p > t)\big]\mathrm{d}t \\ &= \int_0^{+\infty} \left[\mu\left(|f| > t^{\frac{1}{p}}\right) - \mu\left(|g| > t^{\frac{1}{p}}\right)\right]\mathrm{d}t\,.\end{aligned}$$

Substituting $z = t^{\frac{1}{p}}$, the integral becomes

$$\mathcal{I}(p) = p\int_0^{+\infty} \big[\mu(|f| > z) - \mu(|g| > z)\big]z^{p-1}\,\mathrm{d}z = p\int_0^{+\infty} \varphi(z)z^{p-1}\,\mathrm{d}z\,, \tag{A.3}$$

where
$$\varphi(z) = \mu(|f| > z) - \mu(|g| > z)\,.$$
Notice that $\varphi : [0, +\infty) \to \mathbb{R}$ is continuous from the right and that
$$\|f\|_p = \|g\|_p \;\Leftrightarrow\; \mathcal{I}(p) = 0\,.$$
Suppose $a)$ holds. Being $\varphi : [0, +\infty) \to \mathbb{R}$ the difference of two monotone functions, it is differentiable almost everywhere, and hence continuous almost everywhere. Moreover, it is of bounded variation on $[a, +\infty)$ for all $a > 0$ and then of bounded variation in a neighborhood of each $y \in (0, +\infty)$. Notice that the integral in the right-hand side of (A.3) is the Mellin transform of φ, hence
$$\mathcal{I}(p) = 0 \;\Leftrightarrow\; \{\mathcal{M}\varphi\}(p) = 0\,.$$
By [105, Chapter 6.9, Theorem 28], for every $c \in (1, +\infty)$,
$$\frac{1}{2}\left[\varphi(x+0) + \varphi(x-0)\right] = \frac{1}{2\pi i}\lim_{T\to+\infty}\int_{c-iT}^{c+iT} \{\mathcal{M}\varphi\}(p) x^{-p}\,\mathrm{d}p\,, \tag{A.4}$$
where
$$\varphi(x+0) = \lim_{t\to x^+}\varphi(t) \qquad \text{and} \qquad \varphi(x-0) = \lim_{t\to x^-}\varphi(t)\,.$$
By Lemma A.1.3, $\mathcal{M}\varphi$ is holomorphic on $\{w \in \mathbb{C} : \Re(w) > 1\}$. But
$$\{\mathcal{M}\varphi\}(p) = 0 \qquad \text{for all } p \in P\,,$$
and, by $a)$, P has an accumulation point in $\{w \in \mathbb{C} : \Re(w) > 1\}$. Then, by the identity theorem of complex analytic functions,
$$\mathcal{M}\varphi \equiv 0 \qquad \text{on } \{w \in \mathbb{C} : \Re(w) > 1\}.$$
The inversion formula (A.4) then becomes
$$\frac{1}{2}\left[\varphi(x+0) + \varphi(x-0)\right] = \frac{1}{2\pi i}\lim_{T\to+\infty}\int_{c-iT}^{c+iT} 0\cdot\, x^{-p}\,\mathrm{d}p = 0\,.$$
Being $\varphi(x)$ continuous almost everywhere, we have that $\varphi(x) = 0$ for almost every x. The conclusion easily follows.

Assume now that $b)$ holds and that P has no accumulation points in $(1, +\infty)$. If $\|f\|_\infty = 0$ or $\|g\|_\infty = 0$, then either $i)$ or $ii)$ imply that $f = g = 0$ almost everywhere, and the result is achieved. Without loss of generality, we can suppose $\|f\|_\infty \le \|g\|_\infty = 1$. Indeed,
$$\begin{aligned}
\|f\|_p = \|g\|_p &\Leftrightarrow \|f\|_p^p = \|g\|_p^p \\
&\Leftrightarrow \int_E \|g\|_\infty^p \left(\frac{|f|}{\|g\|_\infty}\right)^p \mathrm{d}\mu = \int_E \|g\|_\infty^p \left(\frac{|g|}{\|g\|_\infty}\right)^p \mathrm{d}\mu \\
&\Leftrightarrow \int_E \left(\frac{|f|}{\|g\|_\infty}\right)^p \mathrm{d}\mu = \int_E \left(\frac{|g|}{\|g\|_\infty}\right)^p \mathrm{d}\mu\,.
\end{aligned}$$

In this case,

$$I(p) = \int_0^1 \varphi(z) z^{p-1}\,\mathrm{d}z\,,$$

and $\varphi \in \mathcal{L}^1[0,1]$. By Lemma A.1.1 we have that $\mathcal{I}(p) = 0$ for all $p \in P$ if and only if $\varphi(z) = 0$ for almost every $z \in [0,1]$. By the the right-continuity, we conclude. □

A.3 Construction of the counterexample

In this section we want to show that, in general, the boundedness hypothesis on f and g in Theorem A.0.1, when $b)$ holds, cannot be removed. In the first part we give some definitions to set the problem in a more general frame, then we develop the counterexample. Precisely, we will firstly build a continuous function φ defined on the positive real semiaxis and orthogonal to every monomial (and for linearity to every polynomial). Then, we will prove that this function is continuous and it is of bounded variation on $[0,+\infty)$. So, it can be written as the difference of two strictly decreasing functions; their inverses are the functions we are looking for. To conclude we show, as corollaries of independent interest, that modifying a bit this function φ firstly we can make it smooth, and secondly it could be orthogonal to every rational power of x, with fixed denominator. For an in-depth analysis of this argument see e.g. [97, 99].

Lemma A.3.1. *The function $\varphi : [0,+\infty) \to \mathbb{R}$ defined as*

$$\varphi(x) = e^{-\sqrt[4]{x}} \sin\left(\sqrt[4]{x}\right)$$

is such that

$$\int_0^{+\infty} x^n \varphi(x)\,\mathrm{d}x = 0 \qquad \text{for all } n \in \mathbb{N}\,.$$

Proof. This result was already known by Stieltjes that gave it in his work on continuous fractions [98, page 105]. For the convenience of the reader we sketch a self-contained proof relying only on elementary complex analysis.
Set

$$I_n = \int_0^{+\infty} x^n e^{-(1-i)x}\mathrm{d}x\,.$$

Being $|x^n e^{-(1-i)x}| = x^n e^{-x}$, we have that the integral I_n is well defined for all n. Moreover, letting $z = 1-i$, performing the change of variables $zx = y$ we obtain:

$$I_n = \int_0^{+\infty} x^n e^{-(1-i)x}\,\mathrm{d}x = \int_0^{+\infty} x^n e^{-zx}\,\mathrm{d}x = z^{-n-1}\int_\gamma y^n e^{-y}\,\mathrm{d}y\,,$$

where γ is the half-line starting at the origin, containing the point $1-i$. Consider the triangular path T_N in the complex plane joining the points 0, N, $N-iN$. Being $y^n e^{-y}$ analytic over the interior of T_N the integral along T_N is 0. Moreover:

$$\left|\int_N^{N-iN} y^n e^{-y}\,\mathrm{d}y\right| = \left|\int_0^N (N-it)^n e^{-N+it}\,\mathrm{d}t\right| \le \int_0^N |N^2+t^2|^{\frac{n}{2}} e^{-N}\,\mathrm{d}t \to 0\,,$$

and so

$$\int_0^N y^n e^{-y}\,\mathrm{d}y + \int_N^{N-iN} y^n e^{-y}\,\mathrm{d}y + \int_{N-iN}^0 y^n e^{-y}\,\mathrm{d}y = 0.$$

Then, passing to the limit for N tending to $+\infty$, the first term tends to $\Gamma(n+1)$, the second term tends to 0, and the third tends to $-z^{n+1}I_n$ hence:

$$I_n = z^{-n-1}\Gamma(n+1) = z^{-n-1}n!\,.$$

Then,

$$\begin{aligned} I_n &= n!\cdot(1-i)^{-n-1} = n!\cdot(1+i)^{n+1}\cdot 2^{-n-1} = \\ &= n!\cdot\left[\frac{(1+i)}{\sqrt{2}}\right]^{n+1}\cdot 2^{-n-1}\cdot 2^{\frac{n+1}{2}} = n!\cdot e^{\frac{(n+1)i\pi}{4}}\cdot 2^{-\frac{n+1}{2}}\,. \end{aligned}$$

