10-2006

Optimal Release of Inventory Using Online Auctions: The Two Item Case

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DRAFT 2006-Oct-03 - comments welcome, do not cite or distribute without permission

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In this paper we analyze policies for optimally disposing inventory using online auctions. We assume a seller has a fixed number of items to sell using a sequence of, possibly overlapping, *single-item* auctions. The decision the seller must make is when to start each auction. The decision involves a trade-off between a holding cost for each period an item remains unsold, and a higher expected final price the fewer the number of simultaneous auctions underway. Consequently the seller must trade-off the expected marginal gain for the ongoing auctions with the expected marginal cost of the unreleased items by further deferring their release. We formulate the problem as a discrete time Markov Decision Problem and consider two cases. In the first case we assume the auctions are guaranteed to be successful, while in the second case we assume there is a positive probability that an auction receives no bids. The reason for considering these two cases are that they require different analysis. We derive conditions to ensure that the optimal release policy is a control limit policy in the current price of the ongoing auctions, and provide several illustration of results. The paper focuses on the two item case which has sufficient complexity to raise challenging questions.

Key words: Online Auctions; Price Cannibalization; Strategic Auction Release; Markov Decision Process

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1. Introduction

Though Internet based auctions, or online auctions, have been around since the early dot-com era, it is not until rather recently that their importance in eCommerce has developed. From being mainly regarded as Internet based flea-markets for the *Consumer-to-Consumer* (C2C) markets, their importance and presence in the *Business-to-Consumer* (B2C) markets has and continues to rapidly grow. Today many well-established firms operate online auctions not only as complementary sales channels but also as strategic tools in pricing and product introduction decisions. One common use of online auctions is as alternative salvage channels. Dell, IBM, Sharp, Sears, and Fujitsu-Siemens, use their *eBay stores* to sell leased and returned items, as well as any excess inventory. Some companies and organizations, such as Toshiba, Dell, CompUSA, and Major League Baseball, host their own online auctions. Though there are a plethora of web-sites that host online auctions, the most dominant online auction 'house' is ebay.com. In 2005, eBay had 181 million registered users, and 1.9 billion listings for a total sales volume of over $44 billion, up from 135 million users, 1.4 billion listings and sales volume of $34 billion in 2004 (eBay Annual Report 2005, 2004). To put this in perspective, the *US Census Bureau* estimated the 2005 eCommerce segment of US retail sales to account for close to $88 billion (about 2.4% of total US retail sales; US Census Bureau News CB06-66). In 2005, the 7^{th} largest retailer, *Lowe's*, had an annual sales totaling $43.2 billion (stores.org Top 100 Retailers 2006).

The motivation for this paper stems from discussions with Truition and Marketworks; two companies that assist established firms develop eCommerce strategies and infrastructure for setting up online auctions. One of the main question raised in these discussions was: 'How should a company optimally release items for auction in order to maximize their profit?' The objective of this paper is to provide a model for selling a fixed inventory of goods using a sequence of *single item* auctions, and to provide structural results regarding the optimal release policy. More specifically, how should a seller, given N identical items, optimally release each individual item for auction in order to maximize the total profit. We assume all auction parameters, such as *auction length*, *starting price*, and *bid increment*, have been pre-set and that the only decision to make regards the timing of releasing each item for auction. The trade-off that makes the problem interesting is that, on the one hand, the seller incurs a holding cost for each period an item remains unsold, while on the other hand, the more on-going auctions the seller has, the lower the expected final price in each of those auctions. In other words, competing auctions 'cannibalize' each other. Figure 1 displays the final price of 115 laptop auctions at the eBay store *Dell_Financial_Services*. It shows that the final price is on average decreasing in the number of auctions, suggesting some extent of cannibalization.

The holding cost represents, in addition to the usual components such as cost of capital, insurance, and space, the value deprecation of an item. For example, over the first half of 2006 the average selling price for the specific Dell laptop shown in Figure 1 decreased with about $200, or about $1 per day. When the holding cost is 'low' it will never be optimal

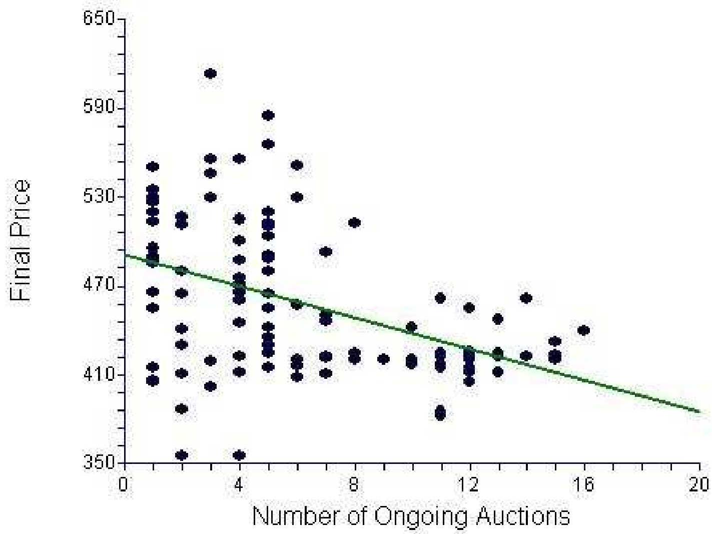


Figure 1: Final price as a function of the number of ongoing auctions of 115 Dell laptop (512MB, 30GB, 1.8MhZ, Intel Pentium 4) auctions at the eBay store *Dell_Financial_Services*. All auctions lasted for three days and took place between 15th of December, 2005, and 30th of June, 2006. The line represents the least square linear regression.

