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DOCTORAL THESIS

New developments in the Renormalization Group

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To the women of my life: Arianna, Cristina, Martina and Rosa.

Declaration of Authorship

I, Alessio Baldazzi, declare that this thesis titled, "New developments in the Renormalization Group" and the work presented in it are my own. Where I have consulted the published work of others, this is always clearly attributed. The thesis is based on the following research papers

- [1] A. Baldazzi, R. Percacci and L. Zambelli, "Functional renormalization and the $\overline{\text{MS}}$ scheme," Phys. Rev. D **103** (2021) no.7, 076012 [arXiv:2009.03255 [hep-th]].
- [2] A. Baldazzi, R. Percacci and L. Zambelli, "Limit of vanishing regulator in the functional renormalization group," Phys. Rev. D **104** (2021) no.7, 076026
- [3] A. Baldazzi, R. B. A. Zinati and K. Falls, "Essential renormalisation group," [arXiv:2105.11482 [hep-th]].
- [4] A. Baldazzi and K. Falls, "Essential Quantum Einstein Gravity," Universe **7** (2021), 294 [arXiv:2107.00671 [hep-th]].

Abstract

New developments in the Renormalization Group

by Alessio Baldazzi

In the first part of the thesis, we review the basics of the Exact Renormalization Group. In the central part, we design a specific choice of renormalization scheme in the context of Functional Renormalization Group to achieve the nonperturbative analogous of the $\overline{\mathrm{MS}}$ scheme's results. Then, we study the properties of a more general family of renormalization schemes, that includes the one we previously analyze, and appears to be useful to eliminate the spurious breaking of symmetries cause by the renormalization scheme. The final part of this thesis consists of a new implementation of the Functional Renormalization Group, based on the Effective Average Action, that allows all possible field redefinitions to simplify the flow equations. Such a simplification is practically useful in reducing the complexity of the computations and has theoretical implications in disentangling the unphysical information due to intrinsic redundancies of the mathematical descriptions of Nature. We show such improvements in the context of the three-dimensional Ising model and the Quantum Einstein Gravity without matter. In particular, using the derivative expansion in both cases we impose renormalization conditions that fix the value of the inessential couplings obtaining only the flow of the essential ones. With such a renormalization scheme, which is called Minimal Essential Scheme, the propagator does not develop additional poles when the truncation of the derivative expansion is increased. This way, we can select the desired universality classes, avoiding encountering instabilities and unitarity violations.

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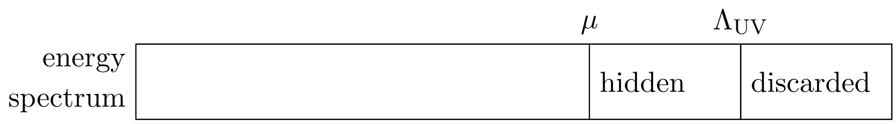
Introduction and Summary

Quantum Field Theory (QFT) provides a fundamental tool to describe physics at high energy (electroweak and strong interaction), phase transitions in particle physics, statistical mechanics and condensed matter physics. This huge versatility is due to the fact that QFT is a very powerful framework to treat systems with a large or infinite number of degrees of freedom (DOFs).
QFT becomes even more powerful when we implement the Renormalization Group (RG), which captures the effective description of systems at different energy scales.

The behavior of systems with a large number of DOFs is completely different from those with a small number of DOFs. Small number DOFs are characterized by an analytic and smooth description. However, this is not the behavior that is observed in the case of phase transitions. The dynamics of phase transitions require some limit, e.g. thermodynamical or continuous limit, since they are characterized by non-analytical behavior of some quantities. When we are dealing with a large number of DOFs, typically it is possible to reduce or "decimate" their number. Suppose that we want to describe a gas with a certain number of moles: since each mole contains 10^{23} particles, the amount of DOFs is computationally unmanageable. However, using the intensive and extensive properties of observables, e.g. pressure and temperature, we can obtain the macroscopic properties knowing few microscopic details, e.g. the mean of the velocities squares. This means that we can reduce the number of DOFs. However, it is natural to ask how much we can reduce it without losing the right effective description. The minimal number corresponds to the number of molecules, or lattice sites, in a volume of size equal to the correlation length, and it tells us how far a DOF is correlated to the other DOFs. Its value depends on the state of the system, e.g. pressure and temperature in the case of gas. In a system like gases at ambient temperature and pressure, the correlation length is comparable to the particle spacing. Therefore, in this case we can assume that the properties of the system are captured by the properties of small clusters of particles. Among the cases where the correlation length is much bigger than the spacing between nearby DOFs there are phase transitions. Using the RG, it is possible to show that systems with many DOFs inside the range of the correlation length develop very special properties. When the number of DOFs inside such a range is low, the details of the interactions are crucial in order to study the dynamics of the system. Instead, when the density is high, the nature of DOFs and their cooperative behavior hide the details of the interactions, that play a secondary role. This implies that different microscopic details, i.e. different interaction structures, can converge to the same behavior during phase transitions. This property is called universality. The RG is a perfect tool to achieve such a description and its basic idea comes from hydrodynamics. In hydrodynamics, we have many DOFs and they are strongly correlated, but we don't need the exact dynamics of all DOFs. What we are interested in are averaged quantities like density, which shows only macroscopic fluctuations. In the process of averaging, we are eliminating all microscopic fluctuations and, therefore, we are reducing the number of DOFs per unit volume. Similarly, the RG replaces the initial microscopic DOFs with a smaller set of macroscopic effective DOFs.

The idea behind the RG can be illustrated in the following example. We start with a physical system and we observe it at scale Λ_{UV}. In order to study its behavior, we need to excite it using some device that can provide energies less or equal to Λ_{UV}, for example photons that scatter against the system. Since there is not enough energy to excite the DOFs with energy greater than Λ_{UV}, or equivalently Physics is insensitive to energy scales greater than Λ_{UV}, we completely discard the Physics beyond such energy value. The observations at scale Λ_{UV} fix the free parameters of our theory.



Then, we make experiments at different scales $\mu \leq \Lambda_{\mathrm{UV}}$ and we discover that DOFs with energy between μ and Λ_{UV} cannot simply be discarded. The radiative corrections due to these modes induce a new effective theory at energy μ, where the initial parameters are modified and new kinds of interaction are induced. The effective description at μ is obtained by *integrating out* the DOFs between μ and Λ_{UV} and such DOFs participate as virtual particles. These DOFs are treated as fluctuations and we average their contributions, like in the hydrodynamics example. Therefore, changing the scale μ below the initial UV scale we observe a different theory which, however, is related to the initial one. In particular, the information about DOFs between μ and Λ_{UV} is encoded into the new structure of interactions that is induced by the radiative corrections of these virtual particles. This way, we can think of such a sector of the energy spectrum as hidden, because its information is averaged and stored into the effective description that we observe at the scale μ. The RG is composed of the transformations that give the new effective theory starting from an initial theory and take into account high energy modes in the hidden sector for low energy processes. The RG teaches us that instead of describing the systems with fixed parameters it is more advantageous to let the parameters "run" along the energy spectrum and obtain a new effective description. From the new parameters at the new energy scale, we can understand what can be neglected or not and reduce the complexity of the effective description. This way of proceeding is strictly connected to the concept of "locality" in the

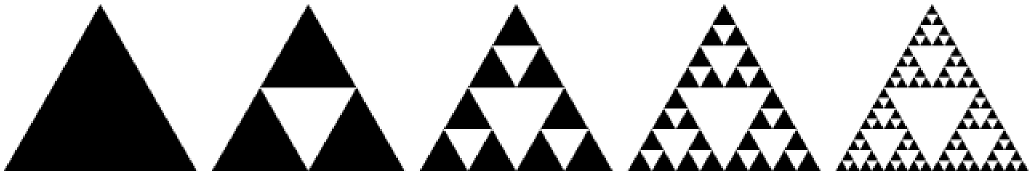


Figure 1: The following figure represents how to build the Sierpiński triangle.

energy spectrum: as we already said, the Physics at a given energy is insensible to DOFs at higher energies. For example, in order to study the Hydrogen energy levels, we don't need to know the structure of the nucleus, but the parameters of the first simply hide the details of the latter. The RG is the machinery that starting from a UV point in the energy spectrum tells us how the effective description evolves changing the energy scale.

Interesting theories, that can be studied inside the RG framework, are those that "look like" the same theory at different scales. These theories are invariant under scale transformations, or equivalently, possess scale symmetry. They are fixed points of the RG flow and describe phase transitions. During phase transitions, because of the large value of the correlation length, the cooperative behavior of fluctuations makes the interaction details negligible and gives rise to structures with all possible sizes, making the system scale invariant. Phase transition can be found in many different systems, natural or artificial [5]. Due to their cooperative behavior, the main characteristics of such phenomena are universality, i.e. the convergence to the same macroscopic behavior regardless of the microscopic details, and scale invariance, which gives rise to anomalous non-integer scaling of observables. This is the typical feature of fractals, which by definition are structures with non-integer scaling dimension. As an example, let's consider the Sierpiński triangle, shown in Figure 1. It may be constructed from an equilateral triangle by repeated removal of triangular subsets[1]. Rescaling the length L of the Sierpiński triangle by $1/2$ implies a rescaling of its "mass" M by a factor $1/3$. This implies the scaling law $M = L^{\Delta}$ with $\Delta = \ln 3/\ln 2 \approx 1,585$. A general fractal does not have a simple symmetric structure as the Sierpiński triangle, but maintains non-integer scaling law. Other examples are given by the coastline of any territory, the sea surface when it is windy, etc.

In this paragraph, we want to make a brief review of how the Renormalization Group was born and developed [6]. After J.J. Thomson discovered the first elementary particle, i.e. the electron [7], many questions arose about its internal structure. Concepts like "self-force" and "self-mass/energy" are the prototype of the modern ideas of radiative effects due to quantum fluctuations. Classical and semi-classical calculations revealed a linear divergence of the self-energy. Then, the Dirac equation in 1928 [8] and the hole

[1]To construct it, start with an equilateral triangle, subdivide it into four smaller congruent equilateral triangles and remove the central triangle. Then repeat the steps of subdivision and removal with each of the remaining smaller triangles infinitely.

theory in 1930 [9] were the first steps toward modern QED: from now on the vacuum will not be empty anymore, but full of degrees of freedom. The vacuum became a dynamical medium containing virtual electron-positron pairs. Weisskopf discovered that screening by induced pairs reduces the linear divergence of the electron self-energy to a logarithmic one[10, 11]. Heisenberg, Dirac, and others studied the electron's charge distribution due to "vacuum polarization", i.e. creation of virtual electron-positron pairs that change the charge distribution. The unscreened "bare charge" was found to be divergent, again logarithmically [12, 13, 14]. Therefore, taking into account the radiative effects due to quantum fluctuations the classical linear divergence is turned into a logarithmic divergence. The great progress in QED came in 1947, with the measurements of the Lamb shift and the electron anomalous moment. After the conference at Shelter Island in the same year, Bethe gave the first theoretical estimation of Lamb shift including radiative corrections [15]. Soon, Schwinger did the same with the electron anomalous moment and, using the quantum corrections to the electron charge, he calculated the corrected electrostatic potential of an electron [16, 17]. At this point, Dyson studied systematically how to calculate radiative corrections using perturbation theory. He expanded the scattering amplitudes as power series in the electron bare charge, where each term involves divergent momentum-space integrals that represent the corrections due to virtual particles. After regularizing them, Dyson showed that mass and charge logarithmic divergence can be removed by the process of renormalization to all orders of perturbation theory [18]. For what regards statistical mechanics, in the thirties Landau built a hydrodynamics effective description of phase transition, which contains some primordial ideas of the RG [19]. In 1953 Stueckelberg and Petermann made some important contributions to the creation of the Renormalization Group studying fixed points properties of statistical systems [20]. Soon, Gell-Mann and Low [21] reformulated Dyson's renormalization program using a functional approach. They discovered that the divergent contributions can be isolated and reabsorbed into multiplicative renormalization constants, that come out from the analysis of the behaviors of the functionals under scale transformations. At this point it was clear that the coupling constants "run" with respect to the energy scale at a rate described by their beta-functions (introduced by Gell-Mann and Low). The transformations with respect to the scale form a group, and the running couplings give a representation of this group, which was named Renormalization Group by Bogoliubov and Shirkov, and the beta-function is a "tangent vector" to the group [22]. Therefore, the Renormalization Group led to the great success of QED: it became possible to reproduce experimental results with an agreement within ten parts in a billion. However, Renormalization was just a mathematical trick to tame divergences coming from loop integrals in Feynman diagrams.

At this point, Ken Wilson's work made Renormalization physical, by uncovering its deep connection with scale transformations. The idea is that scale determines the perception of physical phenomena. When one looks at the sea surface, for example from Molo Audace in Trieste, and it is windy, it is possible to recognize waves with different frequencies. There are big waves with low frequencies and small waves with high frequencies. If we are interested in macroscopic motion along the surface, we can average the small waves contributions. Instead, if we want to look at dynamics at the cm scale, we have to zoom in and we will discover even smaller waves, that were previously discarded. Eventually, in the process of zooming, the structure of the system will change and new DOFs, that previously were hidden, will appear.

We conclude this section making a brief summary of the subjects contained in the following thesis.

Chapter 1 introduces properties of phase transitions like non-analytic behavior of some quantities, universality and scale invariance. Then, we discuss the Landau theory, the prototype of some RG ideas, and we arrive at the concepts of coarse graining and incomplete integration of modes. The chapter is concluded by exploiting general ideas of Wilsonian RG and its exact implementation.

In Chapter 2, we enter into the structure of the exact formulation of the RG. Firstly, we define the space where the RG flow takes place and we analyze how to construct the exact RG equation that encodes the information on the flow. From the flow, we show how to extract the scaling exponents that characterized specific classes of phase transitions. In particular, we classify the scaling exponents in two different independent ways: one by their importance along the flow, relevant and irrelevant, and one by their physical information, essential and inessential. Finally, we review the functional implementations of the RG, i.e. Polchinski's formulation and the Effective Average Action method, that are very powerful approaches to implement the idea of incomplete integration of modes. From the end of this chapter, we refer to the second method as functional RG.

In Chapter 3, we answer the question: is it possible to design a regulator that reproduces the results of dimensional regularization using the functional RG? The answer is positive and we show that such a regulator does not spoil higher-loop results and some non-perturbative features. This regulator depends on different parameters and their limits: in particular, at the end of the limits it vanishes. Therefore, inspired by such a property, we study a one-parameter family of regulators with the same behavior. Both classes of regulators can be collected under the name of vanishing regulators and share interesting properties in preserving symmetries.

Chapter 4 is dedicated to the generalization of the standard functional RG, which we call Essential Renormalization Group. The idea consists of implementing generic field redefinition along the flow. In particular, at every step of the RG we integrate out modes and redefine the fields simultaneously in order to keep the Effective Average Action in the most simple form possible. The field redefinitions are designed in such a way to eliminate all inessential couplings of the Gaussian fixed point. We show that in this way it is possible to avoid that the propagator develops additional poles. Then, we solve the flow and the spectrum for the 3D Ising model at second order in derivative expansion.

In Chapter 5, we apply the same procedure to Einstein Gravity including all diffeomorphic invariant terms at the fourth order in derivative expansion. Using field redefinition of the metric, we find that also in this case the propagator does not develop additional poles coming from curvature square terms. Our results suggest that Newton's constant is the only relevant essential coupling at the Reuter fixed point. Therefore, we conjecture that Quantum Einstein Gravity, the ultraviolet completion of Einstein's theory of General Relativity in the asymptotic safety scenario, has no free parameters in the absence of matter and in particular predicts a vanishing cosmological constant.

Notations

Along the thesis, we will use the following notations.
The integrals for a generic dimension d over positions and momenta will be denoted respectively as follow

$$\int_x \equiv \int \mathrm{d}^d x\,, \qquad\qquad \int_p \equiv \int \frac{\mathrm{d}^d p}{(2\pi)^d}\,. \tag{1}$$

In the case of a specific dimension, we will switch to the standard integral notation.
For measure inside functional integrals, the measure will be written as: $(\mathrm{d}\hat{f})$.
We will implement Euclidean signature.
We adopt the condensed notation for which a dot implies an integral over x such that $X \cdot Y := \int_x X(x)Y(x)$. The generalization to a multi-component field $\phi^A(x)$ is straightforward since the dot would then also imply a sum over the components $X \cdot Y := \sum_A \int_x X_A(x)Y_A(x)$.
We denote Tr the trace of a two-point function and $\mathrm{Tr}X := \int_x X(x,x)$.

Contents

Chapter 1

Critical phenomena and coarse graining

'More is different.'
P.W.Anderson

In this chapter, we start with Section 1.1 giving general features of critical phenomena in statistical mechanics and high energy physics, showing how the concepts of universality and scale invariance arise from phenomenology. In Section 1.2, we introduce the critical exponents and the notion of universality classes. In Section 1.3, we review the mean field approximation and Landau theory obtaining a rough, but easy evaluation of critical exponents [23, 24, 25]. To get better estimations, we introduce the milestone of Renormalization Group: the notion of coarse graining, that is due to Kadanoff and then developed by Wilson [26, 27, 28, 29]. The main topic of Section 1.4 is how to "reduce" or "decimate" the number of DOFs close to the critical point. Then, this idea can be easily generalized to general statistical systems and high energy physics. In fact, the RG is a great "bridge" between many different systems at different energies because of the universal language that is implemented. Finally, in Section 1.5 we conclude by preparing the general ideas to construct an Exact Renormalization Group Equation (ERGE).

1.1 General aspects of phase transitions

Phase transitions appear everywhere in daily life under many different circumstances: the most common example is given by the solid-liquid-vapour phase transitions.

So let's start by taking as an example the water phase diagram in p-T plane. The diagram is composed of regions with only one phase, lines with two coexisting phases and some interesting points, like the *triple point* and the *critical point* that we will discuss further. The line between solid and vapour is called *sublimation curve*, the one between

solid and liquid *melting curve* and the one between liquid and vapour *vapour-pressure curve*. These lines meet at the *triple point*, where all three phases do coexist, and the *vapour-pressure curve* reaches an endpoint called *critical point* ($T_c \sim 647$ K and $p_c \sim 22$ megapascals), contrary to the *sublimation curve* and the *melting curve*. The region with $T > T_c$ and $p < p_c$ is called gaseous phase, while the region with $T \geq T_c$ and $p \geq p_c$ is the supercritical or fluid phase. The phase transitions can happen in two situations: when we cross one of these lines entering into a different region or going through the critical point. An example of the first situation occurs when we cross the *vapour-pressure curve* increasing p at constant T: after the appearance of droplets, the vapour condenses and we observe a discontinuity in the density ρ, since $\rho_{vapour} < \rho_{liquid}$. Another possibility is crossing the same line increasing T at constant p: in this case, the water evaporates and we observe a discontinuity in ρ and the entropy per particle s, since $s_{vapour} > s_{liquid}$. These two possibilities are characterized both by the presence of a discontinuity in $\Delta\rho$ and Δs. However, as soon as the crossing point on the *vapour-pressure curve* approaches the *critical point*, these discontinuities go to zero with a power law, while the compressibility of the gas $\kappa_G = -\frac{1}{V}\frac{\partial V}{\partial p}\Big|_T$ diverges with a power law. Therefore, since there is no difference in the density value between liquid and vapour, at the critical point there is no difference between the two phases and we can continuously change phase going from the liquid phase to vapour phase (and viceversa) passing through the supercritical phase, where the boiling phenomenon (and condensation) does not appear. In fact, approaching the critical point, bubbles of vapour and droplets mix at all scales, from visible to atomic, contrary to points outside the supercritical phase, where small bubbles and droplets are unstable because of the surface tension.
Since the surface tension is vanishing at the critical point and droplets and bubbles with micron sizes are stable, strong light scattering causes *critical opalescence* and the water appears foggy. Since from a macroscopic point of view the main distinction between vapour and liquid is given by the density ρ, the parameter $\rho - \rho_c$ can take any sign in the supercritical phase, while it is strictly positive in the vapour phase (i.e. for $T < T_c$ and below the *vapour-pressure curve*) and strictly negative in the liquid phase (i.e. for $p < p_c$ and above the *vapour-pressure curve*). This means that the $\mathbb{Z}_2$-symmetry, whose action consists in flipping the sign of $\rho - \rho_c$, is broken in liquid-vapour transition. Along the *sublimation curve* and the *melting curve* only transitions that involve discontinuity in ρ and/or s can happen, since there is no critical point at which these curves end and the transitions to the solid phase are associated with the breaking of translational symmetry.

Another example of phase transition is given by the ferromagnetic transition of solids like Fe or Ni. In fact, such materials are characterized by a critical temperature T_c, known as Curie temperature, under which the properties of the solid change. In this case, the role of pressure is substituted by an external magnetic field H, which generically is a vector in 3D space. However, to produce a one-to-one correspondence with the water case, let's suppose the existence of a preferred direction, and so H is just a number along the line spanned by the preferred direction. For $T > T_c$, the material is paramagnetic, while for $T < T_c$ it is ferromagnetic and the T-axis becomes a transition line: crossing such a line at fixed temperature involves a discontinuity in the magnetization M. However, moving along the T-axis for $H = 0$ and approaching T_c from below, the discontinuity goes to zero with a power law and the magnetic susceptibility $\chi = \frac{1}{V}\frac{\partial M}{\partial H}\Big|_T$ diverges with a power law. This means that we can continuously change the orientation of M like the water case: this is simply achieved going through the paramagnetic phase. Moreover, the transition from paramagnetic to ferromagnetic phase is associated with breaking of rotational invariance, or $\mathbb{Z}_2$ if there is a preferred direction, since the spin can assume any configuration when the material is in the paramagnetic phase and cannot in the ferromagnetic phase. The rotational symmetry, or $\mathbb{Z}_2$, acts on the electrons spins, which are interacting between themselves and are coupled to H. The phase transition is the consequence of the magnetism that is caused at atomic level by unpaired electrons spins, since a pair of nearby electrons with aligned spins has lower energy than anti-aligned spins. At high temperature thermal fluctuations prevent order, but, as the temperature is reduced toward the critical temperature, alignment of one magnetic moment causes preferential alignment out to a considerable scale and "bubbles" of aligned spins start to become bigger and bigger. Exactly at T_c, we can observe "bubbles" with all possible sizes, as in the case of water.

As a last example, let's consider the electroweak phase transition. In early universe's moments, the thermal fluctuations were so important to modify the value of Higgs potential parameters. In particular, for $T < T_c$ the minimum of the Higgs field is the trivial one and the spontaneous symmetry breaking does not occur. Then, as the universe expanded and consequently cooled down, the Higgs developed a non-trivial expectation value, that breaks electroweak symmetry $U(1) \times SU(2)$.

At this point, it is clear that phase transitions share some universal properties

- ◇ **discontinuous and/or divergent behaviour of some quantities**
  There are always either discontinuities or divergences. In fact, a phase transition always involves non-analytic properties of thermodynamical variables: since N-finite systems have analytical description, the thermodynamic limit $N \to \infty$ is required. In the case of high energy physics, the thermodynamics limit is substituted by the continuum limit: a quantum field can be seen as a collection of an infinite number of interacting and self-interacting harmonic oscillators.

- ◇ **breaking of symmetries and existence of an order parameter**
  The phase transitions are always associated with the breaking of symmetry and we can always identify an ordered phase, where the symmetry is broken, and a disordered phase, where it is not. Moreover, phase transitions are characterized by an order parameter, which vanishes in the disordered phase and takes a finite value in the ordered one. For example, we find i) $\rho - \rho_c$ (scalar order parameter) for liquid-

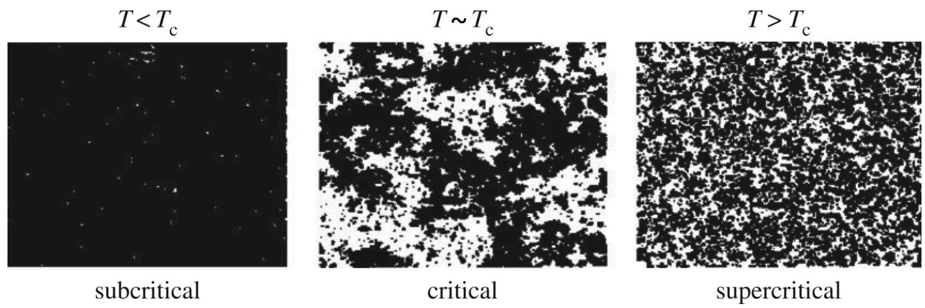


Figure 1.1: Two-dimensional Ising model with its two possible spin states represented as black or white dots (found in [5]). The figure represents snapshots of the spin configurations for three different regimes: (i) $T < T_c$ where the system is ordered with the same spin value, (ii) $T \sim T_c$ where the distribution of the sizes of regions with the same spin value follows a power law, and (iii) $T > T_c$ where the state of individual spins is purely random.

vapour transition, ii) M (scalar order parameter if there is a preferential direction or vector for the isotropic case) for ferromagnetic transition.

◇ **scale invariance and the role of fluctuations**
At the critical point we observe "bubbles" of all sizes and lengths, which means that the material exhibits scale invariance, as we can see in Figure 1.1. We can zoom in the system and observe more and more new structures at all scales, like in a fractal. This feature is due to the strength of fluctuations at the critical point and it is independent of the specific dynamics: in fact, the strong cooperative behavior of the DOFs hides the microscopic details and makes the effective dynamics of different systems convergent to similar features. This property is called universality.

From these considerations, the phase transitions can be classified as first order phase transitions if the order parameter has a discontinuity or as continuous phase transitions if the order parameter change continuously at the transition. In the second case, there is a non-analytic behavior on some other quantity and so we can be more specific relying on Ehrenfest's classification of phase transitions

◇ First-order phase transition: at least one first derivative of the free energy is discontinuous at the transition. Examples: i) The liquid-gas and the liquid-solid transitions where both ρ and s are discontinuous. ii) The ferromagnetic transition at fixed T where M is discontinuous.

◇ Second-order phase transitions: all first derivatives of the free energy are continuous, but at least one second derivative is discontinuous at the transition. Examples: i) The liquid-gas transition at the critical point where there is no jump in ρ, but the specific heat is discontinuous and the compressibility diverges. ii) The ferromagnetic transition at zero external field where M is continuous, but the heat capacity is discontinuous and the susceptibility diverges.

◇ Higher nth-order phase transitions: there is a discontinuity in at least one nth-order derivative of the free energy, while all lower order derivatives are continuous at the transition.

1.2 Critical exponents

In a continuous phase transition, the properties of the materials follow power laws, whose exponents are called critical exponents. Defining the reduced temperature, in terms of the critical temperature, as

$$\tau := \frac{T - T_c}{T_c}\,, \tag{1.1}$$

the power laws for the specific heat C_V, the order parameter m for $\tau < 0$, the susceptibility χ and the critical field, i.e. the dependence of the order parameter m to the external parameter h at T_c, are respectively

$$C_V \sim |\tau|^{-\alpha}\,, \qquad m \sim (-\tau)^{\beta}\,, \qquad \chi \sim |\tau|^{-\gamma}\,, \qquad m \sim |h|^{1/\delta}\,. \tag{1.2}$$

Note that the external parameter for the liquid-vapour case is $(p - p_c)$, while for the ferromagnets is the external magnetic field H.

Finally, we have two more critical exponents that are associated with the behavior of the two-point function. For example, in the water we consider the density-density correlator, in a ferromagnet the spin-spin correlator. Regardless of the type of material, under normal conditions the correlation function decays exponentially, and the characteristic scale of the decaying ξ, called the correlation length, is typically of the same order of molecular distances. Contrary to first-order phase transitions, in a continuous transition the correlation length becomes infinite at T_c

$$\xi \sim |\tau|^{-\nu}\,, \tag{1.3}$$

and the two-point function has the following asymptotic behavior

$$G(|\mathbf{x}|) \sim_{|\mathbf{x}|\to\infty} \frac{\mathrm{e}^{-|\mathbf{x}|/\xi}}{|\mathbf{x}|^{d-2+\eta}} \sim_{T=T_c} \frac{1}{|\mathbf{x}|^{d-2+\eta}}\,, \tag{1.4}$$

where η is called the anomalous dimension. Scale invariance arises because of the divergence of ξ and the anomalous dimension is generated by the presence of configuration states with all possible structure sizes in the system at T_c, see Figure 1.1. As we said in the

Introduction, this is similar to what happens for fractals, the scaling dimension of the quantities does not respect the canonical dimensional scaling.

Each phase transition is identified by a set of critical exponents. Thus, one may expect that phase transitions occur in a long list of different cases with relative critical exponents, since the microscopic DOFs and their dynamics vary from case to case. However, as we already pointed out, critical phenomena are characterized by a striking feature: the critical exponents are not randomly distributed, but they form families that describe different continuous phase transitions, within experimental errors. For example, the critical exponents of the critical point of water and those of the Curie transition in ferromagnets with preferred direction are the same. As we already said, this property is called universality and all the phase transitions with the same critical exponents form a universality class. Ising universality class contains the case of water and ferromagnets with preferred direction. Of course, this feature shows the existence of an underlying common explanation for many phenomena, that must be independent of the microscopic details.

1.3 Mean field approximation and Landau theory

As it is well known, mean field approximation is a powerful tool to study many-body problems. In general, it consists of an approximate treatment of interaction terms so that the Hamiltonian is reduced to an effective one of non-interacting particles immersed in an external field produced by the average interaction with all the other particles. This means that if we start with the following Hamiltonian

$$\mathcal{H} = -K \sum_{\{i,j\}} s_i s_j - \mu H \sum_i s_i \,, \tag{1.5}$$

where we have assumed a preferred direction, two spin state and nearest neighbour interactions, indicated by $\{\}$, the mean field approximation consists in sending $s_i s_j \to \langle s \rangle (s_i + s_j - \langle s \rangle)$, which implies that $\langle s_i s_j \rangle \to \langle s \rangle^2$ and where $\langle s \rangle \equiv \frac{1}{N} \langle \sum_i s_i \rangle$. The same result can be achieved by writing $s_i = \langle s \rangle + \delta s_i$ and keeping only linear terms in the fluctuations δs_i. This way, it is straightforward to find the canonical partition function $\mathcal{Z}_C$ and the associated free energy $\mathcal{F}$

$$\mathcal{F} = -T \ln \mathcal{Z}_C = \frac{zNK}{2} \langle s \rangle^2 - TN \ln \left(2 \cosh \left(\frac{zK\langle s \rangle + \mu H}{T} \right) \right) \,, \tag{1.6}$$

where N is the number of sites and z is the coordination number. Then, the magnetization $M = N\langle s \rangle$ can be found differentiating $\mathcal{F}$ respect to H and in particular

$$\langle s \rangle = \tanh \left(\frac{zK\langle s \rangle}{T} \right) \,. \tag{1.7}$$

The previous equation reproduces phase transition: there is only the trivial solution if $T > T_c \equiv zK$ and an additional non-trivial value of M for $T < T_c$.

In order to find the critical exponents and so the behavior near T_c, we implement Landau idea [19] of expanding (1.6) for small $\langle s \rangle$, small τ and small H

$$\mathcal{F} = \mathcal{F}_0 + V\left(-hm + \frac{1}{2} r\,\tau\, m^2 + \frac{1}{4} u\, m^4 + \ldots\right)\,, \tag{1.8}$$

where the parameters depend on the DOFs and dynamics of the specific example that we consider. However, Equation (1.8) has the advantage that can be used to describe all continuous phase transitions in the same universality class of the ferromagnets simply changing the physical meaning of the parameters. Finally, it is possible to verify that the order parameter and the entropy are continuous at T_c since they are first derivatives of $\mathcal{F}$, while the specific heat is discontinuous and the susceptibility diverges. In short, the critical exponents are

$$\alpha = 0\,, \qquad \beta = \frac{1}{2}\,, \qquad \gamma = -1\,, \qquad \delta = 3\,. \tag{1.9}$$

The theory described above deals with situations where the value of the order parameter is uniform across the system, so in order to find the last two critical exponents, that store information about the fluctuations, we have to consider long wave modes of the order parameter. In order to do this, we have to add a kinetic term $\frac{1}{2}\left(\partial m\right)^2$. This means that we are dealing with a scalar theory with four-point interaction and a temperature-dependent mass. The two-point function is the propagator of a scalar theory and at first order in u, we have

$$\langle m(\mathbf{x}) m(0)\rangle - m^2 \sim \int_q \frac{\mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot\mathbf{q}}}{q^2 + r\,\tau} + o(u) \sim_{|\mathbf{x}|\to\infty} \frac{\mathrm{e}^{-|\mathbf{x}|\sqrt{r\,\tau}}}{|\mathbf{x}|^{d-2}}\,, \tag{1.10}$$

from which we read $\xi^{-2} \sim -r\,\tau$ and

$$\nu = \frac{1}{2}\,, \qquad \eta = 0\,. \tag{1.11}$$

Sufficiently close to the critical point, inspired by Landau theory we can deduce some general relations that hold for the critical exponents. The general free energy can be rewritten as

$$\mathcal{F}/V = f_0 + |\tau|^{d/\theta_\tau} f\left(h\,|\tau|^{-\theta_h/\theta_\tau}\right)\,, \tag{1.12}$$

where f can be designed to satisfy the behavior in the limits $\tau = 0$ and $h = 0$ and θ_τ and θ_h are the scaling exponents of τ and h [1]. In particular, by definition we have $2 - \alpha = d/\theta_\tau$

[1] This means that under a rescaling of lattice spacing by a factor b, $\tau \to b^{\theta_\tau}\tau$ and $h \to b^{\theta_h} h$.

and $2-\alpha-\beta = \theta_\tau/\theta_h$. Using scaling law (1.12), it is possible to find the following relations between the critical exponents that talks about an homogeneous field situation

$$\alpha + 2\beta + \gamma = 2\,, \qquad \beta + \gamma = \beta\,\delta\,. \tag{1.13}$$

Using again the scaling law (1.12), the fluctuation-dissipation theorem and the hyperscaling assumption, i.e. $f \propto \xi^{-d}$, we obtain Fisher's and Josephson's identities

$$\gamma = \nu(2-\eta)\,, \qquad \alpha = 2 - d\,\nu\,. \tag{1.14}$$

Therefore, we start with six critical exponents and we end up with only two independent critical exponents. Typically everything is expressed in terms of η and ν

$$\alpha = 2 - d\,\nu\,, \qquad \beta = \nu\frac{d-2+\eta}{2}\,, \tag{1.15}$$

$$\gamma = \nu(2-\eta)\,, \qquad \delta = \frac{d+2-\eta}{d-2+\eta}\,. \tag{1.16}$$

This choice is related to the fact that ν is typically related to the mass parameter, while anomalous dimension η enters as a correction to the classical scaling of field. This connection becomes more evident when we express them in terms of θ_τ and θ_h

$$\nu = 1/\theta_t\,, \qquad \eta = d + 2 - 2\theta_h\,, \tag{1.17}$$

since the mass parameter is proportional to τ and the field is coupled to h.

Arrived at this point, it is compulsory a consistency check, which is called the Landau-Ginzburg criterion. We have to verify that our original assumptions are justified. In particular, since in mean field approximation $\langle m(\mathbf{x})m(0)\rangle \approx m^2$, we have to verify if the space average of the two-point function in Equation (1.10) is small compared to the space average of the squared mean value of the order parameter. This evaluation gives

$$m^2 V \gg \xi^2 \implies 1 \gg u\,(-r\,\tau)^{\frac{d-4}{2}}\,. \tag{1.18}$$

We can note that the dimension d is crucial for critical behavior. The Landau-Ginzburg criterion shows that fluctuations play an increasingly important role if the dimension of the system is reduced and there are limitations in the applicability of the mean field theory. In particular, inside the Ising universality class

- for $d > 4$ Landau theory reproduces quantitative correct results;
- for $d \leq 4$ the picture is only qualitatively correct: we cannot take the limit $\tau \to 0$, but depending on the microscopic details of the system, stored in r and u, there could be a temperature window where the approximation behaves better. Comparing Table 1.1 and Equations (1.9) and (1.11), we can observe the discrepancy for the Ising model where ω is the first element of a series of critical exponents that stores information about corrections to the Landau theory. We will encounter it later.

	$d = 2$	$d = 3$	$d = 4$
α	0	0.11008(1)	0
β	1/8	0.326419(3)	1/2
γ	7/4	1.237075(10)	1
δ	15	4.78984(1)	3
η	1/4	0.036298(2)	0
ν	1	0.629971(4)	1/2
ω	2	0.82966(9)	0

Table 1.1: In the following table, we insert critical exponents of the Ising model for dimensions 2, 3 and 4.

- ⋄ from the exact calculation of the partition function, for $d = 1$ there is no phase transition at non-zero temperature.

From these considerations, $d = 4$ is the upper critical dimension of this universality class, while $d = 1$ is the lower critical dimension.

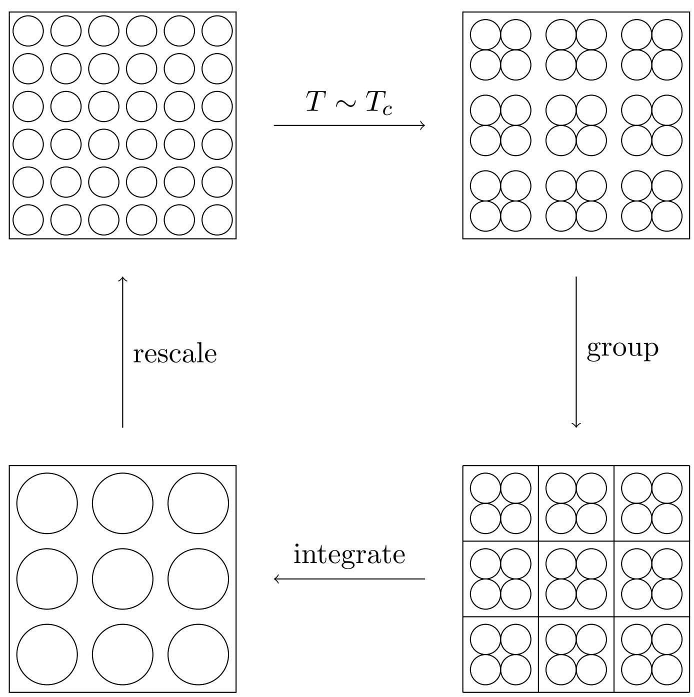
Each universality class will be characterized by these two critical dimensions, upper and lower. The first one establishes the regime where the Landau theory becomes quantitatively incorrect, while the second one where we find a completely wrong result. In fact, decreasing d, fluctuations change the behavior of the thermodynamic quantities close to the transition and we observe the quantitatively different values of the critical exponents, see Table 1.1. By reducing further d, fluctuations become so strong that they wash out the transition altogether and no order can be reached for non-zero temperature.

We conclude this section by stressing the fact that Landau theory is able to capture the properties of continuous phase transitions and gives a not bad estimation of the critical exponents. The simplicity and clearness of this method are the reason why this is the starting point of any calculations concerning about critical phenomena. Therefore, even if for dimensions lower than the upper critical dimension the evaluation of the critical exponents are quantitatively wrong, we ”don’t throw the baby out with the bath water”. For dimensions between the lower and the upper critical dimensions, we can improve our quantitative estimations on the theory including more monomials of the order parameter and its fluctuations to all orders in Equation (1.8). Therefore, we treat the Hamiltonian as an expansion and we obtain corrections to the Landau picture, i.e. exponents like ω. This can be done in a systematic way using the Renormalization Group [30], as we will discuss further.

1.4 Kadanoff idea

As we have seen in the previous section, the correlation length becomes very large near the critical temperature and this implies that for a direct calculation of critical behavior we have to consider all DOFs in a volume of the size of the correlation length. This is due to the fact that inside this volume all DOFs are strongly correlated. Of course, this can become very involved. However, inside the volume of the size of correlation length, all DOFs behave like a single effective DOF. Therefore, Kadanoff came out with the idea of "reducing" or "decimate" the density of DOFs by replacing the large correlation length with a smaller one and changing consistently the effective dynamics. This procedure can be iterated until the correlation length is of the order of unity, measured with the new rescaled effective lattice spacing. This way, we eliminate all strongly coupled DOFs and obtain an effective system with relatively simple uncorrelated DOFs. This method is called *coarse graining*, and it is the base of the RG.

Let's sketch this idea with a concrete visual example. Imagine observing with a microscope a system at a given energy scale Λ_0, which means that we are considering an ensemble of DOFs with given particles spacing a_0, which is proportional to the inverse of the energy scale $a_0 \sim 1/\Lambda_0$. Since physics is insensitive to fluctuations with lengths smaller than a_0 or equivalently energies bigger than Λ_0, for our microscope these DOFs are elementary. As T approaches T_c, ξ becomes bigger, i.e. $\xi > a_0$, and "bubbles" of condensed DOFs start to appear. This is shown in the first line of the graphic reported below.



(1.19)

Since the "bubbles" behaves like single DOF because of the strong correlation, Kadanoff suggested doing three operations:

◇ grouping, or *Kadanoff blocking*, the strongly correlated DOFs to form the effective

DOFs, whose "centers of gravity" are spaced with a new length $a_1 > a_0$, i.e. there is a new effective scale energy $\Lambda_1 \sim 1/a_1$;

- ◇ integrating out the fluctuations inside each effective DOF, or equivalently all the modes with length between a_1 and a_0 or with energy between Λ_0 and Λ_1;
- ◇ rescaling the new length/energy in order to make the system as similar as possible to the starting not strongly correlated elementary system.

In particular, the third operation is simply achieved by zooming out with the microscope. At the end of this operation, the new scale a_1 will become the elementary length. This way, we are "hiding" the microscopic/elementary details, obtaining the desired effective description. We use the verb "to hide" because at the new scale we can not observe directly the previous DOFs, but the integrated information is stored/hidden inside the new effective dynamics.

Now we will take a quantitative example and we will present the example given by Wilson in [29], making some little changes. Take a plane lattice with lattice spacing a_0, where at each site we have a spin with two configurations and the Hamiltonian given in Equation (1.5). Since we are close to T_c, all the spins, contained in a square region with b^2 spins [2], will be strongly correlated and we can suppose that in each b^2-block only two states are present: all up or all down. Then we define the renormalized block spins such that their magnitude is ± 1: in particular, the Fourier transform of the spin field $\sigma_{\mathbf{q}} = \zeta \sigma^{(1)}_{b\mathbf{q}}$. Due to the rescaling of the new effective variable, the field can receive corrections to its canonical dimension or equivalently can develop the anomalous dimension discussed in Section 1.2. In this case it is given by the following relation

$$\eta = d + 2 - 2\frac{\ln \zeta}{\ln b}\,. \tag{1.20}$$

The Hamiltonian of the block spins is also constructed in such a way to have the same form of (1.5) with the only substitution in the interaction coupling: $K \to K_1 = f(K)$. Reducing the dimension of the lattice implies

$$\xi_1(K) \equiv \xi\left[f(K)\right] = \frac{1}{b}\xi(K)\,. \tag{1.21}$$

At the critical point, $K_c = f(K_c)$ so that $\xi\left[f(K_c)\right]$ is infinite. Near T_c, or equivalently K_c, we have $f(K) - K_c = f'(K_c)\,(K - K_c)$, and using the Equation (1.21) and the behavior of ξ near T_c, we get

$$\nu = \frac{\ln b}{\ln f'(K_c)}\,. \tag{1.22}$$

[2]In [29] $b = 2$ and this is what is represented in the diagram (1.19). Not specifying the value of b is done to stress the renormalization scheme dependence.

Note that the choice of the value b and the shape of the block spins represent the renormalization scheme: this means that in the RHS of (1.20) and (1.22) the scheme dependence must cancel in order to match with the physical quantity on the LHS. Of course, apart from toy models we have to rely on approximations and so we introduce some spurious scheme dependence in the evaluation of η and ν.

For example, we can approximate this system with the Gaussian model. This approximation consists in smoothing the delta functions in the canonical partition function

$$\mathcal{Z} = \prod_m \int (\mathrm{d}s_m) 2\delta(s_n^2 - 1) \mathrm{e}^{(K/T)\sum_{n,i} s_n s_{n+\hat{\imath}}} \to \prod_m \int (\mathrm{d}s_m) \mathrm{e}^{-\frac{1}{2} b s_m^2 + (K/T)\sum_{n,i} s_n s_{n+\hat{\imath}}} \,. \tag{1.23}$$

This mathematical trick to transform the sums into integrals can be interpreted as the presence of noise in the evaluation of the particles configurations.
Completing the square and using the Fourier transform of the spin variable $\sigma_{\mathbf{q}} = \sum_n \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\hat{n}} s_n$, we get

$$\mathcal{Z} = \int (\mathrm{d}\sigma_{\mathbf{q}}) \mathrm{e}^{-\frac{K}{2T}\int_{\mathbf{q}} \left[\sum_i |\exp(\mathrm{i}q_i) - 1|^2 + r\right] \sigma_{\mathbf{q}} \sigma_{-\mathbf{q}}} \,, \tag{1.24}$$

where $r = (bT - 2dK)/K$. We can immediately see that for $T_c = 2dK/b$ the parameter r is zero as in the Landau theory. Since we are interested in long-range behavior, we expand in q and then, because of practical reasons, change the integration from $-\pi < q_i < \pi$ to $0 < |\mathbf{q}| < 1$. After the expansion in momenta and keeping only the leading term, we obtain that $\xi^{-2} \sim r$ and the same scaling exponents of Landau theory. Now let's apply the Kadanoff idea, i.e. we integrate out modes with $1/b < |\mathbf{q}| < 1$ in such a way that

$$\mathrm{e}^{-\mathcal{H}_1[\sigma^{(1)}]/T} = \int (\mathrm{d}\sigma_>) \mathrm{e}^{-\mathcal{H}[\sigma_>]/T} \,. \tag{1.25}$$

Since there is no interaction between the modes, the Hamiltonian maintains the same form apart from the fact that there is a different integration range. Now we do some scale changes designed to make the effective Hamiltonian look as much like the original as possible. Defining $|\mathbf{q}_1| \equiv b|\mathbf{q}|$, such that $0 < |\mathbf{q}_1| < 1$, the effective variable $\sigma_{\mathbf{q}} = \zeta \sigma^{(1)}_{\mathbf{q}_1} = \zeta \sigma^{(1)}_{b\mathbf{q}}$, where $\zeta \equiv b^{1+d/2}$ and $r_1 = b^2 r$ or equivalently $\xi_1 = \xi/b$, we can bring the Hamiltonian in the same form. At this point using Equations (1.20) and (1.22), we find the critical exponents of Landau theory, i.e. $\eta = 0$ and $\nu = 1/2$.

The situation becomes more interesting if we consider the following modification

$$\mathcal{H}[\sigma] = \frac{1}{2}\int_{\mathbf{q}} (q^2 + r)\sigma_{\mathbf{q}}\sigma_{-\mathbf{q}} + u \int_{\mathbf{q}_1}\int_{\mathbf{q}_2}\int_{\mathbf{q}_3} \sigma_{\mathbf{q}_1}\sigma_{\mathbf{q}_2}\sigma_{\mathbf{q}_3}\sigma_{-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3} \,. \tag{1.26}$$

Using the same procedure, for $d < 4$ and at the leading order in u

$$\zeta_1 = b^{1+d/2} \,, \qquad r_1 = b^2\left[r + 3c\frac{u}{1+r}\right] \,, \qquad u_1 = b^{4-d}\left[u - 9c\frac{u^2}{(1+r)^2}\right] \,, \tag{1.27}$$

where $c = b^2 \int_{1/b<|\mathbf{q}|<1} 1$. Iterating and keeping always the leading term in u, we obtain the effective interaction that describes the behavior of the modes with $0 < |\mathbf{q}| < b^{-\ell-1}$

$$r_{\ell+1} = b^2 \left[r_\ell + 3c \frac{u_\ell}{1+r_\ell} \right] , \qquad u_{\ell+1} = b^{4-d} \left[u_\ell - 9c \frac{u_\ell^2}{(1+r_\ell)^2} \right] . \tag{1.28}$$

We can note that inside our approximation there are two *fixed points*, i.e. values of the couplings that do not change after the coarse graining iteration, i.e. $r_\star = r_{\ell+1} = r_\ell$ and $u_\star = u_{\ell+1} = u_\ell$. One is the Gaussian fixed point where $r_\star = u_\star = 0$ and the other one is the Wilson-Fisher (WF) fixed point. The first one is already treated in the Gaussian model, while the latter one is a new feature due to the interaction term. The approximation of keeping only the leading terms in u is justified for dimensions very close to $d = 4$, in fact for infinitesimal $\epsilon = 4 - d$ [31] the WF fixed point is located at

$$u_\star = \frac{1}{9c}(b^{4-d} - 1) \sim \frac{\ln b}{9c}\epsilon , \qquad r_\star = -\frac{3b^2 c}{1-b^2} u_\star \sim -\frac{b^2 c \ln b}{3(1-b^2)}\epsilon . \tag{1.29}$$

Note that the position of the fixed point is scheme-dependent: this remains true even if more sophisticated tools are implemented, as we will see later. Studying the stability of the WF fixed point, we have

$$\begin{pmatrix} r_{\ell+1} - r_\star \\ u_{\ell+1} - u_\star \end{pmatrix} \sim M \begin{pmatrix} r_\ell - r_\star \\ u_\ell - u_\star \end{pmatrix} , \quad \text{where } M = \begin{pmatrix} b^2 \left[1 - \frac{3cu_\star}{(1+r_\star)^2} \right] & \frac{b^2 3c}{1+r_\star} \\ \frac{b^\epsilon 18c(u_\star)^2}{(1+r_\star)^3} & b^\epsilon \left[1 - \frac{18cu_\star}{(1+r_\star)^2} \right] \end{pmatrix} ,$$

and the eigenvalues are

$$\lambda_1 = b^2 \left(1 - \epsilon \frac{\ln b}{3} \right) , \qquad \lambda_2 = 1 - \epsilon \ln b . \tag{1.30}$$

For sufficiently large n, using the fact that $r \propto \tau$ and $\lambda_2 < 1$, we have $r_{\ell+n} - r_\star \propto \lambda_1^n (T - T_c)$ and $u_{\ell+n} - u_\star \propto \lambda_1^n (T - T_c)$. These relations imply that $r_{\ell+n+1}(T + T_c\tau/\lambda_1) - r_\star = r_{\ell+n}(T + T_c\tau) - r_\star$ and $u_{\ell+n+1}(T + T_c\tau/\lambda_1) - u_\star = u_{\ell+n}(T + T_c\tau) - u_\star$. Using the scaling rule for the effective interactions

$$\xi(r_{\ell+n}, u_{\ell+n}) = b^{-\ell-n} \xi(r_0(T - T_c), u_0(T - T_c)) \tag{1.31}$$

and the previous consideration, it holds

$$b^{-\ell-n-1} \xi(\tau/\lambda_1) = b^{-\ell-n} \xi(\tau) . \tag{1.32}$$

Using the scaling law of ξ, we find Equation (1.22) with $f'(T_c) = \lambda_1$. Using (1.20) and (1.22), we find

$$\eta = o(\epsilon^2) , \qquad \nu = \frac{1}{2} + \frac{\epsilon}{12} + o(\epsilon^2) . \tag{1.33}$$

where the lowest order contribution to η is $\sim \epsilon^2$, since it receives contribution from diagram proportional to u^2.

We conclude this section by pointing out the fact that even if the procedure depends on scheme choices, which together with the approximation affects some quantities like the fixed point position, it is possible to extract physical information. Then, the idea consists in refining step by step consistently and systematically.

1.5 Wilsonian Renormalization Group

Along this chapter, we concentrated more on statistical mechanics system and in particular, we observe the Renormalization machinery on Hamiltonians $\mathcal{H}$. For the rest of the thesis, we will treat continuum system whose dynamics is described by actions, since we are interested also in high energy physics.

As we have seen in the previous section, the Wilsonian Renormalization Group consists in the calculation of the functional integral by an iterative procedure where at each step the functional integral involves only modes with momenta contained in a given finite range. In particular, suppose that we observe with the microscope at length $\sim 1/\Lambda_0$, meaning that we start with a theory with bare action S_0 at some UV cutoff Λ_0. The modes of the field can be classified by the wave vector or, equivalently, by the momentum q, and by construction all modes has $|q| \leq \Lambda_0$. Then, we zoom out and we want to know how the effective dynamics look like. Zooming out means that we are observing the system at a different energy scale $\Lambda_1 = \Lambda_0/b$, with $b > 1$. All the modes contained in the *momentum shell* $\Lambda_1 < |q| < \Lambda_0$ represents fluctuations which we cannot see with current sensitivity of the microscope, i.e. $\sim 1/\Lambda_1$. We denote modes in this momentum shell as *fast* modes, χ_f, and modes outside as *slow* modes, χ_s. Proceeding in the same way as in Equation (1.25), the action at scale Λ_1, denoted with $S_1[\chi_s]$, is called Wilsonian Effective Action and is defined as

$$\mathrm{e}^{-S_1[\chi_s]} = \int (\mathrm{d}\hat{\chi}_f)\, \mathrm{e}^{-S_0[\hat{\chi}]} \equiv \int_{\Lambda_1<|q|<\Lambda_0} (\mathrm{d}\hat{\chi})\, \mathrm{e}^{-S_0[\hat{\chi}]}\,, \tag{1.34}$$

where we have split the measure in the path integral, i.e. $(\mathrm{d}\hat{\chi}) = (\mathrm{d}\hat{\chi}_s)(\mathrm{d}\hat{\chi}_f)$. Note that by construction, the partition function obtained by the functional integral over $\hat{\chi}_s$ of $\mathrm{e}^{S_1[\chi_s]}$ is equal to the original partition function, which together with the physical information remains untouched. Whereas S_0 is generally chosen to be a simple local functional of the fields, for example take the scalar $\mathbb{Z}_2$-invariant theory

$$S_0 = \int_x \left(\frac{1}{2}\partial_\mu\chi\partial_\mu\chi + \frac{1}{2}m^2\chi^2 + \frac{1}{4!}\lambda\chi^4 \right)\,, \tag{1.35}$$

the action S_1 is composed by all manner of complicated interactions induced by the fluctuations, e.g.

$$S_1 = \int_x \left(\frac{Z(b)}{2} \partial_\mu \chi \partial_\mu \chi + \frac{1}{2} m_1(b)^2 \chi^2 + \frac{1}{4!} \lambda_1(b) \chi^4 + \dots \right) . \tag{1.36}$$

The Wilsonian Effective Action S_1 represents the new action of the slow modes χ_s and it encodes the effective dynamics at the new scale. Thus, it is true that the microscope at the current scale $\sim 1/\Lambda_1$ is insensitive to the fast fluctuations, but the effective description still maintains some information about the details of the microscopic theory. The new couplings that appear in the interactions inside S_1 encode the integrated information about fast modes. Then, as we said in the previous section, the strategy consists in the iteration of such a procedure. The integration, weighted with the factor e^{-S_1}, over modes contained in the momentum shell $\Lambda_2 < |q| < \Lambda_1$, with $\Lambda_2 = \Lambda_1/b = \Lambda_0/b^2$, produces another Wilsonian Effective Action S_2 for the modes with $|q| < \Lambda_2$. The procedure is iterated until one reaches some infrared, macroscopic scale. From this point of view the Renormalization Group is composed by transformations, defined by different scheme choices [3],

$$U(S_0) = S_1 \quad \to \quad U(S_1) = S_2 \quad \to \quad U(S_2) = S_3 \quad \dots \to \quad U(S_{n-1}) = S_n \ . \tag{1.37}$$

Note that the result consists of a sequence of Wilsonian Effective Actions $S_0, S_1, S_2, \dots S_n$ that are valid at decreasing cutoffs $\Lambda_0, \Lambda_1, \Lambda_2, \dots \Lambda_n$, but all produce the same IR physics by construction. This sequence of actions is the Wilsonian Renormalization Group. Contrary to Perturbative Renormalization Group, the Wilsonian implementation is a "functional renormalization" in the sense that at each step one renormalizes the whole action functional and not just one or a few couplings. Note that for the moment we have only discussed the grouping and integrating steps of the diagram (1.19). The rescaling step consists in redefining the momenta so that they satisfy the original constraint, i.e. $q' \equiv q/b < \Lambda_0$. This means that after zooming out we change units in such a way that the new scale appears as the original initial scale. Since we want S_1 to look as similar to S_0 as possible, another part of the rescaling consists of the normalization of the kinetic term. We will discuss further this point along the thesis.

Each step of this procedure can be discrete like in the previous example or continuous. The advantage of the latter case consists of the fact that it is possible to construct a differential equation that describes the evolution of the action along an infinitesimal change of the energy scale, i.e. Λ to $\Lambda - \delta\Lambda$. This is achieved by taking the number b close to 1 and, thus, making the difference between successive Wilsonian Effective Actions arbitrarily small. After defining the "RG time" $t := -\ln \frac{\Lambda}{\Lambda_0}$, the exact Renormalization Group equation (ERGE) reads

$$\lim_{b \to 1} \frac{S_i - S_{i+1}}{1 - 1/b} \equiv \frac{\mathrm{d}S_t}{\mathrm{d}t} = \mathcal{T}[S_t] \,, \tag{1.38}$$

[3] As we already stress out in the previous section, for example the choice of b represents part of the scheme choice.

which is also called the *flow equation*. Note that the minus sign in the definition of the RG time is conventional: it means that we are studying the flow from UV to IR.

The flow equation defines a *beta functional*, which describes the flow of the whole action. We want to stress the following points

- ◇ since each step of the renormalization machinery involves an integration over a finite range of momenta, the flow equation is finite;
- ◇ the Wilsonian Renormalization Group provides the non-perturbative definition of Renormalization Group, since no assumptions have been made on the action;
- ◇ in a loop expansion the difference $S_i - S_{i+1}$ is calculated summing the contributing Feynman diagrams where the integration over loop momenta is restricted to a shell of thickness $\delta q = (1 - 1/b)\, q$. This means that each loop will give a contribution to some power of $\delta q/q = (1 - 1/b)$. Taking the limit of continuous RG transformations, higher loops becomes negligible and the Wilsonian RG will take the form of a one-loop equation [32]. This apparent simplicity comes with a prize: the flow equation gives the evolution of infinitely many couplings at the same time;
- ◇ in Equation (1.34) Λ_1 acts as the UV cutoff of the modes that remain to be integrated out, but it can also be seen as the IR cutoff of the modes that have already been integrated out. From this point of view, the Wilsonian Effective Action would seem to depend on two cutoffs. However, as we shall see, this is not a severe problem, since we will not be interested in calculating the action but rather its beta functional, which is UV finite. Thus, it is not necessary to specify the UV cutoff. In Chapter 2, we will discuss that using an IR cutoff has other advantages.

Quoting Wilson [27], the general flow equation (1.38) can be seen as the analogous equation to the second Newton's law

$$\frac{\mathrm{d}p}{\mathrm{d}t} = F(x,p)\,. \tag{1.39}$$

The idea is that given an initial theory described by S_0, there is a corresponding dynamics varying the RG time. Like Newtonian dynamics, different scenarios can happens along the flow, but the most simple case consists in reaching a stationary point or more precisely a *fixed point*, i.e. an action $S_\star$ that satisfies the conditions $U(S_\star) = S_\star$ in the discrete case and

$$\mathcal{T}[S_\star] = 0 \tag{1.40}$$

in the continuous case.
From this point of view, QFTs are trajectories in the infinite space of possible actions[4],

[4]In Chapter 2, we will be more precise after defining the *theory space*.

while fixed points correspond to CFTs, since they are invariant/stationary under the RG transformation, and they represents a specific universality class.

As we have seen with the WF fixed point found in Section 1.4, the RHS of Equation (1.38) is affected by some arbitrariness due to the choice of the renormalization scheme. This means that the form of $S_\star$ depends on this choice, see the fixed point position in Equation (1.29). The scheme dependence is not the only arbitrariness that we have to deal with: we can parameterize our theory in different ways, redefining the couplings and the fields. Of course, it is useful to work with variables that have direct physical meaning, like the order parameter. However, sometimes a change of variable can clear the picture of our description. We will discuss this point further in Chapter 4. Therefore, in order to extract some universal quantities, like the critical exponents, the physical information must be disentangled by the scheme dependence and the redundancies due to the freedom to choose the variables. It is also important to keep in mind that the scheme dependence is not harmless. Hiding is not the only result of the scheme dependence. Choosing some particular scheme can wash out some fixed points and other possible configurations of the Wilsonian Effective Action. This means that the scheme dependence choice must be done carefully and according to the system that we want to analyze. In other words, the transformation operator $\mathcal{T}$ in Equation (1.38) depends on the Wilsonian Effective Action and the scheme dependence must be designed for the particular choice of the theory. We will discuss this further and more precisely in Chapter 2 and in particular, we will see that this remains true also for other versions of the ERG.

Chapter 2

Exact Renormalization Group

'πάντα ρεῖ'
Heraclitus of Ephesus

The idea of this chapter consists of a review of the basic notions of ERG [33, 34] with additional comments. Several concepts, that we encounter, will be treated and generalized in Chapter 4. In Section 2.1, we introduce the generating functionals, which are useful objects for QFT computations. In Section 2.2, we define the space parameterized by the couplings of theories, which will be the stage of the dynamics described by ERG (1.38). Following [34, 35], in Section 2.3 we specify the structure of ERG (1.38) and we obtain the translation in mathematical language of the operations reported in diagram (1.19). In Section 2.4, we present the Wilsonian ERG [29] and the sharp cutoff version of Wegner [32]. In Section 2.5 we analyze the properties of the fixed points presenting some exact results of Wegner [35]: in particular, we classify the elements of the spectrum of the fixed points. In Section 2.6, we show how the spectrum does not depend on the parameterization of the couplings. In Sections 2.7 and 2.8, we move to the functional implementations of the RG encountering two formulations: the Polchinski formulation, that is directly connected to the Wilsonian ERG, and the EAA, that is connected by the Legendre transform. In both cases, the Local Potential Approximation (LPA) is presented, however only for the EAA method there are explicit details of numerical methods to solve it in $d = 3$. Finally, in Section 2.9 we conclude by stressing some points on the effects of renormalization scheme, parameterization of the action and renormalization conditions to the flow equation.

Note that every time we talk about quantum fluctuations, which are used in high energy physics, we also include statistical fluctuations, which instead are used in statistical systems like ferromagnets. In fact, the different implementations of the RG, and especially the functional implementations, are characterized by the flexibility to be applied to many different systems, from high energy to low energy physics.

2.1 Generating functionals in QFT

In QFT all physical information is stored in correlation functions. In the path-integral formalism, these are functionals $\hat{\mathcal{O}}[\hat{\chi}]$ of the quantum field $\hat{\chi}$ averaged over all possible field configurations (quantum fluctuations), in which each configuration is weighted with e^{-S}. Therefore, the most general objects which we wish to compute are expectation values of *observables* $\hat{\mathcal{O}}$ given by

$$\langle \hat{\mathcal{O}} \rangle := \mathcal{N} \int (\mathrm{d}\hat{\chi})\ \hat{\mathcal{O}}[\hat{\chi}]\ \mathrm{e}^{-S[\hat{\chi}]} \,, \tag{2.1}$$

where $\mathcal{N}^{-1} = \int (\mathrm{d}\hat{\chi})\, \mathrm{e}^{-S[\hat{\chi}]}$ and $\hat{\mathcal{O}}[\hat{\chi}] = \hat{\mathcal{O}}$ is an observable expressed as generic functional of the fields $\hat{\chi}$. For example, one could be interested in an n-point function of the field in which case

$$\hat{\mathcal{O}}[\hat{\chi}] = \prod_{i=1}^{n} \hat{\chi}(x_i) \,, \tag{2.2}$$

but one could also be interested in products of composite operators at different points in space.

The exact definition of the path integral measure depends on the regularization, and typically it is defined by

$$\int (\mathrm{d}\hat{\chi})\, \mathrm{e}^{-\frac{1}{2}\hat{\chi}\cdot M_\Lambda \cdot \hat{\chi}} = 1 \,, \tag{2.3}$$

where Λ is the ultraviolet cutoff which we will formally take to infinity or to some scale much greater than all relevant physical scales. The two-point function $M_\Lambda(x_1, x_2)$ can be understood as a metric on $\mathcal{M}$ which is independent of the field $\hat{\chi}$ and should diverge in the continuum limit, namely

$$\lim_{\Lambda \to \infty} M_\Lambda \to \infty \,. \tag{2.4}$$

In the simplest case, $M_\Lambda(x_1, x_2) = \alpha \Lambda^2 \delta(x_1, x_2)$, where α is a positive constant.

We are at liberty to change the integration variable in the functional integral (2.1)

$$\langle \hat{\mathcal{O}}' \rangle = \mathcal{N}' \int (\mathrm{d}\hat{\chi}')\ \hat{\mathcal{O}}'[\hat{\chi}']\ \mathrm{e}^{-S'[\hat{\chi}']} \,, \tag{2.5}$$

provided the exponential factor transform as a density

$$\mathrm{e}^{-S[\hat{\chi}']} = \mathrm{e}^{-S'[\hat{\chi}'[\hat{\chi}]]} \det \frac{\delta \hat{\chi}[\hat{\chi}']}{\delta \hat{\chi}'} \,. \tag{2.6}$$

Note that (2.5) is equivalent to the previous definition (2.1), since the observable $\mathcal{O}$ transform as a scalar, i.e. $\mathcal{O}'[\hat{\chi}'] = \mathcal{O}[\hat{\chi}]$. The physical information that we measure is given by generalized integral and this means that our description of the physical world possesses some redundancies by construction. We will discuss more precisely the implications of these redundancies in the next sections.

In practice, the computation of correlation functions is facilitated by the introduction of suitable generating functionals. For example, the generating functional $\mathcal{Z}[j]$ of the all correlation functions for the field $\hat{\chi}$ is given by

$$\mathcal{Z}[j] := \langle \mathrm{e}^{j\cdot\hat{\chi}} \rangle = \mathcal{N} \int (\mathrm{d}\hat{\chi})\ \mathrm{e}^{j\cdot\hat{\chi}}\, \mathrm{e}^{-S[\hat{\chi}]}\,, \tag{2.7}$$

while the generating functional $\mathcal{W}[j]$ of the connected correlation functions for the field $\hat{\chi}$ is given by

$$\mathrm{e}^{\mathcal{W}[j]} := \mathcal{Z}[j]\,, \tag{2.8}$$

where $j \cdot \hat{\chi}$ is a source term for the field $\hat{\chi}$. In presence of the source, expectation values are given by

$$\langle \hat{\mathcal{O}} \rangle_j = \mathrm{e}^{-\mathcal{W}[j]} \langle \mathrm{e}^{j\cdot\hat{\chi}} \hat{\mathcal{O}} \rangle\,, \tag{2.9}$$

and they reduce to (2.1) by taking $j = 0$. In practice, given (2.7) and (2.8), source-dependent expectation values can be computed as

$$\langle \hat{\mathcal{O}} \rangle_j = \mathrm{e}^{-\mathcal{W}[j]} \hat{\mathcal{O}} \left[\hat{\chi} \left[\frac{\delta}{\delta j} \right] \right] \mathrm{e}^{\mathcal{W}[j]}\,. \tag{2.10}$$

For example, the n-point functions is obtained by

$$\langle \prod_{i=1}^{n} \hat{\chi}(x_i) \rangle = \mathrm{e}^{-\mathcal{W}[j]} \prod_{i=1}^{n} \hat{\chi} \left[\frac{\delta}{\delta j(x_i)} \right]\ \mathrm{e}^{\mathcal{W}[j]} \Bigg|_{j=0}\,. \tag{2.11}$$

Given $\mathcal{W}[j]$, other generating functionals, related to $\mathcal{W}[j]$ by transformations and/or the addition of further sources, can be considered[1]. For example, the one-particle irreducible (1PI) Effective Action $\Gamma[\chi]$ is obtained by the Legendre transform

$$\Gamma[\chi] = -\mathcal{W}[j] + \chi \cdot j\,, \tag{2.12}$$

where $\chi = \langle \hat{\chi} \rangle_j$ is the mean parameterized field. Equivalently, $\Gamma[\chi]$ can be defined by the solution to the integro-differential equation

$$\mathrm{e}^{-\Gamma[\chi]} = \langle \mathrm{e}^{(\hat{\chi}-\chi)\cdot \frac{\delta}{\delta\chi}\Gamma[\chi]} \rangle\,, \tag{2.13}$$

with χ-dependent expectation values given by

$$\langle \hat{\mathcal{O}}[\hat{\chi}] \rangle_\chi = \mathrm{e}^{\Gamma[\chi]} \langle \mathrm{e}^{(\hat{\chi}-\chi)\cdot \frac{\delta}{\delta\chi}\Gamma[\chi]} \hat{\mathcal{O}}[\hat{\chi}] \rangle\,. \tag{2.14}$$

Note that taking functional derivatives with respect to $\frac{\delta}{\delta\chi}\Gamma[\chi]$ and evaluating on shell, i.e. $\frac{\delta}{\delta\chi}\Gamma[\chi] = 0$, we obtain all the connected n-point functions.

In Chapter 4, we will change the notation in order to generalize all this construction.

[1]We will explore more this freedom in Chapter 4.

2.2 Theory space

Since the action is the dynamical variable of the Wilsonian RG, let's analyze this element. The Wilsonian Effective Action S_t can be expressed in terms of an operator basis $\{\mathcal{O}_i\}$, that generically are monomials of the fields and their derivatives. In fact, the action is a quasi-local functional of the fields, since typically the expansion does not have a finite number of terms. Then, each operator is weighted by a scale-dependent parameter, or *coupling*, $\{g_i\}$

$$S_t[\hat{\chi}] = \sum_i g_i(t)\, \mathcal{O}_i[\hat{\chi}]\,, \tag{2.15}$$

where $\{\mathcal{O}_i\}$ shares the same symmetry properties of S_t. As we note in Section 1.5, even if we start with a simple action at initial scale Λ_0, quantum fluctuations will generate *all* possible operators compatible with the symmetry of our system. Therefore, the idea is starting from the symmetry, then writing down a complete operator basis compatible to that symmetry and finally writing S_t as in Equation (2.15).
From dimensional analysis, each coupling has a canonical dimension and we can define dimensionless couplings using the energy scale Λ

$$\tilde{g}_i := \Lambda^{-\Delta_i}\, g_i\,, \tag{2.16}$$

where Δ_i is the canonical dimension associated to g_i. The space spanned by $\{\tilde{g}_i\}$ is called *theory space*, which can be treated as a manifold (for a discussion on the geometry of theory space see [36, 37, 38]), and we can perform change of coordinates redefining the couplings. A first classification of the couplings is done respect to the classical dimension

- ⋄ *canonically relevant* if $\Delta_i > 0$;
- ⋄ *canonically irrelevant* if $\Delta_i < 0$;
- ⋄ *canonically marginal* if $\Delta_i = 0$.

Of course, this classification does not take into account the effect of quantum fluctuations that can add quantum contributions to the classical scaling and change the nature of the couplings. Plugging Equation (2.15) into Equation (1.38), we can extract the evolution of all the couplings of the Wilsonian Effective Action respect to the RG time

$$\frac{\mathrm{d}g_i}{\mathrm{d}t} = \beta_i(g_j)\,. \tag{2.17}$$

This is the dimensionful version of the flow equation, but since we want to study trajectories in theory space, we move to the dimensionless version. The dimensionless beta functions that describe the evolution or the running of the corresponding dimensionless couplings have the following structure

$$\frac{\mathrm{d}\tilde{g}_i}{\mathrm{d}t} = \tilde{\beta}_i(\tilde{g}_j) = -\Delta_i \, \tilde{g}_i + \alpha(\tilde{g}_j) \,, \tag{2.18}$$

where the first term on the RHS is the classical contribution and the second term represents the quantum fluctuations contributions. These two contributions together represent the transformation operator $\mathcal{T}$ on RHS of (1.38) in theory space. Since the beta functions do not depend explicitly on t, this system of equations is autonomous. The $\tilde{\beta}_i$ can be viewed as a vectorfield on theory space describing the evolution of the system and a single trajectory represents a theory. Therefore, QFTs are represented by trajectories in theory space, while we find CFTs at fixed points, which are the points in the theory space where all the dimensionless beta functions are zero, i.e. $\mathcal{T}[S_\star] = 0$. Trivial examples of CFT are the Gaussian or *free theories* , where all interaction couplings, i.e. operator terms that contain more than two fields, are zero. The perturbative approach to QFTs investigates the theories at an infinitesimal distance from the Gaussian theories.

Another classification of couplings is based on their connection to redundancies that are present in QFTs by construction. In particular, a coupling is defined

- *essential* if it enters into physical observables and, as a consequence, cannot be eliminated by a field redefinition;

- *inessential* if it can be eliminated from the action by a field redefinition.

The prototypical example of inessential coupling is the wave function renormalization constant, which is eliminated by the normalization of the kinetic term. The wave function renormalization constant is just the simplest example, but there are infinitely many ways of redefining the field and therefore in the action (2.15) there are infinitely many inessential couplings[2]. Consequently, any change in the couplings[3] $g_i \to g_i + \delta g_i$ which is equivalent to a field redefinition gives a theory that is physically equivalent to the original theory. Put differently, there are directions in theory space along which all physical quantities remain unchanged. These directions form "sub-manifolds of constant physics" in theory space. Locally in theory space, we can therefore work in a coordinate system $\{g_i\} = \{\lambda_a, \zeta_\alpha\}$ adapted to these sub-manifolds where λ_a are the essential couplings which will appear in expressions for the physical observables (2.1). The remaining couplings ζ_α are therefore the inessential couplings. It is well known that the S-matrix is not affected by field redefinitions. Thus, if a term in the action can be eliminated by a field redefinition, it does not have any effect on physically measurable quantities such as cross sections.

[2]It would be convenient to define the physical theory space as the quotient of the space of the functionals S_t by the field reparameterizations.

[3]Here we are using δ to denote a variation of the couplings keeping field variables fixed.

Since the variation $\zeta_\alpha \to \zeta_\alpha + \delta\zeta_\alpha$ is equivalent to a field redefinition, let's consider the effect on the Lagrangian of an infinitesimal field redefinition $\hat{\chi} \to \hat{\chi} + \delta\hat{\chi}$, with $\delta\hat{\chi} = f(\hat{\chi}, \partial\hat{\chi}, \ldots)$, where the dots stand for terms containing second and higher derivatives of the field. Varying the action and integrating by parts as usual, at tree level we obtain [39, 40]

$$\mathcal{L}_t \to \mathcal{L}_t + \left(\frac{\delta\mathcal{L}_t}{\delta\hat{\chi}} - \partial_\mu \frac{\delta\mathcal{L}_t}{\delta\partial_\mu\hat{\chi}} + \ldots\right)\delta\hat{\chi} + \partial_\mu\left(\frac{\delta\mathcal{L}_t}{\delta\partial_\mu\hat{\chi}}\,\delta\hat{\chi} + \ldots\right). \tag{2.19}$$

The first term is proportional to the equation of motion (EOM) and the second one is a total derivative. Thus, at tree level the action transforms as

$$S_t \to S_t + \delta\hat{\chi} \cdot \frac{\delta S_t}{\delta\hat{\chi}}\,. \tag{2.20}$$

We can see that any additional term in the action vanishes on-shell, or equivalently any term proportional to the EOM can be eliminated by performing a field redefinition.

From this consideration, we can define inessential coupling ζ "at tree level" as follow

$$\zeta\frac{\partial}{\partial\zeta}S_t = \Phi[\hat{\chi}] \cdot \frac{\delta S_t}{\delta\hat{\chi}}\,, \tag{2.21}$$

where Φ is a quasi-local functional of $\hat{\chi}$. The quantum contributions to the previous equations come from the form of $\mathcal{T}$ in Equation (1.38) and, consequently, depend on the renormalization scheme choice. However, if we are close to the Gaussian theory, then there is always a hierarchy between the tree-level contribution and the quantum contribution. This means that sufficiently close to the Gaussian theory, we can assume that the inessential couplings will be associated with terms proportional to the EOM. This situation is similar to the hierarchy present in Equation (2.18) between the classical term and the quantum contribution when we are close to the free theories.

Typically one can think that the classification in essential and inessential does not depend on the form of the action, like the canonical classification coming from the classical dimension. As we already said, the full dimension depends on the action since the corrections depend on the form of the interactions. In a different way, also the concepts of essential/inessential couplings are not given a priori, but come from the form of the action. In particular, for a given action the inessential couplings are simply defined by Equation (2.21). This means that couplings, that in some cases are inessential, can be essential in different cases and viceversa.
To clarify this point, let's consider a Gaussian theory at order $2n$

$$S_{\mathrm{G,2n}} := \frac{\zeta_{2n}}{2}\int_x \hat{\chi}\prod_{i=1}^{n}\left(-\partial^2 + m_i^2\right)\hat{\chi}\,. \tag{2.22}$$

The propagator is

$$\Pi = \frac{1}{\zeta_{2n}} \prod_{i=1}^{n} \frac{1}{p^2 + m_i^2} , \tag{2.23}$$

and using the residue method or the induction proof, it is possible to show that

$$\Pi = \frac{1}{\zeta_{2n}} \sum_{i=1}^{n} \frac{c_i^{(n)}}{p^2 + m_i^2} , \qquad \text{where } c_i^{(n)} \equiv \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{m_j^2 - m_i^2} . \tag{2.24}$$

From this equation, it is manifest that the Gaussian theory at order $2n$ describes the propagation of n massive particles with masses m_i even if there is a single scalar field in the theory. Moreover, looking at the structure of the $c_i^{(n)}$, for $n \geq 2$ there are either ghosts or tachyons. It is clear that ζ_{2n} is inessential, while the mass parameters m_i are not. This means that the couplings associated with the operators $\int_x \hat{\chi}(-\partial^2)^s\hat{\chi}$ are essential for $0 \leq s < n$, since they contain the mass parameters of the theory and are not proportional to EOM. The coupling associated to $\int_x \hat{\chi}(-\partial^2)^n\hat{\chi}$, or equivalently ζ_{2n}, is inessential, since its value can be set to any non-zero value using the Equation (2.20). Typically, it is canonically normalized to one and it cannot be set to zero because this would eliminate the tree level action and change completely the nature of the theory. Finally, the operators $\int_x \hat{\chi}(-\partial^2)^s\hat{\chi}$ with $s > n$ can be written in terms of operators already contained in the tree level action and therefore are inessential. To reiterate, if we are studying the "standard" Gaussian theory ($n = 1$), the coefficient of $\int_x \hat{\chi}(-\partial^2)\hat{\chi}$ is inessential, but it is not if we are considering a higher derivative Gaussian theory ($n > 1$). We will discuss again this subject in Chapter 5, when we will discuss gravity theories.

There is also the trivial case for $n = 0$, which means that the action is just a mass term and the EOM is trivial, i.e. $\hat{\chi} = 0$. This point is called *high temperature point* and describes a theory where the mass parameter dominates and nothing propagates. Note that in this theory everything is inessential.

Note that even if we are dealing with a single scalar field, we have to say "Gaussian theories": Gaussian theory means that only quadratic terms in the fields are present, but it does not specify the order of the kinetic operator.
For the rest of the thesis, the term Gaussian theory is understood to be the case $n = 1$ and in the other cases we will talk about higher derivative Gaussian theory.

2.3 Generators of the Renormalization group

In this section, we will review the construction of ERGEs following Wegner [35]. As we have discussed in Chapter 1, the RG machinery is composed of operations summarized in diagram (1.19) and starting from those considerations we construct the general RG approach to construct the ERGE.

The RG must be composed by two operations

◇ **dilatation transformation**
Under dilatation the positions, and consistently the momenta, transform as $x \to \mathrm{e}^{\omega}\, x$ and $p \to \mathrm{e}^{-\omega}\, p$, while the variable transforms as

$$\hat{\chi}(x) \to \mathrm{e}^{-\Delta_\chi \omega} \hat{\chi}(\mathrm{e}^{-\omega} x) = \hat{\chi} - \omega\,(x \cdot \partial_x + \Delta_\chi)\,\hat{\chi} + o(\omega^2)\,, \tag{2.25}$$

where $\Delta_\chi = \frac{d-2}{2}$ is the classical dimension of the field $\hat{\chi}$, and the action

$$S_t[\hat{\chi}] \to S_t[\hat{\chi}] + \omega\, \mathcal{T}_{dil} \cdot S_t[\hat{\chi}] + o(\omega^2)\,, \tag{2.26}$$

$$\mathcal{T}_{\mathrm{dil}} \cdot S[\hat{\chi}] = -\psi_{\mathrm{dil}}(x) \frac{\delta}{\delta \hat{\chi}(x)} \cdot S_t[\hat{\chi}]\,, \tag{2.27}$$

$$\psi_{\mathrm{dil}} := \Lambda^{-\Delta_\chi} \left(-x_\mu \partial_\mu \hat{\chi}(x) - \frac{d-2}{2} \hat{\chi}(x) \right)\,, \tag{2.28}$$

where in ψ_{dil} the first term accounts for the rescaling of the coordinates and the second accounts for the rescaling of the field. In particular, if we have a term $\Xi[\varphi] = O(\varphi^n, \partial^s)$ in the action, such that $\Xi[\varphi]$ has canonical dimension $n(d-2)/2 + s - d$, one can show that

$$\psi_{\mathrm{dil}} \cdot \frac{\delta}{\delta \varphi} \Xi[\varphi] = -\left(n(d-2)/2 + s - d\right)\, \Xi[\varphi]\,. \tag{2.29}$$

The operator (2.28) applied to a functional of the field $\hat{\chi}$, counts its canonical dimension. In Appendix A, we show the steps to obtain Equation (2.29).

This transformation is simply the mathematical translation of the rescaling operation shown in the diagram (1.19).

◇ **transformation of the variables**
Suppose

$$\hat{\chi} \to \hat{\chi} + \epsilon \Phi[\hat{\chi}]\,, \tag{2.30}$$

where ϵ is infinitesimal and Φ is a quasi-local functional of the field $\hat{\chi}$, which depends on S_t. The action S_t transforms as

$$S_t[\hat{\chi}] \to S_t[\hat{\chi}] + \epsilon \Phi[\hat{\chi}] \cdot \frac{\delta S_t[\hat{\chi}]}{\delta \hat{\chi}}\,, \tag{2.31}$$

and the functional measure

$$(\mathrm{d}\hat{\chi}) \to (\mathrm{d}\hat{\chi}) \left(1 + \epsilon \frac{\delta \Phi[\hat{\chi}]}{\delta \hat{\chi}} \right)\,. \tag{2.32}$$

From the previous equations, we obtain

$$\mathcal{Z}[0] \to \int (\mathrm{d}\hat{\chi}) \exp\left[-S_t[\hat{\chi}] - \epsilon \left(\Phi[\hat{\chi}] \cdot \frac{\delta S_t[\hat{\chi}]}{\delta \hat{\chi}} - \mathrm{Tr} \frac{\delta \Phi}{\delta \hat{\chi}} \right) \right]\,. \tag{2.33}$$

From this equation it is clear that $\mathcal{Z}$ is invariant under the transformation

$$S_t[\hat{\chi}] \to S_t[\hat{\chi}] + \epsilon \left(\Phi[\hat{\chi}] \cdot \frac{\delta S_t[\hat{\chi}]}{\delta \hat{\chi}} - \mathrm{Tr} \frac{\delta \Phi[\chi]}{\delta \hat{\chi}} \right) . \tag{2.34}$$

At this point we can complete the definition of inessential couplings given in Equation (2.21) by stating that ζ is inessential if

$$\zeta \frac{\partial}{\partial \zeta} S_t = \Phi[\hat{\chi}] \cdot \frac{\delta S_t}{\delta \hat{\chi}} - \mathrm{Tr} \frac{\delta \Phi[\chi]}{\delta \hat{\chi}} , \tag{2.35}$$

where Φ is a quasi-local functional of $\hat{\chi}$. Note that the first term on the RHS, or tree level term as we denote it in the previous section, is proportional to the EOM, while the second term on the RHS represents the quantum contributions to the definition of inessential couplings.

Then, we define the operator that produces this transformation

$$\mathcal{T}_{\mathrm{tra}}[\Phi] \cdot S[\hat{\chi}] := \Phi[\hat{\chi}] \cdot \frac{\delta S[\hat{\chi}]}{\delta \hat{\chi}} - \mathrm{Tr} \frac{\delta \Phi[\chi]}{\delta \hat{\chi}} . \tag{2.36}$$

This operation is the mathematical translation of the Kadanoff blocking and integration seen in diagram (1.19). Thus, in this operation the scheme dependence choice is contained, since it represents how we perform the blocking and integration procedures. In considering transformations we have in mind dilatation-free transformations, i.e. transformations that leave unchanged the lengths of the system. We require this because we want the two operations to be independent.

Therefore, the ERGE, introduced in Equation (1.38), will have the following structure

$$\frac{\mathrm{d}S_t}{\mathrm{d}t} = \mathcal{T}[\Phi] \cdot S_t[\hat{\chi}] = \mathcal{T}_{\mathrm{dil}} \cdot S_t[\hat{\chi}] + \mathcal{T}_{\mathrm{tra}}[\Phi] \cdot S_t[\hat{\chi}] , \tag{2.37}$$

where the first term on the RHS acts linearly on S_t, while the second one on the RHS does not since Φ depends on S_t.
In [35], Wegner observes that the independence of Φ respect to S_t normally leads to severe UV divergences and, consequently, non-linearity of the ERGE introduced by $\mathcal{T}_{\mathrm{tra}}[\Phi]$ is "essential". However, within a certain region in theory space we are free to choose the dependence of Φ on S_t. We will also see that this choice is not harmless: in fact, it can exclude the investigation on some sub-manifolds of the theory space. For example, if we start with the Gaussian theory, we cannot reach the higher derivative fixed point. We will discuss further this point.

2.4 Wilsonian exact RG equation

In [35], it is discussed how to choose Φ in order to reproduce the ERGE discovered by Wilson in [29][4]. In particular, it holds

$$\Phi_{\mathrm{Wil}} = \frac{\partial \alpha}{\partial t}\left(\hat{\chi} - \frac{\delta S_t}{\delta \hat{\chi}}\right), \tag{2.38}$$

where the function α is introduced to separate small momenta modes from large momenta modes. The idea is to choose a shape for this function such that slow modes are frozen, i.e. their contribution to the functional integral is negligible, while fast modes are completely integrated out. Equivalently, we can say that the strategy consists in an *incomplete integration* in which large momenta are more completely integrated than small momenta. For this reason, this function is called *cutoff function.* Wilson's choice is

$$\alpha(t) = p^2(\mathrm{e}^{2t} - 1) + \rho(t)\,, \qquad \rho(0) = 0\,, \tag{2.39}$$

where the function ρ allows the normalization of the kinetic term. This way, Wilson generalizes the idea of Gell-Mann and Low of making the transformation

$$\hat{\chi}(x) \to Z\,\hat{\chi}(x)\,, \tag{2.40}$$

where Z is the wave function renormalization constant, to the new transformation [41, 42]

$$\hat{\chi}(x) \to \int_{x'} Z(x,x')\,\hat{\chi}(x')\,. \tag{2.41}$$

The kernel $Z(x,x')$ implements the separation of modes, weighting them in different ways, and introduces a non-locality, that makes the method a really efficient calculational tool. Wilson introduces the idea of a smooth integration of modes, that is achieved through the smooth choice of the function α. The smoothness implies that there is a contribution coming from slow modes, which must be negligible respect to the fast mode contribution. This function contains information about the renormalization scheme choice, since it tells us how we separate fast modes from slow modes. Using Φ_{Wil}, the Wilson's ERGE reads

$$\frac{\mathrm{d}S_t}{\mathrm{d}t} = \mathcal{T}_{\mathrm{dil}} \cdot S_t + \mathrm{Tr}\left[\frac{\partial \alpha}{\partial t}\left(\frac{\delta^2 S_t}{\delta\hat{\chi}\delta\hat{\chi}} - \frac{\delta S_t}{\delta\hat{\chi}}\frac{\delta S_t}{\delta\hat{\chi}} + \hat{\chi}\frac{\delta S_t}{\delta\hat{\chi}}\right)\right]. \tag{2.42}$$

The trace is typically done in momentum space, where the function α is a local function, see Equation (2.39). In particular, the choice of the function ρ is important, since not all choices reproduce fixed point solutions. As we already said and we will repeat, the RG scheme is not harmless. In [43], it is shown that at the fixed point solution

$$\left.\frac{\partial \rho}{\partial t}\right|_{\star} = 1 - \frac{\eta}{2}\,. \tag{2.43}$$

[4]To obtain exactly the same send $\mathcal{H}_t \to -S_t$.

