Can quantum fluctuations differentiate between standard and unimodular gravity?

Gustavo P. de Brito,[1,*] Oleg Melichev,[2,†] Roberto Percacci,[2,‡] and Antonio D. Pereira[3,4,§]

[1]*CP3-Origins, University of Southern Denmark, Campusvej 55, DK-5230 Odense M, Denmark*

[2]*International School for Advanced Studies, via Bonomea 265, I-34136 Trieste, Italy and INFN, Sezione di Trieste, Italy*

[3]*Institute for Mathematics, Astrophysics and Particle Physics (IMAPP), Radboud University, Heyendaalseweg 135, 6525 AJ Nijmegen, The Netherlands*

[4]*Instituto de Física, Universidade Federal Fluminense, Campus da Praia Vermelha, Av. Litorânea s/n, 24210-346, Niterói, RJ, Brazil*

We formally prove the existence of a quantization procedure that makes the path integral of a general diffeomorphism-invariant theory of gravity, with fixed total spacetime volume, equivalent to that of its unimodular version. This is achieved by means of a partial gauge fixing of diffeomorphisms together with a careful definition of the unimodular measure. The statement holds also in the presence of matter. As an explicit example, we consider scalar-tensor theories and compute the corresponding logarithmic divergences in both settings. In spite of significant differences in the coupling of the scalar field to gravity, the results are equivalent for all couplings, including non-minimal ones.

I. INTRODUCTION

General Relativity (GR) is in perfect agreement with all experimental data. Even if this is confirmed by the next observational campaigns, another important issue remains open. Classical GR has several equivalent formulations that may differ when quantum effects are taken into account. An interesting example is unimodular gravity (UG) [1–10], that we define as a theory of gravity constrained by

$$\sqrt{|g|} = \omega \ , \tag{1}$$

where ω is a fixed volume form [1]. UG was advocated to be better suited to address some potential conceptual problems, such as the cosmological constant problem and the problem of time in

* gustavo@cp3.sdu.dk

† omeliche@sissa.it

‡ percacci@sissa.it

§ A.Duarte@science.ru.nl and adpjunior@id.uff.br

[1] One often just takes $\omega = 1$, which justifies the name, but this may not be applicable globally, so we stick to the more general definition.

quantum gravity, offering, in this viewpoint, some advantages [11–23], see, however, [24–26]. At the classical level, GR and UG have the same equations of motion, but the nature of the cosmological constant term differs: it is a Lagrangian parameter, in the first case, an integration constant in the second. Choosing the integration constant of UG to have the same value as the Lagrangian parameter of GR, will give the same local dynamics. There is however a subtlety that prevents a full equivalence between the two theories at global level. We can view (1) as a partial gauge-fixing for GR. Integrating both sides we get

$$\int \mathrm{d}^4x\sqrt{|g|} = \int \mathrm{d}^4x\,\omega\ , \tag{2}$$

The total volume of spacetime[2] is diffeomorphism-invariant and therefore a physical observable. Since (2) is a statement about an observable, it cannot be viewed as a gauge condition. Thus the total volume is a physical degree of freedom in GR, but not in UG. One must conclude that GR has one degree of freedom more than UG - just one, not one per spacetime point [8]. Thus, classical UG is equivalent to a version of classical GR where the total volume of spacetime is held fixed. This can be seen in the Lagrangian formalism by adding to the action a term

$$\frac{\Lambda}{8\pi G}\left(V - \int \mathrm{d}^4x\sqrt{g}\right) \tag{3}$$

where Λ has to be thought of as a Lagrange multiplier enforcing that the spacetime volume is equal to V[3]. In this paper we shall not discuss the global, large scale properties and when we say "GR" we shall implicitly mean "GR with fixed total volume".

It is then natural to ask whether this "almost equivalence" holds also for the quantum versions of the theories. The unimodularity condition (1) restricts the invariance group from diffeomorphisms (Diff) to special (volume-preserving) diffeomorphisms (SDiff) [4], and one may expect that this could lead to different quantum theories. For example, if we choose a unimodular gauge for GR, this requires a Faddeev-Popov determinant, while in the quantization of UG is defined as an integral over unimodular metrics[5], this condition is present *ab initio* and no Faddeev-Popov determinants are necessary.

In recent years, there appeared in the literature conflicting statements about the equivalence, or lack thereof, between GR and UG at the quantum level, see, e.g., [17, 18, 21–25, 28–45]. We

[2] When the volume is infinite, one has to regulate it by "putting the system in a box" and impose (2) on the regulated system.

[3] This point of view has been used by Hawking in Euclidean quantum gravity, where he interpreted the resulting partition function as the "volume canonical ensemble", see [27]

[4] In the recent literature, the group SDiff is often referred to as TDiff, where T stands for "transverse".

[5] There are alternative formulations of UG and its quantization as, e.g., in [22, 28] which may have a different gauge-fixing structure than the one adopted in this work.

believe that some of these contradictions may be just due to different quantization procedures. In this work, we prove in general, based on formal path integral arguments, that there exists a quantization procedure that preserves the "almost equivalence" between these theories. The proof goes through for any *Diff*-invariant action and in this sense extends beyond ordinary GR. Of course, there may be other definitions of the quantum theories that break the equivalence, but in the absence of other independent arguments in their favor, we think that the one we describe here is more natural. Our argument is in the same spirit as the one presented in [24, 25] and extends the results of [23, 36, 46] beyond one-loop order. We should remark that both GR and UG are not renormalizable in perturbation theory and the formal path integrals should be ultraviolet (UV) regularized. Our formal proof relies on the use of the background field method, but we leave the parameterization of the metric, i.e., the way that we split the full metric in background and fluctuating parts, generic. Hence, this also extends previous results [23, 36] which made explicit use of the so-called exponential split of the metric to impose the unimodularity condition [30].

Our proof of equivalence is given initially for pure gravity and one may again worry that as soon as matter degrees of freedom are introduced, the equivalence would fall apart. This is due to the different vertex structures. In GR, the determinant of the metric produces infinitely many vertices between gravitons and matter fields that are absent in UG. Hence, Feynman rules are different in the two settings and one might expect that it is very unlikely that in the computation of an observable, miraculous cancellations lead to equivalent results. Yet, there are results in the literature explicitly showing that this happens, see, e.g., [37, 38, 47]. In fact, we shall see that our formal proof of equivalence extends also to the case when matter is present.

