Undergraduate Student Research Internship Report:

Analysis of Credit Risk and Single / Two Factor Model

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ABSTRACT

Since 2008, businesses and banks must manage and track more risk than ever before. Financial risk management helps companies and banks decrease the risk of investment and trade. Additionally, financial risk management gives a guide on how to forecast and manage the risk efficiently. More specifically, the three major risks are market risk, credit risk, and operational risk. This report will focus on the credit risk: introducing the definition of credit risk, single factor model, the relationship between β coefficient and default probability, and the relationship of m coefficient and default probability. Using the single factor model, we will extend the definition and application to the double factor model. Furthermore, the coding will be provided.

TABLE OF CONTENTS

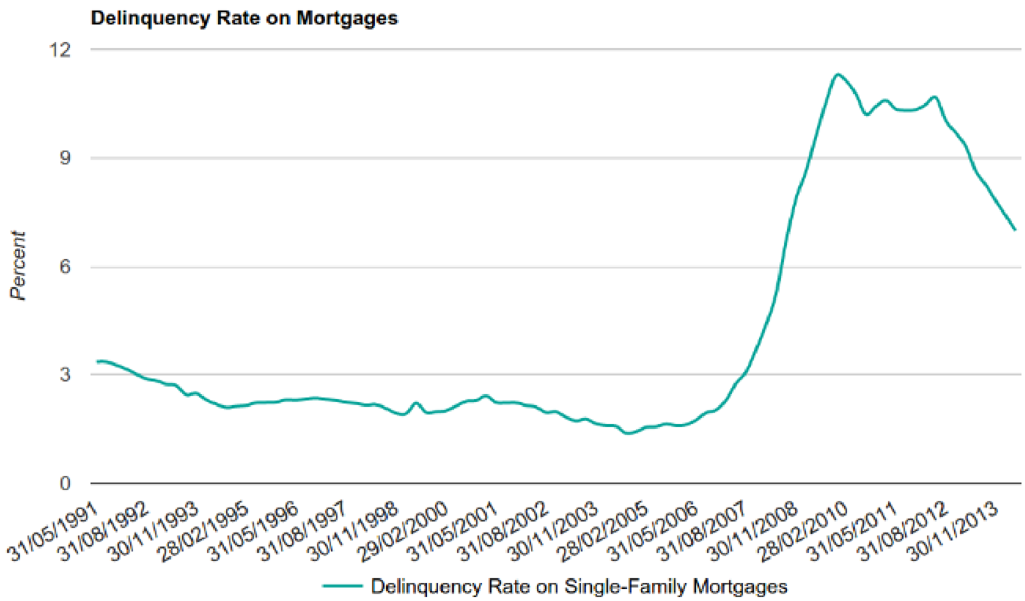
CREDIT RISK

In Financial Risk Management, credit is defined as an economic obligation. For example, we would like to purchase a good. However, we do not have enough money to pay for that good. In this case, the credit can help us to obtain the good in advance and we are bound to pay back the money later. Under the definition of credit, we are also curious about the outcome when borrowers fail to repay. We consider this situation as a default. Furthermore, we consider the credit risk as economic loss from default or changes in ratings. From a quantitative perspective, for any individual and financial institution, if their credit points decrease, banks will re-evaluate their credit risk, and always, their default risk would rise. In general, the low credit points will result in bad consequences for them, such that they can not get a loan from the bank.

Credit risk strictly sticks with market risk, since every risk is not independent, for example, if the society suffers an economic recession, the unemployment rate has dramatically increased, then individuals and companies may not be able to pay back loans, then their credit points will be lower than before and the direct result will be they can not get loans again.

The situation may increase the credit risk

Individuals and financial institutions may suffer bankruptcy or lose an incredible amount of money. Hence, they will fail to pay back the loan to the banks. Secondly, many debtors' collateral value is not worth that much money, hence, they might default, this also may increase the credit risk. Thirdly, due to inflation, a country may increase the interest rate, in this case, most people are not able to afford the increased interest, which may lead them to fail to repay the loan. For example, in July 2022, the Bank of Canada raised the interest rate by 100 basis points to solve the high inflation issue. Since most people's salaries did not improve, their default risk may increase. Another example is in 2008, during the global financial crisis, in the United States of America, the Delinquency rate on mortgages was almost 12%, this means every 100 households that paid the mortgage will result in 12 people paid late or unpaid.