So,

$$I_{4p+3}\in\mathbb{R}\quad\text{for all } p\in\mathbb{N}\,,$$

and then

$$\Im\left(I_{4p+3}\right) = 0\quad\text{for all } p\in\mathbb{N}\,,$$

so that

$$0 = \Im(I_{4p+3}) = \int_0^{+\infty} x^{4p+3}e^{-x}\Im(e^{ix})\,\mathrm{d}x = \int_0^{+\infty} x^{4p+3}e^{-x}\sin(x)\,\mathrm{d}x\quad\text{for all } p\in\mathbb{N}\,.$$

Letting $x = u^{\frac{1}{4}}$, we arrive to

$$\int_0^{+\infty} u^p e^{-\sqrt[4]{u}}\sin(\sqrt[4]{u})\,\mathrm{d}u = 0\quad\text{for all } p\in\mathbb{N}\,.$$

The function

$$\varphi(x) = e^{-\sqrt[4]{x}}\sin(\sqrt[4]{x})$$

has the requested properties. □

Lemma A.3.2. *The function φ defined in Lemma A.3.1 belongs to $BV([0,+\infty))$.*

Proof. Observe preliminarily that $\varphi(0) = 0$ and φ tends to 0 at infinity; moreover,

$$\varphi'(x) = \frac{e^{-\sqrt[4]{x}}\cos\left(\sqrt[4]{x}\right)}{4x^{3/4}} - \frac{e^{-\sqrt[4]{x}}\sin\left(\sqrt[4]{x}\right)}{4x^{3/4}} = \frac{\sqrt{2}e^{-\sqrt[4]{x}}}{4x^{3/4}}\sin\left(\frac{\pi}{4} - \sqrt[4]{x}\right),$$

and so

$$\varphi'(x) = 0 \;\Leftrightarrow\; \frac{\sqrt{2}e^{-\sqrt[4]{x}}}{4x^{3/4}}\sin\left(\frac{\pi}{4} - \sqrt[4]{x}\right) = 0 \;\Leftrightarrow\; \sqrt[4]{x} = \frac{\pi}{4} + k\pi \quad \text{for } k \in \mathbb{N}.$$

The second derivative of φ is given by

$$\varphi''(x) = \frac{e^{-\sqrt[4]{x}}\left(3\sin\left(\sqrt[4]{x}\right) - (2\sqrt[4]{x} + 3)\cos\left(\sqrt[4]{x}\right)\right)}{16\,x^{7/4}}.$$

Letting

$$x_n = \left(\frac{\pi}{4} + n\pi\right)^4,$$

we see that $(\varphi''(x_n))_n$ has alternating signs, since

$$\varphi''(x_n) = (-1)^{n+1}\frac{256\sqrt{2}e^{\pi\left(-\left(n+\frac{1}{4}\right)\right)}}{(4\pi n + \pi)^6}.$$

So, the total variation of φ is the series of variations between each stationary point plus the variation between 0 and the first stationary point. Writing $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$, we have

$$\begin{aligned}
V_{\mathbb{R}^+}(\varphi) &= \frac{e^{-\frac{\pi}{4}}}{\sqrt{2}} + \sum_{n\geq 0}|\varphi(x_{n+1}) - \varphi(x_n)| \\
&= \frac{e^{-\frac{\pi}{4}}}{\sqrt{2}} + \sum_{n\in 2\mathbb{N}}|\varphi(x_{n+1}) - \varphi(x_n)| + \sum_{n\in 2\mathbb{N}+1}|\varphi(x_{n+1}) - \varphi(x_n)| \\
&= \frac{e^{-\frac{\pi}{4}}}{\sqrt{2}} + \sum_{n\in\mathbb{N}}|\varphi(x_{2n+1}) - \varphi(x_{2n})| + \sum_{n\in\mathbb{N}}|\varphi(x_{2n+2}) - \varphi(x_{2n+1})| \\
&= \frac{e^{-\frac{\pi}{4}}}{\sqrt{2}} + \sum_{n\in\mathbb{N}}\left| -\frac{e^{-\frac{1}{4}\pi(8n+5)}}{\sqrt{2}} - \frac{e^{-\frac{1}{4}\pi(8n+1)}}{\sqrt{2}}\right| + \sum_{n\in\mathbb{N}}\left|\frac{e^{-\frac{1}{4}\pi(8n+9)}}{\sqrt{2}} + \frac{e^{-\frac{1}{4}\pi(8n+5)}}{\sqrt{2}}\right| \\
&= \frac{e^{-\frac{\pi}{4}}}{\sqrt{2}}\left[\sum_{n=0}^{\infty}\left(e^{-2\pi n} + \frac{2e^{-2\pi n}}{e^{\pi}} + \frac{e^{-2\pi n}}{e^{2\pi}}\right) + 1\right] \\
&= \frac{e^{-\frac{\pi}{4}}}{\sqrt{2}}\left[\left(1 + \frac{2}{e^{\pi}} + \frac{1}{e^{2\pi}}\right)\sum_{n=0}^{\infty}\left(e^{-2\pi n}\right) + 1\right] \\
&= \frac{e^{-\frac{\pi}{4}}}{\sqrt{2}} \cdot \left[\frac{(1 + e^{\pi})^2}{e^{2\pi}} \cdot \frac{1}{1 - e^{-2\pi}} + 1\right] = \sqrt{2} \cdot \frac{e^{\frac{3}{4}\pi}}{e^{\pi} - 1}
\end{aligned}$$

We are now ready to construct the counterexample. Define

$$\phi(t) = P(t,+\infty) + \frac{1}{(t+1)^2}\,, \qquad \psi(t) = N(t,+\infty) + \frac{1}{(t+1)^2}\,,$$

where $P(t,+\infty)$ and $N(t,+\infty)$ are, respectively, the positive and the negative variation of φ on $(t,+\infty)$. The functions ϕ and ψ are positive, strictly decreasing, bounded, and achieve their maximum in 0. Moreover,

$$\lim_{t\to+\infty}\phi(t) = \lim_{t\to+\infty}\psi(t) = 0\,,$$

and

$$\phi(t) - \psi(t) = \varphi(t)\,.$$

Restricting the codomain of ϕ to $(0,\phi(0)]$ and that of ψ to $(0,\psi(0)]$, we obtain two invertible functions

$$\hat{\phi} : [0,+\infty) \to (0,\phi(0)]\,, \qquad \hat{\psi} : [0,+\infty) \to (0,\psi(0)]\,.$$

Moreover, their inverses are also non negative decreasing functions. Define

$$f = \hat{\phi}^{-1} : (0,\phi(0)] \to [0,+\infty)\,, \qquad g = \hat{\psi}^{-1} : (0,\psi(0)] \to [0,+\infty)\,,$$

and notice that

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} g(x) = +\infty\,.$$

Extend f and g to all $\mathbb{R}$ by setting them equal to 0 outside their domain, and call them $\tilde{f}$ and $\tilde{g}$. These are the functions we are looking for. Indeed, we will now prove that $\mu(|\tilde{f}| > \alpha)$ does not coincide with $\mu(|\tilde{g}| > \alpha)$ for a.e. $\alpha \geq 0$. By contradiction suppose that

$$\mu(|\tilde{f}| > \alpha) = \mu(|\tilde{g}| > \alpha) \qquad \text{for a.e. } \alpha \geq 0\,.$$

Being $\tilde{f}$ and $\tilde{g}$ non-negative and being their level sets coincident with those of f and g, we have

$$\mu(f > \alpha) = \mu(g > \alpha) \qquad \text{for a.e. } \alpha \geq 0\,.$$

But f and g are monotonically strictly decreasing, so $\{f > \alpha\} = [0, f^{-1}(\alpha))$ and $\{g > \alpha\} = [0, g^{-1}(\alpha))$, hence

$$f^{-1}(\alpha) = g^{-1}(\alpha) \qquad \text{for a.e. } \alpha \geq 0\,.$$

Being $f^{-1} = \phi$ and $g^{-1} = \psi$,

$$\phi(\alpha) = \psi(\alpha) \qquad \text{for a.e. } \alpha \geq 0\,.$$

Recall then the definition of ϕ and ψ to obtain

$$P(\alpha,+\infty)+\frac{1}{(\alpha+1)^2}=N(\alpha,+\infty)+\frac{1}{(\alpha+1)^2}\qquad \text{for a.e. } \alpha\geq 0\,,$$

so

$$P(\alpha,+\infty)=N(\alpha,+\infty)\qquad \text{for a.e. } \alpha\geq 0\,,$$

and then

$$\varphi(\alpha)=P(\alpha,+\infty)-N(\alpha,+\infty)=0\qquad \text{for a.e. } \alpha\geq 0\,,$$

finding a contradiction. The proof is then completed. □

In the following corollary, we want to extend Lemma A.3.1 to find a continuous function orthogonal to every fractional power of x with fixed denominator.