Moreover, since ρ is connected to reparameterization invariance, there is a corresponding inessential coupling, which is associated to the following operator

$$\mathcal{O}_{\mathrm{par}} = \mathrm{Tr}\left[\frac{\delta S_t}{\delta\hat{\chi}}\frac{\delta S_t}{\delta\hat{\chi}} - \frac{\delta S_t}{\delta\hat{\chi}\delta\hat{\chi}} + \hat{\chi}\frac{\delta S_t}{\delta\hat{\chi}}\right]. \tag{2.44}$$

For more details see [44].

In [32] Wegner presents his version of the ERGE

$$\frac{\mathrm{d}S_t}{\mathrm{d}t} = \mathcal{T}_{\mathrm{dil}}\cdot S_t + \lim_{t\to 0}\frac{1}{2t}\mathrm{Tr}'\left[\ln\left(\frac{\delta^2 S_t}{\delta\hat{\chi}\delta\hat{\chi}}\right) - \frac{\delta S_t}{\delta\hat{\chi}}\frac{\delta S_t}{\delta\hat{\chi}}\left(\frac{\delta^2 S_t}{\delta\hat{\chi}\delta\hat{\chi}}\right)^{-1}\right], \tag{2.45}$$

where the primes in the trace represent the fact that we integrate modes contained only in the momentum shell $(\Lambda - \delta\Lambda, \Lambda)$. A detailed presentation is contained in [45]. This equation is the first equation where it is implemented the idea of a *sharp cutoff*, which means that slow modes are completely separated from fast modes and slow modes do not contribute at all to the functional integral during the infinitesimal RG step. The problem with sharp cutoffs is that typically they introduced non local terms in position space, so smooth cutoffs are preferred.

2.5 Fixed points and scaling exponents

In this section, we will follow [35] and present the properties of fixed points using the general ERGE. Fixed points of the exact RG are found by looking at t-independent solutions of Equation (2.37) such that the fixed point action $S_\star$ obeys

$$\mathcal{T}\left[\Phi_\star\right]\cdot S_\star = \mathcal{T}_{\mathrm{dil}}\cdot S_\star + \mathcal{T}_{\mathrm{tra}}\left[\Phi_\star\right]\cdot S_\star = 0\,, \tag{2.46}$$

which in general defines a relationship between $\Phi_\star$ and $S_\star$. Thus, the action generated by the dilation transformation

$$S_\star + \epsilon\,\mathcal{T}_{\mathrm{dil}}\cdot S_\star \tag{2.47}$$

are equal to the action generated by the transformation

$$S_\star + \epsilon\,\mathcal{T}_{\mathrm{tra}}\left[-\Phi_\star\right]\cdot S_\star\,. \tag{2.48}$$

There is a special class of fixed point, that is called *trivial fixed points*, where Equation (2.46) is satisfied by actions that satisfies $\mathcal{T}_{\mathrm{dil}}\cdot S_\star = \mathcal{T}_{\mathrm{tra}}\left[\Phi_\star\right]\cdot S_\star = 0$. The main examples of trivial fixed points consist in the Gaussian fixed points (GFP), where all essential couplings are zero [5]

$$S_{\mathrm{GFP,2n}} := \frac{1}{2}\int_x \hat{\chi}\left(-\partial^2\right)^n\hat{\chi}\,. \tag{2.49}$$

[5] Having in mind what we already said into Section 2.2, we have set $\zeta_{2n} = 1$.

Analogously to the Gaussian theories, for the rest of the thesis, the term GFP is understood to be the case $n = 1$ and in the other cases we will talk about higher derivative Gaussian fixed point. The trivial case $n = 0$ will be denoted by *high temperature fixed point.*

Now, we want to study the properties of a fixed point solution and in order to do that we perturb the fixed point. Suppose to add eigenperturbations, such that $S_t = S_\star + \delta S_\star$ and $\Phi = \Phi_\star + \delta\Phi$, where

$$\delta S_t = \epsilon \sum_i \mathrm{e}^{\lambda_i t}\mathcal{O}_i \,, \qquad \delta\Phi = \epsilon \sum_i \mathrm{e}^{\lambda_i t}\Omega_i \,, \tag{2.50}$$

for an infinitesimal ϵ. Then, the eigenvalue equation is obtained from Equation (2.37)

$$\epsilon \sum_i \mathrm{e}^{\lambda_i t}\lambda_i\mathcal{O}_i = \mathcal{T}\left[\Phi_\star + \delta\Phi\right] \cdot (S_\star + \delta S_t) = \epsilon \sum_i \mathrm{e}^{\lambda_i t}\left\{\mathcal{T}[\Phi_\star]\cdot\mathcal{O}_i + \mathcal{T}_{\mathrm{tra}}[\Omega_i]\cdot S_\star\right\} + o(\epsilon^2)\,.$$

Since every term in the sum is linearly independent, we obtain all the eigenvalue equations projecting on the eigenperturbation basis.
In particular, defining the scaling exponents $\theta := -\lambda$, we classify the eigenvalues as

- ◇ *relevant* if $\theta > 0$: these directions are IR attractive;
- ◇ *irrelevant* if $\theta < 0$: these directions are IR repulsive;
- ◇ *marginal* if $\theta = 0$: at linear level no dependence on the RG time.

At GFP, using Equation (2.51) it holds that $\theta_i = \Delta_i$, and, consequently, the concept of canonical dimension coincide with the notion of scaling exponents.
Since a fixed point solution represents a universality class, the θs are the scaling exponents that characterize that universality class. The critical exponent ν is given by minus the inverse of the most relevant eigenvalue[6] (see Equation (1.17)). The other critical exponent η is the eigenvalue associated with the eigenperturbation linear in the field (see Equation (1.17)). Finally, all the other eigenvalues represent the corrections to these two exponents: in particular among them we also find ω, i.e. the lowest irrelevant eigenvalue, introduced in Table 1.1.

Then, let's consider a particular perturbation $S_\star \to S_\star + \epsilon\, \mathrm{e}^{\lambda t}\mathcal{T}_{\mathrm{tra}}[\phi]\cdot S_\star$ and $\Phi_\star \to \Phi_\star + \epsilon\, \mathrm{e}^{\lambda t}\hat{\Omega}[\phi]$, where we relax the condition on $\hat{\Omega}$ to be dilatation-free. The eigenvalue problem reads

$$\epsilon\, \mathrm{e}^{\lambda t}\lambda\, \mathcal{T}_{\mathrm{tra}}[\phi]\cdot S_\star = \mathcal{T}\left[\Phi_\star + \epsilon\, \mathrm{e}^{\lambda t}\hat{\Omega}[\phi]\right]\cdot\left(S_\star + \epsilon\, \mathrm{e}^{\lambda t}\mathcal{T}_{\mathrm{tra}}[\phi]\cdot S_\star\right) + o(\epsilon^2)\,, \tag{2.51}$$

that gives

$$\lambda\, \mathcal{T}_{\mathrm{tra}}[\phi]\cdot S_\star = \mathcal{T}_{\mathrm{tra}}\left[K\left[\Psi_\star - \psi_{\mathrm{dil}};\phi\right] + \hat{\Omega}[\phi]\right]\cdot S_\star + o(\epsilon)\,, \tag{2.52}$$

[6]Excluding the trivial one, which is related to the vacuum energy and can be disregarded for systems without gravity.

where

$$K\left[\Phi_1;\Phi_2\right] := \Phi_1 \frac{\delta\Phi_2}{\delta\hat{\chi}} - \Phi_2 \frac{\delta\Phi_1}{\delta\hat{\chi}} \,. \tag{2.53}$$

Choosing

$$\hat{\Omega} = \alpha\,\phi - K\left[\Phi_\star + \psi_{\rm dil};\phi\right] \,, \tag{2.54}$$

we can change the eigenvalue λ to any value α we want. Since for infinitesimal ϵ the action is invariant under the additional term $\mathcal{T}_{\rm tra}[\phi]\cdot S_\star$ and λ is completely scheme dependent, the eigenperturbations of the form

$$\mathcal{T}_{\rm tra}[\Phi]\cdot S_\star \,, \tag{2.55}$$

for any quasi-local functional, are called *redundant*.

Le's return to the general eigenvalue problem

$$\epsilon\,{\rm e}^{\lambda t}\,\lambda\,\hat{\mathcal{O}} = \mathcal{T}\left[\Psi_\star + \epsilon\,{\rm e}^{\lambda t}\hat{\Omega}\right]\cdot\left(S_\star + \epsilon\,{\rm e}^{\lambda t}\hat{\mathcal{O}}\right) \,. \tag{2.56}$$

Then we choose $\{\mathcal{O}_i\}$ such that they form a complete set of linearly independent non-redundant eigenoperators, i.e. linearly independent of any redundant operator, so that

$$\hat{\mathcal{O}} = \sum \alpha_i \mathcal{O}_i + \mathcal{T}_{\rm tra}[\phi]\cdot S_\star \tag{2.57}$$

vanishes only for $\alpha_i = 0$ and $\mathcal{T}_{\rm tra}[\phi]\cdot S_\star = 0$. In the same way we expand $\hat{\Omega}$

$$\hat{\Omega} = \sum \alpha_i \Omega_i + \bar{\Omega}[\phi] \,. \tag{2.58}$$

Inserting these ansatzs inside the Equation (2.56), we obtain that any operator $\hat{\mathcal{O}}$ with at least one non-vanishing α_i is determined uniquely, modulo additional redundant operator, and its eigenvalue is independent of Ω_i, i.e. it is scheme independent. These operators are called *scaling* operators.

Therefore, Wegner [35] has shown that eigenperturbations fall into two classes:

- ⋄ *scaling*: the corresponding θ does not depend on the RG scheme;
- ⋄ *redundant*: the corresponding θ does depend on the RG scheme.

Thus, redundant eigenperturbations store no physical information, conversely the scaling operators have scheme independent universal scaling exponents and are physical perturbations of the fixed point. Moreover, since couplings associated with redundant operators have scaling laws that depend completely on the RG scheme choice, to study a fixed point we need only the scaling exponents coming from the couplings associated with scaling operators.

Close to the generic GFP, the redundant operators are given by operators of the form

$$\frac{1}{2}\int_x \Phi[\hat{\chi}]\left(-\partial^2\right)^n \hat{\chi} + \text{quantum corrections}\,. \tag{2.59}$$

Close to the GFP, there is a hierarchy between the first term, i.e. the tree-level term, and the second term, which comes from the quantum corrections. Therefore, at the generic GFP all operator $\int_x \hat{\chi}\left(-\partial^2\right)^s \hat{\chi}$ with $s \geq n$ is redundant and, since their value is completely scheme dependent, we can set to zero all the terms with $s > n$ and canonically normalize to one the term with $s = n$. As we said for Gaussian theories, we cannot set it to zero because in this way we would kill the tree-level term and change the universality class.

We see that the concepts of this section are in one-to-one correspondences with those of Section 2.2. The classical dimensions, that are just the eigenvalues at the GFP, are generalized to the scaling exponent θ, and the concepts of essential/inessential valid for QFTs are transposed to the concepts of scaling/redundant at the CFTs or fixed points. As for the previous concepts, there is no connection between relevant/irrelevant/marginal and scaling /redundant. Usually essential and inessential are also used for the couplings associated with scaling and redundant operators at fixed points, so we will also adopt this interchangeable notation.

In this section, we have considered the general case to study the properties of scaling and redundant operators without relying on approximations. However, from a practical point of view, we are obliged to rely on approximations. The unavoidable approximation consists of the truncation of the expansion in Equation (2.15), since for practical reason we can deal only with a finite number of terms in computations. The approximations usually bring some RG scheme dependence on the scaling operator's scaling exponents. From a geometrical point of view, the sub-manifold of constant physics, given by the inessential directions, mixes with the essential directions.

We conclude this section by stressing again the fact that the concept of inessential coupling depends on the action choice and therefore, as we have seen at the generic GFP, operators that are redundant at one fixed point can be scaling at another fixed point and viceversa. This means that our initial choice can exclude some parts of the theory space. In order to stress this dependence, in Chapter 4 we will use the following notation

$$\mathcal{T}(S_\star)\Phi[\hat{\chi}] \equiv \mathcal{T}_{\text{tra}}[\Phi]\cdot S_\star\,. \tag{2.60}$$

2.6 Properties of critical exponents

In this section, we will present how the critical exponents or the scaling exponents arise from the beta functions of the couplings [40]. The idea is to follow the same procedure of Equation (2.51). Inserting in Equation (2.18) $\tilde{g}^i = \tilde{g}^i_\star + y^i$, for small y we obtain

$$\frac{\mathrm{d}y^k}{\mathrm{d}t} = \tilde{\beta}^k(\tilde{g}^i - \tilde{g}^i_\star) = \sum_i M^k{}_i\, y^i + \sum N^k{}_{ij}\, y^i\, y^j + o(y^3)\,. \tag{2.61}$$

where

$$M^k{}_i := \frac{\partial \tilde{\beta}^k}{\partial \tilde{g}^i}\Big|_\star , \qquad N^k{}_{ij} := \frac{1}{2}\frac{\partial^2 \tilde{\beta}^k}{\partial \tilde{g}^i\,\partial \tilde{g}^j}\Big|_\star . \tag{2.62}$$

Keeping the linear part and diagonalizing, $z^a = \left(S^{-1}\right)^a{}_i\, y^i$ and $(S^{-1})^a{}_i\, M^i{}_j\, S^j{}_b = -\theta_a \delta^a_b$, we reobtain the scaling laws for the eigenperturbation that we introduced in the previous section

$$z^a = C^a\, \mathrm{e}^{-\theta_a\, t} . \tag{2.63}$$

It is important to note that in general a fixed point cannot be said to be a *IR fixed point* or an *UV fixed point*: in fact, there will be IR attractive/UV repulsive directions, from which it can be approached in the IR, and IR repulsive/UV attractive directions, from which it can be approached in the UV.

Now let's analyze the properties of the critical exponents under a reparameterization of the theory space. Suppose that we change coordinates in theory space

$$\tilde{g}'^i = \tilde{g}'^i(\tilde{g}^j) , \tag{2.64}$$

which implies that the dimensionless beta functions transform as a vector field

$$\tilde{\beta}'^i = \left(J^{-1}\right)^i{}_j \tilde{\beta}^j , \qquad \text{where } J^i{}_j = \frac{\partial \tilde{g}^i}{\partial \tilde{g}'^j} \text{ and } \left(J^{-1}\right)^i{}_j = \frac{\partial \tilde{g}'^i}{\partial \tilde{g}^j} . \tag{2.65}$$

The stability matrix transforms as follow

$$M'^i{}_j = \left(J^{-1}\right)^i{}_a M^a{}_b J^b{}_j + J^b{}_j \frac{\partial^2 \tilde{g}'^i}{\partial \tilde{g}^b \partial \tilde{g}^a} \tilde{\beta}^a , \tag{2.66}$$

which implies that the scaling exponents are independent of the choice of coordinates in the theory space.

As we already said in the previous section, inessential couplings are not required to tend to a fixed value, since they have scaling laws completely dependent on the scheme and, consequently, do not affect physical observables.

2.7 Polchinski ERGE

In this section, we will present the ERGE discovered by Polchinski in [46]. In this new approach, we start from the Wilson idea of incomplete integration of modes and then we construct the flow in such a way that the flow equation is functional. This means that we are interested not in the flow of all couplings at once, but in the functional flow of the operators that enter into the action. Therefore, we expand the action as follow

$$S_t[\hat{\chi}] = \sum_\alpha \mathcal{O}_\alpha[\hat{\chi}, t] , \tag{2.67}$$

contrary to Equation (2.15). For this reason, Polchinski's ERGE is the first example of *functional ERGE* (FERGE). Of course, the functional RG flow contains the flow of all the couplings, since the operators $\mathcal{O}_\alpha[\hat{\chi}, t]$ can be Taylor-expanded in monomials of the fields and their derivative recovering Equation (2.15). A trivial example is given by the potential operator that contains all the coupling of monomials of the fields by definition.
The idea of functional RG is to condense many, usually infinite terms, of the expansion in Equation (2.15) and explore non-perturbative regions of the theory space, or equivalently regions far from the GFPs.

Also for this case, we will present the scalar case to avoid technicalities. The starting point is the modification of the kinetic term. In particular, we define[7]

$$\Delta_\Lambda(p^2) = p^{-2} C_{\mathrm{UV}}(p^2/\Lambda^2)\,, \tag{2.68}$$

such that it is a massless propagator whose momentum p is cutoff by an UV cutoff Λ. We require that the cutoff function C_{UV} is analytic at $p^2 = 0$ and that $C_{\mathrm{UV}}(0) = 1$ (so that physics is unchanged at scales much less than Λ), and C_{UV} goes to zero as $p^2 \to \infty$ sufficiently rapidly. Then, we modify the generating function as follow

$$\mathcal{Z}_\Lambda[j] = \mathcal{N} \int (\mathrm{d}\hat{\chi})\; \mathrm{e}^{-S(\hat{\chi},\Lambda) + j\cdot\hat{\chi}}\,, \tag{2.69}$$

$$S(\hat{\chi}, \Lambda) := \frac{1}{2} \hat{\chi} \cdot \Delta_\Lambda^{-1} \cdot \hat{\chi} + S_\Lambda^{\mathrm{int}}(\hat{\chi})\,. \tag{2.70}$$

Taking the logarithmic derivative respect to Λ, we obtain

$$\Lambda \frac{\mathrm{d}Z}{\mathrm{d}\Lambda} = -\mathcal{N} \int (\mathrm{d}\hat{\chi}) \left(\frac{1}{2} \hat{\chi} \cdot \Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \Delta_\Lambda^{-1} \cdot \hat{\chi} + \Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} S_{\mathrm{int}} \right) \mathrm{e}^{-S(\hat{\chi},\Lambda) + j\cdot\hat{\chi}}\,, \tag{2.71}$$

and then choosing[8]

$$\Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} S_{\mathrm{int}} = \frac{1}{2} \frac{\partial S_{\mathrm{int}}}{\partial \hat{\chi}} \cdot \Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \Delta_\Lambda \cdot \frac{\partial S_{\mathrm{int}}}{\partial \hat{\chi}} - \frac{1}{2} \mathrm{Tr}\; \frac{\partial^2 S_{\mathrm{int}}}{\partial \hat{\chi} \partial \hat{\chi}} \cdot \Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \Delta_\Lambda\,, \tag{2.72}$$

the integrand in the functional integral becomes a total derivative

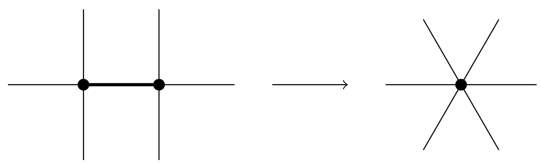
$$\Lambda \frac{\mathrm{d}Z}{\mathrm{d}\Lambda} = \mathcal{N}' \int (\mathrm{d}\hat{\chi})\; \Lambda \frac{\partial}{\partial \Lambda} \Delta_\Lambda \cdot \frac{\partial}{\partial \hat{\chi}} \cdot \left(\Delta_\Lambda^{-1} \cdot \hat{\chi} + \frac{1}{2} \frac{\partial}{\partial \hat{\chi}} \right) \mathrm{e}^{-S(\hat{\chi},\Lambda) + j\cdot\hat{\chi}}\,, \tag{2.73}$$

where $\mathcal{N}' = \mathcal{N} \exp \mathrm{Tr}\; \ln \Delta_\Lambda$, and we have

$$\Lambda \frac{\mathrm{d}\mathcal{Z}_\Lambda}{\mathrm{d}\Lambda} = 0\,. \tag{2.74}$$

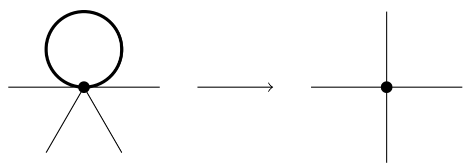
[7]In [46], Polchinski puts the mass into the definition of Δ_Λ.
[8]For a more detailed computation, see [47]

From these passages, it is clear that if we reduce Λ and simultaneously we change S_Λ^{int} as in Equation (2.72), $\mathcal{Z}_\Lambda$ and its derivatives, i.e. n-point functions, do not change. Equation (2.72) is the Polchinski version of the ERGE, which is UV-finite since all momentum integrals are regulated by the C_{UV}.
As modes are removed from the propagator, compensating terms must be added to S_Λ^{int} and, therefore, a simple action in UV can turn into an involved one, like in the Wilsonian case. From a diagrammatic point of view we have the following interpretation for the first term in Equation (2.72)



(2.75)

and for the second term



(2.76)

which shows how more involved interaction terms are generated along the RG flow. The Wilsonian Effective Action is achieved using the following identification

$$S_\Lambda^{\text{Wil}} = \frac{1}{2}\hat{\chi} \cdot \Delta_\Lambda^{-1} \cdot \hat{\chi} + S_\Lambda^{\text{int}} . \tag{2.77}$$

Note that on the RHS of Equation (2.72), there is not the dilatation term. This is due to the fact that we are presenting the dimensionful version of the flow equation, analogous to Equation (2.17). Rescaling every quantity by Λ produces on the RHS the dilatation term and so the dimensionless version of the flow equation, analogous to Equation (2.18) and (2.37) and which is needed to find the fixed points.
After inserting the dilatation term, Wilson's flow equation (2.42) is identical to (2.72), after the transformation [48, 49]

$$\hat{\chi} \to \sqrt{C_{\text{UV}}}\hat{\chi} . \tag{2.78}$$

In order to solve Equation (2.72), we have to rely on approximations. In the following subsection, we will present the Local Potential approximation.

2.7.1 Local potential approximation in Polchinski ERGE

Consider only the potential operator, $S_\Lambda^{\mathrm{int}} = \int_x V(\hat{\chi}, \Lambda)$ disregarding all higher derivative terms. Then, the flow equation reads

$$\Lambda \frac{\mathrm{d}V}{\mathrm{d}\Lambda} = \frac{\alpha}{\Lambda^2} \left(\frac{\partial V}{\partial \hat{\chi}} \right)^2 - \gamma \Lambda^{d-2} \frac{\partial^2 V}{\partial \hat{\chi} \partial \hat{\chi}} \,, \tag{2.79}$$

where $\alpha = -C'_{\mathrm{UV}}(0)$ and $\gamma = -\int_p C'_{\mathrm{UV}}(p^2)$. Rescaling the variables $\hat{\chi} \to \sqrt{\gamma}\,\Lambda^{d/2-1} \hat{\chi}$ and $V \to (\gamma/\alpha) \Lambda^d\, V$, we have

$$\partial_t V = -d\, V + \frac{d-2}{2}\, \hat{\chi}\, \frac{\partial V}{\partial \hat{\chi}} + \left(\frac{\partial V}{\partial \hat{\chi}} \right)^2 - \frac{\partial^2 V}{\partial \hat{\chi} \partial \hat{\chi}} \,, \tag{2.80}$$

where $t = \ln \Lambda/\Lambda_0$, for some arbitrary initial scale Λ_0 [9]. Note that the structure is analogous to Equation (2.37): in fact, on the RHS we find the dilatation contribution (2.27) linear in the potential and the "blocking contribution", analogous to (2.36) and which instead is not linear in the potential. It is interesting to introduce a Gibbsian-like measure $\mu := \exp\left[-V(\hat{\chi}, t)\right]$, and the previous flow equation becomes [50]

$$\partial_t \mu = -d\, \mu\, \ln \mu + \frac{d-2}{2}\, \hat{\chi}\, \frac{\partial \mu}{\partial \hat{\chi}} - \mu\, \frac{\partial^2 \mu}{\partial \hat{\chi} \partial \hat{\chi}} \,. \tag{2.81}$$

Looking for fixed points, i.e. solutions such that $\partial_t \mu_\star = 0$, we find three trivial fixed points

- $\mu_\star = 1$ or equivalently $V_\star = 0$, which is the Gaussian fixed point;
- $\mu_\star = \exp\left[-\frac{1}{2}\hat{\chi}^2 + \frac{1}{d}\right]$ or equivalently $V_\star = \frac{1}{2}\hat{\chi}^2 - \frac{1}{d}$, which is the High temperature fixed point[10];
- $\mu_\star = 0$, which is called the Low temperature fixed point[11].

More interesting are non-trivial solutions which give zero because of cancellation between the dilatation part and the transformation part. In dimensions lower than 4, we can find the WF fixed point and linearizing around this solution we can evaluate the critical exponents of the Ising model. As soon as we decrease the dimension lower than 3, more and more non-trivial fixed points appear until we reach $d = 2$, where we have infinite universality classes [51]. In order to find the critical exponents, the flow equation must be solved numerically and then perturbed to find the spectrum. For example, in [52] for $d = 3$ it is found: $\nu = 0.687(1)$, $\omega = 0.595(1)$ and $\eta = 0$. To improve the evaluation of the critical

[9] Note that respect to Equation (1.38) and (2.37), there is no minus sign in the RG time definition. This means that we are looking to flow from the IR to the UV region.

[10] Note that the potential is composed only by the mass term which dominates upon all the other terms.

[11] Note that this FP can be seen only reparameterizing the flow equation in terms of μ.

exponents, we have to increase the truncation and add new operators in the approximation of the action [48, 49, 53]

$$S_\Lambda^{\rm int} = \int_x V(\hat\chi, \Lambda) + \frac{1}{2}\int_x z(\hat\chi, \Lambda)\partial_\mu\hat\chi\partial_\mu\hat\chi + \dots \,. \tag{2.82}$$

This kind of expansion is called *derivative expansion* and it will be discussed further in the next section.

2.8 Effective Average Action method

In this section, we will present another version of functional ERGE.

As we discussed in Chapter 1, the Kadanoff-Wilson's idea of coarse graining consists of mapping actions onto other actions at different scales. Then the actions obtained are the actions of the modes that have not yet been integrated out in the partition function. This remains true also for the functional implementation of Polchinski. The drawback of working with the Wilsonian Effective Action is that these actions are very abstract objects, since they enter into functional integrals and their meaning is not "transparent". Instead of computing this sequence of Hamiltonians or actions, we can compute the Gibbs free energy or Effective Action Γ of the fast modes that have already been integrated out. The Effective Action contains all the quantum information about the system and it is more manageable, since all fluctuations have been integrated out. The idea is to build a one-parameter family of models, indexed by a scale k such that

$$\lim_{k\to\Lambda_0}\Gamma_k = S_0\,, \qquad\qquad \lim_{k\to 0}\Gamma_k = \Gamma\,, \tag{2.83}$$

where Λ_0 is the initial UV scale. Since we want to decouple the slow modes of the action in the partition function, a very convenient implementation of this idea is to give them a large mass. Therefore, we define the k-dependent generating functional

$$\mathrm{e}^{\mathcal{W}_k[j]} = \mathcal{Z}_k[j] = \mathcal{N}_k \int (\mathrm{d}\hat\chi)\, \mathrm{e}^{-S[\hat\chi]-\Delta S_k[\hat\chi]+j\cdot\hat\chi}\,, \tag{2.84}$$

$$\Delta S_k = \frac{1}{2}\int_x \hat\chi\, \mathcal{R}_k\,[\Delta]\,\hat\chi\,, \tag{2.85}$$

where $\mathcal{N}_k^{-1} = \int(\mathrm{d}\hat\chi)\,\mathrm{e}^{-S[\hat\chi]-\Delta S_k[\hat\chi]}$ and $\mathcal{R}_k$ is called *regulator* and it is an additive modification of the inverse two-point function. The regulator is therefore a function of a single momentum p, or rather its modulus $z = p^2$, depending on an scale k. The regulator must implement the coarse graining and therefore it contains the information on the scheme dependence, or equivalently on how we decouple UV modes and IR modes. It is typically assumed to satisfy the following conditions:

- ◇ to be positive (must suppress modes);

- $\diamond$ to be monotonically increasing with k, for all z;
- $\diamond$ to be monotonically decreasing with z, for all k;
- $\diamond$ $\lim_{k\to 0} \mathcal{R}_k(z) = 0$ for all z;
- $\diamond$ for $z > k^2$, $\mathcal{R}_k$ goes to zero sufficiently fast, e.g. as an exponential.

The first three conditions are obvious properties of a cutoff. The fourth guarantees that the path integral reproduces the standard partition function for $k = 0$. The fifth condition ensures that high momentum modes are integrated out unsuppressed and guarantees the UV convergence of the RHS of the flow equation. For certain purposes, one may sometimes forgo the last two conditions and consider cutoffs that either do not decrease very fast for large momenta or even diverge when $z \to 0$. These five conditions are useful in that they provide a clear physical interpretation for the coarse graining implemented by the regulator, and they ensure control on the UV and IR endpoints of the momentum integrals.

The k-dependent expectation value of observables is given by

$$\langle \hat{\mathcal{O}} \rangle_{j,k} = \mathcal{N}_k \, \mathrm{e}^{-\mathcal{W}_k[j]} \int (\mathrm{d}\hat{\chi}) \; \hat{\mathcal{O}}[\hat{\chi}] \; \mathrm{e}^{-S[\hat{\chi}] - \Delta S_k[\hat{\chi}] + j\cdot\hat{\chi}} \,, \tag{2.86}$$

$$\mathrm{e}^{\mathcal{W}_k[j]} = \langle \mathrm{e}^{j\cdot\hat{\chi}} \rangle_k \,, \tag{2.87}$$

which are identical to (2.1) and (2.8) taking $k \to 0$. Defining $\chi := \langle \hat{\chi} \rangle_{j,k}$, we take to Legendre transform as we do for the Effective Action. In this case we subtract the additional regulator piece and we define the Effective Average Action (EAA)

$$\Gamma_k\,[\chi] = -\mathcal{W}_k[j] + j \cdot \chi - \Delta S_k[\chi] \,. \tag{2.88}$$

Another interesting way to define the EAA is to modify Equation (2.13) into the following integro-differential equation [12]

$$\mathrm{e}^{-\Gamma_k[\chi]} = \langle \mathrm{e}^{(\hat{\chi}-\chi)\cdot \frac{\delta}{\delta\chi}\Gamma_k[\chi] - \frac{1}{2}(\hat{\chi}-\chi)\cdot\mathcal{R}_k(\hat{\chi}-\chi)} \rangle \,, \tag{2.89}$$

with χ and k-dependent expectation values given by

$$\langle \hat{\mathcal{O}}[\hat{\chi}] \rangle_{\chi,k} = \mathrm{e}^{\Gamma_k[\chi]} \langle \mathrm{e}^{(\hat{\chi}-\chi)\cdot \frac{\delta}{\delta\chi}\Gamma_k[\chi] - \frac{1}{2}(\hat{\chi}-\chi)\cdot\mathcal{R}_k(\hat{\chi}-\chi)} \hat{\mathcal{O}}[\hat{\chi}] \rangle \,. \tag{2.90}$$

We can see that taking one derivative with respect to $\mathcal{R}_k$ and evaluating at $k \to 0$ and on shell, i.e. $\frac{\delta}{\delta\chi}\Gamma[\chi] = 0$, gives the connected 2-point function. From this point of view, $\mathcal{R}_k$ acts like a source for the connected $2n$-point functions. From the second way of defining EAA, we can note that $\mathcal{R}_k$ is a mathematical tool as the source j. Its role consists in adding to the new ERG the scheme dependence encoded in the "blocking" procedure.

[12] Note that there is no k subscript in the angle parenthesis.

Using (2.88) or (2.89) and taking the logarithmic derivative respect to k gives the Wetterich-Morris FERGE [54, 55, 56, 57]

$$\frac{\mathrm{d}\Gamma_k}{\mathrm{d}t} = \frac{\hbar}{2}\mathrm{Tr}\left(\frac{\delta^2\Gamma_k}{\delta\chi\delta\chi} + \mathcal{R}_k\right)^{-1} \cdot \frac{\mathrm{d}\mathcal{R}_k}{\mathrm{d}t}\,, \tag{2.91}$$

where $t := \log(k/k_0)$ for an arbitrary initial scale k_0[13] and we have re-inserted $\hbar$ to stress the fact that the RHS represents the quantum contribution.

Equation (2.91) gives the evolution of EAA under the change of the parameter k, or equivalently under the RG evolution. In particular, by construction the EAA satisfies the conditions (2.83). We note that the trace on the RHS of (2.91) is IR and UV-finite, and that the flow equation contains no reference to a bare action or UV physics. We refer to [34, 40, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68] for reviews of this equation and its applications.

The RHS of Equation (2.91) does not contain the dilatation term, because we are presenting the dimensionful version of the flow equation, as we did for the Polchinski FERGE. In order to insert the dilatation term on the RHS, we have to express every quantity in dimensionless variables using k, as we did in Equation (2.16), and the logarithmic derivative on the RHS of (2.91) generates the dilatation operator of Equation (2.27). This process is needed to analyze the fixed points, i.e. $\partial_t\Gamma_\star = 0$.

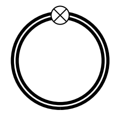
The notation $\Gamma_k[\phi]$ emphasizes the important dependence of this functional on the scale k, but Γ_k also depends on the shape of the cutoff function $\mathcal{R}_k$. The notation $\Gamma[\phi, \mathcal{R}_k]$ would thus be more appropriate, and could be replaced by a functional equation where the derivatives with respect to k are replaced by functional derivatives with respect to $\mathcal{R}_k$.

In a nutshell, the Functional Renormalization Group (FRG) is a convenient way of implementing Wilson's idea of integrating out modes one momentum shell at the time. At its core lies a choice of a regulator function $\mathcal{R}_k$ that suppresses the contribution of low momentum modes to the path integral. Thus, as k decreases more and more fluctuations are integrated out. The expectation value at scale k is therefore a precursor of the true expectation value, obtained at $k = 0$, and the EAA a precursor of the Effective Action. In Section 1.5, we point out the role of the new cutoff Λ_1 after the RG step. The scale k plays the analogous role, since it is a UV cutoff for the slow modes in the Wilson-Polchinski formulation (analogous to Λ_1 in Equation (1.34)), but it also plays the role of an IR cutoff in the EAA method since Γ_k is the Effective Action of the fast modes.

The contribution to the functional integral of a momentum shell of thickness Δk can be written as a loop expansion. The ℓ-loop term is of order $(\Delta k/k)^\ell$, so that the continuous FRGE ($\Delta k/k \to 0$) looks like a one-loop equation, as we expect [32]. The one-loop nature of the FRGE is manifest in the presence of a single trace (momentum integration). In fact,

[13] Note that respect to Equation (1.38) and (2.37), there is no minus sign in the RG time definition. This means that we are looking to flow from the IR to the UV region.

the FRGE can be represented graphically as

$$\partial_t \Gamma_k = \frac{1}{2}$$  , (2.92)

where the double line represents the full propagators and the crossed circle represents the insertion of the regulator $\partial_t \mathcal{R}_k$.
The counterpart of this simplicity is that the equation is only exact if one takes into account *all* possible terms in the action. Since it is practically impossible to solve the exact equation, its effectiveness hinges crucially on a good choice of approximation and of the regulator. There are three main systematic expansion schemes. We briefly recall their definition, and then discuss the relation among them, and to standard perturbation theory.

Loop expansion. This is an expansion in powers of $\hbar$ [69, 70]. We write for the EAA

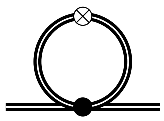
$$\Gamma_k[\chi] = S_\Lambda[\chi] + \sum_{L=1}^{n} \hbar^L \Gamma_{L,k}[\chi] \ . \tag{2.93}$$

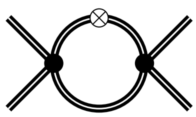
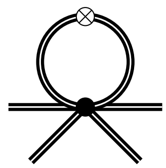
Inserting (2.93) in the the flow equation (2.91) one can reproduce the usual beta functions of perturbation theory. First, introducing S_Λ in the RHS of (2.91), one calculates the one-loop beta functional $\partial_t \Gamma_{1,k}$. Integrating over k from Λ to k' gives the one-loop EAA $\Gamma_{1,k'}$, and using this in the RHS of (2.91) one calculates the two-loop beta functional $\partial_t \Gamma_{2,k}$. The procedure can be iterated. Since in many cases the loop expansion coincides with the expansion in the marginal coupling constant, this approximation scheme is very close to standard weak-coupling perturbation theory.

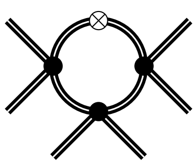
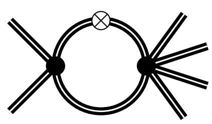
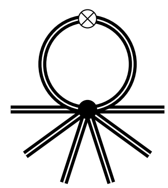
Vertex expansion. The EAA can be Taylor-expanded in powers of the field [71, 72, 73]

$$\Gamma_k[\chi] = \sum_n \int_{p_1} \dots \int_{p_n} \Gamma_k^{(n)}(p_1, \dots, p_n)\chi(p_1)\dots\chi(p_n) \,, \tag{2.94}$$

where p_n are the external momenta. By functionally differentiating (2.91) one obtains an infinite sequence of flow equations for the n-point functions $\Gamma_k^{(n)}$. The vertex expansion consists in truncating this sequence at some finite order. The first three equations of the sequence for a $\mathbb{Z}_2$-invariant scalar theory can be represented graphically as follows

$$\partial_t \Gamma_k^{(2)} = -\frac{1}{2}$$  , (2.95a)

$$\partial_t \Gamma_k^{(4)} = 3$$  $$-\frac{1}{2}$$  , (2.95b)

$$\partial_t \Gamma_k^{(6)} = -45$$  $$+\,15$$  $$-\frac{1}{2}$$  . (2.95c)

Here the black dots represent full vertices. The vertex expansion is clearly a good approximation in weak field situations, and is widely used in particle physics, where one generally deals with just a few quanta of the field. In this approximation, one retains the full momentum dependence.

Derivative expansion. When one is interested in low energy phenomena, one can expand the action in powers of derivatives. This is close to many applications of the effective field theory approach. For a single scalar field the expansion starts with [48, 49, 74, 75]

$$\Gamma_k[\chi] = \int_x \left(V_k(\chi) + \frac{1}{2} z_k(\chi)(\partial\chi)^2 + O(\partial^4) \right) \tag{2.96}$$

where V_k and z_k are arbitrary functions of the field. Inserting it in (2.91) one obtains flow equations for V_k, z_k etc. This is complementary to the vertex expansion, because one retains the full field dependence, but only the lowest powers of momentum.

These expansions give rise to different forms of perturbation theory, where different parameters are assumed to be small, and a statement that is perturbative in one expansion is generally nonperturbative in the others. For example, the leading order of the derivative expansion, which is the Local Potential Approximation (LPA) [74], consists in retaining in (2.96) only the running potential V_k and to put $z_k = 1$. The beta function of the potential that can be obtained in this way from the FRGE contains information about infinitely many orders of the vertex expansion, and to all loop orders. If furthermore the potential is assumed to be a finite polynomial, then one is working simultaneously in the derivative and vertex expansion. Similarly, truncating the vertex expansion to a finite order gives n-point functions that include all orders of the derivative expansion and of the loop expansion, and the EAA calculated at a given order of the loop expansion contains information that includes all orders of the derivative and vertex expansion. In practice, in applications of perturbative quantum field theory to particle physics, one generally considers two-, three- and four-point functions, at a finite order of the loop expansion, and therefore one is working simultaneously in the vertex and in the loop expansion. This is what we shall refer to as *standard perturbation theory* in Chapter 3.

In Section 2.5, we defined redundant and scaling operators at fixed points for the Wilsonian Effective Action. In Chapter 4 we will define the redundant operators in the EAA method obtaining the analogous definition to Equation (2.35), however at this stage we can follow an analogous idea of Equation (2.21). In fact, sufficiently close to the Gaussian fixed points, i.e.

$$\Gamma_{\mathrm{GFP}} = \frac{\zeta_{2n}}{2} \int_x \chi(-\partial^2)^n \chi \,, \tag{2.97}$$

the "tree level" definition of inessential coupling is

$$\zeta \frac{\partial}{\partial \zeta} \Gamma_k = \Phi[\chi] \cdot \frac{\delta \Gamma_k}{\delta \chi} \,, \tag{2.98}$$

where Φ is a functional of χ. This form is precisely the definition of inessential couplings for the Effective Action [39] and it is analogous to the "tree level" definition of inessential couplings for the Wilsonian Effective Action in Equation (2.21). Therefore, as we already said for the Wilsonian Effective Action, at the GFP operators $\frac{1}{2}\int_x \chi(-\partial^2)^s \chi$ with $s \geq n$ are redundant and we can set to zero the couplings associated with those having $s > n$ and normalized to one ζ_{2n}.
In Chapter 4, we will be more precise and generalize the definition (2.98) to include a nonlinear part that comes from the "blocking contribution", analogous to the second term in Equation (2.36), and allows to reach all the points in theory space.
Close to the Gaussian fixed point, in the standard procedure of perturbative QFT, renormalization conditions set the values of the inessential coupling present in free theories. For example, at the standard GFP ($n = 1$) we impose $\zeta_2 = 1$ to fix the wave function renormalization. In Chapter 4, we will discuss the connection between inessential couplings and renormalization conditions.

2.8.1 Local potential approximation in EAA method

As we said previously, the derivative expansion is a convenient tool to treat statistical systems. In this subsection, we will discuss briefly the trivial fixed point coming from this approximation of the EAA and then we will present a genuine functional method to calculate the critical exponents at the non-trivial fixed points.

Let's consider $\mathbb{Z}_2$-invariant scalar field theory. At LPA, we set to $z_k = 1$ in Equation (2.96) and the flow equation reads

$$\partial_t V_k = \frac{1}{2(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty \mathrm{d}z \, z^{\frac{d}{2}-1} \frac{\partial_t \mathcal{R}_k}{z + \mathcal{R}_k(z) + V_k^{(2)}} \,, \tag{2.99}$$

where the super-script (n) on functions of the field denotes their n-th derivative. Defining the dimensionless rescaled variables

$$\tilde{\chi} := k^{-(d-2)/2}\sqrt{2(4\pi)^{d/2}\Gamma(d/2+1)}\chi\,, \qquad v_t(\tilde{\chi}) := k^{-d}(4\pi)^{d/2}\Gamma(d/2+1)V_k(\chi)\,, \tag{2.100}$$

and writing the regulator as $\mathcal{R}_k =: k^2 R(y)$ with $y := z/k^2$, the dimensionless flow equation reads

$$\partial_t v_t = -d\, v_t + \frac{d-2}{2}\tilde{\chi} v_t^{(1)} + \frac{d}{2}\int_0^\infty \mathrm{d}y\, y^{\frac{d}{2}-1}\frac{R(y) - y\,R'(y)}{y + R(y) + v_t^{(2)}}\,. \tag{2.101}$$

Assuming $\mathbb{Z}_2$-symmetry and Taylor-expanding the potential

$$v_t(\tilde{\chi}) = \sum_n \frac{\tilde{\lambda}_{2n}(t)}{(2n)!}\tilde{\chi}^{2n}\,, \tag{2.102}$$

we can derive infinitely many beta functions $\tilde{\beta}_{2n} = \partial_t\tilde{\lambda}_{2n}$. This way, we are implementing the derivative expansion and the vertex expansion simultaneously. These are obtained by expanding both sides of (2.101) in powers of the field and equating the coefficients. For arbitrary regulator, and in any dimension, for the first few couplings this leads to

$$\tilde{\beta}_0 = -\,d\tilde{\lambda}_0 + \frac{d}{2}\int_0^\infty \mathrm{d}y\, y^{\frac{d}{2}-1}\frac{R(y) - y\,R'(y)}{y + R(y) + \tilde{\lambda}_2}\,, \tag{2.103a}$$

$$\tilde{\beta}_2 = -\,2\tilde{\lambda}_2 - \lambda_4\frac{d}{2}\int_0^\infty \mathrm{d}y\, y^{\frac{d}{2}-1}\frac{R(y) - y\,R'(y)}{\left(y + R(y) + \tilde{\lambda}_2\right)^2}\,, \tag{2.103b}$$

$$\tilde{\beta}_4 = (d-4)\tilde{\lambda}_4 + \frac{d}{2}\int_0^\infty \mathrm{d}y\, y^{\frac{d}{2}-1}\frac{R(y) - y\,R'(y)}{\left(y + R(y) + \tilde{\lambda}_2\right)^2}\left(\frac{6\lambda_4^2}{\left(y + R(y) + \tilde{\lambda}_2\right)} - \lambda_6\right), \tag{2.103c}$$

$$\tilde{\beta}_6 = (2d-6)\tilde{\lambda}_6 + \frac{d}{2}\int_0^\infty \mathrm{d}y\, y^{\frac{d}{2}-1}\frac{R(y) - y\,R'(y)}{\left(y + R(y) + \tilde{\lambda}_2\right)^2}\left(\frac{-90\lambda_4^3}{\left(y + R(y) + \tilde{\lambda}_2\right)^2} + \frac{30\lambda_4\lambda_6}{\left(y + R(y) + \tilde{\lambda}_2\right)} - \lambda_8\right). \tag{2.103d}$$

Note that they coincide with the Equation (2.92) and the first three equations of the vertex expansion, namely Equations (2.95), when the n-point functions are evaluated at zero momentum.

At this point we have written the RG equation in form of Equation (2.37) and we can analyze the fixed points, i.e. $v_\star$ such that $\partial_t v_\star = 0$. From the potential beta functional, it is manifest the presence of the trivial fixed point

$$v_\star = \frac{1}{2}\int_0^\infty \mathrm{d}y\, y^{\frac{d}{2}-1}\frac{R(y) - y\,R'(y)}{y + R(y)} \tag{2.104}$$

which is the Gaussian fixed point, where all the couplings, except the vacuum energy $\tilde{\lambda}_0$, has zero value[14]. In order to find the high temperature fixed point, we have to change the kinetic term since starting with the expansion (2.96) excludes from the beginning some universality classes. In particular, we can modify Equation (2.96) in the following way

$$\Gamma_k[\chi] = \int_x \left(V_k(\chi) + \frac{1}{2}\,\chi\, h_k(\Delta)\,\chi \right) . \tag{2.105}$$

The flow equations reads

$$\partial_t v_t = -d\, v_t + \frac{d-2}{2}\tilde{\chi} v_t^{(1)} + \frac{d}{2}\int_0^\infty \mathrm{d}y\, y^{\frac{d}{2}-1} \frac{R(y) - y\,R'(y)}{\tilde{h}(y) + R(\tilde{h}(y)) + v_t^{(2)}} , \tag{2.106}$$

$$\partial_t \tilde{h}_t(y) + (d - 2\Delta_\chi)\tilde{h}_t(y) - 2\,y\partial_y\,\tilde{h}_t(y) = 0 , \tag{2.107}$$

where $h_k(z) = k^{d-2\Delta_\chi}\,\tilde{h}_t(y)$ and Δ_χ is the dimension of χ. This way, we can explore other universality classes, changing Δ_χ [15].

Returning to Equation (2.101), for the non-trivial fixed point, we have to specify the regulator and the dimension. We choose the optimized or Litim regulator

$$R(y) = (1-y)\Theta(1-y) , \tag{2.108}$$

where $\Theta(1-y)$ is the Heaviside theta function and which has been argued to provide *optimized* results, in a certain class of models and truncations [76, 77]. Contrary to the Wilsonian Effective Action, we can also use sharp cutoff under some circumstances since the difficulties induced by the sharp limit may be circumvented by considering the Legendre transformation [55]. In fact, the first derivation of an ERGE for the Effective Action has been carried out with a sharp cutoff in [78].

Since we are interesting in reproducing the critical exponents in the third column of Table 1.1, we set $d = 3$ and we obtain the fixed point equation for the potential

$$0 = -3v_\star + \frac{1}{2}\tilde{\chi}\, v_\star^{(1)} + \frac{1}{1 + v_\star^{(2)}} . \tag{2.109}$$

In order to solve this equation, we can analyze the dimensionless beta functions (2.103) truncating at given power of the field and look at values of $\tilde{\lambda}$s such that $\tilde{\beta}$s are zero. The functional result of the derivative expansion at LPA is achieved when we take the

[14]Someone can complain about the fact that actually there is a cancellation between the classical contribution and the quantum contribution, but this cancellation happens only at the level of vacuum energy which does not matter for systems without gravity.

[15]Solving for $\partial_t h_\star = 0$, we find $h_\star = y^{(d-2\Delta_\chi)/2}$. Requiring analyticity at $y = 0$, $\Delta_\chi = (d-2n)/2$. Note that $n = 0$ corresponds to the high temperature fixed point, while $n \geq 2$ to higher derivative Gaussian fixed points.

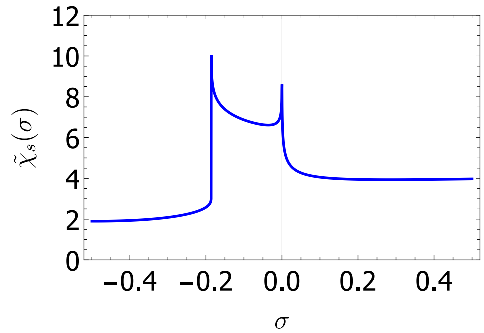


Figure 2.1: Spike-plot at LPA in $d = 3$.

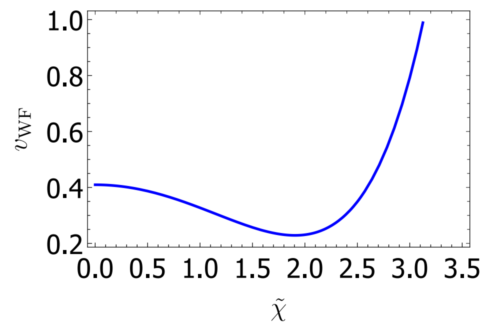


Figure 2.2: WF potential at LPA.

infinite limit for the truncation. Then depending on the problem that we want to solve, the convergence can be faster or slower.

Now we will explore two methods used to solve functionally the fixed point equation for the potential. The first method is called *shooting from the origin* and it works as follow[16]. We set generic conditions at zero value of the field $\tilde{\chi}$, i.e. $\{v_\star(0), v_\star^{(1)}(0)\}$, and we integrate numerically the equation. We get a family of solutions labelled by the initial conditions. Since we want a $\mathbb{Z}_2$ invariant theory or equivalently $v_\star$ even, the solutions are labelled by $\{v_\star(0)\}$. This value can be translated into $\sigma_\star := v_\star^{(2)}(0)$ using the fixed point equation, and in particular $v_\star(0) = (3 + 3\sigma_\star)^{-1}$. The reason under this choice is that $\sigma_\star$ represents the value of the mass parameter at the fixed point solution.

As the differential equation is not linear, we expect that for a generic initial condition there is a finite domain of the solution since at a value of the field, that we denote with $\tilde{\chi}_s(\sigma_\star)$, the solution blows up. We can plot $\tilde{\chi}_s(\sigma_\star)$ in function of the parameter $\sigma_\star$ and this is shown in Figure 2.1. This technique is sometimes referred to as *spike-plot* because globally well-defined solutions, namely divergences in $\varphi_s(\sigma)$, appear as spikes [75, 80, 81, 82]. As for the Polchinski formulation, we can treat d as a continuous parameter and as soon as we decrease the dimension below three new fixed points appear [81]. In case $d = 3$ for the Litim regulator, we have the GFP at $\sigma_{\mathrm{GFP}} = 0$ and the WF fixed point at $\sigma_{\mathrm{WF}} = -0.18606$. The profile of the potential at the WF solution is plotted in Figure 2.2.

Another method, complementary to the previous one, is the *shooting from infinity* method. In this case we solve iteratively the fixed point equation starting from the classical solution[17] $v_\star = \mathrm{A}\tilde{\chi}^6$. This way, we have an asymptotic evaluation $v_\star$ which reads

$$v_\star \sim_{\tilde{\chi}\to\infty} \mathrm{A}\tilde{\chi}^6 + \frac{1}{150\mathrm{A}\tilde{\chi}^4} - \frac{1}{6300\mathrm{A}^2\tilde{\chi}^8} + O\left(\tilde{\chi}^{-12}\right) + \dots \tag{2.110}$$

and depends on the parameter A. Now, we solve numerically the flow equation starting from

[16] For a very good review see [79].

[17] We call it classical because it is the solution that we get neglecting the quantum part.

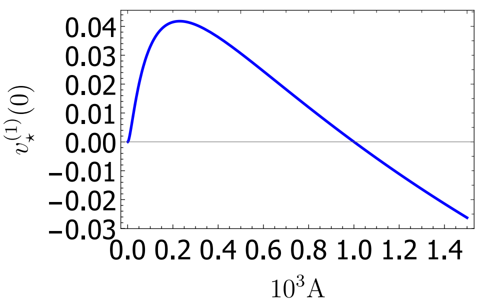


Figure 2.3: $v_\star^{(1)}(0)$, found from the shooting from infinity, as a function of the asymptotic parameter A.

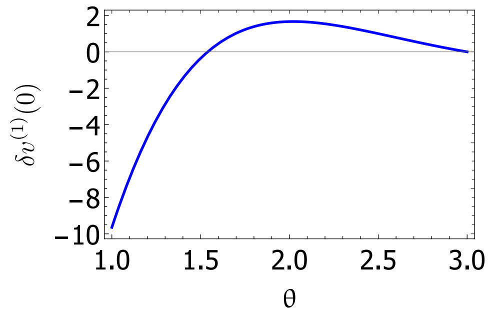


Figure 2.4: $\delta v^{(1)}(0)$, found from the shooting from infinity, as a function of the scaling exponents θ.

a large value of the field[18]. Also in this case we obtain a family of solutions parameterized by A. This time the important "detector" parameter is $v_\star^{(1)}(0)$, because when it is zero we find a globally-defined solution that satisfies the parity condition. The plot of $v_\star^{(1)}(0)$ in function of A is plotted in Figure 2.3 and shows that the curve intersects the abscissa axis twice: for $\mathrm{A}_{\mathrm{GFP}} = 0$ which is the GFP, and for $\mathrm{A}_{\mathrm{WF}} = 0.0010$ which is the WF fixed point. The profile of the WF potential is very close to the one found with the shooting from the origin for field value smaller than ~ 3.3 (under this value the mismatch is smaller than 10^{-3}), and then for greater values is becomes closer and closer to $\mathrm{A}_{\mathrm{WF}}\tilde{\chi}^6$.

In order to find the spectrum of fixed points, we perturb our solution

$$v_t = v_\star + \epsilon\, \mathrm{e}^{-\theta t}\, \delta v\,, \tag{2.111}$$

and the eigenvalues problem is

$$(\theta - 3)\delta v + \frac{1}{2}\tilde{\chi}\, \delta v^{(1)} - \frac{\delta v^{(2)}}{\left(1 + v_\star^{(2)}\right)^2} = 0\,. \tag{2.112}$$

Plugging the potential of the GFP, we simply obtain $\delta v_{\mathrm{GFP}} = \tilde{\chi}^{2(3-\theta)}$. Requiring analyticity and we find $\theta_n = 3 - \frac{n}{2}$ for all $n \in \mathbb{N}$, and in particular odd values of n correspond to odd parity eigenperturbations and even values correspond to even eigenperturbations. It is clear that the critical exponents coincide with the canonical dimensions of the all (even/odd) monomials that enter into the potential.

More interesting is to plug the potential of the WF fixed point: in this case we use

[18] In our case we took $\tilde{\chi}_{\max} = 30$: in general this value must be chosen large enough to be in the asymptotic region, where the asymptotic evaluation matches better with the solution.

again the shooting from infinity method. The asymptotic behavior of δv reads

$$\delta v \sim_{\tilde{\chi}\to\infty} \tilde{\chi}^{6-2\theta} - \frac{(5-2\theta)(6-2\theta)}{4500 A_{\mathrm{WF}}^2 \tilde{\chi}^{4+2\theta}} - \frac{(5-2\theta)(6-2\theta)}{94500 A_{\mathrm{WF}}^3 \tilde{\chi}^{8+2\theta}} + \dots . \tag{2.113}$$

We then solve the eigenvalue equation (2.112) starting from the large value of the field. After this process, we get again a family of solutions parameterized by θ. Note that δv can be normalized as we want since it satisfies a linear equation. In particular, this implies that the eigenvalue problem is over-constrained, since we impose two conditions at large value of the field, one parity condition at zero and the normalization of the eigenperturbations. The result of the over-constrained equation gives a discrete spectrum.
To impose even parity, we plotted $\delta v^{(1)}(0)$ as a function of θ and we look to the points where it becomes zero, while for odd parity, we have to look for values of θ where $\delta v(0)$ becomes zero. In Figure 2.4 there are the only two point for positive θ where $\delta v^{(1)}(0) = 0$: the trivial eigenvalue associated to the constant perturbation ($\theta = 3$) and the relevant scaling exponent equal to 1.5395, whose inverse gives ν.
Finally, at LPA we find that $\nu = 0.6496$, $\omega = 0.6557$ from the even spectrum and $\eta = 0$ from the odd spectrum[19].

At this point we have two "orthogonal" possibilities: either we change the regulator and we study the regulator dependence minimizing its effects[20] [76, 77, 83, 84, 85, 86] or we increase the truncation and we include the flow of z_k in Equation (2.96).

2.8.2 Relation with Polchinski formulation

In [87, 88] it is shown how the Polchinski formulation of the FRG is connected to the EAA formulation. In this subsection, we will present the idea developed in these works.

As we have in Section 2.7, the Polchinski's version of the FERGE is

$$\Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} S_\Lambda^{\mathrm{int}} = \frac{1}{2} \frac{\partial S_\Lambda^{\mathrm{int}}}{\partial \hat{\chi}} \cdot \Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \Delta_{\mathrm{UV}} \cdot \frac{\partial S_\Lambda^{\mathrm{int}}}{\partial \hat{\chi}} - \frac{1}{2} \mathrm{Tr}\, \frac{\partial^2 S_\Lambda^{\mathrm{int}}}{\partial \hat{\chi} \partial \hat{\chi}} \cdot \Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \Delta_{\mathrm{UV}} \,, \tag{2.114}$$

$$S_\Lambda^{\mathrm{Wil}} = \frac{1}{2} \hat{\chi} \cdot \Delta_{\mathrm{UV}}^{-1} \cdot \hat{\chi} + S_\Lambda^{\mathrm{int}} \,, \tag{2.115}$$

while the EAA version of the FERGE can be recast in the following form

$$\Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \Gamma_\Lambda^{\mathrm{int}} = -\frac{1}{2} \mathrm{Tr} \left(1 + \Delta_{\mathrm{IR}} \cdot \frac{\partial^2 \Gamma_\Lambda^{\mathrm{int}}}{\partial \chi \partial \chi}\right)^{-1} \cdot \Delta_{\mathrm{IR}}^{-1} \Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \Delta_{\mathrm{IR}} \,, \tag{2.116}$$

$$\Gamma_\Lambda^{\mathrm{tot}} = \frac{1}{2} \chi \cdot \Delta_{\mathrm{IR}}^{-1} \cdot \chi + \Gamma_\Lambda^{\mathrm{int}} \,, \tag{2.117}$$

[19] The most relevant odd eigenvalue is $-5/2$ and it is associated to the operator $\tilde{\chi}$. This means that the field scale classically.

[20] We will discuss more precise this point in Chapter 4.

where $\Delta_{\mathrm{IR}}^{-1} := \Delta + \mathcal{R}_\Lambda(\Delta)$. Note that the field has the hat for the Wilsonian Effective Action since it is the object that enters into the functional integral, while there are no hats in the EAA since we are dealing with expectation value.

The spectrum at LPA has been analyzed in [89, 90] and there is a remarkable degree of coincidence. This is due to the fact that they are related by a change of variables. Providing that

$$C_{\mathrm{UV}} + C_{\mathrm{IR}} = 1\,, \tag{2.118}$$

where $C_{\mathrm{IR}}(p^2/\Lambda) = p^2 \Delta_{\mathrm{IR}}(p^2/\Lambda^2)$, then we have the Legendre transform relation to pass from one formulation to another

$$S_\Lambda^{\mathrm{int}}[\hat{\chi}] = \Gamma_\Lambda^{\mathrm{int}}[\chi] + \frac{1}{2}(\hat{\chi} - \chi) \cdot \Delta_{\mathrm{IR}}^{-1} \cdot (\hat{\chi} - \chi)\,. \tag{2.119}$$

Note that this is an exact statement which transform the corresponding flow equations into each other [88]. If we truncate at LPA, Equation (2.119) gives the same relation between the Wilsonian potential and the effective potential. Therefore, the Polchinski's FERGE and the Wetterich-Morris FERGE are two realizations of the same exact RG.

We conclude this section by noting that in the Polchinski formulation the flow equation for the potential is given by Equation (2.80), while with EAA Equation (2.101) seems to possess more freedom in the choice of the regulator $\mathcal{R}_k$. However, the EAA Γ_k contains all the information about our system, i.e. RG flow, existence of a fixed point, computations of correlation functions, etc, while the Wilsonian Effective Action must still be put in the functional integral. Therefore, from this point of view the EAA is more manageable than the Wilsonian Effective Action given by the Polchinski's FRG, but the freedom comes with a cost: we have to be good at extracting the physical part from the non-physical part and disentangling the renormalization scheme dependence.

2.9 Outline

In this chapter, we have translated the general ideas given in Chapter 1: in particular, we specify the form of the ERG and we see its different implementations. We have also defined the space where the RG flow takes place, giving an introductory guide for its structure. In particular, we have seen the notions of essential and inessential couplings and their connection to the action form and the renormalization scheme that we choose. In Section 2.8, we mention the connection between the renormalization condition of perturbative QFT and inessential couplings, i.e. the normalization condition of the standard kinetic term that sets to one the value of the wave function renormalization constant. This procedure close to the Gaussian universality class is just one of the possible renormalization conditions that we can set, since we have more freedom in the parameterization than just rescaling the field variable by a constant factor. At a generic point of theory space, there is always a certain amount of freedom to set the value of the inessential couplings using renormalization conditions. Moreover, inside different universality classes the role of kinetic term is assigned to different operators, like for the high temperature point or higher derivative Gaussian theories. This means that we are choosing a subset of universality classes when we set to one the coupling in front of the desired kinetic operator. Therefore, the renormalization conditions, together with the renormalization scheme, are not harmless. As we said for the renormalization scheme, also the renormalization conditions can exclude some regions of the theory space.

In order to clarify better these points, let's make some examples. Let's start with the parameterization of the action, or equivalently the choice of coordinate system in the theory space. The specific choice that we make can hide some information. In the LPA flow in Polchinski formulation, the low temperature fixed point is visible only after the change from V to the Gibbsian-like measure. For the Wetterich-Morris equation, something similar happens: in fact, when we decide the truncation of our theory to be LPA with the standard kinetic term, we are excluding all the other universality classes with kinetic term of different order. The renormalization conditions, that we impose, have the same effect, because, when we choose to set to one a particular coupling among all the possible couplings associated with different kinetic operators, we are excluding regions of the theory space. For the cutoff as well, there is the same impact on the theory space. Parameterization, renormalization conditions and renormalization scheme, or for simplicity we can say the RG scheme choices, project our investigation on a submanifold of theory space. As we already said, the RG scheme choice must be designed for the specific cases that we want to analyze. Since this exclusion seems inevitable, we must look at the RG scheme choice as a viewfinder. At this point, one may be worried about the RG method. Quoting Fisher in [42], it may happen that a badly chosen, or "unfocused", RG scheme exhibits no fixed point. Then, the idea would simply be to re-examine the physics/mathematics more closely, or "adjust the viewfinder", to choose the formulation most accurately reflecting the physical situation of interest.

In Chapter 3 we will discuss two example of how the regulator choice in the FRG cannot be harmless. The first case can happens when we design the regulator in such a way to achieve some particular results: then generally the use of such a regulator is limited to specific cases [1]. The second situation occurs when the regulator choice breaks some symmetries of the system: in this case physical information is hidden and unphysical features arise and give a completely wrong qualitative and quantitative picture [2, 79].

We conclude this chapter by giving an hint about the generalization of Equation (2.41). In this equation, we can find Wilson's idea of incomplete integration of modes, which generalizes the Gell-Mann and Low idea used in perturbative QFT. The natural next step is to generalize even more Equation (2.41) to the following transformation [42]

$$\chi(x) \to Z(\chi(x))\,\chi(x)\,. \tag{2.120}$$

Of course, this changes the n-point vertex structure

$$\begin{aligned}\Gamma^{(n)} \to & Z^n(0)\Gamma^{(n)} + \sum Z^{(1)}\,\Gamma^{(n-1)}\,\cdot\delta + \sum Z^{(2)}\,\Gamma^{(n-2)}\,\cdot\delta\cdot\delta + \dots \\ & + \sum Z^{(n-2)}\,\Gamma^{(2)}\,\cdot\underbrace{\delta\cdot\dots\cdot\delta}_{n-2\text{ times}}\,.\end{aligned} \tag{2.121}$$

where δ denotes delta functions.

In Chapter 4, we will see that this kind of transformations can be done along the RG flow using a generalized version of the FERGE (2.91) and they shed new light on the problem of inessential couplings. This way, it is possible not only to simplify the equations from a practical point of view, but also to achieve a physical spectrum that contains only scaling operators. Also in the contest of EFT, the standard procedure consists of simplifying the description using field redefinitions. Let's consider the following action [91]

$$S = \int_x \left(\frac{1}{2}\partial_\mu\chi\partial_\mu\chi + \frac{1}{2}m^2\chi^2 + \frac{\lambda}{4!}\chi^4 + \frac{c_1}{\Lambda^2}\chi^3\partial^2\chi + \frac{c_2}{\Lambda^2}\chi^6 + \dots\right)\,. \tag{2.122}$$

Then we perform the field redefinition

$$\chi(x) \to \chi + \frac{\alpha_1}{\Lambda^2}\chi^3 + \dots\,, \tag{2.123}$$

and the new action reads

$$S = \int_x\left(\frac{1}{2}\partial_\mu\chi\partial_\mu\chi + \frac{1}{2}m^2\chi^2 + \left(\frac{\lambda}{4!} + \frac{\alpha_1\,m^2}{\Lambda^2}\right)\chi^4 + \frac{(c_1-\alpha_1)}{\Lambda^2}\chi^3\partial^2\chi + \left(\frac{c_2}{\Lambda^2} + \frac{\alpha_1\,\lambda}{3!\Lambda^2}\right)\chi^6 + \dots\right)\,.$$

The two actions give the same S-matrix, but in the latter we can eliminate the $\chi^3\partial^2\chi$ operator, putting $\alpha = c_1$, at the cost of redefining the coefficients of the χ^4 and χ^6 operators. Of course, computations are easier in this way, because we have a smaller number of independent operators.

Chapter 3

Vanishing regulators in FRG

'In nihil ab nihilo quam cito recidimus.'
ancient Roman epigraph

Dimensional regularization [92], together with modified minimal subtraction ($\overline{\text{MS}}$) [1], is the most widely used regularization and renormalization method in particle physics. It owes its popularity mainly to its simplicity and to the fact that it respects gauge invariance, one of the cornerstones of particle physics models. It is also remarkably selective: in the language of momentum cutoffs, it extracts only the logarithmic divergences, which for most applications turn out to contain the important information (in particular, the beta functions of the marginal couplings). However, in its standard implementation, dimreg is a purely perturbative device, and it works only in even dimensions.

On the other hand, the FRG[2] is a convenient way of implementing Wilson's idea of incomplete integrating. At its core lies a choice of the regulator function $\mathcal{R}_k$ that suppresses the contribution of low momentum modes to the path integral.

In Section 2.8 we have defined the standard perturbative theory in the FRG. However, a different implementation of the weak coupling expansion is possible in presence of background fields, since perturbation theory can then account for the full dependence of the vertices on the latter variables. This is what might be called *functional perturbation theory*. The application of dimreg to such functional methods [93, 94, 95] has been recently revived in the study of conformal field theories [96]. In these respects, our study could be interpreted as an attempt to extend these methods to the nonperturbative domain.

In summary, as we have said in Section 2.8, one of the most interesting features of the FRGE is the availability of various approximation schemes that sometimes allow us to follow the flow of infinitely many couplings in a single stroke and to go beyond standard perturbation theory. On the other hand, the arbitrariness in the choice of the regulator

[1] In the FRG one follows the flow of renormalized quantities. Therefore, for a meaningful comparison, we have to supplement dimreg by a renormalization prescription.

[2] From now on, we will choose the FRG based on the EAA method.

means that much of the information contained in the flow is unphysical. One has to learn to extract physical information from it.

Since the strengths and weaknesses of the FRGE and of dimreg/$\overline{\text{MS}}$ are quite complementary, it would be useful to transfer some of the strengths of one method to the other, or at least to use them in a complementary way, so as to overcome the respective weaknesses. The main question that we shall address is the following: is there a choice of regulator that reproduces the beta functions of the $\overline{\text{MS}}$ scheme in the standard perturbative domain? We provide here a positive answer to this question: we show that by bending the standard rules and procedures of the FRG it is possible to reproduce the results of dimreg/$\overline{\text{MS}}$, at least up to two loops. For this reason we will talk about a *pseudo-regulator* that, upon use in the FRG equation, reproduces the beta functions of $\overline{\text{MS}}$. In particular, such a regulator must be implemented with a limit that makes it vanishing.

Therefore, it is natural to explore other classes of vanishing regulator. The main motivation for this study comes from another issue that arises in certain applications of the FRG. The central idea is simple and can be stated in great generality. Suppose that the action at the microscopic level is invariant under certain transformations. Since the symmetry reflects physical properties of the system, one would like to maintain it in the course of the RG flow. However, for technical reasons, it may be difficult to construct a regulator that has the symmetry, and in this case the EAA will not have it either. To be more precise, the classical symmetry of the bare action is translated into a "quantum" symmetry of the EAA, which is deformed by the presence of the regulator. The latter symmetry is only implicitly determined, as the corresponding regulator-dependent Ward identity cannot in general be analytically and exactly solved [97]. This will give rise to unpleasant complications. Intuitively, we may try to minimize the breaking of the symmetry by making the regulator as "small" as possible. Let us make this notion a bit more precise. For dimensional reasons, we can write the regulator as

$$\mathcal{R}_k(z) = k^2 R_a(y) = k^2 a R(y) \tag{3.1}$$

and R is a dimensionless function of the dimensionless variable $y = z/k^2$, that is assumed to satisfy the normalization condition $R(0) = 1$ and a is a positive real number. [3] In many applications it is convenient to choose a shape function R depending on some of the parameters appearing in the ansatz adopted for the EAA. The most common example is the insertion of an overall wave function renormalization factor Z_k. Using regulator of the form (3.1), we shall mainly neglect these subtleties, as in most of our studies we will truncate the EAA to a scale-dependent local effective potential, and will be concerned with the limit $a \to 0$, which we call the limit of vanishing cutoff. [4] One expects that in this limit the spurious effects due to the breaking of the symmetry by the regulator can be removed,

[3] Consider a fixed shape function R, such that $R(y) = 0$ for $y > 1$. The limit $a \to \infty$ is expected to completely remove from the path integral all the fluctuations with momenta $q^2 < k^2$. This is often referred to as the sharp cutoff limit. Numerically optimal results are usually obtained for $a \approx 1$.

[4] Thus, vanishing cutoff should not be misinterpreted as $k \to 0$.

or at least minimized. It may seem that this limit is trivial, because for $a = 0$ there is no cutoff, and the RHS of the exact FRG equation (2.91) vanishes, but we shall see that some important physical information remains available even in this limit.

Even though many of the challenges and properties of the vanishing-regulator limit can be expected to characterize large families of shape functions R, we mainly focus on the following regulator choice

$$R_a(y) = a(1-y)\Theta(1-y)\,, \tag{3.2}$$

as in several interesting cases it is hardly feasible to study the vanishing regulator limit without having first specified a shape function. The reasons for this are explained in Section 3.7.3 and further discussed in Section 4.9.

In Section 3.1 we state the problem in a precise way, in the most straightforward and simplified setting: the case of a linear scalar field theory in the LPA. The solution of the problem and our pseudo-regulator are given in Section 3.2. We also explore some of the intrinsic freedom in the construction of the pseudo-regulator, and we exhibit a one-parameter family of regulators that continuously connects the results of standard FRG regulators with those of the $\overline{\text{MS}}$ pseudo-regulator.

In Section 3.3, we account for the inclusion of the field's anomalous dimension. This transition only requires minor generalizations of the pseudo-regulator, allowing for some more free parameters, which come along with corresponding forms of *RG improvement* in the one-loop flow equations. Section 3.4 further shows that the same pseudo-regulator is appropriate for the order ∂^2 of the derivative expansion. This discussion offers us the chance to address two exploratory applications of the $\overline{\text{MS}}$ functional RG equations. The first is the description of nonperturbative critical phenomena, namely two-dimensional multicritical scalar theories. We perform this study with the main goal to test the physical content of the RG improvement, which is the imprint of the FRG origin of our $\overline{\text{MS}}$ equations. The second application is provided by nonlinear $O(N)$ models in two dimensions, whose interest in this context lies in the interplay between nonlinearly realized symmetries and the FRG equations. In Section 3.4.2 we address the $O(N+1)$ nonlinear sigma model also implementing the vanishing regulator (3.2). This is an example of a system where the regulator breaks the symmetry of the theory (respecting only the subgroup $O(N)$) but in the limit of vanishing regulators the symmetry is seen to be restored.

An even more general truncation is needed to reproduce the two-loop $\overline{\text{MS}}$ beta functions in massive four-dimensional ϕ^4 theory (the perturbatively renormalizable linear $O(N)$ model). This is discussed in Section 3.5. This exercise serves as a proof that by means of the FRG and our pseudo-regulator one can, by considering large-enough truncations, obtain $\overline{\text{MS}}$ flow equations which are beyond a one-loop form.

In Section 3.6 we explore the role of dimensionality in our construction. In fact, while dimreg/$\overline{\text{MS}}$ is usually at work in an even number of dimensions d, the FRG equations can be obtained and applied for continuous d. We show that the latter feature can be preserved while taking the limit from the FRG to $\overline{\text{MS}}$.

In Section 3.7 we deal with the $\mathbb{Z}_2$-invariant scalar field theory in $d \geq 2$ Euclidean dimensions, and its RG fixed point (representing the Ising universality class). We find that the main features of the WF fixed point remain accessible in the limit of vanishing regulator, but the best approximation (after this limit is taken and among all possible polynomial truncations of the potential) for the correlation-length critical exponent ν is obtained with the simplest truncation, that only involves relevant couplings (the mass and the quartic coupling). There we also discuss the relation between the vanishing-a limit of (3.2) and the constant (momentum-independent) regulator, as well as the subtleties concerning the application of vanishing regulators in an even number of dimensions. In Section 3.8 we discuss a similar problem that arises in applications of the background field method. It is generally the case that the regulator breaks the symmetry of the classical action consisting of equal and opposite shifts of the background and fluctuation fields. Also this symmetry is seen to be restored in the limit of vanishing regulators. We conclude in Section 4.9 with a brief discussion of our results and some outlooks. Some auxiliary formulas and analyzes are provided in Appendices B , C and D.

3.1 Statement of the problem

In Section 2.8, we state the conditions that are generally imposed on a regulator. However, they are not needed in the derivation of the FRG equation, which would keep its exact one-loop form for any regulator choice. For the dimensionless *cutoff profile* $R(y) := k^{-2}\mathcal{R}_k(p^2)$ with $y = p^2/k^2$, the following are typical families of choices

$$R(y) = \frac{y^w}{e^{y^w} - 1} \ , \qquad\qquad R(y) = (1-y)^w \Theta(1-y) \ . \tag{3.3}$$

The second choice for $w = 1$ coincides with the Litim regulator reported in (2.108). Note that k plays the role of an *infrared* cutoff: its effect is to give a mass of order k to the modes with $\sqrt{z} < k$, and no mass to the modes with $\sqrt{z} > k$.

In order to extract useful information from the exact equation one has to approximate it in some way. For definiteness, let us focus on a single scalar field in the LPA. Then, Equation (2.99) can be rewritten as

$$\partial_t V_k = \frac{1}{2(4\pi)^{d/2}} Q_{d/2}\left[\frac{\partial_t \mathcal{R}_k}{P_k + V_k^{(2)}}\right] \ , \tag{3.4}$$

where $P_k(\Delta) := \Delta + \mathcal{R}_k(\Delta)$ [5]and

$$Q_n\left[W\right] := \frac{1}{\Gamma(n)} \int_0^\infty \mathrm{d}z\, z^{n-1} W(z) \tag{3.5}$$

[5]This definition is valid when we treat LPA.

is the momentum integral translated into Mellin transform. Assuming $\mathbb{Z}_2$-symmetry and Taylor-expanding the potential

$$V_k(\chi) = \sum_n \frac{\lambda_{2n}(k)}{(2n)!}\chi^{2n} , \tag{3.6}$$

we can derive infinitely many beta functions $\beta_{2n} = \partial_t \lambda_{2n}$. The previous expansion is the dimensionfull version of Equation (2.102). Analogously to Section 2.8.1, by expanding both sides of (3.4) in powers of the field and equating the coefficients, we obtain the dimensionfull version of (2.103), without the rescaling (2.100). For arbitrary regulator, and in any dimension, for the first few couplings this leads to

$$\beta_0 = \frac{1}{2(4\pi)^{d/2}} Q_{d/2}\left[\frac{\partial_t \mathcal{R}_k}{P_k+\lambda_2}\right] , \tag{3.7a}$$

$$\beta_2 = -\frac{1}{2(4\pi)^{d/2}} \lambda_4 Q_{d/2}\left[\frac{\partial_t \mathcal{R}_k}{(P_k+\lambda_2)^2}\right], \tag{3.7b}$$

$$\beta_4 = \frac{1}{2(4\pi)^{d/2}} \left(6\lambda_4^2 Q_{d/2}\left[\frac{\partial_t \mathcal{R}_k}{(P_k+\lambda_2)^3}\right] - \lambda_6 Q_{d/2}\left[\frac{\partial_t \mathcal{R}_k}{(P_k+\lambda_2)^2}\right]\right), \tag{3.7c}$$

$$\beta_6 = \frac{1}{2(4\pi)^{d/2}} \left(-90\lambda_4^3 Q_{d/2}\left[\frac{\partial_t \mathcal{R}_k}{(P_k+\lambda_2)^4}\right] + 30\lambda_4\lambda_6 Q_{d/2}\left[\frac{\partial_t \mathcal{R}_k}{(P_k+\lambda_2)^3}\right] - \lambda_8 Q_{d/2}\left[\frac{\partial_t \mathcal{R}_k}{(P_k+\lambda_2)^2}\right]\right) . \tag{3.7d}$$

We note that these are one-loop beta functions, since no resummation is involved. In order to have more explicit formulae, we can use the optimized regulator (2.108), that gives

$$Q_n\left[\frac{\partial_t \mathcal{R}_k}{(P_k+\lambda_2)^\ell}\right] = \frac{2}{\Gamma(n+1)} \frac{k^{2(n+1)}}{(k^2+\lambda_2)^\ell} . \tag{3.8}$$

Then, the first beta functions are

$$\beta_0 = \frac{k^{d+2}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+1\right)} \frac{1}{(k^2+\lambda_2)} , \tag{3.9a}$$

$$\beta_2 = -\frac{k^{d+2}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+1\right)} \frac{\lambda_4}{(k^2+\lambda_2)^2} , \tag{3.9b}$$

$$\beta_4 = \frac{k^{d+2}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+1\right)} \left(\frac{6\lambda_4^2}{(k^2+\lambda_2)^3} - \frac{\lambda_6}{(k^2+\lambda_2)^2}\right), \tag{3.9c}$$

$$\beta_6 = \frac{k^{d+2}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+1\right)} \left(-90\frac{\lambda_4^3}{(k^2+\lambda_2)^4} + 30\frac{\lambda_4\lambda_6}{(k^2+\lambda_2)^3} - \frac{\lambda_8}{(k^2+\lambda_2)^2}\right) . \tag{3.9d}$$

One can also calculate the beta functions of this theory at one loop using dimreg/$\overline{\text{MS}}$. The corresponding expressions read

$$\beta_0 = \frac{(-1)^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)(4\pi)^{d/2}}\lambda_2^{d/2} \ , \tag{3.10a}$$

$$\beta_2 = \frac{(-1)^{d/2}}{\Gamma\left(\frac{d}{2}\right)(4\pi)^{d/2}}\lambda_4\lambda_2^{d/2-1} \ , \tag{3.10b}$$

$$\beta_4 = \frac{(-1)^{d/2}}{(4\pi)^{d/2}}\left(3\lambda_4^2\frac{\lambda_2^{d/2-2}}{\Gamma\left(\frac{d}{2}-1\right)} + \lambda_6\frac{\lambda_2^{d/2-1}}{\Gamma\left(\frac{d}{2}\right)}\right) \ , \tag{3.10c}$$

$$\beta_6 = \frac{(-1)^{d/2}}{(4\pi)^{d/2}}\left(15\lambda_4^3\frac{\lambda_2^{d/2-3}}{\Gamma\left(\frac{d}{2}-2\right)} + 15\lambda_4\lambda_6\frac{\lambda_2^{d/2-2}}{\Gamma\left(\frac{d}{2}-1\right)} + \lambda_8\frac{\lambda_2^{d/2-1}}{\Gamma\left(\frac{d}{2}\right)}\right) . \tag{3.10d}$$

Note that the massless limit $\lambda_2 \to 0$ is finite for a fixed even dimension because of the Gamma functions in the denominators. One can even derive a functional perturbative beta function for V_k, analogous to (3.4) in [96]. (We shall discuss this in Section 3.2.1). The beta functions obtained by the two procedures are strikingly different. In the beta functions derived from the FRG, the dimension is carried by k, and there are denominators that automatically produce decoupling when one crosses the mass threshold $k^2 = \lambda_2$. In the beta functions of dimreg the dimension is always carried by powers of λ_2, and threshold effects are not accounted for. In fact such beta functions are only valid at energies much higher that λ_2.

Nevertheless, there is a relationship between these two sets of beta functions. To see this, note that for a generic regulator, the Q-functional with $\ell = n+1$ and $\lambda_2 = 0$ (which is dimensionless) is universal, i.e.

$$Q_n\left[\frac{\partial_t \mathcal{R}_k}{P_k^{n+1}}\right] = \frac{2}{\Gamma(n+1)} \ , \tag{3.11}$$

independently of the shape of the regulator. The reason for this is that in this case the integrand is a total derivative

$$\int_0^\infty \mathrm{d}z z^{n-1}\frac{\partial_t \mathcal{R}_k}{P_k^{n+1}} = 2\int_0^\infty \mathrm{d}y y^{n-1}\frac{R(y) - y\,R'(y)}{(y+R(y))^{n+1}} = \frac{2}{n}\int_0^\infty \mathrm{d}y \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{y}{y+R(y)}\right)^n . \tag{3.12}$$

The universal result will hold even if the regulator does not satisfy all the requirements that are listed in Section 2.8: it is enough that $R(\infty) = 0$ and $R(0) > 0$.

In the presence of a mass λ_2, we can expand the Q-functional for $k^2 > \lambda_2$

$$Q_n\left[\frac{\partial_t \mathcal{R}_k}{(P_k+\lambda_2)^\ell}\right] = \sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\ell+j)}{\Gamma(\ell)\Gamma(j+1)}\lambda_2^j Q_n\left[\frac{\partial_t \mathcal{R}_k}{P_k^{j+\ell}}\right]. \tag{3.13}$$

We see that the term $j = n - \ell + 1$ in the sum is universal and equal to

$$\frac{2(-1)^{n-\ell+1}}{\Gamma(\ell)\Gamma(n-\ell+2)}\lambda_2^{n-\ell+1} \ . \tag{3.14}$$

The beta functions of dimensional regularization consist exactly of all these universal terms, all the remaining ones being set simply to zero.

The main question we wish to address is whether the beta functions (3.10) can be directly obtained from the FRGE. This will be the case provided $\mathcal{R}_k$ is such that

$$Q_n\left[\frac{\partial_t \mathcal{R}_k}{(P_k+\lambda_2)^\ell}\right] = \frac{2(-1)^{n-\ell+1}}{\Gamma(\ell)\Gamma(n-\ell+2)}\lambda_2^{n-\ell+1} \ . \tag{3.15}$$

Thus the question becomes one about the existence of a regulator that gives (3.15). It is immediately clear that any standard regulator, satisfying the criteria given in Section 2.8, cannot fulfil this requirement. To understand why, it is sufficient to consider the case $\lambda_2 = 0$, in which case the requirement (3.15) becomes

$$Q_n\left[\frac{\partial_t \mathcal{R}_k}{P_k^\ell}\right] = \frac{2}{\Gamma(n+1)}\delta_{\ell,n+1} \ . \tag{3.16}$$

This implies that

$$\frac{1}{\Gamma(n)}\int_0^\infty \mathrm{d}y\, y^{n-1}\frac{R - y\,R'}{(y+R)^\ell} = \frac{1}{\Gamma(n+1)}\delta_{\ell,n+1} \ . \tag{3.17}$$

Using integration by parts and the standard properties of regulators, we obtain

$$\left(1 - \frac{\ell-1}{n}\right)Q_n\left[\frac{\mathcal{R}_k}{P_k^\ell}\right] = \left(\frac{\ell}{n+1} - \frac{\ell-1}{n}\right)\frac{\delta_{\ell,n+1}}{\Gamma(n+1)} \tag{3.18}$$

that, for $\ell \neq n+1$, gives $Q_n\left[\frac{\mathcal{R}_k}{P_k^\ell}\right] = 0$. Since the integrand in this Q-functional is positive, this implies $\mathcal{R}_k = 0$. While $\mathcal{R}_k$ cannot be identically vanishing, it appears possible to reproduce $\overline{\text{MS}}$ beta functions by giving up some of the requirements that are usually made of regulators and taking the $\mathcal{R}_k \to 0$ limit in a suitable way, as we shall discuss in the next section.

3.2 The $\overline{\text{MS}}$ pseudo-regulator

The desired pseudo-regulator depends, in addition to the scale k, also on a dimensionless parameter ϵ and a mass μ, which play a similar role as the ϵ and μ parameters of dimreg

$$\mathcal{R}_k(z) = \lim_{\epsilon\to 0} z\left[\left(\frac{zk^2}{\mu^4}\right)^\epsilon - 1\right] \ , \tag{3.19}$$

or equivalently

$$R(y) = \lim_{\epsilon \to 0} y \left[\left(\frac{y}{\tilde{\mu}^4} \right)^{\epsilon} - 1 \right] , \tag{3.20}$$

where $\tilde{\mu} = \mu/k$. A derivation and an explanation of this ansatz are given in Appendix B. Calculations have to be performed with a finite positive ϵ and the limit $\epsilon \to 0$ must be taken at the end of all calculations. Note that expanding for small ϵ

$$\mathcal{R}_k(z) = \epsilon z \log \left(\frac{k^2 z}{\mu^4} \right) + O(\epsilon^2) . \tag{3.21}$$

The function (3.19) grossly violates the defining properties of a regulator, as spelled out in Section 2.8. Aside from the fact that it vanishes in the limit $\epsilon \to 0$, it is a growing function of z and goes to zero for $z \to 0$. Nevertheless, it does what we asked for. Calculating the Q-functional, we obtain

$$\begin{aligned} Q_n \left[\frac{\partial_t \mathcal{R}_k}{(P_k + m^2)^\ell} \right] &= k^{2(n-\ell+1)} \frac{2}{\Gamma(n)} \int_0^\infty \mathrm{d}y \, y^{n-1} \frac{R - y\partial_y R - (\tilde{\mu}/2)\partial_{\tilde{\mu}} R}{(y + R + \tilde{m}^2)^\ell} \\ &= \lim_{\epsilon \to 0} \frac{2\epsilon}{1+\epsilon} m^{2(n-\ell+1)} \left(\frac{\mu^2}{km} \right)^{\frac{2n\epsilon}{1+\epsilon}} \frac{\Gamma\left(1 + \frac{n}{1+\epsilon}\right) \Gamma\left(\ell - 1 - n + \frac{n\epsilon}{1+\epsilon}\right)}{\Gamma(n)\Gamma(\ell)} . \end{aligned}$$

Here we introduced the more conventional notation m^2 for the mass parameter λ_2, and defined $\tilde{m} = m/k$. The integral in this Q-functional is convergent for $\ell > (n+1+\epsilon)/(1+\epsilon)$ and is defined elsewhere by analytic continuation. In the limit $\epsilon \to 0$ it goes to zero except at the points where the second Gamma function in the numerator has a pole, namely when $\ell - n - 1$ is zero or a negative integer. This way, we recover (3.15). We note that for $n \leq \ell - 2$ the final result (3.15) is identically zero because of the presence of the Gamma function on the denominator. Since ℓ is an integer, n must be integer in order to have a non zero result: since, for the beta functions of the LPA, $n = d/2$, this implies that only in even dimensions we get a non zero result. This agrees with the standard lore that dimreg only works in even dimensions.

