Scuola Internazionale Superiore di Studi Avanzati

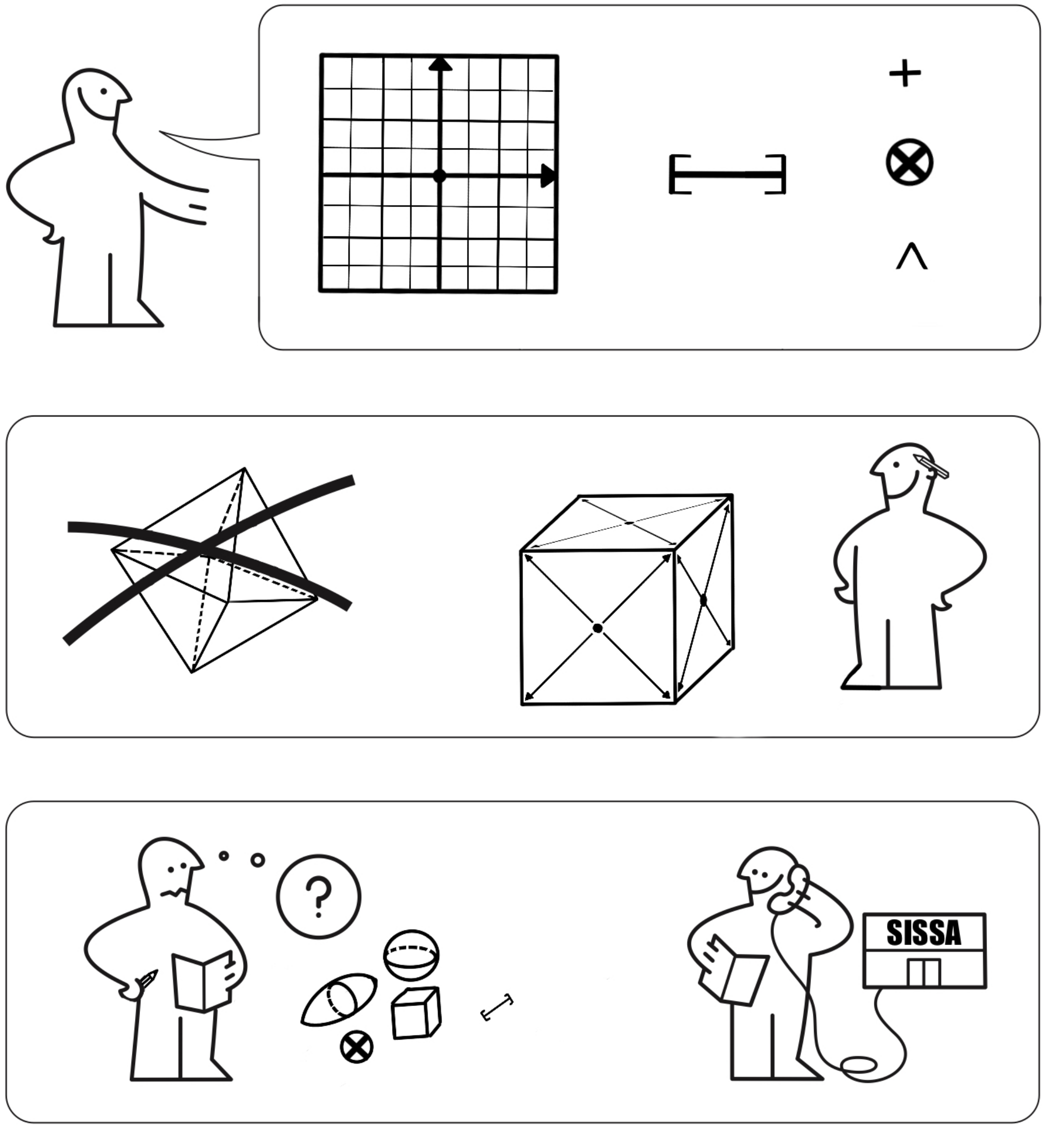
Mathematics Area - PhD course in Geometry

The Handbook of Zonoid Calculus

Author:
Léo Mathis

Supervisor:
Antonio Lerario

Academic year: 2021-2022


SISSA

Abstract

In this work, we present a new method of computation that we call *zoonoid calculus*. It is based on a particular class of convex bodies called *zonoids* and on a representation of zonoids using random vectors. Concretely, this is a recipe to build multilinear maps on spaces of zonoids from multilinear maps on the underlying vector spaces. We call this recipe the *fundamental theorem of zonoid calculus* (FTZC).

Using this and the wedge product in the exterior algebra we build the *zonoid algebra*, that is a structure of algebra on the space of convex bodies of the exterior algebra of a vector space. We show how this relates to the notion of *mixed volume* on one side and to random determinants on the other. This produces new inequalities on expected absolute determinants. We also show how this applies in two detailed examples: fiber bodies and non centered Gaussian determinants. We then use FTZC to produce a new function on zonoids of a complex vector space that we call the *mixed J-volume*.

We uncover a link between the zonoid algebra and the algebra of valuations on convex bodies. We prove that the *wedge product of zonoids* extends Alesker's product of smooth valuations.

Finally we apply the previous results to integral geometry in two different context. First we show how, in Riemannian homogeneous spaces, the expected volume of random intersections can be computed in the zonoid algebra. We use this to produce a new inequality modelled on the Alexandrov–Fenchel inequality, and to compute formulas for random intersection of real submanifolds in complex projective space. Secondly, we prove how a *Kac-Rice type* formula can relate to the zonoid algebra and a certain *zonoid section*. We use this to study the expected volume of random submanifolds given as the zero set of a *random function*. We again produce an inequality on the densities of expected volume modelled on the Alexandrov–Fenchel inequality, as well as a general *Crofton formula* in Finsler geometry.

Contents

Introduction

Introduction

The word *calculus* comes from the latin for *pebbles*, in its most general sense, it denotes “a method of computation or calculation in a special notation” [65]. In this work, we present a new “method of computation” involving special convex-shaped pebbles called *zonoids*.

In this introduction, we will give a general idea of the work presented here and give the main results and ideas. The statements of the results are sometime simplified to be readable here. When in doubt, the reader should always refer to the proper statement that is in the text.

Zonoids

Zonoids are a particular class of convex bodies. They are limits of special polytopes called *zonotopes* that are *finite Minkowski sum of segments*, i.e. centrally symmetric polytopes with all faces centrally symmetric, see Figure 1, a precise definition is given in Section 1.2.1.

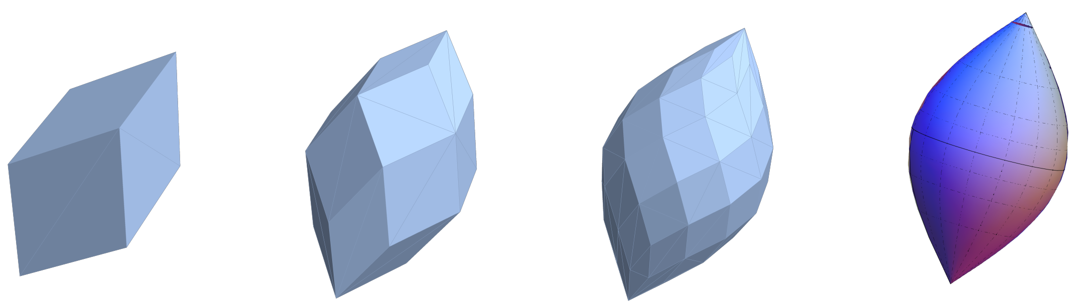


Figure 1: A sequence of zonotopes and its limit: a zonoid.

Zonoids appear in different context in convex geometry, historically with the *projection body* [80, Section 10.9] and Shephard’s problem [80, Section 10.11]. But also in many other domains such as statistics with the *lift zonoid* [50, 67], probability [66, 86], quantum information theory [13], control theory [94], and even physics of solids [59].

In particular zonoids are deeply linked with integral transforms and measure theory. Indeed there is a bijection between the non negative measures on the unit sphere of a Euclidean space $\mathbb{V}$ and the zonoids in $\mathbb{V}$ (up to translation). This bijection, called the *cosine transform*, associates to the dirac delta measure supported on the points $\pm x$ and of total weight $r \geq 0$ the segment $(r/2)[-x, x]$ and then is extended by linearity where the sum on the space of convex bodies is the Minkowski sum. This connection is explained in detail in Section 1.2.3.

In this work we advocate for another point of view introduced by Richard Vitale in [86] that uses random vectors. If X is a random vector in $\mathbb{V}$ that admits a first moment and $X_1, \ldots, X_N, \ldots$ are iid copies of X then one can construct the *Vitale zonoid* associated to X as:

$$\mathbb{E}[0,X] := \lim_{N\to\infty} \frac{1}{N}\left([0,X_1]+\cdots+[0,X_N]\right)$$

where the sum is the Minkowski sum (Definition 1.1.4). One can show that all zonoids can be obtained in this way (Proposition 1.2.30), we explain this construction in detail in Section 1.2.2. Although the correspondence between random vectors and zonoids is not one to one, it allows more flexibility than the measures on the sphere. This flexibility will allow to build algebraic structures on the space on zonoids and introduce the *zonoid calculus.* Moreover the simplicity of this point of view will allow to do computations sometime rather easily.

Zonoid Calculus

The main result in this context is the *Fundamental Theorem of Zonoid Calculus* (FTZC, Theorem 2.1.16), this is a joint work with Peter Bürgisser, Paul Breiding and Antonio Lerario [26]. It allows to build multilinear maps on the spaces of zonoids from a multilinear map on the underlying vector spaces, this is based on a tensor product of zonoids that apppears in [13].

We denote by $\mathscr{Z}(\mathbb{V})$ the *space of zonoids* in $\mathbb{V}$, the precise definition is given in Section 1.2.

Theorem 2.1.16. *Let $M : \mathbb{V}_1\times\cdots\times\mathbb{V}_k\to\mathbb{W}$ be a multilinear map between finite dimensional real vector spaces. Then there exists a unique continuous map*

$$\widehat{M} : \mathscr{Z}(\mathbb{V}_1)\times\cdots\times\mathscr{Z}(\mathbb{V}_k)\to\mathscr{Z}(\mathbb{W})$$

that is linear in each variable and that is such that for every $x_1\in\mathbb{V}_1,\ldots,x_k\in\mathbb{V}_k$ we have

$$\widehat{M}\left([0,x_1],\ldots,[0,x_k]\right) = [0, M(x_1,\ldots,x_k)].$$

Note that the actual statement of Theorem 2.1.16 involves the centered segments $\frac{1}{2}[-x_i,x_i]$ rather than $[0,x_i]$, the map is then extended to the non centered case in Definition 2.1.17. We omitted this subtlety here to increase readability.

There is nothing mysterious in this construction: the hypotheses and multilinearity determine the map on finite Minkowski sum of segments, namely *zonotopes*, then if we show that this is well defined and continuous, it determines the map on zonoids. In fact this idea may have been implicit or hidden in some argument involving zonoids in the past. However, having it explicitly stated like this is of great help and allows to uncover this construction in already known convex geometry operations as well as building new ones.

For example, if we consider the multilinear map

$$\det : (\mathbb{R}^m)^m \to \mathbb{R}$$

then it is not difficult to show (Theorem 2.2.6) that for all zonoids $K_1,\ldots,K_m$ in $\mathbb{R}^m$, we have that $\widehat{\det}(K_1,\ldots,K_m)\subset\mathbb{R}$ is a segment of length $m!\mathrm{MV}(K_1,\ldots,K_m)$ where MV denotes the mixed volume (Proposition 1.1.22).

A less trivial example is given in Section 2.5 where we explain, in a joint work with Chiara Meroni [61], how the construction of *fiber bodies*, a generalization of the *fiber polytopes* introduced by Louis J. Billera and Bernd Sturmfels [22], falls into the framework of zonoid calculus and FTZC.

Theorem 2.5.22. *The fiber body of a zonoid is a zonoid. If K is a zonoid in $\mathbb{R}^{n+m}$ then*

$$\widehat{F_\pi}(K,\ldots,K) = (n+1)!\,\Sigma_\pi(K)$$

where Σ_π denotes the fiber body with respect to an orthogonal projection $\pi:\mathbb{R}^{n+m}\to\mathbb{R}^n$ (Definition 2.5.2) and F_π is defined in Definition 2.5.21.

This allows, for example, to give a new explicit formula for the fiber body of a zonotope in Corollary 2.5.23, that generalizes a formula for the fiber body of a cube [22, Theorem 4.1].

The map $\widehat{M}$ behaves well under the Vitale construction as it satisfies for *independent* random vectors $X_1, \ldots, X_k$,

$$\widehat{M}\left(\mathbb{E}[0, X_1], \ldots, \mathbb{E}[0, X_k]\right) = \mathbb{E}[0, M(X_1, \ldots, X_k)].$$

Together with the example of the determinant given above, this allows to link random determinant and convex geometry, generalizing what was done by Richard Vitale in [86]. In a joint work with Peter Bürgisser, Paul Breiding and Antonio Lerario [26] we show the following.

Corollary 2.2.26. *Let $1 \leq k \leq m$, let $X_1, \ldots, X_k$ be independent integrable random vectors of $\mathbb{R}^m$ and let $M := (X_1, \ldots, X_k) \in \mathbb{R}^{m \times k}$ be the random matrix whose columns are the vectors X_i. We have*

$$\mathbb{E}\sqrt{\det(M^t M)} = \frac{m!}{(m-k)!\kappa_{m-k}} \mathrm{MV}(\mathbb{E}[0, X_1], \ldots, \mathbb{E}[0, X_k], B_m[m-k])$$

where $B_m := B(\mathbb{R}^m)$ is the unit ball, $B_m[m-k]$ denotes that it is repeated $m-k$ times in the argument of the mixed volume MV *and $\kappa_{m-k} := \mathrm{vol}_{m-k}(B_{m-k})$.*

The application of zonoid calculus to absolute random determinants is explained in Section 2.2.3. In particular, applying the Alexandrov–Fenchel inequality (Lemma 1.1.25) we get the following new inequality for random determinants.

Corollary 2.2.29. *Let $X_1, X_1', X_2, X_2', Y_1, \ldots, Y_{m-2}$ be independent random vectors of $\mathbb{R}^m$ that admit first moments and such that X_1 and X_1', respectively X_2 and X_2', have the same law. We have*

$$\left(\mathbb{E}|\det(X_1, X_2, Y_1, \ldots, Y_{m-2})|\right)^2 \geq \left(\mathbb{E}|\det(X_1, X_1', Y_1, \ldots, Y_{m-2})|\right)\left(\mathbb{E}|\det(X_2, X_2', Y_1, \ldots, Y_{m-2})|\right).$$

And similarly for the Brunn-Minkowski inequality, see Corollary 2.2.28.

In Section 2.6 we study this in the context of *Gaussian vectors*. Studying the Vitale zonoid associated to a non centered Gaussian vector, we can show that it is very close to an ellipsoid.

Theorem 2.6.7. *Let $c \in \mathbb{R}^m$ and let ξ be a standard Gaussian vector in $\mathbb{R}^m$. There is an ellipsoid $\mathcal{E} \subset \mathbb{R}^m$ such that*

$$b_\infty \mathcal{E} \subset \mathbb{E}[0, \xi + c] \subset \mathcal{E}$$

where $b_\infty := \min\{\varphi_\infty(\cos(t), \sin(t)) \,|\, t \in [0, 2\pi]\} \sim 0.989\ldots$, with φ_∞ defined in (2.6.5).

The ellipsoid is precisely defined in the proper statement of Theorem 2.6.7 below and estimates and asymptotics of its volume are computed. Moreover, applying this to random determinants, it gives an estimate of the expected absolute determinants of a non centered Gaussian matrix in terms of mixed volume of ellipsoids which corresponds to the centered case and was proved by Zakhar Kabluchko and Dmitry Zaporozhets in [46].

Theorem 2.6.13. *Let $0 < k \leq m$ and let $X_1, \ldots, X_k \in \mathbb{R}^m$ be independent non degenerate Gaussian vectors and consider the random matrix $\Gamma := (X_1, \ldots, X_k)$ whose columns are the vectors X_i. There are ellipsoids $\mathcal{E}_1, \ldots, \mathcal{E}_k$ such that*

$$(b_\infty)^k \alpha_{m,k} \mathrm{MV}\left(\mathcal{E}_1, \ldots, \mathcal{E}_k, B_m[m-k]\right) \leq \mathbb{E}\sqrt{\det\left(\Gamma^t \Gamma\right)} \leq \alpha_{m,k} \mathrm{MV}\left(\mathcal{E}_1, \ldots, \mathcal{E}_k, B_m[m-k]\right)$$

where $B_m[m-k]$ denotes the unit ball $B_m \subset \mathbb{R}^m$ repeated $m-k$ times in the argument of the mixed volume MV *and $\alpha_{m,k} := \frac{m!}{(2\pi)^{k/2}(m-k)!\kappa_{m-k}}$.*

All these results can be seen as operations in an algebra that we construct and call the *zonoid algebra*. Given a real finite dimensional vector space $\mathbb{V}$ it is defined as

$$\mathscr{A}(\mathbb{V}) := \bigoplus_{k=0}^{m} \mathscr{Z}(\Lambda^k \mathbb{V})$$

with the sum being the Minkowski sum and the product is constructed from the wedge product in the exterior algebra of $\mathbb{V}$ and using FTZC:

$$\cdot \wedge \cdot : \mathscr{Z}(\Lambda^k \mathbb{V}) \times \mathscr{Z}(\Lambda^l \mathbb{V}) \to \mathscr{Z}(\Lambda^{k+l} \mathbb{V}).$$

In practice, using the Vitale construction, this product, that we call *wedge product of zonoids*, satisfies for X, Y independent random vectors:

$$(\mathbb{E}[0, X]) \wedge (\mathbb{E}[0, Y]) = \mathbb{E}[0, X \wedge Y].$$

The zonoid algebra is introduced and explained in Section 2.2. Note that this is not properly speaking an *algebra* since the Minkowski sum doesn't have an inverse. However, it is not difficult to embed algebraically the space of zonoids $\mathscr{Z}(\mathbb{V})$ into a vector space $\widehat{\mathscr{Z}}(\mathbb{V})$ via a Grothendieck construction. The space $\widehat{\mathscr{Z}}(\mathbb{V})$, that we call the space of *virtual zonoids*, consists of formal differences $K - L$ with $K, L \in \mathscr{Z}(\mathbb{V})$. This construction is purely algebraic and in particular does not give any topology on the vector space of virtual zonoids.

There are, for zonoids, essentially two ways to realize $\widehat{\mathscr{Z}}(\mathbb{V})$ as a topological vector space. The first one is to embed it as a subspace of the continuous functions on the sphere with the help of the *support function*. The second choice uses the correspondence between the zonoids and measures on the sphere to identify the space of virtual zonoids with the space of *even signed measures* on the unit sphere. We have two maps:

$$\begin{aligned} h : \widehat{\mathscr{Z}_0}(V) &\to C(S(\mathbb{V})) \\ K - L &\mapsto h_K - h_L \end{aligned} \qquad\qquad \begin{aligned} \mu : \widehat{\mathscr{Z}_0}(V) &\to \mathcal{M}(S(\mathbb{V})) \\ K - L &\mapsto \mu_K - \mu_L \end{aligned}$$

where the subscript in $\mathscr{Z}_0$ indicates that we consider *centered* zonoids (i.e. with the center of symmetry at the origin) and h_K is the *support function* of K and μ_K its *generating measure*. When on $C_{even}(S(\mathbb{V}))$ we consider the topology given by the supremum norm $\|\cdot\|_\infty$ and on $\mathcal{M}(S(\mathbb{V}))$ the *weak-$*$* topology we get two *different* topologies on the space $\widehat{\mathscr{Z}_0}(\mathbb{V})$ that both coincide, on the cone of zonoids $\mathscr{Z}_0(\mathbb{V}) \subset \widehat{\mathscr{Z}_0}(\mathbb{V})$, with the standard topology given by the *Hausdorff distance*.

Having a vector space can come handy and allows to use the tools of linear algebra. However topological considerations here are very subtle and one should be careful when making continuity statements. See for example Proposition 1.2.54 that summarizes some continuity and non continuity properties of the cosine transform (which corresponds to the passage from measures to continuous functions in the above identification).

Nevertheless, we show that Corollary 2.2.26 is a particular case of a more general statement in the zonoid algebra.

Theorem 2.2.24. *Let $c_1, \ldots, c_k \in \mathbb{N}$ such that $c := c_1 + \ldots + c_k \leq m$, let $X_1 \in \mathbb{R}^{m \times c_1}, \ldots, X_k \in \mathbb{R}^{m \times c_k}$ be independent and integrable and let $M := (X_1, \ldots, X_k)$ be the random $m \times c$ matrix whose columns are the matrices X_i. Then we have*

$$\mathbb{E}\sqrt{\det(M^t M)} = \ell(\mathbb{E}[0, Y_1] \wedge \cdots \wedge \mathbb{E}[0, Y_k])$$

where $\ell(\cdot)$ is the length *or first intrinsic volume (Definition 1.2.31) and Y_i is the random vector in $\Lambda^{c_i}\mathbb{R}^m$ that is the image of X_i under the map $\mathbb{R}^{m \times c_i} \to \Lambda^{c_i}\mathbb{R}^m$ that maps $(x_1, \ldots, x_{c_i})$ to $x_1 \wedge \cdots \wedge x_{c_i}$ for all $x_1, \ldots, x_{c_i} \in \mathbb{R}^m$.*

There is a particular subalgebra of the zonoid algebra where most of the future computations will take place. Recall that the Grassmannian of k dimensional vector subspaces of a vector space $\mathbb{V}$, denoted $G_k(\mathbb{V})$, embeds in the projective space of $\Lambda^k\mathbb{V}$ via the Plücker embedding. Thus signed measures on the Grassmannian $G_k(\mathbb{V})$ can be seen as a subspace of the even signed measures of $S(\Lambda^k\mathbb{V})$.

Zonoids in $\Lambda^k\mathbb{V}$ that have as generating measure a measure supported on the Grassmannian $G_k(\mathbb{V})$ will be called *Grassmannian zonoids*. It is not difficult to see that they form a subalgebra of $\mathscr{A}(\mathbb{V})$

that we call the *Grassmannian zonoid algebra* or simply *Grassmannian algebra*. One may think of it as a product structure on the spaces of (signed) measures on the Grassmannian, although it is good to keep in mind the point of view of random vectors, where the expression for the (wedge) product is rather simple. We introduce and study Grassmannian zonoids in Section 2.2.2. We derive a few formulas and lemmas that will be useful in the next chapters. In particular Lemma 2.2.17 for the computation of *length* in the Grassmannian algebra that will be extensively used for computations in the last chapter.

Finally, in a complex vector space $\mathbb{C}^n \cong \mathbb{R}^{2n}$ we consider the (real and complex) multilinear map *complex determinant*:

$$\det_{\mathbb{C}} : \left(\mathbb{R}^{2n}\right)^n \cong \left(\mathbb{C}^n\right)^n \to \mathbb{C} \cong \mathbb{R}^2.$$

Mimicking the relation between mixed volume and real determinant, but this time with *complex determinant*, we define a new function on zonoids of $\mathbb{C}^n$ that we call the *mixed J–volume* (here J refers to the complex structure) and that is given for all $K_1, \ldots, K_n \in \mathscr{Z}(\mathbb{C}^n)$, by:

$$\mathrm{MV}^J(K_1, \ldots, K_n) := \frac{1}{n!} \ell \left(\widehat{\det}_{\mathbb{C}}(K_1, \ldots, K_n) \right)$$

where ℓ denotes the *length* or first intrinsic volume. The *J–volume* of a zonoid $K \in \mathscr{Z}(\mathbb{C}^n)$ is then defined to be $\mathrm{vol}_n^J(K) := \mathrm{MV}^J(K, \ldots, K)$.

It is not difficult to prove that the mixed J-volume shares some similar symmetries with the classical mixed volume. In fact when restricted to a Lagrangian subspace it is equal to the classical volume. In general, however, they are different, indeed note that the mixed J-volume takes n arguments in a space of *real* dimension $2n$.

The mixed J-volume shares many similarities with another function of degree n on convex bodies of $\mathbb{C}^n$ called *Kazarnovskii's pseudo volume*. However similar, we show that, surprisingly enough, they are not the same. In fact we show in Section 3.1.2 that the mixed J-volume, defined on zonoids, extends to polytope but does *not* extend continuously to all convex bodies. We introduce and study in detail the J-volume in Section 2.3.

Valuations

A *valuation* on convex bodies is a map ϕ defined on convex bodies with values on a semigroup such that for all convex bodies K, L it satisfies

$$\phi(K) + \phi(L) = \phi(K \cup L) + \phi(K \cap L)$$

whenever $K \cup L$ is a convex body. Valuations have been extensively studied since the time of Dehn who solved Hilbert's third problem using valuations [34]. In this work, we mostly focus on *continuous real translation invariant valuations*. McMullen proved that the space $val(\mathbb{V})$ of continuous real translation invariant valuations on convex bodies of a real vector space $\mathbb{V}$ of dimension m decomposes as

$$val(\mathbb{V}) = \bigoplus_{k=0}^{m} val_k(\mathbb{V})$$

where $val_k(\mathbb{V})$ are the valuations of *degree* k, i.e. the valuations ϕ such that for all convex body K and all $t \geq 0$ we have $\phi(tK) = t^k \phi(K)$, see Proposition 3.1.3 below.

More recently, in the 2000s, Semyon Alesker made two major breakthroughs in the study of valuations. First he proved in [3] that the spaces $val_k(\mathbb{V})$, when further decomposed in even and odd valuations, is an irreducible representation of $Gl(\mathbb{V})$. In infinite dimension, this means that every invariant subspace is dense. Using this he was able to answer positively to a conjecture by McMullen about the density of the subspace spanned by mixed volumes.

The second result was the construction in [5] of a product structure on a dense subspace of $val(\mathbb{V})$, namely the *smooth valuations*, that turns it into a graded algebra. In Section 3.3.2, we will show that our wedge product of zonoids can extend Alesker's product to a larger subspace of even valuations and that this is a special case of a recent extension constructed by Nguyen-Bac Dang and Jian Xiao in [33].

The idea is very simple. Given a signed measure μ on the Grassmannian $G_k(\mathbb{V})$ one can construct an even valuation ϕ_μ in $val_k(\mathbb{V})$ by letting, for every convex body K

$$\phi_\mu(K) := \int_{G_k(\mathbb{V})} \mathrm{vol}_k(K|E)\,\mathrm{d}\mu(E)$$

where $K|E$ denotes the orthogonal projection of K onto the subspace $E \in G_k(\mathbb{V})$. Recalling that signed measures on the Grassmnnian correspond to Grassmannian zonoids in the terminology introduced above, this defines a map Φ from the *Grassmannian algebra* to the space of *translation invariant continuous real even valuations*:

$$\Phi : K \mapsto \phi_{\mu_K}.$$

Next we show that the kernel of this map, which turns out to be the same as the *kernel of the cosine transform* $\mathfrak{M}(k, \mathbb{V})$, is a closed ideal for the wedge product of zonoids. Thus the wedge product of zonoids gives a well defined product on the image of Φ. It is then not difficult to show that this extends Alesker's product of smooth valuations by showing that it is a special case of Dang and Xiao's extension. Moreover we show that other operations on valuations, such as *convolution* and a certain duality, descend from operations on the Grassmannian algebra. We explain this in Section 3.3 in a setting that does not depend on the choice of an Euclidean structure.

Theorem 3.3.18. *The product defined by the map Φ and the wedge product of zonoids extends Alesker's product of smooth valuations.*

The interest of this point of view is that, with zonoid calculus, the expressions appear simpler. In fact we show that the valuations in the image of Φ take a special form on zonoids. For this we introduce the *exponential* of a zonoid, that is given for every zonoid K in $\mathbb{V}$ by

$$e^K := \sum_{k=0}^{m} \frac{1}{k!} K^{\wedge k}$$

where $m = \dim \mathbb{V}$. This is a semigroup morphism between the zonoids with Minkowski sum and the Grassmannian zonoids with the wedge product. Next, if $\mathbb{V}$ is endowed with a scalar product $\langle \cdot, \cdot \rangle$, it induces, using FTZC, a bilinear form on the space of zonoids that we denote $(\cdot, \cdot)$. Concretely, with the Vitale construction, this bilinear form gives for random vectors X and Y:

$$(\mathbb{E}\underline{X}, \mathbb{E}\underline{Y}) = \mathbb{E}|\langle X, Y \rangle|$$

where $\mathbb{E}\underline{X} = \mathbb{E}[0, X] + \frac{1}{2}\{-\mathbb{E}X\}$ is the centered version of the Vitale construction. With this, we show in Proposition 3.3.10 that if A is a (virtual) Grassmannian zonoid with generating measure μ, then for every zonoid K we have:

$$\phi_\mu(K) = (A, e^K).$$

The author's hope is that this point of view of zonoid calculus and random vectors will help with the computation in the algebra of valuations. In the meantime, it certainly helps with the computation in the Grassmannian algebra. Indeed certain operations, such as taking the length, can be done at the level of valuations. Thus if one is interested in evaluating the length in the Grassmannian algebra, one can work instead in the algebra of valuations which is smaller (it is the quotient by $\mathfrak{M}(\mathbb{V})$). In some cases this is a considerable reduction of complexity. For example in the case of a complex vector space, the space of unitary invariant Grassmannian zonoids (i.e. unitary invariant measures on the real Grassmannian) is infinite dimensional while it was proved by Alesker (again!) in [3] that the space of unitary invariant valuation is finite dimensional, generated, as an algebra, by two elements, see Example 3.3.20 below.

Integral geometry

In the last chapter of this work, Chapter 4, we apply zonoid calculus to integral geometry, more precisely to random intersection problems. Typically, the setting is as follows. Given $X_1, \ldots, X_k$ independent *random* submanifolds of a Riemannian manifold M, we want to evaluate the quantity

$$\mathbb{E}\,\mathrm{vol}(X_1 \cap \cdots \cap X_k) = ?$$

where the volume is the Riemannian volume in the appropriate dimension. In particular if the sum of the codimensions of the submanifolds adds up to the dimension of M we want to evaluate the *average number of points* in the intersection. Of course one has to specify how to take *random submanifolds* in M.

Homogeneous spaces

We will first assume that M is a compact Riemannian homogeneous spaces. This means that there is a compact Lie group G that acts on M by isometries. Then, if $X \subset M$ is a fixed submanifold and g is a random element of G taken with the probability defined by the normalized Haar measure, it defines a random submanifold by taking $g \cdot X$.

Now, to such a random submanifold, one can associate a Grassmannian zonoid in the cotangent space at a point $o \in M$. The idea is the following: take a point $x \in X$ and move it to o using the group action. Then consider the normal space of X at o and its orbit under the *isotropy group* $H := Fix_G(o)$. This defines a measure (that is H-invariant) on the Grassmannian $G_c(T_o^*M)$ where c is the codimension of the submanifold X. Then we average over all $x \in X$ and normalize by the quotient of volumes $\mathrm{vol}(X)/\mathrm{vol}(M)$. We call the corresponding Grassmannian zonoid K_X. The main result, that is a joint work with Peter Bürgisser, Paul Breiding and Antonio Lerario, is that this zonoid computes the volume of random intersections.

Theorem 4.1.4. *Let $X_1, \dots, X_n \subset M$ be submanifolds, such that $c := \sum_{i=1}^n \mathrm{codim}(X_i) \leq m = \dim(M)$, and let $K_{X_1}, \dots, K_{X_n}$ be their associated zonoids. Let $g_1, \dots, g_n$ be independent random elements of G taken with the normalized Haar measure. Then*

$$\frac{1}{\mathrm{vol}_m(M)} \mathbb{E}\left[\mathrm{vol}(g_1X_1 \cap \cdots \cap g_nX_n)\right] = \ell(K_{X_1} \wedge \cdots \wedge K_{X_n}).$$

In particular, in the case where $c = m$ we obtain

$$\mathbb{E}\#(g_1X_1 \cap \cdots \cap g_nX_n) = \mathrm{vol}_m(M)\, \ell(K_{X_1} \wedge \cdots \wedge K_{X_n}).$$

In the last case this can be reformulated as an equality in the zonoid algebra with a nice cohomological flavor as follows.

$$K_{X_1} \wedge \cdots \wedge K_{X_n} = \Big(\mathbb{E}\#(g_1X_1 \cap \cdots \cap g_nX_n)\Big) K_{\{o\}}$$

where $K_{\{o\}}$ is the zonoid associated to a point.

The link between wedge product of zonoids and mixed volumes allows then to interpret the Alexandrov-Fenchel inequality in a new inequality in the context of random intersection.

Theorem 4.1.12. *Let $X, Y, Z_3 \dots, Z_m \subset M$ be hypersurfaces. Let $g_1, \dots, g_m$ be independent and uniform in G and denote the random surface $Z := g_3Z_3 \cap \cdots \cap g_mZ_m$. We have*

$$\mathbb{E}\#(g_1X \cap g_2Y \cap Z) \geq \sqrt{\mathbb{E}\#(g_1X \cap g_2X \cap Z)\ \mathbb{E}\#(g_1Y \cap g_2Y \cap Z)}.$$

Next, we use Theorem 4.1.4 to study intersection of *real* submanifolds of $\mathbb{C}\mathrm{P}^n$. In particular, using the reduction to valuations we produce the following formula that generalizes Bézout and which, to the knowledge of the author, is new.

Theorem 4.1.26. *Let $n \geq 2$, let $X \hookrightarrow \mathbb{C}\mathrm{P}^n$ be a real submanifold of real codimension 2, and consider $d_X := \mathbb{E}\#(X \cap g\mathbb{C}\mathrm{P}^1)$ where g is random and uniform in $U(n+1)$ and $\Delta_X := d_X - \frac{(n-1)!}{\pi^{n-1}} \mathrm{vol}_{2n-2}(X)$. Then, if $g_1, \dots, g_n$ are uniform and independent random element in $U(n+1)$, we have*

$$\mathbb{E}\#(g_1X \cap \cdots \cap g_nX) = \sum_{k=0}^n \binom{n}{k} \frac{c_k}{4^k(n-1)^k} \Delta_X^k d_X^{n-k}$$

where $c_k := \sum_{j=0}^k \binom{k}{j}\binom{2j}{j} 2^{k-j}$.

If X is a complex irreducible hypersurface then $\Delta_X = 0$ and d_X is the *degree* of X in which case the formula gives Bézout. Note that this formula could also presumably be proven by means of a *kinematic formula in complex space forms* proved by Andreas Bernig, Joseph Fu and Gil Solanes in [21], see the discussion at the end of Section 4.1.2.

For example, in the case where $n = 2$ and X is a real surface in $\mathbb{C}\mathrm{P}^2$, this gives

$$\mathbb{E}\#(g_1 X \cap g_2 X) = d_X^2 + 2 d_X \Delta_X + \frac{9}{8}\Delta_X^2.$$

Then we show how to construct the zonoids K_X in the special case where X is a *Schubert variety* in the Grassmannian. Schubert varieties are particular subvarieties of the Grassmannian indexed by Young diagrams. We explain in Section 4.1.3 how these Schubert diagrams can also be used to describe the normal and tangent space at a point of a Schubert variety. We then explain how Peter Bürgisser and Antonio Lerario in [29] and Antonio Lerario and the author in [54] use the invariance of the zonoid K_X to compute its volume.

In Section 4.1.4 we put this in a more general perspective. Given a group action on a vector space $\mathbb{V}$ and a subspace $\Sigma \subset \mathbb{V}$, we show how the convex bodies of $\mathbb{V}$ invariant under the group action can be identified, under suitable conditions on the action and the subspace, with the convex bodies on Σ. The map that goes from one to the other is the orthogonal projection $\pi : \mathbb{V} \to \Sigma$. We study the necessary conditions in detail and give some examples where this situation appeared, see Theorem 4.1.38 below.

Kac-Rice formula and zonoids

In the very last part of this work, Section 4.2, we present a joint work with Michele Stecconi where we study a more general setting. In this case M is any Riemannian manifold and $X := f^{-1}(0)$ is the zero set of a random C^1 map $f : M \to \mathbb{R}^k$. We have again to specify how do we want to take the *random map*, that we call random *field*, f. We collect a list of hypotheses that will make the Kac-Rice formula work in Definition 4.2.3 and we call those the *z-KRok* hypotheses for "zonoid-Kac-Rice ok".

There are 4 *z-KRok* hypotheses. The first one ensures that 0 is almost surely a regular value of f so that $X = f^{-1}(0)$ does indeed define a random submanifold while *z-KRok*-ii and iii are some regularity and absolute continuity assumptions. Then the difficulty in building the zonoid with respect to the previous case is that there is no *isotropy group*. You would like to be able to fix a point $p \in M$ and look at *all the random submanifold X that passes through p*. The problem being that the event "$p \in X$" or equivalently "$f(p) = 0$" has probability zero and thus conditioning to it is not well defined. This problem can be solved with the concept of *regular conditional probability*, see Definition 4.2.1. The condition *z-KRok*-iv ensures that there is a regular conditional probability that satisfies some continuity and finiteness assumption.

Hence, given a random field $f : M \to \mathbb{R}^k$ that is *z-KRok* (i.e. satisfies the *z-KRok* hypotheses), writing $f = (f^1, \dots, f^k)$, we can consider, for a given $p \in M$, the random vector of $\Lambda^k T_p^* M$ given by

$$(\mathrm{d}_p f^1 \wedge \cdots \wedge \mathrm{d}_p f^k | f(p) = 0)$$

where $(\,\cdot\,| f(p) = 0)$ denotes the conditioning defined by *z-KRok*-iv. The precise meaning is given in Section 4.2.1. Then we can define, for every $p \in M$, a Grassmannian zonoid in $\Lambda^k T_p^* M$ associated to the *z-KRok* field f by letting

$$\zeta_f(p) := \left([0, \mathrm{d}_p f^1 \wedge \cdots \wedge \mathrm{d}_p f^k] | f(p) = 0\right).$$

This zonoid will then, as in the previous case, allow to evaluate expectations of volumes. For this we will use a *Kac-Rice formula*, see (4.2.23) which is based on one that was proved by Michele Stecconi in [82]. We obtain from it the main theorem of this section.

Theorem 4.2.30. *Let $f : M \to \mathbb{R}^k$ be* z-KRok *and consider the random submanifold $X := f^{-1}(0)$. Then for all open set $U \subset M$ we have*

$$\mathbb{E}\left[\mathrm{vol}(X \cap U)\right] = \int_U \ell(\zeta_f(p))\, \mathrm{d}M(p)$$

where ℓ is the length *or first intrinsic volume.*

We also show two important features of the zoonoid section. The first one is the *pull-back property*, Lemma 4.2.17. In its simplest form, it states that given a submanifold $S \subset M$ that is almost surely transversal to $X = f^{-1}(0)$ then the restriction of f to S is again *z-KRok* and its associated zonoid section is the pull back of ζ_f. Unfortunately this condition of being *almost surely transversal* to the field f cannot be removed and there are some *z-KRok* fields that admit submanifold that do *not* satisfy this, see (4.2.17). It would be convenient to have at least a sufficient condition to avoid these cases but for now it is not clear to the author what this should be.

Nevertheless, this pull back property is helpful to compute the zonoid section in some examples. We also use it to prove, in Proposition 4.2.21, that if M is endowed with *any* Riemannian metric then there exists a *z-KRok* field $f : M \to \mathbb{R}$ such that the zonoid section is given by the unit balls. We call such fields *AT-fields* after Robert J. Adler and Jonathan E. Taylor [2] and they play a key role in the proof of the Alpha formula.

The second property of the zonoid section is the fact that independent intersections correspond to wedge products of the zonoids. More precisely, if $f : M \to \mathbb{R}^k$ and $f' : M \to \mathbb{R}^l$ are two independent *z-KRok* fields then we construct a third field $(f, f') : M \to \mathbb{R}^{k+l}$ whose zero set is the intersection of the previous two zero set. We show in Lemma 4.2.26 that (f, f') is again *z-KRok* and that we have for all $p \in M$

$$\zeta_{(f,f')}(p) = \zeta_f(p) \wedge \zeta_{f'}(p).$$

This allows to compute the expectation of number of points in intersection of independent *z-KRok* fields in terms of mixed volumes in Corollary 4.2.32. Once again, we can then interpret the Alexandrov-Fenchel inequality in this context to produce a *Kac-Rice Alexandrov-Fenchel* inequality (KRAF).

Theorem 4.2.33. *Let $g_1, \ldots, g_{m-2}, f_1, f_1', f_2, f_2' : M \to \mathbb{R}$ be independent* z-KRok *fields, such that f_1' is distributed as f_1 and f_2' is distributed as f_2. Let $Y := (g_1)^{-1}(0) \cap \ldots \cap (g_{m-2})^{-1}(0)$, $X_i := (f_i)^{-1}(0)$ and $X_i' := (f_i')^{-1}(0)$, $i = 1, 2$. Then we have for all open subset $U \subset M$:*

$$\mathbb{E}\left[\#(X_1 \cap X_2 \cap Y \cap U)\right] \geq \int_U \sqrt{\delta_{\#X_1 \cap X_1' \cap Y}(p) \cdot \delta_{\#X_2 \cap X_2' \cap Y}(p)}\, \mathrm{d}M(p)$$

where for $i = 1, 2$ we wrote

$$\delta_{\#X_i \cap X_i' \cap Y}(p) := \ell(\zeta_{f_i}(p) \wedge \zeta_{f_i'}(p) \wedge \zeta_{g_1}(p) \wedge \cdots \wedge \zeta_{g_{m-2}}(p)).$$

Note that, unlike in Theorem 4.1.12, we cannot directly relate this to the product of the expectation of number of points on the right hand side. In fact by Hölder's inequality, one sees that the right hand side is smaller or equal than $\sqrt{\mathbb{E}\left[\#(X_1 \cap X_1' \cap Y \cap U)\right] \mathbb{E}\left[\#(X_2 \cap X_2' \cap Y \cap U)\right]}$.

The zonoid section ζ_f contains more information than just the expectation of volume and one could also count the number of points of intersection of independent *z-KRok* fields *with sign*, see Corollary 4.2.36 and Theorem 4.2.35.

Finally we conclude this section, chapter and work by interpreting our results in the context of *Finsler geometry*. The choice, for each point $p \in M$ of a norm $F_p : T_pM \to \mathbb{R}$ that satisfies further regularity assumptions is called a *Finsler structure* on M. Here these regularities will not always be satisfied but, at least in this introduction, we will use the term *Finsler* anyway. In Finsler geometry, the notion of *length* of smooth curves is well defined, if $\gamma : [0, 1] \to M$ is a C^1 curve then, one simply let

$$\ell^F(\gamma) := \int_0^1 F_{\gamma(t)}(\dot{\gamma}(t))\mathrm{d}t.$$

In our case, given a *z-KRok* field $f : M \to \mathbb{R}$, the zonoid $\zeta_f(p)$ defines a (semi) norm on T_pM, by the *support function*. If $X := f^{-1}(0)$, we obtain then a Finsler structure on M that we denote F^X. Concretely, this is given for every $p \in M$ and every $v \in T_pM$ by

$$F_p^X(v) := \frac{\rho_{f(p)}(0)}{2}\, \mathbb{E}\left[|\mathrm{d}_p f(v)| \,\middle|\, f(p) = 0\right].$$

With this, we are able to show a *Crofton formula* in Finsler geometry.

Theorem 4.2.43. *Let $f : M \to \mathbb{R}$ be* z-KRok *and consider the random hypersurface $X := f^{-1}(0)$. Let $\gamma : [0,1] \to \mathbb{R}$ be a C^1 curve such that X is transversal to γ almost surely. Then*

$$\mathbb{E}\#(\gamma \cap X) = 2\,\ell^{F^X}(\gamma).$$

Furthermore, there is a notion of volume in Finsler geometry called the *Holmes-Thompson volume*, see Definition 4.2.44. We show that this can be obtained using zonoid calculus in Lemma 4.2.45 to obtain a more general Crofton formula for higher dimensional submanifold.

Theorem 4.2.46. *Let $1 \leq k \leq m$, let $f_1, \ldots, f_k : M \to \mathbb{R}$ be iid* z-KRok *fields, let $X_i := f_i^{-1}(0)$ and let $X^{(k)} := X_1 \cap \cdots \cap X_k$. Let $\iota : S \hookrightarrow M$ be an embedded submanifold of dimension k such that $X^{(k)}$ is transversal to S almost surely, then we have*

$$\mathbb{E}\#\left(S \cap X^{(k)}\right) = k!\kappa_k \operatorname{vol}_k^{F^{X_1}}(S)$$

where $\operatorname{vol}_k^{F^{X_1}}$ denotes the Holmes–Thompson volume for the Finsler structure F^{X_1}.

Disclaimer on vocabulary

The names *theorem, proposition, lemma*, etc. . . are attributed to the results in accordance to their position in this work and in no way to their importance in overall mathematics. The name "theorem" is reserved to *original* results that the author is at least a coauthor of and that he considers of particular importance *among the other results*, something that the reader can bring back home.

The name "lemma" is given to results that are used to prove theorems or that have more the flavor of a tool *in this work*.

Acknowledgments

During my educational process I received so much support from so many people and this work is dedicated to all of them.

I wish, above all, to thank my supervisor Antonio Lerario. It is clear that nothing could have happened without him. I thank him for making me discover beautiful mathematics and a beautiful city. I am grateful that he always showed explicitly his support and affection and that he considered me not only as a student but also as a collaborator from the beginning.

This experience would have not been the same without my big brother Michele Stecconi. His unique view on life reflects on his way of doing mathematics and he keeps inspiring me for both. I truly value our time together and hope we will have plenty more in the future.

I thank all my friends and co-PhDs here in Trieste and in particular Mario and Stefano who were always here for me during these four years.

Before coming to SISSA I had the most enriching experience at the ENS Lyon. This was firstly due to the kindness and strong desire to transmit mathematics of the whole educational team. I would like in particular to mention Bruno Sevennec and Jean-Claude Sikorav who were always waiting in *salle passerelle* for us to ask questions. I also thank Marie Lhuissier and Olga Paris-Romaskevich for our time together in particular at the *séminaire de la détente*. Their approach to mathematics is not only inspiring but also of public interest. Of course my time at the school was memorable also thanks to Cameron Alan and Quentin, although this was not always about mathematics.

Before that, a lot of teachers believed in me and each and every one of them contributed to me arriving where I am today. I have a special thought to Oleg Ogievetsky who was the first to respond to my desire to see what was research in mathematics about.

Bien sûr les premières personnes à avoir toujours cru en moi et supporté mes actions dans tous les sens du termes sont mes parents. Je les remercie de tout mon coeur et je suis incroyablement chanceux d'être né là où je suis né.

Finally my most tender thoughts go to Juliette who supported me in so many ways during these four years. I am so glad that this adventure was done with you at my side, I owe you a lot.

Notation

Convex bodies	
∂K	boundary of K
$\mathbb{E}\underline{X}$	Vitale zonoid associated to X
$\mathcal{F}_k(P)$	faces of dimension k of P
$\mathscr{G}(k,\mathbb{V})$	Grassmannian zonoids in $\Lambda^k\mathbb{V}$
$\mathscr{G}_0(k,\mathbb{V})$	$\mathscr{G}(k,\mathbb{V})\cap\mathscr{Z}_0(\Lambda^k\mathbb{V})$
$\mathscr{G}(\mathbb{V})$	Grassmannian algebra of $\mathbb{V}$
h_K	support function of K
$\bar{h}_K$	h_K restricted to the unit sphere
K^u	$\{x\in K \,\vert\, \langle u,x\rangle = h_K(u)\}$
$\mathscr{K}(\mathbb{V})$	convex bodies of $\mathbb{V}$
$\mathscr{K}_0(\mathbb{V})$	$K\in\mathscr{K}(\mathbb{V})$ such that $(-1)K=K$
$\widehat{\mathcal{K}}$	Grothendieck group/vector space of $\mathcal{K}$
ℓ	length or first intrisic volume
μ_K	generating measure of K
MV	mixed volume
$\mathcal{N}_K(x)$	normal cone of K at x
$\mathcal{P}(\mathbb{V})$	polytopes of $\mathbb{V}$
$val^+(\mathbb{V})$	C^0 real even translation invariant valuations
$val^+_k(\mathbb{V})$	elements of $val^+(\mathbb{V})$ of degree k
V_k	k-th intrisic volume
$[x,y]$	$\{(1-t)x+ty \,\vert\, t\in[0,1]\}$
$\underline{x}$	$\frac{1}{2}[-x,x]$
$\mathscr{Z}(\mathbb{V})$	zonoids of $\mathbb{V}$
$\mathscr{Z}_0(\mathbb{V})$	centered zonoids of $\mathbb{V}$: $\mathscr{Z}(\mathbb{V})\cap\mathscr{K}_0(\mathbb{V})$
Sets and spaces	
$*$	Hodge star operator
$B(\mathbb{V})$	unit ball of $\mathbb{V}$
B_m	$B(\mathbb{R}^m)$
$G_k(\mathbb{V})$	Grassmannian of k vector subspaces of $\mathbb{V}$
κ_m	m dimensional volume of B_m
$\mathcal{M}(S(\mathbb{V}))$	even signed measures on $S(\mathbb{V})$
$\mathcal{M}^+(S(\mathbb{V}))$	even non negative measures on $S(\mathbb{V})$
$S(\mathbb{V})$	unit sphere of $\mathbb{V}$
S^m	$S(\mathbb{R}^{m+1})$
s_m	m dimensional volume of S^m
$\mathcal{S}^G$	elements of $\mathcal{S}$ fixed by the action of G
$\mathbb{V}^*$	dual space of $\mathbb{V}$
Probability	
$X\in\mathcal{M}$	X is a random element of $\mathcal{M}$
$\mathbb{E}[f(X)]$	integral of f w.r.t. the law of X
$P(A)$	probability of the event A
Functions and function spaces	
$\Vert f\Vert_\infty$	sup of $\vert f(x)\vert$ over the domain of definition of f
$C(\mathcal{S})$	continuous functions $f:\mathcal{S}\to\mathbb{R}$
$C_{even}(S(\mathbb{V}))$	continuous $f:S(\mathbb{V})\to\mathbb{R}$ such that $f(-x)=f(x)$
$C^1(M,N)$	C^1 functions from M to N
d_pf	differential of f at p
J_pf	Jacobian of f at p
∇f	gradient of f

Chapter 1

Convex bodies and zonoids

Throughout this chapter, $\mathbb{V}$ will denote an Euclidean space of dimension $m < \infty$. We will write $\mathbb{V}^*$ for the dual space of $\mathbb{V}$, that is $\mathbb{V}^* := \mathrm{Hom}(\mathbb{V}, \mathbb{R})$. We will write $\langle \cdot, \cdot \rangle$ to denote both the scalar product in $\mathbb{V}$ and the pairing $\mathbb{V}^* \times \mathbb{V} \to \mathbb{R}$.

We consider an Euclidean space in order to be able to talk about *volumes* but we will try to avoid to identify $\mathbb{V}^* \cong \mathbb{V}$ as much as possible.

In this chapter we recall a few facts from convex geometry. The results being standard we will not give a proof most of the time but we will indicate a precise reference each time. The standard reference in convex geometry is Rolf Schneider's [80].

1.1 Basics of convex geometry

1.1.1 Convex bodies and representing functions

We start with the most basic definition.

Definition 1.1.1. A subset $C \subset \mathbb{V}$ is called *convex* if for any pair of points $x, y \in C$ the segment $[x, y]$ is contained in C, that is if for all $t \in [0, 1]$, we have $(1-t)x + ty \in C$.

Definition 1.1.2. A *convex body* is a non empty compact convex set. The set of convex bodies of $\mathbb{V}$ will be denoted $\mathscr{K}(\mathbb{V})$. We also define $\mathscr{K}_0(\mathbb{V}) := \{C \in \mathscr{K}(\mathbb{V}) \,|\, (-1)C = C\}$ the convex bodies symmetric with respect to the origin.

The space $\mathscr{K}(\mathbb{V})$ is a monoid with a scalar multiplication with the following operations.

Definition 1.1.3. The *scalar multiplication*: $\forall \lambda \in \mathbb{R}$, $\lambda C := \{\lambda x \,|\, x \in C\}$.

Definition 1.1.4. The *Minkowski sum*: $C + D := \{x + y \,|\, x \in C,\ y \in D\}$.

Note that $\mathscr{K}_0(\mathbb{V}) \subset \mathscr{K}(\mathbb{V})$ is a submonoid, i.e. it is closed under the two operations just defined. Moreover, the neutral element for the Minkowski sum is $\{0\}$, the convex body consisting of only one point: the origin.

Remark 1.1.5. Note also that the Minkowski sum is monotone with respect to inclusion, that is if $K \subset K'$ and $L \subset L'$ are convex bodies then $K + L \subset K' + L'$.

There is a natural distance on $\mathscr{K}(\mathbb{V})$ induced by the norm on $\mathbb{V}$.

Definition 1.1.6. The *Hausdorff distance* is given for all $K, L \in \mathscr{K}(\mathbb{V})$ by

$$\mathrm{d}(K, L) := \inf \{r \geq 0 \,|\, K \subset L + rB(\mathbb{V});\ L \subset K + rB(\mathbb{V})\}$$

where $B(\mathbb{V})$ is the unit ball of $\mathbb{V}$. The norm of a convex body K is then defined to be

$$\|K\| := \mathrm{d}(\{0\}, K) = \inf \{r > 0 \,|\, K \subset rB(\mathbb{V})\}$$

in other words it is the radius of the smallest ball containing K.

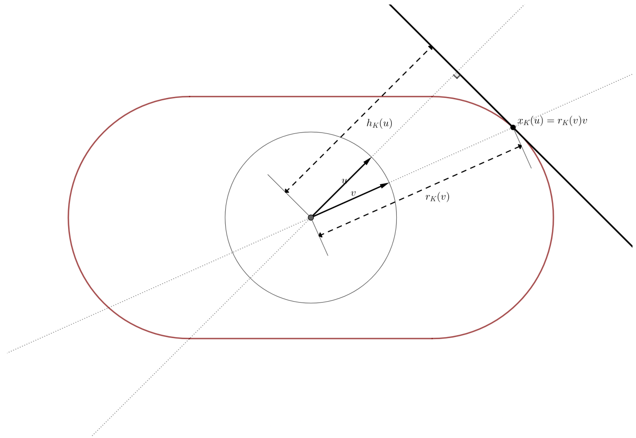


Figure 1.1: Support and radial functions

Note that this norm satisfies the triangle inequality with the Minkowski sum. The space $\mathscr{K}(\mathbb{V})$ will always be considered endowed with the topology induced by this distance. The Hausdorff distance depends on the choice of a norm on $\mathbb{V}$, but one can check that the induced topology doesn't.

We now introduce two functions that characterize a convex body K. The first one is natural to consider but the second one will turn out to be easier to work with, see Figure 1.1.

Definition 1.1.7. The *radial function* $r_K : \mathbb{V} \to \mathbb{R}$ is given for all $v \in \mathbb{V}$ by

$$r_K(v) := \sup \{ r > 0 \mid rv \in K \} .$$

If K contains the origin in its interior, it defines a norm on $\mathbb{V}$ such that K is the unit ball of this norm. It is given by

$$\|v\|_K := \inf \left\{ r > 0 \,\middle|\, \frac{v}{r} \in K \right\} = \frac{1}{r_K(v)}.$$

Definition 1.1.8. The *support function* $h_K : \mathbb{V}^* \to \mathbb{R}$ is given for all $u \in \mathbb{V}^*$ by

$$h_K(u) := \sup \{ \langle u, x \rangle \mid x \in K \} .$$

The support function turns out to characterize the convex body meaning that one can reconstruct a convex body from its support function. The following is [80, p.44].

Proposition 1.1.9. *Let $K \in \mathscr{K}(\mathbb{V})$ then*

$$K = \{ x \in \mathbb{V} \mid \langle u, x \rangle \leq h_K(u) \, \forall u \in \mathbb{V}^* \} .$$

One can characterize the support functions, the following is [80, Theorem 1.7.1].

Proposition 1.1.10. *A function $h : \mathbb{V}^* \to \mathbb{R}$ is the support function of a convex body if and only if it is* sublinear*, that is if for every $\lambda \geq 0$ and every $u, u' \in \mathbb{V}^*$, $h(\lambda u) = \lambda h(u)$ and $h(u + u') \leq h(u) + h(u')$.*

Example 1.1.11. The support function of the unit ball $B(\mathbb{V})$ is $h_{B(\mathbb{V})}(u) = \|u\|$.

Example 1.1.12. Let $x \in \mathbb{V}$, the support function of the point $\{x\}$ is $h_{\{x\}}(u) = \langle u, x \rangle$. The support function of the segment $\underline{x} := \frac{1}{2}[-x, x]$ is $h_{\underline{x}}(u) = \frac{1}{2}|\langle u, x \rangle|$. The support function of the segment $[0, x]$ is $h_{[0,x]}(u) = \max\{0, \langle u, x \rangle\}$.

Let us mention that the support and radial function are *dual* to each other.

Definition 1.1.13. Let $K \in \mathscr{K}(\mathbb{V})$ have non empty interior. Its *polar body* is defined to be

$$K^\circ := \{ u \in \mathbb{V}^* \mid h_K(u) \leq 1 \} \in \mathscr{K}(\mathbb{V}^*).$$

By [80, Theorem 1.6.2] we have that $K^{\circ\circ} = K$ if, in addition, K contains the origin. Moreover the following is [80, Lemma 1.7.13]

Proposition 1.1.14. *Let $K \in \mathscr{K}(\mathbb{V})$ containing the origin in its interior and let $K^\circ \in \mathscr{K}(\mathbb{V}^*)$ be its polar body then for all $v \in \mathbb{V}$ we have $h_{K^\circ}(\mathbb{V}) = \|v\|_K$.*

In that case, we have, by duality, $h_K = \|\cdot\|_{K^\circ}$. In other words, the support function defines a norm on $\mathbb{V}^*$ that is the dual norm to $\|\cdot\|_K$.

The support function turns out to be very handy when manipulating convex bodies.

Proposition 1.1.15. *Let $K, L \in \mathscr{K}(\mathbb{V})$ and let $\lambda \geq 0$. The following holds.*

(i) $h_{\lambda K + L} = \lambda h_K + h_L$*;*

(ii) $K \subset L$ *if and only if* $h_K \leq h_L$*;*

(iii) Let $T : \mathbb{V} \to \mathbb{W}$ be a linear map and $T^t : \mathbb{W}^* \to \mathbb{V}^*$ be its transpose we have $h_{T(K)} = h_K \circ T^t$;

(iv) $\mathrm{d}(K, L) = \sup\{|h_K(u) - h_L(u)| \mid u \in S(\mathbb{V}^*)\}$.

Proof. Items (i) and (iii) are a direct consequence of the definition of the support function in Definition 1.1.8. Item (ii) follows from the expression of K from its support function in Proposition 1.1.9. Finally, item (iv) is [80, Lemma 1.8.14]. □

We have the following bound for the support function.

Proposition 1.1.16. *Let $K \in \mathscr{K}(\mathbb{V})$ and let $u \in \mathbb{V}^*$, we have $h_K(u) \leq \|u\|\|K\|$ with equality if and only if K is a ball.*

Proof. As mentioned above, $\|K\|B(\mathbb{V})$ is the smallest ball containing K, thus by Proposition 1.1.15-(ii) we have $h_K \leq \|K\| h_{B(\mathbb{V})}$. The result follows from the identity $h_{B(\mathbb{V})}(u) = \|u\|$. □

Remark 1.1.17. If we write $\bar{h}_K$ for the restriction of h_K to the unit sphere $S(\mathbb{V}^*)$, item (iv) in the last proposition reads $\mathrm{d}(K, L) = \|\bar{h}_K - \bar{h}_L\|_\infty$ where $\|\cdot\|_\infty$ is the supremum norm on the continuous functions on $S(\mathbb{V}^*)$, i.e.

$$\|f\|_\infty := \sup\{|f(u)| \mid u \in S(\mathbb{V}^*)\}.$$

This means that the map $\bar{h}. : \mathscr{K}(\mathbb{V}) \to C(S(\mathbb{V}^*))$ that maps $K \mapsto \bar{h}_K$ is a linear isometric embedding. Its image is a convex cone inside the vector space $C(S(\mathbb{V}^*))$ (endowed with the supremum norm). We will study in more detail this point of view in Section 1.2.4.

Since the Hausdorff distance corresponds to the supremum norm, we have that a sequence of convex bodies converges if and only if their support functions (restricted to the sphere) converge uniformly. Uniform convergence can be tricky to prove, luckily, for support functions, pointwise convergence turns out to be sufficient, the following is [80, Theorem 1.8.15].

Lemma 1.1.18. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of convex bodies and let $h : \mathbb{V}^* \to \mathbb{R}$ be such that h_{K_n} converges pointwise to h. Then h is the support function of a convex body K and $K_n \to K$ in the Hausdorff distance topology.*

Let us make the following definition.

Definition 1.1.19. Let $K \in \mathscr{K}(\mathbb{V})$ and let $u \in \mathbb{V}^* \setminus \{0\}$. The *face* of K in the direction u is $K^u := \{x \in K \mid \langle u, x\rangle = h_K(u)\}$. If K^u consists of a single point we say that K is *strictly convex in the direction* u and we denote this point $x_K(u)$. Finally we say that K is *strictly convex* if it is strictly convex in every direction.

This is usually called an *exposed* face and the notion of face is strictly weaker in some cases but we will abuse notation and make no distinction here.

An additional property of the support function is the following which is [80, Corollary 1.7.3]. We denote by ∇h the gradient of the function $h : \mathbb{V}^* \to \mathbb{R}$.

Proposition 1.1.20. *Let $K \in \mathscr{K}(\mathbb{V})$ and let $u \in \mathbb{V}^* \setminus \{0\}$. The support function h_K is differentiable at u if and only if K is strictly convex in the direction u. In that case we have*

$$\nabla h_K(u) = x_K(u).$$

where recall from Definition 1.1.19 that $x_K(u)$ is the point such that $K^u = \{x_K(u)\}$.

Let us also mention that the faces of a convex body satisfy some additive properties, the following is [80, Theorem 1.7.5.(c)].

Proposition 1.1.21. *Let $K, L \in \mathscr{K}(\mathbb{V})$ and let $u \in S(\mathbb{V}^*)$, we have*

$$(K+L)^u = K^u + L^u.$$

1.1.2 Mixed volume and related concepts

The function *volume* on $\mathscr{K}(\mathbb{V})$ is homogeneous of *degree* m (where recall that $m = \dim \mathbb{V}$) meaning that for all $K \in \mathscr{K}(\mathbb{V})$ and all $t \geq 0$ we have $\mathrm{vol}_m(tK) = t^m \mathrm{vol}_m(K)$. Minkowski proved that it can be *polarized* and gives rise to a *multilinear* form. The following is [80, Theorem 5.1.7].

Proposition 1.1.22 (and Definition)**.** *There is a nonnegative symmetric continuous function, called the* mixed volume*, $\mathrm{MV} : \mathscr{K}(\mathbb{V})^m \to \mathbb{R}$ such that for all $K_1, \ldots, K_l \in \mathscr{K}(\mathbb{V})$ and all $t_1, \ldots, t_m$ we have*

$$\mathrm{vol}_m(t_1K_1 + \cdots + t_lK_l) = \sum_{i_1,\ldots,i_m=1}^{l} t_{i_1} \cdots t_{i_m} \mathrm{MV}(K_{i_1}, \ldots, K_{i_m}).$$

The mixed volume satisfies the following properties, for details and proofs see [80, Section 5.1].

Proposition 1.1.23. *Let $K_1, \ldots, K_m, K, L \in \mathscr{K}(\mathbb{V})$, the mixed volume satisfies the following.*

(i) *For all permutation σ, $\mathrm{MV}(K_{\sigma(1)}, \ldots, K_{\sigma(m)}) = \mathrm{MV}(K_1, \ldots, K_m)$;*

(ii) *for all $\lambda \geq 0$, we have $\mathrm{MV}(\lambda K + L, K_2, \ldots, K_m) = \lambda \mathrm{MV}(K, K_2, \ldots, K_m) + \mathrm{MV}(L, K_2, \ldots, K_m)$;*

(iii) *we have $\mathrm{MV}(K, \ldots, K) = \mathrm{vol}_m(K)$;*

(iv) *we have $\mathrm{MV}(K_1, \ldots, K_m) > 0$ if and only if there are segments $[x_1, y_1] \subset K_1, \ldots, [x_m, y_m] \subset K_m$ such that the vectors $y_1 - x_1, \ldots, y_m - x_m$ are linearly independent.*

In the following we denote $\mathrm{MV}(K[k], \ldots)$ for the repetition of K, k times in the argument of the mixed volume.

A particular case is the mixed volume with the unit ball.

Definition 1.1.24. Let $K \in \mathscr{K}(\mathbb{V})$ and let $0 \leq k \leq m$. The k-th *intrinsic volume* of K is defined to be

$$\mathrm{V}_k(K) := \frac{\binom{m}{k}}{\kappa_{m-k}} \mathrm{MV}(K[k], B(\mathbb{V})[m-k]).$$

Note that we have $\mathrm{V}_m = \mathrm{vol}_m$ and V_0 is constant equal to one. Moreover, by Proposition 1.1.22 the intrinsic volumes are the coefficients of the polynomial describing the volume of a neighbourhood of a convex body $K \in \mathscr{K}(\mathbb{V})$, that is for all $t \geq 0$, we have

$$\mathrm{vol}_m(K + tB(\mathbb{V})) = \sum_{k=0}^{m} \mathrm{V}_k(K) \kappa_{m-k} t^{m-k}.$$

This formula is called *Steiner's polynomial.*

Mixed volume and intrisic volumes are the main objects of key inequalities in convex geometry. One of the most important (if not *the* most important) is known as the *Alexandrov–Fenchel inequality* (AF), see [80, Theorem 7.3.1].

Lemma 1.1.25 (AF)**.** *Let $K_3, \dots, K_m \in \mathscr{K}(\mathbb{V})$ and let us denote by $\mathfrak{K}$, the tuple $(K_3, \dots, K_m)$. For all convex bodies $K, L \in \mathscr{K}(\mathbb{V})$ we have*

$$\mathrm{MV}(K, L, \mathfrak{K}) \geq \sqrt{\mathrm{MV}(K, K, \mathfrak{K})\mathrm{MV}(L, L, \mathfrak{K})}.$$

Another inequality bounds from below the volume of the Minkowski sum of two convex bodies and is known as the *Brunn–Minkowski inequality* (BM). It has several equivalent forms and we chose to present here the multiplicative one, see [80, p.372 (e)].

Lemma 1.1.26 (BM)**.** *Let $K_0, K_1 \in \mathscr{K}(\mathbb{V})$. For all $t \in [0, 1]$, we have*

$$\mathrm{vol}_m((1-t)K_0 + tK_1) \geq \mathrm{vol}_m(K_0)^{1-t}\,\mathrm{vol}_m(K_1)^t.$$

There exists a local, measure theoretic version of the intrinsic volumes and Steiner's polynomial. To define it, let us introduce the following.

Definition 1.1.27. Let $K \in \mathscr{K}(\mathbb{V})$ and recall that ∂K denotes its boundary. The *normal cone* at $x \in \partial K$ is defined to be

$$\mathscr{N}_K(x) := \{u \in \mathbb{V}^* \,|\, \langle u, y - x\rangle \leq 0, \, \forall y \in K\}.$$

If K has a smooth boundary then the normal cone is the half line spanned by the outer normal. We can now define the surface area measure, see [89] or [80, Section 4.2].

Definition 1.1.28. Let $K \in \mathscr{K}(\mathbb{V})$. The $(m-1)$–*surface area measure* (or just surface area measure), is the nonnegative measure $\mathcal{S}_{m-1}(K, \cdot)$ on the unit sphere $S(\mathbb{V}^*)$ given for all Borelian $\eta \subset S(\mathbb{V})$ by

$$\mathcal{S}_{m-1}(K, \eta) := \mathcal{H}^{m-1}\left(\{x \in \partial K \,|\, \mathscr{N}_K(x) \cap \eta \neq \emptyset\}\right)$$

where $\mathcal{H}^{m-1}$ denotes the Hausdorff measure.

There are other surface area measure $\mathcal{S}_k(K, \cdot)$, $0 \leq k \leq m-1$ that can be obtained by means of a Steiner-type formula, see [80, (4.27)], for $t \geq 0$, this gives

$$\mathcal{S}_{m-1}(K + tB(\mathbb{V}), \cdot) = \sum_{k=0}^{m-1} \binom{m-1}{k} t^{m-1-k} \mathcal{S}_k(K, \cdot).$$

Similarly there is a mixed version of the surface area measure. The following is [80, Theorem 5.1.7].

Proposition 1.1.29 (and Definition)**.** *There is a symmetric map $\mathcal{MS}$, called the* mixed area measure*, from $\mathscr{K}(\mathbb{V})^{m-1}$ into the space of finite Borel measures on $S(\mathbb{V}^*)$ such that for all $K_1, \dots, K_l \in \mathscr{K}(\mathbb{V})$ and all $t_1, \dots, t_l \geq 0$ we have*

$$\mathcal{S}_{m-1}(t_1K_1 + \cdots + t_lK_l, \cdot) = \sum_{i_1, \dots, i_{m-1}=1}^{l} t_{i_1} \cdots t_{i_{m-1}} \mathcal{MS}(K_{i_1}, \dots, K_{i_{m-1}}, \cdot).$$

The mixed area measure satisfies similar properties as the mixed volume, for details and proofs see [80, Section 5.1].

Proposition 1.1.30. *Let $K_1, \dots, K_{m-1}, K, L \in \mathscr{K}(\mathbb{V})$, and let $\lambda \geq 0$. The mixed area measure satisfies the following:*

(i) For all permutation σ, $\mathcal{MS}(K_{\sigma(1)}, \dots, K_{\sigma(m-1)}) = \mathcal{MS}(K_1, \dots, K_{m-1}, \cdot)$;

(ii) we have $\mathcal{MS}(\lambda K + L, K_2, \dots, K_{m-1}, \cdot) = \lambda \mathcal{MS}(K, K_2, \dots, K_{m-1}, \cdot) + \mathcal{MS}(L, K_2, \dots, K_{m-1}, \cdot)$;

(iii) we have $\mathcal{MS}(K, \dots, K, \cdot) = \mathcal{S}_{m-1}(K, \cdot)$;

(iv) for all $0 \leq k \leq m-1$, we have $\mathcal{S}_k(\lambda K, \cdot) = \lambda^k \mathcal{S}_k(K, \cdot)$.

Moreover the mixed area measure allows to compute the mixed volume, the following is still [80, Theorem 5.1.7].

Proposition 1.1.31. *Let $K_1, \dots, K_m \in \mathscr{K}(\mathbb{V})$, we have*

$$\mathrm{MV}(K_1, \dots, K_m) = \frac{1}{m} \int_{S(\mathbb{V}^*)} h_{K_1}(u) \mathcal{MS}(K_2, \dots, K_{m-1}, \mathrm{d}u).$$

1.1.3 Curved bodies and polytopes

Let us illustrate the concepts of the previous section in two cases where computations are doable.

Curved bodies

Recall that if a support function is differentiable its gradient restricted to the sphere parametrizes the boundary of the convex body, see Proposition 1.1.20.

Definition 1.1.32. A convex body K is called *curved* if h_K is C^2 and the map

$$x_K := (\nabla h_K)|_{S(\mathbb{V}^*)} : S(\mathbb{V}^*) \to \partial K$$

that maps $u \mapsto x_K(u)$ the point of ∂K that admits u as an outer normal, is a C^1 diffeomorphism. The inverse of x_K is the *Gauss map* $u_K : \partial K \to S(\mathbb{V}^*)$ which is such that for all $x \in \partial K$, $u_K(x)$ is the outer unit normal of K at x.

Note that the property of being curved implies, in particular, that the convex body has nonempty interior and that its boundary ∂K is a (closed) C^2 hypersurface of $\mathbb{V}$. Curved bodies can be dealt with by using functional analysis tools on their support functions. They are dense in the space of convex bodies, for more details see [80, Section 2.5] where curved bodies are called C^2_+.

Let us observe that for all $u \in S(\mathbb{V}^*)$ the tangent space $T_uS(\mathbb{V}^*)$ and the tangent space $T_{x_K(u)}\partial K$ can both be identified with $u^\perp$. Thus the differential $D_u x_K$ can be seen as an endomorphism of $u^\perp$, and as such is selfadjoint, see [80, p.116].

Definition 1.1.33. The *reverse Weingarten map* is the selfadjoint operator $\overline{W}_u := D_u x_K : u^\perp \to u^\perp$. Its eingenvalues denoted $r_1 \geq \cdots \geq r_{n-1} > 0$, are called the *principal radii of curvature* of K at u.

Then the surface area measure of a curved body admits a density that can be expressed in terms of the principal radii of curvatures, see [80, (4.26)].

Proposition 1.1.34. *Let $K \in \mathscr{K}(\mathbb{V})$ be a curved body and let $\eta \subset S(\mathbb{V}^*)$ be a Borelian. We have for all $0 \leq k \leq m-1$*

$$\mathcal{S}_k(K,\eta) = \int_\eta s_k(K,u)\mathrm{d}u$$

where $s_k(K,u)$ denotes the normalized k-th elementary symmetric polynomial in the principal radii of curvature of K at u, i.e.

$$s_k(K,u) = \binom{m-1}{k}^{-1} \sum_{1\leq i_1 \leq \cdots \leq i_k \leq m-1} r_{i_1}\cdots r_{i_k}.$$

In particular we have

$$\mathrm{vol}_m(K) = \frac{1}{m}\int_{S(\mathbb{V}^*)} h_K(u) s_{m-1}(K,u)\mathrm{d}u.$$

Polytopes

At the other extreme there is another very important dense subset of convex bodies.

Definition 1.1.35. A convex body P is a *polytope* if it is the intersection of finitely many half spaces. The space of polytopes of $\mathbb{V}$ will be denoted $\mathcal{P}(\mathbb{V})$.

Polytopes are handy because they are given by a *finite* amount of data. Indeed according to the definition, for all $P \in \mathcal{P}(\mathbb{V})$ there exist $u_1, \ldots, u_l \in S(\mathbb{V}^*)$ and $a_1, \ldots, a_l \in \mathbb{R}$ such that the polytope P can be written $P = \bigcap_{i=1}^l \{x \in \mathbb{V} \mid \langle u_i, x\rangle \leq a_i\}$. Thus they often can be handled with *combinatoric* tools.

Recall the definition of a face in Definition 1.1.19 and of the normal cone in Definition 1.1.27.

Definition 1.1.36. Let $P \in \mathcal{P}(\mathbb{V})$, we denote by $\mathcal{F}_k(P)$ the set of k-dimensional faces of P. If $F \in \mathcal{F}_k(P)$ we denote by $\mathcal{N}_P(F) := \mathcal{N}_P(x)$ where x is any point in the relative interior of F. Finally for all $F \in \mathcal{F}_k(P)$, we define its *external angle*

$$\Theta(F,P) := \mathrm{vol}_{n-k-1}(\mathcal{N}_P(F) \cap S(\mathbb{V}^*))/s_{n-k-1}.$$

where $s_n := \mathrm{vol}_n(S^n)$.

We can express the intrinsic volumes of a polytope in terms of volume of its faces and external angles. The following is [80, (4.23)].

Proposition 1.1.37. *For any $P \in \mathcal{P}(\mathbb{V})$ and all $0 \leq k \leq m$ we have*

$$\mathrm{V}_k(P) = \sum_{F \in \mathcal{F}_k(P)} \Theta(F,P)\,\mathrm{vol}_k(F).$$

Similarly one can explicitly compute the surface area measure.

Proposition 1.1.38. *Let $P \in \mathcal{P}(\mathbb{V})$ and for all $F \in \mathcal{F}_{m-1}(P)$, let $u_P(F)$ denote the outer unit normal of P at any point in the relative interior of F. Then we have*

$$\mathcal{S}_{m-1}(P,\cdot) = \sum_{F \in \mathcal{F}_{m-1}(P)} \mathrm{vol}_{m-1}(F)\delta_{u_P(F)}(\cdot).$$

Proof. From the definition of the surface area measure in Definition 1.1.28 we see that if $\eta \subset S(\mathbb{V}^*)$ is a Borelian then $\mathcal{S}_{m-1}(P,\eta) = \sum \mathrm{vol}_{m-1}(F)$ where the sum runs over all $F \in \mathcal{F}_{m-1}(P)$ such that $\mathcal{N}_P(F)$ is spanned by a vector in η. Since the half line $\mathcal{N}_P(F)$ is spanned by $u_P(F)$, this proves the result. □

1.2 Zonoids, definition(s)

We now come to our main object of study, namely *zonoids*, which are a special class of convex bodies. When dealing with zonoids there are essentially four different points of view that we will detail in this section.

First we will see the most geometric definition which builds zonoids using the Minkowski sum of segments.

The second approach is the one introduced by Vitale in [86] which builds zonoids using random segments. This approach plays a central role in this work and will often be our favourite choice.

The third point of view uses measures on the sphere and is classical when dealing with zonoids, it is for example extensively used in [80].

Finally we can see zonoids as (continuous) functions on the sphere through their support functions. This part is less developed, as characterizing such functions is a difficult problem (as we will see in the next section).

As we mentioned, the second approach will often be the one we adopt in the following. However it is the plurality of points of view that makes the space of zonoid rich and interesting. Switching from one point of view to the other can often make proofs and/or computations easier and having many of them is a great chance.

Let us also mention that there is a fifth point of view that characterizes zonoids as the *range of vector valued measures*. This will not be discussed here since the author did not work on it. Maybe it deserves more attention and can lead to something interesting, in the meantime one can refer to [24].

In the following, we will denote, for every $x \in \mathbb{V}$

$$\underline{x} := \tfrac{1}{2}[-x,x]. \tag{1.2.1}$$

1.2.1 Zonoids as limits of zonotopes

Definition 1.2.1. A convex body $K \in \mathscr{K}(\mathbb{V})$ is called a *zonotope* if it can be expressed as a finite sum of segments, that is if there exist $x_1, y_1, \dots, x_N, y_N \in \mathbb{V}$ such that $K = [x_1, y_1] + \cdots + [x_N, y_N]$. A *zonoid* is a limit (in the Hausdorff distance topology) of zonotopes. The space of zonoids in $\mathbb{V}$ will be denoted $\mathscr{Z}(\mathbb{V})$, moreover we let $\mathscr{Z}_0(\mathbb{V}) := \mathscr{Z}(\mathbb{V}) \cap \mathscr{K}_0(\mathbb{V})$.

Note that segments are centrally symmetric, more precisely, for all $x, y \in \mathbb{V}$, we have

$$[x, y] = \underline{z} + \tfrac{1}{2}\{c\}$$

where $z := (x - y)$ and $c := x + y$ (recall notation (1.2.1) for the segment $\underline{z}$). Observing that $\{c\} + \{c'\} = \{c + c'\}$ for all $c, c' \in \mathbb{V}$ this means that for every zonotope K, there exist $z_1, \dots, z_N, c \in \mathbb{V}$ such that $K = \underline{z_1} + \cdots + \underline{z_N} + \frac{1}{2}\{c\}$. In other words, this means that every zonotope, and thus every zonoid, has a center of symmetry.