to have more than one auction underway. This is because the fewer the number of ongoing auctions the higher the expected final price for each of the auctions, due to cannibalization. Therefore, the optimal release policy is to wait until the current auction is completed before releasing the next item; that is, to hold N non-overlapping *sequential* auctions. On the other hand, if the holding cost is 'high' then it will never be optimal to delay the release of an item and instead all items should be released simultaneously. The reason for this is that the additional holding cost from deferring will exceed the gain in expected final price by having fewer ongoing auctions. The optimal release policy is to hold N *simultaneous* auctions (all overlapping and note that this is different from one N-item auction). We will mainly be interested in situations where the holding cost has some strategic implication and the optimal policy is not one of the extreme policies. Furthermore, we will provide conditions that show that the optimal release policy is dependent on the state of the ongoing auctions or *closed loop*, in contrast to a state independent or *open loop* policy. The two extreme open loop policies are the sequential respectively simultaneous release policies discussed above. Figure 2 below highlights the above discussion. The figure depicts the expected total profit as a function of the per period per item holding cost for a numerical example of two items and auction length of three periods (see Section 3.4). We see that for low holding costs the optimal policy is to release the auctions sequentially, while for high costs, the optimal policy is to release the auctions simultaneously. For cases in between, the optimal policy depends on the price in the current auction and dominates the open loop policies.

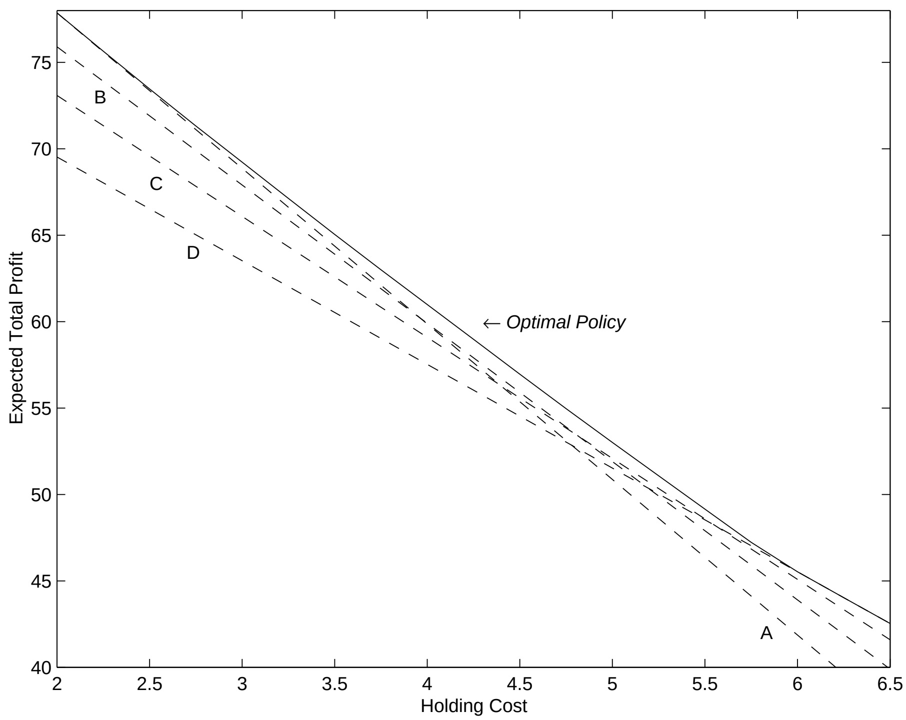


Figure 2: Two item, three period numerical example of expected total profit as a function of the per item per period holding cost. The four dashed lines represent the open loop policies; non-overlapping sequentially release (A), release with two day overlap (B), release with one day overlap (C), and simultaneously release (D). The solid line that lays above them represents the total expected profit for the optimal policy, which is closed loop. See Section 3.4 for details regarding formulation and computation.

The problem we are addressing falls into one of the four categories of open research areas described by Pinker, Seidmann, and Vakrat (2003). Namely how could (or should) a firm integrate online auctions into their business model. Similarly, the Internet auctions review by Bajari and Hortacsu (2004) states that more research needs to be done regarding 'the analysis of markets with multiple simultaneous auctions'. Our hope is that this paper can provide a structural framework, insights and results regarding this issue.

Though it may appear to be an oversimplification, this paper considers the two item case. The two item case provides sufficient complexity to be both interesting and give rise to some surprising results. Furthermore, this will enable the discussion to focus on the trade-off between releasing and deferring the release and not become convoluted by the combinatorial complexity of the N item case. It should also be noted that this problem has not yet been addressed in the existing auction theory or inventory literature. Previous research has mainly focused on the analysis of an isolated single auction, either *single-item* or *multi-item* auctions, and in the case of multiple auctions only considered non-overlapping sequential auctions. And in particular, most research questions have been with regard to the impact of various auction parameters, such as, starting price, reserve price, bid increment, auction length, and lot size. The novelty of this paper is that it provides a framework for analyzing the issue of strategic timing of auctions when auctions compete or 'cannibalize' on each other.

1.1 Related Literature

In recent years auction theory has come to play an important role in the management science and revenue management field, resulting in a wide spectrum of applications of auction theory. However, given the voluminous literature on inventory management and dynamic pricing, relatively little has been written with regards to inventory management using online auctions. Two papers that consider the impact auctions have on the inventory re-ordering policy are Vulcano and van Ryzin (2004), and Huh and Janakiraman (2006). Vulcano and van Ryzin focus on how a seller should optimally choose the auction format and how this decision will affect the optimal inventory re-ordering policy. They formulate the problem as an infinite horizon dynamic program and show what the joint auction-format and replenishment policy is. Huh and Janakiraman show that using auctions as a sales channel will satisfy conditions to ensure that (s, S) policies are optimal. Vulcano and van Ryzin have previously analyzed a problem that is similar to the one we address (Vulcano, van Ryzin, and Maglaras 2003). There they consider a seller, who given a fixed inventory and fixed time-horizon, has to optimally 'auction' off the goods . The underlying 'auction' mechanism they consider is in the spirit of *priceline.com* where people place 'bids' and sellers can choose to accept or reject the offers. They model each *multi-item* 'auction' as a separate period and perform the symmetric equilibrium analysis for each period (auction). Another related paper is Pinker, Seidmann, and Vakrat (2001), where they analyze the problem of optimally disposing a given inventory using a sequence of non-overlapping *multi-item* online auctions, and solve for the symmetric equilibrium analysis given uniform valuations. Their objective is to categorize the optimal number of multi-item auctions and the optimal unit to release in each auction. In contrast to these papers, we permit the auctions to overlap and analyze the auction dynamics as a Markov chain.