Credit Risky Securities

Let us introduce several credit risky securities, they are corporate debt securities, sovereign debt, credit derivatives and structured credit products.

1. Corporate debt securities: it is usually described as bonds and bank loans; they all belong to fixed income.
2. Sovereign debt: It is a tool to measure the currency and is usually issued by local government and government-controlled companies.
3. Credit derivatives: it is a contract, and this value depends on the creditworthiness.
4. Structured credit products are individual and commercial mortgages and loans. For example, student loans for school, credit cards etc.

In conclusion, these credit risky securities have a common feature: use credit to measure its value. Nevertheless, credit risky securities may suffer a big problem in the economic recession due to people's income and society's employment rate.

Economic Balance Sheet: Asset, Equity, and Debt

Before we go through the credit risk, we need to be familiar with the several definitions of the economic balance sheet, they have two categories which are Assets and Liabilities. Now from the graph, we know that the value of the firm(A_t) belongs to the Assets category, Equity(E_t) and Debt(D_t) belong to Liabilities. We also have the balance sheet formula $A_t = E_t + D_t$

Asset	Liability
Asset (A_t)	Equity(E_t)
	Debt(D_t)

In the assets, A_t is an asset of the firm, E_t is capital invested by the firm's shareholders, and D_t is a contract to pay a fixed amount of money. The *equity ratio* is equal to $\frac{E_t}{A_t}$ while we denote the $\frac{A_t}{E_t}$ as the *leverage ratio*.

Definition of Default and its Application

In Malz, A. M. (2011), we refer to default as the failure to pay a financial obligation. The probability of default is denoted by PD. Credit exposure means the lenders may suffer a loss with default situations.

Loss Given Default (LGD), is the amount the creditor loses in the event of a default. We can express exposure as "exposure = recovery + LGD" where the recovery amount is part of the money owed that the creditors receive in the event of bankruptcy. We usually denote recovery as R located in [0,1]. In addition, we can also know that R= $\frac{recovery}{exposure} = \frac{exposure-LGD}{exposure}$ = 1- $\frac{LGD}{exposure}$ (Note that we always treat the LGD as a known parameter).

Expected loss is the expected value of the credit loss. The formula presented as

EL= PD×(1-R)×exposure = PD× LGD. Note that LGD and recovery rate are conditional expectations, that is

E[loss|default]=$\frac{Expected\ loss}{PD}$ = LGD, with this formula, we can conclude several points:

1. In this formula, PD can not be zero.

2. As default probability becomes very low, the LGD can be very large

3. LGD is conditional expectation

SINGLE FACTOR MODELS

Single factor model can be considered as a type of structural model, which is related to the risk of credit loss. We set the logarithmic asset return is $A_T = log(\frac{A_T - A_t}{A_t})$ .Note we have T= t + d, d is denoted as time interval.

Now we can state the single factor model, that a firm's asset return could be represented by $a_T = \beta m + \sqrt{1-\beta^2}\,\varepsilon_i$. In this formula , we indicate that the β correlation factor; m is the correlation between default and the economic performance; ε_i is an idiosyncratic risk. We can consider that m and ε are standard normal, these are $m \sim N(0, 1)$ and $\varepsilon \sim N(0, 1)$,i,i.d.

And we can conclude that $E[a_T] = \beta E[m] + \sqrt{1-\beta^2}E[\varepsilon_i] = \beta \cdot 0 + \sqrt{1-\beta^2} \cdot 0 = 0$, $Var[A_T] = \beta^2 var[m] + (1-\beta^2)var[\varepsilon_i] = 1$. And since the a_T~N(0,1) is standard normal distributed, we can apply Z-value tables, and the calculation is presented as $Z = \frac{X-\mu}{\sigma}$ we can say that $PD = P[a_T \le k] \Leftrightarrow k = \phi^{-1}(PD)$, where PD is default probability, $\phi^{-1}(*)$ is the quantile function of standard normal distribution and $\phi^{-1}(PD)$ is PD-th quantile of a_T . Moreover, we can denote the k as default threshold.