As an explicit check, we shall consider gravity non-minimally coupled to a scalar field and show that the one loop UV divergences are the same for GR and UG. This disagrees with [43], who claimed that a particular dimensionless combination of couplings, called Δ, has different beta functions in the two settings. In our calculation, the beta functions turn out to be the same. What is perhaps more important, we find that the beta functions of Δ are gauge-dependent, which may at least in part explain the discrepancy. Furthermore, the implementation of the unimodularity condition adopted in [43] is different from the one we use in this paper. We therefore think that the question whether different formulations of quantum UG can lead to different physical predictions than GR remains still open.

The paper is structured as follows: In Sect. II, we define the path integral of diffeomorphism-invariant theories and formally show that it is possible to partially fix the gauge so as to reduce it to the one of UG. In Sect. III, we perform an explicit computation in order to verify the claim of

Sect. II in the presence of matter. We consider scalar-tensor theories with a non-minimal coupling. We then proceed to the calculation of the one-loop beta functions in these theories both in the full diffeomorphism and special diffeomorphism invariant cases. In particular, we compute the running of Δ and discuss how it depends on the choice of gauge. We also display the results of the running of Δ in GR with linear parameterization of the metric, in a generic linear covariant gauge, and compare our findings with the available literature. We collect our conclusions and perspectives in Sect IV. Appendix A is mainly intended for users of the functional renormalization group and explains how to extract the logarithmic terms of the beta functions. Appendix B contains some long expressions that are omitted in the main text.

II. EQUIVALENCE OF PATH INTEGRALS

The starting point of our analysis is the (Euclidean)[6] path integral defined by a gravitational action $S_{\rm Diff}(g_{\mu\nu})$, $g_{\mu\nu} = g_{\mu\nu}(\bar{g};h)$ being the metric, $\bar{g}_{\mu\nu}$ a fixed background metric and $h_{\mu\nu}$ the fluctuating field which is integrated over. The fluctuating field $h_{\mu\nu}$ does not need to be small, i.e., a perturbation around $\bar{g}_{\mu\nu}$. Moreover, the split of the full metric $g_{\mu\nu}$ in background and fluctuating parts is also general, not being restricted to the standard additive (linear) split. The action is assumed to be invariant under diffeomorphisms (but it is not restricted to be the Einstein-Hilbert action), and so is the functional measure $\mathcal{D}h_{\mu\nu}$. Formally, the path integral is expressed as

$$\mathcal{Z}_{\rm Diff} = \int \frac{\mathcal{D}h_{\mu\nu}}{V_{\rm Diff}} {\rm e}^{-S_{\rm Diff}[g(\bar{g};h)]} \,. \tag{4}$$

The factor $V_{\rm Diff}$ stands for the volume of the diffeomorphism group.

In most practical calculations within a continuum quantum-field theoretic setting, a gauge-fixing term must be introduced in (4). This is typically achieved by the Faddeev-Popov procedure. The redundancy is generated by vector fields ϵ^μ which can be decomposed as

$$\epsilon^\mu = \epsilon^\mu_{\rm T} + \nabla^\mu \phi \,, \tag{5}$$

with $\nabla_\mu \epsilon^\mu_{\rm T} = 0$ and ∇_μ the covariant derivative defined with respect to the metric $g_{\mu\nu}$. The transverse vectorfields $\epsilon^\mu_{\rm T}$ generate the group *SDiff* of special (volume-preserving) diffeomorphisms.

Instead of introducing a single gauge-fixing condition for the entire group of diffeomorphism, we introduce two different conditions, first breaking *Diff* to *SDiff*, and then breaking *SDiff*. This strategy has been discussed and worked out in a different way in [24], see also [25] and [48] for

[6] The Euclidean signature is not essential at this stage and the results could be equally deduced in the Lorentzian case.

a general discussion of partial gauge fixing. In the first step we choose a gauge-fixing functional $\mathcal{F}(g)$ and insert the standard Faddeev-Popov identity given by

$$1 = \Delta_{\mathcal{F}}(g) \int \mathcal{D}\phi \, \delta(\mathcal{F}(g^{\phi})) \,, \tag{6}$$

where $\Delta_{\mathcal{F}}(g)$ denotes the Faddeev-Popov determinant. The notation g^{ϕ} denotes the transformation of the metric generated by the longitudinal vectorfield $\nabla_{\mu}\phi$:

$$\delta_{\phi} g_{\mu\nu} = 2\nabla_{\mu}\nabla_{\nu}\phi \,. \tag{7}$$

We can now plug (6) in (4) leading to

$$\mathcal{Z}_{\text{Diff}} = \int \frac{\mathcal{D}h_{\mu\nu}}{V_{\text{Diff}}} \left(\Delta_{\mathcal{F}}(g) \int \mathcal{D}\phi \, \delta(\mathcal{F}(g^{\phi})) \right) \mathrm{e}^{-S_{\text{Diff}}[g(\bar{g};h)]} \,. \tag{8}$$

Following the standard steps we now use the gauge invariance of the measure, of the Faddeev-Popov determinant and the action and redefine the integration variable, to get

$$\mathcal{Z}_{\text{Diff}} = \int \frac{\mathcal{D}\phi \mathcal{D}h_{\mu\nu}}{V_{\text{Diff}}} \Delta_{\mathcal{F}}(g) \delta(\mathcal{F}(g)) \, \mathrm{e}^{-S_{\text{Diff}}[g(\bar{g};h)]} \,. \tag{9}$$

In [23, 36] it was shown that

$$V_{\text{Diff}} = \text{Det}\left(-\nabla^2\right) \times V_{\text{SDiff}} \times \int \mathcal{D}\phi \,, \tag{10}$$

where V_{SDiff} denotes the volume of the *SDiff* group. Hence,

$$\mathcal{Z}_{\text{Diff}} = \int \frac{\mathcal{D}h_{\mu\nu}}{V_{\text{SDiff}}} \frac{1}{\text{Det}\left(-\nabla^2\right)} \Delta_{\mathcal{F}}(g) \delta(\mathcal{F}(g)) \mathrm{e}^{-S_{\text{Diff}}[g(\bar{g};h)]} \,. \tag{11}$$

An explicit example of this first stage of gauge fixing is the unimodular gauge defined by

$$\mathcal{F}(g) = \det g_{\mu\nu} - \omega^2(x) \,, \tag{12}$$

$\omega(x)$ being a fixed scalar density. The delta function in (11) enforces that the full dynamical metric is unimodular. The corresponding Fadeev-Popov determinant is

$$\Delta_{\mathcal{F}}(g) = \text{Det}\left(\omega^2(x)(-\nabla^2)\right) \,. \tag{13}$$