Corollary A.3.3. *Fix $q\in\mathbb{N}\setminus\{0\}$. There exists a continuous function $\varphi_q:(0,+\infty)\to\mathbb{R}$, not identically equal to 0 , such that*

$$\int_0^{+\infty} x^{\frac{n}{q}}\varphi_q(x)\,\mathrm{d}x=0\qquad \text{for all } n\in\mathbb{N}\,.$$

Proof. Define I_n as before. We have

$$\int_0^{+\infty} x^{4p+3}e^{-x}\sin(x)\,\mathrm{d}x=0\quad \text{for all } p\in\mathbb{N}\,.$$

Letting $x=u^{\frac{1}{4q}}$ we arrive to

$$\int_0^{+\infty} u^{\frac{p}{q}}e^{-\sqrt[4q]{u}}\sin(\sqrt[4q]{u})\,u^{\frac{1-q}{q}}\,\mathrm{d}u=0\quad \text{for all } p\in\mathbb{N}\,.$$

The function

$$\varphi(x)=e^{-\sqrt[4q]{x}}\sin(\sqrt[4q]{x})\,x^{\frac{1-q}{q}}$$

is the one we were looking for. □

The aim of the subsequent lemma is to show that, if we multiply the functions $\varphi(x)$ and $\varphi_q(x)$, found respectively in Lemma A.3.1 and Corollary A.3.3, by a suitable power of x, we obtain two new functions that maintain the same property of orthogonality but are arbitrarily regular. We achieve this result applying Faà di Bruno's formula.

Lemma A.3.4. *Let $w\in\mathcal{C}^\infty(\mathbb{R})$ and $0<\alpha<1$. Then the function $g_n:[0,+\infty)\to\mathbb{R}$,*

$$g_n(x)=x^n w(x^\alpha)\,,$$

is of class $\mathcal{C}^n$ on $[0,+\infty)$, with $g_n^{(j)}(0)=0$ for all $j=0,1,2,\dots,n$.

Proof. A central tool of this proof will be Faà di Bruno's formula that we will recall briefly. Let w and u be $\mathcal{C}^m$ real valued functions such that the composition $w \circ u$ is defined; then $(w \circ u)(x)$ is of class $\mathcal{C}^m$ and for $x > 0$ we have

$$(w \circ u)^{(j)}(x) = j! \sum_{k=1}^{j} \left[\frac{w^{(k)}(u(x))}{k!} \sum_{h_1+\cdots+h_k=j} \frac{u^{(h_1)}(x)}{h_1!} \cdots \frac{u^{(h_k)}(x)}{h_k!} \right],$$

or, setting $(k_1, \ldots, k_j) = K$ e $(1, \ldots, j) = J$,

$$(w \circ u)^{(j)}(x) = \sum_{K \cdot J = j} \frac{j!}{k_1! \cdots k_j!} \, w^{(k_1+\cdots+k_j)}(u(x)) \cdot \prod_{n=1}^{j} \left(\frac{u^{(n)}(x)}{n!} \right)^{k_n}$$

For a proof of this formula look at [92]. We have that

$$g_n^{(j)}(x) = \sum_{h=0}^{j} \binom{j}{h} n(n-1) \cdots (n-h+1) x^{n-h} \left[w(x^\alpha) \right]^{(n-h)},$$

and each term of this sum is of the form

$$C \, x^{n-h} [w(x^\alpha)]^{(n-h)}, \tag{A.5}$$

where C is a real number depending on j, h and n. Now we use the Faà di Bruno's formula to express the derivatives of w. In our case $u(x) = x^\alpha$ and so

$$u^{(h)}(x) = (\alpha)_h \, x^{\alpha-h} \qquad \text{where} \qquad (\alpha)_h = \alpha(\alpha-1) \cdots (\alpha-h+1).$$

Consequently

$$u^{(h_1)}(x) \cdots u^{(h_k)}(x) = (\alpha)_{h_1} (\alpha)_{h_2} \cdots (\alpha)_{h_k} x^{\alpha-h_1} x^{\alpha-h_2} \cdots x^{\alpha-h_k},$$

and if $h_1 + \cdots + h_k = j$,

$$u^{(h_1)}(x) \cdots u^{(h_k)}(x) = C(h_1, \ldots h_k) x^{k\alpha-j}.$$

So, applying Faà di Bruno's formula to (A.5), each term has the form

$$x^{n-h} \sum_{k=1}^{n-h} C_k w^{(k)}(x^\alpha) \cdot x^{k\alpha-(n-h)} = \sum_{k=1}^{n-h} C_k w^{(k)}(x^\alpha) \cdot x^{k\alpha}.$$

To conclude observe that

$$g_n^{(j)}(x) = \sum_{k=1}^{j} C'_k w^{(k)}(x^\alpha) \, x^{k\alpha},$$

and apply the theorem on the limit of the derivative. □

As a consequence of Lemma A.3.4, we have that the function $\varphi : [0, +\infty) \to \mathbb{R}$ in the statement of Lemma A.3.1 can be chosen to be arbitrarily regular (but not $\mathcal{C}^\infty$). For example, taking

$$\varphi(x) = e^{-\sqrt[4]{x}} \sin(\sqrt[4]{x}) x^n, \qquad n \in \mathbb{N},$$

by Lemma A.3.4 choosing $w(x) = e^{-x} \sin(x)$ and $\alpha = \frac{1}{4}$, we see that $\varphi(x)$ is of class $\mathcal{C}^n$. The same reasoning choosing the same w and $\alpha = \frac{1}{4q}$ allows to conclude that also the function φ_q is of class $\mathcal{C}^n$ if multiplied by x^{n+1}.

A.4 Final remarks

As a direct consequence of Theorem A.0.1, we have the following.

Corollary A.4.1. *Suppose $\mu(E) < +\infty$, $f \in \mathcal{L}^p(E)$ for all $p \geq 1$ and $P = (p_j)_{j=1}^\infty$ is a sequence of distinct real numbers $p_j \geq 1$. Let C be a non negative constant, and suppose that either $a)$ or $b)$ holds. Then, the following two conditions are equivalent:*

$i)$ $\left(\dfrac{1}{\mu(E)}\right)^{\frac{1}{p}} \|f\|_p = C$ *for all* $p \in P$;

$ii)$ $|f(x)| = C$ *for a.e.* $x \in E$.

Indeed, Theorem A.0.1 applies taking as g a constant function.

Theorem A.0.1 remains valid assuming the existence of an accumulation point of P in $(0, 1]$ and supposing $f, g \in \mathcal{L}^p(E)$ for all $p > 0$. In this case, $\|f\|_p$ is formally defined as before, although this is not a norm anymore. To prove this, first notice that, without loss of generality, we can always assume that $f, g \geq 0$. Let $\delta \in (0, 1]$ be an accumulation point of P. For all $p \in P \cap \left(\frac{\delta}{2}, 2\right)$,

$$\begin{aligned} \int_E |f|^p \mathrm{d}\mu = \int_E |g|^p \mathrm{d}\mu \quad \Leftrightarrow \quad & \left(\int_E |f^{\frac{\delta}{2}}|^{\frac{2p}{\delta}} \mathrm{d}\mu\right)^{\frac{\delta}{2p}} \\ = & \left(\int_E |g^{\frac{\delta}{2}}|^{\frac{2p}{\delta}} \mathrm{d}\mu\right)^{\frac{\delta}{2p}} \quad \Leftrightarrow \quad \|f^{\frac{\delta}{2}}\|_{\frac{2p}{\delta}} = \|g^{\frac{\delta}{2}}\|_{\frac{2p}{\delta}} . \end{aligned}$$

The set $\widetilde{P} = \{\frac{2p}{\delta} : p \in P \cap \left(\frac{\delta}{2}, 2\right)\}$ is contained in $(1, +\infty)$ and has an accumulation point there. We can now apply the second part of Theorem A.0.1 to find that

$$\mu(|f^{\frac{\delta}{2}}| > \alpha) = \mu(|g^{\frac{\delta}{2}}| > \alpha) \qquad \text{for all } \alpha \geq 0,$$

and so $\mu(|f| > \alpha) = \mu(|g| > \alpha)$ for all $\alpha \geq 0$.