Proposition 1.2.2. *Let $K \in \mathscr{Z}(\mathbb{V})$, there exists a unique $K_0 \in \mathscr{Z}_0(\mathbb{V})$ and a unique point $o(K) \in \mathbb{V}$ such that*

$$K = K_0 + \frac{1}{2}\{o(K)\}.$$

Definition 1.2.3. The point $o(K) \in K$ which is the symmetric of the origin with respect to the center of symmetry of K (or equivalently 2 times the center of K) will be called the *pole* of K. Elements of $\mathscr{Z}_0(\mathbb{V})$ will be called *centered* zonoids.

By Proposition 1.2.2, we have the decomposition

$$\mathscr{Z}(\mathbb{V}) = \mathscr{Z}_0(\mathbb{V}) \oplus \mathbb{V}$$

as monoids. Indeed if $K, L \in \mathscr{Z}(\mathbb{V})$ and $K = K_0 + \frac{1}{2}\{o(K)\}$, $L = L_0 + \frac{1}{2}\{o(L)\}$ then we have $K + L = K_0 + L_0 + \frac{1}{2}\{o(K) + o(L)\}$. In particular $o(K + L) = o(K) + o(L)$. This simple observation will allow us to treat the centered zonoid and the pole separately.

Zonotopes

Zonotopes are polytopes that, as mentioned above, are centrally symmetric. However not all centrally symmetric polytopes are zonotopes. The following characterization of zonotopes is [80, Theorem 3.5.2].

Proposition 1.2.4. *A polytope is a zonotope if and only if all its two dimensional faces are centrally symmetric.*

In particular all centrally symmetric polytopes in $\mathbb{R}^2$ are zonotopes. By approximation we have that all centrally symmetric bodies in dimension 2 are zonoids.

Corollary 1.2.5. *In dimension 2 we have $\mathscr{Z}_0(\mathbb{R}^2) = \mathscr{K}_0(\mathbb{R}^2)$.*

In general for $m > 2$, the inclusion $\mathscr{Z}_0(\mathbb{R}^m) \subset \mathscr{K}_0(\mathbb{R}^m)$ is strict. Let us describe more precisely the face structure of zonotopes, the reader can also refer to [62]. Without loss of generality we can assume that the zonotope is centered: let $x_1, \dots, x_N \in \mathbb{V}$, we let

$$K := \underline{x_1} + \cdots + \underline{x_N} \in \mathscr{Z}_0(\mathbb{V}). \tag{1.2.2}$$

By Proposition 1.1.21 we have that every face of K is again a zonotope. More precisely we have the following.

Proposition 1.2.6. *Let K be the zonotope given by (1.2.2) and let $u \in S(\mathbb{V}^*)$. We have*

$$K^u = \sum_{x_i \in u^\perp} \underline{x_i} + \frac{1}{2}\left\{ \sum_{x_i \notin u^\perp} \epsilon_i x_i \right\}$$

where $\epsilon_i := \operatorname{sign}\langle u, x_i \rangle$.

Proof. Apply Proposition 1.1.21 and note that $\underline{x_i}^u$ is either the point $\frac{\epsilon_i}{2}\{x_i\}$ if $x_i \notin u^\perp$ or $\underline{x_i}^u = \underline{x_i}$ if $x_i \in u^\perp$. □

Thus faces that are parallel are translate of the same *vectorial face*. Let us be more precise introducing a few notations. We denote by $G_k(\mathbb{V})$ the Grassmannian of (vectorial) k-planes of $\mathbb{V}$.

Definition 1.2.7. Let $P \in \mathcal{P}(\mathbb{V})$ be a polytope and let $F \in \mathcal{F}_k(P)$, we denote by $E_F \in G_k(\mathbb{V})$ the vector space parallel to the affine span of F. Moreover we define

$$G_k(P) := \{E \in G_k(\mathbb{V}) \,|\, \text{there exists a } k\text{-dim. face } F \text{ of } P \text{ such that } E = E_F\}.$$

Note that if K is the zonotope defined by (1.2.2) then, by Proposition 1.2.6, this takes the form

$$G_k(K) = \{E \in G_k(\mathbb{V}) \,|\, \text{there exist linearly independent } x_{i_1}, \dots, x_{i_k} \in E\}. \tag{1.2.3}$$

Definition 1.2.8. Let K be the zonotope defined by (1.2.2) and let $E \in G_k(K)$, we define the *vectorial face* of K parallel to E to be

$$F(E, K) := \sum_{x_i \in E} \underline{x_i}.$$

By Proposition 1.2.6 above, for every face $F \in \mathcal{F}_k(K)$ there is $c \in \mathbb{V}$ such that

$$F = F(E_F, K) + \tfrac{1}{2}\{c\}. \tag{1.2.4}$$

Moreover c is a combination of the summands x_i with coefficient in $\{0, \pm 1\}$.

The face structure of zonotopes imply an important property of their external angles (recall Definition 1.1.36).

Lemma 1.2.9. *Let $K \in \mathscr{Z}_0(\mathbb{V})$ be the zonotope defined by* (1.2.2) *and let $E \in G_k(K)$. The external angle satisfies*

$$\sum_{E_F = E} \Theta(K, F) = 1$$

where the sum runs over all the faces $F \in \mathcal{F}_k(K)$ such that $E_F = E$.

Proof. We give a proof similar to what can be found in the proof of [26, Theorem 6.15]. Given $E \in G_k(K)$, pick a nonzero $u \in E^\perp$. By the above discussion, the face K^u is a translate of $\sum_{x_i \in u^\perp} \underline{x_i}$. In addition, if F is a face of K such that $E_F = E$, F is a translate of $F(K, E) = \sum_{x_i \in E} \underline{x_i}$. Since $E \subset u^\perp$, the face K^u contains a translate of F. Moreover, $\dim(F) = k$ and it follows that $\dim(K^u) \geq k$ which implies $\dim(\mathcal{N}_K(K^u)) \leq m - k$. In other words we proved that if $E \in G_k(K)$ and $u \in E^\perp$, then $\dim(\mathcal{N}_K(K^u)) \leq m - k$.

We now show that for almost all $u \in E^\perp$ we have $\dim(\mathcal{N}_K(K^u)) = m - k$. Indeed, for this it is enough to write $E^\perp \subseteq \bigcup_{u \in E^\perp} \mathcal{N}_K(K^u)$, thus the set $\left\{u \in E^\perp \mid \dim(\mathcal{N}_K(K^u)) < m - k\right\}$ is contained in a finite union of cones of dimension at most $m - k - 1$.

Let us now consider the unit sphere $S(E^\perp) \subset S(\mathbb{V}^*)$, and denote by $\mathcal{F}$ the set of faces F of K such that $E_F \supset E$ and $\dim(\mathcal{N}_K(F)) < m - k$. Then, by the above reasoning,

$$\left\{u \in S(E^\perp) \,|\, \dim(K^u) < m - k\right\} \subseteq \bigcup_{F \in \mathcal{F}} \mathcal{N}_K(F) \cap S(E^\perp).$$

Each set $\mathcal{N}_K(F) \cap S(E^\perp)$ with $F \in \mathcal{F}$ has dimension at most $n - k - 2$. Since the set $\mathcal{F}$ is finite, it implies, as above, that $\left\{u \in S(E^\perp) \,|\, \dim(K^u) < m - k\right\}$ is contained in a finite union of sets of dimension at most $m - k - 2$, and in particular it has measure zero in $S(E^\perp)$. It follows that $\left\{u \in S(E^\perp) \,|\, \dim(K^u) = k\right\} \subset S(E^\perp)$ has full measure. Letting u vary in $S(E^\perp)$, the set $\{K^u\}$ exhausts all k–dimensional faces F with $E_F = E$ and therefore:

$$\sum_{E_F = E} \Theta(K, F) = \sum_{E_F = E} \frac{\mathrm{vol}_{m-k-1}(\mathcal{N}_K(F) \cap S(\mathbb{V}^*))}{\mathrm{vol}_{m-k-1}(S^{m-k-1})} = \frac{\mathrm{vol}_{m-k-1}(S^{m-k-1})}{\mathrm{vol}_{m-k-1}(S^{m-k-1})} = 1.$$

This concludes the proof. □

It is possible that an analogous to Lemma 1.2.9 exists for *curved zonoids*, i.e. zonoids that are curved bodies in the sense of Definition 1.1.32. It could be a result similar to [10, Theorem 2 and 3]. As far as the author knows, it remains an open problem.

This allows another expression for the intrinsic volumes of zonotopes.

Corollary 1.2.10. *Let $K \in \mathscr{Z}(\mathbb{V})$ be a zonotope and let $0 \leq k \leq m$. The k-th intrinsic volume of K is given by*

$$\mathrm{V}_k(K) = \sum_{E \in G_k(K)} \mathrm{vol}_k(F(E, K)).$$

Proof. Since all parallel faces are translate of each other (see (1.2.4)), the formula in Proposition 1.1.37 yields $\mathrm{V}_k(K) = \sum_{E \in G_k(K)} \mathrm{vol}_k(F(E,K)) \sum_{E_F = E} \Theta(K, F)$ where the internal sum runs over all the faces F such that $E_F = E$. The result then follows from Lemma 1.2.9. □

Remark 1.2.11. Note that if K is the zonotope defined by (1.2.2) then the first intrisic volume takes the following simple form:

$$\mathrm{V}_1(K) = \sum_{i=1}^{N} \|x_i\|.$$

Generalized and virtual zonoids

By definition of zonoids, $\mathscr{Z}_0(\mathbb{V})$ is a closed subset of $\mathscr{K}_0(\mathbb{V})$ (in the topology induced by the Hausdorff distance). Let us describe a larger class of convex bodies.

Definition 1.2.12. A (centrally symmetric) convex body $K \in \mathscr{K}_0(\mathbb{V})$ is called a *generalized zonoid* if there exist zonoids $L_1, L_2 \in \mathscr{Z}_0(\mathbb{V})$ such that $K + L_1 = L_2$.

It turns out that the set of generalized zonoids is not a closed subset of $\mathscr{K}_0(\mathbb{V})$. The following is [80, Corollary 3.5.7]

Proposition 1.2.13. *The set of generalized zonoids form a dense subset of $\mathscr{K}_0(\mathbb{V})$, that is for every $K \in \mathscr{K}_0(\mathbb{V})$ there is a sequence of generalized zonoids (K_n) such that $K_n \to K$ in the Hausdorff distance.*

If K is a generalized zonoid, it can be thought of as the *difference* of two zonoids. In fact there is a way to consider the group generated by differences of zonoids (or convex bodies) via the construction of the *Grothendieck group*, see [51].

Indeed, the set of centered zonoids $\mathscr{Z}_0(\mathbb{V})$ with the Minkowski sum forms a commutative monoid with the so called *cancellation rule*, that is if $A, B, C \in \mathscr{Z}_0(\mathbb{V})$ are such that $A + B = C + B$ then $A = C$. This implies that the monoid $\mathscr{Z}_0(\mathbb{V})$ embedds into a commutative group that we denote $\widehat{\mathscr{Z}_0}(\mathbb{V})$ that satisfies some universal property, see [51, p.39].

Concretely, each pair of zonoids $K, L \in \mathscr{Z}_0(\mathbb{V})$ gives rise to an element of the Grothendieck group that we denote by $K - L \in \widehat{\mathscr{Z}_0}(\mathbb{V})$ and is such that if $K', L' \in \mathscr{Z}_0(\mathbb{V})$ then $K - L = K' - L'$ in $\widehat{\mathscr{Z}_0}(\mathbb{V})$ if and only if $K + L' = K' + L$ in $\mathscr{Z}_0(\mathbb{V})$. The Grothendieck construction and the cancellation rule ensures that $\widehat{\mathscr{Z}_0}(\mathbb{V})$ is a well defined commutative group. To each zonoid $K \in \mathscr{Z}_0(\mathbb{V})$ corresponds the element $K - \{0\} \in \widehat{\mathscr{Z}_0}(\mathbb{V})$. By a slight abuse of notation, we will still write $K = K - \{0\}$ and we denote $-K := \{0\} - K$.

In our case we also have the multiplication by a nonegative scalar $\lambda \geq 0$ on $\mathscr{Z}_0(\mathbb{V})$. This operation carries on to the Grothendieck group by letting $\lambda(K - L) := \lambda K - \lambda L$. Moreover if we let $(-1)(K - L) := (L - K)$ we obtain a multiplication by all scalars that turns $\widehat{\mathscr{Z}_0}(\mathbb{V})$ into a vector space. Similarly one can construct the vector space $\widehat{\mathscr{K}_0}(\mathbb{V})$.

Definition 1.2.14. The vector space of *virtual zonoids* is denoted $\widehat{\mathscr{Z}_0}(\mathbb{V})$ and is a subspace of the *virtual symmetric bodies* denoted $\widehat{\mathscr{K}_0}(\mathbb{V})$. Moreover we define $\widehat{\mathscr{Z}}(\mathbb{V}) := \widehat{\mathscr{Z}_0}(\mathbb{V}) \oplus \mathbb{V}$ where the sum is intended as vector spaces.

Note that the expression $K - L \in \widehat{\mathscr{Z}}(\mathbb{V})$ with $K, L \in \mathscr{Z}_0(\mathbb{V})$ is not unique since for all $A \in \mathscr{Z}_0(\mathbb{V})$ we have $K - L = (K + A) - (L + A)$.

Remark 1.2.15. The subset $\mathscr{Z}_0(\mathbb{V}) \subset \widehat{\mathscr{Z}_0}(\mathbb{V})$ is a convex cone. Similarly for $\mathscr{K}_0(\mathbb{V}) \subset \widehat{\mathscr{K}_0}(\mathbb{V})$.

Remark 1.2.16. This abstract construction does not give a topology on the vector spaces. In fact two different topologies can be defined on $\widehat{\mathscr{Z}_0}(\mathbb{V})$ that coincide on $\mathscr{Z}_0(\mathbb{V})$ with the topology given by the Hausdorff distance. These will be discussed in the next sections.

1.2.2 Zonoids as average segment: Vitale's construction

Let us recall some elements of probability theory. If $\mathcal{M}$ is a measurable space, that is a set endowed with a σ-algebra, *a random element* X of $\mathcal{M}$ is a measurable map $X : \Omega \to \mathcal{M}$ from some probability space Ω, that is a measurable space endowed with a probability measure. The *law* of X is the push forward of this probability measure on $\mathcal{M}$. Probabilists are very rarely actually interested in the source space Ω and we will make no exception. We thus introduce some convenient notation that is adapted to this and was invented by Michele Stecconi.

Definition 1.2.17. If X is a random element of $\mathcal{M}$ a measurable space, we write $X \underline{\in} \mathcal{M}$.

If $\mathcal{M}$ is a topological space we always consider it endowed with the σ-algebra of the *Borelians* generated by the open sets. If $X \underline{\in} \mathcal{M}$ and $f : \mathcal{M} \to \mathbb{R}$ is a measurable map integrable with respect to the law of X, we write $\mathbb{E}f(X)$ for the integral of f with respect to the law of X. Finally we say that a property $\mathscr{P}$ holds *almost surely* if it happens with probability one, that is if the set $\{\omega \in \Omega \,|\, X(\omega) \text{ has property } \mathscr{P}\}$ has measure one with respect to the probability measure on Ω.

Let us now turn to the specific case of convex bodies.

Definition 1.2.18. Let $\Lambda \underline{\in} \mathscr{K}(\mathbb{V})$ be a random convex body such that $\mathbb{E}\|\Lambda\| < \infty$. Then we define $\mathbb{E}\Lambda \in \mathscr{K}(\mathbb{V})$ to be the convex body whose support function is given for all $u \in \mathbb{V}^*$ by

$$h_{\mathbb{E}\Lambda}(u) := \mathbb{E}h_\Lambda(u).$$

The fact that it indeed defines the support function of a convex body follows from the finiteness assumption and Proposition 1.1.16 and from the characterization of support functions in Proposition 1.1.10.

There is a strong law of large number for compact sets proved by Zvi Artstein and Richard A. Vitale in [11] that gives to this convex body a geometrical meaning. The following is [11, Theorem, p.880].

Proposition 1.2.19. *Let $\Lambda_1, \dots, \Lambda_N, \dots \underline{\in} \mathscr{K}(\mathbb{V})$ be independent identically distributed (iid) random convex bodies, then we have*

$$\frac{1}{N}(\Lambda_1 + \cdots + \Lambda_N) \xrightarrow[N\to\infty]{} \mathbb{E}\Lambda_1$$

almost surely.

Remark 1.2.20. Note that in the previous proposition we have a sequence of *random* convex bodies, but the limit object $\mathbb{E}\Lambda$ is *deterministic*.

Since the sum of zonoids is a zonoid and the space of zonoids is closed in the space of convex bodies we obtain the following.

Proposition 1.2.21. *Let $\Lambda \underline{\in} \mathscr{Z}(\mathbb{V})$ and $\Lambda_0 \underline{\in} \mathscr{Z}_0(\mathbb{V})$ be such that $\mathbb{E}\|\Lambda\|, \mathbb{E}\|\Lambda_0\| < \infty$, then $\mathbb{E}\Lambda \in \mathscr{Z}(\mathbb{V})$ and $\mathbb{E}\Lambda_0 \in \mathscr{Z}_0(\mathbb{V})$.*

In the following we will mainly consider two examples constructed from a random vector.

Definition 1.2.22. Let $X \underline{\in} \mathbb{V}$ be a random vector, we say that X is *integrable* if $\mathbb{E}\|X\| < \infty$. In such case, the (centered) zonoid $\mathbb{E}\underline{X} \in \mathscr{Z}_0(\mathbb{V})$ will be called the *Vitale zonoid* associated to X.

The support function of the Vitale zonoid associated to a random integrable vector $X\in\mathbb{V}$ is computed using Example 1.1.12. We obtain for all $u \in \mathbb{V}^*$

$$h_{\mathbb{E}\underline{X}}(u) = \frac{1}{2}\mathbb{E}|\langle u, X\rangle|. \tag{1.2.5}$$

Remark 1.2.23. Note that the Vitale zonoid depends only on the *law* of X.

From an integrable random vector one can also consider the zonoid $\mathbb{E}[0, X]$. It turns out that this is just a translate of the Vitale zonoid associated to X.

Proposition 1.2.24. *Let $X\in\mathbb{V}$ be integrable, we have*

$$\mathbb{E}[0, X] = \mathbb{E}\underline{X} + \tfrac{1}{2}\{\mathbb{E}X\}.$$

In particular $o(\mathbb{E}[0, X]) = \mathbb{E}X$.

Proof. From Example 1.1.12, we have for all $u \in \mathbb{V}^*$

$$h_{\mathbb{E}[0,X]}(u) = \mathbb{E}\{\max(0, \langle u, X\rangle)\}. \tag{1.2.6}$$

It is thus enough to note that for all $t \in \mathbb{R}$, we have $\max(0, t) = \frac{1}{2}(|t| + t)$ and use (1.2.5) and the fact that for all $x \in \mathbb{V}$, $h_{\{x\}} = \langle \cdot, x\rangle$. □

Remark 1.2.25. This implies that $\mathbb{E}[0, X] = \mathbb{E}\underline{X}$ if and only if $\mathbb{E}X = 0$ in particular this is the case if X is symmetric, that is if X and $-X$ have the same law.

Let us have a look at some examples.

Example 1.2.26. Let $x_1, \dots, x_N \in \mathbb{V}$ and let $X\in\mathbb{V}$ be the random vector that is equal to Nx_i with probability $\frac{1}{N}$. Then using (1.2.5) and (1.2.6) we obtain

$$\mathbb{E}\underline{X} = \sum_{i=1}^{N} \underline{x_i} \qquad\qquad \mathbb{E}[0, X] = \sum_{i=1}^{N} [0, x_i].$$

Moreover, Let $\tilde{X}\in\mathbb{V}$ be equal to x_i/p_i with probability p_i for any choice of $0 < p_i < 1$ such that $\sum_{i=1}^N p_i = 1$. Then we have again $\mathbb{E}\underline{\tilde{X}} = \mathbb{E}\underline{X}$ and $\mathbb{E}[0, \tilde{X}] = \mathbb{E}[0, X]$.

Example 1.2.27. Let $\xi\in\mathbb{V}$ be a standard Gaussian vector, that is the law of ξ admits the density given for all $x \in \mathbb{V}$ by $\frac{1}{(2\pi)^{m/2}} \exp(-\|x\|^2/2)$ (see Section 2.6.1). Then we have

$$\mathbb{E}\underline{X} = \frac{1}{\sqrt{2\pi}} B(\mathbb{V}).$$

Indeed for all $u \in \mathbb{V}^*$ with $\|u\| = 1$ the random variable $\langle u, \xi\rangle\in\mathbb{R}$ is a standard Gaussian variable and thus we have $\mathbb{E}|\langle u, \xi\rangle| = \sqrt{\frac{2}{\pi}}$ and the result follows from (1.2.5)

We see from Example 1.2.26 that the Vitale zonoid does not uniquely determine the integrable random vector. This defines an equivalence relation on the integrable random vectors of $\mathbb{V}$ known as the *zonoid equivalence* that was studied by Ilya Molchanov, Michael Schmutz, and Kaspar Stucki in [66].

Definition 1.2.28. We say that two integrable random vectors $X, Y\in\mathbb{V}$ are *zonoid equivalent* if $\mathbb{E}\underline{X} = \mathbb{E}\underline{Y}$.

A simple characterization of zonoid equivalence is proved in [66]. The following is [66, Theorem 2].

Proposition 1.2.29. *Let $X, Y\in\mathbb{V}$ be integrable. X is zonoid equivalent to Y if and only if for every measurable $f : \mathbb{V} \to \mathbb{R}$ that is nonnegative, positively homogeneous and even we have*

$$\mathbb{E}f(X) = \mathbb{E}f(Y).$$

The application $X \mapsto \mathbb{E}\underline{X}$ is thus not injective. However it was proved in [86, Theorem 3.2] that it is surjective, that is, every zonoid can be obtained as the Vitale zonoid associated to some random vector. Because of the importance of this result we include a proof of it.

Proposition 1.2.30. *Let $K \in \mathscr{Z}_0(\mathbb{V})$, there exists $X \in \mathbb{V}$ integrable such that $K = \mathbb{E}\underline{X}$*

Proof. If K is a zonotope then the random vector is given in Example 1.2.26. Else, there is a sequence of zonotopes K_i such that $K_i \to K$. Since K is centered, we can also assume that K_i is centered for all i and write $K_i = \underline{x_1^{(i)}} + \cdots + \underline{x_{N_i}^{(i)}}$. We define the random vector $\tilde{X}_i \in \mathbb{V}$ that is equal to $x_j^{(i)}/\|x_j^{(i)}\|$ with probability $p_j^{(i)} := \|x_j^{(i)}\|/\mathrm{V}_1(K_i)$. Recall from Remark 1.2.11 that $\mathrm{V}_1(K_i) = \sum_{j=1}^{N_i} \|x_j^{(i)}\|$ and thus $\sum_{j=1}^{N_i} p_j^{(i)} = 1$ and this indeed defines a probability.

Now we have $\tilde{K}_i := \mathbb{E}\underline{\tilde{X}_i} = \frac{1}{\mathrm{V}_1(K_i)} K_i$. Moreover the sequence of random vector $\tilde{X}_i$ is uniformly bounded. Thus , up to taking a subsequence, we can assume that $\tilde{X}_i$ converges weak-$*$ to some integrable random vector $\tilde{X} \in \mathbb{V}$, we let $\tilde{K} := \mathbb{E}\underline{\tilde{X}}$. Since $\tilde{X}_i$ converges weak-$*$ to $\tilde{X}$, it follows that $h_{\mathbb{E}\underline{\tilde{X}_i}}$ converges pointwise to $h_{\mathbb{E}\underline{\tilde{X}}}$ and thus, by Lemma 1.1.18, we have $\tilde{K}_i \to \tilde{K}$.

But by assumption and continuity of first intrisic volume we know that $\tilde{K}_i$ converges to $\frac{1}{\mathrm{V}_1(K)} K$ and thus $K = \mathrm{V}_1(K)\tilde{K} = \mathbb{E}\underline{X}$ with $X := \mathrm{V}_1(K)\tilde{X} \in \mathbb{V}$. $\square$

The characterization of the zonoid equivalence in Proposition 1.2.29 shows that the following is well defined.

Definition 1.2.31. Let $X \in \mathbb{V}$ be an integrable random vector, let $c \in \mathbb{V}$ and consider $K := \mathbb{E}\underline{X} + \frac{1}{2}\{c\}$. The *length* of K is defined to be

$$\ell(K) := \mathbb{E}\|X\|.$$

Following a similar idea as in the proof of Proposition 1.2.30, we can prove that the length is something we already encountered.

Proposition 1.2.32. *For all zonoid $K \in \mathscr{Z}(\mathbb{V})$ we have $\ell(K) = \mathrm{V}_1(K)$.*

Proof. We assume without loss of generality that K is centered. Let K_i be a sequence of zonotope with $K_i \to K$. Let $\tilde{X}_i$ be as in the proof of Proposition 1.2.30 in such a way that $\tilde{K}_i := \mathbb{E}\underline{\tilde{X}_i} = \frac{1}{\mathrm{V}_1(K_i)} K_i$ and $\tilde{X}_i$ converges weak-$*$ to $\tilde{X}$ with $K = \mathrm{V}_1(K)\mathbb{E}\underline{\tilde{X}}$. Note that $\|\tilde{X}_i\| = 1$ almost surely and thus $\|\tilde{X}\| = 1$ almost surely and in particular $\mathbb{E}\|\tilde{X}\| = 1$. Now $K = \mathbb{E}\underline{X}$ with $X = \mathrm{V}_1(K)\tilde{X}$ and thus $\mathbb{E}\|X\| = \mathrm{V}_1(K)$ which is what we wanted. $\square$

Since we will use the function *length* a lot on zonoids, we will continue to use this name and notation despite the equality just shown to emphasize that we think about Definition 1.2.31.

Next we see that the Vitale construction behaves well with linear transformation.

Proposition 1.2.33. *Let $\Lambda \in \mathscr{K}(\mathbb{V})$ be a random convex body with $\mathbb{E}\|\Lambda\| < \infty$ and let $T : \mathbb{V} \to \mathbb{W}$ be a linear map. Then $\mathbb{E}\|T(\Lambda)\| < \infty$ and we have*

$$\mathbb{E}T(\Lambda) = T(\mathbb{E}\Lambda).$$

In particular if $X \in \mathbb{V}$ is an integrable random vector

$$\mathbb{E}\underline{T(X)} = T(\mathbb{E}\underline{X}) \qquad\qquad \mathbb{E}[0, T(X)] = T(\mathbb{E}[0, X]).$$

Proof. The finiteness condition follows from the fact that $T(B(\mathbb{V})) \subset \|T\|_{op} B(\mathbb{W})$ where $\|\cdot\|_{op}$ is the operator norm. Next, by definition of $\mathbb{E}\Lambda$ and by Proposition 1.1.15-(iii), we have for all $u \in \mathbb{W}^*$, $h_{\mathbb{E}T(\Lambda)}(u) = \mathbb{E}h_{T(\Lambda)}(u) = \mathbb{E}h_\Lambda(T^t(u)) = h_{\mathbb{E}\Lambda}(T^t(u))$. Applying again Proposition 1.1.15-(iii) we get $h_{\mathbb{E}T(\Lambda)}(u) = h_{T(\mathbb{E}\Lambda)}(u)$ which is what we wanted. $\square$

Next we show how to obtain the Minkowski sum with Vitale's construction.

Proposition 1.2.34 (The Bernoulli trick)**.** *Let $X_0, X_1 \in \mathbb{V}$ be integrable and $\epsilon \in \{0,1\}$ be a Bernoulli random variable of parameter $0 \le t \le 1$ independent of the pair (X,Y), that is $\epsilon = 0$ with probability $(1-t)$ and equal to 1 with probability t. Let $X_t := (1-\epsilon)X_0 + \epsilon X_1$, then we have*

$$\mathbb{E}\underline{X_t} = (1-t)\mathbb{E}\underline{X_0} + t\mathbb{E}\underline{X_1}.$$

Proof. Writing down the support function, we have, using the independence assumption, for all $u \in \mathbb{V}^*$: $h_{\mathbb{E}\underline{X_t}}(u) = (1-t)h_{\mathbb{E}\underline{X_0}}(u) + th_{\mathbb{E}\underline{X_1}}(u)$ which is what we wanted. □

Remark 1.2.35. It can be useful at this point to emphasize that the operation of *taking the (centered) segment* is not linear, that is

$$\underline{x+y} \neq \underline{x} + \underline{y}$$

and thus $\mathbb{E}\underline{X+Y} \neq \mathbb{E}(\underline{X}+\underline{Y}) = \mathbb{E}\underline{X} + \mathbb{E}\underline{Y}$ in general. This remark may seem trivial since $\underline{x+y}$ is one dimensional and $\underline{x}+\underline{y}$ is (in general) of dimension 2, however, in the Vitale construction one could be tempted to assume linearity without thinking about it. If, while manipulating Vitale zonoids, you find yourself proving something that you think shouldn't be true, try to see if somewhere along the way you assumed that $\underline{x+y} = \underline{x} + \underline{y}$.

If $t \in \mathbb{R}$ and $x \in \mathbb{V}$ then we have $\underline{tx} = |t|\underline{x}$ and thus if $X \in \mathbb{V}$ is integrable we have $\mathbb{E}\underline{tX} = |t|\mathbb{E}\underline{X}$. This can be generalized in the following way.

Proposition 1.2.36. *Let $X \in \mathbb{V}$ be integrable and let $\rho \in \mathbb{R}$ be independent of X and integrable. Then we have*

$$\mathbb{E}\underline{\rho X} = \mathbb{E}|\rho|\, \mathbb{E}\underline{X}.$$

Proof. Let $u \in \mathbb{V}^*$, then $h_{\mathbb{E}\underline{\rho X}}(u) = \frac{1}{2}\mathbb{E}|\rho|\,|\langle u, X\rangle|$. Using the independence gives the result. □

We illustrate how this can help to compute the Vitale zonoid in the next example.

Example 1.2.37. Let $U \in S(\mathbb{V})$ be uniform on the unit sphere. Then we have

$$\mathbb{E}\underline{U} = \frac{1}{\sqrt{2\pi}\rho_m} B(\mathbb{V})$$

where $\rho_m = \mathbb{E}\|\xi\|$ with $\xi \in \mathbb{V}$ standard Gaussian vector, that is

$$\rho_m := \frac{\sqrt{2}\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} = \frac{m\kappa_m}{\sqrt{2\pi}\kappa_{m-1}}. \tag{1.2.7}$$

Indeed, ξ have the same law as $\|\xi\|U$ with ξ independent of U. As observed in Remark 1.2.23, the Vitale zonoid only depends on the law of the random vector and thus $\mathbb{E}\underline{\xi} = \mathbb{E}\underline{\|\xi\|U} = \mathbb{E}\|\xi\|\underline{U} = \rho_m \mathbb{E}\underline{U}$. The first term has been computed in Example 1.2.27 and it gives what we claimed.

Examples 1.2.27 and 1.2.37 gives another way to compute the length of a zonoid.

Proposition 1.2.38. *Let $K \in \mathscr{Z}(\mathbb{V})$, let $\xi \in \mathbb{V}^*$ be a standard Gaussian vector and let $U \in S(\mathbb{V}^*)$ be uniform on the unit sphere $S(\mathbb{V}^*)$. We have*

$$\ell(K) = \sqrt{2\pi}\mathbb{E}h_K(\xi) = \sqrt{2\pi}\rho_m\, \mathbb{E}h_K(U).$$

where the definition of ρ_m is given in (1.2.7)*.*

Proof. First of all, we can assume that K is centered since ξ is symmetric. Suppose now $X \in \mathbb{V}$ is independent of ξ and such that $K = \mathbb{E}\underline{X}$. then $\mathbb{E}h_K(\xi) = \frac{1}{2}\mathbb{E}|\langle \xi, X\rangle| = \mathbb{E}h_{\mathbb{E}\underline{\xi}}(X)$. By Example 1.2.27, we have $h_{\mathbb{E}\underline{\xi}}(X) = \frac{1}{\sqrt{2\pi}}\|X\|$ which gives the first equality. The second one is deduced similarly. □

Corollary 1.2.39. *The length is increasing with respect to inclusion, i.e. if $K, L \in \mathscr{Z}(\mathbb{V})$ are such that $K \subset L$ then $\ell(K) \le \ell(L)$.*

Proof. It is enough to apply Proposition 1.1.15-(*ii*) to the formula in Proposition 1.2.38. □

These provide an inequality between the length and the norm.

Proposition 1.2.40. *Let $K \in \mathscr{Z}_0(\mathbb{V})$, we have*

$$2\|K\| \ \leq \ \ell(K) \ \leq \sqrt{2\pi}\rho_m \|K\|,$$

with equality on the left hand side if and only if K is a (centrally symmetric) segment, and equality holding on the right hand side if and only if K is a ball.

Proof. Let $X \in \mathbb{V}$ be such that $K = \mathbb{E}\underline{X}$. Then by applying Cauchy-Schwartz to (1.2.5), we get for all $u \in S(\mathbb{V}^*)$, $h_K(u) \leq \frac{1}{2}\ell(K)$ and this proves the first inequality as well as the equality case. For the second inequality, we apply Proposition 1.1.16 and 1.2.38. The equality in Proposition 1.1.16 happens if and only if K is a ball. □

Let us conclude this paragraph by mentioning that there is a variant of the Vitale construction introduced by Karl Mosler called the *lift Zonoid*. Given a random vector $X \in \mathbb{V}$, consider $(1, X) \in \mathbb{R} \times \mathbb{V}$. Then the *lift zonoid* of X is $\mathbb{E}\underline{(1, X)} \in \mathscr{Z}_0(\mathbb{R} \times \mathbb{V})$. Unlike the Vitale zonoid, the lift zonoid characterizes the law of the random vector X, it has numerous applications in probability and statistics, see [67]

1.2.3 Zonoids as measures: the classical viewpoint

It is most common to approach centered zonoids with even measures on the sphere. This point of view is extensively used in [80] for example. We recall it and describe how this approach relates to Vitale's construction. In the following, we denote the space of even signed measures on the uit sphere $S(\mathbb{V})$ by $\mathcal{M}(S(\mathbb{V}))$ and the cone of non negative measures $\mathcal{M}^+(S(\mathbb{V}))$. Recall that $\mathcal{M}(S(\mathbb{V}))$ is dual to the space of even continuous functions on the sphere, that we denote by $C_{even}(S(\mathbb{V}))$. In accordance to this, we will write for every $\mu \in \mathcal{M}(S(\mathbb{V}))$ and for every $f \in C_{even}(S(\mathbb{V}))$:

$$\langle \mu, f \rangle := \int_{S(\mathbb{V})} f \mathrm{d}\mu.$$

Moreover, recall that the space $\mathcal{M}(S(\mathbb{V}))$ admits a topology called the *weak-$*$ topology* that is such that a sequence μ_n converges to μ weak-$*$ if and only if for every $f \in C_{even}(S(\mathbb{V}))$, we have $\langle \mu_n, f \rangle \to \langle \mu, f \rangle$.

The starting point is the following which is [80, Theorem 3.5.3].

Proposition 1.2.41. *For every centered zonoid $K \in \mathscr{Z}_0(\mathbb{V})$ there is a unique $\mu_K \in \mathcal{M}^+(S(\mathbb{V}))$ such that*

$$h_K(u) = \int_{S(\mathbb{V})} |\langle u, x \rangle| \ \mathrm{d}\mu_K(x). \tag{1.2.8}$$

Definition 1.2.42. Given a centered zonoid $K \in \mathscr{Z}_0(\mathbb{V})$ the measure μ_K is called the *generating measure* of K. If $X \in \mathbb{V}$ is integrable, we write $\mu_X := \mu_{\mathbb{E}\underline{X}}$.

The generating measure of a Vitale zonoid can be computed.

Proposition 1.2.43. *Let $X \in V$ be integrable, then μ_X is the measure such that for every continuous function $f : S(\mathbb{V}) \to \mathbb{R}$ we have*

$$\int_{S(\mathbb{V})} f \mathrm{d}\mu_X := \frac{1}{2}\mathbb{E}\left\{ \|X\| f\left(\frac{X}{\|X\|}\right) \mathbb{1}_{X \neq 0} \right\}$$

Proof. The function $x \mapsto \|x\| f\left(\frac{x}{\|x\|}\right) \mathbb{1}_{x \neq 0}$ is a one homogeneous continuous function on $\mathbb{V}$. Thus, by Proposition 1.2.29, the term on the right only depends on the zonoid $\mathbb{E}\underline{X}$. To see that it satisfies Proposition 1.2.41 apply it to $f = |\langle u, \cdot \rangle|$ for any $u \in \mathbb{V}^*$. □

One can build the zonoid whose generating measure is the surface area measure of a given convex body, see [80, (5.80)].

Definition 1.2.44. Let $K \in \mathscr{K}(\mathbb{V})$. The *projection body* of K is the zonoid $\Pi K \in \mathscr{Z}_0(\Lambda^{m-1}\mathbb{V})$ whose support function is given for all $w \in \Lambda^{m-1}\mathbb{V}^*$ by

$$h_{\Pi K}(w) = \frac{1}{2}\int_{S(\mathbb{V}^*)} |w \wedge u| \, \mathrm{d}\mathcal{S}_{m-1}(K)(u).$$

Similarly one can define the *mixed projection body*. Using the formulas for the volume involving the surface area measure, one finds for all $w \in S(\mathbb{V}^*)$ ([80, (5.80)])

$$h_{\Pi K}(w) = \mathrm{vol}_{m-1}(K|w^{\perp})$$

where $(K|w^{\perp})$ denotes the orthogonal projection of K onto $w^{\perp}$ identifying $\Lambda^{m-1}\mathbb{V}^* \cong \mathbb{V}$ with the volume form. It turns out that every zonoid is a projection body. This is called Minkowski's existence and uniqueness Theorem, see [80, Theorem 8.1.1 and 8.22 and Section 10.9]

Proposition 1.2.45. *For every centered zonoid $K \in \mathscr{Z}_0(\Lambda^{m-1}\mathbb{V})$ there is a unique $L \in \mathscr{K}_0(\mathbb{V})$ such that $K = \Pi L$.*

One can express the length (recall that this is how we call the first intrinsic volume see Definition 1.2.31) of a zonoid as the total mass of its generating measure.

Proposition 1.2.46. *For all $K \in \mathscr{Z}_0(\mathbb{V})$ we have*

$$\ell(K) = 2\mu_K(S(\mathbb{V})).$$

Proof. Let $X \in V$ integrable such that $K = \mathbb{E}\underline{X}$ then by Proposition 1.2.43 we have that

$$\mu_K(S(\mathbb{V})) = \int_{S(\mathbb{V})} 1 \, \mathrm{d}\mu_K = \frac{1}{2}\mathbb{E}\left\{\|X\| \, \mathbb{1}_{X\neq 0}\right\} = \frac{1}{2}\mathbb{E}\|X\|,$$

which is what we wanted. □

The main tool to consider the link between zonoids and measures is the following.

Definition 1.2.47. The *cosine transform* is the linear map $\mathrm{H} : \mathcal{M}(S(\mathbb{V})) \to C_{even}(S(\mathbb{V}^*))$ given for all $\mu \in \mathcal{M}(S(\mathbb{V}))$ and $u \in S(\mathbb{V}^*)$ by

$$\mathrm{H}(\mu)(u) := \int_{S(\mathbb{V})} |\langle u, x\rangle| \, \mathrm{d}\mu(x).$$

The image of the cosine transform will be denoted by $\mathrm{H}(\mathbb{V})$ and the image of $\mathcal{M}^+(S(\mathbb{V})))$ by $\mathrm{H}^+(\mathbb{V})$.

It was proven by Bolker in [24] that H is continuous on $\mathcal{M}^+(S(\mathbb{V})))$. In fact we will prove in the next section that the restriction of H to the non negative measures is a homeomorphism between $\mathcal{M}^+(S(\mathbb{V})))$ with the weak-$*$ topology and $\mathrm{H}^+(\mathbb{V})$ with the supremum norm. By Proposition 1.1.15-(*iv*), it implies the following.

Proposition 1.2.48. *Let $K \in \mathscr{Z}_0(\mathbb{V})$ and let $K_n \in \mathscr{Z}_0(\mathbb{V})$ be a sequence of centered zonoids. We have $K_n \to K$ if and only if $\mu_{K_n} \to \mu_K$ weak-$*$.*

With this point of view of the cosine transform, a (symmetric) convex body $K \in \mathscr{K}_0(\mathbb{V})$ is a zonoid if and only if its support function (restricted to the sphere) is in the image of the cosine transform $\mathrm{H}^+(\mathbb{V})$. When one considers instead *signed* measure the Hahn Jordan decomposition will allow us to extend this point of view.

Proposition 1.2.49 (Hahn–Jordan decomposition)**.** *For every $\mu \in \mathcal{M}(S(\mathbb{V}))$, there are unique non-negative measures $\mu_+, \mu_- \in \mathcal{M}^+(S(\mathbb{V}))$ such that $\mu = \mu_+ - \mu_-$ and such that $(\mu_+ + \mu_-)(S(\mathbb{V}))$ is minimal.*

Corollary 1.2.50. *Let $K \in \mathscr{K}_0(\mathbb{V})$, then K is a generalized zonoid if and only if there is a signed measure $\mu_K \in \mathcal{M}(S(\mathbb{V}))$ such that $H(\mu_K) = h_K|_{S(\mathbb{V}^*)}$.*

Proof. Suppose there is $A, B \in \mathscr{Z}_0(\mathbb{V})$ such that $K + A = B$. Then $h_K = h_B - h_A$, thus by linearity of the cosine transform $\mu_K = \mu_B - \mu_A$. Conversely if $H(\mu_K) = h_K|_{S(\mathbb{V}^*)}$ then, by the Hahn–Jordan decomposition, there are $\alpha, \beta \in \mathcal{M}^+(S(\mathbb{V}))$ such that $\mu_K = \beta - \alpha$. Then consider the zonoid A, respectively B, whose generating measure is α, respectively β. By the linearity of the cosine transform they satisfy $K + A = B$ and thus K is a generalized zonoid. □

Note that the Hahn–Jordan decomposition is unique, but the writing $\alpha = \mu - \nu$ for a signed measure α is not. In fact for all $\mu, \nu, \mu', \nu' \in \mathcal{M}(S(\mathbb{V}))$ we have $\mu - \nu = \mu' - \nu'$ if and only if $\mu + \nu' = \mu' + \nu$. By this simple remark we see that there is a bijection between the space of centered virtual zonoids $\widehat{\mathscr{Z}_0}(\mathbb{V})$ and the space of signed measure $\mathcal{M}(S(\mathbb{V}))$. This gives a first topology on $\widehat{\mathscr{Z}_0}(\mathbb{V})$.

Definition 1.2.51. The pull back by the bijection $\widehat{\mathscr{Z}_0}(\mathbb{V}) \to \mathcal{M}(S(\mathbb{V}))$, $K - L \mapsto \mu_K - \mu_L$ of the weak-$*$ topology on $\mathcal{M}(S(\mathbb{V}))$ will also be called the *weak-$*$ topology on* $\widehat{\mathscr{Z}_0}(\mathbb{V})$.

Remark 1.2.52. By Proposition 1.2.48, we see that the restriction of the weak-$*$ topology on $\widehat{\mathscr{Z}_0}(\mathbb{V})$ on the cone $\mathscr{Z}_0(\mathbb{V})$ coincides with the topology given by the Hausdorff distance.

1.2.4 Support functions of zonoids

In this section we detail the point of view of support functions. Let us recall that we denote by $C_{even}(S(\mathbb{V}^*))$ the space of continuous even functions on the unit sphere of $\mathbb{V}^*$. On this space we have the *supremum norm* given for all $f \in C_{even}(S(\mathbb{V}^*))$ by

$$\|f\|_\infty := \sup\{|f(u)| \mid u \in S(\mathbb{V}^*)\}.$$

As observed in Remark 1.1.17, we have an embedding

$$h_\cdot : \mathscr{K}_0(\mathbb{V}) \hookrightarrow C_{even}(S(\mathbb{V}^*)) \tag{1.2.9}$$

that sends $K \mapsto \bar{h}_K = h_K|_{S(\mathbb{V}^*)}$. By by Proposition 1.1.15-(*iv*) this is an isometry whose image is a convex cone and the image of $\mathscr{Z}_0(\mathbb{V})$ is also a cone contained and closed in the previous one.

This map extends to an injective map on virtual symmetric bodies (recall Definition 1.2.14) by mapping $K - L \in \widehat{\mathscr{K}_0}(\mathbb{V})$ to $(h_K - h_L)|_{S(\mathbb{V}^*)} \in C_{even}(S(\mathbb{V}^*))$. This allows to define a norm on $\widehat{\mathscr{K}_0}(\mathbb{V})$ and thus on $\widehat{\mathscr{Z}_0}(\mathbb{V})$.

Definition 1.2.53. Let $K - L \in \widehat{\mathscr{K}_0}(\mathbb{V})$. We define its *norm* to be

$$\|K - L\| := \mathrm{d}(K, L).$$

By Proposition 1.1.15-(*iv*) this makes the map $h_\cdot : \widehat{\mathscr{K}_0}(\mathbb{V}) \to C_{even}(S(\mathbb{V}^*))$ an isometry and, in particular, a homeomorphism on its image. Moreover, note that for all $K \in \mathscr{K}_0(\mathbb{V})$, $\|K\| = \|K - \{0\}\|$ coincides with the norm of convex bodies already defined in Definition 1.1.6.

In the previous section, we saw that a convex body is a zonoid if and only if its support function restricted to the sphere belongs to a certain subspace $\mathrm{H}^+(\mathbb{V})$ that is the image of the linear operator called the *cosine transform*, (Definition 1.2.47). In other words, the image of $\mathscr{Z}_0(\mathbb{V})$ by the embedding (1.2.9) is $\mathrm{H}^+(\mathbb{V})$ and the image of $\widehat{\mathscr{Z}_0}(\mathbb{V})$ is what we called $\mathrm{H}(\mathbb{V})$.

Thus the relation between the weak-$*$ topology on $\widehat{\mathscr{Z}_0}(\mathbb{V})$ defined in the previous section (Definition 1.2.51) and the topology given by the norm just defined is described by this operator: the cosine transform. We already announced (and we will prove it below) that these two topologies coincide on the cone $\mathscr{Z}_0(\mathbb{V})$, since they both coincide with the topology given by the Hausdorff distance. However, since the weak-$*$ topology on the space of measures is in general not metrizable, they must be different on the whole space $\widehat{\mathscr{Z}_0}(\mathbb{V})$.

We now prove some continuity result of the cosine transform. The space of signed measure $\mathcal{M}(S(\mathbb{V}))$ is considered endowed with the weak-$*$ topology (see previous section) and the space of even continous functions $C_{even}(S(\mathbb{V}^*))$ with the supremum norm. The following is similar to [26, Theorem 2.26].

Proposition 1.2.54. *The cosine transform* $\mathrm{H}\colon \mathcal{M}(S(\mathbb{V})) \to C_{even}(S(\mathbb{V}^*))$ *satisfies the following properties.*

(i) H *is injective.*

(ii) $\mathrm{H}(\mathbb{V})$ *is a dense subspace of* $C_{even}(S(\mathbb{V}^*))$*.*

(iii) There exists $c_m > 0$ *such that for all* $\mu \in \mathcal{M}^+(S(\mathbb{V}))$*,* $c_m\|\mathrm{H}(\mu)\|_\infty \leq \mu(S(\mathbb{V})) \leq \|\mathrm{H}(\mu)\|_\infty$*.*

(iv) H *is sequentially continuous.*

(v) The restriction $\mathrm{H}\colon \mathcal{M}^+(S(\mathbb{V})) \to \mathrm{H}^+(\mathbb{V})$ *is a homeomorphism.*

(vi) The inverse $\mathrm{H}^{-1} : \mathrm{H}(\mathbb{V}) \to \mathcal{M}(S(\mathbb{V}))$ *is not sequentially continuous for* $m > 1$*.*

Proof of Proposition 1.2.54. Assertions (i) and (ii) are in (the proof of) [80, Theorem 3.5.4] and item (iii) is just Proposition 1.2.40 restated in our context using the expression of the length as total mass proved in Proposition 1.2.46.

As for assertion (iv): because H is linear it suffices to prove sequential continuity at 0. Suppose that (μ_i) converges to 0 in $\mathcal{M}(S(\mathbb{V}))$ in the weak-$*$ topology. Let $h_i := \mathrm{H}(\mu_i)$, in such a way that $h_i(u) = \int_{S(\mathbb{V})} |\langle u, x\rangle|\, \mu_i(\mathrm{d}x)$, and in particular $h_i(u) \to 0$ for all $u \in S(\mathbb{V}^*)$. So we have pointwise convergence of the h_i. Since those are signed measure, h_i are not necessarily support functions and we cannot apply Lemma 1.1.18. We are going to show that $h_i \to 0$ uniformly. Recall that every measure μ_i has a unique Hahn–Jordan decomposition $\mu_i = \mu_{i,+} - \mu_{i,-}$ (see Proposition 1.2.49), where $\mu_{i,+}, \mu_{i,-} \in \mathcal{M}^+(S(\mathbb{V}))$. We define $|\mu_i| := \mu_{i,+} + \mu_{i,-}$. The Banach–Steinhaus Theorem (e.g., see [74]) implies that $\kappa := \sup_i(|\mu_i|(S(\mathbb{V}))) < \infty$. Therefore, we have for any $u \in S(\mathbb{V}^*)$,

$$|h_i(u)| \leq \int_{S(\mathbb{V})} |\langle u, x\rangle|\, \mathrm{d}|\mu_i|(x) \leq |\mu_i|(S(\mathbb{V})) \leq \kappa,$$

and hence $\sup_i \|h_i\|_\infty \leq \kappa$. Moreover, for $u_1, u_2 \in S(\mathbb{V}^*)$,

$$|h_i(v_1) - h_i(v_2)| \leq \int_{S(\mathbb{V})} |\langle v_1 - v_2, x\rangle|\, \mathrm{d}|\mu_i|(x) \leq \kappa\|v_1 - v_2\|.$$

The Arzelà-Ascoli Theorem (e.g., see [74]) implies that (h_i) has a uniformly convergent subsequence (h_{i_j}). Thus $h_{i_j} \to 0$ uniformly since $h_{i_j} \to 0$ pointwise. By the same argument we see that any subsequence of (h_i) has a subsequence that uniformly converges to 0. This implies that $h_i \to 0$ uniformly. Therefore, it follows that the map H is sequentially continuous.

For assertion ($\mathbb{V}$), Bolker [24, Theorem 5.2] showed that $\mathrm{H}\colon \mathcal{M}^+(S(\mathbb{V})) \to C_{even}(S(\mathbb{V}^*))$ is continuous. So we only need to show that the inverse $\mathrm{H}^{-1} : \mathrm{H}^+(\mathbb{V}) \to \mathcal{M}^+(S(\mathbb{V}))$ is continuous. For this it suffices to show that H^{-1} is sequentially continuous on $\mathrm{H}^+(\mathbb{V})$, because the topology on $\mathrm{H}^+(\mathbb{V})$ is given by a norm. We take a sequence $(h_i) \subset \mathrm{H}^+(\mathbb{V})$ that converges to $h \in \mathrm{H}^+(\mathbb{V})$. Let $(\mu_i) \subset \mathcal{M}^+(S(\mathbb{V}))$ be such that $\mathrm{H}(\mu_i) = h_i$ and let μ be a measure with $\mathrm{H}(\mu) = h$. We have to show that μ_i converges weak-$*$ to μ. For this, we fix $f \in C_{even}(S(\mathbb{V}^*))$ and show that $\langle \mu_i - \mu, f\rangle \to 0$. This would imply that $\mu_i - \mu \to 0$. Let $\varepsilon > 0$, by assertion (ii), there are $x_1, \ldots, x_N \in S(\mathbb{V})$ and $t_1, \ldots, t_N \in \mathbb{R}$ such that the function $g(u) := \sum_{k=1}^N t_k|\langle u, x_k\rangle|$ in $C_{even}(S(\mathbb{V}^*))$ satisfies $\|f - g\|_\infty < \varepsilon/(2c)$. We decompose

$$\langle \mu_i - \mu, f\rangle = \langle \mu_i, f - g\rangle + \langle \mu_i - \mu, g\rangle + \langle \mu, f - g\rangle. \tag{1.2.10}$$

The sequence of real numbers $\|h_i\|_\infty$ converges to $\|h\|_\infty$ and is thus bounded so that there is $c > 0$ such that $\sup_i \|h_i\|_\infty \leq c$ and $\|h\|_\infty \leq c$. An upper bound for the absolute value of third term in (1.2.10) is $|\langle \mu, f - g\rangle| = |\int_{S(\mathbb{V})}(f(x) - g(x))\, \mathrm{d}\mu(x)| \leq \mu(S(\mathbb{V}))\|f - g\|_\infty$ where we used the Euclidean structure to identify $\mathbb{V}^* \cong \mathbb{V}$ and the fact that μ is nonnegative. By assertion (iii) and by taking $c = c_m$, this is bounded by $c\|f - g\|_\infty < \varepsilon/2$. We get the same bound for the first term. The middle term equals $\sum_{k=1}^N t_k(h_i(x_k) - h(x_k))$ and, by the pointwise convergence already proven, converges to zero for $i \to \infty$. Therefore, $\limsup_i |\langle \mu_i - \mu, f\rangle| \leq \varepsilon$ which proves assertion ($\mathbb{V}$).

Assertion (vi) relies on the noncontinuity of the tensor product of zonoids that will be proven below, see [26, Theorem 2.26-(6)]. $\square$

We thus proved that there are two different topologies on the space of virtual zonoids that coincide on the cone of zonoids. It is unclear for now if there are others.

Open problem 1. Characterize the topologies on $\widehat{\mathscr{Z}_0}(\mathbb{V})$, respectively $\widehat{\mathscr{K}}(\mathbb{V})$, that coincide with the Hausdorff distance topology on $\mathscr{Z}_0(\mathbb{V})$, respectively $\mathscr{K}(\mathbb{V})$.

The zonoid problem

So far we saw several ways to *build* zonoids but we didn't see how to *characterize* zonoids. Indeed one could ask: given a convex body $K \subset \mathbb{V}$, how can I recognize if it's a zonoid or not? We already observed that the first necessary condition is that K is centrally symmetric. To determine further characterization is in general a difficult problem and remains, in its whole generality, open.

Open problem 2 (The zonoid problem). Is there an algorithm that, given $K \in \mathscr{K}_0(\mathbb{R}^m)$, determines if $K \in \mathscr{Z}_0(\mathbb{R}^m)$?

Of course the answer to this question depends on a definition of an *algorithm*. Let us give some partial answers. The first is that, as was observed in Corollary 1.2.5, this question is trivial in dimension 2 and all centrally symmetric convex bodies are zonoids. When $m = \dim(\mathbb{V}) > 2$ however, the inclusion $\mathscr{Z}_0(\mathbb{V}) \subset \mathscr{K}_0(\mathbb{V})$ is strict. Indeed since a polytope is a zonoid if and only if it is a zonotope ([80, Corollary 3.5.7]) then Proposition 1.2.4 provides examples of centrally symmetric polytopes that are not zonoids.

Example 1.2.55. The octahedron in $\mathbb{R}^3$ is in $\mathscr{K}_0(\mathbb{R}^3) \setminus \mathscr{Z}_0(\mathbb{R}^3)$.

This is not specific to polytopes and for example, Rolf Schneider in [80, p.203] provides a one dimensional family of curved convex bodies in $\mathscr{K}_0(\mathbb{R}^3) \setminus \mathscr{Z}_0(\mathbb{R}^3)$.

Wolfgang Weil proved in [88] that an answer to the zonoid problem cannot be *strictly local*. More precisely he proved the following, see also [80, p.204].

Proposition 1.2.56. *For all $m > 2$, there exists a convex body $K \in \mathscr{K}_0(\mathbb{R}^m) \setminus \mathscr{Z}_0(\mathbb{R}^m)$, arbitrarily smooth, that has the following property. For each $u \in S^{m-1}$ there exist a zonoid $Z \in \mathscr{Z}_0(\mathbb{R}^m)$ and a neighbourhood U of u in S^{m-1} such that the boundaries of K and Z coincide at all points where the exterior unit normal vector belongs to U.*

Weil conjectured a weaker characterization along great spheres that turned out to be true if and only if the dimension is even. For a little *tour d'horizon* of zonoid characterization, see [80, p.204 note 9].

An equivalent formulation of the zonoid problem would be the following: given an even function $h : S(\mathbb{V}^*) \to \mathbb{R}$, that is a support function of a convex body, is there an algorithm to determines if $h \in \mathrm{H}^+(\mathbb{V})$? Weil's result shows that it cannot be answered by means of only local quantities such as derivatives.

This is in contrast to the case of convex bodies in general. Indeed if $h \in C_{even}(S(\mathbb{V}^*))$ is C^2, then the condition of h being the support function of a convex body can be characterized by a positivity condition of its Hessian at each point which is a local condition.

We mention one last characterization that is due to Yossi Lonke in [58] and that will be used in the next section. Let us first make a definition.

Definition 1.2.57. Let $e \in S^{m-1}$, we define $O(e) \subset O(m)$ to be the subgroup of the orthogonal group of $\mathbb{R}^m$ that fixes e, i.e. $O(e) = \{g \in O(m) \,|\, g(e) = e\} \cong O(m-1)$. Moreover, we say that $g \in O(e)$ is *uniform* if its law is the normalized Haar measure on the compact Lie group $O(e)$. If $K \in \mathscr{K}(\mathbb{R}^m)$, we define for each $e \in S^{m-1}$

$$\mathcal{S}_e K := \mathbb{E} g_e(K)$$

where $g_e \in O(e)$ uniform.

Note that $\mathcal{S}_e K$ is a solid of revolution, that is, for all $g \in O(e)$, $g(\mathcal{S}_e K) = \mathcal{S}_e K$. The following is [58, Theorem 1].

Proposition 1.2.58 (Lonke's criterion)**.** *Let $K \in \mathscr{K}_0(\mathbb{R}^m)$, then $K \in \mathscr{Z}_0(\mathbb{R}^m)$ if and only if for every $e \in S^{m-1}$ we have*

$$\mathcal{S}_e K \in \mathscr{Z}_0(\mathbb{R}^m)$$

In other words, $h_K \in \mathrm{H}^+(\mathbb{R}^m)$ if and only if for all $e \in S^{m-1}$, $\mathbb{E}(h_K \circ g_e^t) \in \mathrm{H}^+(\mathbb{R}^m)$ with $g_e \in O(e)$ uniform.

This characterization is nice and explicit but is this *an algorithm*? Some people would argue that the only conditions that can be tested by an algorithm are the *semi algebraic* ones. In the next section we prove that the condition of being a zonoid is definable in an o-minimal structure larger that the semi algebraic one.

1.3 Tameness of zonoids

In this section we examine how *nice* the condition "being a zonoid" is in $\mathbb{R}^m$. This is a joint work with Antonio Lerario and, in particular, this section will be a shortened and introductory version of [55]. We invite the reader to refer to it for more details.

1.3.1 o-minimal structures

Polynomials are a *finite* data and thus easier to handle for a mathematician or a computer. Thus the subsets defined by polynomial equations, namely the *algebraic* sets are easier to study. Unfortunately, this class of sets is not stable by projection. One has also to consider the following.

Definition 1.3.1. A subset $X \subset \mathbb{R}^m$ is called *basic semialgebraic* if it is defined by a finite number of polynomial equalities and inequalities, i.e. if there are $P_1, \dots, P_a, Q_1, \dots, Q_b \in \mathbb{R}[x_1, \dots, x_m]$ such that $X = \{x \in \mathbb{R}^m \,|\, P_i(x) = 0,\ 1 \le i \le a \text{ and } Q_j(x) \ge 0,\ 1 \le j \le b\}$. A set is *semialgebraic* if it is a boolean combination (i.e. finite intersection, union and complementary) of basic semialgebraic sets. Finally, if X, Y are semialgebraic, a function $f : X \to Y$ is said to be *semialgebraic* if its graph is semialgebraic in $X \times Y$.

The first observation is that, in dimension 1, the semialgebraic sets of $\mathbb{R}$ are finite unions of points and intervals (possibly unbounded). In higher dimension, semialgebraic sets still have many finiteness properties such as bounded number of connected components or bounded homology. Moreover, projections of semialgebraic sets are semialgebraic and, finally, all first order formulas involving semialgebraic objects define semialgebraic sets (this means that every set that you define with quantifiers and semialgebraic sets and functions is again semialgebraic).

With this collection of nice properties (and many more), working in the *semialgebraic category* often makes proofs easier. However, they are not stable with integration, i.e. the partial integration of a semialgebraic function is not necessarily semialgebraic. This makes it difficult to work with conditions such as Proposition 1.2.58.

One can generalize the notion of semialgebraicity and give a list of axioms for collections of subsets in order to have similar properties.

Definition 1.3.2. An o-minimal structure is a collection $\mathscr{O}_m$ of subsets of $\mathbb{R}^m$ for all $m \ge 0$ such that

1. $\mathscr{O}_m$ is a *boolean algebra* of $\mathbb{R}^m$, i.e. $\emptyset, \mathbb{R}^m \in \mathscr{O}_m$ and if $A, B \in \mathscr{O}_m$ then so does $A \cap B$ and $A \cup B$;
2. if $A \in \mathscr{O}_m$ then $\mathbb{R} \times A, A \times \mathbb{R} \in \mathscr{O}_{m+1}$;
3. we have for all $m \ge 1$, $\{(x_1, \dots, x_m) \in \mathbb{R}^m \,|\, x_1 = x_m\} \in \mathscr{O}_m$;
4. if $A \in \mathscr{O}_{m+1}$ and $\pi : \mathbb{R}^{m+1} \to \mathbb{R}^m$ is the projection on the first m coordinates, then $\pi(A) \in \mathscr{O}_m$;
5. we have $\left\{(x, y) \in \mathbb{R}^2 \,|\, x < y\right\} \in \mathscr{O}_2$;
6. the sets in $\mathscr{O}_1$ are precisely the finite union of points and intervals.

A set $A \in \mathscr{O}_m$ is said to be *definable* in the o-minimal structure. A function $f : A \to B$ between two definable sets is called *definable* if its graph is definable in $A \times B$.

It turns out that sets that are *definable* in some o-minimal structure share many properties with the semialgebraic sets. This is why definable sets of functions are also called *tame*, as opposed to wild.

A subset of $\mathbb{R}^m$ is called *semianalytic* if it is locally defined by a finite number of analytic equalities and inequalities. This does not define an o-minimal structure since this is not stable by projections. One can then consider these together with the projections of semianalytic sets to form the *subanalytic* sets and functions. Finally, *globally subanalytic functions* are functions that are subanalytic in the ambient projective space and it gives a well defined an o-minimal structure, see [85] or [16] for a more rigorous definition.

Definition 1.3.3. We denote by $\mathbb{R}_{\mathrm{an}}$ the o-minimal structure generated by the globally subanalytic sets and by $\mathbb{R}_{\mathrm{an,exp}}$ the o-minimal structure generated by the globally subanalytic sets together with the graph of the exponential function.

If P is a globally subanalytic set, following [31], we denote by $\mathscr{C}(P)$ the $\mathbb{R}$-algebra of real valued functions generated by all globally subanalytic functions on X and all the functions of the form $x \mapsto \log f(x)$, where $f : X \to (0, \infty)$ is globally subanalytic. A function in $\mathscr{C}(X)$ is called a *constructible* function. Notice that functions definable in $\mathbb{R}_{\mathrm{an}}$ are constructible and that constructible functions are definable in $\mathbb{R}_{\mathrm{an,exp}}$.

In the sequel we will simply say that a set or a function definable in $\mathbb{R}_{\mathrm{an}}$ is *subanalytic* (omitting the word "global").

We will use the following crucial result [31, Theorem 1.3], see also [32] and [57].

Proposition 1.3.4. *Let P be subanalytic and $F \in \mathscr{C}(P \times \mathbb{R}^m)$. Suppose that for all $p \in P$ the function $F(p, \cdot) : \mathbb{R}^m \to \mathbb{R}$ is integrable. Then the the function $I(F) : P \to \mathbb{R}$ defined by $I(F)(p) := \int_{\mathbb{R}^m} F(p, x)\mathrm{d}x$ is constructible, and in particular definable in $\mathbb{R}_{\mathrm{an,exp}}$.*

Remark 1.3.5. In the case $F : P \times \mathbb{R}^m \to \mathbb{R}^m$ is semialgebraic, then the parametrized integral function $I(F)$ is definable in a structure strictly smaller than $\mathbb{R}_{\mathrm{an,exp}}$, see [47].

Let us make the following definition.

Definition 1.3.6. For every continuous $h : \mathbb{R}^m \to \mathbb{R}$ and $e \in S^{m-1}$ we define $\mathcal{S}_e h := \mathbb{E}(h \circ g_e^t)$ where $g_e \in O(e)$ uniform.

This definition is made in such a way that for all $K \in \mathscr{K}_0(\mathbb{R}^m)$ and $e \in S^{m-1}$ we have $h_{\mathcal{S}_e K} = \mathcal{S}_e h_K$.

The following result will allow us to use Lonke's criterion.

Corollary 1.3.7. *If $h : P \times S^{m-1} \to \mathbb{R}$ is constructible, the function $(e, p, u) \mapsto \mathcal{S}_e h(p, u)$ is also constructible.*

Proof. Consider a subanalytic function $F : S^{m-1} \times O(m-1) \to O(m)$ such that for almost all $e \in S^{m-1}$ the function $F(e, \cdot)$ is a subanalytic isomorphism between $O(m-1)$ and $O(e)$. (Since we are only requiring that F is definable, such function can also be defined piecewise.) Then we can write:

$$\mathcal{S}_e h(p, u) = \int_{O(m-1)} h(p, (F(e, \tilde{g})(u)) \,|\det JF(e, \cdot)| d\tilde{g},$$

where $d\tilde{g}$ is the normalized Haar measure on $O(m-1)$. Since the integrand is constructible, the result follows by applying Theorem 1.3.4 after noticing that there is a diffeomorphism, definable in $\mathbb{R}_{\mathrm{an}}$, between an open dense of subset of $O(m-1)$ and $\mathbb{R}^m$, $m = n(n-1)/2$ (for instance one can take the restriction of the Riemannian exponential map at the identity on an appropriate subanalytic domain). □

1.3.2 Tame families of convex bodies and zonoids

Definition 1.3.8. Let P be a subanalytic set. A subanalytic family of convex bodies in $\mathbb{R}^m$ is a subanalytic set $T \subset P \times \mathbb{R}^m$ such that for every $p \in P$ the set

$$K_p := \{x \in \mathbb{R}^m \,|\, (p, x) \in T\}$$

is a convex body. For $p \in P$, we will denote by h_p the support function of K_p (instead of h_{K_p}).

If $T \subset P \times \mathbb{R}^m$ is a subanalytic family of convex bodies, the function $H : T \to \mathbb{R}$, given by $H(p,u) = h_p(u)$, is subanalytic. Moreover, if P and $T \subseteq P \times \mathbb{R}^m$ are subanalytic, denoting by $T_p := \{x \in \mathbb{R}^m \,|\, (p,x) \in T\}$, it is immediate to see that the following sets are subanalytic:

(i) $\mathscr{K}(P) := \{p \in P \,|\, T_p \text{ is a convex body}\}$;

(ii) $\mathscr{K}_0(P) := \{p \in P \,|\, T_p \text{ is a centrally symmetric convex body, centered at the origin}\}$.

Theorem 1.3.9 (Lerario–M)**.** *Let P be a subanalytic set and let $\{K_p \,|\, p \in P\}$ be a subanalytic family of convex bodies in $\mathbb{R}^m$. Then the set $\mathscr{Z}_0(P) := \{p \in P \,|\, K_p \text{ is a zonoid}\}$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$.*

Idea of the proof. Using the notation introduced in the previous section, one needs to show that the set $\{p \in P \,|\, h_p \in \mathrm{H}^+(\mathbb{R}^m)\}$ is definable. It turns out that one can always invert the cosine transform of definable function in a *distributional sense*, so one has to show that the condition "$H^{-1}(h_p)$ is a nonnegative measure" is definable among the family of distributions $\mathrm{H}^{-1}(h_p)$.

The first step is to reduce the problem to a one dimensional one using Lonke's criterion. Indeed fixing $e \in S^{m-1}$ the function $\mathcal{S}_e h(u)$ is a function of $\langle u, e\rangle$.

The second ingredient is an expression of the inverse of the cosine transform using another integral operator called the *Radon transform.* For smooth functions this allows to express the inverse of the cosine transform as a differential operator.

Then one considers that subanalytic functions are piecewise smooth and the pieces are definable. Thus in dimension one a differential operator in the sense of distribution has a simple form: where the function is smooth it is the classical differentiation, at points where it is not smooth there are some *jumps* that produce deltas and derivative of deltas.

Since the points where a subanalytic function is not smooth form a definable sets, one shows that "being a nonnegative measure" at those points is a definable condition. □

Chapter 2

Zonoid calculus

In this chapter, we introduce explain and illustrate *zonoid calculus*. Concretely this is a recipe to build multilinear maps on the spaces of zonoids from multilinear maps on the underlying vector spaces. We call this recipe the *Fundamental Theorem of Zonoid Calculus* (FTZC), see Theorem 2.1.16 below. It is based on a tensor product of zonoids that we define in Definition 2.1.3.

We then give a particular focus on the bilinear map given by the wedge product. Using this, we build the *zonoid algebra* and *Grassmannian algebra* in Section 2.2. We investigate what happens in the case of a complex vector space and construct a new function on zonoids that we call *mixed J-volume* in Section 2.3. Finally we investigate some operations on the zonoid algebra that will be useful later in Section 2.4 and we illustrate the zonoid algebra with two detailed examples in Section 2.5 and Section 2.6.

Once again, in this chapter, $\mathbb{V}$ will denote a real vector space of dimension $m < \infty$. Sections 2.1, 2.2, 2.2.3 and 2.3 are a joint work with Peter Bürgisser, Paul Breiding and Antonio Lerario (abbreviated B.B.L.M).

2.1 FTZC

Consider a linear map $T : \mathbb{V} \to \mathbb{W}$. As was observed previously, the image of a convex body in $\mathbb{V}$ is a convex body in $\mathbb{W}$, in other words, T induces a map $\widehat{T} : \mathscr{K}(\mathbb{V}) \to \mathscr{K}(\mathbb{W})$. This map satisfies some properties collected in the next result.

Proposition 2.1.1. *Let $T : \mathbb{V} \to \mathbb{W}$ be a linear map and let $\widehat{T} : \mathscr{K}(\mathbb{V}) \to \mathscr{K}(\mathbb{W})$ the induced map on convex bodies, that is for all $K \in \mathscr{K}(\mathbb{V})$, $\widehat{T}(K) = \{T(x) \,|\, x \in K\}$. The following are satisfied.*

(i) the map $\widehat{T}$ is linear, i.e for all $K, L \in \mathscr{K}(\mathbb{V})$ and $t \geq 0$, we have $\widehat{T}(tK + L) = t\widehat{T}(K) + \widehat{T}(L)$;

(ii) the map $\widehat{T}$ is continuous;

(iii) for all $x \in \mathbb{V}$, we have $\widehat{T}(\underline{x}) = \underline{T(x)}$ and $\widehat{T}([0, x]) = [0, T(x)]$;

(iv) the map $\widehat{T}$ is increasing with respect to inclusion, that is if $K \subset L$ then $\widehat{T}(K) \subset \widehat{T}(L)$.

(v) the map $\widehat{T}$ extends uniquely to a linear map on virtual convex bodies that we also denote by $\widehat{T} : \widehat{\mathscr{K}}(\mathbb{V}) \to \widehat{\mathscr{K}}(\mathbb{W})$ and this extension is continuous.

Proof. Item (i) and (ii) are a consequence of Proposition 1.1.15-(iii). Item (iii) and (iv) follow from the definition of the map $\widehat{T}$. As for item (v), the linear extension is defined as $\widehat{T}(K-L) := \widehat{T}(K)-\widehat{T}(L)$. One can check that this is well defined. Continuity of this map follows again from Proposition 1.1.15-(iii). $\square$

Consider now a multilinear map $M : \mathbb{V}_1 \times \cdots \times \mathbb{V}_k \to \mathbb{W}$. One would like to have a similar result and be able to produce a map $\widehat{M} : \mathscr{K}(\mathbb{V}_1) \times \cdots \times \mathscr{K}(\mathbb{V}_k) \to \mathscr{K}(\mathbb{W})$ that is multilinear, continuous and maybe also satisfies the other properties of Proposition 2.1.1.

Unfortunately a pointwise definition as before would not work here. In particular, the multilinear image of convex bodies need not to be convex.

However, restricting to zonoids one can define such a map, this is the main result on this section that we call the *fundamental theorem of zonoid calculus* (FTZC), Theorem 2.1.16. It relies on a notion of tensor product of zonoids which already appeared in a work of Guillaume Aubrun and Cécilia Lancien [13, Definition 3.2].

2.1.1 Tensor product of zonoids

In order to define the tensor product in terms of Vitale zonoids we need the following lemma.

Lemma 2.1.2. *Let $X, X' \in V$ and $Y, Y' \in \mathbb{W}$ be integrable random vectors such that (X, X') is independent of (Y, Y'). We have: if X is zonoid equivalent to X' and Y zonoid equivalent to Y' then $X \otimes Y$ is zonoid equivalent to $X' \otimes Y'$.*

Proof. Fixing $y \in \mathbb{W}$, consider the linear map $\tau_y : \mathbb{V} \to \mathbb{V} \otimes \mathbb{W}$ defined by $\tau_y(x) := x \otimes y$. Then, for all $u \in \mathbb{V}^* \otimes \mathbb{W}^*$, we have

$$h_{\mathbb{E}\underline{X \otimes Y}}(u) = \tfrac{1}{2}\mathbb{E}|\langle X, (\tau_Y)^t(u)\rangle| = \mathbb{E}\left[h_{\mathbb{E}\underline{X}}\left((\tau_Y)^t(u)\right)\right]. \tag{2.1.1}$$

where in the second equality we used the independence of X and Y and where in the last term, the expectation is on the random vector Y. This shows that the zonoid $\mathbb{E}\underline{X \otimes Y}$ only depends on $\mathbb{E}\underline{X}$ and not on the random vector. A symmetric argument shows the same for Y and we get the result. □

This shows that the following is well defined.

Definition 2.1.3. The tensor product of zonoids

$$\cdot \otimes \cdot : \mathscr{Z}(\mathbb{V}) \times \mathscr{Z}(\mathbb{W}) \to \mathscr{Z}(\mathbb{V} \otimes \mathbb{W})$$

is defined for all $X \in \mathbb{V}$ and $Y \in \mathbb{W}$ integrable and *independent* by

$$\mathbb{E}\underline{X} \otimes \mathbb{E}\underline{Y} := \mathbb{E}\underline{X \otimes Y}$$

and for all $K \in \mathscr{Z}_0(\mathbb{V})$ and $L \in \mathscr{Z}_0(\mathbb{W})$, and all $x \in \mathbb{V}, y \in \mathbb{W}$ by

$$\left(K + \tfrac{1}{2}\{x\}\right) \otimes \left(L + \tfrac{1}{2}\{y\}\right) := K \otimes L + \tfrac{1}{2}\{x \otimes y\}.$$

Remark 2.1.4. Notice that in the proof of the previous lemma, more precisely in (2.1.1) we have proved that for every centered zonoid $K \in \mathscr{Z}_0(\mathbb{V})$ and every integrable $\mathbb{E}\underline{Y}$ we have for all $u \in \mathbb{V}^* \otimes \mathbb{W}^*$:

$$h_{K \otimes \mathbb{E}\underline{Y}}(u) = \mathbb{E}h_K\left((\tau_Y)^t(u)\right) \tag{2.1.2}$$

where recall that for all $y \in \mathbb{W}$ and all $x \in \mathbb{V}$, we define $\tau_y(x) = x \otimes y$. More precisely, in our case, we get $(\tau_y)^t = \mathrm{Id}_{\mathbb{V}^*} \otimes \langle \cdot, y\rangle : \mathbb{V}^* \otimes \mathbb{W}^* \to \mathbb{V}^*$. Equivalently, one could write

$$K \otimes \mathbb{E}\underline{Y} = \mathbb{E}\widehat{\tau_Y}(K).$$

Notice that for this formula makes sense even if K is not a zonoid and this could be a definition of a tensor product of a general convex body with a centered zonoid.

Let us give right away some examples.

Example 2.1.5. Let $x \in \mathbb{V}$ and $y \in \mathbb{W}$, we have

$$\underline{x} \otimes \underline{y} = \underline{x \otimes y}, \qquad\qquad [0, x] \otimes [0, y] = [0, x \otimes y].$$

Indeed, the centered case is done using the fact that $\underline{x} = \mathbb{E}\underline{X}$ where $X = x$ with probability one. For the non centered segment, one compute it using that $[0, x] = \underline{x} + \frac{1}{2}\{x\}$.

Similarly one can prove the following.

Proposition 2.1.6. *Let $X\in\mathbb{V}$ and $Y\in\mathbb{W}$ be independent and integrable. We have*

$$\mathbb{E}[0,X]\otimes\mathbb{E}[0,Y]=\mathbb{E}[0,X\otimes Y]$$

Proof. This is a straightforward computation using the fact that $\mathbb{E}[0,X]=\mathbb{E}\underline{X}+\frac{1}{2}\{\mathbb{E}X\}$ (Proposition 1.2.24). □

Example 2.1.7. Let $a,b\in\mathbb{N}$, we define the *Segre Zonoid* to be the tensor $B(\mathbb{R}^a)\otimes B(\mathbb{R}^b)$. If we identify the space of matrices $\mathbb{R}^a\otimes\mathbb{R}^b$ to the space of linear maps $M:\mathbb{R}^b\to\mathbb{R}^a$, then the support function of the Segre zonoid can be expressed as the first intrinsic volume of the ellipsoid defined by the matrix, more precisely for all $M\in\mathbb{R}^a\otimes\mathbb{R}^b$ we have:

$$h_{B(\mathbb{R}^a)\otimes B(\mathbb{R}^b)}(M)=\ell\left(M(B(\mathbb{R}^b)\right)$$

Indeed, let us first notice that in this identification, given $y\in\mathbb{R}^b$ and $M\in\mathbb{R}^a\otimes\mathbb{R}^b$ we have that $(\tau_y)^t(M)=M(y)$, where recall the expression of $(\tau_y)^t$ in Remark 2.1.4. Let $Y\in\mathbb{R}^b$ be integrable such that $B(\mathbb{R}^b)=\mathbb{E}\underline{Y}$. Then, from (2.1.2) we get

$$h_{B(\mathbb{R}^a)\otimes B(\mathbb{R}^b)}(M)=\mathbb{E}\|M(Y)\|.$$

We recognize on the right hand side the length of $\mathbb{E}\underline{M(Y)}=M(\mathbb{E}\underline{Y})=M(B(\mathbb{R}^b))$ which proves the claim.

This tensor product satisfies at least some of the desired properties.