Sometimes one needs the Q-functionals for $n \leq 0$. One can obtain them by observing that

$$Q_n \left[\frac{\partial_t \mathcal{R}_k}{(P_k + m^2)^\ell} \right] = \frac{(-1)^j}{\Gamma(n+j)} \int_0^\infty \mathrm{d}z z^{n+j-1} \left(\frac{\mathrm{d}}{\mathrm{d}z} \right)^j \frac{\partial_t \mathcal{R}_k(z)}{(P_k(z) + m^2)^\ell} , \tag{3.22}$$

where j is an integer such that $n + j > 0$. Evaluating this expression for the pseudo-regulator, we get

$$Q_n \left[\frac{\partial_t \mathcal{R}_k}{(P_k + m^2)^\ell} \right] = 2\, \delta_{-n,0}\, \delta_{\ell,1} . \tag{3.23}$$

This agrees with the analytic continuation of (3.15).

3.2.1 The effective potential in the LPA

We complete the discussion of the LPA approximation of a scalar theory by giving the functional equation for the potential

$$\partial_t v_t = -dv_t + \left(\frac{d}{2} - 1\right) \tilde{\chi} v_t' + c_d \left(-v_t^{(2)}\right)^{\frac{d}{2}} \ , \tag{3.24}$$

where $v_t = k^{-d} V_k$, $\tilde{\chi} = k^{1-\frac{d}{2}} \chi$ and $c_d = \frac{1}{(4\pi)^{d/2} \Gamma\left[\frac{d}{2}+1\right]}$. This agrees with the beta functional in $d = 4$ discussed in [96]. For comparison, the optimized regulator leads to the form

$$\partial_t v_t = -dv_t + \left(\frac{d}{2} - 1\right) \tilde{\chi} v_t' + c_d \frac{1}{1 + v_t^{(2)}} \ , \tag{3.25}$$

which coincides with Equation (2.109) modulo a constant rescaling of the quantities. We observe that (3.24) picks exactly the terms of the expansion of (3.25) with the right power of $v_t^{(2)}$ to give a dimension-d operator. Equations (3.24) and (3.25) are one-loop results, and in this sense can be said to be perturbative, but they contain infinitely many terms of the vertex expansion and thus are not perturbative in the standard sense.

Equation (3.24) can be applied only to even dimensions, so it does not admit the Wilson-Fisher fixed point as a solution in $d = 3$. This was to be expected, since dimreg only works in even dimensions. However we anticipate that generalizations to continuous d (including also odd integers) are possible, and will be discussed in Section 3.6. Equation (3.24) has been used in [98, 99, 100, 101, 102] to obtain several new results on statistical models. In $d = 2$ the corresponding fixed-point equation has the critical Sine-Gordon solution

$$V_\star = -\frac{m^2}{8\pi} \cos\left(\sqrt{8\pi}\chi\right) \ , \tag{3.26}$$

where m is an arbitrary mass. This result holds independently of the shape of the regulator [81]. A related question is whether this pseudo-regulator can reproduce some of the (multi)-critical theories in $d = 2$. It turns out that the answer is positive, as we shall discuss in greater detail in Section 3.4.1, where we consider a larger truncation.

3.2.2 A first generalization

In the definition (3.19) we have used an external, arbitrary mass scale μ. One could use instead a dimensionful coupling of the theory. In particular, in a massive theory, one could use m instead of μ. In the discussion of the two-loop beta functions, it will be convenient to actually use a mixture of the two. Therefore, let us generalize the pseudo-regulator to

$$\mathcal{R}_k(z) = \lim_{\epsilon \to 0} z \left[\left(\frac{zk^2}{m^{2b} \mu^{4-2b}} \right)^{\epsilon} - 1 \right] \ . \tag{3.27}$$

Note that m is a running parameter, so when we evaluate the Q-functional (3.15) it gives rise to an additional term depending on the beta function of the mass $\beta_{m^2} = \partial_t m^2$

$$Q_n\left[\frac{\partial_t \mathcal{R}_k}{(P_k+m^2)^\ell}\right] = \frac{2(-1)^{n-\ell+1}}{\Gamma(\ell)\Gamma(n-\ell+2)}\left(1-\frac{b}{2}\frac{\beta_{m^2}}{m^2}\right) m^{2(n-\ell+1)} \; . \tag{3.28}$$

The term with the beta function of the mass is a higher-loop effect, so at one loop this pseudo-regulator still reproduces the result of dimreg.

We note that the above discussion could be generalized replacing m by any combination of couplings with the dimension of mass. This would give rise to additional beta functions in the RHS of (3.28) and may be useful in higher-loop calculations.

In the massless case ($m = 0$) one has to set $b = 0$ and introduce by hand an IR regulator in the Q-functionals

$$Q_n\left[\frac{\partial_t \mathcal{R}_k}{P_k^\ell}\right] \mapsto Q_n\left[\frac{\partial_t \mathcal{R}_k}{(P_k+\mu^2)^\ell}\right] \; . \tag{3.29}$$

The limit $\mu \to 0$ has to be taken in the very end. Note that this IR regulator mass is not necessarily equal to the dimreg parameter μ, but we will not need this degree of generality, so the same mass will be used in both rôles. Then we obtain

$$Q_n\left[\frac{\partial_t \mathcal{R}_k}{(P_k+\mu^2)^\ell}\right] = \frac{2(-1)^{n-\ell+1}}{\Gamma(\ell)\Gamma(n-\ell+2)}\mu^{2(n-\ell+1)} \; . \tag{3.30}$$

As we already said above, this formula gives zero for $n < l-1$. Taking the limit for $\mu \to 0$ we get zero for $n > l+1$. So the result is

$$Q_n\left[\frac{\partial_t \mathcal{R}_k}{P_k^\ell}\right] = \lim_{\mu\to 0} Q_n\left[\frac{\partial_t \mathcal{R}_k}{(P_k+\mu^2)^\ell}\right] = \frac{2\delta_{\ell,n+1}}{\Gamma(n+1)} \; . \tag{3.31}$$

We note that only one combination of ℓ and n gives a non-vanishing result, which corresponds to the universal result of (3.11).

3.2.3 Interpolation with the optimized regulator

The Q-functionals for the optimized regulator (2.108) have been given in (3.8). Now let us consider the following one-parameter family of regulators (3.2). For $a \neq 1$ they violate the normalization condition, however such a condition is not needed to derive the flow equation (2.91) and the universal result in Equation (3.11). Moreover, the parameter a is used to optimize the results [84, 85, 86].

The corresponding Q-functionals are given by

$$Q_n\left[\frac{\partial_t \mathcal{R}_k}{(P_k+m^2)^\ell}\right] = \frac{2a}{(a+\tilde{m}^2)^\ell}\frac{k^{2(n-\ell+1)}}{\Gamma(n+1)}\; {}_2F_1\left(\ell, n, n+1, -\frac{1-a}{a+\tilde{m}^2}\right) \; , \tag{3.32}$$

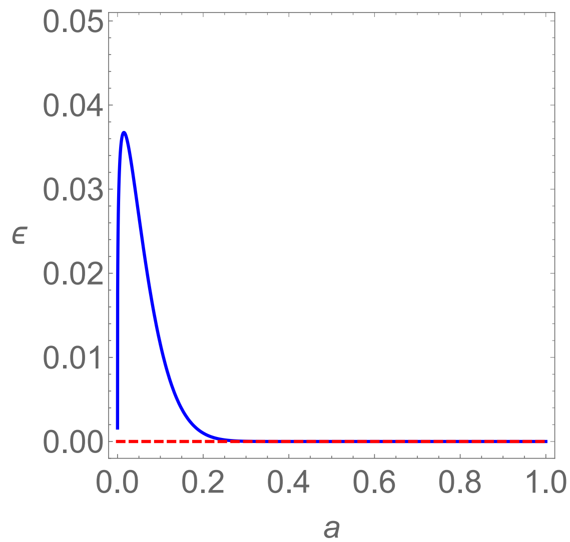


Figure 3.1: Blue, continuous curve: a path that reproduces the beta functions of dimensional regularization. Red, dashed curve: the limit of vanishing regulator. For a more detailed discussion see Section 3.7.

which coincides with Equation (3.8) for $a = 1$. If $\ell < n+1$ and $\tilde{m} > 0$, these are monotonically increasing functions of a, which are equal to (3.8) for $a = 1$ and decrease monotonically to zero when $a \to 0$. If $\ell > n+1$ and $\tilde{m} > 0$, they grow as functions of a and they go to zero when $a \to 0$. For $\tilde{m} = 0$ they are monotonic functions on the interval $0 < a < 1$, and in particular, we obtain

$$\lim_{a\to 0} Q_n\left[\frac{\partial_t \mathcal{R}_k}{P_k^\ell}\right] = \begin{cases} 0 \quad \text{for} \quad \ell < n+1 \ , \\ 1 \quad \text{for} \quad \ell = n+1 \ , \\ \infty \quad \text{for} \quad \ell > n+1 \ . \end{cases} \tag{3.33}$$

Note that the Q-functionals $\ell = n+1$, $\tilde{m} = 0$ are independent of a and equal to (3.11): thus, the universality of these Q-functionals is not spoiled by the regulator not being normalized. Clearly, in the massless case the beta functions will not be finite. [6] For this reason an additional regularizing device is needed to make sense of vanishing regulators inside the flow equations. On the other hand, if we set $a = 0$ with a generic mass parameter, the regulator vanishes identically and so do all the beta functions, including the universal ones. This means that the limit $a \to 0$ is not continuous. We would like to find a way to obtain at least the universal beta functions also for $a = 0$. One can achieve this by introducing an additional parameter ϵ. Consider the following interpolating regulator

[6]Note that these are infrared divergences: in the massive case all Q-functionals go to zero for $a \to 0$.

$\mathcal{R}_k = k^2 R(y, \tilde{m}^2, \epsilon, a)$, with

$$R(y, \tilde{m}^2, \epsilon, a) = \Big(a + (1-a)\tilde{\mu}^{-2(2-b)\epsilon}\tilde{m}^{-2b\epsilon} y^{1+\epsilon} - y\Big)\,\Theta\left(1 - \frac{a}{a+\epsilon} y\right) \ . \tag{3.34}$$

For $\epsilon \to 0$ it reduces to (3.2) and for $a \to 0$ it reduces to (3.27). Thus we can go continuously from the optimized regulator (2.108) to the pseudo-regulator (3.27) reproducing dimreg by following the blue curve shown in Figure 3.1. This way, the limit $a \to 0$ can be made continuous. The price one pays is that for $\epsilon \neq 0$ one does not have a good regulator in the sense of Section 2.8. In any case we obtain the desired result that all the non-universal beta functions go continuously to zero, while the universal ones remain constant.

Another possibility, instead, consists of taking the limits in the inverse order. In fact, in this case we can not even talk about the parameter ϵ and try to take the limit $a \to 0$ along the path $\epsilon = 0$ (red, dashed) in Figure 3.1. Then, since we know that behavior of the Q-functionals in Equation (3.32), we will take the limit $a \to 0$ inside the expressions of critical exponents. Thus, contrary to the dimreg case, we do not take the limit of the flow but we solve the flow and then we take the limit, i.e. red-dashed path in Figure 3.1. In Section 3.7, we show that this procedure do not eliminate the physical information about fixed point and then in Sections 3.4.2 and 3.8 we implement such a limit in order to cure symmetry breaking due to the choice of the regulator.

3.3 Beyond the LPA

In any quantum field theory application, and in the FRG framework as well, the choice of a regularization scheme should be tailored to a specific model and computation. In fact, although it is possible to devise regulators which remove all possible divergences altogether, much of the simplicity and power of the FRG comes from the possibility to adopt less drastic choices. More minimalistic regularization schemes allow for analytic, rather than numerical, computations, thus rendering the optimization process of such schemes simpler and more transparent.

In the process of relaxing the approximations used to solve the exact FRG equations, it is thus inevitable to reconsider the regulator choice. In this section we discuss the adjustment of the $\overline{\text{MS}}$ pseudo-regulator to the transition from the LPA to the inclusion of the running wave function renormalization. In the following, after the construction of a more general family of pseudo-regulators, we discuss its application to scalar field theory. We show how these pseudo-regulators are appropriate for investigations within the LPA$'$ approximation, which differs from the LPA only for the inclusion of a field- and momentum-independent wave function renormalization factor. The next layer of complexity, namely the derivative expansion at order ∂^2 including the field-dependence of the wave function renormalization, will be addressed in Section 3.4.

From the point of view of standard perturbation theory, the step from the LPA to the LPA$'$ already involves the resummation of an infinite class of Feynman diagrams – those self-energy-like one-particle-reducible corrections to the internal propagator lines which are accounted for by a non-trivial field's anomalous dimension – and therefore goes beyond finite-order perturbative calculations.

3.3.1 Rôle of the wave function renormalization

If the kinetic term in the action contains a non-trivial wave function renormalization factor $Z_k \neq 1$, one usually includes this global factor inside $\mathcal{R}_k$

$$\mathcal{R}_k(z) \mapsto Z_k \mathcal{R}_k(z) \; . \tag{3.35}$$

There are several reasons in favor of this choice. First of all, it allows to take over the regulators already working in the LPA, as the relevant regularized kinetic term is then in the functional form $z+\mathcal{R}_k(z)$. Furthermore, it is motivated by the desired invariance under rigid rescalings of the fields, also called reparameterizations. In other words, it allows to remove Z_k from the flow equations by simply rescaling the fields according to their quantum dimension

$$d_\chi = \frac{d-2+\eta_t}{2} \; , \tag{3.36}$$

where

$$\eta_t = -\partial_t \log Z_k \tag{3.37}$$

is the field anomalous dimension. While the former motivation is just a matter of convenience, the latter is deeper and less arbitrary. In fact, this choice is the one that minimizes the spurious breaking of reparameterization invariance due to the truncation of the exact FRG equation [34, 48, 53, 103].

Following the choice of (3.35). the flow equations receive further RG resummations encoded in the appearance of η_t on the RHS as

$$\partial_t \mathcal{R}_k(z) \mapsto Z_k \left(\partial_t \mathcal{R}_k(z) - \eta_t \mathcal{R}_k(z) \right) \; . \tag{3.38}$$

While the first term on the RHS gives rise to the Q-functionals already discussed in Section 3.1, the second term leads to the following new integrals

$$Q_n\left[\frac{\mathcal{R}_k}{(P_k+m^2)^\ell}\right] = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\ell+j)}{\Gamma(\ell)\Gamma(j+1)} m^{2j}\, Q_n\left[\frac{\mathcal{R}_k}{P_k^{j+\ell}}\right] \; . \tag{3.39}$$

Also for these new Q-functionals we see that the term $j = n - \ell + 1$ has no explicit k dependence, but it is not universal. For instance, the first example of regulator of Equation (3.3) with $w = 1$ would give

$$Q_n\left[\frac{\mathcal{R}_k}{P_k^{n+1}}\right] = \begin{cases} \log(2) & n=1\,, \\ \log\left(\frac{4}{3}\right) & n=2\,, \\ \frac{1}{2}\log\left(\frac{32}{27}\right) & n=3\,, \end{cases} \tag{3.40}$$

while the optimized regulator (2.108) leads to

$$Q_n\left[\frac{\mathcal{R}_k}{P_k^{n+1}}\right] = \frac{1}{\Gamma(n+2)} \ . \tag{3.41}$$

This exemplifies the arbitrariness in the construction of an $\overline{\text{MS}}$ pseudo-regulator for calculations beyond the LPA.

If we straightforwardly apply the recipe (3.35), we obtain a divergent result

$$Q_n\left[\frac{\mathcal{R}_k}{(P_k+m^2)^\ell}\right] = \lim_{\epsilon\to 0}\left\{\frac{(-m^2)^{n-\ell+1}}{\Gamma(\ell)\Gamma(n-\ell+2)}\left(1-\frac{n}{n+1}\right)\frac{1}{\epsilon}\right\} \ . \tag{3.42}$$

Therefore, including the wave function renormalization in the pseudo-regulator requires some additional work. In the following we explore a family of pseudo-regulators which achieve the goal of reproducing one-loop $\overline{\text{MS}}$ results, plus RG resummations, in the $\epsilon \to 0$ limit.

3.3.2 An extended family of pseudo-regulators

The first requirement on a new pseudo-regulator which is appropriate for the LPA$'$, is that it reduces to the pseudo-regulator we have adopted for the LPA in the $Z_k \to 1$ limit. [7] Hence we consider a generalization of (3.19) which amounts to the introduction of two new parameters $Z_0 > 0$ and σ

$$\mathcal{R}_k(z) = Z_0 Z_k^{\sigma\epsilon}\left[\left(\frac{k^2}{\mu^4}\right)^\epsilon z^{1+\epsilon} - z\right] \ . \tag{3.43}$$

While the most common choice, as in (3.35), would be $Z_0 = 1$ and $\sigma = 1/\epsilon$, we prefer to keep the two variables arbitrary for the time being. We define

$$G_k(q^2) = \left(Z_k q^2 + V_k^{(2)} + \mathcal{R}_k(q^2)\right)^{-1} \tag{3.44}$$

the regularized propagator. With our pseudo-regulator this reads

$$G_k = \frac{1}{Z_0\left(\frac{Z_k^\sigma k^2}{\mu^4}\right)^\epsilon z^{1+\epsilon} - (Z_0 Z_k^{\sigma\epsilon} - Z_k)z + V_k^{(2)}}, \tag{3.45}$$

from which it is manifest that having a vanishing $(Z_0 Z_k^{\sigma\epsilon} - Z_k)$ would tremendously simplify the task of evaluating the loop integrals. Though we restrain from this simplifying assumption, we still assume that this difference is small. We calculate all loop integrals by means of their Taylor series in this difference around zero.

[7] For simplicity we set $b = 0$.

Then the generic Q-functional becomes

$$
\begin{aligned}
&Q_n\left[G_k^\ell \partial_t \mathcal{R}_k\right]= \qquad (3.46)\\
&=\frac{\epsilon Z_0 Z_k^{\sigma\epsilon}}{(1+\epsilon)\Gamma(n)\Gamma(\ell)}\sum_{p=0}^{\infty}\frac{(Z_0 Z_k^{\sigma\epsilon}-Z_k)^p}{\Gamma(p+1)}\left(Z_0\left(\frac{Z_k^\sigma k^2}{\mu^4}\right)^{\epsilon}\right)^{-\frac{n+p}{1+\epsilon}}\\
&\left[(2-\sigma\eta_t)\Gamma\left(\frac{n+p}{1+\epsilon}+1\right)\Gamma\left(\ell+p-\frac{n+p}{1+\epsilon}-1\right)\left(Z_0\left(\frac{Z_k^\sigma k^2}{\mu^4}\right)^{\epsilon}\right)^{-1}\left(V_k^{(2)}\right)^{\frac{n+p}{1+\epsilon}-\ell-p+1}\right.\\
&\left.+\sigma\eta_t\Gamma\left(\frac{n+p+1}{1+\epsilon}\right)\Gamma\left(\ell+p-\frac{n+p+1}{1+\epsilon}\right)\left(Z_0\left(\frac{Z_k^\sigma k^2}{\mu^4}\right)^{\epsilon}\right)^{-\frac{1}{1+\epsilon}}\left(V_k^{(2)}\right)^{\frac{n+p+1}{1+\epsilon}-\ell-p}\right].
\end{aligned}
$$

From this expression it can be clearly seen that σ cannot diverge for vanishing ϵ (as a comparison of (3.35) and (3.43) would suggest) or both terms would also diverge. On the other hand choosing a vanishing σ in this limit would remove any Z_k and η_t dependence, thus reproducing the same results of the LPA pseudo-regulator. Finally, choosing σ to stay constant in the $\epsilon \to 0$ limit leads to

$$
\begin{aligned}
Q_n\left[G_k^\ell \partial_t \mathcal{R}_k\right]&=\frac{Z_0^{-n}\left(-V_k^{(2)}\right)^{n-\ell+1}}{\Gamma(n)\Gamma(\ell)\Gamma(n-\ell+2)}\sum_{p=0}^{\infty}\frac{\Gamma(n+p)}{\Gamma(p+1)}\left(1-\frac{Z_k}{Z_0}\right)^p\left[(2-\sigma\eta_t)+\frac{\sigma\eta_t(n+p)}{n+p+1}\right]\\
&=\frac{Z_0^{-n}\left(-V_k^{(2)}\right)^{n-\ell+1}}{\Gamma(\ell)\Gamma(n-\ell+2)}\left[(2-\sigma\eta_t)\frac{Z_0^n}{Z_k^n}+\sigma\eta_t\frac{n}{n+1}{}_2F_1\left(1+n,1+n,2+n;1-\frac{Z_k}{Z_0}\right)\right].
\end{aligned}
\tag{3.47}
$$

Summarizing we have

$$
Q_n\left[G_k^\ell \partial_t \mathcal{R}_k\right]=\frac{Z_k^{-n}\left(-V_k^{(2)}\right)^{n-\ell+1}}{\Gamma(\ell)\Gamma(n-\ell+2)}\left(2-\sigma\eta_t(1+H_0)\right), \tag{3.48a}
$$

$$
Q_n\left[G_k^\ell G_k' \partial_t \mathcal{R}_k\right]=-\frac{n}{n-1}\frac{Z_k^{1-n}\left(-V_k^{(2)}\right)^{n-\ell-1}}{\Gamma(\ell+2)\Gamma(n-\ell)}\left(2-\sigma\eta_t(1+H_1)\right), \tag{3.48b}
$$

$$
Q_n\left[G_k^\ell G_k'' \partial_t \mathcal{R}_k\right]=\frac{2n}{n-2}\frac{Z_k^{2-n}\left(-V_k^{(2)}\right)^{n-\ell-2}}{\Gamma(\ell+3)\Gamma(n-\ell-1)}\left(2-\sigma\eta_t(1+H_2)\right), \tag{3.48c}
$$

where primes denote differentiation with respect to z, and we introduced the following notations

$$H_0(n, Z_k, Z_0) = -\frac{n}{n+1}\left(\frac{Z_k}{Z_0}\right)^n {}_2F_1\left(n+1, n+1, n+2; 1-\frac{Z_k}{Z_0}\right), \tag{3.49a}$$

$$H_1(n, Z_k, Z_0) = -\frac{n-1}{n}\frac{Z_0}{Z_k} + \frac{n-1}{n+1}\left(1-\frac{Z_k}{Z_0}\right)\left(\frac{Z_k}{Z_0}\right)^{n-1} {}_2F_1\left(n+1, n+1, n+2; 1-\frac{Z_k}{Z_0}\right), \tag{3.49b}$$

$$\begin{aligned} H_2(n, Z_k, Z_0) = {} & \frac{n-2}{n}\left(1-\frac{2n-1}{n-1}\frac{Z_k}{Z_0}\right)\left(\frac{Z_0}{Z_k}\right)^2 \\ & - \frac{n-2}{n+1}\left(1-\frac{Z_k}{Z_0}\right)^2\left(\frac{Z_k}{Z_0}\right)^{n-2} {}_2F_1\left(n+1, n+1, n+2; 1-\frac{Z_k}{Z_0}\right). \end{aligned} \tag{3.49c}$$

The dependence of the H_i functions on Z_k/Z_0 signals the expected breaking of reparameterization invariance, which translates in non-autonomous flow equations for the dimensionless renormalized couplings. An autonomous flow can be recovered in special cases: besides the $\sigma \to 0$ limit, other interesting choices are

$$\lim_{Z_0 \to 0} H_i(n) = 0 \ , \tag{3.50}$$

$$\lim_{Z_0 \to \infty} H_i(n) = -1 \ . \tag{3.51}$$

For instance, for $Z_0 \to 0$ we obtain

$$Q_n\Big[G_k^\ell \partial_t \mathcal{R}_k\Big] = \frac{Z_k^{-n}\left(-V_k^{(2)}\right)^{n-\ell+1}}{\Gamma(l)\Gamma(n-\ell+2)}\,(2-\sigma\eta_t) \ , \tag{3.52a}$$

$$Q_n\Big[G_k^\ell G_k' \partial_t \mathcal{R}_k\Big] = -\frac{n}{n-1}\frac{Z_k^{1-n}\left(-V_k^{(2)}\right)^{n-\ell-1}}{\Gamma(l+2)\Gamma(n-\ell)}\,(2-\sigma\eta_t) \ , \tag{3.52b}$$

$$Q_n\Big[G_k^\ell G_k'' \partial_t \mathcal{R}_k\Big] = \frac{2n}{n-2}\frac{Z_k^{2-n}\left(-V_k^{(2)}\right)^{n-\ell-2}}{\Gamma(l+3)\Gamma(n-\ell-1)}\,(2-\sigma\eta_t) \ . \tag{3.52c}$$

From (3.48) we see that the second case suppresses the RG improvement terms, in the same way as setting $\sigma = 0$, and therefore gives back the LPA result. Thus, in summary, the proper way to use the pseudo-regulator (3.43) is to first evaluate the integrals, then take the limit $\epsilon \to 0$ and finally the limit $Z_0 \to 0$.

Finally, it is worth stressing that the previous identities are not restricted to the LPA$'$ truncation. For truncations where Z_k depends on fields and/or momentum [8], the relevant

[8] In this chapter, we will use a different notation for the field and/or momentum dependent wave function renormalization.

wave function renormalization factor appearing inside the pseudo-regulator is to be identified with Z_k evaluated at preferred values of momentum and fields, for instance minimizing the potential and the inverse propagator. In the simplest cases the latter are vanishing values. Then the simple propagator G_k of (3.44), and the loop integrals given in the previous equations, would arise after a polynomial expansion of Z_k to obtain derivative vertices which are local in field space and in spacetime.

3.4 The derivative expansion at order ∂^2

The pseudo-regulators introduced in the previous section are also apt for application to a larger class of truncations which accounts for a possible field dependence of the wave function renormalization, the order ∂^2 of the derivative expansion. While this kind of more elaborate approximation is often an optional for many models, it is in some cases a necessity already as a zeroth order approach, such as for instance in the applications to nonlinear sigma models or for conformal field theories in two dimensions. For this reason, in this section we address these two examples. They allow us to account for a trivial generalization of the LPA$'$ formulas given in the previous section, and also to discuss more subtle points about the scope of an $\overline{\text{MS}}$ pseudo-regulator, such as its applicability to strongly interacting field theories and to models with nonlinear symmetries.

3.4.1 Multicritical models

We consider the following truncation of Γ_k

$$\Gamma_k[\chi] = \int_x \left(V_k(\chi) + \frac{1}{2} z_k(\chi) \partial_\mu \chi \partial^\mu \chi \right) \,, \tag{3.53}$$

where the wave function renormalization constant is $Z_k := z_k(0)$. This kind of ansatz is general enough to capture the emergence of a tower of multicritical χ^{2p} scalar field theories below the fractional upper critical dimensions $d_p = 2p/(p-1)$, and to provide good estimates of their properties in $d = 2$ [75, 104, 105]. As these conclusions apply to conventional FRG regulator choices, it is interesting to check whether these nice results can be obtained even with an $\overline{\text{MS}}$ pseudo-regulator.

The flow equations of the functions V_k and z_k for the pseudo-regulator (3.43) can be obtained from those presented in Appendix C for the more general case of $O(N)$ models[9]. We are interested in the case $d = 2$ and $N = 1$, therefore we rescale the field

$$\chi = Z_k^{-1/2} \tilde{\chi} \,, \quad \eta_t = -\partial_t \log Z_k \tag{3.54}$$

[9]Note that for the $O(N)$ model we use $Z_k(\rho(\chi))$ instead of $z_k(\chi)$, however for $N = 1$ we have $z_k(\chi) = Z_k(\chi) = \tilde{Z}_k(\chi)$.

	this work	opt. [75, 105]	hom. [104]	exact [106]
η_2	0.25	0.2132	0.309	0.25
ν_2	0.666667	...	0.863	1
η_3	0.111111	0.1310	0.200	0.15
ν_3	0.5625	...	0.566	0.556
η_4	0.0625	0.0910	0.131	0.1
ν_4	0.533333	...	0.545	0.536
η_5	0.04	0.0679	0.0920	0.0714
ν_5	0.520833	...	0.531	0.525
η_6	0.0277778	0.0522	0.0679	0.0535714
ν_6	0.514286	...	0.523	0.519
η_7	0.0204082	...	0.0521	0.0416667
ν_7	0.510417	...	0.517	0.514
η_8	0.015625	...	0.0412	0.0333333
ν_8	0.507937	...	0.514	0.511
η_9	0.0123457	...	0.0334	0.0272727
ν_9	0.50625	...	0.511	0.509
η_{10}	0.01	...	0.0277	0.0227273
ν_{10}	0.505051	...	0.509	0.508
η_{11}	0.00826446	...	0.0233	0.0192308
ν_{11}	0.504167	...	0.508	0.506

Table 3.1: Estimates of the critical exponents η_p and ν_p for the two dimensional χ^{2p} multicritical scalar models. The first three columns present FRG estimates: the first obtained with the $\overline{\text{MS}}$ pseudo-regulator, the second with the optimized regulator of (2.108), the third with a homogeneous regulator. Finally, the last column shows the exact results, from CFT methods.

and introducing the dimensionless renormalized functions

$$v_t(\tilde{\chi}) = k^{-2} V_k(\chi) \ , \quad \zeta_t(\tilde{\chi}) = Z_k^{-1} z_k(\chi) \ , \tag{3.55}$$

these flow equations read

$$\partial_t v_t = -\, 2v_t + \frac{\eta_t}{2} \tilde{\chi}\, v_t^{(1)} - \frac{1}{4\pi} \left(1 - \sigma \frac{\eta_t}{2}\right) \zeta_t^{-1} v_t^{(2)} \ , \tag{3.56a}$$

$$\partial_t \zeta_t = \eta_t \zeta_t + \frac{\eta}{2} \tilde{\chi}\, \zeta_t^{(1)} + \frac{1}{8\pi} \left(1 - \sigma \frac{\eta_t}{2}\right) \left[-2\frac{\zeta_t^{(2)}}{\zeta_t} + 3 \left(\frac{\zeta_t^{(1)}}{\zeta_t} \right)^2 \right] \ . \tag{3.56b}$$

Now we search for the scaling solutions, i.e. $\partial_t v_\star = \partial_t \zeta_\star = 0$, for this system of equations. Setting the $\mathbb{Z}_2$-parity and normalizations conditions

$$v_\star(0) = 0 \ , \qquad\qquad v_\star^{(2)}(0) = \tilde{m}^2 \ , \tag{3.57a}$$

$$\zeta_\star(0) = 1 \ , \qquad\qquad \zeta_\star^{(1)}(0) = 0 \ , \tag{3.57b}$$

the previous system of equations has the following family of fixed points

$$v_\star = -\frac{2-\sigma\eta_\star}{16\pi}\tilde{m}^2 \cos\left(\frac{2}{\sqrt{\eta_\star}} \arctan\sqrt{\frac{\Phi^2}{1-\Phi^2}}\right) \ , \tag{3.58a}$$

$$\zeta_\star = \left(1-\Phi^2\right)^{-1} \ , \tag{3.58b}$$

$$\Phi = \sqrt{\frac{4\pi\eta_\star}{2-\sigma\eta_\star}}\,\tilde{\chi} \ . \tag{3.58c}$$

It is remarkable that with the $\overline{\text{MS}}$ pseudo-regulator the scaling solutions can be written in closed form: usually in the FRGE they are only known numerically. Depending on the sign of m^2, the fixed-point potential can have a maximum or a minimum for zero field.

Note that $\zeta_\star$ diverges for $\Phi^2 = 1$. In order to have a potential $v_\star$ which is smooth at this point and is bounded from below we impose

$$v_\star^{(n)}\Big|_{\Phi^2\to 1} = \text{finite} \ , \quad \forall n \tag{3.59a}$$

$$\lim_{\tilde{\chi}\to\infty} v_\star = +\infty \ . \tag{3.59b}$$

From (3.59a) we get the quantization rule

$$\sin\frac{\pi}{\sqrt{\eta_\star}} = 0 \ , \tag{3.60}$$

such that

$$\upeta \equiv \eta_\star = \frac{1}{p^2} \ , \quad p = 1,2,3\ldots \ , \tag{3.61}$$

while (3.59b) can be fulfilled by adjusting the sign of $\tilde{m}^2$ (while the modulus remains free)

$$\tilde{m}^2 \lesseqgtr 0 \quad \text{if} \quad (-1)^{1/\sqrt{\eta_\star}} = \pm 1 \ . \tag{3.62}$$

This way, $v_\star$ acquires the typical shape of a $(p-1)$-critical potential.

To compute the critical exponents associated to these fixed points, we linearize the RG flow around them and look for eigenperturbations. In other words, we insert $v_t \to v_\star + e^{-\uptheta t}\,\delta v$, $\zeta_t \to \zeta_\star + e^{-\uptheta t}\,\delta\zeta$ and $\eta = \eta_\star + \delta\eta$ into (3.56), and expand them to first order in the perturbations δv, $\delta\zeta$ and $\delta\eta$. For $\delta\eta \neq 0$ the corresponding $\delta\zeta$ is complex and furthermore singular at $\Phi^2 = 1$. We therefore impose $\delta\eta = 0$. In this simplified case the linearized equations read

$$-\theta\,\delta v = -2\,\delta v + \frac{\eta_\star}{2}\tilde{\chi}\,\delta v^{(1)} - \frac{1}{4\pi}\left(1-\sigma\frac{\eta_\star}{2}\right)\,\zeta_\star^{-1}\left[\delta v^{(2)} - v_\star^{(2)}\,\frac{\delta\zeta}{\zeta_\star}\right]\,, \tag{3.63a}$$

$$-\theta\,\delta\zeta = \eta_\star\,\delta\zeta + \frac{\eta_\star}{2}\tilde{\chi}\,\delta\zeta^{(1)} + \frac{1}{4\pi}\left(1-\sigma\frac{\eta_\star}{2}\right)\left[-\frac{\delta\zeta^{(2)}}{\zeta_\star} + 3\frac{\zeta_\star^{(1)}\,\delta\zeta^{(1)}}{\zeta_\star^2} + \left(\frac{\zeta_\star^{(2)}}{\zeta_\star} - 3\left(\frac{\zeta_\star^{(1)}}{\zeta_\star}\right)^2\right)\frac{\delta\zeta}{\zeta_\star}\right]. \tag{3.63b}$$

The condition of fixed η results in LPA-like perturbations with vanishing $\delta\zeta$.
Besides the trivial solutions

$$\theta = 2\,, \qquad \delta v = 1\,, \tag{3.64a}$$

$$\theta = 2 - \frac{\eta_\star}{2}\,, \qquad \delta v = \tilde{\chi}\,, \tag{3.64b}$$

$$\theta = 0\,, \qquad \delta v = v_\star\,, \tag{3.64c}$$

we find the even eigenperturbations

$$\theta = 2 - 2\eta_\star\, n^2\,, \qquad \delta v = \cos\left(\sqrt{\frac{4-2\theta}{\eta_\star}}\arctan\sqrt{\frac{\Phi^2}{1-\Phi^2}}\right)\,, \tag{3.65}$$

and the odd eigenperturbations

$$\theta = 2 - 2\eta_\star\left(n+\frac{1}{2}\right)^2\,, \qquad \delta v = \sqrt{\frac{2-\sigma\eta}{8\pi(2-\theta)}}\sin\left(\sqrt{\frac{4-2\theta}{\eta_\star}}\arctan\sqrt{\frac{\Phi^2}{1-\Phi^2}}\right), \tag{3.66}$$

where $\eta_\star$ assumes its fixed-point value (3.61). Enforcing regularity of δv at the pole of ζ requires $n = 1, 2, 3\ldots$. From the largest even parity eigenvalue ($n = 1$), excluding the unit operator, we get the critical exponent ν

$$\nu = \frac{1}{2-2\eta}\,. \tag{3.67}$$

In Table 3.1 we summarize these results for the critical exponents η and ν and compare them to FRG estimates obtained by means of the optimized regulator and with the homogeneous regulator, as well as with the exact values. Comparing (3.61) with the exact result

$$\eta = \frac{3}{(p+1)(p+2)} \tag{3.68}$$

we see that for large p our result is off by a factor 3, whereas ν correctly tends to 1/2.

3.4.2 The nonlinear σ model

Addressing the nonlinear σ model with the $\overline{\mathrm{MS}}$ pseudo-regulator requires only a simple generalization of the truncation we just studied, to account for a multiplet of fields, rather than a single one. We therefore start from the following truncation of Γ_k for a $O(N)$-invariant multiplet of scalars

$$\Gamma_k[\chi] = \int_x \left(U_k(\rho) + \frac{1}{2} Z_k(\rho) \partial_\mu \chi^a \partial^\mu \chi^a + \frac{1}{4} Y_k(\rho) \partial_\mu \rho \partial^\mu \rho \right) \ , \tag{3.69}$$

where the N fields χ^a are in the fundamental representation of $O(N)$, and $\rho = \chi^a \chi^a/2$ is the corresponding local invariant. We further define the radial wave function renormalization

$$\tilde{Z}_k(\rho) = Z_k(\rho) + \rho Y_k(\rho) \ . \tag{3.70}$$

In Appendix C we show the flow equations of this model in the present truncation. For the especially interesting case $d = 2$, with the pseudo-regulator we obtain

$$\partial_t U_k = -\frac{1}{4\pi} \left(\frac{U_k^{(1)} + 2\rho U_k^{(2)}}{\tilde{Z}_k} + (N-1) \frac{U_k^{(1)}}{Z_k} \right) \ , \tag{3.71a}$$

$$\partial_t \tilde{Z}_k = -\frac{\left(\tilde{Z}_k^{(1)} + 2\rho \tilde{Z}_k^{(2)}\right)}{4\pi \tilde{Z}_k} - (N-1) \frac{\left(Z_k^{(1)} + \rho Y_k^{(1)}\right)}{4\pi Z_k} + \frac{3\rho \left(\tilde{Z}_k^{(1)}\right)^2}{4\pi \tilde{Z}_k^2} + (N-1) \frac{\rho Z_k^{(1)} \left(Y_k - Z_k^{(1)}\right)}{2\pi Z_k^2} \ . \tag{3.71b}$$

Here we suppressed the RG improvement by setting $\sigma = 0$; the effect of a non-vanishing σ will be addressed in a moment.

As it stands, this action could still describe a linear model. If we make the assumptions

$$Z_k(\rho) = \frac{Z_k}{g_k^2} \ , \qquad \tilde{Z}_k(\rho) = \frac{1}{g_k^2} \left(\frac{1}{Z_k} - 2\rho \right)^{-1} , \qquad U_k = -h_k \sqrt{\frac{1}{Z_k} - 2\rho} \ , \tag{3.72}$$

the EAA becomes

$$\Gamma_k[\chi] = \int \mathrm{d}^2 x \left[\frac{Z_k}{2g_k^2} \left(\delta^{ab} + \frac{\chi^a \chi^b}{\frac{1}{Z_k} - 2\rho} \right) \partial_\mu \chi^a \partial^\mu \chi^b - h_k \sqrt{\frac{1}{Z_k} - 2\rho} \ , \right] \ , \tag{3.73}$$

which describes a nonlinear σ model with values in a sphere S^N of radius $Z_k^{-1/2}$ and coupled to an external source h_k [24]. In this case the symmetry group is extended to $O(N+1)$. Inserting this ansatz in the flow equations (3.71) one deduces the correct one-loop beta functions

$$\partial_t g_k = -\frac{N-1}{4\pi} g_k^3 \ , \qquad \eta_k = -\partial_t \log Z_k = \frac{N}{2\pi} g_k^2 \ , \qquad \partial_t h_k = 0 \ . \tag{3.74}$$

Thus, the flow equations (3.71) maintain the form of the ansatz in (3.72), that is to say, they preserve the nonlinearly realized $O(N+1)/O(N)$ symmetry. This might appear trivial as we are applying one-loop RG equations, but it is not so for two reasons. First, this compatibility extends beyond one-loop order as we observe in the following by the inclusion of the RG improvement. Second, because this conclusion does not hold for finite ϵ, i.e. within the realm of ordinary FRG computations. In fact, it is well known that the FRG regulator, being a deformation of the two point function of the N fields, explicitly breaks the nonlinear part of the $O(N+1)$ symmetry. For this reason, most FRG applications to nonlinear sigma models have adopted different formulations based on the background field method [66, 107, 108, 109, 110, 111, 112, 113, 114].

Let's then turn to the RG improvement, which leads us beyond the one-loop approximation. To this end we should note that the pseudo-regulator in (3.43) has a factor $Z_k^{\sigma\epsilon}$, but here Z_k should be replaced by $Z_k g_k^{-2}$ to be compatible with the ansatz of (3.72). Then, with this little adjustment of the pseudo-regulator, for a generic σ we get

$$\partial_t g_k = -\frac{(N-1)g_k^3}{4\pi + \sigma g_k^2} , \qquad\qquad \eta_k = \frac{2N g_k^2}{4\pi + \sigma g_k^2} , \tag{3.75}$$

and $\partial_t h_k = 0$. Even though the previous flow equations hold in $d = 2$, it is possible to apply them in $d = 2+\varepsilon$ by simply augmenting them with their ε-dependent canonical dimensional part. In so doing, one can recover the ε-expansion description of the non-trivial fixed point which exists for $\varepsilon > 0$. We defer this discussion to the end of Section 3.6.

As we anticipate at the beginning of this chapter and in the end of Section 3.2.3, another possibility to recover flow equations (3.74) is to use the regulator (3.2) and so follow the red-dashed path in Figure 3.1.

A standard cutoff term

$$\Delta S_k(\chi) = \frac{Z_k}{2g_k^2} \int \mathrm{d}^2x\, \chi^a \mathcal{R}_k(-\partial^2)\chi^a \tag{3.76}$$

breaks $O(N+1)$ invariance, while preserving $O(N)$. Therefore, if we start at some scale k with an EAA of the form (3.73), the flow will immediately generate $O(N+1)$-violating terms, and thus will take place in the larger theory space parameterized by (3.69).

This can be seen already by projecting the flow generated by the ansatz (3.73) on the local potential, i.e. by considering (C.1). For non-vanishing a and for field-dependent wave function renormalizations, the choice $U_k = 0$ is not preserved by the RG flow. However, in the $a \to 0$ limit it becomes a consistent ansatz, as in $\partial_t U_k$ the RHS behaves like $a \log a$ when $a \to 0$. Let us then inspect the flow of the wave function renormalizations with $U_k = 0$. Inserting the previous ansatz in the flow equation (C.2) for $\tilde{Z}_k(\rho)$ [10], in the limit

[10] Note that now $Z_k(\rho = 0) = Z_k/g_k^2$, so inside the formulae for the Q-functionals we must send $Z_k \to Z_k/g_k^2$ and $\eta_t \to \eta_t + 2\partial_t g_k/g_k$.

$a \to 0$ we obtain

$$-\frac{2Z_k\partial_t g_k}{g_k^3(1-2Z_k\rho)}-\frac{Z_k\eta_t}{g_k^2(1-2Z_k\rho)^2}=\frac{Z_k}{4\pi}\frac{(2\partial_t g_k+(\eta_t-2)g_k)(2(N-1)Z_k\rho+1)}{g_k(1-2Z_k\rho)^2}+o(a). \tag{3.77}$$

As the functional ρ dependence on both sides of the equation is comparable, this equation can be algebraically solved for $\partial_t g_k$ and η_t, resulting in Equations (3.75) with $\sigma = 1$. These are the correct one-loop beta functions, augmented by RG resummations due to the dependence of the regulator on Z_k and g_k. The same result can be derived by considering the flow equation for $Z_k(\rho)$. Thus, within the present truncation the nonlinearly realized $O(N+1)/O(N)$ symmetry is preserved by taking the limit $a \to 0$.

The assumption $U_k = 0$, although justified by the observation that only a trivial potential is compatible with the nonlinearly realized symmetry, can be easily relaxed as long as this explicit symmetry-breaking term is treated as an external source. The simplest of such terms is a linear coupling to the $O(N+1)/O(N)$ variation of χ^a, i.e. χ^{N+1}, is reported in the third identity of (3.72). This ansatz, comprehending an arbitrary source h_k, is compatible with the flow equation in the case of the $\overline{\mathrm{MS}}$ pseudo-regulator. This linear term can also be used to construct an exact FRG equation which manifestly preserves the full $O(N+1)$ symmetry for every regulator function $\mathcal{R}_k$, see Appendix of [2]. For the present standard FRG implementation and regularization scheme, the ansatz (3.72) is not compatible with the flow equation of the potential, neither for $a \neq 0$ nor in the $a \to 0$ limit. Only by assuming that h_k be a function of a vanishing faster than a itself, closure of the order ∂^2 RG flow on the ansatz (3.73) is recovered. To understand this phenomenon it is necessary to study how the modified master equation for the $O(N+1)/O(N)$ symmetry behaves in the $a \to 0$ limit. In Appendix of [2] we show how the construction of a non-vanishing potential term for the nonlinear sigma model is a complicated problem which requires the simultaneous solution of both the flow equation and the modified master equation. As explained in Appendix of [2], in solving this problem the $a \to 0$ limit is of limited use.

3.5 The two-loop beta functions

In the previous sections we have shown that the $\epsilon \to 0$ limit of the FRG beta functions for the $\overline{\mathrm{MS}}$ pseudo-regulator reduces them to well-known $\overline{\mathrm{MS}}$ one-loop RG equations, possibly up to a resummation. We have shown this in the LPA, in the LPA$'$ and at order ∂^2 of derivative expansion. In this section we show how to reproduce the two-loop result in four dimensions, by considering larger truncations and by taking the $\epsilon \to 0$ limit in a suitable way. Although the computation of the beta function of the quartic coupling was discussed by several authors already, see Refs. [69, 70, 115, 116, 117, 118, 119, 120], part of the arguments adopted in those works do not apply to the $\overline{\mathrm{MS}}$ pseudo-regulator, which is not an IR regulator. Furthermore, we crucially rely on analytic continuation of divergent integrals, such that parametric limits are allowed to not commute, whereas standard FRG regulators render all integrals convergent. In addition, we also compute the two-loop running of the mass.

We closely follow the notations[11] and the arguments of the first FRG work addressing this task, namely [69]. We therefore focus on the linear $O(N)$ models with bare action

$$S[\phi] = \int_x \left\{ \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a + U_\Lambda(\rho) \right\} \ , \tag{3.78a}$$

$$U_\Lambda(\rho) = \bar{m}^2\rho + \frac{\bar{\lambda}}{2}\rho^2 \ , \tag{3.78b}$$

and we proceed with the solving strategy given in Equation (2.93). Note that compared to (3.6), we have changed the notation to $\lambda_2 = \bar{m}^2$ and $\lambda_4 = 3\bar{\lambda}$, and the bars denote bare couplings. In a massless scheme such as $\overline{\mathrm{MS}}$, the two-loop beta function of the quartic coupling is universal and mass independent, such that it is usually possible to assume $\bar{m}^2 = 0$ right from the start. We instead focus on a massive theory in the symmetric regime for technical reasons. In fact, we are going to adopt an FRG pseudo-regulator which does not regulate IR divergences. This does not prevent us from analyzing the massless theory though, as we are allowed to take the $\bar{m}^2 \to 0$ limit of any IR safe quantity after the loop integrals are computed.

In our regularization scheme, it is furthermore essential to account for the k-dependence of the renormalized mass parameter m^2, or else the correct two-loop beta function would not be reproduced. In fact, as the latter contributes to the running of λ in any mass-dependent scheme, it does so also in our computations at non-vanishing ϵ. Interestingly, this contribution will survive the $\epsilon \to 0$ limit, if the latter is taken carefully enough. This computation will thus serve as an example of a more general mechanism, according to which the super-renormalizable and the non-renormalizable sectors of a theory, which show a non-trivial running in any mass-dependent scheme, do feed back into the running of renormalizable operators even in a massless scheme such as $\overline{\mathrm{MS}}$, provided the mass-thresholds effects are correctly accounted for [12].

As discussed in the beginning of this chapter, a two-loop result involves arbitrarily-high orders of the derivative expansion. We therefore cannot use an ansatz such as (2.96), or its multi-field generalization. Instead, we must make the ansatz

$$\Gamma_k[\phi] = \int_x \left\{ U_k(\rho) + \frac{1}{2}\partial_\mu\phi^a\, Z_k\left(\rho, -\partial^2\right)\partial^\mu\phi^a + \frac{1}{4}\partial_\mu\rho\, Y_k\left(\rho, -\partial^2\right)\partial^\mu\rho \right\} \ . \tag{3.79}$$

Equivalently, expressing the field in the angular (Goldstone) modes, and in the radial (massive), mode $\sqrt{\rho}$, one finds that their wave function renormalizations are given by the

[11] In particular, note that in this section we will use ϕ instead of χ.
[12] This mechanism has been observed also in [121]

function Z_k and

$$\tilde{Z}_k\left(\rho,q^2\right)=Z_k\left(\rho,q^2\right)+\rho Y_k\left(\rho,q^2\right)\ , \tag{3.80}$$

respectively. In order to compute the beta function of the quartic coupling it may seem sufficient to stop at the fourth order of the vertex expansion, and hence assume that U_k is quadratic in ρ and Z_k is linear. However, in the vertex expansion, the beta function of $\Gamma^{(n)}$ involves also $\Gamma^{(n+1)}$ and $\Gamma^{(n+2)}$, so we need U_k up to order ρ^3 and Z_k up to order ρ^2. In general we shall use the following terminology for the expansion of these functions

$$U_k(\rho)=\sum_{n=1}^{\infty}\frac{u_n}{n!}Z_k^n k^{d-n(d-2)}(\rho-\rho_0)^n\ , \tag{3.81a}$$

$$Z_k(\rho,q^2)=\sum_{n=0}^{\infty}\frac{z_n(q^2/k^2)}{n!}Z_k^{n+1}k^{-n(d-2)}(\rho-\rho_0)^n\ , \tag{3.81b}$$

$$\tilde{Z}_k(\rho,q^2)=\sum_{n=0}^{\infty}\frac{\tilde{z}_n(q^2/k^2)}{n!}Z_k^{n+1}k^{-n(d-2)}(\rho-\rho_0)^n\ , \tag{3.81c}$$

where $Z_k = Z_k(\rho_0,0)$ is the wave function renormalization and ρ_0 is the minimum of the potential. In any scheme, the effective potential is already renormalized at one loop. The functions $Z_k(\rho,q^2)$ and $\tilde{Z}_k(\rho,q^2)$ also receive one-loop radiative corrections in mass-dependent schemes. However, the one-loop contributions to these functions are field dependent corrections, and therefore correspond to radiatively generated momentum-dependent vertices. In any scheme the field-independent part of the wave function renormalizations, i.e. the fields anomalous dimensions, receive corrections from the two-loop order on. These well known perturbative facts are recovered from the FRG equations, straightforwardly in mass-dependent schemes, and with a little care also for mass-independent schemes, as we show in this section.

Although the FRGE looks like a one-loop equation, this is only true as long as one uses the full propagators and vertices (double lines and black dots in Equations (2.95)). The full propagators and vertices can be expanded in loops giving rise to infinite series that can be represented in terms of standard Feynman diagrams. This introduces resummations of perturbative diagrams of two kinds. The first is the so called *spectral adjustment* of the regulator, i.e. the possible dependence of the regulator on the couplings of the theory, most commonly the wave function renormalization Z_k (as already discussed in Section 3.3.1). This produces terms depending on the field's anomalous dimensions. The second source of resummations is provided by the mass thresholds, which in a functional setup may also depend on the point of expansion in the space of field amplitudes,

$$\kappa=k^{2-d}Z_k\rho_0\ , \tag{3.82}$$

at which we define local couplings. These include the perturbatively renormalizable ones

$$m^2\equiv k^2u_1=Z_k^{-1}U_k^{(1)}(\rho_0)\ ,\qquad \lambda\equiv u_2=k^{d-4}Z_k^{-2}U_k^{(2)}(\rho_0)\ , \tag{3.83a}$$

as well as the non-renormalizable ones

$$u_n = k^{n(d-2)-d} Z_k^{-n} U_k^{(n)}(\rho_0) \ , \qquad n \geq 3 \ , \tag{3.83b}$$

$$z_n(y) = k^{n(d-2)} Z_k^{-(n+1)} Z_k^{(n)}(\rho_0, k^2 y) \ , \quad n \geq 1 \ , \tag{3.83c}$$

$$\tilde{z}_n(y) = k^{n(d-2)} Z_k^{-(n+1)} \tilde{Z}_k^{(n)}(\rho_0, k^2 y) \ , \quad n \geq 1 \ . \tag{3.83d}$$

If one were to suppress both these portals towards higher order corrections, the FRGE would boil down to a pure one-loop result. However, thanks to these two contributions, higher loops are generated by solving the RG equations and constructing the RG trajectory, i.e. in the process of renormalizing the theory.

Let us now address the task of integrating the $d = 4$ flow equations from the UV initial condition $\Gamma_{k=\Lambda} = S$ down to $k < \Lambda$ [13]. We recall that for the bare theory of (3.78) the loop expansion corresponds to the expansion in the coupling λ. From now on it is more convenient to think in this way. To reproduce perturbation theory, we need to compute the RG vector field in vicinity of the Gaussian fixed point up to next-to-leading order in λ. This is tantamount to integrating the flow order by order in a Taylor expansion for small λ. We first input the initial condition Γ_Λ on the right hand sides of the RG equations. This produces a one-loop beta function for the renormalizable couplings m^2 and λ. On the other hand, the RG equations radiatively generate further couplings, namely those whose t-derivative at this initial point is non-vanishing. By considering the Feynman diagrams mentioned above, we deduce a power counting for the radiatively generated couplings in terms of λ.
The first few couplings to be generated, and the corresponding orders of magnitude in terms of the initial quartic coupling, are

$$\eta = O\left(\lambda^2\right), \qquad z_1 = O\left(\lambda^2\right), \qquad \tilde{z}_1 = O\left(\lambda^2\right), \tag{3.84a}$$

$$u_3 = O\left(\lambda^3\right), \qquad z_2 = O\left(\lambda^3\right), \qquad \tilde{z}_2 = O\left(\lambda^3\right), \tag{3.84b}$$

$$u_4 = O\left(\lambda^4\right), \qquad z_1^2 = O\left(\lambda^4\right), \qquad \tilde{z}_1^2 = O\left(\lambda^4\right). \tag{3.84c}$$

Similar relations hold for u_n and z_n, with progressively higher powers of λ for higher n. Thus after an infinitesimal RG step from $k = \Lambda$ to $k = \Lambda - \delta k$, the effective average action changes and the perturbative expansion of the FRG vector field correspondingly adjusts. To compute the most general form of β_λ along such a flow, which is exact at the order λ^3 (still within a local expansion around vanishing fields), we can use the power counting of

[13] Here the limit $\Lambda \to \infty$ is allowed as part of the regularization choice, and should not be confused with the possibility to remove a UV cutoff, thus defining a UV complete theory. The latter question is instead emerging when trying to take such a limit at for a fixed IR action $\Gamma_{k=0}$.

(3.84) to eliminate the higher order terms. This results in

$$
\begin{aligned}
\beta_{m^2} =& -\frac{k^2}{16\pi^2}\left[(N-1)\lambda\, l^4_{1,0}(0) + 3\lambda\, l^4_{0,1}(2\lambda\kappa) + (N-1)\langle z_1\rangle^6_{1,0}(0) + \langle \tilde{z}_1\rangle^6_{0,1}(2\lambda\kappa)\right] \\
& + \eta m^2 + k^2\left(2\kappa + \partial_t\kappa\right)\lambda + O(\lambda^3)\ , \qquad (3.85) \\
\beta_\lambda =& \frac{N-1}{16\pi^2} l^4_{2,0}(0)\lambda^2 + \frac{9}{16\pi^2} l^4_{0,2}(2\lambda\kappa)\lambda^2 \\
& - \frac{N-1}{16\pi^2} l^4_{1,0}(0) u_3 - \frac{5}{16\pi^2} l^4_{0,1}(2\lambda\kappa) u_3 + \left(2\kappa + \partial_t\kappa\right) u_3 \\
& + \frac{N-1}{8\pi^2}\lambda\langle z_1\rangle^6_{2,0}(0) + \frac{3}{8\pi^2}\lambda\langle \tilde{z}_1\rangle^6_{0,2}(2\lambda\kappa) - \frac{N-1}{16\pi^2}\langle z_2\rangle^6_{1,0}(0) - \frac{1}{16\pi^2}\langle \tilde{z}_2\rangle^6_{0,1}(2\lambda\kappa) \\
& + 2\eta\lambda + O(\lambda^4)\ . \qquad (3.86)
\end{aligned}
$$

In these equations the threshold functions $l^4_{n,m}$ and the averages $\langle z_n\rangle^6_{n,m}$ denote one-loop integrals over virtual momenta, with momentum-independent and -dependent vertices respectively. The precise definitions can be found in Appendix (D.24). The λ dependence of mass thresholds should also be expanded, for instance

$$
l^d_{0,n}(2\lambda\kappa) = l^d_{n,0}(0) - 2n\lambda\kappa\, l^d_{n+1,0}(0) + O\left(\lambda^2\right)\ . \qquad (3.87)
$$

However this would bring corrections only for non-vanishing κ, which is not generated at the two-loop order. We can thus set $\kappa = 0$ in Equations (3.85) and (3.86). While the contribution of the non-renormalizable couplings u_3, $z_{1,2}$ and $\tilde{z}_{1,2}$ is obvious in any mass-dependent scheme, one might expect that it would not be present in $\overline{\mathrm{MS}}$, since all dimensionful integrals, in absence of mass thresholds, need to vanish in the $\epsilon \to 0$ limit. This expectation is however incorrect, because when the computation is performed at non-vanishing mass, and the $m \to 0$ limit is taken after the $\epsilon \to 0$ limit, the contribution of the beta functions of the mass and of the non-renormalizable couplings inside β_λ attains a finite non-vanishing value.

To illustrate the details of this mechanism, we should first choose a specific form of the ϵ-dependent pseudo-regulator which is suitable for the present computation. We adopt the following function [14]

$$
\mathcal{R}_k(z) = \left[\left(\frac{k^2}{\mu^{4-2b}M^{2b}}\right)^{\epsilon}\left(z+M^2\right)^{1+\epsilon} - \left(z+m^2\right)\right]. \qquad (3.88)
$$

Here μ is a k-independent momentum scale and M plays the role of a regularized mass, which is assumed to be k-dependent. M should be an analytic function of m, such that

[14] As η is vanishing at one loop in this model, we discard the precise form of the Z_0 and Z_k dependence of the pseudo-regulator: in fact, the η dependence appearing on the RHS of the flow equations through the regularization only contributes to the RHS of the flow from three loops on, and so we can safely replace $Z_k \to 1$.

the $m \to 0$ limit smoothly removes also M. [15] The precise form of M making all relevant integrals finite and ensuring the k independence of the beta functions in the $\epsilon \to 0$ limit is derived in Appendix D.1. Imposing the analiticity requirement on M fixes $b = 1$.

We now turn to the beta function of (3.86), where on the RHS we organized different kinds of contributions on different lines. The first line provides the one-loop expression, as well as also a first type of higher order contribution, due to the RG improvement of the pseudo-regulator. More specifically, the threshold functions are responsible for the appearance of β_{m^2} on the RHS of β_λ, as

$$l^4_{0,2}(2\lambda\kappa) = l^4_{2,0}(0) + O\left(\lambda^2\right) = 1 - \frac{\beta_{m^2}}{2m^2} + O\left(\lambda^2\right) = 1 - \frac{N+2}{32\pi^2}\lambda + O\left(\lambda^2\right) \ . \tag{3.89}$$

As already anticipated, the ratio β_{m^2}/m^2 attains a finite mass-independent value. In fact, the one-loop $\overline{\text{MS}}$ result for β_{m^2} is recovered also with the present pseudo-regulator. Even the two-loop $\overline{\text{MS}}$ coefficient for β_{m^2} can be correctly reproduced, although this requires a careful choice of the function M^2, which is described in Appendix D.2.5.

The second line of (3.86) encodes the effect of u_3, which is generated by the flow equation itself, as detailed in Appendix D.2. It is a general feature of the FRG equations that solving the flow equation for u_3 as a function of λ, at leading order in λ, is equivalent to setting u_3 at its λ-dependent fixed point value. With the present pseudo-regulator this value of the sextic coupling reads

$$u_3^{(1-loop)} = \frac{N+26}{32\pi^2}\lambda^3 \frac{k^2}{m^2} \ . \tag{3.90}$$

This illustrates a second mechanism that generates two-loop terms, even with the $\overline{\text{MS}}$ pseudo-regulator. In fact, despite all momentum integrals appearing in the beta function of u_3, and any other non-renormalizable couplings, being dimensionful and thus vanishing in the $\epsilon \to 0$ limit, some of the integrals appearing in the solution of the flow and fixed-point equations for these couplings are dimensionless and therefore survive in $\overline{\text{MS}}$. In other words, the flow equations should be solved before the $\epsilon \to 0$ limit is taken. Then, replacing (3.90) in the second line of (3.86) produces further λ^3 terms in the beta function.

A similar fate applies to the third line of (3.86), although the computational details this time are somewhat more intricate. This is due to the momentum dependence of the non-renormalizable couplings appearing inside Z and $\tilde{Z}$. In the process of solving the flow equations for these couplings at leading order in λ, and plugging the solution in (3.86), the

[15] If M depended also on other couplings aside from m, their contributions would only appear from three loops.

following momentum averages are generated

$$\langle z_1\rangle^6_{2,0}(0) = -8(16\pi^2)\lambda^2 A \;, \tag{3.91a}$$
$$\langle \tilde{z}_1\rangle^6_{0,2}(0) = -4(16\pi^2)(N+8)\lambda^2 A \;, \tag{3.91b}$$
$$\langle z_2\rangle^6_{1,0}(0) = 32(16\pi^2)\lambda^3 B \;, \tag{3.91c}$$
$$\langle \tilde{z}_2\rangle^6_{0,1}(0) = 8(16\pi^2)(N+26)\lambda^3 B \;. \tag{3.91d}$$

Here A and B are dimensionless double momentum integrals whose precise form is given in Appendix D.2.3. Although these are two-loop integrals, they involve only one copy of $\partial_t\mathcal{R}_k$, because one of them disappears in the process of solving the flow equations for the non-renormalizable couplings. As a consequence, the $1/\epsilon^2$ pole of the integrals is not balanced by the ϵ factor coming from the single $\partial_t\mathcal{R}_k$. Thus both A and B exhibit a $1/\epsilon$ pole [16]. Despite this divergence, the flow equation itself is finite, at least at order λ^3, as in fact the only appearance of A and B on the RHS of (3.86) is through the combination $A+B$, in which the $1/\epsilon$ poles cancel. The final result of this process is therefore

$$A + B = \frac{1}{2\,(16\pi^2)^2} \;. \tag{3.92}$$

Also for these terms, taking the $\epsilon \to 0$ limit too early, i.e. before the flow for Z and $\tilde{Z}$ is solved and fed back inside β_λ, would fail to unveil higher order corrections.

Putting all these contributions together, the truncated beta function of (3.86) in the $\epsilon \to 0$ limit reduces to

$$\beta_\lambda = \frac{N+8}{16\pi^2}\lambda^2 - \frac{2(5N+22)}{(16\pi^2)^2}\lambda^3 + 2\eta\lambda \;. \tag{3.93}$$

We next turn to the computation of the anomalous dimension. Following [69], we split η in the sum

$$\eta = \eta^{(a)} + \eta^{(b)} \;, \tag{3.94}$$

the two terms on the RHS being the contributions of the momentum-independent and -dependent parts of the wave function renormalizations, respectively. Notice that both contributions would vanish in a truncation neglecting the field dependence of the wave function renormalizations, as the vacuum expectation value κ vanishes at this order. Thus both terms are entirely due to the four-point function $\Gamma^{(4)}_k$. In the first part, $\eta^{(a)}$ is proportional to the derivative couplings at zero momenta

$$\eta^{(a)} = \frac{1}{16\pi^2} l^4_{1,0}(0)\left[(N-1)z_1(0) + \tilde{z}_1(0)\right] \;. \tag{3.95}$$

[16] Incidentally, neither A nor B would be divergent within a strict derivative expansion where the RG-generated momentum dependence of Z and $\tilde{Z}$ is truncated to its power series expansion around $p^2 = 0$, because in the latter case the two-loop integrals would exhibit only a $1/\epsilon$ pole. However this truncation would not reproduce the full two-loop beta function, but just part of the $O(\lambda^3)$ contributions.

The $O(\lambda^2)$ solution of the flow equation gives

$$z_1(0) = \frac{1}{3(16\pi^2)}\frac{k^2}{m^2}\lambda^2 \ , \qquad \tilde{z}_1(0) = \frac{(N+8)}{6(16\pi^2)}\frac{k^2}{m^2}\lambda^2 \ , \tag{3.96}$$

such that the first contribution to the anomalous dimension reads

$$\eta^{(a)} = -\frac{(N+2)}{2(16\pi^2)}\lambda^2 \ . \tag{3.97}$$

The second part of the anomalous dimension is instead proportional to the non-trivial momentum dependence of $\Gamma_k^{(4)}$. Again taking the limit for $\epsilon \to 0$ at the end of the nesting process, we find

$$\eta^{(b)} = \frac{(N+2)}{(16\pi^2)^2}\lambda^2 \ . \tag{3.98}$$

Thus, the whole two-loop anomalous dimension is recovered

$$\eta = \eta^{(a)} + \eta^{(b)} = \frac{(N+2)}{2(16\pi^2)^2}\lambda^2 \ . \tag{3.99}$$

Inserting in (3.93) we finally arrive at

$$\beta_\lambda = \frac{N+8}{16\pi^2}\lambda^2 - \frac{9N+42}{(16\pi^2)^2}\lambda^3 \ . \tag{3.100}$$

This is the universal part of the beta function at two loops. With different mass-dependent regulators one would obtain additional non-universal terms depending on the mass. The contributions to the beta function from three loops up is known not to be universal. In our approach this regulator-dependence arises at least from two sources: the freedom of inserting other couplings in the pseudo-regulator, as discussed in Section 3.2.2 and footnote 15, and the contributions coming from Z_k, as mentioned in Section 3.3.1 and footnote 14.

A similar treatment of (3.85) leads to

$$\beta_{m^2} = m^2\left[\frac{(N+2)}{16\pi^2}\lambda - \frac{(N+2)}{4(16\pi^2)^2}\lambda^2\left((1+2f_1)(N+2) - 8\sqrt{3}\pi + 70\right)\right]. \tag{3.101}$$

Some more details are reported in Appendix D.2.5. We observe that the two-loop term is not universal, and that the $\overline{\text{MS}}$ result can be reproduced by suitably fixing the parameter f_1, which enters the pseudo-regulator (3.88) through the choice of M.

3.6 Generalization to continuous dimensions

Despite the fact that $\overline{\rm MS}$ is limited to applications in an even number of dimensions, the pseudo-regulator we discussed lends itself to generalizations to any continuous d, thanks to the intimate relation that exists between dispersion relations and the dimensionality of spacetime. Consider the following pseudo-regulator

$$\mathcal{R}_k(z) = Z_0 Z_k^{\sigma\epsilon}\left[\mu^{2(1-\alpha)}\left(\frac{k^2}{\mu^4}\right)^{\epsilon} z^{\alpha+\epsilon} - z\right] , \tag{3.102}$$

which generalizes (3.43) in that the regularized propagator is now an homogeneous function of momentum with power $\alpha+\epsilon$, rather than $1+\epsilon$. This allows to correspondingly generalize the formulae (3.48) for the Q-functionals, whenever the dimension of the momentum integrals, after having factored out all μ dependence, is a nonnegative integer. In fact, in this case the $\epsilon \to 0$ and the $Z_0 \to 0$ limits give

$$Q_n\left[G_k^{\ell}\partial_t\mathcal{R}_k\right] = 2\mu^{2n\left(1-\frac{1}{\alpha}\right)}\frac{\Gamma\left(\frac{n}{\alpha}\right) Z_k^{-\frac{n}{\alpha}}\left(-V_k^{(2)}\right)^{\frac{n}{\alpha}-\ell+1}}{\Gamma(n)\Gamma(\ell)}\frac{\left(1-\frac{\sigma\eta}{2}\right)}{\Gamma\left(\frac{n}{\alpha}-\ell+2\right)} , \tag{3.103a}$$

$$Q_n\left[G_k^{\ell}G_k'\partial_t\mathcal{R}_k\right] = -\frac{\alpha^2\Gamma\left(\frac{n-1}{\alpha}+2\right)\mu^{2(n-1)\left(1-\frac{1}{\alpha}\right)}}{(n-1)\Gamma(n)}\frac{2Z_k^{1-\frac{n}{\alpha}}\left(-V_k^{(2)}\right)^{\frac{n-1}{\alpha}-\ell}}{\Gamma(\ell+2)}\frac{\left(1-\frac{\sigma\eta}{2}\right)}{\Gamma\left(\frac{n-1}{\alpha}-\ell+1\right)} , \tag{3.103b}$$

$$\begin{aligned} Q_n\left[G_k^{\ell}G_k''\partial_t\mathcal{R}_k\right] = {}& \frac{\alpha^2\left(2\alpha-\frac{(\alpha-1)(\ell+2)}{\frac{n-2}{\alpha}+2}\right)\Gamma\left(\frac{n-2}{\alpha}+3\right)\mu^{2(n-2)\left(1-\frac{1}{\alpha}\right)}}{(n-2)\Gamma(n)} \\ & \times\frac{2Z_k^{2-\frac{n}{\alpha}}\left(-V_k^{(2)}\right)^{\frac{n-2}{\alpha}-\ell}}{\Gamma(\ell+3)}\frac{\left(1-\frac{\sigma\eta}{2}\right)}{\Gamma\left(\frac{n-2}{\alpha}-\ell+1\right)} , \end{aligned} \tag{3.103c}$$

where the α-dependent arguments of the Gamma functions in the denominators are positive integers. Recall that for a scalar field theory and in the derivative expansion the index n takes the values $d/2+l$ with $l = 0, 1, 2, \dots$. Hence, if α is a continuous power, these formulae are applicable to continuous d.

We illustrate the use of this generalized pseudo-regulator, by addressing the description of the Wilson-Fisher fixed point for $2 < d < 4$, for the linear $O(N)$ models. We focus on the flow equations we presented in Appendix C within the derivative expansion, namely Equations (C.1) and (C.2). As we expect the effective potential to play a dominant role in the description of the Wilson-Fisher fixed point, we demand that the corresponding quantum contributions be non-vanishing in our regularization scheme. Specifically, the first kind of Q-functional, given in (3.103a), is non-trivial in $2 < d < 4$ only if $d/2\alpha$ is a positive integer. Under this assumption the flow equations of the derivative expansion become[17]

$$
\begin{aligned}
\partial_t u &= -du + (d-2+\eta)\tilde\rho\, u^{(1)} \\
&\quad + \frac{\alpha\tilde\mu^{d\left(1-\frac{1}{\alpha}\right)}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+1\right)}\left[\tilde\zeta^{-\frac{d}{2\alpha}}\left(-u^{(1)}-2\tilde\rho u^{(2)}\right)^{\frac{d}{2\alpha}} + (N-1)\zeta^{-\frac{d}{2\alpha}}\left(-u^{(1)}\right)^{\frac{d}{2\alpha}}\right] , \qquad (3.104a) \\
\partial_t\tilde\zeta &= \eta\,\tilde\zeta + (d-2+\eta)\tilde\rho\,\tilde\zeta^{(1)} \\
&\quad - \frac{\tilde\mu^{d\left(1-\frac{1}{\alpha}\right)}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\left[(\tilde\zeta^{(1)}+2\tilde\rho\tilde\zeta^{(2)})\tilde\zeta^{-\frac{d}{2\alpha}}\left(-u^{(1)}-2\tilde\rho u^{(2)}\right)^{\frac{d}{2\alpha}-1} + (N-1)\tilde\zeta^{(1)}\zeta^{-\frac{d}{2\alpha}}\left(-u^{(1)}\right)^{\frac{d}{2\alpha}-1}\right] \\
&\quad - \frac{(d+2(\alpha-6))(d-2\alpha)\tilde\mu^{d\left(1-\frac{1}{\alpha}\right)}}{6\alpha(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\tilde\rho(3u^{(2)}+2\tilde\rho u^{(3)})\tilde\zeta^{(1)}\tilde\zeta^{-\frac{d}{2\alpha}}\left(-u^{(1)}-2\tilde\rho u^{(2)}\right)^{\frac{d}{2\alpha}-2} \\
&\quad - (N-1)\frac{d(d+2)(d^2-4\alpha^2)\tilde\mu^{d\left(1-\frac{1}{\alpha}\right)}}{24\alpha(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+2\right)}\tilde\rho u^{(2)}\zeta^{(1)}\zeta^{-\frac{d}{2\alpha}}\left(-u^{(1)}\right)^{\frac{d}{2\alpha}-2} \\
&\quad + (N-1)\frac{(d-2\alpha)\tilde\mu^{d\left(1-\frac{1}{\alpha}\right)}}{\alpha(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}u^{(2)}(\tilde\zeta-\zeta)\zeta^{-\frac{d}{2\alpha}}\left(-u^{(1)}\right)^{\frac{d}{2\alpha}-2} , \qquad (3.104b)
\end{aligned}
$$

where u, ζ and $\tilde\zeta$ are the dimensionless renormalized counterparts of the U_k, Z_k and $\tilde Z_k$ of Equations (3.69) and (3.70), defined in analogy to Equation (3.55). Here for notational simplicity, we dropped the RG improvement by setting $\sigma = 0$; the full equations contain the factor $(1-\sigma\eta/2)$ in front of every quantum contribution. The $\tilde\mu$ dependence of these flow equations can be cancelled by a further rescaling of all dimensionful quantities with respect to $\tilde\mu$, which casts the RG equations in a genuine $\overline{\mathrm{MS}}$ form.

We look for fixed points of the previous flow equations by means of a small-fields polynomial expansion

$$
u_\star = \sum_{i=0}\frac{\lambda^\star_{2i}}{i!}\tilde\rho^i , \qquad \tilde\zeta_\star = 1 + \sum_{i=1}\frac{\tilde z^\star_{2i}}{i!}\tilde\rho^i , \qquad \zeta_\star = 1 + \sum_{k=i}\frac{z^\star_{2i}}{i!}\tilde\rho^i . \qquad (3.105)
$$

We find the Gaussian fixed point for every value of α and a non-trivial fixed point only for $\alpha = d/4$, located at

$$
\lambda^\star_4 = \frac{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)(4-d)}{8+N}\tilde\mu^{4-d} , \qquad (3.106)
$$

with $\eta = 0$, and all others couplings being vanishing. We note that with standard regulators the dimensionless potential of the WF fixed point has a non-trivial minimum, but the dimensionful mass (deduced from the limit $k \to 0$) is zero, in accordance with the fact that the theory is scale invariant at quantum level. With the $\overline{\mathrm{MS}}$ pseudo-regulator the "dimensionless mass" λ_2 is zero even for finite k. The same phenomenon happens also in the functional perturbative approach [96].

[17] Note that we have dropped the t subscript.

For $\alpha = d/4$ the power of $\tilde{\mu}$ appearing in the Equations (3.104) is $(d-4)$. Therefore, the rescaling, which maps these equations into those of $\overline{\text{MS}}$, is effectively declaring the dimensionality of the couplings to be the one expected in four dimensions, the rescaling factors differing only by powers of $(k/\tilde{\mu})^{4-d}$. Furthermore, the value $\alpha = d/4$ is precisely the one that makes the quartic interaction marginal for continuous d. In fact, the effective kinetic term of the regularized theory has a dispersion relation z^α, which changes the dimensionality of the scalar field from $(d-2)/2$ to $(d-2\alpha)/2$. In other words, within the present truncation, our one-loop-like equations for ϕ^4 theory are able to detect the Wilson-Fisher fixed point in continuous d only when the pseudo-regulator turns d into an "effective upper critical dimension". This interpretation is also consistent with the apparent absence of the multicritical models with ρ^p interactions, for $p > 2$. In fact, the effective upper critical dimensions for these models would be at $d = 2p\alpha/(p-1)$, which is not compatible with our simplifying assumption of an integer $d/2\alpha$.

The stability matrix at the fixed point of (3.106) is triangular and the eigenvalues are

$$\lambda_i^v = -d + i(d-2) + i\left[1 + \frac{6(i-2)}{N+8}\right](4-d) \ , \tag{3.107a}$$

$$\lambda_i^\zeta = i(d-2) + \left[i + 1 + \frac{3(2i^2-2i-3)}{N+8}\right](4-d) \quad + \frac{3i}{2(N+8)}(4-d)^2 \ . \tag{3.107b}$$

By setting $d = 4 - \varepsilon$ we recognize that this prediction agrees with the usual first order of the ε expansion. For instance, for $N = 1$ we get

$$\lambda^v = \left(-2 + \frac{\varepsilon}{3}, \varepsilon, 2 + 3\varepsilon, 4 + \frac{19}{3}\varepsilon, 6 + 11\varepsilon \ldots\right) \ , \tag{3.108a}$$

$$\lambda^\zeta = \left(2, 4 + \frac{4}{3}\varepsilon, 6 + 4\varepsilon, 8 + 8\varepsilon \ldots\right) \ . \tag{3.108b}$$

Notice that ε here should not be confused with the ϵ of Equation (3.102), the latter having been removed by the limit $\epsilon \to 0$. Also, the one-loop predictions of (3.107) become exact in the $N \to \infty$ limit.

Order ε^2 corrections affect the estimate of ω_i in (3.107b) but are missing in (3.107a). This is related to the fact that the fixed point value of $\eta = 0$ is vanishing in this truncation, such that the RG resummations triggered by the dependence of $\mathcal{R}_k$ on Z_k are ineffective at the fixed point. In fact, improvements on the estimate of λ's can be obtained by allowing for the feedback of other couplings in the pseudo-regulator. For instance, it is natural to allow for the replacement of the mass parameter μ^2 with the running λ_2 through a tunable parameter b [18], and write

$$\mathcal{R}_k(z) = Z_0 Z_k^{\sigma\epsilon}\left[\mu^{2(1-\alpha)}\left(\frac{k^2}{\mu^{2(2-b)}\lambda_2^b}\right)^{\epsilon} z^{\alpha+\epsilon} - z\right] . \tag{3.109}$$

[18] We cannot replace every occurrence of μ^2 with λ_2, otherwise singularities of the form β_2/ϵ would arise in the beta functions. The insertion of λ_2 in the pseudo-regulator must preserve the cancellation of such poles.

This would result in a different RG improvement of Equations (3.104), where each quantum term is now multiplied by the factor $(1 - b\beta_2/(2\lambda_2) - \sigma\eta/2)$, which leads to the following b-dependent quartic coupling evaluated at the fixed point and critical exponents

$$\lambda_4^\star = \frac{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)(4-d)}{N+8-\frac{b}{2}(N+2)(4-d)}\tilde{\mu}^{4-d} \ , \tag{3.110a}$$

$$\lambda_i^v = -d + i(d-2) + i\left[1 + \frac{6(i-2)}{N+8}\right](4-d) + \delta_{i,4}\frac{b}{2}\left(\frac{N+2}{N+8}\right)(4-d)^2 \ , \tag{3.110b}$$

$$\lambda_i^\zeta = i(d-2) + \left[i + 1 + \frac{3(2i^2-2i-3)}{N+8}\right](4-d) + \frac{3i}{2(N+8)}(4-d)^2 \ . \tag{3.110c}$$

Getting better estimates of the critical exponent η and the correlation-length exponent ν does instead require a larger truncation, accounting for at least part of the two-loop contributions, as discussed in Section 3.5. This kind of more elaborate analysis of the Wilson-Fisher fixed point by means of the $\overline{\mathrm{MS}}$ pseudo-regulator is left for future studies.

We reiterate that the flow equations (3.104)) have been obtained under the assumptions that $2 < d < 4$ and that $d/2\alpha$ is a positive integer. In even dimensions the equations have additional terms. In fact, taking the $d \to 2$ limit in (3.104) would miss relevant contributions which are present in the flow equations studied in Section 3.4.1. The latter can be reproduced by the α-generalized $\overline{\mathrm{MS}}$ pseudo-regulator, by applying (3.103) directly in $d = 2$ and taking the $\alpha \to 1$ limit. We conclude the presentation of such a pseudo-regulator stressing the fact that ϵ is any real number in $(0, 2)$, but in order to better estimation of scaling exponents or other CFTs (for $d < 3$ or $\epsilon \in (1, 2)$) we need to larger truncations. The limitations are due to the fact that even dimension behaves differently and the limit is not smooth in terms of the parameter α. Such a difference between even and odd dimensions will be found also in Section 3.7.4 when we analyze another class of vanishing regulators.

Finally, let us comment on the extension of the nonlinear sigma model of Section 3.4.2 to dimension $d > 2$. Instead of just using the ε-expansion, it is possible to use directly the generalized $\overline{\mathrm{MS}}$ pseudo-regulator (3.102). As a result, in $d = 2 + \varepsilon$ we recover the well-known non-trivial fixed point

$$g_*^2 = \frac{2\pi\varepsilon}{N-1} + \frac{\pi\sigma\varepsilon^2}{(N-1)^2} + O\left(\varepsilon^3\right) \ , \tag{3.111a}$$

$$\nu^{-1} = \varepsilon - \frac{\sigma\varepsilon^2}{2(N-1)} + O\left(\varepsilon^3\right) \ , \tag{3.111b}$$

$$\eta = \frac{\varepsilon}{N-1} + O\left(\varepsilon^3\right) \ . \tag{3.111c}$$

Here it is possible to adjust σ ($\sigma = -2$) to get the full two-loop result for ν, but not for η. The latter correction would arise by considering a truncation where Z_k depends also on the momenta, as is discussed in Section 3.5.

3.7 The Ising universality class with vanishing regulator

Returning to the beta functions (3.7), we note that if we set $\lambda_2 = 0$, the beta functions of the relevant couplings go to zero, those of the marginal couplings are independent of a and those of the irrelevant couplings diverge in the limit of vanishing regulator. Given this rather singular behavior, one may fear that all physical information gets lost in this limit. Actually, this is not so, as we intend to show in $d = 3$, where the system is known to have a non-trivial fixed point. As we anticipated in Section 3.2.3, taking the limit $a \to 0$ of the flow is meaningless, however solving the flow and then taking the limit do not spoil the fixed point information.

3.7.1 The WF fixed point: relevant couplings and vanishing regulator

In order to make our point it will be enough, as a first step, to consider a truncation that contains only the relevant couplings. Since we are in $d = 3$ we truncate the series in Equation (3.6) to $n = 2$. Defining the dimensionless variables

$$\tilde{\lambda}_{2n} := k^{-d+n(d-2)}\lambda_{2n}\,, \tag{3.112}$$

the beta functions are

$$\tilde{\beta}_2 = -2\tilde{\lambda}_2 - \frac{a\tilde{\lambda}_4}{6\pi^2\left(a+\tilde{\lambda}_2\right)^2}\,{}_2F_1\left(2,\frac{3}{2},\frac{5}{2};\frac{a-1}{a+\tilde{\lambda}_2}\right), \tag{3.113a}$$

$$\tilde{\beta}_4 = -\tilde{\lambda}_4 + \frac{a\tilde{\lambda}_4^2}{\pi^2\left(a+\tilde{\lambda}_2\right)^3}\,{}_2F_1\left(3,\frac{3}{2},\frac{5}{2};\frac{a-1}{a+\tilde{\lambda}_2}\right). \tag{3.113b}$$

Expanding the RHS at fisrt order in $\tilde{\lambda}_2$, the WF fixed point is now located at

$$\tilde{\lambda}_2^\star = \frac{2a^3\left(1-\frac{\arctan\left(\frac{\sqrt{1-a}}{\sqrt{a}}\right)}{\sqrt{(1-a)a}}\right)}{a^2\left(2a-1-\frac{\arctan\left(\frac{\sqrt{1-a}}{\sqrt{a}}\right)}{\sqrt{(1-a)a}}\right)+16(1-a)\,{}_2F_1\left(\frac{3}{2},3;\frac{5}{2};\frac{a-1}{a}\right)}\,, \tag{3.114a}$$

$$\tilde{\lambda}_4^\star = \frac{\pi^2a^2}{{}_2F_1\left(\frac{3}{2},3;\frac{5}{2};\frac{a-1}{a}\right)}\,. \tag{3.114b}$$

If we expand the critical couplings around $a = 0$

$$\tilde{\lambda}_2^\star \sim_{a\to 0} -\frac{2a}{5} + o\left(a^{3/2}\right)\,, \qquad \tilde{\lambda}_4^\star \sim_{a\to 0} \frac{16\pi\sqrt{a}}{3} + o(a^{3/2})\,. \tag{3.115}$$

Thus the WF fixed point merges with the Gaussian fixed point. Note that since $\tilde{\lambda}_2 = \tilde{m}^2$ is linear in a for $a \to 0$ at the WF fixed point, the Q-functional (3.32) does not go to zero and this entails that the quantum/statistical contribution to the critical exponents will be non-trivial for $a \to 0$.

In fact, the position of the fixed point is not physically significant, as we already see in different cases. If we consider the stability matrix at the non-trivial fixed point

$$M = \left(\frac{\partial \tilde{\beta}_i}{\partial \tilde{\lambda}_j} \right)_\star = \begin{pmatrix} -\frac{5}{3} & \dfrac{4a\, {}_2F_1\left(\frac{3}{2},3;\frac{5}{2};\frac{a-1}{a}\right)\left(1-\frac{\tan^{-1}\left(\frac{\sqrt{1-a}}{\sqrt{a}}\right)}{\sqrt{(1-a)a}}\right)}{\pi^2 a^2\left(2a-\frac{\tan^{-1}\left(\frac{\sqrt{1-a}}{\sqrt{a}}\right)}{\sqrt{(1-a)a}}-1\right)+16(1-a)\, {}_2F_1\left(\frac{3}{2},3;\frac{5}{2};\frac{a-1}{a}\right)} \\ 0 & 1 \end{pmatrix} \tag{3.116}$$

we see that the component $(1,2)$ of M goes to zero for $a \to 0$ and so the stability matrix becomes diagonal. The eigenvalues of M, that is minus the critical exponents θ_i, are actually independent of a, in particular $\nu = (\theta_1)^{-1} = 0.6$. We see that even though the WF fixed point collapses towards the Gaussian one, it keeps its distinct character in the limit $a \to 0$ and a different critical exponent ν. In fact, the numerical value is not very bad, considering the drastic approximation.

3.7.2 The WF fixed point in the LPA with vanishing regulator

Let us now treat the potential as a whole [48]. At LPA we obtain the beta functional in Equation (3.4). Using the regulator (3.2), setting $d = 3$ and rescaling (2.100) the beta function of the dimensionless potential v becomes[19]

$$\partial_t v = -3v + \frac{1}{2}\tilde{\chi} v^{(1)} + \frac{a}{a + v^{(2)}}\ {}_2F_1\left(1, \frac{3}{2}, \frac{5}{2}; \frac{a-1}{a + v^{(2)}}\right) . \tag{3.117}$$

We look for even scaling solutions shooting from the origin with initial condition $v^{(2)}(0)$ and $v^{(1)}(0) = 0$. There are only two values of $v^{(2)}(0)$ which can be identified as fixed-point solutions: $v^{(2)}(0) = 0$, that corresponds to the Gaussian fixed point, and some negative value that corresponds to the WF fixed point. As in the preceding section, for decreasing values of a, the WF fixed point moves towards the Gaussian one. We see that also in the functional treatment, the WF fixed point collapses into the Gaussian one.

This is confirmed by shooting from infinity. The potential for the WF solution has the following asymptotic behavior for large field[20]

$$v = \mathrm{A}\tilde{\chi}^6 + a\left(\frac{1}{150\mathrm{A}\tilde{\chi}^4} - \frac{2a+3}{31500\mathrm{A}^2\tilde{\chi}^8} + \frac{8a^2+12a+15}{8505000\mathrm{A}^3\tilde{\chi}^{12}} - \frac{a}{67500\mathrm{A}^3\tilde{\chi}^{14}} + O\left(\mathrm{A}^{-4}\tilde{\chi}^{-16}\right) \right). \tag{3.118}$$

[19] Note that we have dropped the t subscript.

[20] This expression coincides with Equation (2.110) for $a = 1$.