In the last part of this section we consider the special case of ℓ^p spaces and show that Theorem A.0.1 can be generalized in this setting. We recall that, for a sequence $A = (a_n)_n$, the ℓ^p norms are defined as follows:

$$\|A\|_p = \left(\sum_{n=0}^{\infty} |a_n|^p \right)^{\frac{1}{p}}, \qquad \|A\|_\infty = \sup_n |a_n| .$$

Here is our result.

Theorem A.4.2. *Let $A = (a_n)_n$ and $B = (b_n)_n$ be two sequences of real numbers in ℓ^1. If $P = (p_j)_{j=1}^{\infty}$ is a sequence of distinct real numbers $p_j \geq 1$ having an accumulation point in $(1, +\infty]$ and*

$$\|A\|_p = \|B\|_p \quad \text{for all } p \in P ,$$

then the sequences

$$|A| = (|a_n|)_n \qquad \text{and} \qquad |B| = (|b_n|)_n$$

can be obtained one from the other by permutation, appending or removing some zeroes.

Proof. Suppose that the accumulation point is finite. Choose $X = \mathbb{N}$, $\mathcal{A} = \mathcal{P}(\mathbb{N})$ and μ the counting measure. By Theorem A.0.1, we have that

$$\#\left(|A| > \alpha\right) = \#\left(|B| > \alpha\right) \qquad \text{for all } \alpha \geq 0 .$$

Without loss of generality we can suppose that $a_n, b_n > 0$ for all $n \in \mathbb{N}$. Being $(a_n)_n$ and $(b_n)_n$ absolutely convergent, we can rearrange them in such a way that A and B are non increasing without modifying the ℓ^p norms, thus obtaining $\hat{A} = (\hat{a}_n)_n$ and $\hat{B} = (\hat{b}_n)_n$, respectively. Clearly

$$\#\left(A > \alpha\right) = \#\left(\hat{A} > \alpha\right) \quad \text{and} \quad \#\left(B > \alpha\right) = \#\left(\hat{B} > \alpha\right) .$$

If $\hat{a}_n = \hat{b}_n$ for all n, then the theorem is proved. Assume by contradiction that $\hat{A} \neq \hat{B}$ and let $\bar{n}$ be the smallest index such that $\hat{a}_{\bar{n}} \neq \hat{b}_{\bar{n}}$. Suppose for instance that $\hat{a}_{\bar{n}} > \hat{b}_{\bar{n}}$ and choose

$$\alpha = \frac{\hat{a}_{\bar{n}} + \hat{b}_{\bar{n}}}{2} .$$

With this choice we have

$$\#\left(\hat{A} > \alpha\right) \geq \bar{n} > \#\left(\hat{B} > \alpha\right) ,$$

a contradiction. Suppose now that $+\infty$ is the only accumulation point of P. We have that

$$\lim_j \|\hat{A}\|_{p_j} = \|\hat{A}\|_\infty = \hat{a}_0 ,$$

and similarly for $\hat{B}$, obtaining that $\hat{a}_0 = \hat{b}_0$. Define $\hat{A}_1 = (\hat{a}_n)_{n\geq 1}$ and $\hat{B}_1 = (\hat{b}_n)_{n\geq 1}$ and notice that:

$$\|\hat{A}_1\|_{p_j}^{p_j} = \sum_{n=0}^{\infty} \hat{a}_n^{p_j} - \hat{a}_0^{p_j} = \|\hat{A}\|_{p_j}^{p_j} - \hat{a}_0^{p_j} = \|\hat{B}\|_{p_j}^{p_j} - \hat{b}_0^{p_j} = \|\hat{B}_1\|_{p_j}^{p_j} ,$$

and consequently $\|\hat{A}_1\|_{p_j} = \|\hat{B}_1\|_{p_j}$ for all j. By the same reasoning we will obtain that

$$\lim_j \|\hat{A}_1\|_{p_j} = \|\hat{A}_1\|_\infty = \hat{a}_1 ,$$

and the same for $\hat{B}_1$, to obtain $\hat{a}_1 = \hat{b}_1$. Proceeding by induction, we conclude that $\hat{A} = \hat{B}$. $\square$

Notice that, choosing for example $p_j = j!$ for all $j \geq 1$, the condition $\|\hat{A}\|_{p_j} = \|\hat{B}\|_{p_j}$ for all j implies that $\hat{A} = \hat{B}$, but in this case the series in (A.1) converges. Notice moreover that Theorem A.4.2 remains valid if 1 is the only accumulation point of P, supposing that (A.1) holds.

The problem of the necessity of condition (A.1) has been solved in 2020 by Erdély (see [38]) in the following sense:

Theorem A.4.3. *There is a finite Borel measure μ on $E := [0,1]$ with $0 < \mu(E) < +\infty$ and there are two functions $f, g \in \mathcal{L}(E)$ such that $\|f\|_p = \|g\|_p$ for all $p \in P$ but*

$$\mu(\{x \in E : |f(x)| < \alpha\}) \neq \mu(\{x \in E : |g(x)| < \alpha\})$$

for at least one value of $\alpha \geq 0$.

We want to conclude this chapter showing we can find two real functions having coincident norms for an arbitrary finite set of p values in $[1, +\infty)$. Let us start by proving the following theorem.

Theorem A.4.4. *For all $n > 0$, given n distinct, positive real numbers $p_1, \ldots, p_n$ greater or equal to 1 there exist a polynomial $\eta(x)$ such that*

$$\int_0^1 \eta(x)\, x^{p_i}\, \mathrm{d}x = 0 \qquad \textit{for all } i = 1 \ldots n.$$

Proof. Define, for every $c \in (-1, 0)$ and $x \in [0,1]$,

$$\eta(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(s-p_1)(s-p_2)\cdots(s-p_n)}{(s+1)(s+2)\cdots(s+n+1)(s+n+2)} x^{-s}\, \mathrm{d}s\, .$$

η is the inverse Mellin transform of the function

$$R(s) = \frac{(s-p_1)(s-p_2)\cdots(s-p_n)}{(s+1)(s+2)\cdots(s+n+1)(s+n+2)} = \frac{P(s)}{Q(s)}$$

having as zeroes exactly $p_1, \dots, p_n$. We want now to prove $\eta(x)$ is a polynomial. Let us consider the semicircular path γ_j having as a diameter the line segment from $c - ji$ to $c + ji$ and closed to the left. By the residue theorem

$$\lim_{j\to+\infty} \int_{\gamma_j} \frac{P(s)}{Q(s)} x^{-s}\,\mathrm{d}s = \int_{c-i\infty}^{c+i\infty} \frac{P(s)}{Q(s)} x^{-s}\,\mathrm{d}s\,.$$

since the integral on the circular arc tends to 0 as the radius tends to 0 for $|s| \to +\infty$. On the other hand

$$\lim_{j\to\infty} \int_{\gamma_j} \frac{P(s)}{Q(s)} x^{-s}\,\mathrm{d}s = \sum_{k=1}^{n+2} \mathrm{Res}\left(\frac{P(s)x^{-s}}{Q(s)}, s=-k\right)$$
$$= \sum_{k=1}^{n+2} \frac{P(-k)}{Q'(-k)} x^k = \sum_{k=1}^{n+2} (-1)^{k-1} \frac{P(-k)}{(k-1)!(n+2-k)!} x^k,$$

that is a polynomial in x. Now, by repeating the same construction made in Section A.3 that leads to $\tilde{f}$ and $\tilde{g}$ starting from φ for the function η to obtain two functions having equals p-norms for $p = p_1, \dots, p_n$. $\square$

Appendix B

Dini derivatives of continuous functions

In this second appendix we want to inquire the behavior of Dini derivatives of continuous functions. Beside the intrinsic interest the inspiration for this study arose firstly from the study of lower and upper solutions (see e.g. the conditions [29, Definition 3.1, Chapter 1, Section 3] but also Proposition 2.2.5 in Chapter 2 of this thesis) and secondly by the possibility to further extend the results presented in Chapter 1 for the Poincaré-Bohl Theorem for bounded open sets with even more irregular boundary.