Proposition 2.1.8. *The tensor product of zonoids is associative, it is linear and positively homogeneous in each variable. Moreover, on centered zonoids, the tensor product is monotonically increasing in each variable; that is, if $K_1\subset K_2$ and $L_1\subset L_2$ then $K_1\otimes L_1\subset K_2\otimes L_2$.*

Proof. The associativity and homogeneity follow from the definition and the associativity and homogeneity of the classical tensor product. For linearity, we use the Bernoulli trick (Proposition 1.2.34). Indeed let $X_0,X_1\in\mathbb{V}$, $Y\in\mathbb{W}$ independent of X,X' and let $\epsilon\in\{0,1\}$ be a Bernoulli variable of parameter $1/2$ independent of X,X' and Y. Then by Proposition 1.2.34, if $Z=(1-\epsilon)2X_0+\epsilon 2X_1$ we then have $\mathbb{E}\underline{Z}=\mathbb{E}\underline{X_0}+\mathbb{E}\underline{X_1}$. Moreover, $Z\otimes Y=(1-\epsilon)2X_0\otimes Y+\epsilon 2X_1\otimes Y$ so, by the same argument, $\mathbb{E}\underline{Z\otimes Y}=\mathbb{E}\underline{X_0\otimes Y}+\mathbb{E}\underline{X_1\otimes Y}=\mathbb{E}\underline{X_0}\otimes\mathbb{E}\underline{Y}+\mathbb{E}\underline{X_1}\otimes\mathbb{E}\underline{Y}$. But by definition of the tensor product and by the independence of the variables, $\mathbb{E}\underline{Z\otimes Y}=\mathbb{E}\underline{Z}\otimes\mathbb{E}\underline{Y}=(\mathbb{E}\underline{X_0}+\mathbb{E}\underline{X_1})\otimes\mathbb{E}\underline{Y}$. This shows linearity in the first variable, the same argument shows linearity in the second variable. Monotonicity is a consequence of (2.1.2) and Proposition 1.1.15-(*ii*). □

Example 2.1.9. Let $x_1,\dots,x_N\in\mathbb{V}$ and $y_1,\dots,y_M\in\mathbb{W}$, we have

$$\left(\sum_{i=1}^N\underline{x_i}\right)\otimes\left(\sum_{i=1}^M\underline{y_j}\right)=\sum_{i=1}^N\sum_{j=1}^M\underline{x_i\otimes y_j},$$
$$\left(\sum_{i=1}^N[0,x_i]\right)\otimes\left(\sum_{i=1}^M[0,y_j]\right)=\sum_{i=1}^N\sum_{j=1}^M[0,x_i\otimes y_j].$$

Indeed, this is a conscequence of the multilinearity and Example 2.1.5.

We now show how the length and norm behave well under the tensor product.

Proposition 2.1.10. *For all $K\in\mathscr{Z}(\mathbb{V})$ and $L\in\mathscr{Z}(\mathbb{W})$ we have*

$$\ell(K\otimes L)=\ell(K)\ell(L).$$

Moreover, if K and L are centered then

$$\|K\otimes L\|\leq 2\sqrt{m}\,\|K\|\|L\|.$$

Proof. Since the length is translation invariant, we can suppose for the first assertion that $K = \mathbb{E}\underline{X}$ and $L = \mathbb{E}\underline{Y}$ with independent random vector $X \in \mathbb{V}$ and $Y \in \mathbb{W}$. Then, by Definition 1.2.31, $\ell(K \otimes L) = \mathbb{E}\|X \otimes Y\| = \mathbb{E}\|X\|\,\mathbb{E}\|Y\| = \ell(K)\ell(L)$, which is what we want.

For the norm inequality, assume $\mathbb{V} = \mathbb{R}^m$ and $\mathbb{W} = \mathbb{R}^n$ and w.l.o.g. $m \leq n$. Recall that the nuclear norm of a matrix $M \in \mathbb{R}^{m\times n}$ is defined as the sum of its singular values. The corresponding unit ball B_{nuc} equals the convex hull of the rank one matrices $\mathbb{V} \otimes \mathbb{W}$ such that $v \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$ have norm one, e.g., see [35]. If we denote by B the unit ball with respect to the Frobenius norm, we get $B \subset \sqrt{m} B_{\text{nuc}}$, where we used that $m \leq n$. We obtain

$$\|K \otimes L\| = \max_{u\in B} h_{K\otimes L}(u) \;\leq\; \sqrt{m} \max_{u\in B_{\text{nuc}}} h_{K\otimes L}(u) = \tfrac{1}{2}\sqrt{m} \max_{u} \mathbb{E}|\langle u, X \otimes Y\rangle|.$$

But for $u = \mathbb{V} \otimes \mathbb{W}$ with unit vectors v, w, we have

$$\mathbb{E}|\langle \mathbb{V} \otimes \mathbb{W}, X \otimes Y\rangle| = \mathbb{E}|\langle v, X\rangle| \cdot \mathbb{E}|\langle w, Y)\rangle| = 4h_K(\mathbb{V})h_L(\mathbb{W}) \;\leq\; 4\|K\| \cdot \|L\|.$$

Using the convexity of $h_{K\otimes L}$, implies the second assertion. ☐

It is straightforward to extend the tensor product of zonoids to a bilinear map between spaces of virtual zonoids.

Proposition 2.1.11 (Tensor product of virtual zonoids)**.** *The tensor product of zonoids from Definition 2.1.3 uniquely extends to a bilinear map* $\cdot \otimes \cdot : \widehat{\mathscr{Z}}(\mathbb{V}) \times \widehat{\mathscr{Z}}(\mathbb{W}) \to \widehat{\mathscr{Z}}(\mathbb{V} \otimes \mathbb{W})$*. The resulting tensor product of virtual zonoids is associative.*

Proof. The only possible way to define it is by setting

$$(K_1 - K_2) \otimes (L_1 - L_2) := (K_1 \otimes L_1 + K_2 \otimes L_2) - (K_1 \otimes L_2 + K_2 \otimes L_1).$$

Using the multilinearity of the tensor product of zonoids, it is straightforward to check that this is well defined and defines a bilinear map. The associativity follows from the associativity of the tensor product of zonoids. ☐

Notice that we haven't talked about continuity yet. We will prove that the tensor product is continuous on zonoids but that this is not true for its extension to virtual zonoids. In order to do that, it is convenient to use the point of view of measures introduced in Section 1.2.3.

Recall that the *Segre map* is the map $S(\mathbb{V}) \times S(\mathbb{W}) \to S(\mathbb{V} \otimes \mathbb{W})$ that sends $(u, v) \mapsto u \otimes v$.

Definition 2.1.12. We define the map $\widetilde{T} : \mathcal{M}(S(\mathbb{V})) \times \mathcal{M}(S(\mathbb{W})) \to \mathcal{M}(S(\mathbb{V} \otimes \mathbb{W}))$ to be the tensor product of measures composed with the pushforward of the Segre map.

Proposition 2.1.13. *Let* $K \in \mathscr{Z}_0(\mathbb{V})$ *and* $L \in \mathscr{Z}_0(\mathbb{W})$*, we have*

$$\mu_{K\otimes L} = \widetilde{T}(\mu_K, \mu_L).$$

Proof. Since the map $\widetilde{T}$ is bilinear we can assume without loss of generality that μ_K and μ_L are probability measures; the general case then follows by homogeneity and linearity. In that case, by Proposition 1.2.43, $K = \mathbb{E}\underline{X}$ where $X \in S(\mathbb{V}) \subset \mathbb{V}$ is a random vector of law μ_K. Similarly, $L = \mathbb{E}\underline{Y}$, where $Y \in S(\mathbb{W}) \subset \mathbb{W}$ is a random vector of law μ_L, that we can assume to be independent of X. By definition, we have $K \otimes L = \mathbb{E}\underline{X \otimes Y}$. The law of $X \otimes Y$ is the pushforward by the Segre map of the tensor product of measures $\mu_K \otimes \mu_L \in \mathcal{M}^+(S(\mathbb{V}) \times S(\mathbb{W}))$, and this concludes the proof. ☐

One could take Proposition 2.1.13 as the definition of the tensor product of zonoids, as it may appear simpler. This simplicity however entirely relies on the fact that the tensor product on vectors (the Segre map) sends the product of spheres to the sphere. When later in Section 2.1.2, we will deal with multilinear maps that do not have this property, the point of view of random vectors will be easier to handle.

We see that the tensor product of zonoid is well understood in the point of view of measures and in the point of view of the Vitale zonoid. However for the two other point of view this is less clear and this is the reason why it is not obvious how to generalize this tensor product to general convex bodies.

In particular we lack a description of the support function of the tensor product in terms of the support functions of the arguments. The best we have is (2.1.2) which express the support function of the tensor product in terms of the support function of the first variable. We saw in different proofs how useful was this expression but it still requires to know the Vitale construction for the second argument.

Open problem 3. Find an expression of $h_{K\otimes L}$ directly in terms of h_K and h_L.

Continuity of the tensor product

Let us start by proving an inequality.

Lemma 2.1.14. *Let $K_1, K_2 \in \mathscr{Z}_0(\mathbb{V})$ and $L \in \mathscr{Z}_0(\mathbb{W})$ and let $n := \dim \mathbb{W}$, we have*

$$\|K_1 \otimes L - K_2 \otimes L\| \;\leq\; \sqrt{2\pi}\rho_n \|L\| \|K_1 - K_2\|,$$

where recall the definition of ρ_n in (1.2.7).

Proof. Let $u \in \mathbb{V}^* \otimes \mathbb{W}^*$ be such that $\|u\| = 1$ and $\|K_1 \otimes L - K_2 \otimes L\| = |h_{K_1\otimes L}(u) - h_{K_2\otimes L}(u)|$. Let $L = \mathbb{E}\underline{Y}$. From (2.1.2) we get

$$h_{K_1\otimes L}(u) - h_{K_2\otimes L}(u) = \mathbb{E}\left(h_{K_1} - h_{K_2}\right)((\tau_Y)^t(u)),$$

hence,

$$\begin{aligned} |h_{K_1\otimes L}(u) - h_{K_2\otimes L}(u)| \;&\leq\; \mathbb{E}\left|\left(h_{K_1} - h_{K_2}\right)(((\tau_Y)^t(u))\right| \\ &\leq\; \|h_{K_1} - h_{K_2}\|_\infty \,\mathbb{E}\|(\tau_Y)^t(u))\| \\ &\leq\; \|K_1 - K_2\| \;\mathbb{E}\|Y\| = \|K_1 - K_2\| \;\ell(L) \end{aligned}$$

where, for the third inequality, we used the fact that $\|(\tau_y)^t\|_{op} = \|\tau_y\|_{op} = \|y\|$ with $\|\cdot\|_{op}$ the operator norm. Applying Proposition 1.2.40 completes the proof. $\square$

The main continuity properties are summarized in the following result.

Proposition 2.1.15. *Suppose that $\dim \mathbb{V}, \dim \mathbb{W} \geq 2$. Then, the tensor product map satisfies the following.*

(i) $\cdot\otimes\cdot : \mathscr{Z}(\mathbb{V}) \times \mathscr{Z}(\mathbb{W}) \to \mathscr{Z}(\mathbb{V}\otimes\mathbb{W})$ *is continuous. More specifically, for $K_1, K_2 \in \mathscr{Z}_0(\mathbb{V})$ and $L_1, L_2 \in \mathscr{Z}_0(\mathbb{W})$, we have*

$$d_H\left(K_1 \otimes L_1, K_2 \otimes L_2\right) \;\leq\; \sqrt{2\pi}\left(\rho_m\|K_2\| + \rho_n\|L_1\|\right)\left(\mathrm{d}\left(K_1, K_2\right) + \mathrm{d}\left(L_1, L_2\right)\right),$$

where $m := \dim \mathbb{V}$ and $n := \dim \mathbb{W}$;

(ii) the extension to virtual zonoids $\cdot\otimes\cdot : \widehat{\mathscr{Z}}(\mathbb{V}) \times \widehat{\mathscr{Z}}(\mathbb{W}) \to \widehat{\mathscr{Z}}(\mathbb{V}\otimes\mathbb{W})$ *with the norm topology on both sides is not sequentially continuous, but separately (i.e., componentwise) continuous;*

(iii) the extension to virtual zonoids $\cdot\otimes\cdot : \widehat{\mathscr{Z}}(\mathbb{V}) \times \widehat{\mathscr{Z}}(\mathbb{W}) \to \widehat{\mathscr{Z}}(\mathbb{V}\otimes\mathbb{W})$ *with the weak–$*$ topology on both sides is sequentially continuous.*

Proof. In this whole proof, without loss of generality, we assume that the zonoids are centered. For proving (1), recall that $\mathrm{d}(K, L) = \|K - L\|$. From the multilinearity of the tensor product and the triangle inequality of the norm, we get

$$\|K_1 \otimes L_1 - K_2 \otimes L_2\| \;\leq\; \|K_1 \otimes L_1 - K_2 \otimes L_1\| + \|K_2 \otimes L_1 - K_2 \otimes L_2\|.$$

Combined with Lemma 2.1.14, this yields

$$\|K_1 \otimes L_1 - K_2 \otimes L_2\| \;\leq\; \sqrt{2\pi}\rho_m\|L_1\|\|K_1 - K_2\| + \sqrt{2\pi}\rho_n\|K_2\|\|L_1 - L_2\|,$$

which proves the first assertion.

As for (2), the separate continuity follows directly from Lemma 2.1.14. To prove that it is not (sequential) continuous, we begin with a general observation. Let $\varphi\colon E \times F \to G$ be a bilinear map of real normed vector spaces. Then φ is (sequential) continuous if and only if it has finite operator norm:

$$\|\varphi\|_{\mathrm{op}} := \sup_{\|x\|\leq 1, \|y\|\leq 1} \|\varphi(x,y)\| \ < \ \infty.$$

We show now that the tensor product of virtual (centered) zonoids has infinite operator norm. It suffices to prove this for $\mathbb{V} = \mathbb{W} = \mathbb{R}^2$. Consider the sequence of vectors $a_n := (n, 1)$, $b_n := (n, 0)$ and the corresponding sequence of segments $\underline{a_n}$, $\underline{b_n}$ in $\mathbb{R}^2$. This defines the sequence of virtual zonoids $\underline{a_n} - \underline{b_n} \in \widehat{\mathscr{Z}}(\mathbb{R}^2)$. It is immediate to check that

$$\|\underline{a_n} - \underline{b_n}\| = \mathrm{d}\left(\underline{a_n}, \underline{b_n}\right) = \frac{1}{2}.$$

Consider $P_n := (\underline{a_n} - \underline{b_n}) \otimes (\underline{a_n} - \underline{b_n}) \in \widehat{\mathscr{Z}}(\mathbb{R}^2 \otimes \mathbb{R}^2)$. It suffices to show that $\lim_{n\to\infty} \|P_n\| = \infty$. For this, we compute

$$a_n \otimes a_n = \begin{bmatrix} n^2 & n \\ n & 1 \end{bmatrix},\ b_n \otimes b_n = \begin{bmatrix} n^2 & 0 \\ 0 & 0 \end{bmatrix},\ a_n \otimes b_n = \begin{bmatrix} n^2 & 0 \\ n & 0 \end{bmatrix},\ b_n \otimes a_n = \begin{bmatrix} n^2 & n \\ 0 & 0 \end{bmatrix}.$$

Their inner product with the matrix $w_n := \begin{bmatrix} 1 & -n \\ -n & 0 \end{bmatrix}$ is given by

$$\langle a_n \otimes a_n, w_n\rangle = -n^2,\ \langle b_n \otimes b_n, w_n\rangle = n^2,\ \langle a_n \otimes b_n,\ w_n\rangle = \langle b_n \otimes a_n, w_n\rangle = 0,$$

We obtain $h_{\underline{a_n\otimes a_n}}(w_n) = \frac{1}{2}|\langle a_n \otimes a_n, w_n\rangle| = \frac{n^2}{2}$, $h_{\underline{b_n\otimes b_n}}(w_n) = \frac{n^2}{2}$, and $h_{\underline{a_n}\otimes\underline{b_n}}(w_n) = h_{\underline{b_n}\otimes\underline{a_n}}(w_n) = 0$. Therefore,

$$h_{P_n}\left(\frac{w_n}{\|w_n\|}\right) = \frac{h_{P_n}(w_n)}{\|w_n\|} = \frac{n^2}{\|w_n\|}. \tag{2.1.3}$$

But since $\|w_n\| = \sqrt{1+2n^2}$, (2.1.3) explodes as $n \to \infty$ and this completes the proof of the second item.

For item (3) we recall from Proposition 2.1.13 that at the level of measures, the tensor product of (centered) zonoids equals the tensor product of measures composed with the pushforward of the Segre map. The pushforward of a measure under a continuous map is weak–$*$ continuous. Mapping two measures to their product measure is sequentially continuous by [23, Theorem 2.8]. This finishes the proof for the third assertion. □

2.1.2 Multilinear maps induced on zonoids

We are now ready to state and prove our main result.

Theorem 2.1.16 (The Fundamental Theorem of Zonoid Calculus, B.B.L.M.)**.** *Let $M : \mathbb{V}_1 \times \cdots \times \mathbb{V}_k \to \mathbb{W}$ be a multilinear map. There is a unique multilinear map*

$$\widehat{M} : \widehat{\mathscr{Z}_0}(\mathbb{V}_1) \times \cdots \times \widehat{\mathscr{Z}_0}(\mathbb{V}_k) \to \widehat{\mathscr{Z}_0}(\mathbb{W}),$$

that is continuous on zonoids and such that for every $x_1 \in \mathbb{V}_1, \ldots, x_k \in \mathbb{V}_k$

$$\widehat{M}\left(\underline{x_1}, \ldots, \underline{x_k}\right) = \underline{M(x_1, \ldots, x_k)}.$$

Moreover, $\widehat{M}$ sends zonoids to zonoids and is continuous in each variable for the norm topology and sequentially continuous for the weak-$*$ topology.

Proof. To show existence, we rely on the universal property of tensor product: there is a unique linear map $L : \mathbb{V}_1 \otimes \cdots \otimes \ \mathbb{V}_k \to \ \mathbb{W}$ such that $L(x_1 \otimes \cdots \otimes x_k) = M(x_1, \ldots, x_k)$. Consider the induced linear continuous map $\widehat{L} : \widehat{\mathscr{Z}_0}(\mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_k) \to \widehat{\mathscr{Z}_0}(\mathbb{W})$. For $(K_1, \ldots, K_p) \in \widehat{\mathscr{Z}_0}(\mathbb{V}_1) \times \cdots \times \widehat{\mathscr{Z}}(\mathbb{V}_k)$, we define the map $\widehat{M}$ by

$$\widehat{M}(K_1, \ldots, K_k) := \widehat{L}(K_1 \otimes \cdots \otimes K_k).$$

This is the composition of the linear map $\widehat{L}$ with the multilinear tensor product defined in Definition 2.1.3, therefore it is multilinear.

Restricting to zonoids, we see that $\widehat{M}(\mathscr{Z}_0(\mathbb{V}_1) \times \cdots \times \mathscr{Z}_0(\mathbb{V}_k)) \subseteq \mathscr{Z}_0(\mathbb{W})$. The asserted formula for the image of $\widehat{M}$ on tuples of segments is a direct consequence of Example 2.1.5 and the definition of the map $\widehat{M}$.

Since $\widehat{L}$ is continuous and the tensor product map from Proposition 2.1.11 is separate continuous, $\widehat{M}$ is separate continuous. Similarly, $\widehat{M}|_{\mathscr{Z}_0(\mathbb{V}_1)\times\cdots\times\mathscr{Z}_0(\mathbb{V}_k)}$ is continuous since the tensor product map on zonoids is continuous (Proposition 2.1.8) and $\widehat{L}$ is continuous (Proposition 2.1.1).

For the uniqueness of the map $\widehat{M}$ we argue as follows. Since by definition, any zonoid in $\mathbb{V}_i$ can be approximated by symmetric segments and the values of $\widehat{M}$ are determined on tuples of segments, the componentwise continuity of $\widehat{M}$ determines $\widehat{M}$ on $\mathscr{Z}_0(\mathbb{V}_1) \times \cdots \times \mathscr{Z}_0(\mathbb{V}_k)$. In turn, this determines $\widehat{M}$ by multilinearity.

By Proposition 2.1.8, the tensor product of zonoids preserves the componentwise order. Moreover, $\widehat{L}$ preserves the order as shown in Proposition 2.1.1. This implies that $\widehat{M}$ preserves the componentwise order.

The last statement about the sequential continuity with respect to the weak-$*$ topology follows from Proposition 2.1.15-(*iii*). □

Let us extend this map to non centered zonoids. We use the same symbol to denote it.

Definition 2.1.17. Let $M : \mathbb{V}_1 \times \cdots \times \mathbb{V}_k \to \mathbb{W}$ be a multilinear map. We define the multilinear map $\widehat{M} : \widehat{\mathscr{Z}}(\mathbb{V}_1) \times \cdots \times \widehat{\mathscr{Z}}(\mathbb{V}_k) \to \widehat{\mathscr{Z}}(\mathbb{W})$ to be given for all $K_1 \in \mathscr{Z}_0(\mathbb{V}_1), \ldots, K_1 \in \mathscr{Z}_0(\mathbb{V}_1)$ and $x_1 \in \mathbb{V}_1, \ldots, x_k \in \mathbb{V}_k$ by

$$\widehat{M}\left(K_1 + \tfrac{1}{2}\{x_1\}, \ldots, K_k + \tfrac{1}{2}\{x_k\}\right) := \widehat{M}\left(K_1, \ldots, K_k\right) + \tfrac{1}{2}\{M(x_1, \ldots, x_k)\}.$$

Note that with this definition we have

$$\widehat{M}\left([0, x_1], \ldots, [0, x_k]\right) = [0, M(x_1, \ldots, x_k)].$$

In fact this could be a justification of why we carry this $\frac{1}{2}$ term everywhere.

Finally, we show that FTZC (Theorem 2.1.16) behaves well with the Vitale construction.

Proposition 2.1.18. *Let $M : \mathbb{V}_1 \times \cdots \times \mathbb{V}_k \to \mathbb{W}$ be a multilinear map and let $X_1 \in \mathbb{V}_1, \ldots, X_k \in \mathbb{V}_k$ be integrable independent random vectors. Then, $M(X_1, \ldots, X_k) \in \mathbb{W}$ is integrable and we have*

$$\widehat{M}\left(\mathbb{E}\underline{X_1}, \ldots, \mathbb{E}\underline{X_k}\right) = \mathbb{E}\underline{M(X_1, \ldots, X_k)}$$
$$\widehat{M}\left(\mathbb{E}[0, X_1], \ldots, \mathbb{E}[0, X_k]\right) = \mathbb{E}[0, M(X_1, \ldots, X_k)].$$

Proof. The fact that it is integrable follows from the fact that M has a finite operator norm and integrability and independence of the random vectors. Moreover let us note that the second equality follows from the first and Definition 2.1.17 using the fact that $\mathbb{E}[0, X_i] = \mathbb{E}\underline{X_i} + \frac{1}{2}\{\mathbb{E}X_i\}$.

To prove the first equality, consider, as in the proof of Theorem 2.1.16, the linear map $L : \mathbb{V}_1 \otimes \cdot \otimes V_k \to \mathbb{W}$ induced by the universal mapping property of the tensor product. Then recall that $\widehat{M}$ is the composition of the tensor product with $\widehat{L}$. We obtain

$$\widehat{M}\left(\mathbb{E}\underline{X_1}, \ldots, \mathbb{E}\underline{X_k}\right) = \widehat{L}\left(\mathbb{E}\underline{X_1} \otimes \cdots \otimes \mathbb{E}\underline{X_k}\right) = \widehat{L}\left(\mathbb{E}\underline{X_1 \otimes \cdots \otimes X_k}\right).$$

Moreover, we have $\widehat{L}\left(\mathbb{E}\underline{X_1 \otimes \cdots \otimes X_k}\right) = \mathbb{E}\underline{L(X_1 \otimes \cdots \otimes X_k)} = \mathbb{E}\underline{M(X_1, \ldots, X_k)}$ which is what we wanted. □

Open problem 4. Is the image of generalized zonoids in Theorem 2.1.16 a generalized zonoid? What is the preimage of zonoids?

2.2 The zonoid algebra

For this section, recall that an Euclidean structure on $\mathbb{V}$ induces an Euclidean structure on the exterior product $\Lambda^k\mathbb{V}$ given for all $v_1, \dots, v_k, w_1, \dots, w_k \in \mathbb{V}$ by

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle := \det(\langle v_i, w_j \rangle)_{1\leq i,j\leq k}. \tag{2.2.1}$$

Equivalently this can be seen as an identification $\Lambda^k(\mathbb{V}^*) \cong (\Lambda^k\mathbb{V})^*$.

2.2.1 The algebra

In the exterior algebra, we have a collection of bilinear maps $\mathrm{wedge}_{k,l} : \Lambda^k\mathbb{V} \times \Lambda^l\mathbb{V} \to \Lambda^{k+l}\mathbb{V}$ that is given by $\mathrm{wedge}_{k,l}(a, b) = a \wedge b$. We consider the bilinear map induced on zonoids and if $A \in \mathscr{Z}(\Lambda^k V), A' \in \mathscr{Z}(\Lambda^l V)$ we write

$$A \wedge A' := \widehat{\mathrm{wedge}}_{k,l}(A, A'). \tag{2.2.2}$$

We will call this operation the *wedge product of zonoids*. Using Proposition 2.1.18 we have for X and Y *independent* integrable random vectors:

$$\mathbb{E}\underline{X} \wedge \mathbb{E}\underline{Y} = \mathbb{E}\underline{X \wedge Y}; \qquad \mathbb{E}[0, X] \wedge \mathbb{E}[0, Y] = \mathbb{E}[0, X \wedge Y]. \tag{2.2.3}$$

Remark 2.2.1. Note that the wedge product on centered zonoids is commutative, this follows from (2.2.3) and the fact that $\underline{x} = \underline{-x}$.

Definition 2.2.2. The *zonoid algebra* and respectively the *centered zonoid algebra* are defined as vector spaces to be

$$\widehat{\mathscr{A}}(\mathbb{V}) := \bigoplus_{k=0}^{m} \widehat{\mathscr{Z}}(\Lambda^k\mathbb{V}), \qquad \widehat{\mathscr{A}_0}(\mathbb{V}) := \bigoplus_{k=0}^{m} \widehat{\mathscr{Z}_0}(\Lambda^k\mathbb{V}),$$

and endowed with the multiplication given by the wedge product of zonoids. Similarly we define

$$\mathscr{A}(\mathbb{V}) := \bigoplus_{k=0}^{m} \mathscr{Z}(\Lambda^k\mathbb{V}) \subset \widehat{\mathscr{A}}(\mathbb{V}), \qquad \mathscr{A}_0(\mathbb{V}) := \bigoplus_{k=0}^{m} \mathscr{Z}_0(\Lambda^k\mathbb{V}) \subset \widehat{\mathscr{A}_0}(\mathbb{V}).$$

We will abuse notation and often also refer to those as the (centered) zonoid algebra. Finally, an element of $\widehat{\mathscr{Z}}(\Lambda^k\mathbb{V}) \subset \widehat{\mathscr{A}}(\mathbb{V})$ will be said to have *degree* k.

Remark 2.2.3. Definition 2.2.2 defines the zonoid algebra as an algebra, but remember that if we want a topology we have essentially two choices: the norm topology and the weak-$*$ topology (see Section 1.2.4) and they have different implication in terms of continuity of the wedge product, see Theorem 2.1.16. If instead we restrict to $\mathscr{A}(\mathbb{V})$ with the Hausdorff distance topology, the wedge product is continuous.

Note that we have, as algebras,

$$\widehat{\mathscr{A}}(\mathbb{V}) \cong \widehat{\mathscr{A}_0}(\mathbb{V}) \oplus \Lambda\mathbb{V}$$

where $\Lambda\mathbb{V} := \bigoplus_{k=0}^{m} \Lambda^k\mathbb{V}$. Therefore we can often reduce to the study of the centered case.

Proposition 2.2.4. *The algebra $\mathscr{A}(\mathbb{V})$ is an associative real graded algebra and the subalgebra $\mathscr{A}_0(\mathbb{V})$ is in addition commutative. The wedge maps of zonoids*

$$\mathscr{Z}(\Lambda^{d_1}\mathbb{V}) \times \cdots \times \mathscr{Z}(\Lambda^{d_p}\mathbb{V}) \to \mathscr{Z}(\Lambda^{d_1+\cdots+d_p}V),\ (A_1, \dots, A_p) \mapsto A_1 \wedge \cdots \wedge A_p$$

are continuous. These maps preserve the inclusion order of zonoids: if we have $A'_j \subset A_j$ for zonoids in $\mathscr{Z}(\Lambda^{d_j}V)$, then

$$A'_1 \wedge \cdots \wedge A'_p \ \subset\ A_1 \wedge \cdots \wedge A_p.$$

Moreover, the wedge product of zonoids does not increase the length:

$$\ell(A_1 \wedge \cdots \wedge A_p) \ \leq\ \ell(A_1) \cdots \ell(A_p).$$

Proof. The associativity follows from the associativity of the wedge product and (2.2.3). The distributivity is equivalent to say that the wedge product is bilinear which is guaranteed by Theorem 2.1.16. The gradedness follows from the definition of $\mathscr{A}(V)$. The multiplicative unit lies in $\widehat{\mathscr{Z}}(\Lambda^0 V) = \widehat{\mathscr{Z}}(\mathbb{R}) \simeq \mathbb{R}$. The commutativity of the wedge product of centered zonoids was already observed in Remark 2.2.1.

The continuity of the wedge product of zonoids and the monotonicity with respect to inclusion are features of Theorem 2.1.16.

For the length inequality, we use that the antisymmetrization map $\otimes_j \Lambda^{d_j} V \to \Lambda^{d_1+\cdots+d_p} V$ is an orthogonal projection. Hence $\ell(A_1 \wedge \cdots \wedge A_p) \leq \ell(A_1 \otimes \cdots \otimes A_p) = \ell(A_1) \cdots \ell(A_p)$ by Proposition 2.1.10. □

Here is an immediate yet important observation about wedge products of zonoids.

Lemma 2.2.5. *Let $K \in \mathscr{Z}(\mathbb{V})$, we have $K^{\wedge k} = 0$ for all $k > \dim(K)$.*

Proof. Without loss of generality, we can assume that K is centered. Let us write $K = \mathbb{E}\underline{X}$. By (2.2.3) we have $K^{\wedge k} = \mathbb{E}\underline{X_1 \wedge \cdots \wedge X_k}$, where $X_1, \ldots, X_k$ are independent copies of X. With probability one we have that X belongs to the linear span of K which is of the same dimension of K, and so with probability one the X_i are linearly dependent. Hence, $X_1 \wedge \cdots \wedge X_k = 0$ almost surely, so that $K^{\wedge k} = 0$. □

The zonoid algebra is deeply linked with the notion of mixed volume and intrisic volumes.

Theorem 2.2.6 (B.B.L.M.)**.** *For every $K_1, \ldots, K_m \in \mathscr{Z}(\mathbb{V})$, we have*

$$K_1 \wedge \cdots \wedge K_m = m!\, \underline{\mathrm{MV}(K_1, \ldots, K_m)} + \frac{1}{2}\left\{\det(o(K_1), \ldots, o(K_m))\right\} \in \mathscr{Z}(\Lambda^m \mathbb{V})$$

where recall that $o(K_i) \in \mathbb{V}$ is the pole*, that is the unique point such that $K_i + \frac{1}{2}\{-o(K_i)\} \in \mathscr{Z}_0(\mathbb{V})$. In particular composed with the length we get*

$$\ell(K_1 \wedge \cdots \wedge K_m) = m!\, \mathrm{MV}(K_1, \ldots, K_m). \tag{2.2.4}$$

Moreover, if $K \in \mathscr{Z}(\mathbb{V})$ we have for every $0 \leq k \leq m$:

$$\ell(K^{\wedge k}) = k!\, \mathrm{V}_k(K) \tag{2.2.5}$$

where recall that V_k denotes the k-th intrisic volume (Definition 1.1.24).

Proof. Let us first assume that the zonoids are centered, that is that $o(K_i) = 0\ \forall i$. Then, let $x_1, \ldots, x_k \in \mathbb{V}$ and notice that $\det(x_1, \ldots, x_k) = \mathrm{vol}_m(\underline{x_1} + \cdots + \underline{x_m})$. Moreover, by the properties of the mixed volume (Proposition 1.1.23), we have $\mathrm{vol}_m(\underline{x_1} + \cdots + \underline{x_m}) = m!\mathrm{MV}(\underline{x_1}, \ldots, \underline{x_m})$. Since the mixed volume is continuous and linear in each variable, and since in dimension 1 we have $\underline{x+y} = \underline{x} + \underline{y}$, this implies the result for centered zonoids. For the non centered case, simply apply Definition 2.1.17.

For the second part, since the length is translation invariant, we can assume that K is centered. Moreover for simplicity, let us assume $\mathbb{V} = \mathbb{R}^m$ and write $B_m := B(\mathbb{R}^m)$. Let us take the independent integrable random vectors $X_1, \ldots, X_k, Y_1, \ldots, Y_{m-k} \in \mathbb{V}$ such that $K = \mathbb{E}\underline{X_i}$ and Y_j are Gaussian of variance $\sqrt{2\pi}$ in such a way that $B_m = \mathbb{E}\underline{Y_j}$. By the first part of the proposition, we can write

$$\mathrm{MV}(K[k], B_m[m-k]) = \frac{1}{m!}\, \ell(K^{\wedge k} \wedge B_m^{\wedge(m-k)}) = \frac{1}{m!}\, \mathbb{E}|X_1 \wedge \cdots \wedge X_k \wedge Y_1 \wedge \cdots \wedge Y_{m-k}|.$$

Using the independence, we first integrate over the Y_j while leaving the X_i fixed. By orthogonal invariance of $Y := Y_1 \wedge \cdots \wedge Y_{m-k}$, we can assume that the space spanned by the X_i is the span of a fixed orthonormal frame $e_1, \ldots, e_k$. Then, $X_1 \wedge \cdots \wedge X_k = \|X_1 \wedge \cdots \wedge X_k\|\, e_1 \wedge \cdots \wedge e_k$ and so, using that Y is independent of the X_i:

$$\mathrm{MV}(K[k], B_m[m-k]) = \frac{c}{m!}\, \mathbb{E}\|X_1 \wedge \cdots \wedge X_k\| = \frac{c}{m!}\, \ell(K^{\wedge k}),$$

with the constant $c := \mathbb{E}_Y \|e_1 \wedge \cdots \wedge e_k \wedge Y\|$. In order to determine this constant, we use that $\|e_1 \wedge \cdots \wedge e_k \wedge Y\| = \|\tilde{Y}_1 \wedge \ldots \wedge \tilde{Y}_k\|$, where $\tilde{Y}_j$ denotes the orthogonal projection of Y_j onto the orthogonal complement $\mathbb{R}^{m-k}$ of $\mathbb{R}^k = \mathrm{Span}\{e_1, \ldots, e_k\}$. Since the unit ball B_{m-k} is the projection of the unit ball B_m we have that $B_{m-k} = \mathbb{E}\underline{\tilde{Y}_j}$, we obtain with the first part of the proposition:

$$\mathbb{E}|\tilde{Y}_1 \wedge \cdots \wedge \tilde{Y}_{m-k}| = \ell\big(B_{m-k}^{\wedge(m-k)}\big) = (m-k)! \operatorname{vol}_{m-k}(B_{m-k}).$$

We therefore conclude that $\mathrm{MV}(K[k], B_m[m-k]) = \frac{1}{m!} \operatorname{vol}_{m-k}(B_{m-k})\ell(K^{\wedge k})$, which finishes the proof. $\square$

Thus we can think of the wedge product of zonoids as an *intermediate term*, when computing the mixed volume, and the study of the zonoid algebra, the study of these intermediate terms.

When computing products of zonoids in $\mathbb{V}$ we obtain a particular class of elements.

Definition 2.2.7. An element of the zonoid algebra $A \in \mathscr{Z}(\Lambda^k\mathbb{V})$ is called *fully decomposable* if there are $K_1, \ldots, K_m \in \mathscr{Z}(\mathbb{V})$ such that we can write $A = K_1 \wedge \cdots \wedge K_k$.

As we see in Theorem 2.2.6, only fully decomposable zonoids are involved in the computation of mixed volumes. It turns out that they span a proper subalgebra of the zonoid algebra that we describe in the next section.

2.2.2 Grassmannian zonoids

We will call *simple vectors* of $\Lambda^k\mathbb{V}$ all vectors of the form $x_1 \wedge \cdots \wedge x_k$ with $x_1, \ldots, x_k \in \mathbb{V}$. Recall that we denote by $G_k(\mathbb{V})$ the Grassmannian of k vectorial subspaces of $\mathbb{V}$. Recall that the Grassmannian embeds in the projective space of $\Lambda^k\mathbb{V}$ via the Plücker embedding that sends $E \in G_k(\mathbb{V})$ to the class $[e_1 \wedge \cdots \wedge e_k] \in \mathrm{P}(\Lambda^k\mathbb{V})$, where $e_1, \ldots, e_k$ is a basis of E. In particular the set of simple vectors in $\Lambda^k\mathbb{V}$ can be viewed as the cone over the Grassmannian and a measure on $G_k(\mathbb{V})$ can be identified with an even measure on $S(\mathbb{V})$ supported on the simple vectors. Similarly functions on the Grassmannian can (and will) be identified with even homogeneous functions on simple vectors.

Let us begin with a notation.

Definition 2.2.8. Let $E \in G_k(\mathbb{V})$, we define the segment

$$\underline{E} := \underline{e_1 \wedge \cdots \wedge e_k} \subset \Lambda^k\mathbb{V}$$

where $e_1, \ldots, e_k$ is an orthonormal basis of E.

This allows to make the following definition.

Definition 2.2.9. A zonoid $K \in \mathscr{Z}(\Lambda^k\mathbb{V})$ is a *Grassmannian zonotope* if there exists subspaces $E_1, \ldots, E_n \in G_k(\mathbb{V})$, scalars $\lambda_1, \ldots, \lambda_n \geq 0$ and a simple vector $c = c_1 \wedge \cdots \wedge c_k \in \Lambda^k\mathbb{V}$ such that

$$K = \lambda_1 \underline{E_1} + \cdots + \lambda_n \underline{E_n} + \tfrac{1}{2}\{c\}$$

A *Grassmannian zonoid* is a limit of Grassmannian zonotopes. We denote the set of Grassmannian zonoids in $\Lambda^k\mathbb{V}$ by $\mathscr{G}(k, \mathbb{V}) \subset \mathscr{Z}(\Lambda^k\mathbb{V})$, and we define the set of centered Grassmannian zonoids to be $\mathscr{G}_0(k, \mathbb{V}) := \mathscr{G}(k, \mathbb{V}) \cap \mathscr{Z}_0(\Lambda^k\mathbb{V})$. Similarly we define the *virtual Grassmannian zonoids* $\widehat{\mathscr{G}}(k, \mathbb{V})$ and *virtual centered Grassmannian zonoids* $\widehat{\mathscr{G}_0}(k, \mathbb{V})$.

Remark 2.2.10. For $k \in \{0, 1, m-1, m\}$, where recall $m := \dim \mathbb{V}$, all zonoids are Grassmannian.

Remark 2.2.11. Note that for all k, $\mathscr{G}(k, \mathbb{V})$ is closed under Minkowski addition and multiplication by a non negative scalar.

From the random vector and measure point of view Grassmannian zonoids can be characterized.

Lemma 2.2.12. *Let $K \in \mathscr{Z}_0(\Lambda^k \mathbb{V})$. The following are equivalent.*

(i) $K \in \mathscr{G}_0(k, \mathbb{V})$;

(ii) There is an integrable random vector $X \in \Lambda^k V$ that is almost surely simple, i.e. such that almost surely $X = X_1 \wedge \cdots \wedge X_k$ (the vectors $X_1, \ldots, X_k$ can be dependent*), such that $K = \mathbb{E}\underline{X}$*

(iii) There is a random subspace $E \in G_k(\mathbb{V})$ and a random scalar $\rho \in \mathbb{R}$ such that $\rho \geq 0$ almost surely (E and ρ be dependent) and such that $K = \mathbb{E}\rho\underline{E}$.

(iv) The support of the measure $\mu_K \in \mathcal{M}^+(S(\mathbb{V}))$ is contained in the intersection of $S(\mathbb{V})$ with the set of simple vectors, i.e. $\mu_K \in \mathcal{M}^+(G_k(\mathbb{V}))$.

Proof. The equivalence $(ii) \iff (iv)$ follows from Proposition 1.2.43. The equivalence $(i) \iff (ii)$ follows from the fact that Hausdorff convergence of zonoids corresponds to weak−$*$ convergence of measures which is Proposition 1.2.48.

To see $(ii) \Rightarrow (iii)$, let $\rho := \|X_1 \wedge \cdots \wedge X_k\|$ and $E := \mathrm{Span}(X_1, \ldots, X_k)$ if $X_1 \wedge \cdot \wedge X_k \neq 0$ and whatever if $X_1 \wedge \cdots \wedge X_k = 0$.

Finally one can see that $(iii) \Rightarrow (ii)$ by choosing a measurable map that sends $E \in G_k(\mathbb{V})$ to a simple vector $e_1 \wedge \cdots \wedge e_k$ and thus we get a random vector $X := \rho e_1 \wedge \cdots \wedge e_k \in \Lambda^k \mathbb{V}$. By definition, X is almost surely simple and $K = \mathbb{E}\underline{X}$ which concludes the proof. □

Finally from this we get the following.

Lemma 2.2.13. *The wedge product of two Grassmannian zonoids is a Grassmannian zonoid.*

Proof. If the zonoids are centered, this follows from Lemma 2.2.12-(ii) and (2.2.3). The noncentered case follows then from the fact that the wedge product of two simple vectors is again simple. □

This allows to define this special subalgebra of the zonoid algebra.

Definition 2.2.14. The *Grassmannian zonoid algebra* and the *centered Grassmannian zonoid algebra* are defined to be

$$\widehat{\mathscr{G}}(\mathbb{V}) := \bigoplus_{k=0}^{d} \widehat{\mathscr{G}}(k, \mathbb{V}); \qquad \widehat{\mathscr{G}_0}(\mathbb{V}) := \bigoplus_{k=0}^{d} \widehat{\mathscr{G}_0}(k, \mathbb{V})$$

which are subalgebra of $\widehat{\mathscr{A}}(\mathbb{V})$ and $\widehat{\mathscr{A}_0}(\mathbb{V})$ respectively. Similarly we define

$$\mathscr{G}(\mathbb{V}) := \bigoplus_{k=0}^{d} \mathscr{G}(k, \mathbb{V}); \qquad \mathscr{G}_0(\mathbb{V}) := \bigoplus_{k=0}^{d} \mathscr{G}_0(k, \mathbb{V})$$

we will again sometimes abuse notation and call those (centered) Grassmannian zonoid algebra moreover we will sometimes use simply the term *Grassmannian algebra* to denote any of them.

One can describe the wedge product operation directly in terms of measure. As one can see, the description is significantly more involved compared to (2.2.3) and this is precisely why we often prefer the point of view of random vectors.

Proposition 2.2.15. *Let $A \in \widehat{\mathscr{G}_0}(k, \mathbb{V})$ and $A' \in \widehat{\mathscr{G}_0}(l, \mathbb{V})$ where $k + l \leq d$, and let $\mu := \mu_A$ and $\mu' := \mu_{A'}$. Then $\mu \wedge \mu'$ is the (signed) measure such that for all continuous $f : G_{k+l}(\mathbb{V}) \to \mathbb{R}$ we have*

$$\langle \mu \wedge \mu', f \rangle = \frac{1}{2} \int_{G_k(\mathbb{V}) \times G_l(\mathbb{V}) \setminus \Xi} |E \wedge F| \, f(E + F) \, \mathrm{d}(\mu \otimes \mu')(E, F), \tag{2.2.6}$$

where $|E \wedge F| := \|e_1 \wedge \cdots \wedge e_k \wedge f_1 \wedge \cdots \wedge f_l\|$ with $e_1, \ldots, e_k$ an orthonormal basis of E and $f_1, \ldots, f_l$ an orthonormal basis of F, and Ξ is the subset where the planes E and F intersect non transversally.

Proof. Since both sides are bilinear we can assume the measures are probabilities. In that case $A = \mathbb{E}\underline{X}$ with X following the law μ (remember that we identify measures on the Grassmannian with even measures on the sphere supported on simple vectors) and $A' = \mathbb{E}\underline{X'}$ with X' following the law μ' and that we can assume independent of X. Then from Proposition 1.2.43 we get

$$\langle \mu \wedge \mu', f\rangle = \int_{G_{k+l}(\mathbb{V})} f \, \mathrm{d}(\mu \wedge \mu') = \frac{1}{2}\mathbb{E}\left[\|X \wedge X'\| f\left(\frac{X \wedge X'}{\|X \wedge X'\|}\right) \mathbb{1}_{X \wedge X' \neq 0}\right].$$

If X represents in Plücker a subspace E and X' a subspace F and $X \wedge X' \neq 0$ then the unit vector $\frac{X \wedge X'}{\|X \wedge X'\|}$ represents the subspace $E + F$. Moreover then $\|X \wedge X'\| = |E \wedge F|$. Finally note that $X \wedge X' = 0$ if and only if E and F intersect non transversally and the result follows. $\square$

We have of course that $\widehat{\mathscr{G}}_0(\mathbb{V})$ with the weak-$*$ topology is homeomorphic to $\bigoplus_{k=0}^{m} \mathcal{M}(G_k(\mathbb{V}))$. With this point of view, one can see the centered Grassmannian algebra as an algebra structure on the space of signed measures of the Grassmannian, with the product given by (2.2.6). As we said before, the reason why we do not adopt this point of view is because the definition in terms of random vectors is much simpler and more flexible. For example the associativity of the wedge product is immediate from (2.2.3) while it is less obvious on (2.2.6).

A particular case of Grassmannian zonoids are the *fully decomposable* ones (Definition 2.2.7). This corresponds to the case in Lemma 2.2.12-(ii) where $X_1, \ldots, X_k$ are *independent* random vectors.

Proposition 2.2.16. *Finite sums of zonoids of fully decomposable zonoids are dense in* $\mathscr{G}(k, \mathbb{V})$. *Hence the set* $\{K_1 \wedge \cdots \wedge K_k \mid K_1, \ldots, K_k \in \mathscr{Z}(\mathbb{V})\}$ *spans a sequentially dense subspace in the virtual Grassmannian zonoids* $\widehat{\mathscr{G}}(k, \mathbb{V})$ *in both the norm and weak-$*$ topology.*

Proof. It is enough to show the centered case. By definition, any zonoid in $\mathscr{G}_0(k, \mathbb{V})$ is the limit of finite sums of segments of the form $\underline{x_1 \wedge \cdots \wedge x_k}$. It is then enough to see that such segments are fully decomposable. Indeed we have $\underline{x_1 \wedge \cdots \wedge x_k} = \underline{x_1} \wedge \cdots \wedge \underline{x_k}$. $\square$

Next we see that the length of a Grassmannian zonoid can be computed inside the Grassmannian algebra.

Lemma 2.2.17. *Let* $C \in \mathscr{G}(k, \mathbb{V})$. *Then we have*

$$\ell(C) = \frac{1}{(m-k)!\kappa_{m-k}} \ell\left(C \wedge B(\mathbb{V})^{\wedge(m-k)}\right)$$

where recall $\kappa_d := \mathrm{vol}_d(B(\mathbb{R}^d))$.

Proof. Since the length is translation invariant, we can assume C is centered. Moreover let us assume $\mathbb{V} = \mathbb{R}^m$ and write $B_m := B(\mathbb{R}^m)$. Let $C = \mathbb{E}\underline{X_1 \wedge \cdots \wedge X_k}$, let $b \in \mathbb{R}^m$ be a Gaussian vector of mean 0 and variance $\sqrt{2\pi}$ in such a way that $B_m = \mathbb{E}\underline{b}$ and let $b_1, \ldots, b_{m-k}$ be iid copies of b that are independent of $X_1 \wedge \cdots \wedge X_k$. Then, using the independence of the random variables and the fact that $b_1 \wedge \cdots \wedge b_{m-k}$ is orthogonal invariant, we have

$$\begin{aligned}\ell\left(C \wedge B_d^{\wedge(d-k)}\right) &= \mathbb{E}\|X_1 \wedge \cdots \wedge X_k \wedge b_1 \wedge \cdots \wedge b_{d-k}\| \\ &= \mathbb{E}\|X_1 \wedge \cdots \wedge X_k\| \cdot \mathbb{E}\|e_1 \wedge \cdots \wedge e_k \wedge b_1 \wedge \cdots \wedge b_{m-k}\|,\end{aligned}$$

where $e_1, \ldots, e_m$ denotes the standard basis of $\mathbb{R}^m$. We obtain

$$\ell\left(C \wedge B_m^{\wedge(m-k)}\right) = \ell(C) \cdot \mathbb{E}\|\pi(b_1) \wedge \cdots \wedge \pi(b_{m-k})\|$$

where $\pi : \mathbb{R}^m \to \mathbb{R}^{m-k}$ is the orthogonal projection onto $\mathrm{Span}(e_{k+1}, \ldots, e_m)$. Then it remains only to see, using Theorem 2.2.6, that $\mathbb{E}\|\pi(b_1) \wedge \cdots \wedge \pi(b_{m-k})\| = \ell\left(\pi(B_m)^{\wedge(m-k)}\right) = \ell\left((B_{m-k})^{\wedge(m-k)}\right) = (m-k)!\kappa_{m-k}$. $\square$

Let us give two examples that show how this can be useful to do computations.

Example 2.2.18. If $B_m := B(\mathbb{R}^m)$ is the unit ball we have

$$\ell(B_m) = m\frac{\kappa_m}{\kappa_{m-1}}.$$

Indeed by the previous lemma we have $\ell(B_m) = \frac{1}{(m-1)!\kappa_{m-1}}\ell(B_m^{\wedge m})$. Then the result follows by (2.2.4).

Example 2.2.19. Let $O(\mathbb{V})$ be the orthogonal group of $\mathbb{V}$. Note that $O(\mathbb{V})$ acts on $G_k(\mathbb{V})$ transitively. Let $g \in O(\mathbb{V})$ be a random element that has as law the normalized Haar measure on $O(\mathbb{V})$; we will say that g is *uniform*. Let $E \in G_k(\mathbb{V})$ be any fixed k-subspace of $\mathbb{V}$. We have

$$\mathbb{E}g\underline{E} = \frac{(m-k)!\kappa_{m-k}}{m!\kappa_m}B(\mathbb{V})^{\wedge k}.$$

Indeed both sides admit as generating measure a (nonnegative) measure on $G_k(\mathbb{V})$ that is invariant under the action of $O(\mathbb{V})$. Since such a measure is unique up to a scalar multiple, it is enough to compute the length of both zonoids to prove the equality. On the left hand side we have $\ell(g\underline{E}) = \ell(\underline{E}) = 1$ and thus the zonoid on the left have length 1. On the right hand side, by Lemma 2.2.17, we have

$$\ell(B(\mathbb{V})^{\wedge k}) = \frac{1}{(m-k)!\kappa_{m-k}}\ell(B(\mathbb{V})^{\wedge m}) = \frac{m!\kappa_m}{(m-k)!\kappa_{m-k}} \tag{2.2.7}$$

Note that this gives an example of a Grassmannian zonoid that is fully decomposable although it was not immediately clear from the definition $\mathbb{E}g\underline{E}$.

In the particular case of the powers of a zonoid in $\mathbb{V}$ the support function on simple vectors have a simple interpretation.

Lemma 2.2.20. *Let $K \in \mathscr{Z}_0(\mathbb{V})$ be a centered zonoid and let $u = u_1 \wedge \cdots \wedge u_k \in \Lambda^k\mathbb{V}^*$ be a simple vector. We have*

$$h_{K^{\wedge k}}(u_1 \wedge \cdots \wedge u_k) = \frac{\|u_1 \wedge \cdots \wedge u_k\|}{2}\, k!\,\mathrm{vol}_k(\pi_u(K))$$

where $\pi_u : \mathbb{V} \to \mathrm{Span}(u_1, \ldots, u_k)$ denotes the orthogonal projection, identifying $\mathbb{V}^* \cong \mathbb{V}$ with the Euclidean structure.

Proof. Let $X \in \mathbb{V}$ be such that $K = \mathbb{E}\underline{X}$ and let $X_1, \ldots, X_k$ be iid copies of X. Then we have

$$\begin{aligned} h_{K^{\wedge k}}(u) &= \frac{1}{2}\mathbb{E}|\langle X_1 \wedge \cdots \wedge X_k, u_1 \wedge \cdots \wedge u_k\rangle| \\ &= \frac{\|u_1 \wedge \cdots \wedge u_k\|}{2}\mathbb{E}\|\pi_u(X_1) \wedge \cdots \wedge \pi_u(X_k)\| \\ &= \frac{\|u_1 \wedge \cdots \wedge u_k\|}{2}\ell(\pi_u(K)^{\wedge k}). \end{aligned}$$

Finally, by Theorem 2.2.6, we have $\ell(\pi_u(K)^{\wedge k}) = k!\,\mathrm{vol}_k(\pi_u(K))$ which concludes the proof. □

In the case $k = m-1$, this completely determines the support function, since all vectors are simple in $\Lambda^{m-1}\mathbb{V}$. In this case in Lemma 2.2.20 we find the support function of the *projection body* of K, see Definition 1.2.44.

Corollary 2.2.21. *Let $K \in \mathscr{Z}(\mathbb{V})$ and let $\Pi K \in \mathscr{Z}_0(\Lambda^{m-1}\mathbb{V})$ denotes its projection body. We have*

$$K^{\wedge(m-1)} = \frac{(m-1)!}{2}\,\Pi K.$$

This implies that the surface area measure is the wedge product of the generating measure; this was also proven in [80, Theorem 5.3.3].

Corollary 2.2.22. *Let $K \in \mathscr{Z}_0(\mathbb{V})$ and identify $\Lambda^{m-1}\mathbb{V} \cong \mathbb{V}^*$ with the Euclidean structure. We have*

$$\mathcal{S}_{m-1}(K) = \frac{4}{(m-1)!}\,\mu_K^{\wedge(m-1)}.$$

This last example together with Minkowski's uniqueness Theorem (Proposition 1.2.45) shows that the map $K \mapsto K^{\wedge(m-1)}$ is injective. Thus the following.

Proposition 2.2.23. *For all $1 \leq k \leq m-1$, the map $\mathscr{Z}_0(\mathbb{V}) \to \mathscr{G}(k, \mathbb{V})$; $K \mapsto K^{\wedge k}$ is injective.*

Note that, for $k = m-1$, it is not surjective since some zonoids in $\Lambda^{m-1}\mathbb{V}$ are the projection body of centrally symmetric bodies that are not zonoids [80, Theorem 8.2.2].

The Alexandrov–Fenchel inequality (Lemma 1.1.25) also interprets as a statement in the Grassmannian zonoid algebra. More precisely it states that if $A \in \mathscr{Z}(\Lambda^k\mathbb{V})$ is a *fully decomposable* zonoid (Definition 2.2.7) then for all $K, L \in \mathscr{Z}(\mathbb{V})$ we have

$$\ell(K \wedge L \wedge A)^2 \geq \ell(K \wedge K \wedge A)\ell(L \wedge L \wedge A). \tag{2.2.8}$$

This immediately rises the question if this still holds more generally in the (Grassmannian) zonoid algebra. The following conjecture is more a shot in the dark than something based on hard evidence.

Conjecture 5. The Alexandrov–Fenchel inequality (2.2.8) is still true if $A \in \mathscr{G}(k, \mathbb{V})$ but is not true in general for $A \in \mathscr{Z}(\Lambda^k\mathbb{V})$.

Finally let us remark that the Brunn–Minkowski inequality (Lemma 1.1.26) can also be seen as an inequality in the Grassmannian zonoid algebra. More precisely, it states that for all $K_0, K_1 \in \mathscr{Z}(\mathbb{V})$ and all $t \in [0,1]$ we have

$$\ell\big(((1-t)K_0 + tK_1)^{\wedge m}\big) \geq \ell(K_0^{\wedge m})^{(1-t)}\ell(K_1^{\wedge m})^t.$$

2.2.3 Random determinants and mixed volume

In this section we explain how statements in the (Grassmannian) zonoid algebra such as Theorem 2.2.6 interpret in terms of expected absolute determinants in the spirit of [86]. We start with the most general form.

Theorem 2.2.24 (B.B.L.M.). *Let $c_1, \ldots, c_k \in \mathbb{N}$ such that $c := c_1 + \ldots + c_k \leq m$, consider $X_1 \in \mathbb{R}^{m\times c_1}, \ldots, X_k \in \mathbb{R}^{m\times c_k}$ independent and integrable and let $M := (X_1, \ldots, X_k) \in \mathbb{R}^{m\times c}$ be the random matrix whose columns are the matrices X_i. Then we have*

$$\mathbb{E}\sqrt{\det(M^tM)} = \ell(\mathbb{E}\underline{Y_1} \wedge \cdots \wedge \mathbb{E}\underline{Y_k})$$

where $Y_k \in \Lambda^{c_i}\mathbb{R}^m$ is the image of X_i under the map $\mathbb{R}^{m\times c_i} \to \Lambda^{c_i}\mathbb{R}^m$ that maps $(x_1, \ldots, x_{c_i})$ to $x_1 \wedge \cdots \wedge x_{c_i}$ for all $x_1, \ldots, x_{c_i} \in \mathbb{R}^m$ in particular $\mathbb{E}\underline{Y_i} \in \mathscr{G}_0(c_i, \mathbb{R}^m) \subset \mathscr{Z}_0(\Lambda^{c_i}\mathbb{R}^m)$.

Proof. First of all, let us remark that the right hand side is equal to $\mathbb{E}\|Y_1 \wedge \cdots \wedge Y_k\|$. Then the rest is a matter of translating in terms of matrices what we saw before in the zonoid algebra. Let us write $M := (Z_1, \ldots, Z_c)$ with $Z_i \in R^m$ in such a way that $X_i = (Z_{\tilde{c}_{i-1}+1}, \ldots, Z_{\tilde{c}_i})$ where $\tilde{c}_0 := 0$ and $\tilde{c}_i = c_1 + \cdots + c_i$ for $0 < i \leq k$. Note that, by construction, $Y_1 \wedge \cdots \wedge Y_k = Z_1 \wedge \cdots \wedge Z_c$. Finally it is enough to see that

$$\det(M^tM) = \det(\langle Z_i, Z_j\rangle_{1\leq i,j\leq c})$$

which, by definition of the Euclidean structure on $\Lambda^c\mathbb{R}^m$, see (2.2.1), is equal to $\|Z_1 \wedge \cdots \wedge Z_c\|^2 = \|Y_1 \wedge \cdots \wedge Y_k\|^2$ and this concludes the proof. □

Remark 2.2.25. Note that, since the length is translation invariant, we can replace on the right hand side, each $\mathbb{E}\underline{Y_i}$ by $\mathbb{E}[0, Y_i]$.

In the case where $c_1 + \cdots + c_k = m$ the matrix M is a square matrix and we obtain

$$\mathbb{E}|\det(M)| = \ell(\mathbb{E}\underline{Y_1} \wedge \cdots \wedge \mathbb{E}\underline{Y_k}). \tag{2.2.9}$$

Another special case is when the blocks X_i are just columns, in this case this gives a generalization of Vitale's Theorem [86, Theorem 3.2].

Corollary 2.2.26. *Let $1 \leq k \leq m$, let $X_1, \ldots, X_k \in \mathbb{R}^m$ be independent and integrable and consider $M := (X_1, \ldots, X_k) \in \mathbb{R}^{m \times k}$, the random matrix whose columns are the vectors X_i. We have*

$$\mathbb{E}\sqrt{\det(M^t M)} = \frac{m!}{(m-k)!\kappa_{m-k}} \mathrm{MV}(\mathbb{E}\underline{X_1}, \ldots, \mathbb{E}\underline{X_k}, B_m[m-k])$$

where recall that $B_m := B(\mathbb{R}^m)$ is the unit ball and $B_m[m-k]$ denotes that it is repeated $m-k$ times in the argument of the mixed volume MV*. In particular, if all columns are identically distributed we have*

$$\mathbb{E}\sqrt{\det(M^t M)} = k! \mathrm{V}_k(\mathbb{E}\underline{X_1}).$$

where recall that V_k denotes the k-th intrisic volume (Definition 1.1.24).

Proof. By Theorem 2.2.24, we have $\mathbb{E}\sqrt{\det(M^t M)} = \ell(\mathbb{E}\underline{X_1} \wedge \cdots \wedge \mathbb{E}\underline{X_k})$. Applying Lemma 2.2.17 we find $\ell(\mathbb{E}\underline{X_1} \wedge \cdots \wedge \mathbb{E}\underline{X_k}) = \frac{1}{(m-k)!\kappa_{m-k}} \ell(\mathbb{E}\underline{X_1} \wedge \cdots \wedge \mathbb{E}\underline{X_k} \wedge B_m^{\wedge m-k})$, the result follows from (2.2.4). $\square$

In the case $k = m$, M is a square matrix and we obtain

$$\mathbb{E}|\det(M)| = m! \, \mathrm{MV}(\mathbb{E}\underline{X_1}, \ldots, \mathbb{E}\underline{X_m}).$$

If in addition all columns are identically distributed, we find Vitale's Theorem [86, Theorem 3.2]:

$$\mathbb{E}|\det(M)| = m! \, \mathrm{vol}_m(\mathbb{E}\underline{X_1}). \tag{2.2.10}$$

Example 2.2.27 (Centered Gaussian vectors and ellipsoids). Recall that a Gaussian vector $X \in \mathbb{R}^m$ is a random vector such that for all $u \in \mathbb{R}^m$, $\langle u, X\rangle \in \mathbb{R}$ is a Gaussian variable, see Section 2.6.1. If it is centered, i.e. if $\mathbb{E}X = 0$ then there is a linear map $T : \mathbb{R}^m \to \mathbb{R}^m$ such that X has the same law as $T(\xi)$ where $\xi \in \mathbb{R}^m$ is a standard Gaussian vector. The variance of X is then given by $\sqrt{T^t T}$, moreover, by Proposition 1.2.33 we have that $\mathbb{E}\underline{X} = \frac{1}{\sqrt{2\pi}} T(B_m)$ where $B_m = B(\mathbb{R}^m)$. In other words the Vitale zonoid of a centered Gaussain vector is an ellipsoid.

In the case where T is invertible, the ellipsoid $\mathcal{E}_X := T(B_m)$ is also called the *dispersion ellipsoid* of X (see [46, (1.1)]) and can be described, using Proposition 1.1.9 for $K = T(B_m)$, by:

$$\mathcal{E}_X = \left\{ x \in \mathbb{R}^m \mid \langle u, T^{-1}(x)\rangle \leq 1 \, \forall u \in S^{m-1} \right\}.$$

Now let $X_1, \ldots, X_k \in \mathbb{R}^m$, $k \leq m$, be independent centered Gaussian vectors, let $M := (X_1, \ldots, X_k) \in \mathbb{R}^{m \times k}$ be the Gaussian matrix whose columns are the X_i and write $\mathcal{E}_i := \mathcal{E}_{X_i}$ in such a way that we have $\mathbb{E}\underline{X_i} = (2\pi)^{-1/2}\mathcal{E}_i$. Applying Corollary 2.2.26, we find a new proof of the result of Zakhar Kabluchko and Dmitry Zaporozhets, [46, Theorem 1.1], namely :

$$\mathbb{E}\sqrt{\det(M^t M)} = \frac{m!}{(m-k)!\kappa_{m-k}(2\pi)^{\frac{k}{2}}} \mathrm{MV}(\mathcal{E}_1, \ldots, \mathcal{E}_k, B_m[m_k]).$$

We will see later in Section 2.6, in a more developed example what happens when the Gaussian vector is not centered.

Corollary 2.2.26 allows also to translate the Brunn–Minkowski (Lemma 1.1.26) and Alexandrov–Fenchel (Lemma 1.1.25) inequalities in term of random matrices.

Corollary 2.2.28 (BM for random determinants)**.** *Let $X_0, X_1 \in \mathbb{R}^m$ be integrable, let $\epsilon \in \{0, 1\}$ be a Bernoulli variable of parameter $t \in [0, 1]$, i.e. $\epsilon = 1$ with probability t, independent of X_0, X_1 and let $X_t := (1-\epsilon)X_0 + \epsilon X_1$. Let $M_t \in \mathbb{R}^{m \times m}$ be a random square matrix whose columns are iid copies of X_t. We have*

$$\mathbb{E}|\det M_t| \geq \left(\mathbb{E}|\det M_0|\right)^{(1-t)} \left(\mathbb{E}|\det M_1|\right)^t.$$

Corollary 2.2.29 (AF for random determinants)**.** *Let $X_1, X_1', X_2, X_2', Y_1, \ldots, Y_{m-2} \in \mathbb{R}^m$ be independent and integrable and such that X_1 and X_1' (respectively X_2 and X_2') have the same law. We have*

$$\left(\mathbb{E}|\det(X_1, X_2, Y_1, \ldots, Y_{m-2})|\right)^2 \geq \left(\mathbb{E}|\det(X_1, X_1', Y_1, \ldots, Y_{m-2})|\right) \left(\mathbb{E}|\det(X_2, X_2', Y_1, \ldots, Y_{m-2})|\right).$$

Remark 2.2.30. Note that the generalization of Alexandrov–Fenchel for Grassmannian zonoids, that is the first part of Conjecture 5, is equivalent to the fact that the previous result is still valid if the Y_i are *dependent*.

Remark 2.2.31. The equality case of Alexandrov–Fenchel for zonoids was described in [79]. From this, one can deduce the equality case for random determinants in Corollary 2.2.29.

We conclude by an example.

Example 2.2.32. Let $Z_1, \dots, Z_n \in \mathbb{C}^n$ be integrable random vectors and $L := (Z_1, \dots, Z_n) \in \mathbb{C}^{n\times n}$ be the random complex matrix whose columns are the complex vectors Z_j. We show how to "compute" $\mathbb{E}(|\det(L)|^2)$ with Theorem 2.2.24 (in the next section we will examine the case $\mathbb{E}|\det(L)|$, see (2.3.1) below). To this end, we decompose $Z_j = X_j + iY_j$ with real random vectors $X_j, Y_j \in \mathbb{R}^n$ (possibly dependent) and where $i = \sqrt{-1}$. We let $m := 2n$ and we consider the random real matrix given by $M = (M_1, \dots, M_n) \in \mathbb{R}^{m\times m}$, where $M_j = \left(\begin{smallmatrix} X_j & -Y_j \\ Y_j & X_j \end{smallmatrix}\right)$. It satisfies the hypothesis of Theorem 2.2.24 with all $c_j = 2$, $k = n$ and $c = m$. Observe that $|\det(L)|^2 = |\det(M)|$. If we define the integrable random vector

$$Q_j := \begin{pmatrix} X_j \\ Y_j \end{pmatrix} \wedge \begin{pmatrix} -Y_j \\ X_j \end{pmatrix} \in \Lambda^2(\mathbb{R}^m),$$

we get then by (2.2.9) that

$$\mathbb{E}(|\det L|^2) = \ell(\mathbb{E}\underline{Q_1} \wedge \cdots \wedge \mathbb{E}\underline{Q_n}).$$

2.3 Mixed J-volume

In this section we assume that our vector space $\mathbb{V}$ is endowed with a complex structure, that is a linear endomorphism $J : \mathbb{V} \to \mathbb{V}$ that is such that $J^2 = -\operatorname{Id}_{\mathbb{V}}$. This implies that the dimension of $\mathbb{V}$ is even and we let $m =: 2n$ and it turns $\mathbb{V}$ into a complex vector space by letting for all $x \in \mathbb{V}$ and $a, b \in \mathbb{R}$, $(a + ib) \cdot x := ax + bJ(x)$.

Given a complex structure, we can mimic all the construction of the zonoid algebra with *complex* linearity (which in particular implies real linearity). In particular mimicking (2.2.4) with *complex* wedge product leads to a new multilinear function on zonoids that we call the *mixed J-volume* and denote by MV^J. It takes n zonoids in a $2n$–dimensional real vector space, while the ordinary mixed volume is instead a function of $2n$ arguments. This notion turns out to be closely related but different from Kazarnovskii's pseudovolume (see Definition 2.3.17 below).

Before going into more details, let us describe the setting more precisely. We assume that J is an isometry for the Euclidean structure of $\mathbb{V}$. This implies that the complex bilinear form $\mathbb{V} \times \mathbb{V} \to \mathbb{C}$, $(x, y) \mapsto \langle x, y\rangle - i\langle x, Jy\rangle$ is a hermitian form.

In such a setting, a *Lagrangian plane* is a subspace $E \in G_n(\mathbb{V})$ such that $JE = E^\perp$, i.e. such that the bilinear form $\langle \cdot, J\cdot\rangle$ is identically zero on E. In other words we have $\mathbb{V} = E \oplus JE$. Thus, the choice of an orthonormal basis on E induces an isomorphism $\mathbb{V} \cong \mathbb{C}^n$ that sends E to $\mathbb{R}^n$.

We denote $\Lambda^k_{\mathbb{C}}(\mathbb{V})$ the complex exterior algebra and, given vectors $v_1, \dots, v_k \in \mathbb{V}$, we denote by $v_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} v_k \in \Lambda^k_{\mathbb{C}}(\mathbb{V})$ their complex exterior product. Note that this construction depends on the choice of the complex structure J, however we prefer the notation with "$\mathbb{C}$" that we find easier to read. Note that the hermitian structure on $\mathbb{V}$ implies an hermitian structure on the complex exterior algebra in a similar fashion as (2.2.1). In particular, taking the real part, it induces an Euclidean norm on each $\Lambda^k_{\mathbb{C}}(\mathbb{V})$ and consequently we have a length functional $\ell : \mathscr{Z}(\Lambda^k_{\mathbb{C}}(\mathbb{V})) \to \mathbb{R}$; see Definition 1.2.31.

Finally, for simplicity, we will reduce our study to the centered case, that is to $\mathscr{Z}_0(\mathbb{V})$.

2.3.1 The mixed J–volume of zonoids

The complex wedge product is, in particular, a real multilinear map. Therefore we can apply Theorem 2.1.16 to obtain a well-defined notion of complex product of virtual zonoids.

Definition 2.3.1. Consider the (real and) complex multilinear map $F \colon \mathbb{V}^n \to \Lambda^n_{\mathbb{C}}(\mathbb{V})$ defined by the complex wedge $F(v_1, \dots, v_n) := v_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} v_n$. For any $K_1, \dots, K_n \in \widehat{\mathscr{Z}_0}(\mathbb{V})$, we define:

$$K_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} K_n := \widehat{F}(K_1, \dots, K_n).$$

The next definition uses this construction to define the mixed J–volume, this is to be compared with (2.2.4).

Definition 2.3.2. We define the *mixed J–volume* $\mathrm{MV}^J : \widehat{\mathscr{Z}_0}(\mathbb{V})^n \to \mathbb{R}$ to be the $\mathbb{R}$–multilinear map given, for all $K_1, \ldots, K_n \in \widehat{\mathscr{Z}_0}(\mathbb{V})$, by:

$$\mathrm{MV}^J(K_1, \ldots, K_n) := \frac{1}{n!}\ell\left(K_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} K_n\right).$$

The *J–volume* of a zoonoid $K \in \widehat{\mathscr{Z}_0}(\mathbb{V})$ is defined to be:

$$\mathrm{vol}_n^J(K) := \mathrm{MV}^J(K, \ldots, K).$$

Remark 2.3.3. Notice that, since $\Lambda^{2n}(\mathbb{V}) \simeq \mathbb{R}$ is of real dimension one, zonoids in $\Lambda^{2n}(\mathbb{V})$ are just segments. By contrast, the top complex exterior power $\Lambda^n_{\mathbb{C}}(\mathbb{V}) \simeq \mathbb{C}$ is of real dimension two and centered zonoids in this space are more than segments (in fact they are precisely the centrally symmetric convex bodies; see [80, Theorem 3.5.2]). Thus $K_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} K_n$ is a zonoid in $\Lambda^n_{\mathbb{C}}(\mathbb{V}) \simeq \mathbb{R}^2$. Then taking its length loses some information. However, one can see using Definition 2.3.1, that if one of the K_i is invariant under the $U(1)$ action on $\mathbb{V}$, then $K_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} K_n$ is also $U(1)$ invariant and hence must be a disc centered at the origin which in this case is completely determined by its length. We compute the length of a disc in Lemma 2.3.5 below.

Let us study some of the properties of the mixed $J-$volume. On some classes of zonoids of the complex space $\mathbb{V}$ it behaves particularly well. The first case is when $\mathbb{V} = \mathbb{C}^n$ and all the zonoids are contained in the real $n-$plane $\mathbb{R}^n \subset \mathbb{C}^n$. In that case, we will show that the mixed $J-$volume is equal to the classical mixed volume (see Proposition 2.3.6 (2)).

Next, we consider complex discs.

Definition 2.3.4. Let $z \in \mathbb{V}$. We define Δ_z to be the closed centered disc of radius $|z|$ in the complex line $\mathbb{C}z \cong \mathbb{R}^2$.

In order to describe a random vector representing Δ_z, let us introduce the following notation. For $\theta \in \mathbb{R}$ we denote by $e^{\theta J} : \mathbb{V} \to \mathbb{V}$ the linear operator

$$e^{\theta J} := \cos(\theta)\,\mathrm{Id}_{\mathbb{V}} + \sin(\theta) J.$$

We then have the following lemma.

Lemma 2.3.5. *Let $\theta \in [0, 2\pi]$ be a uniformly distributed random variable and $z \in \mathbb{V}$ nonzero. Consider the random vector $X_z \in \mathbb{V}$ defined by $X_z := \pi e^{\theta J} z$. Then:*

$$\Delta_z = \mathbb{E}\underline{X_z}; \qquad\qquad \ell(\Delta_z) = \pi\|z\|.$$

Proof. Since for every $\theta \in [0, 2\pi]$ the vector $e^{\theta J}z$ belongs to $\mathbb{C}z$, we have $h_{\mathbb{E}\underline{X_z}}(u) = 0$ for every $u \in (\mathbb{C}z)^\perp$. This implies that $\mathbb{E}\underline{X_z}$ is contained in $\mathbb{C}z$. Moreover, we have $\mathbb{E}|\langle e^{\theta J}z, z\rangle| = \|z\|^2\,\mathbb{E}|\cos\theta| = \frac{2}{\pi}\|z\|^2$. This implies for $\lambda \in \mathbb{C}$ that

$$h_{\mathbb{E}\underline{X_z}}(\lambda z) = \frac{1}{2}\mathbb{E}|\langle X_z, \lambda z\rangle| = \frac{1}{2}\pi|\lambda|\,\mathbb{E}|\langle e^{\theta J}z, z\rangle| = |\lambda|\|z\|^2 = \|z\| \cdot |\lambda z|.$$

On the other hand, $h_{\Delta_z}(\lambda z) = \|z\| \cdot |\lambda z|$, hence the first assertion follows. The second statement follows immediately from the fact that $\|X_z\| = \pi\|z\|$ almost surely. □

Proposition 2.3.6 (Properties of the mixed J–volume)**.** *The following properties hold:*

(i) The mixed J–volume of zonoids $\mathrm{MV}^J : \mathscr{Z}_0(\mathbb{V})^n \to \mathbb{R}$ is symmetric, multilinear, and monotonically increasing in each variable.

(ii) Let $E \in G_n(\mathbb{V})$ be a Lagrangian plane and let $K_1, \ldots, K_n \in \mathscr{Z}_0(E) \subset \mathscr{Z}_0(\mathbb{V})$. Then:

$$\mathrm{MV}^J(K_1, \ldots, K_n) = \mathrm{MV}(K_1, \ldots, K_n).$$

(iii) Let $T : \mathbb{V} \to \mathbb{V}$ be a $\mathbb{C}$–linear transformation (i.e., such that $TJ = JT$), and denote by $\det_{\mathbb{C}}(T)$ its complex determinant. Then, for all $K_1, \ldots, K_n \in \mathscr{Z}_0(\mathbb{V})$,

$$\mathrm{MV}^J(TK_1, \ldots, TK_n) = |\det_{\mathbb{C}}(T)| \, \mathrm{MV}^J(K_1, \ldots, K_n).$$

(iv) For every $z_1, \ldots, z_n \in \mathbb{V}$ we have $\mathrm{MV}^J(\Delta_{z_1}, \cdots, \Delta_{z_n}) = \frac{\pi^n}{n!} |z_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} z_n|$

(v) For every $\theta \in \mathbb{R}$ and every $K_1, \ldots, K_n \in \mathscr{Z}_0(\mathbb{V})$ we have

$$\mathrm{MV}^J(e^{\theta J} K_1, K_2, \ldots, K_n) = \mathrm{MV}^J(K_1, \ldots, K_n).$$

(vi) Let $K_1, \ldots, K_n \in \mathscr{Z}_0(\mathbb{V})$, we have $\mathrm{MV}^J(K_1, \ldots, K_n) > 0$ if and only if there are $z_1 \in K_1, \ldots, z_n \in K_n$ such that $z_1, \ldots, z_n$ form a $\mathbb{C}$–basis of $\mathbb{V}$.

Proof. Multilinearity of MV^J follows from the definition and Theorem 2.1.16. To see that MV^J is symmetric, given zoonoids $K_1, \ldots, K_n$ in $\mathbb{V}$, let $X_1, \ldots, X_n \in \mathbb{V}$ be independent integrable random vectors such that $K_j = \mathbb{E}\underline{X_j}$. We have

$$\begin{aligned} K_1 \wedge_{\mathbb{C}} K_2 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} K_n &= \mathbb{E}\underline{X_1 \wedge_{\mathbb{C}} X_2 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} X_n} \\ &= \mathbb{E}\underline{-X_2 \wedge_{\mathbb{C}} X_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} X_n} \\ &= \mathbb{E}\underline{X_2 \wedge_{\mathbb{C}} X_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} X_n} \\ &= K_2 \wedge_{\mathbb{C}} K_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} K_n. \end{aligned}$$

The same argument gives symmetry in each pairs of variables. The fact that the mixed J–volume is monotonically increasing in each variable is a direct cosequence of the definition, Theorem 2.1.16 and the monotonicity of the length (Corollary 1.2.39) .

Let us prove point (ii). Let $K_1, \ldots, K_n \in \mathscr{Z}_0(E) \subset \mathscr{Z}_0(\mathbb{V})$ and let $X_1, \ldots, X_n \in E$ be independent and such that $K_j = \mathbb{E}\underline{X_j}$, $1 \leq j \leq n$. By Definition 2.3.2 the mixed J–volume is $\mathrm{MV}^J(K_1, \ldots, K_n) = \frac{1}{n!}\ell\left(K_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} K_n\right) = \frac{1}{n!}\mathbb{E}\|X_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} X_n\|$. We have, by Lemma 2.3.12 below, $\|X_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} X_n\| = \|X_1 \wedge \cdots \wedge X_n\|$. We get $\mathrm{MV}^J(K_1, \ldots, K_n) = \frac{1}{n!}\mathbb{E}\|X_1 \wedge \cdots \wedge X_n\|$. We conclude from (2.2.4) that the latter is equal to $\mathrm{MV}(K_1, \ldots, K_n)$.

