Travelling Wave Solutions on a Cylindrical Geometry

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Abstract

Fluid equations are generally quite difficult and computationally-expensive to solve. However, if one is primarily interested in how the *surface* of the fluid deforms, we can re-formulate the governing equations purely in terms of free surface variables [1]. Reformulating the equations in such a way drastically cuts down on computational cost, and may be useful in areas such as modelling blood flow. Here, we study one such free-boundary formulation on a cylindrical geometry.

Introduction

The free-boundary formulation by Blyth and Părău [1] considers an irrotational, inviscid, and incompressible, cylindrical jet of fluid. They use this to model axially-symmetric period waves on the surface of the jet. Here, we re-create and verify their results.

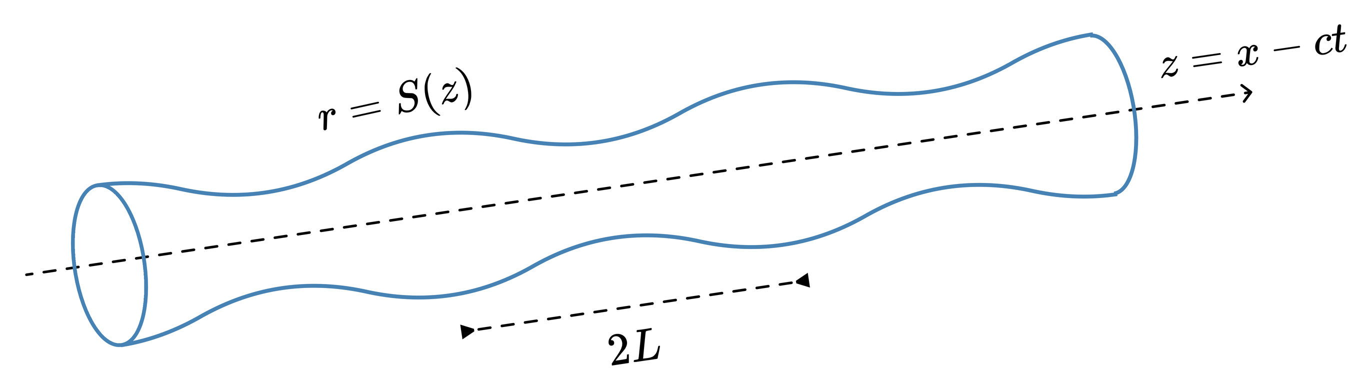

Figure 1: Periodic flow geomtery in the travelling frame

In the travelling frame, $(x, t) \to (z = x - ct, t)$ the governing equation at the free cylindrical surface is

$$\int_{-L}^{L} kS\left[(1+S_z^2)\left(\frac{c^2}{2} - \frac{1}{S(1+S_z^2)^{1/2}} + \frac{S_{zz}}{(1+S_z^2)^{3/2}} + \frac{B}{2S^2} + \mathcal{E}\right)\right]^{1/2} \times \left(K_1(kb)I_1(kS) - I_1(kb)K_1(kS)\right) e^{ikz}\, dz = 0 \quad (1)$$

where $k \in \mathbb{Z}^+$ and

unknowns $\begin{cases} S & \text{surface displacement} \\ c & \text{wave speed} \end{cases}$

B magnetic bond number

I_1 and K_2 modified Bessel functions

b central rod radius

Numerical Method

Define solution as truncated Fourier series:

$$S(z) \approx \sum_{n=0}^{N} a_n e^{in\pi z/L} \xrightarrow[\text{part}]{\text{real}} \sum_{n=0}^{N} a_n \cos n\pi z/L \quad (2)$$

The *N+1* unknown Fourier coefficients, along with the unknown wave-speed c results in a total of *N+2* unknowns for each point on the bifurcation branch:

$$[c, a_0, a_1, a_2, \cdots, a_N] \quad (3)$$

The surface condition (1) gives us *N* equations, one for each value of $k \in [1, N]$ and we define 2 more equations:

$$\begin{aligned} a_0 - 1 &= 0 \\ |a_1 - \alpha| &= 0 \end{aligned} \quad (4)$$

where α is a parameter we set in order to step up the bifurcation branch and compute solutions of increasing amplitude.

N+2 equations for N+2 unknowns ⇒ solved using Newton's method in MATLAB

Results

We compute periodic travelling wave solutions in two different regimes by varying a combination of the magnetic bond number B and half-domain length L.

Figures 2 and 3 show wave profiles without and with Wilton ripples (respectively) along with their bifurcation branches, **where each point corresponds to a unique wave profile**.