Dini derivatives take their names after Ulisse Dini, who introduced them in 1878, cf. [35]; let us recall their standard notations

$$D_+f(x) = \liminf_{h\to 0^+} \frac{f(x+h)-f(x)}{h}\,, \qquad D^+f(x) = \limsup_{h\to 0^+} \frac{f(x+h)-f(x)}{h}\,,$$
$$D_-f(x) = \liminf_{h\to 0^-} \frac{f(x+h)-f(x)}{h}\,, \qquad D^-f(x) = \limsup_{h\to 0^-} \frac{f(x+h)-f(x)}{h}\,.$$

Here, and in the rest of the chapter, we assume that $f : I \to \mathbb{R}$ is defined on some open interval $I \subseteq \mathbb{R}$. A fundamental step in the study of Dini derivatives was achieved in the first quarter of the twentieth century by Denjoy [34] for continuous functions, Young [106] for measurable functions, and Saks [93] for arbitrary ones. The Denjoy–Young–Saks theorem states that at each point x, except for a set of measure zero, one of the following four alternatives holds:

1. f has a finite derivative at x;
2. $D_-f(x) = D^+f(x) \in \mathbb{R}\,,\ \ D^-f(x) = +\infty\,,\ D_+f(x) = -\infty\,;$
3. $D^-f(x) = D_+f(x) \in \mathbb{R},\ D^+f(x) = +\infty\,,\ D_-f(x) = -\infty\,;$

4. $D^- f(x) = D^+ f(x) = +\infty\,, \quad D_- f(x) = D_+ f(x) = -\infty\,.$

Denjoy also explicitly constructed a continuous function realizing each of the previous four conditions on a perfect set of positive Lebesgue measure; a highly remarkable result, in consideration of the fact that continuous functions can exhibit very pathological behaviors (see, e.g., [62]). We refer to the book by Bruckner [20] for an extensive study of Dini derivatives and a more complete historical account.

In this chapter, for any function $f : I \to \mathbb{R}$, we are interested in studying the set

$$V_f := \{x \in I : D_- f(x) < D^+ f(x)\}\,.$$

It should be noticed that, in the above mentioned example by Denjoy, the set V_f is totally disconnected, i.e., it does not contain any nontrivial interval. The main question is: *how large can this set be?*

It is well known that there exist *non-continuous* functions $f : \mathbb{R} \to \mathbb{R}$ for which $V_f = \mathbb{R}$ (see for instance [58], where the function $f : \mathbb{R} \to \mathbb{R}$ has a dense graph in $\mathbb{R}^2$). On the contrary, we will prove that there are no *continuous* functions with such a property. To be more precise, let us introduce the following class of functions.

Definition B.0.1. *We say that a function $f : I \to \mathbb{R}$ is* upper well behaved *if for every compact interval J contained in I there is a $x_J \in J$ such that $f(x_J) = \max f(J)$.*

Clearly, every continuous function is upper well behaved. On the other hand, one can easily find examples of upper well behaved functions which are nowhere continuous (e.g., the well known Dirichlet function).

Here is our first result.

Theorem B.0.2. *If $f : I \to \mathbb{R}$ is upper well behaved, then the set V_f is totally disconnected.*

Our theorem complements Denjoy's example of a continuous function, for which $\mu(V_f) > 0$; it suggests that, if f is continuous, the set V_f should be "small", in some sense. Some questions then arise:

Q1. *If $f : I \to \mathbb{R}$ is continuous, or even upper well behaved, is the set V_f of first Baire category?*

Q2. *If $I = (a, b)$ and $f : I \to \mathbb{R}$ is continuous, can $\mu(V_f)$ be equal to $b - a$?*

We will also show that the set V_f can be preassigned, at least in the class of totally disconnected *closed* sets; taking, e.g., $I = \mathbb{R}$, for any given totally disconnected closed set $\mathcal{V} \subseteq \mathbb{R}$, there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $V_f = \mathcal{V}$. This will be a consequence of Lemma B.1.2 below.

Let us now investigate on the possibility for a function $f : \mathbb{R} \to \mathbb{R}$ to be such that $V_f = \mathbb{R}$ and, at the same time, to be continuous at some points of its domain. We will prove the following.

Theorem B.0.3. *For any totally disconnected closed set $A \subseteq \mathbb{R}$, there exists a function $f : \mathbb{R} \to \mathbb{R}$, whose set of continuity points coincides with A, such that $V_f = \mathbb{R}$, and more precisely*

$$D_- f(x) = -\infty \quad \textit{and} \quad D^+ f(x) = +\infty\,, \quad \textit{for every } x \in \mathbb{R}\,.$$

Recall that a Smith–Volterra–Cantor set is a totally disconnected closed set C, contained in $[0,1]$, having any assigned Lebesgue measure $\mu(C) \in]0,1[$. Iterating its construction on any interval $[n, n+1]$, with $n \in \mathbb{Z}$, we could have a totally disconnected closed set A with "almost full" measure.

A further question now arises:

Q3. *If $V_f = \mathbb{R}$, can the function $f : \mathbb{R} \to \mathbb{R}$ be continuous on a dense set of points?*

The proof of Theorem B.0.2 and Theorem B.0.3 are provided in the next section. They are based on the knowledge that every monotone function is differentiable almost everywhere, and on some simple properties of continued fractions.

B.1 Proofs

We denote by μ be the Lebesgue measure on $\mathbb{R}$.

Proof of Theorem B.0.2. By contradiction, let $[a,b] \subseteq V_f$, with $a < b$. Let $(x_n)_n$ be a sequence in $[a,b]$ such that $f(x_n) \to \inf f([a,b])$. Passing if necessary to a subsequence, we can assume that $x_n \to \check{x}$, for some $\check{x} \in [a,b]$. We have two cases.

<u>Case 1</u>: $\check{x} \in [a,b)$. We will prove that f is increasing in $(\check{x}, b]$, hence almost everywhere differentiable there, a contradiction.

By contradiction, let α, β in $(\check{x}, b]$ be such that $\alpha < \beta$ and $f(\alpha) > f(\beta)$. Being $\check{x} < \alpha$ and $f(\alpha) > \inf f([a,b])$, there exists n such that $x_n < \alpha$ and $f(x_n) < f(\alpha)$. Since f is upper well behaved, there is a $\hat{x} \in [x_n, \beta]$ such that $f(\hat{x}) = \max f([x_n, \beta])$. Being $f(\hat{x}) \geq f(\alpha) > \max\{f(x_n), f(\beta)\}$, it has to be $\hat{x} \in (x_n, \beta)$, whence $D_- f(\hat{x}) \geq 0 \geq D^+ f(\hat{x})$, a contradiction, since $\hat{x} \in V_f$.

<u>Case 2</u>: $\check{x} = b$. One proves in an analogous way that f is decreasing in $[a,b)$, hence almost everywhere differentiable there, a contradiction.

The proof is thus completed. □

Remark B.1.1. *If we define a function $f : I \to \mathbb{R}$ to be* lower well behaved *when $(-f)$ is upper well behaved, then it can be proved that the set*

$$\Lambda_f := \{x \in I : D^- f(x) > D_+ f(x)\}$$

is totally disconnected.

Let us now go for the proof of Theorem B.0.3. In the following, we allow an *interval* to be reduced to a single point. It will be useful to consider the function $F_\sharp : \mathbb{R} \to [0,1]$ defined as

$$F_\sharp(x) = \begin{cases} 2\sqrt{x(1-x)}\,, & \text{if } x \in [0,1]\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

We first need to prove the following two lemmas.