The graph of PD and k 's relationship- Strictly follow the Z-value table, in the example , we only select 0.1, 0.05,0.01 and 0.001:

π	**k**
0.1	-1.28
0.05	-1.65
0.01	-2.33
0.001	-3.09

Conditional and Unconditional Default Probability

Conditional default probability: $PD(m) = \phi(\frac{k-\beta m}{\sqrt{1-\beta^2}})$, however, we would like to find out the

Unconditional Default Probability, these are the steps to find how to make it happen:

1. Consider the loss level as a random variable X=x.Also, we consider x$\in$ $[0, 1]$.
2. We set the PD = loss level , loss level and market factor return

$$x(m) = p(m) = \phi(\frac{k-\beta m}{\sqrt{1-\beta^2}})$$

Now we can find m by giving a specific loss level of x, that is,

$$\phi^{-1}(x) = \frac{k-\beta m}{\sqrt{1-\beta^2}} \Rightarrow \phi^{-1}(x)\sqrt{1-\beta^2} = k-\beta m$$

$$\beta m=k- \phi^{-1}(x)\sqrt{1-\beta^2} \Rightarrow m= \frac{k-\phi^{-1}(x)\sqrt{1-\beta^2}}{\beta}$$

3. Now we can consider P[loss level] is equal to the P[m], we also know that the market factor is a standard normal, that is ,

$$P[X \le x] = \phi(m) = \phi\left(\frac{k - \phi^{-1}(x)\sqrt{1-\beta^2}}{\beta}\right)$$

4. We re-do step 1-3 to obtain the distribution of x .

Relationship between Beta Coefficient(β) and Default Probability

In Single factor model , we have found that $a_i = \beta_i \overline{m} + \sqrt{1 - \beta_i^2}\,\varepsilon_i$ where i=1,2,… Now we introduce a new notation, k, default threshold and we denote the probability default as π, then the $\pi = \phi(k)$. Now we are trying to figure out the Beta(β) default probability ‘s relationship, we have several steps,

1. Get default threshold k(numeric value)
2. Get value $\overline{m}$, remain this fixed
3. State this formula, p(m)= $\phi\left(\frac{k_i - \beta_i m}{\sqrt{1-\beta_i^2}}\right)$, i= 1,2…,note this is the conditional cumulative default probability function , also this satisfies the normal distribution
4. Coding presentation in R:

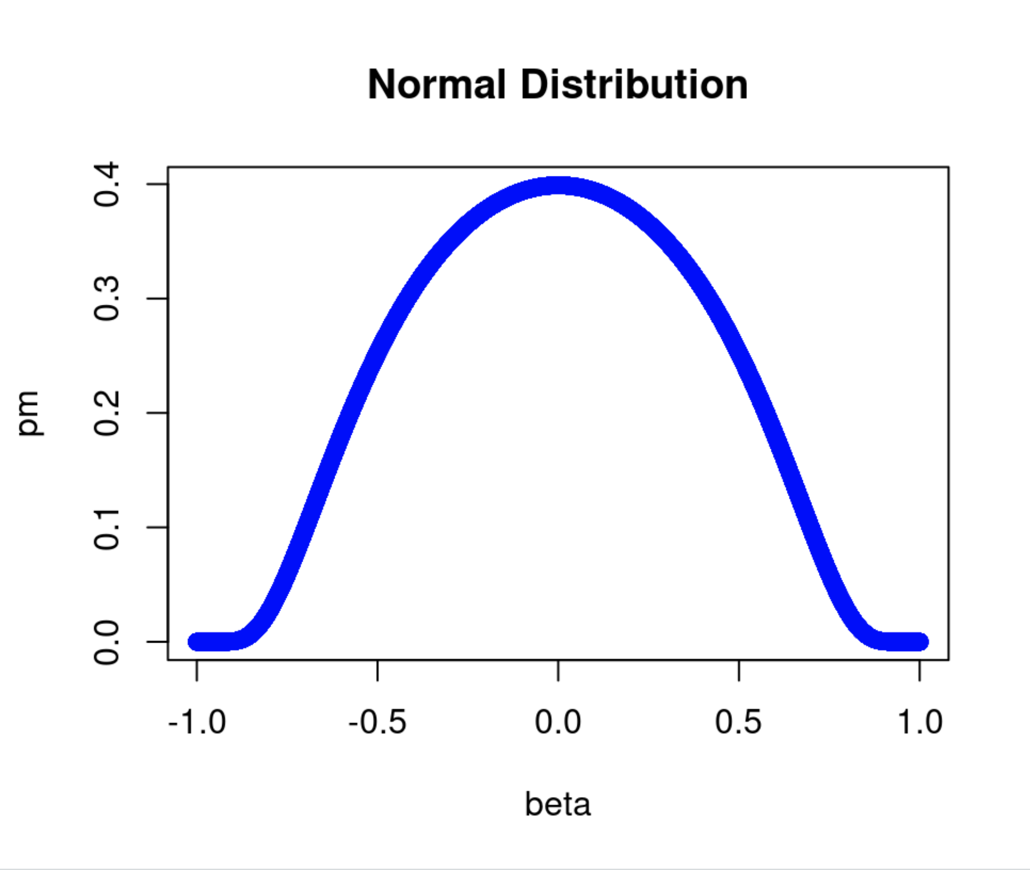
```
beta <- seq(-0.99999, 0.99999, by = .00001)

# we can not set up -1 or 1 since dominator can not be zero

m<-0.9 # make this value fixed

pm <- dnorm(beta, mean = -beta*m, sd = sqrt(1-beta^2))# pm means conditional cumulative
default probability
```

```
plot(beta,pm, main = "Normal Distribution", col = "blue")
```