The above four papers all use a game theoretic approach. A paper which uses a different analysis methodology is Bertsimas, Hawkins and Perakis (2003). The problem they address is how a seller should optimally set the auction control parameters *starting price*, *reserve price* and *auction length*, in order to maximize revenue. They model the problem as a Markov Decision Problem and with empirical data from over 17,000 eBay auctions determine the optimal parameters. Bapna, Goes, and Gupta (2003) also address the issue of optimal auction control parameters in a revenue management context. The main focus of their analysis is to highlight the importance and structural implication of the *bid increment* in a *first-price multi-item* auction. Using data from 90 online auctions they empirically validate their findings. The two common elements of the above literature is that they focus on the optimal setting of auction parameters and analyze each auction in isolation. In contrast, we model the optimal release or timing of auctions given fixed auction parameters and a dynamic interaction between competing auctions.

A paper that does analyze the dynamics between competing auctions is Peters and Severinov (2002). They take the other extreme case and consider all auctions to be simultaneously released. They present a model with M bidders and N *single-item* auctions, where both M and N are fixed, and derive the Bayesian-Nash equilibrium for the final price of the N auctions. In particular they show that the final price will be the same for all auctions. Though they are implicitly assuming an online setting, there is nothing explicit in their model that incorporates the special dynamics of online auctions, such as the arrival rate of bidders or fixed auction dead-line.

Overview of Paper

The remaining paper is organized as follows. In Section 2 we state the problem and formulate it as an MDP model. In Section 3 we discuss the case when the auctions are guaranteed to be successful, and hence the seller only has to list an item once. While in Section 4 we discuss the case when there is a positive probability an auction receives zero bids and the seller has to re-list items from unsuccessful auctions. In Section 5 we summarize our conclusions and provide ideas for future research. All proofs are included in Appendix 1. For ease of discussion, both the seller and bidders will be referred to as he, with no gender bias intended.

We write *increasing* (*decreasing*) instead of *non-decreasing* (*non-increasing*). In addition, we will refer to *non-overlapping sequentially* released auctions as simply *sequentially* released auctions, and write *auction* instead of *online auction*.

2. Model Formulation

Before we formulate the MDP model we begin with a brief description of the underlying auction dynamics. We will not provide a rigors framework for how auctions can be modeled as stochastic processes, but only sketch out the main ideas regarding the stochastic elements driving the price transitions. For a detailed discussion regarding modeling auctions by stochastic processes, see Segev, Beam, and Shantikumar (2001), and the PhD thesis Odegaard (2007). Though we are not explicitly modeling a specific online auction or seller, it may be illustrative to consider the eBay auction format and, for instance, the eBay store *Dell_Financial_Services*.

2.1 A General Framework for Modeling Online Auctions

We assume a seller has two auctions underway. Each auction i, $i = 1, 2$, is at time t defined by its *current price* (or *bid*) $X_{i,t}$ and its *elapsed auction time* $Y_{i,t}$. The state of the auctions at time t will be written as $[X_1, Y_1; X_2, Y_2]_t$. The auctions are ordered according to elapsed time ($Y_1 \geq Y_2$). We assume potential bidders arrive according to a non-stationary Poisson process with rate λ_t. After arriving, a bidder observes the state of the auctions. If an arriving bidder's valuation $V \leq \min\{X_1, X_2\}$ he leaves the auction site. If $\min\{X_1, X_2\} < V \leq \max\{X_1, X_2\}$ he places a bid $b_t(V, (X_j, Y_j))$ in the lower priced auction j, $j = argmin\{X_1, X_2\}$. If $V > X_2 \geq X_1$ he places a bid $b_t(V, [X_1, Y_1]) > X_1$ in the first auction, since it will end sooner and has a lower price. If, however, $V > X_1 > X_2$ then a fraction α of the bidders will place a bid $b_t(V, [X_1, Y_1]) > X_1$ in auction 1, while $1 - \alpha$ will place a bid $b_t(V, [X_2, Y_2]) > X_2$ in auction 2. The reason why some bidders will choose to participate in the more expensive auction 1 is because they prefer to receive the item sooner and thus trade-off the expected final price and time between the two auctions. After a bidder places a bid, the current price of that auction is updated according to the governing auction rules. When the next bidder arrives, s time-units later, he observes either $[X_1', Y_1'; X_2, Y_2']_{t+s}$ or $[X_1, Y_1'; X_2', Y_2']_{t+s}$. Where $Y_i' = \min\{Y_i + s, \tau\}$, τ the fixed auction length, and $X_i' = f(b_t(V, [X_i, Y_i]))$, $i = 1, 2$, for a well-defined function $f(\cdot)$. This process continues until the first auction is over, $Y_1 = \tau$, at which point the seller announces the winner of that auction, awards the item, and collects the payment. The dynamics for the remaining time of the second auction follows a similar but much simpler structure. For the remainder of the paper we assume that the transitions of X_i can be summarized by either a transition probability matrix or a conditional cumulative distribution function, that may be the result of the above process. Though we are considering a general auction setting there are some specific 'behavioral' assumptions we impose. These assumptions, or rather their mathematical translation, will ensure that certain structural results will follow. A more detail discussion regarding the specifics of this will follow.