In order to prove (iii), let $K_1, \ldots, K_n \in \mathscr{Z}_0(\mathbb{V})$ be zonoids and let again $X_1, \ldots, X_n \in \mathbb{V}$ be independent and such that $K_j = \mathbb{E}\underline{X_j}$. Then $TK_j = \mathbb{E}\underline{T(X_j)}$ and we have

$$\begin{aligned} \mathrm{MV}^J(TK_1, \ldots, TK_n) &= \frac{1}{n!}\mathbb{E}|TX_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} TX_n| \\ &= \frac{1}{n!}\mathbb{E}|(\det_{\mathbb{C}}(T))X_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} X_n| \\ &= \frac{1}{n!}|\det_{\mathbb{C}}(T)|\mathbb{E}|X_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} X_n| \\ &= |\det_{\mathbb{C}}(T)|\mathrm{MV}^J(K_1, \ldots, K_n). \end{aligned}$$

To show point (iv) we use Lemma 2.3.5 and write, for $\theta_1, \ldots, \theta_n \in [0, 2\pi]$ independent and uniformly distributed:

$$\begin{aligned} \mathrm{MV}^J(\Delta_{z_1}, \ldots, \Delta_{z_n}) &= \frac{1}{n!}\ell(\Delta_{z_1} \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} \Delta_{z_n}) \\ &= \frac{1}{n!}\ell\left(\mathbb{E}\underline{\pi e^{J\theta_1} z_1} \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} \mathbb{E}\underline{\pi e^{J\theta_1} z_n}\right) \\ &= \frac{1}{n!}\mathbb{E}\left|(\pi e^{J\theta_1} z_1) \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} (\pi e^{J\theta_n} z_n)\right| = \frac{\pi^n}{n!}|z_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} z_n|, \end{aligned}$$

which is what we wanted.

For point (v), let again $X_1, \dots, X_n \in \mathbb{V}$ be independents and such that $K_j = \mathbb{E}\underline{X_j}$. Then $e^{\theta J} K_j = \mathbb{E}\underline{e^{\theta J} X_j}$ and we have

$$\begin{aligned}\mathrm{MV}^J(e^{\theta J} K_1, K_2, \dots, K_n) &= \frac{1}{n!}\mathbb{E}|e^{\theta J} X_1 \wedge_{\mathbb{C}} X_2 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} X_n| \\ &= \frac{1}{n!}\mathbb{E}|X_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} X_n| = \mathrm{MV}^J(K_1, \dots, K_n).\end{aligned}$$

Finally for the last item, let again $X_1, \dots, X_n \in \mathbb{V}$ be independents and such that $K_j = \mathbb{E}\underline{X_j}$. Then $\mathrm{MV}^J(K_1, K_2, \dots, K_n) = 0$ if and only if $X_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} X_n = 0$ almost surely. This is true if and only if the $\mathbb{C}$-span of $X_1, \dots, X_n$ is contained almost surely in a complex hyperplane and thus $K_1, \dots, K_n$ are contained in a complex hyperplane. In other words, for all choice $z_1 \in K_1, \dots, z_n \in K_n$, the vectors $z_1, \dots, z_n$ are $\mathbb{C}$-linearly dependent and this concludes the proof. □

Conjecture 6. The mixed J-volume satisfy an *Alexandrov–Fenchel inequality*.

Maybe one could try to prove this conjecture with the method of *combinatorial atlas* as explained in [30] as it seems adapted to the algebraic nature of the J-volume of zonoids and the combinatorial nature of the J-volume of zonotopes.

2.3.2 Random complex determinants and J-volume

Here we state and prove a complex version of Theorem 2.2.24 which gives a way to describe the expectation of the modulus of the determinant of a random complex matrix with independent blocks (to compare with Example 2.2.32 where we used Theorem 2.2.24 to compute instead the expectation of the *square* of the modulus of the determinant). To do that, mimicking the definition of the wedge product of zonoids (2.2.2), we associate with each complex wedge product $\wedge_{\mathbb{C}} \colon \Lambda^k_{\mathbb{C}}\mathbb{V} \times \Lambda^l_{\mathbb{C}}\mathbb{V} \to \Lambda^{k+l}_{\mathbb{C}} V$ the componentwise continuous bilinear map

$$\wedge_{\mathbb{C}} : \widehat{\mathscr{Z}_0}\big(\Lambda^k_{\mathbb{C}}\mathbb{V}\big) \times \widehat{\mathscr{Z}_0}\big(\Lambda^l_{\mathbb{C}}\mathbb{V}\big) \to \widehat{\mathscr{Z}_0}\big(\Lambda^{k+l}_{\mathbb{C}} V\big).$$

induced from it by Theorem 2.1.16. We then have the following.

Theorem 2.3.7 (B.B.L.M.). *Let $c_1, \dots, c_k \in \mathbb{N}$ be such that $c := c_1 + \dots + c_k \le n$, consider $X_1 \in \mathbb{C}^{n\times c_1}, \dots, X_k \in \mathbb{C}^{n\times c_k}$ independent integrable and let $M := (X_1, \dots, X_k) \in \mathbb{C}^{n\times c}$ be the random complex matrix whose columns are the matrices X_i. Then we have*

$$\mathbb{E}\sqrt{\det(M^*M)} = \ell(\mathbb{E}\underline{Y_1} \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} \mathbb{E}\underline{Y_k})$$

where $M^* = \overline{M}^t$ denotes the adjoint of M and $Y_k \in \Lambda^{c_i}\mathbb{C}^m$ is the image of X_i under the map $\mathbb{C}^{m\times c_i} \to \Lambda^{c_i}_{\mathbb{C}}\mathbb{C}^m$ that maps $(z_1, \dots, z_{c_i})$ to $z_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} z_{c_i}$ for all $z_1, \dots, z_{c_i} \in \mathbb{C}^m$ in particular $\mathbb{E}\underline{Y_i} \in \mathscr{Z}_0(\Lambda^{c_i}_{\mathbb{C}}\mathbb{C}^m)$.

Proof. First note that M^*M is hermitian and in particular its determinant is real and nonnegative thus $\sqrt{\det(M^*M)}$ is well defined. Then the proof is the same as the proof of Theorem 2.2.24 noticing that M^*M is the Gramm matrix of the columns of M for the hermitian form and thus $\sqrt{\det(M^*M)}$ gives the norm of the complex wedge of the columns of M. □

As before we can derive some special cases. First, in the case where $c_1 + \cdots + c_k = n$ the matrix M is a square complex matrix and we obtain

$$\mathbb{E}|\det_{\mathbb{C}}(M)| = \ell(\mathbb{E}\underline{Y_1} \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} \mathbb{E}\underline{Y_k}).$$

Another special case is when the blocks X_i are just columns, in this case it relates to the J-volume.

Corollary 2.3.8. *Let $1 \le k \le n$, let $X_1, \dots, X_k \in \mathbb{C}^n$ be independent and integrable and consider $M := (X_1, \dots, X_k) \in \mathbb{C}^{n\times k}$, the random complex matrix whose columns are the complex vectors X_i. We have*

$$\mathbb{E}\sqrt{\det(M^*M)} = \frac{n!}{(n-k)!\kappa^J_{n-k}}\mathrm{MV}^J(\mathbb{E}\underline{X_1}, \dots, \mathbb{E}\underline{X_k}, B_{2n}[n-k])$$

where $\kappa^J_{n-k} := \mathrm{vol}^J_{n-k}(B_{2(n-k)})$ (computed in Corollary 2.3.9 below).

Proof. Once again, the proof is the same as Corollary 2.2.26, we just need to prove that

$$\ell(\mathbb{E}\underline{X_1} \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} \mathbb{E}\underline{X_k}) = \frac{n!}{(n-k)!\kappa^J_{n-k}} \mathrm{MV}^J(\mathbb{E}\underline{X_1}, \ldots, \mathbb{E}\underline{X_k}, B_{2n}[n-k]).$$

We let $\xi_1, \ldots, \xi_{n_k} \in \mathbb{C}^n \cong \mathbb{R}^{2n}$ be iid Gaussian vectors of mean 0 and variance $\sqrt{2\pi}$ in such a way that $\mathbb{E}\underline{\xi_j} = B_m$, we assume that they are independent of the X_i and we let $e_1, \ldots, e_n \in \mathbb{C}^n$ be a unitary basis. Then we have

$$\begin{aligned}\mathrm{MV}^J(\mathbb{E}\underline{X_1}, \ldots, \mathbb{E}\underline{X_k}, B_{2n}[n-k]) &= \frac{1}{n!}\ell(\mathbb{E}\underline{X_1} \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} \mathbb{E}\underline{X_k} \wedge_{\mathbb{C}} \xi_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} \xi_{n-k}) \\ &= \frac{1}{n!}\mathbb{E}\|X_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} X_k\| \, \mathbb{E}|e_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} e_k \wedge_{\mathbb{C}} \xi_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} \xi_{n-k}|\end{aligned}$$

Where we used the unitary invariance of $\xi_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} \xi_{n-k}$ and the independence of the random vectors. Denoting by $\pi : \mathbb{C}^n \to \mathrm{Span}\{e_{k+1}, \ldots, e_n\} \cong \mathbb{C}^{n-k}$ the orthogonal projection, we get

$$\begin{aligned}\mathrm{MV}^J(\mathbb{E}\underline{X_1}, \ldots, \mathbb{E}\underline{X_k}, B_{2n}[n-k]) &= \frac{1}{n!}\ell(\mathbb{E}\underline{X_1} \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} \mathbb{E}\underline{X_k})\mathbb{E}\|\pi(\xi_1) \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} \pi(\xi_{n-k}))\\ &= \frac{1}{n!}\ell(\mathbb{E}\underline{X_1} \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} \mathbb{E}\underline{X_k})(n-k)! \, \mathrm{vol}^J_{n-k}(B_{2(n-k)}\end{aligned}$$

which is what we wanted. ☐

In the case where $k = n$, M is a square complex matrix and we obtain

$$\mathbb{E}|\det_{\mathbb{C}}(M)| = n! \, \mathrm{MV}^J(\mathbb{E}\underline{X_1}, \ldots, \mathbb{E}\underline{X_n}). \tag{2.3.1}$$

If in addition all columns are identically distributed, we find:

$$\mathbb{E}|\det_{\mathbb{C}}(M)| = n! \ \mathrm{vol}^J_n(\mathbb{E}\underline{X_1}). \tag{2.3.2}$$

As an application, we compute the J–volume of balls.

Corollary 2.3.9. *The J–volume of the unit ball $B_{2n} \subset \mathbb{C}^n$ equals:*

$$\kappa^J_n := \mathrm{vol}^J_n(B_{2n}) = \frac{\pi^n}{2^{n^2}} \prod_{j=1}^n \binom{2j}{j}.$$

Proof. Let $Z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ be a random vector filled with independent standard complex Gaussians $z_j = \frac{1}{\sqrt{2}}(\xi_{j,1} + i\xi_{j,2})$, that is, $\xi_{j,1}, \xi_{j,2}, \ldots, \xi_{n,1}, \xi_{n,2}$ are independent standard real Gaussians. We claim that

$$\mathbb{E}\underline{Z} = \frac{1}{2\sqrt{\pi}} B_{2n}.$$

Indeed, let $u \in \mathbb{C}^n$. By definition, we have $h_{\mathbb{E}\underline{Z}}(u) = \frac{1}{2}\mathbb{E}|\langle Z, u\rangle|$. Using the $U(n)$–invariance of Z, we can then assume that $u = \|u\|e_1$ where e_1 is the first vector of the standard basis of $\mathbb{C}^n$. We obtain

$$h_{\mathbb{E}\underline{U}}(u) = \frac{1}{2}\mathbb{E}\left|\mathrm{Re}(Z^T\overline{u})\right| = \frac{1}{2\sqrt{2}}\|u\| \, \mathbb{E}|\xi_{1,1}| = \frac{1}{2\sqrt{2}}\|u\|\sqrt{\frac{2}{\pi}} = \frac{1}{2\sqrt{\pi}}\|u\|.$$

which is what we wanted.

Let now $M \in \mathbb{C}^{n \times n}$ be a random complex matrix whose columns are i.i.d. copies of Z. Then, the complex version of Vitale's Theorem, that is (2.3.2), gives

$$\mathbb{E}|\det_{\mathbb{C}}(M)| = \frac{n!}{(2\sqrt{\pi})^n} \ \mathrm{vol}^J_n(B_{2n}).$$

Now, we show that $\mathbb{E}|\det(M)| = \prod_{j=1}^n \Gamma(j + \frac{1}{2})/\Gamma(j)$. Indeed, note that $|\det M|$ has the same law as $\det(W)^{\frac{1}{2}}$ where $W = MM^*$ is a complex Wishart matrix. Following [28, p. 83-84], we see that $\det(W)$

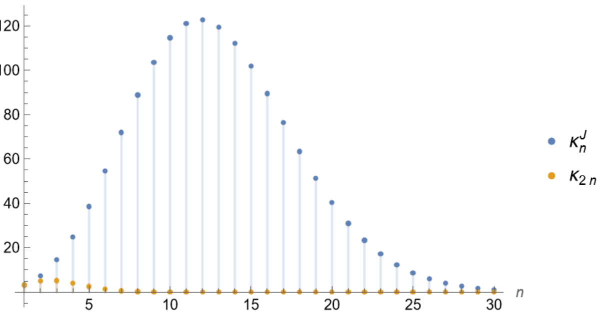


Figure 2.1: κ_n^J and κ_{2n} as a function of n

is distributed as $\frac{1}{2^n}\chi_{2n}^2 \cdot \chi_{2n-2}^2 \cdots \chi_2^2$, where each χ_{2j}^2 denotes a chi–square distribution with $2j$ degrees of freedom and the χ_{2j}^2 are independent. Therefore, $|\det(M)|$ has the same law as $\frac{1}{2^{n/2}}\chi_{2n} \cdot \chi_{2n-2} \cdots \chi_2$. Recall from (1.2.7) that $\mathbb{E}\chi_{2j} = \frac{\sqrt{2}\Gamma(j+\frac{1}{2})}{\Gamma(j)}$. Using independence, we have proved that:

$$\kappa_n^J = \mathrm{vol}_n^J(B_{2n}) = \frac{(4\pi)^{n/2}}{n!} \prod_{j=1}^{n} \frac{\Gamma(j+\frac{1}{2})}{\Gamma(j)}. \tag{2.3.3}$$

The half integer value of the Gamma function are computed in Lemma A.3 and we get

$$\frac{\Gamma(j+\frac{1}{2})}{\Gamma(j)} = j\,\frac{\pi^{\frac{1}{2}}}{2^{2j}}\binom{2j}{j}.$$

Thus the product gives

$$\prod_{j=1}^{n} \frac{\Gamma\left(j+\frac{1}{2}\right)}{\Gamma(j)} = n!\,\frac{\pi^{\frac{n}{2}}}{2^{n(n+1)}} \prod_{j=1}^{n} \binom{2j}{j}.$$

Reintroducing in (2.3.3) gives the result. □

Notice that applying Corollary 2.3.9 when $n = 1$ we get $\mathrm{vol}_1^J(B_2) = \pi = \mathrm{vol}_2(B_2)$, but already when $n = 2$ we get $\mathrm{vol}_2^J(B_4) = \frac{3\pi^2}{4} \neq \frac{\pi^2}{2} = \mathrm{vol}_4(B_4)$. In general vol_n^J and vol_{2n} are different, starting by the fact that the first is homogeneous of degree n while the other is of degree $2n$, see also Figure 2.1.

2.3.3 The J–volume of polytopes

In this section we show that it is possible to extend the notion of J–volume to polytopes in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. To do so, we give an alternative formula for the J–volume of zonotopes that makes sense for any polytope (Theorem 2.3.14). To obtain this formula, we rely on the special property of normal cones of a zonotope that is given by Lemma 1.2.9. We will see later that this connects to the theory of valuations. However, as we will see, it is not possible to continuously extend the J–volume continuously from polytopes to all convex bodies.

As a first step, we will give in Proposition 2.3.13 below an alternative way of writing the J–volume of zonotopes. This involves the following quantity, from now on we will denote by $G(k, m) := G_k(\mathbb{R}^m)$.

Definition 2.3.10. For every $E \in G(n, 2n)$, we define

$$\sigma^J(E) := |E \wedge JE| = |e_1 \wedge \cdots \wedge e_n \wedge Je_1 \wedge \cdots \wedge Je_n| \in [0, 1]$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of E.

One can check that this definition does not depend on the choice of an orthonormal basis. Moreover, σ^J is invariant under the action of $U(n)$ on $G(n, 2n)$. Note that $\sigma^J(E) = 1$ if and only if E is *Lagrangian*, and $\sigma^J(E) = 0$ if and only if E contains a complex line.

Remark 2.3.11. Denoting by $\theta_1(E) \leq \cdots \leq \theta_{\lfloor \frac{n}{2} \rfloor}(E)$ the *Kähler angles* of E, introduced by Tasaki in [84], one can check that

$$\sigma^J(E) = \prod_{j=1}^{\lfloor \frac{n}{2} \rfloor} (\sin\theta_i(E))^2.$$

In general, $\sigma^J(E)$ can be computed using the following lemma.

Lemma 2.3.12. *Let $z_1, \ldots, z_n \in \mathbb{C}^n$ be $\mathbb{R}$-linearly independent and denote by $E \in G(n, 2n)$ its real span. Then, writing $z_j = x_j + iy_j$ with $x_j, y_j \in \mathbb{R}^n$, we have*

$$|z_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} z_n| = \left\| \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \wedge \cdots \wedge \begin{bmatrix} x_n \\ y_n \end{bmatrix} \right\| \cdot \sigma^J(E)^{\frac{1}{2}}.$$

Proof. Consider the matrices $X = (x_1, \ldots, x_n) \in \mathbb{R}^{n\times n}$ and $Y = (y_1, \ldots, y_n) \in \mathbb{R}^{n\times n}$ with the columns x_j, y_j. One can check that $\det \left(\begin{smallmatrix} X & -Y \\ Y & X \end{smallmatrix}\right) = |\det(X + iY)|^2$, see [17, Lemma 5]. In particular, we can write

$$\begin{aligned} |z_1 \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} z_n|^2 = |\det(X+iY)|^2 &= \left| \det \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \right| \\ &= \left| \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \wedge \cdots \wedge \begin{bmatrix} x_n \\ y_n \end{bmatrix} \wedge \begin{bmatrix} -y_1 \\ x_1 \end{bmatrix} \wedge \cdots \wedge \begin{bmatrix} -y_n \\ x_n \end{bmatrix} \right| \\ &= \left\| \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \wedge \cdots \wedge \begin{bmatrix} x_n \\ y_n \end{bmatrix} \right\|^2 \cdot |e_1 \wedge \cdots \wedge e_n \wedge Je_1 \wedge \cdots Je_n|, \end{aligned}$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for E, and the last equality follows from:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \wedge \cdots \wedge \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \left\| \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \wedge \cdots \wedge \begin{bmatrix} x_n \\ y_n \end{bmatrix} \right\| \cdot e_1 \wedge \cdots \wedge e_n.$$

The conclusion follows from Definition 2.3.10. □

Recall the notation from Section 1.2.1: for a k-dimensional face F of a polytope P in $\mathbb{R}^m$, we denote by $E_F \in G(k, m)$ the vector space parallel to the affine span of F. For $0 \leq k \leq m$ recall Definition 1.2.7 $G_k(P) := \{E \in G(k, m) \,|\, \text{there exists a } k\text{-dim. face } F \text{ of } P \text{ such that } E = E_F\}$ and its particular form (1.2.3) in the case of a zonotope $P = \underline{x_1} + \cdots + \underline{x_n}$ for some $x_1, \ldots x_n \in \mathbb{R}^m$. Recall also in that case Definition 1.2.8 of the "vectorial" face $F(E, P) = \sum_{x_i \in E} \underline{x_i}$.

The next result gives an explicit expression of the J–volume of a zonotope.

Proposition 2.3.13. *Let $P \subset \mathbb{C}^n$ be a centered zonotope. Then*

$$\mathrm{vol}_n^J(P) = \sum_{E \in G_n(P)} \mathrm{vol}_n(F(E, P)) \cdot \sigma^J(E)^{\frac{1}{2}}.$$

Proof. We write $P = \sum_{j=1}^p \underline{z_j}$. By definition, we have $\mathrm{vol}_n^J(P) = \frac{1}{n!}\ell(P^{\wedge_{\mathbb{C}} n})$. Using multilinearity and Theorem 2.1.16, we can write

$$\left(\sum_{j=1}^p \underline{z_j} \right) \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} \left(\sum_{j=1}^p \underline{z_j} \right) = \sum_{1 \leq j_1, \ldots, j_n \leq p} \underline{w_{j_1, \ldots, j_n}},$$

where $w_{j_1,\ldots,j_n} := z_{j_1} \wedge_{\mathbb{C}} \cdots \wedge_{\mathbb{C}} z_{j_n}$. Therefore, using the linearity of the length,

$$\mathrm{vol}_n^J(P) = \frac{1}{n!} \sum_{1 \leq j_1, \ldots, j_n \leq p} |w_{j_1,\ldots,j_n}| = \sum_{j_1 < \ldots < j_n} |w_{j_1,\ldots,j_n}|.$$

We may assume the sum runs only over the $j_1 < \ldots < j_n$ such that the real span $E_{j_1,\ldots,j_n}$ of $z_{j_1}, \ldots, z_{j_n}$ has dimension n. We write $z_j = x_j + iy_j$ with $x_j, y_j \in \mathbb{R}^n$ and use Lemma 2.3.12 to obtain

$$|w_{j_1,\ldots,j_n}| = \left\| \begin{bmatrix} x_{j_1} \\ y_{j_1} \end{bmatrix} \wedge \cdots \wedge \begin{bmatrix} x_{j_n} \\ y_{j_n} \end{bmatrix} \right\| \cdot \sigma^J(E_{j_1,\ldots,j_n})^{\frac{1}{2}}.$$

Combining this and exchanging the order of summation, we arrive at

$$\mathrm{vol}_n^J(P) = \sum_{E\in G_n(P)} \sigma^J(E)^{\frac{1}{2}} \cdot \sum_{E_{j_1,\dots,j_n}=E} \left\| \begin{bmatrix} x_{j_1} \\ y_{j_1} \end{bmatrix} \wedge \cdots \wedge \begin{bmatrix} x_{j_n} \\ y_{j_n} \end{bmatrix} \right\|, \tag{2.3.4}$$

where for fixed $E \in G_n(P)$, the second sum runs over all $j_1 < \ldots < j_n$ such that $E = E_{j_1,\dots,j_n}$. Shephard's formula [81, (57)] applied to the zonotope $F(E,P)$ (see Definition 1.2.8) tells us that

$$\sum_{E_{j_1,\dots,j_n}=E} \left\| \begin{bmatrix} x_{j_1} \\ y_{j_1} \end{bmatrix} \wedge \cdots \wedge \begin{bmatrix} x_{j_n} \\ y_{j_n} \end{bmatrix} \right\| = \mathrm{vol}_n(F(E,P)).$$

Substituting this into (2.3.4) gives the statement. □

We now turn to a key result of this section. Based on the property of the external angle of a zonotope given by Lemma 1.2.9 we can give an alternate formula for the J–volume of zonotopes.

Theorem 2.3.14 (B.B.L.M.). *Let $P \subset \mathbb{C}^n$ be a zonotope. Then*

$$\mathrm{vol}_n^J(P) = \sum_{F\in\mathcal{F}_n(P)} \mathrm{vol}_n(F) \cdot \Theta(P,F) \cdot \sigma^J(E_F)^{\frac{1}{2}},$$

where recall that $\mathcal{F}_n(P)$ denotes the set of n–dimensional faces of P.

Proof. We will prove that the right hand side in this theorem is equal to the right hand side in Proposition 2.3.13. Let $z_1, \dots, z_p \in \mathbb{C}^n$ be such that $P = \sum_{j=1}^p \frac{1}{2}[-z_j, z_j]$ and let $E \in G_n(P)$. As we discussed in Section 1.2.1 (see (1.2.4)), all the faces F of P such that $E_F = E$ are translates of the vectorial face $F(E,P) = \sum_{z_j\in E} \underline{z_j}$, we can thus write:

$$\sum_{F\in\mathcal{F}_n(P)} \mathrm{vol}_n(F)\Theta(P,F)\sigma^J(E_F)^{\frac{1}{2}} = \sum_{E\in G_n(P)} \mathrm{vol}_n(F(E,P))\sigma^J(E_F)^{\frac{1}{2}} \sum_{E_F=E} \Theta(P,F).$$

But, by Lemma 1.2.9 $\sum_{E_F=E} \Theta(P,F) = 1$ and this gives what we wanted. □

We now note that the formula in Theorem 2.3.14 still makes sense for polytopes that are not zonotopes. We use this to define the J–volume on polytopes.

Definition 2.3.15. Let P be polytope in $\mathbb{C}^n$. We define its J–volume to be

$$\mathrm{vol}_n^J(P) := \sum_{F\in\mathcal{F}_n(P)} \mathrm{vol}_n(F) \cdot \Theta(P,F) \cdot \sigma^J(E_F)^{\frac{1}{2}}$$

where $\mathcal{F}_n(P)$ denotes the set of n–dimensional faces of P.

We will later study the J–volume in the framework of the theory of valuations on polytopes. For now, let us show some properties of the J–volume on polytopes.

Proposition 2.3.16. *The J–volume on $\mathcal{P}(\mathbb{C}^n)$ has the following properties.*

(i) The valuation vol_n^J is n–homogeneous and $U(n)$–invariant.

(ii) Let $P \subset \mathbb{R}^n \subset \mathbb{C}^n$ be a polytope. Then $\mathrm{vol}_n^J(P) = \mathrm{vol}_n(P)$.

Proof. The first item follows from the $U(n)$-invariance of σ^J and Definition 2.3.15. For the second item, if P is of dimension less than n, both volumes are zero and there is nothing to prove. If $\dim(P) = n$, its only face of dimension n is P itself and $E_P = \mathbb{R}^n$. Moreover $\sigma^J(\mathbb{R}^n) = 1$. Finally since $N_P(P) = (\mathbb{R}^n)^\perp$ we have $\Theta(P,P) = 1$. The claim follows with Definition 2.3.15. □

We will see later that it is not possible to extend the J–volume continuously on the whole $\mathscr{K}(\mathbb{C}^n)$. In particular, Andreas Bernig observed that this implies that the J–volume is not increasing on polytopes since this property would imply the extension.

We conclude with the notion of *Kazarnovskii's pseudovolume* [48]. We use the expression found in [4] in the proof of his Proposition 3.3.1. The normalization constant can be determined using the fact that it agrees with the classical volume on $\mathbb{R}^n \subset \mathbb{C}^n$ just like the J–volume.

Definition 2.3.17. The Kazarnovskii's pseudovolume vol_n^K is given for any polytope $P \subset \mathbb{C}^n$ by the formula

$$\mathrm{vol}_n^K(P) = \sum_{F\in\mathcal{F}_n(P)} \mathrm{vol}_n(F)\cdot\Theta(P,F)\cdot\frac{1}{(\kappa_n)^2}\,\mathrm{vol}_{2n}\left(B(E_F)+JB(E_F)\right),$$

where recall that $\mathcal{F}_n(P)$ denotes the set of n-dimensional faces of P, $B(E_F)$ denotes the unit ball of E_F, and $\kappa_n := \mathrm{vol}_n(B(\mathbb{R}^n))$.

In our setting we prove the following, to be compared to Definition 2.3.15.

Proposition 2.3.18. *For any polytope $P \in \mathcal{P}(\mathbb{C}^n)$ the Kazarnovskii pseudovolume is given by*

$$\mathrm{vol}_n^K(P) = \sum_{F\in\mathcal{F}_n(P)} \mathrm{vol}_n(F)\cdot\Theta(P,F)\cdot\sigma^J(E_F),$$

where $\mathcal{F}_n(P)$ denotes the set of n-dimensional faces of P;

Proof. We need to prove that for any $E \in G(n,2n)$ we have

$$\mathrm{vol}_{2n}(B(E)+JB(E)) = (\omega_n)^2\sigma^J(E). \tag{2.3.5}$$

Using (2.2.4) we write

$$\begin{aligned}\mathrm{vol}_{2n}(B(E)+JB(E)) &= \frac{1}{(2n)!}\ell\left(\left(B(E)+JB(E)\right)^{\wedge 2n}\right)\\ &= \frac{1}{(2n)!}\sum_{j=0}^{2n}\binom{2n}{j}\ell\left((B(E))^{\wedge j}\wedge(JB(E))^{\wedge(2n-j)}\right)\end{aligned}$$

where we used that ℓ is linear. Since $\dim(B(E)) = \dim(JB(E)) = n$, we see from Lemma 2.2.5 that $(B(E))^{\wedge j} = 0$ whenever $j > n$ and that $(JB(E))^{\wedge(2n-j)} = 0$ whenever $j < n$. In other words, only the index $j = n$ contributes to the sum and we get

$$\mathrm{vol}_{2n}(B(E)+JB(E)) = \frac{1}{(n!)^2}\,\ell\left((B(E))^{\wedge n}\wedge(JB(E))^{\wedge n}\right). \tag{2.3.6}$$

Next let $X\in E$ be such that $\mathbb{E}\underline{X} = B(E)$ (for instance, a Gaussian vector in E of variance $\sqrt{2\pi}$) and let $X_1,\dots,X_n$ be i.i.d. copies of X. Let $e_1,\dots,e_n$ be an orthonormal basis of E. Note that we have

$$X_1\wedge\cdots\wedge X_n = \pm\|X_1\wedge\cdots\wedge X_n\|\;e_1\wedge\cdots\wedge e_n.$$

With this in mind, (2.3.6) gives

$$\begin{aligned}\mathrm{vol}_{2n}(B(E)+JB(E)) &= \frac{1}{(n!)^2}\,\mathbb{E}|X_1\wedge\cdots\wedge X_n\wedge JX_{n+1}\wedge\cdots\wedge X_{2n}|\\ &= \frac{1}{(n!)^2}\left(\mathbb{E}\|X_1\wedge\cdots\wedge X_n\|\right)^2\sigma^J(E)\end{aligned}$$

Again, by (2.2.4), we have $\mathbb{E}\|X_1\wedge\cdots\wedge X_n\| = n!\,\mathrm{vol}_n(B(E)) = n!\omega_n$ and this gives (2.3.5), which concludes the proof. $\square$

2.4 Operations in the zonoid algebra

In this section we describe several operations that one can do in the zonoid algebra and in the Grassmannian algebra. It will be of great help in the next chapter when studying the valuations induced from zonoids.

2.4.1 Hodge star

In the exterior algebra of $\mathbb{V}$ there is a collection of linear maps called the Hodge star denoted

$$* : \Lambda^k \mathbb{V} \to \Lambda^{m-k} \mathbb{V}^*$$

that is given for all $x \in \Lambda^k \mathbb{V}$ and $y \in \Lambda^{m-k} \mathbb{V}$ by

$$\langle *x, y \rangle := \langle \omega_{\mathbb{V}^*}, x \wedge y \rangle.$$

Where $\omega_{\mathbb{V}^*} \in \Lambda^m \mathbb{V}^*$ denotes the volume form of $\mathbb{V}^*$. Note that the Euclidean structure determines the volume form only up to sign, so we fix one. This choice will not matter for centered zonoids.

It can be shown that the Hodge star is a linear isometry and that it is an involution up to sign, in the sense that for all $x \in \Lambda^k \mathbb{V}$, $* * x = (-1)^k x$.

Moreover, the Hodge star preserves the simple vectors and sends a vector space $E \in G_k(\mathbb{V})$ to its orthogonal $E^\perp \in G_{m-k}(\mathbb{V}^*)$.

The Hodge star induces a map in the zonoid algebras and since it preserves the Grassmannian/simple vectors also on the Grassmannian zonoid algebras:

$$* : \widehat{\mathscr{A}}(\mathbb{V}) \to \widehat{\mathscr{A}}(\mathbb{V}^*); \qquad\qquad * : \widehat{\mathscr{G}}(\mathbb{V}) \to \widehat{\mathscr{G}}(\mathbb{V}^*).$$

Note that it reverse the grading in the sense that if A is an element of degree k then $*A$ is of degree $m-k$. Moreover note that when restricted to centered zonoid we have $** = \mathrm{Id}_{\widehat{\mathscr{A}_0}(\mathbb{V})}$.

This allows us to define a dual to the wedge product on the zonoid algebra.

Definition 2.4.1. Let $A, A' \in \widehat{\mathscr{A}_0}(\mathbb{V})$, we define their *convolution* to be

$$A \vee A' := * \left((*A) \wedge (*A') \right).$$

This operation respects the *dual grading*, i.e. if A is of degree $m-k$ and A' of degree $m-l$ then $A \vee A'$ is of degree $m-(k+l)$ and the neutral element is $\underline{\omega_{\mathbb{V}}}$ where $\omega_{\mathbb{V}} \in \Lambda^m \mathbb{V}$ is the volume form.

Because the Hodge star preserves the Grassmannian zonoids, $\widehat{\mathscr{G}_0}(\mathbb{V})$ is also a subalgebra of $\widehat{\mathscr{A}_0}(\mathbb{V})$, for the convolution $\vee$.

2.4.2 The pairing

Definition 2.4.2. Consider the bilinear map $\mathbb{V}^* \times \mathbb{V} \to \mathbb{R}$. By FTZC (Theorem 2.1.16), it induces a map $\widehat{\mathscr{Z}_0}(\mathbb{V}^*) \times \widehat{\mathscr{Z}_0}(\mathbb{V}) \to \widehat{\mathscr{Z}_0}(\mathbb{R})$. Composed with the length this defines a bilinear form:

$$\widehat{\mathscr{Z}_0}(\mathbb{V}^*) \times \widehat{\mathscr{Z}_0}(\mathbb{V}) \to \mathbb{R}.$$

For all $A \in \widehat{\mathscr{Z}_0}(\mathbb{V}^*)$, $B \in \widehat{\mathscr{Z}_0}(\mathbb{V})$, we denote their image by this bilinear form by $(A, B) \in \mathbb{R}$.

For Vitale zonoids this pairing is quite explicit. Indeed, by Proposition 2.1.18, for all $U \in \mathbb{V}^*$ and $X \in \mathbb{V}$ integrable and independent we have

$$(\mathbb{E}\underline{U}, \mathbb{E}\underline{X}) = \mathbb{E}|\langle U, X \rangle|. \tag{2.4.1}$$

From this we obtain the following.

Lemma 2.4.3. *For all $K \in \mathscr{Z}_0(\mathbb{V})$ and $u \in \mathbb{V}^*$, we have*

$$\begin{aligned}(\underline{u}, K) &= 2h_K(u);\\ (B(\mathbb{V}^*), K) &= 2\ell(K).\end{aligned}$$

Proof. The first statement is a direct consequence of (2.4.1) letting $U = u$ almost surely. The second one follows from the first and Proposition 1.2.38, considering U to be a standard Gaussian vector. □

Corollary 2.4.4. *The pairing defined in Definition 2.4.2 is non degenerate.*

Proof. Suppose that $K - L \in \widehat{\mathscr{Z}_0}(\mathbb{V})$ is such that $(\cdot, K - L)$ is identically zero on $\widehat{\mathscr{Z}_0}(\mathbb{V}^*)$. In particular for all $u \in \mathbb{V}^*$, $h_{K-L}(u) = h_K(u) - h_L(u) = 0$. This implies that $K - L = 0$ in $\widehat{\mathscr{Z}_0}(\mathbb{V})$. A similar argument shows it for the first variable □

Fixing $L \in \mathscr{Z}(\mathbb{V}^*)$ we thus get a linear map $(L, \cdot) : \widehat{\mathscr{Z}}(\mathbb{V}) \to \mathbb{R}$. By FTZC the pairing is continuous in each variable for the strong topology so this defines an element of the dual space. Since it is non degenerate, this means that it defines an injective map

$$\widehat{\mathscr{Z}_0}(\mathbb{V}^*) \to \widehat{\mathscr{Z}_0}(\mathbb{V})^* \tag{2.4.2}$$

where on the right we have the dual of the vector space $\widehat{\mathscr{Z}_0}(\mathbb{V})$ with norm topology. The continuity of this map is related to the continuity of the pairing as a function of two variable, which we saw in general is not always guaranteed and depends on the choice of topology. The next lemma gives another interpretation of the pairing which will make the choice of topology natural.

Lemma 2.4.5. *Let $B \in \widehat{\mathscr{Z}_0}(\mathbb{V})$ and $A \in \widehat{\mathscr{Z}_0}(\mathbb{V}^*)$ and recall that $\mu_A \in \mathcal{M}(S(\mathbb{V}^*))$ denotes the generating measure of A and $\bar{h}_B \in C(S(\mathbb{V}^*))$ the restriction of the support function of B to the unit sphere $S(\mathbb{V}^*)$. We have*

$$\langle \mu_A, \bar{h}_B \rangle = \int_{S(\mathbb{V}^*)} h_B(u) \mathrm{d}\mu_A(u) = \frac{1}{4}(A, B).$$

Proof. Suppose $A = \mathbb{E}\underline{U}$ for some integrable $U \in \mathbb{V}^*$. Then by Proposition 1.2.43, we have that $\langle \mu_A, \bar{h}_B \rangle = \frac{1}{2}\mathbb{E}h_B(U)$. Now assume that $K = \mathbb{E}\underline{X}$ for some $X \in \mathbb{V}$ integrable and independent from U, we obtain $\langle \mu_A, \bar{h}_B \rangle = \frac{1}{4}\mathbb{E}|\langle U, X\rangle|$ and we conclude by (2.4.1). □

Corollary 2.4.6. *The pairing*

$$\widehat{\mathscr{Z}_0}(\mathbb{V}^*) \times \widehat{\mathscr{Z}_0}(\mathbb{V}) \to \mathbb{R}$$

is continuous when $\widehat{\mathscr{Z}_0}(\mathbb{V}^*)$ is endowed with the weak-$*$ topology and $\widehat{\mathscr{Z}}(\mathbb{V})$ with the norm topology.

Remark 2.4.7. Maybe at this point the reader start feeling dizzy about the choices of topology. There is nothing really mysterious here. The space $\mathscr{Z}_0(\mathbb{V})$ with the norm topology is a subspace of $C(S(\mathbb{V}^*))$. This subspace is dense in the even continuous functions and thus its dual $\mathscr{Z}_0(\mathbb{V})^*$ is $C_{even}(S(\mathbb{V}^*))^*$ which is again the space of signed even measures $\mathcal{M}(S(\mathbb{V}^*))$. Then, Lemma 2.4.5 tells us that the map (2.4.2) takes a (virtual) zonoid and gives its generating measure. This is of course continuous in the weak-$*$ topology since it is the very definition of this topology.

Next we see how this pairing behaves in the zonoid algebra with the wedge product and the Hodge star.

Proposition 2.4.8. *Let $A, A' \in \widehat{\mathscr{A}_0}(\mathbb{V}^*)$, $B, B' \in \widehat{\mathscr{A}_0}(\mathbb{V})$ remember that $\omega_{\mathbb{V}} \in \Lambda^m\mathbb{V}$ denotes the volume form of $\mathbb{V}$ and consider 1 and any other scalar as part of $\Lambda^0\mathbb{V} = \mathbb{R}$. The following holds.*

(i) $(A, B) = (*B, *A)$;

(ii) $(A \wedge A', B) = (A, *A' \vee B)$ *and* $(A, B \wedge B') = (A \vee *B', B)$;

(iii) $(*B, B') = \left(\underline{\omega_{\mathbb{V}^*}}, B \wedge B'\right) = (*B \vee *B', \underline{1})$

Proof. Point (i) follows from the fact that $|\langle u, x\rangle| = |\langle *x, *u\rangle|$.

Point (ii) is derived from the identity $|\langle x \wedge y, z\rangle| = |\langle x, *(y \wedge *z)\rangle|$.

To prove point (iii) write $*B = \underline{\omega_{\mathbb{V}^*}} \vee *B$ or $B' = B' \wedge \underline{1}$ and apply point (ii). □

Remark 2.4.9. Point (iii) could be reformulated by saying that $(*B, B')$ is the (length of) the term of degree m in $B \wedge B'$ or the term of degree 0 in $*B \vee *B'$.

2.4.3 Invariant zonoids: measure theoretic approach

The standard action $Gl(\mathbb{V}) \curvearrowright \mathbb{V}$ induces an action on the exterior algebra given for all $g \in Gl(\mathbb{V})$ and simple vectors $x_1 \wedge \cdots \wedge x_k \in \Lambda^k\mathbb{V}$ by

$$g \cdot (x_1 \wedge \cdots \wedge x_k) := g(x_1) \wedge \cdots \wedge g(x_k). \tag{2.4.3}$$

In particular this action preserves simple vectors. In other words this gives a group morphism

$$\rho^{\wedge k} : Gl(\mathbb{V}) \to Gl(\Lambda^k\mathbb{V}).$$

In our case, this map preserves orthogonality in the sense that the image of the the orthogonal group $O(\mathbb{V})$ is contained in $O(\Lambda^k\mathbb{V})$. This is because for all $g \in Gl(\mathbb{V})$, $(\rho^{\wedge k}(g))^t = \rho^{\wedge k}(g^t)$.

This action induces an action on the zonoid algebra $Gl(\mathbb{V}) \curvearrowright \widehat{\mathscr{A}}(\mathbb{V})$ and on the Grassmannian algebras $\widehat{\mathscr{G}}(\mathbb{V}), \widehat{\mathscr{G}_0}(\mathbb{V})$. From (2.4.3), we see that it satisfies for all $A, B \in \widehat{\mathscr{G}}(\mathbb{V})$

$$g \cdot (A \wedge B) = (g \cdot A) \wedge (g \cdot B). \tag{2.4.4}$$

Moreover we have the following.

Proposition 2.4.10. *Let $A \in \widehat{\mathscr{A}_0}(\mathbb{V}^*), B \in \widehat{\mathscr{A}_0}(\mathbb{V})$, for all $g \in Gl(\mathbb{V})$ we have*

$$(A, g \cdot B) = (g^t \cdot A, B).$$

Thus if $g \in O(\mathbb{V})$ $(g \cdot A, g \cdot B) = (A, B)$ and in particular $\ell(g \cdot B) = \ell(B)$.

Of course the action of $Gl(\mathbb{V})$ on the Grassmannian algebra splits into $\widehat{\mathscr{G}_0}(\mathbb{V}) = \bigoplus_{k=0}^m \widehat{\mathscr{G}_0}(k, \mathbb{V})$. We will see in the next section that for $1 < k < m-1$, $\widehat{\mathscr{G}_0}(k, \mathbb{V})$ admits a closed $Gl(\mathbb{V})$-invariant subspace.

Let us say a word about invariant zonoids.

Definition 2.4.11. Let G be a group acting on a space $\mathcal{S}$. We denote by $\mathcal{S}^G$ the subset of the points fixed by the action of G, that is $\mathcal{S}^G := \{x \in \mathcal{S} \,|\, g \cdot x = x \,\forall g \in G\}$.

Let $H \subset O(\mathbb{V})$ be a closed subgroup and consider the action on $S(\Lambda^k\mathbb{V})$ induced by the $Gl(\mathbb{V})$ action (2.4.3). By definition of the quotient topology we have an identification of the spaces of continuous functions $C(S(\Lambda^k\mathbb{V}))^H \cong C\left(S(\Lambda^k\mathbb{V})/H\right)$. Dualizing we get

$$\mathcal{M}(S(\Lambda^k\mathbb{V}))^H \cong \mathcal{M}\left(S(\Lambda^k\mathbb{V})/H\right).$$

In particular, since $O(\mathbb{V})$, and thus H, preserves the Grassmannian, we have

$$(\widehat{\mathscr{G}_0}(k, \mathbb{V})^H, \text{weak-}*) \cong (\mathcal{M}\left(G_k(\mathbb{V})/H\right), \text{weak-}*)$$

and

$$(\widehat{\mathscr{G}_0}(k, \mathbb{V})^H, \|\cdot\|) \hookrightarrow (C\left(S(\Lambda^k\mathbb{V})/H\right), \|\cdot\|_\infty)$$

Remark 2.4.12. There is a unique normalized Haar measure on H, recall that a random element $g \in \mathbb{V}$ is called *uniform* if its law is this Haar measure of probability. This induces a projection

$$pr_H : \widehat{\mathscr{A}}(\mathbb{V}) \to \widehat{\mathscr{A}}(\mathbb{V})^H$$

given for all $K \in \mathscr{Z}(\Lambda^k\mathbb{V})$ by

$$pr_H(K) := \mathbb{E} g \cdot K$$

where $g \in H$ is uniform. This projection preserves the Grassmannian algebra because of (2.4.3).

The space $G_k(\mathbb{V})/H$ is the space of H-orbits. For every $x \in G_k(\mathbb{V})$ we have a point in the orbit space $H \cdot x \in G_k(\mathbb{V})/H$. The Dirac delta measures $\delta_{H\cdot x}$ are dense in $\mathcal{M}\left(G_k(\mathbb{V})/H\right) \cong \mathcal{M}\left(G_k(\mathbb{V})\right)^H$ they are the analogous of the segments for the invariant zonoids.

Definition 2.4.13. For every $x \in G_k(\mathbb{V})$ we define $\underline{H \cdot x} \in \mathscr{G}(k, \mathbb{V})$ to be the zonoid whose generating measure is $\delta_{H\cdot x} \in \mathcal{M}\left(G_k(\mathbb{V})/H\right) \cong \mathcal{M}\left(G_k(\mathbb{V})\right)^H$

One can see that we have

$$\underline{H \cdot x} = pr_H(\underline{x}).$$

Because of (2.4.4), the product of invariant zonoids is again invariant. Thus the wedge product defines an algebra structure on $\bigoplus_k \mathcal{M}\left(G_k(\mathbb{V})/H\right)$. The structure constants of this algebra are given by the measures

$$\mu_{\underline{H\cdot x}\wedge\underline{H\cdot y}} = \delta_{H\cdot x} \wedge \delta_{H\cdot y}$$

for all $x \in G_k(\mathbb{V})$ and $y \in G_l(\mathbb{V})$.

Example 2.4.14. If we let $H = O(\mathbb{V})$ be the whole group then the action is transitive on every Grassmannian and the orbits spaces are all reduced to a point. Indeed we have

$$\widehat{\mathscr{G}_0}(\mathbb{V})^{O(\mathbb{V})} = \bigoplus_{k\geq 0} \mathbb{R}B(\mathbb{V})^{\wedge k} \cong \mathbb{R}[t]/(t^{m+1}).$$

Example 2.4.15. Let $m = 2n$ and let $\mathbb{V} = \mathbb{R}^{2n} = \mathbb{C}^n$. Consider the action of $U(n)$ on $G_k(\mathbb{R}^{2n})$. Then Tasaki showed in [84] that, for $1 \leq k \leq n$, given $E \in G_k(\mathbb{R}^{2n})$, a complete invariant of the orbit $U(n) \cdot E$ are the *Kähler angles* $2\pi \geq \theta_1(E) \geq \cdots \geq \theta_{\lfloor \frac{k}{2} \rfloor}(E) \geq 0$. Hence the orbit space is

$$G_k(\mathbb{R}^{2n})/U(n) \cong \left\{ \theta \in [0, 2\pi]^{\lfloor \frac{k}{2} \rfloor} \,|\, \theta_1 \geq \cdots \geq \theta_{\lfloor \frac{k}{2} \rfloor} \right\}.$$

Hence the vector spaces $\widehat{\mathscr{G}_0}(k, \mathbb{C}^n)^{U(n)} \cong \mathcal{M}(G_k(\mathbb{R}^{2n})/U(n))$ are one dimensional for $k = 0, 1, m-1, m$ and infinite dimensional if $1 < k < m - 1$.

Example 2.4.16. Let $\mathbb{V} = \mathbb{R}^a \otimes \mathbb{R}^b$ assuming $a \leq b$ and consider the action of $H = O(a) \times O(b)$ on $\mathbb{V}$. If we identify $\mathbb{R}^a \otimes \mathbb{R}^b$ with the $a \times b$ rectangular matrices then this action is given for all $(g_1, g_2) \in O(a) \times O(b)$ and all $M \in \mathbb{R}^a \otimes \mathbb{R}^b$ by $(g_1, g_2) \cdot M = g_1 M g_2^t$. The singular value decomposition (svd) tells us that the complete invariants of this orbits are the *singular values* of M: $\sigma_1(M) \geq \cdots \geq \sigma_a(M) \geq 0$. Note that instead of ordering the singular values we can consider them as elements of the quotient space $\mathbb{R}^a/\mathcal{E}^a$ where $\mathcal{E}^a := \mathfrak{S}_a \rtimes (\mathbb{Z}/2)^a$ is acting on $\mathbb{R}^a$ by permutation of coordinates and change of sign of coordinates. Thus we have

$$\widehat{\mathscr{Z}_0}(\mathbb{R}^a \otimes \mathbb{R}^b)^{O(a)\times O(b)} \cong \widehat{\mathscr{Z}_0}(\mathbb{R}^a)^{\mathcal{E}^a}.$$

It is not clear, at least to the author, what are the orbit spaces on the higher Grassmannians.

Open problem 7. Describe the orbit spaces $G_k(\mathbb{R}^a \otimes \mathbb{R}^b)/O(a) \times O(b)$ for $k > 1$.

In Section 4.1.4, we will see another approach to invariant zonoids that is more adapted to the computation of geometric quantities.

2.4.4 The kernel of the cosine transform

We already introduced the *cosine transform* in Definition 1.2.47. One can generalize that replacing measures on the sphere by measures on the Grassmannian and looking at its cosine transform as a function on the Grassmannian.

In our context this corresponds to a map that takes the generating measure $\mu_A \in \mathcal{M}(G_k(\mathbb{V}))$ of a virtual (centered) Grassmannian zonoid $A \in \widehat{\mathscr{G}_0}(\mathbb{V})$ and maps it to $h_A|_{G_k(\mathbb{V}^*)}$. Where recall that even functions on the sphere of the exterior algebra induce functions on the Grassmannian via the Plücker embedding. In the following let us write

$$g_A := h_A|_{G_k(\mathbb{V}^*)} : G_k(\mathbb{V}^*) \to \mathbb{R}.$$

It was first conjectured by Georges Matheron in [60] that this is still injective and he proved the cases $k = 1, m-1$ (which are clear in our context given the injectivity of the classical cosine transform). However, this was disproved by Goodey and Weil who showed in [43, Theorem 2.1] that its kernel is non trivial for $1 < k < m$.

Definition 2.4.17. We define the *kernel of the cosine transform* (KoCT) to be

$$\mathfrak{M}(k, \mathbb{V}) := \left\{ A \in \widehat{\mathscr{G}}_0(k, \mathbb{V}) \,|\, g_A \equiv 0 \right\}.$$

Moreover we write

$$\mathfrak{M}(\mathbb{V}) := \bigoplus_{k=0}^{m} \mathfrak{M}(k, \mathbb{V}).$$

Proposition 2.4.18. *The subspace $\mathfrak{M}(\mathbb{V})$ is an ideal of $\widehat{\mathscr{G}}_0(\mathbb{V})$ invariant by $*$ and closed in the weak–$*$ topology.*

Proof. By definition, $\mathfrak{M}(k, \mathbb{V})$ is a subspace of $\widehat{\mathscr{G}}_0(k, \mathbb{V})$, to prove it is an ideal, we need to show that given $A \in \widehat{\mathscr{G}}_0(k, \mathbb{V})$ and $\kappa \in \mathfrak{M}(l, \mathbb{V})$ their wedge product is in the KoCT. Suppose $A = \mathbb{E}\underline{X} - \mathbb{E}\underline{Y}$ and $\kappa = \mathbb{E}\underline{\alpha} - \mathbb{E}\underline{\beta}$ for some random integrable simple vectors X, Y, α, β all independents. Let $u_1, \ldots, u_{k+l} \in \mathbb{V}^* \cong \mathbb{V}$ and write $w := u_1 \wedge \cdots \wedge u_{k+l}$. Then we have

$$h_{\kappa \wedge A}(w) = \frac{1}{2} \left(\mathbb{E}|\langle \alpha \wedge X, w \rangle| + \mathbb{E}|\langle \beta \wedge Y, w \rangle| - \mathbb{E}|\langle \beta \wedge X, w \rangle| - \mathbb{E}|\langle \alpha \wedge Y, w \rangle| \right).$$

We now observe that $|\langle x \wedge y, z \rangle| = |\langle x, *(y \wedge *z) \rangle|$. Using this identity and the independence we obtain

$$h_{\kappa \wedge A}(w) = \mathbb{E}\left[h_\kappa(*(X \wedge *w)) \right] - \mathbb{E}\left[h_\kappa(*(Y \wedge *w)) \right].$$

Since X, Y, and w are (almost surely) simple vectors, so are $*(X \wedge *w)$ and $*(Y \wedge *w)$. Moreover since κ is in the KoCT, by definition, h_κ vanishes on simple vectors. Thus we have proved that $h_{\kappa \wedge A}(w) = 0$ for all w simple, i.e $\kappa \wedge A$ is in the KoCT.

To see that it is closed, let us write for all $E \in G_k(\mathbb{V})$, $f_E := |\langle E, \cdot \rangle| \in C(G_k(\mathbb{V}))$. Then we have by definition

$$\mathfrak{M}(k, \mathbb{V}) = \bigcap_{E \in G_k(\mathbb{V})} \left\{ A \in \widehat{\mathscr{G}}(k, \mathbb{V}) \,|\, \int_{G_k(\mathbb{V})} f_E \, \mathrm{d}\mu_A = 0 \right\}.$$

By definition of the weak–$*$ topology, each of the sets on the right are closed and thus the intersection is closed.

The fact that it is Hodge star invariant follows from the fact that the Hodge star preserves the simple vectors. □

Remark 2.4.19. Since it is Hodge star invariant, $\mathfrak{M}(\mathbb{V})$ is also an ideal for the convolution $\vee$ (see Definition 2.4.1).

Next we see that the KoCT is related to the pairing from Section 2.4.2.

Proposition 2.4.20. *Let $A \in \widehat{\mathscr{G}}_0(\mathbb{V}^*)$. Then $A \in \mathfrak{M}(\mathbb{V}^*)$ if and only if for every $B \in \widehat{\mathscr{G}}_0(\mathbb{V})$ we have $(A, B) = 0$.*

Proof. We can assume A is of degree k. Let $w \in \Lambda^k \mathbb{V}$ be simple. Then, by Lemma 2.4.3, we have $(A, \underline{w}) = 2h_A(w)$. If $(A, \cdot) \equiv 0$ on Grassmannian zonoids, in particular it vanishes on segments and thus $A \in \mathfrak{M}(k, V)$. Now suppose $A \in \mathfrak{M}(k, V)$ and let $B = \mathbb{E}\underline{X} - \mathbb{E}\underline{Y} \in \widehat{\mathscr{G}}_0(k, \mathbb{V})$ with X, Y almost surely simple. We have $(A, B) = 2\mathbb{E}\left[h_A(X) - h_A(Y) \right] = 0$ and this proves the result. □

Remark 2.4.21. This last result means that the pairing $(\cdot,\cdot)$ when restricted to Grassmannian zonoids is *degenerate*: there are nonzero elements A such that $(A,\cdot)$ is identically zero (on Grassmannian zonoids). In particular it implies that the pairing on the whole zonoid algebra is not positive definite.

Of course if one has an algebra and a closed ideal, one immediately wants to take a quotient. Indeed we shall do so in Chapter 3 and see how $\widehat{\mathscr{G}_0}(\mathbb{V}^*)/\mathfrak{M}(\mathbb{V}^*)$ injects in the space of even continuous translation invariant valuations on $\mathscr{K}(\mathbb{V})$, see Corollary 3.3.8.

Finally since the $Gl(\mathbb{V})$ action on the exterior algebra preserves simple vectors we have the following.

Proposition 2.4.22. *The subspace $\mathfrak{M}(\mathbb{V}) \subset \widehat{\mathscr{G}}(\mathbb{V})$ is a $Gl(\mathbb{V})$-invariant subspace.*

Proof. If $A \in \mathfrak{M}(k,\mathbb{V})$, $g \in Gl(\mathbb{V})$ and $E \in G_k(\mathbb{V}^*)$ then $g_{g\cdot A}(E) = g_A(g^t \cdot E) = 0$. □

2.4.5 Radon Transform and wedge with balls

We introduce the following integral transform.

Definition 2.4.23. The *Radon Transform* is the linear map $R : C_{even}(S(\mathbb{V})) \to C_{even}(S(\mathbb{V}^*))$ given for all $f \in C_{even}(S(\mathbb{V}))$ and $u \in \mathbb{V}^*$ by

$$(Rf)(u) := \mathbb{E}[f(X_u)]$$

where $X_u \in S(u^\perp)$ is uniform.

We connect this transform with an operation on the zonoid algebra.

Definition 2.4.24. We define the map $\beta : \widehat{\mathscr{Z}_0}(\mathbb{V}) \to \widehat{\mathscr{Z}_0}(\Lambda^{m-1}\mathbb{V})$ given for all $A \in \widehat{\mathscr{Z}_0}(\mathbb{V})$ by

$$\beta(A) := A \wedge B(\mathbb{V})^{\wedge(m-2)}.$$

We can show the following.

Proposition 2.4.25. *For all $A \in \widehat{\mathscr{Z}_0}(\mathbb{V})$, identifying with the Hodge star $*(\Lambda^{m-1}\mathbb{V}^*) = \mathbb{V}$, we have*

$$\bar{h}_{*\beta(A)} = \frac{(m-1)!\kappa_{m-1}}{2} R\bar{h}_A$$

where recall that $\bar{h}_A$ denotes the support function restricted to the sphere.

Proof. We can assume $A = K = \mathbb{E}\underline{X} \in \mathscr{Z}_0(\mathbb{V})$, moreover, let $b_1, \ldots, b_{m-1} \in \mathbb{V}$ be iid independent of X such that $\mathbb{E}\underline{b_i} = B(\mathbb{V})$. Then for all $x \in S(\mathbb{V})$ we have

$$\begin{aligned} h_{*\beta K}(x) &= \frac{1}{2}\mathbb{E}|\langle x, *(X \wedge b_1 \wedge \cdots \wedge b_{m-2})\rangle| \\ &= \frac{1}{2}\mathbb{E}|\langle X, *(x \wedge b_1 \wedge \cdots \wedge b_{m-2})\rangle| \\ &= \mathbb{E}\left[h_K(*(x \wedge b_1 \wedge \cdots \wedge b_{m-2}))\right] \end{aligned}$$

where in the second line we used the properties of the Hodge star and in the last line the independence of the variables. Next we want to compute the law of the random vector $Y := *(x \wedge b_1 \wedge \cdots \wedge b_{m-2})$. Assume that $\|x\| = 1$, we see that $x \wedge b_1 \wedge \cdots \wedge b_{m-2} = \pi_x(b_1) \wedge \cdots \wedge \pi_x(b_{m-2})$ where π_x is the orthogonal projection on $x^\perp$. Thus $Y \in x^\perp$ almost surely and Y is invariant under the action of $O(x^\perp)$. Since h_A is homogeneous, we can replace Y by anything zonoid equivalent to it (see Proposition 1.2.29). By what we just said Y is zonoid equivalent to a constant times X_x uniform on the unit sphere of $x^\perp$. To compute this constant we see that

$$\mathbb{E}\|Y\| = \ell(\pi_x(B(\mathbb{V}))^{\wedge(m-2)}) = \ell(B_{m-1}^{\wedge(m-2)}).$$

We use Lemma 2.2.17 to compute it, giving

$$\ell(B_{m-1}^{\wedge(m-2)}) = \frac{1}{2}\ell(B_{m-1}^{\wedge(m-1)}) = \frac{(m-1)!\kappa_{m-1}}{2}$$

which gives what we wanted. □

Remark 2.4.26. Note that for all $g \in O(\mathbb{V})$ and all $K \in \mathscr{Z}_0(\mathbb{V})$ we have $\beta(g \cdot K) = g \cdot \beta(K)$.

Example 2.4.27. Let $x \in \mathbb{V}$ we get

$$* \,\beta(\underline{x}) = \frac{(m-2)!\kappa_{m-2}}{2} \|x\| \, B(x^{\perp}).$$

Indeed by the previous remark $\beta(\underline{x})$ is invariant under the action of $O(x^{\perp})$ thus to prove the equality we claim it is enough to compute the length.

$$\begin{aligned}
\ell(\underline{x} \wedge B(\mathbb{V})^{\wedge(m-2)}) &= \frac{1}{2}\ell(\underline{x} \wedge B(\mathbb{V})^{\wedge(m-1)}) \\
&= \frac{(m-1)!\kappa_{m-1}}{2}\|x\| \\
&= \frac{(m-2)!\kappa_{m-2}}{2}\|x\|\ell(B_{m-1})
\end{aligned}$$

where we used Lemma 2.2.17 in the case $k = m-1$ for the first equality and in the other direction in the case $k = 1$ for the second equation and in the third equality we used the expression for the length of balls computed in Example 2.2.18.

The Radon transform has been extensively studied and in particular is known to be injective. Thus it proves the following.

Proposition 2.4.28. *For all $0 \leq k \leq m-2$, the map $\widehat{\mathscr{Z}_0}(\mathbb{V}) \to \widehat{\mathscr{G}_0}(k+1, \mathbb{V})$ given for all $K \in \mathscr{Z}_0(\mathbb{V})$ by $K \mapsto K \wedge B(\mathbb{V})^{\wedge k}$ is injective.*

Open problem 8. Is the previous proposition still true if we replace $B(\mathbb{V})$ by any curved/full dimensional zonoid of $\mathbb{V}$?

2.5 Example 1: Fiber bodies

In this section we give an example where the zonoid algebra naturally appears and helps with computations. This is a joint work with Chiara Meroni and this section is based on [61].

In this section we consider the Euclidean vector space $\mathbb{R}^{n+m}$ endowed with the standard Euclidean structure and we let $E \subset \mathbb{R}^{n+m}$ be a subspace of dimensions n. Denote by F its orthogonal complement, in such a way that $\mathbb{R}^{n+m} = E \oplus F$. Let $\pi : \mathbb{R}^{n+m} \to E$ be the orthogonal projection onto E. Throughout this section we will canonically identify the Euclidean space with its dual.

If $K \in \mathscr{K}(\mathbb{R}^{n+m})$ we write K_x for the orthogonal projection onto F of the fiber of $\pi|_K$ over x, namely

$$K_x := \{y \in F \,|\, (x, y) \in K\}.$$

The *fiber body* of K with respect to π is the *average* of the fibers of K under this projection:

$$\Sigma_\pi K = \int_{\pi(K)} K_x \, \mathrm{d}x. \tag{2.5.1}$$

Where this integral is defined in a similar way as the *expected* convex body in Definition 1.2.18. We will make it more rigorous below.

Such a notion was introduced for polytopes by Billera and Sturmfels in [22]. It has been investigated in many different contexts, from combinatorics such as in [12] to algebraic geometry and even tropical geometry in the context of polynomial systems [38, 37, 83].

We propose here to study the fiber body of zonoids. After a quick investigation of the fiber body of general convex bodies, we will show how the fiber body of a zonoid can be computed using zonoid calculus.

2.5.1 Definition and first considerations

Let us first define the fiber body of a general convex body in $\mathscr{K}(\mathbb{R}^{n+m})$ and study some basic properties. We will need the following notion.

Definition 2.5.1. A map $\gamma : \pi(K) \to F$ such that for all $x \in \pi(K)$, $\gamma(x) \in K_x$ is called a *section of* π. When there is no ambiguity on the map π we will simply say that γ is a *section*.

Using this notion we are now able to define our main object of study for this section.

Definition 2.5.2. The *fiber body* of K with respect to the projection π is the convex body

$$\Sigma_\pi K := \left\{ \int_{\pi(K)} \gamma(x)\, \mathrm{d}x \,|\, \gamma : \pi(K) \to F \text{ measurable section} \right\} \in \mathscr{K}(F).$$

Here $\mathrm{d}x$ denotes the integration with respect to the n–dimensional Lebesgue measure on E. We say that a section γ *represents* $y \in \Sigma_\pi K$ if $y = \int_{\pi(K)} \gamma(x)\mathrm{d}x$.

Remark 2.5.3. Note that, with this setting, if $\pi(K)$ is of dimension $< n$, then its fiber body is $\Sigma_\pi K = \{0\}$.

This definition of fiber bodies, that can be found for example in [37] under the name *Minkowski integral*, extends the classic construction of fiber polytopes [22], up to a constant. Here, we choose to omit the normalization $\frac{1}{\mathrm{vol}_n(\pi(K))}$ in front of the integral used by Billera and Sturmfels in order to make apparent the *degree* of the map Σ_π seen in (2.5.2). This degree becomes clear with the notion of *mixed fiber body*, see [37, Theorem 1.2].

Proposition 2.5.4. *For any $\lambda \in \mathbb{R}$ we have $\Sigma_\pi(\lambda K) = \lambda|\lambda|^n \Sigma_\pi K$. In particular if $\lambda \geq 0$*

$$\Sigma_\pi(\lambda K) = \lambda^{n+1}\Sigma_\pi K. \tag{2.5.2}$$

Proof. If $\lambda = 0$ it is clear that the fiber body of $\{0\}$ is $\{0\}$. Suppose now that $\lambda \neq 0$ and let $\gamma : \pi(K) \to F$ be a section. We can define another section $\tilde{\gamma} : \pi(\lambda K) \to F$ by $\tilde{\gamma}(x) := \lambda\gamma\left(\frac{x}{\lambda}\right)$. Using the change of variables $y = x/\lambda$, we get that

$$\int_{\lambda\pi(K)} \tilde{\gamma}(x)\ \mathrm{d}x = \lambda|\lambda|^n \int_{\pi(K)} \gamma(y)\ \mathrm{d}y.$$

This proves that $\Sigma_\pi \lambda K \subseteq \lambda|\lambda|^n \Sigma_\pi K$. Repeating the same argument for λ^{-1} instead of λ, the other inclusion follows. $\square$

Corollary 2.5.5. *If $K \in \mathscr{K}_0(\mathbb{R}^{n+m})$ then $\Sigma_\pi K \in \mathscr{K}_0(F)$.*

Proof. Apply the previous proposition with $\lambda = -1$ to get $\Sigma_\pi((-1)K) = (-1)\Sigma_\pi K$. If K is centrally symmetric with respect to the origin then $(-1)K = K$ and the result follows. $\square$

As a consequence of the definition, it is possible to deduce a formula for the support function of the fiber body. This is the rigorous version of equation (2.5.1).

Proposition 2.5.6. *For any $u \in F$ we have*

$$h_{\Sigma_\pi K}(u) = \int_{\pi(K)} h_{K_x}(u)\mathrm{d}x. \tag{2.5.3}$$

Proof. By definition

$$h_{\Sigma_\pi K}(u) = \sup\left\{ \int_{\pi(K)} \langle u, \gamma(x)\rangle\ \mathrm{d}x \,|\, \gamma \text{ measurable section} \right\} \leq \int_{\pi(K)} h_{K_x}(u)\mathrm{d}x.$$

To obtain the equality, it is enough to show that there exists a measurable section $\gamma_u : \pi(K) \to F$ with the following property: for all $x \in \pi(K)$ the point $\gamma_u(x)$ maximizes the linear form $\langle u, \cdot\rangle$ on K_x. In other words for all $x \in \pi(K)$, $\langle u, \gamma_u(x)\rangle = h_{K_x}(u)$. This is due to [14, Proposition 2.1]. $\square$

Remark 2.5.7. In the terminology of Section 1.2.2 and Definition 1.2.18 we could write this last result as $\Sigma_\pi K = \mathrm{vol}_n(\pi(K))\mathbb{E}K_X$ where $X \in \pi(K)$ is uniformly distributed.

A similar result can be shown for the faces of the fiber body.

Definition 2.5.8. Let $K \in \mathscr{K}(\mathbb{R}^{n+m})$, if $\mathcal{U} = \{u_1, \dots, u_k\}$ is an ordered family of vectors of $\mathbb{R}^{n+m}$, we write

$$K^{\mathcal{U}} := (\cdots (K^{u_1})^{u_2} \cdots)^{u_k}$$

where recall that K^u denotes the face of K in the direction u, see Definition 1.1.19.

In the following, we show that the face of the fiber body is, in some sense, the fiber body of the faces.

Lemma 2.5.9. *Let $\mathcal{U} = \{u_1, \dots, u_k\}$ be a an ordered family of linearly independent vectors of F, take $y \in \Sigma_\pi K$ and let $\gamma : \pi(K) \to F$ be a section that represents y. Then $y \in (\Sigma_\pi K)^{\mathcal{U}}$ if and only if $\gamma(x) \in (K_x)^{\mathcal{U}}$ for almost all $x \in \pi(K)$. In particular we have that*

$$(\Sigma_\pi K)^{\mathcal{U}} = \left\{ \int_{\pi(K)} \gamma(x)\mathrm{d}x \,|\, \gamma \textit{ section such that } \gamma(x) \in (K_x)^{\mathcal{U}} \textit{ for all } x \right\}. \tag{2.5.4}$$

Proof. Suppose first that $\mathcal{U} = \{u\}$. Assume that $\gamma(x)$ is not in $(K_x)^u$ for all x in a set of non–zero measure $\mathscr{O} \subset \pi(K)$. Then there exists a measurable function $\xi : \pi(K) \to F$ with $\langle u, \xi \rangle \geq 0$ and $\langle u, \xi(x) \rangle > 0$ for all $x \in \mathscr{O}$, such that $\tilde{\gamma} := \gamma + \xi$ is a section (for example you can take $\tilde{\gamma}(x)$ to be the nearest point on K_x of $\gamma(x) + u$). Let $\tilde{y} := \int_{\pi(K)} \tilde{\gamma}$. Then $\langle u, \tilde{y} \rangle = \langle u, y \rangle + \int_{\pi(K)} \langle u, \xi \rangle > \langle u, y \rangle$. Thus y does not belong to the face $(\Sigma_\pi K)^u$.

Suppose now that y is not in the face $(\Sigma_\pi K)^u$. Then there exists $\tilde{y} \in \Sigma_\pi K$ such that $\langle u, \tilde{y} \rangle > \langle u, y \rangle$. Let $\tilde{\gamma}$ be a section that represents $\tilde{y}$. It follows that $\int_{\pi(K)} \langle u, \tilde{\gamma} \rangle > \int_{\pi(K)} \langle u, \gamma \rangle$. This implies the existence of a set $\mathscr{O} \subset \pi(K)$ of non–zero measure where $\langle u, \tilde{\gamma}(x) \rangle > \langle u, \gamma(x) \rangle$ for all $x \in \mathscr{O}$. Thus for all $x \in \mathscr{O}$, $\gamma(x)$ does not belong to the face $(K_x)^u$.

In the case $\mathcal{U} = \{u_1, \dots, u_{k+1}\}$ we can apply inductively the same argument. Replace $\Sigma_\pi K$ by $(\Sigma_\pi K)^{\{u_1,\dots,u_k\}}$ and u by u_{k+1}, and use the representation of $(\Sigma_\pi K)^{\{u_1,\dots,u_k\}}$ given by (2.5.4). $\square$

Using the same strategy in the proof of Proposition 2.5.6 we obtain the following formula.

Lemma 2.5.10. *For every $u, v \in F$, $h_{(\Sigma_\pi K)^u}(v) = \int_{\pi(K)} h_{(K_x)^u}(v)\, \mathrm{d}x$.*

By definition, a point y of the fiber body $\Sigma_\pi K$ is the integral $y = \int_{\pi(K)} \gamma(x)\mathrm{d}x$ of a *measurable* section γ. Thus γ can be modified on a set of measure zero without changing the point y, i.e. y only depends on the L^1 class of γ. It is natural to ask what our favourite representative in this L^1 class will be and how regular can it be. In the case where K is a polytope, γ can always be chosen continuous. However if K is not a polytope and if y belongs to the boundary of $\Sigma_\pi K$, a continuous representative may not exist. This is due to the fact that, in general, the map $x \mapsto K_x$ is only upper semicontinuous, see [49, Section 6].

Example 2.5.11. Consider the function $f : S^1 \to \mathbb{R}$ such that

$$f(x,y) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

and let $K \in \mathscr{K}(\mathbb{R}^3)$ be the convex hull of the graph of f, see Figure 2.2. Let $\pi : \mathbb{R}^3 \to \mathbb{R}$ be the projection on the first coordinate $\pi(x, y, z) = x$. Then the point $p \in \Sigma_\pi K \subset \mathbb{R}^2$ maximizing the linear form associated to $(y, z) = (1, 0)$ must have only non–continuous sections. This can be proved using the representation of a face given by (2.5.4).

Open problem 9. What regularity can we ask to the section needed to represent all points of the fiber body?

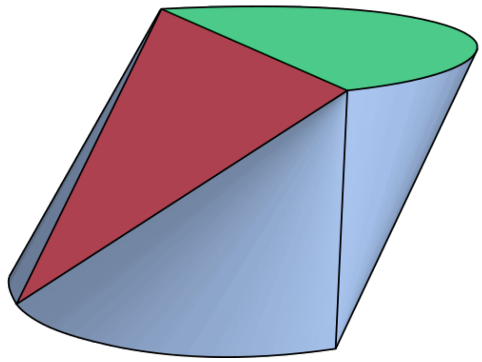


Figure 2.2: The convex body of Example 2.5.11. In its boundary there are 2 green half–discs, 2 red triangles and 4 blue cones.

2.5.2 Fiber bodies of curved bodies

In this section we are interested in the case where K is a curved body in the sense of Definition 1.1.32. This, a priori, does not concerns zonoids nor zonoid calculus but the author found the result short and interesting enough to be included. We prove Theorem 2.5.14 which is a formula to compute support function of the fiber body directly in terms of the support function of K, without having to compute those of the fibers.

If K is a curved body then in particular it is full–dimensional and its boundary is a C^2 hypersurface. Moreover we have the following which is [80, p.116], where curved convex bodies are said to be "of class C^2_+" and the differential $\mathrm{d}_v\nabla h_K$ is denoted by $\overline{W}_v$.

Lemma 2.5.12. *Let $K\subset\mathbb{R}^{n+m}$ be a curved convex body and let $v\in S^{n+m-1}$. Then the differential $\mathrm{d}_v\nabla h_K$ is a symmetric positive definite automorphism of $v^\perp$.*

Recall that in the curved case the gradient of the support function gives is equal to the face, see Proposition 1.1.20. The following gives an expression for the face of the fiber body. This is to be compared with the case of polytopes which is given in [38, Lemma 11].

Lemma 2.5.13. *If K is a curved convex body and $u\in F$ with $\|u\|=1$, then*

$$\nabla h_{\Sigma_\pi K}(u)=\int_E \nabla h_K(u+\xi)\cdot J_{\psi_u}(\xi)\ \mathrm{d}\xi$$

where $\psi_u: E\to E$ is given by $\psi_u(\xi)=(\pi\circ\nabla h_K)(u+\xi)$ and $J_{\psi_u}(\xi)$ denotes its Jacobian (i.e. the determinant of its differential) at the point ξ.

Proof. From (2.5.4) we have that $\nabla h_{\Sigma_\pi K}(u)=\int_{\pi(K)}\gamma_u(x)\mathrm{d}x$, where $\gamma_u(x)=\nabla h_{K_x}(u)$. Assume that $x=\psi_u(\xi)$ is a change of variables. We get $\gamma_u(x)=(\gamma_u\circ\pi\circ\nabla h_K)(u+\xi)=\nabla h_K(u+\xi)$ and the result follows.

It remains to prove that it is indeed a change of variables. Note that $\nabla h_K(u+\xi)=\nabla h_K(v)$ where $v=\frac{u+\xi}{\|u+\xi\|}\in S^{n+m-1}$. The differential of the map $\xi\mapsto E$ maps E to $(E+\mathbb{R}u)\cap v^\perp$. Moreover ∇h_K restricted to the sphere is a C^1 diffeomorphism by assumption. Thus it only remains to prove that its differential $\mathrm{d}_v\nabla h_K$ sends $(E+\mathbb{R}u)\cap v^\perp$ to a subspace that does not intersect $\ker(\pi|_{v^\perp})$. To see this, note that $\ker(\pi|_{v^\perp})^\perp=(E+\mathbb{R}u)\cap v^\perp$. Moreover, by the previous lemma, we have that $\langle w,\mathrm{d}_v\nabla h_K\cdot w\rangle=0$ if and only if $w=0$. Thus if $w\in\ker(\pi|_{v^\perp})^\perp$ and $w\neq 0$, then $\pi(\mathrm{d}_v\nabla h_K\cdot w)\neq 0$. Putting everything together, this proves that $\mathrm{d}_\xi\psi_u$ has no kernel which is what we wanted. □

As a direct consequence we derive a formula for the support function.

Theorem 2.5.14 (M–Meroni). *Let $K\subset\mathbb{R}^{n+m}$ be a curved convex body. Then the support function of $\Sigma_\pi K$ is for all $u\in F$*

$$h_{\Sigma_\pi K}(u)=\int_E\langle u,\nabla h_K(u+\xi)\rangle\cdot J_{\psi_u}(\xi)\ \mathrm{d}\xi$$

where $\psi_u : E \to E$ *is given by* $\psi_u(\xi) = (\pi \circ \nabla h_K)(u+\xi)$ *and* $J_{\psi_u}(\xi)$ *denotes its Jacobian at the point* ξ*.*

Proof. Apply the previous lemma to $h_{\Sigma_\pi K}(u) = \langle u, \nabla h_{\Sigma_\pi K}(u)\rangle$. □

Assume that the support function h_K is *algebraic*, i.e. it is a root of some polynomial equation. Then, the integrand in Lemma 2.5.13 and in Theorem 2.5.14 is also algebraic. Indeed, it is simply $\nabla h_K(u+\xi)$ times the Jacobian of ψ_u which is a composition of algebraic functions. We can generalize this concept in the direction of *D–modules* (see [95], or [77] for a text with a view towards applied nonlinear algebra). One can define what it means for a D–ideal of the Weyl algebra D to be *holonomic.* Then a function is holonomic if its annihilator, a D–ideal, is holonomic. Intuitively, this means that such function satisfies a system of linear homogeneous differential equations with polynomial coefficients, plus a suitable dimension condition. Holonomicity can be seen as a generalization of algebraicity which is closed under integration. We say that a convex body K is *holonomic* if its support function h_K is holonomic. In this setting, the fiber body satisfies the following property.

Corollary 2.5.15. *If K is a curved holonomic convex body, then its fiber body is again holonomic.*

Proof. We prove that the integrand in Theorem 2.5.14 is a holonomic function of u and ξ. Then the result follows from the fact that the integral of a holonomic function is holonomic [77, Proposition 2.11]. If h_K is holonomic then $\nabla h_K(u+\xi)$ is a holonomic function of u and ξ, as well as its scalar product with u. It remains to prove that the Jacobian of ψ_u is holonomic. But ψ_u is the projection of a holonomic function and thus holonomic, so the result follows. □

It is probable that the assumption of being curved is not needed for Corollary 2.5.15 but we needed it to write down the formula in Theorem 2.5.14.