Lemma B.1.2. *Let A be a totally disconnected closed set. Then, there exists a nonnegative continuous function $\sigma_A : \mathbb{R} \to \mathbb{R}$ such that:*

- *σ_A is differentiable on $\mathbb{R} \setminus A$;*
- *for all $x \in A$ one has $D_-\sigma_A(x) = -\infty$ and $D^+\sigma_A(x) = +\infty$;*
- *$\sigma_A(x) = 0$ if and only if $x \in A$.*

Proof. We first prove the result in the case when A is bounded. Without loss of generality we can assume that $A \subseteq (0,1)$. Since A is closed, its complement in $(0,1)$ can be written as an at most countable union of pairwise disjoint open intervals $U_n = (a_n, b_n)$, with $n \geq 1$. We will treat in detail only the case when there are infinitely many of them (in the other case A has only finitely many points, and the proof is much easier). We can then write

$$A = (0,1) \setminus \bigcup_{n \geq 1} U_n\,.$$

We define $R_1 = [0,1]$ and, for every $n \geq 2$,

$$R_n = [0,1] \setminus \bigcup_{j=1}^{n-1} U_j\,.$$

The following properties hold true:

- $U_n \subseteq R_n$, for every $n \geq 1$;

- $R_1 \supseteq R_2 \supseteq \cdots \supseteq R_n \supseteq \cdots$;
- $\bigcap_{n\geq 1} R_n = A \cup \{0,1\}$.

Moreover, for $n \geq 2$ the set R_n is the union of n pairwise disjoint closed intervals

$$R_n = S_{n,1} \cup S_{n,2} \cup \cdots \cup S_{n,n} \,.$$

We set $S_{1,1} = R_1 = [0,1]$. For every $n \geq 1$ there exists an integer $H(n) \in \{1,\ldots,n\}$ such that $U_n \subseteq S_{n,H(n)}$. For simplicity, let us introduce the notation

$$\rho_n = \mu\big(S_{n,H(n)}\big)\,.$$

Note that, since A is totally disconnected, we have

$$\lim_n \rho_n = 0\,. \tag{B.1}$$

We define the function $\tilde{\sigma}_A : \mathbb{R} \to \mathbb{R}$ as

$$\tilde{\sigma}_A(x) = \sum_{n=1}^{\infty} \sqrt{\rho_n}\, F_\sharp\left(\frac{x-a_n}{b_n-a_n}\right).$$

Notice that, for each $x \in \mathbb{R}$, the above sum has at most one non-zero addend. It is clear that $\tilde{\sigma}_A(x) \geq 0$ for all $x \in \mathbb{R}$, and that

$$A = \{x \in (0,1) : \tilde{\sigma}_A(x) = 0\}\,.$$

If $x \in (0,1) \setminus A$, then $x \in U_n$ for some n, hence $\tilde{\sigma}_A$ is differentiable there. However, $\tilde{\sigma}_A(x) = 0$ for every $x \in \mathbb{R} \setminus (0,1)$. We thus need to modify $\tilde{\sigma}_A$ outside some interval $[\delta, 1-\delta]$, with $\delta \in (0,1)$, containing A in its interior. It is indeed possible to find a function $\sigma_A : \mathbb{R} \to \mathbb{R}$, which coincides with $\tilde{\sigma}_A$ on $[\delta, 1-\delta]$, and is continuously differentiable on $(-\infty, \delta] \cup [1-\delta, +\infty)$, being strictly positive there, and

$$\sigma_A(x) = 1 \quad \text{for every } x \in (-\infty, 0] \cup [1, +\infty)\,.$$

This function $\sigma_A : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R} \setminus A$ and it is such that

$$A = \{x \in \mathbb{R} : \sigma_A(x) = 0\}\,.$$

We would like to prove that, for any $x \in A$, the function σ_A is continuous at x, with $D_-\sigma_A(x) = -\infty$ and $D^+\sigma_A(x) = +\infty$.

Suppose then $x \in A$, and so $\sigma_A(x) = 0$. For every $n \geq 1$ we can find an index $N(x,n) \in \{1,\ldots,n\}$ such that $x \in S_{n,N(x,n)}$. Let us first focus our attention on a right neighborhood of x. We consider two cases.

Case 1. Assume $\inf\{y \in A : y > x\} > x$. Then $x = a_n$, for a certain index n. In particular, $U_n \cup \{x\} = [a_n, b_n)$ is a right neighborhood of x, and it is easily seen that $\lim_{y\to x^+} \sigma_A(y) = 0$ and $D^+\sigma_A(x) = +\infty$.

Case 2. Assume $\inf\{y \in A : y > x\} = x$. In this case, $S_{n,N(x,n)}$ contains a right neighborhood of x, for every $n \geq 1$, and

$$S_{n,N(x,n)} \cap \{y \in (x,1) : y \notin A\} = \bigcup_{j\in J_n} U_j \, ,$$

where J_n is an infinite set of integers, such that

$$\lim_n (\min J_n) = +\infty \, . \tag{B.2}$$

We first prove that σ_A is continuous from the right at x. Fix $\varepsilon > 0$. By (B.1) and (B.2), there exists $\bar{n} \geq 1$ such that

$$n \geq \bar{n} \quad \Rightarrow \quad \rho_j < \varepsilon^2 \quad \text{for every } j \in J_n \, . \tag{B.3}$$

For any $y \in S_{\bar{n},N(x,\bar{n})} \cap (x,1)$ we have that, either $y \in A$, hence $\sigma_A(y) = 0$, or $y \in U_j$ for a certain $j \in J_{\bar{n}}$; in this case, by (B.3),

$$\sigma_A(y) = \sqrt{\rho_j}\; F_\sharp\left(\frac{y - a_j}{b_j - a_j}\right) \leq \sqrt{\rho_j} < \varepsilon \, .$$

We have thus proved that $0 \leq \sigma_A(y) < \varepsilon$ for every y in a right neighborhood of x, and so $\lim_{y\to x^+} \sigma_A(y) = 0$.

We now prove that $D^+\sigma_A(x) = +\infty$. We claim that there exists a strictly increasing sequence $(n_k)_k$ of positive integers such that

$$S_{n_k,H(n_k)} = S_{n_k,N(x,n_k)} \, . \tag{B.4}$$

Indeed, set $n_1 = 1$. Then, for some $m \geq 2$ we know that it will be

$$S_{2,N(x,2)} = S_{3,N(x,3)} = \cdots = S_{m,N(x,m)} \neq S_{m+1,N(x,m+1)}$$

if and only if the sets $U_1 \, , U_2 \, , \ldots , U_{m-1}$ have an empty intersection with $S_{2,N(x,2)}$, while $U_m \subseteq S_{2,N(x,2)}$. We see that in this case $S_{m,H(m)} = S_{m,N(x,m)}$; such an m is denoted by n_2. Then, one proceeds inductively: assume that n_k has been defined, for a certain $k \geq 2$; for some $m \geq n_k + 1$ it will be

$$S_{n_k+1,N(x,n_k+1)} = S_{n_k+2,N(x,n_k+2)} = \cdots = S_{m,N(x,m)} \neq S_{m+1,N(x,m+1)}$$

if and only if the sets $U_{n_k} \, , \, U_{n_k+1} \, , \, \ldots , \, U_{m-1}$ have an empty intersection with $S_{n_k+1,N(x,n_k+1)}$, while $U_m \subseteq S_{n_k+1,N(x,n_k+1)}$. We see that $S_{m,H(m)} = S_{m,N(x,m)}$; such an m is denoted by n_{k+1}.

We have thus defined the sequence $(n_k)_k$ for which (B.4) holds. Denote by $\hat{x}_{n_k}$ the midpoints of the intervals U_{n_k}. Since, by (B.4),

$$x \in S_{n_k,N(x,n_k)} = S_{n_k,H(n_k)} \quad \text{and} \quad \hat{x}_{n_k} \in U_{n_k} \subseteq S_{n_k,H(n_k)}\,,$$

it has to be $[x, \hat{x}_{n_k}] \subseteq S_{n_k,H(n_k)}$, hence $\hat{x}_{n_k} - x \le \rho_{n_k}$. Then, by (B.1),

$$D^+\sigma_A(x) \ge \lim_k \frac{\sigma_A(\hat{x}_{n_k}) - \sigma_A(x)}{\hat{x}_{n_k} - x} \ge \lim_k \frac{\sqrt{\rho_{n_k}}}{\rho_{n_k}} = \lim_k \frac{1}{\sqrt{\rho_{n_k}}} = +\infty\,.$$

A similar argument shows that $\lim_{y \to x^-} \sigma_A(y) = 0$ and $D_-\sigma_A(x) = -\infty$, so that the proof is completed, in the case when A is bounded.