2.2 The Problem

We are considering a seller who, over a planning horizon T, intends to sell two identical items using a pair of single-item auctions. Each auction is assumed to have the same fixed time-length $\tau < \infty$. We divide τ into a sequence of discrete periods such that each auction period coincide with the length of the discrete time periods that constitute T. The seller decides at the start of each period whether or not to release an item for auction. It is important to emphasize that an ongoing auction does not have to be completed before the next auction is started; auctions may overlap each other. We model the seller's problem as a discrete time *Markov Decision Problem* (MDP) with the objective of maximizing expected total profit. Two cases will be considered. In the first case we assume the auctions are guaranteed to be successful and hence the seller only has to list an item once. Since the seller only has two items, the seller is faced with a finite planning horizon 2τ. In the second case, we assume there is a positive probability that an auction is unsuccessful, meaning that no bids arrived, and that the seller has to re-list unsold items. Consequently the seller is faced with an infinite planning horizon. The reason for separating the two cases is that they require different models and analyses.

We will throughout the paper assume two fundamental aspects regarding the seller. The first is that the seller would only be interested in selling via auctions if the accumulated holding cost over the duration of an auction is compensated by the expected final price. We summarize this in the following lemma and refer to it as *the positive expected profit* assumption.

Lemma *If the expected revenue from an auction does not compensate the holding cost accumulated over the auction duration, then it is optimal to immediately dispose of the items.*

The second assumption is that the seller is vigilant in keeping track of how many items he has released for auction and how many are remaining, and that he has no reason to speculatively hold inventory. Summarized as follows and referred to as *the vigilant seller* assumption.

Lemma *If the price dynamics of an auction are independent of time and the holding cost is positive then it will always be optimal to have at least one auction underway while there still is remaining inventory.*

In other words, if there are no auctions underway but the seller still has items remaining he should always start at least one auction. Thus at the start of the planning horizon, he should always start at least one auction. This lemma is equivalent to Lemma 1 in Pinker, Seidmann and Vakrat (2001), where a proof is provided.

2.3 Markov Decision Problem Formulation

To formulate the seller's problem as an MDP, we require the following elements.

Decision Epochs, $t = 0, 1, \ldots, T$

We assume discrete time periods of equal length and that decisions are made at the beginning of each period. We are implicitly regarding T as a fixed number of days and assume decisions are made at the start of each day. However, for a general framework where, for instance, decisions are made more frequently, say hourly, T could be increased to reflect the appropriate planning horizon. We consider two cases, $T < \infty$ and $T = \infty$.

State Space

At each decision epoch t, the system state, $S = ([X_1, Y_1; X_2, Y_2], Z)$, consists of the state of each auction, $[X_i, Y_i]_{i=1,2}$, and the number of ongoing auctions Z. Each auction i, $i = 1, 2$, is defined by the pair of random variables *current price* (*bid*), X_i, and *elapsed auction time*, Y_i. We will consider both discrete and continuous prices. For the discrete case $X_i \in \{0, p, p+k, p+2k, \ldots, P\}$, where p, k and P are positive, finite integers. While for the continuous case $X_i \in \{0\} \cup [p, P]$, where $[p, P] \subset \Re_+$. In both cases, p is the starting price of the auction, P the upper limit of what any bidder would be willing to bid, and for discrete prices, k is the price-increment. We assume Y_i is discrete and finite, $Y_i \in \{0, 1, \ldots, \tau\} \cup \{\delta\}$, where $\tau < \infty$. The symbol δ is used to indicate that the auction is completed and the item awarded. We will interchangeably use the notation X_{i,Y_i} and (X_i, Y_i) to denote the state of an auction. For instance, $X_{i,\tau}$ is the final price of auction i.

At the start of an auction $Y_i = 0$ and $X_i = 0$. For each additional period an auction is *underway* Y_i increases by one. When an auction has successfully been completed, that is

$X_{i,\tau} \geq p$, the item is awarded and payment received. In this case, the state of the auction evolves as follows, for $p \leq p_i \leq p_i'$,

$$\ldots \Longrightarrow (p_i, \tau - 1) \Longrightarrow (p_i', \tau) \Longrightarrow \Delta_i \Longrightarrow \Delta_i \Longrightarrow \ldots$$

where $\Delta_i = (X_i, \delta)$. If an auction is unsuccessful, that is $X_{i,\tau} = 0$, the auction returns to the initial state $(0, 0)$, that is the transitions follow,

$$\ldots \Longrightarrow (0, \tau - 1) \Longrightarrow \begin{cases} (q, \tau) & \text{w. prob.} \Pr\{X_{i,\tau} = q \mid X_{i,\tau-1} = 0\} \\ (0, \tau) \equiv (0, 0) & \text{w. prob.} \Pr\{X_{i,\tau} = 0 \mid X_{i,\tau-1} = 0\} \end{cases}$$

The reader familiar with auctions or auction theory, may notice that we have not included a *reserve price*. Including a reserve price will not change our analysis, and omitting it simplifies matters. We let Δ denote the (absorbing) state when both items have been sold, $\Delta = ([\Delta_1; \Delta_2], 0)$.

Though it may appear redundant we include a counter Z of the number of ongoing auctions. The number of ongoing auctions at time t will be defined by Z_t. In order to avoid issues with Z_t in decision epochs where an auction will be started by the vigilant seller assumption, we define Z_t to be the number of ongoing auctions in the *instantenous moment* before decision epoch t, before any price jumps have occurred and before the seller has made a non-trivial or relevant decision. For instance, at the start of the planning horizon $Z_0 = 1$.

As a minor notational convention, we will avoid double parenthesis for functions where the state space is the only argument, that is we write $f([X_1, Y_1; X_2, Y_2], Z)$ instead of the strictly correct $f(([X_1, Y_1; X_2, Y_2], Z))$.