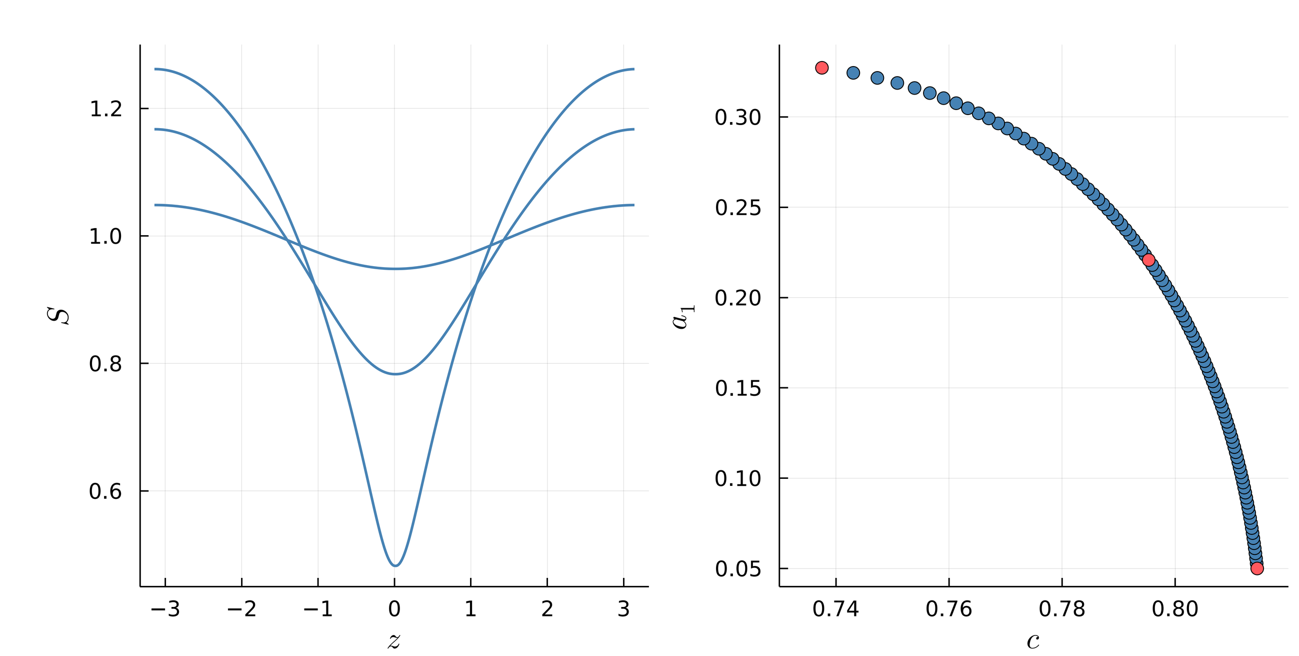

Figure 2: Solution branch (right) corresponding to B=1.5 and L=π, along with (left) three representive wave profiles

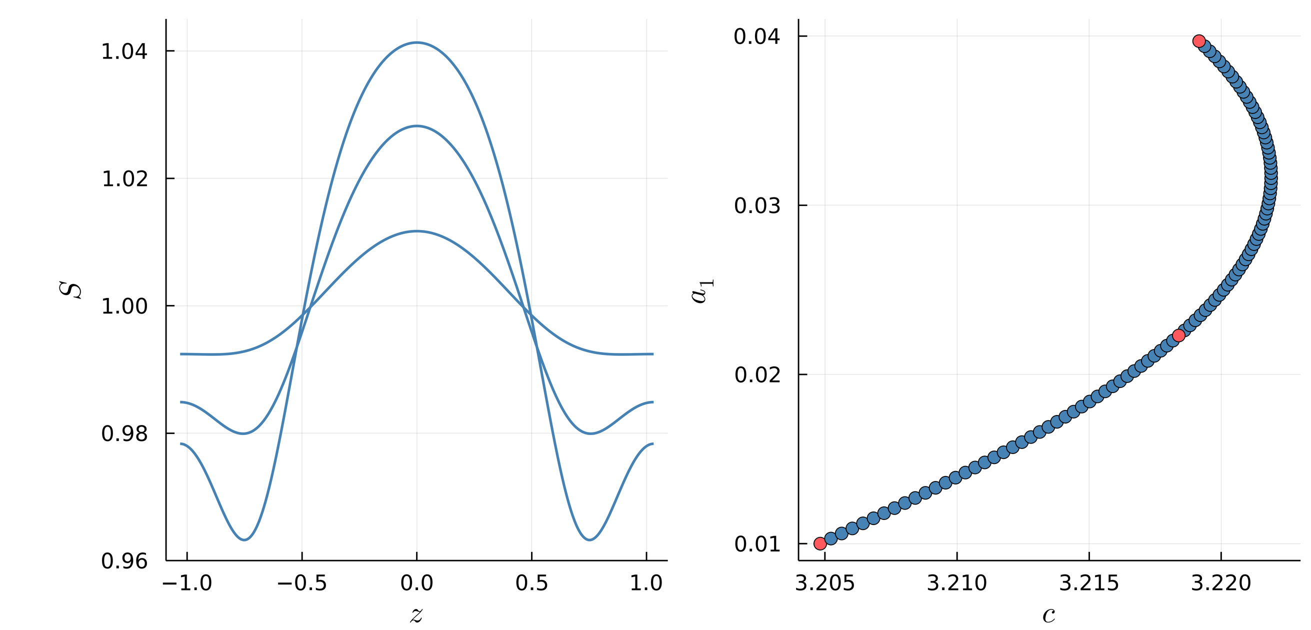

Figure 3: Solution branch (right) corresponding to *B*=30 and *L*=1.0305, along with (left) three representive wave profiles

Conclusions & Future Work

Our solutions agree with those achieved by Blyth and Părău, and therefore we have successfully re-produced their results. This work is now well-poised to be expanded upon. Future work includes:

- Stability analysis on this cylindrical jet formulation (on-going)
- Modifying the formulation by including a flexural term to model elastic blood-vessel walls

Acknowledgments

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References

[1] Ablowitz, M., Fokas, A., & Musslimani, Z. (2006). On a new non-local formulation of water waves. *Journal of Fluid Mechanics*, 562, 313-343.

[2] Blyth, M., & Părău, E. (2019). The Nonlocal Ablowitz-Fokas-Musslimani Water-Wave Method for Cylindrical Geometry. *SIAM Journal on Applied Mathematics*, *79*(3), 743–753.