The contribution due to $\omega^2(x)$ in (13) can be absorbed in a normalization factor of the path integral and thereby it is harmless. Finally, by plugging (13) into (11), yields

$$\mathcal{Z}_{\text{Diff}} = \int \frac{\mathcal{D}h_{\mu\nu}}{V_{\text{SDiff}}} \delta(\det g_{\mu\nu} - \omega^2(x)) \mathrm{e}^{-S_{\text{Diff}}[g(\bar{g};h)]} \,. \tag{14}$$

Due to the presence of the delta functional in (14), the action in the Boltzmann factor collapses to its unimodular counterpart, i.e., $S_{\mathrm{Diff}}[g(\bar{g};h)] \to S_{\mathrm{SDiff}}[g(\bar{g};h)]$ where factors of $\sqrt{g}$ are replaced by $\omega(x)$ and when expanded in $h_{\mu\nu}$, the constraint $\mathcal{F}(g) = 0$ must be imposed. Eq.(14) is the path integral of UG with the unimodular measure $(\mathcal{D}h_{\mu\nu})_{\mathrm{UG}}$ defined by

$$(\mathcal{D}h_{\mu\nu})_{\mathrm{UG}} \equiv \mathcal{D}h_{\mu\nu}\, \delta(\det g_{\mu\nu} - \omega^2(x))\,, \tag{15}$$

i.e,

$$\mathcal{Z}_{\mathrm{Diff}} = \int \frac{(\mathcal{D}h_{\mu\nu})_{\mathrm{UG}}}{V_{\mathrm{SDiff}}} \mathrm{e}^{-S_{\mathrm{SDiff}}[g(\bar{g};h)]} \equiv \mathcal{Z}_{\mathrm{SDiff}}\,. \tag{16}$$

One particular parameterization which is well-suited for the implementation of the unimodularity condition is the exponential split

$$g_{\mu\nu} = \bar{g}_{\mu\kappa}(\mathrm{e}^{h})^{\kappa}{}_{\nu}\,. \tag{17}$$

Unimodularity of $g_{\mu\nu}$ is achieved by requiring the background to be unimodular $(\det\bar{g} = \omega^2(x))$ and that the flucuations $h_{\mu\nu}$ are traceless[7].

In order to complete the gauge-fixing procedure, one applies again the Faddeev-Popov method for a gauge condition which fixes the *SDiff* invariance. This is achieved, e.g., by taking the standard linear covariant gauges in quantum gravity and applying the transverse projector to it. We refer to [23, 30, 31, 33, 36, 44, 45, 50] for more details.

We remark that eq.(16) does not rely on the specific form of the gravitational action. Morever, if matter interactions were included (also of arbitrary form), the equivalence would still hold. In this case, the matter action $S_{\mathrm{M}}^{\mathrm{Diff}}(\varphi,\psi,A)$ is mapped to $S_{\mathrm{M}}^{\mathrm{SDiff}}(\varphi,\psi,A)$ with the replacement $\sqrt{g} \to \omega$ and fluctuations satisfying the constraint defined by the delta functional in (15). Thus, we expect that gravity-matter systems in a full diffeomorphism-invariant setting are equivalent, quantum-mechanically, to gravity-matter systems in the unimodular framework.

III. NON-MINIMAL COMPARISONS IN SCALAR-TENSOR THEORIES

This section is devoted to the explicit calculation of one-loop divergences in gravity-matter systems, illustrating the quantum equivalence between *Diff*− and *SDiff*− invariant theories. In

[7] Another efficient method is the "densitized" parameterization, see, e.g., [22, 49]. If one opts for less efficient implementations, the unimodularity condition becomes difficult to implement in practical calculations. Nevertheless, for the partial gauge-fixing associated with the gauge freedom (7), there seem to be no generation of quartic ghost terms [48] due to the fact that one just introduces a ghost-antighost pair.

particular, we focus on scalar-tensor theories including non-minimal couplings between gravity and the scalar field.

This system has been discussed recently in [43], where it was claimed that through consideration of a suitable dimensionless combination of couplings Δ, it is possible to distinguish GR from UG. This would seem to contradict our results. While for us the beta function of Δ is the same in the two theories, it turns out to be gauge dependent, thus weakening the significance of this test. We should also stress that in [43], the authors employ a different way of implementing the unimodularity condition, and we do not exclude the existence of quantization schemes that break the equivalence between GR and UG.

A. Action

The beta functions of GR coupled to a scalar have been derived previously in, e.g., [51] for the general class of actions

$$S[\phi, g] = \int \mathrm{d}^d x \sqrt{g} \left(V(\phi) - F(\phi) R + \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi \right) . \tag{18}$$

This includes an arbitrary potential V and arbitrary non-minimal couplings parametrized by the function F. If one expands V and F in Taylor series in ϕ, with the additional assumption of invariance under $\phi \to -\phi$,

$$V(\phi) = \mathcal{V} + \frac{1}{2} m^2 \phi^2 + \lambda \phi^4 \dots , \quad \mathcal{V} = \frac{\Lambda}{8\pi G_N} \tag{19}$$

$$F(\phi) = Z_N + \frac{1}{2} \xi \phi^2 + \dots , \quad Z_N = \frac{1}{16\pi G_N} \tag{20}$$

We are especially interested in the dimensionless couplings ξ and λ, whose leading one-loop beta functions are universal, i.e., indenpendent of the regularization scheme, and in dimensionless ratios of the dimensionful couplings, such as $G_N \Lambda$, $G_N m^2$, Λ/m^2, since their beta functions are also known to be less gauge- and parameterization-dependent. In [51], the beta functions were computed by the use of the functional renormalization group (FRG) equation which is based on a cutoff-like regularization. Thus, power-law divergences are also taken into account. In [43], on the other hand, the authors employed dimensional regularization which is blind to the power-law divergences[8]. For a direct comparison, we would have to extract from the FRG the "universal" contributions, i.e., those related to logarithmic running. This is discussed in Appendix A 1. In the

[8] The relation between dimensional regularization and cutoff regularization in the context of functional renormalization is discussed in [64]

next section we directly extract the beta functions from the logarithmic divergences, calculated with heat kernel methods.