Actions

The only non-trivial decision facing the seller is to decide when to release an item provided that the current auction has not yet been successfully completed. In other words, in states when $Y_i < \tau$ and $Y_j = 0$, $i \neq j$. Under all other conditions, the seller either does not have any decision to make or will release an item due to the vigilant seller assumption. At each decision epoch, the actions $a = 1$ corresponds to releasing the remaining item, and $a = 0$ not to release it. Furthermore, because the items are identical, one can without loss of generality, define the remaining item to be item 2. Consequently the action space is,

$$A_{([p_1,t_1;p_2,t_2],z)} = \begin{cases} \{0, 1\} & t_1 < \tau \text{ and } t_2 = 0 \\ \{0\} & t_1 = \tau, \delta \text{ or } t_2 > 0 \end{cases}$$

In Figure 3 a simple example, with $X_i = 0, 1$ and $\tau = 2$, illustrates how the system state may evolve. States enclosed in a box indicate situations with non-trivial decisions. Transitions due to the non-trivial decision of releasing the second item are represented by the dotted lines. Transitions due to not releasing or releasing due to the vigilant seller assumption are represented by the solid lines. Note that there are four possible loops: $([0,0;0,0],1) \rightleftharpoons ([0,1;0,1],2)$, $([0,0;0,0],1) \rightleftharpoons ([0,1;0,0],1)$, $([0,1;0,0],1) \rightleftharpoons ([0,1;0,0],1)$, and $([1,\delta;0,0],1) \rightleftharpoons ([1,\delta;0,1],1)$. And that there is one absorbing state $\Delta = ([1,\delta_1;1,\delta_2],0)$.

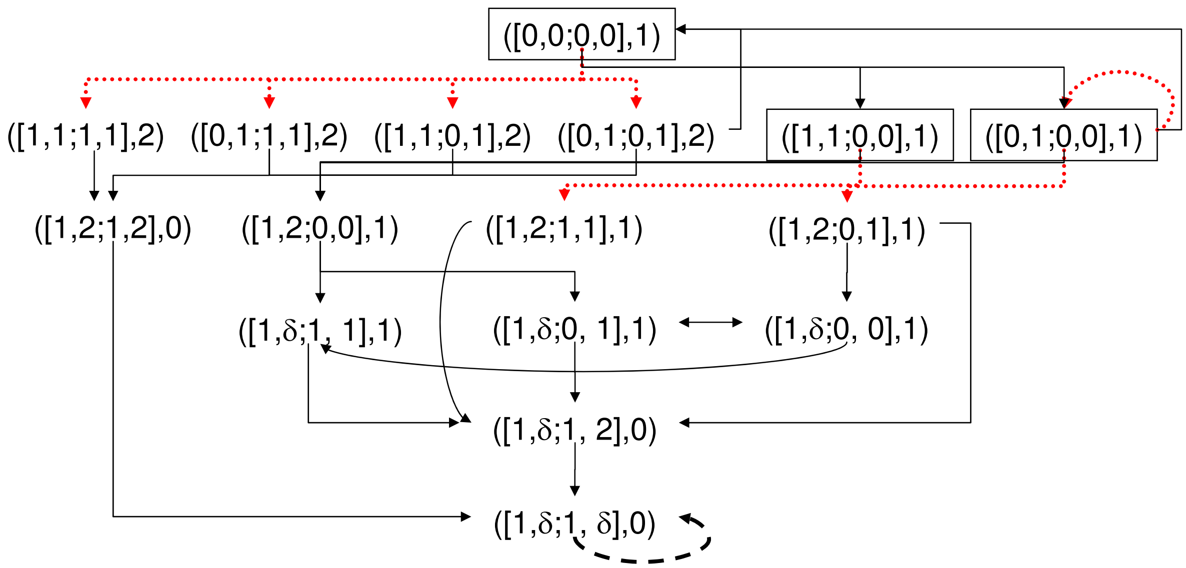


Figure 3: Example of system state transitions when $X_i = 0, 1$ and $\tau = 2$. States enclosed in a box indicate situations with non-trivial decisions. Solid lines represents transitions due to not releasing or release by vigilant seller assumption; dotted lines represents transitions due to non-trivial release decisions; dashed line represent absorbing cycle.

Rewards

For each period in which an item has not been sold, the seller incurs a positive holding cost h. When an auction is successfully completed the seller receives the payment and awards the item. After an item has been sold and the state $(X_i, Y_i) = \Delta_i$, $i = 1, 2$, the seller will in perpetuity neither incur any cost nor receive any payment for that item. Let $r_t(s)$ denote the reward in period t given a state $s \in S$. It is given by,

$$r_t([p_1, t_1; p_2, t_2], z) = p_1 1_{\{t_1 = \tau\}} - h 1_{\{t_1 < \tau\}} + p_2 1_{\{t_2 = \tau\}} - h 1_{\{t_2 < \tau\}}$$

Transition Probabilities

Each period in which an auction is underway the price increments follow the dynamics of an *exogenously* given stochastic process. In other words, we assume that there is some underlying bidder behavior model, such as the one described above, and more importantly that one can summarize the distribution of the one period price transitions. For discrete prices, these are represented by the following transition probability matrices,

$$\Pi_1 = \begin{pmatrix} \pi_{0,0|1} & \pi_{0,p|1} & \cdots & \pi_{0,P|1} \\ 0 & \pi_{p,p|1} & \cdots & \pi_{p,P|1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pi_{P,P|1} \end{pmatrix} \qquad \Pi_2 = \begin{pmatrix} \pi_{0,0|2} & \pi_{0,p|2} & \cdots & \pi_{0,P|2} \\ 0 & \pi_{p,p|2} & \cdots & \pi_{p,P|2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pi_{P,P|2} \end{pmatrix}$$