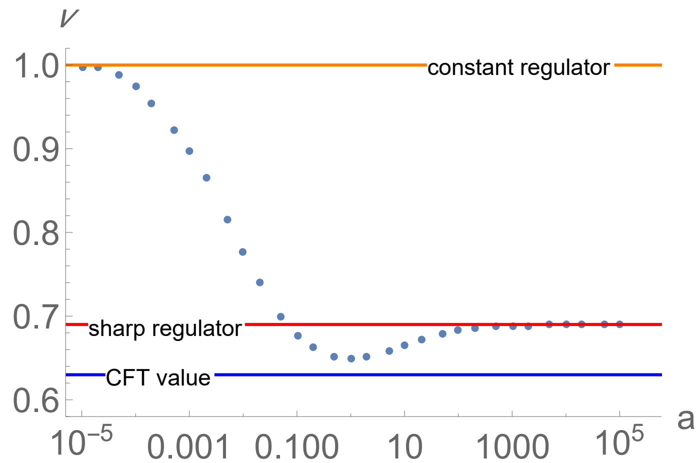


Figure 3.2: The dots represent the values of the critical exponent ν as a function of a. For comparison we have drawn the values of ν for the sharp regulator and the constant (mass) regulator, as well as the conformal-bootstrap value [124]. This figure extends Figure 12 in [77] to low values of a.

The free parameter A can be fixed as function of a by requiring $\mathbb{Z}_2$ symmetry for vanishing field [79]. We find that in the limit $a \to 0$, A tends to A ≈ 0.0015. [21]

The scaling exponents θ_i are obtained by linearizing the flow equation around the fixed point and calculating the spectrum of eigenperturbations. The analysis has to be done numerically. For the Gaussian fixed point the spectrum is independent of a. Figure 3.2 gives ν of the WF fixed point as a function of a for $10^{-5} < a < 10^5$. As expected, the best value is obtained for $a \approx 1$, while in the limit of vanishing regulator ν appears to approach $\nu = 1$. Besides the correlation-length exponent $\nu = (\theta_1)^{-1}$, we also find positive eigenvalues, as reported in Table 3.2.

For vanishing a all the scaling exponents are odd integers. This coincides with the spectrum of the $O(N)$ model in the limit of large N, which is known, and we have checked, to be independent of a [122, 123].

3.7.3 Vanishing regulators and constant regulators

At this point it is relevant to recall that the critical exponent $\nu = 1$ is known to result also from the LPA equations for a constant regulator, often called a “Callan-Symanzik”, [77]

$$\mathcal{R}_k = k^2. \tag{3.119}$$

[21] The asymptotic parameter is A $= 0.001$ for $a = 1$ and it increases monotonically for $a \to 0$.

	θ_1	θ_2	θ_3	θ_4	θ_5
$a = 1$	1.539	-0.656	-3.180	-5.912	-8.796
$a = 0$	1	-1	-3	-5	-7

Table 3.2: The first few critical exponents at the Wilson-Fisher fixed, point computed in the local potential approximation for the regulator (3.2). We report the most common choice $a = 1$ and the limiting case of the vanishing regulator.

This observation points at a more general result which we detail in this section.

So far we have first solved the fixed point equations for generic a and then sent $a \to 0$. On the other hand, we are now going to argue that when the vanishing-regulator limit is taken on the LPA beta functions, i.e. before integrating the flow, it results, in general non-even d, in the flow equations of the constant regulator.

The first way of reaching this conclusion is based on a redefinition of the RG scale k. Suppose that in addition to the parameter a we also introduce a parameter b rescaling the cutoff k

$$\mathcal{R}_k(z) = a\left(bk^2 - z\right)\Theta\left(bk^2 - z\right) . \tag{3.120}$$

This rescaling can be motivated as follows. First of all, it should not change the scaling solutions. Furthermore, we can define an "effective" cutoff scale k_{eff} as the momentum scale where the cutoff term $\mathcal{R}_k$ becomes comparable to the kinetic term. If we decrease a, the effective cutoff scale also decreases. It was suggested in [77] that the decrease of a should be compensated by choosing b so that at some conventional scale $z_0 < k^2$, the regulator is normalized: $\mathcal{R}_k(z_0) = k^2$. This fixes $b = \frac{1}{a} + \frac{z_0}{k^2}$, leading to the regulator

$$\mathcal{R}_k(z) = \left(k^2 - a\left(z - z_0\right)\right)\Theta\left(k^2 - a\left(z - z_0\right)\right) . \tag{3.121}$$

Now we see that in the limit $a \to 0$, the regulator becomes a constant as in (3.119). The latter leads to the dimensionless flow equation

$$\partial_t v = -dv + \left(\frac{d}{2} - 1\right)\tilde{\chi}v^{(1)} + \frac{\pi\left(1 + v^{(2)}\right)^{\frac{d}{2}-1}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)\sin\left(\frac{d\pi}{2}\right)} . \tag{3.122}$$

In $d = 3$ and after the rescaling $v \to v/(4\pi)$ and $\tilde{\chi} \to \tilde{\chi}/\sqrt{4\pi}$ this takes the simple form

$$\partial_t v = -3v + \frac{1}{2}\tilde{\chi}v^{(1)} - \sqrt{1 + v^{(2)}} . \tag{3.123}$$

This argument can actually be easily generalized to arbitrary shape functions R, as defined in (3.1). We first include the parameter b in the regulator, to account for the possibility to rescale k

$$\mathcal{R}_k(z) = bk^2 a\, R(y/b). \tag{3.124}$$

Then we choose $b = 1/a$ such that the regulator becomes

$$\mathcal{R}_k(z) = k^2\, R(ay). \tag{3.125}$$

Then the $a \to 0$ limit of (3.124), results in the constant regulator[22].

An alternative way of arguing that the $a \to 0$ limit reduces the LPA flow equation for the regulator (3.2) to the constant regulator case (3.122) is by performing an a-dependent rescaling. Namely, by redefining

$$\tilde{\chi} = a^{(d-2)/4}\hat{\chi}\ , \tag{3.126a}$$

$$v(\tilde{\chi}) = a^{d/2}\hat{v}(\hat{\chi}) + a\frac{1}{(d-2)(4\pi)^{d/2}\Gamma(1+d/2)}\ , \tag{3.126b}$$

in the flow equations for the regulator (3.2) and then taking the $a \to 0$ limit at fixed $\hat{\chi}$ and $\hat{v}$, we again find (3.122). For instance in $d = 3$ this rescaling entails that the prefactor a in (3.117) goes away.

Both kind of arguments however are applicable only for non-exceptional d. In particular, in some cases removing the momentum dependence of the regulator by sending $a \to 0$, as in (3.121) and (3.125), is not possible, because the $a \to 0$ limit and the momentum integral cannot be exchanged. This happens whenever the integral corresponding to the constant regulator is divergent. In fact, the momentum integral leading to (3.122) is convergent only for $d < 2$. [23]

If in the scalar LPA we adopt the constant regulator in $d \geq 2$, using analytic continuation as a tool for the definition of the momentum integral, the result has a meromorphic structure with poles for even values of d. On the other hand, if we try to directly take the limit $a \to 0$ with the regulator (3.2), and expand the Q-functionals (3.32), with $n = d/2$, d even and $\tilde{m} = 0$, in a around $a = 0$, there appear terms with $\log a$. As a consequence, we expect that the vanishing-regulator limit of the LPA flow equation will enjoy special properties in even dimensions. As a matter of fact, if analytic continuation is not adopted in the definition of the loop integrals, the arguments we just outlined point to the conclusion that the vanishing-regulator limit does not need to reproduce the constant-regulator case in the whole range $d \geq 4$.

[22]By comparing this with the original regulator in (3.1) we see that we have effectively cast the regulator as a function of $k_{\rm eff}^2 = ak^2$, rather then of k itself, and then considered $k_{\rm eff}$ as a independent.

[23]However the UV divergence in $2 \leq d < 4$ affects only the field-independent part of the effective potential and in these cases it could be removed by implementing the standard subtraction. Notice that this subtraction would introduce an IR divergence in $d = 2$. For some values of d the limit $a \to 0$ cannot be taken at the level of the integrands. In these cases we have first of all to compute the integrals, and this requires to specify the shape function R. This is the main reason why we focus on the special regulator choice of (3.2). More general results holding for arbitrary shape functions can be deduced once the field-theory model, the number of Euclidean dimensions d and the truncation of the EAA is specified.

3.7.4 Beta functions in two and four dimensions

As we argued at the end of the previous section, in the case of even dimensions the limit of vanishing regulators has a more intricate structure. Therefore, in this section we analyze these special cases in more detail.

We start with $d = 2$, where the flow equation of the LPA reads

$$\partial_t v = -2v + \frac{a}{4\pi(1-a)} \log\left(\frac{1+v^{(2)}}{a+v^{(2)}}\right) . \tag{3.127}$$

Defining

$$v(\tilde{\chi}) = a\hat{v}(\tilde{\chi}) \tag{3.128}$$

and simplifying a factor a from the flow equation, in the $a \to 0$ limit we are left with

$$\partial_t \hat{v} = -2\hat{v} - \frac{1}{4\pi} \log a - \frac{1}{4\pi} \log\left(1 + \hat{v}^{(2)}\right) . \tag{3.129}$$

The potential must be shifted by a factor that contains $\log a$, i.e. $\hat{v} \to \hat{v} - \frac{1}{8\pi} \log a$, in order to eliminate this divergent term for the limit $a \to 0$. We observe that the coefficient of the $\log a$ term matches exactly the coefficient of the $1/\epsilon$ pole of the expansion of (3.122) for $d = 2 + \epsilon$. [24] The finite logarithmic contribution coincides with the one in (3.122). Therefore, up to a field-independent shift of the potential, in $d = 2$ the vanishing-regulator limit agrees with the constant regulator.

We then turn to the LPA in $d = 4$. We first truncate the potential to a polynomial expansion around vanishing fields as in (3.6). For continuity with the previous sections, we also turn to the dimensionless couplings defined in (3.112). By considering the leading contributions to the beta functions $\tilde{\beta}_{2n}$ for vanishing a, we construct an ansatz based on the following scaling

$$\hat{\lambda}_2 = a^{-1}\tilde{\lambda}_2 \ , \tag{3.130a}$$
$$\hat{\lambda}_4 = \log(a)\tilde{\lambda}_4 \ , \tag{3.130b}$$
$$\hat{\lambda}_{2n} = a^{n-2}(\log a)^n \tilde{\lambda}_{2n}, \quad n > 2 \ . \tag{3.130c}$$

Assuming the $\hat{\lambda}_{2n}$ couplings can be kept fixed in the $a \to 0$ limit, results in the following

[24] This correspondence between $\log a$ singularities of the flow equations for the regulator (3.2) and $1/\epsilon$ poles of (3.122) holds also in higher even dimensions.

set of beta functions

$$\partial_t \hat{\lambda}_2 = -2\hat{\lambda}_2 + \frac{\hat{\lambda}_4}{16\pi^2}\left[1 + \frac{1+\log\left(1+\hat{\lambda}_2\right)}{\log a}\right], \tag{3.131a}$$

$$\partial_t \hat{\lambda}_4 = \frac{1}{\log a}\left[\frac{3}{16\pi^2}\frac{\hat{\lambda}_4^2}{1+\hat{\lambda}_2} + \frac{1}{16\pi^2}\hat{\lambda}_6\right], \tag{3.131b}$$

$$\partial_t \hat{\lambda}_6 = 2\hat{\lambda}_6 - \frac{15}{16\pi^2}\frac{\hat{\lambda}_4^3}{(1+\hat{\lambda}_2)^2} + \frac{1}{16\pi^2}\hat{\lambda}_8 + \frac{\hat{\lambda}_8}{16\pi^2}\frac{1+\log\left(1+\hat{\lambda}_2\right)}{\log a} + \frac{15}{16\pi^2}\frac{\hat{\lambda}_4\hat{\lambda}_6}{(1+\hat{\lambda}_2)\log a}\ , \tag{3.131c}$$

and similar results for higher couplings. Notice that terms of order $(\log a)^{-1}$ could be neglected as sub-leading in all beta functions apart for the second one, where such a term is in fact the leading one.

In order to include the beta functions of all couplings in a functional treatment, we turn to the task of including the definitions (3.130) in a rescaling of the effective potential. It is impossible to achieve this goal by a two-parameters rescaling of the kind studied in the previous sections. However, (3.130c) trivially lends itself to a functional rescaling. Hence, we can treat the first two couplings on a special footing, and embed the remaining ones in a functional which is related to higher derivatives of $v(\chi)$.

First, to simplify notations, it is convenient to define

$$\tilde{\rho} = \tilde{\chi}^2/2\ , \qquad\qquad u(\tilde{\rho}) = v(\tilde{\chi})\ . \tag{3.132}$$

Next, we define

$$f(\tilde{\rho}) = u^{(1)}(\tilde{\rho}) - \tilde{\lambda}_2 - \frac{\tilde{\lambda}_4}{3}\tilde{\rho}\ . \tag{3.133}$$

So by construction $f(0) = f'(0) = 0$, while $f^{(n)}(0) \propto \lambda_{2(n+1)}$. The functional flow equation for f can be obtained from the functional equation for $u^{(1)}$ by

$$\partial_t f(\tilde{\rho}) = \partial_t u^{(1)}(\tilde{\rho}) - \tilde{\beta}_2 - \frac{\tilde{\beta}_4}{3}\tilde{\rho}\ , \tag{3.134}$$

and then replacing $u^{(1)}$ through the definition (3.133). The identities $\partial_t f_k(0) = \partial_t f_k^{(1)}(0) = 0$ also follow from this definition. By the rescaling

$$f(\tilde{\rho}) = \frac{a}{\log a}\hat{f}(\hat{\rho})\ , \qquad\qquad \tilde{\rho} = a\log a\,\hat{\rho}\ , \tag{3.135}$$

together with the previous definitions of $\hat{\lambda}_2$ and $\hat{\lambda}_4$, we recover the full tower of relations (3.130). By inserting the previous definitions in the flow equation for $u^{(1)}(\rho)$ one can deduce the following functional flow equation

$$\partial_t \hat{f}(\hat{\rho}) = -2\hat{f}(\hat{\rho}) + 2\hat{\rho}\hat{f}^{(1)}(\hat{\rho}) + \frac{3}{16\pi^2}\hat{f}^{(1)}(\hat{\rho}) + \frac{1}{8\pi^2}\hat{\rho}\hat{f}^{(2)}(\hat{\rho}) + \frac{5}{16\pi^2}\hat{\rho}\hat{f}^{(2)}(0) - \frac{1}{16\pi^2}\frac{\hat{\rho}\hat{\lambda}_4^2}{1+\hat{\lambda}_2} - \frac{1}{16\pi^2}\hat{\lambda}_4 \log\Big(1+\hat{\lambda}_2\Big) + \frac{1}{16\pi^2}\hat{\lambda}_4 \log\Big(1+\hat{\lambda}_2+\hat{\lambda}_4\hat{\rho}\Big)\,. \tag{3.136}$$

This functional flow generates the leading terms in Equation (3.131c), and similar beta functions for the higher-order couplings, upon truncating it to a polynomial ansatz regular at the origin. However, we stress again that (3.136) does not include Equations (3.131a) and (3.131b), which therefore have to be supplemented to exhaust the LPA flow equations.

These flow equations are different from those of a constant regulator. In fact, the latter are formally UV divergent. More specifically, in $\tilde{\beta}_{2n}$ the contribution linear in $\tilde{\lambda}_{2n+2}$ corresponds to a momentum integral with dimension 2 which is not regularized by the constant regulator (3.119). Similar discrepancies arise in $d = 6, 8, \ldots$. The flow equation for the constant regulator in $d = 4 - \epsilon$ reads

$$\partial_t v = -4v + 2\tilde{\rho}\, v^{(1)} + \frac{\left(2\tilde{\rho}v^{(2)} + v^{(1)} + 1\right)\left[\log\left(2\tilde{\rho}v^{(2)} + v^{(1)} + 1\right) - 1\right]}{16\pi^2} + \frac{\left(2\tilde{\rho}v^{(2)} + v^{(1)} + 1\right)}{16\pi^2}\left[\gamma - \log(4\pi) - \frac{2}{\epsilon}\right]. \tag{3.137}$$

The second line in this equation arises from the expansion of the sine in the denominator of (3.122). It provides contributions to the $\tilde{\lambda}_{2n+2}$ term inside $\tilde{\beta}_{2n}$. Such terms would be absent in the $\overline{\text{MS}}$ scheme. These $1/\epsilon$ contributions which are divergent in $d = 4$ are a typical product of the analytic continuation adopted in the definition of the integral. Similar contributions which diverge in $d = 4$ are expected also if any other alternative definition is chosen. For instance, if a sharp UV cutoff Λ is introduced, the second line of (3.137) would be replaced by a different expression which is ill-defined in the $\Lambda \to \infty$ limit.

If we perform an *ad hoc* subtraction of the second line, the flow equation (3.137) leads to the following beta functions

$$\tilde{\beta}_2 = -2\tilde{\lambda}_2 + \frac{\tilde{\lambda}_4 \log\left(\tilde{\lambda}_2 + 1\right)}{16\pi^2}\,, \tag{3.138a}$$

$$\tilde{\beta}_4 = \frac{3\tilde{\lambda}_4^2}{16\pi^2\left(\tilde{\lambda}_2 + 1\right)} + \frac{\tilde{\lambda}_6 \log\left(\tilde{\lambda}_2 + 1\right)}{16\pi^2}\,, \tag{3.138b}$$

$$\tilde{\beta}_6 = 2\tilde{\lambda}_6 - \frac{15\tilde{\lambda}_4^3}{16\pi^2\left(\tilde{\lambda}_2 + 1\right)^2} + \frac{\tilde{\lambda}_8 \log\left(\tilde{\lambda}_2 + 1\right)}{16\pi^2} + \frac{15\tilde{\lambda}_6\tilde{\lambda}_4}{16\pi^2\left(\tilde{\lambda}_2 + 1\right)}\,. \tag{3.138c}$$

A comparison with (3.131) immediately reveals several differences. Apart for the scaling (classical) terms, the first two quantum/statistical terms are equal, up to the fact that the λ_2 dependence of the λ_{2n+2} term has been washed away in (3.131) by the $a \to 0$ limit, and up to the crucial $\log a$ dependence of (3.131b). However, all the additional quantum/statistical terms in (3.138) are absent in (3.131).

The peculiar simplicity which the Equations (3.131) attain in the $a \to 0$ limit, together with the $1/\log a$ dependence of (3.131b), raises the question as to whether these beta functions retain enough physical information for being practically useful. As a first step towards addressing this question, we limit ourselves to a simple observation. Namely, as long as the subleading logarithmic a-dependence is retained in (3.131), the ϕ^4-theory beta function and other universal physics is still present. For instance, we can study the WF fixed point in $d = 4-\epsilon$. In order to employ (3.131) in this study, we need to prescribe that the $\epsilon \to 0$ limit be taken before the $a \to 0$ one. This means in practice that the vanishing-regulator limit is taken on the $d = 4$ FRG equations. Had we sent $a \to 0$ in $d < 4$, we would have found different equations for $\hat{\lambda}_{2n}$ and precisely the constant-regulator ones, as already mentioned in Section 3.7.3.

Within the simplest truncation corresponding to retaining only $\tilde{\lambda}_2$ and $\tilde{\lambda}_4$, where we add the classical scaling term $-\epsilon\lambda_4$ to $\hat{\beta}_4$ to account for the shift of dimensionality, the WF fixed point to first order in ϵ is located at

$$\hat{\lambda}_2 = \frac{1}{6}\epsilon\,(1+\log a) \ , \qquad\qquad \hat{\lambda}_4 = \frac{16}{3}\pi^2\epsilon\log a \ . \tag{3.139}$$

These fixed-point couplings have to be interpreted as small, even if they seemingly blow up for $a \to 0$, because the limit $\epsilon \to 0$ has to be taken first. Notice that keeping the sub-leading order-$(\log a)^{-1}$ contribution to $\hat{\beta}_4$ is essential for revealing the fixed point. By computing the corresponding critical exponents, we find the universal one-loop result

$$\theta_1 = 2 - \frac{\epsilon}{3} \ , \qquad \theta_2 = -\epsilon \ . \tag{3.140}$$

3.8 Background field issues

When one splits the field into a classical background and a quantum/statistical fluctuation

$$\chi = \chi_B + \varphi \ , \tag{3.141}$$

the action, being a function of χ, is invariant under the shift symmetry

$$\chi_B \mapsto \chi_B + \epsilon \ , \qquad\qquad \varphi \mapsto \varphi - \epsilon \ . \tag{3.142a}$$

This can be expressed by the identity

$$\frac{\delta S}{\delta \chi_B} - \frac{\delta S}{\delta \varphi} = 0 \ . \tag{3.143}$$

On the other hand, the regulator only depends on the background field and is therefore not invariant under the shift symmetry. In particular in gauge theories, in order to preserve background gauge invariance, the cutoff is usually written as a function of the background covariant derivative: $\mathcal{R}_k(-\bar{D}^2)$. This effect can be mimicked in the scalar case by artificially introducing a dependence of $\mathcal{R}_k$ on χ_B. For example, Morris and collaborators considered regulators of the general form [79]

$$\mathcal{R}_k(z) = (k^2 - k^2 h(\tilde{\chi}_B) - z)\Theta(k^2 - k^2 h(\tilde{\chi}_B) - z) \ . \tag{3.144}$$

The EAA then becomes a functional $\Gamma_k[\varphi, \chi_B]$, i.e. it has a separate dependence on these two arguments. The breaking of the shift symmetry results in a modified Ward identity

$$\frac{\delta \Gamma_k}{\delta \chi_B} - \frac{\delta \Gamma_k}{\delta \varphi} = \frac{1}{2}\mathrm{Tr}\left[\left(\frac{\delta^2 \Gamma_k}{\delta\varphi\delta\varphi} + \mathcal{R}_k\right)^{-1} \frac{\delta \mathcal{R}_k}{\delta \chi_B}\right] \ . \tag{3.145}$$

It has been shown that such background dependence in the regulator can either destroy physical fixed points or create artificial ones [79]. On the other hand, when the FRG equation is solved together with the Ward identity (3.145), the correct physical picture can be reconstructed. While this can be achieved in the scalar case [79], it is much harder in the case of gauge theories, and in particular for gravity [125]. It is therefore desirable to find other ways around this obstacle. The form of the equation (3.145) suggests that in the limit of vanishing regulator the shift symmetry is restored. One would therefore expect that in this limit the aforementioned pathologies should also disappear. In this section we will see how this actually happens in the scalar theory.

We begin by briefly reviewing some results of [79]. We consider the same system as in Section 3.7.2, in $d = 3$, but we use the regulator (3.144). In a single-field approximation one identifies $\chi_B = \varphi$. The corresponding flow equation for the potential reads

$$\partial_t v = -3v + \frac{1}{2}\tilde{\varphi} v^{(1)} + \frac{(1-h)^{3/2}}{1-h+v^{(2)}}\left(1 - h - \frac{1}{2}\partial_t h + \frac{1}{4}\tilde{\varphi} h'\right)\Theta(1-h) \, . \tag{3.146}$$

Two special cases for h have been considered. The first case is $h = \alpha\tilde{\varphi}^2$. In this case, for $\alpha < 0$ the Heaviside theta on the RHS of (3.146) is equal to one. Solving the fixed point equation, one finds that the Gaussian fixed point becomes interacting and an increasing number of fake fixed points appear, as α becomes more negative. For example, Table 3.3 presents the non-trivial fixed points and the associated relevant critical exponents for two negative values of α. In both cases FP_2 is the deformed Gaussian fixed point. For $\alpha > 0$ because of the Heaviside theta function on the RHS of (3.146), $v = A\tilde{\varphi}^6$ for $\tilde{\varphi} > 1/\sqrt{\alpha}$. The Gaussian fixed point is always absent, and for $\alpha > 0.08$ also the WF fixed point disappears.

The second case is $h = \alpha v^{(2)}$. The Gaussian [25] and the WF fixed points always exist, but when α is increased, new fixed points appear near the Gaussian one [26] and move away

[25] Note that the Gaussian fixed point corresponds to the point $(v(0), v^{(1)}(0)) = (1/3, 0)$.

[26] In particular for $\alpha \geq 0.85$ a first additional fixed point appears.

FP	θ1	θ2
1	1.17	-
2	2.11	0.82

$\boldsymbol{\alpha = -1/2}$

FP	θ1	θ2	θ3	θ4
1	0.89	-	-	-
2	2.35	0.76	-	-
3	2.02	1.43	0.60	-
4	2.10	1.69	1.08	0.39

$\boldsymbol{\alpha = -2}$

Table 3.3: The non-trivial fixed-point solutions of (3.146) with $h = \alpha\tilde{\varphi}^2$, and the corresponding relevant critical exponents, for $\alpha = -1/2$ (left panel) and $\alpha = -2$ (right panel). The entries which are left blank correspond to irrelevant deformations. FP_1 is the Wilson-Fisher fixed point, while FP_2 is a "deformed Gaussian" fixed point as it possesses two relevant directions.

from it as α becomes bigger: for example, for $\alpha = 1$ there is a spurious fixed point and for $\alpha = 2$ there are three of them.

As we said in Section 2.9, we encounter a case where the choice of the regulator can destroy physical information and/or generate unphysical features.

In [79] the authors solve the anomalous Ward identity for shift symmetry and show how to recover the physical results. Instead, we shall discuss here the effect of taking the limit of vanishing regulator. To this end, we first introduce the parameter a in (3.144)

$$\mathcal{R}_k(z) = a(k^2 - k^2 h(\tilde{\chi}_B) - z)\Theta(k^2 - k^2 h(\tilde{\chi}_B) - z)\ . \tag{3.147}$$

Within a single-field LPA truncation this leads to the flow equation

$$\partial_t v = -3v + \frac{1}{2}\tilde{\varphi}v^{(1)} + \Theta(1-h)\frac{a\,(1-h)^{3/2}}{a(1-h)+v^{(2)}}\left(1-h-\frac{1}{2}\partial_t h+\frac{1}{4}\tilde{\varphi}h'\right){}_2F_1\left(1,\frac{3}{2},\frac{5}{2};\frac{(a-1)(1-h)}{a(1-h)+v^{(2)}}\right). \tag{3.148}$$

Again, we discuss separately the two choices for the function h.

First case: $\boldsymbol{h = \alpha\tilde{\varphi}^2}$. Following [79] we start with a quadratically-field-dependent regulator. However we slightly depart from that reference in that we find it more convenient to portrait the landscape of fixed points by a different numerical method, a shooting from the origin. This way, we have verified that decreasing a the spurious fixed points disappear and the physical fixed points converge to the origin. This convergence is the same phenomenon that we observed in Sections 3.7.1 and 3.7.2.

At these fixed points, we compute the spectrum of critical exponents with the same method used in [79], namely by shooting from infinity, as we did in Section 3.7.2. This means that we first construct an asymptotic expansion of the fixed-point potential as well as of the eigenfunction of the linearized flow around the fixed point. For $\alpha < 0$ the Heaviside theta on the RHS of equation (3.148) is equal to one, and the potential has the following behavior at infinity

$$
\begin{aligned}
v = {} & \mathrm{A}\tilde{\varphi}^6 + \frac{a|\alpha|^{5/2}}{150\mathrm{A}}|\tilde{\varphi}| + \frac{a|\alpha|^{3/2}(525\mathrm{A} - (3+2a)\alpha^2)}{31500\mathrm{A}^2|\tilde{\varphi}|} \\
& + \frac{a\sqrt{-\alpha}\left(212625\mathrm{A}^2 - \alpha^2(3780a\mathrm{A} + 5670\mathrm{A}) + \left(16a^2 + 24a + 30\right)\alpha^4\right)}{17010000\mathrm{A}^3|\tilde{\varphi}|^3} + O\left(|\tilde{\varphi}|^{-5}\right) \, .
\end{aligned} \tag{3.149}
$$

Shooting on A and on a corresponding asymptotic parameter for the perturbation, and by demanding $\mathbb{Z}_2$ parity at the origin, we determine the location of the fixed point as well as the quantized values of the critical exponents. In the $a \to 0$ limit the latter become independent of α and agree with the spectrum discussed in Section 3.7.2.

For $\alpha > 0$, because of the Heaviside theta one the RHS of (3.148) $v = \mathrm{A}\tilde{\varphi}^6$ for $\tilde{\varphi} > 1/\sqrt{\alpha}$. Therefore for $\tilde{\varphi} < 1/\sqrt{\alpha}$ the potential as a function of $\delta = \left(\frac{1}{\sqrt{\alpha}} - \tilde{\varphi}\right)^{1/2}$ has the following asymptotic behavior

$$
v = \frac{\mathrm{A}}{\alpha^3} - \frac{6\mathrm{A}}{\alpha^{5/2}}\delta^2 + \frac{15\mathrm{A}}{\alpha^2}\delta^4 + \frac{2\sqrt{2}\,a\,\alpha^{13/4}}{75\mathrm{A}}\delta^5 - \frac{135000\mathrm{A}^4 + a^2\alpha^{10}}{6750\alpha^{3/2}\mathrm{A}^3}\delta^6 + o\left(\delta^7\right) \, . \tag{3.150}
$$

Shooting from infinity and decreasing a we recover the Gaussian and the WF fixed points. In particular, for $\alpha = 1/25$ the Gaussian fixed point reappears for $a \lesssim 10^{-2}$, while for $\alpha = 1/9$ the WF fixed point reappears for $a \lesssim 0.35$ and the Gaussian one for $a \lesssim 4 \cdot 10^{-3}$. Also in this case the critical exponents of the Gaussian and WF fixed points approach the values found for vanishing a in Section 3.7.2.

Second case: $\boldsymbol{h = \alpha v^{(2)}}$. We then move on to consider a regulator which depends on the second derivative of the effective potential, through a constant $\alpha > 0$. In this particular case shooting from the origin is not convenient for technical reasons, therefore we shoot from large field values.

This time $v = \mathrm{A}\tilde{\varphi}^6$ for $\tilde{\varphi} > \tilde{\varphi}_c \equiv (30\mathrm{A}\alpha)^{-1/4}$ provided $v^{(2)} > 1/\sqrt{\alpha}$. Below $\tilde{\varphi}_c$ the potential can be expanded in $\delta = \tilde{\varphi}_c - \tilde{\varphi}$ as follows

$$
v = \frac{\mathrm{A}}{(30\mathrm{A}\alpha)^{3/2}} - \frac{6\mathrm{A}}{(30\mathrm{A}\alpha)^{5/4}}\delta + \frac{1}{2\alpha^2}\delta^2 + F(\delta)\,, \tag{3.151}
$$

$$
F = \delta^{16/5}\left(\frac{-25\sqrt{5}\mathrm{A}^{1/10}\alpha^{-17/10}}{88\sqrt{2}\,3^{3/10}a^{2/5}} + \frac{125\;5^{3/4}\mathrm{A}^{-1/20}\alpha^{-53/20}}{5984\sqrt[4]{2}\,3^{17/20}a^{4/5}}\delta^{1/5} - \frac{71875\mathrm{A}^{-1/5}\alpha^{-18/5}}{246445056\;3^{2/5}a^{6/5}}\delta^{2/5} + o(\delta^{3/5})\right). \tag{3.152}
$$

Shooting on A and searching for values which correspond to a vanishing $v^{(1)}(0)$ one can reveal several spurious fixed points at non-vanishing α and a. More and more of them are generated from the Gaussian fixed point for bigger and bigger values of α. We find that decreasing a at fixed $\alpha > 0$ reduces the number of spurious fixed points, and in the $a \to 0$ limit all of them disappear while the Gaussian and the WF fixed points merge. In particular, we verify that also in this case the critical exponents tend to the values obtained in Section 3.7.2 for $a \to 0$.

3.8.1 Ward Identity for the shift symmetry

Going beyond a single-field approximation, i.e. keeping both φ and χ_B as distinct, the LPA truncation becomes [27]

$$\Gamma_k[\varphi, \chi_B] = \int_x \left(\frac{1}{2} \left(\partial_\mu \varphi\right)^2 + \frac{1}{2} \left(\partial_\mu \chi_B\right)^2 + V_k(\varphi, \chi_B) \right) . \tag{3.153}$$

Using the regulator (3.147) the modified Ward identity (3.145) and the flow equation become

$$\partial_{\tilde\varphi} v - \partial_{\tilde\chi_B} v = c_d \frac{h'}{2} \frac{a(1-h)^{d/2}}{a(1-h) + \partial^2_{\tilde\varphi} v} \,{}_2F_1\!\left(1, \frac{d}{2}, \frac{d}{2}+1; -\frac{(1-a)(1-h))}{a(1-h)+\partial^2_{\tilde\varphi} v}\right) , \tag{3.154}$$

$$\partial_t v + d\, v - \frac{(d-2)}{2} \left(\tilde\varphi \partial_{\tilde\varphi} v + \tilde\chi_B \partial_{\tilde\chi_B} v\right) = \tag{3.155}$$

$$c_d \frac{a(1-h)^{d/2}}{a(1-h) + \partial^2_{\tilde\varphi} v} \left(1 - h - \frac{1}{2}\partial_t h + \frac{1}{4}\left(d-2\right) \tilde\varphi h'\right) \,{}_2F_1\!\left(1, \frac{d}{2}, \frac{d}{2}+1; -\frac{(1-a)(1-h))}{a(1-h)+\partial^2_{\tilde\varphi} v}\right),$$

where $c_d = \left((4\pi)^{d/2}\Gamma\left(\frac{d}{2}+1\right)\right)^{-1}$. We rescale all the quantities in the following way

$$\tilde\varphi = a^{(d-2)/4}\, \hat\varphi\,, \qquad \tilde\chi_B = a^{(d-2)/4}\, \hat\chi_B\,, \tag{3.156}$$

$$v(\tilde\varphi) = a^{d/2}\, \hat v(\hat\varphi) + a \frac{1}{(4\pi)^{d/2}(d-2)\,\Gamma\left(\frac{d}{2}+1\right)}\,, \qquad h = a^\gamma\, \hat h\ . \tag{3.157}$$

This set of definitions agrees with the one in (3.126). Here γ depends on the choice of h: for example $\gamma = 1$ for both $h = \alpha\tilde\chi_B^2$ and $h = \alpha v^{(2)}$. For the sake of generality we shall keep γ free for the time being. Expanding for small a and assuming $2 < d < 4$, the Ward identity and the flow equation become

$$\partial_{\hat\varphi} \hat v - \partial_{\hat\chi_B} \hat v = \frac{a^{\gamma+1-d/2}}{d(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\, \hat h' + \cdots\,, \tag{3.158}$$

$$\partial_t \hat v + d\, \hat v - \frac{(d-2)}{2} \left(\hat\varphi \partial_{\hat\varphi} \hat v + \hat\chi_B \partial_{\hat\chi_B} \hat v\right) = \tag{3.159}$$

$$-\frac{a^{\gamma+1-d/2}}{d(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)} \left(\partial_t \hat h + d\, \hat h - \frac{(d-2)}{2} \hat\chi_B\, \hat h'\right) + \frac{\Gamma\left(\frac{d}{2}-1\right)}{(4\pi)^{d/2}} \left(1 + \partial^2_{\hat\varphi} \hat v\right)^{d/2-1} + \cdots\,,$$

where the dots denote quantities that go to zero for $a \to 0$.
From the modified Ward identity we see that to have a well defined vanishing-regulator limit we must demand $\gamma \geq \frac{d}{2} - 1$. If $\gamma > \frac{d}{2} - 1$, $\partial_{\hat\varphi} \hat v = \partial_{\hat\chi_B} \hat v$: this implies that

[27] The mixing term $\partial_\mu \chi_B \partial^\mu \varphi$ is ruled out by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry on the arguments of the EAA.

$\hat{v}(\hat{\varphi}, \hat{\chi}_B) = \hat{v}(\hat{\varphi} + \hat{\chi}_B)$ and so we recover the shift symmetry and the flow equation without background. If $\gamma = \frac{d}{2} - 1$, the modified Ward identity gives

$$\hat{v}(\hat{\varphi}, \hat{\chi}_B) = \hat{v}_s(\hat{\varphi} + \hat{\chi}_B) - \frac{1}{d(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\hat{h}(\hat{\chi}_B)\,. \tag{3.160}$$

Inserting this result into the flow equation, we recover again the equation without background.

3.9 Discussion

Mass-dependent Wilsonian RG schemes, such as for instance momentum subtraction with a sharp UV cutoff, simultaneously achieve the two goals of regularizing a field theory and of defining the heavy modes to be integrated out while constructing an effective description of the system. In these schemes, information about mass thresholds is essential and built in the effective theory at all scales. Mass-independent schemes instead, e.g. dimreg/$\overline{\mathrm{MS}}$, remove the latter piece of information by taking the limit of infinite separation between the physical scales of applicability of the effective field theory and the heavy masses of the underlying microscopic description. It is therefore natural that, in the construction of a mass-independent scheme out of a Wilsonian one, the infinite-separation-of-scales limit also becomes a regularization-removal process.

This is precisely what has been observed in this chapter. More specifically, we have focused on the functional renormalization group equations, a prototypical Wilsonian representation of field theory based on shell-by-shell integration of modes according to a coarse-graining-defining function $\mathcal{R}_k(q^2)$, which acts as a smooth infrared cutoff on modes with momentum $q^2 \ll k^2$. As a matter of fact, we have found that it is possible to achieve a continuous transition from this exact mass-dependent scheme to functional RG equations within $\overline{\mathrm{MS}}$, at the price of taking a parametric limit $\epsilon \to 0$ that in even dimensions also results in the removal of the cutoff: $\mathcal{R}_k \to 0$ [28].

The dependence of $\mathcal{R}_k$ on continuous parameters, such as our ϵ, is allowed and welcome in the FRG setup. In fact, it is often used in FRG applications as a diagnostic tool (weak dependence on such parameters is taken as a sign of a good truncation) or even as a selection criterion for the "best" regulator (e.g. through the principle of minimum sensitivity [85, 86]). However we find that, while taking the $\epsilon \to 0$ limit, the regulator $\mathcal{R}_k$ at some point must leave the domain of acceptable IR Wilsonian cutoffs and violate some of the conditions that define physical coarse grainings. This is quite to be expected, as dimreg is by no means a physical IR cutoff. As such, also the pseudo-cutoff form which should be attained by $\mathcal{R}_k$ for asymptotically small ϵ, our (3.19), defies every interpretation

[28] The generalization to continuous dimensions discussed in Section 3.6 is an exception, as in this case the deformation of the dispersion relation operated by the pseudo-regulator survives the $\epsilon \to 0$ limit.

as a conventional regulator, and is well suited for its goal only when augmented by analytic continuation of the momentum integrals in ϵ.

After the construction of an FRG pseudo-regulator which successfully reproduces the one-loop $\overline{\mathrm{MS}}$ beta functions for vanishing ϵ, see Section 3.2, we have addressed the question as to whether this pseudo-regulator choice and the $\epsilon \to 0$ limit spoil the nonperturbative nature of the exact FRG equation. We have provided reasons to argue for a negative answer. In Sections 3.3 and 3.4 we have first illustrated the physical content of the RG resummations contained in the RG improvement of one-loop beta functions, showing that they account for higher-order perturbative contributions and can even fairly describe some nonperturbative critical phenomena in two dimensions, see for instance Table 3.1.

Moreover, one of the most promising aspects of this research direction, is the possibility to look at the $\overline{\mathrm{MS}}$ limit of FRG equations as a novel way to approach the challenging problem of gauge and nonlinear symmetries. We have limited ourselves to explore these aspects in Section 3.4.2, where we have observed that the $\epsilon \to 0$ limit of the RG equations of a linear $O(N)$ model have the pleasant property of preserving also a nonlinearly realized $O(N+1)$ symmetry. Further systematic studies of this problem are in order, to assess whether taking the $\overline{\mathrm{MS}}$ limit might ease the task of fulfilling Ward-Takahashi identities and master equations (actually $\mathcal{R}_k$-deformed versions of the latter).

Then, inspired by the properties of the pseudo-regulator, we have discussed the effect of an overall suppression of the regulator with a constant factor a, and in particular the limit $a \to 0$, that we called the limit of vanishing regulator.

In the case of the Wilson-Fisher fixed point, we have first studied the first form of the vanishing-regulator limit, by analyzing the a dependence of the fixed-point solution. Decreasing a has the effect of shifting the fixed points towards the Gaussian one, but the scaling exponents remain distinct even in the limit $a \to 0$. Here we have limited our analysis to the leading order of the derivative expansion.

In a polynomial approximation of the potential, the values of the scaling exponents become progressively worse as one increases the order of the polynomial. This is in agreement with the statement in [77] that the radius of convergence of the Taylor expansion of V is proportional to a. We have avoided this problem by also considering the functional treatment (LPA), but in this case one gets the exponent $\nu = 1$, which is worse than for any polynomial and coincides with the upper boundary conjectured in [77].

We have then analyzed the second form of the vanishing-regulator limit, taking it on the LPA beta functional of scalar field theory, finding agreement with the first kind of limit as far as the critical exponents are concerned, although the locations of the fixed point differ. Even though some naive arguments suggest that the limit of vanishing regulator might generally reproduce the results of a constant (momentum independent) mass-like regulator, we have observed that in the LPA this is the case only when the constant-regulator momentum integrals are convergent. As we adopted analytic continuation in the definition of the integrals, this excludes even integer values of $d \geq 4$ (the $d = 2$ case can be reduced to the constant-regulator case by a field-independent shift in the potential). As a consequence, the vanishing-regulator limit remains different from the constant regulator in $d = 4$. We expect this conclusion to hold also in higher even dimensions, if analytic continuation is used, or in the whole range $d \geq 4$ without analytic continuation. It remains to be seen whether these conclusions are robust against enlargements of the truncation. For instance, at the second order of the derivative expansion, there might be a non-trivial interplay between the momentum-derivatives of the regulator and the $a \to 0$ limit, resulting in further differences between the constant and the vanishing regulators.

For all these reasons, it will be quite interesting to systematically study the next order of the derivative expansion, including a field-dependent wave function renormalization $z_k(\chi)$. In Section 3.4.2, this level of approximation has been analyzed only for the two-dimensional nonlinear sigma model, as in this case it is the first non-trivial order of the derivative expansion. It is also known that in the case of quantum critical points the convergence of this expansion requires an increasingly accurate tuning of a. For the three-dimensional Wilson-Fisher fixed point, this tuning process is expected to converge to optimal values within the range $0.5 < a < 1$ [85]. Hence, it appears very unlikely that at the special point $a \to 0$ the derivative expansion might be convergent.

However, we should mention that the amplitude a is only one of an infinite series of free parameters within the regulator $\mathcal{R}_k$. In this work we have not allowed for such a residual freedom, having fixed the regulator to a piece-wise linear form. This choice has been justified as follows. In some circumstances, depending on the theory (or approximation) under study, as well as on the number of Euclidean dimensions d, the argument of the momentum integral might be non-integrable in the $a \to 0$ limit. Nonetheless the integral might allow for a finite $a \to 0$ limit, i.e. the limit and the integral cannot be exchanged. Whenever this happens, one must first clearly define the momentum integrals by choosing a specific shape function and when applicable a unique analytic continuation, and then investigate the possible behavior of these integrals in the parametric $a \to 0$ limit. In all other cases, namely when the $a \to 0$ limit can be brought inside the momentum integrals, one can easily generalize the discussion to arbitrary shape functions R, as done in Section 3.7.3. Still, optimization criteria over the remaining parameters might be essential to obtain accurate results in the vanishing-regulator limit. It might also be possible to take advantage of these additional parameters, with their associated free limiting behavior, to construct alternative flow equations resulting from the vanishing-regulator limit. For instance, in the so-called LPA$'$ truncation, this kind of additional freedom allowed to construct a one-parameter family of $\overline{\text{MS}}$-like schemes within the FRG [1].

In fact, the limit of vanishing regulator shares several features with the more specific case of the $\overline{\text{MS}}$-like pseudo-regulators. In particular in Section 3.5, we observed that the best way of capturing the effect of quantum/statistical fluctuations beyond one loop is not adopting the derivative expansion, but rather accounting for the momentum dependence of vertices as in a vertex expansion. Because of their similarities, it is reasonable to expect that this behavior of $\overline{\text{MS}}$-like pseudo-regulators against the choice of truncation scheme might be shared by the larger class of vanishing regulators.

In spite of the poor results of the $a \to 0$ limit of the LPA for the benchmark case of the Wilson-Fisher fixed point, we think that this limit may be useful in simple approximations, in problems where a symmetry is broken by the regulator. As a first example we have discussed the $O(N+1)$-nonlinear sigma model, in a formulation where the regulator breaks the global symmetry to $O(N)$. In this case we have shown that in the limit of vanishing regulator the beta functions converge to those of the $O(N+1)$-symmetric theory. We have then considered the shift symmetry arising in the background field treatment of a scalar theory. When this symmetry is broken by the regulator, this can either generate unphysical fixed points or, what is worse, destroy a physical fixed point. We have verified that the Ward identities of the shift symmetry are restored in the limit of vanishing regulator, and that all the unphysical features of the flow disappear when a becomes sufficiently small.

It is important to stress the difference between this logic and the following one that is sometimes found in the FRG literature: the RG flow equations are solved first (and independently of the Ward identities) for a parametric family of regulators; then the latter parameters are tuned such that the violation of some finite-dimensional subset of the Ward identities is minimized. This procedure, which is crucially tied to the approximation scheme adopted for solving the RG equations, when applied to the parameter a of (3.1), typically results in some non-vanishing value which is close to the value maximizing the rate of convergence of the chosen truncation scheme ($a \sim 1$). This approach has been studied for instance in the case of conformal symmetry [126]. In this reference the Ward identities for special conformal transformation, either in their quantum or classical form (i.e. regulator dependent or independent respectively), are not solved as functional constraints [29].

By contrast, in the studies we presented in Sections 3.4.2 and 3.8, the ansatz for the EAA included exact solutions of the classical Ward identities for $O(N+1)$ and shift symmetry respectively, which are easy to solve independently from the RG equations. It is thus not surprising that the symmetry breaking induced by the RG flow is minimized for $a \to 0$. In fact, one might expect that the quantum Ward identities reduce to their classical counterparts when $a \to 0$. Thus, because of the different strategy followed in the choice of the initial ansatz for the EAA, while the authors of [126] could only minimize the unavoidable symmetry breaking, in this work we could tune it to zero by taking the limit of vanishing regulators.

It is interesting that a study similar to the one of [126] was performed in [127], where the symmetry expected to emerge at the RG fixed point is supersymmetry rather than conformal symmetry. In this latter work the ansatz for the EAA does include an exact solution of the classical supersymmetric Ward identity. The minimization of the breaking of supersymmetry at the fixed point by means of the optimization of the regulator was also studied, but unfortunately the limit of vanishing regulator was not within the parametric space considered in this reference. In fact, we expect the application of the vanishing-

[29] The truncated modified Ward identity is cast in the form $f(\tilde{\chi}) = 0$, for a certain function f. This equation is not fulfilled, for by any value of a. However it is possible to tune a such that the function f is minimized in an almost $\tilde{\chi}$-independent sense.

regulator limit to supersymmetric models to be interesting and useful.

The main motivation of this work was the hope that vanishing regulators, or perhaps just “sufficiently small regulators”, may be useful also in the application of the FRG to gauge theories and gravity, where the background field method is almost always adopted. Our results suggest that this may be possible, but that the usefulness of this idea may be restricted to the simplest truncations.

Chapter 4

Essential Renormalization Group

'Entia non sunt multiplicanda praeter necessitatem.'
William of Ockham

Our mathematical descriptions of natural phenomena contain redundant, superfluous information which is not present in Nature. This follows since, for any given problem, we always have the basic liberty to re-express the set of dynamical variables in terms of a new, perhaps simpler, set. In this respect, our mathematical models fall into equivalence classes, where two models are considered to be physically equivalent if they are related by a change of variables. Natural phenomena are therefore described by an equivalence class of effective theories rather than a specific model. However, in practice, in order to test our models against experiment, we would like to find those models that reduce the time and effort needed to compute a given physical observable.

The RG provides a framework to iteratively perform a change of variables with the purpose of describing physics at different length scales. This, in practice, translates into a flow in a space spanned by the couplings which parameterize all possible interactions between the physical degrees of freedom. However, due to the aforementioned redundancies, this *theory space* is divided into equivalence classes. As a consequence, we do not have to compute the flow of all coupling constants, but instead, we only need to compute the flow of the *essential couplings*, which are those eventually appearing in expressions for physical observables. The other coupling constants, known as the *inessential couplings*, can take quite arbitrary values since changing them amounts to moving within an equivalence class. It follows, therefore, that an inessential coupling is any coupling for which a change in its value can be reabsorbed by a change of variables. The prototypical example of an inessential coupling is the one related to a simple linear rescaling or renormalization of the dynamical variables, namely, in a field-theoretic language, the wave function renormalization. Actually, it is this transformation that gives the RG its name. However, there is an infinite number of other inessential couplings related to more general, nonlinear changes of variables. As we will show explicitly, one is free to specify the values of all inessential couplings instead of computing their flow. This freedom can then be exploited to simplify or otherwise optimize the calculation of physical quantities of interest. In addition, this has the advantage that we automatically disentangle the physical information from the unphysical redundant content encoded in the inessential couplings. Such a possibility has been advocated independently by G. Jona-Lasinio [42] and by S. Weinberg [39]. Although a perturbative approach has been put forward in [128], so far, no concrete non-perturbative implementation based on general nonlinear changes of variables has been realized.

The purpose of this chapter is to arrive at a concrete scheme of this type, with the explicit aim of reducing the complexity of computations within the framework of K. Wilson's exact RG [27, 28, 29]. We shall refer to this concrete scheme as the *minimal essential scheme*. Essential schemes can be defined more generally as those for which we only compute the running of the essential couplings, having specified renormalization conditions that determine the values of the inessential couplings as functions of the former.

To achieve our aim, in Section 4.1 we first develop the concept of field reparameterizations in quantum field theory (QFT). These changes of variables can be understood geometrically as local *frame transformations* on configuration space. After introducing the notation of a frame transformation for a classical field theory, we present a frame covariant formulation of QFT, where no particular frame is preferred a priori. This way, it becomes manifest that observables are invariant under frame transformations. This leads to a precise definition of an inessential coupling and its conjugate *redundant operator*, whose identification is crucial to the concrete implementation of essential schemes. In the rest of the thesis, we combine this frame covariant formalism with a generalized version of the exact RG.

In Section 4.2 we are led to consider the generalized form of the flow of the EAA which incorporates frame transformations along the RG flow [61]. It is this equation that allows us to implement essential schemes. Moreover, we derive the dimensionless form of the generalized flow equation, where it becomes clear that the cutoff scale k is itself an inessential coupling. We notice that the RG equations we use can be seen as the counterpart of the generalized flow equations for the Wilsonian Effective Action first written down by F. Wegner [35].

In order to make contact with the previous versions of the exact RG, in Section 4.3 we reduce our general equations to the *standard scheme* where only a single inessential coupling, namely the wave function renormalization, is specified.

Having presented the frame covariant formulation of the exact RG, in Section 4.4 we introduce the minimal essential scheme. In this scheme, all the inessential couplings are set to zero at every scale along the RG flow. Several comments are in order. Having a scheme of this type at hand provides practical advantages as well as a clearer physical picture of renormalization. On the practical side, a major improvement of the minimal essential scheme as compared to the standard one is the fact that the form of the propagator maintains a simple form along the RG flow. This ensures that the propagating degrees of freedom are just those of the corresponding free theory. Conceptually, our scheme may also lead to a better understanding of the equivalence of quantum field theories [129, 130, 131] and the universality of statistical physics models at criticality, building on the insights of previous works [35, 39, 42, 120, 132, 133, 134, 135]. Moreover, we further develop and take advantage of the analogy between frame transformations and gauge transformations [133]. Although, for the sake of simplicity, we will treat a single scalar field ϕ, the generalization to theories with other field content is obvious. As such, the scheme which we develop can be exploited in a wide range of areas of theoretical physics where the exact RG is a useful calculation tool.

As we have seen in Section 2.5, F. Wegner proved [35] that, at a fixed point of the RG, critical exponents associated with redundant operators are entirely scheme-dependent. Section 4.5 is then devoted to the discussion of the fixed-point equations and how the corresponding critical exponents can be obtained, contrasting the differences between the standard and (minimal) essential schemes. In particular, we pay attention to the identification of the anomalous dimension, whose computation presents the most substantial differences with respect to the standard case. One of the most prominent results in this section regards the fact that at a fixed point, redundant perturbations are automatically discarded. This makes essential schemes a preferred tool to access only the necessary, essential physical content.

Moving towards actual implementations of essential schemes, it is important to realize that, a priori, the EAA may contain all possible terms compatible with the symmetries of the model under consideration. However, any concrete application of the exact RG relies on approximation schemes that reduce the EAA to a manageable subset of all terms. The celebrated *derivative expansion* [48, 80] consists of approximating $\Gamma_k[\phi]$ by its Taylor expansion in gradients of ϕ. In this manner, in order to obtain approximate beta functions with a finite amount of effort, one typically has to truncate the derivative expansion to a given finite order ∂^s. At each order $s = 0, 2, 4, \dots$ one is able to compute physical quantities, providing estimates which show convergence as s is increased. To date, this program has been carried out in the standard scheme up to order $s = 6$ for the 3D Ising model [85], where furthermore it has been argued that the derivative expansion can have a finite radius of convergence. While at order $s = 0$ the EAA is projected onto the space of effective potentials $V_k(\phi)$ [136, 137], at higher orders, one obtains coupled flow equations for an increasing number of independent functions of the field [80, 84, 85, 138, 139]. Consequently, as the order increases, this program rapidly grows in complexity. The minimal essential scheme reduces this complexity order by order in the derivative expansion. In addition, while there can be spurious effects due to approximations, those arising from inessential couplings will not be present.

To demonstrate the scheme's utility, in Section 4.6 we derive the explicit form of the flow equation at order $s = 2$ of the derivative expansion and in Section 4.7 we apply it to the study of the critical point of the 3D Ising model. In particular, we shall identify the Wilson-Fisher fixed point as a globally-defined scaling solution to the exact RG equations and calculate the values of the universal critical exponents ν, ω and η. These results are obtained by solving the flow equations both functionally and with a polynomial truncation. The numerical estimates we obtained for the critical exponents are found to be in good agreement with respect to the computations performed at order ∂^2 in the standard scheme [139, 140, 141, 142]. The simplifications exemplified by this application of the minimal essential scheme at order $s = 2$ of the derivative expansion are expected at all higher orders. This is demonstrated in Section 4.8 by providing a recipe on how to implement the minimal essential scheme order by order.

We devote Sections 4.9 to a general discussion: here we advocate the possibility of employing non-minimal essential schemes in optimization problems by applying extended principle of minimal sensitivity (PMS) studies [143]. After taking the opportunity to make general considerations about redundant operators and the generalizability of essential schemes, we then discuss the implications entailed for the vertex expansion, defined in Equation (2.94). Appendix E contains a detailed derivation of the frame covariant exact renormalization group equation for the EAA. In the end of Appendix A we show some identities related to the generator of dilatations, which are important to express the exact renormalization flow equations in dimensionless variables. In Appendix F we comment on the connection between the renormalization conditions and inessential couplings for free theories including the high temperature fixed point and higher-derivative theories. Finally, in Appendix G we explicitly calculate the general flow equation at second order in derivative expansion in two different ways, i.e. in momentum space and in position space. Regarding the notation, x and p will be understood to be dimensionfull ad their corresponding dimensionless quantities are y and q respectively.

4.1 Frame transformations in quantum field theory

In this section, we will change the notation for S, $\mathcal{W}$ and Γ in order to stress the field dependence of these quantities: in particular since the notion of frame is central, subscripts will be put on these quantities.

4.1.1 Classical frame transformations

The classical dynamics of a field theory is encoded in an action $S_\chi[\chi]$. This can be considered as a scalar function on the configuration space $\mathcal{M}$ viewed as a manifold, where the points are field configurations $\chi : \mathbb{R}^d \to \mathbb{R}$. In this respect, the values of the dynamical field variable $\chi(x)$ can be considered as a preferred coordinate system for which the action takes a particular form. What distinguishes the variable χ as "the field" is that, typically, it assumes a straightforward physical significance being an easily accessible observable experimentally. From a geometrical point of view, this is equivalent to defining a particular local set of *frames* on $\mathcal{M}$. The classical dynamics is then defined by the principle that the action is stationary, namely

$$\frac{\delta S_\chi}{\delta \chi(x)} = 0\,. \tag{4.1}$$

This provides the equations of motion for the field variable χ. However, it could be the case that the equations of motion are relatively difficult to solve when written in terms of χ and can be simplified by re-expressing the action in terms of different variables $\phi = \phi[\chi]$. Provided the map $\phi[\chi]$ is invertible, such that the inverse map $\chi = \chi[\phi]$ exists, this amounts to choosing a different frame. If this is the case, we can solve the equations of motion for a new action $S_\phi[\phi]$, which is related to the action in the original frame by

$$S_\chi[\chi] = S_\phi[\phi[\chi]]\,. \tag{4.2}$$

The solutions to the two equations of motion are then in a one-to-one correspondence since invertibility ensures that the Jacobian between the two frames is non-singular. To see this correspondence, we observe that (4.1) can be written as

$$\int_{x_1} \frac{\delta \phi(x_1)}{\delta \chi(x)} \frac{\delta S_\phi[\phi]}{\delta \phi(x_1)} = 0\,, \tag{4.3}$$

and, as such, the non-singular nature of the Jacobian implies that

$$\frac{\delta S_\phi[\phi]}{\delta \phi(x)} = 0\,. \tag{4.4}$$

To calculate observables, we should evaluate them on the dynamical shell consisting of points on $\mathcal{M}$ where (4.1) is satisfied. However, one should bear in mind that observables transform as scalars on $\mathcal{M}$, and therefore, they must transform accordingly.

In general the map $\phi[\chi]$ can be nonlinear in the field χ. The imposition that $\phi[\chi]$ is invertible in the vicinity of a constant field configuration also restricts the map to be *quasi-local*. Specifically, quasi-local means that if we expand $\phi[\chi]$ in derivatives of the field, the expansion is analytic and thus we can write

$$\phi(x) \sim \sum_{s=0}^{\infty} L_s(\chi(x), \partial_\mu \chi(x), \dots)\,, \tag{4.5}$$

where $L_s = O(\partial^s)$ are local functions of the field and its derivatives at x, involving s derivatives. If the series terminates at a finite order then we have strict locality.

As an example of a frame transformation, let us consider a generic action involving up to two derivatives of the field

$$S_\chi[\chi] = \int_x \left[\frac{z_\chi(\chi)}{2} (\partial_\mu \chi)(\partial_\mu \chi) + V_\chi(\chi) \right]\,, \tag{4.6}$$

this can be re-expressed in the *canonical frame* where it depends only on a potential $V_\phi(\phi) = V_\chi(\chi(\phi))$, assuming therefore the simpler form

$$S_\phi[\phi] = \int_x \left[\frac{1}{2}(\partial_\mu\phi)(\partial_\mu\phi) + V_\phi(\phi)\right] . \tag{4.7}$$

This is achieved by the following transformation

$$\chi \to \chi(\phi) \,, \quad \frac{\partial\chi(\phi)}{\partial\phi} = \frac{1}{\sqrt{z_\chi(\chi(\phi))}} \,, \tag{4.8}$$

which is the inverse of the transformation

$$\phi \to \phi(\chi) \,, \quad \frac{\partial\phi(\chi)}{\partial\chi} = \sqrt{z_\chi(\chi)} \,. \tag{4.9}$$

Thus, provided $z_\chi(\chi)$ is non-singular, we can transform to the canonical frame where solutions to the equations of motion will be in a one-to-one correspondence. Note that the requirement on z_χ to be not singular rules out some submanifold of the theory space. Therefore, going to the canonical frame selects a particular region in theory space.[1]

More generally, actions in two different frames will transform as scalars on $\mathcal{M}$, where a change of frame is understood as a diffeomorphism from $\mathcal{M}$ to itself. Under an infinitesimal frame transformation $\phi \to \phi + \xi[\phi]$, the action transforms as

$$S[\phi] \to S[\phi] + \xi[\phi] \cdot \frac{\delta}{\delta\phi} S[\phi] \,. \tag{4.10}$$

This result coincides with Equation 2.20, where we consider the classical or tree level contribution of the Wilsonian Effective Action.

The transformation (4.10) is an infinitesimal *classical* frame transformation. It is clear that, with a bit of work, classical field theory can be formulated in a covariant language allowing one the freedom to easily pick different frames to calculate observables. This freedom is analogous to the freedom to pick a particular gauge condition in general relativity, which amounts to picking a set of local frames on spacetime. In the rest of this section, we lift the discussion on frame transformations in order to develop a frame covariant formulation of quantum field theory.

4.1.2 The principle of frame invariance in QFT

In this section, using the concept of frames we will generalized the standard notions given in Section 2.1 to state the principle of frame invariance in QFT.

[1] This aspect is analogous to the parameterization procedure that we discuss in Section 2.9.

Pointing out the frame like the classical case, we rewrite Equation (2.1), i.e. expectation values of *observables* $\hat{\mathcal{O}}$, as

$$\langle \hat{\mathcal{O}} \rangle := \mathcal{N} \int (\mathrm{d}\hat{\chi})\ \hat{\mathcal{O}}_{\hat{\chi}}[\hat{\chi}]\ \mathrm{e}^{-S_{\hat{\chi}}[\hat{\chi}]}\,, \tag{4.11}$$

where $\mathcal{N}^{-1} = \int (\mathrm{d}\hat{\chi})\, \mathrm{e}^{-S_{\hat{\chi}}[\hat{\chi}]}$ and $\hat{\mathcal{O}}_{\hat{\chi}}[\hat{\chi}] = \hat{\mathcal{O}}$ is an observable expressed as functional of the fields $\hat{\chi}$. In the same way, the generating functional $\mathcal{W}_{\hat{\chi}}[j_1]$ of the connected correlation functions for the field $\hat{\chi}$ is given by

$$\mathrm{e}^{\mathcal{W}_{\hat{\chi}}[j_1]} := \langle \mathrm{e}^{j_1 \cdot \hat{\chi}} \rangle = \mathcal{N} \int (\mathrm{d}\hat{\chi})\ \mathrm{e}^{j_1 \cdot \hat{\chi}} \mathrm{e}^{-S_{\hat{\chi}}[\hat{\chi}]}\,, \tag{4.12}$$

where $j_1 \cdot \hat{\chi}$ is a source term for the field $\hat{\chi}$. Note that we consider the current j_1 as a mathematical tool to calculate correlation functions, and therefore, there is also the possibility to couple it to different powers of the field $\hat{\chi}$. The generating functional of connected correlation functions for all the powers of $\hat{\chi}$ up to degree n is

$$\mathrm{e}^{\mathcal{W}_{\hat{\chi}}[j_1, j_2, \ldots j_n]} := \langle \mathrm{e}^{j_1 \cdot \hat{\chi} + j_2 \cdot \hat{\chi} \cdot \hat{\chi} + \ldots + j_n \underbrace{\hat{\chi} \cdot \ldots \cdot \hat{\chi}}_{n \text{ times}}} \rangle\,, \tag{4.13}$$

where j_i is a function of i position arguments. Taking the functional derivative respect to j_i and putting all the currents to zero gives the i-point function. This very simple observation tells us that the standard use of j_1 is a particular choice and we can generalize such a treatment.

Here we are interested in the further generalization of (4.12) where the source J couples instead to a composite operator $\hat{\phi} = \hat{\phi}[\hat{\chi}]$, such that we generate the correlation functions of $\hat{\phi}$ rather than those of $\hat{\chi}$. To ensure that these correlation functions contain the same physical information, we take $\hat{\phi} = \hat{\phi}[\hat{\chi}]$ to define a diffeomorphism from $\mathcal{M}$ to itself, or phrased differently, a frame transformation from the original $\hat{\chi}$-frame to a new $\hat{\phi}$-frame. Therefore, we are led to consider a family of generating functionals

$$\mathrm{e}^{\mathcal{W}_{\hat{\phi}}[J]} := \langle \mathrm{e}^{J \cdot \hat{\phi}} \rangle = \mathcal{N} \int (\mathrm{d}\hat{\chi})\ \mathrm{e}^{J \cdot \hat{\phi}[\hat{\chi}]} \mathrm{e}^{-S_{\hat{\chi}}[\hat{\chi}]}\,, \tag{4.14}$$

for the composite operator $\hat{\phi}[\hat{\chi}]$, which from now on we call the *parameterized field*. In geometrical terms, (4.14) makes sense if we understand $\hat{\phi}(x)$ as a set of scalars on $\mathcal{M}$ labelled by the points in real space x. If we were to introduce purely abstract coordinates on $\mathcal{M}$, then the gradient of $\hat{\phi}(x)$ is a coframe field while the inverse the coframe field is a frame field.

In presence of the source, expectation values are given by

$$\langle \hat{\mathcal{O}} \rangle_J = \mathrm{e}^{-\mathcal{W}_{\hat{\phi}}[J]} \langle \mathrm{e}^{J \cdot \hat{\phi}} \hat{\mathcal{O}} \rangle\,, \tag{4.15}$$

and they reduce to (4.11) by taking $J = 0$. In practice, given (4.14), source-dependent expectation values can be computed as

$$\langle\hat{\mathcal{O}}\rangle_J = \mathrm{e}^{-\mathcal{W}_{\hat{\phi}}[J]}\hat{\mathcal{O}}\left[\hat{\chi}\left[\frac{\delta}{\delta J}\right]\right]\mathrm{e}^{\mathcal{W}_{\hat{\phi}}[J]}\,, \tag{4.16}$$

where $\hat{\chi}[\hat{\phi}]$ is the inverse diffeomorphism of $\hat{\phi}$. Since the observables $\hat{\mathcal{O}}$ are scalars on $\mathcal{M}$, such that

$$\hat{\mathcal{O}} = \hat{\mathcal{O}}_{\hat{\chi}}[\hat{\chi}] = \hat{\mathcal{O}}_{\hat{\phi}}[\hat{\phi}]\,, \tag{4.17}$$

we can thus equivalently write (4.16) as

$$\langle\hat{\mathcal{O}}\rangle_J = \mathrm{e}^{-\mathcal{W}_{\hat{\phi}}[J]}\hat{\mathcal{O}}_{\hat{\phi}}\left[\frac{\delta}{\delta J}\right]\mathrm{e}^{\mathcal{W}_{\hat{\phi}}[J]}\,. \tag{4.18}$$

The source J could be viewed as a physical external field that couples linearly to $\hat{\phi}$. In this interpretation, however, we would be considering a model where $S_{\hat{\chi}}[\hat{\chi}]$ is replaced by $S_{\hat{\chi}}[\hat{\chi}] - J\cdot\hat{\phi}[\hat{\chi}]$, resulting in a physical dependence on the choice of frame. For the rest of the thesis, instead, we will adopt the *principle of frame invariance*, meaning that we will work within a frame covariant (or other words reparameterization, or field-redefinition covariant) formalism where physical quantities are independent of the choice of frame. Consequently, in this formalism all physical couplings, possibly including a coupling $h\cdot\hat{\chi}$ to an external field h, should be part of the action $S_{\hat{\chi}}$, and the source J shall be viewed merely as a device to compute correlation functions such that, after differentiating $\mathcal{W}_{\hat{\phi}}[J]$, we are ultimately interested in taking $J = 0$. Physical quantities are therefore obtained by the frame covariant expression[2]

$$\langle\hat{\mathcal{O}}\rangle = \mathrm{e}^{-\mathcal{W}[J]}\hat{\mathcal{O}}\left[\frac{\delta}{\delta J}\right]\mathrm{e}^{\mathcal{W}[J]}\Big|_{J=0}\,, \tag{4.19}$$

with the final result being a frame invariant quantity. For example the n-point functions is obtained by

$$\langle\prod_{i=1}^{n}\hat{\chi}(x_i)\rangle = \mathrm{e}^{-\mathcal{W}[J]}\prod_{i=1}^{n}\hat{\chi}\left[\frac{\delta}{\delta J(x_i)}\right]\mathrm{e}^{\mathcal{W}[J]}\Big|_{J=0}\,. \tag{4.20}$$

The advantage of working with a frame covariant setup is that the complexity of computing certain physical quantities may be reduced by the choice of a specific frame. For many quantities such as the correlation functions of the physical field $\hat{\chi}$ e.g. (4.20), the specific choice of the frame may simply be $\hat{\phi} = \hat{\chi}$. However, for universal quantities computed in the vicinity of a continuous phase transition in statistical physics, or quantities which are computed at vanishing external field, such as S-matrix elements in particle physics, it

[2] From now on we can suppress the $\hat{\phi}$ subscripts from $\mathcal{W}[J] \equiv \mathcal{W}_{\hat{\phi}}[J]$, $\hat{\mathcal{O}}[\hat{\phi}] \equiv \hat{\mathcal{O}}_{\hat{\phi}}[\hat{\phi}]$ etc. whenever we are discussing a generic frame and no confusion can arise.

may be that the specific choice of $\hat{\phi}$ is non-trivial. What is important is that in principle we can compute any observable in any frame. Then in practice we can exploit the frame where computations become most manageable.

4.1.3 Change of integration variables

In addition to the freedom of fixing a frame by choosing a particular $\hat{\phi}[\hat{\chi}]$ which couples to the source, we are also at liberty to make a change of integration variables in the corresponding functional integral (4.14), as we already discuss in Section 2.1. Under this change of variables, $\hat{\phi}[\hat{\chi}]$ transforms as a set of scalars on $\mathcal{M}$ and $\mathcal{W}_{\hat{\phi}}[J]$ is hence invariant. Of course, we are at liberty to make $\hat{\phi}$ the integration variable and therefore we can equivalently write

$$\mathrm{e}^{\mathcal{W}_{\hat{\phi}}[J]} = \mathcal{N} \int (\mathrm{d}\hat{\phi})\ \mathrm{e}^{-S_{\hat{\phi}}[\hat{\phi}]}\ \mathrm{e}^{J\cdot\hat{\phi}}\,, \tag{4.21}$$

where

$$\mathrm{e}^{-S_{\hat{\phi}}[\hat{\phi}]} = \mathrm{e}^{-S_{\hat{\chi}}[\hat{\chi}[\hat{\phi}]]} \det \frac{\delta\hat{\chi}[\hat{\phi}]}{\delta\hat{\phi}} \tag{4.22}$$

has transformed as a density. However, since these transformations leave $\mathcal{W}[J]$ invariant, it is entirely immaterial whether we perform this transformation (or any other change of integration variables) or not. Furthermore, the expectation value of an observable (i.e. what we mean by $\langle \dots \rangle$) can also be defined in a covariant way as

$$\langle\hat{\mathcal{O}}\rangle := \mathcal{N} \int (\mathrm{d}\hat{\phi})\ \hat{\mathcal{O}}_{\hat{\phi}}[\hat{\phi}]\ \mathrm{e}^{-S_{\hat{\phi}}[\hat{\phi}]}\,, \tag{4.23}$$

which is equivalent to the previous definition (4.11). However, from now on by a frame transformation, we always refer to a change in the field which couples to the source, rather than a change of integration variables.

4.1.4 Effective Actions

As we discuss in Section 2.1, given $\mathcal{W}[J]$, other generating functionals, related to $\mathcal{W}[J]$ by transformations and/or the addition of further sources, can be considered. For example, the one-particle irreducible (1PI) Effective Action $\Gamma[\phi]$ is obtained by the Legendre transform

$$\Gamma_{\hat{\phi}}[\phi] = -\mathcal{W}_{\hat{\phi}}[J] + \phi\cdot J\,, \tag{4.24}$$

where $\phi = \langle\hat{\phi}[\hat{\chi}]\rangle_J$ is the mean parameterized field. Equivalently, $\Gamma[\phi]$ can be defined by the solution to the integro-differential equation

$$\mathrm{e}^{-\Gamma_{\hat{\phi}}[\phi]} = \langle \mathrm{e}^{(\hat{\phi}-\phi)\cdot\frac{\delta}{\delta\phi}\Gamma_{\hat{\phi}}[\phi]}\rangle\,, \tag{4.25}$$

with ϕ-dependent expectation values given by

$$\langle \hat{\mathcal{O}}[\hat{\chi}] \rangle_{\phi} = \mathrm{e}^{\Gamma_{\hat{\phi}}[\phi]} \langle \mathrm{e}^{(\hat{\phi}-\phi)\cdot \frac{\delta}{\delta \phi}\Gamma_{\hat{\phi}}[\phi]} \hat{\mathcal{O}}[\hat{\chi}] \rangle \,. \tag{4.26}$$

As we have seen in Section 2.8, we will be interested in a particular class of generating functionals that generalize the 1PI Effective Action in the presence of an additional source $K(x_1, x_2)$ for two-point functions. In the next section we will identify $K(x_1, x_2)$ with a cutoff function, but for now, we view it simply as an additional source independent of ϕ. Its inclusion leads to a modified Effective Action

$$\mathrm{e}^{-\Gamma[\phi,K]} = \langle \mathrm{e}^{(\hat{\phi}-\phi)\cdot \frac{\delta}{\delta \phi}\Gamma[\phi,K] - \frac{1}{2}(\hat{\phi}-\phi)\cdot K \cdot (\hat{\phi}-\phi)} \rangle \,, \tag{4.27}$$

so that K- and ϕ-dependent expectation values can be defined by

$$\langle \hat{\mathcal{O}} \rangle_{\phi,K} = \mathrm{e}^{\Gamma[\phi,K]} \langle \mathrm{e}^{(\hat{\phi}-\phi)\cdot \frac{\delta}{\delta \phi}\Gamma[\phi,K] - \frac{1}{2}(\hat{\phi}-\phi)\cdot K \cdot (\hat{\phi}-\phi)} \hat{\mathcal{O}} \rangle \,. \tag{4.28}$$

We will also denote the expectation value of an operator $\hat{\mathcal{O}}$ by dropping the hat, such that

$$\mathcal{O}[\phi, K] \equiv \langle \hat{\mathcal{O}} \rangle_{\phi,K} \,. \tag{4.29}$$

Note that Equation (4.27) coincides with Equation (2.89) using $\hat{\phi} = \hat{\chi}$ and identifying the source K with the regulator $\mathcal{R}_k$, which is needed to implement the coarse graining.

4.1.5 Functional identities

An infinite set of identities can be derived systematically by taking successive derivatives of (4.27) and (4.28) with respect to ϕ and K and using the identities obtained from lower derivatives. Here we will obtain those identities which we will make explicit use of in the rest of this chapter. First, taking one derivative of (4.27) with respect to ϕ one finds that

$$(K + \Gamma^{(2)}[\phi, K]) \cdot (\phi - \langle \hat{\phi} \rangle_{\phi,K}) = 0 \,, \tag{4.30}$$

where $\Gamma^{(2)}[\phi, K]$ denotes the second functional derivative of $\Gamma[\phi, K]$ with respect to ϕ. Thus, assuming the invertibility of $K + \Gamma^{(2)}[\phi, K]$, one has that ϕ is again the mean parameterized field

$$\phi = \langle \hat{\phi} \rangle_{\phi,K} \,. \tag{4.31}$$

Taking a further derivative of (4.31) with respect to ϕ one finds that the two-point function is given by

$$\mathcal{G}[\phi, K](x_1, x_2) := \langle (\hat{\phi}(x_1) - \phi(x_1))(\hat{\phi}(x_2) - \phi(x_2)) \rangle_{\phi,K} = \frac{1}{\Gamma^{(2)}[\phi, K] + K}(x_1, x_2) \,. \tag{4.32}$$

Then, varying (4.27) with respect to K at fixed ϕ we obtain the functional identity [54, 55]

$$\delta \Gamma[\phi, K]\big|_{\phi} = \frac{1}{2} \mathrm{Tr}\, \mathcal{G}[\phi, K] \cdot \delta K \,. \tag{4.33}$$

Sending again $\hat{\phi} \to \hat{\chi}$ and $K \to \mathcal{R}_k$, the equation above is just another way of writing Equation (2.91). Taking a functional derivative of (4.28) with respect to ϕ and using the previously derived identities we obtain

$$\langle(\hat{\phi} - \phi)\, \hat{\mathcal{O}}\rangle_{\phi,K} = \mathcal{G}[\phi, K] \cdot \frac{\delta}{\delta\phi}\mathcal{O}[\phi, K]\,. \tag{4.34}$$

There are two special configurations of the source $K(x_1, x_2)$. First, if we take $K = 0$ then $\Gamma[\phi, 0] = \Gamma[\phi]$ is the 1PI Effective Action. If additionally $\Gamma[\phi]$ is evaluated at its stationary point ϕ_{min} the expectation values (4.28) reduce to the frame invariants (4.11). Secondly, if we take $K(x_1, x_2) = M_\Lambda(x_1, x_2)$, where M_Λ is the metric that defines the measure (2.3), then the two-point source term produces a delta function in the path integral as the continuum limit (2.4) is taken, and we have

$$\lim_{\Lambda\to\infty} \Gamma[\phi, M_\Lambda] = S[\phi]\,, \tag{4.35}$$

where $S[\phi] = S_{\hat{\phi}}[\phi]$ is given by (4.22). Furthermore, the expectation values are given by the mean-field expression

$$\lim_{\Lambda\to\infty} \langle\hat{\mathcal{O}}\rangle_{\phi,M_\Lambda} = \hat{\mathcal{O}}[\phi]\,. \tag{4.36}$$

This two limits coincides with the two limits in Equation 2.83, that make $\Gamma[\phi, K]$ a useful generating functional for the exact RG since one can realize Wilson's concept of an incomplete integration by allowing K to interpolate between the two limits.

Therefore, as we suggest in Section 2.8, the addition of K, or $\mathcal{R}_k$, is similar to the addition of the source for the field. In both case, we have a mathematical tool needed to extract the physical information.

4.1.6 Inessential couplings and active frame transformations

Although in a particular frame the microscopic action may assume a relatively simple form, e.g. $S_{\hat{\chi}}[\hat{\chi}] = \int_x \left[\frac{1}{2}(\partial_\mu\hat{\chi})(\partial_\mu\hat{\chi}) + \frac{1}{2}m^2\hat{\chi}^2 + \frac{1}{4!}\lambda\hat{\chi}^4\right]$, the generating functionals will typically be very complicated. As a consequence of this, expanding the generating functionals in a typical operator basis, we will find an infinite set of non-vanishing coupling constants g_i. Different choices of the operator basis in terms of which we expand the generating functionals, therefore, correspond to different coordinate systems on theory space. In a frame covariant formalism, we are free to make frame transformations without affecting physical observables even though the form of the generating functionals will change. Consequently, any change in the coupling constants[3] $g_i \to g_i + \delta g_i$ which is equivalent to a frame transformation gives a theory that is physically equivalent to the original theory. As we said in Section 2.2, there are directions in theory space along which all physical quantities remain unchanged. Therefore, we arrived to the same point, but with a different prospective.

[3]Here we are using δ to denote a variation with respect to the couplings keeping field variables fixed.

Again, locally in theory space, we can work in a coordinate system $\{g_i\} = \{\lambda_a, \zeta_\alpha\}$ adapted to these sub-manifolds where λ_a are the essential couplings which will appear in expressions for the physical observables (4.11) and ζ_α are the inessential couplings. It follows that changing the values of the inessential couplings $\zeta \to \zeta + \delta\zeta$ is equivalent to the change induced by a local frame transformation

$$\hat{\phi}[\hat{\chi}] \to \hat{\phi}[\hat{\chi}] - \hat{\xi}[\hat{\chi}] + O(\hat{\xi}^2)\,, \tag{4.37}$$

where $\hat{\xi}[\hat{\chi}] = \hat{\Phi}[\hat{\chi}]\ \zeta\delta\zeta$. For the generating functionals $\mathcal{W}[J]$, $\Gamma[\phi]$ and $\Gamma[\phi, K]$ one finds that they transform respectively as

$$\mathcal{W}[J] \to \ \mathcal{W}[J] - J \cdot \xi[J] + O(\xi^2)\,, \tag{4.38}$$

$$\Gamma[\phi] \to \ \Gamma[\phi] + \xi[\phi] \cdot \frac{\delta}{\delta\phi}\Gamma[\phi] + O(\xi^2)\,, \tag{4.39}$$

$$\Gamma[\phi, K] \to \ \Gamma[\phi, K] + \xi[\phi, K] \cdot \frac{\delta}{\delta\phi}\Gamma[\phi, K] \ - \mathrm{Tr}\,\mathcal{G}[\phi, K] \cdot \frac{\delta}{\delta\phi}\xi[\phi, K] \cdot K \ + O(\xi^2)\,, \tag{4.40}$$

where $\xi[J]$, $\xi[\phi]$ and $\xi[\phi, K]$ are expectation values

$$\xi[J] = \langle\hat{\xi}[\hat{\chi}]\rangle_J\,, \tag{4.41}$$

$$\xi[\phi] = \langle\hat{\xi}[\hat{\chi}]\rangle_\phi\,, \tag{4.42}$$

$$\xi[\phi, K] = \langle\hat{\xi}[\hat{\chi}]\rangle_{\phi,K}\,. \tag{4.43}$$

In Equation (4.40) the form of the term involving the trace comes from using the identity (4.34) putting $\hat{\mathcal{O}} = \hat{\xi}$.

In the case of the 1PI Effective Action $\Gamma[\phi]$ we note that (4.39) has the same form as the classical frame transformation (4.10). This means that a derivative of $\Gamma[\phi]$ with respect to an inessential coupling gives

$$\zeta\frac{\partial}{\partial\zeta}\Gamma[\phi] = \Phi[\phi] \cdot \frac{\delta}{\delta\phi}\Gamma[\phi]\,, \tag{4.44}$$

for some functional $\Phi[\phi]$. We see explicitly that the frame transformation is proportional to the equation of motion as in the classical case. This is the origin of the statement that one can use the equations of motion to calculate the running of essential couplings [39]. However, in what follows we will work with the EAA, which has the form of $\Gamma[\phi, K]$ where K is chosen to be a cutoff function. In this case, therefore, we have that

$$\zeta\frac{\partial}{\partial\zeta}\Gamma[\phi, K] = \Phi[\phi, K] \cdot \frac{\delta}{\delta\phi}\Gamma[\phi, K] \ - \mathrm{Tr}\,\mathcal{G}[\phi, K] \cdot \frac{\delta}{\delta\phi}\Phi[\phi, K] \cdot K\,. \tag{4.45}$$

As we anticipated with the tree level version in Equation (2.98), this transformation includes a loop term in addition to the tree-level term which vanishes on the equation of motion. The operator on the RHS of (4.45) is the *redundant operator* conjugate to the inessential coupling ζ. Every inessential coupling is therefore conjugate to a redundant operator which is in turn determined by some (quasi-)local field $\Phi(x)$ which characterizes the frame transformation. From a geometrical point of view, a derivative with respect to an inessential coupling can be understood as an "averaged" Lie derivative. While $\Gamma[\phi]$ is in this sense a scalar, the averaged Lie derivative of $\Gamma[\phi, K]$ is nonlinear due to the presence of K. From this point of view, (4.45) can be understood as an *active frame transformation* (or active reparameterization), where the functional form of $\Gamma[\phi, K]$ is modified leaving ϕ and K fixed. An active frame transformation is therefore equivalent to a change in the values of the inessential couplings keeping the essential couplings fixed. Different frames are therefore fully characterized by specifying values of the inessential couplings. The analogy with gauge fixing in general relativity is then clear: the frame transformations are analogous to gauge transformations while conditions that specify the inessential couplings are analogous to gauge fixing conditions.

4.1.7 Passive frame transformations

Instead of active frame transformations, we can consider *passive frame transformations*, namely those which are characterized by simply expressing $\Gamma[\phi, K]$ in terms of different variables. These will not be simply related to active frame transformations since, for a nonlinear function $\Phi[\phi] \neq \langle \Phi[\hat{\phi}] \rangle$. However, if we consider a linear frame transformation of the form

$$\hat{\phi}'' = c \cdot \hat{\phi}' \,, \tag{4.46}$$

where c is a field independent two-point function, one has that $\phi'' = c \cdot \phi'$. From this property, we have the simple identity

$$\Gamma_{\hat{\phi}'}[\phi', c^T \cdot K \cdot c] = \Gamma_{\hat{\phi}''}[c \cdot \phi', K] \,, \tag{4.47}$$

where c^T is the transpose of c. These linear passive frame transformations will help us to make contact with more standard derivations of the exact RG equation and clarify the transition from dimensionless to dimensionful variables. More generally, they expose the fact that a linear transformation of K and ϕ which keeps $\phi \cdot K \cdot \phi$ invariant is equivalent to a frame transformation.

4.2 Frame covariant flow equation

We will now write down the RG flow equations for a frame covariant EAA, which will be the generalization of Equation (2.91). These will take a generalized form which will allow us to make arbitrary frame transformations along an RG trajectory. The equations can be written both in dimensionful variables, where the cutoff scale k is made explicit or in dimensionless variables, where we work in units of k and hence all the quantities including the coordinates $y := kx$ are dimensionless. The dimensionful version (4.54), along with more general flow equations which incorporate field redefinitions along the flow, has been derived previously in [61].

4.2.1 Dimensionful covariant flow

In dimensionful variables, the frame covariant effective average action is obtained by introducing a cutoff scale k in two independent manners. Firstly, we identify $K = \mathcal{R}_k$ with an additive IR cut off $\mathcal{R}_k$ which suppresses fluctuations below momentum scales $p^2 \simeq k^2$ and vanishes in the ultraviolet (UV) for momenta $p^2 \gg k^2$. In position space the regulator is a function of the Bochner-Laplacian $\Delta = -\partial_\mu \partial_\mu$ such that we have

$$\mathcal{R}_k(x_1, x_2) = k^2\, R(\Delta/k^2)\delta(x_1, x_2) = k^2 \int_p R(p^2/k^2)\, \mathrm{e}^{\mathrm{i}p_\mu(x_1^\mu - x_2^\mu)}\,, \tag{4.48}$$

where $R(p^2/k^2)$ is the dimensionless cutoff function which vanishes in the limit $p^2/k^2 \to \infty$, while for $p^2/k^2 \to 0$ it has a non-zero limit $R(0) > 0$, ensuring the suppression of IR modes. Secondly, one allows the parameterized field $\hat{\phi}$ itself to depend on k. This leads to the following frame covariant effective average action

$$\mathrm{e}^{-\Gamma_k[\phi]} := \langle \mathrm{e}^{(\hat{\phi}_k - \phi)\cdot \frac{\delta}{\delta\phi}\Gamma_k[\phi] - \frac{1}{2}(\hat{\phi}_k - \phi)\cdot \mathcal{R}_k \cdot (\hat{\phi}_k - \phi)} \rangle\,, \tag{4.49}$$

which is the Effective Action (4.27), where the source for the two-point functions K is now specified to be given by the cutoff function $\mathcal{R}_k$ and where $\hat{\phi} = \hat{\phi}_k[\hat{\chi}]$ is the k-dependent parameterized field. Therefore an equivalent definition is

$$\Gamma_k[\phi] = \Gamma_{\hat{\phi}_k}[\phi, \mathcal{R}_k]\,, \tag{4.50}$$

where the k dependence of $\Gamma_k[\phi]$ comes from both the k dependence of the regulator $\mathcal{R}_k$ and the parameterized field $\hat{\phi}_k$. We can then define k- and ϕ-dependent expectation in the usual manner, namely

$$\langle \hat{\mathcal{O}} \rangle_{\phi,k} = \mathrm{e}^{\Gamma_k[\phi]} \langle \mathrm{e}^{(\hat{\phi}_k - \phi)\cdot \frac{\delta}{\delta\phi}\Gamma_k[\phi] - \frac{1}{2}(\hat{\phi}_k - \phi)\cdot \mathcal{R}_k \cdot (\hat{\phi}_k - \phi)} \hat{\mathcal{O}} \rangle\,, \tag{4.51}$$

such that in this case the general identity (4.31) implies

$$\phi = \langle \hat{\phi}_k \rangle_{\phi,k}\,. \tag{4.52}$$

Here we anticipate that letting the parameterized field $\hat{\phi}_k$ to be itself k-dependent, allows for the possibility of eliminating all the inessential coupling constants from the set of independent running couplings. This, in a nutshell, will be what we define later as an *essential scheme*. In this respect, we recognize that the redundant operators assume the following form

$$\zeta \frac{\partial}{\partial \zeta} \Gamma_k[\phi] = \Phi_k[\phi] \cdot \frac{\delta}{\delta\phi} \Gamma_k[\phi] - \mathrm{Tr}\, \mathcal{G}_k[\phi] \cdot \frac{\delta}{\delta\phi} \Phi_k[\phi] \cdot \mathcal{R}_k\,, \tag{4.53}$$

where $\mathcal{G}_k[\phi] = (\Gamma_k^{(2)}[\phi] + \mathcal{R}_k)^{-1}$ is the IR regularized propagator. The exact RG flow equation obeyed by the frame covariant EAA (4.49) is then given by

$$\boxed{\left(\partial_t + \Psi_k[\phi] \cdot \frac{\delta}{\delta\phi}\right) \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr}\, \mathcal{G}_k[\phi] \left(\partial_t + 2 \cdot \frac{\delta}{\delta\phi} \Psi_k[\phi]\right) \cdot \mathcal{R}_k}\,, \tag{4.54}$$

where $t := \log(k/k_0)$, with k_0 some physical reference scale, and

$$\Psi_k[\phi] := \langle \partial_t \hat{\phi}_k[\hat{\chi}] \rangle_{\phi,k} \tag{4.55}$$

is the *RG kernel* which can be a general quasi-local functional of the field ϕ. The flow equation (4.54) follows directly from using (4.33), which accounts for the k dependence of $\mathcal{R}_k$, while the remaining terms arise due to the k-dependence of $\hat{\phi}_k$, which therefore assume the form of an infinitesimal frame transformation.
Note that Equation (4.55) implements the idea of [42] discussed in Equation (2.120). In Appendix E we give a more detailed derivation of (4.54) which generalizes the derivation of the flow for the EAA presented in [54].