2.5.3 Fiber bodies of zonoids

We saw in Corollary 2.5.5 that the fiber body of a centrally symmetric convex body is again centrally symmetric. However, it is not clear from the definition, nor from (2.5.3) that the fiber body of a zonoid is a zonoid. Indeed, if $K \in \mathscr{Z}_0(\mathbb{R}^{n+m})$, the section K_x is not in general a zonoid.

We will show however that it is the case and that the operation of taking the fiber body is actually an instance of zonoid calculus. Let us first introduce some of the tools used by Esterov in [37].

Definition 2.5.16. For any $u \in F$ define $T_u := \mathrm{Id}_E \oplus \langle u, \cdot\rangle : E \oplus F \to E \oplus \mathbb{R}$.

Definition 2.5.17. Let $C \in \mathscr{K}(E \oplus \mathbb{R})$. The *shadow volume* $V_+(C)$ of C is defined to be the integral of the maximal function on $\pi(C) \subset E$ such that its graph is contained in C, i.e.

$$V_+(C) = \int_{\pi(C)} \varphi(x)\mathrm{d}x,$$

where $\varphi(x) = \sup\{t \,|\, (x,t) \in C\}$. In particular if $(-1)C = C$, then the shadow volume is $V_+(C) = \frac{1}{2}\mathrm{vol}_{n+1}(C)$.

The *shadow volume* can then be used to express the support function of the fiber body.

Lemma 2.5.18. *For* $u \in F$ *and* $K \in \mathscr{K}(\mathbb{R}^{n+m})$*, we have*

$$h_{\Sigma_\pi K}(u) = V_+\left(T_u(K)\right).$$

In particular if $(-1)K = K$*,*

$$h_{\Sigma_\pi K}(u) = \frac{1}{2}\mathrm{vol}_{n+1}\left(T_u(K)\right). \tag{2.5.5}$$

Proof. We also denote by $\pi : E \oplus \mathbb{R} \to E$ the projection onto E. The shadow volume is the integral on $\pi(T_u(K)) = \pi(K)$ of the function $\varphi(x) = \sup\{t \,|\, (x,t) \in T_u(K)\} = \sup\{\langle u, y\rangle \,|\, (x,y) \in K\} = h_{K_x}(u)$. Thus the result follows from Proposition 2.5.6. □

Remark 2.5.19. Note that if $m = 2$ then T_u is the projection onto the hyperplane spanned by E and u. In that case (2.5.5) is the formula for the support function of the *projection body* ΠK of K at Ju, where J is a rotation by $\pi/2$ in F, see [80, Section 10.9]. Thus in that case, $\Sigma_\pi K$ is the projection of ΠK onto F rotated by $\pi/2$.

At this point, the reader familiar with zonoid calculus starts to see the signs that indicates its presence such as volume and projection body. But before going any further, let us note that Esterov shows in [37] that the map $\Sigma_\pi : \mathscr{K}(\mathbb{R}^{n+m}) \to \mathscr{K}(F)$ comes from another map, which is (Minkowski) multilinear in each variable: the *mixed fiber body*. The following is [37, Theorem 1.2].

Proposition 2.5.20. *There is a unique symmetric multilinear map*

$$\mathrm{M}\Sigma_\pi : \left(\mathscr{K}(\mathbb{R}^{n+m})\right)^{n+1} \to \mathscr{K}(F)$$

such that for all $K \in \mathscr{K}(\mathbb{R}^{n+m})$, $\mathrm{M}\Sigma_\pi(K, \dots, K) = \Sigma_\pi(K)$.

As announced, we will show that on zonoids this maps comes from FTZC (Theorem 2.1.16). To describe the corresponding multilinear map, let us observe that the splitting $\mathbb{R}^{n+m} = E \oplus F$ induces a splitting

$$\Lambda^k(E \oplus F) = \bigoplus_{a+b=k} \Lambda^{a,b}(E \oplus F) \tag{2.5.6}$$

where $\Lambda^{a,b}(E \oplus F) \cong \Lambda^a E \otimes \Lambda^b F$. This induces a collection of multilinear maps

$$F_\pi^{a,b} : (E \oplus F)^{a+b} \to \Lambda^a E \otimes \Lambda^b F \tag{2.5.7}$$

given by the wedge product $(E \oplus F)^k \to \Lambda^k(E \oplus F)$ composed with the orthogonal projection on one of the components of the splitting (2.5.6).

In our context we will be interested in the case where $(a, b) = (n, 1)$.

Definition 2.5.21. We write $F_\pi := F_\pi^{n,1}$, more explicitely, it is given for all $x_1, \dots, x_{n+1} \in E$ and $y_1, \dots, y_{n+1} \in F$ by:

$$F_\pi\left((x_1, y_1), \dots, (x_{n+1}, y_{n+1})\right) := \sum_{i=1}^{n+1} (-1)^{n+1-i} (x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_{n+1}) y_i$$

where $x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_{n+1}$ denotes the determinant of the chosen vectors omitting x_i.

We are now able to prove the main result of this section, here stated in the language of the zonoid calculus introduced in Section 2.1.2 and Theorem 2.1.16.

Theorem 2.5.22 (M–Meroni)**.** *The mixed fiber body of a zonoid is a zonoid. Moreover, if $K_1, \dots K_{n+1} \in \mathscr{Z}_0(\mathbb{R}^{n+m})$ then*

$$\widehat{F_\pi}(K_1, \dots, K_{n+1}) = (n+1)!\, \mathrm{M}\Sigma_\pi(K_1, \dots, K_{n+1})$$

where $\widehat{F_\pi} : \mathscr{Z}_0(\mathbb{R}^{n+m})^{n+1} \to \mathscr{Z}_0(F)$ denotes the maps induced on zonoids from F_π by FTZC (Theorem 2.1.16).

Proof. Suppose first that all zonoids are equal that is $K_i = K$ for all i and let $K = \mathbb{E}\underline{X}$ and $u \in F$. Note that $T_u(K) = \mathbb{E}\underline{T_u(X_1)}$. Thus by (2.5.5) and Vitale's Theorem (2.2.10) we get

$$h_{\Sigma_\pi K}(u) = \tfrac{1}{2} \mathrm{vol}_{n+1}\left(\mathbb{E}\underline{T_u(X)}\right) = \tfrac{1}{2} \tfrac{1}{(n+1)!} \mathbb{E}|T_u(X_1) \wedge \cdots \wedge T_u(X_{n+1})| \tag{2.5.8}$$

where $X_1, \dots, X_{n+1} \in \mathbb{R}^{n+m}$ are iid copies of X.

Now let us write $X_i := (\alpha_i, \beta_i)$ with $\alpha_i \in E$ and $\beta_i \in F$. Then

$$\begin{aligned}
|T_u(X_1) \wedge \cdots \wedge T_u(X_{n+1})| &= |(\alpha_1, \langle u, \beta_1\rangle) \wedge \cdots \wedge (\alpha_{n+1}, \langle u, \beta_{n+1}\rangle)| \\
&= \left| \sum_{i=1}^{n+1} (-1)^{n+1-i} (\alpha_1 \wedge \cdots \wedge \widehat{\alpha_i} \wedge \cdots \wedge \alpha_{n+1}) \langle u, \beta_i\rangle \right| \\
&= \left| \langle u, F_\pi\left((\alpha_1, \beta_1), \dots, (\alpha_{n+1}, \beta_{n+1})\right)\rangle \right|.
\end{aligned}$$

Reintroducing this in (2.5.8) we obtain:

$$h_{\Sigma_\pi K}(u) = \frac{1}{(n+1)!}\frac{1}{2}\mathbb{E}\left|\langle u, F_\pi\left(X_1,\ldots,X_{n+1}\right)\rangle\right|$$

which is what we want. The general case follows by expanding $\Sigma_\pi(t_1K_1+\cdots+t_{n+1}K_{n+1})$. $\square$

Applying it to zonotopes, this allows to generalize [22, Theorem 4.1].

Corollary 2.5.23. *For all $z_1,\ldots,z_N\in\mathbb{R}^{n+m}$, the fiber body of the zonotope $\sum_{i=1}^N \underline{z_i}$ is the zonotope given by*

$$\Sigma_\pi\left(\sum_{i=1}^N \underline{z_i}\right) = (n+1)! \sum_{1\leq i_1<\cdots<i_{n+1}\leq N} \underline{F_\pi(z_{i_1},\ldots,z_{i_{n+1}})}. \tag{2.5.9}$$

Proof. This follows from the defining property of $\widehat{F}_\pi$, namely $\widehat{F}_\pi(\underline{z_{i_1}},\ldots,\underline{z_{i_{n+1}}}) = \underline{F_\pi(z_{i_1},\ldots,z_{i_{n+1}})}$, the fact that if any two z_i are equal this is zero and the symmetry of $\widehat{F}_\pi$ which both follow from its definition. $\square$

Formula (2.5.9) was implemented by Chiara Meroni for `OSCAR 0.8.2-DEV` [71] and `SageMath 9.2` [75] and is available at `https://mathrepo.mis.mpg.de/FiberZonotopes`.

Remark 2.5.24. In Definition 2.5.21, we have defined the map F_π as the composition of the wedge product and a projection. It follows that the (mixed) fiber body enters the framework of the zonoid algebra. More precisely if $K\in\mathscr{Z}_0(\mathbb{R}^{n+m})$ then $\Sigma_\pi K$ is a projection of $\frac{1}{(n+1)!}K^{\wedge(n+1)}\in\mathscr{Z}_0(\Lambda^{n+1}\mathbb{R}^{n+m})$.

Considering the splitting (2.5.6), the fiber body can be seen as a particular case of a collection of maps $\widehat{F}_\pi^{a,b}:\mathscr{Z}_0(E\oplus F)^{a+b}\to\mathscr{Z}_0(\Lambda^aE\otimes\Lambda^bF)$ induced from the maps (2.5.7). Indeed from Definition 2.5.21, it corresponds to the case $(a,b)=(n,1)$. Moreover from Corollary 2.2.21 we see that if $a+b=n+m-1$, i.e $(a,b)=(n,m-1)$ or $(n-1,m)$ then we obtain a projection of the *projection body* ΠK. It remains the question to what correspond the other cases. Is there something that already appeared in convex geometry? A rapid count of dimensions shows that in most of the other cases, the target space $\Lambda^aE\otimes\Lambda^bF$ has a bigger dimension than the source space $\mathbb{R}^{n+m}$. There are however exceptions for instance $n=m=2$ and $a=b=1$. In that case we obtain a bilinear map $F_\pi^{1,1}:(\mathbb{R}^4)^2\to\mathbb{R}^2\otimes\mathbb{R}^2\cong\mathbb{R}^4$.

Open problem 10. Investigate, in the case $n=m=2$, the map

$$\widehat{F}_\pi^{1,1}:\mathscr{Z}_0(\mathbb{R}^4)^2\to\mathscr{Z}_0(\mathbb{R}^4).$$

From this point of view and using basic zonoid calculus techniques, we can deduce the following.

Proposition 2.5.25. *Let $K\in\mathscr{Z}_0(\mathbb{R}^{n+m})$, for all $L_1,\ldots,L_{m-1}\in\mathscr{Z}_0(F)$, we have*

$$\mathrm{MV}(L_1,\ldots,L_{m-1},K[n+1]) = \frac{(n+1)!m!}{(m+n)!}\mathrm{MV}(L_1,\ldots,L_{m-1},\Sigma_\pi K)$$

where on the left the mixed volume is on the whole space $\mathbb{R}^{n+m}$ while on the right we consider the mixed volume on the subspace F.

Proof. Let $X\in\mathbb{R}^{n+m}$ and $Y_1,\ldots,Y_{n-1}\in F$ be integrable all independent such that $K=\mathbb{E}\underline{X}$ and $L_i=\mathbb{E}\underline{Y_i}$ and let $X_1,\ldots,X_{n+1}$ be iid copies of X. Then by (2.2.4) we have

$$\mathrm{MV}(L_1,\ldots,L_{m-1},K[n+1]) = \frac{1}{(m+n)!}\mathbb{E}|Y_1\wedge\cdots\wedge Y_{m-1}\wedge X_1\wedge\cdots\wedge X_{n+1}|.$$

In the decomposition (2.5.6), $Y_1\wedge\cdots\wedge Y_{m-1}\in\Lambda^{0,m-1}\mathbb{R}^{n+m}$ thus in the wedge product above, the only non zero term in the decomposition of $X_1\wedge\cdots\wedge X_{n+1}$ is the one in $\Lambda^{n,1}\mathbb{R}^{n+m}$. In other words

$$\mathbb{E}|Y_1\wedge\cdots\wedge Y_{m-1}\wedge X_1\wedge\cdots\wedge X_{n+1}| = \mathbb{E}|Y_1\wedge\cdots\wedge Y_{m-1}\wedge F_\pi(X_1,\ldots,X_{n+1})|.$$

By Theorem 2.5.22, this yields

$$\mathrm{MV}(L_1, \ldots, L_{m-1}, K[n+1]) = \frac{(n+1)!}{(n+m)!}\ell(L_1 \wedge \cdots \wedge L_{m_1} \wedge \Sigma_\pi K)$$

and the result follows by (2.2.4). □

Next, we illustrate how Theorem 2.5.22 can be useful to explicitely compute the fiber body.

Definition 2.5.26. Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$ and let $D_i := B(e_i^\perp)$ be the unit disc in $e_i^\perp \cong \mathbb{R}^2$. We define the *dice* to be the zonoid $\mathscr{D} := D_1 + D_2 + D_3$. See Figure 2.3a.

The *dice* is a special case of what is called a *discotope*, that is a finite Minkowski sum of discs. For more on discotopes see [1, 42] or [61, Section 5.2].

Consider the projection $\pi := \langle e_1, \cdot \rangle : \mathbb{R} \oplus \mathbb{R}^2 \to \mathbb{R}$. Even in this simple example the fibers of the dice under this projection can be tricky to describe. However, using zonoid calculus, one can compute explicitly the fiber body without much effort (see Figure 2.3b).

(a) (b)

Figure 2.3: Left: the dice. Right: its fiber body.

Proposition 2.5.27. *With respect to this projection π, the fiber body of $\mathscr{D}$ is*

$$\Sigma_\pi(\mathscr{D}) = D_1 + \frac{\pi}{4}\left(\underline{e_2} + \underline{e_3}\right) + \frac{1}{2}\Lambda$$

where Λ is the convex body whose support function is given by

$$h_\Lambda(u_2, u_3) = \frac{1}{2}\int_0^\pi \sqrt{\cos(\theta)^2\,(u_2)^2 + \sin(\theta)^2\,(u_3)^2}\;\mathrm{d}\theta.$$

Proof. First of all let us note that by expanding the mixed fiber body $\mathrm{M}\Sigma_\pi(\mathscr{D}, \mathscr{D})$ we have

$$\Sigma_\pi(\mathscr{D}) = \Sigma_\pi(D_1) + \Sigma_\pi(D_2) + \Sigma_\pi(D_3) + 2\left(\mathrm{M}\Sigma_\pi(D_1, D_2) + \mathrm{M}\Sigma_\pi(D_1, D_3) + \mathrm{M}\Sigma_\pi(D_2, D_3)\right).$$

Now let $\sigma_1(\theta) := (0, \cos(\theta), \sin(\theta))$, $\sigma_2(\theta) := (\cos(\theta), 0, \sin(\theta))$ and $\sigma_3(\theta) := (\cos(\theta), \sin(\theta), 0)$ in such a way that $h_{D_i}(u) = \frac{\pi}{2}\mathbb{E}|\langle u, \sigma_i(\theta)\rangle|$ (see Lemma 2.3.5).

We then want to use Theorem 2.5.22 to compute all the summands of the expansion. Using this and Proposition 2.1.18 we have that $\mathrm{M}\Sigma_\pi(D_i, D_j) = \frac{\pi^2}{8}\mathbb{E}\underline{F_\pi(\sigma_i(\theta), \sigma_j(\phi))}$ with $\theta, \phi \in [0, 2\pi]$ uniform and independent. In our case, $F_\pi(x, y) = (x_1y_2 - y_1x_2, x_1y_3 - y_1x_3)$. We obtain

$$F_\pi(\sigma_1(\theta), \sigma_1(\phi)) = 0,\ F_\pi(\sigma_2(\theta), \sigma_2(\phi)) = (0, \sin(\phi - \theta)),$$
$$F_\pi(\sigma_3(\theta), \sigma_3(\phi)) = (\sin(\phi - \theta), 0),\ F_\pi(\sigma_1(\theta), \sigma_2(\phi)) = -\cos(\phi)\cdot(\cos(\theta), \sin(\theta)),$$
$$F_\pi(\sigma_1(\theta), \sigma_3(\phi)) = -\cos(\phi)\cdot(\cos(\theta), \sin(\theta)),\ F_\pi(\sigma_2(\theta), \sigma_3(\phi)) = (\cos(\theta)\sin(\phi), \sin(\theta)\cos(\phi)).$$

Computing the support function $h_{\pi^2/8\mathbb{E}\underline{F_\pi(\sigma_i(\theta),\sigma_j(\phi))}}(u) = (\pi/4)^2\mathbb{E}|\langle u, F_\pi(\sigma_i(\theta),\sigma_j(\phi))\rangle|$ and using that $\mathbb{E}|\cos(\phi)| = 2/\pi$, we get

$$\Sigma_\pi(D_1) = 0; \quad \Sigma_\pi(D_2) = \frac{\pi}{4}\,\underline{e_2}; \quad \Sigma_\pi(D_3) = \frac{\pi}{4}\,\underline{e_3};$$
$$\mathrm{M}\Sigma_\pi(D_1, D_2) = \mathrm{M}\Sigma_\pi(D_1, D_3) = \frac{1}{4}D_1$$

It only remains to compute $\mathrm{M}\Sigma_\pi(D_2, D_3)$. We have

$$h_{\mathrm{M}\Sigma_\pi(D_2,D_3)}(u) = \frac{1}{2}\frac{\pi^2}{8}\mathbb{E}|\langle u, F_\pi(\sigma_2(\theta),\sigma_3(\phi))\rangle| = \frac{\pi^2}{16}\mathbb{E}|u_2\cos(\theta)\sin(\phi) + u_3\sin(\theta)\cos(\phi)|.$$

We use then the independence of θ and ϕ to find

$$h_{\mathrm{M}\Sigma_\pi(D_2,D_3)}(u) = \frac{\pi}{8}\mathbb{E}\sqrt{\cos(\theta)^2\,(u_2)^2 + \sin(\theta)^2\,(u_3)^2} = \frac{1}{4}h_\Lambda(u)$$

Puting back together everything we obtain the result. □

Remark 2.5.28. In the case where $u_2 \neq 0$ we have

$$h_\Lambda(u) = |u_2|E\left(\sqrt{1 - \left(\frac{u_3}{u_2}\right)^2}\right)$$

where $E(s) = \int_0^{\pi/2}\sqrt{1 - s^2\sin(\theta)^2}\mathrm{d}\theta$ is the complete elliptic integral of the second kind. This function is not semialgebraic thus the example of the dice (which is semialgebraic, see [61, section 5.3]) shows that the fiber body of a semialgebraic convex body is not necessarily semialgebraic. However E is holonomic. This suggests as mentioned before that the curved assumption in Corollary 2.5.15 may not be needed.

Conjecture 11. Let M be a multilinear map between finite dimensional vector spaces and let $\widehat{M}$ be the induced map on zonoids by FTZC. Then $\widehat{M}$ maps holonomic bodies to holonomic bodies.

2.6 Example 2: Gaussian zonoids

This example is less about zonoid calculus and more about an application of the generalizations of Vitale's Theorem presented in Section 2.2.3.

We will see how the point of view of convex geometry can help the study of random determinants. For example suppose we have two integrable vectors $X, Y \in \mathbb{V}$ such that $\mathbb{E}\underline{X} \subset \mathbb{E}\underline{Y}$. This implies numerous inequalities in random matrices, for instance for any $X_2, \ldots X_m \in \mathbb{V}$ integrable, independent of X, Y we have $\mathbb{E}|\det(X, X_2, \ldots, X_m)| \leq \mathbb{E}|\det(Y, X_2, \ldots, X_m)|$. But also many more, for rectangle matrices, complex determinants, etc. The statement $\mathbb{E}\underline{X} \subset \mathbb{E}\underline{Y}$ is a concise way to express all these inequalities.

In this example, we illustrate this with Gaussian vectors.

2.6.1 Gaussian vectors

Recall that a Gaussian random variable of mean $c \in \mathbb{R}$ and variance $\sigma > 0$ is a random real number whose law has the density $t \mapsto \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-(t-c)^2/(2\sigma^2))$. Moreover, one can see the Dirac delta measure δ_c as a degenerate Gaussian variable obtained from the general case letting $\sigma \to 0$.

Definition 2.6.1. Let $X \in \mathbb{V}$. We say that X is *Gaussian* is for every $u \in \mathbb{V}^* \setminus \{0\}$, the random variable $\langle u, X\rangle \in \mathbb{R}$ is Gaussian. We say that X is *non degenerate* if for all $u \in \mathbb{V}^* \setminus \{0\}$, the random variable $\langle u, X\rangle \in \mathbb{R}$ is non degenerate.

If a Gaussian vector $X\in\mathbb{R}^m$ is non degenerate, then there is a positive definite $\Sigma \in \mathbb{R}^{m\times m}$ and a vector $c \in \mathbb{R}^m$ such that the law of X admits the density

$$x \mapsto \frac{1}{\det(2\pi\Sigma)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\langle (x-c), \Sigma^{-1}(x-c)\rangle\right). \tag{2.6.1}$$

In that case, c is the *mean* of X while Σ is its *variance* (sometime also called covariance matrix). We say that a Gaussian vector is *centered* if its mean is $c = 0$, otherwise we call it *non centered.*

In this section we want to study the Vitale zonoid associated to Gaussian vectors.

Definition 2.6.2. A zonoid $K \in \mathscr{Z}_0(\mathbb{V})$ is called a *Gaussian zonoid* if there is a Gaussian vector $X\in\mathbb{V}$ such that $K = \mathbb{E}\underline{X}$.

In the following, we assume, for simplicity, that $\mathbb{V} = \mathbb{R}^m$ and that all Gaussian vectors are non degenerate.

A particular case is the *standard Gaussian vector* $\xi \in \mathbb{R}^m$ which admits a density given for all $x \in \mathbb{R}^m$ by $\rho(x) = (2\pi)^{-\frac{m}{2}} \exp\left(-\frac{\|x\|^2}{2}\right)$ i.e. the centered Gaussian vector with variance $\Sigma = \mathrm{Id}$.

One can prove, using for example the general expression of the density of a Gaussian vector (2.6.1), that for every (non degenerate) Gaussian vector $X\in\mathbb{R}^m$ there is a linear map $M : \mathbb{R}^m \to \mathbb{R}^m$ and a vector $c \in \mathbb{R}^m$ such that X has the same law as $M(c+\xi)$. In that case, X has mean $M(c)$ and variance M^tM.

We use this fact and Proposition 1.2.33 to reduce our study to the case where the Gaussian vector is of the form $c + \xi$, i.e. has variance the identity, hence the following definition.

Definition 2.6.3. For every $c \in \mathbb{R}^m$ we define

$$G(c) := \mathbb{E}\underline{c+\xi}$$

where $\xi\in\mathbb{R}^m$ is a standard Gaussian vector.

Hence, a convex body $K \subset \mathbb{R}^m$ is a (non degenerate) Gaussian zonoid if and only if there exists a linear map $M : \mathbb{R}^m \to \mathbb{R}^m$ and a vector $c \in \mathbb{R}^m$ such that $K = M(G(c))$.

We already saw in Example 1.2.27 that, in the case $c = 0$, we have:

$$G(0) = \frac{1}{\sqrt{2\pi}} B_m$$

where recall that $B_m = B(\mathbb{R}^m)$ denotes the unit ball. Thus centered Gaussian zonoids are linear images of the unit ball, that is ellipsoids.

2.6.2 Non centered Gaussian zonoids

In general, we can compute the support function of $G(c)$ explicitly, see Figure 2.4. Note that, by Proposition 1.2.33 and the invariance of ξ, $G(c)$ is invariant by $O(c^\perp)$, the stabilizer of c in the orthogonal group $O(m)$, that is $G(c)$ is a solid of revolution around the axis spanned by $c \in \mathbb{R}^m$.

Proposition 2.6.4. *Let $c \in \mathbb{R}^m \setminus \{0\}$ and let us write every $u \in \mathbb{R}^m$ as $u = (x, y) \in \mathbb{R} \times \mathbb{R}^{m-1}$ with $x = \langle u, c/\|c\|\rangle$ and $y \in c^\perp$. Then the support function of $G(c)$ is given by*

$$h_{G(c)}(x, y) = \frac{\sqrt{x^2(1 + \|c\|^2) + \|y\|^2}}{\sqrt{2\pi}} e^{\frac{-x^2\|c\|^2}{2(x^2(1+\|c\|^2)+\|y\|^2)}} + \frac{x\|c\|}{2} \operatorname{erf}\left(\frac{x\|c\|}{\sqrt{2}\sqrt{x^2(1 + \|c\|^2) + \|y\|^2}}\right)$$

where $\operatorname{erf}(t) := \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} \mathrm{d}s$ *is the* error function.

Proof. The random variable $\langle (x, y), c+\xi\rangle$ is a Gaussian variable with mean $x\|c\|$ and variance equal to $\sqrt{x^2(1 + \|c\|^2) + \|y\|^2}$. Computing the first absolute moment of a Gaussian gives the result, see [53, (3)] or [90]. $\square$

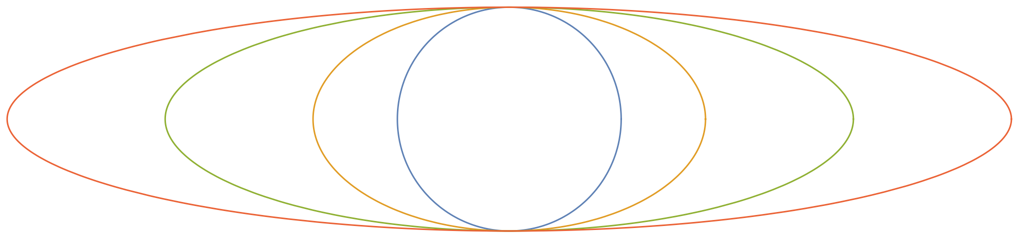


Figure 2.4: The Gaussian eye: the zonoids $G(c)$ for $\|c\| = 0, 1, 2$ and 3.

Proposition 2.6.5. *The map $G : \mathbb{R}^m \to \mathscr{Z}_0(\mathbb{R}^m)$ given by $c \mapsto G(c)$ is continuous. Moreover for all $c \neq 0$ the map $t \mapsto G(tc)$ is strictly increasing with respect to inclusion on $t > 0$.*

Proof. Continuity follows from the fact that the function $h_{G(\cdot)}(\cdot) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ given by $(c, u) \mapsto h_{G(c)}(u)$ is continuous and Lemma 1.1.18.

For the second part, we can assume without loss of generality that $\|c\| = 1$. It is enough to show that given a fixed non zero point $(x, y) \in \mathbb{R} \times \mathbb{R}^{m-1}$, the function $t \mapsto h_{G(tc)}(x, y)$ is strictly increasing. We get from Proposition 2.6.4:

$$\frac{\mathrm{d}}{\mathrm{d}t} h_{G(tc)}(x, y) = \frac{t x^2 e^{\frac{-x^2 t^2}{2(x^2(1+t^2)+\|y\|^2)}}}{\sqrt{2\pi}\sqrt{x^2(1+t^2)+\|y\|^2}} + \frac{x}{2}\operatorname{erf}\left(\frac{xt}{\sqrt{2}\sqrt{x^2(1+t^2)+\|y\|^2}}\right)$$

which is positive on $t > 0$ and this concludes the proof. $\square$

For $c \neq 0$ the Gaussian zonoid $G(c)$ is not an ellipsoid. However we shall show that it remains close to one, in a certain sense that we describe in Theorem 2.6.7 below, see also Figure 2.5. In order to state the main result, let us first introduce a few definitions.

First we define $\lambda : \mathbb{R} \to \mathbb{R}$ to be given for all $s \in \mathbb{R}$ by

$$\lambda(s) := \sqrt{1+s^2}\, e^{\frac{-s^2}{2(1+s^2)}} + s\sqrt{\frac{\pi}{2}}\operatorname{erf}\left(\frac{s}{\sqrt{2}\sqrt{1+s^2}}\right). \tag{2.6.2}$$

Note that, by Proposition 2.6.4 and following the same notation, for all $c \in \mathbb{R}^m$, we have

$$h_{G(c)}(1, 0) = \frac{\lambda(\|c\|)}{\sqrt{2\pi}}. \tag{2.6.3}$$

Using that and we find a first naive bound.

Proposition 2.6.6. *For all $c \neq 0$, the zonoid $G(c)$ is contained in the following cylinder:*

$$G(c) \subset \frac{1}{\sqrt{2\pi}} B(c^\perp) + \sqrt{\frac{2}{\pi}}\frac{\lambda(\|c\|)}{\|c\|}\, \underline{c}$$

and in particular we have

$$\ell(G(c)) \leq \frac{(m-1)\kappa_{m-1}}{\kappa_{m-2}\sqrt{2\pi}} + \sqrt{\frac{2}{\pi}}\lambda(\|c\|).$$

Proof. We use the fact that $h_{G(c)}(x, y) \leq h_{G(c)}(0, y) + h_{G(c)}(x, 0)$ and then use (2.6.3) and the expression of the support function in Proposition 2.6.4 to get the first statement. The second statement follows from the linearity of the length and the computation of the length of a ball of dimension $m-1$, see Example 2.2.18. $\square$

Then we define the constant

$$a_\infty := e^{-\frac{1}{2}} + \sqrt{\frac{\pi}{2}}\operatorname{erf}\left(\frac{1}{\sqrt{2}}\right) \sim 1.462\ldots \tag{2.6.4}$$

and the function $\varphi_\infty : \mathbb{R}^2 \to \mathbb{R}$ given for all $(x,z) \in \mathbb{R}^2$ by

$$\varphi_\infty(x,z) := \frac{\sqrt{x^2 + a_\infty^2 z^2}}{a_\infty} \exp\left(\frac{-x^2}{2(x^2 + a_\infty^2 z^2)}\right) + \frac{x\sqrt{\pi}}{a_\infty\sqrt{2}} \operatorname{erf}\left(\frac{x}{\sqrt{2}\sqrt{x^2 + a_\infty^2 z^2}}\right). \tag{2.6.5}$$

Finally, given $c \in \mathbb{R}^m$, we define the linear map $T_c : \mathbb{R}^m \to \mathbb{R}^m$ that is the identity if $c = 0$ and that sends $c \mapsto \lambda(\|c\|)c$ and is the identity on $c^\perp$ if $c \neq 0$. In other words, in an orthonormal basis $e_1, \ldots, e_m$ such that $c = \|c\|e_1$, it is given by the matrix:

$$T_c := \begin{pmatrix} \lambda(\|c\|) & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \tag{2.6.6}$$

Theorem 2.6.7 (M.). *For all $c \in \mathbb{R}^m$, we have*

$$b_\infty T_c\left(\frac{1}{\sqrt{2\pi}}B_m\right) \subset G(c) \subset T_c\left(\frac{1}{\sqrt{2\pi}}B_m\right)$$

where $b_\infty := \min\{\varphi_\infty(\cos(t), \sin(t)) \,|\, t \in [0, 2\pi]\} \sim 0.989\ldots$

Proof. If $c = 0$, $G(0)$ is equal to the upper bound and there is nothing to prove. Thus we can assume without loss of generality that $c = se_1$ where e_1 is the first standard basis vector of $\mathbb{R}^m$ and $s > 0$. Let $\widetilde{G}(s) := \sqrt{2\pi}T_{se_1}^{-1}G(se_1)$. The idea of the proof is to show that the map $s \mapsto \widetilde{G}(s)$ is strictly decreasing with respect to inclusion for $s > 0$. Once this is established, it is enough to show that the limit object $\widetilde{G}(\infty)$ exists and contains a ball of radius b_∞.

Let us first show that $s \mapsto \widetilde{G}(s)$ is decreasing. Let $(x,z) \in \mathbb{R}^2$ and consider the function

$$\varphi_{x,z}(s) := \frac{\sigma(s)}{\sqrt{2\pi}} e^{-\frac{\tau^2(s)}{2}} + \frac{\mu(s)}{2} \operatorname{erf}\left(\frac{\tau(s)}{\sqrt{2}}\right)$$

where

$$\mu(s) := \frac{sx}{\lambda(s)}; \qquad \sigma(s) := \sqrt{\frac{s^2+1}{\lambda^2(s)}x^2 + 2\pi z^2}; \qquad \tau(s) := \frac{\mu(s)}{\sigma(s)}.$$

Then $h_{\widetilde{G}(s)}(u) = \varphi_{x,\|y\|}(s)$ where, as in Proposition 2.6.4, $x = \langle u, e_1\rangle$ and y is the orthogonal projection of u onto $e_1^\perp$. It is enough to show that for all $x \in \mathbb{R}$ and $z \geq 0$, the function $s \mapsto \varphi_{x,z}(s)$ is decreasing on $s > 0$. One can check that $\varphi_{0,z} = |z|$ and $\varphi_{x,0} = |x|$ which are constants in s. Moreover since $\varphi_{\pm x, \pm z} = \varphi_{x,z}$, we can assume $x, z > 0$. Thus in the following we fix $x, z > 0$ and omit them in the notation, writing $\varphi := \varphi_{x,z}$.

Consider now the change of variable $\tilde{s} := \frac{s}{\sqrt{s^2+1}}$. One can write $\varphi(s)$ as a function of the variable $\tilde{s}$. Since $\tilde{s}$ is strictly increasing on $s > 0$ it is enough to show that φ is decreasing in $\tilde{s}$ on $0 < \tilde{s} < 1$. Writing φ' for the derivative of φ with respect to $\tilde{s}$ at $\tilde{s}$, we obtain for $0 < \tilde{s} < 1$ (we omit the dependence on s in the notation):

$$\frac{(1-\tilde{s}^2)\tilde{s}\lambda^2}{x \operatorname{erf}\left(\frac{\tilde{s}}{\sqrt{2}}\right) \operatorname{erf}\left(\frac{\tau}{\sqrt{2}}\right)} \varphi' = \rho\left(\frac{\tilde{s}}{\sqrt{2}}\right) - \rho\left(\frac{\tau}{\sqrt{2}}\right) \tag{2.6.7}$$

where

$$\rho(t) := t\frac{\operatorname{erf}'(t)}{\operatorname{erf}(t)} = \frac{2te^{-t^2}}{\sqrt{\pi}\operatorname{erf}(t)}.$$

One can show that $\rho(t)$ is strictly decreasing for $t > 0$, see [9, Lemma 2.1]. Moreover, since $x, z > 0$, we have

$$\tau = \frac{x}{\sqrt{x^2 + 2\pi(1-\tilde{s}^2)\lambda^2 z^2}}\tilde{s} < \tilde{s}.$$

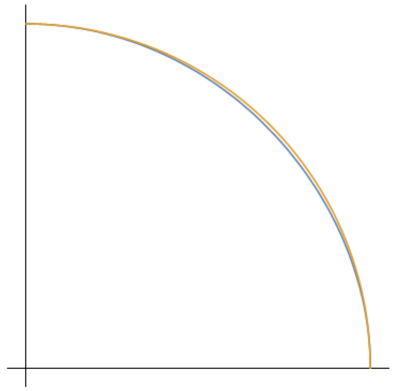


Figure 2.5: The boundary of $\widetilde{G}(\infty)$ and $\widetilde{G}(0) = B_2$ in the positive orthant

And thus $\rho\left(\frac{\tilde{s}}{\sqrt{2}}\right) < \rho\left(\frac{\tau}{\sqrt{2}}\right)$. The coefficient in front of φ' in (2.6.7) is positive on $0 < \tilde{s} < 1$ and thus this shows that $\varphi' < 0$ on $0 < \tilde{s} < 1$. In definitive we have shown that for all $s > 0$ the map $s \mapsto \widetilde{G}(s)$ is (strictly) decreasing with respect to inclusion.

We now note that for all fixed $(x, z) \in \mathbb{R}^2$ and as $s \to \infty$, $\varphi_{x,z}(s)$ tends to $\varphi_\infty(x, z)$ defined in (2.6.5). Writing as before $u = (x, y) \in \mathbb{R} \times \mathbb{R}^{m-1}$, by Lemma 1.1.18, the function $\varphi_\infty(x, \|y\|)$ is the support function of a zonoid that we denote by $\widetilde{G}(\infty)$. By what we just proved, for all $s > 0$ we have

$$\widetilde{G}(\infty) \subset \widetilde{G}(s) \subset \widetilde{G}(0).$$

Since T_0 is the identity, we have that $\widetilde{G}(0) = B_m$. Moreover, $\widetilde{G}(\infty)$ contains a ball of radius b_∞ since it is the minimum of its support function on the sphere. Mapping everything through $\frac{1}{\sqrt{2\pi}} T_{se_1}$ (which preserves inclusion) gives the result. □

From Theorem 2.6.7 and the fact that $\det(T_c) = \lambda(\|c\|)$, we get as an immediate corollary an estimate on the volume of the Gaussian zonoids $G(c)$.

Corollary 2.6.8. *For every $c \in \mathbb{R}^m$ we have*

$$(b_\infty)^m \frac{\lambda(\|c\|)}{(2\pi)^{\frac{m}{2}}} \kappa_m \leq \mathrm{vol}_m(G(c)) \leq \frac{\lambda(\|c\|)}{(2\pi)^{\frac{m}{2}}} \kappa_m$$

where recall the definition of λ in (2.6.2) *and of b_∞ in* Theorem 2.6.7.

The function λ is explicit and is expressed in terms of special functions. However, one can get a simpler expression in the asymptotic cases $s \to 0$ and $s \to \infty$. In these cases we have the following expansions:

$$\lambda(s) = 1 + s^2 + O\left(s^4\right); \qquad \lambda(s) = a_\infty s + O\left(\frac{1}{s}\right) \tag{2.6.8}$$

where recall the constant a_∞ defined in (2.6.4). We see that, when c is close to 0, the volume of $G(c)$ tends to the upper bound in Corollary 2.6.8 and the lower bound is far from being sharp. In fact in that case we have a better estimate from below that comes from the following inequality.

Lemma 2.6.9. *For any $t \geq 0$ we have*

$$t\,\mathrm{erf}(t) \geq \frac{1}{\sqrt{\pi}}\left(1 - e^{-t^2}\right).$$

Proof. It is enough to see that $t\,\mathrm{erf}(t) \geq \frac{1}{\sqrt{\pi}} \int_0^t 2se^{-s^2} \mathrm{d}s$. □

For all $c \in \mathbb{R}^m$ we define $L_c : \mathbb{R}^m \to \mathbb{R}^m$ to be the identity if $c = 0$ and to be the map that sends $c \mapsto \sqrt{1 + \|c\|^2}c$ and is the identity on $c^\perp$ if $c \neq 0$. In other words, in an orthonormal basis $e_1, \ldots, e_m$ such that $c = \|c\|e_1$, it is given by the matrix:

$$L_c := \begin{pmatrix} \sqrt{1+\|c\|^2} & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

Proposition 2.6.10. *For all $c \in \mathbb{R}^m$, we have*

$$L_c\left(\frac{1}{\sqrt{2\pi}}B_m\right) \subset G(c).$$

Proof. Applying Lemma 2.6.9 to the support function of $G(c)$ computed in Proposition 2.6.4 we find

$$h_{G(c)}(x,y) \geq \frac{1}{\sqrt{2\pi}}\sqrt{x^2(1+\|c\|^2) + \|y\|^2}.$$

The right hand side is equal to $\frac{1}{\sqrt{2\pi}}\|L_c(u)\|$ when, as before, $x = \langle u, c/\|c\|\rangle$ and y is the orthogonal projection of u onto $c^\perp$. Since $L_c = L_c^t$, by Proposition 1.1.15–(iii), this is the support function of $L_c\left(\frac{1}{\sqrt{2\pi}}B_m\right)$ and the result follows by Proposition 1.1.15–(ii). $\square$

Noting that $\det(L_c) = \sqrt{1+\|c\|^2}$, we get the following.

Corollary 2.6.11. *For every $c \in \mathbb{R}^m$, we have:*

$$\frac{\kappa_m}{(2\pi)^{\frac{m}{2}}}\sqrt{1+\|c\|^2} \leq \mathrm{vol}_m(G(c)).$$

Combining this result with Corollary 2.6.8 and the expansion (2.6.8) we find the following.

Proposition 2.6.12. *When $\|c\| \to 0$, we have*

$$1 + \frac{1}{2}\|c\|^2 + O\left(\|c\|^4\right) \leq \frac{(2\pi)^{\frac{m}{2}}}{\kappa_m}\mathrm{vol}_m\left(G(c)\right) \leq 1 + \|c\|^2 + O\left(\|c\|^4\right).$$

2.6.3 Random Gaussian determinants

We now apply the previous results and the results of Section 2.2.3 to estimate the expectation of the absolute determinant of a random matrix whose columns are non centered Gaussian vectors. As mentioned in Example 2.2.27 the centered case was proved by Zakhar Kabluchko and Dmitry Zaporozhets in [46] to be equal to the mixed volume of ellipsoids. We now show that the non centered case is not far from that. This is an application of Corollary 2.2.26.

Theorem 2.6.13 (M.)**.** *Let $0 < k \leq m$ and let $X_1, \ldots, X_k \in \mathbb{R}^m$ be independent Gaussian vectors such that $X_i = M_i(c_i + \xi_i)$ with $M_i : \mathbb{R}^m \to \mathbb{R}^m$ a linear map, $c_i \in \mathbb{R}^m$ fixed vectors and ξ_i iid standard Gaussian vectors of $\mathbb{R}^m$. Consider the random matrix $\Gamma := (X_1, \ldots, X_k)$ whose columns are the vectors X_i and define the ellipsoids $\mathcal{E}_i := (M_i \circ T_{c_i})(B_m)$ for $i = 1, \ldots, k$ where $B_m \subset \mathbb{R}^m$ is the unit ball and recall the definition of T_c in (2.6.6). We have*

$$(b_\infty)^k \alpha_{m,k}\mathrm{MV}\left(\mathcal{E}_1, \ldots, \mathcal{E}_k, B_m[m-k]\right) \leq \mathbb{E}\sqrt{\det\left(\Gamma^t\Gamma\right)} \leq \alpha_{m,k}\mathrm{MV}\left(\mathcal{E}_1, \ldots, \mathcal{E}_k, B_m[m-k]\right)$$

where $B_m[m-k]$ denotes the unit ball $B_m \subset \mathbb{R}^m$ repeated $m-k$ times in the argument of the mixed volume MV*, $\alpha_{m,k} := \frac{m!}{(2\pi)^{k/2}(m-k)!\kappa_{m-k}}$ and b_∞ is defined in Theorem 2.6.7.*

Proof. We have $\mathbb{E}\underline{X_i} = M_i(G(c_i))$. Since the maps M_i preserve inclusion, Theorem 2.6.7 gives the inclusions $b_\infty(M_i \circ T_{c_i})\left(\frac{1}{\sqrt{2\pi}}B_m\right) \subset M_i(G(c_i)) \subset (M_i \circ T_{c_i})\left(\frac{1}{\sqrt{2\pi}}B_m\right)$. Then we apply Corollary 2.2.26 and the result follows by the monotonicity of the mixed volume. □

This result is to be compared with [46, Theorem 1.1] which says that, in the centered case (i.e. the case where $c_i = 0$ for all i) this is equal to the upper bound. In some sense one can interpret Theorem 2.6.7 by saying that, for each *non centered* Gaussian vector of the form $X = M(\xi + c)$, there is a *centered* Gaussian vector $Y = (M \circ T_c)(\xi)$ such that, for random determinants, X is "trapped" between $b_\infty Y$ and Y.

As before, in the case where some c_i are close to zero, the lower bound in Theorem 2.6.13 is not very good. Applying Proposition 2.6.10 and building the ellipsoids with L_c instead of T_c we get a better estimate.

Proposition 2.6.14. *Let $0 < l \leq k \leq m$ and let $X_1, \dots, X_k \in \mathbb{R}^m$ be independent Gaussian vectors such that $X_i = M_i(c_i + \xi_i)$ with $M_i : \mathbb{R}^m \to \mathbb{R}^m$ a linear map, $c_i \in \mathbb{R}^m$ fixed vectors and ξ_i iid standard Gaussian vectors of $\mathbb{R}^m$. Consider the random matrix $\Gamma := (X_1, \dots, X_k)$ whose columns are the vectors X_i and define the ellipsoids $\mathcal{E}'_i := (M_i \circ L_{c_i})(B_m)$ for $i = 1, \dots, l$ and $\mathcal{E}_i := (M_i \circ T_{c_i})(B_m)$ for $i = l+1, \dots, k$. We have*

$$(b_\infty)^{k-l}\alpha_{m,k}\mathrm{MV}\left(\mathcal{E}'_1, \dots, \mathcal{E}'_l, \mathcal{E}_{l+1}, \dots, \mathcal{E}_k, B_m[m-k]\right) \leq \mathbb{E}\sqrt{\det\left(\Gamma^t\Gamma\right)}$$

where $\alpha_{m,k} := \frac{m!}{(2\pi)^{k/2}(m-k)!\kappa_{m-k}}$ and b_∞ is defined in Theorem 2.6.7.

Chapter 3

Zonoids and valuations

In this chapter we introduce the theory of valuations on convex bodies. The goal of this chapter is then to show that we can extend the product of smooth (even) valuations introduced by Semyon Alesker using the wedge product of zonoids.

We first introduce the general theory in Section 3.1. We then build the *exponential of zonoids* in Section 3.2 that will be used in the next section. Finally, in Section 3.3, we explain how to build a map from the Grassmannian algebra to the algebra of even continuous real translation invariant valuations such that the kernel is a ideal. We then show in Theorem 3.3.18 that the product thus defined extends Alesker's product by showing that this is a special case of a recent extension built by Nguyen-Bac Dang and Jian Xiao in [33].

As before, $\mathbb{V}$ denotes an Euclidean space of dimension $m < \infty$.

3.1 Theory of valuations

Let us start with the general definition.

Definition 3.1.1. A *valuation* on a subclass $\mathcal{C} \subset \mathscr{K}(\mathbb{V})$ of convex bodies with values on a semi group $(\mathcal{S}, +)$ is a map $\phi : \mathcal{C} \to \mathcal{S}$, such that for all $K, L \in \mathcal{C}$ such that $K \cup L, K \cap L \in \mathcal{C}$ we have

$$\phi(K) + \phi(L) = \phi(K \cup L) + \phi(K \cap L) \tag{3.1.1}$$

We will be particularly interested in real valuations on convex bodies or polytopes that are translation invariant and sometimes continuous.

3.1.1 Translation invariant continuous even valuations

Definition 3.1.2. We denote by $val^+(\mathbb{V})$ the space of translation invariant continuous real valuations on $\mathscr{K}(\mathbb{V})$ that are even (i.e $\phi(K) = \phi((-1) \cdot K)$. We write $val_k^+(\mathbb{V}) \subset val^+(\mathbb{V})$ the valuations that are positively homogeneous of degree k.

McMullen proved that translation invariant continuous valuation can only have integer degree of homogeneity. More precisely, we have the following, see [80, (6.22)].

Proposition 3.1.3. *We have*

$$val^+(\mathbb{V}) = \bigoplus_{k=0}^{m} val_k^+(\mathbb{V}).$$

Moreover, $val_k^+(\mathbb{V})$ is one dimensional for $k = 0, m$ and infinite dimensional otherwise.

There is a classical norm on the space of valuations given for all $\phi \in val^+(\mathbb{V})$ by

$$\|\phi\| := \sup \{\phi(K) \mid K \subset B(\mathbb{V})\}.$$

We will call the topology induced by this norm on $val^+(\mathbb{V})$ the *standard topology.* Endowed with this norm, $val^+(\mathbb{V})$ is a Banach space, see [80, Section 6.5].

Since the space $val^+_m(\mathbb{V})$ is one dimensional spanned by vol_m if we restrict a valuation of degree k to a subspace of dimension k it is a multiple of the volume vol_k on that space. This gives rise to the following definition.

Definition 3.1.4. for a valuation $\phi \in val^+_k(\mathbb{V})$, its *Klain function* is the map $\mathrm{Kl}_\phi : G_k(\mathbb{V}) \to \mathbb{R}$ such that for all $E \in G_k(\mathbb{V})$ we have

$$\phi|_E = \mathrm{Kl}_\phi(E)\,\mathrm{vol}_k(\cdot)$$

It turns out that the Klain function is continuous and uniquely determine the valuation, see [80, Theorem 6.4.11].

Proposition 3.1.5. *The map $val^+_k(\mathbb{V}) \to C(G_k(\mathbb{V}))$, $\phi \mapsto \mathrm{Kl}_\phi$ is injective.*

Thus another possible topology on $val^+_k(\mathbb{V})$ is given by pulling back the sup norm topology by the Klain embedding. In other words we define

$$\|\phi\|_{\mathrm{Kl}} := \|\,\mathrm{Kl}_\phi\,\|_\infty.$$

We call the induced topology the *Klain topology* on $val^+(\mathbb{V})$, for more on this topology one can check [19, Section 3].

The fact that the Klain function detemines the valuation implies that to check equality of valuations, it is enough to check it on zonoids.

Lemma 3.1.6. *Let $\phi_1, \phi_2 \in val^+(\mathbb{V})$ then $\phi_1 = \phi_2$ if and only if for all $K \in \mathscr{Z}_0(\mathbb{V})$, $\phi_1(K) = \phi_2(K)$.*

Proof. It is enough to see that for every $\phi \in val^+_k(\mathbb{V})$ and every $E \in G_k(\mathbb{V})$, we have

$$\mathrm{Kl}_\phi(E) = \phi\left(\underline{e_1} + \cdots + \underline{e_k}\right)$$

where $e_1, \dots, e_k$ is an orthonormal basis of E. □

Moreover for a valuation $\phi \in val^+_k(\mathbb{V})$ we have

$$\mathrm{Kl}_\phi(E) = \frac{1}{\kappa_k}\phi(B(E)).$$

Thus, since $B(E) \subset B(\mathbb{V})$ we have that there is a constant $0 < c < \infty$ such that for all $\phi \in val^+(\mathbb{V})$

$$\|\phi\|_{\mathrm{Kl}} \leq c\|\phi\|. \tag{3.1.2}$$

Example 3.1.7. One can check that for all $K, L \in \mathscr{K}_0(\mathbb{V})$ we have $K + L = K \cup L + K \cap L$. It follows that any linear function is a valuation of degree one.

Example 3.1.8. Let $L_1, \dots, L_k \in \mathscr{K}_0(\mathbb{V})$ and define the function $\phi_{L_1,\dots,L_k} : \mathscr{K}(\mathbb{V}) \to \mathbb{R}$ that is given for all $K \in \mathscr{K}_0(\mathbb{V})$ by

$$\phi_{L_1,\dots,L_k}(K) := \mathrm{MV}(K[m-k], L_1, \dots, L_k).$$

Then $\phi_{L_1,\dots,L_k} \in val^+(m-k, \mathbb{V})$. In particular the intrinsic volumes V_k are valuations.

Example 3.1.9. If $\phi \in val^+(\mathbb{V})$ is a valuation and $L \in \mathscr{K}_0(\mathbb{V})$ then the map

$$K \mapsto \phi(K + L)$$

is also a valuation. In particular $\mathrm{vol}_m(\cdot + L)$ is a valuation.

Another example that will be of particular importance to us is the following.

Example 3.1.10. For every signed measure $\mu \in \mathcal{M}(G_k(\mathbb{V}))$, we can associate a valuation $\phi_\mu \in val^+_k(\mathbb{V})$ given for all $K \in \mathscr{K}(\mathbb{V})$ by

$$\phi_\mu(K) = \int_{G_k(\mathbb{V})} \mathrm{vol}_k(K|E)\,\mathrm{d}\mu(E)$$

where $(K|E)$ denotes the orthogonal projection of K onto E.

3.1.2 Valuations on polytopes and non extendability of J-volume

On polytopes we have a weaker notion of continuity that corresponds to continuity on the set of polytopes that have parallel faces.

Definition 3.1.11. A valuation $\phi\colon \mathcal{P}(\mathbb{V}) \to \mathbb{R}$ is said to be *weakly continuous* if for every finite set $U = \{u_1, \dots, u_r\} \subset S(\mathbb{V}^*)$ of unit vectors positively spanning $\mathbb{V}^*$, i.e., $\sum_i \mathbb{R}_+ u_i = \mathbb{V}$, the function

$$(t_1, \dots, t_r) \mapsto \phi\left(\{v \in V \mid \langle v, u_i\rangle \le t_i,\, i = 1, \dots, r\}\right)$$

is continuous on the set $(t_1, \dots, t_r)$ for which the argument of ϕ is nonempty.

One can check that a continuous valuation is weakly continuous. The general form of weakly continuous, translation invariant valuations on $\mathcal{P}(V)$ was described in [63]. In particular, applying [63, Theorem 1] to the J-volume (Definition 2.3.15) we get.

Proposition 3.1.12. *The map* $\mathrm{vol}_n^J : \mathcal{P}(\mathbb{C}^n) \to \mathbb{R}$ *is a weakly continuous, translation invariant valuation.*

The valuation vol_n^J is a special case of an *angular valuation*, see [87].

Definition 3.1.13. Let $f : G_k(\mathbb{V}) \to \mathbb{R}$ be a measurable function. The *angular valuation* associated to f is $\phi_f : \mathcal{P}(\mathbb{V}) \to \mathbb{R}$, defined for all $P \in \mathcal{P}(\mathbb{V})$ by

$$\phi_f(P) := \sum_{F \in \mathcal{F}_k(P)} \mathrm{vol}_k(F) \cdot \Theta(F, P) \cdot f(E_F),$$

where recall the notation of Section 1.1.3 on polytopes.

It is known [63] that ϕ_f is a weakly continuous valuation.

The possibility of continuously extending an angular valuation from polytopes to convex bodies was studied by Wannerer in [87]. The following is [87, Theorem 1.2]. We recall that an even function on the sphere of $\Lambda^k \mathbb{V}$ induces a function on the Grassmannian $G_k(\mathbb{V})$ via the Plücker embedding.

Proposition 3.1.14. *The angular valuation* $\phi_f\colon \mathcal{P}(\mathbb{V}) \to \mathbb{R}$ *can be extended to a continuous valuation on* $\mathscr{K}(\mathbb{V})$*, if and only if* f *is the restriction to* $G_k(\mathbb{V})$ *of a homogeneous quadratic polynomial on* $\Lambda^k \mathbb{V}$*.*

We see from Definition 2.3.15 that the J-volume is the angular valuation associated to the function $(\sigma^J)^{1/2} : G_n(\mathbb{C}^n) \to \mathbb{R}$.

If $n = 1$, then σ^J is constant and equal to 1. The previous proposition implies that in this case we can extend vol_1^J to a continuous valuation on $\mathscr{K}(\mathbb{C})$. In fact one can see that in that case we obtain the classical first intrinsic volume. If $n \ge 2$, however, it is not possible as we will show next.

Corollary 3.1.15. *If* $n \ge 2$*, there is no continuous valuation on* $\mathscr{K}(\mathbb{C}^n)$ *that is equal to* vol_n^J *on* $\mathcal{P}(\mathbb{C}^n)$*.*

Proof. Using the notation of Proposition 3.1.14, we have $\mathrm{vol}_n^J = \phi_{(\sigma^J)^{1/2}}$. We identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and let J be the standard complex structure on it. Consider the homogeneous quadratic polynomial $p\colon \Lambda^n \mathbb{R}^{2n} \to \Lambda^{2n} \mathbb{R}^{2n}$, $w \mapsto w \wedge Jw$. From Definition 2.3.10, $\sigma^J(w) = |p(w)|$ for $w \in G(n, 2n)$ (in the Plücker embedding). Suppose there were a homogeneous quadratic polynomial $q\colon \Lambda^n \mathbb{R}^{2n} \to \mathbb{R}$ such that we have $|p(w)|^{\frac{1}{2}} = q(w)$ for all $w \in G(n, 2n)$. Let us show that this leads to a contradiction, which will complete the proof by Proposition 3.1.14. First of all, we notice that $q(w)$ must be a nonnegative polynomial and that we have $|p(w)| = q(w)^2$ on $G(n, 2n)$.

Let $e_1, \dots, e_n \in \mathbb{C}^n$ be the standard basis, so that $|e_1 \wedge \cdots \wedge e_n \wedge Je_1 \wedge \cdots \wedge Je_n| = 1$. We define the curve $w(\theta) := (\cos(\theta)e_1 + \sin(\theta)Je_2) \wedge e_2 \wedge \cdots \wedge e_n$ in $G(n, 2n)$ for $\theta \in [0, \pi]$. This curve interpolates between a Lagrangian plane (for $\theta = 0$) and a plane, which contains a complex line (for $\theta = \pi$). We have that

$$\begin{aligned} p(w(\theta)) &= (\cos(\theta)e_1 + \sin(\theta)Je_2) \wedge e_2 \wedge \cdots \wedge e_n \wedge (\cos(\theta)Je_1 - \sin(\theta)e_2) \wedge e_2 \wedge \cdots \wedge e_n \\ &= \cos(\theta)^2 (e_1 \wedge \cdots \wedge e_n \wedge Je_1 \wedge \cdots \wedge Je_n), \end{aligned}$$

and so $|p(w(\theta))| = \cos(\theta)^2$. If we have $|p(w(\theta))| = q(w(\theta))^2$, then $q(w(\theta)) = \cos(\theta)$, because q is nonnegative. Since q is a quadratic form and by the definition of $w(\theta)$, there are $a, b, c \in \mathbb{R}$ such that $q(w(\theta)) = a\cos(\theta)^2 + b\cos(\theta)\sin(\theta) + c\sin(\theta)^2$ for all θ. Thus, we have an equality of functions $a\cos(\theta)^2 + b\cos(\theta)\sin(\theta) + c\sin(\theta)^2 = \cos(\theta)$. It can be checked that such an equality is not possible, so our assumption was wrong and $(\sigma^J)^{1/2}$ cannot be the restriction of the square of a quadratic form to $G(n, 2n)$ and this completes the proof. □

Remark 3.1.16. Note that for all continuous $f : G_k(\mathbb{V}) \to \mathbb{R}$ there is a continuous extension of ϕ_f to zonoids given for all $K \in \mathscr{Z}_0(\mathbb{V})$ by

$$\phi_f(K) := \int_{G_k(\mathbb{V})} f \,\mathrm{d}\mu_{K^{\wedge k}}.$$

Open problem 12. Let $d \geq 4$ be even. What is the biggest subclass $\mathcal{C}_d \subset \mathscr{K}_0(\mathbb{V})$ containing $\mathscr{Z}_0(\mathbb{V})$ such that, for every f that is the restriction of a polynomial of degree d on $\Lambda^k\mathbb{V}$ to $G_k(\mathbb{V})$, the angular valuation ϕ_f extends continuously to $\mathcal{C}_d$?

3.1.3 Modern theory of valuation

In the modern theory of valuations there are two breakthroughs made by Alesker that we will present here. The first one is the irreducibility theorem and the second one is the discovery of a product on a subclass of valuations that respects the grading.

These results concern the study of the action of $Gl(\mathbb{V}^*)$ on $val^+(\mathbb{V})$ given for all $g \in Gl(\mathbb{V}^*)$, $\phi \in val^+(\mathbb{V})$ and $K \in \mathscr{K}(\mathbb{V})$ by

$$(g \cdot \phi)(K) := \phi(g^t(K)). \tag{3.1.3}$$

It is in general more common to consider the action of $Gl(\mathbb{V})$ given by $(g \cdot \phi) = \phi \circ g^{-1}$ but this makes no difference since the map $g \mapsto g^{-t}$ is a Lie group isomorphism between $Gl(\mathbb{V}^*)$ and $Gl(\mathbb{V})$. Moreover, it will be clear later that the action given by (3.1.3) is more natural in our context.

The following is [80, Theorem 6.5.1].

Proposition 3.1.17 (Alesker's irreducibility Theorem)**.** *The representation of $Gl(\mathbb{V}^*)$ on $val_k^+(\mathbb{V})$ is irreducible for every $0 \leq k \leq m$ meaning that each invariant subspace is dense in the standard topology.*

Note that because of inequality (3.1.2), a subspace dense in the standard topology is also dense in the Klain topology.

Example 3.1.18. The valuations of the form $\phi_{L_1,\dots,L_k}$ of Example 3.1.8 span a dense subspace of $val^+_{m-k}(\mathbb{V})$. Indeed, it is enough to note that for all $g \in Gl(\mathbb{V}^*)$

$$g \cdot \phi_{L_1,\dots,L_k} = |\det(g)|^{-1}\phi_{g^{-t}(L_1),\dots,g^{-t}(L_k)}.$$

Thus the subspace spanned by such valuations is $Gl(\mathbb{V}^*)$ invariant. By Alesker's irreducibility theorem, this is also dense.

In order to define the product and convolution of valuations, we need to restrict to a certain subclass.

Definition 3.1.19. A valuation $\phi \in val^+(\mathbb{V})$ is called *smooth* if the map $Gl(\mathbb{V}^*) \to val^+(\mathbb{V})$ given by $g \mapsto (g \cdot \phi)$ is smooth. We denote the subspace of smooth valuations by $val^{+,\infty}(\mathbb{V}) \subset val^+(\mathbb{V})$. Siminarly we write $val_k^{+,\infty}(\mathbb{V}) \subset val_k^+(\mathbb{V})$.

Example 3.1.20. Consider the inclusion $C^\infty_{even}(G_k(\mathbb{V})) \hookrightarrow \mathcal{M}(G_k(\mathbb{V}))$ that identifies a smooth function f on the Grassmannian with the measure whose density is f. Then we can consider the valuation ϕ_f as in Example 3.1.10: it is given for all $K \in \mathscr{K}(\mathbb{V})$ by

$$\phi_f(K) := \int_{G_k(\mathbb{V})} \mathrm{vol}_k(K|E) f(E)\,\mathrm{d}E$$

where $K|E$ denotes the orthogonal projection of K onto E and $\mathrm{d}E$ denotes integration with respect to the normalized Haar measure. It turns out that such valuation are smooth and furthermore that every smooth valuation admits such a form, see [20].

Example 3.1.21. If $L_1, \dots, L_k \in \mathscr{K}_0(\mathbb{V})$ are curved with smooth boundary then the valuation $\phi_{L_1,\dots,L_k}$ defined in Example 3.1.8 is smooth.

In [5], Semyon Alesker constructed a product of smooth valuations.

Proposition 3.1.22 (Alesker's product)**.** *There is a bilinear map $val_k^{+,\infty}(\mathbb{V}) \times val_l^{+,\infty}(\mathbb{V}) \to val_{k+l}^{+,\infty}(\mathbb{V})$ that is continuous for the standard topology and that turns the space $val^{+,\infty}(\mathbb{V})$ into a graded algebra. For every k, the product $val_k^{+,\infty}(\mathbb{V}) \times val_{m-k}^{+,\infty}(\mathbb{V}) \to val_m^{+,\infty}(\mathbb{V}) = \mathbb{R}\,\mathrm{vol}_m$ is a perfect pairing. We denote the product of two valuations ϕ_1, ϕ_2 by $\phi_1 \wedge \phi_2$.*

We will not need a precise description of this product since we will rather use the point of view of convolution that has a simpler characterization. The perfect pairing gives rise to a duality in this graded algebra, see [4, Theorem 1.2.1].

Proposition 3.1.23 (Poincaré-Alesker duality)**.** *There is an involution isomorphism of vector spaces $* : val_k^{+,\infty}(\mathbb{V}) \xrightarrow{\sim} val_{m-k}^{+,\infty}(\mathbb{V}^*)$ uniquely determined by the fact that for every $E \in G_{m-k}(\mathbb{V}^*)$ we have*

$$\mathrm{Kl}_{*\phi}(E) = \mathrm{Kl}_{\phi}(E^{\perp}).$$

With this duality we can define a convolution product.

Definition 3.1.24. The convolution $\vee : val_{m-k}^{+,\infty}(\mathbb{V}) \times val_{m-l}^{+,\infty}(\mathbb{V}) \to val_{m-(k+l)}^{+,\infty}(\mathbb{V})$ is defined for all valuations ϕ_1, ϕ_2 by

$$\phi_1 \vee \phi_2 := *((*\phi_1) \wedge (*\phi_2))$$

The convolution just defined is uniquely determined by continuity (for standard topology) and the fact that for every $L_1, \dots, L_k, L'_1, \dots, L'_l \in \mathscr{K}_0(\mathbb{V})$ we have (see [33, Definition 1.12] and also [20])

$$\phi_{L_1,\dots,L_k} \vee \phi_{L'_1,\dots,L'_k} = \phi_{L_1,\dots,L_k,L'_1,\dots,L'_k}$$

where recall the definition of the valuation $\phi_{L_1,\dots,L_k}$ in Example 3.1.8.

The intrinsic volumes are valuations that are invariant by the $O(\mathbb{V}^*)$ action (3.1.3). In fact Hadwiger showed that these are the only one, see [80, Theorem 6.4.14].

Proposition 3.1.25 (Hadwiger's Theorem)**.** *For all $0 \leq k \leq m$, we have*

$$val^{+}(\mathbb{V})^{O(\mathbb{V}^*)} = \mathbb{R}\mathrm{V}_k.$$

A remarkable refinement of this Theorem was given by Alesker, see [80, Theorem 6.5.3].

Proposition 3.1.26. *Let $H \subset O(\mathbb{V}^*)$ be a compact subgroup. The space $val^{+}(\mathbb{V})^{H}$ is finite dimensional if and only if H acts transitively on the sphere $S(\mathbb{V}^*)$.*

3.1.4 $\mathcal{P}-$positive valuations

Recently Nguyen-Bac Dang and Jian Xiao in [33] extended the product and convolution of smooth valuations to a larger subspace of $val^{+}(\mathbb{V})$ (they do not restrict to even valuations but we will).

Definition 3.1.27. A valuation $\phi \in val_k^{+}(\mathbb{V})$ is $\mathcal{P}-$positive if there exists a Radon measure M on $\mathscr{K}(\mathbb{V})^{m-k}$ such that for every convex body K we have

$$\phi(K) = \int_{\mathscr{K}(\mathbb{V})^{m-k}} \mathrm{MV}(K[k], L_1, \dots, L_{m-k}) \mathrm{d}M(L_1, \dots, L_{m-k}). \tag{3.1.4}$$

We denote by $\mathcal{P}_k^{+} \subset val_k^{+}(\mathbb{V})$ the cone of $\mathcal{P}-$positive valuations and by $\mathcal{P}_k \subset val_k^{+}(\mathbb{V})$ the subspace it generates, i.e $\mathcal{P}_k = \mathcal{P}_k^{+} - \mathcal{P}_k^{+}$.

Radon measures M on $\mathscr{K}(\mathbb{V})^{m-k}$ such that (3.1.4) is finite for every $K \in \mathscr{K}(\mathbb{V})$ (and thus give rise to a well defined $\mathcal{P}-$positive valuation) are characterized by the fact that

$$\int_{\mathscr{K}(\mathbb{V})^{m-k}} \mathrm{MV}(B(\mathbb{V})[k], L_1, \ldots, L_{m-k}) \mathrm{d}M(L_1, \ldots, L_{m-k}) < \infty. \tag{3.1.5}$$

The smooth valuations are dense in $\mathcal{P}_k$, see [33, Theorem 2.19]. Moreover, one can then show that there is a well defined extension of the convolution product on the completion of the space $\mathcal{P}_k$ for a certain norm, see [33, Section 2.3].

Proposition 3.1.28. *There is a well defined extension of the convolution product of smooth valuations that is such that if M_1, respectively M_2, is a Radon measure on $\mathscr{K}(\mathbb{V})^{k_1}$, respectively $\mathscr{K}(\mathbb{V})^{k_2}$, that satisfy* (3.1.5) *and if $\phi_1 \in \mathcal{P}^+_{m-k_1}$, respectively $\phi_2 \in \mathcal{P}^+_{m-k_2}$ is its associated $\mathcal{P}-$positive valuation then $\phi_1 \vee \phi_2 \in \mathcal{P}^+_{m-(k_1+k_2)}$ is the $\mathcal{P}-$positive valuation associated to the product measure $M_1 \otimes M_2$ on $\mathscr{K}(\mathbb{V})^{k_1+k_2}$, in other words, it is given for all $K \in \mathscr{K}(\mathbb{V})$ by*

$$(\phi_1 \vee \phi_2)(K) = \int \mathrm{MV}(K[m-(k_1+k_2)], L_1, \ldots, L_{k_1+k_2})\, \mathrm{d}M_1(L_1, \ldots, L_{k_1}) \mathrm{d}M_2(L_{k_1+1}, \ldots, L_{k_1+k_2})$$

where the integration is over $\mathscr{K}(\mathbb{V})^{k_1} \times \mathscr{K}(\mathbb{V})^{k_2} = \mathscr{K}(\mathbb{V})^{k_1+k_2}$.

3.2 The exponential map on zonoids

Before going on to the part where we show the link between the Grassmannian zonoid algebra and valuations (that the reader may have guessed at this point), we need a last little tool of zonoid calculus. We present in this section an exponential map from zonoids to the Grassmannian zonoid algebra and show how this shares similarities with Peter McMullen's polytope algebra [64].

3.2.1 Exponential and logarithm

Definition 3.2.1. We define the map $\exp : \widehat{\mathscr{Z}_0}(\mathbb{V}) \to \widehat{\mathscr{G}_0}(\mathbb{V})$ to be given for all $A \in \widehat{\mathscr{Z}_0}(\mathbb{V})$ by

$$\exp(A) := \sum_{k=0}^{m} \frac{1}{k!} A^{\wedge k}.$$

We also write $e^A := \exp(A)$.

Note that exp maps the cone $\mathscr{Z}_0(\mathbb{V})$ to the cone $\mathscr{G}_0(\mathbb{V})$. Moreover, since the wedge product is commutative on centered zonoids, we have the following.

Proposition 3.2.2. *The map* $\exp : (\widehat{\mathscr{Z}_0}(\mathbb{V}), +) \to (\mathscr{G}_0(\mathbb{V}), \wedge)$ *is a group morphism, i.e. we have $e^{\{0\}} = 1$ and for all $A, A' \in \widehat{\mathscr{Z}_0}(\mathbb{V})$ we have*

$$e^{A+A'} = e^A \wedge e^{A'}. \tag{3.2.1}$$

In particular every element in the image of the exponential is invertible for the wedge product and the inverse of e^A is e^{-A}. Note that moreover for the $Gl(\mathbb{V})$ action defined in Section 2.4.3 the exponential is a $Gl(\mathbb{V})$ morphism, meaning that for all $g \in Gl(\mathbb{V})$ and all $A \in \widehat{\mathscr{Z}_0}(\mathbb{V})$ we have

$$\exp(g \cdot A) = g \cdot \exp(A). \tag{3.2.2}$$

Proposition 3.2.3. *The image of the exponential map spans a sequentially dense subspace of $\widehat{\mathscr{G}_0}(\mathbb{V})$ for both the weak-$*$ and norm topology.*

Proof. It is enough to see that we can obtain all segments of degree k for all $k = 1, \dots, m$. Indeed if $x \in \mathbb{V}$ then $e^{\underline{x}} = 1 + \underline{x}$ thus,

$$\underline{x} = e^{\underline{x}} - e^{\{0\}}.$$

We go by induction, suppose we can obtain all segments of degree$\leq k-1$ and let $x_1, \dots, x_k \in \mathbb{V}$ then

$$e^{\underline{x_1}+\cdots+\underline{x_k}} = (1+\underline{x_1}) \wedge \cdots \wedge (1+\underline{x_k}) = \{\text{segments of deg} \leq k-1\} + \underline{x_1 \wedge \cdots \wedge x_k}$$

and this proves the induction. $\square$

One can invert algebraically the exponential but in order to do so we need to restrict to a subclass of Grassmannian zonoids.

Definition 3.2.4. We define

$$\widehat{\mathscr{G}_0}(\mathbb{V})_1 := \bigoplus_{k=1}^{m} \widehat{\mathscr{G}_0}(k, \mathbb{V})$$

i.e. elements of $\widehat{\mathscr{G}_0}(\mathbb{V})_1$ are elements of $\widehat{\mathscr{G}_0}(\mathbb{V})$ that have no term of degree 0.

Note that $\widehat{\mathscr{G}_0}(\mathbb{V})_1$ is a subalgebra and that $1+\widehat{\mathscr{G}_0}(\mathbb{V})_1$ is closed under the wedge product operation. Moreover note that for all $A \in \widehat{\mathscr{Z}_0}(\mathbb{V})$, $e^A \in 1 + \widehat{\mathscr{G}_0}(\mathbb{V})_1$.

Definition 3.2.5. We define the map $\log : 1 + \widehat{\mathscr{G}_0}(\mathbb{V})_1 \to \widehat{\mathscr{G}_0}(\mathbb{V})_1$ to be given for all $A \in \widehat{\mathscr{G}_0}(\mathbb{V})_1$ by

$$\log(1+A) := \sum_{k \geq 1} \frac{(-1)^k}{k} A^{\wedge k}.$$

Once again, because our algebra is commutative, we have for all $A, B \in \widehat{\mathscr{G}_0}(\mathbb{V})_1$,

$$\log((1+A) \wedge (1+B)) = \log(1 + (A + B + A \wedge B)) = \log(1+A) + \log(1+B)$$

and for all $A \in \widehat{\mathscr{Z}_0}(\mathbb{V})$

$$\log(e^A) = A.$$

Note that all these definition could be made using the convolution $\vee$ instead of the wedge product. If we define for all $A \in \widehat{\mathscr{Z}_0}(\Lambda^{m-1}\mathbb{V})$

$$e_A := \sum_{k=0}^{m} \frac{1}{k!} A^{\vee k}$$

with the convention that $A^{\vee 0} = \underline{\omega}$ where $\omega \in \Lambda^m \mathbb{V}$ is the volume form. Then since $A^{\vee k} = *((*A)^{\wedge k})$, we have

$$e_A = * e^{*A}.$$

3.2.2 Zonoid vs polytope algebra

Peter McMullen, in [64] introduced the *polytope algebra*. Formally it is the group generated by $[P]$ for all $P \in \mathcal{P}(\mathbb{V})$ with the product defined for all $P, Q \in \mathcal{P}(\mathbb{V})$ by

$$[P] \cdot [Q] := [P + Q] \tag{3.2.3}$$

and with the relations

(i) $[P + \{x\}] = [P]$ $\forall x \in \mathbb{V}$;

(ii) $[P] + [Q] = [P \cup Q] + [P \cap Q]$, $\forall P, Q \in \mathcal{P}(\mathbb{V})$ such that , $P \cup Q \in \mathcal{P}(\mathbb{V})$

The relation (i) means that the polytope algebra is actually generated by the translation class of polytopes while in (ii) we recognize the valuation property (3.1.1).

Similarly as what we did in the Grassmannian algebra, one can define the exponential and logarithm as power series in a subspace of the polytope algebra, see [64, Lemma 18]. The logarithm of a class $[P]$ can be thought of as the polytope P itself and thus the map $P \mapsto [P]$ can be thought of as an exponential map, the reader can check [27] to see how this statements can be made more rigorous in terms of support function. This is coherent with the definition of the product in the polytope algebra (3.2.3).

In the Grassmannian algebra we saw a very similar situation. If we consider the image of the exponential $\exp : \mathscr{Z}_0(\mathbb{V}) \to \mathscr{G}_0(\mathbb{V})$ (which spans a dense subspace by Proposition 3.2.3) then for each centered zonoid $K \in \mathscr{Z}_0(\mathbb{V})$ we have a class e^K in the algebra. It indeed satisfies for all $K, L \in \mathscr{Z}_0(\mathbb{V})$, $e^K \wedge e^L = e^{K+L}$ which is the analogous of (3.2.3).

Moreover the centered zonoids $\mathscr{Z}_0(\mathbb{V})$ can be thought of as the translation classes of the space of all zonoids $\mathscr{Z}(\mathbb{V})$ which would be the analogous of relations (i) in the polytope algebra.

It remains to see if we have an analogous of the relations (ii) in the zonoid algebra. That is, do we have $e^K + e^L = e^{K\cup L} + e^{K\cup L}$ whenever $K, L \in \mathscr{Z}_0(\mathbb{V})$ are such that $K \cap L$ and $K \cup L \in \mathscr{Z}_0(\mathbb{V})$? This amounts to ask if the map $\mathscr{Z}_0(\mathbb{V}) \to \mathscr{Z}_0(\Lambda^k\mathbb{V})$ given by $K \mapsto K^{\wedge k}$ is a valuation.

For $k = 0, 1, m-1, m$ we already know that this is the case. Indeed $k = 0, 1$ are trivial, $k = m$ is the volume and $k = m-1$ is the projection body (see Corollary 2.2.21). However for all the other k, the map $K \mapsto K^{\wedge k}$ is not the restriction of a continuous translation invariant valuation $\mathscr{K}(\mathbb{V}) \to \mathscr{K}(\Lambda^k\mathbb{V})$. Indeed our map is equivariant (see (3.2.2)) and recently Jacob Henkel and Thomas Wannerer showed in a yet unpublished work that such equivariant valuation do not exist for $1 < k < m-1$. It could be however that the exponential is a valuation on the set of zonoids only.

Despite this fact, we will see below in Section 3.3 how $\widehat{\mathscr{G}_0}(\mathbb{V}^*)$ is deeply linked to the algebra $val^+(\mathbb{V})$.

Finally, William Fulton and Bernd Sturmfels showed in [41] that the Chow ring of a toric variety embedds into the polytope algebra. The Chow ring of a toric variety computes the complete intersection of divisors in a toric variety, see also [25] for an extension of this fact to a more general algebra generated by convex bodies. We will see in Chapter 4 that the Grassmannian zonoid algebra will help compute *random* intersection in real manifolds, which strengthen the link between polytope and zonoid algebra.

3.2.3 Exponential of a polytope

One can extend the definition of the exponential to polytopes in a similar fashion as the extension of J-volume that was presented in Section 2.3.3. This is based on the property of the external angle of a zonotope showed in Lemma 1.2.9. However, we will show here that it fails to satisfy the axiom of a group morphism (3.2.1).

As for J-volume we start by giving a formula for zonotopes. Recall all the notation introduced in Section 1.1.3 and Section 1.2.1. The reader is also encouraged to compare this section to Section 2.3.3.

Proposition 3.2.6. *Let $x_1, \ldots, x_n \in \mathbb{V}$ and let $K := \sum_{i=1}^n \underline{x_i}$ be a zonotope. Then for all $0 \le k \le m$ we have*

$$\frac{1}{k!} K^{\wedge k} = \sum_{E \in G_k(K)} \mathrm{vol}_k(F(E, K))\, \underline{E}$$

where recall that $G_k(K)$ denotes the k-subspaces parallel to k-faces of K (Definition 1.2.7), $F(E, K)$ is the vectorial face defined in Definition 1.2.8 and $\underline{E} \subset \Lambda^k\mathbb{V}$ is the segment defined by a representent in Plücker of the subspace E (Definition 2.2.8).