From the graph above, we are pretty sure that the beta coefficient with conditional probability is bell-shaped, due to m being fixed the highest conditional probability is less than 0.4, still, we need to be careful that beta can not be -1 or 1, since the denominator part can not be zero.

5. How to use the probability of default to estimate the beta, firstly we have the formula

$p(m) = \phi\left(\frac{k_i - \beta_i m}{\sqrt{1-\beta_i^2}}\right)$, let's define the p(m) as the conditional probability of default, CPD,

6. In this case, we easily apply the Beta coefficient in the programming, the basic idea to apply in the coding is to remain k fixed to get the conditional default probability then to get the Beta coefficient. The exact derivation is: denote the conditional probability of default as CPD, then we know that

CPD= $\phi(\frac{k-\beta m}{\sqrt{1-\beta^2}}) \Rightarrow Z = \phi^{-1}(CPD) = \frac{k-\beta m}{\sqrt{1-\beta^2}} \Rightarrow Z = \frac{k-\beta m}{\sqrt{1-\beta^2}} \Rightarrow (k - \beta m)^2 = Z^2(1 - \beta^2)$

$\Rightarrow k^2 + k^2 m^2 - 2k\beta m = Z^2 - Z^2\beta^2 \Rightarrow (m^2 + Z^2)\beta^2 - 2km\beta + k^2 - Z^2$ by using quadratic formula: $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$, let $a = (m^2 + Z^2)$, $b =- 2km$, $c = k^2 - Z^2$, insert the a,b,c value in to the quadratic formula: $\frac{2km + \sqrt{4k^2m^2 - 4(m^2+Z^2)(k^2-Z^2)}}{2(m^2+Z^2)}$ =β , in addition to this we know that

$k = \phi^{-1}(PD)$ (*) , we can also substitute (*) into quadratic formula:

$\frac{2\phi^{-1}(PD)m + \sqrt{4[\phi^{-1}(PD)]^2 m^2 - 4(m^2+Z^2)([\phi^{-1}(PD)]^2 - Z^2)}}{2(m^2+Z^2)}$, where Z= $\phi^{-1}(CPD)$, m and z can not be zero.

Conclusion

As for the single factor model, we only have one factor that can not be realistic, in fact, the more factors we have , the more accurate we can achieve. In the financial risk management, as we used to talk the risk, the major risks never exists independently, we rely on each other, and so does the single factor model, we need to add more possible factors in it, such as more assets and more companies, after figured out their correlation, wc can calculate the Beta more accurate.We will also extend the double factor model and show how it exact works.

TWO FACTOR MODEL

Introduction

Since we already have the single- factor model, we can now extend the idea to two factors. Suppose you have the two companies, i and j, then they are the pairs of companies, the market factor is β_1 andβ_2, m~N(0,1) and ε_i ~N(0,1) , base on single factor model theory, we can extend as two- factor models, we can set up them as $m_j = \beta_1 m_i + \sqrt{1-\beta_1^2}\,\varepsilon_i$, where m_i, m_j and ε_i are N(0,1) and iid.