Let us now consider the case when A is unbounded both from below and from above. We can define a bilateral sequence $(x_n)_{n\in\mathbb{Z}}$ of points, not belonging to A, such that $x_{n+1} - x_n \ge 1$ for every $n \in \mathbb{Z}$. Define $A_n = A \cap [x_n, x_{n+1}]$, for every $n \in \mathbb{Z}$. Notice that A_n is closed, totally disconnected and bounded, for every $n \in \mathbb{Z}$. Applying the above procedure with A_n instead of A, we obtain the corresponding functions σ_{A_n}, which we denote by σ_n. Notice that, by construction, for every n we have that

$$\sigma_n(x_n) = 1\,, \quad \sigma_n(x_{n+1}) = 1\,, \quad \text{and} \quad \sigma_n'(x_n) = \sigma_n'(x_{n+1}) = 0\,.$$

We define the function $\sigma_A : \mathbb{R} \to \mathbb{R}$ as

$$\sigma_A(x) = \sigma_n(x)\,, \quad \text{for every } \ n \in \mathbb{Z} \ \text{ and } \ x \in [x_n, x_{n+1}]\,,$$

It is readily verified that σ_A well-defined, continuous on all $\mathbb{R}$, and differentiable on $\mathbb{R} \setminus A$.

The cases when A is unbounded only from below or only from above can be obtained adapting the procedure adopted in the previous two cases. □

Lemma B.1.3. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function, and define $f(x) = \psi(x) \cdot \mathscr{R}(x)$, where*

$$\mathscr{R}(x) = \begin{cases} 1\,, & \text{if } x = 0 \text{ or } x \in \mathbb{R} \setminus \mathbb{Q}\,, \\ 2 - \dfrac{1}{p}\,, & \text{if } x \in \mathbb{Q} \setminus \{0\} \text{ and } |x| = \dfrac{p}{q} \text{ with } \gcd(p,q) = 1\,. \end{cases}$$

Then, the set of continuity points of f coincides with the set of zeros of ψ; moreover,

- *if $\psi(x) \neq 0$, then $D_-f(x) = -\infty$ and $D^+f(x) = +\infty$;*
- *if $\psi(x) = 0$, then $D_-f(x) = 2D_-\psi(x)$ and $D^+f(x) = 2D^+\psi(x)$.*

Proof. The result is proved by means of the theory of continued fractions, for which we refer to [82]. We fix $x \in \mathbb{R}$ and consider two cases.

Case 1: $\psi(x) \neq 0$. It is easy to prove that f is not continuous at these points.

If $x \in (0, +\infty) \setminus \mathbb{Q}$, let $(c_n(x))_{n\in\mathbb{N}}$ be the sequence of convergents of the continued fraction representing x. Define

$$x_n^+ = \frac{a_{2n}}{b_{2n}} = c_{2n}(x)\,, \qquad x_n^- = \frac{a_{2n+1}}{b_{2n+1}} = c_{2n+1}(x)\,.$$

The sequence $(x_n^+)_n$ converges to the right while $(x_n^-)_n$ converges to the left to x. Since the fractions $c_n(x)$ are in lowest terms, we have

$$\frac{f(x_n^+) - f(x)}{x_n^+ - x} = \frac{\left(2 - \dfrac{1}{a_{2n}}\right)\psi(c_{2n}(x)) - \psi(x)}{c_{2n}(x) - x} \to +\infty\,,$$

because the numerator tends to $\psi(x) > 0$ as $n \to +\infty$. Analogously,

$$\frac{f(x_n^-) - f(x)}{x_n^- - x} = \frac{\left(2 - \dfrac{1}{a_{2n+1}}\right)\psi(c_{2n+1}(x)) - \psi(x)}{c_{2n+1}(x) - x} \to -\infty\,,$$

Hence, $D^+f(x) = +\infty$ and $D_-f(x) = -\infty$.

If $x \in (0, +\infty) \cap \mathbb{Q}$, let $x = \frac{p}{q}$ with $\gcd(p, q) = 1$, and define, for every $n \in \mathbb{N}$,

$$y_n^+ = \frac{p}{q} + \frac{1}{(2q)^n} = \frac{2^n p q^{n-1} + 1}{2^n q^n}\,, \qquad y_n^- = \frac{p}{q} - \frac{1}{(2q)^n} = \frac{2^n p q^{n-1} - 1}{2^n q^n}\,.$$

For every $n \geq 2$, the fractions are reduced to lowest terms, while their numerators tend to infinity as $n \to +\infty$. So,

$$\frac{f(y_n^+) - f(x)}{y_n^+ - x} = \frac{\left(2 - \dfrac{1}{2^n p q^{n-1} + 1}\right)\psi\left(y_n^+\right) - \left(2 - \dfrac{1}{p}\right)\psi\left(x\right)}{(2q)^{-n}} \to +\infty\,,$$

because the numerator tends to $\frac{1}{p}\psi(x) > 0$ as $n \to +\infty$. Analogously,

$$\frac{f(y_n^-) - f(x)}{y_n^- - x} = -\frac{\left(2 - \dfrac{1}{2^n p q^{n-1} - 1}\right)\psi\left(y_n^-\right) - \left(2 - \dfrac{1}{p}\right)\psi\left(x\right)}{(2q)^{-n}} \to -\infty\,.$$

Hence, $D^+f(x) = +\infty$ and $D_-f(x) = -\infty$. We have thus proved the conclusion, in this case, for every $x > 0$.

A similar argument leads to the conclusion when $x < 0$. Finally, if $x = 0$, we define, for every $n \geq 1$,

$$z_n^+ = \frac{n+1}{n^2}, \qquad z_n^- = -\frac{n+1}{n^2},$$

so that

$$\frac{f(z_n^\pm) - f(0)}{z_n^\pm - 0} = \frac{\left(2 - \dfrac{1}{n+1}\right)\psi\left(z_n^\pm\right) - \psi\left(0\right)}{z_n^\pm} \to \pm\infty,$$

since $\psi(0) > 0$, hence proving again that $D^+ f(0) = +\infty$ and $D_- f(0) = -\infty$.

Case 2: $\psi(x) = 0$. The continuity of f at x is trivial, since

$$\psi(y) \leq f(y) \leq 2\psi(y), \qquad \text{for every } y \in \mathbb{R}. \tag{B.5}$$

The function

$$r_x(y) = \frac{\psi(y) - \psi(x)}{y - x} = \frac{\psi(y)}{y - x}$$

is continuous in its domain $\mathbb{R} \setminus \{x\}$, and

$$r_x(y)(y - x) \geq 0, \qquad \text{for every } y \in \mathbb{R} \setminus \{x\}. \tag{B.6}$$

Moreover,

$$D^+\psi(x) = \limsup_{y \to x^+} r_x(y), \qquad D_-\psi(x) = \liminf_{y \to x^-} r_x(y).$$

Correspondingly, we can find two sequences of irrational numbers $(\xi_n^-)_n$ in $(-\infty, x)$ and $(\xi_n^+)_n$ in $(x, +\infty)$ such that $\lim_n \xi_n^\pm = x$ and

$$\lim_n r_x(\xi_n^+) = D^+\psi(x), \qquad \lim_n r_x(\xi_n^-) = D_-\psi(x).$$

We now assume $x > 0$. Recalling the notation $(c_n(\zeta))_n$ for the sequence of the convergents of the continued fraction representing $\zeta \notin \mathbb{Q}$, we can find two sequences of positive rational numbers $(\zeta_n^\pm)_n$ such that

$$\zeta_n^- = c_{2\kappa(n)+1}(\xi_n^-) = \frac{\gamma_n^-}{\delta_n^-} \quad \text{and} \quad \zeta_n^+ = c_{2\kappa(n)}(\xi_n^+) = \frac{\gamma_n^+}{\delta_n^+},$$

where the choice $\kappa(n) > n$ is such that $|\xi_n^\pm - \zeta_n^\pm| < n^{-1}$, $|r_x(\xi_n^\pm) - r_x(\zeta_n^\pm)| < n^{-1}$, and $\gamma_n^\pm > n$. In particular, we can ensure that $\lim_n \zeta_n^\pm = x$ and

$$\lim_n r_x(\zeta_n^+) = D^+\psi(x), \qquad \lim_n r_x(\zeta_n^-) = D_-\psi(x).$$