Proof. We have

$$\begin{aligned}\frac{1}{k!} K^{\wedge k} &= \sum_{i_1 < \cdots < i_k} \underline{x_{i_1} \wedge \cdots \wedge x_{i_k}} \\ &= \sum_{E \in G_k(K)} \left(\sum \|x_{i_1} \wedge \cdots \wedge x_{i_k}\| \right) \underline{E}\end{aligned}$$

where the internal sum on the second line runs over the $i_1 < \cdots < i_k$ such that $x_{i_1}, \ldots, x_{i_k} \in E$. We then just apply Shephard's formula [81, Equation (57)] to see that this is equal to $\mathrm{vol}_k(F(E, K))$. $\square$

Of course this sum only makes sense for zonotopes since it refers to the vectorial face. However, using Lemma 1.2.9, we find another expression.

Proposition 3.2.7. *Let $x_1, \ldots, x_n \in \mathbb{V}$ and let $K := \sum_{i=1}^n \underline{x_i}$ be a zonotope. Then for all $0 \leq k \leq m$ we have*

$$\frac{1}{k!} K^{\wedge k} = \sum_{F \in \mathcal{F}_k(K)} \mathrm{vol}_k(F) \Theta(K, F)\, \underline{E_F}.$$

Proof. The proof is the same as in Theorem 2.3.14. $\square$

This allows us to carry over the definition of exp on polytopes.

Definition 3.2.8. for all $0 \leq k \leq m$ we define $\epsilon_k : \mathcal{P}(\mathbb{V}) \to \mathscr{G}_0(k, \mathbb{V})$ for all $P \in \mathcal{P}(\mathbb{V})$ by

$$\epsilon_k(P) := \sum_{F \in \mathcal{F}_k(P)} \mathrm{vol}_k(F) \Theta(P, F)\, \underline{E_F}.$$

Moreover we let $\exp : \mathcal{P}(\mathbb{V}) \to \mathscr{G}_0(\mathbb{V})$ be defined as $\exp := \sum_{k=0}^m \epsilon_k$.

Note that, in particular, ϵ_1 is a map that associate to each polytope a zonotope and that is the identity on zonotopes. The case $k = m - 1$ is something we already encountered.

Lemma 3.2.9. *For every $P \in \mathcal{P}(\mathbb{V})$ we have*

$$\epsilon_{m-1}(P) = \frac{1}{2} \Pi P.$$

Proof. This follows from the definition of the projection body in Definition 1.2.44 and the expression of the surface area measure of a polytope in Proposition 1.1.38. $\square$

Remark 3.2.10. In general, we can show with the same proof that $\ell(\epsilon_k(P)) = \mathrm{V}_k(P)$.

By Proposition 1.1.21, the k-faces of a sum $P_1 + P_2$ can be decomposed into the sum of i-faces of P_1 and $(k-i)$-faces of P_2 for $0 \leq i \leq k$, it follows that there is a mixed version of $\epsilon_k : \mathcal{P}(\mathbb{V})^k \to \mathscr{G}_0(k, \mathbb{V})$ that is Minkowski multilinear and such that $\epsilon_k(P) = \epsilon_k(P, \ldots, P)$ for all $P \in \mathcal{P}(\mathbb{V})$. For the exponential to be a group morphism we would need for all $0 \leq i \leq k$ that $\binom{k}{i} \epsilon_k(P_1[i], P_2[k-i]) = \epsilon_i(P_1) \wedge \epsilon_{k-i}(P_2)$. Unfortunately this is impossible because of the next result.

Lemma 3.2.11. *Let $P \subset \mathbb{V}$ be a centrally symmetric polytope that is not a zonotope. Then*

$$\epsilon_{m-1}(P) \neq \epsilon_{m-1}(\epsilon_1(P))$$

Proof. This follows from Lemma 3.2.9 and the unicity of the solution to the Minkowski problem [80, Theorem 8.1.1]. $\square$

3.3 Valuation associated to a zonoid

In this section we describe how the algebra $\widehat{\mathscr{G}_0}(\mathbb{V}^*)$ surjects on a dense subalgebra of $val^+(\mathbb{V})$. From the point of view of measures, the correspondence between our convolution product and the convolution of valuations was described in [20]. We show here how this correspondence connects to the exponential just defined and hope to convince the reader that the flexibility of the point of view of random vectors helps to simplify the expressions and computations.

3.3.1 Valuations with Crofton measures

We start with the definition of the main object of this section. Remember that to each $A \in \widehat{\mathscr{G}_0}(k, \mathbb{V})$ there is a generating (signed) measure associated $\mu_A \in \mathcal{M}(G_k(\mathbb{V}))$, see Section 2.6.

Definition 3.3.1. For every $A \in \widehat{\mathscr{G}_0}(\mathbb{V}^*)$, we define $\phi_A : \mathscr{K}(\mathbb{V}) \to \mathbb{R}$ to be the map given for any $K \in \mathscr{K}(\mathbb{V})$ by

$$\phi_A(K) := \int_{G_k(\mathbb{V}^*)} \mathrm{vol}_k(K|\tilde{E}) \mathrm{d}\mu_A(E) \tag{3.3.1}$$

where $\tilde{E} \cong E$ under the identification $\mathbb{V}^* \cong \mathbb{V}$ given by the Euclidean structure and $K|\tilde{E}$ denotes the orthogonal projection of K onto E. We also define a mixed version, that we also denote ϕ_A, that is defined for all $K_1, \ldots, K_k \in \mathscr{K}(\mathbb{V})$ by

$$\phi_A(K_1, \ldots, K_k) := \int_{G_k(\mathbb{V}^*)} \mathrm{MV}(K_1|\tilde{E}, \ldots, K_k|\tilde{E}) \mathrm{d}\mu_A(E).$$

Finally if $A \in \widehat{\mathscr{G}_0}(\mathbb{V})$ and $A := \sum_{i=0}^m A_i$ with $A_i \in \widehat{\mathscr{G}_0}(i, \mathbb{V})$ we define $\phi_A := \sum_{i=0}^m \phi_{A_i}$.

The measure μ_A is somtimes called a *Crofton measure* for the valuation ϕ_A.

Remark 3.3.2. The choice in the definition of taking a measure on the dual space and then identify the subspaces $\tilde{E} \cong E$ may seem strange and arbitrary. However we will see that this definition is independent of the choice of the Euclidean structure which should convince the reader of its naturality.

Lemma 3.3.3. *For every $A \in \widehat{\mathscr{G}_0}(k, \mathbb{V}^*)$, the map ϕ_A belongs to $val_k^+(\mathbb{V})$.*

Proof. Suppose $K_n \to K$ in $\mathscr{K}(\mathbb{V})$, the projection onto a subspace and the volume are continuous thus $\mathrm{vol}_k(K_n|\tilde{E}) \to \mathrm{vol}_k(K|\tilde{E})$. Moreover since K_n converges, it follows that $\mathrm{vol}_k(K_n|\tilde{E})$ is bounded uniformly on $\tilde{E}$ and thus we obtain $\phi_A(K_n) \to \phi_A(K)$ by dominated convergence. □

We denote by

$$\Phi : \widehat{\mathscr{G}_0}(\mathbb{V}^*) \to val^+(\mathbb{V})$$

the map given for all $A \in \widehat{\mathscr{G}_0}(\mathbb{V}^*)$ by $\Phi(A) := \phi_A$. We denote the image of Φ by

$$\mathscr{V}(\mathbb{V}) := \Phi(\widehat{\mathscr{G}_0}(\mathbb{V}^*)) \subset val^+(\mathbb{V}) \tag{3.3.2}$$

and we write $\mathscr{V}^+(\mathbb{V}) := \Phi(\mathscr{G}_0(\mathbb{V}^*))$, $\mathscr{V}(k, \mathbb{V}) := \Phi(\widehat{\mathscr{G}_0}(k, \mathbb{V}^*))$ and $\mathscr{V}^+(k, \mathbb{V}) := \Phi(\mathscr{G}_0(k, \mathbb{V}^*))$.

It follows from the definition that Φ is linear. Moreover we will prove later that it is a $Gl(\mathbb{V})$ map for the representation (3.1.3). Let us prove something a bit weaker for now. It is easier in this case to identify $\mathbb{V}^* \cong \mathbb{V}$ and consider the action given by the inverse rather than the transpose.

Proposition 3.3.4. *For all $0 \leq k \leq m$, the space $\mathscr{V}(k, \mathbb{V})$ is $Gl(\mathbb{V})$–invariant under the action given by $(g\tilde{\cdot}\phi)(K) := \phi(g^{-1}K)$ and thus it is also invariant for the action given by* (3.1.3).

Proof. Let $\phi := \phi_A$ for $A \in \widehat{\mathscr{G}}(k, \mathbb{V}^*)$. Then identifying $\mathbb{V}^* \cong \mathbb{V}$ we have:

$$\phi_A(g^{-1}K) := \int_{G_k(\mathbb{V})} \mathrm{vol}_k(g^{-1}K|E) \mathrm{d}\mu_A(E).$$

Now we prove that

$$\mathrm{vol}_k(g^{-1}K|E) = \frac{1}{\|g(E)\|} \mathrm{vol}_k(K|g(E)) \tag{3.3.3}$$

Where $\|g(E)\| := \|g(e_1) \wedge \cdots \wedge g(e_k)\|$ for $e_1, \ldots, e_k$ an orthornormal basis of E. We use the decomposition $g = QR$ where Q is orthogonal and R preserves E. Then $\mathrm{vol}_k(g^{-1}K|E) = |\det(R^{-1}|_E)| \mathrm{vol}_k(Q^{-1}K|E)$. Next we note that $|\det(R^{-1}|_E)| = |\det(R|_E)|^{-1} = \|R(E)\|^{-1} = \|g(E)\|^{-1}$. Moreover since Q is orthogonal and $R(E) = E$, we have $Q^{-1}K|E = K|Q(E) = K|g(E)$ and this proves (3.3.3). Now considering the measure μ' such that for all $f \in C(G_k(\mathbb{V}))$ we have

$$\int_{G_k(\mathbb{V})} f(E) \mathrm{d}\nu(E) := \int_{G_k(\mathbb{V})} \frac{1}{\|g(E)\|} f(g(E)) \mathrm{d}\mu_A(E).$$

It follows that

$$(g \cdot \phi_A)(K) = \int_{G_k(\mathbb{V})} \mathrm{vol}_k(K|g(E))\mathrm{d}\nu(E) = \phi_{A'}(K)$$

where A' is the zonoid generated by ν. $\square$

From Alesker's irreducibility Theorem (Proposition 3.1.17) we deduce then the following.

Corollary 3.3.5. *The subspace $\mathscr{V}(\mathbb{V}) \subset val^+(\mathbb{V})$ is dense for the standard topology.*

The following is a key result that will allow us, in some sense, to identify Φ with the cosine transform on the Grassmannian. Recall from Section 2.4.4 that for every $A \in \widehat{\mathscr{G}_0}(k, \mathbb{V}^*)$, we write $g_A = h_A|_{G_k(\mathbb{V})}$ for the restriction of the support function on the Grassmannian.

Lemma 3.3.6. *For every $A \in \widehat{\mathscr{G}_0}(k, \mathbb{V}^*)$ we have*

$$\mathrm{Kl}_{\phi_A} = g_A.$$

Proof. Let $F \in G_k(\mathbb{V})$, we want to compute $\mathrm{Kl}_{\phi_A}(F)$. To do this consider $K \in \mathscr{K}(F)$. We need only to observe that in that case for every $E \in G_k(\mathbb{V})$, $\mathrm{vol}_k(K|\tilde{E}) = \mathrm{vol}_k(K)|\langle E, F\rangle|$. Then using (3.3.1) and (1.2.8), we find $\phi_A(K) = \mathrm{vol}_k(A) \cdot h_A(F)$. $\square$

It follows that we have

$$\|\phi_A\|_{\mathrm{Kl}} \leq \|A\|.$$

In other words, the map

$$\Phi : (\widehat{\mathscr{G}_0}(\mathbb{V}^*), \|\cdot\|) \to (val^+(\mathbb{V}), \|\cdot\|_{\mathrm{Kl}})$$

is continuous. The cosine transform can be seen as the map $\mathrm{Id} : (\widehat{\mathscr{G}_0}(\mathbb{V}^*), \text{weak-}*) \to (\widehat{\mathscr{G}_0}(\mathbb{V}^*), \|\cdot\|)$. Since it is sequentially continuous (Proposition 1.2.54-(*iv*)), we have that

$$\Phi : (\widehat{\mathscr{G}_0}(\mathbb{V}^*), \text{weak-}*) \to (val^+(\mathbb{V}), \|\cdot\|_{\mathrm{Kl}})$$

is sequentially continuous.

Remark 3.3.7. Note that from the definition we have immediately that weak–$*$ convergence implies pointwise convergence of the corresponding valuation. However sequential continuity is stronger.

Recall the definition of the kernel of the cosine transform (KoCT) in Section 2.4.4. It follows from Lemma 3.3.6 that this corresponds to the kernel of Φ.

Corollary 3.3.8. *The kernel of the map Φ is the KoCT, i.e. for all $A \in \widehat{\mathscr{G}_0}(\mathbb{V}^*)$ we have*

$$\phi_A \equiv 0 \iff A \in \mathfrak{M}(\mathbb{V}^*).$$

This allows us to identify the spaces

$$\mathscr{V}(\mathbb{V}) \cong \widehat{\mathscr{G}_0}(\mathbb{V}^*)/\mathfrak{M}(\mathbb{V}^*).$$

Since $\mathfrak{M}(\mathbb{V}^*)$ is an ideal, the wedge product is a well defined product on the quotient. Moreover remember that it is preserved by the Hodge star and thus is also an ideal for the convolution, see Section 2.4.4. This implies that the following is well defined.

Definition 3.3.9. Let $A, A' \in \mathcal{G}(\mathbb{V}^*)$, we define

$$\phi_A \wedge \phi_{A'} := \phi_{A\wedge A'} \qquad *\phi_A := \phi_{*A} \qquad \phi_A \vee \phi_{A'} := \phi_{A\vee A'}.$$

As the reader may expect, we will see that this notation does not contradict the one introduced in the previous section with Alesker's product. For this it will be handy to use the zonoid calculus to prove it on zonoids and then use Lemma 3.1.6 to conclude.

3.3.2 Exponential and valuations

On zonoids the valuations of $\mathscr{V}(\mathbb{V})$ take a very nice form. Recall the definition of the exponential of a zonoid in Section 3.2 and the definition of the pairing in Section 2.4.2.

Proposition 3.3.10. *For every $A \in \widehat{\mathscr{G}}_0(\mathbb{V}^*)$ and $K \in \mathscr{Z}_0(\mathbb{V})$ we have*

$$\phi_A(K) = (A, e^K). \tag{3.3.4}$$

Proof. Since both sides are linear in A, we can assume that A is of degree k and that μ_A is a probability measure on $G_k(\mathbb{V}^*)$. Let $Y = Y_1 \wedge \cdots \wedge Y_k$ be of law μ_A and let E_Y be the k–space represented by Y in Plücker. Let $K = \mathbb{E}\underline{X}$ with $X \in \mathbb{V}$ integrable and let $X_1, \ldots, X_k$ be iid copies of X that we assume independent of Y. We have by (2.2.4)

$$\phi_A(K) = \frac{1}{k!} \int_{G_k(\mathbb{V}^*)} \mathbb{E}\|\pi_E(X_1) \wedge \cdots \wedge \pi_E(X_k)\| \mathrm{d}\mu_A(E)$$

where π_E is the orthogonal projection onto $\tilde{E}$. Since Y is of law μ_A, we can rewrite it as

$$\begin{aligned}
\phi_A(K) &= \frac{1}{k!}\mathbb{E}\|\pi_{E_Y}(X_1) \wedge \cdots \wedge \pi_{E_Y}(X_k)\| \\
&= \frac{1}{k!}\mathbb{E}|\langle Y, X_1 \wedge \cdots \wedge X_k\rangle| \\
&= (A, \frac{1}{k!}K^{\wedge k}).
\end{aligned}$$

Since A is of degree k, the last term is equal to (A, e^K) and this concludes the proof. □

Note that, as promised in Remark 3.3.2, the expression (3.3.4) is independent of the choice of any Euclidean structure.

Lemma 3.3.11. *Let $A_1, A_2 \in \widehat{\mathscr{G}}(\mathbb{V}^*)$. Then, $\phi_{A_1} = \phi_{A_2}$ if and only if for every $C \in \widehat{\mathscr{G}}(\mathbb{V}^*)$, $\ell(A_1 \wedge C) = \ell(A_2 \wedge C)$.*

Proof. We can assume A_1 and A_2 are of degree k. Let us write $\phi_i := \phi_{A_i}$. From Lemma 3.1.6 $\phi_1 = \phi_2$ if and only if for all $K \in \mathscr{Z}_0(\mathbb{V})$, $\phi_1(K) = \phi_2(K)$. From Proposition 3.3.10 this is the case if and only if $(A_1 - A_2, e^K) = 0$ for all $K \in \mathscr{Z}_0(\mathbb{V})$. Since they are of degree k, we get that $\phi_1 = \phi_2$ if and only if $\ell((A_1 - A_2) \wedge *(K^{\wedge k})) = 0$ for all $K \in \mathscr{Z}_0(\mathbb{V})$. The result follows by the (sequential) density of the image of the exponential, that is Proposition 3.2.3. □

Corollary 3.3.12. *For every $g \in Gl(\mathbb{V}^*)$, $A \in \widehat{\mathscr{G}}_0(\mathbb{V}^*)$, $K \in \mathscr{K}(\mathbb{V})$, we have*

$$\phi_{g \cdot A}(K) = \phi_A(g^t K).$$

In other words, Φ is a $Gl(\mathbb{V})$ map and we have $g \cdot \phi_A = \phi_{g \cdot A}$

Proof. For zonoids this is almost immediate since $(g \cdot A, e^K) = (A, g^t e^K) = (A, e^{g^t K})$. Then we use Lemma 3.1.6 to conclude in general. □

Example 3.3.13. Using Lemma 3.3.6 one can prove that

$$\phi_{B(\mathbb{V}^*)^{\wedge k}} = k!\kappa_k \mathrm{V}_k.$$

We can easily compute another example.

Lemma 3.3.14. *Let $L \in \mathscr{Z}(\mathbb{V})$, then for all $K \in \mathscr{K}(\mathbb{V})$ we have*

$$\phi_{*e^L}(K) = \mathrm{vol}_m(L + K).$$

Proof. As before, by Lemma 3.1.6, it is enough to prove it for K a zonoid. In that case we use the expression of Proposition 3.3.10 and the rules in Proposition 2.4.8 to obtain

$$\phi_{*e^L}(K) = (*e^L, e^K) = (\underline{\omega}, e^L \wedge e^K) = (\underline{\omega}, e^{L+K}) = \frac{1}{m!}\ell\left((K+L)^{\wedge m}\right)$$

where $\omega \in \Lambda^m \mathbb{V}^*$ is the volume form. We conclude by (2.2.4). ☐

With the same proof we find the following, to be compared with Example 3.1.8.

Example 3.3.15. Let $L_1, \ldots, L_k \in \mathscr{Z}_0(\mathbb{V})$. Then for all $K \in \mathscr{K}(\mathbb{V})$ we have:

$$\phi_{*(L_1\wedge\cdots\wedge L_k)}(K) = \frac{m!}{(m-k)!}\mathrm{MV}(K[m-k], L_1, \ldots, L_k).$$

Finally one can compute the length of a Grassmannian zonoid just knowing the associated valuation.

Lemma 3.3.16. *Let $A \in \widehat{\mathscr{G}_0}(k, \mathbb{V}^*)$ and let $\phi_A \in \mathscr{V}(k, \mathbb{V})$ be the associated valuation. We have*

$$\ell(A) = \frac{1}{\kappa_k}\phi_A(B(\mathbb{V})).$$

Proof. The unit ball is a zonoid thus, by Proposition 3.3.10 and because A is of degree k, we have $\phi_A(B(\mathbb{V})) = \frac{1}{k!}(A, B(\mathbb{V})^{\wedge k})$. Using the rules of the pairing Proposition 2.4.8 we find $\phi_A(B(\mathbb{V})) = \frac{1}{k!}\ell(*A \wedge B(\mathbb{V})^{\wedge k})$ and by Lemma 2.2.17 $\frac{1}{k!}\ell(*A \wedge B(\mathbb{V})^{\wedge k}) = \kappa_k \ell(*A) = \kappa_k \ell(A)$ which is what we wanted. ☐

Let us now show that the valuations obtained from zonoids are a particular case of $\mathcal{P}$-positive valuations in the sense of Definition 3.1.27.

Proposition 3.3.17. *Let $X_1 \wedge \cdots \wedge X_k \in \Lambda^k \mathbb{V}^*$ be integrable and let $A := \mathbb{E}\underline{X_1 \wedge \cdots \wedge X_k} \in \mathscr{G}_0(k, \mathbb{V}^*)$. Let us write $X_{k+1} \wedge \cdots \wedge X_m := *(X_1 \wedge \cdots \wedge X_k) \in \Lambda^{m-k}\mathbb{V}$. Then for every convex body $K \subset \mathbb{V}$, we have:*

$$\phi_A(K) = \frac{1}{k!}\mathbb{E}\left[\mathrm{MV}\left(K[k], \underline{X_{k+1}}, \ldots, \underline{X_m}\right)\right].$$

In particular, $\mathscr{V}^+(k, \mathbb{V}) \subset \mathcal{P}_k^+$ and thus $\mathscr{V}(k, \mathbb{V})$ is a subspace of $\mathcal{P}_k$.

Proof. Both sides define a continuous even translation invariant valuation of degree k. Thus it is enough to prove the equality for zonoids, in which case it follows from Proposition 3.3.10. ☐

It is now not difficult to show that our product and convolution correspond to those already defined on valuations.

Theorem 3.3.18 (M.)**.** *The convolution $\vee$ on $\mathscr{V}(\mathbb{V})$ in Definition 3.3.9 corresponds to the convolution product of $\mathcal{P}$-positive valuations from Proposition 3.1.28, and the Hodge star $*$ to the Poincaré–Alesker duality. Thus the wedge product $\wedge$ corresponds to the usual product of valuations.*

Proof. The duality follows from Proposition 3.1.23 and Lemma 3.3.6.

Let us prove the convolution. Let $A := \mathbb{E}\underline{X_1 \wedge \cdots \wedge X_k} \in \mathscr{G}_0(k, \mathbb{V}^*)$, $B := \mathbb{E}\underline{Y_1 \wedge \cdots \wedge Y_l} \in \mathscr{G}_0(l, \mathbb{V}^*)$. Let us write $X_{k+1} \wedge \cdots \wedge X_m := *(X_1 \wedge \cdots \wedge X_k) \in \Lambda^{m-k}\mathbb{V}$ and $Y_{l+1} \wedge \cdots \wedge Y_m := *(Y_1 \wedge \cdots \wedge Y_l) \in \Lambda^{m-l}\mathbb{V}$. By definition $\phi_A \vee \phi_B = \phi_{A\vee B}$ and

$$A \vee B = *((*A) \wedge (*B)) = *\mathbb{E}\underline{X_{k+1} \wedge \cdots \wedge X_m \wedge Y_{l+1} \wedge \cdots \wedge Y_m}.$$

Thus by Proposition 3.3.17 we have for every convex body K

$$(\phi_A \vee \phi_B)(K) = \frac{1}{(k+l-m)!}\mathbb{E}\left[\mathrm{MV}\left(K[k+l-m], \underline{X_{k+1}}, \ldots, \underline{X_m}, \underline{Y_{l+1}}, \ldots, \underline{Y_m}\right)\right]$$

which is what we wanted. ☐

Later, we will want to evaluate length of wedge products in the Grassmannian algebra. Lemma 3.3.11 tells us that for this purpose we can reduce to the quotient $\widehat{\mathscr{G}_0}(\mathbb{V}^*)/\mathfrak{M}(\mathbb{V}^*) \cong val^+(\mathbb{V})$. The last result Theorem 3.3.18 tells us that this equality is also an equality of algebras.

This quotient can reduce the ring significantly specially in the case of invariant zonoids. Since the map Φ is a $Gl(\mathbb{V}^*)$ map, if $H \subset O(\mathbb{V}^*)$ is a closed subgroup, it maps $\widehat{\mathscr{G}_0}(\mathbb{V}^*)^H$ to $val^+(\mathbb{V})^H$.

Example 3.3.19. Using Example 3.3.13 and Example 2.4.14 we get Hadwiger's Theorem (Proposition 3.1.25) and we get that as algebra

$$val^+(\mathbb{V})^{O(\mathbb{V}^*)} \cong \mathbb{R}[t]/(t^{m+1})$$

where the generator is $t = \phi_{B(\mathbb{V})}$. In this case there is no reduction and $\mathfrak{M}(\mathbb{V}^*)^{O(\mathbb{V}^*)} = \{0\}$.

Example 3.3.20. Let $H = U(n)$ acting on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Since $U(n)$ acts transitively on the sphere S^{2n-1}, by Proposition 3.1.26, $val^+(\mathbb{C}^n)^{U(n)}$ is finite dimensional. As an algebra, it was described by Joseph Fu, see [40, Theorem 3.1]:

$$val^+(\mathbb{C}^n)^{U(n)} \cong \mathbb{R}[t,s]/(f_{n+1}, f_{n+2}),$$

where $\deg t = 1$, $\deg s = 2$, and

$$\log(1 + t + s) = \sum_{i\geq 1} f_i(t,s)$$

is the decomposition into homogeneous components. The generators are given (up to multiples) by $t = \phi_{B(\mathbb{R}^{2n})}$ and $s = \phi_{P_n}$ where $P_n \in \mathscr{G}_0(2, \mathbb{C}^n)$ is the Grassmannian zonoid whose generating measure is the uniform measure on $\mathbb{C}\mathrm{P}^n \subset G_2(\mathbb{R}^{2n})$. In that case this is a considerable reduction, since the algebra $\widehat{\mathscr{G}_0}(\mathbb{C}^n)^{U(n)}$ is *infinite* dimensional, see Example 2.4.15. Note also that, as was explained in Section 3.2, $\log(1 + B(\mathbb{R}^{2n}) + P_n)$ is a well defined element of the Grassmannian algebra. It is unclear to the author what exactly is happening here.

Chapter 4

Integral geometry

In this chapter we apply the techniques developed in the previous chapters to the study of random intersection in Riemannian manifolds. The general setting is as follows. Let M be a Riemannian manifold and let $X_1, \dots, X_d \hookrightarrow M$ be independent *random* submanifolds. We want to compute in general the quantity

$$\mathbb{E}\,\mathrm{vol}(X_1 \cap \cdots \cap X_d) \tag{4.0.1}$$

where the volume is the Riemannian volume in the appropriate dimension. For example when we have $\sum_{i=1}^d \mathrm{codim}(X_i) = \dim M$ we should expect a finite number of points in the intersection and we want to compute the number

$$\mathbb{E}\#(X_1 \cap \cdots \cap X_d)$$

We will see this in two different situation. The first case is a joint work with Peter Bürgisser, Paul Breiding and Antonio Lerario (abbreviated B.B.L.M.) and is still partly ongoing. We let M be a compact Riemannian homogeneous space meaning that its group of isometries G acts transitively. The random submanifolds considered are then built taking a fixed submanifold $\tilde{X} \hookrightarrow M$ and a uniform random $g \in G$ and considering $X = g \cdot X$. We will see that in that case one can associate to each such X a zonoid $K_X \in \mathscr{G}_0(\mathbb{V}^*)$ where $\mathbb{V}^* = T_o^*M$ is the cotangent at a point M. This zonoid computes (4.0.1) in the sense that we have:

$$\mathbb{E}\,\mathrm{vol}(X_1 \cap \cdots \cap X_d) = \ell(K_{X_1} \wedge \cdots \wedge K_{X_d}).$$

In some sense the Grassmannian zonoid algebra and the algebra of valuation behave as a *probabilistic Chow ring* on Riemannian homogeneous spaces. We will see how in that case the action of the isotropy group and the invariance of the zonoid will help to reduce the complexity.

The second case is a joint work with Michele Stecconi. In this situation M is a Riemannian manifold and $X = f^{-1}(0)$ where $f \in C^1(M, \mathbb{R}^k)$ is a random differentiable function whose law satisfy some suitable condition. In that case we build a zonoid in each cotangent space at each point, i.e. we build a section $\zeta : p \mapsto \zeta(p) \in \mathscr{G}(T_p^*, M)$ of the fiber bundle whose fibers are spaces of zonoids in the cotangent space. Then we show that for each open set $U \subset M$ we have

$$\mathbb{E}\,\mathrm{vol}(X_1 \cap \cdots \cap X_d \cap U) = \int_U \ell(\zeta_{X_1}(p) \wedge \cdots \wedge \zeta_{X_d}(p))\mathrm{d}M(p)$$

where $\mathrm{d}M(p)$ indicates integration with respect to the Riemannian density on M. In other words the map $U \mapsto \mathbb{E}\,\mathrm{vol}(X_1 \cap \cdots \cap X_d \cap U)$ is a measure on M which is absolutely continuous with respect to the Riemannian density and whose density is given by $p \mapsto \ell(\zeta_{X_1}(p) \wedge \cdots \wedge \zeta_{X_d}(p))$. We will see how this interprets nicely in the context of Finsler geometry.

4.1 Homogeneous spaces

In this section, the global approach presented in Section 4.1.1 and the probabilistic Schubert calculus of Section 4.1.3 is a joint work with Peter Bürgisser, Paul Breiding and Antonio Lerario. The reduction to valuations (Proposition 4.1.15 and 4.1.16), the computations in $\mathbb{C}\mathrm{P}^n$ presented in Section 4.1.2 and the geometry of invariant bodies in Section 4.1.4 are the work of the author.

In this section, G will be a compact Lie group, endowed with a left and right invariant Riemannian metric. We denote by $\mathrm{Id} \in G$ the identity and by $L_g : G \to G$ the left translation by $g \in G$. If $H \subseteq G$ is a closed Lie subgroup, the homogeneous space $M := G/H$ inherits a Riemannian structure, by declaring the quotient map $p : G \to G/H$ to be a Riemannian submersion (this is well defined since H acts on G by isometries). We let $m := \dim M$. We denote by $o \in M$ the image of Id under the projection map and by $L_g : M \to M$ the map induced by left translation by an element $g \in G$. For the left translations, we will use use the notation $L_g(x) = g \cdot x$ (when the base point is clear from the context, we will simply denote by g_* the differential of L_g). Finally, we denote by $g_x^* : T^*_{g(x)}M \to T^*_x M$ the pull-back given for all $\alpha \in T^*_{g(x)}M$ by $g_x^*(\alpha) := \alpha \circ g_*$. We refer the reader to [45] for more details on these constructions.

Below, when talking about "volume" of a submanifold of a Riemannian manifold, we will mean the Riemannian volume (for the induced Riemannian structure). When a smooth manifold has finite volume, we can turn it into a probability space by normalizing the volume to 1; we will call the resulting distribution the *uniform distribution*. Recall also that on compact Lie groups there is a unique normalized Haar measure and that we call random elements with such law *uniform*.

4.1.1 Submanifolds and associated zonoid

In the following, submanifolds are assumed to be relatively compact and of finite volume. The Riemannian structure induced on a submanifold $X \subset M$ gives rise to a measure and a random point $x \in X$ will be called *uniform* if its law is this measure normalized to be a probability measure.

In all this section, the *isotropy group* H is the subgroup of G that fixes $o \in M$. Thus we have an action of H on T_oM by $g \cdot v := g_*(v)$ and on T_o^*M by $g \cdot \alpha := g^*(\alpha)$.

Definition 4.1.1. In the following we will write $\mathbb{V} := T_oM$ and $\mathbb{V}^* := T_o^*M$.

Recall, from Section 2.4.3, that the linear action of H on $\mathbb{V}^*$ induces an action on all the exterior powers $\Lambda^k\mathbb{V}^*$. Moreover, G acts by isometries on M and thus the action of H is orthogonal. This implies that H acts on the Grassmannians $G_c(\mathbb{V}^*)$. For the following definition, recall also that given a subspace $E \in G_c(\mathbb{V}^*)$, we denote by $\underline{E}$ the segment of length 1 in $\Lambda^c\mathbb{V}^*$ that is supported by a representant of E in the Plücker embedding, see Definition 2.2.8.

Definition 4.1.2. Let $X \subset M$ be a submanifold of codimension c and let $x \in X$ and let $g \in G$ be such that $g(o) = x$. Then we define the following Grassmannian zonoid of $\mathbb{V}^*$.

$$K_X(x) := \frac{\mathrm{vol}_{m-c}(X)}{\mathrm{vol}_m(M)} \mathbb{E}\underline{h^* g^* N_x X} \in \mathscr{G}(c, \mathbb{V}^*) \tag{4.1.1}$$

where $h \in H$ is uniform. Moreover we let

$$K_X := \mathbb{E} K_X(\xi) \in \mathscr{G}(c, \mathbb{V}^*)$$

where $\xi \in X$ is uniform.

Note that this definition is independent of the choice of $g \in G$. Moreover observe that we have

$$\ell(K_X) = \frac{\mathrm{vol}_{m-c}(X)}{\mathrm{vol}_m(M)}.$$

Example 4.1.3. Points are submanifolds of codimension $m = \dim M$ and, as one can see from the definition, for all $x \in M$ we have

$$K_{\{x\}} = K_{\{o\}} = \frac{1}{\mathrm{vol}_m(M)} \underline{\omega_o} \in \mathscr{Z}_0(\Lambda^m\mathbb{V}^*) \tag{4.1.2}$$

where $\omega_o \in \Lambda^m\mathbb{V}^*$ is the volume form at o.

In general, products of these zonoids compute the average volume of intersection. This is based on a kinematic formula in Riemannian homogeneous spaces proved by Ralph Howard in [45] and generalized by Peter Bürgisser and Antonio Lerario in [29]. The following is [29, Theorem 7.2] translated in our context.

Theorem 4.1.4 (B.B.L.M.)**.** *Let $X_1, \ldots, X_n \subset M$ be submanifolds, such that $c := \sum_{i=1}^n \operatorname{codim}(X_i) \leq m$ where recall that $m = \dim(M)$, and let $K_{X_1}, \ldots, K_{X_n}$ be their associated zonoids defined in Definition 4.1.2. Let $g_1, \ldots, g_n \in G$ be independent and uniform. Then*

$$\frac{1}{\operatorname{vol}_m(M)} \mathbb{E}\left[\operatorname{vol}_{m-c}(g_1 X_1 \cap \cdots \cap g_n X_n)\right] = \ell(K_{X_1} \wedge \cdots \wedge K_{X_n}). \tag{4.1.3}$$

In particular, in the case where $c = m$ we obtain

$$\mathbb{E}\#(g_1 X_1 \cap \cdots \cap g_n X_n) = \operatorname{vol}_m(M)\, \ell(K_{X_1} \wedge \cdots \wedge K_{X_n}). \tag{4.1.4}$$

In this case, since $K_{X_1} \wedge \cdots \wedge K_{X_n} \in \mathscr{Z}_0(\Lambda^m \mathbb{V}^*) \cong \mathbb{R}$ is a segment, it is determined by its length thus, this is equivalent to

$$K_{X_1} \wedge \cdots \wedge K_{X_n} = \Big(\mathbb{E}\#(g_1 X_1 \cap \cdots \cap g_n X_n)\Big) K_{\{o\}} \in \mathscr{Z}_0(\Lambda^m \mathbb{V}^*), \tag{4.1.5}$$

where $K_{\{o\}}$ is the zonoid associated to a point, see (4.1.2).

Notice the nice cohomological flavour of (4.1.5).

There is an important class of submanifolds for which the associated zonoid is easier to compute. This notion of was introduced in [29] in the case of hypersurfaces.

Definition 4.1.5. A submanifold $X \subset M$ is said to be *cohomogeneous* if for all $x_1, x_2 \in X$, there is $g \in G$ such that $g(x_1) = x_2$ and

$$g^* N_{x_2} X = N_{x_1} X.$$

With the help of our Riemannian structure one can see that, since G acts by isometries, this is equivalent to ask the same property on *tangent spaces* rather than normal spaces. Moreover, one can see that a submanifold X is cohomogeneous if and only if for every $x_1, x_2 \in X$ and every $g_1, g_2 \in G$ such that $g_i(o) = x_i$, $i = 1, 2$, the subspaces $(g_1)^* N_{x_1} X$ and $(g_2)^* N_{x_2} X$ are in the same H-orbit in $G_c(\mathbb{V}^*)$. This implies that in that case, the zonoid $K_X(x)$ does not depend on x.

Proposition 4.1.6. *Let $X \subset M$ be a cohomogeneous submanifold. Then for every $x \in X$ we have*

$$K_X = K_X(x).$$

Example 4.1.7. Orbits of subgroups of G are cohomogeneous submanifolds of G. In fact if $H = \{\mathrm{Id}\}$ is trivial one can see that those are the only cohomogeneous submanifold.

Example 4.1.8 (The sphere). Let us consider the example $M = S^m$ as a sanity check. The sphere is a Riemannian homogeneous manifold that can be presented as

$$S^m \cong O(m+1)/O(m).$$

In that case the isotropy group is the whole orthogonal group, $H \cong O(\mathbb{V}^*)$ and thus every submanifold is cohomogeneous. Similarly in the real projective space $\mathbb{R}\mathrm{P}^m$.

Let $X \subset S^m$ be a submanifold of codimension c. Its associated zonoid is given by:

$$K_X = \frac{\operatorname{vol}_{m-c}(X)}{s_{m-c}} \frac{1}{(2\pi)^c} B(\mathbb{V}^*)^{\wedge c} \tag{4.1.6}$$

where recall that $\kappa_d = \operatorname{vol}_d(B(\mathbb{R}^d))$ and where $s_m := \operatorname{vol}_m(S^m)$. Indeed, because the isotropy group is the whole orthogonal group, the random segment in (4.1.1) has the same law as $\underline{g \cdot E}$ where $g \in O(\mathbb{V}^*)$ is uniform and $E \in G_c(\mathbb{V}^*)$ is any fixed subspace. In that case, it was computed in Example 2.2.19

that $\mathbb{E}\underline{g \cdot E} = \frac{(m-c)!\kappa_{m-c}}{m!\kappa_m} B(\mathbb{V}^*)^{\wedge c}$. Then we use the identity $m!\kappa_m s_m = 2(2\pi)^m$, see Lemma A.4. In particular if $X = \mathcal{H}$ is an hyperplane section, we find

$$K_{\mathcal{H}} = \frac{1}{2\pi} B(\mathbb{V}^*). \tag{4.1.7}$$

Using zonoid calculus (see (2.2.4)), we see that m-th power of this zonoid gives

$$\ell(K_{\mathcal{H}}^{\wedge m}) = \frac{1}{(2\pi)^m} m!\kappa_m = \frac{2}{s_m}$$

which is coherent with the fact that m independent random hyperplanes intersect (on the sphere) almost surely on exactly two points. Note that on the sphere, since $\mathscr{G}_0(k, \mathbb{V}^*)^{O(\mathbb{V}^*)} = \mathbb{R}_{\geq 0} B(\mathbb{V})^{\wedge k}$ all zonoids obtained from Definition 4.1.2 are proportional. In fact if $X \subset S^m$ is a hypersurface we see from Definition 4.1.2 that we have $K_X = \frac{\mathrm{vol}_{m-1}(X)}{s_{m-1}} K_{\mathcal{H}}$. More generally, if X is of codimension c, by (4.1.6) and (4.1.7):

$$K_X = \frac{\mathrm{vol}_{m-c}(X)}{s_{m-c}} K_{\mathcal{H}}^{\wedge c}.$$

In particular, if $\gamma \hookrightarrow S^m$ is a curve, we find by letting $c = m - 1$

$$\ell(K_\gamma \wedge K_{\mathcal{H}}) = \frac{1}{s_m} \frac{\mathrm{vol}_1(\gamma)}{\pi}.$$

Using our results, this gives

$$\mathbb{E}\#(\gamma \cap g\mathcal{H}) = \frac{\mathrm{vol}_1(\gamma)}{\pi}. \tag{4.1.8}$$

where $g \in O(m+1)$ is uniform. This is the well known *Crofton formula*. Similarly one can prove with this technique the fact that the random intersection of submanifolds is proportional to the product of volumes and compute the constant.

Of course everything is easier on the sphere because, as was observed in Example 2.4.14, the algebra $\widehat{\mathscr{G}}(\mathbb{V}^*)^{O(\mathbb{V}^*)} \cong \mathbb{R}[t]/(t^{m+1})$ is very simple. In general, not all zonoids obtained from cohomogeneous submanifolds are proportional.

Definition 4.1.9. We say that two cohomogeneous submanifolds $X, Y \subset M$ of codimension c are *of the same H-type* if for one (and thus any) $(x, y) \in X \times Y$ and $g, g' \in G$ such that $g(o) = x$, $g'(o) = y$, we have that $g^*(N_x X)$ and $(g')^*(N_y Y)$ are in the same H-orbit in $G_c(\mathbb{V}^*)$.

Proposition 4.1.10. *Let $X, Y \subset M$ be cohomogeneous submanifolds of codimension c. Then X and Y are of the same H-type if and only if*

$$K_X = \frac{\mathrm{vol}_{m-c}(X)}{\mathrm{vol}_{m-c}(Y)} K_Y.$$

Proof. Recall from Section 2.4.3 that we can identify invariant Grassmannian zonoids with measures on the orbit space : $\mathscr{G}_0(c, \mathbb{V}^*)^H \cong \mathcal{M}^+(G_c(\mathbb{V}^*)/H)$. Zonoids obtained from Definition 4.1.2 all have as generating measure a Dirac delta measure on this orbit space. These have the same support if and only if they are of the same H-type and this concludes the proof. □

Remember that for zonoids in $\mathbb{V}^*$ (i.e. of degree 1), the length of the wedge product corresponds to the mixed volume, see (2.2.4). Thus, in the particular case of hypersurfaces, Theorem 4.1.4 gives the following.

Corollary 4.1.11. *Let $X_1, \ldots, X_m \hookrightarrow M$ be hypersurfaces and let $K_{X_1}, \ldots, K_{X_m} \in \mathscr{Z}_0(\mathbb{V}^*)$ be their associated zonoids defined in Definition 4.1.2. Let $g_1, \ldots, g_m \in G$ be independent and uniform. Then*

$$\mathbb{E}\#(g_1 X_1 \cap \cdots \cap g_m X_m) = m! \,\mathrm{vol}_m(M)\, \mathrm{MV}(K_{X_1}, \ldots, K_{X_m}).$$

This allows to interpret the Alexandrov-Fenchel inequality (Lemma 1.1.25) as an inequality in random intersection.

Theorem 4.1.12 (Kinematic AF, B.B.L.M.)**.** *Let $X, Y, Z_3 \dots, Z_m \subset M$ be hypersurfaces. Let $g_1, \dots, g_m \in G$ be independent and uniform and denote the random surface $Z := g_3 Z_3 \cap \cdots \cap g_m Z_m$. We have*

$$\mathbb{E}\#(g_1 X \cap g_2 Y \cap Z) \geq \sqrt{\mathbb{E}\#(g_1 X \cap g_2 X \cap Z)\ \mathbb{E}\#(g_1 Y \cap g_2 Y \cap Z)}.$$

Remark 4.1.13. Note that a positive answer to Conjecture 5 would mean that we can replace Z by any submanifold of dimension 2 translated uniformly by G.

Recall that the the c-th intrisic volume of a zonoid $K \in \mathbb{V}^*$ is given by $\mathrm{V}_c(K) = \ell(K^{\wedge c})/c!$, see (2.2.5). Thus, in the case of self intersection, Theorem 4.1.4 gives the following.

Corollary 4.1.14. *Let $X \subset M$ be a hypersurface and let $g_1, \dots, g_c \in G$ be uniform and independent. Then we have*

$$\mathbb{E}\,\mathrm{vol}_{m-c}(g_1 X \cap \cdots \cap g_c X) = \mathrm{vol}_m(M) c!\, \mathrm{V}_c(K_X)$$

where recall that V_c denotes the c-th intrinsic volume.

Recall the link between the Grassmannian algebra and the algebra of valuations that was detailed in Section 3.3 and recall Definition 3.3.1 where we define the (translation invariant continuous real even) valuation $\phi_A \in val^+(\mathbb{V})$ associated to a Grassmannian zonoid $A \in \mathscr{G}_0(\mathbb{V}^*)$ (equivalently to a measure on the Grassmannian). Because of Lemma 3.3.11 to compute the length of wedge product of Grassmannian zonoids such as in (4.1.3), one can reduce to the algebra of valuation $\mathscr{V}(\mathbb{V}) \cong \widehat{\mathscr{G}_0}(\mathbb{V}^*)/\mathfrak{M}(\mathbb{V}^*)$, see (3.3.2). In fact, in the light of Lemma 3.3.16 and with the product of valuations defined in Definition 3.3.9 (which extends Alesker's product by Theorem 3.3.18) we can rewrite (4.1.3) as follows.

Proposition 4.1.15. *Let $X_1, \dots, X_n \subset M$ be submanifolds of codimension $c_i := \mathrm{codim}(X_i)$, such that $c := \sum_{i=1}^n c_i \leq m$. and let $\phi_{X_i} \in \mathscr{V}^+(c_i, \mathbb{V})^H \subset val^+_{c_i}(\mathbb{V})^H$ be the valuation associated to the zonoid K_{X_i} using Definition 3.3.1. Let $g_1, \dots, g_n \in G$ be independent and uniform. Then we have*

$$\mathbb{E}\left[\mathrm{vol}_{m-c}(g_1 X_1 \cap \cdots \cap g_n X_n)\right] = \frac{\mathrm{vol}_m(M)}{\kappa_c} (\phi_{X_1} \wedge \cdots \wedge \phi_{X_n})(B(\mathbb{V})).$$

Similarly intersection of independent copies of a hypersurface with a fixed submanifold can be interpreted as a valuation operation on the zonoids.

Proposition 4.1.16. *Let $X \hookrightarrow M$ be a hypersurface and let $Y \hookrightarrow M$ be a submanifold of dimension c (and thus of codimension $m-c$) and let $\phi_Y \in \mathscr{V}(c, \mathbb{V})^H$ be the valuation associated to K_Y from Definition 3.3.1. Let $g_1, \dots, g_c \in G$ be independent and uniform. Then we have*

$$\mathbb{E}\#(Y \cap g_1 X \cap \cdots \cap g_c X) = \mathrm{vol}_m(M) c!\, (*\phi_Y)(K_X)$$

where recall $* : \mathscr{V}(m-c, \mathbb{V}) \to \mathscr{V}(c, \mathbb{V}^*)$ denotes Poincaré-Alesker duality (see Proposition 3.1.23 and Definition 3.3.9).

Proof. First note that the left hand side is equal to $\mathbb{E}\#(gY \cap g_1 X \cap \cdots \cap g_c X)$ where $g \in G$ is uniform, independent of $g_1, \dots, g_c$. Now, by (4.1.3), this is equal to $\mathrm{vol}_m(M)\ell(K_Y \wedge K_X^{\wedge c})$. Using Proposition 2.4.8 we find

$$\mathbb{E}\#(gY \cap g_1 X \cap \cdots \cap g_c X) = \mathrm{vol}_m(M)(*K_Y, K_X^{\wedge c}).$$

We conclude by (3.3.4). □

To conclude, let us note that we have seen that to each cohomogeneous submanifold of codimension c corresponds an H-orbit in the Grassmannian $G_c(\mathbb{V}^*)$. It is not clear if for each orbit there exists a cohomogeneous submanifold with the normal in this orbit.

Open problem 13. Given an orbit $H \cdot E \in G_c(\mathbb{V}^*)/H$, is there a cohomogeneous submanifold X of codimension c such that for all $x \in X$ and all $g \in G$ such that $g(o) = x$, we have $g^* N_x X \in H \cdot E$?

4.1.2 A case study: complex projective space

Consider the case $M = \mathbb{C}\mathrm{P}^n \cong U(n+1)/(U(n)\times U(1))$. In homogeneous coordinates $[z_0 : \cdots : z_n]$, we can assume that $o = [1 : 0 : \cdots : 0]$. In that case the tangent space identifies $\mathbb{V} \cong \mathbb{C}^n$ and the action of the isotropy group H on it is just the regular action of $U(n)$ on $\mathbb{C}^n$. The Fubini-Study metric is precisely the one that corresponds to the usual scalar product in this identification.

The first observation is the following.

Proposition 4.1.17. *Every real hypersurface $X \subset \mathbb{C}\mathrm{P}^n$ is cohomogeneous. Its associated zonoid is*

$$K_X = \mathrm{vol}_{2n-1}(X)\frac{2^{n-1}(n-1)!}{\pi^{n-1}\binom{2n}{n}}B(\mathbb{V}^*)$$

Proof. Because $U(n)$ acts transitively on $S(\mathbb{V}^*) \cong S^{2n-1}$, X is cohomogeneous. Using the fact that $\mathrm{vol}_{2n}(\mathbb{C}\mathrm{P}^n) = \pi^n/n!$ we find that its zonoid is given by

$$K_X = \mathrm{vol}_{2n-1}(X)\frac{n!}{\pi^n}\,\mathbb{E}\underline{U} \tag{4.1.9}$$

where $U \in S^{2n-1}$ is uniform. We can then use Example 2.2.19 in the case $k = 1, m = 2n$ to find that

$$\mathbb{E}\underline{U} = \frac{\kappa_{2n-1}}{2n\kappa_{2n}}B(\mathbb{V}^*)$$

Then we use the identity $\kappa_{2n-1}/\kappa_{2n} = 2^n\pi/(n\binom{2n}{n})$, see (A.1), and reintroduce in (4.1.9) to get the result. □

Before giving the second example let us define a zonoid (the most careful readers will notice that it already appeared in Example 3.3.20).

Definition 4.1.18. In the identification $\mathbb{V}^* \cong \mathbb{C}^n$, consider the first complex coordinate line $\mathbb{C} \subset \mathbb{V}^*$ seen as a real plane. Then we define

$$P_n := \mathbb{E}\underline{g\cdot\mathbb{C}} \in \mathscr{G}_0(2, \mathbb{V}^*)$$

where $g \in U(n)$ is uniform and where recall that $\underline{g\cdot\mathbb{C}}$ denotes the segment supported by a Plücker coordinate of $g\cdot\mathbb{C}$, see Definition 2.2.8.

Since the group $U(n)$ acts transitively on the complex Grassmannians, we have that complex subspaces in $\mathbb{C}\mathrm{P}^n$ are cohomogeneous. We have the following.

Proposition 4.1.19. *For every $1 \le k \le n$, the complex plane $\mathbb{C}\mathrm{P}^{n-k} \hookrightarrow \mathbb{C}\mathrm{P}^n$ is a cohomogeneous submanifold (of real codimension $2k$). For $k = 1$, the zonoid associated to the complex hyperplane is given by*

$$K_{\mathbb{C}\mathrm{P}^{n-1}} = \frac{n}{\pi}\,P_n.$$

Using integral geometry we can then compute the length of the powers of the zonoid P_n.

Corollary 4.1.20. *For every $1 \le k \le n$ we have*

$$\frac{1}{k!}\ell((P_n)^{\wedge k}) = \frac{1}{n^k}\binom{n}{k}. \tag{4.1.10}$$

Proof. First notice that if $g_1, \ldots, g_k \in U(n+1)$ are independent and uniform then the intersection $g_1\mathbb{C}\mathrm{P}^{n-1}\cap\cdots\cap g_k\mathbb{C}\mathrm{P}^{n-1}$ is almost surely a $\mathbb{C}\mathrm{P}^{n-k}$. In particular $\mathbb{E}\,\mathrm{vol}_{2n-2k}(g_1\mathbb{C}\mathrm{P}^{n-1}\cap\cdots\cap g_k\mathbb{C}\mathrm{P}^{n-1}) = \pi^{n-k}/(n-k)!$. By (4.1.3) we obtain

$$\left(\frac{n}{\pi}\right)^k \ell((P_n)^{\wedge k}) = \frac{n!}{\pi^n}\frac{\pi^{n-k}}{(n-k)!}.$$

Dividing by $k!$ gives the result. □

Notice that P_n is of degree $2 > 1$ thus Corollary 4.1.20 is *not* the computation of the k-th intrinsic volume. From the previous result we obtain the following.

Corollary 4.1.21. *For all $1 \leq k \leq n$ we have*

$$K_{\mathbb{C}\mathrm{P}^{n-k}} = (K_{\mathbb{C}\mathrm{P}^{n-1}})^{\wedge k}$$

Proof. The zonoid $K_{\mathbb{C}\mathrm{P}^{n-k}} \in \mathscr{G}_0(2k, \mathbb{V}^*)$ has a generating measure that is a multiple of the uniform measure on the complex Grassmannian, thus there is a constant $c \geq 0$ such that $K_{\mathbb{C}\mathrm{P}^{n-k}} = c\,(P_n)^{\wedge k}$. To compute this constant, notice that, by definition of the associated zonoid, we have

$$\ell(K_{\mathbb{C}\mathrm{P}^{n-k}}) = \frac{\mathrm{vol}_{2n-2k}(\mathbb{C}\mathrm{P}^{n-k})}{\mathrm{vol}_{2n}(\mathbb{C}\mathrm{P}^{n})} = \frac{n!}{(n-k)!}\frac{1}{\pi^k}.$$

Then we obtain $c = \frac{n!}{(n-k)!\pi^k \ell(P_n^{\wedge k})}$ and we conclude by Corollary 4.1.20. □

Once again, because $U(n)$ acts transitively on the complex Grassmannians, every complex submanifolds in $\mathbb{C}\mathrm{P}^n$ is cohomogeneous and they all are of the same $U(n)$-type (in the sense of Definition 4.1.9). In complex algebraic geometry, the volume of a subvariety is given by its degree, more precisely if $X \hookrightarrow \mathbb{C}\mathrm{P}^n$ is a complex irreducible variety of complex codiension k then we have (see [68, Section 5.C]):

$$\mathrm{vol}_{2n-2k}(X) = \deg(X)\ \mathrm{vol}_{2n-2k}(\mathbb{C}\mathrm{P}^{n-k}) = \deg(X)\,\frac{\pi^{n-k}}{(n-k)!}. \tag{4.1.11}$$

It follows that the associated zonoids are integer multiples of the zonoids associated to to the complex planes. More precisely we have the following.

Proposition 4.1.22. *Let $X \hookrightarrow \mathbb{C}\mathrm{P}^n$ be a complex irreducible variety of complex codiension k and degree d, we have*

$$K_X = d\,K_{\mathbb{C}\mathrm{P}^{n-k}}.$$

Proof. Since X and $\mathbb{C}\mathrm{P}^{n-k}$ are of the same $U(n)$-type, it follows from Proposition 4.1.10 and (4.1.11). □

Thus we realize the cohomology ring of $\mathbb{C}\mathrm{P}^n$ as a subring of $\widehat{\mathscr{G}_0}(\mathbb{V}^*)^{U(n)}$.

As detailed in Example 2.4.15, Hiroyuki Tasaki proved in [84] that the $U(n)$-orbit in the *real* Grassmannian $G_c(\mathbb{C}^n)$ are characterized by the *Kähler angles* $2\pi \geq \theta_1(E) \geq \cdots \geq \theta_{\lfloor \frac{c}{2} \rfloor}(E) \geq 0$. Let us remind the reader that these are the angles between a plane and its image under multiplication by the complex structure. In particular for a complex plane they are all zero and for a Lagrangian plane they are all $\pi/2$.

As observed in Example 2.4.15 there are many more zonoids in $\mathscr{G}_0(\mathbb{V}^*)^{U(n)}$ as it is infinite dimensional, in particular there is a different zonoid for each Kähler angle. However, as shown in Proposition 4.1.15, we only care about the valuation induced by zonoids. In the particular case of $U(n)$ invariant valuations, as explained in Example 3.3.20, they are exactly generated by the two previous examples. That is, for each Grassmannian zonoid $K \in \mathscr{G}_0(c, \mathbb{V}^*)^{U(n)}$, there are real numbers $x_{j,k}$ such that

$$\phi_K = \sum_{j+2k=c} x_{j,k}\,(\phi_{B(\mathbb{V}^*)})^{\wedge j} \wedge (\phi_{P_{n-1}})^{\wedge k}.$$

Be careful that in general the coefficients $x_{j,k}$ can be negative even when we start with a zonoid and not a virtual zonoid. In particular for $c = 2$ we can compute these coefficients with a simple linear system.

Proposition 4.1.23. *Let $n \geq 2$ and let $K \in \mathscr{G}_0(2, \mathbb{V})^{U(n)}$. The coefficients $x_{\mathbb{R}}, x_{\mathbb{C}} \in \mathbb{R}$ such that*

$$\phi_K = x_{\mathbb{R}}\,\phi_{B(\mathbb{V}^*)^{\wedge 2}} + x_{\mathbb{C}}\,\phi_{\mathbb{C}\mathrm{P}^{n-1}}$$

where $\phi_{\mathbb{C}\mathrm{P}^{n-1}}$ is the valuation associated to $K_{\mathbb{C}\mathrm{P}^{n-1}} = (n/\pi)P_n$, are given by

$$\begin{cases} x_{\mathbb{R}} = & \frac{1}{(2\pi)^2(n-1)}\left(\frac{\pi}{n}\ell(K) - d_K\right) \\ \\ x_{\mathbb{C}} = & \frac{1}{2(n-1)}\left((2n-1)d_K - \frac{\pi}{n}\ell(K)\right) \end{cases}$$

where $d_K = \frac{\pi^n}{n!}\ell(K \wedge K_{\mathbb{C}\mathrm{P}^1})$ and where recall that $K_{\mathbb{C}\mathrm{P}^1} = (n/\pi)^{n-1}P_n^{\wedge(n-1)}$.

Proof. By Lemma 3.3.11 and using the fact that $\phi_{\mathbb{C}\mathrm{P}^{n-1}} = (n/\pi)\phi_{P_n}$, we have that

$$\begin{array}{ccccc} \ell(K) & = & x_{\mathbb{R}}\ell\left(B(\mathbb{V}^*)^{\wedge 2}\right) & + & x_{\mathbb{C}}(n/\pi)\ell(P_n) \\ \ell\left(K \wedge K_{\mathbb{C}\mathrm{P}^{n-1}}^{\wedge(n-1)}\right) & = & x_{\mathbb{R}}(n/\pi)^{n-1}\ell\left(B(\mathbb{V}^*)^{\wedge 2} \wedge P_n^{\wedge(n-1)}\right) & + & x_{\mathbb{C}}(n/\pi)^n\ell(P_n^{\wedge n}) \end{array}$$

The length of the powers of the unit ball was computed in (2.2.7) and we find

$$\ell(B(\mathbb{V}^*)^{\wedge 2}) = (2n)(2n-1)\frac{\kappa_{2n}}{\kappa_{2n-2}} = (2\pi)(2n-1)$$

where in the second equality, we used $\kappa_{2n} = \pi^n/n!$. Next, using the useful Lemma 2.2.17 and the length of powers of P_n computed in (4.1.10), we have

$$\ell\left(B(\mathbb{V}^*)^{\wedge 2} \wedge P_n^{\wedge(n-1)}\right) = (2\pi)\ell\left(P_n^{\wedge(n-1)}\right) = (2\pi)\frac{n!}{n^{n-1}}.$$

Then using $\ell(P_n) = 1$ and $\ell(P_n^{\wedge n}) = n!/n^n$, and the fact that $K_{\mathbb{C}\mathrm{P}^{n-1}}^{\wedge(n-1)} = K_{\mathbb{C}\mathrm{P}^1}$, we find that $x_{\mathbb{R}}$ and $x_{\mathbb{C}}$ are solution of the linear system

$$\begin{cases} 2\pi^2(2n-1) & x_{\mathbb{R}} & + & x_{\mathbb{C}} & = & \frac{\pi}{n}\ell(K) \\ 2\pi^2 & x_{\mathbb{R}} & + & x_{\mathbb{C}} & = & \frac{\pi^n}{n!}\ell(K \wedge K_{\mathbb{C}\mathrm{P}^1}) \end{cases}.$$

Inverting the system gives the result. □

Applied to zonoids associated to submanifolds of codimension 2 this gives the following.

Proposition 4.1.24. *Let $n \geq 2$, let $X \hookrightarrow \mathbb{C}\mathrm{P}^n$ be a submanifold of real codimension 2, let $\phi_X \in val_2^+(\mathbb{V})$ be the valuation associated to the zonoid K_X. Then the coefficients $x_{\mathbb{R}}, x_{\mathbb{C}} \in \mathbb{R}$ such that*

$$\phi_X = x_{\mathbb{R}}\,\phi_{B(\mathbb{V}^*)^{\wedge 2}} + x_{\mathbb{C}}\,\phi_{\mathbb{C}\mathrm{P}^{n-1}}$$

are given by

$$\begin{cases} x_{\mathbb{R}} = & \frac{1}{(2\pi)^2(n-1)}\left(d_X - \frac{(n-1)!}{\pi^{n-1}}\mathrm{vol}_{2n-2}(X)\right) \\ \\ x_{\mathbb{C}} = & \frac{1}{2(n-1)}\left((2n-1)d_X - \frac{(n-1)!}{\pi^{n-1}}\mathrm{vol}_{2n-2}(X)\right) \end{cases} \tag{4.1.12}$$

where

$$d_X := \mathbb{E}\#(X \cap g\mathbb{C}\mathrm{P}^1)$$

with $g \in U(n+1)$ uniform.

Proof. We simply apply the previous result to $K = K_X$, noticing that $\frac{\pi}{n}\ell(K) = \frac{(n-1)!}{\pi^{n-1}}\mathrm{vol}_{2n-2}(X)$ and that, by (4.1.3) we have $\frac{\pi^n}{n!}\ell(K \wedge K_{\mathbb{C}\mathrm{P}^1}) = \mathbb{E}\#(X \cap g\mathbb{C}\mathrm{P}^1)$. □

Remark 4.1.25. In particular if X is an irreducible complex hypersurface of degree d then $d_X = d$ and, by (4.1.11), we have also $\frac{(n-1)!}{\pi^{n-1}}\mathrm{vol}_{2n-2}(X) = d$. Thus we find $x_{\mathbb{R}} = 0$ and $x_{\mathbb{C}} = d$ which coincides with what was found in Proposition 4.1.22.

This interprets nicely in terms of self intersections.

Theorem 4.1.26 (M.)**.** *Let $n \geq 2$, let $X \hookrightarrow \mathbb{C}\mathrm{P}^n$ be a submanifold of real codimension 2, and let $d_X := \mathbb{E}\#(X \cap g\mathbb{C}\mathrm{P}^1)$ where $g \in U(n+1)$ uniform and $\Delta_X := d_X - \frac{(n-1)!}{\pi^{n-1}} \mathrm{vol}_{2n-2}(X)$. Then, if $g_1, \ldots, g_n \in U(n+1)$ are uniform and independent, we have*

$$\mathbb{E}\#(g_1X \cap \cdots \cap g_nX) = \sum_{k=0}^{n} \binom{n}{k} \frac{c_k}{4^k(n-1)^k} \Delta_X^k d_X^{n-k}$$

where

$$c_k := \sum_{j=0}^{k} \binom{k}{j}\binom{2j}{j} 2^{k-j}. \tag{4.1.13}$$

Proof. By our general formula (4.1.4), we have that $\mathbb{E}\#(g_1X \cap \cdots \cap g_nX_n) = (\pi^n/n!)\ell(K_X^{\wedge n})$. By Lemma 3.3.11 and (4.1.12), we can replace K_X by $x_{\mathbb{R}}B(\mathbb{V})^{\wedge 2} + x_{\mathbb{C}}K_{\mathbb{C}\mathrm{P}^{n-1}}$ where $x_{\mathbb{R}}, x_{\mathbb{C}} \in \mathbb{R}$ are given by (4.1.12). Next we note that

$$x_{\mathbb{R}} = \frac{1}{(2\pi)^2(n-1)}\Delta_X$$
$$x_{\mathbb{C}} = d_X + \frac{1}{2(n-1)}\Delta_X.$$

Then we have:

$$x_{\mathbb{R}}B(\mathbb{V})^{\wedge 2} + x_{\mathbb{C}}K_{\mathbb{C}\mathrm{P}^{n-1}} = d_X K_{\mathbb{C}\mathrm{P}^{n-1}} + \frac{\Delta_X}{2(n-1)}\left(K_{\mathbb{C}\mathrm{P}^{n-1}} + \frac{1}{2\pi^2}B(\mathbb{V}^*)^{\wedge 2}\right)$$

We obtain the following.

$$\begin{aligned}\mathbb{E}\#(g_1X \cap \cdots \cap g_nX) &= \frac{\pi^n}{n!}\ell\left((x_{\mathbb{R}}B(\mathbb{V})^{\wedge 2} + x_{\mathbb{C}}K_{\mathbb{C}\mathrm{P}^{n-1}})^{\wedge n}\right) \\ &= \frac{\pi^n}{n!}\sum_{k=0}^{n}\binom{k}{n}\frac{\Delta_X^k d_X^{n-k}}{2^k(n-1)^k}\sum_{j=0}^{k}\binom{k}{j}\frac{1}{2^j\pi^{2j}}\ell\left(K_{\mathbb{C}\mathrm{P}^j} \wedge B(\mathbb{V}^*)^{\wedge 2j}\right)\end{aligned} \tag{4.1.14}$$

where, in the second equality, we used the fact that $K_{\mathbb{C}\mathrm{P}^{n-1}}^{\wedge(n-j)} = K_{\mathbb{C}\mathrm{P}^j}$ (Corollary 4.1.21). Then we use our favourite lemma: Lemma 2.2.17 to compute

$$\begin{aligned}\ell\left(K_{\mathbb{C}\mathrm{P}^j} \wedge B(\mathbb{V}^*)^{\wedge 2j}\right) &= (2j)!\frac{\pi^j}{j!}\ell\left(K_{\mathbb{C}\mathrm{P}^j}\right) \\ &= \binom{2j}{j}\frac{\pi^{2j}}{\pi^n}n!\end{aligned}$$

where for the first equality we used $\kappa_{2j} = \pi^j/j!$ and for the second equality the fact that $\ell\left(K_{\mathbb{C}\mathrm{P}^j}\right) = (n!\pi^j)/(j!\pi^n)$. Reintroducing in (4.1.14) above, we get

$$\mathbb{E}\#(g_1X \cap \cdots \cap g_nX) = \sum_{k=0}^{n}\binom{k}{n}\frac{\Delta_X^k d_X^{n-k}}{2^k(n-1)^k}\sum_{j=0}^{k}\binom{k}{j}\binom{2j}{j}\frac{1}{2^j}.$$

In the internal sum, we recognize $c_k/2^k$ and this gives what we wanted. □

The sequence c_k defined in (4.1.13) is listed in the Online Encyclopedia of Integer Sequences (OEIS) [70] as [A081671], the first terms starting from $k = 0$ are $1, 4, 18, 88, 454, 2424, \ldots$

In the case where X is a complex irreducible hypersurface of degree d, as noticed in Remark 4.1.25, we have $d_X = d$ and $\Delta_X = 0$. Theorem 4.1.26 then tells us that the self intersection $\mathbb{E}\#(g_1X \cap \cdots \cap g_nX) = d^n$ which in that case is given by Bézout's Theorem. When X is close to a complex irreducible hypersurface, the quantity Δ_X is small and we can see Theorem 4.1.26 as a perturbation of Bézout:

$$\mathbb{E}\#(g_1X \cap \cdots \cap g_nX) = d_X^n + \frac{n}{n-1}d_X^{n-1}\Delta_X + \frac{9n}{16}d_X^{n-2}\Delta_X^2 + O(\Delta_X^3).$$

The formula in Theorem 4.1.26 could also presumably be obtained by means of a kinematic formula in $\mathbb{C}\mathrm{P}^n$ as developped by Andreas Bernig, Joseph Fu and Gil Solanes in [21]. Let us say a word about this point of view. There is a notion of *valuations on manifolds* developped by Semyon Alesker (see [6]), those are functions on the submanifolds with corner (also called differentiable polyhedra) of M, that satisfy the valuation property (3.1.1). Similarly as for the case of convex valuations in Proposition 3.1.26, the space of $U(n+1)$ invariant valuation in $\mathbb{C}\mathrm{P}^n$ is *finite dimensional*. The *Euler characteristic* χ is an example of such valuation (which in dimension 0 coincide with the number of points). The *kinematic formula* in that case says that for every submanifolds $X, Y \subset \mathbb{C}\mathrm{P}^n$ and if $g \in U(n+1)$, we have (see [21, Section 2.3])

$$\mathbb{E}\left[\chi\left(X \cap gY\right)\right] = \sum_{i,j} c_{ij}\phi_i(X)\phi_j(Y) \tag{4.1.15}$$

where ϕ_i is a basis of the $U(n+1)$ invariant valuations on $\mathbb{C}\mathrm{P}^n$ and $c_{i,j} \in \mathbb{R}$ are constants that does not depend on X and Y. By taking X of codimension 2 and $Y = g_1X \cap \cdots \cap g_{n-1}X$ we can iterate (4.1.15) to obtain a formula for the number of points $\mathbb{E}\#(g_1X \cap \cdots \cap g_nX)$. Next, one can relate valuations on $\mathbb{C}\mathrm{P}^n$ to the valuations on $\mathbb{C}^n$ by means of a *transfer principle*, see [21, Section 2.4]. It would then remain to compute the constants which could require some work but in principle is doable.

4.1.3 Probabilistic Schubert Calculus

We now study the case of the real Grassmannian. We fix $0 \leq a \leq b$ integers and consider the homogeneous space $M = G(a, a+b) = G_a(\mathbb{R}^{a+b}) \cong O(a+b)/(O(a)\times O(b))$. This is a smooth manifold of dimension ab. We fix our favourite point to be $o := \mathbb{R}^a$ the plane spanned by the a first coordinates in $\mathbb{R}^{a+b}$. The tangent space at o is given by $\mathbb{V} = T_oG(a, a+b) = \mathrm{Hom}(\mathbb{R}^a, (\mathbb{R}^a)^\perp)$. We sometime identify $\mathbb{V} \cong \mathbb{R}^a \otimes \mathbb{R}^b$ that we think of as the space of $a \times b$ real matrices. The action the isotropy group $H = O(a) \times O(b)$ is given for every $\varphi \in \mathbb{V}$ and every $(g, h) \in O(a) \times O(b)$ by

$$(g, h) \cdot \varphi = g \circ \varphi \circ h^t$$

where we used the natural identification $((\mathbb{R}^a)^\perp)^* = \mathbb{R}^b$ where $\mathbb{R}^b \subset \mathbb{R}^{a+b}$ is the space spanned by the last b coordinates. We use the standard scalar product on $\mathbb{V}$. It is such that the Plücker embedding (see the beginning of Section 2.2.2):

$$Pl : G(a, a+b) \hookrightarrow \mathrm{P}(\Lambda^a\mathbb{R}^{a+b})$$

is a Riemannian immersion. Indeed fixing an orthonormal basis $e_1, \ldots, e_a$ of $\mathbb{R}^a$ and an orthonormal basis $f_1, \ldots, f_b$ of $(\mathbb{R}^a)^\perp \cong \mathbb{R}^b$, then an orthonormal basis of $\mathbb{V}$ is given by the maps $\varphi_{i,j}$, $1 \leq i \leq a$, $1 \leq j \leq b$, that are given by

$$\varphi_{i,j}(e_k) = \delta_{i,k} f_j. \tag{4.1.16}$$

A simple computation yields

$$\mathrm{d}_o Pl(\varphi_{i,j}) = e_1 \wedge \cdots \wedge e_{i-1} \wedge f_j \wedge e_{i+1} \wedge \cdots \wedge e_a$$

which is again an orthonormal frame and this proves the claim.

We want, in this section, to apply our integral geometry and zonoid calculus framework to the intersection of *Schubert varieties*. These are special subvarieties of the Grassmannian that play a central role in the understanding of these spaces. It is convenient to index them by *Young diagrams*. Let us make a few definitions.

Definition 4.1.27. A *Young diagram* that fits in $a \times b$ is a collection of boxes in the grid with a rows and b columns such that if a box is selected then all the boxes on the left of the same row and above in the same columns are also selected, see Figure 4.1. We sometime omit to say that a Young diagram fits in $a \times b$ if it is clear from the context or if it has no importance. If λ is a Young diagram, we call the *codimension* of λ the number of boxes in the Young diagram and we denote it by $|\lambda|$.

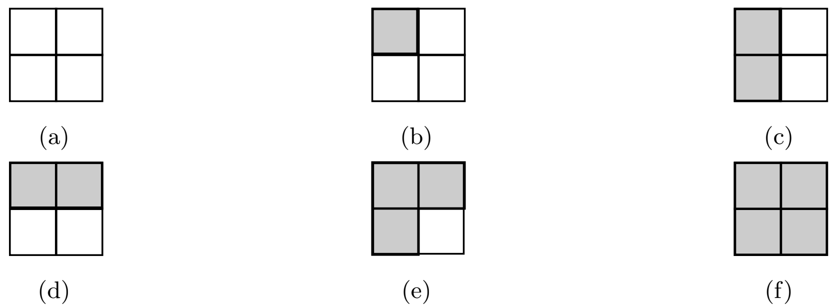


Figure 4.1: All the Young diagrams that fit in 2×2: one of codimension $0, 1, 3$ and 4 and two of codimension 2.

We think of the $a \times b$ grid with the rows labelled from top to bottom and the columns labelled from left to right. In this context if λ is a Young diagram that fits in $a \times b$, for each $(i, j) \in a \times b$ we write $(i, j) \in \lambda$ if the box in the ith row and jth column belongs to λ. Moreover, if $(i, j) \in \lambda$ is such that $(i + 1, j), (i, j + 1) \notin \lambda$, we say that it is an *outer corner* of λ.

Remark 4.1.28. We can think of the vector space $\mathbb{V} \cong \mathbb{R}^a \otimes \mathbb{R}^b$ as the space spanned by the boxes in the $a \times b$ grid. To the box (i, j) we can associate the map $\varphi_{i,j}$ given by (4.1.16).

Finally in $\mathbb{R}^{a+b}$ we fix the complete flag:

$$\{0\} =: \mathbb{W}_0 \subset \mathbb{W}_1 \subset \cdots \subset \mathbb{W}_{a+b} := \mathbb{R}^{a+b}$$

where for all $1 \leq k \leq a + b$, $\mathbb{W}_k := \mathbb{R}^k$ is the space spanned by the k first coordinates. We are now ready to define our favourite varieties for this section.

Definition 4.1.29. To every Young diagram λ that fits in $a \times b$, we associate the Schubert variety $\Omega_\lambda \subseteq G(a, a + b)$, defined by:

$$\Omega_\lambda := \{E \in G(a, a + b) \mid \dim(E \cap \mathbb{W}_{b-j+i}) \geq i \quad \forall (i, j) \in \lambda\}.$$

Each $\Omega_\lambda \subseteq G(a, a + b)$ is a subvariety of codimension $|\lambda|$, the codimension of λ. It is possibly singular but of finite volume.

One can check that if $E \in G(a, a+b)$ is such that $\dim(E \cap \mathbb{W}_{b-j+i}) \geq i$ for some $(i, j) \in a \times b$ then, because of the inclusion condition of the spaces $\mathbb{W}_k$, it is automatic that the same holds for $(i - 1, j)$ and $(i, j - 1)$. It follows that only the boxes at the outer corners of the Young diagram are needed to define the Schubert variety. In other words, if λ is a Young diagram that fits in $a \times b$ with outer corners $(m_1, p_1), \ldots, (m_r, p_r)$ then we have

$$\Omega_\lambda = \{E \in G(a, a + b) \mid \dim(E \cap \mathbb{W}_{b-p_i+m_i}) \geq m_i \quad 1 \leq i \leq r\}.$$

The smooth part of Ω_λ is obtained by replacing inequalities by equalities in this definition:

$$\Omega_\lambda^{sm} = \{E \in G(a, a + b) \mid \dim(E \cap \mathbb{W}_{b-p_i+m_i}) = m_i \quad 1 \leq i \leq r\}.$$

In the complex setting, Schubert varieties form a basis of the cohomology spaces of the Grassmannian. Hence intersection problems in the complex Grassmannian can be reduced to computing the cup product of cohomology classes of Schubert varieties. At the level of Young diagrams, the laws that rules these products are known as *Schubert calculus*.

In the real setting, in general, the cohomology ring can only compute intersections modulo 2. We will try here to apply thecniques developed above to compute *average random intersection*. For this we will first show that they are cohomogeneous and thus we need to understand better the tangent and normal spaces of Schubert varieties.

The following lemma gives a description of tangent spaces to Schubert varieties (at smooth points). In the case of simple Schubert varieties, i.e. varieties associated to a Young diagram with a unique outer corner, it was proved by László Fehér and Ákos Matszangosz in [39, Proposition 4.3]. Recall that the tangent space at $E \in G(a, a+b)$ is identified with the morphisms $E \to E^\perp$. Moreover, let us identify in the following $\mathbb{R}^{a+b}$ with its dual using the Euclidean structure and in particular we consider $E^\perp$ as a subspace of $\mathbb{R}^{a+b}$.

Lemma 4.1.30. *Let λ be a Young diagram that fits in $a \times b$ and $\Omega_\lambda \subset G(a, a+b)$ be the corresponding Schubert variety. Denote by $(m_1, p_1), \ldots, (m_r, p_r)$ the outer corners of λ. We have for all $E \in \Omega_\lambda^{sm}$:*

$$T_E\Omega_\lambda^{sm} = \left\{\varphi : E \to E^\perp \,|\, \varphi\left(E \cap \mathbb{W}_{b-p_i+m_i}\right) \subset E^\perp \cap \mathbb{W}_{b-p_i+m_i} \quad \forall\, i = 1, \ldots, r\right\} \tag{4.1.17}$$

Proof. First suppose $r = 1$ so that $\Omega_\lambda^{sm} := \{E \in G(a, a+b) \,|\, \dim(E \cap \mathbb{W}_{b-p+m}) = m\}$. We prove that the right hand side of (4.1.17) is included in the left hand side. The equality follows by a count of dimensions (this can also be seen, a posteriori, using (4.1.20)).

Let $\varphi \in \mathrm{Hom}(E, E^\perp)$ and consider the curve $\gamma_\varphi : (-\varepsilon, \varepsilon) \to G(a, a+b)$ given by

$$\gamma_\varphi(t) = \{x + t\varphi(x) \,|\, x \in E\} \in G(a, a+b).$$

Then γ_φ satisfies $\gamma_\varphi(0) = E$ and $\dot\gamma_\varphi(0) = \varphi$.

Now suppose $\varphi\left(E \cap \mathbb{W}_{b-p+m}\right) \subset E^\perp \cap \mathbb{W}_{b-p+m}$. Then we have that the image of $E \cap \mathbb{W}_{b-p+m}$ under the map $\mathrm{Id} + t\varphi$ is included in $\gamma_\varphi(t) \cap \mathbb{W}_{b-p+m}$ and, since dimension can only locally decrease, and, for small $t \in (-\epsilon, \epsilon)$, the map $\mathrm{Id} + t\varphi$ is invertible, it means that for t small enough we have $\dim\left(\gamma_\varphi(t) \cap \mathbb{W}_{b-p+m}\right) = m$, i.e. $\gamma_\varphi(t) \in \Omega_\lambda^{sm}$. Thus $\varphi \in T_E\Omega_\lambda^{sm}$ and we have the inclusion we wanted.