B. Dynamical gravitons: UG or GR in exponential parametrization

We start from GR in the exponential parametrization (17) and follow the procedure of [52]. We decompose the metric fluctuation in its irreducible spin 2, 1 and 0 components:

$$h_{\mu\nu} = h^{\rm TT}_{\mu\nu} + \bar\nabla_\mu\xi_\nu + \bar\nabla_\nu\xi_\mu + \bar\nabla_\mu\bar\nabla_\mu\sigma - \frac{1}{4}\bar g_{\mu\nu}\bar\nabla^2\sigma + \frac{1}{4}\bar g_{\mu\nu}h\,, \tag{21}$$

with $\bar\nabla^\mu h^{\rm TT}_{\mu\nu} = 0$, $\bar\nabla^\mu\xi_\mu = 0$ and $\bar g^{\mu\nu}h_{\mu\nu} = h$. A redefinition of the fields σ and ξ_μ is performed in order to cancel the Jacobian generated by the York decomposition (21),

$$\xi'_\mu = \sqrt{-\bar\nabla^2 - \frac{\bar R}{4}}\xi_\mu\,, \qquad \text{and} \qquad \sigma' = \sqrt{-\bar\nabla^2}\sqrt{-\bar\nabla^2 - \frac{\bar R}{3}}\sigma\,. \tag{22}$$

We take the background metric $\bar g_{\mu\nu}$ to be a four-dimensional Euclidean maximally symmetric space. Then we choose the "unimodular physical gauge", which consists of setting to zero the spin one field ξ'_μ and the spin-0 field h. With these choices, the gauge fixed Hessian is

$$\begin{aligned}\tilde S^{(2)}_{\rm grav} = \int {\rm d}^4x\sqrt{\bar g}\Bigg[&\frac{1}{4}F(\bar\phi){h^{TT}}_{\mu\nu}\left(-\bar\nabla^2 + \frac{\bar R}{6}\right)h^{TT\mu\nu} - \frac{3}{32}F(\bar\phi)\sigma'(-\bar\nabla^2)\sigma' \\ &- \frac{3}{4}F'(\bar\phi)\delta\phi\sqrt{(-\bar\nabla^2)\left(-\bar\nabla^2 - \frac{\bar R}{3}\right)}\sigma' + \frac{1}{2}\delta\phi\left(-\bar\nabla^2 + V''(\bar\phi) - F''(\bar\phi)\bar R\right)\delta\phi\Bigg].\end{aligned} \tag{23}$$

As a further simplification we note that defining [9]

$$\sigma'' = \sigma' + 4\frac{F'(\bar\phi)}{F(\bar\phi)}\sqrt{\frac{-\bar\nabla^2 - \frac{\bar R}{3}}{-\bar\nabla^2}}\delta\phi\,, \tag{24}$$

the gauge fixed Hessian becomes diagonal,

$$\begin{aligned}S^{(2)}_{\rm grav} = \int {\rm d}^4x\sqrt{\bar g}\Bigg[&\frac{1}{4}F(\bar\phi){h^{TT}}_{\mu\nu}\left(-\bar\nabla^2 + \frac{\bar R}{6}\right)h^{TT\mu\nu} - \frac{3}{32}F(\bar\phi)\sigma''(-\bar\nabla^2)\sigma'' \\ &+ \frac{1}{2}\delta\phi\left(-\bar\nabla^2 + V''(\bar\phi) - F''(\bar\phi)\bar R + 3\frac{F'(\bar\phi)^2}{F(\bar\phi)}\left(-\bar\nabla^2 - \frac{\bar R}{d-1}\right)\right)\delta\phi\Bigg].\end{aligned} \tag{25}$$

The unimodular physical gauge produces Faddeev-Popov ghost determinants

$$\Delta_{\rm FP=}\sqrt{{\rm det}_0(-\bar\nabla^2)}\sqrt{{\rm det}_1\left(-\bar\nabla^2 - \frac{\bar R}{4}\right)}, \tag{26}$$

[9] This change of variables has a trivial Jacobian.

with the subscripts 0 and 1 denoting the spin of the fields that the corresponding operators act on. Thus the one-loop partition function reads

$$Z = \mathrm{e}^{-S_{\mathrm{grav}}[\bar{\phi},\bar{g}]} \frac{\sqrt{\det \Delta_1}}{\sqrt{\det \Delta_2}\sqrt{\det \Delta_S}} \,, \tag{27}$$

where $\Delta_1 = -\bar{\nabla}^2 - \frac{\bar{R}}{4}$, $\Delta_0 = -\bar{\nabla}^2$ and

$$\Delta_S = -\bar{\nabla}^2 + E_S \,, \qquad E_S = \frac{FV'' - (F'^2 + FF'')\bar{R}}{F + 3F'^2} \,. \tag{28}$$

This agrees with the standard result for GR with a cosmological constant, except for the appearance of the additional scalar determinant.

Consider now the same calculation in UG. The trace fluctuation h is absent from the degrees of freedom from the start and it is therefore not necessary to fix the corresponding gauge. The *SDiff* gauge can be fixed again by setting $\xi' = 0$. Altogether this produces the Faddeev-Popov determinant

$$\Delta_{\mathrm{FP}}^{\mathrm{UG}} = \sqrt{\det{}_1 \left(-\bar{\nabla}^2 - \frac{\bar{R}}{4} \right)} . \tag{29}$$

On the other hand, as discussed in [23, 36], the factorization of the volume of *SDiff* produces an additional determinant $\sqrt{\det(-\bar{\nabla}^2)}$ which cancels the determinant coming from the integration over σ', so that the final result is again exactly (27). Notably, such an equivalence holds irrespective of the choice of $F(\phi)$.

In a standard perturbative approach, the beta functions can be read off from the logarithmic divergences. The one-loop effective action is

$$\Gamma = S + \frac{1}{2}\mathrm{Tr}\log \Delta_2 - \frac{1}{2}\mathrm{Tr}\log \Delta_1 + \frac{1}{2}\mathrm{Tr}\log \Delta_S \,, \tag{30}$$

and its divergent parts can be obtained from

$$\Gamma_{\mathrm{div}} = -\frac{1}{2}\frac{1}{16\pi^2} \log\left(\frac{\Lambda^2}{\mu^2}\right) \int \mathrm{d}^4x \sqrt{\bar{g}} \left[b_4(\Delta_2) - b_4(\Delta_1) + b_4(\Delta_S) \right] \,, \tag{31}$$

with Λ standing for an ultraviolet cutoff and μ being a reference scale. The first two contributions in (31) only give terms of order R^2 and are not relevant for the beta functions of interest. For Δ_S we have

$$\begin{aligned} -\frac{1}{2}\frac{1}{16\pi^2} b_4(\Delta_S) &= -\frac{1}{2}\frac{1}{16\pi^2}\left(\frac{1}{2}E_S^2 - \frac{1}{6}\bar{R}\,E_S + O(\bar{R}^2) \right) \\ &= -\frac{1}{64\pi^2}\frac{V''^2}{\left(1 + 3\frac{F'^2}{F}\right)} + \frac{1}{192\pi^2}\frac{1 + 6F'' + 9\frac{F'^2}{F}}{\left(1 + 3\frac{F'^2}{F}\right)} V''\bar{R} + O(\bar{R}^2) \,. \end{aligned} \tag{32}$$