Now the question arises as to how $\Psi_k[\phi]$ should be determined. Evidently, we can arrive at a closed flow equation for $\Gamma_k[\phi]$ by specifying $\Psi_k[\phi]$ to be determined by $\Gamma_k[\phi]$ in some explicit manner. This is the approach pursued in other works [144, 145] in order to describe bound states through flowing bosonization and exploited in [146, 147, 148, 149] to describe hadronization in QCD. The alternative, which we shall pursue, is instead to specify renormalization conditions that constrain the form of $\Gamma_k[\phi]$ by fixing the values of the inessential couplings and solve the flow equation for the essential couplings and for parameters appearing in $\Psi_k[\phi]$ to determine the form of the frame transformation.

Let us note that, if we wish to impose a symmetry on $\Gamma[\phi]$ under some transformation of ϕ such as $\phi \to -\phi$, then one should impose that $\Psi_k[\phi]$ transforms in the same way as ϕ. This requirement grants that the RG flow preserves the symmetry of the theory. Thus, if we want that $\Gamma_k[-\phi] = \Gamma_k[\phi]$, we should then impose that $\Psi_k[-\phi] = -\Psi_k[\phi]$.

As a final comment, let us now consider the limits $k \to 0$ and $k = \Lambda \to \infty$. In the limit $k \to 0$ the regulator $R_k(x_1, x_2)$ vanishes and thus we recover the 1PI Effective Action $\Gamma_0[\phi] = \Gamma[\phi]$ where $\hat{\phi}[\hat{\chi}] = \hat{\phi}_0[\hat{\chi}]$. In the opposite limit instead, making reference to Equation (2.3), we can identify $M_\Lambda(x_1, x_2)$ by

$$\mathcal{R}_\Lambda(x_1, x_2) \sim M_\Lambda(x_1, x_2)\,. \tag{4.56}$$

Thus, $\Gamma_{k=\Lambda}[\phi] \sim S_{\hat{\phi}_\infty}[\phi]$ where $S_{\hat{\phi}_\infty}$ is given by Equation (4.22). After giving an initial condition for the flow at $k = \Lambda$, the flow equation will then evolve towards the 1PI Effective Action while transforming the frame from $\hat{\phi}_\Lambda$ to $\hat{\phi}_0$.

4.2.2 Dimensionless covariant flow

In order to uncover RG fixed points, we need to work in units of the cutoff scale k such that the RG flow, expressed in terms of dimensionless couplings $\tilde{g}_i$, obey an autonomous set of equations, as Equation (2.18). The passage to dimensionless variables can be done either by a passive frame transformation or by an active one. The active way, however, is more elegant and makes it also evident that the scale k itself is simply an inessential coupling. To this end we define

$$\mathrm{e}^{-\Gamma_t[\varphi]} = \langle \mathrm{e}^{(\hat{\varphi}_t-\varphi)\cdot\frac{\delta}{\delta\varphi}\Gamma_t[\varphi]-\frac{1}{2}(\hat{\varphi}_t-\varphi)\cdot R\cdot(\hat{\varphi}_t-\varphi)} \rangle \,, \tag{4.57}$$

where we use φ to denote the dimensionless fields and the subscript t instead of k to emphasize that there is no explicit dependence on k. In (4.57) the dimensionless regulator $R = R(\Delta)$ is understood as a function of the dimensionless Laplacian viewed as a two point function $\Delta(y_1, y_2) := -\partial^2_{y_1}\delta(y_1 - y_2)$ where y_1 and y_2 are dimensionless coordinates.

The expectation values of observables are given by

$$\langle \hat{\mathcal{O}} \rangle_{\varphi,t} = \mathrm{e}^{\Gamma_t[\varphi]} \langle \mathrm{e}^{(\hat{\varphi}_t-\varphi)\cdot\frac{\delta}{\delta\varphi}\Gamma_t[\varphi]-\frac{1}{2}(\hat{\varphi}_t-\varphi)\cdot R\cdot(\hat{\varphi}_t-\varphi)} \hat{\mathcal{O}} \rangle \,. \tag{4.58}$$

From Equation (2.28), we recall that the generator of dilatations ψ_{dil} is defined as

$$\psi_{\mathrm{dil}}(y) := -y_\mu \partial_\mu \varphi(y) - \frac{d-2}{2}\varphi(y) \,, \tag{4.59}$$

in which the first term accounts for the rescaling of the coordinates and the second accounts for the rescaling of the field. Then, we define the dimensionless RG kernel ψ_t as

$$\psi_t^{\mathrm{tot}}[\varphi] := \psi_t[\varphi] + \psi_{\mathrm{dil}}[\varphi] := \langle \partial_t \hat{\varphi}_t[\hat{\chi}] \rangle_{\varphi,t} \,, \tag{4.60}$$

where ψ_t^{tot} denotes the total dimensionless RG kernel incorporating the dilatation step of the RG transformation, given by ψ_{dil}, and the reparameterization along the RG flow, given by ψ_t. The dimensionless flow equation is given by

$$\left(\partial_t + \psi_t^{\mathrm{tot}}[\varphi] \cdot \frac{\delta}{\delta\varphi}\right)\Gamma_t[\varphi] = \mathrm{Tr}\frac{1}{\Gamma_t^{(2)}[\varphi] + R} \cdot \frac{\delta}{\delta\varphi}\psi_t^{\mathrm{tot}}[\varphi] \cdot R \,. \tag{4.61}$$

The form of (4.61) makes it clear that an RG transformation is nothing but an active frame transformation which includes a dilatation step where the conjugate inessential coupling is k itself. This is inline with the observations made in [150] that show a direct relation between the flow of EAA and the anomaly due to the breaking of scale invariance.

To arrive at a more familiar form of the trace, we notice that the following identity holds

$$\mathrm{Tr}\frac{1}{\Gamma_t^{(2)}[\varphi] + R} \cdot \frac{\delta}{\delta\varphi}\psi_{\mathrm{dil}}[\varphi] \cdot R = \frac{1}{2}\mathrm{Tr}\frac{1}{\Gamma_t^{(2)}[\varphi] + R} \cdot \dot{R} \,, \tag{4.62}$$

where

$$\dot{R}(\Delta) := 2(R(\Delta) - \Delta R'(\Delta)) = \partial_t \mathcal{R}_k|_{k=1} \,, \tag{4.63}$$

which we prove in Appendix A. Using (4.62), it is then straightforward to show that (4.61) is (4.54) recast in dimensionless variables. In particular, the passive transformation (4.46) is given by

$$\hat{\varphi}(y) = k^{-(d-2)/2}\hat{\phi}(k^{-1}y) =: (c_{\text{dil}} \cdot \hat{\phi})(y)\,, \tag{4.64}$$

and thus $c_{\text{dil}}(y, x_1) = k^{-(d-2)/2}\delta(k^{-1}y - x_1)$. The form of Equation (4.59) then results from differentiating (4.64).

Finally, let us then denote a dimensionless redundant operator by

$$\boxed{\zeta\frac{\partial}{\partial\zeta}\Gamma_t = \mathcal{T}(\Gamma_t)\Phi[\varphi] := \Phi[\varphi]\cdot\frac{\delta}{\delta\varphi}\Gamma_t[\varphi] - \text{Tr}\frac{1}{\Gamma_t^{(2)}[\varphi] + R}\frac{\delta}{\delta\varphi}\Phi[\varphi]\cdot R}\,, \tag{4.65}$$

where $\mathcal{T}(\Gamma_t)$ is understood as a Γ_t-dependent linear operator which acts on $\Phi[\varphi]$. Then the flow equation can be concisely written as

$$\boxed{-\partial_t\Gamma_t[\varphi] = \mathcal{T}(\Gamma_t)(\psi_t[\varphi] + \psi_{\text{dil}}[\varphi])}\,. \tag{4.66}$$

Note two important points

- ◇ we recover the same structure of Equation (2.37);
- ◇ the form of the previous equation makes it explicit that the RG flow is simply a frame transformation.

4.2.3 Relation to Wilsonian flows

Let us end this section by making contact with generalized flow equations for the Wilsonian Effective Action. If we relax the constraints on $\mathcal{R}_k$ such that we no longer view it as a regulator, one can obtain the flow equations for the Wilsonian Effective Action S_k by taking the limit $\mathcal{R}_k \to \infty$. In particular, replacing the $\mathcal{R}_k \to \alpha\mathcal{R}_k$ and taking $\alpha \to \infty$ while denoting $\Gamma_k[\phi] \to S_k[\phi]$, the generalized flow equation (4.54) reduces to

$$\left(\partial_t + \Psi_k[\phi]\cdot\frac{\delta}{\delta\phi}\right)S_k[\phi] = \text{Tr}\,\frac{\delta}{\delta\phi}\Psi_k[\phi]\,, \tag{4.67}$$

apart from a vacuum term which we neglect, while a redundant operator is given by

$$\zeta\frac{\partial}{\partial\zeta}S_k[\phi] = \Phi\cdot\frac{\delta}{\delta\phi}S_k[\phi] - \text{Tr}\,\frac{\delta}{\delta\phi}\Phi[\phi]\,, \tag{4.68}$$

which coincides with Equation (2.36). These are the expressions for the generalized flow equation and redundant operators first written down in [35] and analyzed in Sections 2.3 and 2.4. The reason we obtain the flow for the Wilsonian Effective Action in the limit $\mathcal{R}_k \to \infty$ is simple: this is due to the fact that the regulator term induces a delta function in the functional integral such that $\Gamma_{\hat{\phi}_k}[\phi, K] \to S_{\hat{\phi}_k}[\phi]$.

The flow equation (4.67) has been used to demonstrate scheme independence to different degrees [120, 133, 134, 135]. However, in the flow equation (4.67), one has to introduce a UV-cuff into $\Psi_k[\phi]$ in order to regularize the trace. One advantage of the flow equations (4.54) is that the regulator $\mathcal{R}_k$ is disentangled from the RG kernel $\Psi_k[\phi]$, meaning that the trace will be regularized for any $\Psi_k[\phi]$ provided $\mathcal{R}_k$ decreases fast enough in the large momentum limit. Therefore, in the EAA formulation we have two independent RG scheme ingredients to tune in order to analyze the a particular physical system.

4.3 The standard scheme

4.3.1 Wetterich-Morris flow

As an example, in this section, we focus on the simple case where one eliminates only a single inessential coupling, namely the wave function renormalization Z_k which is conjugate to the redundant operator $\mathcal{T}(\Gamma_k)\varphi$. The removal of Z_k then introduces the anomalous dimension of the field,

$$\eta_k = -\partial_t \log(Z_k)\,, \tag{4.69}$$

and it is a necessary step to uncover fixed points with a non-zero anomalous dimension. As with the transition to dimensionless variables, Z_k can be eliminated by an active frame transformation or by a passive transformation. By either method, we arrive at the Wetterich-Morris equation in the presence of a non-zero anomalous dimension [54, 55]. By the active method, this is achieved by simply setting

$$\Psi_k[\phi] = -\frac{1}{2}\eta_k \phi\,, \tag{4.70}$$

from which we can infer that

$$\hat{\phi}_k = Z_k^{1/2}\hat{\phi}_0\,, \tag{4.71}$$

where we choose to impose $Z_0 = 1$ as the boundary condition. Following the passive route instead, we begin with the EAA $\Gamma_{\hat{\phi}_0,k}[\phi_0] = \Gamma[\phi_0, Z_k\mathcal{R}_k]$ which is given explicitly by

$$\mathrm{e}^{-\Gamma_{\hat{\phi}_0,k}[\phi_0]} = \langle \mathrm{e}^{(\hat{\phi}_0-\chi_0)\cdot\frac{\delta}{\delta\phi_0}\Gamma_{\hat{\phi}_0,k}[\phi_0]+\frac{Z_k}{2}(\hat{\phi}_0-\chi_0)\cdot\mathcal{R}_k\cdot(\hat{\phi}_0-\chi_0)}\rangle\,. \tag{4.72}$$

The flow equation is now given by

$$\partial_t\Gamma_{\hat{\phi}_0,k}[\phi_0] = \frac{1}{2}\mathrm{Tr}\frac{1}{\Gamma^{(2)}_{\hat{\phi}_0,k}[\phi_0] + Z_k\mathcal{R}_k}\cdot\partial_t(Z_k\mathcal{R}_k)\,, \tag{4.73}$$

which is the standard form of the Wetterich-Morris equation, apart from making the dependence on the wave function renormalization explicit. Then we make the passive change of frames (4.46) to eliminate Z_k from the flow equation by setting $\phi_0 = Z_k^{-1/2}\phi$, where

(4.47) implies that $\Gamma_k[\phi] = \Gamma_{\hat{\phi}_0,k}[Z_k^{-1/2}\phi]$. The flow equation (4.73) can then be recast in the form

$$\left(\partial_t - \frac{1}{2}\eta_k\phi \cdot \frac{\delta}{\delta\phi}\right)\Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\,\mathcal{G}_k[\phi] \cdot (\partial_t\mathcal{R}_k - \eta_k\mathcal{R}_k)\,, \tag{4.74}$$

which is now manifestly independent of Z_k and is equal to (4.54) with Ψ_k given by (4.70). The fact that the terms proportional to η_k in (4.74) have the form of a redundant coupling then simply reflects the fact that Z_k was inessential. In dimensionless variables the flow equation (4.74) is given by (4.61) where $\psi_t = -\frac{1}{2}\eta_k\varphi$.

4.3.2 Renormalization conditions

We have arrived at the flow equation (4.74) without having specified the inessential coupling Z_k. This means that we have the freedom to impose a renormalization condition that constrains the form of $\Gamma_k[\phi]$ by fixing the value of one coupling to some fixed value. Solving the flow equation (4.74) under the chosen renormalization then determines η_k as a function of the remaining couplings. In terms of $\Gamma_{\hat{\phi}_0,k}[\phi_0]$, this is equivalent to identifying Z_k with one coupling. A typical choice is to expand the $\Gamma_{\hat{\phi}_0,k}[\phi_0]$ in fields and in derivatives and then identify Z_k with the coefficient of the term $\frac{1}{2}\int_x(\partial_\mu\phi_0)(\partial_\mu\phi_0)$. In terms of $\Gamma_k[\phi]$ this fixes the coefficient of $\int_x(\partial_\mu\phi)(\partial_\mu\phi)$ to be 1/2. However, this choice is not unique. Analogously to Equation (2.96), one can instead expand $\Gamma_k[\phi]$ only in derivatives such that

$$\Gamma_k[\phi] = \int_x\left[V_k(\phi) + \frac{1}{2}z_k(\phi)(\partial_\mu\phi)(\partial_\mu\phi)\right] + O(\partial^4)\,, \tag{4.75}$$

where $V_k(\phi)$ and $z_k(\phi)$ are functions of the field and then choose the renormalization condition

$$z_k(\bar{\phi}) = 1\,, \tag{4.76}$$

for a single constant value of the field $\phi(x) = \bar{\phi}$. The essential scheme which we present in the next sections is based on renormalization conditions that generalize (4.76).

Before arriving at this generalization, let us first scrutinize the choice (4.76) for the renormalization condition to trace the reasoning behind it. To this end we note that $z_k(\bar{\phi})$ is the inessential coupling conjugate to the redundant operator (4.65) in the case where $\Phi = \frac{1}{2}\varphi$, as it is clear from (4.74), namely

$$\frac{1}{2}\mathcal{T}(\Gamma_t)\varphi = \frac{1}{2}\varphi \cdot \frac{\delta}{\delta\varphi}\Gamma_t[\varphi] - \frac{1}{2}\mathrm{Tr}\,\mathcal{G}_t[\varphi] \cdot R\,. \tag{4.77}$$

In general, the redundant operator is a complicated functional of φ since it depends on the form of $\Gamma_t[\varphi]$. However, at the Gaussian fixed point $\Gamma_{\mathrm{GFP}} \equiv \mathcal{K}$ with

$$\mathcal{K}[\varphi] := \frac{1}{2}\int_y(\partial_\mu\varphi)(\partial_\mu\varphi)\,, \tag{4.78}$$

one has that (4.77) reduces to the free action itself

$$\frac{1}{2}\mathcal{T}(\mathcal{K})\varphi = \frac{1}{2}\int_y (\partial_\mu\varphi)(\partial_\mu\varphi) + \text{constant}\,, \tag{4.79}$$

apart from a vacuum term. The fact that $\mathcal{K}$ is invariant under shifts $\varphi(y) \to \bar{\varphi} + \varphi(y)$ then reveals why we were free to choose the renormalization point $\bar{\varphi}$. Thus any of the renormalization conditions (4.76) will fix the same inessential coupling at the Gaussian fixed point. As we elaborate on in Sections 2.8 and 2.9 and Appendix F, one can also fix inessential couplings at an alternative free fixed point by imposing an alternative renormalization condition to eliminate Z_k. This makes it clear that the renormalization condition (4.76) is intimately related to the kinematics of the Gaussian fixed point (4.78). Here we are discussing only a single inessential coupling. However, as we have anticipated in Section 2.2, in general there is an infinite number of inessential couplings and we would like to impose renormalization conditions to eliminate all of them. We may then ask whether there is a practical way to do so. In the next section, we will present the minimal essential scheme which achieves this aim.

4.4 Minimal essential scheme

Our aim in this section is to find a scheme that imposes a renormalization condition for each inessential coupling ζ_α by fixing them to some prescribed values. In order to solve the flow equations when applying multiple renormalization conditions, we allow ψ_t to depend on a set of *gamma functions* $\{\gamma_\alpha\}$, where we must include one gamma function for each renormalization condition. The gamma functions, along with the beta functions for the remaining running couplings, are then found to be functions of the remaining couplings. For example, instead of fixing $\psi_t = -\frac{1}{2}\eta_k\varphi$, as in the standard scheme where we apply a single renormalization condition, we can instead choose $\psi_t = \gamma_1(t)\varphi + \gamma_2(t)\varphi^3$ and then impose two renormalization conditions which fixes the values of two inessential couplings. Solving the flow equation under these conditions, the gamma functions will then be determined as functions of the remaining running couplings. In general, we can write

$$\psi_t[\varphi] = \sum_\alpha \gamma_\alpha(t)\Phi_\alpha[\varphi]\,, \tag{4.80}$$

where the $\{\Phi_\alpha[\varphi]\}$ are a set of linearly independent local operators, one for each renormalization condition which we impose. In essential schemes we include all possible local operators in the set $\{\Phi_\alpha[\varphi]\}$. Applying a renormalization condition for each Φ_α would then fix the value of all inessential couplings. For this purpose, we wish to find a practical set of renormalization conditions that generalize the one applied in the standard scheme. Following the logic of the last section, we therefore choose the renormalization conditions such that we fix the values of the inessential couplings at the Gaussian fixed point. Inserting $\Gamma_{\rm GFP} = \mathcal{K}$ into (4.65), the redundant operators at the Gaussian fixed point are given by

$$\mathcal{T}(\mathcal{K})\Phi_\alpha = \Phi_\alpha \cdot \Delta\varphi \, - \mathrm{Tr}\, \frac{R}{\Delta + R} \cdot \frac{\delta}{\delta\varphi}\Phi_\alpha[\varphi] \, . \tag{4.81}$$

Then, in the minimal essential scheme we write the EAA such that it depends only on the essential couplings λ by specifying the ansatz[4]

$$\Gamma_t[\varphi] = \mathcal{K} + \sum_a \lambda_a(t) e_a[\varphi] \, , \tag{4.82}$$

where $\{e_a[\varphi]\}$ are a set of operators which are linearly independent of the redundant operators (4.81) and together with the latter form a complete basis. Without loss of generality we can assume that the couplings behave as $\lambda_a(t) = \mathrm{e}^{-\theta_{\rm G} t}\lambda_a(0) + \dots$ in the vicinity of the Gaussian fixed point, in which case $e_a[\varphi]$ are the *scaling operators* at the Gaussian fixed point, $\theta_{\rm G}$ the corresponding Gaussian critical exponents and the essential couplings $\lambda_a(t)$ are the *scaling fields* [35].

The task of distinguishing the scaling operators from redundant operators at the Gaussian fixed point is made simpler by the following observation: if Φ_α is a homogeneous function of the field of degree n, then the first term in (4.81) is a homogeneous function of degree $n+1$, while the second term is a homogeneous function of degree $n-1$. It follows from this structure that if $\{e_a[\varphi]\}$ are a set of operators which are linearly independent of $\Phi_\alpha \cdot \Delta\varphi$, they will also be linearly independent of $\mathcal{T}(\mathcal{K})\Phi_\alpha$. In other words, when identifying the scaling operators at the Gaussian fixed point, we can neglect the second term in (4.81) which is understood as a loop correction. To see this clearly, let us first assume that the scaling operators $e_a[\varphi]$ are linearly independent of $\Phi_\alpha \cdot \Delta\varphi$ such that

$$\sum_\alpha c_\alpha \Phi_\alpha \cdot \Delta\varphi + \sum_a c_a e_a[\varphi] = 0 \, , \tag{4.83}$$

if and only if $c_\alpha = 0$ and $c_a = 0$. Then we can expand the redundant operator as

$$\mathcal{T}(\mathcal{K})\Phi_\alpha = \sum_\beta \tilde{\Upsilon}_{\alpha\beta}\Phi_\beta[\varphi] \cdot \Delta\varphi + \sum_a \tilde{\upsilon}_{\alpha a} e_a[\varphi] \, , \tag{4.84}$$

where $\tilde{\Upsilon}_{\alpha\beta}$ and $\tilde{\upsilon}_{\alpha a}$ are numerical coefficients. Then one can show that the eigenvalues of the matrix with components $\tilde{\Upsilon}_{\alpha\beta}$ will all be equal to one and thus $\tilde{\Upsilon}$ is an invertible matrix. To see that the eigenvalues of $\tilde{\Upsilon}$ are all equal to one, let's first consider the simple example where $\{\Phi_\alpha\} = \{\Phi_1, \Phi_2\} = \{\varphi, \varphi^3\}$ for which Υ has the form

$$\Upsilon = \begin{pmatrix} 1 & 0 \\ \tilde{\Upsilon}_{21} & 1 \end{pmatrix} , \tag{4.85}$$

[4]Here we neglect the vacuum energy term since it is independent of φ.

where Υ_{21} is in general non-zero. The zero component follows from the fact that $\mathcal{T}(\mathcal{K})\varphi$ is linear in the field and therefore involves no term of the form $\varphi^3 \cdot \Delta\varphi$. The form of the matrix $\tilde{\Upsilon}$ is preserved in the general case by working in the basis where $\{\Phi_\alpha\} = \{\Phi_{\alpha_0}, \Phi_{\alpha_1}, \dots\}$, with α_n labelling each linearly independent local operator with n powers of the field. For $n = 1$ we have $\Phi_{\alpha_1} = \{\varphi, \Delta\varphi, \dots\}$, while for $n = 2$ we have $\Phi_{\alpha_2} = \{\varphi^2, \varphi\Delta\varphi, (\partial_\mu\varphi)^2, \dots\}$, with the ellipses denoting terms involving four or more derivatives. Then the matrix Υ has the form

$$\tilde{\Upsilon} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \tilde{\Upsilon}_{21} & 1 & 0 & \cdots \\ \tilde{\Upsilon}_{31} & \tilde{\Upsilon}_{32} & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} , \tag{4.86}$$

which has all eigenvalues equal to one.

Having set the renormalization conditions at the Gaussian fixed point, we know that the couplings λ_a will be the essential couplings in the vicinity of the Gaussian fixed point. However, away from the Gaussian fixed point, the form of the redundant operators will change[5]. Expanding the redundant operators for a general action of the form (4.82) we will obtain

$$\mathcal{T}(\Gamma_t)\Phi_\alpha[\varphi] = \sum_\beta \Upsilon_{\alpha\beta}(\lambda)\Phi_\beta[\varphi] \cdot \Delta\varphi + \sum_b \upsilon_{\alpha b}(\lambda) e_b[\varphi] , \tag{4.87}$$

where $\Upsilon_{\alpha\beta}(\lambda)$ and $\upsilon_{\alpha b}(\lambda)$ are functions of the essential couplings and reduce to $\Upsilon_{\alpha\beta}(0) = \tilde{\Upsilon}_{\alpha\beta}$ and $\upsilon_{\alpha b}(0) = \tilde{\upsilon}_{\alpha b}$ at the Gaussian fixed point. At any point where $\Upsilon_{\alpha\beta}(\lambda)$ is invertible, the operators $\mathcal{T}(\Gamma_t)\Phi_\alpha[\varphi]$ and $e_b[\varphi]$ will be linearly independent. The points for which Υ is not invertible form a disconnected hyper-surface consisting of all points in the essential theory space (i.e. the space spanned by the essential couplings λ_a), where

$$\det \Upsilon(\lambda) = 0 . \tag{4.88}$$

On the hyper-surface (4.88), the flow will typically be singular. Therefore, adopting the minimal essential scheme puts a restriction on which physical theories we can have access to. However, it is intuitively clear that this restriction has a physical meaning since the theories in question are those that share the kinematics of the Gaussian fixed point. In fact, a remarkable consequence of the minimal essential scheme is that the propagator evaluated at any constant value of the parameterized field $\varphi(x) = \bar{\varphi}$ will be given by

$$\mathcal{G}_t[\bar{\varphi}] = \frac{1}{q^2 + v_t^{(2)}(\bar{\varphi}) + R(q^2)} , \tag{4.89}$$

where $v_t^{(2)}(\bar{\varphi})$ is the second derivative of a dimensionless potential. This simple form follows since by integration by parts $\int_x (\varphi - \bar{\varphi})\Delta^{s/2}(\varphi - \bar{\varphi}) = \int_x \varphi\Delta^{s/2}\varphi$ for even integers $s \geq 2$.

[5] As we already said, the RG scheme can exclude regions of the theory space.

Let us hasten to point out that this does not imply that the propagator for the physical field $\hat{\chi}$ is of this form, but only that the propagator can be brought into this form by a frame transformation. In particular, the form (4.89) does not exclude the possibility that $\hat{\chi}$ develops an anomalous dimension η, namely that the connected two-point function of $\hat{\chi}$ scales as $\sim p^{-2+\eta}$.

4.5 Fixed points

As we have seen in Section 2.5, in the vicinity of fixed points one can obtain universal scaling exponents which are independent of the renormalization conditions which define different schemes. However, there are also critical exponents associated with redundant operators which are entirely scheme dependent. In this section we will show the contrast features of essential schemes with those of the standard scheme in these respects.

4.5.1 Fixed points and scaling exponents

Fixed points of the exact RG are uncovered by looking at t-independent solutions of (4.61) such that the fixed point action $\Gamma_\star$ obeys

$$\left(\psi_\star^{\text{tot}}[\varphi] \cdot \frac{\delta}{\delta\varphi}\right) \Gamma_\star[\varphi] = \text{Tr} \frac{1}{\Gamma_\star^{(2)}[\varphi] + R} \cdot \frac{\delta}{\delta\varphi} \psi_\star^{\text{tot}}[\varphi] \cdot R \,, \tag{4.90}$$

which in general defines a relationship between $\psi_\star$ and $\Gamma_\star$.

The critical exponents associated with the fixed point are then found by perturbing the fixed point solution $\Gamma_\star$ by adding a small perturbation $\delta\Gamma_t = \Gamma_t - \Gamma_\star$ and similarly perturbing $\psi_\star$ by

$$\delta\psi_t = \left.\frac{\delta\psi_t}{\delta\Gamma_t}\right|_{\Gamma_t = \Gamma_*} \delta\Gamma_t \,, \tag{4.91}$$

and studying the linearized flow equation for $\delta\Gamma_t$ which is given by

$$-\partial_t \delta\Gamma_t = \left(\frac{\delta\mathcal{T}(\Gamma_\star)}{\delta\Gamma_t} \psi_\star^{\text{tot}}\right) \delta\Gamma_t + \mathcal{T}(\Gamma_\star)\delta\psi_t \,. \tag{4.92}$$

The critical exponents θ are then defined by looking for eigenperturbations which are of the form

$$\delta\Gamma_t = \epsilon\, \mathrm{e}^{-t\theta} \mathcal{O}[\varphi] \,, \quad \delta\psi_t = \epsilon\, \mathrm{e}^{-t\theta} \Omega[\varphi] \,, \tag{4.93}$$

where $\mathcal{O}[\varphi]$ and $\Omega[\varphi]$ are t-independent. Depending on the sign of θ, one refers to the operator $\mathcal{O}[\varphi]$ as *relevant* ($\theta > 0$), *irrelevant* ($\theta < 0$) or *marginal* ($\theta = 0$). We note that the functional form of $\mathcal{O}[\varphi]$ will depend on the frame and hence on the scheme. Physically, we know however that they must be the expectation value of the same observable $\hat{\mathcal{O}}$. Wegner [35] has shown that eigenperturbations fall into two classes: redundant eigenperturbations where $\mathcal{O}[\varphi]$ is a redundant operator, and therefore multiplied by an inessential coupling, and scaling operators which are linearly independent of the former (i.e. the analogue of $e_a[\varphi]$). It is possible to perform the analogous demonstration of Wegner about redundant and scaling operators [151] , discussed in Section 2.5, applied to Equation (4.90). At the Gaussian fixed point, the redundant operators are some linear combination of the redundant operators (4.81). More generally, the redundant operators at any fixed point, which have the form [6]

$$\mathcal{O}_\Phi[\varphi] = \mathcal{T}(\Gamma_\star)\Phi[\varphi] \,, \tag{4.94}$$

have critical exponents θ which are entirely scheme dependent.
Redundant eigenperturbations carry no physics and should be disregarded. Conversely, the scaling operators have scheme independent universal scaling exponents and are physical perturbations of the fixed point.

In the standard scheme, one removes only a single inessential coupling and thus one will have an infinite number of redundant eigenperturbations which must be disregarded. In essential schemes instead, all inessential couplings are removed and thus we automatically disregard all redundant eigenperturbations.

4.5.2 The redundant perturbation due to shifts

Actually, there remains one redundant operator which is not automatically disregarded in the minimal essential scheme, namely the one for which $\Phi[\varphi] = 1$. The reason for this is that the Gaussian action is invariant under constant shifts of the field $\varphi \to \varphi + \text{constant}$. Happily, this redundant operator can be treated exactly and hence it is nonetheless simple to disregard it. In fact, it is straightforward to show that $\mathcal{O}_{\text{shift}}[\varphi] := \mathcal{O}_{\Phi=1}[\varphi]$ is always an eigenperturbation independently of the scheme, where

$$\mathcal{O}_{\text{shift}}[\varphi] = 1 \cdot \frac{\delta}{\delta\varphi}\Gamma_\star[\varphi] \,, \tag{4.95a}$$

$$\Omega_{\text{shift}}[\varphi] = 1 \cdot \frac{\delta}{\delta\varphi}\psi_\star[\varphi] + \theta - \frac{d-2}{2} \quad . \tag{4.95b}$$

To see that this will always be an eigenoperator, we can replace the field in the fixed point equation by $\varphi \to \varphi + \epsilon$ and expand to first order in ϵ. This gives an identity obeyed by the fixed point action from which the solution (4.95) to the linearized flow follows immediately. In the standard scheme where $\psi_t[\varphi] = -\eta_k \frac{1}{2}\varphi$ it follows directly from (4.95b) that $\theta = \frac{d-2+\eta_\star}{2}$. In the minimal essential scheme, in order to fully determine $\psi_t[\varphi]$, we can impose that

$$\psi_t^{(1)}[0] = 0 \,, \tag{4.96}$$

[6] Analogously to Equations (2.55) and (2.60).

and then determine θ by setting $\varphi = 0$ in (4.95b). In general, one then obtains

$$\theta = -1 \cdot \frac{\delta}{\delta\varphi}\psi_\star[\varphi] + \frac{d-2}{2}\bigg|_{\varphi=0} . \tag{4.97}$$

However (4.96) is only one choice and it is clear that by imposing a different condition, θ can take any value.

4.5.3 The anomalous dimension

Let us now discuss a scaling operator associated with the anomalous dimension. In the standard scheme, one introduces the parameter η_k via the choice of the RG kernel. At a fixed point $\eta_k = \eta_\star = \eta$ is the anomalous dimension where we use η to represent the universal critical exponent rather than $\eta_\star$ which is a parameter introduced in the RG kernel *only* in the standard scheme. The fact that $\eta = \eta_\star$ is the value of the universal exponent comes about because in the standard scheme there is a scaling relation between $\eta_\star$ and the scaling exponent for the operator $\mathcal{O} = \int_x \varphi$. To see this, we note that given a solution $\Gamma_k[\phi]$ to the flow equation (4.74), the EAA defined as $\Gamma_k[\phi] + Z_k^{-1/2} \int_x h\phi$ is still a solution to (4.74), provided h is independent of k and ϕ. It is then evident that h is nothing but a physical external field that couples to $\hat{\chi}$ in the microscopic action. At a fixed point, this means that there is always an eigenperturbation of this form. In dimensionless variables, the eigenperturbation is given by

$$\delta\Gamma_t = \epsilon\, \mathrm{e}^{-t\frac{d+2-\eta_\star}{2}} \int_y \varphi\,, \tag{4.98}$$

and thus we see there is a scaling exponent given by $\theta = \frac{d+2-\eta_\star}{2}$. Thus, along with the other scaling exponents, $\theta = \frac{d+2-\eta_\star}{2}$ will be a universal quantity. However, the simple form $\mathcal{O}[\varphi] = \int_x \varphi$ originates from the simple linear relation between $\hat{\phi}$ and $\hat{\chi}$ typical of the standard scheme and from the fact that in any frame a physical source must couple to one and the same field $\hat{\chi}[\hat{\phi}]$. In a general scheme, the relation between $\hat{\phi}$ and $\hat{\chi}$ will be nonlinear and hence to compute η we must instead look for an eigenperturbation of the form

$$\delta\Gamma_t = \epsilon \int_y \langle c_{\text{dil}} \cdot \hat{\chi} \rangle_{\varphi,t} \equiv \epsilon\, \mathrm{e}^{-t\frac{d+2-\eta}{2}} \int_y \chi[\varphi]\,, \tag{4.99}$$

where $\chi[\varphi] = \varphi$ only in the frame associated with the standard scheme. If we impose a symmetry on the fixed point action under $\varphi \to -\varphi$ then we will have that $\chi[-\varphi] = -\chi[\varphi]$. Apart from this characteristic, there is nothing that distinguishes $\frac{d+2-\eta}{2}$ from any other scaling exponent. Thus to compute η we must look at odd eigenperturbations of an even fixed point action. A related point, that has been recognized in [152], is that while η_k approaches the particular value η at a fixed point, independently of the renormalization condition, this is not true of the gamma functions appearing in ψ_t whenever ψ_t is nonlinear.

4.6 The minimal essential scheme at order ∂^2

We will now derive the flow equation in the minimal essential scheme at order ∂^2 in the derivative expansion. This is achieved by expanding the action as in (4.75) and neglecting the higher derivative terms. However, in the minimal essential scheme the renormalization condition (4.76) is generalized such that

$$z_k(\phi) = 1\,, \tag{4.100}$$

for *all* values of the field and all scales k. Thus, we go from fixing a single coupling in the standard scheme to fixing a whole function of the field in the essential one. To close the flow equations under this renormalization condition, we set the RG kernel to

$$\Psi_k[\phi] = F_k(\phi(x))\,, \tag{4.101}$$

where $F_k(\phi(x))$ is a function of the fields (without derivatives) constrained such that we can solve the flow equation under the renormalization condition (4.100). Therefore, working at order ∂^2 the ansatz for the EAA is simply given by

$$\Gamma_k[\phi] = \int_x \left[V_k(\phi) + \frac{1}{2}(\partial_\mu\phi)(\partial_\mu\phi)\right]\,. \tag{4.102}$$

Inserting (4.102) and (4.101) into (4.54) the LHS is given by

$$\partial_t\Gamma_k[\phi] + \int_x \frac{\delta\Gamma_k[\varphi]}{\delta\phi(x)}F_k(\phi(x)) = \int_x \left[\partial_t V_k(\phi) + F_k(\phi)V_k^{(1)}(\phi) + F_k^{(1)}(\phi)\,(\partial_\mu\phi)\,(\partial_\mu\phi)\right]\,, \tag{4.103}$$

where the super-script (n) on functions of the field denotes their n-th derivative. These terms depend on $F_k(\phi)$ and thus, instead of solving for $\partial_t V_k(\phi)$ and $\partial_t z_k(\phi)$, we will instead solve for $\partial_t V_k(\phi)$ and $F_k(\phi)$. To find the equations for $\partial_t V_k$ and F_k, in Appendix G we expand the trace on the RHS of the flow equation (4.54) with the action given by (4.102) and field renormalization (4.101) up to order ∂^2. The result is given by

$$\partial_t V_k = -F_k\,V_k^{(1)} + \frac{1}{2(4\pi)^{d/2}}Q_{d/2}\left[G_k\left(\partial_t\mathcal{R}_k + 2F_k^{(1)}\mathcal{R}_k\right)\right]\,, \tag{4.104a}$$

$$\begin{aligned} F_k^{(1)} = {} & \frac{\left(V_k^{(3)}\right)^2}{2(4\pi)^{d/2}}\left(Q_{d/2}\left[G_k^2 G_k'\left(\partial_t\mathcal{R}_k + 2F_k^{(1)}\mathcal{R}_k\right)\right] + Q_{d/2+1}\left[G_k^2 G_k''\left(\partial_t\mathcal{R}_k + 2F_k^{(1)}\mathcal{R}_k\right)\right]\right) \\ & - \frac{V_k^{(3)}F_k^{(2)}}{(4\pi)^{d/2}}\left(Q_{d/2}\left[G_k G_k'\mathcal{R}_k\right] + Q_{d/2+1}\left[G_k G_k''\mathcal{R}_k\right]\right)\,, \end{aligned} \tag{4.104b}$$

where we introduced the following quantities

$$P_k(z) = z + \mathcal{R}_k(z) \ , \tag{4.105}$$

$$G_k = \left(P_k + V_k^{(2)}\right)^{-1} \ , \tag{4.106}$$

$$Q_n[W] = \frac{1}{\Gamma(n)} \int_0^\infty \mathrm{d}z \, z^{n-1} \, W(z) \ , \tag{4.107}$$

and the primes on G_k indicate derivatives with respect to the momentum squared z.

4.7 Wilson-Fisher Fixed point

Let us now exemplify the minimal essential scheme at order ∂^2 by studying the 3D Ising model in the vicinity of the Wilson-Fisher fixed point.

4.7.1 Flow equations in $d = 3$

To this end, we specialize the study of Equations (4.104) to the case $d = 3$. In the following, we make use of the cutoff function [76]

$$\mathcal{R}_k(z) = (k^2 - z)\Theta(k^2 - z) \, , \tag{4.108}$$

where $\Theta(k^2 - z)$ is the Heaviside theta function. This choice of the cutoff function leads to a particularly simple closed form of Equations (4.104). Being interested in critical scaling solutions of the RG flow, we make the transition to dimensionless variables such that the dimensionless field is given by $\varphi = k^{-\frac{1}{2}}\phi$ and the dimensionless functions are defined by $v = k^{-3}V$ and $f = k^{-\frac{1}{2}}F$. The equations (4.104) then read

$$\partial_t v_t(\varphi) + 3v_t(\varphi) - \frac{1}{2}\left[\varphi - 2f_t(\varphi)\right] v_t^{(1)}(\varphi) = b\, \frac{1 + \frac{2}{5} f_t^{(1)}(\varphi)}{1 + v_t^{(2)}(\varphi)} \, , \tag{4.109a}$$

$$- f_t^{(1)}(\varphi) = \frac{b}{2} \frac{\left[v_t^{(3)}(\varphi)\right]^2}{\left[1 + v_t^{(2)}(\varphi)\right]^4} \, . \tag{4.109b}$$

The constant b takes the value $b = 1/(6\pi^2)$, but we note that b can also be set to any positive real value $b \to \kappa^2 b$ since this is equivalent to performing the redefinitions $v_t(\varphi) \to v_t(\kappa\varphi)/\kappa^2$, $f_t(\varphi) \to f_t(\kappa\varphi)/\kappa$ and then rescaling the field by $\varphi \to \varphi/\kappa$. Choosing b to take other values can be useful for numerical purposes, while all our results are presented for $b = 1/(6\pi^2)$. Let us stress at this point that equations (4.109) have a simpler form as compared to the analogous equations [75] in the standard scheme using (4.108). In particular, in the minimal essential scheme, the Q-functionals (4.107) are simple rational functions of $v^{(2)}$ and $v^{(3)}$, whereas in the standard scheme they involve transcendental functions.

4.7.2 Scaling solutions

In the minimal essential scheme, scaling solutions are given by k-independent solutions $v(\varphi)$ and $f(\varphi)$ to Equations (4.109), which therefore solve the following system of ordinary differential equations

$$3v(\varphi) - \frac{1}{2}\varphi\, v^{(1)}(\varphi) + f(\varphi)v^{(1)}(\varphi) = b\,\frac{1+\frac{2}{5}f^{(1)}(\varphi)}{1+v^{(2)}(\varphi)}\,, \tag{4.110a}$$

$$-\, f^{(1)}(\varphi) = \frac{b}{2}\frac{\left[v^{(3)}(\varphi)\right]^2}{\left[1+v^{(2)}(\varphi)\right]^4}\,. \tag{4.110b}$$

We notice that differentiating the first equation with respect to φ, yields an equation for $v^{(3)}$ which is expressed in terms of lower derivatives of v and f. Once this expression for $v^{(3)}$ is substituted into the second equation, the system reduces to a second-order differential one. The so-obtained equation for f turns out to be quadratic in $f^{(2)}$. Solving algebraically for $f^{(2)}$ we therefore have two roots. We thus conclude that any solution of (4.110) can be characterized by a set of four initial conditions along with the choice of one of the roots.

We are interested in globally-defined solutions $v(\varphi) = v_\star(\varphi)$ and $f(\varphi) = f_\star(\varphi)$ to (4.110) which are well-defined for all values of $\varphi \in \mathbb{R}$. These solutions correspond to fixed points of the RG. Furthermore the $\mathbb{Z}_2$ symmetry of the Ising model demands that $v_\star(\varphi)$ and $f_\star(\varphi)$ should be even and odd functions respectively. Looking at the behaviour of any putative fixed-point solution in the large-field limit one realizes that if a globally-defined solution exists, then for $\varphi \to \pm\infty$ it must behave as

$$v(\varphi) = \mathrm{A_V}\,\varphi^6 + O(\varphi^5)\,, \tag{4.111}$$

$$f(\varphi) = \pm\mathrm{A_F} \, + O(\varphi^{-9})\,, \tag{4.112}$$

with all the higher-order terms being determined as functions of $\mathrm{A_V}$ and $\mathrm{A_F}$. The previous equations represents the next order respect to the LPA expansion of the potential in Equation (2.110). On the other hand, to ensure the correct parity of the corresponding scaling solution, one finds that, by studying the equations (4.110), it is necessary and sufficient to impose the conditions[7]

$$\{v^{(1)}(0) = 0,\ f^{(1)}(0) = 0\}\,, \tag{4.113}$$

which are obtained by expanding (4.110) around $\varphi = 0$. In particular, we notice that (4.113) and (4.110) imply that $f(0) = 0$. Thus, the expansion at infinity gives us two free parameters which must be chosen such that at $\varphi = 0$ the conditions (4.113) are met. We expect at most a countable number of acceptable fixed point solutions to Equations (4.110). As expected we have found only two, namely the Gaussian and the Wilson-Fisher fixed points.

[7]Equivalently, the conditions $\{f(0) = 0,\ f^{(1)}(0) = 0\}$ imply that $v^{(1)}(0) = 0$.

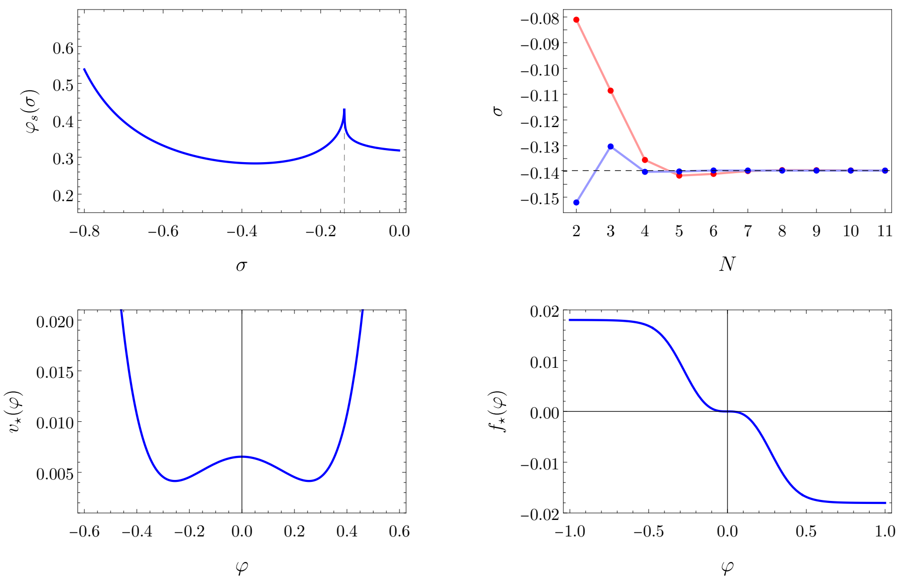


Figure 4.1: In the top-left panel, we show the singular values $\varphi_s(\sigma)$ as a function of σ. The spike located at $\sigma_\star = -0.13967$ represents the Wilson-Fisher universality class. The value of $\sigma_\star = v_\star^{(2)}(0)$ obtained from the expansion around $\rho = 0$ (red) and the expansion around the minimum $\bar{\rho}_\star$ (blue) as a function of the truncation order N is showed in the top-right panel where the dashed line represents the corresponding functional value obtained from the spike-plot. The globally-defined fixed-point effective potential $v_\star(\varphi)$ and RG kernel $f_\star(\varphi)$ corresponding to the Wilson-Fisher fixed point solution are given in the bottom panels respectively.

In order to show this result, we can numerically solve the equations (4.110) for different initial conditions at $\varphi = 0$. This is convenient since, by imposing (4.113), we are left with only one boundary condition which we can take to be the dimensionless mass squared $\sigma := v^{(2)}(0)$. In addition to σ we also have to choose the root for $f^{(2)}$. The two roots can be distinguished by noticing that in the limit $\sigma \to 0$, one root displays the Gaussian fixed point while the other does not. By setting the initial conditions at $\varphi = 0$ we are therefore left with two one-parameter families of solutions.

As the above reasoning dictates, one immediately realizes that only a countable number of solutions exist globally for all values of $\varphi \in \mathbb{R}$. Generic solutions which start at $\varphi = 0$ end at a singularity located at a finite value of the field $\varphi = \varphi_s(\sigma)$, as we discussed in Section 2.8.1. We can therefore plot the function $\varphi_s(\sigma)$ to find those values $\sigma_\star$ for which $\varphi_s(\sigma)$ diverges: these are the values for which the corresponding solution of Equations (4.110) are globally defined. In Figure 4.1 (first panel) we show the result of this search for well-defined scaling solutions selecting the root which possesses the Gaussian fixed point

and scanning σ in the range $-1 < \sigma < 0$.
The Wilson-Fisher fixed point solution is found at

$$\sigma_\star = -0.13967\,. \tag{4.114}$$

In passing, we observe that the family of solutions which include the Gaussian fixed point also displays Wilson-Fisher fixed point, while we have detected no spike in the other family.

In order to corroborate the spike-plot analysis, we search for scaling solutions by expanding $v_\star(\varphi)$ and $f_\star(\varphi)$ in powers of the fields up to a finite order N. For this purpose it is convenient to re-express $v_\star$ and $f_\star$ in terms of the manifest $\mathbb{Z}_2$ invariant $\rho(\varphi) \equiv \frac{1}{2}\varphi^2$. Expanding around $\rho = 0$ to order N we can write v and f as

$$v_\star(\varphi) = \sum_{n=0}^{N} \lambda^\star_{2n}\rho^n\,, \tag{4.115a}$$

$$f_\star(\varphi) = \varphi \sum_{n=1}^{N-1} \gamma^\star_{2n+1}\rho^n\,, \tag{4.115b}$$

(such that $v_\star(\varphi)$ is even and $f_\star(\varphi)$ is odd), while expanding around the minimum $\bar\rho_\star = \frac{1}{2}\varphi^2_{\min\star}$ of the fixed-point potential, our truncations are given by

$$v_\star(\varphi) = \bar\lambda^\star_0 + \sum_{n=2}^{N} \bar\lambda^\star_{2n}\left(\rho - \bar\rho^\star\right)^n\,, \tag{4.116a}$$

$$f_\star(\varphi) = \varphi \sum_{n=0}^{N-1} \bar\gamma^\star_{2n+1}\left(\rho - \bar\rho^\star\right)^n\,. \tag{4.116b}$$

The Equations (4.110), expanded in ρ around $\rho = 0$ ($\rho = \bar\rho_\star$) reduce to algebraic equations for the couplings $\lambda_{2n\star}$ ($\bar\lambda_{2n\star}$ and $\bar\rho_\star$) and the fixed point values $\gamma_{2n\star}$ ($\bar\gamma$). Solving these algebraic solutions we find approximate scaling solutions at each order N which converge, as N is increased, to the corresponding scaling solution we obtained numerically from the spike-plot. In particular, the values of $\sigma_\star = v^{(2)}_\star(0)$ found at each order N in the two expansions is plotted in Figure 4.1 (second panel) and are seen to converge to the functional value (4.114). We thus conclude that the approximate solutions at order N converge to the globally-defined numerical solutions as $N \to \infty$.

We close this section by a remark: in the spike-plot approach, the task of integrating the scaling equations to find a globally defined solution involves fine tuning σ. In practice, to obtain the global functions $v_\star(\varphi)$ and $f_\star(\varphi)$, we have taken advantage of the asymptotic solutions (4.111) and (4.112) and of the expansion around the minimum (4.116). Specifically, in order to determine values for $\mathrm{A_F}$ and $\mathrm{A_V}$ we can match the $v(\varphi)$ and $\frac{\partial v(\varphi)}{\partial\rho}$ for values of the field where the expansion around the minimum and the large field one overlap. This determines

$$\mathrm{A_V} \approx 1.35\,, \tag{4.117}$$
$$\mathrm{A_F} \approx -0.018\,. \tag{4.118}$$

Although the expansions of $f(\varphi)$ do not perfectly overlap, a suitable Padé approximant to the large field expansion eventually matches the expansion around the minimum. In the future we plan to perform the functional treatment using the shooting from infinity method discussed in Section 2.8.1. The corresponding globally-defined functions $v_\star(\varphi)$ and $f_\star(\varphi)$ at the Wilson-Fisher fixed point are plotted in the third and fourth panels of Figure 4.1. An in-depth analysis of global fixed points and their relation to local expansions has been given in [153, 154].

4.7.3 Eigenperturbations

To obtain the critical exponents for the Wilson-Fisher fixed point we solve the flow equations (4.109) in the vicinity of the scaling solution. Functionally, perturbations of the scaling solution

$$\delta v_t(\varphi) = v_t(\varphi) - v_\star(\varphi)\,, \qquad\qquad \delta f_t(\varphi) = f_t(\varphi) - f_\star(\varphi) \tag{4.119}$$

obey the linearized flow equation

$$\begin{aligned}\partial_t \delta v_t(\varphi) = {}& \frac{1}{2}\left[\varphi - 2f_\star(\varphi)\right]\delta v_t^{(1)}(\varphi) - 3\delta v_t(\varphi)\\ & - v_\star^{(1)}(\varphi)\delta f_t(\varphi) + \frac{2b\,\delta f_t^{(1)}(\varphi)}{5\left[1+v_\star^{(2)}(\varphi)\right]} - \frac{b\left[5+2f_\star^{(1)}(\varphi)\right]\delta v_t^{(2)}(\varphi)}{5\left[1+v_\star^{(2)}(\varphi)\right]^2}\,, \end{aligned} \tag{4.120a}$$

$$-\delta f_t^{(1)}(\varphi) = \frac{b\,v_\star^{(3)}(\varphi)\,\delta v_t^{(3)}(\varphi)}{\left[1+v_\star^{(2)}(\varphi)\right]^4} - \frac{2b\left[v_\star^{(3)}(\varphi)\right]^2\delta v_t^{(2)}(\varphi)}{\left[1+v_\star^{(2)}(\varphi)\right]^5}\,. \tag{4.120b}$$

Similarly to the fixed point equations (4.110), these can be converted into second order differential equations. We note that, since $v_\star(\varphi)$ is an even function, and $f_\star(\varphi)$ is an odd function, one can consider even and odd perturbations $\delta v_t(\varphi)$ separately. In order to find the spectrum of scaling exponents θ_n we can express a general perturbation as a sum of its eigenperturbations[8]

$$\delta v_t(\varphi) = \sum_n C_n \mathrm{e}^{-\theta_n t} \mathcal{O}_n(\varphi) , \tag{4.121a}$$

$$\delta f_t(\varphi) = \sum_n C_n \mathrm{e}^{-\theta_n t} \Omega_n(\varphi) , \tag{4.121b}$$

where C_n are undetermined constants that parameterize the perturbations of the fixed point and n runs over the spectrum of eigenperturbations. For each n the functions Ψ_n and Ω_n obey a pair of coupled second order differential equations which depend on θ_n. The sum is justified by the fact that the spectrum θ_n is quantized. To show this, first we consider the large field limit $\varphi \to \infty$ where we determine that

$$\mathcal{O}_n = A_n \varphi^{6-2\theta_n} + 6\left(\theta_n - \frac{1}{2}\right)^{-1} \mathrm{A_V} B_n \varphi^5 \dots \;, \tag{4.122}$$

$$\Omega_n = B_n + \dots \tag{4.123}$$

up to subleading terms. This introduces two parameters A_n and B_n for each eigenperturbation. Considering the behaviour around $\varphi = 0$, for even and odd perturbations we have that $\mathcal{O}_n^{(1)}(0) = 0$ and $\mathcal{O}_n(0) = 0$ respectively. Furthermore the linearity of the equations allows us to normalize even and odd perturbations by $\mathcal{O}_n(0) = 1$ and $\mathcal{O}_n^{(1)}(0) = 1$. Imposing that the RG kernel vanishes at vanishing field (4.96) then enforces that $\Omega_n(0) = 0$ for either parity. On the other hand $\Omega_n^{(1)}(0) = 0$ follows automatically from (4.120b) since $v_\star(\varphi)$ is even (and hence $v_\star^{(3)}(0) = 0$). Therefore we need to satisfy three independent boundary conditions at $\varphi = 0$ to ensure the correct parity, while we only have two free parameters A_n and B_n. As a result, the allowed values of θ_n must be quantized to satisfy all three boundary conditions.

4.7.4 Scaling exponents

In order to compute the scaling exponents ν and ω we look at even eigenperturbations. Here we shall use t-dependent generalizations of the expansions (4.115) and (4.116) to compute the exponents at order N in both expansions. The couplings λ_{2n}, $\bar{\lambda}_{2n}$ and $\bar{\rho}$ are now k-dependent with beta functions

$$\partial_t \lambda_{2n} = \beta_{2n}(\lambda) , \tag{4.124a}$$

$$\partial_t \bar{\lambda}_{2n} = \bar{\beta}_{2n}(\bar{\lambda}, \bar{\rho}) , \tag{4.124b}$$

$$\partial_t \bar{\rho} = \beta_{\bar{\rho}}(\bar{\lambda}, \bar{\rho}) , \tag{4.124c}$$

[8] This is a slight abuse of notation since earlier we denoted eigenperturbations of the fixed point action as $\mathcal{O}$ while $\mathcal{O}_n$ are perturbations of the fixed point potential.

and similarly $\gamma_{2n} = \gamma_{2n}(\lambda)$ and $\bar{\gamma}_{2n} = \bar{\gamma}_{2n}(\bar{\lambda}, \bar{\rho})$ are also determined as functions of the couplings. The critical exponents obtained from the expansion around $\varphi = 0$ are obtained from eigenvalues of the stability matrix

$$M^{\text{even}}_{nm} = \left.\frac{\partial \beta_{2n}}{\partial \lambda_{2m}}\right|_{\lambda=\lambda^\star}, \tag{4.125}$$

where $\lambda_\star$ denotes the values of the couplings at the Wilson-Fisher fixed point. Similarly, by defining $\bar{\lambda}_2 := \bar{\rho}$ and $\bar{\beta}_2 := \beta_{\bar{\rho}}$, the stability matrix for the expansion around the minimum is defined by

$$\bar{M}^{\text{even}}_{nm} = \left.\frac{\partial \bar{\beta}_{2n}}{\partial \bar{\lambda}_{2m}}\right|_{\bar{\lambda}=\bar{\lambda}^\star}. \tag{4.126}$$

The critical exponents are equal to minus the eigenvalues of the stability matrix. In particular, the critical exponent $-1/\nu$ is identified with the sole relevant eigenvalue (ignoring the vacuum energy), which has a negative real part, while the correction-to-scaling exponent ω is identified with the irrelevant eigenvalue with the smallest positive real part. The values of these exponents at different orders N up to $N = 11$ are shown in Figure 4.2 (second and third panels). We observe that the critical exponents converge towards as the order N is increased and in general the expansion around the minimum converges faster with respect to the one around zero. At order $N = 11$ in the expansion around the minimum we find that

$$\nu = 0.6271\,, \tag{4.127}$$

$$\omega = 0.8350\,. \tag{4.128}$$

In order to compute the scaling exponent η we look at odd perturbations $\delta v_t(\varphi)$ and even perturbations $\delta f_t(\varphi)$. This introduces a set of beta functions for couplings that multiply odd functions of the field and which, though vanishing at the Wilson-Fisher fixed point, exhibit non-zero scaling exponents. These exponents have been computed in using the exact RG in [155].

These odd perturbations also include the redundant perturbation due to shifts (4.95). Imposing (4.96), which implies $\Omega_{\text{shift}}(0) = 0$, we then have that the critical exponent (4.97) is given by $\theta_{\text{shift}} = 1/2$ since $1 \cdot \frac{\delta}{\delta\varphi}\psi_\star[0] = f^{(1)}_\star(0) = 0$. Thus (4.95) reduces to $\mathcal{O}_{\text{shift}} = \int_x v^{(1)}_\star(\varphi)$ and $\Omega_{\text{shift}} = f^{(1)}_\star(\varphi)$. Of course there is nothing physical about the value 1/2 since we can obtain any value for the scaling exponent θ_{shift} by instead considering the perturbation of $f_\star$ where $\Omega_{\text{shift}} = f^{(1)}_*(\varphi)+c$ for any value of c which leads to $\theta_{\text{shift}} = 1/2+c$. This is equivalent to choosing a condition other than $f_t(0) = 0$. In any case, this redundant perturbation is easily identified and discarded. To calculate the anomalous dimension η, we again use expansions around vanishing field and around the minimum of the potential

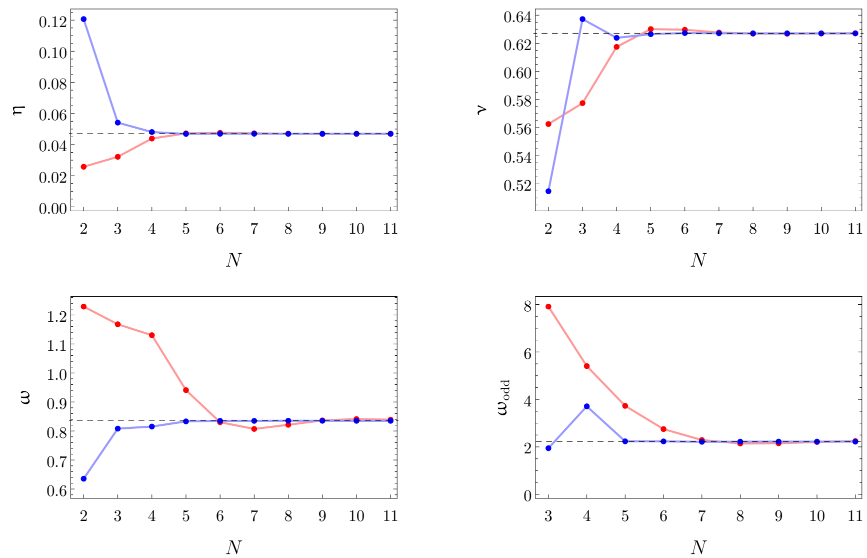


Figure 4.2: Critical exponents η (top-left panel), ν (top-right panel), ω (bottom-left panel), ω_{odd} (bottom-right panel), as a function of the truncation order N for the expansions around $\rho = 0$ (red) and the expansion around the minimum of the potential $\bar{\rho}$ (blue). Dashed lines represent the numerical values given in the main text.

$v_\star(\varphi)$. At order N in the expansion around $\varphi = 0$, we expand $\delta v_t(\varphi)$ and $\delta f_t(\varphi)$ as

$$\delta v_t(\varphi) = \varphi \sum_{n=0}^{N-1} \lambda_{2n+1} \rho^n \,, \tag{4.129a}$$

$$\delta f_t(\varphi) = \varphi^2 \sum_{n=0}^{N-1} \gamma_{2n+2} \rho^n \,, \tag{4.129b}$$

while the expansion around the minimum is written as

$$\delta v_t(\varphi) = \varphi \sum_{n=0}^{N-1} \bar{\lambda}_{2n+1} \left(\frac{1}{2}\varphi^2 - \bar{\rho}^\star \right)^n \,, \tag{4.130a}$$

$$\delta f_t(\varphi) = \varphi^2 \sum_{n=0}^{N-1} \bar{\gamma}_{2n+2} \left(\frac{1}{2}\varphi^2 - \bar{\rho}^\star \right)^n \,, \tag{4.130b}$$

and we notice that these expansions ensure that the boundary condition (4.96) is satisfied. With these forms of the perturbations, the linearized equations (4.120) are odd. One can then factor out a power of φ to obtain even equations which can be expanded in the $\mathbb{Z}_2$ invariant ρ around $\rho = 0$ and $\bar{\rho}^\star$. The linearized equations expanded around $\rho = 0$ ($\rho = \bar{\rho}^\star$)

can then be solved for β_{2n+1} and γ_{2n+2} which are both linear in λ_{2n+1}. We then obtain the critical exponents from the stability matrices

$$M^{\text{odd}}_{nm} = \left.\frac{\partial\beta_{2n+1}}{\partial\lambda_{2m+1}}\right|_{\lambda=\lambda^\star}, \tag{4.131a}$$

$$\bar{M}^{\text{odd}}_{nm} = \left.\frac{\partial\bar{\beta}_{2n+1}}{\partial\bar{\lambda}_{2m+1}}\right|_{\lambda=\lambda^\star}, \tag{4.131b}$$

at each order N in the two expansions. In the spectrum of odd eigenperturbations we find a single relevant positive critical exponents (disregarding θ_{shift}) which we identify as $(5-\eta)/2$ in accordance with (4.99). As with ν and ω we find that the numerical value of η converges $N \to \infty$. The values of η at orders $N = 2$ to $N = 11$ are plotted in the first panel of Figure 4.2. At order $N = 11$ we find

$$\eta = 0.0470\,. \tag{4.132}$$

We have also confirmed that this value η is independent of the boundary condition (4.96). From the eigenperturbation associated to the scaling exponent $(5-\eta)/2$, which is identified with the eigenperturbation (4.99), we can reconstruct the microscopic field χ from the composite operator ϕ.
The convergence of the least irrelevant eigenvalue $\omega_{\text{odd}} = -\theta$ associated to an odd perturbation shows a slower convergence than η. At order $N = 11$ in the expansion around the minimum the first three digits have converged to

$$\omega_{\text{odd}} = 2.22\,. \tag{4.133}$$

One can also consider solving the linearized equations for perturbations with both even and odd parts obtaining a stability matrix from which ν, ω, η and ω_{odd} can all be obtained with the same values obtained from treating the perturbations separately.

4.8 Higher orders of derivative expansion

Having demonstrated the minimal essential scheme at order ∂^2, let us now discuss how it can be generalized to higher orders in the derivative expansion. Within the standard scheme, the EAA Γ_k at order ∂^4 in the derivative expansion can be expressed as [84, 139, 140]

$$\begin{aligned}\Gamma_k = \int_x \Big\{ & V_k(\rho) + \frac{1}{2} z_k(\rho)\,(\partial_\mu\phi\,\partial_\mu\phi) \\ & + W^a_k(\rho)\,(\Delta\phi)^2 + W^b_k(\rho)\phi\Delta\phi\,(\partial_\mu\phi\,\partial_\mu\phi) + W^c_k(\rho)\,(\partial_\mu\phi\,\partial_\mu\phi)^2 \Big\}\,,\end{aligned} \tag{4.134}$$

where the three functions $W^i_k(\rho)$, with $i = a, b, c$ are linearly independent with respect to integration by parts.

We notice that both $W_k^a(\rho)$ and $W_k^b(\rho)$ are in the form of $\Phi \cdot \Delta\phi$, and hence in the minimal essential scheme the EAA reduces to

$$\Gamma_k = \int_x \left\{ V_k(\rho) + \frac{1}{2} \left(\partial_\mu \phi \, \partial_\mu \phi\right) + W_k(\rho) \left(\partial_\mu \phi \, \partial_\mu \phi\right)^2 \right\} , \tag{4.135}$$

which involves only two functions, namely the effective potential $V_k(\rho)$ and $W_k(\rho) \equiv W_k^c(\rho)$. In order to cope with the essential program, we generalize the RG kernel (4.101) to allow for terms involving up to two derivatives, namely

$$\Psi_k(x) = F_0(\phi) + F_{2,a}(\phi)\Delta\phi + \phi F_{2,b}(\phi) \left(\partial_\mu \phi \, \partial_\mu \phi\right) . \tag{4.136}$$

Inserting the ansatz (4.135) into the LHS of the flow equation (4.54), we note that this produces all of the terms at fourth order in the derivative expansion, namely

$$\begin{aligned} \partial_t \Gamma_k + \int_x \frac{\delta \Gamma_k}{\delta \phi} \Psi_k = \int_x \Big\{ & \partial_t V_k + F_0 V_k^{(1)} + \left[F_0^{(1)} + V_k^{(1)} \phi F_{2,b} + \left(V_k^{(1)} F_{2,a} \right)^{(1)} \right] \left(\partial_\mu \phi \, \partial_\mu \phi\right) \\ & + F_{2,a} \left(\Delta\phi\right)^2 + \phi F_{2,b} \Delta\phi \left(\partial_\mu \phi \, \partial_\mu \phi\right) \\ & + \left[\partial_t W_k + F_0 W_k^{(1)} + 4 W_k F_0^{(1)} \right] \left(\partial_\mu \phi \, \partial_\mu \phi\right)^2 \Big\} + O(\partial^6) . \end{aligned} \tag{4.137}$$

It is easy to generalize this procedure to higher orders in derivative expansion. For example, at order ∂^6 we have to include all possible terms up to four derivatives in the RG kernel

$$\begin{aligned} \Psi_k(x) = {} & F_0 + F_{2,a}\Delta\phi + \phi F_{2,b} \left(\partial_\mu \phi \, \partial_\mu \phi\right) + F_{4,a} \Delta^2 \phi \\ & + F_{4,b} \left(\Delta\phi\right)^2 + F_{4,c} \Delta\phi \left(\partial_\mu \phi \, \partial_\mu \phi\right) + F_{4,d} \left(\partial_\mu \phi \, \partial_\mu \phi\right)^2 + F_{4,e} \left(\partial_\mu \Delta\phi\right) \left(\partial_\mu \phi\right) . \end{aligned} \tag{4.138}$$

This way, we reduce the number of operators in the ansatz for the EAA from 13 to 5. In the following table we show the comparison between the number of operators for Γ_k in the standard and essential schemes.

	standard	essential
LPA	1	1
∂^2	2	1
∂^4	5	2
∂^6	13	5
$\vdots$	$\vdots$	$\vdots$

While at order $s = 0$ (i.e. in the LPA) the minimal essential scheme coincides with the standard scheme, the essential one can be carried out at any order in the derivative expansion, reducing its complexity order by order. At a given order ∂^s, the procedure of minimal essential scheme can be summarized as follows

- ◇ Apart from the canonical kinetic term with coefficient 1/2, eliminate all operators of the form $\Phi \cdot \Delta\phi$ from the ansatz of Γ_k;
- ◇ insert all the possible terms up to order $\partial^{(s-2)}$ into the RG kernel $\Psi_k(x)$;
- ◇ use equation (4.54) to find a set of beta functions for the essential operators which remain in the EAA, plus a set of equations which determine the functions appearing in the RG kernel Ψ_k.

Note that the final number of equations which one must solve at each order of the derivative expansion is the same as in the standard scheme. However, in the minimal essential scheme we obtain beta functions only for the essential couplings. Moreover, since the ansatz for EAA becomes simpler in the minimal essential scheme, the complexity in the calculation is reduced. In particular, the simple form of the propagator (4.89) evaluated at a constant field configuration is guaranteed.

4.9 Discussion

As we have both elucidated and demonstrated, the fact that the values of the inessential couplings are arbitrary can be used to one's advantage in practical QFT computations. This is made possible within the exact RG by the exact flow equation (4.54), derived by allowing the field variables $\hat{\phi}_k$ to themselves depend on the renormalization scale k. This then allows us to solve the flow equation in a scheme where we provide a renormalization condition for every inessential coupling. In these essential schemes, one only has to compute the flow of essential couplings. This has the advantage that the flow of inessential couplings, which cannot carry any physical information and therefore can only distract us from the physics, is automatically disregarded. The focus of this chapter has been on the minimal essential scheme applied to a single scalar field and we have explicitly worked out the details for the derivative expansion. It is clear that these advantages are not restricted to this narrow scope. As such, here we take the opportunity to adopt a broader view of essential schemes and discuss their possible applications.

4.9.1 Non-minimal essential schemes and extended PMS studies

In the minimal essential scheme which we have presented, one sets all inessential couplings to zero apart from the coefficient of the kinetic term, which is fixed to be equal to one half. The motivation of this particular essential scheme is to minimize the complexity of calculations. It is in this sense that the minimal essential scheme is minimal, with the most striking simplification being the minimal form of the propagator (4.89). However, this choice of scheme is just one possibility and it can be that there are other useful schemes where the inessential couplings take non-trivial values. One possibility is instead to look for optimized schemes by applying the principle of minimal sensitivity to a given observable computed in a given approximation. In general terms, the PMS states that optimized schemes are those for which the inessential couplings take the values $\zeta = \zeta_{\rm PMS}$ for which

$$\frac{\partial}{\partial \zeta}\,(\text{observable})\bigg|_{\zeta=\zeta_{\rm PMS}} = 0\,. \tag{4.139}$$

This being the case for all values of ζ only if the observable is computed without making an approximation. In practice, there will be a discrete set of values of $\zeta_{\rm PMS}$ for which (4.139) is satisfied.

It is natural to look for optimized schemes by considering non-minimal variants of the minimal essential scheme, where we continue to specify the values of all inessential couplings but relax the requirement that they take trivial values. In particular, we are free to write the general ansatz

$$\Gamma_t[\varphi] = \sum_a \lambda_a(t) e_a[\varphi] + \Phi_t[\varphi]\cdot\Delta\varphi\,, \tag{4.140}$$

where

$$\Phi_t[\varphi] = \sum_\alpha \zeta_\alpha \Phi_\alpha[\varphi] = \frac{1}{2} z_t(\varphi) + O(\partial^2)\,. \tag{4.141}$$

We thus reintroduce the inessential couplings ζ_α which parameterize $\Phi_t[\varphi]$.[9] To close the flow equation without introducing independent beta functions for the inessential couplings one can set

$$\zeta_\alpha = \zeta_\alpha(\lambda)\,, \tag{4.142}$$

where the functions $\zeta_\alpha(\lambda)$ are prescribed functions of the essential couplings. With the restriction that $\Phi_t[\varphi] = \mathcal{K}$ when $\lambda = 0$, such that we still have the Gaussian fixed point in the canonical form[10], we are otherwise largely free to pick the functions $\zeta_\alpha(\lambda)$. Different prescriptions which specify every inessential coupling are *non-minimal essential schemes*. At order ∂^2 in the derivative expansion non-minimal essential schemes correspond to solving two flow equations which depend on three functions $v_t(\varphi)$, $z_t(\varphi)$, and $f_t(\varphi)$ by choosing $z_t(\varphi)$ to be completely determined by the potential $v_t(\varphi)$. In particular, the flow equa-

[9]Here we are making a slight abuse of notation since we have not properly identified λ_a and ζ_α as essential and inessential couplings respectively. We ignore these subtleties for the purpose of this discussion.

[10]One can, of course, choose a non-canonical form of the Gaussian fixed point but there would seem no particular practical advantage in doing so.

tions (4.104) generalizes to the following system of equations

$$\partial_t V_k + F_k\, V_k^{(1)} = \frac{1}{2(4\pi)^{d/2}} Q_{d/2}\left[G_k\left(\partial_t R_k + 2F_k^{(1)} R_k\right)\right], \tag{4.143}$$

$$\begin{aligned}
\partial_t z_k + z_k^{(1)}\, F_k + 2 z_k\, F_k^{(1)} = &- \frac{z_k^{(2)}}{2(4\pi)^{d/2}}\, Q_{d/2}\left[G_k^2\left(\partial_t R_k + 2F_k^{(1)} R_k\right)\right] \\
&+ \frac{\left(z_k^{(1)}\right)^2}{(4\pi)^{d/2}}\left\{\frac{2d+1}{2} Q_{d/2+1}\left[G_k^3\left(\partial_t R_k + 2F_k^{(1)} R_k\right)\right]\right. \\
&\qquad + \frac{(d+2)(d+4)}{4} Q_{d/2+2}\left[G_k^2 G_k'\left(\partial_t R_k + 2F_k^{(1)} R_k\right)\right] \\
&\qquad \left. + \frac{(d+2)(d+4)}{4} Q_{d/2+3}\left[G_k^2 G_k''\left(\partial_t R_k + 2F_k^{(1)} R_k\right)\right]\right\} \\
&+ \frac{z_k^{(1)}\, V_k^{(3)}}{(4\pi)^{d/2}}\left\{2\, Q_{d/2}\left[G_k^3\left(\partial_t R_k + 2F_k^{(1)} R_k\right)\right]\right. \\
&\qquad + (d+2) Q_{d/2+1}\left[G_k^2 G_k'\left(\partial_t R_k + 2F_k^{(1)} R_k\right)\right] \\
&\qquad \left. + (d+2) Q_{d/2+2}\left[G_k^2 G_k''\left(\partial_t R_k + 2F_k^{(1)} R_k\right)\right]\right\} \\
&+ \frac{\left(V_k^{(3)}\right)^2}{(4\pi)^{d/2}}\left\{Q_{d/2}\left[G_k^2 G_k'\left(\partial_t R_k + 2F_k^{(1)} R_k\right)\right]\right. \\
&\qquad \left. + Q_{d/2+1}\left[G_k^2 G_k''\left(\partial_t R_k + 2F_k^{(1)} R_k\right)\right]\right\} \\
&- 2\frac{z_k^{(1)} F_k^{(2)}}{(4\pi)^{d/2}}\left\{Q_{d/2}\left[G_k^2\left(\partial_t R_k + 2F_k^{(1)} R_k\right)\right]\right. \\
&\qquad + \frac{(d+2)}{2} Q_{d/2+1}\left[G_k G_k'\left(\partial_t R_k + 2F_k^{(1)} R_k\right)\right] \\
&\qquad \left. + \frac{(d+2)}{2} Q_{d/2+2}\left[G_k G_k''\left(\partial_t R_k + 2F_k^{(1)} R_k\right)\right]\right\} \\
&- 2\frac{V_k^{(3)} F_k^{(2)}}{(4\pi)^{d/2}}\left\{Q_{d/2}\left[G_k G_k' R_k\right] + Q_{d/2+1}\left[G_k G_k'' R_k\right]\right\}.
\end{aligned} \tag{4.144}$$

Although the complexity of calculations is increased with respect to the minimal essential scheme one can look for optimized schemes by applying the PMS. For example, one can study the dependence of the universal scaling exponents at a non-trivial fixed point to determine values $\zeta_\alpha(\lambda_\star) = \zeta_\alpha^{\mathrm{PMS}}$ which satisfy the PMS criteria

$$\frac{\partial}{\partial \zeta_\alpha(\lambda_\star)} \theta(\zeta^{\mathrm{PMS}}) = 0\,. \tag{4.145}$$

Since there is an infinite number of inessential couplings, we can in principle attempt to locate an extremum (4.145) in an infinite-dimensional space. In practice we can vary a finite number of the inessential couplings for example by letting $z_t(\varphi) = z_\star(\varphi) + O((\lambda - \lambda_\star)^2)$ and choosing $z_\star(\varphi)$ to be a finite order polynomial. It is therefore possible to make extended field-dependent PMS studies which are not possible in the standard scheme. This may lead to a better determination of physical quantities at a fixed order in the derivative expansion than those obtained in the standard scheme [139]. Thus a natural next step in the application of essential schemes is to perform an extended PMS study of the Ising critical exponents at order ∂^2.

4.9.2 Redundancies and symmetries

As well as arriving at a practical scheme for the exact RG our work also clarifies some important conceptual points. In particular, regarding the existence of redundant operators, it is abundantly clear that there is one redundant operator for each inessential coupling. F. Wegner has proved by linearizing the flow equations around a given fixed point, the inessential couplings do not appear in the linearized beta functions of the essential couplings [35]. Physically, we know it must be true since it is this property that ensures that universal scaling exponents are independent of the unphysical inessential couplings. The underlying mathematical reason is that there is a symmetry associated with each inessential coupling which together form a group (the group of frame transformations) that has closed Lie algebra. However, when making approximations, this property may be lost if the symmetries are broken and therefore a spurious dependence on the inessential couplings may arise. In particular, if this property does not hold, the criteria that an operator be an eigenperturbation and a redundant operator will seemingly overconstrain the eigenvalue problem [125]. To see this clearly, imagine we have one essential coupling λ and one inessential coupling ζ obeying the following system of linearized beta functions $\partial_t \lambda = M_{\lambda\lambda}\lambda + M_{\lambda\zeta}\zeta$ and $\partial_t \zeta = M_{\zeta\lambda}\lambda + M_{\zeta\zeta}\zeta$. Then if $M_{\lambda\zeta} = 0$, it is clear that the redundant operator conjugate to ζ is an eigenperturbation since letting ζ be non-zero does not cause λ to run. On the other hand, if in an approximation $M_{\lambda\zeta} \neq 0$, then the redundant operator will not be an eigenperturbation. This can then lead one to conclude that redundant eigenperturbations are rare since there must be a symmetry in order to satisfy both criteria. However, this apparent rareness is an artefact of making approximations, since it is the closed nature of the Lie algebra associated with frame invariance that provides the required infinite number of symmetries independently of the scheme. In an essential scheme, this problem is avoided by fiat since the redundant perturbations are disregarded. It may be fruitful nonetheless to find approximation schemes that preserve frame covariance, such that physical quantities are scheme independent at each order of the approximation scheme. Some progress in this direction has been made at second order of the derivative expansion for a variant of the Wilsonian Effective Action [103, 156].

4.9.3 Generalizability

The minimal essential scheme and the non-minimal variants can be straightforwardly generalized to theories with different field content, symmetries and the inclusion of fermionic fields. Given the many applications of the exact RG to a wide array of physical systems, we can expect that essential schemes can be useful both in reducing complexity and in order to find optimized schemes to compute observables. In particular, the application of essential schemes to gauge theories could reduce spurious dependence on gauge fixing parameters and background fields, since these are both examples of inessential couplings. Moreover, we mention here that essential schemes can possibly shed light on the issue of generalizing the exact RG to problems involving boundaries. In particular, removing inessential coupling from the boundary action may help to preserve general boundary conditions along the RG flow.

4.9.4 Vertex expansion

Our focus in this chapter has been on the simplifications that arise at each order in the derivative expansion, however, essential schemes can also be applied in other systematic approximation schemes. One such a scheme is the vertex expansion where the EAA is expanded in terms of the n-point functions $\Gamma_k^{(n)}[0]$ to some finite order, discussed in Section 2.8. If we approximate Γ_k as depending on up to N powers of the field then we should include up to $N-1$ powers of the field in Ψ_k in order to solve the flow equation in an essential scheme. This can allow us to account for the full momentum dependence while keeping N finite. For example, to ensure that the two-point function takes the simple form $-\partial^2 + m^2$ we should include a term $-\frac{1}{2}\eta_k(\Delta)\phi$ in Ψ_k which accounts for the general linear field reparameterization. In fact, a scheme that removes all redundant operators from the two-point function in this manner has been put forward in [157]. The minimal essential scheme, applied consistently to a vertex expansion, would generalize this scheme by removing all redundant operators from the higher n-point functions include in the approximation.

Chapter 5

Essential Quantum Einstein Gravity

'Truth is ever to be found in simplicity,
and not in the multiplicity and confusion of things.'
Isaac Newton
'Everything should be made as simple
as possible, but no simpler.'
Albert Einstein

Wilson's exact Renormalization Group (RG) [29] provides a framework to construct a consistent quantum field theory (QFT) that describes gravity. This possibility, known as asymptotic safety, relies on the gravitational couplings exhibiting an ultraviolet (UV) fixed point that allows the UV cut-off to be removed while keeping physical quantities finite [158]. The theory can then be defined as a perturbation of the fixed point along a renormalizable trajectory that leaves the UV fixed point and evolves towards the infrared (IR), where it is identified with the renormalized theory. In this framework [40, 159], the number of free dimensionless parameters is one fewer than the number of relevant couplings at the fixed point, which parameterize the UV critical surface formed from all renormalizable trajectories.

So far, the evidence suggests that there is such a fixed point, known as the Reuter fixed point [160, 161, 162, 163, 164], and that it possesses three relevant couplings in pure gravity [165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176]. However, not all couplings need to reach a fixed point for the theory to be asymptotically safe, since one has the freedom to perform field reparameterizations which can be used to eliminate the so-called inessential couplings from the RG equations [158]. The inessential couplings do not appear in expressions for observables, such as cross sections and reaction rates, and, therefore, can take different values without affecting the physics. As we already said, couplings fall into two classes: the essential couplings λ_a which enter into expressions for observables and the inessential couplings ζ_α which are scheme dependent and unphysical. Consequently, the scaling behaviour of inessential couplings is entirely scheme dependent and they must not be included in the set of relevant couplings [35]. It follows that a coupling that may appear relevant could turn out to be inessential and, therefore, does not contribute to the counting of free parameters. Although the potential existence of inessential couplings has been pointed out [40, 125, 159, 177, 178], they have been almost universally ignored in investigations of asymptotic safety. In particular, attempts to find a suitable fixed point have required fixed points for all gravitational couplings, included in a given approximation, instead of incorporating field reparameterizations into the RG equations and checking which of the couplings are inessential. Here we shall remedy this oversight by incorporating field reparameterizations in the gravitational RG equations which allow us to eliminate the inessential couplings from the flow equations. To do so we will utilize the essential RG approach, which has been put forward in Chapter 4 and in [3], where we only compute the flow of the essential couplings.

Our strategy will be to adapt the minimal essential scheme devised in [3], in the context of scalar field theories, to remove all inessential couplings in pure gravity. This can be carried out order by order in the derivative expansion, where only terms with up to s-derivatives of the fields are included in the Effective Action. At each order s the minimal essential scheme is implemented by identifying the inessential couplings at a Gaussian fixed point of the theory and fixing their values, such that one obtains beta functions for the essential couplings only. An important point is that this scheme involves a specification of the kinematical degrees of freedom, since it assumes that the degrees of freedom are those of the Gaussian fixed point. This implies that the minimal essential scheme can break down a finite distance from the Gaussian fixed point and, thus, cannot describe all possible non-perturbative behaviour. However, one can then instead identify inessential couplings at other points in theory space, which, while technically more involved, would allow the essential RG to describe all regions of theory space.

For a scalar field there are Gaussian fixed points associated to kinetic operators $(-\partial^2)^{s/2}$ for every even integer s, which involve different degrees of freedom. As such, there is a minimal essential scheme associated to each Gaussian fixed point, that is physically distinct since they are associated with different degrees of freedom. Within a given minimal essential scheme, the RG flow is then constrained to the physical theory space associated to those degrees of freedom. In other words, the minimal essential scheme restricts the RG flow to a universality class that contains the corresponding Gaussian fixed point. Although, typically, RG studies are concerned with the universality class involving the Gaussian fixed point for which $s = 2$, one can also study universality classes associated to higher derivative theories [99, 100]. When one utilizes the minimal essential scheme for $s = 2$, the Gaussian fixed points for higher derivative theories are excluded. Therefore, this choice is not without physical consequences since by adopting a minimal essential scheme we focus our attention on possible fixed points in a specific universality class rather than attempting to find all possible fixed points.

For quantum gravity, we will consider the universality class of quantum Einstein gravity (QEG) meaning that it is associated to the quantization of the physical degrees of freedom associated to Einstein's theory of gravity. As such, in this chapter by the Gaussian fixed point (GFP) in the context of gravity we refer to the one associated to the linearized Einstein-Hilbert action unless otherwise stated. Here we should point out that we mean something more specific (but perhaps more deserving of the name) by QEG than the more broad definition given, e.g., in [179]. In particular, we do not only specify the fields and symmetries, in terms of which we parameterize the theory, but also the physical degrees of freedom. For example, a quantization of higher derivative gravity [180] can be carried out by quantizing the metric assuming diffeomorphism invariance, but it is a quantization of more degrees of freedom than Einstein's theory. This shift of emphasis to the physical degrees of freedom and the physical essential couplings will bring our investigation of asymptotic safety closer to the original formulation [158] by S. Weinberg: a move which has been strongly encouraged recently [181].

To set the stage, in Section 5.1 we give a short review of the essential RG technique, which generalizes the usual approach to the exact (aka the non-perturbative functional) RG for the effective average action (EAA) by allowing for field to be reparameterized along the RG flow. In Section 5.2 we revisit S. Weinberg's formulation of asymptotic safety which emphasizes the manner in which essential couplings enter expressions for observables. In Section 5.3 we derive the generalized flow equation for quantum gravity that takes into account the freedom to reparameterize the quantum metric along the RG flow. In fact, the flow equation will contain a new ingredient: the RG kernel. This quantity encodes the description about how the fields are reparameterized along the flow. Then, we write down a systematic derivative expansion of the diffeomorphism invariant part of the EAA and the covariant RG kernel. In particular, we expand the EAA to fourth order in derivatives and the RG kernel to second order. In Section 5.4, we analyze the GFP properties: in particular, from this analysis we determine that the vacuum energy is inessential. The advantage of studying the GFP consists of the fact that it is a free fixed point and the results can be obtained without relying on approximations. After having found the inessential couplings at the GFP, in Section 5.5 we discuss the properties of the universality class that contains the GFP and all the trajectories that have the same essential couplings. In such a subspace of the theory space, the propagator evaluated on conformally flat spacetime possesses the same form as the one at the GFP. Up to order four in the derivative expansion (apart from the topological Gauss–Bonnet term) only Newton's constant is essential in this universality class. In particular, any fixed point on these trajectories has the degrees of freedom of the GFP. In Section 5.6, we study the RG flow of QEG in the minimal essential scheme at orders two and four of the derivative expansion. Our investigation confirms the existence of the Reuter fixed point: this implies that higher derivative terms coming from the operators $\sqrt{\det g}\, R^2$ and $\sqrt{\det g}\, R_{\mu\nu}R^{\mu\nu}$ are inessential in the universality class of the GFP. Moreover, this means that, contrary to the expectations based on perturbative renormalizability, the existence of the Reuter fixed point in the minimal essential scheme suggests that a possible

UV-completion of the gravitational theory does not require additional degrees of freedom. In Section 5.7 we draw our conclusions and discuss the outlook for future investigations of quantum gravity using the essential RG. The derivation of RG equations in the minimal essential scheme for QEG at fourth order in the derivative expansion for a generic dimension and a generic regulator cutoff are presented in Appendix H.1.