Π_z, $z = 1, 2$, is the one-period transition probability matrix for an individual auction when there are z auctions underway, for $Y_i = t_i < \tau$, $z = 1, 2$, and $p_i \leq q$,

$$\Pr\{X_{i,t_i+1} = q \mid X_{i,t_i} = p_i, Z = z\} = \pi_{p_i,q|z} \tag{1}$$

In the case of continuous prices, we assume the price transition dynamics can be represented by a conditional cumulative distribution function, for $Y_i = t_i < \tau$, $z = 1, 2$, and $w \leq x$,

$$\Pr\{X_{i,t_i+1} \leq x \mid X_{i,t_i} = w, Z = z\} = F^z_{X_{i,t_i+1}|X_{i,t_i}}(x|w) = \int_w^x f^z_{X_{i,t_i+1}|X_{i,t_i}}(q|w)dq \tag{2}$$

Where $f^z_{X_{i,t_i+1}|X_{i,t_i}}(\cdot|w)$, $z = 1, 2$, is the one-period conditional transition probability density function for an individual item when there are z auctions underway and $X_{i,t_i} = w$.

Using the Chapman-Kolmogorov equations (cf.*Chapter 4.2*, Ross 1996), the n-period transition probabilities for a single auction can be derived. To illustrate, assume prices are discrete and we are interested in the two- and three-period transition probability of auction 1, and that there are z_1, z_2, and z_3 auctions underway in the first, second, and third period respectively, for $Y_1 = t_1 \leq \tau - 3$,

$$\Pr\{X_{1,t_1+2} = q \mid X_{1,t_1} = p_1, Z_t = z_1, Z_{t+1} = z_2\} = \sum_{j=p_1}^{q} \pi_{p_1,j|z_1}\pi_{j,q|z_2}$$

$$\Pr\{X_{1,t_1+3} = q \mid X_{1,t_1} = p_1, Z_t = z_1, Z_{t+1} = z_2, Z_{t+2} = z_3\} = \sum_{j=p_1}^{q}\sum_{k=j}^{q} \pi_{p_1,j|z_1}\pi_{j,k|z_2}\pi_{k,q|z_3}$$

Consequently, to derive the probability distribution of the final price we simply multiply the transition probability matrices accordingly. For instance, suppose $\tau = 3$ then the top row in $\Pi_{z_1}\Pi_{z_2}\Pi_{z_3} \equiv \Pi_{z_1 \cdot z_2 \cdot z_3}$ provides the unconditional probability distribution of the final price for an item with z_1, z_2, and z_3 auctions in the first, second, and third period respectively. For continuous prices and $Y_1 = t_1 \leq \tau - 3$,

$$f^{z_1 \cdot z_2}_{X_{1,t_1+2}|X_{1,t_1}}(x|w) = \int_w^x f^{z_2}_{X_{1,t_1+2}|X_{1,t_1+1}}(x|u) f^{z_1}_{X_{1,t_1+1}|X_{1,t_1}}(u|w)du$$

$$f^{z_1 \cdot z_2 \cdot z_3}_{X_{1,t_1+3}|X_{1,t_1}}(x|w) = \int_w^x \int_w^v f^{z_3}_{X_{1,t_1+3}|X_{1,t_1+2}}(x|v) f^{z_2}_{X_{1,t_1+2}|X_{1,t_1+1}}(v|u) f^{z_1}_{X_{1,t_1+1}|X_{1,t_1}}(u|w)dudv$$

2.4 Bidding Behavior Assumptions

We will next provide some additional assumptions regarding the transition probabilities. These assumptions, which represents some specific bidding behavior, will ensure that certain structural results will follow. The assumptions should not be regarded as categorical statements about all bidders, but rather as a statistical reflection of what the bidding behavior is like in the majority of auctions. In order to validate our assumptions we collected auction data at the eBay store *Dell_Financial_Services* from December 2005 to August 2006. Dell Financial Services (DFS), a sub-sidary of Dell, provides leasing programs for Dell products. After products are returned DFS will refurbish them and sell via their own auction site as well as their eBay store. Due to space limitations will we not provide detailed statistical validation and refer the reader to the PhD thesis Odegaard (2007).

When two auctions are underway we assume that the auction prices evolve independently. That is, the price in one auction does not affect the transition dynamics of the other auction. In other words, for discrete prices and $Y_i = t_i < \tau$, $i = 1, 2$,

$$\begin{aligned} \Pr\{X_{i,t_i+1} = q \mid X_{1,t_1} = p_1, X_{2,t_2} = p_2, Z = 2\} &= \Pr\{X_{i,t_i+1} = q \mid X_{i,t_i} = p_i, Z = 2\} \\ &= \pi_{p_i,q|2} \end{aligned}$$

While for continuous prices and $Y_i = t_i < \tau$, $i = 1, 2$,

$$\begin{aligned} \Pr\{X_{i,t_i+1} \leq x \mid X_{1,t_1} = w_1, X_{2,t_2} = w_2, Z = 2\} &= \Pr\{X_{i,t_i+1} \leq x \mid X_{i,t_i} = w_i, Z = 2\} \\ &= F^2_{X_{i,t_i+1}|X_{i,t_i}}(x|w_i) \end{aligned}$$

Implicitly this assumes that bidders choose a bid-amount only based on the current price and elapsed auction time of the auction they are placing a bid in. Consequently, with two auctions underway, the transition probability for the system state is simply the product of the individual transition probabilities. For discrete prices and $t_1, t_2 < \tau$, we have $\Pr\{([q, t_1 + 1; r, t_2 + 1], z') \mid ([p_1, t_1; p_2, t_2], 2)\} = (\pi_{p_1,q|2})(\pi_{p_2,r|2})$. Therefore the main interesting system state transition probabilities, $\pi(s'|s, a)$, are for those states $s \in S$ such that $A_s = \{0, 1\}$. For discrete prices, $s = ([p_1, t_1; 0, 0], 1)$ and $t_1 < \tau$,

$$\pi(s'|s,a) = \begin{cases} \pi_{p_1,q|1} & a = 0 \qquad s' = ([q, t_1 + 1; 0, 0], 1) \\ (\pi_{p_1,q|2})(\pi_{0,r|2}) & a = 1 \qquad s' = ([q, t_1 + 1; r, 1], 1 + 1_{\{t_1+1<\tau\}}) \end{cases}$$

The case for continuous prices is defined similarly.