Inserting (19) and (20) and expanding in powers of ϕ we obtain the beta functions

$$\beta_{\mathcal{V}}=\frac{m^4}{32\pi^2} \tag{33}$$

$$\beta_{m^2}=\frac{3\lambda}{2\pi^2}m^2-\frac{6Gm^4\xi^2}{\pi} \tag{34}$$

$$\beta_{\lambda}=\frac{9\lambda^2}{2\pi^2}-\frac{72Gm^2\lambda\xi^2}{\pi} \tag{35}$$

$$\beta_{G}=-\frac{G^2m^2(1+6\xi)}{6\pi} \tag{36}$$

$$\beta_{\xi}=\frac{\lambda(1+6\xi)}{4\pi^2}+\frac{Gm^2\xi^2(1-12\xi)}{\pi} \tag{37}$$

We note that the leading terms are the same as for the pure scalar theory, discussed in Appendix A. Here we only kept correction terms linear in G (with the exception of the beta function of G itself). The explicit one-loop computation reported above leads to the same results in GR and UG, since (31) is the same in both cases. Moreover, we have kept only the contributions generated by the universal Q-functionals. The remaining terms are associated to power divergences and are not universal. The conclusion of this explicit calculation agrees with our statement in Sec. II. In particular, the non-minimal scalar-gravity coupling does not change this conclusion.

As is well-known, quantum-gravity contributions to matter beta functions can be gauge dependent. The "unimodular physical gauge" can be obtained from the standard two-parameter linear covariant gauge condition for Diff-invariant theories, namely

$$\bar{\nabla}^{\nu}h_{\nu\mu}-\frac{1+\beta}{4}\bar{\nabla}_{\mu}h=\alpha b_{\mu}\,, \tag{38}$$

with b_μ being a fixed function, by taking the limits $\alpha\to 0$ and $\beta\to-\infty$. For generic α, β, the beta functions $\beta_{\mathcal{V}}$ and β_G are left unchanged, while the others become,

$$\begin{aligned}\beta_{m^2}&=\frac{3m^2\lambda}{2\pi^2}+\frac{2Gm^4(4\alpha-3(2+(3-\beta)\xi)^2)}{(3-\beta)^2\pi}\,,\\ \beta_{\lambda}&=\frac{9\lambda^2}{2\pi^2}-\frac{8Gm^2\lambda(12-4\alpha+24(3-\beta)\xi+9(3-\beta)^2\xi^2)}{(3-\beta)^2\pi}\,,\\ \beta_{\xi}&=\frac{\lambda(1+6\xi)}{4\pi^2}-\frac{Gm^2}{12\pi}\mathcal{F}(\alpha,\beta,\xi)\,.\end{aligned} \tag{39}$$

The contribution $\mathcal{F}(\alpha,\beta,\xi)$ is lengthy and collected in the Appendix B. In the limit $\beta\to-\infty$, the beta functions turn out to be α-independent. It is also worth mentioning that the first two of these beta functions are also independent of another parameter that can be introduced in the definition of the measure, namely the use of a densitized metric as a quantum field, see, e.g., [49].

As a side comment, at one-loop order, the universal gravitational correction to the quartic coupling λ at vanishing non-minimal coupling is negative, irrespective of the values of the gauge parameter β, provided that[10] $\alpha < 3$,

$$\beta_\lambda\Big|_{\text{grav}} = -\lambda \frac{32Gm^2}{\pi} \frac{3-\alpha}{(3-\beta)^2} \,. \tag{40}$$

At $\alpha = 3$ or $\beta \to \pm\infty$, the contribution vanishes at one loop. In particular, in the unimodular physical gauge, the gravitational contribution vanishes at vanishing ξ. However, in such a gauge, if the non-minimal coupling is included, the contribution is always negative at leading order in G. Hence, such a contribution can balance the non-gravitational contribution to the one-loop running of λ - which is positive and leads to the well-known triviality problem. In order to circumvent the issues due to the gauge dependence, and of the non-universal power-law terms, one will have to to compute a gauge invariant physical observable possibly along the lines of [53].

C. A universal beta function?

In [43], it was argued that the dimensionless combination of couplings

$$\Delta = \frac{(Gm^2)^2}{\lambda} \,, \tag{41}$$

has a universal beta function and carries a physical meaning. By quantizing UG in the presence of non-minimally coupled scalar fields, the authors claim that the results differ in GR and UG. More precisely, taking into account the differences in notation, their result for UG is

$$\beta_\Delta^{\text{UG}} = \Delta \frac{-9\lambda + 2\pi Gm^2(-4 - 24\xi + 180\xi^2)}{6\pi^2} \tag{42}$$

while their result for GR is

$$\beta_\Delta^{\text{GR}} = \Delta \frac{-9\lambda + 2\pi Gm^2(-4 + 156\xi + 180\xi^2)}{6\pi^2} \,. \tag{43}$$

Hence, Δ would be a physical quantity able to distinguish GR and UG if the scalar field is non-minimally coupled to the gravitational field.

Using our previous calculations, we cannot distinguish UG and GR non-minimally coupled to scalars at one-loop simply because the path integrals are the same. In particular, in the unimodular physical gauge, we obtain

$$\beta_\Delta = \Delta \frac{-9\lambda + 2Gm^2\pi(-1 - 6\xi + 180\xi^2)}{6\pi^2} \,, \tag{44}$$

[10] In an Euclidean setting, α has to be non-negative.

which differs from either of the results above. These discrepancies may be ascribed to the fact that we are using a different parameterizarion of the metric and a different implementation of the unimodularity condition.