5.1 Essentials of the Essential Renormalization Group

In this section we review the essential RG approach [3] seen in Chapter 4 in order to move to the gravity case.

The essential RG is a method to eventually compute (2.1) that makes use of the generalized exact RG equation for the EAA, which depends on the RG scale k. The EAA obtains a dependence on the RG scale k from two sources. First, the EAA depends on k due to the presence of a momentum-dependent IR cut-off (4.48), which implements the coarse-graining procedure, cutting off low momentum modes in the functional integral (4.49) that defines the EAA. The second source of k-dependence comes from the liberty to perform field reparameterizations along the flow parameterized by a k-dependent diffeomorphism $\hat{\phi}_k[\hat{\chi}]$ of configuration space which we integrate over in (2.1). This is achieved by considering a generating functional for correlation functions of the k-dependent fields $\hat{\phi}_k[\hat{\chi}]$ rather than the k-independent fields $\hat{\chi}$. Explicitly this functional is the generalized EAA action $\Gamma_k[\phi]$ defined by the functional integro-differential equation (4.49). In the limit $k \to 0$ the cut-off vanishes and EAA reduces to the one-part irreducible Effective Action $\Gamma[\phi] = \Gamma_0[\phi]$ for the field $\hat{\phi}_0$. In the opposing limit $k \to \infty$ the EAA reduces to the bare action written in terms of the fields $\hat{\phi}_\infty$. Let us note that we could additionally make a change of integration variables in the RHS of (4.49) which keeps $\Gamma_k[\phi]$ invariant provided we make this change everywhere including in the measure. Here we are keeping the integration variables $\hat{\chi}$ and the bare action $S[\hat{\chi}]$ k-independent, such that the k-dependence comes only from the regulator and the composite fields $\hat{\phi}_k[\hat{\chi}]$. Since we are ultimately interested in computing observables (4.51) at vanishing regulator and on the equations of motion for $\Gamma_0[\phi]$ we will recover (2.1) independently of the regulator and the parameterization $\hat{\phi}_k[\hat{\chi}]$.

In general, $\Gamma_k[\phi]$ will depend on all couplings compatible with the symmetries of the theory. As we have seen in Chapter 4, in the essential RG, the utility of $\hat{\phi}_k[\hat{\chi}]$ is that we may choose to reparameterize the field to fix the values of inessential couplings which, by definition, are simply those couplings that depend on the form of $\hat{\phi}_k[\hat{\chi}]$. Since observables (2.1) are invariant under a change in $\hat{\phi}_k[\hat{\chi}]$, they do not depend on the inessential couplings. Any scheme which fixes or otherwise specifies the flow of inessential couplings is an essential scheme. Thus, in an essential scheme we can compute only the flow of essential couplings $\lambda_a(k)$, i.e. those which ultimately enter into observables (2.1).

The generalized flow equation satisfied by $\Gamma_k[\phi]$ is given by Equation (4.54), where the RG kernel Ψ_k takes into account the k-dependent field reparameterizations.

By choosing $\Psi_k[\phi]$ we can specify the flow of inessential couplings ζ, which are defined in Equation (4.53). Within perturbation theory, in the vicinity of a Gaussian fixed point the second term in the RHS of Equation (4.53) will be sub-leading, since it is loop correction being proportional to Planck's constant $\hbar$. In general, there will be an inessential coupling associated to every linearly independent quasi-local field $\Phi_k[\phi]$ which generates an independent field reparameterization. Although the possible field reparameterizations $\Phi_k[\phi]$ are themselves independent of the position in theory space, it is important to stress again that the redundant operator depends on the EAA $\Gamma_k[\phi]$ and, thus, the identification of inessential couplings will depend on the form of the EAA. Thus, couplings which may be inessential at one fixed point can be essential at others. As an example at the GFP

$$\Gamma_{\mathrm{GFP}}[\phi] = \frac{\zeta}{2} \int_x \phi(-\partial^2)\phi \,, \tag{5.1}$$

the coefficient of ζ of the kinetic term is inessential. This can be understood since on the equations of motion $\partial^2\phi = 0$ the kinetic term vanishes. Changing the value of ζ corresponds to moving along a line of equivalent fixed points. However, if we consider the fourth order GFP

$$\Gamma_{\mathrm{GFP4}}[\phi] = \frac{\zeta}{2} \int_x \phi(-\partial^2)^2\phi \,, \tag{5.2}$$

the operator $\frac{1}{2}\int_x \phi(-\partial^2)\phi$ is not redundant since it does not vanish on the equations of motion $(\partial^2)^2\phi = 0$ for (5.2). Here we also see the connection between inessential couplings and the number of degrees of freedom. For the fourth order theory we have two propagating degrees of freedom which are massless at the fixed point (5.2). By adding the term with two derivatives, the action becomes

$$\Gamma_k[\phi] = \frac{\zeta}{2} \int_x \phi(-\partial^2)(-\partial^2 + m^2)\phi \,, \tag{5.3}$$

where m^2 is an essential coupling being identified as a mass for one of the degrees of freedom. Let us also note that at the GFP (5.1) the higher order term $\int_x \phi(-\partial^2)^2\phi$ is redundant since it vanishes on the equations of motion $\partial^2\phi = 0$. This reflects the fact that by starting with only one propagating degree of freedom we cannot gain more degrees of freedom along the RG flow. We will observe that this example shares some similarities with Einstein theory of pure gravity vs High derivative pure gravity.

Since the terms involving $\Psi_k[\phi]$ in (4.54) have the form of a redundant operator, the liberty to choose $\Psi_k[\phi]$ is precisely the liberty to specify the flow of all inessential couplings. Thus, for each inessential coupling we specify an RG condition, understood as a constraint on the form of $\Gamma_k[\phi]$ along the RG flow, then we solve the flow equation under this condition for the beta functions of the essential couplings and gamma functions which parameterize $\Psi_k[\phi]$. Different essential schemes correspond to different sets of RG conditions for the inessential couplings. From a geometric point of view, we can think of reparameterizations as local frame transformations on configuration space that are analogous to gauge transformations [3, 133]. RG conditions are, therefore, analogous to gauge fixing conditions which fix a particular frame, as with gauge conditions we typically want to find RG conditions that minimize the complexity of a given observable.

Since the form of the redundant operators (4.53) depend on $\Gamma_k[\phi]$ in practice the simplest scheme to implement is the minimal essential scheme which sets all inessential couplings at the GFP (5.1) to zero (apart from the coefficient of the kinetic term which is canonically normalized). One can show that this is achieved by setting all terms in the $\Gamma_k[\phi]$, which can be brought into the form $\int_x \Phi\, \partial^2 \phi$ by an integration by parts, to zero. In other words, in the minimal essential scheme we put to zero any term in $\Gamma_k[\phi]$ that vanishes when we apply the equations of motion at the GFP apart from the canonically normalized kinetic term (5.1) itself. Thus, by adopting the minimal essential scheme we trade a non-linear dependence on $z_k(\phi)$ in the flow equation for a linear dependence of $F_k(\phi)$. More generally, in the minimal essential scheme the $\mathcal{G}_k[\phi]$ evaluated at any constant value of the field $\phi(x) = \bar{\phi}$ has the form

$$\mathcal{G}_k[\bar{\phi}] = \frac{1}{-\partial^2 + \mathcal{R}_k + V_k^{(2)}(\bar{\phi})}\,, \tag{5.4}$$

where $V_k^{(2)}(\bar{\phi})$ is the second derivative of the potential.

The simplified form of the dimensionful propagator (5.4) or equivalently the dimensionless one (4.89), which continues to hold at any order of the derivative expansion, produces simplifications in practical calculations, and maintains a form that manifestly contains only physical degrees of freedom which are present at the GFP (5.1). This implies, for example, the absence of ghosts and tachyons, and it constraints our theory to stay in the subspace of the theory space where the degrees of freedom are the same of the GFP. As we will see, these features can also be guaranteed for the graviton propagator. What cannot be guaranteed is that there also exists other fixed points apart from the GFP in this subspace. Thus, by adopting the minimal essential scheme, we limit our search for additional fixed points by constraining the propagating degrees of freedom.

5.2 Weinberg's Formulation of Asymptotic Safety

Having reviewed the essential RG, let us now discuss the criteria of asymptotic safety as formulated by Weinberg [158] and how it is realized by solving the flow equation for the EAA within an essential scheme. The criteria necessitate that we have a UV-complete QFT where there is no UV cut-off, which is achieved if the theory lies on an RG trajectory that originates from a UV fixed point. However, as has been emphasized recently [181], Weinberg's formulation is more precise since it is concentrated on the absence of unphysical UV divergences in physical quantities, such as reaction rates, rather than on the behaviour of correlation functions of fields $\hat{\phi}$. This is important since correlation functions depend on inessential couplings ζ_α. In a scheme where we do not specify the values of inessential couplings but compute their flow, we are at the very least making our life harder unnecessarily. In the worst-case, an inessential coupling may not reach a fixed point and thus in such a scheme asymptotic safety could be obscured. In an essential scheme, we only compute the flow of the essential couplings and, thus, avoid these matters.

The divergences, which are absent in asymptotic safety, are those we expect to appear if we only have an effective theory that involves an artificial UV cut-off $\Lambda_{\rm UV}$, characterizing our ignorance of physics on small distances $\ell < 1/\Lambda_{\rm UV}$. An effective theory will break down as energies approach the cut-off scale and we will, therefore, encounter unphysical divergences. In an asymptotically safe theory, such divergences should be absent since we have sent $\Lambda_{\rm UV} \to \infty$. In fact, the form of the flow Equation (4.54) assumes that the limit $\Lambda_{\rm UV} \to \infty$ has been taken and would take a modified form if an independent UV cut-off were introduced [55, 88]. Asymptotic safety requires that as we take some characteristic energy scale $E \to \infty$ observables (such as reaction rates) scale as

$$\mathcal{O} \sim E^D \,, \tag{5.5}$$

where D is the dimension of $\mathcal{O}$. This means in particular that dimensionless quantities will not diverge even when we take $E \to \infty$ and, thus, at high energies the theory is scale invariant. Note that asymptotic safety is a rather generic requirement that we impose to be "reasonably sure" that there are no divergences in physical quantities related to the theory breaking down at a finite energy scale. On the one hand, asymptotic safety does not rule out *all* divergent behaviour, since unobservable correlation functions can diverge at finite energies even if the theory is well defined at all energies. On the other hand, asymptotic safety does *not* guarantee that a theory is physically acceptable since, for example, there can be asymptotically safe theories that are not unitary [182], the simple example being a free theory with four derivatives.

If we were handed the full quantum Effective Action Γ and computed observables from it directly, the coupling constants entering the expression for an observable would be the essential couplings $\lambda_a^{\rm phys.} \equiv \lambda_a(0)$ evaluated at $k = 0$. One may then wonder what the link is to a fixed point of the exact RG obtained in the opposing limit $k \to \infty$. In particular one may worry that observables can depend on additional energy scales E_n in addition to the scale E which we take to infinity. To understand the connection, note that if we supply an initial condition for the flow at a scale $k = \mu$, the flow equation supplies a function

$$\lambda_a^{\rm phys.} = \lambda_a^{\rm phys.}(\lambda_b(\mu), \mu) \,, \tag{5.6}$$

since by integrating the flow for a given initial condition we will obtain λ_a when we arrive to $k = 0$. Therefore, we can write any observable which depends on energy scales E and $\{E_n\}$ and the physical couplings $\lambda_a^{\rm phys.}$ as a function

$$\mathcal{O} = \mathcal{O}(E, \lambda_a(\mu), \mu, E_n) \,, \tag{5.7}$$

where $\mathcal{O}(E, \lambda_a(\mu), \mu, E_n)$ is independent of μ by construction meaning. On the other hand, dimensional analysis means that we can also write

$$\mathcal{O} = \mu^D \tilde{\mathcal{O}}(E/\mu, \tilde{\lambda}_a(\mu), E_n/\mu)\,, \tag{5.8}$$

where $\tilde{\lambda}_a(\mu) = \mu^{-d_a}\lambda_a(\mu)$ are the dimensionless couplings and d_a is the mass dimension of the coupling λ_a. Generically, the functions for the dimensionless observables $\tilde{\mathcal{O}}$ will be finite for finite values of its arguments, while if one argument were to diverge then generically we expect $\tilde{\mathcal{O}}$ to become singular. Now, since $\mathcal{O}$ is independent of μ, we can set $\mu = E$, such that

$$\mathcal{O} = E^D \tilde{\mathcal{O}}(1, \tilde{\lambda}_a(E), E_n/E)\,. \tag{5.9}$$

Then, it is clear that the limit $E \to \infty$ only exists if the limit $\lim_{\mu\to\infty} \tilde{\lambda}_a(\mu)$ exists. If a subset of the dimensionless essential couplings $\tilde{\lambda}_a(E)$ diverges at some finite $E = \Lambda_{\text{UV}}$, then we expect the observable to be singular at this point. However, if all the couplings $\tilde{\lambda}_a(\mu)$ remain finite for $\mu \to \infty$, such that they reach a UV fixed point $\lim_{\mu\to\infty} \tilde{\lambda}_a(\mu) = \lambda_a^\star$, then

$$\lim_{E\to\infty} \mathcal{O} = E^D \tilde{\mathcal{O}}(1, \tilde{\lambda}_a^\star, 0)\,, \tag{5.10}$$

which is exactly the requirement of asymptotic safety. An important point is that, since the RHS of (5.8) is independent of μ, if we would send E_n to infinity instead of E we could then identify $\mu = E_n$ and reach the conclusion that $\mathcal{O} \sim E_n^D$ as $E_n \to \infty$.

Crucially, it is only the essential couplings that need to attain a UV fixed point. In fact, inessential couplings ζ_α are simply not present in physical observables (5.7) and, therefore, their behaviour is not restricted a priori. All of these remarks apply to asymptotically safe theories in general, in the remainder of this chapter we will develop the formalism to investigate asymptotic safety in quantum gravity within an essential scheme.

5.3 Generalized Flow Equation and Essential Schemes for Quantum Gravity

In this section, we will derive the generalized flow equation for quantum gravity from which we use to apply the essential RG method in order to investigate asymptotic safety. This construction generalizes the formalism introduced in [160] by allowing for the field redefinitions at the heart of the essential RG. For quantum gravity the EAA is denoted $\Gamma_k[f; \bar{g}]$, where $f = \{g_{\mu\nu}, c^\mu, \bar{c}_\mu\}$ denotes the set of mean fields, $g_{\mu\nu}$ is the (mean) metric, and c^μ and $\bar{c}_\mu$ are the (mean) anti-commuting ghost and anti-ghost. In addition to the mean fields, $\Gamma_k[f; \bar{g}]$ also depends on an auxiliary background metric $\bar{g}_{\mu\nu}$ in order to conserve background covariance. The EAA for gravity is defined analogously to the case of the scalar field (4.49) through the functional integral

$$e^{-\Gamma_k[f,\bar{g}]} = \int (\mathrm{d}\hat{\chi})\; e^{-S[\hat{\chi};\bar{g}]} e^{(\hat{f}_k[\hat{\chi}]-f)\cdot\frac{\delta}{\delta f}\Gamma_k[f;\bar{g}]} e^{-\frac{1}{2}(\hat{f}_k[\hat{\chi}]-f)\cdot\mathcal{R}_k[\bar{g}]\cdot(\hat{f}_k[\hat{\chi}]-f)}\,, \tag{5.11}$$

where $\hat{\chi}$ are a set of fields which parameterize the fields $\hat{f}_k[\hat{\chi}] = \{\hat{g}_{\mu\nu\,k}[\hat{\chi}], \hat{c}^{\mu}_{k}[\hat{\chi}], \hat{\bar{c}}_{\mu\,k}[\hat{\chi}]\}$, such that the latter defines a k-dependent diffeomorphism of the configuration space to itself. Formally, since the configuration space involves the ghost fields, it is a super-manifold. The background field dependence enters in two places. First, the action $S[\hat{\chi};\bar{g}]$ includes gauge fixing and ghost terms and secondly the cut-off $\mathcal{R}_k[\bar{g}]$ depends on covariant derivatives and a tensor structure which are built from the background metric. Similarly to the case of the scalar field, it follows from (5.11) that

$$f = \langle \hat{f}_k \rangle_{f,k}\,, \tag{5.12}$$

where the expectation value of any functional of the fields $\hat{\mathcal{O}}[\hat{\chi}]$ is defined by

$$\langle \hat{\mathcal{O}} \rangle_{f,k} := e^{\Gamma_k[f,\bar{g}]} \int (\mathrm{d}\hat{\chi})\; e^{-S[\hat{\chi};\bar{g}]} e^{(\hat{f}_k[\hat{\chi}]-f)\cdot\frac{\delta}{\delta f}\Gamma_k[f;\bar{g}]} e^{-\frac{1}{2}(\hat{f}_k[\hat{\chi}]-f)\cdot\mathcal{R}_k[\bar{g}]\cdot(\hat{f}_k[\hat{\chi}]-f)} \hat{\mathcal{O}}[\hat{\chi}]\,. \tag{5.13}$$

The generalized flow equation for $\Gamma_k[f;\bar{g}]$ is given by

$$\left(\partial_t + \Psi_k[f;\bar{g}]\cdot\frac{\delta}{\delta f}\right)\Gamma_k[f;\bar{g}] = \frac{1}{2}\mathrm{STr}\,\mathcal{G}_k[f;\bar{g}]\left(\partial_t + 2\cdot\frac{\delta}{\delta f}\Psi_k[f;\bar{g}]\right)\cdot\mathcal{R}_k[\bar{g}]\,, \tag{5.14}$$

where $f = \langle \hat{f} \rangle$ are the mean fields and $\mathcal{G}_k[f,\bar{g}]$ denotes the propagator

$$\mathcal{G}_k[f;\bar{g}] := \left(\frac{\delta}{\delta f}\Gamma_k[f;\bar{g}]\frac{\overleftarrow{\delta}}{\delta f} + \mathcal{R}_k[\bar{g}]\right)^{-1}\,, \tag{5.15}$$

with $\overleftarrow{\delta}$ signifying that the derivative acts to the left. The $\cdot$ implies a continuous matrix multiplication including sum over all field components and integration over spacetime. The STr denotes a supertrace in the same sense with a minus sign inserted for anti-commuting fields. For gravity the RG kernel now has component for each field $\Psi_k = \{\Psi^{g}_{\mu\nu}, \Psi^{c\mu}, \Psi^{\bar{c}}_{\mu}\}$, such that $\Psi_k = \langle \partial_t \hat{f}_k \rangle_{f,k}$. By setting $\Psi_k = 0$ we obtain the flow equation for gravity derived in [160], however in this case we would have to also compute the flow of inessential couplings. Using the background field method [183], one is ultimately interested in identifying $\bar{g}_{\mu\nu} = g_{\mu\nu}$ and setting $c^{\mu} = 0 = \bar{c}_{\nu}$. It is, therefore, convenient to write the action as

$$\Gamma_k[g, c, \bar{c}; \bar{g}] = \bar{\Gamma}_k[g] + \hat{\Gamma}_k[g, c, \bar{c}; \bar{g}]\,, \tag{5.16}$$

where

$$\bar{\Gamma}_k[g] \equiv \Gamma_k[g,0,0;g] \implies \hat{\Gamma}_k[g,0,0;g] = 0 \tag{5.17}$$

is a diffeomorphism invariant action and $\hat{\Gamma}_k[g,c,\bar{c};\bar{g}]$ contains terms which depend on the ghosts and the two metrics separately, including the ghost and gauge fixing terms. The diffeomorphism invariant action has the derivative expansion

$$\bar{\Gamma}_k[g] = \int_x \sqrt{\det g}\left\{\frac{\rho_k}{8\pi} - \frac{1}{16\pi G_k}R + a_k R^2 + b_k R_{\mu\nu}R^{\mu\nu} + c_k E + O(\partial^6)\right\} . \tag{5.18}$$

Here G_k and ρ_k are the running Newton's constant and vacuum energy, respectively, and a_k, b_k and c_k multiply the $O(\partial^4)$ terms with $E = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2$. It will also be useful to define the cosmological constant as

$$\Lambda_k := \rho_k G_k\,, \tag{5.19}$$

since it is this combination that appears in the canonically normalized propagator. In four dimensions the integral $\int \mathrm{d}^4x\sqrt{\det g}E$ is a topological invariant, so c_k will not enter into any derivative of $\bar{\Gamma}_k[g]$ and, as such, c_k does not appear in any beta function [176, 184].

At a non-trivial fixed point required by asymptotic safety, the RG flow of dimensionless couplings in units of k will become independent of k. As such, it is convenient to define the dimensionless couplings

$$\tilde{G}(t) := k^{d-2}G_k\,, \quad \tilde{\rho}(t) := k^{-d}\rho_k\,, \quad \tilde{\Lambda}(t) := \tilde{G}(t)\tilde{\rho}(t)\,, \tag{5.20}$$

where we will omit to make the t-dependence of the dimensionless couplings explicit in the following.

Here we shall use the commonly used background field approximation where $\hat{\Gamma}_k[g,c,\bar{c};\bar{g}]$ is approximated by a BRST invariant action consisting of the background covariant gauge fixing and ghost terms. In particular, we shall take

$$\hat{\Gamma}_k[g,c,\bar{c};\bar{g}] = \frac{1}{2}\int_x \sqrt{\det \bar{g}}\left(F^\nu \bar{g}_{\mu\nu}F^\mu + \bar{c}_\mu \mathcal{Q}^\mu{}_\nu c^\nu\right)\,, \tag{5.21}$$

where, to simplify calculations, we adopt background covariant harmonic gauge

$$F^\mu = \frac{\sqrt{2}}{\kappa_k}\left(\bar{g}^{\mu\lambda}\bar{g}^{\nu\rho} - \frac{1}{2}\bar{g}^{\nu\mu}\bar{g}^{\rho\lambda}\right)\bar{\nabla}_\nu g_{\lambda\rho}\,, \tag{5.22}$$

and κ_k denotes the dimensionful coupling

$$\kappa_k \equiv \sqrt{32\pi G_k}\ . \tag{5.23}$$

The ghosts operator is then given by

$$\mathcal{Q}^{\mu}{}_{\nu}c^{\nu} \equiv \mathcal{L}_c F^{\mu} = \frac{\sqrt{2}}{\kappa_k}\left(\bar{g}^{\mu\lambda}\bar{g}^{\nu\rho} - \frac{1}{2}\bar{g}^{\nu\mu}\bar{g}^{\rho\lambda}\right)\bar{\nabla}_{\nu}(g_{\rho\sigma}\nabla_{\lambda}c^{\sigma} + g_{\lambda\sigma}\nabla_{\rho}c^{\sigma})\,. \tag{5.24}$$

In the background field approximation, we will choose $\Psi^{c\mu} = 0 = \Psi^{\bar{c}}_{\mu}$, while we choose the RG kernel for the metric to be given by

$$\Psi^{g}_{\mu\nu}[g] = \gamma_g g_{\mu\nu} + \gamma_R R g_{\mu\nu} + \gamma_{Ricci} R_{\mu\nu} + O(\partial^4)\,, \tag{5.25}$$

where γ_i with $i = \{g, R, Ricci\}$ are the 'gamma functions' which, along with the beta functions, will be determined as functions of the couplings that appear in $\bar{\Gamma}_k[g]$. Each gamma function allows us to impose a renormalization condition which fixes the flow of an inessential coupling. Thus, retaining three gamma functions allows us to impose three renormalization conditions which are constraints on the form of $\bar{\Gamma}_k[g]$ that we impose along the RG flow. We note that γ_g is dimensionless while γ_R and γ_{Ricci} have mass dimension -2, thus we define dimensionless gamma functions $\tilde{\gamma}_R := k^2\gamma_R$ and $\tilde{\gamma}_{Ricci} := k^2\gamma_{Ricci}$. As with the derivative expansion for a scalar field, if we work at order ∂^s in the derivative expansion, we include all terms of order ∂^{s-2} in the RG kernel (5.25).

In our approximation the flow for the diffeomorphism invariant action $\bar{\Gamma}_k[g]$ is given by

$$\left(\partial_t + \Psi^{g}_{k}\cdot\frac{\delta}{\delta g}\right)\bar{\Gamma}_k = \frac{1}{2}\mathrm{Tr}\,\mathcal{G}^{gg}_{k}\left(\partial_t + 2\cdot\frac{\delta}{\delta g}\Psi^{g}_{k}\right)\cdot\mathcal{R}^{gg}_{k} - \mathrm{Tr}\,\mathcal{G}^{\bar{c}c}_{k}\cdot\partial_t\mathcal{R}^{\bar{c}c}_{k}\,, \tag{5.26}$$

where

$$\mathcal{G}^{\bar{c}c}_{k} := \frac{1}{K_{\bar{c}c}\cdot\Delta_{\mathrm{gh}} + \mathcal{R}^{\bar{c}c}_{k}}\,, \tag{5.27}$$

$$\mathcal{G}^{gg}_{k} := \frac{1}{\frac{\delta^2\bar{\Gamma}_k}{\delta g\delta g} + K_{gg}\cdot\Delta_{\mathrm{gf}} + \mathcal{R}^{gg}_{k}}\,, \tag{5.28}$$

Δ_{gh} and Δ_{gf} denote the differential operators

$$\Delta_{\mathrm{gh}}{}^{\mu}{}_{\nu} = -\delta^{\mu}_{\nu}\nabla^2 - R^{\mu}{}_{\nu}\,, \qquad (\Delta_{\mathrm{gf}})_{\mu\nu}{}^{\rho\lambda} = \nabla_{\mu}\nabla_{\nu}g^{\rho\lambda} - 2\delta^{(\rho}_{(\nu}\nabla_{\mu)}\nabla^{\lambda)}\,, \tag{5.29}$$

and

$$K^{\mu\nu,\alpha\beta}_{gg} := \frac{1}{2\kappa_k^2}\sqrt{\det g}\left(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta}\right)\,, \qquad K^{\mu\nu}_{\bar{c}c} := \frac{\sqrt{2}}{\kappa_k}\sqrt{\det g}\,g_{\mu\nu}\,. \tag{5.30}$$

We then choose the regulators to be of the form

$$\mathcal{R}^{gg}_{k}[g] = K_{gg}R_k(\Delta)\,, \qquad \mathcal{R}^{\bar{c}c}_{k}[g] = K_{c\bar{c}}R_k(\Delta)\,, \tag{5.31}$$

where $\Delta = -g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the Laplacian.

The redundant operators for $\bar{\Gamma}_k[g]$ are given by

$$\mathcal{T}[\Phi^g] := \Phi^g \cdot \frac{\delta}{\delta g}\bar{\Gamma}_k - \mathrm{Tr}\,\mathcal{G}_k^{gg} \cdot \frac{\delta}{\delta g}\Phi^g \cdot \mathcal{R}_k^{gg}\,, \tag{5.32}$$

where Φ^g are symmetric covariant tensors composed of the metric, curvature tensors and their covariant derivatives, e.g., $\Phi^g_{\mu\nu} = g_{\mu\nu}, Rg_{\mu\nu}, R_{\mu\nu}$.

The minimal essential scheme for quantum gravity, which we will further elaborate on in Sections 5.4 and 5.5, closely follows the perturbative scheme put forward in [185, 186]. The scheme puts to zero any term that vanishes when the vacuum Einstein equations

$$R_{\mu\nu} = 0 \tag{5.33}$$

apply apart from the Einstein–Hilbert term itself. The reasoning is that the fixed point where $\tilde{G} = 0$ and $\tilde{\Lambda} = 0$ is the analog of the GFP for a scalar field theory. This means that can we set to zero both $a_k = 0$ and $b_k = 0$, while leaving c_k non-zero since this term is topological in $d = 4$. As with the GFP (5.1) in the scalar field theory the fact that any operator that vanishes on the equations of motion (5.33) can be removed by a field redefinition is a property of the fixed point where $\tilde{G} = 0$ and $\tilde{\Lambda} = 0$. A higher derivative Gaussian fixed point, more analogous to the fourth order fixed point (5.2), is achieved by instead writing

$$a_k = -\frac{1+\omega}{3\lambda}\,, \qquad b_k = \frac{1}{\lambda}\,, \qquad c_k = \frac{1-2\theta}{2\lambda}\,, \tag{5.34}$$

and sending $\lambda \to 0$. At this fixed point the degrees of freedom are those of Stelle's higher derivative gravity rather than Einstein gravity. Furthermore, since the equations of motion for higher derivative gravity do not imply (5.33) the couplings a_k and b_k (or equivalently λ and ω) are essential at the higher derivative Gaussian fixed point.

Here we concentrate on Einstein gravity where a_k and b_k are inessential in the vicinity of the GFP where $\tilde{G} = 0$ and $\tilde{\Lambda} = 0$. Thus, after setting $a_k = 0$ and $b_k = 0$ and neglecting all terms with more than four derivatives in $\bar{\Gamma}_k$, while retaining γ_g, γ_R, and γ_{Ricci} we expand the Equation (5.14) to order ∂^4 to obtain five flow equations from the independent tensor structures present in (5.18) using off-diagonal heat kernel techniques [187]. The evaluation of the traces and the resulting flow equations are presented in Appendix H.1 . The equations are presented for arbitrary cut-off function $R_k(\Delta)$ and in arbitrary dimension d neglecting terms proportional c_k in the traces: this is justified in $d = 4$ since in this case the corresponding invariant is topological. From now on, we will take $d = 4$. For explicit calculations, we will use the Litim cut-off function

$$R_k(\Delta) = (k^2 - \Delta)\Theta(k^2 - \Delta)\,, \tag{5.35}$$

where $\Theta(x)$ is the Heaviside theta function.

5.4 The Vacuum Energy Is Inessential

Having set a_k and b_k to zero, we can solve the equations for γ_R and γ_{Ricci}, given in (H.23) and (H.24). What is less clear is which renormalization condition we should apply to freeze the inessential coupling associated to γ_g, that must be some combination of G_k and ρ_k. As has been pointed out in [177], to find a non-trivial fixed point in gravity must actually require $\tilde{G}$ to have a fixed point. In fact, by rescaling the metric, or, in other words, choosing a system of units, one cannot set $G_k = 1$ and $k = 1$ simultaneously. This is still the case even with γ_g present since one does not find a non-trivial fixed point for $\tilde{\Lambda}$ if we try to fix the condition that $G_k = G_0$. The reason is that the beta function for $\tilde{\Lambda}$ still depends on $k^2 G_0$ which diverges as $k \to \infty$. However, one still has the freedom to apply one RG condition afforded by the presence of γ_g. What is evident is that the dimensionless inessential coupling will still need a fixed point value. Thus, we should instead fix a dimensionless coupling to one value along the RG flow. However, one finds that doing so can prevent the GFP from being present itself. For example if we set $\tilde{G} = 1$ or $\tilde{\Lambda} = 1$, the GFP, which in dimensionless variables is at $\tilde{G} = 0$ and $\tilde{\Lambda} = 0$, cannot be attained. This is a consequence of the fact that with a specific renormalization condition we cannot explore all universality classes contained in the theory space. In particular, since we will consider trajectories inside the subspace of theory space which contains the GFP, we will take into account the values of $\tilde{G}$ and $\tilde{\Lambda}$ at the GFP. Therefore, to determine which dimensionless coupling we should fix, we analyze the GFP to understand which particular combination $\tilde{G}$ and $\tilde{\Lambda}$ is inessential. However, we should understand this limit as the approach to a free theory on flat spacetime where $\bar{\Gamma}_k[g]$ reduces to the linearized Einstein-Hilbert action. To see this limit properly we have to decompose the metric as[1]

$$\hat{g}_{\mu\nu} = \mathfrak{g}_{\mu\nu} + \kappa_k \hat{\phi}_{\mu\nu}\,, \tag{5.36}$$

where $\mathfrak{g}_{\mu\nu}$ is a flat metric. We call $\hat{\phi}_{\mu\nu}$ the graviton field since it is a fluctuation around a flat metric $\mathfrak{g}_{\mu\nu}$, allowing one to define asymptotic states as free gravitons. In the parameterization (5.36), it becomes clear that κ_k is the coupling constant that measures the strength of self interactions of the graviton. The GFP corresponds to the theory where $\kappa_k = 0$. As we shall show later γ_R and γ_{Ricci} are both proportional to κ_k^2. Taking a derivative of (5.36) with respect to t, we obtain

$$\partial_t \hat{g}_{\mu\nu} = \frac{1}{2}\eta_N \kappa_k \hat{\phi}_{\mu\nu} + \kappa_k \partial_t \hat{\phi}_{\mu\nu} + O(\kappa_k^2)\,, \tag{5.37}$$

where $\eta_N := \partial_t \log G_k$. The factor of κ_k ensures that the field $\hat{\phi}_{\mu\nu}$ is canonically normalized. The expectation value of $\partial_t \hat{\phi}_{\mu\nu}$ is, therefore, given by

$$\Psi^{\phi}_{\mu\nu} \equiv \langle \partial_t \hat{\phi}_{\mu\nu}(k) \rangle = \frac{1}{\kappa_k}\Psi^{g}_{\mu\nu} - \frac{1}{2}\eta_N \phi_{\mu\nu} + O(\kappa_k)\,, \tag{5.38}$$

[1] We stress that this decomposition is independent of the split $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, which is purely technical.

where $\phi_{\mu\nu} = \langle \hat{\phi}_{\mu\nu} \rangle$, and inserting the expression for Ψ^g we obtain

$$\Psi^\phi_{\mu\nu} = \gamma_{\rm shift}\mathfrak{g}_{\mu\nu} - \frac{1}{2}\eta_\phi \phi_{\mu\nu} + O(\kappa_k)\,, \tag{5.39}$$

where

$$\eta_\phi := \eta_N - 2\gamma_g\,, \qquad \gamma_{\rm shift} := \frac{\gamma_g}{\kappa_k} \tag{5.40}$$

are the anomalous dimension of the graviton field and the gamma function related to a shift of the graviton field by a constant. Imposing that $\gamma_{\rm shift}$ is finite when $\kappa_k = 0$, we deduce that $\gamma_g = 0$ at the GFP. Defining

$$K_{\phi\phi} := \kappa_k^2 K_{gg}, \qquad \mathcal{R}^{\phi\phi}_k := \kappa_k^2 \mathcal{R}^{gg}_k\,, \tag{5.41}$$

the flow Equation (5.14) can be rewritten as

$$\left(\partial_t|_\phi + \Psi^\phi_k \cdot \frac{\delta}{\delta\phi}\right)\bar{\Gamma}_k = \frac{1}{2}{\rm Tr}\,\mathcal{G}^{\phi\phi}_k\left(\partial_t + 2\cdot\frac{\delta}{\delta\phi}\Psi^\phi_k\right)\cdot\mathcal{R}^{\phi\phi}_k - {\rm Tr}\,\mathcal{G}^{\bar{c}c}_k\cdot\partial_t\mathcal{R}^{\bar{c}c}_k\,, \tag{5.42}$$

where the canonically normalized regularized propagator is

$$\mathcal{G}^{\phi\phi}_k = \frac{1}{\frac{\delta^2\bar{\Gamma}_k}{\delta\phi\delta\phi} + K_{\phi\phi}\cdot\Delta_{\rm gf} + \mathcal{R}^{\phi\phi}_k}\,. \tag{5.43}$$

Inserting $g_{\mu\nu} = \mathfrak{g}_{\mu\nu} + \kappa_k\phi_{\mu\nu}$ into $\bar{\Gamma}_k[g]$ and then expanding in κ_k, we find that at the GFP the EAA has the form

$$\bar{\Gamma}^{\rm GFP}_k := \frac{1}{2}\phi\cdot K_{\phi\phi}[\mathfrak{g}]\cdot(\Delta[\mathfrak{g}] - \Delta_{\rm gf}[\mathfrak{g}])\cdot\phi + k^4\frac{1}{8\pi}\int {\rm d}^4x\sqrt{\det\mathfrak{g}}\,\tilde{\rho}_{\rm GFP}\,, \tag{5.44}$$

where we anticipate that for $\kappa_k = 0$ the vacuum energy is $\rho_k = k^4\tilde{\rho}_{\rm GFP}$ and $\tilde{\rho}_{\rm GFP}$ denotes the dimensionless fixed point value for the vacuum energy, which we will determine shorty. Inserting (5.44) into the LHS of (5.42) we obtain

$$\left(\partial_t|_\phi + \Psi^\phi_k\cdot\frac{\delta}{\delta\phi}\right)\bar{\Gamma}_{GFP} = \frac{1}{2\pi}k^4\int {\rm d}^4x\sqrt{\det\mathfrak{g}}\,\tilde{\rho}_{\rm GFP} - \frac{1}{2}\eta_\phi\phi\cdot K[\mathfrak{g}]\cdot(\Delta[\mathfrak{g}] - \Delta_{\rm gf}[\mathfrak{g}])\cdot\phi\,, \tag{5.45}$$

while on the RHS we have, using that $\gamma_g = 0$,

$$\frac{1}{2}{\rm Tr}\,\mathcal{G}^{\phi\phi}_k\left(\partial_t + 2\cdot\frac{\delta}{\delta\phi}\Psi^\phi_k\right)\cdot\mathcal{R}^{\phi\phi}_k - {\rm Tr}\,\mathcal{G}^{\bar{c}c}_k\cdot\partial_t\mathcal{R}^{\bar{c}c}_k = k^4\int_0^\infty {\rm d}z\,z\frac{-3\eta_\phi R_k(z) + \partial_t R_k(z)}{16\pi^2(R_k(z)+z)}\,, \tag{5.46}$$

which is independent of $\phi_{\mu\nu}$ and, as such, we find that $\eta_\phi = 0$ at the GFP which together with $\gamma_g = 0$ implies $\eta_N = 0$. We then see that the GFP value of the dimensionless vacuum energy is

$$\tilde{\rho}_{\text{GFP}} = \frac{1}{8\pi} \int_0^\infty \mathrm{d}z\, z\, \frac{\partial_t R_k(z)}{z + R_k(z)}\,. \tag{5.47}$$

Using the Litim cut-off we obtain the value

$$\tilde{\rho}_{\text{GFP}} = \frac{1}{8\pi}\,. \tag{5.48}$$

We conclude that the GFP is characterized uniquely by $\tilde{G} = 0$, $\eta_N = 0$, $\gamma_g = 0$ and a scheme dependent value for the dimensionless vacuum energy $\tilde{\rho} = \tilde{\rho}_{\text{GFP}}$. The fact that $\eta_N = 0$ means that we arrive at the GFP in dimensionless variables when $G_k \to G_0$ is a constant, such that $\tilde{G}$ vanishes as $\tilde{G} \sim k^2 G_0$ in the limit $k \to 0$. Thus the GFP is an IR fixed point for $\tilde{G}$.

Two remarks are in order concerning the vacuum energy. First, let us note that we could also choose a more general cut-off scheme allowing for different cut-off functions for the ghosts and gravitons in such a manner that $\tilde{\rho}_{\text{GFP}}$ would vanish [188]. At the exact level no physics should depend on the choice of cut-off so the value of $\tilde{\rho}_{\text{GFP}}$ should be of no significance. Secondly, we note that it may seem we could satisfy the flow with $\rho_k = k^4 \tilde{\rho}_{\text{GFP}} + \rho_0$ allowing for a non zero cosmological constant, since ρ_0 is a constant of integration that will not appear in (5.45). However, only with $\rho_0 = 0$ do we have a fixed point.

Now, away from the GFP, γ_g needs not be equal to zero, so we can now write the linearized ansatz for γ_g around the GFP as

$$\gamma_g = w_1 \left(\tilde{\rho} - \frac{1}{8\pi} \right) + w_2 \tilde{G} + \dots\,, \tag{5.49}$$

where w_1 and w_2 are free parameters which we are free to choose and the dots are nonlinear terms in the expansion around the GFP. Expanding the beta functions for $\tilde{G}$ and $\tilde{\rho}$ we obtain

$$\partial_t \tilde{G} = 2\tilde{G} + \dots\,, \tag{5.50}$$

$$\partial_t \tilde{\rho} = \left(\frac{w_1}{3\pi} - 4 \right) \left(\tilde{\rho} - \frac{1}{8\pi} \right) + \left(\frac{w_2}{3\pi} + \frac{38}{24\pi^2} \right) \tilde{G} + \dots \tag{5.51}$$

and, thus, we see that the linearized flow of $\tilde{G}$ around the GFP is scheme independent, while the linearized flow of $\tilde{\rho}$ is scheme dependent. Since scheme dependence is the hallmark of an inessential coupling, we can conclude that Newton's coupling $\tilde{G}$ is an essential coupling in the vicinity of the GFP, while $\tilde{\rho}$ is inessential. We are free to specify the flow for

$\tilde{\rho}$ instead of computing it and we can freely choose the corresponding scaling dimension rather than assuming it should have dimension 4. In fact, we can even make the vacuum energy, which canonically is the most relevant coupling, an irrelevant coupling simply by choosing $w_1 > 12\pi$. Let us stress that these are exact statements since we are at the GFP and terms at order ∂^6 arise at two loops.

A remarkable consequence of the vacuum energy being inessential is that we may simply choose that $\rho_{k=0} = 0$ and, thus, the vanishing of the vacuum energy is achieved by a renormalization condition. Thus, at least in pure gravity, there is no fine tuning problem related to the cosmological constant once we apply this condition. However, this condition dictates the vanishing of the cosmological constant and by imposing it we are restricting which theories we can have access to. This suggests that there is a universality class of quantum gravity where the cosmological constant is zero. This universality class possesses the IR GFP where G_0 is a constant and $\rho_0 = 0$. Although there may be other universality classes where the cosmological constant is non-zero, here we will explore this one to see if there is also a non-trivial fixed point that can be used to define the interacting QFT.

Before ending this section, let us stress two points regarding the interpretation of couplings in gravity and the possible (imperfect) analogies we can make with couplings in, e.g., ϕ^4-theory. First, despite appearances, G_k is not the inverse wave function renormalization, but a coupling, more analogous to the interaction coupling λ in ϕ^4-theory. Starting from the standard parameterization of ϕ^4-theory and sending $\phi \to \frac{1}{\lambda}\phi$

$$S = \frac{1}{\lambda}\int_x \left(\frac{Z}{2}\partial_\mu\phi\partial_\mu\phi + \frac{Z}{2}m^2\phi^2 + \frac{Z^2}{4!}\phi^4\right) . \tag{5.52}$$

In particular, while the wave function renormalization is an inessential coupling in ϕ^4-theory, G_k is an essential coupling like λ. Secondly, again despite appearances, $\Lambda_k = \rho_k G_k$ is not a mass squared. A more clear interpretation of the vacuum energy comes if we choose to parameterize the metric, such that $\rho_k\sqrt{\det g}$ is linear in the field σ which parameterizes conformal fluctuations [189]. This can be achieved by setting $g_{\mu\nu} = \left(1 + \frac{\sigma}{d}\right)^{\frac{2}{d}} \mathfrak{g}_{\mu\lambda}(e^h)^\lambda{}_\nu$, where $h^\mu_\mu = 0$, such that $\rho_k\sqrt{\det g} = \rho_k\sqrt{\det \mathfrak{g}}\left(1 + \frac{\sigma}{d}\right)$ is linear in σ. The fact that ρ_k couples also to the purely vacuum term is what is crucial for ρ_k to be inessential. Thus ρ_k, rather than being analogous to a mass in a scalar theory which is essential, can better be interpreted as a constant source which couples linearly to the field in the broken phase of diffeomorphisms.

5.5 Minimal Essential Scheme for Quantum Einstein Gravity

Since the vacuum energy is inessential coupling at the GFP, we can fix it by a renormalization condition. In particular, we can pick a condition which ensures that we are in the universality class which possesses the GFP and removes the vacuum energy from the set of couplings we must compute the flow of. We will adopt the simplest RG condition of this type which sets

$$\tilde{\rho}(t) \equiv k^{-4}\rho_k = \tilde{\rho}_{\rm GFP} \tag{5.53}$$

for all scales $k = k_0 e^t$. The RG condition (5.53) identifies the vacuum energy with cut-off scale $\rho_k = k^4\tilde{\rho}_{\rm GFP}$. Having applied (5.53), then the dimensionless product $\tau_k := \rho_k G_k^2$ is given by

$$\tau_k = \tilde{\rho}_{\rm GFP}\tilde{G}(t)^2 \tag{5.54}$$

and, therefore, the flow of $\tilde{G}(t)$ completely determines the flow of τ_k. In classical general relativity, in the absence of matter, τ_k is the only meaningful coupling since one can rescale the metric. This can be seen explicitly from the flow equation by observing that, when the RHS is neglected, the beta function for τ_k is independent of γ_g. More generally, when $k = 0$ it is evident that only dimensionless ratios couplings can be essential since a rescaling of the metric will change the values of dimensionful couplings. As such, τ_0 is the physical cosmological constant in Planck units which can be considered as an observable, which vanishes in the universality class we are considering.

Let us stress, however, that although ρ_k will vanish at $k = 0$, its presence in the action is still needed to consistently solve the flow equation for non-zero k. If we would simply neglect the flow of ρ_k entirely, then G_k would appear to be inessential since we could instead use γ_g to dictate the flow of G_k instead.

In addition to (5.53), we specify an infinite set of renormalization conditions which exclude all terms that are dependent on the Ricci curvature $R_{\mu\nu}$ from the ansatz from $\bar{\Gamma}_k$ apart from the Einstein–Hilbert action and the topological Gauss–Bonnet term. This defines the minimal essential scheme for quantum gravity. At second order in curvature, the most general diffeomorphism invariant action can be written as

$$\bar{\Gamma}_k[g] = \int {\rm d}^4x\sqrt{\det g}\left\{\frac{\rho_k}{8\pi} - \frac{1}{16\pi G_k}R + RW_{R,k}(\Delta)R + R_{\mu\nu}W_{Ricci,k}(\Delta)R^{\mu\nu} + c_k E\right\} \tag{5.55}$$

and, hence, in the minimal essential scheme we set $W_{R,k}(\Delta) = 0 = W_{Ricci,k}(\Delta)$. Since, furthermore, all the higher terms depend only on the Weyl curvature $C_{\mu\nu\rho\lambda}$, the propagator evaluated on any conformally flat spacetime, i.e., those where $C_{\mu\nu\rho\lambda} = 0$, is just that of classical general relativity [186]. Consequently, the regularized propagator evaluated on a conformally flat background takes the form

$$\mathcal{G}_k^{gg} = K_{gg}^{-1} \cdot \frac{1}{\Delta - 2k^4 G_k\tilde{\rho}_{\rm GFP} + R_k(\Delta)} + O(R_{\mu\nu})\,. \tag{5.56}$$

This ensures that the theory at $k = 0$ describes massless gravitons only.

Here, we shall only consider pure gravity. However, in [185], general arguments for spin 0, 1/2, 1, 3/2, and 2 fields suggest that terms which would modify the propagator by introducing new poles are redundant in the vicinity of the GFP. For example if we consider a scalar tensor theory

$$\bar{\Gamma}_k[g,\phi] = -\frac{1}{16\pi G_k}\int \mathrm{d}^4x\sqrt{\det g}\left[-\frac{1}{16\pi G_k}R + \frac{1}{2}(\nabla_\mu\phi\nabla^\mu\phi) + V_k(\phi)\right], \tag{5.57}$$

then we can still use both the equations of motion for the metric and the scalar to remove inessential couplings. Out of all terms with up to four derivatives, the only additional terms which do not vanish when the equations of motion apply are

$$\int \mathrm{d}^4x\sqrt{\det g}W_k(\phi)(\nabla_\mu\phi\nabla^\mu\phi)^2 + \int \mathrm{d}^4x\sqrt{\det g}C_k(\phi)E\,, \tag{5.58}$$

neither of which enter the propagator evaluated on a conformally flat spacetime and for constant values of ϕ.

The fact that we can remove the terms which lead to extra poles in the propagator along the RG flow indicates that the poles encountered in other schemes are spurious [190]. However, let us stress that nothing is wrong with using a scheme where the form factors do not vanish, such that the propagator at $k=0$ with $\rho_0 = 0$ has the form

$$\mathcal{G}_0 = K_{gg}^{-1}\cdot\frac{1}{Z(\Delta)\Delta} + O(R_{\mu\nu\rho\lambda})\,, \tag{5.59}$$

where $Z(\Delta)$ is a wave function renormalization factor related to the form factors $W_{R,k}(\Delta)$ and $W_{Ricci,k}(\Delta)$, which have been computed in various approximations in [173, 191, 192, 193, 194, 195] and the physical implications for scattering amplitudes have been discussed in [196, 197, 198, 199]. (In principle there can be another independent wave function renormalization related to the scalar degree of freedom that is introduced in theories such as $f(R)$ gravity. For simplicity, we discuss the case where there is only one wave function renormalization which implies a linear relation between $W_{R,k}(\Delta)$ and $W_{Ricci,k}(\Delta)$). There are two cases, either Z introduces new poles into the propagator, or it does not. In the latter case, we can remove Z by a reparameterization since it must be an entire function, and thus it is just a momentum-dependent wave function renormalization. In this case, we will find the same physics as in the minimal essential scheme, namely, although the field redefinition would modify the vertices of the theory, the propagator would return to the minimal form (5.56). The case where Z is not an entire function corresponds to a universality class not accessible to the minimal essential scheme for pure gravity. In particular, it would include particles other than the massless graviton. Thus, on one hand, the minimal essential scheme for quantum gravity, like its counterpart of scalar field theories [3], does put a restriction on what physics we can access by following the corresponding RG flow. On the other hand, this is a feature of the scheme, and not a bug, since the restricted theory space has a physical meaning, describing the interactions of gravitons which are fluctuation around a flat spacetime. Moreover, there is no reason why these fluctuations can not be strongly interacting, in particular $\tilde{G}$ can become of order unity.

It may of course be that this universality class, which only includes a massless graviton, does not contain a suitable UV fixed point and that one would need more degrees of freedom to describe a consistent theory of quantum gravity. For example, it could be the case that one would need the extra degrees of freedom which are present in higher derivative gravity and are needed to make the theory perturbatively renormalizable, or that one would need to add an $\sqrt{\det g}R^2$ which includes an extra scalar degree of freedom in addition to the graviton. Here we will test the hypothesises that these extra degrees of freedom are not necessary for non-perturbative renormalizability.

5.6 The Reuter Fixed Point in the Derivative Expansion

To test the aforementioned hypothesis, the minimal essential scheme can be carried out at each order in the derivative expansion. Here we will study the RG flow at order ∂^2, where the action is the Einstein–Hilbert action with (5.53), and at order ∂^4 where the action takes the form

$$\bar{\Gamma}_k[g] = \int \mathrm{d}^4x \sqrt{\det g} \left\{ k^4 \frac{\tilde{\rho}_{\mathrm{GFP}}}{8\pi} - \frac{1}{16\pi G_k} R + c_k E + O(\partial^6) \right\} , \tag{5.60}$$

with the only order ∂^4 term being the topological one. At order ∂^2 we set γ_R and γ_{Ricci} to zero, along with all higher-order terms in Ψ_k, and expand the flow equation for (5.60) to order ∂^2 solving for

$$\gamma_g = \gamma_g(\tilde{G}) \,, \qquad \partial_t \tilde{G} = \beta_{\tilde{G}}(\tilde{G}) \,, \tag{5.61}$$

which are functions of $\tilde{G}$ alone. At order ∂^4 we include all order ∂^4 tensor structures in the flow equations but solve for γ_R and γ_{Ricci} instead of the running of the higher derivative couplings a_k and b_k, which are set to zero. Thus, at order ∂^4 the minimal essential flow is characterized by five dimensionless functions of $\tilde{G}$, namely

$$\gamma_g = \gamma_g(\tilde{G}) \,, \quad \partial_t \tilde{G} = \beta_{\tilde{G}}(\tilde{G}) \,, \tag{5.62}$$

$$\tilde{\gamma}_R = \tilde{\gamma}_R(\tilde{G}) \,, \quad \tilde{\gamma}_{Ricci} = \tilde{\gamma}_{Ricci}(\tilde{G}) \,, \quad \partial_t c_k = \beta_c(\tilde{G}) \,. \tag{5.63}$$

Let us stress that calculation is vastly simpler than the calculation where higher derivative couplings a_k and b_k do not vanish [176, 184] and that the final form of the beta and gamma functions only depend on one coupling rather than four in the standard scheme.

As a first check we can analyze the behaviour around the GFP at $G_k = 0$ to see how the universal one-loop divergencies are accounted for. In particular, the one-loop divergencies encountered in dimensional regularization in our chosen gauge with $\Lambda = 0$ are given by three terms

$$\Gamma_{\text{div}} = \frac{1}{d-4}\frac{1}{(4\pi)^2}\int_x \sqrt{\det g}\left[\frac{1}{60}R^2 + \frac{7}{10}R_{\mu\nu}R^{\mu\nu} + \frac{53}{45}E\right]. \tag{5.64}$$

Upon replacing $\frac{1}{d-4} \to \log(k/k_0)$ and taking a derivative with respect to k the same three terms will appear in the flow equation on the RHS of Equations (H.23)–(H.25) , respectively. However, the terms that would renormalize a_k and b_k are, instead, absorbed into γ_R and γ_{Ricci}, while c_k will still be renormalized. Expanding in G_k we find that

$$\gamma_R = -\frac{11}{30\pi}G_k + O(G_k^2)\,, \quad \gamma_{Ricci} = \frac{7}{10\pi}G_k + O(G_k^2)\,, \quad \beta_c = \frac{1}{(4\pi)^2}\frac{53}{45} + O(G_k)\,, \tag{5.65}$$

which precisely account for the divergences (5.64).

The non-perturbative beta functions $\beta_{\tilde{G}}(\tilde{G})$ at orders ∂^2 and ∂^4 are plotted in Figure 5.1 and are seen to closely agree for $\tilde{G}$ in the plotted region. At both orders there exists a UV fixed point where

$$\tilde{G} = \tilde{G}_\star = 0.6538 \quad \text{at } O(\partial^2)\,, \tag{5.66}$$
$$\tilde{G} = \tilde{G}_\star = 0.6275 \quad \text{at } O(\partial^4)\,, \tag{5.67}$$

which we can identify as the Reuter fixed point [160, 161]. The Reuter fixed point splits the phase diagram of quantum gravity into a weakly coupled and strongly coupled regions for $0 < \tilde{G} < \tilde{G}_\star$ and $\tilde{G} > \tilde{G}_\star$, respectively. The critical exponent at the Reuter fixed point

$$\theta = -\frac{\partial \beta_{\tilde{G}}}{\partial \tilde{G}}(\tilde{G}_\star) \tag{5.68}$$

is given by

$$\theta = 2.3129 \quad \text{at } O(\partial^2)\,, \tag{5.69}$$
$$\theta = 2.3709 \quad \text{at } O(\partial^4)\,, \tag{5.70}$$

which can be compared to the canonical scaling dimension of $\theta_{\text{can}} = 2$ which is obtained at one-loop, and, therefore, receives a small correction. This suggests that the Reuter fixed point is weakly non-perturbative [171, 200].

The gamma function $\gamma_g(\tilde{G})$, plotted in Figure 5.2 at orders ∂^2 and ∂^4, also appears stable between the two approximations and is approximately linear in weakly coupled phase. At the Reuter fixed point γ_g takes the values

$$\gamma_g^\star = -1.1605 \quad \text{at } O(\partial^2)\,, \tag{5.71}$$
$$\gamma_g^\star = -1.1062 \quad \text{at } O(\partial^4)\,. \tag{5.72}$$

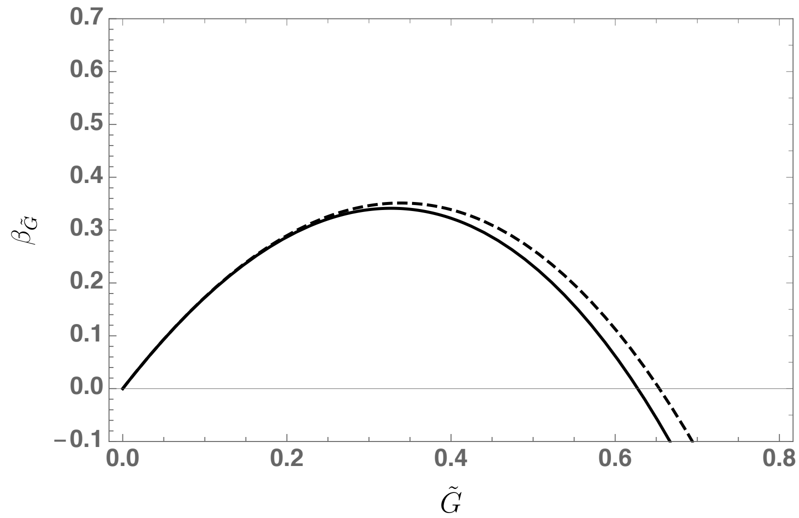


Figure 5.1: The beta function for Newton's constant in the Einstein-Hilbert approximation (dashed line) and the order ∂^4 approximation (solid line).

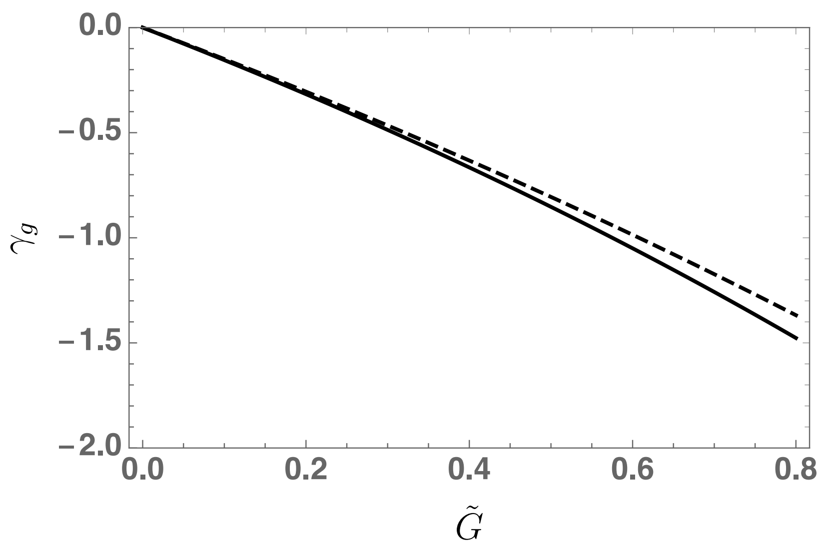


Figure 5.2: The gamma function γ_g in the Einstein-Hilbert approximation (dashed line) and the order ∂^4 approximation (solid line).

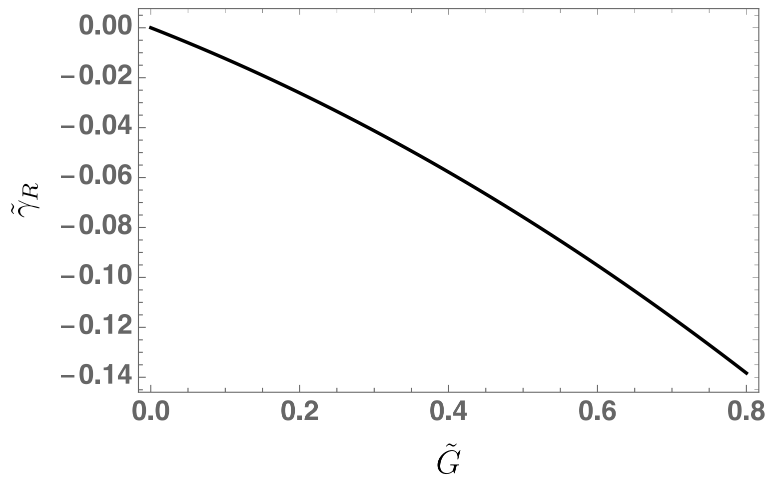


Figure 5.3: The gamma function $\tilde{\gamma}_R$ in the order ∂^4 approximation.

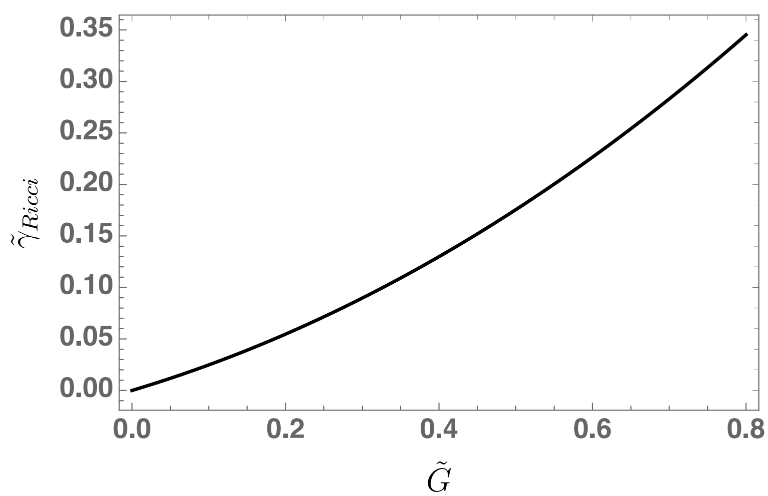


Figure 5.4: The gamma function $\tilde{\gamma}_{Ricci}$ in the order ∂^4 approximation.

The stability between the orders can be understood by looking at the gamma functions $\gamma_R(\tilde{G})$ and $\gamma_{Ricci}(\tilde{G})$, which are zero at order ∂^2, and remain small at order ∂^4 in the region $0 < G < \tilde{G}_\star$, as can be seen in Figures 5.3 and 5.4. At the Reuter fixed point γ_R and γ_{Ricci} take the values

$$\tilde{\gamma}^\star_R = -0.10079 \quad \text{at } O(\partial^4)\,, \tag{5.73}$$

$$\tilde{\gamma}^\star_{Ricci} = 0.24150 \quad \text{at } O(\partial^4)\,. \tag{5.74}$$

Thus, we observe a remarkable stability as the order of the approximation is increased. At order ∂^4 we also find the beta function of c_k which is plotted in Figure 5.5.

Let us stress that the values of the gamma functions are not universal quantities and will depend on the RG scheme. We note that at $\tilde{G} \approx 3$ the beta functions $\beta_{\tilde{G}}(\tilde{G})$ calculated at orders ∂^2 and ∂^4 begin to differ substantially. This indicates that the derivative expansion may not converge in the strong coupling phase $\tilde{G} > \tilde{G}_\star$. However, since we undoubtedly live in a weakly coupled phase, this should have few phenomenological consequences.

Finally, we note that at the Reuter fixed point the redundant operators (5.32) are given by

$$\mathcal{T}[g_{\mu\nu}] = \int_x \sqrt{\det g}\left(-0.0079673\,k^4 - 0.028948k^2\,R\right)\,, \tag{5.75}$$

at order ∂^2, and by

$$\begin{aligned}
\mathcal{T}[g_{\mu\nu}] &= \int_x \sqrt{\det g}\,\big(-0.0079428\,k^4 - 0.030241k^2\,R \qquad (5.76)\\
&\qquad -0.0028418\,R^2 + 0.0048354\,R_{\mu\nu}R^{\mu\nu} - 0.00072986\,E\big)\,,\\
\mathcal{T}[Rg_{\mu\nu}] &= \int_x \sqrt{\det g}\left(-0.0016664\,k^6 - 0.0073873k^4\,R - 0.029686\,k^2\,R^2\right)\,, \qquad (5.77)\\
\mathcal{T}[R_{\mu\nu}] &= \int_x \sqrt{\det g}\,\big(-0.0016664\,k^6 - 0.00055327k^4\,R \qquad (5.78)\\
&\qquad -0.016115\,k^2\,R^2 + 0.033644\,k^2\,R_{\mu\nu}R^{\mu\nu} + 0.00041544\,k^2\,E\big)\,,
\end{aligned}$$

at order ∂^4. It is straightforward to show that these operators (5.75) and (5.76) are linearly independent of the terms in the Reuter fixed point action and form a complete basis at orders ∂^2 and ∂^4, respectively. This confirms that the RG conditions which we choose to fix the inessential couplings at the GFP continue to fix the values of the inessential couplings at the interacting Reuter fixed point.

5.7 Discussion and Outlook

We have investigated the non-perturbative renormalizability of gravity [158] taking care to disregard the running of inessential couplings for the first time. The consequences of

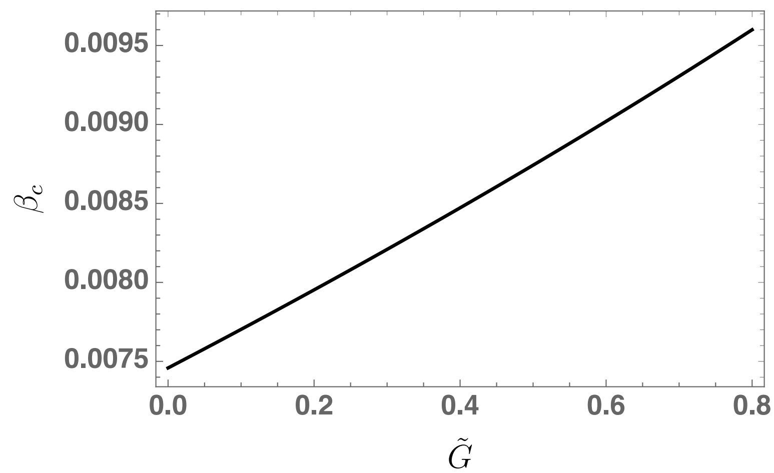


Figure 5.5: The beta function $\beta_c = \partial_t c_k$ in the order ∂^4 approximation.

doing so are profound: not only are calculations much simpler in the minimal essential scheme, but we also reveal that only Newton's constant is essential and relevant in our approximation.

Although this conclusion could change by including higher-order terms, this seems unlikely since all higher-order terms are canonically irrelevant and, thus, the quantum correction to their scaling dimensions would have to be large. Additionally, the stability of the fixed point going from order ∂^2 to order ∂^4 indicates that our approximations do not miss another relevant coupling. Moreover, the Goroff–Sagnotti term, which is the only ∂^6 term that is independent of the Ricci curvature, has been found to be irrelevant at the Reuter fixed point [201]. As a result we expect that the qualitative picture obtained here at order $s = 4$ will not change as we go to higher orders. Ultimately, this can be confirmed by systematically increasing the order of the derivative expansion. This program will be technically simpler within the minimal essential scheme since there will be less terms in the EAA than in the standard approach [184], which does not remove redundant operators. Furthermore, it has been argued that additional poles in the propagator can prevent the convergence of the derivative expansion in quantum gravity [195]. However, in the minimal essential scheme we can avoid such poles and thus we expect to see convergence of the derivative expansion as is observed for scalar field theories [85].

Apart from strengthening the evidence in favour of the existence of the Reuter fixed point, we can also give a straightforward argument in favour of the theory being unitary, since the terms that contain four derivatives are redundant. This property will be true of all higher derivatives if the fixed point can be found in the minimal essential scheme, which assumes their absence from the beginning. Consequently, the minimal essential scheme provides a framework to address some of the open problems for the asymptotic safety program [181, 202] which concern the form of the propagator. We should stress that by using the minimal essential scheme we can dictate which physical degrees of freedom we are attempting to renormalize, and, thus, ensure that we are dealing with a unitary theory, rather than searching in a space of theories littered with non-unitary ones. In calculations that retain terms outside of those in the minimal essential scheme, we expect to find many fixed points which lie in different universality classes. In fact, studies that include many powers of the Riemann curvature have found fixed points with as many as four relevant directions [203].

Perhaps most profoundly, we have identified the vacuum energy as inessential coupling which agrees with other arguments [204]. The fact that it is true at the GFP makes this a property of perturbative quantum gravity. One can show that the contributions proportional to w_1 and w_2 in the linearized beta function (5.51) come from the terms proportional to Ψ_k in the RHS of the flow Equation (5.14) and terms proportional to $\tilde{\rho}_{\rm GFP}$ from the LHS of the flow equation. Reinstating powers of Planck's constant $\hbar$, one sees that both contributions vanish when $\hbar = 0$. This means that the inessential nature of the vacuum energy is a quantum effect. In a scheme where $\tilde{\rho}_{\rm GFP} = 0$ the classical term in the redundant operator would vanish at the GFP, but the contributions due to Ψ_k in the RHS of the flow equation mean ρ_k is anyway inessential. The elementary understanding of this effect is that a rescaling of field $\hat{g} \to \Omega \hat{g}$ produces an infinite factor $\sim \prod_x \Omega$ in the functional measure which when regularized will renormalize the vacuum energy [188]. In the flow equation for the EAA, this manifests in the term proportional to γ_g in the RHS of the flow equation. Thus, simply by renormalizing the quantum metric field, we can adjust the renormalization of the vacuum energy. Since, in the universality class we have investigated, the vacuum energy is inessential both at the GFP and the Reuter fixed point, no physical meaning can be attributed to its flow. However, there can be other universality classes, both for pure gravity and for gravity coupled to matter, where the cosmological constant is essential and its flow has physical consequences [205, 206, 207].

Since there is only one relevant essential coupling at the Reuter fixed point, it would appear that the vanishing of the cosmological constant in Planck units $\tau_0 = \Lambda_k G_k|_{k=0}$ at $k = 0$ must be a prediction of the Reuter fixed point. Thus, if a different scheme would find a non-vanishing τ_0 it would be a contradiction that could only be explained as an artefact of an approximation. To investigate this, one can refrain from fixing the renormalization of $\tilde{\rho}$, as in the minimal essential scheme, but instead only assume γ_g vanishes at $\tilde{G} = 0$. Then, expanding $\partial_t \tau_k$ around $\tilde{G} = 0$ while keeping τ_k, one finds at order ∂^2 that

$$\partial_t \tau_k = -\frac{14 \tilde{G} \tau_k}{3\pi} + O(G^2)\,, \tag{5.79}$$

which implies that τ_0 could take a non-zero value. Studying the full beta functions with $\gamma_g = 0$ one finds trajectories leaving the Reuter fixed point and ending at any value $\tau_0 < 0$, contradicting the minimal essential scheme. However, going to order ∂^4 one finds instead that

$$\partial_t \tau_k = -\frac{328\tau_k^2}{3(20\pi - 7\tau_k)} + O(G^2)\,, \tag{5.80}$$

which only vanishes at $\tau_k = 0$ and thus no contradiction with the minimal essential scheme can occur. Thus, the vanishing of the observable τ_0 appears to be a robust prediction of the Reuter fixed point.

Here we have only treated pure gravity and thus to properly address the cosmological constant problem we should understand the situation when matter is coupled to gravity [208]. In fact, arguably there was never a cosmological constant problem in pure gravity since if we adopt dimensional regularization only terms proportional to ρ_k would renormalize ρ_k and we can simply set $\rho_k = 0$. What will remain true even in the presence of matter is that there is an inessential coupling related to a rescaling of the spacetime metric. This might shed new light on the cosmological constant problem [209].

This work can be extended in several directions. A crucial test is to make sure that the qualitative picture is stable when the form of the cut-off function is modified. Moreover, to obtain the best numerical estimate of the critical exponent θ, the principle of minimal sensitivity (PMS) can be applied by studying the dependence of θ on unphysical parameters, such as those which enter a class of cut-off functions or the values of inessential couplings, such as the vacuum energy. The PMS selects the value of θ where this dependence is minimal (for a recent application of the PMS to the critical exponents of the Ising model see [85]). Furthermore, the dependence on the parameterization of the metric tensor and the choice of gauge [210, 211, 212, 213] can also be investigated within the minimal essential scheme. In the background field approximation, we neglect the running of these parameters, while a proper treatment of these parameters should identify them with inessential couplings since they cannot enter expressions for observables. Thus, going beyond the background field approximation, the minimal essential scheme should include extra gamma functions in order to impose renormalization conditions for each unphysical parameter. As an alternative, one can use diffeomorphism and parameterization invariant exact renormalization equations, such as those based on the geometrical effective action [214, 215] or the background independent exact renormalization group [216].

The finding that there appears to be only one relevant essential coupling in QEG is an encouraging sign for attempts to make contact with other methods which can be used to investigate asymptotic safety. In particular, it would be very interesting if perturbative methods based on expansions around two dimensions [188, 217, 218] could also calculate the critical exponent θ by performing a two-loop calculation in the minimal essential scheme. Additionally, the value of θ can be computed in lattice and tensor model approaches to quantum gravity [219, 220, 221, 222].

Conclusion

'A rose by any other name would smell as sweet.'
William Shakespeare

Any description of Nature that we write down as a mathematical model will always depend on how we choose to parameterize or label physical objects (whether we make this decision consciously or not). On the other hand, Nature does not depend on how we label things. However, taking the attitude that "any parameterization will do" is not practical since solving a model is typically simpler by parameterizing the physics in a particular way. A better attitude is to first identify which parameters of the model are inessential and tune them to simplify the task of solving the model. K. Wilson's exact Renormalization Group embodies the attitude to physics in which one does not write down a model but rather computes the model by solving a flow equation. In essential schemes, we adopt both attitudes such that we are not solving for the inessential couplings but only the for essential ones. This way, what we solve for is not the mathematical model but only those physical quantities we are ultimately interested in. This distinction is very clear when we compute critical exponents at a critical point. In both the standard scheme and essential schemes we will get a spectrum of critical exponents. However, it is the spectrum of the latter that will only contain critical exponents which characterize a physical scaling law realized in Nature. As such, one should bear in mind that in the standard scheme not all critical exponents will be physical and that if we assume that they are, we can come to incorrect conclusions. In particular, there is nothing to prevent an inessential coupling to appear relevant in some schemes, as we have seen in the case of the vacuum energy in Chapter 5, and therefore to give an incorrect counting of the number of relevant couplings at a non-trivial fixed point.

Appendices

Appendix A

Properties of the dilatation operator

In this Appendix we present the main passages in order to demonstrate Equation (2.29), which is related to ψ_{dil}, and identity (4.62), needed to find the dimensionless version of the flow equation for EAA given in Equation (4.66). Let us show that the term $-y^\mu \partial_\mu$ in ψ_{dil}, given in (4.59), counts the number of derivatives. Denoting

$$\partial_r = \partial_{\mu_1} ... \partial_{\mu_r} \,, \tag{A.1}$$

then if

$$\Phi[\varphi] = \Phi(\varphi(y), \partial_{\mu_1}\varphi(y), ...) = O(\partial^s) \,, \tag{A.2}$$

such that

$$\Xi[\varphi] = \int_y \Phi[\varphi] \,, \tag{A.3}$$

we have that

$$\sum_r r \frac{\partial \, \Phi}{\partial \, \partial_r \varphi(x)} \partial_r \varphi(x) = s \Phi(x) \,. \tag{A.4}$$

Additionally we have that

$$[\partial_r, y^\mu \partial_\mu] = r \partial_r \,, \tag{A.5}$$

which can be proved by induction. Then using the above identities and integrating by parts we have that

$$\begin{aligned} y^\mu \partial_\mu \varphi \cdot \frac{\delta}{\delta \varphi} \int_y \Phi(y) &= \int_y \sum_r \frac{\partial \, \Phi}{\partial \, \partial_r \varphi(y)} \partial_r y^\mu \partial_\mu \varphi(x) \\ &= s \int_y \Phi + \int_y \sum_r \frac{\partial \, \Phi}{\partial \, \partial_r \varphi(y)} y^\mu \partial_\mu \partial_r \varphi(y) \\ &= s \int_y \Phi + \int_y y^\mu \partial_\mu \Phi = (s - d) \int_y \Phi \,. \end{aligned}$$

Finally adding this contribution to the multiplicative contribution of $\psi_{\rm dil}$ we obtain Equation (2.29) .

Let us now prove the identity (4.62)

$$\mathrm{Tr}\frac{1}{\Gamma_t^{(2)}[\varphi]+R}\cdot\frac{\delta}{\delta\varphi}\psi_{\rm dil}[\varphi]\cdot R=\frac{1}{2}\mathrm{Tr}\frac{1}{\Gamma_t^{(2)}[\varphi]+R}\cdot\dot{R}\,. \tag{A.6}$$

In order to lighten the notation we drop the spacetime indexes, but it is clear that $\partial_y\, y = \partial_q\, q = d$. Starting from the RHS of identity (4.62) we have

$$\begin{aligned}\mathrm{Tr}\frac{1}{\Gamma_t^{(2)}[\varphi]+R}\cdot\frac{\delta}{\delta\varphi}\psi_{\rm dil}[\varphi]\cdot R &= \int_{y_1,y_2,y_3}\mathcal{G}(y_1,y_2)\,\frac{\delta\psi_{dil}(y_3)}{\delta\phi(y_2)}\,R(y_3,y_1)\\ &=\int_{y_1,y_2}\mathcal{G}(y_1,y_2)\,\left(-y_3\,\partial_{y_3}\delta(y_3-y_2)-\frac{d-2}{2}\delta(y_3-y_2)\right)\,R(y_3,y_1)\\ &=\int_{y_1,y_2}\mathcal{G}(y_1,y_2)\,\left(y_2\,\partial_{y_2}+d-\frac{d-2}{2}\right)\,R(y_2,y_1)\\ &=\int_{y_1,y_2}\int_q\mathcal{G}(y_1,y_2)\,\left(-\mathrm{i}y_2\,q+\frac{d}{2}+1\right)\,R(q^2)e^{-\mathrm{i}q(y_2-y_1)}\,.\end{aligned}$$

Then we can rewrite the non-trivial part of the previous expression as

$$\begin{aligned}\int_{y_1,y_2}\int_q\mathcal{G}(y_1,y_2)\,(\mathrm{i}y_2\,q)\,R(q^2)e^{-\mathrm{i}q(y_2-y_1)} &=\frac{1}{2}\int_{y_1,y_2}\int_q\mathcal{G}(y_1,y_2)\,\mathrm{i}\,(y_2\,-y_1)\,q\,R(q^2)e^{-\mathrm{i}q(y_2-y_1)}\\ &=\frac{1}{2}\int_{y_1,y_2}\int_q\mathcal{G}(y_1,y_2)\,q\,R(q^2)\,\left(-\,\partial_q e^{-iq(y_2-y_1)}\right)\\ &=\frac{1}{2}\int_{y_1,y_2}\int_q\mathcal{G}(y_1,y_2)\,\partial_q\,\left(q\,R(q^2)\right)\,e^{-\mathrm{i}q(y_2-y_1)}\\ &=\frac{1}{2}\int_{y_1,y_2}\int_q\mathcal{G}(y_1,y_2)\,\left(d\,R(q^2)+q\,\partial_q R(q^2)\right)\,e^{-\mathrm{i}q(y_2-y_1)}\,,\end{aligned}$$

where in the first passage we just write y_2 as $(y_2+y_2)/2$ and then in the second term we exchange y_1 and y_2 using the symmetry of the propagator and send $q\to -q$. So putting everything together

$$\begin{aligned}\int_{y_1,y_2}\int_q\mathcal{G}(y_1,y_2)\,\left(iy_2\,q-\frac{d}{2}+1\right)\,R(q^2)e^{-\mathrm{i}q(y_2-y_1)} &=\int_{y_1,y_2}\int_q\mathcal{G}(y_1,y_2)\,\left(1-q^2\partial_{q^2}\right)\,R(q^2)e^{-\mathrm{i}q(y_2-y_1)}\\ &=\frac{1}{2}\mathrm{Tr}\frac{1}{\Gamma_t^{(2)}[\varphi]+R}\cdot\dot{R}\,,\end{aligned}$$

where $\dot{R}(\Delta):=2(R(\Delta)-\Delta R'(\Delta))$, given in Equation (4.63).

Appendix B

Derivation of the pseudo-regulator

In this section we present a derivation of the functional form of the $\overline{\text{MS}}$ pseudo-regulator in Equation (3.27). We want to obtain the result (3.15) from the FRGE. Inspired by the deformation of integrands which takes place in dimensional regularization, we consider the following family of regulators

$$\mathcal{R}_k(z) = \mu^{-2\epsilon} F(k, \mu, m, \epsilon) z^{1+\epsilon} - z \ , \tag{B.1}$$

where F is an arbitrary dimensionless function and μ is a classical arbitrary mass parameter. The second term in (B.1) is there to cancel the original inverse propagator of the bare theory. Note that $\partial_t \mathcal{R}_k(z)$ has two contributions: one coming from the explicit dependence of F on k and another one proportional to $\beta_{m^2} = \partial_t m^2$. Assuming that $\partial_t F \propto \epsilon$ and using the following identity

$$\Gamma\left[-n+\epsilon\right] = \frac{(-1)^n}{\Gamma[n+1]} \frac{1}{\epsilon} + O(\epsilon^0) \ , \tag{B.2}$$

the Mellin transform of the first term inside $\partial_t \mathcal{R}_k$ is

$$\frac{\partial_t F}{F^{1+\frac{n}{1+\epsilon}}} \left(\frac{\mu}{m}\right)^{\frac{2n\epsilon}{1+\epsilon}} \frac{\Gamma\left(1+\frac{n}{1+\epsilon}\right)\Gamma(l-n-1+\frac{n\epsilon}{1+\epsilon})}{(1+\epsilon)\Gamma(n)\Gamma(l)\left(m^2\right)^{-(n-l+1)}} = \frac{\partial_t F}{\epsilon\, F^{1+n}} \frac{\left(-m^2\right)^{n-l+1}}{\Gamma(l)\Gamma(n-l+2)} + O(\epsilon) \ . \tag{B.3}$$

So taking the limit for $\epsilon \to 0$, we find (3.15) if

$$\partial_t F(k, \mu, m, \epsilon) = 2\epsilon F(k, \mu, m, \epsilon)^{1+n} \ , \tag{B.4}$$

that is

$$F = 1 + \epsilon \log\left(\frac{k^2}{\mu^{2-2b} m^{2b}}\right) + O(\epsilon^2) \approx \left(\frac{k^2}{\mu^{2-2b} m^{2b}}\right)^{\epsilon} . \tag{B.5}$$

With this F , the second piece of $\partial_t \mathcal{R}_k$ proportional to β_{m^2} can be calculated in the same way. This agrees with (3.27), of which (3.19) is a special case, corresponding to $b = 0$. As described in the main text, different choices of b only affect higher-order corrections.

Appendix C

Flow equations at order ∂^2 for the linear $O(N)$ model

The flow equations for U_k and for $\tilde{Z}_k$, which is defined in (3.70), read

$$\partial_t U_k = \frac{Q_{\frac{d}{2}}\left[G_1 \partial_t \mathcal{R}_k\right] + (N-1) Q_{\frac{d}{2}}\left[G_0 \partial_t \mathcal{R}_k\right]}{2(4\pi)^{d/2}} \;, \tag{C.1}$$

$$\begin{aligned}
\partial_t \tilde{Z}_k = & -\frac{\left(\tilde{Z}_k^{(1)} + 2\rho \tilde{Z}_k^{(2)}\right)}{2(4\pi)^{d/2}} Q_{\frac{d}{2}}\left[G_1^2 \partial_t \mathcal{R}_k\right] - (N-1)\frac{\left(Z_k^{(1)} + \rho Y_k^{(1)}\right)}{2(4\pi)^{d/2}} Q_{\frac{d}{2}}\left[G_0^2 \partial_t \mathcal{R}_k\right] \\
& + \frac{2\rho\left(\tilde{Z}_k^{(1)}\right)^2}{(4\pi)^{d/2}}\left\{\frac{2d+1}{2} Q_{\frac{d}{2}+1}\left[G_1^3 \partial_t \mathcal{R}_k\right] + \frac{(d+2)(d+4)}{4}\left(Q_{\frac{d}{2}+2}\left[G_1^2 G_1' \partial_t \mathcal{R}_k\right] + Q_{\frac{d}{2}+3}\left[G_1^2 G_1'' \partial_t \mathcal{R}_k\right]\right)\right\} \\
& + \frac{2\rho\left(3U_k^{(2)} + 2\rho U_k^{(3)}\right)^2}{(4\pi)^{d/2}}\left(Q_{\frac{d}{2}}\left[G_1^2 G_1' \partial_t \mathcal{R}_k\right] + Q_{\frac{d}{2}+1}\left[G_1^2 G_1'' \partial_t \mathcal{R}_k\right]\right) \\
& + \frac{2\rho \tilde{Z}_k^{(1)}\left(3U_k^{(2)} + 2\rho U_k^{(3)}\right)}{(4\pi)^{d/2}}\left\{(d+2)\left(Q_{\frac{d}{2}+1}\left[G_1^2 G_1' \partial_t \mathcal{R}_k\right] + Q_{\frac{d}{2}+2}\left[G_1^2 G_1'' \partial_t \mathcal{R}_k\right]\right) + 2Q_{\frac{d}{2}}\left[G_1^3 \partial_t \mathcal{R}_k\right]\right\} \\
& + (N-1)\frac{\rho Y_k}{(4\pi)^{d/2}}\left(2U_k^{(2)} Q_{\frac{d}{2}}\left[G_0^3 \partial_t \mathcal{R}_k\right] + d Z_k^{(1)} Q_{\frac{d}{2}+1}\left[G_0^3 \partial_t \mathcal{R}_k\right]\right) \\
& + (N-1)\frac{2\rho\left(Z_k^{(1)}\right)^2}{(4\pi)^{d/2}}\left\{\frac{(d+2)(d+4)}{4}\left(Q_{\frac{d}{2}+2}\left[G_0^2 G_0' \partial_t \mathcal{R}_k\right] + Q_{\frac{d}{2}+3}\left[G_0^2 G_0'' \partial_t \mathcal{R}_k\right]\right) + \frac{1}{2} Q_{\frac{d}{2}+1}\left[G_0^3 \partial_t \mathcal{R}_k\right]\right\} \\
& + (N-1)\frac{2\rho\left(U_k^{(2)}\right)^2}{(4\pi)^{d/2}}\left(Q_{\frac{d}{2}}\left[G_0^2 G_0' \partial_t \mathcal{R}_k\right] + Q_{\frac{d}{2}+1}\left[G_0^2 G_0'' \partial_t \mathcal{R}_k\right]\right) \\
& + (N-1)\frac{2\rho Z_k^{(1)} U_k^{(2)}}{(4\pi)^{d/2}}(d+2)\left(Q_{\frac{d}{2}+1}\left[G_0^2 G_0' \partial_t \mathcal{R}_k\right] + Q_{\frac{d}{2}+2}\left[G_0^2 G_0'' \partial_t \mathcal{R}_k\right]\right) \;,
\end{aligned} \tag{C.2}$$

where we introduce the following notations

$$G_0 = \left(Z_k q^2 + \mathcal{R}_k(q^2) + U_k^{(1)} \right)^{-1}, \tag{C.3a}$$

$$G_1 = \left(\tilde{Z}_k q^2 + \mathcal{R}_k(q^2) + U_k^{(1)} + 2\rho U_k^{(2)} \right)^{-1}, \tag{C.3b}$$

for the Goldstone-bosons and radial-mode propagators. The equations presented above are the flow equations at order ∂^2 of the derivative expansion for the linear $O(N)$ model, which can be found for instance in Ref. [138]. These descend from the exact FRG equation upon specifying the truncation of (3.69). The beta functional for Z_k is instead presented with slightly different notations in Appendix D.2, more precisely in (D.56).