We assume bids are non-retractable, which for the case of discrete prices implies that Π_z, $z = 1, 2$, are upper-triangular ($\pi_{q,p_i|z} = 0$ for $q < p_i$). While for continuous prices, we require that $F^z_{X_{i,t_i+1}|X_{i,t_i}}(x|w) = 0$ for $x < w$, $z = 1, 2$. Consequently, the current price of an auction is increasing. Though strictly speaking on, for instance, eBay, bidders may retract a bid, it is very rare.

Transition probabilities are assumed to be *stationary* with respect to both: 1) calender time t, and 2) elapsed auction time Y_i, $i = 1, 2$. We make this simplifying assumption in order to ensure the model is tractable. In reality, the dynamics of X_i may depend on calender time. For instance, at night and weekends there tends to be less bidding activity. We will for simplicity ignore this and strictly consider stationary transition probabilities with regard to calender time. Likewise we will ignore non-stationary transitions with regard to the elapsed auction time. A well-established phenomena of online auctions, is that the price dynamics or bidding behavior is dramatically different toward the end of auctions. Our model can be easily modified to account for this. One reason for this, is because some bidders try to place their bids as close as possible to the end of the auction, thereby leaving no time for others to counter-bid. This is referred to as *sniping*. Roth and Ockenfels (2004), and Shmueli, Russo, and Wolfgang (2004) analyze different aspects regarding non-stationary bidding activity.

The next set of assumptions play a more crucial role in the ensuing analysis. Each is stated in the discrete and continuous prices case.

Assumption 2.1. *The probability of making a jump to the higher prices is increasing in the current price.*
Discrete prices: for $p_i < P$, $i = 1, 2$, $z = 1, 2$,

$$\sum_{q=r}^{P} \pi_{p_i,q|z} \leq \sum_{q=r}^{P} \pi_{p_i+1,q|z} \qquad \forall r \leq P \tag{3}$$

Continuous prices: for $w \leq x \leq P$, $i = 1, 2$, $z = 1, 2$,

$$F^z_{X_{i,t_i+1}|X_{i,t_i}}(x|w) = \int_w^x f^z_{X_{i,t_i+1}|X_{i,t_i}}(q|w)dq \qquad \textit{is decreasing in } w \tag{4}$$

Equivalently,

$$\frac{\partial}{\partial w} F^z_{X_{i,t_i+1}|X_{i,t_i}}(x|w) = \int_w^x \frac{\partial}{\partial w} f^z_{X_{i,t_i+1}|X_{i,t_i}}(q|w)dq - f^z_{X_{i,t_i+1}|X_{i,t_i}}(w|w) \leq 0$$

Assumption 2.1 means that bids are increasing in the current price. In other words, the likelihood of placing a 'high' bid is increasing in the current price. This holds for example if bid increments were independent of the current price. In reality, however, bid increments tend to be decreasing in the current price, and it is therefore not immediate that Assumption 2.1 holds. Empirical evidence supporting Assumption 2.1 and showing that bid increments are decreasing in the current price, can be seen in the sub-figures A through E in Figure 4 below. They depict the price-jumps at 12 hour intervals for the 115 auctions discussed in the Introduction. Each circle represents an individual auction. Auctions along the 45 degree angle are auctions in which the price remained unchanged 12 hours later (no transition took place). The feature supporting our assumption is that in all figures the price-jumps form an upward sloping 'band'. A counter indication to our claim would be if there was a large number of auctions that at low prices ($\approx$ \$0 – \$150) made jumps to the high prices ($\approx$ \$500 – \$600).

Assumption 2.2. *The probability of making a jump to higher prices decreases when there are two ongoing auctions.*
Discrete prices: for $p_i \leq P$, $i = 1, 2$

$$\sum_{q=r}^{P} \pi_{p_i,q|2} \leq \sum_{q=r}^{P} \pi_{p_i,q|1} \qquad \forall r \leq P \tag{5}$$

Continuous prices: for $w \leq P$, $i = 1, 2$

$$F^1_{X_{i,t_i+1}|X_{i,t_i}}(x|w) \leq F^2_{X_{i,t_i+1}|X_{i,t_i}}(x|w) \qquad \forall x \leq P \tag{6}$$