What is perhaps more important is that, even if we stick to our computation scheme, the quantity Δ is gauge dependent. In fact, in the linear covariant gauge (38), the result is

$$\beta_\Delta = \frac{\Delta(-9(3-\beta)^2\lambda - 2Gm^2\pi(48\alpha + \beta^2\mathcal{A}_1(\xi) + 6\beta\mathcal{A}_2(\xi) - 27\mathcal{A}_3(\xi)))}{6\pi^2(3-\beta)^2}\,, \tag{45}$$

with

$$\mathcal{A}_1(\xi) = 1 + 6\xi - 180\xi^2\,, \quad \mathcal{A}_2(\xi) = -1 + 66\xi + 180\xi^2\,, \quad \mathcal{A}_3(\xi) = 5 + 46\xi + 60\xi^2\,. \tag{46}$$

Thus, even in the absence of ξ, the beta function of Δ is gauge dependent and comparing results for GR and UG would be problematic. We also remark that, in the limit $\beta \to \pm\infty$, eq.(45) reduces to (44) irrespective of α.

D. Dynamical gravitons: GR in linear parametrization

So far, the explicit one-loop computations were performed using the exponential parameterization of the metric. In this parameterization, the unimodularity condition simply amounts to removing the trace mode of the gravitational field fluctuation $h_{\mu\nu}$. While field redefinitons, properly done, should not affect the result of physical quantities, there are several subtleties when changing from one parameterization to another in quantum gravity. In this section, we present the one-loop results for the scalar-gravity system with a non-minimal coupling in the so-called linear parameterization, i.e.,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,, \tag{47}$$

in the linear covariant gauges (38). This system was studied, e.g., in [51], but the beta functions were computed with the functional renormalization group and contained also non-universal terms. Here, we select just the universal contributions, that are related to logarithmic divergences. In a general gauge (α, β), the result is completely equivalent to (39) apart from the beta function of the non-minimal coupling β_ξ which reads

$$\beta_\xi = \frac{\lambda(1+6\xi)}{4\pi^2} - \frac{Gm^2}{12\pi}\mathcal{G}(\alpha,\beta,\xi)\,, \tag{48}$$

where the explicit expression for $\mathcal{G}(\alpha,\beta,\xi)$ is reported in Appendix B. In particular, if we take $\alpha \to 0$ and $\beta \to \pm\infty$, we obtain

$$\mathcal{G}(0,\pm\infty,\xi) = 6(-13 + 10\xi^2 + 24\xi^3)\,, \tag{49}$$

which differs from $\mathcal{F}(0, \pm\infty, \xi)$. The beta function of Δ in a general linear covariant gauge and in linear parameterization is the same as (45). Hence, although gauge-dependent, Δ seems to display some kind of universality as far as different choices of parameterization is concerned. This fact is not very surprising given that, the only beta function in the linear parameterizarion that differs from the exponential parameterization at one-loop is β_ξ and it does not enter the definition of β_Δ.

In [54], Kamenshchick and Steinwachs (see, also, [55]) investigated the one-loop divergences of a more general theory than the one considered in this work. In particular, they have considered a scalar-gravity action $S_{\rm KS}[g, \Phi]$ expressed as[11]

$$S_{\rm KS}[g,\Phi] = \int \mathrm{d}^4x\sqrt{g}\left(V(\tilde{\Phi}) - F(\tilde{\Phi})R + \frac{1}{2}g^{\mu\nu}G(\tilde{\Phi})\nabla_\mu\Phi^a\nabla_\nu\Phi_a\right), \tag{50}$$

where $a = 1, \ldots, N$ and N is a positive integer. The functions V and F depend on $\tilde{\Phi} = \sqrt{\delta_{ab}\Phi^a\Phi^b}$. The gauge condition used in [54] is

$$F_\mu = \sqrt{F(\tilde{\Phi})}\left(\bar{\nabla}^\alpha h_{\alpha\mu} - \frac{1}{2}\bar{\nabla}_\mu h - \frac{F'(\tilde{\Phi})}{F(\tilde{\Phi})}n^a\bar{\nabla}_\mu\varphi_a\right), \tag{51}$$

with φ_a the scalar field fluctuations and $n^a = \Phi^a/\tilde{\Phi}$. Unfortunately, our gauge condition (38) is not deformable to this, and therefore we cannot directly compare our results with theirs. However, we can extract from their work the beta function of Δ in the gauge (51). The authors employed the linear parameterization of the metric and the reduction to the a single-scalar non-minimally coupled to gravity is achieved by taking $N \to 1$, $\Phi^a = n^a = 1$ and $\tilde{\Phi} \to \phi$. Moreover, in order to have the same scalar-tensor action we discussed in this work, one has to take $G(\tilde{\Phi}) \to 1$ and their quantity s has to be identified as

$$s = -\frac{F}{F + 3F'^2} = \frac{-1 - 8\pi\xi G\phi^2}{1 + 8\pi\xi G\phi^2(1 + 6\xi)}\,. \tag{52}$$

The beta functionals of V and F are $\beta_V = 2\alpha_1$, $\beta_F = 2\alpha_2$, where α_1 and α_2 are given in their equations (48) and (49). From there we read off

$$\beta_{m^2} = \frac{3\lambda}{2\pi^2}m^2 - \frac{2Gm^4(2 + 4\xi + 3\xi^2)}{\pi}\,, \tag{53}$$

$$\beta_\lambda = \frac{9\lambda^2}{2\pi^2} - \frac{8Gm^2\lambda(2 + 8\xi + 9\xi^2)}{\pi}\,, \tag{54}$$

$$\beta_G = -\frac{G^2m^2(1 + 6\xi)}{6\pi}\,, \tag{55}$$

$$\beta_\xi = \frac{\lambda(1 + 6\xi)}{4\pi^2} - \frac{Gm^2(13 - 16\xi - 39\xi^2 - 36\xi^3)}{3\pi}\,, \tag{56}$$

[11] The functions U, G, V of [54] correspond to $-F$, -1, $-V$ in our notation.

which gives

$$\beta_\Delta = \Delta \frac{-9\lambda + 10\pi G m^2 (5 + 30\xi + 36\xi^2)}{6\pi^2} \,. \tag{57}$$

This confirms once more that the first and last term in the fraction are universal, but not the other ones.

IV. CONCLUSIONS

Disregarding the single global spacetime volume degree of freedom, we have shown at a formal path integral level that the classical equivalence between general (*Diff*–invariant) and unimodular (*SDiff*–invariant) versions of gravity theories, can be maintained at the quantum level[12]. This is true independently of the choice of the action and also in the presence of matter. This was achieved by a careful factorization of the gauge-group volume which produces an extra contribution that is missing in comparison with standard GR calculations. Keeping this in mind, due to such an equivalence, the path integrals lead to the same perturbative results order by order. Moreover, at least within the standard perturbative assumptions, our results are not restricted to an Euclidean setting. As a formal argument, one should verify it with explicit calculations, since the path-integral manipulations deal with ill-defined objects and require a proper regularization.