Appendix D

Two-loop calculation for ϕ^4 theory

D.1 Threshold functions for a mass-dependent pseudo-regulator

In this appendix we detail the computation of the following threshold functions

$$l_{n,0}^d(0) := \frac{nZ_k^n}{2} k^{2n-d} \int_0^\infty \mathrm{d}z \, z^{\frac{d}{2}-1} \frac{\partial_t \mathcal{R}_k(z)}{P_k(z)^{n+1}} \ , \tag{D.1}$$

where $P_k(z) := Z_k z + \mathcal{R}_k(z)$, by means of the mass-dependent pseudo-regulator of (3.88). As we need the result for the computation of the two-loop beta function in four dimensional $\lambda\phi^4$ theory, we content ourselves of the first orders in a perturbative expansion in λ. In particular, we neglect the η dependence appearing on the RHS of the flow equations through the regularization, as it would lead to higher orders in λ. Our pseudo-regulator choice results in simple propagators but a somewhat more convoluted contribution of the differentiated pseudo-regulator:

$$P_k(z) = Z_k \left(\frac{k^2}{\mu^{4-2b} M^{2b}} \right)^{\epsilon} \left(z + M^2 \right)^{1+\epsilon} , \tag{D.2a}$$

$$\partial_t \mathcal{R}_k(z) = 2\epsilon \left(1 - \frac{b\, \partial_t M^2}{2M^2} \right) P_k(z) + (1+\epsilon) \partial_t M^2 \, P_k(z) - \beta_{m^2} \ . \tag{D.2b}$$

The loop integral can then be split into three different kinds of contributions, corresponding to the three pieces of $\partial_t \mathcal{R}_k$

$$\begin{aligned} \frac{2\, l_{n,0}^d}{nk^{2n-d}} = {} & 2\epsilon \left(1 - \frac{b\, \partial_t M^2}{2M^2} \right) \left(\frac{k^2}{\mu^{4-2b} M^{2b}} \right)^{-n\epsilon} \frac{\Gamma\left(n + n\epsilon - \frac{d}{2}\right)}{\Gamma(n+n\epsilon)} \left(M^2 \right)^{\frac{d}{2}-n-n\epsilon} \\ & + (1+\epsilon) \partial_t M^2 \left(\frac{k^2}{\mu^{4-2b} M^{2b}} \right)^{-n\epsilon} \frac{\Gamma\left(n+1+n\epsilon - \frac{d}{2}\right)}{\Gamma(n+1+n\epsilon)} \left(M^2 \right)^{\frac{d}{2}-n-1-n\epsilon} \\ & - \beta_{m^2} \left(\frac{k^2}{\mu^{4-2b} M^{2b}} \right)^{-(n+1)\epsilon} \frac{\Gamma\left(n+1+(n+1)\epsilon - \frac{d}{2}\right)}{\Gamma(n+1+(n+1)\epsilon)} \left(M^2 \right)^{\frac{d}{2}-(1+\epsilon)(n+1)} \ . \end{aligned} \tag{D.3}$$

To extract the $\epsilon \to 0$ asymptotics we make use of the standard expansion

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{\Gamma(n+1)}\left[\frac{1}{\epsilon} - \gamma + h(n)\right] + o(\epsilon) \ , \tag{D.4}$$

where $h(n) = \sum_{i=1}^{n} \frac{1}{i}$. Furthermore, we need to parameterize the possible dependence of M^2 on ϵ. Recalling that for vanishing ϵ also $\mathcal{R}_k$ needs to vanish, i.e. M^2 should reduce to m^2, we can write

$$M^2 = m^2 + \epsilon\, m_1^2(k,m,\mu) + O\left(\epsilon^2\right) \ , \tag{D.5}$$

$$\partial_t M^2 = \beta_{m^2}\left(f_0 + \epsilon F_1(k,m,\mu)\right) + O\left(\epsilon^2\right) \ . \tag{D.6}$$

Here m_1^2 and F_1 are two independent functions and f_0 is a proportionality factor. Thus, we allow for the possibility that $\lim_{\epsilon\to 0}\partial_t M^2 \neq \partial_t \lim_{\epsilon\to 0} M^2$, which can be achieved e.g. by means of the choice

$$M^2 = \left(1 + (f_0 - 1)\int_{\epsilon^2}^{\epsilon^2 \frac{m^2}{\mu^2}} \mathrm{d}s\,\Gamma(s)\right) m^2 + \epsilon\, m_1^2 \ . \tag{D.7}$$

The need for this behavior of M^2 can be appreciated by inspecting the integrals

$$\begin{aligned}
\left.\frac{2\, l_{n,0}^d}{n k^{2n-d}}\right|_{n\leq \frac{d}{2}-1} &= \frac{\beta_{m^2}\left((n+1)f_0 - n\right)\left(-m^2\right)^{\frac{d}{2}-n-1}}{n\Gamma(n+2)\Gamma\left(\frac{d}{2}-n\right)\epsilon} + \frac{2\left(1 - \frac{b\, f_0\, \beta_{m^2}}{2m^2}\right)\left(-m^2\right)^{\frac{d}{2}-n}}{\Gamma(n+1)\Gamma\left(\frac{d}{2}-n+1\right)} \\
&+ \frac{\beta_{m^2}\left(-m^2\right)^{\frac{d}{2}-n-1}\left\{n(1-f_0)\left[\log\left(\frac{k^2}{\mu^{4-2b}m^{2b-2}}\right) + h(n) - h\left(\frac{d}{2}-n-1\right)\right] + f_0 + F_1(k)\right\}}{n\Gamma(n+1)\Gamma\left(\frac{d}{2}-n\right)} \\
&- \frac{\beta_{m^2} m_1^2\left(-m^2\right)^{\frac{d}{2}-n-2}\left((n+1)f_0 - n\right)}{n\Gamma(n+2)\Gamma\left(\frac{d}{2}-n-1\right)} + o(\epsilon) \ .
\end{aligned} \tag{D.8}$$

These exhibit a $1/\epsilon$ pole which can be eliminated by tuning $f_0 \neq 1$. To fulfill this, as well as the condition of removing the renormalization scale k from the beta functions, we set

$$f_0 = \frac{n}{n+1} \ , \tag{D.9}$$

$$F_1 = f_1 + \frac{n}{n+1}\left(h\left(\frac{d}{2}-n-1\right) - h(n) - 1 - \log\left(\frac{k^2}{\mu^{4-2b}m^{2b-2}}\right)\right) \ , \tag{D.10}$$

$$\begin{aligned}
m_1^2 = &\left[f_1 + \frac{n}{n+1}\left(h\left(\frac{d}{2}-n-1\right) - h(n) - 1 - \log\left(\frac{k}{\mu}\right)\right)\right]\beta_{m^2}\log\frac{k}{\mu} \\
&+ (b-1)\frac{n}{2(n+1)} m^2\left(\log\frac{m^2}{\mu^2}\right)^2 + O(\lambda^2) \ .
\end{aligned} \tag{D.11}$$

As a result we have

$$\begin{aligned}
\left.\frac{2\, l_{n,0}^{d}}{n k^{2n-d}}\right|_{n\leq\frac{d}{2}-1} &= \frac{2\left(1-\frac{b\,n\,\beta_{m^2}}{2(n+1)m^2}\right)\left(-m^2\right)^{\frac{d}{2}-n}}{\Gamma(n+1)\Gamma\left(\frac{d}{2}-n+1\right)} + f_1 \frac{\beta_{m^2}\left(-m^2\right)^{\frac{d}{2}-n-1}}{n\Gamma(n+1)\Gamma\left(\frac{d}{2}-n\right)} + o(\epsilon) \\
&= \frac{2\left(-m^2\right)^{\frac{d}{2}-n}}{\Gamma(n+1)\Gamma\left(\frac{d}{2}-n+1\right)} - \frac{\beta_{m^2}}{m^2}\frac{\left(-m^2\right)^{\frac{d}{2}-n}}{\Gamma(n+2)\Gamma\left(\frac{d}{2}-n\right)}\left[\frac{b\,n}{\frac{d}{2}-n} + f_1\frac{n+1}{n}\right] + o(\epsilon)\ . \qquad \text{(D.12)}
\end{aligned}$$

Recall that M^2 must be analytic around $m^2 = 0$. From (D.11) we see that this can be achieved only if $b = 1$.

On the other hand, the remaining loop integrals are harmless, as they read

$$\frac{2\, l_{n,0}^{d}}{n k^{2n-d}} = (f_0 - 1)\frac{\beta_{m^2}}{m^2}\frac{\Gamma\left(n-\frac{d}{2}+1\right)}{\Gamma(n+1)}\left(m^2\right)^{\frac{d}{2}-n} + o(\epsilon)\,, \quad \text{for} \quad n \geq \frac{d}{2}-1 \qquad \text{(D.13)}$$

$$\frac{2\, l_{n,0}^{d}}{n k^{2n-d}} = \frac{2}{\Gamma\left(\frac{d}{2}+1\right)}\left[1 - ((b-1)f_0 + 1)\frac{\beta_{m^2}}{2m^2}\right] + o(\epsilon)\,, \quad \text{for} \quad n = \frac{d}{2}\ . \qquad \text{(D.14)}$$

For completeness we list some of these integrals in the lowest even numbers of dimensions. If $d = 2$ there is no divergent l-function, and in particular

$$l_{1,0}^{2}(0) = 1 - \frac{1}{2}\frac{\beta_{m^2}}{m^2}\ , \qquad \text{(D.15a)}$$

$$l_{n>1,0}^{2}(0) = \frac{\beta_{m^2}}{m^2}\frac{(f_0-1)}{2}\left(\frac{k^2}{m^2}\right)^{n-1}\ . \qquad \text{(D.15b)}$$

If $d = 4$, the function $l_{1,0}^4$ which enters in the determination of β_{m^2} has a pole unless we choose $f_0 = 1/2$ according to (D.9). This results in

$$l_{1,0}^{4}(0) = -\frac{m^2}{k^2} + (1+2f_1)\frac{\beta_{m^2}}{4k^2}\ , \qquad \text{(D.16a)}$$

$$l_{2,0}^{4}(0) = 1 - \frac{\beta_{m^2}}{2m^2}\ , \qquad \text{(D.16b)}$$

$$l_{n>2,0}^{4}(0) = -\frac{1}{4(n-1)}\left(\frac{k^2}{m^2}\right)^{n-2}\frac{\beta_{m^2}}{m^2}\ . \qquad \text{(D.16c)}$$

These equations are easily interpreted by applying them e.g. to a $\lambda\phi^4$ theory within the LPA. To zeroth order in λ, i.e. neglecting β_{m^2} on the RHS, we recover the standard result that integrals with negative mass dimension do not contribute to the one-loop beta functions. Moreover the positive dimensional integral leads to the usual one-loop RG equation for the mass

$$\beta_{m^2} = \frac{N+2}{16\pi^2}\lambda m^2 + O\left(\lambda^2\right)\ . \qquad \text{(D.17)}$$

Further details of the pseudo-regulator choice, such as the coefficient f_1, would affect higher perturbative orders. In fact, in Appendix D.2.5 we show that the latter coefficient is fixed by requiring that β_{m^2} agrees with the $\overline{\mathrm{MS}}$ result also at two loops.

D.2 Two-loop computation

In this appendix we detail the computation of the universal part of the two-loop beta function in ϕ^4 theory in four dimensions.

According to our priority, i.e. the computation of β_λ at order λ^3, we first focus on the flow equation for the effective potential

$$\partial_t U_k = \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\partial_t \mathcal{R}_k(q)}{2} \left[\frac{N-1}{M_0(\rho, q^2)} + \frac{1}{M_1(\rho, q^2)} \right], \tag{D.18a}$$

where

$$M_0\big(\rho, q^2\big) = Z_k\big(\rho, q^2\big) q^2 + \mathcal{R}_k(q) + U_k^{(1)}(\rho) \ , \tag{D.18b}$$

$$M_1\big(\rho, q^2\big) = \tilde{Z}_k\big(\rho, q^2\big) q^2 + \mathcal{R}_k(q) + U_k^{(1)}(\rho) + 2\rho U_k^{(2)}(\rho). \tag{D.18c}$$

From this functional equation, the beta functions of the mass and of the quartic coupling can be derived by differentiation with respect to ρ.

$$\begin{aligned} \partial_t U_k^{(1)}(\rho) = & -2v_d(N-1)k^{d-2} Z_k^{-1} U_k^{(2)}(\rho) l_{1,0}^d(0) - 2v_d(N-1)k^d Z_k^{-1} \langle Z_k^{(1)}(\rho)\rangle_{1,0}^{d+2}(0) \\ & - 2v_d k^{d-2} Z_k^{-1} \left(3U_k^{(2)}(\rho) + 2\rho U_k^{(3)}(\rho)\right) l_{0,1}^d(w) - 2v_d k^d Z_k^{-1} \langle \tilde{Z}_k^{(1)}(\rho)\rangle_{0,1}^{d+2}(w) \end{aligned} \tag{D.19}$$

$$\begin{aligned} \partial_t U_k^{(2)}(\rho) = & \, 2v_d k^{d-4} \left((N-1) Z_k^{-2} \left(U_k^{(2)}(\rho)\right)^2 l_{2,0}^d(0) + Z_k^{-2} \left(3U_k^{(2)}(\rho) + 2\rho U_k^{(3)}(\rho)\right)^2 l_{0,2}^d(w) \right) \\ & + 4v_d(N-1)k^{d-2} Z_k^{-2} U_k^{(2)}(\rho) \langle Z_k^{(1)}(\rho)\rangle_{2,0}^{d+2}(0) \\ & + 4v_d k^{d-2} Z_k^{-2} \left(3U_k^{(2)}(\rho) + 2\rho U_k^{(3)}(\rho)\right) \langle \tilde{Z}_k^{(1)}(\rho)\rangle_{0,2}^{d+2}(w) \\ & + 2v_d(N-1)k^d Z_k^{-2} \langle Z_k^{(1)}(\rho)^2\rangle_{2,0}^{d+4}(0) + 2v_d k^d Z_k^{-2} \langle \tilde{Z}_k^{(1)}(\rho)^2\rangle_{0,2}^{d+4}(w) \\ & - 2v_d(N-1)k^{d-2} Z_k^{-1} U_k^{(3)}(\rho) l_{1,0}^d(0) - 2v_d k^{d-2} Z_k^{-1} \left(5U_k^{(3)}(\rho) + 2\rho U_k^{(4)}(\rho)\right) l_{0,1}^d(w) \\ & - 2v_d(N-1)k^d Z_k^{-1} \langle Z_k^{(2)}(\rho)\rangle_{1,0}^{d+2}(0) - 2v_d k^d Z_k^{-1} \langle \tilde{Z}_k^{(2)}(\rho)\rangle_{0,1}^{d+2}(w) \end{aligned} \tag{D.20}$$

Defining ρ_0 as the field expansion point and

$$w_0 = 2\rho_0 \, U_k^{(2)}(\rho_0) \ , \tag{D.21a}$$

$$P_k = Z_k\left(\rho_0, z\right) z + \mathcal{R}_k(z) + U_k^{(1)}(\rho_0) \ , \tag{D.21b}$$

$$\tilde{P}_k = \tilde{Z}_k\left(\rho_0, z\right) z + \mathcal{R}_k(z) + U_k^{(1)}(\rho_0) \ , \tag{D.21c}$$

the flow equations for the two renormalizable couplings read

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}U_k^{(1)}(\rho_0) = \partial_t U_k^{(1)}(\rho_0) + U_k^{(2)}(\rho_0)\frac{\mathrm{d}}{\mathrm{d}t}\rho_0 \\
&= -2v_d(N-1)k^{d-2}Z_k^{-1}U_k^{(2)}(\rho_0)l_{1,0}^d(0) - 2v_d(N-1)k^d Z_k^{-1}\langle Z_k^{(1)}(\rho_0)\rangle_{1,0}^{d+2}(0), \\
&- 2v_d k^{d-2}Z_k^{-1}\left(3U_k^{(2)}(\rho_0) + 2\rho_0 U_k^{(3)}(\rho_0)\right) l_{0,1}^d(w_0) - 2v_d k^d Z_k^{-1}\langle \tilde{Z}_k^{(1)}(\rho_0)\rangle_{0,1}^{d+2}(w_0) \\
&+ U_k^{(2)}(\rho_0)\frac{\mathrm{d}}{\mathrm{d}t}\rho_0 \ ,
\end{aligned}
\tag{D.22}
$$

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}U_k^{(2)}(\rho_0) = \partial_t U_k^{(2)}(\rho_0) + U_k^{(3)}(\rho_0)\frac{\mathrm{d}}{\mathrm{d}t}\rho_0 \\
&= 2v_d(N-1)k^{d-4}Z_k^{-2}\left(U_k^{(2)}(\rho_0)\right)^2 l_{2,0}^d(0) + 2v_d k^{d-4}Z_k^{-2}\left(3U_k^{(2)}(\rho_0) + 2\rho_0 U_k^{(3)}(\rho_0)\right)^2 l_{0,2}^d(w_0) \\
&+ 4v_d(N-1)k^{d-2}Z_k^{-2}U_k^{(2)}(\rho_0)\langle Z_k^{(1)}(\rho_0)\rangle_{2,0}^{d+2}(0) \\
&+ 4v_d k^{d-2}Z_k^{-2}\left(3U_k^{(2)}(\rho_0) + 2\rho_0 U_k^{(3)}(\rho_0)\right)\langle \tilde{Z}_k^{(1)}(\rho_0)\rangle_{0,2}^{d+2}(w_0) \\
&+ 2v_d(N-1)k^d Z_k^{-2}\langle Z_k^{(1)}(\rho_0)^2\rangle_{2,0}^{d+4}(0) + 2v_d k^d Z_k^{-2}\langle \tilde{Z}_k^{(1)}(\rho_0)^2\rangle_{0,2}^{d+4}(w_0) \\
&- 2v_d(N-1)k^{d-2}Z_k^{-1}U_k^{(3)}(\rho_0)l_{1,0}^d(0) - 2v_d k^{d-2}Z_k^{-1}\left(5U_k^{(3)}(\rho_0) + 2\rho_0 U_k^{(4)}(\rho_0)\right) l_{0,1}^d(w_0) \\
&- 2v_d(N-1)k^d Z_k^{-1}\langle Z_k^{(2)}(\rho_0)\rangle_{1,0}^{d+2}(0) - 2v_d k^d Z_k^{-1}\langle \tilde{Z}_k^{(2)}(\rho_0)\rangle_{0,1}^{d+2}(w_0) \\
&+ U_k^{(3)}(\rho_0)\frac{\mathrm{d}}{\mathrm{d}t}\rho_0 \ .
\end{aligned}
\tag{D.23}
$$

Here we adopted standard notations for the loop integrals

$$
l_{n_1,n_2}^d(w) = -\frac{Z_k^{n_1+n_2}}{2}k^{2(n_1+n_2)-d}\int_0^\infty \mathrm{d}z\, z^{\frac{d}{2}-1}\partial_t\left\{P_k(z)^{-n_1}(\tilde{P}_k(z)+w)^{-n_2}\right\} \ , \tag{D.24a}
$$

$$
\langle D_k(\rho_0)\rangle_{n_1,n_2}^d(w) = -\frac{Z_k^{n_1+n_2}}{2}k^{2(n_1+n_2)-d}\int_0^\infty \mathrm{d}z\, z^{\frac{d}{2}-1}D_k(\rho_0,z)\partial_t\left\{P_k(z)^{-n_1}(\tilde{P}_k(z)+w)^{-n_2}\right\} \ , \tag{D.24b}
$$

and $v_d^{-1} = 2(4\pi)^{d/2}\Gamma(d/2)$. Notice however that our convention for the inverse propagators P_k and $\tilde{P}_k$ slightly departs from the most common choice [69], in that we include the mass parameter $\bar{m}^2$ therein. Furthermore, while ρ_0 is usually chosen as the running minimum of the potential, such that $\rho_0 > 0$ corresponds to a regime of spontaneous symmetry breaking, we instead assume that $U_k^{(1)}(\rho_0) > 0$. We can safely choose $\rho_0 = 0$ for our goals, as no dynamical symmetry breaking is within reach of a two-loop computation in the present model. Equations (D.22) and (D.23) can be rewritten as

$$
\begin{aligned}
\left(\beta_{m^2} - \eta m^2\right) k^{-2} &= \lambda \left((d-2+\eta)\, \kappa + \partial_t \kappa \right) \\
&\quad - 2v_d (N-1) \left(\lambda\, l^d_{1,0}(0) + \langle z_1 \rangle^{d+2}_{1,0}(0) \right) \\
&\quad - 2v_d \left(3\lambda + 2\kappa u_3\right) l^d_{0,1}(2\lambda\kappa) - 2v_d \left\langle \tilde{z}_1 \right\rangle^{d+2}_{0,1} (2\lambda\kappa) \,, \qquad \text{(D.25)} \\
\beta_\lambda &= (d-4+2\eta)\, \lambda + u_3 \left((d-2+\eta)\kappa + \partial_t \kappa \right) \\
&\quad + 2v_d (N-1) \lambda^2 l^d_{2,0}(0) + 2v_d \left(3\lambda + 2\kappa u_3\right)^2 l^d_{0,2}(2\lambda\kappa) \\
&\quad - 2v_d (N-1) u_3 l^d_{1,0}(0) - 2v_d \left(5u_3 + 2\kappa u_4\right) l^d_{0,1}(2\lambda\kappa) \\
&\quad + 4v_d (N-1) \lambda \langle z_1 \rangle^{d+2}_{2,0}(0) \\
&\quad + 4v_d \left(3\lambda + 2\kappa u_3\right) \left\langle \tilde{z}_1 \right\rangle^{d+2}_{0,2} (2\lambda\kappa) \\
&\quad - 2v_d (N-1) \langle z_2 \rangle^{d+2}_{1,0}(0) - 2v_d \langle \tilde{z}_2 \rangle^{d+2}_{0,1} (2\lambda\kappa) \\
&\quad + 2v_d (N-1) \langle z_1^2 \rangle^{d+4}_{2,0}(0) + 2v_d \langle \tilde{z}_1^2 \rangle^{d+4}_{0,2} (2\lambda\kappa) \,, \qquad \text{(D.26)}
\end{aligned}
$$

where $\eta = -\partial_t \log Z_k$ is the field anomalous dimension. As described in the main text, introducing the powercounting of (3.84), which is generated by the flow equation itself, into Equations (D.25) and (D.26), and truncating them to order λ^3, result in the simplified perturbative Equations (3.85) and (3.86) for $d = 4$. In the following we address the $O(\lambda^3)$ contributions arising on the RHS of (3.86), organizing them line by line, as these also correspond to different kinds of corrections.

D.2.1 Mass beta function's contribution

Using the previous pseudo-regulator and the one-loop beta function for m^2 the threshold functions can be expanded at leading order in λ, as in (3.89). Inserting this into the beta function (3.86) we get

$$
\begin{aligned}
\beta_\lambda =& \frac{N+8}{16\pi^2}\lambda^2 - \frac{(N+8)(N+2)}{2(16\pi^2)^2}\lambda^3 + 2\eta\lambda \\
&- \frac{N-1}{16\pi^2} l^4_{1,0}(0) u_3 - \frac{5}{16\pi^2} l^4_{0,1}(2\lambda\kappa) u_3 + 2\kappa u_3 \\
&+ \frac{N-1}{8\pi^2} \lambda \langle z_1 \rangle^6_{2,0}(0) + \frac{3}{8\pi^2} \lambda \langle \tilde{z}_1 \rangle^6_{0,2} (2\lambda\kappa) \\
&- \frac{N-1}{16\pi^2} \langle z_2 \rangle^6_{1,0}(0) - \frac{1}{16\pi^2} \langle \tilde{z}_2 \rangle^6_{0,1} (2\lambda\kappa) \ . \qquad \text{(D.27)}
\end{aligned}
$$

D.2.2 Sextic coupling's contribution

To evaluate the contribution of the sextic coupling generated by the flow equation, it is enough to consider a uniform and field-independent wave function renormalization for all modes, as in the LPA′; that is, it is safe to set $Z_k\left(\rho, q^2\right) = \tilde{Z}_k\left(\rho, q^2\right) = Z_k$ at order $O(\lambda^3)$. The flow of the sextic coupling can be deduced by taking the third derivative of (D.18a)

$$\begin{aligned}\partial_t U_k^{(3)}(\rho) = {} & 4v_d(N-1)\left(k^{d-4}Z_k^{-2}U_k^{(2)}(\rho)U_k^{(3)}(\rho)l_{2,0}^d(0) - k^{d-6}Z_k^{-3}\left(U_k^{(2)}(\rho)\right)^3 l_{3,0}^d(0)\right)\\ & + 4v_d k^{d-4}Z_k^{-2}\left(3U_k^{(2)}(\rho)+2\rho U_k^{(3)}(\rho)\right)\left(5U_k^{(3)}(\rho)+2\rho U_k^{(4)}(\rho)\right)l_{0,2}^d(w)\\ & - 4v_d k^{d-6}Z_k^{-3}\left(3U_k^{(2)}(\rho)+2\rho U_k^{(3)}(\rho)\right)^3 l_{0,3}^d(w)\\ & - 2v_d(N-1)k^{d-2}Z_k^{-1}U_k^{(4)}(\rho)l_{1,0}^d(0) + 2v_d(N-1)k^{d-4}Z_k^{-2}U_k^{(3)}(\rho)U_k^{(2)}(\rho)l_{2,0}^d(0)\\ & - 2v_d k^{d-2}Z_k^{-1}\left(7U_k^{(4)}(\rho)+2\rho U_k^{(5)}(\rho)\right)l_{0,1}^d(w)\\ & + 2v_d k^{d-4}Z_k^{-2}\left(5U_k^{(3)}(\rho)+2\rho U_k^{(4)}(\rho)\right)\left(3U_k^{(2)}(\rho)+2\rho U_k^{(3)}(\rho)\right)l_{0,2}^d(w) \qquad \text{(D.28)}\end{aligned}$$

and evaluating it at $\rho = \rho_0$, such that $w \to w_0$. Using the fact that

$$\partial_t u_3 = (2d-6)u_3 + Z_k^{-3}\left[\partial_t U_k^{(3)}(\rho_0) + U_k^{(4)}(\rho_0)\frac{\mathrm{d}\rho_0}{\mathrm{d}t}\right], \qquad \text{(D.29)}$$

one deduces

$$\partial_t u_3 = (2d-6)u_3 - 4v_d(N+26)\lambda^3 l_{3,0}^d(0) + O\left(\lambda^4\right). \qquad \text{(D.30)}$$

At one loop and for $d = 4$, u_3 is given by the fixed-point solution of the previous equation

$$u_3^{(1-\text{loop})} = \frac{N+26}{16\pi^2}\lambda^3 k^2 Z_k^3 \int_0^\infty \mathrm{d}z\, \frac{z}{P_k^3}\ . \qquad \text{(D.31)}$$

By evaluating the momentum integral with the previous pseudo-regulator we find an expression which is finite in the $\epsilon \to 0$ limit, namely (3.90) in the main text. Now let's compute κ at one loop, by looking for a scaling solution for it, i.e. by solving $\partial_t\kappa = 0$, which gives

$$\beta_{m^2}k^{-2} = \lambda\left(d-2\right)\kappa - 2v_d(N-1)l_{1,0}^d(0)\lambda - 2v_d\left(3\lambda + 2\kappa u_3\right)l_{0,1}^d\left(2\lambda\kappa\right)\ . \qquad \text{(D.32)}$$

Specifying $d = 4$ and using the previous identities we get

$$\frac{N+2}{16\pi^2}\lambda m^2 k^{-2} = 2\lambda\kappa + \frac{(N-1)}{16\pi^2}m^2k^{-2}\lambda + \frac{1}{16\pi^2}3\lambda m^2 k^{-2} + O(\lambda^2)\ . \qquad \text{(D.33)}$$

So as anticipated in the main text, $\kappa = O(\lambda)$ and as such would not affect the $O(\lambda^3)$ of β_λ. Inserting this result for u_3 into the beta function (D.27) we obtain

$$\begin{aligned}\beta_\lambda = {} & \frac{N+8}{16\pi^2}\lambda^2 + \frac{2(5N+22)}{(16\pi^2)^2}\lambda^3 + \frac{N-1}{8\pi^2}\lambda\langle z_1\rangle_{2,0}^6(0) + \frac{3}{8\pi^2}\lambda\langle\tilde{z}_1\rangle_{0,2}^6\left(2\lambda\kappa\right)\\ & - \frac{N-1}{16\pi^2}\langle z_2\rangle_{1,0}^6(0) - \frac{1}{16\pi^2}\langle\tilde{z}_2\rangle_{0,1}^6\left(2\lambda\kappa\right) + 2\eta\lambda\ . \qquad \text{(D.34)}\end{aligned}$$

D.2.3 Wave-function renormalization contribution

Recalling that κ and Z_k can be neglected in the third line of (3.86), as they would give higher order corrections, the wave function renormalization contribution is encoded in the following averages

$$\langle z_1\rangle^6_{2,0}(0) = 16\pi^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} p^2 Z_k^{(1)}(0,p^2) \frac{\partial_t \mathcal{R}_k(p^2)}{P_k(p^2)^3} \, , \tag{D.35a}$$

$$\langle z_2\rangle^6_{1,0}(0) = 8\pi^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} p^2 Z_k^{(2)}(0,p^2) \frac{\partial_t \mathcal{R}_k(p^2)}{P_k(p^2)^2} \, , \tag{D.35b}$$

and similar relations for $\tilde{z}_1$ and $\tilde{z}_2$. Here we should input the momentum dependence of the wave function renormalization as generated at one loop, that is

$$Z_k^{(1)}(\rho_0,p^2) = -\,4\lambda^2\, \frac{I_k(p^2)}{p^2} \, , \qquad \tilde{Z}_k^{(1)}(\rho_0,p^2) = -2(N+8)\lambda^2\, \frac{I_k(p^2)}{p^2} \, , \tag{D.36a}$$

$$Z_k^{(2)}(\rho_0,p^2) = 32\lambda^3\, \frac{J_k(p^2)}{p^2} \, , \qquad \tilde{Z}_k^{(2)}(\rho_0,p^2) = 8(N+26)\lambda^3\, \frac{J_k(p^2)}{p^2} \, , \tag{D.36b}$$

where I_k and J_k are the following one-loop integrals

$$I_k(p^2) = \frac{1}{2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{P_k(q)} \left(\frac{1}{P_k(q+p)} - \frac{1}{P_k(q)} \right) \, , \tag{D.37a}$$

$$J_k(p^2) = \frac{1}{2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{P_k(q)^2} \left(\frac{1}{P_k(q+p)} - \frac{1}{P_k(q)} \right) \, . \tag{D.37b}$$

Nesting these expressions leads to (3.91), where the averages of z_1, $\tilde{z}_1$ and z_2, $\tilde{z}_2$ are respectively proportional to the dimensionless two-loop integrals

$$A = \frac{1}{2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} I_k(p^2) \frac{\partial_t \mathcal{R}_k(p)}{P_k(p)^3} \, , \tag{D.38a}$$

$$B = \frac{1}{2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} J_k(p^2) \frac{\partial_t \mathcal{R}_k(p)}{P_k(p)^2} \, . \tag{D.38b}$$

We first compute $I_k(p^2)$ with the pseudo-regulator (3.88)

$$
\begin{aligned}
I_k(p^2,\epsilon) &= \frac{1}{2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{P_k(q)}\left(\frac{1}{P_k(q+p)}-\frac{1}{P_k(q)}\right)\\
&= \frac{1}{2}\left(\frac{\mu^2 m^2}{k^2}\right)^{2\epsilon}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{\left(q^2+m^2\right)^{1+\epsilon}\left((q+p)^2+m^2\right)^{1+\epsilon}} - (p\to 0)\\
&= \frac{1}{2}\left(\frac{\mu^2 m^2}{k^2}\right)^{2\epsilon}\frac{\Gamma(2+2\epsilon)}{\Gamma(1+\epsilon)^2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\int_0^1 \mathrm{d}x\frac{x^\epsilon(1-x)^\epsilon}{\left(x\,q^2+(1-x)(q+p)^2+m^2\right)^{2+2\epsilon}} - (p\to 0)\\
&= \frac{1}{2}\left(\frac{\mu^2 m^2}{k^2}\right)^{2\epsilon}\frac{\Gamma(2+2\epsilon)}{\Gamma(1+\epsilon)^2}\int_0^1 \mathrm{d}x\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{x^\epsilon(1-x)^\epsilon}{\left(q^2+x(1-x)p^2+m^2\right)^{2+2\epsilon}} - (p\to 0)\\
&= \frac{1}{32\pi^2}\left(\frac{\mu^2}{k^2}\right)^{2\epsilon}\frac{\Gamma(2\epsilon)}{\Gamma(1+\epsilon)^2}\int_0^1 \mathrm{d}x\, x^\epsilon(1-x)^\epsilon\left(1+x(1-x)\frac{p^2}{m^2}\right)^{-2\epsilon} - (p\to 0)\,.
\end{aligned}
\tag{D.39}
$$

Taking the limit for $\epsilon\to 0$ results in the following finite expression

$$
I_k(p^2) = \frac{-1}{32\pi^2}\int_0^1 \mathrm{d}x\log\left(1+x(1-x)\frac{p^2}{m^2}\right) = \frac{1}{16\pi^2}\left[1-\sqrt{\frac{4m^2+p^2}{p^2}}\operatorname{atanh}\left(\sqrt{\frac{p^2}{4m^2+p^2}}\right)\right]\,.
\tag{D.40}
$$

We can then insert this result in the expression (D.38a) for the A coefficient

$$
\begin{aligned}
A &= \epsilon\left(\frac{\mu^2 m^2}{k^2}\right)^{2\epsilon}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{I_k(p^2,\epsilon)}{\left(p^2+m^2\right)^{2+2\epsilon}}\\
&= \frac{\epsilon}{(16\pi^2)^2}\left(\frac{\mu^2 m^2}{k^2}\right)^{4\epsilon}\frac{\Gamma(2\epsilon)}{\Gamma(1+\epsilon)^2}\int_0^1 \mathrm{d}x\, x^\epsilon(1-x)^\epsilon\int_0^\infty \mathrm{d}p\, p^3\frac{\left(m^2+x(1-x)p^2\right)^{-2\epsilon}}{\left(p^2+m^2\right)^{2+2\epsilon}}\\
&- \frac{\epsilon}{(16\pi^2)^2}\left(\frac{\mu^2 m^2}{k^2}\right)^{4\epsilon}\frac{\Gamma(2\epsilon)}{\Gamma(1+\epsilon)^2}\int_0^1 \mathrm{d}x\, x^\epsilon(1-x)^\epsilon\int_0^\infty \mathrm{d}p\, p^3\frac{m^{-4\epsilon}}{\left(p^2+m^2\right)^{2+2\epsilon}}\\
&= \frac{\epsilon}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{\Gamma(2\epsilon)}{\Gamma(1+\epsilon)^2}\int_0^1 \mathrm{d}x\, x^\epsilon(1-x)^\epsilon\frac{1}{4(1-x(1-x))}\\
&\quad\times\left[\frac{\sqrt{\pi}16^\epsilon\Gamma\left(2\epsilon+\frac{1}{2}\right)((1-x)x)^{2\epsilon}(1+2\epsilon-(1-x)x(1-2\epsilon))}{\sin(2\pi\epsilon)\Gamma(2\epsilon+2)((1-x)x-1)^{4\epsilon}}\right.\\
&\quad\left.-\frac{2(1-\epsilon)x(1-x)+(1+2\epsilon-(1-x)x(1-2\epsilon))\,{}_2F_1\left(1,2+2\epsilon;3-2\epsilon;\frac{1}{x-x^2}\right)}{(1-x)^2x^2(1-2\epsilon)(1-\epsilon)}\right]\\
&- \frac{\epsilon}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{\Gamma(2\epsilon)}{\Gamma(1+\epsilon)^2}\int_0^1 \mathrm{d}x\, x^\epsilon(1-x)^\epsilon\frac{1}{4\epsilon(1+2\epsilon)}\,.
\end{aligned}
\tag{D.41}
$$

If we first expand the integrand around $\epsilon = 0$ and then perform the integral over x we find

$$A = \frac{1}{(16\pi^2)^2}\left[-\frac{1}{16\epsilon} + \frac{3 - 2\log\left(\frac{\mu^2}{k^2}\right)}{8} + o(\epsilon)\right] . \tag{D.42}$$

Notice that the coefficient of the pole is equal to one fourth of the coefficient in front of $\log\left(\frac{\mu^2}{k^2}\right)$.

To demonstrate that the $\epsilon \to 0$ limit and the x integration do commute, let's compute the two also in the opposite order. Thus, we first perform the integral over x and then take $\epsilon \to 0$. For notational convenience we split A in four different terms

$$A = a_1 + a_2 + a_3 + a_4 , \tag{D.43}$$

where we define

$$a_1 = \frac{1}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{\epsilon\Gamma(-1+2\epsilon)}{2\Gamma(\epsilon+1)^2}\int_0^1 \mathrm{d}x\frac{((1-x)x)^{\epsilon-1}}{(1+x(1-x))} , \tag{D.44a}$$

$$a_2 = \frac{-1}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{\sqrt{\pi}2^{4\epsilon-3}\Gamma\left(\frac{1}{2}+2\epsilon\right)}{\sin(2\pi\epsilon)(2\epsilon+1)\Gamma(\epsilon+1)^2}\int_0^1 \mathrm{d}x\frac{((1-x)x)^{3\epsilon}(1+2\epsilon-x(1-x)(1-2\epsilon))}{(-1+x(1-x))^{1+4\epsilon}} , \tag{D.44b}$$

$$a_3 = -\frac{1}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{\pi\epsilon}{2\sin(2\pi\epsilon)\Gamma(3-2\epsilon)\Gamma(\epsilon+1)^2}$$
$$\times\int_0^1 \mathrm{d}x\frac{((1-x)x)^{\epsilon}(1+2\epsilon-(1-x)x(1-2\epsilon))\,{}_2F_1\left(1,2(\epsilon+1);3-2\epsilon;\frac{1}{x-x^2}\right)}{x^2(1-x)^2(1-x(1-x))} , \tag{D.44c}$$

$$a_4 = -\frac{1}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{1}{8\epsilon(1+2\epsilon)^2} . \tag{D.44d}$$

By performing the integrals over x and then expanding them around $\epsilon = 0$ they become

$$a_1 = \frac{1}{(16\pi^2)^2}\left[-\frac{1}{2\epsilon} - 1 - \frac{\pi\sqrt{3}}{18} - 2\log\left(\frac{\mu^2}{k^2}\right) + o(\epsilon)\right] , \tag{D.45a}$$

$$a_2 = \frac{1}{(16\pi^2)^2}\left[\frac{1}{16\epsilon} - \frac{1}{8} - \frac{\pi}{36}\left(\sqrt{3}+9i\right) + \frac{1}{4}\log\left(\frac{\mu^2}{k^2}\right) + o(\epsilon)\right] , \tag{D.45b}$$

$$a_3 = \frac{1}{(16\pi^2)^2}\left[\frac{1}{2\epsilon} + 1 + \frac{\pi}{12}\left(\sqrt{3}+3i\right) + 2\log\left(\frac{\mu^2}{k^2}\right) + o(\epsilon)\right] , \tag{D.45c}$$

$$a_4 = \frac{1}{(16\pi^2)^2}\left[-\frac{1}{8\epsilon} + \frac{1}{2} - \frac{1}{2}\log\left(\frac{\mu^2}{k^2}\right) + o(\epsilon)\right] . \tag{D.45d}$$

Combining these results we recover (D.42).

Then we turn to the computation of $J_k(p^2)$

$$
\begin{aligned}
J_k(p^2,\epsilon) &= \frac{1}{2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{P_k(q)^2}\left(\frac{1}{P_k(q+p)}-\frac{1}{P_k(q)}\right)\\
&= \frac{1}{2}\left(\frac{\mu^2 m^2}{k^2}\right)^{3\epsilon}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{(q^2+m^2)^{2+2\epsilon}\,((q+p)^2+m^2)^{1+\epsilon}} - (p\to 0)\\
&= \frac{1}{2}\left(\frac{\mu^2 m^2}{k^2}\right)^{2\epsilon}\frac{\Gamma(3+3\epsilon)}{\Gamma(1+\epsilon)\Gamma(2+2\epsilon)}\int\frac{\mathrm{d}^4 q}{(2\pi)^4}\int_0^1 \mathrm{d}x\frac{x^{2\epsilon+1}(1-x)^{\epsilon}}{(x\,q^2+(1-x)(q+p)^2+m^2)^{3+3\epsilon}} - (p\to 0)\\
&= \frac{1}{2}\left(\frac{\mu^2 m^2}{k^2}\right)^{3\epsilon}\frac{\Gamma(3+3\epsilon)}{\Gamma(1+\epsilon)\Gamma(2+2\epsilon)}\int_0^1 \mathrm{d}x\int\frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{x^{2\epsilon+1}(1-x)^{\epsilon}}{(q^2+x(1-x)p^2+m^2)^{3+3\epsilon}} - (p\to 0)\\
&= \frac{1}{32\pi^2 m^2}\left(\frac{\mu^2}{k^2}\right)^{3\epsilon}\frac{\Gamma(1+3\epsilon)}{\Gamma(1+\epsilon)\Gamma(1+2\epsilon)}\int_0^1 \mathrm{d}x\,\frac{x^{2\epsilon+1}(1-x)^{\epsilon}}{\left(1+x(1-x)\frac{p^2}{m^2}\right)^{1+3\epsilon}} - (p\to 0)\,. \qquad \text{(D.46)}
\end{aligned}
$$

Taking the limit $\epsilon\to 0$ we again find a finite one-loop result

$$
\begin{aligned}
J_k(p^2) &= \frac{1}{32\pi^2}\int_0^1 \mathrm{d}x\left[\frac{x}{x(1-x)p^2+m^2}-\frac{x}{m^2}\right]\\
&= \frac{1}{64\pi^2 m^2}\left[\frac{2m^2}{\sqrt{p^2(4m^2+p^2)}}\log\left(1+\frac{\left(\sqrt{p^2(4m^2+p^2)}+p^2\right)}{2m^2}\right)-1\right]\,, \qquad \text{(D.47)}
\end{aligned}
$$

which enters the computation of the B coefficient through (D.38b). The latter proceeds along the same lines as for A. Namely, we exchange again the p and the x integrals

$$
\begin{aligned}
B &= \epsilon\left(\frac{\mu^2 m^2}{k^2}\right)^{\epsilon}\int\frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{J_k(p^2,\epsilon)}{(p^2+m^2)^{1+\epsilon}}\\
&= \frac{\epsilon}{(16\pi^2)^2}\left(\frac{\mu^2 m^2}{k^2}\right)^{4\epsilon}\frac{\Gamma(1+3\epsilon)}{\Gamma(1+\epsilon)\Gamma(1+2\epsilon)}\int_0^1 \mathrm{d}x\,x^{2\epsilon+1}(1-x)^{\epsilon}\int_0^\infty \mathrm{d}p\,p^3\frac{\left(m^2+x(1-x)p^2\right)^{-1-3\epsilon}}{(p^2+m^2)^{1+\epsilon}}\\
&\quad -\frac{\epsilon}{(16\pi^2)^2}\left(\frac{\mu^2 m^2}{k^2}\right)^{4\epsilon}\frac{\Gamma(1+3\epsilon)}{\Gamma(1+\epsilon)\Gamma(1+2\epsilon)}\int_0^1 \mathrm{d}x\,x^{2\epsilon+1}(1-x)^{\epsilon}\int_0^\infty \mathrm{d}p\,p^3\frac{m^{-2(1+3\epsilon)}}{(p^2+m^2)^{1+\epsilon}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\epsilon}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{\Gamma(1+3\epsilon)}{\Gamma(1+\epsilon)\Gamma(1+2\epsilon)}\int_0^1 \mathrm{d}x\, x^{2\epsilon+1}(1-x)^{\epsilon}\frac{1}{6\epsilon(1-3\epsilon)x^2(1-x)^2} \\
&\quad\times\Bigg(-\frac{\pi\epsilon\Gamma(4\epsilon)x(1-x)(1+3x(1-x))\left(1-\frac{1}{x(1-x)}\right)^{-\epsilon}}{\sin(3\pi\epsilon)\Gamma(1+\epsilon)\Gamma(-1+3\epsilon)\left(x(1-x)-1\right)^{1+3\epsilon}} \\
&\qquad\qquad+\frac{(1-3\epsilon)x(1-x)+\epsilon(1+3x(1-x))\,{}_2F_1\left[1,1+\epsilon,2-3\epsilon;\frac{1}{x-x^2}\right]}{1-(1-x)x}\Bigg) \\
&\quad-\frac{\epsilon}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{\Gamma(1+3\epsilon)}{\Gamma(1+\epsilon)\Gamma(1+2\epsilon)}\int_0^1 \mathrm{d}x\, x^{2\epsilon+1}(1-x)^{\epsilon}\frac{1}{2(\epsilon-1)\epsilon}\ .
\end{aligned}
\tag{D.48}
$$

This time however we are not allowed to take the $\epsilon \to 0$ limit before computing the x integral. In fact, this would result in the wrong answer

$$
B = \frac{1}{(16\pi^2)^2}\int_0^1 \mathrm{d}x\left[\frac{1}{8(1-x)}+\frac{x}{2}\right]\ . \tag{D.49}
$$

In other words, the integral over x does not commute with the $\epsilon \to 0$ limit, and the latter must be taken as the last step of the computation. To perform the integral over x of the ϵ-depedent expressions, we split also B in four different contributions

$$
B = b_1 + b_2 + b_3 + b_4\ , \tag{D.50a}
$$

$$
b_1 = \frac{1}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{\pi\Gamma(4\epsilon)}{2\sin(3\pi\epsilon)\Gamma(\epsilon)^2\Gamma(2\epsilon+2)}\int_0^1 \mathrm{d}x\frac{(1+3x(1-x))(1-x)^{2\epsilon-1}x^{3\epsilon}}{\left(x(1-x)-1\right)^{1+4\epsilon}}\ , \tag{D.50b}
$$

$$
b_2 = \frac{1}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{\Gamma(3\epsilon)}{2\Gamma(\epsilon)\Gamma(2\epsilon+2)}\int_0^1 \mathrm{d}x\frac{(1-x)^{\epsilon-1}x^{2\epsilon}}{(1-x(1-x))}\ , \tag{D.50c}
$$

$$
\begin{aligned}
b_3 &= \frac{1}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{\pi\epsilon}{2\sin(3\pi\epsilon)\Gamma(2-3\epsilon)\Gamma(\epsilon)\Gamma(2\epsilon+2)} \\
&\quad\times\int_0^1 \mathrm{d}x\frac{(1+3x(1-x))(1-x)^{-2+\epsilon}x^{2\epsilon-1}\,{}_2F_1\left(1,\epsilon+1;2-3\epsilon;\frac{1}{x-x^2}\right)}{(1-x(1-x))}\ ,
\end{aligned}
\tag{D.50d}
$$

$$
b_4 = \frac{1}{(16\pi^2)^2}\left(\frac{\mu^2}{k^2}\right)^{4\epsilon}\frac{(1+2\epsilon)}{2(1-\epsilon)(2+3\epsilon)(1+3\epsilon)}\ . \tag{D.50e}
$$

Now we compute the integrals over x and then expand around $\epsilon = 0$, obtaining

$$b_1 = \frac{1}{(16\pi^2)^2}\left[-\frac{1}{48\epsilon} + \frac{9 + 18i\pi - 4\sqrt{3}\pi - 18\log\left(\frac{\mu^2}{k^2}\right)}{216} + o(\epsilon)\right], \tag{D.51a}$$

$$b_2 = \frac{1}{(16\pi^2)^2}\left[\frac{1}{6\epsilon} + \frac{-18 + \sqrt{3}\pi + 36\log\left(\frac{\mu^2}{k^2}\right)}{54} + o(\epsilon)\right], \tag{D.51b}$$

$$b_3 = \frac{1}{(16\pi^2)^2}\left[-\frac{1}{12\epsilon} + \frac{18 - 9i\pi - 36\log\left(\frac{\mu^2}{k^2}\right)}{108} + o(\epsilon)\right], \tag{D.51c}$$

$$b_4 = \frac{1}{4(16\pi^2)^2} + o(\epsilon)\ . \tag{D.51d}$$

The sum of these terms leads to the result

$$B = \frac{1}{(16\pi^2)^2}\left[\frac{1}{16\epsilon} + \frac{1 + 2\log\left(\frac{\mu^2}{k^2}\right)}{8} + o(\epsilon)\right]. \tag{D.52}$$

Also in this case the coefficient of the pole is equal to one fourth of the coefficient in front of $\log\left(\frac{\mu^2}{k^2}\right)$. As a consequence, the sum $A + B$ which determines the wave function renormalization contribution to the two-loop beta function is finite, as given in (3.92), and the third line of (3.86) evaluates to

$$\begin{aligned}
&\frac{N-1}{8\pi^2}\lambda\langle z_1\rangle^6_{2,0}(0) + \frac{3}{8\pi^2}\lambda\langle \tilde{z}_1\rangle^6_{0,2}(2\lambda\kappa) - \frac{N-1}{16\pi^2}\langle z_2\rangle^6_{1,0}(0) - \frac{1}{16\pi^2}\langle \tilde{z}_2\rangle^6_{0,1}(2\lambda\kappa)\\
&= -8(5N+22)(A+B)\lambda^3 = -\frac{4(5N+22)}{(16\pi^2)^2}\lambda^3\ .
\end{aligned} \tag{D.53}$$

D.2.4 Anomalous dimension contribution

Within the truncation accounting for a field dependent wave function renormalization, we define the anomalous dimension as

$$\eta = -\frac{\mathrm{d}}{\mathrm{d}t}\log Z_k(\rho_0) = -Z_k^{-1}(\rho_0)\partial_t Z_k(\rho_0) - Z_k(\rho_0)^{-1}Z_k^{(1)}(\rho_0)\,\frac{\mathrm{d}}{\mathrm{d}t}\rho_0\ . \tag{D.54}$$

Possible differences between this definition and a similar one based on $\tilde{Z}_k(\rho_0)$ are beyond the $O(\lambda^2)$ we are after. Also, the second term on the RHS of (D.54) would not contribute at this perturbative order. Hence, the relevant term which can be deduced from the exact flow equation is

$$\partial_t Z_k(\rho_0) = \frac{1}{2}\lim_{Q^2\to 0}\frac{\partial}{\partial Q^2}\frac{\delta^2}{\delta\phi(Q)\delta\phi(-Q)}\mathrm{Tr}\left\{\partial_t \mathcal{R}_k(q)\left[\Gamma_k^{(2)}(q)+\mathcal{R}_k(q)\right]^{-1}\right\}\Big|_{\rho_0}$$
$$= \frac{1}{2}\lim_{Q^2\to 0}\frac{\partial}{\partial Q^2}\mathrm{Tr}\left\{\partial_t\mathcal{R}_k(q)\left(\Gamma_k^{(2)}(q)+\mathcal{R}_k(q)\right)^{-1}\Big[-\Gamma_k^{(4)}(Q,-Q,q,-q)\left(\Gamma_k^{(2)}(q)+\mathcal{R}_k(q)\right)^{-1}\right.$$
$$\left.+2\Gamma_k^{(3)}(Q,q,-q-Q)\Big(\Gamma_k^{(2)}(q)+\mathcal{R}_k(q)\Big)^{-1}\Gamma_k^{(3)}(-Q,q+Q,-q)\Big(\Gamma_k^{(2)}(q)+\mathcal{R}_k(q)\Big)^{-1}\Big]\right\}\Big|_{\rho_0}. \tag{D.55}$$

As described in the main text we have that the anomalous dimension is given by the sum of two terms, $\eta^{(a)}$ and $\eta^{(b)}$: the first one is the contribution at zero momentum, while in the second one is the momentum contribution.

The flow equation which encodes $\eta^{(a)}$ is the one within the order ∂^2 of derivative expansion, that is

$$\begin{aligned}\partial_t Z_k(\rho) =& -2v_d k^{d-2}Z_k^{-1}\left\{\left[(N-1)Z_k^{(1)}(\rho)+Y_k(\rho)\right]l_{1,0}^d(0)+\left[Z_k^{(1)}(\rho)+2\rho Z_k^{(2)}(\rho)\right]l_{0,1}^d(w)\right\}\\
&+4v_d k^{d-6}\rho\left(U_k^{(2)}(\rho)\right)^2 Q_{2,1}^{d,0}(w)+4v_d k^{d-4}\rho Y_k(\rho)U_k^{(2)}(\rho)Q_{2,1}^{d,1}(w)\\
&+v_d k^{d-2}\rho\left(Y_k(\rho)\right)^2 Q_{2,1}^{d,2}(w)+16v_d k^{d-4}Z_k^{-2}\rho Z_k^{(1)}(\rho)U_k^{(2)}(\rho)l_{1,1}^d(w)\\
&+\frac{8v_d}{d}k^{d-2}Z_k^{-2}\rho\left(Z_k^{(1)}(\rho)\right)^2 l_{1,1}^{d+2}(w)+8v_d k^{d-2}Z_k^{-2}\rho Z_k^{(1)}(\rho)Y_k(\rho)l_{1,1}^{d+2}(w)\\
&+\frac{16v_d}{d}k^{d-4}\rho Z_k^{(1)}(\rho)U_k^{(2)}(\rho)N_{2,1}^d(w)+\frac{8v_d}{d}k^{d-2}\rho Z_k^{(1)}(\rho)Y_k(\rho)N_{2,1}^{d+2}(w)\ .\end{aligned} \tag{D.56}$$

Following Ref. [69] we define the threshold functions

$$N_{n_1,n_2}^d(w) = k^{2(n_1+n_2-1)-d}\int_0^\infty \mathrm{d}z\, z^{\frac{d}{2}}\partial_t\left\{\dot{P}_k\, P_k^{-n_1}(\tilde{P}_k+w)^{-n_2}\right\}\ , \tag{D.57a}$$

$$Q_{n_1,n_2}^{d,\alpha}(w) = k^{2(n_1+n_2-\alpha)-d}\int_0^\infty \mathrm{d}z\, z^{\frac{d}{2}-1+\alpha}\partial_t\left\{\left[\dot{P}_k+\frac{2z}{d}\ddot{P}_k-\frac{4z}{d}P_k^{-1}\dot{P}_k^2\right]P_k^{-n_1}(\tilde{P}_k+w)^{-n_2}\right\}, \tag{D.57b}$$

$$M_{n_1,n_2}^d(w) = k^{2(n_1+n_2-1)-d}\int_0^\infty \mathrm{d}z\, z^{\frac{d}{2}}\partial_t\left\{\dot{P}_k^2 P_k^{-n_1}(\tilde{P}_k+w)^{-n_2}\right\}\ . \tag{D.57c}$$

These quantities are related in the following way

$$Q_{n_1,n_2}^{d,\alpha}(w) = \frac{2n_1-4}{d}M_{n_1+1,n_2}^{d+2\alpha}(w)+\frac{2n_2}{d}M_{n_1,n_2+1}^{d+2\alpha}(w)+\frac{2n_2}{d}\rho Y_k(\rho)N_{n_1,n_2+1}^{d+2\alpha}(w)-\frac{2\alpha}{d}N_{n_1,n_2}^{d+2\alpha-2}(w)\ . \tag{D.58}$$

Taking $\rho\to\rho_0$ and $w\to w_0$ in (D.56) we get the simplified expression

$$\partial_t Z_k(\rho_0) = \frac{8}{d}v_d k^{d-6}\bar{\lambda}^2\rho_0 M_{4,0}^d(0)-2v_d k^{d-2}Z_k^{-1}\left(NZ_k^{(1)}(\rho_0,0)+Y_k(\rho_0,0)\right)l_{1,0}^d(0)\ , \tag{D.59}$$

which in $d=4$ can be rewritten

$$\eta^{(a)} = \frac{1}{8\pi^2} m_4^4 \kappa \lambda^2 + \frac{Z_k^{-2} k^2}{16\pi^2} l_{1,0}^4(0) \left[(N-1) Z_k^{(1)}(\rho_0, 0) + \tilde{Z}_k^{(1)}(\rho_0, 0) \right] - Z_k^{-1} Z_k^{(1)}(\rho_0, 0) \frac{\mathrm{d}}{\mathrm{d}t} \rho_0 \ , \tag{D.60}$$

$$m_n^d = -\frac{Z_k^{n-2}}{2} M_{n,0}^d(0) \ . \tag{D.61}$$

As at the present order and with our pseudo-regulator both κ and m_4^4 vanish, we are left with (3.95). On the other hand, the derivative couplings generated at one loop are

$$Z_k^{(1)}(\rho_0, 0) = -4\lambda^2 Z_k^4 \lim_{p^2 \to 0} \frac{I_k(p^2)}{p^2} = \frac{1}{3(16\pi^2)} Z_k^2 m^{-2} \lambda^2 \ , \tag{D.62a}$$

$$\tilde{Z}_k^{(1)}(\rho_0, 0) = -2(N+8)\lambda^2 Z_k^4 \lim_{p^2 \to 0} \frac{I_k(p^2)}{p^2} = \frac{(N+8)}{6(16\pi^2)} Z_k^2 m^{-2} \lambda^2 \ , \tag{D.62b}$$

which leads to (3.96). Nesting the latter in (3.95) results in the final expression (3.97) for $\eta^{(a)}$.

We then turn to the momentum dependent contribution. As in Ref. [69] we define the latter by subtracting the momentum independent part from the four-point vertex:

$$\begin{aligned}
\Delta_k(Q,-Q,q,-q) &= \Gamma_k^{(4)}(Q,-Q,q,-q) - \Gamma_k^{(4)}(0,0,q,-q) - \Gamma_k^{(4)}(Q,-Q,0,0) - \Gamma_k^{(4)}(0,0,0,0) \\
&= -\lambda^2 \mathrm{diag}\left(2, N+8, \overbrace{2\ldots2}^{N-2}\right) \frac{1}{2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} P_k^{-1}(p) \left[2P_k^{-1}(p) + P_k^{-1}(p-Q-q) \right. \\
&\qquad \left. + P_k^{-1}(p-Q+q) - 2P_k^{-1}(p+Q) - 2P_k^{-1}(p+q) \right] \ .
\end{aligned} \tag{D.63}$$

For a ϕ^4 theory at one loop

$$\begin{aligned}
\lim_{Q^2 \to 0} \frac{\partial}{\partial Q^2} \Delta_k(Q,-Q,q,-q) &= -\lambda^2 \mathrm{diag}\left(2, N+8, \overbrace{2\ldots2}^{N-2}\right) \\
\times \frac{1}{2} \lim_{Q^2 \to 0} \frac{\partial}{\partial Q^2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} P_k^{-1}(p) &\left[P_k^{-1}(p-Q-q) + P_k^{-1}(p-Q+q) - 2P_k^{-1}(p+Q) \right] \ .
\end{aligned} \tag{D.64}$$

To evaluate this expression it is convenient to define

$$H(p^2, Q^2) = P_k^{-1}(p) P_k^{-1}(p+Q) = \left(\frac{\mu^2 m^2}{k^2} \right)^{2\epsilon} \frac{\Gamma(2+2\epsilon)}{\Gamma(1+\epsilon)^2} \int_0^1 \mathrm{d}x \, \frac{x^\epsilon (1-x)^\epsilon}{(p^2 + x(1-x)\, Q^2 + m^2)^{2+2\epsilon}} \ . \tag{D.65}$$

We then need to expand the following function for small Q

$$H\left(p^2,(Q\pm q)^2\right) = H(q^2) + (Q^2 \pm 2Q\cdot q)H'(q^2) + 2\,(Q\cdot q)^2\,H''(q^2) + O(Q^3) \qquad \text{(D.66)}$$
$$\mathrel{\mathop{=}\!\!\!\!\!\int_q} H(q^2) + Q^2H'(q^2) + \frac{1}{2}Q^2q^2H''(q^2) + O(Q^4)\ ,$$

where the second equal sign denotes equivalence upon integration over $q \in \mathbb{R}^4$, and primes denote derivatives with respect to q^2. The anomalous dimension involves the trace of the four-point vertex, which then reads

$$\mathrm{Tr}\lim_{Q^2\to 0}\frac{\partial}{\partial Q^2}\Delta_k(Q,-Q,q,-q) \mathrel{\mathop{=}\!\!\!\!\!\int_{q^2}} -3(N+2)\lambda^2\!\int_p\left[H'(p^2,q^2)+\frac{1}{2}q^2H''(p^2,q^2) - H'(p^2,0)\right] . \qquad \text{(D.67)}$$

This one-loop expression for the momentum dependence of the four-point vertex is to be nested in the momentum dependent part of Equation (D.55), thus obtaining

$$\eta^{(b)} = \frac{1}{2}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\partial_t\mathcal{R}_k(q)}{P_k(q)^2}\mathrm{Tr}\lim_{Q^2\to 0}\frac{\partial}{\partial Q^2}\Delta_k(Q,-Q,q,-q)\ . \qquad \text{(D.68)}$$

For our pseudo-regulator, we can specify all the terms in the integrand according to Equations (D.2) and (D.65). Taking the limit for $\epsilon \to 0$ after all integrals have been performed, we find the result of (3.98).

D.2.5 Two-loop flow of m^2

In this appendix we show that also the two-loop beta function of the mass can be obtained as the $\epsilon \to 0$ limit of the corresponding FRG equation. We start from (D.25), and neglect higher-loop contributions, e.g. by inserting $\kappa = 0$, thus obtaining the simplified result

$$\beta_{m^2} - \eta m^2 = -\frac{k^2}{16\pi^2}\left[(N+2)\lambda\, l^4_{1,0}(0) + (N-1)\langle z_1\rangle^6_{1,0}(0) + \langle\tilde{z}_1\rangle^6_{0,1}(0)\right]\ . \qquad \text{(D.69)}$$

The contribution of the one-loop wave function renormalization is similar to the one discussed in the previous section

$$\langle z_1\rangle^6_{1,0}(0) = 8\pi^2k^{-2}\int\frac{\mathrm{d}^4p}{(2\pi)^4}p^2Z^{(1)}_k(0,p^2)\frac{\partial_t\mathcal{R}_k(p^2)}{P_k(p^2)^2}\ , \qquad \text{(D.70a)}$$

$$\langle\tilde{z}_1\rangle^6_{0,1}(0) = 8\pi^2k^{-2}\int\frac{\mathrm{d}^4p}{(2\pi)^4}p^2\tilde{Z}^{(1)}_k(0,p^2)\frac{\partial_t\mathcal{R}_k(p^2)}{P_k(p^2)^2}\ , \qquad \text{(D.70b)}$$

where $Z_k^{(1)}$ and $\tilde{Z}_k^{(1)}$ are given in Equations (D.36a). Then the two-loop contributions arise by replacing in (D.69) the following expressions

$$l_{1,0}^4(0) = -\frac{m^2}{k^2} + (1 + 2f_1)\,\frac{\beta_{m^2}}{4k^2}\ , \tag{D.71}$$

$$(N-1)\langle z_1 \rangle_{1,0}^6(0) + \langle \tilde{z}_1 \rangle_{0,1}^6(0) = (N+2)\frac{\left(9 - \sqrt{3}\pi\right)}{8\pi^2}\frac{m^2}{k^2}\lambda^2\ , \tag{D.72}$$

$$\eta = \frac{(N+2)}{2(16\pi^2)^2}\lambda^2\ , \tag{D.73}$$

where f_1 is a free regularization parameter as described in Appendix D.1. The combination of these corrections gives (3.101), from which it is apparent that the unique choice

$$f_1 = -\frac{1}{2} + \frac{4\sqrt{3}\pi - 30}{N+2} \tag{D.74}$$

produces the $\overline{\mathrm{MS}}$ two-loop result

$$\partial_t \log m^2 = \frac{(N+2)}{16\pi^2}\lambda - \frac{5(N+2)}{2(16\pi^2)^2}\lambda^2\ . \tag{D.75}$$

Appendix E

Flow equation with general frame transformations

In this Appendix, we present a derivation of Equation (4.54), which generalizes the demonstration of the flow for the EAA presented in [54], and its development is strictly related to the classical derivation of the flow equation in the standard scheme (4.74).

Our scheme for the ERG is based on the idea that the basic degrees of freedom could flow along the RG trajectory. For this purpose, let us consider the generator of the connected correlation functions

$$\mathcal{W}_{\hat{\chi}}[J] := \log \int (\mathrm{d}\hat{\chi})\ \mathrm{e}^{-S_{\hat{\chi}}[\hat{\chi}] + \int_x J(x)\hat{\chi}(x)}\,, \tag{E.1}$$

where J is an external source. We now introduce a scale dependent generalization of Equation (E.1) which depends on an IR cutoff scale k by making two modifications. First we couple a source J to a k-dependent field $\hat{\phi}_k[\hat{\chi}]$ which is a functional of the fundamental field $\hat{\chi}$. The new field $\hat{\phi}_k[\hat{\chi}]$ satisfies the following relations

$$\langle \hat{\phi}_k[\hat{\chi}] \rangle_{\phi,k} = \phi\,, \tag{E.2}$$

$$\langle \partial_t \hat{\phi}_k[\hat{\chi}] \rangle_{\phi,k} = \Psi_k[\phi]\,, \tag{E.3}$$

along with the boundary condition $\hat{\phi}_\Lambda(x) = \hat{\chi}(x)$ supplied at some fixed reference scale Λ. In a second step, we introduce an IR cutoff by adding the following term to the action

$$\Delta S_k[\hat{\phi}_k] = \frac{1}{2}\int_{x_1,x_2} \hat{\phi}_k(x_1)\mathcal{R}_k(x_1,x_2)\hat{\phi}_k(x_2)\,, \tag{E.4}$$

where $\mathcal{R}_k(x_1,x_2)$ is an IR cutoff function which can be chosen arbitrarily, provided it meets few constraints to ensure that the RG flow interpolates between the microscopic theory in the UV and the full effective theory in the IR. These modifications define the k-dependent generating functional

$$\mathrm{e}^{\mathcal{W}_{\hat{\phi}}[J]} := \int (\mathrm{d}\hat{\chi})\ \mathrm{e}^{-S_{\hat{\chi}}[\hat{\chi}] + \int_x J(x)\hat{\phi}_k(x) - \frac{1}{2}\int_{x_1,x_2} \hat{\phi}_k(x_1)\mathcal{R}_k(x_1,x_2)\hat{\phi}_k(x_2)}\,, \tag{E.5}$$

in terms of which the expectation values of arbitrary operators $\mathcal{O}$ can be obtained by differentiating the $\mathcal{W}_{\hat{\phi}}[J]$ as

$$\begin{aligned}\langle\hat{\mathcal{O}}[\hat{\phi}_k]\rangle &= \mathrm{e}^{-\mathcal{W}_{\hat{\phi}}[J]}\hat{\mathcal{O}}\left[\frac{\delta}{\delta J}\right]\mathrm{e}^{\mathcal{W}_{\hat{\phi}}[J]} \\ &= \mathrm{e}^{-\mathcal{W}_{\hat{\phi}}[J]}\int (\mathrm{d}\hat{\chi})\ \hat{\mathcal{O}}[\hat{\phi}_k]\,\mathrm{e}^{-S_{\hat{\chi}}[\hat{\chi}] + \int_x J(x)\hat{\phi}_k(x) - \frac{1}{2}\int_{x_1,x_2} \hat{\phi}_k(x_1)\mathcal{R}_k(x_1,x_2)\hat{\phi}_k(x_2)}\,.\end{aligned} \tag{E.6}$$

In particular, let's denote the k-dependent average (classical) field by

$$\phi(x) = \frac{\delta}{\delta J(x)}\mathcal{W}_{\hat{\phi}}[J]\,, \tag{E.7}$$

so that higher-order derivatives of $\mathcal{W}_{\hat{\phi}}$ are naturally related to correlation functions of $\hat{\phi}_k$. In this respect, the k-dependent connected two-point function can be defined as

$$\mathcal{G}_k(x_1,x_2) \equiv \frac{\delta^2\mathcal{W}_{\hat{\phi}}}{\delta J(x_1)\delta J(x_2)} = \langle\hat{\phi}_k(x_1)\hat{\phi}_k(x_2)\rangle - \phi(x_1)\phi(x_2)\,. \tag{E.8}$$

We now seek a closed RG equation for $\mathcal{W}_{\hat{\phi}}[J]$. For a given choice of $\Psi_k[\phi]$, by differentiating Equation (E.5) with respect to the RG time t we obtain

$$\partial_t\mathcal{W}_{\hat{\phi}}[J] = \int_x \Psi_k[\phi(x)]J(x) - \frac{1}{2}\int_{x_1,x_2}\langle\hat{\phi}_k(x_1)\hat{\phi}_k(x_2)\rangle\,\partial_t\mathcal{R}_k(x_1,x_2) - \int_{x_1,x_2}\langle\partial_t\hat{\phi}_k(x_1)\hat{\phi}_k(x_2)\rangle\mathcal{R}_k(x_1,x_2)\,. \tag{E.9}$$

Using (E.7), differentiating Equation (E.3) with respect to $J(x_2)$

$$-\phi(x_2)\Psi_k[\phi(x_1)] + \langle\partial_t\hat{\phi}_k(x_1)\hat{\phi}_k(x_2)\rangle = \int_{x_3}\frac{\delta\phi(x_3)}{\delta J(x_2)}\frac{\delta\Psi_k[\phi(x_1)]}{\delta\phi(x_3)} = \int_{x_3}\frac{\delta^2\mathcal{W}_{\hat{\phi}}[J]}{\delta J(x_2)\delta J(x_3)}\frac{\delta\Psi_k[\phi(x_1)]}{\delta\phi(x_3)}\,. \tag{E.10}$$

Then we note that by taking advantage of the previous identity and using Equation (E.8) we finally obtain the following closed flow equation

$$\begin{aligned}\partial_t\mathcal{W}_{\hat{\phi}}[J] = &\int_x \Psi_k[\phi(x)]J(x) - \frac{1}{2}\int_{x_1,x_2}\left[\frac{\delta^2\mathcal{W}_{\hat{\phi}}}{\delta J(x_1)\delta J(x_2)} + \phi(x_1)\phi(x_2)\right]\partial_t\mathcal{R}_k(x_1,x_2) \\ &- \int_{x_1,x_2}\left[\phi(x_2)\Psi_k[\phi_k(x_1)] + \int_{x_3}\frac{\delta^2\mathcal{W}_{\hat{\phi}}[J]}{\delta J(x_2)\delta J(x_3)}\frac{\delta\Psi_k[\phi(x_1)]}{\delta\phi(x_3)}\right]\mathcal{R}_k(x_1,x_2)\,.\end{aligned} \tag{E.11}$$

Let us now introduce the effective average action $\Gamma_k[\phi]$ by the following modified Legendre transformation

$$\Gamma_k[\phi] = -\mathcal{W}_{\hat{\phi}}[J] + \int_x J(x)\phi(x) - \frac{1}{2}\int_{x_1,x_2} \phi(x_1)\mathcal{R}_k(x_1,x_2)\phi(x_2)\,, \tag{E.12}$$

which is intended to be a functional of the average field such that

$$\frac{\delta\Gamma_k[\phi]}{\delta\phi(x_1)} = J(x_1) - \int_x \mathcal{R}_k(x_1,x)\phi(x)\,. \tag{E.13}$$

Differentiating Equation (E.13) with respect to $\phi(x_2)$ and Equation (E.7) with respect to $J(x_1)$ yields the following identity

$$\int_x \mathcal{G}_k(x_1,x)(\Gamma^{(2)} + \mathcal{R}_k)(x,x_2) = \delta(x_1 - x_2)\,. \tag{E.14}$$

Taking advantage of Equations (E.13-E.14) and differentiating Equation (E.12) with respect to t, the desired flow of $\Gamma_k[\phi]$ can be finally expressed as in Equation (4.54), namely

$$\begin{aligned}\partial_t\Gamma_k[\phi] + \int_x \frac{\delta\Gamma_k[\phi]}{\delta\phi(x)}\Psi_k[\phi(x)] = \\ \frac{1}{2}\int_{x_1,x_2} \frac{1}{\Gamma_k^{(2)} + \mathcal{R}_k}(x_1,x_2)\left(\partial_t\mathcal{R}_k(x_2,x_1) + 2\int_{x_3}\frac{\delta\Psi_k[\phi(x_3)]}{\delta\phi(x_2)}\mathcal{R}_k(x_3,x_1)\right)\,.\end{aligned} \tag{E.15}$$

One can also express $\Gamma_k[\phi]$ directly as the solution to integro-differential equation

$$e^{-\Gamma_k[\phi]} = \int (\mathrm{d}\hat{\chi})\ \mathrm{e}^{-S_{\hat{\chi}}[\hat{\chi}] + \int_x \frac{\delta\Gamma_k[\phi]}{\delta\phi}(\hat{\phi}_k(x)-\phi(x)) - \frac{1}{2}\int_{x_1,x_2}(\hat{\phi}_k(x_1)-\phi(x_1))\mathcal{R}_k(x_1,x_2)\left(\hat{\phi}_k(x_2)-\phi(x_2)\right)}\,. \tag{E.16}$$

If as $k \to \Lambda$ the regulator diverges, we then have that $\Gamma_\Lambda[\phi] \to S[\phi]$, where we have used the boundary condition $\hat{\phi}_\Lambda = \hat{\chi}$.

In Chapter 4 we focus on the derivative expansion: this means that $\Psi_k[\phi]$ is given by Equation (4.101) at order $O(\partial^2)$, by Equation (4.136) at order $O(\partial^4)$ and by Equation (4.138) at order $O(\partial^6)$. Another possibility is to consider the vertex expansion, where $\Psi_k[\phi]$ is expressed in powers of the field with coefficients depending on the momenta

$$\Psi_k[\phi(x)] = \sum_n \int_{p_1,\ldots,p_n} \Psi_k(p_1,\ldots,p_n)\phi(p_1)\ldots\phi(p_n)\mathrm{e}^{-\mathrm{i}x(p_1+\cdots+p_n)}\,. \tag{E.17}$$

Appendix F

Renormalization conditions in the standard scheme

In this Appendix, we discuss renormalization conditions for the inessential coupling present in free theories. The idea consists of recasting some notions given in Section 2.8.1 using the new point of view developed in Chapter 4. We have seen that in the standard case we impose Equation (4.76) to fix the wave function renormalization but one can ask what happens for the high temperature fixed point or higher-derivatives theories. In fact, another renormalization condition could be to fix one of the couplings appearing in the potential $V_k(\phi)$. For example we could fix

$$V_k^{(2)}(\bar{\phi}) = Ck^2\,. \tag{F.1}$$

However these choices are not inconsequential since they can limit which fixed points can be found. In general terms a given fixed point solution $\Gamma_\star[\varphi]$ can be found only for a subset of all renormalization conditions. In order to be able to find all fixed points one can instead choose to keep $\eta_\star$ arbitrary. A simple example is to look for free fixed points which can be treated exactly. In this case we can write (ignoring the vacuum term)

$$\Gamma_k[\phi] = \frac{1}{2}\phi \cdot k^2 H_k(-\partial^2/k^2) \cdot \phi\,, \tag{F.2}$$

where fixed points are solutions where $H_k(q^2) = H_\star(q^2)$ is independent of k. We arrive at the fixed point equation

$$q^2 \frac{\partial}{\partial q^2} H_\star(q^2) = \left(1 - \frac{1}{2}\eta_\star\right) H_\star(q^2)\,. \tag{F.3}$$

If we impose that $H_\star(q^2)$ should be analytic around $q^2 = 0$ then the only solutions are $H_\star(q^2) = C\left(q^2\right)^{\frac{1}{2}s}$ where $\frac{1}{2}s$ is a non-negative integer given by $s = 2 - \eta_\star$ and thus the values that $\eta_\star$ can take is quantized and C is an undetermined number. In particular, for $s = 2$ the action is given by (4.75) with $V_k = 0$ and $z_k = C$, while for $s = 0$, which corresponds to the high temperature fixed point, we have $V_k = \frac{1}{2}k^2\phi^2$ and $z_k = 0$, with all higher derivative terms zero in both cases. This is of course a convoluted way to arrive at the conclusion that at free fixed points with s derivatives the canonical dimension is given by $(d-s)/2$.

Now suppose we had chosen (4.76), then the only free fixed point that we could have found would be the one where $s = 2$. On the other hand if instead we had imposed (F.1), then we could only have found the high temperature fixed point where $s = 0$. Since the number C is undetermined, if we leave C unspecified in (4.76) (or (F.1)), we see that there are in fact lines of free fixed points parameterized by C. The critical exponents along a given line do not vary, therefore we understand that all fixed points appearing on the same line belong to a single universality class.

Let us now relate this to a frame transformation. If we are at a free fixed point of the form

$$\Gamma_\star = C\frac{1}{2}\varphi \cdot (-\partial^2)^{\frac{1}{2}s} \cdot \varphi\,, \tag{F.4}$$

then making the transformation (4.37) with

$$\epsilon\hat{\xi}[\hat{\chi}] = \frac{1}{2}\hat{\phi}[\hat{\chi}]\delta C \tag{F.5}$$

and using (4.45), we see that (F.4) transforms as

$$\Gamma_\star \to C\frac{1}{2}\varphi \cdot (-\partial^2)^{\frac{1}{2}s} \cdot \varphi + \frac{1}{2}\delta C\varphi \cdot (-\partial^2)^{\frac{1}{2}s} \cdot \varphi + \text{const}\,, \tag{F.6}$$

where the second term comes from the piece proportional to the equation of motion in equation (4.45), while the constant from the trace term. Thus we obtain a new fixed point where the factor $C \to C + \delta C$ and the vacuum energy is shifted. Thus a change in an inessential coupling at the fixed point is equivalent to a frame transformation that merely moves us along the line of fixed points corresponding to the same universality class.

Appendix G

Scalar field's calculation at order ∂^2 in essential scheme

In this Appendix, we specialize the general flow Equation (4.54) to the second order in the derivative expansion, explicitly performing the computations needed to retrieve Equations (4.104). In Subsection G.1 we choose to work in momentum space: this part is more suitable to problems characterized by translational invariance for which the calculations are made easier by the availability of the Fourier transform. In Subsection G.2 instead, by taking advantage of the heat kernel formalism, we perform the same computations in position space, as this provides an alternative framework for problems where the translational invariance is lost, like curved spaces and/or boundaries.

G.1 Momentum space

Hereafter, we adopt the local potential approximation scheme (4.102). Let's consider the following functional derivatives of the EAA Γ_k, namely

$$\Gamma_k^{(2)}(x_1,x_2) \equiv \frac{\delta^2\Gamma_k}{\delta\phi(x_1)\delta\phi(x_2)} = \int_x \left[\partial_\mu\delta_{x,x_1}\partial_\mu\delta_{x,x_2} + V_k^{(2)}\left(\phi(x)\right)\delta_{x,x_1}\delta_{x,x_1}\right] \ , \tag{G.1}$$

$$\frac{\delta\Gamma_k^{(2)}(x_1,x_2)}{\delta\phi(x_3)} = \int_x V_k^{(3)}\left(\phi(x)\right)\delta_{x,x_1}\delta_{x,x_2}\delta_{x,x_3} \ , \tag{G.2}$$

$$\frac{\delta^2\Gamma_k^{(2)}(x_1,x_2)}{\delta\phi(x_3)\delta\phi(x_4)} = \int_x V_k^{(4)}\left(\phi(x)\right)\delta_{x,x_1}\delta_{x,x_2}\delta_{x,x_3}\delta_{x,x_4} \ , \tag{G.3}$$

where by δ_{x_1,x_2} we indicate the d-dimensional Dirac delta, i.e. $\delta(x_1-x_2)$. We now consider the Fourier transform of Equation (G.1) for a constant field configuration which can be expressed as

$$\int_{x_1,x_2} \Gamma_k^{(2)}(x_1,x_2)\mathrm{e}^{\mathrm{i}(p_1x_1+p_2x_2)} = \left(p_1^2 + V_k^{(2)}\right)(2\pi)^d\delta(p_1+p_2)\ , \tag{G.4}$$

$$\int_{x_1,x_2,x_3} \frac{\delta\Gamma_k^{(2)}(x_1,x_2)}{\delta\phi(x_3)}\mathrm{e}^{\mathrm{i}(p_1x_1+p_2x_2+p_3x_3)} = V_k^{(3)}(2\pi)^d\delta(p_1+p_2+p_3)\ , \tag{G.5}$$

$$\int_{x_1,x_2,x_3,x_4} \frac{\delta^2\Gamma_k^{(2)}(x_1,x_2)}{\delta\phi(x_3)\delta\phi(x_4)}\mathrm{e}^{\mathrm{i}(p_1x_1+p_2x_2+p_3x_3+p_4x_4)} = V_k^{(4)}(2\pi)^d\delta(p_1+p_2+p_3+p_4)\ , \tag{G.6}$$

where we have suppressed the spacetime indices in order to lighten the notation. In the same way, we can write

$$\mathcal{R}_k(x_1,x_2) = \int_p \mathcal{R}_k(p)\mathrm{e}^{-\mathrm{i}p(x_1-x_2)}\ , \tag{G.7}$$

$$G_k(x_1,x_2) = \left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}(x_1,x_2) = \int_p G_k(p)\mathrm{e}^{-\mathrm{i}p(x_1-x_2)}\ , \tag{G.8}$$

$$G_k(p) = \left(p^2 + \mathcal{R}_k(p) + V_k^{(2)}\right)^{-1}\ , \tag{G.9}$$

$$\frac{\delta}{\delta\phi(x_2)}\Psi_k(x_1) = F_k^{(1)}(\phi(x_1))\delta_{x_1,x_2} = \int_p F_k^{(1)}(\phi(x_1))\,\mathrm{e}^{-\mathrm{i}p(x_1-x_2)}\,. \tag{G.10}$$

We notice here that while G_k and Ψ_k are functions of the field, the cutoff function $\mathcal{R}_k$ is not. The LHS of Equation (4.54) then reads

$$\partial_t\Gamma_k + \int_x \frac{\delta\Gamma_k[\phi]}{\delta\phi(x)}F_k(\phi(x)) = \int_x\left[\partial_t V_k + F_k^{(1)}(\phi)\left(\partial_\mu\phi\right)\left(\partial_\mu\phi\right) + F_k(\phi)V_k^{(1)}(\phi)\right]\ , \tag{G.11}$$

while the RHS of Equation (4.54) is composed by two terms, namely

$$\begin{aligned}\frac{1}{2}\int_{x_1,x_2} G_k(x_1,x_2)\partial_t\mathcal{R}_k(x_2,x_1) &= \frac{1}{2}\int_{x_1,x_2,p_1,p_2} G_k(p_1)\partial_t\mathcal{R}_k(p_2)\mathrm{e}^{-\mathrm{i}p_1(x_1-x_2)-\mathrm{i}p_2(x_2-x_1)} \\ &= \frac{1}{2}\int_x\int_p G_k(p)\,\partial_t\mathcal{R}_k(p)\ ,\end{aligned} \tag{G.12}$$

$$\begin{aligned}\int_{x_1,x_2,x_3} G_k(x_1,x_2)\frac{\delta}{\delta\phi(x_2)}\Psi_k(x_3)\,\mathcal{R}_k(x_3,x_1) &= \int_{x_1,x_2,p_1,p_2} G_k(p_1)F_k^{(1)}\,\mathcal{R}_k(p_2)\mathrm{e}^{-\mathrm{i}p_1(x_1-x_2)-\mathrm{i}p_2(x_2-x_1)} \\ &= \int_x\int_p G_k(p)F_k^{(1)}\,\mathcal{R}_k(p)\ .\end{aligned} \tag{G.13}$$

Changing then variables in the remaining momentum integrals as $p \to z = p^2$, the RHS of Equation (4.54) can be written as

$$\frac{1}{2}\mathrm{Tr}\frac{1}{\Gamma_k^{(2)} + \mathcal{R}_k}\cdot\left(\partial_t\mathcal{R}_k + 2\frac{\delta}{\delta\phi}\Psi_k\cdot\mathcal{R}_k\right) = \frac{1}{2(4\pi)^{d/2}}\int_x Q_{d/2}\left[G_k\left(\partial_t\mathcal{R}_k + 2F_k^{(1)}\,\mathcal{R}_k\right)\right]\ , \tag{G.14}$$

where the Q-functionals are defined in Equation (4.107). Considering a constant field configuration and equating (G.11) and (G.14) yields the flow equation (4.104a) for the effective potential V_k.