In these two assets, there is a relationship between them β , $\beta = corr(m_i, m_j)$ .Here are some notations:

1. m is a common factor and its standard normal distributed N(0,1)

2. ε_i is an idiosyncratic factor and only affects a company and it's also a standard normally distributed N(0,1).

3. β is asset correlation for i and j.

4. Note that the firm's values and defaults are independent.

Recall that we have the single factor model, then we can create the two company (A and B) with β_1 and β_2

And we have the two companies asset formulas:

$$a_A^i = \beta_1 m_i + \sqrt{1 - \beta^2{}_1}\ \varepsilon_i$$

$$a_B^j = \beta_2 m_j + \sqrt{1 - \beta^2{}_2}\ \varepsilon_j$$ where m_i, m_j and ε_j are N(0,1) iid.

As we talked in single factor model, we have developed the conditional default probability as :

CPD(m)= $\phi\left(\frac{k_i - \beta_i m}{\sqrt{1-\beta_i^2}}\right)$, so we can see the two companies assets

as $CPD_A(m_i) = \phi(\frac{k_1 - m_i\beta_1}{\sqrt{1-\beta_1^2}})$ (1)

$$CPD_B(m_j) = \phi(\frac{k_2 - m_j\beta_2}{\sqrt{1-\beta_2^2}}) \qquad (2)$$

With (1) and (2), we can also derive them in the same way to deal with CPD and correlation β's relationship, firstly, we have the three formulas:

$$CPD_A(m_i) = \phi(\frac{k_1 - m_i\beta_1}{\sqrt{1-\beta_1^2}}) \qquad (A)$$

$$CPD_B(m_j) = \phi(\frac{k_2 - m_j\beta_2}{\sqrt{1-\beta_2^2}}) \qquad (B)$$

$$m_j = \beta_1 m_i + \sqrt{1 - \beta_1{}^2}\ \varepsilon_i \qquad (C)$$

Now we can derive A and B, to find a beta coefficient and conditional probability(CPD)'s relationship, as we used that in the single factor model, in equation A, we have the $CPD_A(m_i) = \phi(\frac{k_1 - m_i\beta_1}{\sqrt{1-\beta_1^2}})$ in this case, we can get the $\phi^{-1}(CPD_A) = \frac{k_1 - m_i\beta_1}{\sqrt{1-\beta_1^2}} \Rightarrow$ $set\ Z_A = \phi^{-1}(CPD_A),\ then\ Z = \frac{k_1 - m_i\beta_1}{\sqrt{1-\beta_1^2}} \Rightarrow$

$(k_1 - m_i\beta_1)^2 = Z^2(1 - \beta_1^2) \Rightarrow (m_i^2 + Z^2)\beta_1^2 - 2k_1m_i\beta_1 + k_1^2 - Z^2 = 0$,

$a = m_i^2 + Z^2,\ b =- 2k_1m_i, c = k_1^2 - Z^2$, again, with quadratic formula we have the final equation of β_1, that is $\beta_1 = \frac{2k_1m_1 \pm \sqrt{4k_1^2m_i^2 - 4(m_i^2+Z^2)(k_1^2-Z^2)}}{2(m_i^2+Z^2)}$, we get the β_1 value, where Z= $\phi^{-1}(CPD_A)$, special note that $m_i^2 + Z^2 \neq 0$, since the denominator can not be zero.

Move to Equation (B) and equation (C), let's find out the relationship between β_2 and CPD_B , in this case, we need to use equation (C), in the equation (B), similar to the β_1's derivation, we can get the β_2 equation, that is, $\beta_2 = \frac{2k_2m_j \pm \sqrt{4k_2^2m_j^2 - 4(m_j^2+Z^2)(k_j^2-Z^2)}}{2(m_j^2+Z^2)}$, where Z= $\phi^{-1}(CPD_B)$ and $m_j^2 + Z^2$ can not be zero.What if we don't have the m_j's value , we can calculate that with equation (C) , that is

$$m_j = \beta_1 m_i + \sqrt{1 - \beta_1^{\ 2}}\,\varepsilon_i .$$

Conclusion

Two-Factor Models indicate that in a market, two companies with two assets have a correlation between each other. We can find out the beta coefficient and CPD relationship more accurately than in a single factor model, however, it’s not as accurate and powerful as the Multi-Factor Model, we will introduce that in the future report.

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