To this end, we have calculated the universal parts of the one loop beta functions of scalars coupled to gravity. In spite of significant differences[13] in the two cases, the beta functions turn out to be the same. We have then compared these results to those of [43], who also made the same comparison. Our beta function for the dimensionless combination Δ, see (41), differs from theirs both for GR and UG. The differences can probably to be ascribed at least in part to the different way they implement unimodularity[14]. A more detailed analysis has shown that the beta function of Δ is actually gauge-dependent, so that it is not a sufficiently good test. There are two terms in the beta function of Δ that are the same in all gauges and are the same across all calculations we could find in the literature, whereas other terms have strong gauge dependence. For the future, it will be important to identify a genuinely universal combination of couplings, or another potential observable that can act as a benchmark.

We conclude with some comments on the cosmological constant. In UG, a "cosmological term" $\frac{\Lambda}{8\pi G}\int d^4x\,\omega$ in the Lagrangian is just an additive, field-independent term that does not affect the

[12] The equivalence of GR and UG in the presence of an independent connection, deserves a separate investigation, which is ongoing.

[13] The theories feature different gauge symmetries and, e.g., different vertices.

[14] They consider a redefinition of the metric which renders unimodularity automatically at the cost of introducing an extra Weyl invariance.

equations of motion and can be absorbed in the overall normalization of the functional integral. The equations of motion of UG are obtained through a constrained variation of the unimodular action and the cosmological constant is regained as an integration constant upon the use of the Bianchi identities. Hence, an initial condition for the value of the cosmological constant must be provided. GR is only (classically) equivalent to UG if we impose that the total volume of spacetime is fixed. In this restricted theory the cosmological term in (3) is a Lagrange multiplier, whose value is ultimately related to the volume through the equations of motion.

Computations of the beta functions performed in the so-called unimodular gauge [52] show that the cosmological constant decouples from the system of beta functions. This resembles simpler calculations involving the functional renormalization group, where a field-independent contribution is generated by the flow and can be cancelled by a proper normalization of the vacuum energy. This suggests that its quartic running is unphysical. This is in line with other hints coming from different directions [56–59]. This and related issues deserve to be investigated further.

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Appendix A: Derivation of beta functions from the Functional Renormalization Group

1. Extracting the universal terms from the FRG

In the literature, the beta functions of gravity, with or without matter, have been often calculated in the Functional Renormalization Group (FRG) framework, see, e.g., [60–63] for reviews on the subject). Since the FRG is based on a momentum cutoff, the beta functions contain terms proportional to powers of the cutoff, that are not seen with other techniques. In this appendix we discuss the way in which one can recover from the FRG the standard one loop beta functions that one would see, e.g. in dimensional regularization. For a more detailed discussion of the relation between the FRG and dimensional regularization we refer to [64].

In the FRG, a cutoff function R_k is introduced by hand in the quadratic part of the action, in order to suppress the contribution to the functional integral of modes with (Euclidean) momenta smaller than a cutoff scale k. This leads to a coarse-grained effective action Γ_k which coincides with the full effective action at $k = 0$. The flowing action Γ_k obeys the flow equation

$$k\frac{\mathrm{d}}{\mathrm{d}k}\Gamma_k \equiv \partial_t\Gamma_k = \frac{1}{2}\mathrm{Tr}\left[(\Gamma_k^{(2)} + R_k)^{-1}\partial_t R_k\right] , \tag{A1}$$

where $\Gamma_k^{(2)}$ is the Hessian constructed from Γ_k. For our present purposes it will be enough to consider the simple case of scalar fields in a background metric, with a Hessian of the form $\Gamma_k^{(2)} = \Delta$ where Δ is a Laplace-type operator:

$$\Delta = -\nabla^2 + E \, ; \quad E = m^2 + 12\lambda\phi^2 - \xi R \, . \tag{A2}$$

This derives from a scalar action containing a potential and a non-minimal coupling to gravity. Then the r.h.s. of the flow equation is a function $W(\Delta)$ that, for constant ϕ, can be evaluated as

$$\mathrm{Tr}W(\Delta) = \frac{1}{(4\pi)^{d/2}}\left[Q_{d/2}(W)B_0(\Delta) + Q_{d/2-1}(W)B_2(\Delta) + \ldots + Q_0(W)B_d(\Delta) + \ldots\right], \tag{A3}$$

with the Q-functionals defined as

$$Q_n(W) = \frac{(-1)^k}{\Gamma(n+k)}\int_0^\infty \mathrm{d}z\, z^{n+k-1}W^{(k)}(z) \, . \tag{A4}$$

In eq.(A4), $n \in \mathbf{R}$, $W^{(k)}(z)$ stands for the k-th derivative of W with respect to z. If $n > 0$, then $k = 0$. Otherwise, k is a positive integer such that $n + k > 0$. The heat kernel coefficients are $B_n(\Delta) = \int \mathrm{d}^d x\sqrt{g}\ \mathrm{Tr}\ b_n(\Delta)$, where

$$b_0 = 1 \, , \qquad b_2 = \frac{R}{6} - E \, ,$$
$$b_4 = \frac{1}{180}\left(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - R^{\mu\nu}R_{\mu\nu} + \frac{5}{2}R^2\right) - \frac{1}{6}RE + \frac{1}{2}E^2 \, . \tag{A5}$$

In the flow equation, we are interested in computing Q-functionals of the form

$$W(z) = \frac{\partial_t R_k(z)}{(P_k(z))^m} \, , \tag{A6}$$

where $P_k(z) = z + R_k(z)$. If, $m = n + 1$, then one can show that

$$Q_n\left(\frac{\partial_t R_k}{P_k^{n+1}}\right) = \frac{2}{\Gamma(n+1)} \, , \tag{A7}$$

is "universal", i.e. independent of the shape of R_k. For certain cutoff schemes the denominator in the function W is $P_k + E$, and

$$Q_n\left(W\right) = Q_n\left(\frac{\partial_t R_k}{(P_k + E)^m}\right) , \tag{A8}$$

are, in general, non-universal quantities. Nevertheless, one can extract universal parts of each Q-functional defined in eq.(A8) by expanding in E:

$$Q_n\left(\frac{\partial_t R_k}{(P_k+E)^m}\right) = Q_n\left(\frac{\partial_t R_k}{P_k^m}\left(1-m\frac{E}{P_k}+\frac{m(m+1)}{2}\frac{E^2}{P_k^2}-\frac{m(m+1)(m+2)}{3!}\frac{E^3}{P_k^3}+\dots\right)\right), \quad \text{(A9)}$$

and exploiting the linearity of the Q-functionals to pick up the contribution which satisfies $n = m+1$.