Now let $r \geq 2$. Then

$$\Omega_\lambda^{sm} = \bigcap_{i=1}^{r} \Omega_{\lambda_i}^{sm}$$

where λ_i is the Young diagram with unique outer corner (m_i, p_i) and thus

$$\Omega_{\lambda_i}^{sm} := \{E \in G(a, b) \,|\, \dim(E \cap \mathbb{W}_{b-p_i+m_i}) = m_i\}.$$

Then for all $E \in \Omega_\lambda^{sm}$ we have $T_E\Omega_\lambda^{sm} = \bigcap_{i=1}^r T_E\Omega_{\lambda_i}^{sm}$ and in the right hand side each term falls into the case already dealt with $r = 1$. The result follows. □

This description of the tangent space given in Lemma 4.1.30 can be interpreted in terms of the Young diagrams once we choose an appropriate basis. More precisely, let λ be a Young diagram that fit in $a \times b$ with outer corners $(m_1, p_1), \ldots, (m_r, p_r)$ with $m_1 \leq \cdots \leq m_r$ (and thus $p_1 \geq \cdots \geq p_r$) and let $E \in \Omega_\lambda^{sm}$. Choose an orthonormal basis $e_1, \ldots, e_a$ of E such that

$$E \cap \mathbb{W}_{b-p_i+m_i} = \mathrm{Span}\{e_1, \ldots, e_{m_i}\} \tag{4.1.18}$$

and choose $f_1, \ldots, f_b$ an orthonormal basis of $E^\perp$ such that

$$E^\perp \cap \mathbb{W}_{b-p_i+m_i} = \mathrm{Span}\{f_1, \ldots, f_{p_i}\}. \tag{4.1.19}$$

Such a basis exists because of the relations of inclusion of the flag and because of the ordering we chose on the outer corners.

Then let $\varphi_{ij} \in \mathrm{Hom}(E, E^\perp)$ be the map defined by (4.1.16). The set $\{\varphi_{ij} \,|\, 1 \leq i \leq a,\ 1 \leq j \leq b\}$ forms an orthogonal basis of $\mathrm{Hom}(E, E^\perp)$. The previous lemma states precisely that, in this basis,

$$T_E\Omega_\lambda^{sm} = \mathrm{Span}\left\{\varphi_{ij} \,|\, 1 \leq i \leq k,\ 1 \leq j \leq n-k,\ (i,j) \notin \lambda\right\}. \tag{4.1.20}$$

Note that the dual to the space $\mathrm{Hom}(E, E^\perp)$ is identified with

$$\mathrm{Hom}(E, E^\perp)^* = \mathrm{Hom}(E^\perp, E)$$

by letting for every $\psi \in \mathrm{Hom}(E^\perp, E)$, and every $\varphi \in \mathrm{Hom}(E, E^\perp)$:

$$\langle \psi, \varphi \rangle := \mathrm{tr}(\psi \circ \varphi).$$

In the light of this, we let $\varphi^*_{i,j} \in \mathrm{Hom}(E^\perp, E)$ be the map given by

$$\varphi^*_{i,j}(f_k) = \delta_{j,k} e_i.$$

They form an orthonormal basis of the normal space $N_E G(a, a+b)$ (the dual basis of $\{\varphi_{i,j}\}$). One can then identify the normal space of the Schubert variety as

$$N_E \Omega_\lambda^{sm} = \mathrm{Span}\left\{\varphi^*_{ij} \,|\, (i,j) \in \lambda\right\}. \tag{4.1.21}$$

Definition 4.1.31. Let λ be a Youg diagram that fits $a \times b$ and let $E \in \Omega_\lambda^{sm}$. An orthonormal basis $\{e_1, \ldots, e_a, f_1, \ldots, f_b\}$ of $\mathbb{R}^{a+b}$ that satisfies (4.1.18) and (4.1.19) is called *adapted* to the couple (λ, E).

Theorem 4.1.32 (M.)**.** *Schubert varieties are cohomogeneous.*

Proof. Let $E, E' \in \Omega_\lambda^{sm}$ and let $e_1, \ldots, e_a, f_1, \ldots, f_b$ be a basis adapted to (λ, E) and $e'_1, \ldots, e'_a, f'_1, \ldots, f'_b$ a basis adapted to (λ, E'). Then we let $g \in O(a+b)$ be the element that satisfies $g(e_i) = e'_i$ and $g(f_j) = f'_j$. Then it is clear using the description (4.1.20) that g_* sends $T_E\Omega_\lambda^{sm}$ to $T_{E'}\Omega_\lambda^{sm}$. □

We can now consider zonoids associated to Schubert varieties.

Definition 4.1.33. Let λ be a Young diagram that fits $a \times b$. We define the *Schubert zonoid* to be $K_\lambda := K_{\Omega_\lambda} \in \mathscr{G}_0(|\lambda|, \mathbb{V}^*) \cong \mathscr{G}_0(|\lambda|, \mathbb{R}^a \otimes \mathbb{R}^b)$ where K_{Ω_λ} is the zonoid associated to the cohomogeneous submanifold $\Omega_\lambda^{sm} \hookrightarrow G(a, a+b)$ from Definition 4.1.2.

Let us describe these zonoids in the identification $\mathbb{V}^* \cong \mathbb{R}^a \otimes \mathbb{R}^b$. Let $e_1, \ldots, e_a$, respectively $f_1, \ldots, f_b$, be an orthonormal basis for $\mathbb{R}^a$, respectively $\mathbb{R}^b$. For each λ Young diagram that fits $a \times b$, we define the simple vector

$$v_\lambda := \bigwedge_{(i,j)\in\lambda} e_i \otimes f_j \in \Lambda^{|\lambda|}\left(\mathbb{R}^a \otimes \mathbb{R}^b\right)$$

where the wedge product is done in lexicographic order (a different ordering will just change the sign and this won't matter in the end). Then, using the description of the normal space (4.1.21) and the definition of the associated zonoid Definition 4.1.2, we see that

$$K_\lambda = \frac{\mathrm{vol}_{ab-|\lambda|}\left(\Omega_\lambda\right)}{\mathrm{vol}_{ab}\left(G(a, a+b)\right)} \mathbb{E}\underline{h \cdot v_\lambda} \tag{4.1.22}$$

where $h \in O(a) \times O(b)$ is uniform.

The volume of the Grassmannian $G(a, a+b)$ can be computing using known formulas for the volume of orthogonal groups and we obtain, see [29, (2.11)]:

$$\mathrm{vol}_{ab}(G(a, a+b)) = \pi^{\frac{ab}{2}} \frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a-1}{2}\right)\cdots\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{a+b}{2}\right)\Gamma\left(\frac{a+b-1}{2}\right)\cdots\Gamma\left(\frac{b+1}{2}\right)}.$$

The first difficulty is to compute in all generality the volume of the Schubert varieties.

Open problem 14. Compute the volume of $\Omega_\lambda \hookrightarrow G(a, a+b)$ for any Young diagram λ that fits $a \times b$.

In the case of the Young diagram consisting of a single box, this was computed by Peter Bürgisser and Antonio Lerario in [29, Theorem 4.2], they obtain:

$$\frac{\mathrm{vol}_{ab-1}\left(\Omega_\square\right)}{\mathrm{vol}_{ab}\left(G(a, a+b)\right)} = \frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)} \tag{4.1.23}$$

We then can compute its associated zonoid.

Proposition 4.1.34. *Let $\lambda := \square$ be the Young diagram consisting of one box in the $a \times b$ grid. Then the Schubert zonoid associated to it is*

$$K_\square = \frac{1}{4\pi} B_a \otimes B_b \in \mathscr{Z}_0(\mathbb{R}^a \otimes \mathbb{R}^b)^{O(a)\times O(b)}.$$

Proof. We have $v_\square = e_1 \otimes f_1$ and thus if $h = (g_1, g_2) \in O(a) \times O(b)$ is uniform then $h \cdot v_\square = g_1(e_1) \otimes g_2(f_1)$ have the same law as $U \otimes V \in \mathbb{R}^a \otimes \mathbb{R}^b$ where $U \in S^{a-1}$ and $V \in S^{b-1}$ are uniform and independent. Thus we have $\mathbb{E}\underline{h \cdot v_\square} = \mathbb{E}\underline{U} \otimes \mathbb{E}\underline{V}$. Using Example 2.2.19 in the case $k = 1, m = a$ we find

$$\mathbb{E}\underline{U} = \frac{\kappa_{a-1}}{a\kappa_a} B_a = \frac{\Gamma\left(\frac{a}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{a-1}{2}\right)} B_a$$

and similarly for $\mathbb{E}\underline{V}$. Reintroducing in (4.1.22) and using (4.1.23) gives the result. $\square$

In [29], it is called the *Segre zonoid*. Using Corollary 4.1.11, we can then express the volume of the tensor of balls as an average intersection number in the Grassmannian.

Corollary 4.1.35. *We have*

$$\mathrm{vol}_{ab}(B_a \otimes B_b) = \frac{(4\pi)^{ab}}{(ab)!\,\mathrm{vol}_{ab}(G(a, a+b))} \mathbb{E}\# \left(g_1\Omega_\square \cap \cdots \cap g_{ab}\Omega_\square\right)$$

where $g_1, \ldots, g_{ab} \in O(a+b)$ are uniform and independent and $\Omega_\square \hookrightarrow G(a, a+b)$ is the Schubert variety associated to the Young diagram consisting of just one box.

To compute volumes of zonoids in $\mathscr{Z}_0(\mathbb{R}^a \otimes \mathbb{R}^b)^{O(a)\times O(b)}$, one can use the symmetry given by the group action to reduce the complexity. Recall that we assumed $a \leq b$ and consider the space

$$\Sigma := \mathrm{Span}\left\{e_i \otimes f_i \,|\, 1 \leq i \leq a\right\} \subset \mathbb{R}^a \otimes \mathbb{R}^b.$$

When thinking of $\mathbb{R}^a \otimes \mathbb{R}^b$ as the space of rectangular $a \times b$ matrices, $\Sigma \cong \mathbb{R}^a$ is the subspace of diagonal matrices. The complete invariants of the $O(a) \times O(b)$ is given by the *singular value decomposition*. Before the statement, let us define

$$\mathcal{E}^a := \mathfrak{S}_a \ltimes (\mathbb{Z}/2)^a.$$

Together with the action $\mathcal{E}^a \curvearrowright \Sigma \cong \mathbb{R}^a$ that acts by permutation and change of sign of coordinates in the basis.

Lemma 4.1.36 (SVD)**.** *For all $x \in \mathbb{R}^a \otimes \mathbb{R}^b$, there is a unique $\sigma(x) = (\sigma_1(x), \ldots, \sigma_a(x)) \in \Sigma \cong \mathbb{R}^a$ such that $\sigma_1(x) \geq \cdots \geq \sigma_a(x) \geq 0$ that is in the same $O(a) \times O(b)$ of x. Moreover in that case*

$$(O(a) \times O(b) \cdot x) \cap \Sigma = \mathcal{E}^a \cdot \sigma(x).$$

We use SVD to reduce the computation of a mixed volume of invariant zonoids.

Lemma 4.1.37. *Let $K_1, \ldots, K_{ab} \in \mathscr{Z}_0(\mathbb{R}^a \otimes \mathbb{R}^b)^{O(a)\times O(b)}$ and let $X_2, \ldots, X_{ab} \in \mathbb{R}^a \otimes \mathbb{R}^b$ be independent random vectors such that $K_i = \mathbb{E}\underline{X_i}$. Then we have*

$$\mathrm{MV}(K_1, K_2, \ldots, K_{ab}) = \frac{2}{(ab)!} \mathbb{E}\left[h_{\pi(K_1)}(Y)\right]$$

where $Y := \sigma(*(X_2 \wedge \cdots \wedge X_{ab})) \in \Sigma$ with $* : \Lambda^{ab-1}\mathbb{R}^a \otimes \mathbb{R}^b \to \mathbb{R}^a \otimes \mathbb{R}^b$ the Hodge star and where $\pi : \mathbb{R}^a \otimes \mathbb{R}^b \to \Sigma$ is the orthogonal projection.

Proof. Suppose that $K = \mathbb{E}\underline{X_1}$ with X_1 independent of the other X_i. Then by basic zonoid calculus we have

$$\mathrm{MV}(K_1, K_2, \ldots, K_{ab}) = \frac{1}{(ab)!} \mathbb{E}\left|\langle X_1, *X_2 \wedge \cdots \wedge X_{ab}\rangle\right| = \frac{2}{(ab)!} \mathbb{E}\left[h_{K_1}(*X_2 \wedge \cdots \wedge X_{ab})\right].$$

Since $K \in \mathscr{Z}_0(\mathbb{R}^a \otimes \mathbb{R}^b)^{O(a)\times O(b)}$, the support function h_K is $O(a) \times O(b)$-invariant. Thus the last term is equal to $\frac{2}{(ab)!}\mathbb{E}\left[h_K(\sigma(*X_2 \wedge \cdots \wedge X_{ab}))\right]$ which is what we wanted $\square$

In terms of measures, if we denote by μ_Y the generating measure of $\mathbb{E}\underline{Y} \subset \Sigma$, we can reformulate the previous result, using Proposition 1.2.43, as follows:

$$\mathrm{MV}(K_1, K_2, \ldots, K_{ab}) = \frac{4}{(ab)!} \int_{S(\Sigma)} h_{K_1} \, \mathrm{d}\mu_Y.$$

We see that we reduced the computation of a volume in $\mathbb{R}^{ab}$ to an integral on the sphere of $\Sigma \cong \mathbb{R}^a$ which is a considerable reduction of dimension although one of course still needs to compute the measures μ_Y.

Peter Bürgisser and Antonio Lerario in [29] and Antonio Lerario with the author in [54] use this to compute asymptoticf of the number

$$\delta_{a-1,a+b} := \mathbb{E}\#\left(g_1\Omega_\square \cap \cdots \cap g_{ab}\Omega_\square\right)$$

where $g_1, \ldots, g_{ab} \in O(a+b)$ are uniform and independent, as $b \to \infty$.

The link with the $O(a) \times O(b)$ invariant zonoids and $\mathcal{E}^a$ invariant zonoids is actualy stronger than just computation of mixed volume. Let us detail this in a more general setting in the following.

4.1.4 Invariant zonoids: geometric approach

In this section let us consider again our m dimensional Euclidean space $\mathbb{V}$. We suppose we have a subspace Σ with a linear Euclidean injection:

$$\iota : \Sigma \hookrightarrow \mathbb{V}.$$

Writing this injection explicitely may seem pedantic but it actually helps to make things more clear when dualizing. We suppose it is Euclidean in the sense that it preserves the Euclidean structure or equivalentely $\iota(S(\Sigma)) \subset S(\mathbb{V})$. We also consider the projection

$$\pi := \iota^t : \mathbb{V}^* \to \Sigma^*.$$

Suppose that we have closed subgroups $H \subset O(\mathbb{V})$ and $\mathcal{E} \subset O(\Sigma)$ such that their action satisfy the following property:

$$\forall \varepsilon \in \mathcal{E}, s \in \Sigma, \exists h \in H \text{ such that } \iota(\varepsilon \cdot s) = h \cdot \iota(s). \tag{$H\mathcal{E}$-1}$$

We have the following.

Theorem 4.1.38 (M.). *The projection π induces a linear map*

$$\widehat{\pi} : \widehat{\mathscr{K}}(\mathbb{V}^*)^H \to \widehat{\mathscr{K}}(\Sigma^*)^{\mathcal{E}}$$

that is continuous in the norm topology. If the action of H satisfies

$$H \cdot \iota(\Sigma) = \mathbb{V} \tag{$H\mathcal{E}$-2}$$

then this is an isometric embedding: $\widehat{\pi} : \widehat{\mathscr{K}}(\mathbb{V}^*)^H \hookrightarrow \widehat{\mathscr{K}}(\Sigma^*)^{\mathcal{E}}$ and for every $A \in \widehat{\mathscr{Z}_0}(\mathbb{V}^*)^H$ we have $\|\pi(A)\| = \|A\|$. Finally, if, in addition, we have the property that for all $s, s' \in \Sigma$:

$$\iota(s') \in H \cdot \iota(s) \quad \iff \quad s' \in \mathcal{E} \cdot s \tag{$H\mathcal{E}$-3}$$

then $\widehat{\pi}$ is an isomorphism of normed vector spaces: $\widehat{\pi} : \widehat{\mathscr{K}}(\mathbb{V}^*)^H \cong \widehat{\mathscr{K}}(\Sigma^*)^{\mathcal{E}}$.

Proof. We need first to prove that if $K \in \mathscr{K}(\mathbb{V}^*)^H$ then $\pi(K) \in \mathscr{K}(\Sigma)^{\mathcal{E}}$. Indeed, let $s \in \Sigma$ and $\varepsilon \in \mathcal{E}$. Then $h_{\pi(K)}(\varepsilon \cdot s) = h_K(\iota(\varepsilon \cdot s))$. By assumption ($H\mathcal{E}$-1), there exists $g \in H$ such that $\iota(\varepsilon \cdot s) = g \cdot \iota(s)$. Thus $h_{\pi(K)}(\varepsilon \cdot s) = h_K(g \cdot \iota(s))$. By H-invariance, this is equal to $h_K(\iota(s)) = h_{\pi(K)}(s)$ and this proves that $\pi(K)$ is indeed $\mathcal{E}$-invariant.

To prove continuity, we need to prove that for every $A \in \widehat{\mathscr{Z}_0}(\mathbb{V}^*)^H$, we have $\|\pi(A)\| \leq \|A\|$. The norm of a (virtual) convex body is given by the supremum of its support function. Using again that $h_{\pi(K)} = h_K \circ \iota$, we obtain

$$\|\pi(A)\| = \sup\{|h_A(x)| \,|\, x \in \iota(S(\Sigma))\}.$$

Since $\iota(S(\Sigma)) \subset S(\mathbb{V})$ we get $\|\pi(A)\| \leq \|A\|$ which is what we wanted.

Now suppose that we have ($H\mathcal{E}$-2). It implies that $H \cdot \iota(S(\Sigma)) = S(\mathbb{V})$. By H-invariance of A we have

$$\|\pi(A)\| = \sup\{|h_A(g \cdot x)| \,|\, x \in \iota(S(\Sigma)),\, g \in H\}$$

which is equal to $\|A\|$ as we claimed and this shows that the map preserves the norm and in particular is injective.

We need to show that if we assume ($H\mathcal{E}$-3) it is invertible. Let $K \in \mathscr{K}(\Sigma^*)^{\mathcal{E}}$. We define the convex body $L \in \mathscr{K}(\mathbb{V}^*)$ by letting for all $x \in \mathbb{V}$:

$$h_L(x) := \sup\{h_K(s) \,|\, s \in \Sigma,\, \iota(s) \in H \cdot x\}. \tag{4.1.24}$$

It is sublinear and thus defines a support function and it is H-invariant thus $L \in \mathscr{K}(\mathbb{V}^*)^H$. Now applying π, we get for all $s \in \Sigma$:

$$h_{\pi(L)}(s) = \sup\{h_K(s') \,|\, s' \in \Sigma,\, \iota(s') \in H \cdot \iota(s)\} = h_K(s)$$

where in the last equality we used the property ($H\mathcal{E}$-3) and the $\mathcal{E}$-invariance of K. Thus we have $\pi(L) = K$ and since it preserves the norm this inverse is unique and the inverse map is continuous. □

Remark 4.1.39. Since π is linear, the map $\widehat{\pi}$ sends zonoids to zonoids. However, be careful that even in the case where we have properties ($H\mathcal{E}$-2) and ($H\mathcal{E}$-3) the restriction to zonoids of the isomorphism $\widehat{\pi}$ is not in general an isomorphism of the spaces of zonoids. In other words there might be zonoids $K \in \mathscr{Z}_0(\Sigma^*)^{\mathcal{E}}$ such that the preimage $\widehat{\pi}^{-1}(K) \in \mathscr{K}(\mathbb{V}^*)^H$ is not a zonoid. In that case it only gives an embedding of normed vector spaces

$$\widehat{\pi} : \widehat{\mathscr{Z}_0}(\mathbb{V}^*)^H \hookrightarrow \widehat{\mathscr{Z}_0}(\Sigma^*)^{\mathcal{E}}$$

Remark 4.1.40. Note that if we have properties ($H\mathcal{E}$-1), ($H\mathcal{E}$-2) and ($H\mathcal{E}$-3) then the inverse of $K \in \mathscr{K}(\Sigma^*)^{\mathcal{E}}$, $L = \widehat{\pi}^{-1}(K)$ has support function given for any $x \in \mathbb{V}$ by $h_L(x) = h_K(s)$ for any $s \in \Sigma$ such that $\iota(s) \in H \cdot x$. Indeed, it is given by (4.1.24) but because of ($H\mathcal{E}$-3), we have that $\iota(s') \in H \cdot \iota(s)$ if and only if $s' \in \mathcal{E}$ and thus, because of the invariance of K, the argument in the sup is constant.

As the reader may expect, the first nontrivial example is given by the $O(a) \times O(b)$ action on $a \times b$ matrices and SVD.

Corollary 4.1.41. *The projection $\pi : \mathbb{R}^a \otimes \mathbb{R}^b \to \Sigma$ on the space of diagonal matrices, induces a linear isometry*

$$\widehat{\pi} : \widehat{\mathscr{K}}(\mathbb{R}^a \otimes \mathbb{R}^b)^{O(a)\times O(b)} \cong \widehat{\mathscr{K}}(\Sigma)^{\mathcal{E}^a}$$

and a map

$$\widehat{\pi} : \widehat{\mathscr{Z}_0}(\mathbb{R}^a \otimes \mathbb{R}^b)^{O(a)\times O(b)} \hookrightarrow \widehat{\mathscr{Z}_0}(\Sigma)^{\mathcal{E}^a}$$

that is an isometric embedding of normed vector space.

Proof. Properties ($H\mathcal{E}$-1),($H\mathcal{E}$-2) and ($H\mathcal{E}$-3), are all features of SVD (Lemma 4.1.36). □

Example 4.1.42. Another example is given by Raman Sanyal and James Saunderson in [76] where they consider the following case. $\mathbb{V} := Sym_n(\mathbb{R})$ is the space of $n \times n$ symmetric matrices and $H = O(n)$ acts by conjugation. $\Sigma = \mathbb{R}^n$ and $\mathcal{E} = \mathfrak{S}_n$ is the group of permutations acting by permutation of coordinate. Then the inclusion $\iota : \mathbb{R}^n \hookrightarrow Sym_n(\mathbb{R})$ is given by the diagonal matrices. We can see that ($H\mathcal{E}$-1) holds because if $x \in \mathbb{R}^n$ and $p \in \mathfrak{S}_n$ then the diagonal matrix $\iota(p \cdot x)$ can be obtained from $\iota(x)$ conjugating by a permutation matrix (which is also orthogonal). Property ($H\mathcal{E}$-2) follows from the fact that every symmetric matrix is diagonalizable in orthonormal basis. Finally property ($H\mathcal{E}$-3) is equivalent to the fact that the only diagonal matrices in the orbit of a diagonal matrix are given by permutation of the entries on the diagonal.

Sanyal and Saunderson call preimages of zonoids by the map $\widehat{\pi}$, *spectral zonoids* and they show that they are not always zonoids. This is an instance of what was anticipated in Remark 4.1.39.

This approach is a priori different from the one presented in Section 2.4.3 that consisted in identifying the orbit spaces and then consider invariant measures as measures on the orbit spaces. In term of measures, the map $\widehat{\pi}$ presented here is dual to the restriction map that take a continuous function on $\mathbb{V}$ and restrict it to Σ, i.e. the map $f \mapsto f \circ \iota$. In particular we have also continuity in the weak-$*$ topology.

Proposition 4.1.43. *If the action satisfies* ($H\mathcal{E}$-1)*, the map* $\widehat{\pi} : \widehat{\mathscr{Z}_0}(\mathbb{V}^*)^H \to \widehat{\mathscr{Z}_0}(\Sigma^*)^{\mathcal{E}}$ *is continuous in the weak-$*$ topology.*

This geometric approach has the advantage to be more adapted to the computation of geometric quantities such as volume or mixed volume. Indeed, suppose we have a continuous map (not necessarily linear)

$$\sigma : \mathbb{V} \to \Sigma$$

such that for all $x \in \mathbb{V}$ we have

$$\iota(\sigma(x)) \in H \cdot x.$$

Then with the exact same proof as Lemma 4.1.37 we can prove the following.

Lemma 4.1.44. *Let* $K_1, \dots, K_m \in \mathscr{Z}_0(\mathbb{V}^*)^H$ *and let* $X_2, \dots, X_m \in \mathbb{V}^*$ *be independent random vectors such that* $K_i = \mathbb{E}\underline{X_i}$*. Then we have*

$$\mathrm{MV}(K_1, K_2, \dots, K_m) = \frac{2}{m!}\mathbb{E}\left[h_{\pi(K_1)}(Y)\right]$$

where $Y := \sigma(*(X_2 \wedge \dots \wedge X_m)) \in \Sigma$ *with* $* : \Lambda^{m-1}\mathbb{V}^* \to \mathbb{V}$ *the Hodge star.*

One could use this and the description by Sanyal and Saunderson in [76] for symmetric matrices to compute random intersection in the *Lagrangian Grassmannian* and this will be the object of future works.

4.2 Kac-Rice and the zonoid section

Results in this section are a joint work with Michele Stecconi.

In the following, M will denote a smooth Riemannian manifold of dimension m. This time the random submanifolds we consider are given by random maps $f \in C^1(M, \mathbb{R}^k)$ that we call *random fields* (where recall the notation $x \in X$ for a random element x of X see Definition 1.2.17). Indeed, if $0 \in \mathbb{R}^k$ is almost surely a regular value of f then the zero set $X := f^{-1}(0)$ defines a random submanifold of M of codimension k.

We will ask additional regularity condition on the law of the random function $f \in C^1(M, \mathbb{R}^k)$ that we call the *z-KRok* conditions (this stands for "zonoid-Kac-Rice ok"). They are a bit technical but the idea is to be able to give a sense, for every $p \in M$, to conditioning to the event $f(p) = 0$ which in general is of probability zero and then to have that the differential $d_p f$ conditioned to $f(p) = 0$ is integrable.

Given a *z-KRok* random field $f \in C^1(M, \mathbb{R}^k)$, we build a zonoid section in the k-th exterior power of the cotangent bundle of M. That is, for every $p \in M$ we build a zonoid $\zeta_X(p) \in \mathscr{Z}_0(\Lambda^k T_p^* M)$. Then we prove that this zonoid section satisfies nice pullback and intersection property. Most importantly, by adapting the *Kac-Rice formula*, we prove that its length computes the density of volume of the random submanifold $X = f^{-1}(0)$. We deduce inequalities for densities of random intersection of *z-KRok* fields.

Given a *z-KRok* random field $f \in C^1(M, \mathbb{R})$, the support function of its zonoid $\zeta_X(p) \in \mathscr{Z}_0(T^*M)$ defines a (semi)norm $h_{\zeta(p)} : T_p M \to \mathbb{R}$. If this norm is regular enough this is called a *Finsler structure* on the manifold M. In the last part we interpret our results in this context to produce a *Crofton formula* in Finsler geometry.

4.2.1 The *z-KRok* condition

In this section we are going to describe a class of random functions for which Kac-Rice formula works well and it can be written in terms of a *field of zonoids* as we will explain later. The space of C^r functions between two manifolds M and N is denoted by $C^r(M,N)$ and we consider it to be a topological space endowed with the weak Whitney topology (see [44]). Spaces of measures are, as usual, considered endowed with the weak-$*$ topology.

The definition of the *z-KRok* conditions are based on a notion in probability that formalizes the notion of conditioning, see [36] or [52, Definition 2.1-1].

Definition 4.2.1. Let $f \in C^1(M,\mathbb{R}^k)$ and let $p \in M$, suppose that the law of $f(p) \in \mathbb{R}^k$ is absolutely continuous with density $\rho_{f(p)} \colon \mathbb{R}^k \to [0,+\infty)$. Then, a *regular conditional probability* of f given $f(p)$ is a function

$$\begin{aligned} \mu(p,\cdot)(\cdot) \colon \mathbb{R}^k \times \mathcal{B} &\to [0,1] \\ (x,B) &\mapsto \mu(p,x)(B) \end{aligned}$$

where $\mathcal{B}$ denotes the Borelians of $C^1(M,\mathbb{R}^k)$, that satisfies the following two properties.

(i) For every Borelian $B \subset C^1(M,N)$, the function $\mu(p,\cdot)(B) \colon \mathbb{R}^k \to [0,1]$ is measurable and for every Borelian $V \subset \mathbb{R}^k$, we have

$$P(f \in B; f(p) \in V) = \int_V \mu(p,x)(B)\, \rho_{f(p)}(x)\mathrm{d}x.$$

(ii) For all $x \in \mathbb{R}^k$, $\mu(p,x)$ is a Borel probability measure on $C^1(M,N)$.

One more geometric definition. For this, recall that our manifold M is Riemannian.

Definition 4.2.2. Let $\varphi : M \to \mathbb{R}^k$ be a differentiable function and write $\varphi = (\varphi^1,\dots,\varphi^k)$. The *Jacobian* of φ at $p \in M$, denoted $J_p\varphi$, is defined to be

$$J_p\varphi := \|\mathrm{d}_p\varphi^1 \wedge \cdots \wedge \mathrm{d}_p\varphi^k\|.$$

This is sometimes called the *normal Jacobian* of φ. We are now ready for our main definition, namely the *z-KRok* hypoteses. The name *z-KRok*[1] stands for “zonoid-Kac-Rice ok”.

Definition 4.2.3 (*z-KRok* hypotheses)**.** Let $f \in C^1(M,\mathbb{R}^k)$ be a random field. We say that f is *z-KRok* if the following properties hold.

z-KRok-i: Almost surely, 0 is a regular value of f.

z-KRok-ii: For any $p \in M$ the law of $f(p) \in \mathbb{R}^k$ is absolutely continuous and thus admits a density $\rho_{f(p)} \colon \mathbb{R}^k \to [0,+\infty)$.

z-KRok-iii: The function $\rho_{f(\cdot)}(\cdot) \colon M \times \mathbb{R} \to \mathbb{R}$ given by $(p,x) \mapsto \rho_{f(p)}(x)$ is continuous at $(p,0)$ for all $p \in M$.

z-KRok-iv: There exists a regular conditional probability $\mu(p,x) \in \mathscr{P}(C^1(M,\mathbb{R}^k))$ of f given $f(p)$ such that the following holds. Let $J_p \cdot \mu(p,x) \in \mathcal{M}^+\left(C^1(M,\mathbb{R}^k)\right)$ be the measure defined for every continuous $\Psi : C^1(M,\mathbb{R}^k) \to \mathbb{R}$ by

$$\langle J_p \cdot \mu(p,x), \Psi\rangle := \langle \mu(p,x), \Psi\, J_p\rangle = \int \Psi(\varphi)\, J_p\varphi \,\mathrm{d}\left(\mu(p,x)\right)(\varphi).$$

Then we ask that $J_p \cdot \mu(p,x)$ is a finite measure and that the function

$$J \cdot \mu : M \times \mathbb{R}^k \to \mathcal{M}^+\left(C^1(M,\mathbb{R}^k)\right)$$

$$(p,x) \mapsto J_p \cdot \mu(p,x)$$

is continuous at $(p,0)$ for all $p \in M$.

[1] The author pronounces it “zkrok” or “zee krok”.

By definition, for every $p \in M, x \in \mathbb{R}^k$, $\mu(p,x)$ is a probability measure on $C^1(M,\mathbb{R}^k)$. We denote by

$$(f|f(p) = x) \in C^1(M,\mathbb{R}^k)$$

a random function that has law $\mu(p,x)$. Note that this is not properly defined as a *random variable* but in the following we will only care about the law. If $\Phi : C^1(M,\mathbb{R}^k) \to \mathbb{R}$ is a $\mu(p,x)$-integrable function then we write

$$\mathbb{E}\left[\Phi(f)|f(p) = x\right] := \mathbb{E}\left[\Phi\big((f|f(p) = x)\big)\right] = \int_{C^1(M,\mathbb{R}^k)} \Phi(\varphi)\, \mathrm{d}\mu(x,p)(\varphi).$$

Similarly if $\Psi : C^1(M,\mathbb{R}^k) \to \mathcal{A}$ is a Borel map to a topological space $\mathcal{A}$, we write

$$(\Psi(f)|f(p) = x) := \Psi\big((f|f(p))\big) \in \mathcal{A}$$

i.e. $(\Psi(f)|f(p) = x)$ has law the push forward of $\mu(x,p)$ by Ψ. In particular, if we write $f = (f^1,\dots,f^k)$, in the following we will be interested in the random vector

$$(\mathrm{d}_p f^1 \wedge \cdots \wedge \mathrm{d}_p f^k | f(p) = 0) \in \Lambda^k T_p^* M. \tag{4.2.1}$$

Notice that the finiteness assumption in *z-KRok*-iv is equivalent to say that the random vector $(\mathrm{d}_p f^1 \wedge \cdots \wedge \mathrm{d}_p f^k | f(p) = 0) \in \Lambda^k T_p^* M$ is integrable. Of course if we have an integrable random vector in an m dimensional vector space we feel the urge of taking the Vitale zonoid and that is what we will do in the next section. But first let us give a few examples of *z-KRok* fields. The first one, still rather general, should help the reader not familiar with the technical probabilistic concepts introduced, to understand better the *z-KRok* conditions.

Proposition 4.2.4. *Let $\mathcal{F} \subset C^1(M,\mathbb{R}^k)$ be a subspace of dimension $n < \infty$ endowed with a scalar product and such that for all $p \in M$, the map $ev_p : \mathcal{F} \to \mathbb{R}^k$, $\varphi \mapsto \varphi(p)$ is surjective. Let $f \in \mathcal{F}$ be a random function whose law admits a* continuous *density $\rho_f : \mathcal{F} \to \mathbb{R}$ such that $\rho_f(0) > 0$ and such that when $\|\varphi\| \to \infty$, we have $\rho_f(\varphi) = O(\|\varphi\|^{-\alpha})$ for some $\alpha > n$. Then f is* z-KRok.

Proof. Let us detail the *z-KRok* conditions one by one.

For the first one, the trick is to use the *parametric transversality theorem*, see [44, Theorem 2.7]. Indeed, consider the function $\Phi : \mathcal{F} \times M \to \mathbb{R}$ given by $\Phi(\varphi,p) = \varphi(p)$. Then its differential at (φ,p) is given by $ev_p \oplus \mathrm{d}_p\varphi$. By assumption this is surjective and thus the map Φ is transversal to zero, i.e. 0 is a regular value of Φ. The parametric transversality theorem then tells us that for almost all $\varphi \in \mathcal{F}$, the map $\varphi \mapsto \varphi(p)$ is transversal to 0, i.e. for almost all $\varphi \in \mathcal{F}$, 0 is a regular value of φ which is what we wanted.

The law of $f(p)$ is the push forward of the law of f by the linear map $ev_p : \mathcal{F} \to \mathbb{R}^k$. Suppose $B \subset \mathbb{R}^k$ is a Borel subset of measure 0. Then $P(f(p) \in B) = P(f \in ev_p^{-1}(B))$. Let us denote

$$\mathcal{F}_p := \ker(ev_p) = \{\varphi \in \mathcal{F} \,|\, \varphi(p) = 0\}.$$

Then the space $ev_p^{-1}(x)$ is an affine subspace parallel to $\mathcal{F}_p$ which, by the surjectivity of ev_p, is of dimension $n-k$. Thus $ev_p^{-1}(B) \cong B \times \mathcal{F}_p$ is of Lebesgue measure zero in $\mathcal{F}$. Since the law of f is, by assumption, absolutely continuous with respect to the Lebesgue measure on f, we obtain that $P(f \in ev_p^{-1}(B)) = 0$ and thus $P(f(p) \in B) = 0$. This proves that the law of $f(p)$ is absolutely continuous with respect to Lebesgue on $\mathbb{R}^k$ and thus admits a density $\rho_{f(p)} : \mathbb{R}^k \to \mathbb{R}$ and this proves the property *z-KRok*-ii.

We can compute this density, we have for all $p \in M$ and $x \in \mathbb{R}^k$:

$$\rho_{f(p)}(x) = \int_{ev_p^{-1}(x)} \rho_f(\varphi)\mathrm{d}\varphi. \tag{4.2.2}$$

To prove continuity, we can use the assumption of the behaviour at infinity of ρ_f and dominated convergence. Indeed, with the Euclidean structure, we can assume $\mathcal{F} = \mathbb{R}^n$. Let $p \in M$, we can assume that $\mathcal{F}_p = \mathbb{R}^{n-k} \subset \mathbb{R}^n$ is the space spanned by the $n-k$ first coordinates. Then we write

$\rho_f(y,x)$ with $y \in \mathbb{R}^{n-k}$ and $x \in \mathbb{R}^k$. Let now $p_j \to p$ and $x_j \to 0$, let $g_j \in O(n)$ be such that $g_j^{-1}(\mathcal{F}_{p_j}) = \mathcal{F}_p = \mathbb{R}^{n-k}$ then we have

$$\rho_{f(p_j)}(x_j) = \int_{\mathbb{R}^{n-k}} \rho_f(g_j(y), x_j)\mathrm{d}y.$$

On $\mathbb{R}^{n-k}$, the function $y \mapsto \|y\|^{-\alpha}$ is integrable at infinity if and only if $\alpha > n-k$. Thus under our assumption $y \mapsto \rho_f(g_j(y), x_j)$ is dominated by an integrable function uniformly on j and by dominated convergence we get *z-KRok*-iii.

We define $\mu(p,x)$ to be the probability measure on $\mathcal{F}$ with support on the affine space $ev_p^{-1}(x)$ that admits the continuous density $\rho_{f,p,x} : ev_p^{-1}(x) \to \mathbb{R}$ that is 0 if $\rho_{f(p)}(x) = 0$ and else is given by

$$\rho_{f,p,x} := \frac{1}{\rho_{f(p)}(x)} \rho_f|_{ev_p^{-1}(x)}. \tag{4.2.3}$$

Then $\mu(p,x)$ defines a regular conditional probability for f given $f(p)$. Now let us note that for all $p \in M$, there exists a constant $c = c(p) > 0$ such that $J_p\varphi \leq c\|\varphi\|^k$. Thus the function $\varphi \mapsto J_p\varphi\rho_f(\varphi)$ is at infinity an $O\left(\|\varphi\|^{-(\alpha-k)}\right)$ and this is integrable on $ev_p^{-1}(x) \cong \mathbb{R}^{n-k}$ if and only if $\alpha > n$ which is precisely our assumption and this gives us the finiteness condition in *z-KRok*-iv. To see the continuity, let $\Psi : \mathcal{F} \to \mathbb{R}$ be a bounded continuous function. Let $p_j \to p$ and $x_j \to 0$, we repeat the argument of the previous item to write

$$\langle J_p \cdot \mu(p_j, x_j), \Psi\rangle = \frac{1}{\rho_{f(p_j)}(x_j)} \int_{\mathbb{R}^{n-k}} \Psi(g_j(y), x_j) J_p(g_j(y), x_j)\, \rho_f(g_j(y), x_j)\mathrm{d}y$$

for some sequence $g_j \in O(n)$ converging to Id. Since $\rho_f(0) > 0$ we get from (4.2.2) that $\rho_{f(p)}(0) > 0$ for every $p \in M$ and we can argue similarly as before: this is dominated by a $O\left(\|\varphi\|^{-(\alpha-k)}\right)$ at infinity which is integrable and we conclude by dominated convergence to obtain *z-KRok*-iv. □

Definition 4.2.5. We will call a random field $f \in \mathcal{F} \subset C^1(M, \mathbb{R}^k)$ that satisfies the hypotheses of Proposition 4.2.4, a random field of $(\mathcal{F}, \alpha)$*-type*. If there exists $n > 0$ such that f is of $(\mathcal{F}, \alpha)$-type for every $\alpha > n$ then we say that f is of $(\mathcal{F}, \infty)$*-type*.

Example 4.2.6 (Gaussian random fields). A smooth *Gaussian random field* (GRF) is a random map $f \in C^\infty(M, \mathbb{R}^k)$ such that for every finite tuple of points $p_1, \ldots, p_j \in M$, the random vector given by $(f(p_1), \ldots, f(p_j)) \in \mathbb{R}^{k \times j}$ is a Gaussian vector (in the sense of Definition 2.6.1). For example, if $\varphi_1, \ldots, \varphi_n \in C^\infty(M, \mathbb{R}^k)$ are fixed smooth functions and if $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are Gaussian variables then

$$f := \lambda_1\varphi_1 + \cdots + \lambda_n\varphi_n \in C^\infty(M, \mathbb{R}^k) \tag{4.2.4}$$

is a GRF. Not all GRF can be built this way, some naturally live in an infinite dimensional space. For a deeper treatment of GRF, the reader can refer to [56].

If $f \in C^\infty(M, \mathbb{R}^k)$ is a GRF and if for all $p \in M$, the support of the law of $f(p) \in \mathbb{R}^k$ is the whole $\mathbb{R}^k$, i.e. if the Gaussian vector $f(p) \in \mathbb{R}^k$ is non degenerate, then f is *z-KRok*. In the finite dimensional case of a GRF of the form (4.2.4), this is given by Proposition 4.2.4, indeed, one can see that such GRF is of $(\mathcal{F}, \infty)$-type for $\mathcal{F} := \mathrm{Span}\{\varphi_1, \ldots, \varphi_n\}$. In all generality, this is [82, Theorem 3.2]. In that case and if $k = 1$, for all $p \in M$, the random vector

$$(\mathrm{d}_p f | f(p) = 0) \in T_p^* M$$

is also a Gaussian vector. In the case of a GRF of the form (4.2.3), it follows from the fact that the restriction of a Gaussian density to a hyperplane is again (the multiple of) a Gaussian density on this hyperplane.

Example 4.2.7 (Random level sets). Let $\varphi \in C^1(M, \mathbb{R}^k)$ be a fixed function and let $\lambda \in \mathbb{R}^k$ be a random vector whose law admits a *continuous* density $\rho_\lambda : \mathbb{R}^k \to \mathbb{R}$. Then the random field

$$f := \varphi - \lambda \in C^1(M, \mathbb{R}^k)$$

is *z-KRok*. Indeed, to see *z-KRok*-i we see that 0 is a critical value of $\varphi - x$ if and only if x is a critical value of φ. By Sard's theorem, those are of Lebesgue measure zero and since the law of λ is absolutely continuous with respect to Lebesgue, this gives *z-KRok*-i.

Then we see that $f(p)$ admits the density given for every $x \in \mathbb{R}^k$ by $\rho_{f(p)}(x) = \rho_\lambda(\varphi(p) - x)$ and this gives *z-KRok*-ii.

z-KRok-iii is a consequence of the continuity of ρ_λ.

Finally to prove *z-KRok*-iv, we let $\mu(p,x)$ be the Dirac delta measure $\mu(p,x) = \delta_{\varphi - \varphi(p) + x}$. One can check that this is a regular conditional probability for f given $f(p)$ and that it satisfies *z-KRok*-iv.

Note that in that case, we have

$$(\mathrm{d}_p f^1 \wedge \cdots \wedge \mathrm{d}_p f^k | f(p) = 0) = \mathrm{d}_p \varphi^1 \wedge \cdots \wedge \mathrm{d}_p \varphi^k \tag{4.2.5}$$

almost surely.

Before going to geometric considerations, we need one more probabilistic technicality. Indeed, it will be useful to have another formulation of the continuity statement in *z-KRok*-iv. This formulation which seems stronger turns out to be equivalent. A similar generalization was proved in [82, Proposition 2.4], and with a slight modification of the same argument we can show the following.

Lemma 4.2.8. *Let $f \in C^1(M, \mathbb{R}^k)$ be a random map satisfying* z-KRok-*i to iii and let $\mu(p,\cdot)(\cdot)$ be a regular conditional probability of f given $f(p)$ such that $J_p \cdot \mu(p,x)$ is a finite measure. The following statements are equivalent:*

(i) (z-KRok-*iv) The function $J \cdot \mu : M \times \mathbb{R}^k \to \mathcal{M}^+(C^1(M, \mathbb{R}^k))$, $(p,x) \mapsto J_p \cdot \mu(p,x)$ is continuous at $(p,0)$ for all $p \in M$. That is, for any bounded continuous function $\Psi : C^1(M, \mathbb{R}^k) \to \mathbb{R}$ and any convergent sequence $(p_n, x_n) \to (p, 0)$ in $M \times \mathbb{R}^k$ we have*

$$\mathbb{E}\left[\Psi(f) J_{p_n} f \middle| f(p_n) = x_n\right] \to \mathbb{E}\left[\Psi(f) J_p f \middle| f(p) = 0\right].$$

(ii) For any sequence of continuous functions $\Psi_n : C^1(M, \mathbb{R}^k) \to \mathbb{R}$ that converges in the compact-open topology to a continuous function $\Psi_0 : C^1(M, \mathbb{R}^k) \to \mathbb{R}$ and any sequence $(p_n, x_n) \to (p_0, 0)$ converging in $M \times \mathbb{R}^k$ such that for all $n \geq 0$, $\Psi_n(f) \leq C J_{p_n} f$ for some $C > 0$, we have that

$$\mathbb{E}\left[\Psi_n(f) \middle| f(p_n) = x_n\right] \to \mathbb{E}\left[\Psi_0(f) \middle| f(p_0) = 0\right]. \tag{4.2.6}$$

Proof. We need only to prove (i) $\Longrightarrow$ (ii).

Assume (i) and let $\Psi_n, p_n, x_n \to \Psi_0, p_0, 0$ as in the statement of (ii). Observe that for all $n \geq 0$, if $J_{p_n}\varphi = 0$, then $\Psi_n(\varphi) = 0$, so that

$$\begin{aligned}
\mathbb{E}\left[\Psi_n(f) \middle| \Psi_n(p_n) = x\right] &= \int_{C^1(M,\mathbb{R}^k)} \Psi_n(\varphi) \mathrm{d}\mu(p_n, x)(\varphi) = \\
&= \int_{C^1(M,\mathbb{R}^k) \setminus \{J_{p_n} = 0\}} \Psi_n(\varphi) \frac{J_{p_n}\varphi}{J_{p_n}\varphi} \mathrm{d}\mu(p_n, x)(\varphi) + \int_{C^1(M,\mathbb{R}^k) \cap \{J_{p_n} = 0\}} \Psi_n(\varphi) \mathrm{d}\mu(p,x)(\varphi) \\
&= \int_{C^1(M,\mathbb{R}^k)} \frac{\Psi_n(\varphi)}{J_{p_n}\varphi} \mathrm{d}\left(J_{p_n} \cdot \mu(p_n, x)\right)(\varphi)
\end{aligned}$$

where on the last line, the integral is over all $C^1(M, \mathbb{R}^k)$ because $\{J_p = 0\}$ is of measure zero for the measure $J_p \cdot \mu(p,x)$.

Let $E(p,x) := \mathbb{E}\left[J_p f | f(p) = x\right]$ be the total mass of the measure $J_p \cdot \mu(p,x)$. By *z-KRok*.iv, the number $E(p,0) \geq 0$ is finite, though notice that it could be zero. The hypothesis (i) implies that $E(p_n, x_n) \to E(p_0, 0)$. If $E(p_0, 0) = 0$, then the limit (4.2.6) holds since

$$\left|\mathbb{E}\left[\Psi_n(f) \middle| f(p_n) = x_n\right]\right| \leq C\, E(p_n, x_n) \to 0 = \mathbb{E}\left[\Psi_0(f) \middle| f(p_0) = 0\right]$$

where the equality on the right holds because we assumed $\Psi_0(\varphi) \leq C J_{p_0}\varphi$.

Assume that $E(p_0,0)>0$, then we can assume that $E(p_n,x_n)>0$ for all $n\in\mathbb{N}$. In this case, the following sequence of probability measures converges:

$$P_n := E(p_n,x_n)^{-1}J_{p_n}\cdot\mu(p_n,x_n) \to P_0 := E(p_0,0)^{-1}J_{p_0}\cdot\mu(p_0,0).$$

Thus by Skorohod's Theorem (See [23, 73]) there exists a sequence of random functions $Y_n, Y_0 \in C^1(M,\mathbb{R}^k)$ defined on a common probability space such that Y_n has law P_n for all $n\geq 0$ and such that $Y_n\to Y_0$ in $C^1(M,\mathbb{R}^k)$ almost surely. Then

$$\begin{aligned}\mathbb{E}\left[\Psi_n(f)\middle|f(p_n)=x_n\right] &= E(p_n,x_n)\int_{C^1(M,\mathbb{R}^k)}\frac{\Psi_n(\varphi)}{J_{p_n}\varphi}dP_n(\varphi)\\ = E(p_n,x_n)\mathbb{E}\left[\frac{\Psi_n(Y_n)}{J_{p_n}\varphi}\right] &\xrightarrow[n\to\infty]{} E(p_0,0)\mathbb{E}\left[\frac{\Psi_0(Y_0)}{J_{p_0}\varphi}\right]\\ &= E(p_0,0)\int_{C^1(M,\mathbb{R}^k)}\frac{\Psi_0(\varphi)}{J_{p_0}\varphi}dP_0(\varphi)\\ &= \mathbb{E}\left[\Psi_0(f)\middle|f(p)=0\right].\end{aligned}$$

Here the limit holds by dominated convergence, since $\frac{\Psi_n(Y_n)}{J_{p_n}\varphi}\leq C$ and $\frac{\Psi_n(Y_n)}{J_{p_n}\varphi}\to\frac{\Psi_0(Y_0)}{J_{p_0}\varphi}$ almost surely. □

4.2.2 The zonoid section

Let us start with a short comment on zonoid bundles. Let $\pi:E\to M$ be a vector bundle of rank k over M. The structure of vector bundle is given by the *trivialization maps* $\chi_U : E|_U \xrightarrow{\sim} U\times\mathbb{R}^k$ which are homeomorphisms that are linear isomorphism on the fibers.

We can define the zonoid bundle $\mathscr{Z}(E)$ whose fiber at a point $p\in M$ is defined to be the space $\mathscr{Z}(E)_p := \mathscr{Z}(E_p)$ where $E_p=\pi^{-1}(p)$ is the fiber of E at p, and whose bundle structure is given by the collection of maps $\widehat{\chi_U} : \mathscr{Z}(E)|_U \xrightarrow{\sim} U\times\mathscr{Z}(\mathbb{R}^k)$ in particular the topology on $\mathscr{Z}(E)$ is the smallest topology that makes all $\widehat{\chi_U}$ homeomorphisms. Similarly one can define $\mathscr{Z}_0(E)$, $\mathscr{G}(k,E)$, $\mathscr{G}(E)$, $\mathscr{G}_0(k,E)$, $\mathscr{G}_0(E)$ and all the virtual counterparts.

Given a fiber bundle $\pi:F\to M$ we denote by $\Gamma(F)$ the space of continuous sections of F, that is $\gamma\in\Gamma(F)$ if and only if $\gamma:M\to F$ is a continuous map such that for every $p\in M$, $\pi(\gamma(p))=p$. In particular a section $\zeta\in\Gamma(\mathscr{Z}(E))$ is the choice of a zonoid at each point p of the manifold M in the vector space E_p such that this zonoid depends *continuously* on the point p. Locally, on the trivialization charts, this is just a continuous map

$$\zeta : U\to\mathscr{Z}(\mathbb{R}^k)$$

The first observation is that, as vector bundles, we have $\mathscr{Z}(E)\cong\mathscr{Z}_0(E)\oplus E$ with the decomposition given by the pole (Proposition 1.2.2) recall that the *pole* of a zonoid $K\in\mathscr{Z}(\mathbb{V})$ is the unique point $o(K)\in\mathbb{V}$ such that $K+\frac{1}{2}\{-o(K)\}\in\mathscr{Z}_0(\mathbb{V})$. Thus we have

$$\Gamma(\mathscr{Z}(E))\cong\Gamma(\mathscr{Z}_0(E))\oplus\Gamma(E) \tag{4.2.7}$$

Therefore we can, as before, treat the pole of a zonoid and the zonoid as separate sections.

Let us also note that a zonoid section $\zeta:M\to\mathscr{Z}_0(E)$ defines a *seminorm* on each fibers E_p^* of the dual through the support function $h_{\zeta(p)}:E_p^*\to\mathbb{R}$.

We are now ready to define the main object of this section.

Definition 4.2.9. Let $f\in C^1(M,\mathbb{R}^k)$ be *z-KRok*. We define $\zeta_f : M\to\mathscr{G}(k,T^*M)$ to be the zonoid section given for all $p\in M$ by

$$\zeta_f(p) := \rho_{f(p)}(0)\mathbb{E}\left([0,\mathrm{d}_pf^1\wedge\cdots\wedge\mathrm{d}_pf^k]|f(p)=0\right)$$

That is the zonoid $\rho_{f(p)}(0)\mathbb{E}[0,Y]$ for the random vector $Y=\left(\mathrm{d}_pf^1\wedge\cdots\wedge\mathrm{d}_pf^k|f(p)=0\right)\in\Lambda^kT_p^*M$, see (4.2.1) .

Similarly, we define $\underline{\zeta_f} : M \to \mathscr{G}_0(k, T^*M)$ to be the section given for all $p \in M$ by

$$\underline{\zeta_f}(p) := \rho_{f(p)}(0)\mathbb{E}\left(\underline{\mathrm{d}_p f^1 \wedge \cdots \wedge \mathrm{d}_p f^k}|f(p) = 0\right).$$

We also denote

$$o_f(p) := o(\zeta_f(p)) = \rho_{f(p)}(0)\mathbb{E}\left(\mathrm{d}_p f^1 \wedge \cdots \wedge \mathrm{d}_p f^k|f(p) = 0\right) \in \Lambda^k T_p^* M.$$

Note that we have for all $p \in M$, $\zeta_f(p) = \underline{\zeta_f}(p) + \frac{1}{2}\{o_f(p)\}$, i.e. in the splitting (4.2.7), we have

$$\zeta_f = \underline{\zeta_f} \oplus \tfrac{1}{2} o_f$$

Note also that, as observed earlier, the finiteness condition in *z-KRok*-iv guarantees that the random vector $\left(\mathrm{d}_p f^1 \wedge \cdots \wedge \mathrm{d}_p f^k|f(p) = 0\right) \in \Lambda^k T_p^* M$ is integrable hence the zonoids are well defined. Note also that o_f is a section of $\Lambda^k T^*M$, i.e. a differential form of degree k on M. Finally, observe that we have for all $p \in M$:

$$\ell\left(\zeta_f(p)\right) = \ell\left(\underline{\zeta_f}(p)\right) = \rho_{f(p)}(0)\mathbb{E}\left[J_p f|f(p) = 0\right]. \tag{4.2.8}$$

Lemma 4.2.10. *The sections ζ_f, $\underline{\zeta_f}$ and o_f are continuous.*

Proof. Since this is a local statement, we can assume $M = \mathbb{R}^m$. Let $p_n \in \mathbb{R}^m$ be a sequence that converges to $p_0 \in \mathbb{R}^m$ and let $Y_n := \left(\mathrm{d}_{p_n} f^1 \wedge \cdots \wedge \mathrm{d}_{p_n} f^k|f(p_n) = 0\right) \in \Lambda^k \mathbb{R}^m$. For any $v \in \Lambda^k \mathbb{R}^m$ we have

$$h_{\underline{\zeta_f}(p_n)}(v) = \frac{\rho_{f(p_n)}(0)}{2}\mathbb{E}|\langle v, Y_n\rangle|.$$

By *z-KRok*-iii we have $\rho_{f(p_n)}(0) \to \rho_{f(p_0)}(0)$. Moreover let us also consider the continuous function $\Psi_v : C^1(M, \mathbb{R}^m) \to \mathbb{R}$ given by $\Psi_v(\varphi) := |\langle v, \mathrm{d}_{p_n}\varphi^1 \wedge \cdots \wedge \mathrm{d}_{p_n}\varphi^k\rangle|$. Then we have

$$\mathbb{E}|\langle v, Y_n\rangle| = \langle \mu(p_n, 0), \Psi_v\rangle$$

where $\mu(p_n, 0) \in \mathcal{M}^+(C^1(M, \mathbb{R}^k))$ is the regular conditional probability of f given $f(p_n)$ from property *z-KRok*-iv. Note that for all $\varphi \in C^1(M, \mathbb{R}^k)$ we have by Cauchy-Schwartz $\Psi_v(\varphi) \le \|v\| J_p\varphi$. Thus we can apply the technical lemma, Lemma 4.2.8 and obtain that

$$h_{\underline{\zeta_f}(p_n)}(v) \to h_{\underline{\zeta_f}(p_0)}(v)$$

i.e., we have pointwise convergence of the support function. We use Lemma 1.1.18 to conclude that $\underline{\zeta_f}(p_n) \to \underline{\zeta_f}(p_0)$ in the Hausdorff distance topology. The case of o_f is done similarly and $\zeta_f = \underline{\zeta_f} \oplus \frac{1}{2} o_f$ follows. □

Let us compute this zonoid in some example, starting with the $(\mathcal{F}, \alpha)$-type, recall Definition 4.2.5.

Proposition 4.2.11. *Let $f \in \mathcal{F} \subset C^1(M, \mathbb{R}^k)$ be a* z-KRok *field of $(\mathcal{F}, \alpha)$-type where $(\mathcal{F}, \alpha)$ satisfy the hypotheses of Proposition 4.2.4. For every $p \in M$ and every $w \in \Lambda^k T_p M$ we have*

$$h_{\underline{\zeta_f}(p)}(w) = \frac{1}{2}\int_{\mathcal{F}_p} \left|(\mathrm{d}_p\varphi^1 \wedge \cdots \wedge \mathrm{d}_p\varphi^k)(w)\right| \rho_f(\varphi)\mathrm{d}\varphi \tag{4.2.9}$$

$$o_f(p)(w) = \int_{\mathcal{F}_p} (\mathrm{d}_p\varphi^1 \wedge \cdots \wedge \mathrm{d}_p\varphi^k)(w)\rho_f(\varphi)\mathrm{d}\varphi$$

where recall that $\mathcal{F}_p = \ker(ev_p) = \{\varphi \in \mathcal{F} \,|\, \varphi(p) = 0\}$ and $\rho_f : \mathcal{F} \to \mathbb{R}$ is the density of the law of $f \in \mathcal{F}$.

Proof. We already did all the work in the proof of Proposition 4.2.4. In particular we computed the measure $\mu(p, x)$ in (4.2.3). Letting $x = 0$ and multiplying by $\rho_{f(p)}(0)$ gives the result. □

Remark 4.2.12. Notice in this case, and if $\lambda \in \mathbb{R}$, $\lambda \neq 0$, using Proposition 4.2.11 and the fact that $\rho_{\lambda f}(\varphi) = \rho_f(\varphi/\lambda)/|\lambda|^d$ we have

$$\zeta_{\lambda f} = \zeta_f.$$

One can also show that this is true in general but we will not do it here.

We apply this to compute the zonoid section in a particular example.

Proposition 4.2.13. *Let $M = \mathbb{R}^m \setminus \{0\}$ and let $f \in C^1(M, \mathbb{R})$ be given for all $p \in \mathbb{R}^m \setminus \{0\}$ by*

$$f(p) := \langle \xi, p \rangle$$

where $\xi \in \mathbb{R}^m$ is a standard Gaussian vector. Then f is z-KRok *and the zoinoid section is given for all $p \in \mathbb{R}^m \setminus \{0\}$ by*

$$\zeta_f(p) = \frac{1}{2\pi} B(p^\perp) \tag{4.2.10}$$

where recall that $B(p^\perp)$ is the unit ball of $p^\perp$.

Proof. The field is *z-KRok* because it is of $(\mathcal{F}, \infty)$-type with $\mathcal{F} = \mathrm{Span}\{\langle e_1, \cdot\rangle, \ldots, \langle e_m, \cdot\rangle\}$ where $e_1, \ldots, e_m$ is the standard basis of $\mathbb{R}^m$. Indeed, using the basis $\langle e_1, \cdot\rangle, \ldots, \langle e_m, \cdot\rangle$, that we declare to be orthonormal, we identify $\mathcal{F} \cong \mathbb{R}^m$. Then we have for all $y \in \mathcal{F} \cong \mathbb{R}^m$,

$$\rho_f(y) = \frac{1}{(2\pi)^{\frac{m}{2}}} e^{\frac{-\|y\|^2}{2}}$$

and this decays faster than any polynomial at infinity.

Now to compute the zonoid, we will use Proposition 4.2.11. First note that under our identification, we have $\mathcal{F}_p = p^\perp$. Now since $\rho_f|_{p^\perp}$ is even, we get that $o_f(p) = 0$ and thus $\zeta_f(p) = \underline{\zeta_f}(p)$. Now we use Proposition 4.2.11 to find for every $v \in \mathbb{R}^m$:

$$h_{\zeta_f(p)}(v) = \frac{1}{2\sqrt{2\pi}} \int_{p^\perp} |\langle v, y\rangle| \frac{e^{\frac{-\|y\|^2}{2}}}{(2\pi)^{\frac{m-1}{2}}} \mathrm{d}y.$$

We recognize the density of a standard Gaussian vector $\tilde{\xi} \in p^\perp \cong \mathbb{R}^{m-1}$. Remembering that we have $\mathbb{E}\underline{\tilde{\xi}} = B(p^\perp)/\sqrt{2\pi}$, we get that

$$\zeta_f(p) = \frac{1}{\sqrt{2\pi}} \mathbb{E}\underline{\tilde{\xi}} = \frac{1}{2\pi} B(p^\perp)$$

which is what we wanted. □

Example 4.2.14 (GRF and Gaussian zonoid section). Let $f \in C^\infty(M, \mathbb{R})$ be a Gaussian random field (GRF). Then, as mentioned in Example 4.2.6, the random vector $(d_p f | f(p) = 0) \in T_p^* M$ is Gaussian and thus $\underline{\zeta_f}(p)$ is a Gaussian zonoid in the sense of Definition 2.6.2. If it is non degenerate, it follows that there are sections $\gamma \in \Gamma(T^*M)$ and $\Lambda \in \Gamma(\mathrm{Hom}(T^*M, T^*M))$ such that

$$\underline{\zeta_f}(p) = \Lambda(p)\big(G(\gamma(p))\big)$$

where recall (Definition 2.6.3) that $G(\gamma(p)) = \mathbb{E}\underline{\xi + \gamma(p)}$ where $\xi \in T_p^* M$ is a standard Gaussian vector (standard for the Riemannian scalar product). In particular, if $f(p)$ is a centered Gaussian variable then $\gamma(p) = 0$ and $\underline{\zeta_f}(p)$ is an ellipsoid. If we are not centered, we can use Theorem 2.6.7. Let $B_p \subset T_p^* M$ be the unit ball for the Riemannian metric, then we have

$$b_\infty (\Lambda(p) \circ T_{\gamma(p)}) \left(\frac{1}{\sqrt{2\pi}} B_p\right) \subset \underline{\zeta_f}(p) \subset (\Lambda(p) \circ T_{\gamma(p)}) \left(\frac{1}{\sqrt{2\pi}} B_p\right)$$

with $T_{\gamma(p)} \in Gl(T_p^* M)$ and $b_\infty \sim 0.989\ldots$ are defined in Theorem 2.6.7.

Example 4.2.15 (Segment section). Let us consider again the case of Example 4.2.7: let $\varphi \in C^1(M, \mathbb{R}^k)$ be a fixed differentiable function and let $\lambda \in \mathbb{R}$ be a random variable with a continuous density $\rho_\lambda : \mathbb{R} \to \mathbb{R}$ and let $f := \varphi - \lambda$. We computed already the "random" vector (4.2.5) and the density $\rho_{f(p)}$. We obtain then for all $p \in M$:

$$\zeta_f(p) = \rho_\lambda(\varphi(p))[0, \mathrm{d}_p\varphi^1 \wedge \cdots \wedge \mathrm{d}_p\varphi^k].$$

In particular, notice that if φ is constant then the zonoid is $\{0\}$ at each point.

A natural question is then to ask what are exactly the zonoid sections that we can obtain from *z-KRok* fields.

Open problem 15. Characterize the zonoid sections $\underline{\zeta_f} \in \Gamma(\mathscr{G}_0(T^*M))$ that come from *z-KRok* fields from Definition 4.2.9.

Later in Proposition 4.2.21, we will see that we can obtain every ellipsoid section. Moreover, note that at least some of the topological obstructions could be solved by replacing $\mathbb{R}^k$ by a co-oriented vector bundle of rank k and taking random *z-KRok* sections in a similar fashion.

A very partial answer to Open Problem 15 is the following.

Proposition 4.2.16. *Let $M = \mathbb{R}^m$ and let $K \in \mathscr{G}_0(k, T_0^*M) \cong \mathscr{G}_0(k, \mathbb{R}^m)$. Then there exists a* z-KRok *field $f \in C^1(M, \mathbb{R}^k)$ such that $\underline{\zeta_f}(0) = K$.*

Proof. If $K = \{0\}$ then this was done in Example 4.2.15. Else, let $Y_1, \ldots, Y_k \in \mathbb{R}^k$ be such that $K = \underline{Y_1 \wedge \cdots \wedge Y_k}$. We can assume that $Y_1 \wedge \cdots \wedge Y_k \neq 0$ almost surely. Let $\lambda \in \mathbb{R}^k$ be independent of $Y_1, \ldots, Y_k$ and with a continuous bounded density $\rho_\lambda : \mathbb{R}^k \to \mathbb{R}$. We also assume that for all $x \in \mathbb{R}^k$, $\rho_\lambda(x) > 0$. Then let us consider the map $T_Y : \mathbb{R}^m \to \mathbb{R}^k$ given for all $p \in \mathbb{R}^m$ by $T_Y(p) := (\langle Y_1, p\rangle, \ldots, \langle Y_k, p\rangle)$. We define the random field $f \in C^1(\mathbb{R}^m, \mathbb{R}^k)$ by

$$f := T_Y - \lambda$$

where λ is intended as the constant function. We shall prove that this is *z-KRok*. First, note that $J_p f = \|Y_1 \wedge \cdots \wedge Y_k\|$ and we assumed that this is almost surely non zero thus almost surely 0 is a regular value and this gives *z-KRok*-i. For *z-KRok*-ii one finds for all $x \in \mathbb{R}^k$:

$$\rho_{f(p)}(x) = \mathbb{E}\left[\rho_\lambda(T_Y(p) - x)\right]. \tag{4.2.11}$$

Since ρ_λ is bounded, the continuity *z-KRok*-iii follows by dominated convergence. Now for *z-KRok*-iv, one finds that a regular conditional probability for f given $f(p)$ is the measure $\mu(p, x)$ such that for every $\Psi : C^1(\mathbb{R}^m, \mathbb{R}^k)$ continuous and bounded, we have

$$\langle \mu(p,x), \Psi\rangle = \mathbb{E}\left[\Psi(T_Y - T_Y(p) + x)\frac{\rho_\lambda(T_Y(p) - x)}{\rho_{f(p)}(x)}\right]. \tag{4.2.12}$$

Note that because of (4.2.11) and because we assumed $\rho_\lambda > 0$ we get that $\rho_{f(p)} > 0$ and this is well defined. Moreover note that because of (4.2.11) this is indeed a probability measure. To check for finiteness, we can apply this with $\Psi = J_p$. Then, since $J_p(T_Y - T_Y(p) + x) = J_p(T_Y) = \|Y_1 \wedge \cdots \wedge Y_k\|$, we get that the total mass of $J_p \cdot \mu(p, x)$ is

$$\mathbb{E}\left[\|Y_1 \wedge \cdots \wedge Y_k\| \frac{\rho_\lambda(T_Y(p) - x)}{\rho_{f(p)}(x)}\right]$$

and this is finite since $\rho_\lambda(T_Y(p) - x)$ is bounded and $Y_1 \wedge \cdots \wedge Y_k$ is integrable. Again, we get continuity by dominated convergence and this proves that f is *z-KRok*.

Now applying (4.2.12) when $x = 0$ and multiplying by $\rho_{f(p)}(0)$, we get that for every $w \in \Lambda^k\mathbb{R}^m$ we have

$$h_{\underline{\zeta_f}(p)}(w) = \frac{1}{2}\mathbb{E}\left[\rho_\lambda(T_Y(p))\,|\langle w, Y_1 \wedge \cdots \wedge Y_k\rangle|\right]$$

Since $T_Y(0) = 0$ almost surely, if we assume that $\rho_\lambda(0) = 1$ then in $p = 0$ we get $\underline{\zeta_f}(0) = K$ and this concludes the proof. $\square$

The Pull-Back property

We will now show one of the most important properties of the zonoid section and *z-KRok* fields, namely that it satisfies a pull-back property. Recall that if N is a smooth manifold and $\varphi : N \to M$ is a C^1 map then for each $q \in N$ the differential is a map $\mathrm{d}_q\varphi : T_qN \to T_{\varphi(q)}M$ and thus its transpose, denoted $\mathrm{d}_q\varphi^*$ is a map between the cotangent spaces:

$$\mathrm{d}_q\varphi^* : T^*_{\varphi(q)}M \to T^*_qN.$$

It is sometimes called the *pull back* of φ at q. Note that it also induces a map between the exterior algebras that we denote in the same way. Moreover, recall that if $X \subset M$ is a submanifold then we say that φ is *transversal* to X if for every $q \in \varphi^{-1}(X) \subset M$ we have that the sum of $T_{\varphi(q)}X$ and the image of $\mathrm{d}_q\varphi$ span the whole space $T_{\varphi(q)}N$. In particular if φ is an embedding this means that the two submanifolds X and $\varphi(N)$ intersect transversally. Note that this implies that the dimension of M is at least the codimension of X

Lemma 4.2.17. *Let $f \in C^1(M, \mathbb{R}^k)$ be* z-KRok*. Let N be a smooth manifold and let $\varphi\colon N \to M$ be a C^1 map such that φ is transversal to $f^{-1}(0)$ almost surely. Then $f \circ \varphi \in C^1(M, \mathbb{R}^k)$ is* z-KRok *and for all $q \in N$ we have*

$$\zeta_{f\circ\varphi}(q) = d_q\varphi^*\zeta_f(\varphi(q)). \tag{4.2.13}$$

Proof. If we know that $f \circ \varphi$ is *z-KRok*, then (4.2.13) is immediate from the definition. The difficulty is to prove that the random map $f \circ \varphi$ is indeed *z-KRok*. Let us detail them one by one. We assume that N is endowed with any Riemannian metric.

Firstly, under the condition that 0 is a regular value of f, we have that 0 is a regular value of $f \circ \varphi$ if and only if φ is transversal to $f^{-1}(0)$ and this proves *z-KRok*-i.

For $q \in N$, the law of $(f \circ \varphi)(q) \in \mathbb{R}^k$ has density given by $\rho_{(f\circ\varphi)(q)} = \rho_{f(\varphi(q))}$ and this gives *z-KRok*-ii.

Since φ is continuous and ρ_f is continuous at $(p, 0)$, it follows that $\rho_{f\circ\varphi}$ is continuous at $(q, 0)$ for any $q \in N$ which proves *z-KRok*-iii.

Let $\mu(p, x) \in \mathscr{P}(C^1(M, \mathbb{R}^k))$ be the regular conditional probability of f given $f(p)$ given by *z-KRok*-iv for f. Thus the function

$$J \cdot \mu : M \times \mathbb{R}^k \to \mathcal{M}^+\left(C^1(M, \mathbb{R}^k)\right)$$

is continuous at $(p, 0)$. Let $\varphi^* : C^1(M, \mathbb{R}^k) \to C^1(N, \mathbb{R}^k)$ be the function given for all functions $\psi \in C^1(M, \mathbb{R}^k)$ by $\varphi^*(\psi) = \psi \circ \varphi$. This is continuous with respect to the C^1 topologies and we define $\nu(q, x) \in \mathscr{P}(C^1(M, \mathbb{R}^k))$ to be the push-forward of the probability measure $\mu(\varphi(q), x)$ via the function φ^*. So $\nu(q, x)$ is the probability such that for every continuous bounded function $\Psi : C^1(M, \mathbb{R}^k) \to \mathbb{R}$, we have

$$\langle \nu(q, x), \Psi\rangle = \langle \mu(q, x), \Psi \circ \varphi^*\rangle = \mathbb{E}\left[\Psi(\varphi^*(f))|f(\varphi(p)) = x\right]$$

where the second equality is simply the definition of the notation introduced after Definition 4.2.3. From this, one can check that $\nu(q, \cdot)(\cdot)$ is a regular conditional probability of $f \circ \varphi$ given $(f \circ \varphi)(q)$ (see Definition 4.2.1). Indeed for every Borelian $B \subset C^1(M, \mathbb{R}^k)$, we see that

$$\nu(q, x)(B) = P\left((f \circ \varphi) \in B|f(\varphi(p)) = x\right)$$

is a Borel measurable function of $x \in \mathbb{R}^k$ and for any Borelian $V \subset \mathbb{R}^k$, we obtain

$$\begin{aligned} P\left(f \circ \varphi \in B; (f \circ \varphi)(q) \in V\right) &= P\left(f \in \varphi^*(B); f(\varphi(q)) \in V\right) \\ &= \int_{\mathbb{R}^k} \mu(\varphi(q), x)(\varphi^*(B))\rho_{f(\varphi(q))}(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^k} \nu(q, x)(B)\rho_{(f\circ\varphi)(q)}(x)\mathrm{d}x, \end{aligned}$$

where the second inequality holds because $\mu(\varphi(q), x)$ is a regular conditional probability of f given $f(\varphi(q))$ and the third is by definition of $\nu(q, x)$. This proves that $\nu(q, x)$ is a regular conditional probability for $f \circ \varphi$ given $(f \circ \varphi)(q)$.

We now have to show that it satisfies *z-KRok*-iv. For the finiteness condition, observe that the Jacobians satisfy for all $q \in N$:

$$J_q(f \circ \varphi) \leq J_{\varphi(q)} f \cdot J_q \varphi. \tag{4.2.14}$$

It follows that the total mass of $J_q \cdot \nu(q,x)$ is bounded by the total mass of $J_{\varphi(q)} \cdot \mu(\varphi(q), x)$ times $J_q\varphi$ and thus is finite.

It remains to prove the continuity of $J_q \cdot \nu(q,x)$ at $(q,0)$ for all $q \in N$. Let $\Psi : C^1(N, \mathbb{R}^k) \to \mathbb{R}$ be bounded and continuous. Let $(q_j, x_j) \to (q_0, 0)$ be a converging sequence in $N \times \mathbb{R}^k$. Then in our notation, we have

$$\langle J_{q_j} \cdot \nu(q_j, x), \Psi \rangle = \mathbb{E}\left[\Psi(f \circ \varphi)\left(J_{q_j}(f \circ \varphi)\right) | (f \circ \varphi)(q_j) = x_j\right] \tag{4.2.15}$$

Now, since the sequence q_j is contained in a compact subset of N and since $J_q\varphi$ is continuous in q, (because φ is C^1) we can bound $\mathrm{d}_{q_j}\varphi$ uniformly on j and thus (4.2.14) implies that there is a constant $C > 0$ such that for all $j \geq 0$,

$$J_q(f \circ \varphi) \leq C\, J_{\varphi(q)} f. \tag{4.2.16}$$

Thus if we define $\Phi_j : C^1(N, \mathbb{R}^k) \to \mathbb{R}$ for all $\psi \in C^1(N, \mathbb{R}^k)$ by

$$\Phi_j(\psi) := \Psi\left(\psi \circ \varphi\right) J_{q_j}(\psi \circ \varphi).$$

We will want to apply Lemma 4.2.8 to the sequences Φ_j, $p_j := \varphi(q_j)$, x_j. By (4.2.16) and since Ψ is bounded, we have that $\Phi_j(\psi) \leq \tilde{C} J_{p_j}\psi$ for some $\tilde{C} > 0$ where $p_j := \varphi(q_j)$. Moreover (4.2.15) becomes:

$$\mathbb{E}\left[\Psi(f \circ \varphi)\left(J_{q_j}(f \circ \varphi)\right) | (f \circ \varphi)(q_j) = x_j\right] = \mathbb{E}\left[\Phi_j(f) | f(p_j) = x_j\right]$$

The sequence Φ_j converges to the function Φ_0 in the compact-open topology of continuous functions on $C^1(M, \mathbb{R}^k)$. Indeed this is equivalent to say that for every converging sequence $\psi_j \to \psi_0$ in $C^1(M, \mathbb{R}^k)$, $\Phi_j(\psi_j) \to \Phi_0(\psi_0)$. Moreover, in $C^1(M, \mathbb{R}^k)$, $\psi_j \to \psi_0$ if and only if $\mathrm{d}_{p_j}\psi_j \to \mathrm{d}_{p_0}\psi_0$ for every converging sequence $p_j \to p_0$ in M, thus, in particular, $J_{p_j}\psi_j \to J_{p_0}\psi_0$, since $J_p\psi$ depends continuously on $\mathrm{d}_p\psi$. This proves that $\Phi_j \to \Phi_0$. Then, applying Lemma 4.2.8 we get

$$\mathbb{E}\left[\Phi_j(f) | f(p_j) = x_j\right] \to \mathbb{E}\left[\Psi(f \circ \varphi) J_{q_0}(f \circ \varphi) | (f \circ \varphi)(q_0) = 0\right],$$

which proves the continuity in *z-KRok*-iv. Thus we proved that the random field $f \circ \varphi$ is *z-KRok* and this concludes the proof. $\square$

Notice that (4.2.13) also implies that for all $q \in N$:

$$\underline{\zeta_{f \circ \varphi}}(q) = d_q\varphi^* \underline{\zeta_f}(\varphi(q)); \qquad\qquad o_{f \circ \varphi}(q) = d_q\varphi^* o_f(\varphi(q)).$$

Example 4.2.18 (Pre-sheaf property). In the case where $N = U \subset M$ is an open subset of M and $\varphi : U \hookrightarrow M$ is just the inclusion, then $f \circ \varphi = f|_U$ is the restriction of f to the subset U. In that case Lemma 4.2.17 tells us that the restriction of a *z-KRok* field to an open subset is again *z-KRok* and since the pull-back of φ is the identity, the zonoid section of $f|_U$ is just the restriction of the zonoid section of f. In fancy words, one would say that the *z-KRok* fields have the *pre-sheaf property.* It is however not clear if they form a *sheaf.* That is if we have two *z-KRok* fields defined on two open subset on M that "agree" (it is not even clear what this should mean here) on the intersection, can we glue them to make a *z-KRok* field on the union? This difficulty, is one of the reason why Open Problem 15 seems to be a difficult problem.

Example 4.2.19 (Submanifolds). More generally, let N be a submanifold of M and let $\varphi : N \hookrightarrow M$ be the inclusion. Then again $f \circ \varphi = f|_N$ is the restriction to N. Thus if N is almost surely transversal to $f^{-1}(0)$, Lemma 4.2.17 tells us that the restriction $f|_N$ is a *z-KRok* field on N.

This time, for all $p \in N$, the pull-back is the projection

$$\mathrm{d}_p\varphi^* : T_p^*M \to T_p^*N.$$

Thus the zonoid $\zeta_{f|_N}(p)$ is the projection of the zonoid $\zeta_f(p)$ onto the subspace T_p^*N by the map $\mathrm{d}_p\varphi^*$. Note that the submanifold needs not to be embedded and this still works if φ is a (Riemannian) immersion.