Finally,

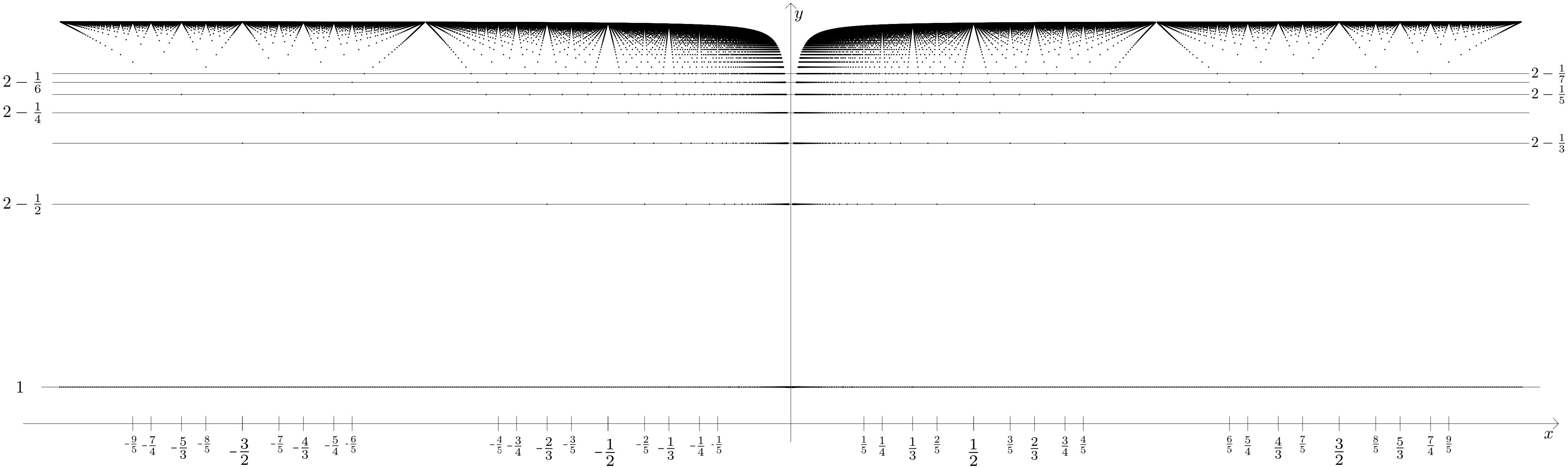
$$\frac{f(\zeta_n^+) - f(x)}{\zeta_n^+ - x} = \frac{f(\zeta_n^+)}{\zeta_n^+ - x} = \left(2 - \frac{1}{\gamma_n^+}\right) \frac{\psi(\zeta_n^+)}{\zeta_n^+ - x} \to 2D^+\psi(x)\,,$$

$$\frac{f(\zeta_n^-) - f(x)}{\zeta_n^- - x} = \frac{f(\zeta_n^-)}{\zeta_n^- - x} = \left(2 - \frac{1}{\gamma_n^-}\right) \frac{\psi(\zeta_n^-)}{\zeta_n^- - x} \to 2D_-\psi(x)\,.$$

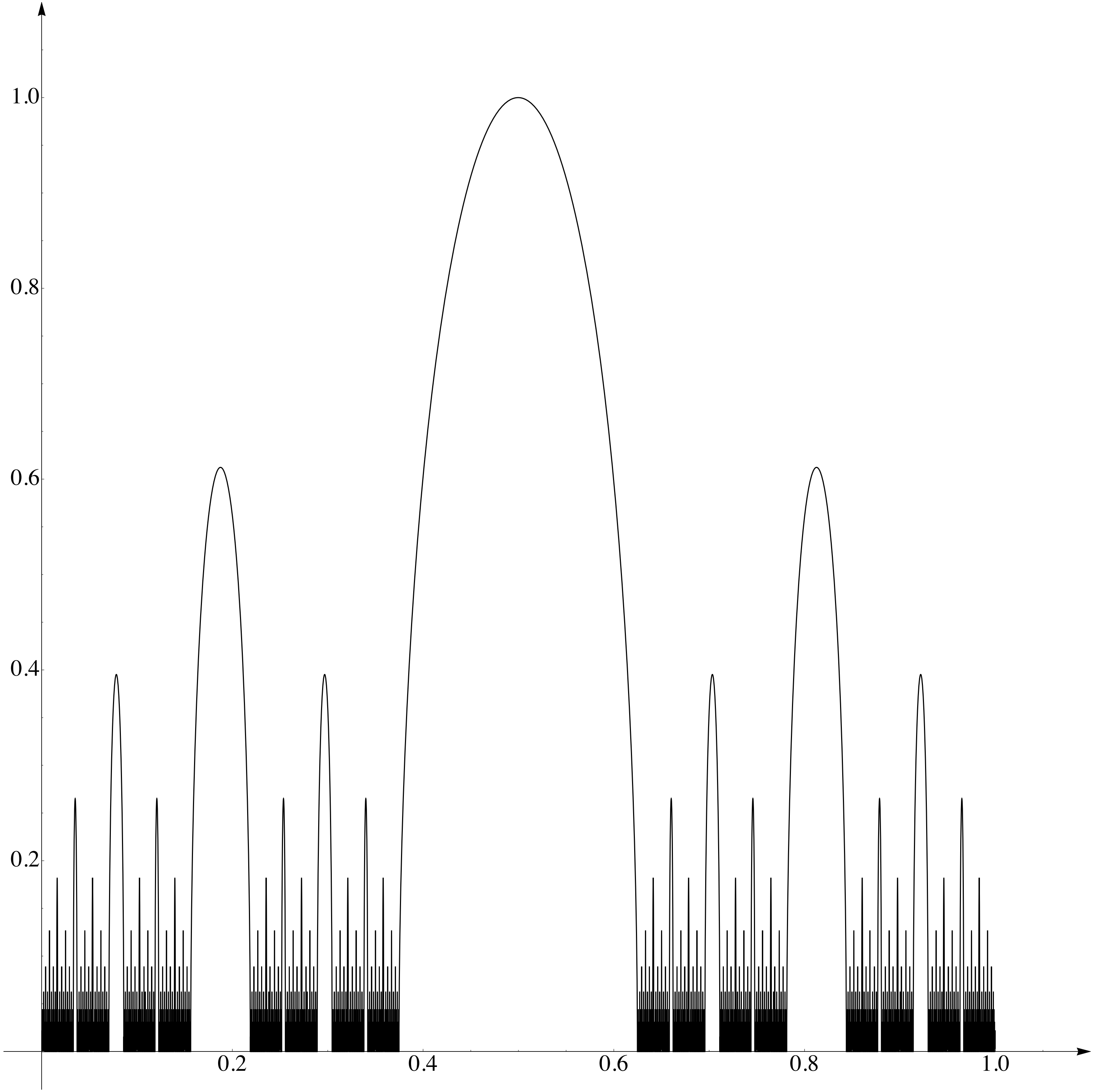
Hence, $D^+f(x) = 2D^+\psi(x)$ and $D_-f(x) = 2D_-\psi(x)$, taking into account (B.5) and (B.6).

The cases when $x < 0$ or $x = 0$ can be carried out similarly. The proof is thus completed. $\square$

The proof of Theorem B.0.3 is now an immediate consequence of Lemma B.1.3, taking as ψ the function σ_A provided by Lemma B.1.2.



The function $\mathscr{R}(x)$ on the interval $[-2, 2]$.



The function $\psi(x)$ restricted to $[0,\ 1]$ when A is the Smith–Volterra–Cantor set.

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