Consider first a "type III" cutoff, see, e.g., [60, 65]. The beta functional is

$$\dot\Gamma_k = \frac{1}{32\pi^2}\left[Q_2\left(\frac{\dot R_k}{P_k}\right)B_0(\Delta)+Q_1\left(\frac{\dot R_k}{P_k}\right)B_2(\Delta)+Q_0\left(\frac{\dot R_k}{P_k}\right)B_4(\Delta)+\dots\right]. \quad \text{(A10)}$$

Only the last term is universal. Thus

$$\dot\Gamma_k\Big|_{\text{univ}} = \frac{2}{32\pi^2}\int \mathrm{d}^4x\sqrt{g}\,b_4(\Delta)\,. \quad \text{(A11)}$$

The relevant terms (up to linear order in R which are not total derivatives) are

$$\begin{aligned} b_4 &\sim \frac{1}{2}E^2-\frac{1}{6}R\,E \\ &\sim \frac{1}{2}m^4+72\lambda^2\phi^4+12\lambda m^2\phi^2+\left(\xi+\frac{1}{6}\right)m^2R+2\lambda(6\xi+1)\phi^2R\,. \end{aligned} \quad \text{(A12)}$$

From here one reads off the beta functions

$$\beta_{\mathcal{V}}=\frac{m^4}{32\pi^2}\,, \quad \text{(A13)}$$

$$\beta_{m^2}=\frac{3\lambda m^2}{2\pi^2}\,, \quad \text{(A14)}$$

$$\beta_{\lambda}=\frac{9\lambda^2}{2\pi^2}\,, \quad \text{(A15)}$$

$$\beta_{Z_N}=\frac{1+6\xi}{96\pi^2}m^2\,, \quad \text{(A16)}$$

$$\beta_{\xi}=\frac{\lambda(1+6\xi)}{4\pi^2}\,. \quad \text{(A17)}$$

The same result can be obtained in a more laborious way using a "type I" cutoff. In this case

$$\dot\Gamma_k=\frac{1}{32\pi^2}\left[Q_2\left(\frac{\dot R_k}{P_k+E}\right)B_0(-\nabla^2)+Q_1\left(\frac{\dot R_k}{P_k+E}\right)B_2(-\nabla^2)+Q_0\left(\frac{\dot R_k}{P_k+E}\right)B_4(-\nabla^2)+\dots\right]. \quad \text{(A18)}$$

The universal terms come from all three pieces in this expression, when one expands in E: the third term in the expansion for Q_2, the second for Q_1 and the leading term for Q_0. In the latter term, $B_4(-\nabla^2)$ is of order R^2 and does not concern us. The rest is

$$\dot\Gamma_k\sim\frac{1}{32\pi^2}\left[Q_2\left(\frac{\dot R_k}{P_k^3}\right)E^2B_0(-\nabla^2)+Q_1\left(\frac{\dot R_k}{P_k^2}\right)(-E)B_2(-\nabla^2)+\dots\right],$$

$$= \frac{1}{32\pi^2} \int \mathrm{d}^4x\sqrt{g}\left[E^2 - 2E\frac{1}{6}R + \dots\right], \tag{A19}$$

which is clearly the same as before, and therefore leads to the same beta functions. Note that the universal parts of the beta functions are those that come from the dimensionless Q-functionals and therefore are independent of k, see eq.A7.

Appendix B: Some General Expressions

In this appendix, we collect some long expressions that were omitted in the main text. In particular, the beta function of the non-minimal coupling ξ depends on the choice of the metric parameterization. Hence, in the exponential parameterization in a general linear covariant gauge (38), the factor $\mathcal{F}$ in eq.(39) is

$$\mathcal{F}(\alpha,\beta,\xi) = -4\frac{\mathcal{F}_1(\alpha,\beta,\xi) - 3\left(\mathcal{F}_2(\beta,\xi) + \mathcal{F}_3(\beta,\xi) + \mathcal{F}_4(\beta,\xi) + \mathcal{F}_5(\beta,\xi) + \mathcal{F}_6(\beta,\xi)\right)}{(3-\beta)^4}\,, \tag{B1}$$

with

$$\begin{aligned}
\mathcal{F}_1(\alpha,\beta,\xi) &= 24\alpha^2 + 2\alpha(\beta^2(24\xi+1) - 18\beta(4\xi+1) - 27)\,,\\
\mathcal{F}_2(\beta,\xi) &= \beta^4\xi^2(12\xi-1)\,,\\
\mathcal{F}_3(\beta,\xi) &= -4\beta^3\xi\left(36\xi^2+9\xi-1\right)\,,\\
\mathcal{F}_4(\beta,\xi) &= \beta^2\left(648\xi^3+342\xi^2+36\xi-2\right)\,,\\
\mathcal{F}_5(\beta,\xi) &= -12\beta\left(108\xi^3+81\xi^2+15\xi+1\right)\,,\\
\mathcal{F}_6(\beta,\xi) &= 9\left(108\xi^3+99\xi^2+12\xi-2\right)\,.
\end{aligned} \tag{B2}$$

As for the linear parameterization, the expression for $\mathcal{G}(\alpha,\beta,\xi)$ in (48) is

$$\mathcal{G}(\alpha,\beta,\xi) = 2\frac{\mathcal{G}_1(\alpha,\beta) + \mathcal{G}_2(\alpha,\beta,\xi) + 3\mathcal{G}_4(\beta,\xi)}{(3-\beta)^4} \tag{B3}$$

with

$$\begin{aligned}
\mathcal{G}_1(\alpha,\beta) &= -3\alpha^2(3(\beta-6)\beta((\beta-6)\beta+18)+259)\,,\\
\mathcal{G}_2(\alpha,\beta,\xi) &= \alpha(2\beta(\beta(3(\beta-12)\beta+24\xi+194)-396)-432\xi+630)\,,\\
\mathcal{G}_3(\beta,\xi) &= -13\beta^4+112\beta^3-458\beta^2+24(\beta-3)^4\xi^3\,,\\
\mathcal{G}_4(\beta,\xi) &= \left(2(\beta(5\beta-42)+117)(\beta-3)^2\xi^2 - 8(\beta(5\beta-12)+27)(\beta-3)\xi + 888\beta - 657\right)\,.
\end{aligned} \tag{B4}$$

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