We now take the second derivative of Equation (4.54) with respect to $\phi(x)$ and $\phi(\bar{x})$, we impose a constant field configuration and then we Fourier transform, so that the LHS reads

$$\int_{x,\bar{x},x_1} \left\{ \delta_{x,x_1}\delta_{\bar{x},x_1}\left[\partial_t V_k^{(2)}(\phi(x_1)) + \Big(F_k\left(\phi(x_1)\right)V_k^{(1)}(\phi(x_1))\Big)^{(2)}\right] + 2F_k^{(1)}(\phi(x_1))\partial_\mu\delta_{x,x_1}\partial_\mu\delta_{\bar{x},x_1}\right\} \mathrm{e}^{\mathrm{i}p_1 x + \mathrm{i}p_2\bar{x}}$$
$$= (2\pi)^d\delta(p_1+p_2)\left[\frac{\delta^2}{\delta\phi(p_1)\delta\phi(-p_1)}\left(\partial_t V_k + F_k\, V_k^{(1)}\right) + 2F_k^{(1)}\, p_1^2\right]. \tag{G.15}$$

Let's now call $\mathbb{T}$ the trace on the RHS of Equation (4.54). Then differentiating with respect to $\phi(x)$ and $\phi(\bar{x})$ yields

$$\mathbb{T}_{x\bar{x}} = -\frac{1}{2}\int_{x_1,x_2,x_3,x_4} G_k(x_1,x_2)\,\frac{\delta^2\Gamma_k^{(2)}(x_2,x_3)}{\delta\phi(x)\delta\phi(\bar{x})}\,G_k(x_3,x_4)\,\partial_t\mathcal{R}_k(x_4,x_1) \tag{G.16}$$
$$-\int_{x_1,x_2,x_3,x_4,x_5} G_k(x_1,x_2)\,\frac{\delta^2\Gamma_k^{(2)}(x_2,x_3)}{\delta\phi(x)\delta\phi(\bar{x})}\,G_k(x_3,x_4)\,\frac{\delta\Psi_k(x_5)}{\delta\phi(x_4)}\,\mathcal{R}_k(x_5,x_1)$$
$$+\frac{1}{2}\int_{x_1,x_2,x_3,x_4,x_5,x_6} G_k(x_1,x_2)\,\frac{\delta\Gamma_k^{(2)}(x_2,x_3)}{\delta\phi(x)}\,G_k(x_3,x_4)\,\frac{\delta\Gamma_k^{(2)}(x_4,x_5)}{\delta\phi(\bar{x})}\,G_k(x_5,x_6)\,\partial_t\mathcal{R}_k(x_6,x_1)$$
$$+\int_{x_1,x_2,x_3,x_4,x_5,x_6,x_7} G_k(x_1,x_2)\,\frac{\delta\Gamma_k^{(2)}(x_2,x_3)}{\delta\phi(x)}\,G_k(x_3,x_4)\,\frac{\delta\Gamma_k^{(2)}(x_4,x_5)}{\delta\phi(\bar{x})}\,G_k(x_5,x_6)\,\frac{\delta\Psi_k(x_7)}{\delta\phi(x_6)}\,\mathcal{R}_k(x_7,x_1)$$
$$+\frac{1}{2}\int_{x_1,x_2,x_3,x_4,x_5,x_6} G_k(x_1,x_2)\,\frac{\delta\Gamma_k^{(2)}(x_2,x_3)}{\delta\phi(\bar{x})}\,G_k(x_3,x_4)\,\frac{\delta\Gamma_k^{(2)}(x_4,x_5)}{\delta\phi(x)}\,G_k(x_5,x_6)\,\partial_t\mathcal{R}_k(x_6,x_1)$$
$$+\int_{x_1,x_2,x_3,x_4,x_5,x_6,x_7} G_k(x_1,x_2)\,\frac{\delta\Gamma_k^{(2)}(x_2,x_3)}{\delta\phi(\bar{x})}\,G_k(x_3,x_4)\,\frac{\delta\Gamma_k^{(2)}(x_4,x_5)}{\delta\phi(x)}\,G_k(x_5,x_6)\,\frac{\delta\Psi_k(x_7)}{\delta\phi(x_6)}\,\mathcal{R}_k(x_7,x_1)$$
$$+\int_{x_1,x_2,x_3} G_k(x_1,x_2)\,\frac{\delta^3\Psi_k(x_3)}{\delta\phi(x)\delta\phi(\bar{x})\delta\phi(x_2)}\,\mathcal{R}_k(x_3,x_1)$$
$$-\int_{x_1,x_2,x_3,x_4,x_5} G_k(x_1,x_2)\,\frac{\delta\Gamma_k^{(2)}(x_2,x_3)}{\delta\phi(x)}\,G_k(x_3,x_4)\,\frac{\delta^2\Psi_k(x_5)}{\delta\phi(\bar{x})\delta\phi(x_4)}\,\mathcal{R}_k(x_5,x_1)$$
$$-\int_{x_1,x_2,x_3,x_4,x_5} G_k(x_1,x_2)\,\frac{\delta\Gamma_k^{(2)}(x_2,x_3)}{\delta\phi(\bar{x})}\,G_k(x_3,x_4)\,\frac{\delta^2\Psi_k(x_5)}{\delta\phi(x)\delta\phi(x_4)}\,\mathcal{R}_k(x_5,x_1)\,.$$

Using equations (G.1) and (G.10) and imposing a constant field configuration we have

$$\begin{aligned}
\mathbb{T}_{x\bar{x}} =& -\frac{1}{2}V_k^{(4)}\delta_{x,\bar{x}}\int_{x_1,x_2} G_k(x_1,x)\; G_k(x,x_2)\;\Big[\partial_t\mathcal{R}_k(x_2,x_1)+2F_k^{(1)}\mathcal{R}_k(x_2,x_1)\Big] \\
&+\frac{1}{2}\left(V_k^{(3)}\right)^2\int_{x_1,x_2} G_k(x_1,x)\,G_k(x,\bar{x})\,G_k(\bar{x},x_2)\;\Big[\partial_t\mathcal{R}_k(x_2,x_1)+2F_k^{(1)}\mathcal{R}_k(x_2,x_1)\Big] \\
&+\frac{1}{2}\left(V_k^{(3)}\right)^2\int_{x_1,x_2} G_k(x_1,\bar{x})\,G_k(\bar{x},x)\,G_k(x,x_2)\;\Big[\partial_t\mathcal{R}_k(x_2,x_1)+2F_k^{(1)}\mathcal{R}_k(x_2,x_1)\Big] \\
&+F_k^{(3)}\delta_{x,\bar{x}}\int_{x_1} G_k(x_1,x)\,\mathcal{R}_k(x,x_1) \\
&-V_k^{(3)}\,F_k^{(2)}\int_{x_1} G_k(x_1,x)\,G_k(x,\bar{x})\,\mathcal{R}_k(\bar{x},x_1) \\
&-V_k^{(3)}\,F_k^{(2)}\int_{x_1} G_k(x_1,\bar{x})\,G_k(\bar{x},x)\,\mathcal{R}_k(x,x_1)\,.
\end{aligned} \tag{G.17}$$

Using then equations (G.8) and (G.7)

$$\begin{aligned}
\mathbb{T}_{x\bar{x}} =& -\frac{1}{2}V_k^{(4)}\delta_{x,\bar{x}}\int_{p_1} G_k(p_1)^2\;\Big[\partial_t\mathcal{R}_k(p_1)+2F_k^{(1)}\mathcal{R}_k(p_1)\Big] \\
&+\frac{1}{2}\left(V_k^{(3)}\right)^2\int_{p_1,p_2} G_k(p_1)\,G_k(p_2)\,G_k(p_1)\,\Big[\partial_t\mathcal{R}_k(p_1)+2F_k^{(1)}\mathcal{R}_k(p_1)\Big]\,\mathrm{e}^{\mathrm{i}x(p_1-p_2)-\mathrm{i}\bar{x}(p_1-p_2)} \\
&+\frac{1}{2}\left(V_k^{(3)}\right)^2\int_{p_1,p_2} G_k(p_1)\,G_k(p_2)\,G_k(p_1)\,\Big[\partial_t\mathcal{R}_k(p_1)+2F_k^{(1)}\mathcal{R}_k(p_1)\Big]\,\mathrm{e}^{-\mathrm{i}x(p_1-p_2)+\mathrm{i}\bar{x}(p_1-p_2)} \\
&+F_k^{(3)}\delta_{x,\bar{x}}\int_{p_1} G_k(p_1)\,\mathcal{R}_k(p_1) \\
&-V_k^{(3)}\,F_k^{(2)}\int_{p_1,p_2} G_k(p_1)\,G_k(p_2)\,\mathcal{R}_k(p_1)\mathrm{e}^{\mathrm{i}x(p_1-p_2)-\mathrm{i}\bar{x}(p_1-p_2)} \\
&-V_k^{(3)}\,F_k^{(2)}\int_{p_1,p_2} G_k(p_1)\,G_k(p_2)\,\mathcal{R}_k(p_1)\mathrm{e}^{-\mathrm{i}x(p_1-p_2)+\mathrm{i}\bar{x}(p_1-p_2)}\,,
\end{aligned} \tag{G.18}$$

and expressing the previous equation in momentum space we obtain

$$\begin{aligned}
\mathbb{T}_{p_1p_2} =& -\frac{1}{2}V_k^{(4)}(2\pi)^d\delta(p_1+p_2)\int_p G_k(p)^2\;\Big[\partial_t\mathcal{R}_k(p)+2F_k^{(1)}\mathcal{R}_k(p)\Big] \\
&+\left(V_k^{(3)}\right)^2(2\pi)^d\delta(p_1+p_2)\int_p G_k(p)\,G_k(p+p_1)\,G_k(p)\;\Big[\partial_t\mathcal{R}_k(p)+2F_k^{(1)}\mathcal{R}_k(p)\Big] \\
&+F_k^{(3)}(2\pi)^d\delta(p_1+p_2)\int_p G_k(p)\,\mathcal{R}_k(p) \\
&-2V_k^{(3)}\,F_k^{(2)}(2\pi)^d\delta(p_1+p_2)\int_p G_k(p)\,G_k(p+p_1)\,\mathcal{R}_k(p)\;.
\end{aligned} \tag{G.19}$$

We then need to expand the previous equation for small p_1; for this purpose, we make use of the following expression

$$f\left((p+p_1)^2\right) = f(p^2) + (p_1^2 + 2\,p_1\cdot p) f'(p^2) + 2\,(p_1\cdot p)^2\, f''(p^2) + O(p_1^3)\,, \tag{G.20}$$

in which primes denote derivatives with respect to p^2. Equating then (G.15) and (G.19), simplifying a common factor $(2\pi)^d \delta(p_1+p_2)$ on both sides and changing variables as $p \to z = p^2$ we obtain

$$\begin{aligned}
&\frac{\delta^2}{\delta\phi(p_1)\delta\phi(-p_1)}\left(\partial_t V_k^{(2)} + F_k\, V_k^{(1)}\right) + 2F_k^{(1)}\, p_1^2 = \\
&- V_k^{(4)} \frac{1}{2(4\pi)^{d/2}} Q_{d/2}\left[G_k^2\,\left(\partial_t \mathcal{R}_k + 2F_k^{(1)}\mathcal{R}_k\right)\right] + F_k^{(3)} \frac{1}{(4\pi)^{d/2}} Q_{d/2}\left[G_k\,\mathcal{R}_k\right] \\
&+ \frac{\left(V_k^{(3)}\right)^2}{(4\pi)^{d/2}} \Big\{ Q_{d/2}\left[G_k^3\,\left(\partial_t \mathcal{R}_k + 2F_k^{(1)}\mathcal{R}_k\right)\right]\, G_k^2 \\
&\qquad + p_1^2 Q_{d/2}\left[G_k'\,\left(\partial_t \mathcal{R}_k + 2F_k^{(1)}\mathcal{R}_k\right)\right] + p_1^2 Q_{d/2+1}\left[G_k''\, G_k^2\,\left(\partial_t \mathcal{R}_k + 2F_k^{(1)}\mathcal{R}_k\right)\right]\Big\} \\
&- V_k^{(3)}\, F_k^{(2)} \frac{2}{(4\pi)^{d/2}} \left\{ Q_{d/2}\left[G_k^2\,\mathcal{R}_k\right] + p_1^2 Q_{d/2}\left[G_k'\, G_k\,\mathcal{R}_k\right] + p_1^2 Q_{d/2+1}\left[G_k''\, G_k\,\mathcal{R}_k\right]\right\} + O(p_1^4)\;.
\end{aligned} \tag{G.21}$$

By finally taking the derivative with respect to p_1^2 and then the limit $p_1 \to 0$, we obtain Equation (4.104b) .

G.2 Position space

We revisit the derivation of Equations (4.104), but now working in position space. In order to lighten the notation, we drop the k subscript and leave it intended throughout the whole section. Let's commence by writing the field as

$$\phi(x) \to \phi + \delta\phi(x)\,, \tag{G.22}$$

where ϕ is now understood as constant and if no argument is shown it means that a function of the field is evaluated at ϕ. Then we write

$$\Gamma^{(2)} + \mathcal{R}_k = G^{-1} + X\,, \tag{G.23}$$

where $G^{-1} = -\partial^2 + \mathcal{R}_k + V^{(2)}$ and we define the following quantities

$$X = V^{(3)}\delta\phi + \frac{1}{2}V^{(4)}\delta\phi^2 + \dots\,, \tag{G.24}$$

$$\Psi^{(1)} = F^{(1)} + Y\,, \tag{G.25}$$

$$Y = F^{(2)}\delta\phi + \frac{1}{2}F^{(3)}\delta\phi^2 + \dots\,. \tag{G.26}$$

The idea now is to expand in $\delta\phi$ and then put the traces into the form $\mathrm{Tr}[\mathcal{O} f(\Delta)]$ and $\mathrm{Tr}[\mathcal{O}^{\mu\nu}\partial_\mu\partial_\nu f(\Delta)]$, where $\mathcal{O}$ are non-derivative operators that might depend on $\delta\phi$ and its derivatives and $f(\Delta)$ is expressed as

$$f(\Delta) = \int_0^\infty \mathrm{d}s \tilde{f}(s) H(s,\Delta)\,, \tag{G.27}$$

where $H(s,\Delta)(x_1,x_2) = \mathrm{e}^{-s\Delta}(x_1,x_2)$ is the heat kernel

$$H(s,\Delta)(x_1,x_2) = \frac{1}{(4\pi s)^{\frac{1}{2}}} e^{-\frac{1}{4s}(x_1-x_2)\cdot(x_1-x_2)}\,. \tag{G.28}$$

By taking advantage of the fact that at $x_1 = x_2$, we have

$$\begin{aligned} H(s,x,x) &= \frac{1}{(4\pi s)^{d/2}}\,, \\ \partial_\mu\partial_\nu H(s,x,x) &= -\frac{\delta_{\mu\nu}}{2(4\pi)^{d/2} s^{d/2+1}}\,, \end{aligned} \tag{G.29}$$

where the derivatives act on the first argument, and therefore one can express the following traces as

$$\mathrm{Tr}[\mathcal{O} f(\Delta)] = \frac{1}{(4\pi)^{d/2}} \int_x \mathcal{O}\, Q_{d/2}[f]\,, \tag{G.30}$$

$$\mathrm{Tr}[\mathcal{O}^{\mu\nu}\partial_\mu\partial_\nu f(\Delta)] = -\frac{1}{2}\frac{1}{(4\pi)^{d/2}} \int_x \mathcal{O}_{\mu\mu}\, Q_{d/2+1}[f]\,, \tag{G.31}$$

where

$$Q_n[f] = \int_0^\infty \mathrm{d}s\, s^{-n} \tilde{f}(s) \tag{G.32}$$

are the equal to the Q-functionals (4.107).

In order to get the flow of the potential V, we then want to set $X = 0$ and $Y = 0$. The LHS of the flow equation (4.54) at constant field is given by

$$\int_x \left[\partial_t V(\phi) + F(\phi) V^{(1)}(\phi)\right]\,, \tag{G.33}$$

while the trace appearing on the RHS of equation (4.54) is given by

$$\begin{aligned} \frac{1}{2}\mathrm{Tr}[(\partial_t \mathcal{R}_k + 2F^{(1)}\mathcal{R}_k)G] &= \int_0^\infty \mathrm{d}s\, \tilde{W}[(\partial_t \mathcal{R}_k + 2F^{(1)}\mathcal{R}_k)G, s]\mathrm{Tr}[H(s)] \\ &= \int_x \frac{1}{2(4\pi)^{d/2}} Q_{d/2}[(\partial_t \mathcal{R}_k + 2F^{(1)}\mathcal{R}_k)G]\,, \end{aligned} \tag{G.34}$$

where we use the heat kernel expansion to calculate the trace. We therefore retrieve Equation (4.104a).

By expanding in $\delta\phi$, one we can find the term which involves $\delta\phi\Delta\delta\phi$ on both the LHS and on the RHS of the flow equation (4.54). On the LHS this yields

$$F^{(1)}(\phi)\,\delta\phi\Delta\delta\phi\ , \tag{G.35}$$

while on the RHS of the flow equation we obtain

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}[(\partial_t\mathcal{R}_k + 2F^{(1)}\mathcal{R}_k{+}2Y\mathcal{R}_k)(G - GXG + GXGXG + ...]\\ = \frac{1}{2}\mathrm{Tr}[(\partial_t\mathcal{R}_k + 2F^{(1)}\mathcal{R}_k)G] - \frac{1}{2}\mathrm{Tr}[XG^2(\partial_t\mathcal{R}_k + 2F^{(1)}\mathcal{R}_k)]\\ + \mathrm{Tr}[Y\mathcal{R}_kG] + \frac{1}{2}\mathrm{Tr}[XGXG^2(\partial_t\mathcal{R}_k + 2F^{(1)}\mathcal{R}_k)]\\ - \mathrm{Tr}[XGY\mathcal{R}_kG] + \dots\ .\end{aligned} \tag{G.36}$$

The terms linear in X and Y do not involve derivatives of $\delta\phi$ so we can ignore them. In order to obtain derivatives of $\delta\phi$ we commute G with X and Y which gives the two terms

$$\frac{1}{2}\mathrm{Tr}[X[G,X]G^2(\partial_t\mathcal{R}_k + 2F^{(1)}\mathcal{R}_k)] - \mathrm{Tr}[X[G,Y]\mathcal{R}_kG]\,. \tag{G.37}$$

Then we use $G = G(\Delta)$ where $\Delta = -\partial^2$ to compute the commutators

$$[G,X] = -[X,\Delta]G'(\Delta) + \frac{1}{2}[[X,\Delta],\Delta]G''(\Delta)\ \ , \tag{G.38}$$

$$[X,\Delta] = X_{,\mu\mu} + 2X_{,\mu}\partial_\mu\ \ , \tag{G.39}$$

$$[[X,\Delta],\Delta] = X_{,\mu\mu\nu\nu} + 4X_{,\mu\mu\nu}\partial_\nu + 4X_{,\mu\nu}\partial_\mu\partial_\nu \tag{G.40}$$

and similarly for Y where the indices after the comma denote derivatives of X with respect to x^μ. The interesting terms are the ones where two derivatives act on X or Y. So the traces we need are

$$\begin{aligned}&\frac{1}{2}\mathrm{Tr}[X(-X_{,\mu\mu}G'(\Delta) + 2X_{,\mu\nu}\partial_\mu\partial_\nu G''(\Delta))G^2(\partial_t\mathcal{R}_k + 2F^{(1)}\mathcal{R}_k)]\\ &\qquad - \mathrm{Tr}[X(-Y_{,\mu\mu}G'(\Delta) + 2Y_{,\mu\nu}\partial_\mu\partial_\nu G''(\Delta))\mathcal{R}_kG] =\\ &= \frac{1}{(4\pi)^{d/2}}\int_x\bigg(-\frac{1}{2}XX_{,\mu\mu}\left(Q_{d/2}[G'G^2(\partial_t\mathcal{R}_k + 2F^{(1)}\mathcal{R}_k)] + Q_{d/2+1}[G''(\partial_t\mathcal{R}_k + 2F^{(1)}\mathcal{R}_k)]\right)\\ &+ XY_{,\mu\mu}\left(Q_{d/2}[G'\mathcal{R}_kG] + Q_{d/2+1}[G''\mathcal{R}_kG]\right)\bigg)\\ &= -\int_x\delta\phi\partial^2\delta\phi\bigg(\frac{1}{2}\left(V^{(3)}\right)^2\left(Q_{d/2}[G'G^2(\partial_t\mathcal{R}_k + 2F^{(1)}\mathcal{R}_k)] + Q_{d/2+1}[G''(\partial_t\mathcal{R}_k + 2F^{(1)}\mathcal{R}_k)]\right)\\ &- V^{(3)}F^{(2)}\left(Q_{d/2}[G'\mathcal{R}_kG] + Q_{d/2+1}[G''\mathcal{R}_kG]\right)\bigg) + O(\delta\phi^3)\,,\end{aligned} \tag{G.41}$$

which upon equating with Equation (G.35) completes the derivation of equation (4.104b).

Appendix H

Gravity's calculation at order ∂^4 in essential scheme

In this appendix we derive the flow equations in minimal essential scheme, i.e. the scheme with renormalization conditions that fix to zero the coefficients of $\sqrt{\det g}\, R^2$ and $\sqrt{\det g}\, R_{\mu\nu}R^{\mu\nu}$. Therefore, in such a scheme the ansatz for EAA at quartic order is simply

$$\bar{\Gamma}_k[g] = \int \mathrm{d}^d x \sqrt{\det g} \left\{ \frac{\rho_k}{8\pi} - \frac{1}{16\pi G_k} R + c_k E \right\} , \tag{H.1}$$

where $\rho_k = \frac{\Lambda_k}{G_k}$ and $E = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2$.
The RG kernel for the quantum metric is given by (5.25), and so the LHS of (4.54) is equal to

$$\begin{aligned} \int \mathrm{d}^d x \sqrt{\det g} \Big\{ \left(\partial_t + \frac{d}{2}\gamma_g\right) \frac{\rho_k}{8\pi} + \left(- \left(\partial_t + \frac{d-2}{2}\gamma_g\right) \frac{1}{16\pi G_k} + (\gamma_{Ricci} + d\gamma_R) \frac{\rho_k}{16\pi} \right) R \\ - \frac{1}{32\pi G_k} (\gamma_{Ricci} + (d-2)\gamma_R) R^2 + \frac{\gamma_{Ricci}}{16\pi G_k} R_{\mu\nu}R^{\mu\nu} + \left(\partial_t + \frac{d-4}{2}\gamma_g\right) c_k E \Big\} . \end{aligned} \tag{H.2}$$

The RHS of (4.54) contains two traces, one coming from the graviton contribution and one from the ghosts contribution and in the following subsections we calculate them, denoting the gravity trace as $\mathbb{T}_{gg}$ and ghost trace as $\mathbb{T}_{\bar{c}c}$.

As a final remark, note that in the calculations reported below we neglect terms proportional c_k in the traces: this is justified in $d = 4$ since in this case the corresponding invariant is topological and so these contributions in RHS of (4.54) vanish in $d = 4$.

H.1 Calculation of gravity trace

In this subsection we calculate the graviton contribution to the quantum part of the flow equation (4.54): in particular, we insert the regulator in such a way that $\Delta \to P_k \equiv \Delta + R_k(\Delta)$, we calculate the Hessian, we expand the argument of the trace to quadratic order in curvature and finally we evaluate the trace using off-diagonal heat kernel techniques [187]. We then choose the regulator to be given by

$$\mathcal{R}_k^{gg} = K_{gg} R_k(\Delta)\,, \tag{H.3}$$

where

$$K_{gg}^{\mu\nu,\alpha\beta} = \frac{1}{2\kappa_k^2}\sqrt{\det g}\left(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta}\right)\,, \tag{H.4}$$

and the following relation holds

$$\partial_t K_{gg} = -\eta_N K_{gg}\,, \tag{H.5}$$

with $\eta_N = \partial_t G_k / G_k$. The Hessian in the gravity sector is

$$\frac{\delta^2 \bar{\Gamma}_k}{\delta g \delta g} + K_{gg}\cdot \Delta_{\text{gf}} + \mathcal{R}_k^{gg} = K_{gg}\cdot\left(P_k + U_0 + U_1\right)\,, \tag{H.6}$$

where

$$U_0 = -2\rho_k G_k\,, \tag{H.7}$$

$$\begin{aligned}(U_1)^{\mu\nu}{}_{\alpha\beta} = {}& \frac{1}{2}R\left(\delta^\mu_\alpha\delta^\nu_\beta + \delta^\mu_\beta\delta^\nu_\alpha - g^{\mu\nu}g_{\alpha\beta}\right) + g^{\mu\nu}R_{\alpha\beta} + R^{\mu\nu}g_{\alpha\beta} - 2\delta^{(\mu}_{(\alpha}R^{\nu)}_{\beta)} - 2R^{(\mu}{}_{(\alpha}{}^{\nu)}{}_{\beta)} \\ & - \frac{d-4}{d-2}\,g_{\alpha\beta}\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right)\,,\end{aligned} \tag{H.8}$$

and the indices in the round brackets are symmetrized. Then the gravitational trace is given by

$$\begin{aligned}\mathbb{T}_{gg} &= \frac{1}{2}\mathrm{Tr}\frac{1}{P_k(\Delta) + U_0 + U_1}\cdot\left((\partial_t - \eta_N)R_k(\Delta) + 2\frac{\delta}{\delta g}\Psi_k^g\cdot\, R_k(\Delta)\right) \\ &= \frac{1}{2}\mathrm{Tr}\left\{\mathcal{G}_k(\Delta) - \mathcal{G}_k(\Delta)^2 U_1 + \mathcal{G}_k(\Delta)^3 U_1^2\right\} \\ &\qquad\qquad \times\left\{(\partial_t - \eta_N)R_k(\Delta) + 2(V^{\mu\nu}\nabla_{(\mu}\nabla_{\nu)} + W_0 + W_1)R_k(\Delta)\right\}\,,\end{aligned} \tag{H.9}$$

where we have written

$$\mathcal{G}_k(\Delta) = \frac{1}{P_k(\Delta) + U_0}\,, \tag{H.10}$$

$$\frac{\delta}{\delta g}\Psi^g_k = V + W = V^{\mu\nu}\nabla_{(\mu}\nabla_{\nu)} + W_0 + W_1\,, \tag{H.11}$$

$$V^{\mu\nu}{}_{\rho\sigma}{}^{\alpha\beta} = \gamma_{Ricci}\left(-\frac{1}{2}\delta^\mu_\rho\delta^\nu_\sigma g^{\alpha\beta} + \delta^\mu_{(\rho} g^{\nu(\alpha}\delta^{\beta)}_{\sigma)} - \frac{1}{2}g^{\mu\nu}\delta^{(\alpha}_{(\rho}\delta^{\beta)}_{\sigma)}\right) + \gamma_R\, g_{\rho\sigma}\left(g^{\mu\alpha}g^{\beta\nu} - g^{\mu\nu}g^{\alpha\beta}\right)\,, \tag{H.12}$$

$$(W_0)_{\rho\sigma}{}^{\alpha\beta} = \gamma_g \delta^{(\alpha}_{(\rho}\delta^{\beta)}_{\sigma)}\,, \tag{H.13}$$

$$(W_1)_{\rho\sigma}{}^{\alpha\beta} = \frac{1}{2}\gamma_{Ricci}\left(\delta^{(\alpha}_{(\rho}R^{\beta)}_{\sigma)} - R^{(\alpha}{}_{(\rho}{}^{\beta)}{}_{\sigma)}\right) + \gamma_R\left(R\,\delta^{(\alpha}_{(\rho}\delta^{\beta)}_{\sigma)} - g_{\rho\sigma}R^{\alpha\beta}\right)\,. \tag{H.14}$$

Defining $\dot{R}_k := (\partial_t - \eta_N)R_k(\Delta)$, $\mathbb{T}_{gg}$ is composed by nine traces, which read

$$\Big\{\frac{1}{2}\mathrm{Tr}\,\mathcal{G}_k\dot{R}_k(\Delta)\,;\; -\frac{1}{2}\mathrm{Tr}\,\mathcal{G}_k^2U_1\dot{R}_k(\Delta)\,;\; \frac{1}{2}\mathrm{Tr}\,\mathcal{G}_k^3U_1^2\dot{R}_k(\Delta)\,;$$
$$\mathrm{Tr}\,\mathcal{G}_kWR_k(\Delta)\,;\; -\mathrm{Tr}\,\mathcal{G}_k^2U_1WR_k(\Delta)\,;\; \mathrm{Tr}\,\mathcal{G}_k^3U_1^2WR_k(\Delta)\,;$$
$$\mathrm{Tr}\,\mathcal{G}_kV^{\mu\nu}\nabla_{(\mu}\nabla_{\nu)}R_k(\Delta)\,;\; -\mathrm{Tr}\,\mathcal{G}_k^2U_1V^{\mu\nu}\nabla_{(\mu}\nabla_{\nu)}R_k(\Delta)\,;\; \mathrm{Tr}\,\mathcal{G}_k^3U_1^2V^{\mu\nu}\nabla_{(\mu}\nabla_{\nu)}R_k(\Delta)\Big\}\,.$$

Defining

$$Q_n[W(\Delta)] := \frac{1}{\Gamma(n)}\int_0^\infty \mathrm{d}z z^{n-1}W(z)\,, \tag{H.15}$$

below we report the evaluation of these traces

- $$\frac{1}{2}\mathrm{Tr}\,\mathcal{G}_k\dot{R}_k(\Delta) = \frac{1}{(4\pi)^{d/2}}\frac{1}{2}\sum_n Q_{d/2-n}\left[\mathcal{G}_k\dot{R}_k\right]\mathrm{tr}A_n$$
$$= \frac{1}{2(4\pi)^{d/2}}\int \mathrm{d}^dx\sqrt{\det g}\left\{\frac{d(d+1)}{2}Q_{d/2}\left[\mathcal{G}_k\dot{R}_k\right] + \frac{d(d+1)}{12}\,R\,Q_{d/2-1}\left[\mathcal{G}_k\dot{R}_k\right]\right.$$
$$\left. + \frac{1}{180}\left(\frac{d(d+1)}{2}\left(\frac{5}{2}R^2 - R_{\mu\nu}R^{\mu\nu}\right) + \frac{d^2-29d-60}{2}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right)Q_{d/2-2}\left[\mathcal{G}_k\dot{R}_k\right]\right\}\,,$$
- $$-\frac{1}{2}\mathrm{Tr}\,\mathcal{G}_k^2U_1\dot{R}_k(\Delta) = -\frac{1}{(4\pi)^{d/2}}\frac{1}{2}\sum_n Q_{d/2-n}\left[\mathcal{G}_k^2\dot{R}_k\right]\mathrm{tr}U_1A_n$$
$$= -\frac{1}{2(4\pi)^{d/2}}\int \mathrm{d}^dx\sqrt{\det g}\left\{\frac{d(d-1)}{2}R\,Q_{d/2}\left[\mathcal{G}_k^2\dot{R}_k\right] + \frac{d(d-1)}{12}\,R^2\,Q_{d/2-1}\left[\mathcal{G}_k^2\dot{R}_k\right]\right\}\,,$$
- $$\frac{1}{2}\mathrm{Tr}\,\mathcal{G}_k^3U_1^2\dot{R}_k(\Delta) = \frac{1}{(4\pi)^{d/2}}\frac{1}{2}\sum_n Q_{d/2-n}\left[\mathcal{G}_k^3\dot{R}_k\right]\mathrm{tr}U_1^2A_n$$
$$= \frac{1}{2(4\pi)^{d/2}}\int \mathrm{d}^dx\sqrt{\det g}\left\{\frac{d^3-5d^2+8d+4}{2(d-2)}R^2 + \frac{d^2-8d+4}{d-2}R_{\mu\nu}R^{\mu\nu} + 3R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right\}Q_{d/2}\left[\mathcal{G}_k^3\dot{R}_k\right]\,,$$

- $\mathrm{Tr}\,\mathcal{G}_k W R_k(\Delta) = \frac{1}{(4\pi)^{d/2}} \sum_n Q_{d/2-n}\left[\mathcal{G}_k R_k\right] \mathrm{tr} W A_n$

$$= \frac{\gamma_g}{(4\pi)^{d/2}} \int \mathrm{d}^d x \sqrt{\det g} \left\{ \frac{d(d+1)}{2} Q_{d/2}\left[\mathcal{G}_k R_k\right] + \frac{d(d+1)}{12} R\, Q_{d/2-1}\left[\mathcal{G}_k R_k\right] \right.$$
$$\left. + \frac{1}{180}\left(\frac{d(d+1)}{2}\left(\frac{5}{2}R^2 - R_{\mu\nu}R^{\mu\nu}\right) + \frac{d^2-29d-60}{2} R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right) Q_{d/2-2}\left[\mathcal{G}_k R_k\right]\right\}$$
$$+ \frac{1}{(4\pi)^{d/2}} \int \mathrm{d}^d x \sqrt{\det g}\, (\gamma_{Ricci} + 2(d-1)\gamma_R)\, \frac{d+2}{4} \left\{ R\, Q_{d/2}\left[\mathcal{G}_k R_k\right] + \frac{1}{6} R^2\, Q_{d/2-1}\left[\mathcal{G}_k R_k\right] \right\} ,$$

- $-\,\mathrm{Tr}\,\mathcal{G}_k^2 U_1 W R_k(\Delta) = -\frac{1}{(4\pi)^{d/2}} \sum_n Q_{d/2-n}\left[\mathcal{G}_k^2 R_k\right] \mathrm{tr} U_1 W A_n$

$$= -\frac{\gamma_g}{(4\pi)^{d/2}} \int \mathrm{d}^d x \sqrt{\det g} \left\{ \frac{d(d-1)}{2} R\, Q_{d/2}\left[\mathcal{G}_k^2 R_k\right] + \frac{d(d-1)}{12} R^2\, Q_{d/2-1}\left[\mathcal{G}_k^2 R_k\right] \right\}$$
$$- \frac{1}{(4\pi)^{d/2}} \int \mathrm{d}^d x \sqrt{\det g} \left\{ \left(\frac{(d+1)}{4}\gamma_{Ricci} + \left(\frac{d(d-1)}{2} - \frac{d-4}{d-2}\right)\gamma_R \right) R^2 \right.$$
$$\left. + \left(-\frac{(d+2)}{4}\gamma_{Ricci} + 2\left(\frac{d-4}{d-2}\right)\gamma_R\right) R_{\mu\nu}R^{\mu\nu} + \frac{3\gamma_{Ricci}}{4} R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \right\} Q_{d/2}\left[\mathcal{G}_k^2 R_k\right] ,$$

- $\mathrm{Tr}\,\mathcal{G}_k^3 U_1^2 W R_k(\Delta) = \frac{1}{(4\pi)^{d/2}} \sum_n Q_{d/2-n}\left[\mathcal{G}_k^3 R_k\right] \mathrm{tr} U_1^2 W A_n$

$$= \frac{\gamma_g}{(4\pi)^{d/2}} \int \mathrm{d}^d x \sqrt{\det g} \left\{ \frac{d^3-5d^2+8d+4}{2(d-2)} R^2 + \frac{d^2-8d+4}{d-2} R_{\mu\nu}R^{\mu\nu} + 3R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \right\} Q_{d/2}\left[\mathcal{G}_k^3 R_k\right] ,$$

- $\mathrm{Tr}\,\mathcal{G}_k V^{\mu\nu}\nabla_{(\mu}\nabla_{\nu)} R_k(\Delta) = \frac{1}{(4\pi)^{d/2}} \sum_n Q_{d/2+1-n}\left[\mathcal{G}_k R_k\right] \left(-\frac{1}{2}\mathrm{tr} V^{\mu}{}_{\mu} A_n + V^{\mu\nu} A_{n-1|\mu\nu}\right)$

$$= -\frac{1}{2(4\pi)^{d/2}} \int \mathrm{d}^d x \sqrt{\det g} \left\{ d(1-d)\left(\frac{d}{4}\gamma_{Ricci} + \gamma_R\right) \left(Q_{d/2+1}\left[\mathcal{G}_k R_k\right] + \frac{R}{6} Q_{d/2}\left[\mathcal{G}_k R_k\right]\right) \right.$$
$$+ \frac{1}{180}\left\{ d(1-d)\left(\frac{d}{4}\gamma_{Ricci} + \gamma_R\right)\left(\frac{5}{2}R^2 - R_{\mu\nu}R^{\mu\nu}\right)\right.$$
$$\left.\left. + \left(\frac{-d^3+31d^2-120}{4}\gamma_{Ricci} + d(1-d)\gamma_R\right) R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \right\} Q_{d/2-1}\left[\mathcal{G}_k R_k\right] \right\}$$
$$+ \frac{1}{(4\pi)^{d/2}} \int \mathrm{d}^d x \sqrt{\det g} \left\{ (1-d)\left(\frac{d}{4}\gamma_{Ricci} + \gamma_R\right) \frac{R}{6}\, Q_{d/2}\left[\mathcal{G}_k R_k\right] \right.$$
$$+ \left\{ \frac{\gamma_{Ricci}}{24}\left(-(d+4)R_{\mu\nu}R^{\mu\nu} + \frac{3d}{2} R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right) \right.$$
$$\left.\left. + \frac{(1-d)}{90}\left(\frac{d}{4}\gamma_{Ricci} + \gamma_R\right)\left(\frac{5}{2}R^2 - R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right)\right\} Q_{d/2-1}\left[\mathcal{G}_k R_k\right] \right\} ,$$

- $$- \operatorname{Tr} \mathcal{G}_k^2 U_1 V^{\mu\nu} \nabla_{(\mu}\nabla_{\nu)} R_k(\Delta) = -\frac{1}{(4\pi)^{d/2}} \sum_n \left(-\frac{1}{2} \mathrm{tr} U_1 V^{\mu}{}_{\mu} A_n + U_1 V^{\mu\nu} A_{n-1|\mu\nu} \right) Q_{d/2+1-n}\left[\mathcal{G}_k^2 R_k\right]$$

$$= \frac{1}{2(4\pi)^{d/2}} \int \mathrm{d}^d x \sqrt{\det g} \left(\frac{-d^3+3d^2-4d+8}{4} \gamma_{Ricci} - (d-4)(d-1)\gamma_R \right) \times \left(R\, Q_{d/2+1}\left[\mathcal{G}_k^2 R_k\right] + \frac{R^2}{6} Q_{d/2}\left[\mathcal{G}_k^2 R_k\right] \right)$$

$$- \frac{1}{(4\pi)^{d/2}} \int \mathrm{d}^d x \sqrt{\det g} \left\{ -\frac{1}{24(d-2)} \left((d^3-5d^2+6d+4)\gamma_{Ricci} + 4(d-3)(d-4)\gamma_R \right) R^2 - \frac{(d-4)}{6(d-2)} \left((d-1)\gamma_{Ricci} + 2\gamma_R \right) R_{\mu\nu}R^{\mu\nu} \right\} Q_{d/2}\left[\mathcal{G}_k^2 R_k\right] \,,$$

and finally

- $$\operatorname{Tr} \mathcal{G}_k^3 U_1^2 V^{\mu\nu} \nabla_{(\mu}\nabla_{\nu)} R_k(\Delta) = \frac{1}{(4\pi)^{d/2}} \sum_n Q_{d/2+1-n}\left[\mathcal{G}_k^3 R_k\right] \left(-\frac{1}{2} \mathrm{tr} U_1^2 V^{\mu}{}_{\mu} A_n + U_1^2 V^{\mu\nu} A_{n-1|\mu\nu} \right)$$

$$= \frac{-1}{2(4\pi)^{d/2}} \int \mathrm{d}^d x \sqrt{\det g} \left\{ \frac{-1}{4(d-2)} \left[(d^4-7d^3+20d^2-28d+24)\gamma_{Ricci} + 4(d-1)(d-4)^2\gamma_R \right] R^2 + \frac{1}{2(d-2)} \left[-(d^3-12d^2+36d-40)\gamma_{Ricci} + 4(d-1)(d-4)^2\gamma_R \right] R_{\mu\nu}R^{\mu\nu} + 3\left(1-\frac{d}{2}\right) \gamma_{Ricci} R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \right\} Q_{d/2+1}\left[\mathcal{G}_k^3 R_k\right] \,.$$

H.2 Calculation of ghost trace

In this subsection we calculate the ghosts contribution to the quantum part of the flow equation (4.54): like in the previous subsection, we insert the regulator in such a way that $\Delta \to P_k \equiv \Delta + R_k(\Delta)$, we calculate the Hessian, we expand the argument of the trace to quadratic order in curvature and finally we evaluate the trace. We then choose the regulator to be given by

$$\mathcal{R}_k^{\bar{c}c} = K_{\bar{c}c} R_k(\Delta) \,, \tag{H.16}$$

where

$$K_{\bar{c}c}^{\mu\nu} = \frac{\sqrt{2}}{\kappa_k} \sqrt{\det g} g_{\mu\nu} \,, \tag{H.17}$$

and the following relation holds

$$\partial_t K_{\bar{c}c} = -\frac{\eta_N}{2} K_{\bar{c}c} \,. \tag{H.18}$$

Since the Hessian in the ghost sector is

$$K_{\bar{c}c} \cdot \Delta_{\mathrm{gh}} + \mathcal{R}_k^{\bar{c}c} = K_{\bar{c}c} \cdot (P_k - Ricci) \,, \tag{H.19}$$

the ghost trace is given by

$$\mathbb{T}_{\bar{c}c} = -\mathrm{Tr}\left(\frac{1}{P_k} + Ricci\frac{1}{P_k^2} + Ricci^2\frac{1}{P_k^3}\right)(\partial_t R_k - \frac{1}{2}\eta_N R_k) = \tag{H.20}$$
$$= -\frac{1}{(4\pi)^{d/2}}\int \mathrm{d}^d x\sqrt{\det g}\left\{d\,Q_{d/2}\left[\frac{(\partial_t R_k - \frac{1}{2}\eta_N R_k)}{P_k}\right] + \frac{d}{6}R\,Q_{d/2-1}\left[\frac{(\partial_t R_k - \frac{1}{2}\eta_N R_k)}{P_k}\right]\right.$$
$$+\frac{1}{180}\left(\frac{5d}{2}R^2 - dR_{\mu\nu}R^{\mu\nu} + (d-15)R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right)Q_{d/2-2}\left[\frac{(\partial_t R_k - \frac{1}{2}\eta_N R_k)}{P_k}\right]$$
$$\left.+R\,Q_{d/2}\left[\frac{\left(\partial_t - \frac{1}{2}\eta_N\right)R_k}{P_k^2}\right] + \frac{1}{6}R^2\,Q_{d/2-1}\left[\frac{\left(\partial_t - \frac{1}{2}\eta_N\right)R_k}{P_k^2}\right] + R_{\mu\nu}R^{\mu\nu}\,Q_{d/2}\left[\frac{\left(\partial_t - \frac{1}{2}\eta_N\right)R_k}{P_k^3}\right]\right\}.$$

H.3 Beta and gamma functions

In this subsection we put all the contributions inside the flow equation together and we write down the beta functions for ρ_k, G_k and c_k and the equations for the gamma functions γ_{Ricci} and γ_R. In order to express everything in the curvature basis $\left(R^2, R_{\mu\nu}R^{\mu\nu}, E\right)$, we have expressed the Riemann tensor square as $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = E + 4R_{\mu\nu}R^{\mu\nu} - R^2$ in the equations contained in H.1 and H.2. From the coefficient of $\sqrt{\det g}$, we can find the beta function of ρ_k by solving

$$\left(\partial_t + \frac{d}{2}\gamma_g\right)\frac{\rho_k}{8\pi} = \frac{1}{(4\pi)^{d/2}}\left\{\frac{d(d+1)}{2}\left(\frac{1}{2}Q_{d/2}\left[\mathcal{G}_k\dot{R}_k\right] + \gamma_g Q_{d/2}\left[\mathcal{G}_k R_k\right]\right)\right. \tag{H.21}$$
$$+\frac{d(d-1)}{2}\left(\frac{d}{4}\gamma_{Ricci} + \gamma_R\right)Q_{d/2+1}\left[\mathcal{G}_k R_k\right]$$
$$\left.-d\,Q_{d/2}\left[\frac{\left(\partial_t - \frac{1}{2}\eta_N\right)R_k}{P_k}\right]\right\}.$$

Note that (H.21) can be also understood as an equation for γ_g: in fact, it is possible to fix the value of $\tilde{\rho}_k$ tuning γ_g. As we discussed in section 5.4, this procedure corresponds to impose a renormalization condition that fix the value of the vacuum energy.

From the coefficient of $\sqrt{\det g}\,R$, we can find the beta function of G_k

$$
\begin{aligned}
-\left(\partial_t+\frac{d-2}{2}\gamma_g\right)\frac{1}{16\pi G_k}+(\gamma_{Ricci}+d\gamma_R)\frac{\rho_k}{16\pi} &= \\
=\frac{1}{(4\pi)^{d/2}}&\left\{\frac{d(d+1)}{12}\left(\frac{1}{2}Q_{d/2-1}\left[\mathcal{G}_k\dot{R}_k\right]+\gamma_g Q_{d/2-1}\left[\mathcal{G}_k R_k\right]\right)\right.\\
&-\frac{d(d-1)}{2}\left(\frac{1}{2}Q_{d/2}\left[\mathcal{G}_k^2\dot{R}_k\right]+\gamma_g Q_{d/2}\left[\mathcal{G}_k^2 R_k\right]\right)\\
&+\frac{1}{48}\left(\left(d^3-3d^2+14d+24\right)\gamma_{Ricci}+4\left(7d^2+3d-10\right)\gamma_R\right)Q_{d/2}\left[\mathcal{G}_k R_k\right]\\
&+\frac{1}{2}\left(\frac{-d^3+3d^2-4d+8}{4}\gamma_{Ricci}-(d-4)(d-1)\gamma_R\right)Q_{d/2+1}\left[\mathcal{G}_k^2 R_k\right]\\
&\left.-\frac{d}{6}Q_{d/2-1}\left[\frac{\left(\partial_t-\frac{1}{2}\eta_N\right)R_k}{P_k}\right]-Q_{d/2}\left[\frac{\left(\partial_t-\frac{1}{2}\eta_N\right)R_k}{P_k^2}\right]\right\}.
\end{aligned}
\tag{H.22}
$$

The coefficient of $\sqrt{\det g}\,R^2$ is

$$
\begin{aligned}
-\frac{1}{32\pi G_k}(\gamma_{Ricci}+(d-2)\gamma_R) &= \\
=\frac{1}{(4\pi)^{d/2}}&\left\{\frac{d^2+21d+40}{240}\left(\frac{1}{2}Q_{d/2-2}\left[\mathcal{G}_k\dot{R}_k\right]+\gamma_g Q_{d/2-2}\left[\mathcal{G}_k R_k\right]\right)\right.\\
&-\frac{d(d-1)}{12}\left(\frac{1}{2}Q_{d/2-1}\left[\mathcal{G}_k^2\dot{R}_k\right]+\gamma_g Q_{d/2-1}\left[\mathcal{G}_k^2 R_k\right]\right)\\
&+\frac{d^3-5d^2+2d+16}{2(d-2)}\left(\frac{1}{2}Q_{d/2}\left[\mathcal{G}_k^3\dot{R}_k\right]+\gamma_g Q_{d/2}\left[\mathcal{G}_k^3 R_k\right]\right)\\
&+\frac{1}{960}\left((d-1)d(d+16)\gamma_{Ricci}+12(d-1)(7d+12)\gamma_R\right)Q_{d/2-1}\left[\mathcal{G}_k R_k\right]\\
&-\frac{\left(d^4-7d^3+32d^2-76d+56\right)\gamma_{Ricci}+4\left(7d^3-27d^2+28d+16\right)\gamma_R}{48(d-2)}Q_{d/2}\left[\mathcal{G}_k^2 R_k\right]\\
&+\frac{d\left(d^3-7d^2+14d-4\right)\gamma_{Ricci}+4(d-1)(d-4)^2\gamma_R}{8(d-2)}Q_{d/2+1}\left[\mathcal{G}_k^3 R_k\right]\\
&\left.-\frac{d+10}{120}Q_{d/2-2}\left[\frac{\left(\partial_t-\frac{1}{2}\eta_N\right)R_k}{P}\right]-\frac{1}{6}Q_{d/2-1}\left[\frac{\left(\partial_t-\frac{1}{2}\eta_N\right)R_k}{P^2}\right]\right\},
\end{aligned}
\tag{H.23}
$$

and the coefficient of $\sqrt{\det g}\, R_{\mu\nu}R^{\mu\nu}$ is

$$\begin{aligned}\frac{\gamma_{Ricci}}{16\pi G_k} = \frac{1}{(4\pi)^{d/2}} \Bigg\{ & \frac{\left(d^2-39d-80\right)}{120}\left(\frac{1}{2}Q_{d/2-2}\left[\mathcal{G}_k\dot{R}_k\right]+\gamma_g Q_{d/2-2}\left[\mathcal{G}_k R_k\right]\right) \\ & +\frac{d^2+4d-20}{d-2}\left(\frac{1}{2}Q_{d/2}\left[\mathcal{G}_k^3\dot{R}_k\right]+\gamma_g Q_{d/2}\left[\mathcal{G}_k^3 R_k\right]\right) \\ & +\frac{1}{480}\left(\left(d^3-45d^2+104d+80\right)\gamma_{Ricci}+4\left(d^2-5d+4\right)\gamma_R\right)Q_{d/2-1}\left[\mathcal{G}_k R_k\right] \\ & +\frac{\left(5d^2-46d+68\right)\gamma_{Ricci}-20(d-4)\gamma_R}{12(d-2)}Q_{d/2}\left[\mathcal{G}_k^2 R_k\right] \\ & +\frac{\left(d^3-12d+8\right)\gamma_{Ricci}-4(d-4)^2(d-1)\gamma_R}{4(d-2)}Q_{d/2+1}\left[\mathcal{G}_k^3 R_k\right] \\ & -\frac{d-20}{60}Q_{d/2-2}\left[\frac{\left(\partial_t-\frac{1}{2}\eta_N\right)R_k}{P_k}\right]-Q_{d/2}\left[\frac{\left(\partial_t-\frac{1}{2}\eta_N\right)R_k}{P^3}\right]\Bigg\}\,. \end{aligned} \tag{H.24}$$

Note that (H.23) and (H.24) are the equations for the gamma functions γ_{Ricci} and γ_R, which are the parameters of the RG kernel that fix to zero the value of the couplings associated to the operators $\sqrt{\det g}\, R^2$ and $\sqrt{\det g}\, R_{\mu\nu}R^{\mu\nu}$.
Finally, from the coefficient of $\sqrt{\det g}\, E$ we can find the beta function of c_k

$$\begin{aligned}\left(\partial_t+\frac{d-4}{2}\gamma_g\right)c_k = \frac{1}{(4\pi)^{d/2}} \Bigg\{ & \frac{d^2-29d-60}{360}\left(\frac{1}{2}Q_{d/2-2}\left[\mathcal{G}_k\dot{R}_k\right]+\gamma_g Q_{d/2-2}\left[\mathcal{G}_k R_k\right]\right) \\ & +3\left(\frac{1}{2}Q_{d/2}\left[\mathcal{G}_k^3\dot{R}_k\right]+\gamma_g Q_{d/2}\left[\mathcal{G}_k^3 R_k\right]\right)-\frac{3\gamma_{Ricci}}{4}Q_{d/2}\left[\mathcal{G}_k^2 R_k\right] \\ & +\frac{(d-4)}{1440}\left(\left(d^2-31d-30\right)\gamma_{Ricci}+4(d-1)\gamma_R\right)Q_{d/2-1}\left[\mathcal{G}_k R_k\right] \\ & -\frac{3}{2}\left(1-\frac{d}{2}\right)\gamma_{Ricci}Q_{d/2+1}\left[\mathcal{G}_k^3 R_k\right]-\frac{(d-15)}{180}Q_{d/2-2}\left[\frac{\left(\partial_t-\frac{1}{2}\eta_N\right)R_k}{P_k}\right]\Bigg\}\,. \end{aligned} \tag{H.25}$$

Bibliography

[1] A. Baldazzi, R. Percacci and L. Zambelli, "Functional renormalization and the $\overline{\mathrm{MS}}$ scheme," Phys. Rev. D **103** (2021) no.7, 076012 [arXiv:2009.03255 [hep-th]].

[2] A. Baldazzi, R. Percacci and L. Zambelli, "Limit of vanishing regulator in the functional renormalization group," Phys. Rev. D **104** (2021) no.7, 076026

[3] A. Baldazzi, R. B. A. Zinati and K. Falls, "Essential renormalisation group," [arXiv:2105.11482 [hep-th]].

[4] A. Baldazzi and K. Falls, "Essential Quantum Einstein Gravity," Universe **7** (2021) no.8, 294 [arXiv:2107.00671 [hep-th]].

[5] Y. Khaluf, E. Ferrante, P. Simoens and C. Huepe, "Scale invariance in natural and artificial collective systems: a review," J. R. Soc. Interface.**14**:20170662.

[6] K. Huang, "A Critical History of Renormalization," Int. J. Mod. Phys. A **28** (2013), 1330050 [arXiv:1310.5533 [physics.hist-ph]].

[7] J. J. Thomson, "Cathode rays," Phil. Mag. Ser. 5 **44** (1897), 293-316 doi:10.1080/14786449708621070

[8] P. A. M. Dirac, "The quantum theory of the electron," Proc. Roy. Soc. Lond. A **117** (1928), 610-624

[9] P. A. M. Dirac, "A Theory of Electrons and Protons," Proc. Roy. Soc. Lond. A **126** (1930) no.801, 360-365

[10] V. Weisskopf, "On the self-energy of the electron," Z. Phys. **89** (1934) no.1-2, 27-39 [erratum: Z. Phys. **90** (1934) no.11-12, 817-818]

[11] V. F. Weisskopf, "On the Self-Energy and the Electromagnetic Field of the Electron," Phys. Rev. **56** (1939), 72-85

[12] W. Heisenberg, "Remarks on the Dirac theory of the positron," Z. Phys. **90** (1934) no.3-4, 209-231 [erratum: Z. Phys. **92** (1934) no.9-10, 692-692]

[13] R. Serber, "Linear modifications in the Maxwell field equations," Phys. Rev. **48** (1935), 49

[14] E. A. Uehling, "Polarization effects in the positron theory," Phys. Rev. **48** (1935), 55-63

[15] H. A. Bethe, "The Electromagnetic shift of energy levels," Phys. Rev. **72** (1947), 339-341

[16] J. S. Schwinger, "On Quantum electrodynamics and the magnetic moment of the electron," Phys. Rev. **73** (1948), 416-417

[17] J. S. Schwinger, "Quantum electrodynamics. 2. Vacuum polarization and selfenergy," Phys. Rev. **75** (1948), 651

[18] F. J. Dyson, "The Radiation theories of Tomonaga, Schwinger, and Feynman," Phys. Rev. **75** (1949), 486-502

[19] L. D. Landau, "On the theory of phase transitions," Zh. Eksp. Teor. Fiz. **7** (1937), 19-32

[20] E. C. G. Stueckelberg de Breidenbach and A. Petermann, "Normalization of constants in the quanta theory," Helv. Phys. Acta **26** (1953), 499-520

[21] M. Gell-Mann and F. E. Low, "Quantum electrodynamics at small distances," Phys. Rev. **95** (1954), 1300-1312

[22] N. N. Bogolyubov and D. V. Shirkov, "INTRODUCTION TO THE THEORY OF QUANTIZED FIELDS," Intersci. Monogr. Phys. Astron. **3** (1959), 1-720

[23] J. Zinn-Justin, "Quantum field theory and critical phenomena," Int. Ser. Monogr. Phys. **113** (2002), 1-1054

[24] J. Zinn-Justin, "Phase transitions and renormalization group," Oxford Univ. Pr. (2007) 452

[25] J. Cardy "Scaling and Renormalization in Statistical Physics," Cambridge University Press, 1996.

[26] L. P. Kadanoff, "Scaling laws for Ising models near T(c)," Physics Physique Fizika **2** (1966), 263-272

[27] K. G. Wilson, "Renormalization group and critical phenomena. 1. Renormalization group and the Kadanoff scaling picture," Phys. Rev. B **4** (1971), 3174-3183

[28] K. G. Wilson, "Renormalization group and critical phenomena. 2. Phase space cell analysis of critical behavior," Phys. Rev. B **4** (1971), 3184-3205

[29] K. G. Wilson and J. B. Kogut, “The Renormalization group and the epsilon expansion,” Phys. Rept. **12** (1974), 75-199

[30] F. J. Wegner, “Corrections to scaling laws,” Phys. Rev. B **5** (1972), 4529-4536

[31] K. G. Wilson and M. E. Fisher, “Critical exponents in 3.99 dimensions,” Phys. Rev. Lett. **28** (1972), 240-243

[32] F. J. Wegner and A. Houghton, “Renormalization group equation for critical phenomena,” Phys. Rev. A **8** (1973), 401-412

[33] S. k. Ma, “Introduction to the renormalization group,” Rev. Mod. Phys. **45** (1973), 589-614

[34] C. Bagnuls and C. Bervillier, “Exact renormalization group equations. An Introductory review,” Phys. Rept. **348** (2001), 91 [arXiv:hep-th/0002034 [hep-th]].

[35] F. J. Wegner, “Some invariance properties of the renormalization group,” J. Phys. C 7 (1974) 2098

[36] H. Sonoda, “Connection on the theory space,” [arXiv:hep-th/9306119 [hep-th]].

[37] B. P. Dolan, “Covariant derivatives and the renormalization group equation,” Int. J. Mod. Phys. A **10** (1995), 2439-2466 [arXiv:hep-th/9403070 [hep-th]].

[38] C. Pagani and H. Sonoda, “Geometry of the theory space in the exact renormalization group formalism,” Phys. Rev. D **97** (2018) no.2, 025015 [arXiv:1710.10409 [hep-th]].

[39] S. Weinberg, in *General Relativity: An Einstein centenary survey*, edited by S. W. Hawking and W. Israel (1979) pp. 790-831.

[40] Percacci, R. *An Introduction to Covariant Quantum Gravity and Asymptotic Safety*; World Scientific: Singapore, 2017.

[41] G. Jona-Lasinio, “The Renormalization Group: A Probabilistic View,” Nuovo Cim. B **26** (1975), 99

[42] G. Jona-Lasinio, in *Proc. Nobel Symp. 24: Collective Properties of Physical Systems*, edited by N. Svartholm (Stockholm, Nobel Foundation; New York, Academic Press, 1973) pp. 38-44.

[43] G. R. Golner, “Nonperturbative Renormalization Group Calculations for Continuum Spin Systems,” Phys. Rev. B **33** (1986), 7863-7866

[44] E. Riedel, G. R. Golner and K. E. Newman, “Scaling Field Representation of Wilson’s Exact Renormalization Group Equation,” Annals Phys. **161** (1985), 178-238

[45] J. Comellas and A. Travesset, "O (N) models within the local potential approximation," Nucl. Phys. B **498** (1997), 539-564 [arXiv:hep-th/9701028 [hep-th]].

[46] J. Polchinski, "Renormalization and Effective Lagrangians," Nucl. Phys. B **231** (1984), 269-295

[47] R. D. Ball and R. S. Thorne, "Renormalizability of effective scalar field theory," Annals Phys. **236** (1994), 117-204 [arXiv:hep-th/9310042 [hep-th]].

[48] T. R. Morris, "Derivative expansion of the exact renormalization group," Phys. Lett. B **329** (1994), 241-248 [arXiv:hep-ph/9403340 [hep-ph]].

[49] T. R. Morris, "Properties of derivative expansion approximations to the renormalization group," Int. J. Mod. Phys. B **12** (1998), 1343-1354 [arXiv:hep-th/9610012 [hep-th]].

[50] J. Generowicz, C. Harvey-Fros and T. R. Morris, "C function representation of the local potential approximation," Phys. Lett. B **407** (1997), 27-32 [arXiv:hep-th/9705088 [hep-th]].

[51] R. Neves, Y. Kubyshin and R. Potting, "Polchinski ERG equation and 2-D scalar field theory," [arXiv:hep-th/9811151 [hep-th]].

[52] A. Hasenfratz and P. Hasenfratz, "Renormalization Group Study of Scalar Field Theories," Nucl. Phys. B **270** (1986), 687-701

[53] J. Comellas, "Polchinski equation, reparameterization invariance and the derivative expansion," Nucl. Phys. B **509** (1998), 662-686 [arXiv:hep-th/9705129 [hep-th]].

[54] C. Wetterich, "Exact evolution equation for the effective potential," Phys. Lett. B **301** (1993), 90-94 [arXiv:1710.05815 [hep-th]].

[55] T. R. Morris, "The Exact renormalization group and approximate solutions," Int. J. Mod. Phys. A **9** (1994), 2411-2450 [arXiv:hep-ph/9308265 [hep-ph]].

[56] M. Bonini, M. D'Attanasio and G. Marchesini, "Perturbative renormalization and infrared finiteness in the Wilson renormalization group: The Massless scalar case," Nucl. Phys. B **409** (1993), 441-464 [arXiv:hep-th/9301114 [hep-th]].

[57] U. Ellwanger, "FLow equations for N point functions and bound states," Z. Phys. C **62** (1994), 503-510 [arXiv:hep-ph/9308260 [hep-ph]].

[58] R. Floreanini and R. Percacci, "The Heat kernel and the average effective potential," Phys. Lett. B **356** (1995), 205-210 [arXiv:hep-th/9505172 [hep-th]].

[59] T. R. Morris, "Elements of the continuous renormalization group," Prog. Theor. Phys. Suppl. **131** (1998), 395-414 [arXiv:hep-th/9802039 [hep-th]].

[60] J. Berges, N. Tetradis and C. Wetterich, "Nonperturbative renormalization flow in quantum field theory and statistical physics," Phys. Rept. **363** (2002), 223-386 [arXiv:hep-ph/0005122 [hep-ph]].

[61] J. M. Pawlowski, "Aspects of the functional renormalisation group," Annals Phys. **322** (2007), 2831-2915 [arXiv:hep-th/0512261 [hep-th]].

[62] O. J. Rosten, "Fundamentals of the Exact Renormalization Group," Phys. Rept. **511** (2012), 177-272 [arXiv:1003.1366 [hep-th]].

[63] H. Gies, "Introduction to the functional RG and applications to gauge theories," Lect. Notes Phys. **852** (2012), 287-348 [arXiv:hep-ph/0611146 [hep-ph]].

[64] B. Delamotte, "An Introduction to the nonperturbative renormalization group," Lect. Notes Phys. **852** (2012), 49-132 [arXiv:cond-mat/0702365 [cond-mat.stat-mech]].

[65] P. Kopietz, L. Bartosch and F. Schütz, "Introduction to the functional renormalization group," Lect. Notes Phys. **798** (2010), 1-380

[66] A. Codello, R. Percacci, L. Rachwał and A. Tonero, "Computing the Effective Action with the Functional Renormalization Group," Eur. Phys. J. C **76** (2016) no.4, 226 [arXiv:1505.03119 [hep-th]].

[67] N. Dupuis, L. Canet, A. Eichhorn, W. Metzner, J. M. Pawlowski, M. Tissier and N. Wschebor, "The nonperturbative functional renormalization group and its applications," Phys. Rept. **910** (2021), 1-114 [arXiv:2006.04853 [cond-mat.stat-mech]].

[68] N. Ohta and L. Rachwal, "Effective Action from the Functional Renormalization Group," [arXiv:2002.10839 [hep-th]].

[69] T. Papenbrock and C. Wetterich, "Two loop results from one loop computations and nonperturbative solutions of exact evolution equations," Z. Phys. C **65** (1995), 519-535 [arXiv:hep-th/9403164 [hep-th]].

[70] A. Codello, M. Demmel and O. Zanusso, "Scheme dependence and universality in the functional renormalization group," Phys. Rev. D **90** (2014) no.2, 027701 [arXiv:1310.7625 [hep-th]].

[71] F. Benitez, J. P. Blaizot, H. Chate, B. Delamotte, R. Mendez-Galain and N. Wschebor, "Non-perturbative renormalization group preserving full-momentum dependence: implementation and quantitative evaluation," Phys. Rev. E **85** (2012), 026707 [arXiv:1110.2665 [cond-mat.stat-mech]].

[72] J. P. Blaizot, R. Mendez-Galain and N. Wschebor, "Non perturbative renormalisation group and momentum dependence of n-point functions (I)," Phys. Rev. E **74** (2006), 051116 [arXiv:hep-th/0512317 [hep-th]].

[73] J. P. Blaizot, R. Mendez-Galain and N. Wschebor, "Non perturbative renormalization group and momentum dependence of n-point functions. II.," Phys. Rev. E **74** (2006), 051117 [arXiv:hep-th/0603163 [hep-th]].

[74] G. Zumbach, "The Local potential approximation of the renormalization group and its applications," Phys. Lett. A **190** (1994), 225-230

[75] N. Defenu and A. Codello, "Scaling solutions in the derivative expansion," Phys. Rev. D **98** (2018) no.1, 016013 [arXiv:1711.01809 [hep-th]].

[76] D. F. Litim, "Optimized renormalization group flows," Phys. Rev. D **64** (2001), 105007 [arXiv:hep-th/0103195 [hep-th]].

[77] D. F. Litim, "Critical exponents from optimized renormalization group flows," Nucl. Phys. B **631** (2002), 128-158 [arXiv:hep-th/0203006 [hep-th]].

[78] J. F. Nicoll and T. S. Chang, "An Exact One Particle Irreducible Renormalization Group Generator for Critical Phenomena," Phys. Lett. A **62** (1977), 287-289

[79] I. H. Bridle, J. A. Dietz and T. R. Morris, "The local potential approximation in the background field formalism," JHEP **03** (2014), 093 [arXiv:1312.2846 [hep-th]].

[80] T. R. Morris, "On truncations of the exact renormalization group," Phys. Lett. B **334** (1994), 355-362 [arXiv:hep-th/9405190 [hep-th]].

[81] A. Codello, "Scaling Solutions in Continuous Dimension," J. Phys. A **45** (2012), 465006 [arXiv:1204.3877 [hep-th]].

[82] T. Hellwig, A. Wipf and O. Zanusso, "Scaling and superscaling solutions from the functional renormalization group," Phys. Rev. D **92** (2015) no.8, 085027 [arXiv:1508.02547 [hep-th]].

[83] D. F. Litim, "Mind the gap," Int. J. Mod. Phys. A **16** (2001), 2081-2088 [arXiv:hep-th/0104221 [hep-th]].

[84] L. Canet, B. Delamotte, D. Mouhanna and J. Vidal, "Nonperturbative renormalization group approach to the Ising model: A Derivative expansion at order partial**4," Phys. Rev. B **68** (2003), 064421 [arXiv:hep-th/0302227 [hep-th]].

[85] I. Balog, H. Chaté, B. Delamotte, M. Marohnic and N. Wschebor, "Convergence of Nonperturbative Approximations to the Renormalization Group," Phys. Rev. Lett. **123** (2019) no.24, 240604 [arXiv:1907.01829 [cond-mat.stat-mech]].

[86] G. De Polsi, I. Balog, M. Tissier and N. Wschebor, "Precision calculation of critical exponents in the $O(N)$ universality classes with the nonperturbative renormalization group," Phys. Rev. E **101** (2020) no.4, 042113 [arXiv:2001.07525 [cond-mat.stat-mech]].

[87] T. R. Morris, "Equivalence of local potential approximations," JHEP **07** (2005), 027 [arXiv:hep-th/0503161 [hep-th]].

[88] T. R. Morris and Z. H. Slade, "Solutions to the reconstruction problem in asymptotic safety," JHEP **11** (2015), 094 [arXiv:1507.08657 [hep-th]].

[89] D. F. Litim, "Universality and the renormalisation group," JHEP **07** (2005), 005 [arXiv:hep-th/0503096 [hep-th]].

[90] C. Bervillier, "O(N) Wilson-Polchinski exact renormalization group equation: Leading and next-to-leading orders in the derivative expansion," J. Phys. Condens. Matter **17** (2005), S1929 [arXiv:hep-th/0501087 [hep-th]].

[91] A. V. Manohar, "Introduction to Effective Field Theories," [arXiv:1804.05863 [hep-ph]].

[92] H. Georgi, "Effective field theory," Ann. Rev. Nucl. Part. Sci. **43** (1993), 209-252

[93] I. Jack and H. Osborn, "Two Loop Background Field Calculations for Arbitrary Background Fields," Nucl. Phys. B **207** (1982), 474-504

[94] C. Ford, I. Jack and D. Jones, "The Standard model effective potential at two loops," Nucl. Phys. B **387** (1992), 373-390 [arXiv:hep-ph/0111190 [hep-ph]].

[95] S. P. Martin, "Two Loop Effective Potential for a General Renormalizable Theory and Softly Broken Supersymmetry," Phys. Rev. D **65** (2002), 116003 [arXiv:hep-ph/0111209 [hep-ph]].

[96] A. Codello, M. Safari, G. P. Vacca and O. Zanusso, "Functional perturbative RG and CFT data in the ϵ-expansion," Eur. Phys. J. C **78** (2018) no.1, 30 [arXiv:1705.05558 [hep-th]].

[97] U. Ellwanger, "Flow equations and BRS invariance for Yang-Mills theories," Phys. Lett. B **335**, 364-370 (1994) [arXiv:hep-th/9402077 [hep-th]].

[98] A. Codello, M. Safari, G. P. Vacca and O. Zanusso, "New universality class in three dimensions: The critical Blume-Capel model," Phys. Rev. D **96** (2017) no.8, 081701 [arXiv:1706.06887 [hep-th]].

[99] M. Safari and G. P. Vacca, "Multicritical scalar theories with higher-derivative kinetic terms: A perturbative RG approach with the ϵ-expansion," Phys. Rev. D **97** (2018) no.4, 041701 [arXiv:1708.09795 [hep-th]].

[100] M. Safari and G. P. Vacca, "Uncovering novel phase structures in $\Box^k$ scalar theories with the renormalization group," Eur. Phys. J. C **78** (2018) no.3, 251 [arXiv:1711.08685 [hep-th]].

[101] A. Codello, M. Safari, G. P. Vacca and O. Zanusso, "Leading order CFT analysis of multi-scalar theories in d¿2," Eur. Phys. J. C **79** (2019) no.4, 331 [arXiv:1809.05071 [hep-th]].

[102] A. Codello, M. Safari, G. P. Vacca and O. Zanusso, "Multi-critical multi-field models: a CFT approach to the leading order," Universe, arXiv:1905.01086 [hep-th].

[103] H. Osborn and D. Twigg, "Reparameterisation Invariance and RG equations: Extension of the Local Potential Approximation," J. Phys. A **42** (2009), 195401 [arXiv:0901.0450 [hep-th]].

[104] T. R. Morris, "The Renormalization group and two-dimensional multicritical effective scalar field theory," Phys. Lett. B **345** (1995), 139-148 [arXiv:hep-th/9410141 [hep-th]].

[105] N. Defenu and A. Codello, "The fate of $O(N)$ multi-critical universal behaviour," [arXiv:2005.10827 [cond-mat.stat-mech]].

[106] A. B. Zamolodchikov, "Conformal Symmetry and Multicritical Points in Two-Dimensional Quantum Field Theory. (In Russian)," Sov. J. Nucl. Phys. **44** (1986), 529-533

[107] A. Codello and R. Percacci, "Fixed Points of Nonlinear Sigma Models in $d > 2$," Phys. Lett. B **672** (2009), 280-283 [arXiv:0810.0715 [hep-th]].

[108] R. Percacci and O. Zanusso, "One loop beta functions and fixed points in Higher Derivative Sigma Models," Phys. Rev. D **81** (2010), 065012 [arXiv:0910.0851 [hep-th]].

[109] M. Fabbrichesi, R. Percacci, A. Tonero and O. Zanusso, "Asymptotic safety and the gauged SU(N) nonlinear σ-model," Phys. Rev. D **83** (2011), 025016 [arXiv:1010.0912 [hep-ph]].

[110] F. Bazzocchi, M. Fabbrichesi, R. Percacci, A. Tonero and L. Vecchi, "Fermions and Goldstone bosons in an asymptotically safe model," Phys. Lett. B **705** (2011), 388-392 [arXiv:1105.1968 [hep-ph]].

[111] R. Flore, A. Wipf and O. Zanusso, "Functional renormalization group of the nonlinear sigma model and the $O(N)$ universality class," Phys. Rev. D **87** (2013) no.6, 065019 [arXiv:1207.4499 [hep-th]].

[112] R. Flore, "Renormalization of the Nonlinear O(3) Model with Theta-Term," Nucl. Phys. B **870** (2013), 444-459 [arXiv:1208.5948 [hep-th]].

[113] R. Percacci and M. Safari, "Functional renormalization of N scalars with O(N) invariance," Phys. Rev. D **88** (2013), 085007 [arXiv:1306.3918 [hep-th]].

[114] M. Safari, "One-loop functional renormalization group flow of scalar theories with electroweak symmetry," Phys. Rev. D **90** (2014) no.6, 065009 [arXiv:1406.0800 [hep-th]].

[115] M. Bonini, G. Marchesini and M. Simionato, "Beta function and infrared renormalons in the exact Wilson renormalization group in Yang-Mills theory," Nucl. Phys. B **483** (1997) 475 [hep-th/9604114].

[116] A. Bonanno and D. Zappala, "Two loop results from the derivative expansion of the blocked action," Phys. Rev. D **57** (1998) 7383 [hep-th/9712038].

[117] M. Pernici and M. Raciti, "Wilsonian flow and mass independent renormalization," Nucl. Phys. B **531** (1998) 560 [hep-th/9803212].

[118] P. Kopietz, "Two loop beta function from the exact renormalization group," Nucl. Phys. B **595** (2001) 493 [hep-th/0007128].

[119] D. Zappala, "Perturbative and nonperturbative aspects of the proper time renormalization group," Phys. Rev. D **66** (2002) 105020 [hep-th/0202167].

[120] S. Arnone, A. Gatti, T. R. Morris and O. J. Rosten, "Exact scheme independence at two loops," Phys. Rev. D **69** (2004) 065009 [hep-th/0309242].

[121] H. Gies, R. Sondenheimer, A. Ugolotti and L. Zambelli, "Scheme dependence of asymptotically free solutions," Eur. Phys. J. C **79** (2019) no.6, 463 [arXiv:1901.08581 [hep-th]].

[122] M. D'Attanasio and T. R. Morris, "Large N and the renormalization group," Phys. Lett. B **409** (1997), 363-370 [arXiv:hep-th/9704094 [hep-th]].

[123] D. F. Litim, E. Marchais and P. Mati, "Fixed points and the spontaneous breaking of scale invariance," Phys. Rev. D **95** (2017) no.12, 125006 [arXiv:1702.05749 [hep-th]].

[124] F. Kos, D. Poland and D. Simmons-Duffin, "Bootstrapping Mixed Correlators in the 3D Ising Model," JHEP **11** (2014), 109 [arXiv:1406.4858 [hep-th]].

[125] J. A. Dietz and T. R. Morris, "Redundant operators in the exact renormalisation group and in the f(R) approximation to asymptotic safety," JHEP **07** (2013), 064 [arXiv:1306.1223 [hep-th]].

[126] I. Balog, G. De Polsi, M. Tissier and N. Wschebor, "Conformal invariance in the nonperturbative renormalization group: a rationale for choosing the regulator," Phys. Rev. E **101**, no.6, 062146 (2020) [arXiv:2004.02521 [cond-mat.stat-mech]].

[127] H. Sonoda, "Phase structure of a three-dimensional Yukawa model," Prog. Theor. Phys. **126**, 57-80 (2011) [arXiv:1102.3974 [hep-th]].

[128] D. Anselmi, "A General Field-Covariant Formulation Of Quantum Field Theory," Eur. Phys. J. C **73** (2013) no.3, 2338 [arXiv:1205.3279 [hep-th]].

[129] J. S. R. Chisholm, "Change of variables in quantum field theories," Nucl. Phys. **26** (1961) no.3, 469-479

[130] S. Kamefuchi, L. O'Raifeartaigh and A. Salam, "Change of variables and equivalence theorems in quantum field theories," Nucl. Phys. **28** (1961), 529-549

[131] M. C. Bergere and Y. M. P. Lam, "Equivalence Theorem and Faddeev-Popov Ghosts," Phys. Rev. D **13** (1976), 3247-3255

[132] R. D. Ball, P. E. Haagensen, J. I. Latorre and E. Moreno, "Scheme independence and the exact renormalization group," Phys. Lett. B **347** (1995), 80-88 [arXiv:hep-th/9411122 [hep-th]].

[133] J. I. Latorre and T. R. Morris, "Exact scheme independence," JHEP **11** (2000), 004 [arXiv:hep-th/0008123 [hep-th]].

[134] S. Arnone, A. Gatti and T. R. Morris, "Exact scheme independence at one loop," JHEP **05** (2002), 059 [arXiv:hep-th/0201237 [hep-th]].

[135] O. J. Rosten, "Scheme independence to all loops," J. Phys. A **39** (2006), 8141-8156 [arXiv:hep-th/0511107 [hep-th]].

[136] J. F. Nicoll, T. S. Chang and H. E. Stanley, "Approximate Renormalization Group Based on the Wegner-Houghton Differential Generator," Phys. Rev. Lett. **33** (1974), 540-543

[137] M. Reuter, N. Tetradis and C. Wetterich, "The Large N limit and the high temperature phase transition for the phi**4 theory," Nucl. Phys. B **401** (1993), 567-590 [arXiv:hep-ph/9301271 [hep-ph]].

[138] N. Tetradis and C. Wetterich, "Critical exponents from effective average action," Nucl. Phys. B **422** (1994), 541-592 [arXiv:hep-ph/9308214 [hep-ph]].

[139] L. Canet, B. Delamotte, D. Mouhanna and J. Vidal, "Optimization of the derivative expansion in the nonperturbative renormalization group," Phys. Rev. D **67** (2003), 065004 [arXiv:hep-th/0211055 [hep-th]].

[140] L. Canet, "Optimization of field-dependent nonperturbative renormalization group flows," Phys. Rev. B **71** (2005), 012418 [arXiv:hep-th/0409300 [hep-th]].

[141] D. F. Litim and D. Zappala, "Ising exponents from the functional renormalisation group," Phys. Rev. D **83** (2011), 085009 [arXiv:1009.1948 [hep-th]].

[142] A. Bonanno and D. Zappala, "Towards an accurate determination of the critical exponents with the renormalization group flow equations," Phys. Lett. B **504** (2001), 181-187 [arXiv:hep-th/0010095 [hep-th]].

[143] P. M. Stevenson, "Optimized Perturbation Theory," Phys. Rev. D **23** (1981), 2916

[144] H. Gies and C. Wetterich, "Renormalization flow of bound states," Phys. Rev. D **65** (2002), 065001 [arXiv:hep-th/0107221 [hep-th]].

[145] S. Floerchinger and C. Wetterich, "Exact flow equation for composite operators," Phys. Lett. B **680** (2009), 371-376 [arXiv:0905.0915 [hep-th]].

[146] M. Mitter, J. M. Pawlowski and N. Strodthoff, "Chiral symmetry breaking in continuum QCD," Phys. Rev. D **91** (2015), 054035 [arXiv:1411.7978 [hep-ph]].

[147] J. Braun, L. Fister, J. M. Pawlowski and F. Rennecke, "From Quarks and Gluons to Hadrons: Chiral Symmetry Breaking in Dynamical QCD," Phys. Rev. D **94** (2016) no.3, 034016 [arXiv:1412.1045 [hep-ph]].

[148] A. K. Cyrol, M. Mitter, J. M. Pawlowski and N. Strodthoff, "Nonperturbative quark, gluon, and meson correlators of unquenched QCD," Phys. Rev. D **97** (2018) no.5, 054006 [arXiv:1706.06326 [hep-ph]].

[149] W. j. Fu, J. M. Pawlowski and F. Rennecke, "QCD phase structure at finite temperature and density," Phys. Rev. D **101** (2020) no.5, 054032 [arXiv:1909.02991 [hep-ph]].

[150] T. R. Morris and R. Percacci, "Trace anomaly and infrared cutoffs," Phys. Rev. D **99** (2019) no.10, 105007 [arXiv:1810.09824 [hep-th]].

[151] A. Rançon and K. Falls, in preparation.

[152] T. L. Bell and K. G. Wilson, "Nonlinear Renormalization Groups," Phys. Rev. B **10** (1974), 3935-3944

[153] D. F. Litim and E. Marchais, "Critical $O(N)$ models in the complex field plane," Phys. Rev. D **95** (2017) no.2, 025026 [arXiv:1607.02030 [hep-th]].

[154] A. Jüttner, D. F. Litim and E. Marchais, "Global Wilson–Fisher fixed points," Nucl. Phys. B **921** (2017), 769-795 [arXiv:1701.05168 [hep-th]].

[155] D. F. Litim and L. Vergara, "Subleading critical exponents from the renormalization group," Phys. Lett. B **581** (2004), 263-269 [arXiv:hep-th/0310101 [hep-th]].

[156] H. Osborn and D. E. Twigg, "Remarks on Exact RG Equations," Annals Phys. **327** (2012), 29-73 [arXiv:1108.5340 [hep-th]].

[157] A. A. Lisyansky and D. Nicolaides, "Exact renormalization group equation for systems of arbitrary symmetry free of redundant operators," Journal of applied physics, **83** (1998) no.11, 6308-6310

[158] Weinberg, S. Ultraviolet Divergences In Quantum Theories Of Gravitation. In *General Relativity*; Hawking, S.W., Israel, W., Eds.; Cambridge University Press: Cambridge, UK, 1979; pp. 790–831.

[159] Reuter, M.; Saueressig, F. *Quantum Gravity and the Functional Renormalization Group: The Road towards Asymptotic Safety*; CUP: Cambridge, UK, 2019.

[160] M. Reuter, "Nonperturbative evolution equation for quantum gravity," Phys. Rev. D **57** (1998), 971-985 [arXiv:hep-th/9605030 [hep-th]].

[161] W. Souma, "Nontrivial ultraviolet fixed point in quantum gravity," Prog. Theor. Phys. **102** (1999), 181-195 [arXiv:hep-th/9907027 [hep-th]].

[162] M. Reuter and F. Saueressig, "Renormalization group flow of quantum gravity in the Einstein-Hilbert truncation," Phys. Rev. D **65** (2002), 065016 [arXiv:hep-th/0110054 [hep-th]].

[163] O. Lauscher and M. Reuter, "Ultraviolet fixed point and generalized flow equation of quantum gravity," Phys. Rev. D **65** (2002), 025013 [arXiv:hep-th/0108040 [hep-th]].

[164] D. F. Litim, "Fixed points of quantum gravity," Phys. Rev. Lett. **92** (2004), 201301 [arXiv:hep-th/0312114 [hep-th]].

[165] O. Lauscher and M. Reuter, "Flow equation of quantum Einstein gravity in a higher derivative truncation," Phys. Rev. D **66** (2002), 025026 [arXiv:hep-th/0205062 [hep-th]].

[166] P. F. Machado and F. Saueressig, "On the renormalization group flow of f(R)-gravity," Phys. Rev. D **77** (2008), 124045 [arXiv:0712.0445 [hep-th]].

[167] A. Codello, R. Percacci and C. Rahmede, "Ultraviolet properties of f(R)-gravity," Int. J. Mod. Phys. A **23** (2008), 143-150 [arXiv:0705.1769 [hep-th]].

[168] A. Codello, R. Percacci and C. Rahmede, "Investigating the Ultraviolet Properties of Gravity with a Wilsonian Renormalization Group Equation," Annals Phys. **324** (2009), 414-469 [arXiv:0805.2909 [hep-th]].

[169] D. Benedetti, P. F. Machado and F. Saueressig, "Asymptotic safety in higher-derivative gravity," Mod. Phys. Lett. A **24** (2009), 2233-2241 [arXiv:0901.2984 [hep-th]].

[170] D. Benedetti, P. F. Machado and F. Saueressig, "Taming perturbative divergences in asymptotically safe gravity," Nucl. Phys. B **824** (2010), 168-191 [arXiv:0902.4630 [hep-th]].

[171] K. Falls, D. F. Litim, K. Nikolakopoulos and C. Rahmede, “A bootstrap towards asymptotic safety,” [arXiv:1301.4191 [hep-th]].

[172] K. Falls, D. F. Litim, K. Nikolakopoulos and C. Rahmede, “Further evidence for asymptotic safety of quantum gravity,” Phys. Rev. D **93** (2016) no.10, 104022 [arXiv:1410.4815 [hep-th]].

[173] T. Denz, J. M. Pawlowski and M. Reichert, “Towards apparent convergence in asymptotically safe quantum gravity,” Eur. Phys. J. C **78** (2018) no.4, 336 [arXiv:1612.07315 [hep-th]].

[174] K. Falls, C. R. King, D. F. Litim, K. Nikolakopoulos and C. Rahmede, “Asymptotic safety of quantum gravity beyond Ricci scalars,” Phys. Rev. D **97** (2018) no.8, 086006 [arXiv:1801.00162 [hep-th]].

[175] K. G. Falls, D. F. Litim and J. Schröder, “Aspects of asymptotic safety for quantum gravity,” Phys. Rev. D **99** (2019) no.12, 126015 [arXiv:1810.08550 [gr-qc]].

[176] K. Falls, N. Ohta and R. Percacci, “Towards the determination of the dimension of the critical surface in asymptotically safe gravity,” Phys. Lett. B **810** (2020), 135773 [arXiv:2004.04126 [hep-th]].

[177] R. Percacci and D. Perini, “Should we expect a fixed point for Newton’s constant?,” Class. Quant. Grav. **21** (2004), 5035-5041 [arXiv:hep-th/0401071 [hep-th]].

[178] C. F. Steinwachs, “Towards a unitary, renormalizable and ultraviolet-complete quantum theory of gravity,” *Front. Phys.* **2020**, *8*, 185, [arXiv:2004.07842 [hep-th]].

[179] M. Reuter and F. Saueressig, “Quantum Einstein Gravity,” New J. Phys. **14** (2012), 055022 [arXiv:1202.2274 [hep-th]].

[180] K. S. Stelle, “Renormalization of Higher Derivative Quantum Gravity,” Phys. Rev. D **16** (1977), 953-969

[181] J. F. Donoghue, “A Critique of the Asymptotic Safety Program,” Front. in Phys. **8** (2020), 56 [arXiv:1911.02967 [hep-th]].

[182] O. J. Rosten, “Relationships Between Exact RGs and some Comments on Asymptotic Safety,” [arXiv:1106.2544 [hep-th]].

[183] L. F. Abbott, “Introduction to the Background Field Method,” Acta Phys. Polon. B **13** (1982), 33 CERN-TH-3113.

[184] B. Knorr, “The derivative expansion in asymptotically safe quantum gravity: general setup and quartic order,” [arXiv:2104.11336 [hep-th]].

[185] D. Anselmi, "Absence of higher derivatives in the renormalization of propagators in quantum field theories with infinitely many couplings," Class. Quant. Grav. **20** (2003), 2355-2378 [arXiv:hep-th/0212013 [hep-th]].

[186] D. Anselmi, "Properties Of The Classical Action Of Quantum Gravity," JHEP **05** (2013), 028 [arXiv:1302.7100 [gr-qc]].

[187] K. Groh, F. Saueressig and O. Zanusso, "Off-diagonal heat-kernel expansion and its application to fields with differential constraints," [arXiv:1112.4856 [math-ph]].

[188] K. Falls, "Physical renormalization schemes and asymptotic safety in quantum gravity," Phys. Rev. D **96** (2017) no.12, 126016 [arXiv:1702.03577 [hep-th]].

[189] K. Falls, "Renormalization of Newton's constant," Phys. Rev. D **92** (2015) no.12, 124057 [arXiv:1501.05331 [hep-th]].

[190] S. P. De Alwis, "Higher Derivative Corrections to Lower Order RG Flow Equations," [arXiv:1809.04671 [hep-th]].

[191] N. Christiansen, D. F. Litim, J. M. Pawlowski and A. Rodigast, "Fixed points and infrared completion of quantum gravity," Phys. Lett. B **728** (2014), 114-117 [arXiv:1209.4038 [hep-th]].

[192] N. Christiansen, B. Knorr, J. M. Pawlowski and A. Rodigast, "Global Flows in Quantum Gravity," Phys. Rev. D **93** (2016) no.4, 044036 [arXiv:1403.1232 [hep-th]].

[193] N. Christiansen, B. Knorr, J. Meibohm, J. M. Pawlowski and M. Reichert, "Local Quantum Gravity," Phys. Rev. D **92** (2015) no.12, 121501 [arXiv:1506.07016 [hep-th]].

[194] B. Knorr, C. Ripken and F. Saueressig, "Form Factors in Asymptotic Safety: conceptual ideas and computational toolbox," Class. Quant. Grav. **36** (2019) no.23, 234001 [arXiv:1907.02903 [hep-th]].

[195] B. Knorr and M. Schiffer, "Non-Perturbative Propagators in Quantum Gravity," Universe **7** (2021) no.7, 216 [arXiv:2105.04566 [hep-th]].

[196] A. Bonanno, T. Denz, J. M. Pawlowski and M. Reichert, "Reconstructing the graviton," [arXiv:2102.02217 [hep-th]].

[197] T. Draper, B. Knorr, C. Ripken and F. Saueressig, "Finite Quantum Gravity Amplitudes: No Strings Attached," Phys. Rev. Lett. **125** (2020) no.18, 181301 [arXiv:2007.00733 [hep-th]].

[198] T. Draper, B. Knorr, C. Ripken and F. Saueressig, "Graviton-Mediated Scattering Amplitudes from the Quantum Effective Action," JHEP **11** (2020), 136 [arXiv:2007.04396 [hep-th]].

[199] A. Platania and C. Wetterich, “Non-perturbative unitarity and fictitious ghosts in quantum gravity,” Phys. Lett. B **811** (2020), 135911 [arXiv:2009.06637 [hep-th]].

[200] A. Eichhorn, S. Lippoldt, J. M. Pawlowski, M. Reichert and M. Schiffer, “How perturbative is quantum gravity?,” Phys. Lett. B **792** (2019), 310-314 [arXiv:1810.02828 [hep-th]].

[201] H. Gies, B. Knorr, S. Lippoldt and F. Saueressig, “Gravitational Two-Loop Counterterm Is Asymptotically Safe,” Phys. Rev. Lett. **116** (2016) no.21, 211302 [arXiv:1601.01800 [hep-th]].

[202] A. Bonanno, A. Eichhorn, H. Gies, J. M. Pawlowski, R. Percacci, M. Reuter, F. Saueressig and G. P. Vacca, “Critical reflections on asymptotically safe gravity,” Front. in Phys. **8** (2020), 269 [arXiv:2004.06810 [gr-qc]].

[203] Y. Kluth and D. F. Litim, “Fixed Points of Quantum Gravity and the Dimensionality of the UV Critical Surface,” [arXiv:2008.09181 [hep-th]].

[204] H. W. Hamber and R. Toriumi, “Inconsistencies from a Running Cosmological Constant,” Int. J. Mod. Phys. D **22** (2013) no.13, 1330023 [arXiv:1301.6259 [hep-th]].

[205] A. Adeifeoba, A. Eichhorn and A. Platania, “Towards conditions for black-hole singularity-resolution in asymptotically safe quantum gravity,” Class. Quant. Grav. **35** (2018) no.22, 225007 [arXiv:1808.03472 [gr-qc]].

[206] A. Bonanno, R. Casadio and A. Platania, “Gravitational antiscreening in stellar interiors,” JCAP **01** (2020), 022 [arXiv:1910.11393 [gr-qc]].

[207] A. Platania, “From renormalization group flows to cosmology,” Front. in Phys. **8** (2020), 188 [arXiv:2003.13656 [gr-qc]].

[208] A. Eichhorn, “An asymptotically safe guide to quantum gravity and matter,” Front. Astron. Space Sci. **5** (2019), 47 [arXiv:1810.07615 [hep-th]].

[209] S. Weinberg, “The Cosmological Constant Problem,” Rev. Mod. Phys. **61** (1989), 1-23

[210] H. Gies, B. Knorr and S. Lippoldt, “Generalized Parametrization Dependence in Quantum Gravity,” Phys. Rev. D **92** (2015) no.8, 084020 [arXiv:1507.08859 [hep-th]].

[211] N. Ohta, R. Percacci and A. D. Pereira, “Gauges and functional measures in quantum gravity I: Einstein theory,” JHEP **06** (2016), 115 [arXiv:1605.00454 [hep-th]].

[212] N. Ohta, R. Percacci and A. D. Pereira, “Gauges and functional measures in quantum gravity II: Higher derivative gravity,” Eur. Phys. J. C **77** (2017) no.9, 611 [arXiv:1610.07991 [hep-th]].

[213] G. P. De Brito, N. Ohta, A. D. Pereira, A. A. Tomaz and M. Yamada, "Asymptotic safety and field parametrization dependence in the $f(R)$ truncation," Phys. Rev. D **98** (2018) no.2, 026027 [arXiv:1805.09656 [hep-th]].

[214] J. M. Pawlowski, "Geometrical effective action and Wilsonian flows," [arXiv:hep-th/0310018 [hep-th]].

[215] I. Donkin and J. M. Pawlowski, "The phase diagram of quantum gravity from diffeomorphism-invariant RG-flows," [arXiv:1203.4207 [hep-th]].

[216] K. Falls, "Background independent exact renormalisation," Eur. Phys. J. C **81** (2021) no.2, 121 [arXiv:2004.11409 [hep-th]].

[217] H. Kawai and M. Ninomiya, "Renormalization Group and Quantum Gravity," Nucl. Phys. B **336** (1990), 115-145

[218] R. Martini, A. Ugolotti, F. Del Porro and O. Zanusso, "Gravity in $d = 2 + \epsilon$ dimensions and realizations of the diffeomorphisms group," [arXiv:2103.12421 [hep-th]].

[219] H. W. Hamber, "Scaling Exponents for Lattice Quantum Gravity in Four Dimensions," Phys. Rev. D **92** (2015) no.6, 064017 [arXiv:1506.07795 [hep-th]].

[220] J. Laiho, S. Bassler, D. Coumbe, D. Du and J. T. Neelakanta, "Lattice Quantum Gravity and Asymptotic Safety," Phys. Rev. D **96** (2017) no.6, 064015 [arXiv:1604.02745 [hep-th]].

[221] R. Loll, "Quantum Gravity from Causal Dynamical Triangulations: A Review," Class. Quant. Grav. **37** (2020) no.1, 013002 [arXiv:1905.08669 [hep-th]].

[222] A. Eichhorn, J. Lumma, A. D. Pereira and A. Sikandar, "Universal critical behavior in tensor models for four-dimensional quantum gravity," JHEP **02** (2020), 110 [arXiv:1912.05314 [gr-qc]].