We can have some submanifold that are not almost surely transversal to $f^{-1}(0)$. For example, consider the random field $f \in C^1(\mathbb{R}^2, \mathbb{R})$ defined for all $(x, y) \in \mathbb{R}^2$ by

$$f(x,y) = (ax+b)^2 + cy, \tag{4.2.17}$$

for suitable $a, b, c \in \mathbb{R}$ this is *z-KRok* and one can see that $f^{-1}(0)$ is almost surely *not* transversal to the curve $\{y = 0\}$. It is not clear in general, what condition to impose on the random field to avoid these pathological cases.

Corollary 4.2.20. *Let $M = S^m \subset \mathbb{R}^{m+1}$ and let $f \in C^1(S^m, \mathbb{R})$ be given for all $p \in S^m \subset \mathbb{R}^{m+1}$ by*

$$f(p) = \langle \xi, p \rangle$$

where $\xi \in \mathbb{R}^{m+1}$ is a standard Gaussian. Then f is z-KRok *and we have for all $p \in S^m$*

$$\zeta_f(p) = \frac{1}{2\pi} B(T_p^* S^m). \tag{4.2.18}$$

Proof. It is enough to consider S^m as a submanifold of $\mathbb{R}^{m+1} \setminus \{0\}$. Then f is the restriction of the *z-KRok* field defined in Proposition 4.2.13. The zero set of this field is a random uniform hyperplane and thus the sphere is almost surely transversal to it. As we discussed in Example 4.2.19, this implies that $\zeta_f(p)$ is the projection onto $T_p^*S^m$ of the previous zonoid section on $\mathbb{R}^{m+1}$ computed in (4.2.10). But in our case we can identify $T_p^*S^m \cong p^\perp$ and this gives the result. □

From this, one can deduce the general case of any Riemannian manifold, this gives a partial answer to Open Problem 15.

Proposition 4.2.21. *For any Riemannian manifold M, there exists a* z-KRok *field $f \in C^1(M, \mathbb{R})$ such that for all $p \in M$, the zonoid section is given by*

$$\zeta_f(p) = \frac{1}{2\pi} B(T_p^* M) \tag{4.2.19}$$

where $B(T_p^*M)$ is the unit ball of T_p^*M for the Riemannian metric.

Proof. Let us first do it for $\mathbb{R}^m$ with the standard Euclidean metric. Consider the map $\tilde{\varphi} : \mathbb{R} \to \frac{1}{\sqrt{m}} S^1$, that is given for all $t \in \mathbb{R}$ by $\tilde{\varphi}(t) := \frac{1}{\sqrt{m}}(\cos(\sqrt{m}t), \sin(\sqrt{m}t))$. Computing for all $t \in \mathbb{R}$, $\|\mathrm{d}_t\tilde{\varphi}\| = 1$, it follows that $\tilde{\varphi}$ is an isometric immersion. Now consider the map φ that is the direct sum of m copies of $\tilde{\varphi}$:

$$\varphi := \tilde{\varphi} \oplus \cdots \oplus \tilde{\varphi} : \mathbb{R}^m \to \left(\tfrac{1}{\sqrt{m}} S^1\right)^m.$$

As the direct sum of isometric immersion, φ is also an isometric immersion. Now it remains to see that $\left(\frac{1}{\sqrt{m}} S^1\right)^m$ is a submanifold of S^{2m-1}. Thus φ is an isometric immersion of $\mathbb{R}^m$ into S^{2m-1}. Now on the sphere S^{2m-1} we consider the *z-KRok* field $\tilde{f}$ given by Corollary 4.2.20, that is the scalar product with a standard Gaussian vector. As we already observed, $\tilde{f}^{-1}(0)$ is almost surely transversal to any map and in particular to φ. Thus we can use the pull-back property and get that the random field $f := \tilde{f} \circ \varphi$ is *z-KRok* and that the zonoid section ζ_f is the pull-back by φ of the zonoid section of $\zeta_{\tilde{f}}$. Since φ is isometric, the pull-back is an isometry, and the result for the zonoid section follows from (4.2.18) and this concludes the case $M = \mathbb{R}^m$.

Now if M is any manifold we use Nash's isometric embedding Theorem [69] to get an isometric embedding $M \hookrightarrow \mathbb{R}^n$ and proceed as before. □

Of course the $(2\pi)^{-1}$ in (4.2.19) is only cosmetic because we started with any Riemannian metric, but because of the situation on the sphere on Corollary 4.2.20, it seems more natural, see also Example 4.2.25 where this appears naturally.

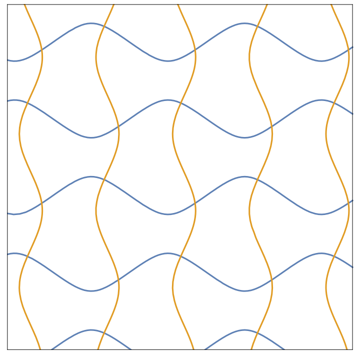


Figure 4.2: The zero sets of two sample of an AT field for $\mathbb{R}^2$ in the square $[-5,5]^2$

Definition 4.2.22. A *z-KRok* random field satisfying (4.2.19), will be called an *Adler-Taylor field*, abbreviated *AT field*, for the Riemannian manifold M.

Remark 4.2.23. Note that from the construction of the AT field in the proof of (4.2.19), we can assume that given any submanifold $S \subset M$ an AT field f for M is almost surely transversal to S. Moreover, by the pull-back property and Example 4.2.19 we see that the restriction of f to S, i.e. the *z-KRok* field $f|_S$ is an AT field for S.

Remark 4.2.24. Following the proof of Proposition 4.2.21, we see that an AT field for $\mathbb{R}^m$ is given for all $p = (p_1, \ldots, p_m) \in \mathbb{R}^m$ by

$$f(p) = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \xi_i \cos(\sqrt{m}\, p_i) + \xi_i' \sin(\sqrt{m}\, p_i)$$

where $\xi_1, \ldots, \xi_m, \xi_1', \ldots, \xi_m' \in \mathbb{R}$ are iid standard Gaussian variables, see Figure 4.2. Note that this is an eigenfunction of the Laplacian of eigenvalue m.

Let's consider the case in Lemma 4.2.17 where $N = M$ and $\varphi : M \to M$ is an isometric diffeomorphism. Then Lemma 4.2.17 tells us that if the field f is invariant by φ, meaning that f and $f \circ \varphi$ have the same law then the zonoid section is also invariant under φ. In the case where we have a Lie group G acting by isometries on M, then if f is invariant under this action, so is the zonoid section ζ_f. With this point of view we could find the cases covered in the previous section.

Example 4.2.25 (Kostlan polynomials). Let $M = S^m$ and let $\mathcal{F}^{(d)}$ be the space of homogeneous polynomials of degree d in $m+1$ variables restricted to the sphere. We choose the basis of $\mathcal{F}^{(d)}$ that is given by the monomials

$$\varphi_\alpha(x) := \sqrt{\binom{d}{\alpha}}\, x^\alpha$$

where $\alpha = (\alpha_0, \ldots, \alpha_m) \in \mathbb{N}^{m+1}$ is a multiindex with $|\alpha| := \alpha_0 + \cdots + \alpha_m = d$ and where recall that $\binom{d}{\alpha} := d!/(\alpha_0! \cdots \alpha_m!)$ and $x^\alpha = x_0^{\alpha_0} \cdots x_m^{\alpha_m}$. We endow the space $\mathcal{F}^{(d)}$ with the scalar product that makes this basis orthonormal. Then, as the reader can check, the action of the orthogonal group $O(m+1)$ on the sphere S^m induces an action on $\mathcal{F}^{(d)}$ that is also orthogonal, i.e. that preserves this scalar product. We define the random field $f_{m,d} \in C^1(S^m, \mathbb{R})$ by

$$f_{m,d} := \sum_{|\alpha|=d} \xi_\alpha \varphi_\alpha$$

where $\xi_\alpha \in \mathbb{R}$ are iid standard Gaussian variables, i.e $f_{m,d}$ is a standard Gaussian vector of $\mathcal{F}^{(d)}$. The random polynomials $f_{m,d}$ are called *Kostlan polynomials*. They are of course *z-KRok* for all $d \geq 1$ since they are of $(\mathcal{F}^{(d)}, \alpha)$-type for any α big enough. Now because of what we said, this random field is invariant by the action of $O(m+1)$ on S^m. Thus, by the pull-back property, as observed above, the zonoid section is also invariant by this action. It follows that at each point they are balls of the same radius. To compute this radius, we place ourselves at the north pole $p_N = (1, 0, \ldots, 0) \in S^m$. Then at this point, the kernel of the evaluation map is given by

$$\mathcal{F}^{(d)}_{p_N} = \left(\varphi_{(d,0,\ldots,0)}\right)^{\perp}$$

where recall $\varphi_{(d,0,\ldots,0)}(x) = x_0^d$. We need now to compute the differential of the elements on this subspace. The only monomials of the basis whose differential at p_N will be non zero are

$$\varphi_j(x) = \sqrt{d}\, x_0^{d-1} x_j$$

where we abused notation in the index indicating j instead of the appropriate multiindex. To compute its differential on the sphere at p_N, one can compute it on $\mathbb{R}^{n+1}$ and kill the component in $\mathrm{d}x_0$. We obtain

$$\mathrm{d}_{p_N}\varphi_j = \sqrt{d}\, \mathrm{d}x_j. \tag{4.2.20}$$

Now, we use (4.2.9). The density of $f_{m,d}$ restricted to $\mathcal{F}^{(d)}_{p_N}$ is the density of a standard Gaussian vector on this space multiplied by $1/\sqrt{2\pi}$. Now remember that the Vitale zonoid of a standard Gaussian vector is the ball of radius $1/\sqrt{2\pi}$. In the light of this and (4.2.20), we obtain at this point and thus at every $p \in S^m$:

$$\zeta_{f_{m,d}}(p) = \frac{\sqrt{d}}{2\pi}\, B\left(T_p^* S^m\right).$$

Note that this implies that the Kostlan polynomials $f_{m,d}$ are an AT field for the standard metric on S^m rescaled by $1/\sqrt{d}$.

Independent intersection and wedge product

Let us give one last important property of the zonoid section. If $f_1 \in C^1(M, \mathbb{R}^k)$ and $f_2 \in C^1(M, \mathbb{R}^l)$ are two *z-KRok* fields, one can build another random field $f := (f_1, f_2) \in C^1(M, \mathbb{R}^{k+l})$, given for all $p \in M$ by $f(p) = (f_1(p), f_2(p))$. Note that in that case, the zero set of f is the intersection of the previous two zero sets: $f^{-1}(0) = f_1^{-1}(0) \cap f_2^{-1}(0)$. In the case where f_1 and f_2 are independent, we show that the zonoid section of the new field is the wedge product of the previous zonoid sections.

Lemma 4.2.26. *Let $f_1 \in C^1(M, \mathbb{R}^k)$ and $f_2 \in C^1(M, \mathbb{R}^l)$ be independent* z-KRok *fields with $k + l \leq m$. Then the random field $f := (f_1, f_2) \in C^1(M, \mathbb{R}^{k+l})$ is* z-KRok *and we have for all $p \in M$*

$$\zeta_f(p) = \zeta_{f_1}(p) \wedge \zeta_{f_2}(p). \tag{4.2.21}$$

Proof. First let us prove that f is *z-KRok*. We proceed, as usual, to prove the *z-KRok* conditions one by one.

By definition, zero is a critical value of f if and only if there exists $p \in M$ such that $f(p) = 0$ and the differential $\mathrm{d}_p f$ is not surjective. Since $f = 0$ if and only if $f_1(p) = 0$ and $f_2(p) = 0$ and since the differential of f is the direct sum of the differential of f_1 and the differential of f_2, this can only happen if zero is a critical value of f_1 or f_2. Since they are *z-KRok* and independent this is of probability zero and this proves *z-KRok*-i.

Since f_1 and f_2 are independent we have that, for all $p \in M$, the law of $f(p) \in \mathbb{R}^{k+l}$ admits the density given for all $x_1 \in \mathbb{R}^k$, $x_2 \in \mathbb{R}^l$ by:

$$\rho_{f(p)}(x_1, x_2) = \rho_{f_1(p)}(x_1)\rho_{f_2(p)}(x_2).$$

This proves that f satisfies *z-KRok*-ii and iii.

As usual, the most delicate property to prove is *z-KRok*-iv. For all $p \in M$ and $x_1 \in \mathbb{R}^k, x_2 \in \mathbb{R}^l$, let $\mu_i(p, x_i)$ be the regular conditional probability of f_i given $f_i(p)$, $i = 1, 2$ given by *z-KRok*-iv. Identifying the spaces $C^1(M, \mathbb{R}^{k+l}) \cong C^1(M, \mathbb{R}^k) \oplus C^1(M, \mathbb{R}^l)$, we define the probability measure:

$$\mu(p, (x_1, x_2)) := \mu_1(p, x_1) \otimes \mu_2(p, x_2). \tag{4.2.22}$$

Then this is a regular conditional probability for f given $f(p)$. Indeed, it is enough to check property (i) in Definition 4.2.1 on products of Borelians, and in this case this follows from the independence of

f_1 and f_2. Now, since $J_p f \leq J_p f_1\, J_p f_2$ we get that the total mass of $J_p \cdot \mu(p,(x_1,x_2))$ is bounded by the product of the total masses of $J_p \cdot \mu(p,x_1)$ and $J_p \cdot \mu_2(p,x_2)$ and thus is finite. To prove continuity, let $\Phi : C^1(M,\mathbb{R}^{k+l}) \to \mathbb{R}$ be continuous and bounded and consider sequences $p_j \to p_0$ in M, $(x_1)_j \to 0$ in $\mathbb{R}^k$ and $(x_2)_j \to 0$ in $\mathbb{R}^l$. Then, if we write $\mu^j := \mu(p_j,((x_1)_j,(x_2)_j))$ and $\mu_i^j := \mu_i(p_j,(x_i)_j)$ for $i = 1,2$, we have for all $j \geq 0$

$$\begin{aligned}\langle J_{p_j} \cdot \mu^j, \Phi\rangle &= \int_{C^1(M,\mathbb{R}^k)} \int_{C^1(M,\mathbb{R}^l)} \Phi(\varphi_1,\varphi_2) J_{p_j}(\varphi_1,\varphi_2)\, \mathrm{d}\mu_2^j(\varphi_2) \mathrm{d}\mu_1^j(\varphi_1)\\ &= \int_{C^1(M,\mathbb{R}^k)} \Psi_j(\varphi_1)\, \mathrm{d}\mu_1^j(\varphi_1)\end{aligned}$$

where

$$\Psi_j(\varphi_1) := \int_{C^1(M,\mathbb{R}^l)} \Phi(\varphi_1,\varphi_2) J_{p_j}(\varphi_1,\varphi_2)\, \mathrm{d}\mu_2^j(\varphi_2).$$

Since Φ is bounded, we have:

$$\Phi(\varphi_1,\varphi_2) J_{p_j}(\varphi_1,\varphi_2) \leq \Phi(\varphi_1,\varphi_2) J_{p_j}\varphi_1 J_{p_j}\varphi_2 \leq C\, J_{p_j}\varphi_2$$

where $C \geq 0$ depends only on φ_1. We apply the technical lemma (Lemma 4.2.8) to get that $\Psi_j \to \Psi_0$. Similarly, we have that

$$\Psi_j(\varphi_1) \leq \left(\int_{C^1(M,\mathbb{R}^l)} \Phi(\varphi_1,\varphi_2) J_{p_j}\varphi_2 \mathrm{d}\mu_2^j(\varphi_2) \right) J_{p_j}\varphi_1$$

Since the expression between parentheses converges and since Φ is bounded we obtain that $\Psi_j(\varphi_1) \leq C J_{p_j}\varphi_1$ for some $C > 0$ and we can again apply Lemma 4.2.8 to get that

$$\langle J_{p_j} \cdot \mu^j, \Phi\rangle = \langle J_{p_j} \cdot \mu_1^j, \Psi_j\rangle \to \langle J_{p_0} \cdot \mu_1^0, \Psi_0\rangle = \langle J_{p_0} \cdot \mu^0, \Phi\rangle$$

which is what we wanted and this finishes to prove that f is *z-KRok*.

It remains only to compute the zonoid. To do that, note that for every $p \in M$, we have:

$$\mathrm{d}_p f^1 \wedge \cdots \wedge \mathrm{d}_p f^{k+l} = (\mathrm{d}_p f_1^1 \wedge \cdots \wedge \mathrm{d}_p f_1^k) \wedge (\mathrm{d}_p f_2^1 \wedge \cdots \wedge \mathrm{d}_p f_2^l).$$

Now because the regular conditional probability of f given $f(p)$ is the product of the previous ones (see (4.2.22)) we obtain that

$$(\mathrm{d}_p f^1 \wedge \cdots \wedge \mathrm{d}_p f^{k+l} | f(p) = 0) = (\mathrm{d}_p f_1^1 \wedge \cdots \wedge \mathrm{d}_p f_1^k | f_1(p) = 0) \wedge (\mathrm{d}_p f_2^1 \wedge \cdots \wedge \mathrm{d}_p f_2^l | f_2(p) = 0)$$

where the two random vectors on the right hand side are independent. Multiplying both sides by $\rho_{f(p)}(0) = \rho_{f_1(p)}(0)\rho_{f_2(p)}(0)$ and taking the associated zonoid gives the result. □

Note again that (4.2.21) also implies that for all $p \in M$ we have

$$\underline{\zeta_f}(p) = \underline{\zeta_{f_1}}(p) \wedge \underline{\zeta_{f_2}}(p); \qquad\qquad o_f(p) = o_{f_1}(p) \wedge o_{f_2}(p).$$

4.2.3 The Alpha formula and density of volume

We now proceed to use the zonoid section we defined to compute expected volumes of the zero set of *z-KRok* fields. We will use the following version of Kac-Rice formula to deduce all our results. This is obtained as a particular case of [82]. The only differences with more standard statements of Kac-Rice formula are in the hypotheses, in particular the statement below is almost identical to [15, Theorem 6.7]. If $f \in C^1(M,\mathbb{R}^m)$ is *z-KRok* (where recall $m = \dim(M)$), then, in the language of [82, Definition 2.1], the pair $(f,\{0\})$ is a *KROK* couple. Thus we get the following which is [82, Theorem 4.1].

Lemma 4.2.27 (α-Kac-Rice formula)**.** *Let* $f \in C^1(M, \mathbb{R}^m)$ *be a* z-KRok *field. Let*

$$\alpha : C^1(M, \mathbb{R}^m) \times M \to \mathbb{R}$$

be a Borel measurable function. Then

$$\mathbb{E}\left[\sum_{p \in f^{-1}(0)} \alpha(f, p)\right] = \int_M \delta_f^\alpha(p)\,\mathrm{d}M(p).$$

Where $\mathrm{d}M(p)$ *indicates integration with respect to the Riemannian volume form on* M *and where*

$$\delta_f^\alpha(p) := \rho_{f(p)}(0)\,\mathbb{E}\left[\alpha(f,p) J_p f \middle| f(p) = 0\right].$$

The name "Kac-Rice formula" is often used to denote also a more general version which allows to deal with the case in which $f^{-1}(0)$ is not zero dimensional, see [15, Theorem 6.8]. The additional flexibility provided by Lemma 4.2.29 below is crucial for us, since we want to be able to compute average volume (and other quantities) of random submanifolds $f^{-1}(0)$ of arbitrary codimension.

Let us prove a lemma that will be a key point in this reduction with AT fields.

Lemma 4.2.28. *Let* $g_1, \dots, g_m \in C^1(M, \mathbb{R})$ *be i.i.d. AT fields for* M *and define the random discrete subset* $\Sigma := g_1^{-1}(0) \cap \dots \cap g_m^{-1}(0)$*. Let* $\alpha : M \to \mathbb{R}$ *be Borel with compact support. Then we have*

$$\int_M \alpha(p)\,\mathrm{d}M(p) = \frac{s_m}{2}\,\mathbb{E}\left[\sum_{p \in \Sigma} \alpha(p)\right]$$

where recall that s_m *is the volume of the unit sphere* $S^m \subset \mathbb{R}^{m+1}$*.*

Proof. We apply the α-Kac-Rice Formula (Lemma 4.2.27) to $f = (g_1, \dots, g_m) \in C^1(M, \mathbb{R}^m)$, that is *z-KRok* because of Lemma 4.2.26, and with a function $\alpha(f, p) = \alpha(p)$ that only depends on p. We obtain:

$$\mathbb{E}\left[\sum_{p \in \Sigma} \alpha(p)\right] = \int_M \rho_{f(p)}(0)\,\mathbb{E}\left[J_p f \middle| f(p) = 0\right]\,\alpha(p)\,\mathrm{d}M(p).$$

As observed in (4.2.8) we have

$$\rho_{f(p)}(0)\mathbb{E}\left[J_p f \middle| f(p) = 0\right] = \ell\left(\zeta_f(p)\right) = \ell\left(\zeta_{g_1}(p) \wedge \dots \wedge \zeta_{g_m}(p)\right)$$

where the second equality holds because of the formula for the zonoid section of independant *z-KRok* fields computed in Lemma 4.2.26. Since g_i are AT fields for M, we have for all $p \in M$, $\zeta_{g_1}(p) = (2\pi)^{-1} B(T_p^* M)$. Remembering the basic zonoid calculus fact that the length of the wedge product of m zonoids is $m!$ times their mixed volume (see (2.2.4)) we obtain:

$$\rho_{f(p)}(0)\mathbb{E}\left[J_p f \middle| f(p) = 0\right] = \frac{m! \kappa_m}{(2\pi)^m}.$$

Using the identity $m! \kappa_m = 2(2\pi)^m / s_m$ (see Lemma A.4) this gives what we wanted. □

Lemma 4.2.29 (Alpha Formula)**.** *Let* $1 \le k \le m$*, let* $f \in C^1(M, \mathbb{R}^k)$ *be a* z-KRok *random field and define the random submanifold* $X := f^{-1}(0)$*. Let* $\alpha : C^1(M, \mathbb{R}^k) \times M \to \mathbb{R}$ *be a Borel measurable function. Then*

$$\mathbb{E}\left[\int_X \alpha(f, p)\,\mathrm{d}X(p)\right] = \int_M \delta_f^\alpha(p)\,\mathrm{d}M(p) \tag{4.2.23}$$

where $\mathrm{d}N(p)$ *denotes integration with respect to the Riemannian volume form on* N *and where*

$$\delta_f^\alpha(p) := \rho_{f(p)}(0)\,\mathbb{E}\left[\alpha(f,p) J_p f \middle| f(p) = 0\right]. \tag{4.2.24}$$

Proof. We are going to prove that, with little work, this is a natural consequence of Lemma 4.2.27. This method of proof shows how it's always possible to reduce everything to the zero dimensional case, thanks to the construction, by Adler and Taylor [2], of Gaussian fields that represent the Riemannian structure, our AT fields defined earlier (Definition 4.2.22).

Let $g_1, \dots, g_{m-k} \in C^1(M, \mathbb{R})$ be iid *z-KRok* fields independent of f that are also AT fields for M. Let us define $F := (f, g_1, \dots, g_{m-k}) \in C^1(M, \mathbb{R}^m)$. By Lemma 4.2.26, we have that F is *z-KRok*. Let $X := f^{-1}(0)$ and $Y := g_1^{-1}(0) \cap \dots \cap g_{m-k}^{-1}(0)$. We start with the left hand side of (4.2.23). Since f is independent of the g_i, we can integrate first in g_i pretending f is fixed. As we already observed, the restriction of an AT field to a submanifold is an AT field (see Remark 4.2.23), we can thus apply Lemma 4.2.28 to X to deduce the following:

$$\mathbb{E}\left[\int_X \alpha(f,p)\,\mathrm{d}X(p)\right] = \frac{s_{m-k}}{2}\mathbb{E}\left[\sum_{p \in Y \cap X} \alpha(f,p)\right]$$

Now we apply Lemma 4.2.27 to the right hand side of this equation with the function $\alpha(F,p) := \alpha(f,p)$ that depends only on the first factor of $F = (f, g_1, \dots, g_{m-k})$ to find:

$$\mathbb{E}\left[\int_X \alpha(f,p)\,\mathrm{d}X(p)\right] = \frac{s_{m-k}}{2}\int_M \delta_F^\alpha(p) dM(p).$$

It remains only to show that $\frac{s_{m-k}}{2}\delta_F^\alpha = \delta_f^\alpha$. In order to do that, first observe that, since f and $(g_1, \dots, g_{m-k})$ are independent, we have for all $p \in M$:

$$\rho_{F(p)}(0) = \rho_f(p)(0)\,\rho_{(g_1,\dots,g_{m-k})(p)}(0).$$

Moreover, recall that in the proof of the formula for independent intersection (Lemma 4.2.26) we proved that the regular conditional probability of the direct sum of two independent *z-KRok* fields is the product of the two regular conditional probabilities, see (4.2.22). In probabilistic notation, this means that we can use Fubini's Theorem to write for all $p \in M$:

$$\begin{aligned}\delta_F^\alpha(p) &= \rho_{F(p)}(0)\,\mathbb{E}\left[\alpha(f,p)J_pF | F(p) = 0\right] \\ &= \rho_f(p)(0)\,\rho_{g(p)}(0)\mathbb{E}\left[\alpha(f,p)\mathbb{E}\left[J_p(f,g)|g(p)=0\right]\middle| f(p)=0\right]\end{aligned} \tag{4.2.25}$$

where we wrote $g := (g_1, \dots, g_{m-k})$. By definition, we have

$$J_p(f,g) = \|\mathrm{d}_p f^1 \wedge \dots \wedge \mathrm{d}_p f^k \wedge \mathrm{d}_p g_1 \wedge \dots \wedge \mathrm{d}_p g_{m-k}\|.$$

Moreover, we have $\rho_{g(p)}(0) = \rho_{g_1(p)}(0) \cdots \rho_{g_{m-k}(p)}(0)$. Let us write $b_i := \rho_{g_i(p)}(0)(\mathrm{d}_p g_i | g_i(p) = 0)$. Using the independence of the g_i we find

$$\rho_{g(p)}(0)\mathbb{E}\left[J_p(f,g)|g(p)=0\right] = \mathbb{E}\|\mathrm{d}_p f^1 \wedge \dots \wedge \mathrm{d}_p f^k \wedge b_1 \wedge \dots \wedge b_{m-k}\| \tag{4.2.26}$$

Now g_i are AT fields so, by definition, we have $\mathbb{E}\underline{b_i} = B(T_p^*M)/(2\pi)$. This allows us to rewrite (4.2.26) as follows.

$$\rho_{g(p)}(0)\mathbb{E}\left[J_p(f,g)|g(p)=0\right] = \frac{1}{(2\pi)^{m-k}}\ell\left(\underline{\mathrm{d}_p f^1 \wedge \dots \wedge \mathrm{d}_p f^k} \wedge (B_m)^{\wedge(m-k)}\right)$$

where, abusing notation, we wrote B_m for the unit ball $B(T_p^*M)$. Now we can compute this using basic zonoid calculus and in particular our favourite lemma: Lemma 2.2.17, to get:

$$\rho_{g(p)}(0)\mathbb{E}\left[J_p(f,g)|g(p)=0\right] = \frac{(m-k)!\kappa_{m-k}}{(2\pi)^{m-k}}\ell\left(\underline{\mathrm{d}_p f^1 \wedge \dots \wedge \mathrm{d}_p f^k}\right) = \frac{(m-k)!\kappa_{m-k}}{(2\pi)^{m-k}}J_p f.$$

Using the identity $(m-k)!\kappa_{m-k}s_{m-k} = 2(2\pi)^{m-k}$ (see Lemma A.4) and reintroducing in (4.2.25) gives what we want.

□

Expected volume density

Of course we would like to relate the alpha formula to the zonoid section defined earlier. The first case and the one we will use the most is the following.

Theorem 4.2.30 (M–Stecconi)**.** *Let $f \in C^1(M, \mathbb{R}^k)$ be* z-KRok *and consider the random submanifold of codimension k given by $X := f^{-1}(0)$. Then for all open set $U \subset M$ we have*

$$\mathbb{E}\left[\mathrm{vol}_{m-k}(X \cap U)\right] = \int_U \ell(\zeta_f(p))\, \mathrm{d}M(p)$$

where $\zeta_f(p)$ is the zonoid section defined in Definition 4.2.9.

Proof. This is a direct application of the alpha formula Lemma 4.2.29 with $\alpha(f,p) = 1_U(p)$ where 1_U is the indicator function of the subset U. ☐

A way of seeing this result is to say that the measure on M given by $U \mapsto \mathbb{E}\left[\mathrm{vol}_{m-k}(X \cap U)\right]$ admits a density and this density is $p \mapsto \ell(\zeta_f(p))$. Applying this to the random intersections of *z-KRok* fields (see Lemma 4.2.26) we get the following.

Corollary 4.2.31. *Let $f_1 \in C^1(M, \mathbb{R}^{c_1}), \dots, f_k \in C^1(M, \mathbb{R}^{c_k})$ be independent* z-KRok *fields such that we have $c := c_1 + \cdots + c_k \leq m$ and consider the random submanifolds $X_i := f_i^{-1}(0)$, $i = 1, \dots, k$. Then for any open subset $U \subset M$ we have*

$$\mathbb{E}\left[\mathrm{vol}_{m-c}(X_1 \cap \cdots \cap X_k \cap U)\right] = \int_U \ell(\zeta_{f_1}(p) \wedge \cdots \wedge \zeta_{f_k}(p))\, \mathrm{d}M(p).$$

Specializing again to the case where all the codimensions are one we get the following.

Corollary 4.2.32. *Let $f_1 \in C^1(M, \mathbb{R}), \dots, f_k \in C^1(M, \mathbb{R})$ be independent* z-KRok *fields and consider the random hypersurfaces $X_i := f_i^{-1}(0)$, $i = 1, \dots, k$. Then for any open subset $U \subset M$ we have*

$$\mathbb{E}\left[\mathrm{vol}_{m-k}(X_1 \cap \cdots \cap X_k \cap U)\right] = \frac{m!}{(m-k)!\kappa_{m-k}} \int_U \mathrm{MV}\left(\zeta_{f_1}(p), \dots, \zeta_{f_k}(p), B(T_p^*M)[m-k]\right)\, \mathrm{d}M(p)$$

where MV *denotes the mixed volume (see Section 1.1.2) and $B(T_p^*M)[m-k]$ denotes that the convex body $B(T_p^*M)$ is repeated $m-k$ times in the argument. In particular, when $k = m$ we get*

$$\mathbb{E}\left[\#(X_1 \cap \cdots \cap X_m \cap U)\right] = m! \int_U \mathrm{MV}\left(\zeta_{f_1}(p), \dots, \zeta_{f_m}(p)\right)\, \mathrm{d}M(p). \tag{4.2.27}$$

Proof. Once again, we apply our useful lemma, Lemma 2.2.17 to find that

$$\ell(\zeta_{f_1}(p) \wedge \cdots \wedge \zeta_{f_k}(p)) = \frac{1}{(m-k)!\kappa_{m-k}} \ell(\zeta_{f_1}(p) \wedge \cdots \wedge \zeta_{f_k}(p) \wedge B_m^{\wedge m-k})$$

where, abusing notation, we wrote $B_m := B(T_p^*M)$. Then the result follows from the previous one (Corollary 4.2.31) and the fact that the wedge product of m zonoids is equal to $m!$ times their mixed volume (see (2.2.4)). ☐

Now the Alexandrov-Fenchel inequality can be interpreted in terms of random intersection (see (2.2.8)). We call the following the Kac-Rice Alexandrov-Fenchel inequality, abbreviated KRAF.

Theorem 4.2.33 (KRAF,M–Stecconi)**.** *Let $g_1, \dots, g_{m-2}, f_1, f_1', f_2, f_2' \in C^1(M, \mathbb{R})$ be independent* z-KRok *fields, such that f_1' is distributed as f_1 and f_2' is distributed as f_2. Let $Y := (g_1)^{-1}(0) \cap \ldots \cap (g_{m-2})^{-1}(0)$, $X_i := (f_i)^{-1}(0)$ and $X_i' := (f_i')^{-1}(0)$, $i = 1, 2$. Then we have for all open subset $U \subset M$:*

$$\mathbb{E}\left[\#(X_1 \cap X_2 \cap Y \cap U)\right] \geq \int_U \sqrt{\delta_{\# X_1 \cap X_1' \cap Y}(p) \cdot \delta_{\# X_2 \cap X_2' \cap Y}(p)}\, \mathrm{d}M(p)$$

where for $i = 1, 2$ we wrote

$$\delta_{\# X_i \cap X_i' \cap Y}(p) := \ell(\zeta_{f_i}(p) \wedge \zeta_{f_i'}(p) \wedge \zeta_{g_1}(p) \wedge \cdots \wedge \zeta_{g_{m-2}}(p)).$$

Remark 4.2.34. Note that a positive answer to Conjecture 5 would mean that we can replace the random field $(g_1, \dots, g_{m-2})$ by any *z-KRok* field $g \in C^1(M, \mathbb{R}^{m-2})$ independent of the other random fields.

Current of expected integration

We are now going to show that the differential form $o_f \in \Gamma(\Lambda^k T_p^* M)$, defined in Definition 4.2.9, computes the average of integration of differential forms. If M is orientable, given a differentiable form $\eta \in \Gamma(\Lambda^d T^* M)$ of degree d and an oriented submanifold $S \subset M$ of dimension d, one can integrate η on S. Formally, if $\iota : S \hookrightarrow M$ is the inclusion, then the pull-back $\mathrm{d}\iota^* \eta \in \Gamma(\lambda^d S)$ defines a top degree differential form that can be integrated on S (we insist that S and M are assumed oriented). We denote

$$\eta|_S := \mathrm{d}\iota^* \eta.$$

In local coordinates, if for $q \in S$, we have $\nu_1(q), \ldots, \nu_{m-d}(q)$ an orthonormal basis of $N_q S \subset T_q^* M$ then the integral is given by

$$\int_S \eta|_S = \int_S \nu_1(q) \wedge \cdots \wedge \nu_c(q) \wedge \eta(q) \, \mathrm{d}S(q) \tag{4.2.28}$$

where, using the orientation, the map $q \mapsto \nu_1(q) \wedge \cdots \wedge \nu_{m-d}(q) \wedge \eta(q) \in \Lambda^m T_q^* M \cong \mathbb{R}$ is considered as a real valued function.

We are now ready to give a precise statement.

Theorem 4.2.35 (M–Stecconi). *Assume that M is orientable. Let $1 \le k \le m$, let $f \in C^1(M, \mathbb{R}^k)$ be a* z-KRok *field, let $X := f^{-1}(0)$ and let $\eta \in \Gamma(\Lambda^{m-k} T_p^* M)$ be a continuous differential form of degree $m-k$. Then we have*

$$\mathbb{E}\left[\int_X \eta|_X\right] = \int_M o_f \wedge \eta.$$

Proof. We will apply the Alpha formula Lemma 4.2.29 with $\alpha_\eta : C^1(M, \mathbb{R}^k) \times M \to \mathbb{R}$ given by 0 if $J_p\varphi = 0$ and else by

$$\alpha_\eta : (\varphi, p) \mapsto \frac{1}{J_p \varphi} \mathrm{d}_p \varphi^1 \wedge \cdots \wedge \mathrm{d}_p \varphi^k \wedge \eta$$

where we used the orientation to identify $\Lambda^m T_p^* M \cong \mathbb{R}$. Note that α_η is uniformly bounded in φ, indeed for all $(\varphi, p) \in C^1(M, \mathbb{R}^k) \times M$, we have $|\alpha_\eta(\varphi, p)| \le \|\eta(p)\|$. In the left hand side of the Alpha formula (4.2.23), we get

$$\mathbb{E}\left[\int_X \alpha(f, p) \, \mathrm{d}X(p)\right] = \mathbb{E}\left[\int_X \nu(p) \wedge \eta \, \mathrm{d}X(p)\right]$$

where $\nu(p) := \frac{1}{J_p \varphi} \mathrm{d}_p \varphi^1 \wedge \cdots \wedge \mathrm{d}_p \varphi^k$. Now it is enough to see that $\nu(p) \in \Lambda^k T_p^* M$ represents the subspace $N_p X \subset T_p^* M$ in Plücker coordinates and thus applying (4.2.28) we get:

$$\mathbb{E}\left[\int_X \alpha(f, p) \, \mathrm{d}X(p)\right] = \mathbb{E}\left[\int_X \eta|_X\right].$$

Now it remains to compute the density $\delta_f^{\alpha_\eta}$. By definition (see (4.2.24)) this is given for all $p \in M$ by:

$$\begin{aligned}
\delta_f^{\alpha_\eta}(p) &= \rho_f(0) \mathbb{E}\left[\alpha(f, p) J_p f \mid f(p) = 0\right] \\
&= \rho_f(0) \mathbb{E}\left[\mathrm{d}_p \varphi^1 \wedge \cdots \wedge \mathrm{d}_p \varphi^k \wedge \eta \mid f(p) = 0\right] \\
&= \rho_f(0) \mathbb{E}\left[\mathrm{d}_p \varphi^1 \wedge \cdots \wedge \mathrm{d}_p \varphi^k \mid f(p) = 0\right] \wedge \eta
\end{aligned}$$

where on the third equality we used the linearity of the map $\nu \mapsto \nu \wedge \eta$. Now we recognize in the last line $o_f(p) \wedge \eta$ and this concludes the proof. □

This formula when $k = m$ allows to do an expected sign count of points of intersection. The following is a direct consequence of the previous result.

Corollary 4.2.36. *Let M be oriented and let $f_1, \dots, f_m \in C^1(M, \mathbb{R})$ be independent* z-KRok *fields and let $X_i := f_i^{-1}(0)$, $i = 1, \dots, m$. Then for all open set $U \subset M$, we have*

$$\mathbb{E}\left[\#_\pm(X_1 \cap \dots \cap X_m \cap U)\right] = \int_U o_{f_1} \wedge \dots \wedge o_{f_m}$$

where $\#_\pm(X_1 \cap \dots \cap X_m \cap U)$ is the number of points of intersections in U counted with sign, i.e.

$$\#_\pm(X_1 \cap \dots \cap X_m \cap U) = \sum_{p \in X_1 \cap \dots \cap X_m \cap U} \mathrm{sign}(\mathrm{d}_p f_1 \wedge \dots \wedge \mathrm{d}_p f_m).$$

Note that under the hypotheses of Corollary 4.2.36, for all $p \in M$, the zonoid $\zeta_{f_1}(p) \wedge \dots \wedge \zeta_{f_m}(p)$ is a zonoid in $\Lambda^m T_p^* M \cong \mathbb{R}$ and thus a segment in $\mathbb{R}$. This segment contains the information of both the count of points of intersection with or without sign. Indeed, its length gives the count without sign by Theorem 4.2.30 and its center is $\frac{1}{2} o_{f_1} \wedge \dots \wedge o_{f_m}$ which, as we just showed, computes the count with sign.

What does the zonoid section know?

We have seen two cases of the Alpha formula (Lemma 4.2.29) where the density δ_f^α was a function of the zonoid section ζ_f. We can ask what are the conditions on the function α for this to be the case.

Proposition 4.2.37. *Let $\alpha : C^1(M, \mathbb{R}^k) \times M \to \mathbb{R}$ be a measurable function that is given for every $(\varphi, p) \in C^1(M, \mathbb{R}^k) \times M$ by 0 if $J_p\varphi = 0$ and else by:*

$$\alpha(\varphi, p) = (J_p\varphi)^{-1} T(\mathrm{d}_p\varphi^1 \wedge \dots \wedge \mathrm{d}_p\varphi^k) + (J_p\varphi)^{-1} F(\mathrm{d}_p\varphi^1 \wedge \dots \wedge \mathrm{d}_p\varphi^k)$$

where $T : \Lambda^k T^* M \to \mathbb{R}$ is linear on the fibers and $F : \Lambda^k T^* M \to \mathbb{R}$ is positively homogeneous on the fibers. Then for every z-KRok *field $f \in C^1(M, \mathbb{R}^k)$ and every $p \in M$, the density $\delta_f^\alpha(p)$ is a function of the zonoid $\zeta_f(p)$.*

Proof. Let $f \in C^1(M, \mathbb{R}^k)$ be *z-KRok* and let $p \in M$. By definition, see (4.2.24), the density is given by

$$\delta_f^\alpha(p) = \rho_{f(p)}(0)\mathbb{E}\left[T\left(\mathrm{d}_p f^1 \wedge \dots \wedge \mathrm{d}_p f^k\right) | f(p) = 0\right] + \rho_{f(p)}(0)\mathbb{E}\left[F\left(\mathrm{d}_p f^1 \wedge \dots \wedge \mathrm{d}_p f^k\right) | f(p) = 0\right].$$

The first summand gives

$$\rho_{f(p)}(0)\mathbb{E}\left[T(\mathrm{d}_p f^1 \wedge \dots \wedge \mathrm{d}_p f^k) | f(p) = 0\right] = T\left(\rho_{f(p)}(0)\mathbb{E}\left[\mathrm{d}_p f^1 \wedge \dots \wedge \mathrm{d}_p f^k | f(p) = 0\right]\right) = T(o_f(p)).$$

For the second term, if we call $Y := \rho_{f(p)}(0)(\mathrm{d}_p f^1 \wedge \dots \wedge \mathrm{d}_p f^k | f(p) = 0)$ then we have tautologically

$$\rho_{f(p)}(0)\mathbb{E}\left[F(\mathrm{d}_p f^1 \wedge \dots \wedge \mathrm{d}_p f^k) | f(p) = 0\right] = \mathbb{E}\left[F(Y)\right]$$

But since F is positively homogeneous, by Proposition 1.2.29, this does not depend on the random vector Y but this is a function of the zonoid $\mathbb{E}\underline{Y} = \underline{\zeta_f}(p)$ and this concludes the proof. □

The current of expected integration treated in Theorem 4.2.35 is the case where $F \equiv 0$ and T is the map given for each $p \in M$ and for each $\beta \in \Lambda^k T_p^* M$ by $T(\beta) = \beta \wedge \eta(p)$. The case of density of volume covered in Theorem 4.2.30 is the case where $T \equiv 0$ and F is the norm (given by the Riemannian structure).

In the latter, we actually need less than the zonoid section and we can reduce to *valuations*. For each point p, recall that the space $val^+(T_p M)$ denotes the translation invariant continuous valuations on convex bodies of $T_p M$, see Section 3.1. Recall that to each Grassmannian zonoids $\zeta \in \mathscr{G}(k, T_p^* M)$ (equivalentely to each measure on the Grassmannian $G_k(T_p^* M)$) we associate a valuation in $val^+(T_p M)$ and that this map is an algebra map, see Section 3.3. Thus for each *z-KRok* field $f \in C^1(M, \mathbb{R}^k)$, the zonoid section $\zeta_f \in \Gamma(\widehat{\mathscr{G}}(k, T^* M))$ gives rise to a section $\phi_f \in \Gamma(val_k^+(TM))$ where $val_k^+(TM)$ is the (infinite dimensional) vector bundle whose fiber at the point $p \in M$ is $val_k^+(T_p M)$. Now since length of wedge product of Grassmannian zonoids can be evaluated in the algebra of valuations (see Lemma 3.3.16), we can argue as in Proposition 4.1.15 to obtain the following.

Theorem 4.2.38 (M–Stecconi)**.** *Let $f \in C^1(M, \mathbb{R}^k)$ be* z-KRok*, let $X := f^{-1}(0)$ and let $\phi_f \in \Gamma(val_k^+(TM))$ be the valuation section associated to the zonoid section $\underline{\zeta_f}$ as explained just above. Then for every open set $U \subset M$, we have*

$$\mathbb{E}\left[\mathrm{vol}_{m-k}\left(X \cap U\right)\right] = \frac{1}{\kappa_k} \int_U \phi_f(p)(B(T_pM))\, \mathrm{d}M(p).$$

Moreover, let $f_1 \in C^1(M, \mathbb{R}^{c_1}), \dots, f_n \in C^1(M, \mathbb{R}^{c_n})$ be independent z-KRok *fields that satisfy $c := \sum_{i=1}^n c_i \leq m$, let $X_i := f_i^{-1}(0)$ and let ϕ_{f_i} be the valuation section associated to the zonoid section $\underline{\zeta_{f_i}}$, $i = 1, \dots, n$, then for every open set $U \subset M$, we have:*

$$\mathbb{E}\left[\mathrm{vol}_{m-c}\left(X_1 \cap \cdots \cap X_n \cap U\right)\right] = \frac{1}{\kappa_c} \int_U \left(\phi_{f_1}(p) \wedge \cdots \wedge \phi_{f_n}(p)\right)(B(T_pM))\, \mathrm{d}M(p).$$

Similarly as we did in the end of Section 4.1.2, we can relate this to the theory of *valuations on manifolds*, see [6]. Indeed, Alesker showed in [6, Section 3.1] that the space of smooth valuations on a manifold M admits a filtration whose corresponding graded algebra is the vector bundle of valuations on TM, i.e. the vector bundle whose fiber at a point $p \in M$ are the smooth valuations on the convex bodies of T_pM and this construction is compatible with the various products of smooth valuations defined. It is clear that the link between the theory presented here and the theory of valuations on manifold runs deeper than this and understanding it better will be the subject of future works of the author.

4.2.4 Crofton formula in Finsler geometry

A *Finsler* structure on a manifold M is the choice, for every point $p \in M$, of a norm F_p on each tangent space T_pM that depends continuously on p and such that the unit ball in T_pM is curved for all p. This gives a well defined notion of length of curves. Indeed, given $\gamma : [0,1] \to M$ a smooth curve, one defines

$$\ell^F(\gamma) := \int_0^1 F_{\gamma(t)}(\dot{\gamma}(t))\mathrm{d}t.$$

In our case, the choice of a full dimensional convex body in each cotangent space induces a norm on the tangent space. Indeed if $\zeta(p) \subset T_p^*M$ is a convex body containing the origin in its interior then $h_{\zeta(p)} : T_pM \to \mathbb{R}$ defines a norm (not necessarily symmetric). In our case the convex body can be taken centrally symmetric (by taking the centered the zonoid section, $\underline{\zeta_f}$) but it is not always full dimensional, let alone curved. In the case where it is not full dimensional it only defines a *semi norm.*Note that Finsler structure such that the dual of the unit balls are zonoids are sometimes called *hypermetric*.

Definition 4.2.39. We call *semi Finsler* structure a collection of semi norm $F_p : T_pM \to \mathbb{R}$ such that there exists $\zeta(p) \subset T_p^*M$ a convex body containing the origin depending continuously on $p \in M$ such that $F_p = h_{\zeta(p)}$.

Remark 4.2.40. The convex body $\zeta(p) \subset T_p^*M$ is contained in an hyperplane $v^\perp$, with $v \in T_pM$ if and only if $h_{\zeta(p)}(v) = 0$. For the semi Finsler structure, it means that travelling along the direction v is *free* and curves that passes at p tangent to v have locally length zero. In the case where the zonoid is $\zeta(p) = \underline{\zeta_f}(p)$, the centered zonoid section of a *z-KRok* field, this happens if and only if $(\mathrm{d}_p f | f(p) = 0)$ is in $v^\perp$ almost surely.

The (centered) zonoid section of a *z-KRok* field with value in $\mathbb{R}$ defines a semi Finsler structure.

Definition 4.2.41. Let $f \in C^1(M, \mathbb{R})$ be a *z-KRok* field and condsider the random hypersurface given by $X := f^{-1}(0)$. We denote by F^X the semi Finsler structure induced by $\underline{\zeta_f}(\cdot)$ that is the centered zonoid section of f defined in Definition 4.2.9. More explicitely, this is defined for all $p \in M$ and all $v \in T_pM$ by

$$F_p^X(v) := \frac{\rho_{f(p)}(0)}{2}\, \mathbb{E}\left[|\mathrm{d}_p f(v)| \,\middle|\, f(p) = 0 \right]. \tag{4.2.29}$$

Remark 4.2.42. Note that we abused notation and wrote F^X while a priori this could depend on the choice of f.

Our previous results interpret in this context as follows. This is to be compared with the classical Crofton formula (4.1.8).

Theorem 4.2.43 (Crofton formula in Finsler geometry, M–Stecconi)**.** *Let* $f \in C^1(M, \mathbb{R})$ *be* z-KRok *and consider the random hypersurface* $X := f^{-1}(0)$*. Let* $\gamma : [0,1] \to \mathbb{R}$ *be a* C^1 *curve such that* X *is transversal to* γ *almost surely. Then*

$$\mathbb{E}\#(\gamma \cap X) = 2\,\ell^{F^X}(\gamma).$$

Proof. Consider the random field $f \circ \gamma \in C^1([0,1], \mathbb{R})$ and apply the pull-back property Lemma 4.2.17. By (4.2.13), we have

$$h_{\zeta_{f\circ\gamma}(t)}(\partial_t) = h_{\zeta_{f(\gamma(t))}}(\dot\gamma(t)).$$

Remember that $\zeta_{f\circ\gamma}(t)$ lives in a space of dimension 1 (formally the cotangent to $[0,1]$ at t) and thus its length is given by

$$\begin{aligned}\ell(\zeta_{f\circ\gamma}(t)) &= h_{\zeta_{f\circ\gamma}(t)}(\partial_t) + h_{\zeta_{f\circ\gamma}(t)}(-\partial_t)\\ &= h_{\zeta_{f(\gamma(t))}}(\dot\gamma(t)) + h_{\zeta_{f(\gamma(t))}}(-\dot\gamma(t))\\ &= 2h_{\underline{\zeta_{f(\gamma(t))}}}(\dot\gamma(t)) = 2F^X(\dot\gamma(t)).\end{aligned}$$

Applying Theorem 4.2.30 to the random *z-KRok* field $f \circ \gamma$, we obtain

$$\mathbb{E}\#(f\circ\gamma)^{-1}(0) = \int_0^1 \ell\left(\zeta_{f\circ\gamma}(t)\right)\,\mathrm{d}t = 2\int_0^1 F^X(\dot\gamma(t))\,\mathrm{d}t$$

We recognize on the right $2\ell^{F^X}(\gamma)$. To conclude, note that $(f\circ\gamma)^{-1}(0) = \gamma^{-1}(\gamma \cap X)$ and thus $\#(f\circ\gamma)^{-1}(0) = \#(\gamma \cap X)$. □

Constructions of Finsler structures that admits a Crofton formula are known for random hyperplanes in projective space, see [18, 72, 78]. Moreover, a more general result very similar to Theorem 4.2.43 can be found in [8, Theorem A] although the *z-KRok* hypothesis is significantly less restrictive and the construction of the metric F^X explicit with (4.2.29).

Unlike for the length, there are several definitions of volume in Finsler manifolds. One way to define k-dimensional volumes of submanifolds is to define a k-density, that is a nonnegative homogeneous function φ_k on the simple vectors of $\Lambda^k TM$. The k-densities satisfy a pull-back property and thus, given an embedded submanifold $\iota : S \hookrightarrow M$, $\iota^*\varphi_k$ defines a density (in the classical sense) and can be integrated. The k-volume of S is then defined to be

$$\mathrm{vol}_{\varphi_k}(S) := \int_S \iota^*\varphi_k.$$

See [7] for the possible choices of k-densities and more details. One of the most common choice is the *Holmes-Thompson density.* To define it, it is convenient for us to use the Riemannian metric on our manifold M and assume that the semi Finsler structure is given at each point $p \in M$ by a convex body $\zeta(p) \subset T_p^*M$ in such a way that $F_p = h_{\zeta(p)}$.

Definition 4.2.44. The k^{th} *Holmes-Thompson* density φ_k^{ht} is given for all $p \in M$, and all simple vectors $v = v_1 \wedge \cdots \wedge v_k \in \Lambda^k T_pM$

$$\varphi_k^{ht}(v_1 \wedge \cdots \wedge v_k) := \frac{\|v_1 \wedge \cdots \wedge v_k\|}{\kappa_k}\,\mathrm{vol}_k(\pi_v(\zeta(p)))$$

where $\|\cdot\|$ is the norm on $\Lambda^k T_pM$ induced by the Riemannian structure, π_v is the orthogonal projection onto $Span(v_1, \ldots, v_k)$ (identifying the space and its dual) and vol_k is the k-dimensional volume for the Riemannian metric in T_pM.

The reader can refer to [7, p.19]. One can also show that this definition doesn't depend on the choice of the Riemannian metric, however, in our case this becomes clear with the next lemma.

Lemma 4.2.45. *If the semi Finsler structure is given at each point $p \in M$ by a zonoid $\zeta(p) \subset T_p^*M$ then the Holmes–Thompson density is given by*

$$\varphi_k^{ht} = \frac{2}{k!\kappa_k} h_{\zeta(p)^{\wedge k}}.$$

Proof. This is a consequence of the definition and Lemma 2.2.20. ☐

With this notion, we obtain a Crofton formula for higher dimensional volumes.

Theorem 4.2.46 (M–Stecconi)**.** *Let $1 \le k \le m$, let $f_1, \dots, f_k \in C^1(M, \mathbb{R})$ be iid* z-KRok *fields, let $X_i := f_i^{-1}(0)$ and let $X^{(k)} := X_1 \cap \dots \cap X_k$. Let $\iota : S \hookrightarrow M$ be an embedded submanifold of dimension k such that $X^{(k)}$ is transversal to S almost surely, then we have*

$$\mathbb{E}\# \left(S \cap X^{(k)} \right) = k!\kappa_k \operatorname{vol}_k^{F^{X_1}}(S)$$

where $\operatorname{vol}_k^{F^{X_1}}$ denotes the Holmes–Thompson volume for the semi Finsler structure F^{X_1} defined by (4.2.29).

Proof. The proof is almost identical to the proof of Theorem 4.2.43 but let us repeat it, if only to compute the constant. Let $f^{(k)} := (f_1, \dots, f_k) \in C^1(M, \mathbb{R}^k)$ and consider $f^{(k)} \circ \iota \in C^1(S, \mathbb{R}^k)$. Since S is almost surely transversal to $X^{(k)} = (f^{(k)})^{-1}(0)$, by the pull-back property (Lemma 4.2.17) it is *z-KRok* and we have for all $q \in S$

$$\zeta_{f^{(k)} \circ \iota}(q) = \mathrm{d}_q\iota^* \zeta_{f^{(k)}}(\iota(q)) = \mathrm{d}_q\iota^* \left((\zeta_{f_1}(\iota(q)))^{\wedge k} \right) = \left(\mathrm{d}_q\iota^* \zeta_{f_1}(\iota(q)) \right)^{\wedge k}.$$

where the second equality holds because $f^{(k)} := (f_1, \dots, f_k)$ and $f_1, \dots, f_k$ are iid and the third equality is by definition of the linear maps induced in the exterior algebra. We fix a Riemannian structure on S such that ι is a Riemannian embedding and we let $\omega_q \in \Lambda^k T^q S$ be the choice of a volume form (if S is not orientable we can work locally). Now we note that $\zeta_{f^{(k)} \circ \iota}(q)$ lives in the one dimensional space $\Lambda^k T_q S$ thus its length is given by:

$$\begin{aligned}
\ell \left(\zeta_{f^{(k)} \circ \iota}(q) \right) &= h_{\zeta_{f^{(k)} \circ \iota}(q)}(\omega_q) + h_{\zeta_{f^{(k)} \circ \iota}(q)}(-\omega_q) \\
&= h_{\zeta_{f^{(k)}}(\iota(q))}(\mathrm{d}_q\iota(\omega_q)) + h_{\zeta_{f^{(k)}}(\iota(q))}(\mathrm{d}_q\iota(-\omega_q)) \\
&= 2h_{\underline{\zeta_{f^{(k)}}}(\iota(q))}(\mathrm{d}_q\iota(\omega_q)) \\
&= 2h_{\underline{\zeta_{f_1}}(\iota(q))^{\wedge k}}(\mathrm{d}_q\iota(\omega_q)) = k!\kappa_k \varphi_k^{ht}(\mathrm{d}_q\iota(\omega_q)).
\end{aligned}$$

Where here φ_k^{ht} denotes the Holmes Thompson density for the semi finsler structure defined by $\underline{\zeta_{f_1}}$. To conclude, we note that $\#(f^{(k)} \circ \iota)^{-1}(0) = \#(S \cap X^{(k)})$ and thus applying (4.2.27) to the *z-KRok* field $(f^{(k)} \circ \iota)$ we get

$$\mathbb{E}\#(S \cap X^{(k)}) = \int_S \ell \left(\zeta_{f^{(k)} \circ \iota}(q) \right) \mathrm{d}S(q) = k!\kappa_k \int_S \varphi_k^{ht}(\mathrm{d}_q\iota(\omega_q)) \, \mathrm{d}S(q)$$

which is what we wanted. ☐

In the case where the random field is an AT field for M we obtain the following Crofton formula in Riemannian geometry.

Corollary 4.2.47. *Let $f \in C^1(M, \mathbb{R})$ be an AT field for M, let $f_1, \dots, f_k$ be iid copies of f and let $X^{(k)} := f_1^{-1}(0) \cap \dots \cap f_k^{-1}(0)$. Then for any submanifold $S \hookrightarrow M$ of dimension k, we have*

$$\mathbb{E}\# \left(S \cap X^{(k)} \right) = \frac{2}{s_k} \operatorname{vol}_k(S)$$

where $\operatorname{vol}_k(S)$ denotes the Riemmannian volume of S.

Proof. As observed in Remark 4.2.23, any submanifold is almost surely transversal to the zero set of the AT field. By definition, the Finsler structure obtained from an AT field is the Riemannian metric multiplied by $1/(2\pi)$. Now the result follows from Theorem 4.2.46 and the identity $k!s_k\kappa_k = 2(2\pi)^k$, see Lemma A.4. □

If we consider the submanifold S in Theorem 4.2.46 to be again random, given by *z-KRok* fields, we obtain the following funny formula.

Corollary 4.2.48. *Let $f_1, \dots, f_k, g_1, \dots, g_{m-k} \in C^1(M, \mathbb{R})$ be independent* z-KRok *fields with $f_1, \dots, f_k$, respectively $g_1, \dots, g_{m-k}$, identically distributed. Consider $X^{(k)} := (f_1)^{-1}(0) \cap \dots \cap (f_k)^{-1}(0)$ and $Y^{(m-k)} := (g_1)^{-1}(0) \cap \dots \cap (g_{m-k})^{-1}(0)$. Then we have*

$$k!\kappa_k \mathbb{E}\left[\mathrm{vol}_k^{F^X}\left(Y^{(m-k)}\right)\right] = (m-k)!\kappa_{m-k}\mathbb{E}\left[\mathrm{vol}_{m-k}^{F^Y}\left(X^{(k)}\right)\right]$$

where $\mathrm{vol}_k^{F^X}$, respectively $\mathrm{vol}_{m-k}^{F^Y}$, denotes the Holmes-Thompson volume for the semi Finsler structure defined by $\underline{\zeta_{f_1}}$, respectively by $\underline{\zeta_{g_1}}$.

Proof. Applying the previous result Theorem 4.2.46 successively to $f_1, \dots, f_k$, fixing $Y^{(m-k)}$ and to $g_1, \dots, g_{m-k}$ fixing $X^{(k)}$, we get, using the independence assumption, that both sides are equal to $\mathbb{E}\#(X^{(k)} \cap Y^{(m-k)})$. □

Appendix A

Gamma function, spheres and balls

In this appendix, we present the Gamma function and formulas involving this function and volumes of spheres and balls. The reader can refer to the Wikipedia pages "n-sphere" [93], "Gamma function" [91] and "Multiplication theorem" [92].

Definition A.1. The *Gamma function* is the function given for all complex number z with non negative real part by:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, \mathrm{d}t.$$

In what follows we will only be interested in the value at real points. The Gamma fuction is a famous analytic generalization of the factorial as it satisfies for all non negative integers $n \geq 0$:

$$\Gamma(n+1) = n!.$$

More generally, we have for all $x > 0$:

$$\Gamma(x+1) = x\Gamma(x).$$

Another useful formula is the following.

Lemma A.2 (Legendre duplication formula)**.** *For all $x \geq 0$, we have:*

$$\Gamma(x)\Gamma\left(x + \frac{1}{2}\right) = 2^{1-2x} \sqrt{\pi}\, \Gamma(2x)$$

Applying it to $x = m$ an integer, we deduce the value of the Gamma function at half-integers points.

Lemma A.3. *For all $m \geq 0$, integer, we have:*

$$\Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^m} \frac{(2m)!}{m!}$$

The volume of spheres and balls can be expressed with the Gamma function. Recall that $B_m = B(\mathbb{R}^m)$ is the unit ball of $\mathbb{R}^m$ and that we denote $S^m = S(\mathbb{R}^{m+1})$ the unit sphere in $\mathbb{R}^{m+1}$. Moreover, recall that we denote their volume by

$$\kappa_m := \mathrm{vol}_m(B_m); \qquad s_m := \mathrm{vol}_m(S^m).$$

We have for all $m \geq 0$:

$$\kappa_m = \frac{2\pi^{\frac{m}{2}}}{m\Gamma\left(\frac{m}{2}\right)} \qquad s_m = \frac{2\pi^{\frac{m+1}{2}}}{\Gamma\left(\frac{m+1}{2}\right)}$$

Note that in even dimension, this gives for all $n \geq 0$:

$$\kappa_{2n} = \frac{\pi^n}{n!}; \qquad s_{2n} = 2(2\pi)^n \frac{n!}{(2n)!}$$

and in odd dimension, for all $n \geq 0$:

$$\kappa_{2n+1} = 2(2\pi)^n \frac{n!}{(2n+1)!}; \qquad s_{2n+1} = 2\frac{\pi^{n+1}}{n!}.$$

We find that the quotient gives for all $n \geq 0$

$$\frac{\kappa_{2n}}{\kappa_{2n-1}} = \frac{1}{2\pi 2^n}\binom{2n}{n}; \qquad \frac{s_{2n+1}}{s_{2n}} = \frac{\pi}{2^n}\binom{2n}{n}. \tag{A.1}$$

Using Legendre duplication formula we also find the following.

Lemma A.4. *For all $m \geq 0$, we have:*

$$m!\kappa_m s_m = 2(2\pi)^m.$$

Bibliography

[1] Karim A. Adiprasito and Raman Sanyal. Whitney numbers of arrangements via measure concentration of intrinsic volumes. *arXiv:1606.09412*, 2016. (Cited on p. 78)

[2] R. J. Adler and J. E. Taylor. *Random fields and geometry*. Springer Monographs in Mathematics. Springer, New York, 2007. (Cited on p. 15, 133)

[3] Semyon Alesker. Description of translation invariant valuations on convex sets with solution of p. mcmullen's conjecture. *Geometric & Functional Analysis GAFA*, 11(2):244–272, Jul 2001. (Cited on p. 11, 12)

[4] Semyon Alesker. Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations. *J. Differential Geom.*, 63(1):63–95, 2003. (Cited on p. 64, 91)

[5] Semyon Alesker. The multiplicative structure on continuous polynomial valuations. *Geometric & Functional Analysis GAFA*, 14(1):1–26, Feb 2004. (Cited on p. 11, 91)

[6] Semyon Alesker. Theory of valuations on manifolds, ii. *Advances in Mathematics*, 207(1):420–454, 2006. (Cited on p. 110, 137)

[7] J. C. Álvarez Paiva and A. C. Thompson. Volumes on normed and Finsler spaces. In *A sampler of Riemann-Finsler geometry*, volume 50 of *Math. Sci. Res. Inst. Publ.*, pages 1–48. Cambridge Univ. Press, Cambridge, 2004. (Cited on p. 138, 139)

[8] Juan-Carlos Àlvarez-Paiva and Gautier Berck. Finsler surfaces with prescribed geodesics, 2010. (Cited on p. 138)

[9] Horst Alzer. Error function inequalities. *Advances in Computational Mathematics*, 33:349–379, 10 2010. (Cited on p. 82)

[10] Rafik Aramyan. A flag representation for a n-dimensional convex body. *The Journal of Geometric Analysis*, 29(3):2998–3009, Jul 2019. (Cited on p. 28)

[11] Zvi Artstein and Richard A. Vitale. A Strong Law of Large Numbers for Random Compact Sets. *The Annals of Probability*, 3(5):879 – 882, 1975. (Cited on p. 29)

[12] Christos A. Athanasiadis, Jesús A. De Loera, Victor Reiner, and Francisco Santos. Fiber polytopes for the projections between cyclic polytopes. *European Journal of Combinatorics*, 21(1):19–47, 2000. (Cited on p. 71)

[13] Guillaume Aubrun and Cécilia Lancien. Zonoids and sparsification of quantum measurements. *Positivity*, 20(1):1–23, 2016. (Cited on p. 7, 8, 42)

[14] Robert J. Aumann. Integrals of set-valued functions. *Journal of Mathematical Analysis and Applications*, 12(1):1–12, 1965. (Cited on p. 72)

[15] Jean-Marc Azais and Mario Wschebor. *Level sets and extrema of random processes and fields*. John Wiley & Sons, Inc., Hoboken, NJ, 2009. (Cited on p. 131, 132)

[16] J. L. Barbosa, L. Birbrair, M. do Carmo, and A. Fernandes. Globally subanalytic cmc surfaces in $\mathbb{R}^3$. *Electronic Research Announcements*, 21(0):186–192, 2014. (Cited on p. 39)

[17] S. Basu, A. Lerario, E. Lundberg, and C. Peterson. Random fields and the enumerative geometry of lines on real and complex hypersurfaces. *Math. Ann.*, 374(3-4):1773–1810, 2019. (Cited on p. 62)

[18] Andreas Bernig. Valuations with crofton formula and finsler geometry. *Advances in Mathematics*, 210(2):733–753, 2007. (Cited on p. 138)

[19] Andreas Bernig and Dmitry Faifman. Valuation theory of indefinite orthogonal groups. *Journal of Functional Analysis*, 273(6):2167–2247, 2017. (Cited on p. 88)

[20] Andreas Bernig and Joseph H. G. Fu. Convolution of convex valuations. *Geometriae Dedicata*, 123(1):153–169, Dec 2006. (Cited on p. 90, 91, 95)

[21] Andreas Bernig, Joseph H. G. Fu, and Gil Solanes. Integral geometry of complex space forms. *Geometric and Functional Analysis*, 24(2):403–492, Apr 2014. (Cited on p. 14, 110)

[22] Louis J. Billera and Bernd Sturmfels. Fiber polytopes. *Annals of Mathematics*, 135(3):527–549, 1992. (Cited on p. 8, 9, 71, 72, 77)

[23] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics. Wiley & Sons, 2nd edition, 1999. (Cited on p. 46, 122)

[24] Ethan D. Bolker. A class of convex bodies. *Transactions of the American Mathematical Society*, 145:323–345, 1969. (Cited on p. 25, 34, 36)

[25] Ana María Botero. The convex-set algebra and intersection theory on the toric riemann-zariski space. *Acta Mathematica Sinica, English Series*, 38(3):465–486, Mar 2022. (Cited on p. 94)

[26] Paul Breiding, Peter Bürgisser, Antonio Lerario, and Léo Mathis. The zonoid algebra, generalized mixed volumes, and random determinants. *Advances in Mathematics*, 402:108361, 2022. (Cited on p. 8, 9, 27, 35, 36)

[27] Michel Brion. The structure of the polytope algebra. *Tohoku Mathematical Journal*, 49(1):1 – 32, 1997. (Cited on p. 94)

[28] P. Bürgisser and F. Cucker. *Condition*, volume 349 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2013. The geometry of numerical algorithms. (Cited on p. 60)

[29] P. Bürgisser and A. Lerario. Probabilistic Schubert calculus. *J. Reine Angew. Math.*, 760:1–58, 2020. (Cited on p. 14, 103, 113, 114, 115)

[30] Swee Hong Chan and Igor Pak. Introduction to the combinatorial atlas, 2022. (Cited on p. 59)

[31] Raf Cluckers and Daniel J. Miller. Stability under integration of sums of products of real globally subanalytic functions and their logarithms. *Duke Math. J.*, 156(2):311–348, 2011. (Cited on p. 39)

[32] G. Comte, J. M Lion, and J. P Rolin. Nature log-analytique du volume des sous-analytiques. *Illinois Journal of Mathematics*, 44(4):884 – 888, 2000. (Cited on p. 39)

[33] Nguyen-Bac Dang and Jian Xiao. Positivity of valuations on convex bodies and invariant valuations by linear actions. *The Journal of Geometric Analysis*, 31(11):10718–10777, Nov 2021. (Cited on p. 11, 87, 91, 92)

[34] M. Dehn. Ueber den rauminhalt, September 1901. (Cited on p. 11)

[35] H. Derksen. On the nuclear norm and the singular value decomposition of tensors. *Found. Comput. Math.*, 16(3):779–811, 2016. (Cited on p. 44)

[36] R. M. Dudley. *Real Analysis and Probability*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2 edition, 2002. (Cited on p. 118)

[37] Aleksandr Esterov. On the existence of mixed fiber bodies. *Moscow Mathematical Journal*, 8(3):433–442, 2008. (Cited on p. 71, 72, 75, 76)

[38] Alexander Esterov and Askold Khovanskii. Elimination theory and newton polytopes. *Functional Analysis and Other Mathematics*, 2(1):45–71, 2008. (Cited on p. 71, 74)

[39] L. M. Fehér and Á. K. Matszangosz. Real solutions of a problem in enumerative geometry. *Period. Math. Hungar.*, 73(2):137–156, 2016. (Cited on p. 112)

[40] Joseph H.G. Fu. Structure of the unitary valuation algebra. *Journal of Differential Geometry*, 72(3):509 – 533, 2006. (Cited on p. 100)

[41] William Fulton and Bernd Sturmfels. Intersection theory on toric varieties. *Topology*, 36:335–353, 1994. (Cited on p. 94)

[42] Fulvio Gesmundo and Chiara Meroni. The geometry of discotopes, 2021. (Cited on p. 78)

[43] Paul Goodey and Ralph Howard. Processes of flats induced by higher dimensional processes. *Advances in Mathematics*, 80(1):92–109, 1990. (Cited on p. 69)

[44] Morris W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original. (Cited on p. 118, 119)

[45] R. Howard. The kinematic formula in Riemannian homogeneous spaces. *Mem. Amer. Math. Soc.*, 106(509):vi+69, 1993. (Cited on p. 102, 103)

[46] Zakhar Kabluchko and Dmitry Zaporozhets. Random determinants, mixed volumes of ellipsoids, and zeros of gaussian random fields. *Journal of Mathematical Sciences*, 199, 06 2012. (Cited on p. 9, 55, 84, 85)

[47] Tobias Kaiser. Integration of semialgebraic functions and integrated Nash functions. *Math. Z.*, 275(1-2):349–366, 2013. (Cited on p. 39)

[48] B. Ya. Kazarnovskiĭ. “Newton polyhedra” of generalized functions. *Izv. Ross. Akad. Nauk Ser. Mat.*, 68(2):53–70, 2004. (Cited on p. 64)

[49] Askold Khovanskii. Completions of convex families of convex bodies. *Mathematical Notes*, 91, 2012. (Cited on p. 73)

[50] Gleb Koshevoy and Karl Mosler. The Lorenz zonoid of a multivariate distribution. *J. Amer. Statist. Assoc.*, 91(434):873–882, 1996. (Cited on p. 7)

[51] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002. (Cited on p. 28)

[52] D. Leao, Jr., Marcelo D. Fragoso, and R. C. Ruffino. Regular conditional probability, disintegration of probability and radon spaces. *Proyecciones (Antofagasta, On line)*, 23(1):15–29, May 2017. (Cited on p. 118)

[53] F. C. Leone, L. S. Nelson, and R. B. Nottingham. The folded normal distribution. *Technometrics*, 3(4):543–550, 1961. (Cited on p. 80)

[54] Antonio Lerario and Léo Mathis. Probabilistic schubert calculus: Asymptotics. *Arnold Mathematical Journal*, 7(2):169–194, Jun 2021. (Cited on p. 14, 115)

[55] Antonio Lerario and Léo Mathis. On tameness of zonoids. *Journal of Functional Analysis*, 282(4):109341, 2022. (Cited on p. 38)

[56] Antonio Lerario and Michele Stecconi. Differential topology of gaussian random fields, 2019. (Cited on p. 120)

[57] J. M Lion and J. P Rolin. Intégration des fonctions sous-analytiques et volumes des sous-ensembles sous-analytiques. *Annales de l'institut Fourier*, 48(3):755–767, 1998. (Cited on p. 39)

[58] Yossi Lonke. A characterisation of zonoids, 2020. (Cited on p. 37)

[59] D. Magisano, F. S. Liguori, L. Leonetti, and G. Garcea. Minkowski plasticity in 3D frames: decoupled construction of the cross-section yield surface and efficient stress update strategy. *Internat. J. Numer. Methods Engrg.*, 116(7):435–464, 2018. (Cited on p. 7)

[60] Georges Matheron. Un théorème d'unicité pour les hyperplans poissoniens. *Journal of Applied Probability*, 11(1):184–189, 1974. (Cited on p. 69)

[61] Léo Mathis and Chiara Meroni. Fiber convex bodies, 2021. (Cited on p. 8, 71, 78, 79)

[62] Peter McMullen. On zonotopes. *Transactions of the American Mathematical Society*, 159:91–109, 1971. (Cited on p. 26)

[63] Peter McMullen. Weakly continuous valuations on convex polytopes. *Arch. Math. (Basel)*, 41(6):555–564, 1983. (Cited on p. 89)

[64] Peter McMullen. The polytope algebra. *Advances in Mathematics*, 78(1):76–130, 1989. (Cited on p. 92, 93, 94)

[65] Merriam-Webster. Calculus — Merriam-Webster online dictionary, 2022. `https://www.merriam-webster.com/dictionary/calculus` [accessed 08-July-2022]. (Cited on p. 7)

[66] Ilya Molchanov, Michael Schmutz, and Kaspar Stucki. Invariance properties of random vectors and stochastic processes based on the zonoid concept. *Bernoulli*, 20(3):1210–1233, 2014. (Cited on p. 7, 30)

[67] Karl Mosler. *Multivariate dispersion, central regions and depth*, volume 165 of *Lecture Notes in Statistics*. Springer-Verlag, Berlin, 2002. The lift zonoid approach. (Cited on p. 7, 33)

[68] David Mumford. *Algebraic Geometry I: Complex Projective Varieties*. Classics in Mathematics. Springer Berlin Heidelberg, 1995. (Cited on p. 107)

[69] John Nash. The imbedding problem for riemannian manifolds. *Annals of Mathematics*, 63(1):20–63, 1956. (Cited on p. 128)

[70] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, 1964. Published electronically at `http://oeis.org`. (Cited on p. 109)

[71] Oscar – open source computer algebra research system, version 0.8.2-dev, 2022. (Cited on p. 77)

[72] J. C. Álvarez Paiva and E. Fernandes. Gelfand transforms and crofton formulas. *Selecta Mathematica*, 13(3):369, Feb 2008. (Cited on p. 138)

[73] K.R. Parthasarathy. *Probability Measures on Metric Spaces*. Ams Chelsea Publishing. Academic Press, 2005. (Cited on p. 122)

[74] W. Rudin. *Functional analysis*. McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973. (Cited on p. 36)

[75] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.2)*, 2021. `https://www.sagemath.org`. (Cited on p. 77)

[76] Raman Sanyal and James Saunderson. Spectral polyhedra, 2020. (Cited on p. 116, 117)

[77] Anna-Laura Sattelberger and Bernd Sturmfels. D-modules and holonomic functions. *arXiv:1910.01395*, 2019. (Cited on p. 75)

[78] R. Schneider. Crofton formulas in hypermetric projective finsler spaces. *Archiv der Mathematik*, 77(1):85–97, Jul 2001. (Cited on p. 138)

[79] Rolf Schneider. On the Aleksandrov–Fenchel inequality involving zonoids. *Geometriae Dedicata*, 27:113–126, 1988. (Cited on p. 56)

[80] Rolf Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 151 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, expanded edition, 2014. (Cited on p. 7, 19, 20, 21, 22, 23, 24, 25, 26, 28, 33, 34, 36, 37, 53, 54, 57, 74, 76, 87, 88, 90, 91, 95)

[81] G. C. Shephard. Combinatorial properties of associated zonotopes. *Canadian J. Math.*, 26:302–321, 1974. (Cited on p. 63, 95)

[82] Michele Stecconi. Kac-rice formula for transverse intersections. *Analysis and Mathematical Physics*, 12(2):44, Feb 2022. (Cited on p. 14, 120, 121, 131)

[83] Bernd Sturmfels and Josephine Yu. Tropical implicitization and mixed fiber polytopes. In *Software for algebraic geometry*, pages 111–131. Springer, 2008. (Cited on p. 71)

[84] Hiroyuki Tasaki. Generalization of Kähler angle and integral geometry in complex projective spaces. In *Steps in differential geometry (Debrecen, 2000)*, pages 349–361. Inst. Math. Inform., Debrecen, 2001. (Cited on p. 62, 68, 107)

[85] Lou van den Dries. o-minimal structures and real analytic geometry. In *Current developments in mathematics, 1998 (Cambridge, MA)*, pages 105–152. Int. Press, Somerville, MA, 1999. (Cited on p. 39)

[86] Richard A. Vitale. Expected absolute random determinants and zonoids. *Annals of Applied Probability*, 1:293–300, 1991. (Cited on p. 7, 9, 25, 31, 54, 55)

[87] Thomas Wannerer. On the extendability by continuity of angular valuations on polytopes. *J. Funct. Anal.*, 279(8):108665, 25, 2020. (Cited on p. 89)

[88] Wolfgang Weil. Blaschkes problem der lokalen charakterisierung von zonoiden. *Archiv der Mathematik*, 29(1):655–659, Dec 1977. (Cited on p. 37)

[89] Wolfgang Weil. On surface area measures of convex bodies. *Geometriae Dedicata*, 9(3):299–306, Sep 1980. (Cited on p. 23)

[90] Wikipedia contributors. Folded normal distribution — Wikipedia, the free encyclopedia, 2022. `https://en.wikipedia.org/wiki/Folded_normal_distribution` [accessed 18-January-2022]. (Cited on p. 80)

[91] Wikipedia contributors. Gamma function — Wikipedia, the free encyclopedia, 2022. `https://en.wikipedia.org/wiki/Gamma_function` [accessed 21-June-2022]. (Cited on p. 141)

[92] Wikipedia contributors. Multiplication theorem — Wikipedia, the free encyclopedia, 2022. `https://en.wikipedia.org/wiki/Multiplication_theorem` [accessed 21-June-2022]. (Cited on p. 141)

[93] Wikipedia contributors. n-sphere — Wikipedia, the free encyclopedia, 2022. `https://en.wikipedia.org/wiki/N-sphere` [accessed 21-June-2022]. (Cited on p. 141)

[94] H. S. Witsenhausen. A remark on reachable sets of linear systems. *IEEE Trans. Automatic Control*, AC–17(4):547, 1972. (Cited on p. 7)

[95] Doron Zeilberger. A holonomic systems approach to special functions identities. *Journal of Computational and Applied Mathematics*, 32(3):321–368, 1990. (Cited on p. 75)