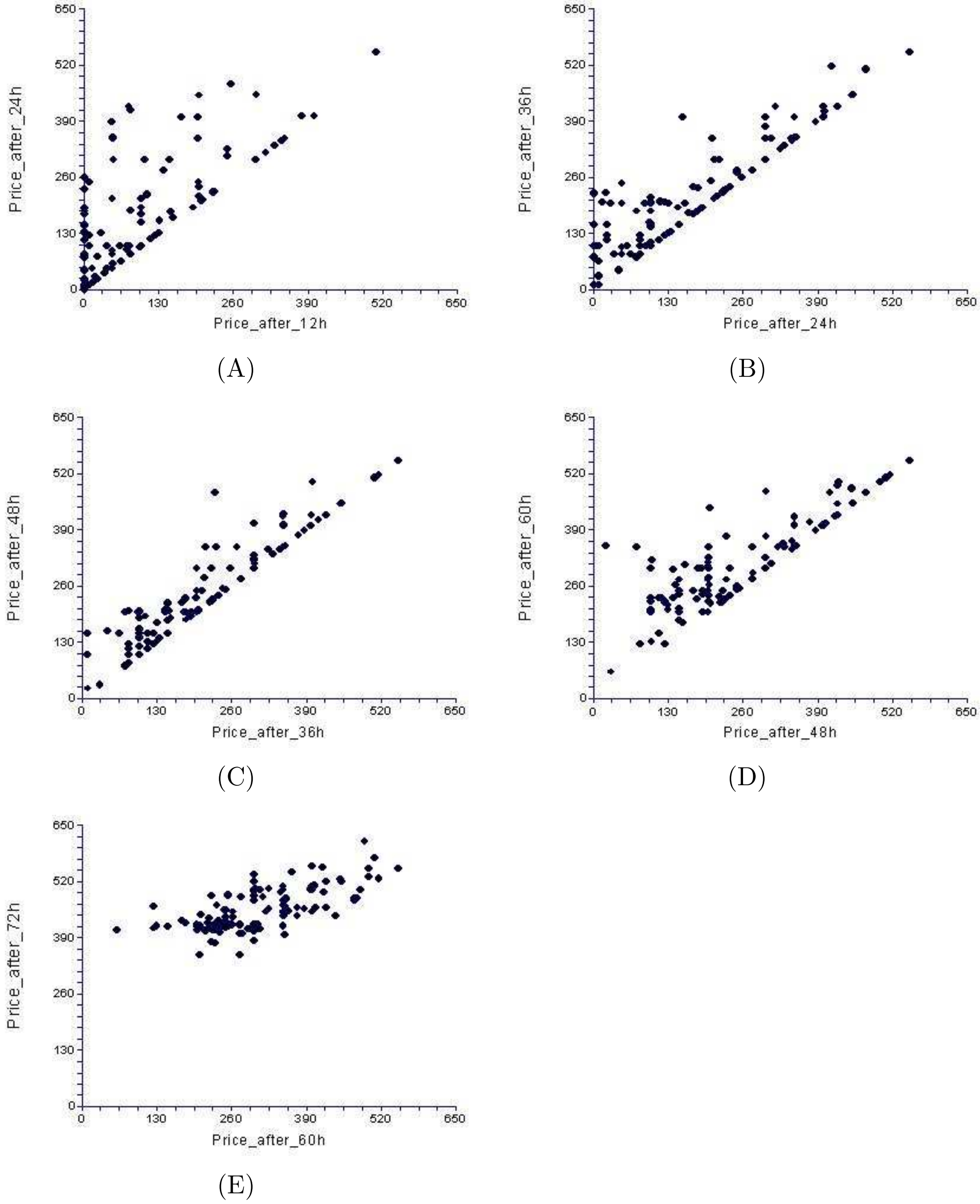


Figure 4: Price transitions at 12 hour intervals for 115 Dell laptop (512MB, 30GB, 18.MhZ, Intel Pentium 4) auctions. The horizontal axis represents the price at various 12 hour intervals, while the vertical axis represents the price 12 hours later. Each circle represents a three day auction that took place at the eBay store *Dell_Financial_Services* between 15th of December, 2005, to 30th of June, 2006.

This assumption formalizes how we model the cannibalization effect. In other words, with two ongoing auctions, each auction will experience more 'modest' price-transitions. Empirical evidence for this claim appeared in Figure 1 of the Introduction, where we saw that when there are more ongoing auctions the final price tends to be lower.

Assumption 2.3. *The difference, in probability of making jumps to the higher prices, between having one versus two ongoing auctions, is decreasing in the current price.*
Discrete prices: for $p_i < P$, $i = 1, 2$

$$\sum_{q=r}^{P}(\pi_{p_i,q|1} - \pi_{p_i,q|2}) \geq \sum_{q=r}^{P}(\pi_{p_i+1,q|1} - \pi_{p_i+1,q|2}) \qquad \forall r \leq P \tag{7}$$

Continuous prices: for $w \leq P$, $i = 1, 2$

$$F^2_{X_{i,t_i+1}|X_{i,t_i}}(x|w) - F^1_{X_{i,t_i+1}|X_{i,t_i}}(x|w) \qquad \text{is decreasing in } w \tag{8}$$

Equivalently,

$$\frac{\partial}{\partial w}F^2_{X_{i,t_i+1}|X_{i,t_i}}(x|w) \leq \frac{\partial}{\partial w}F^1_{X_{i,t_i+1}|X_{i,t_i}}(x|w)$$

This states that the cannibalization effect is diminishing in the current price. In other words the closer the current price is to the upper bound P the less of a difference there will be between having one or two auctions underway. Qualitatively, we see in the graphs of Figure 4, that the closer the price is to $P \approx 620$ the less 'room' there is for the price-transitions, and hence the less cannibalization there can be.

2.5 Examples

At this point it may be natural to inquire about the existence of transition probability matrices and conditional cumulative distribution functions, that satisfy the above assumptions. We next provide conditions under which of some common probability distributions satisfy them. Namely, Uniform - discrete and continuous, Bernoulli, and Exponential. In addition, we later illustrate the assumptions and implications with numerical examples.

Discrete Uniform

Without loss of generality let $p = k = 1$. Suppose that in periods when there is only one auction underway there is an equal probability of jumping to any of the remaining prices, for $p_i \leq P$, $\pi_{p_i,q|1} = \pi_{p_i} = 1/(P + 1 - p_i)$ for all $q \in [p_i, P]$. Furthermore, suppose when two auctions are underway the probability of remaining at the same price increase with κ and that the probability of jumping to P decrease with κ, as shown in the transition probability matrices below.

$$\Pi_1^{\mathrm{U}} = \begin{pmatrix} \frac{1}{P+1} & \frac{1}{P+1} & \cdots & \frac{1}{P+1} & \frac{1}{P+1} \\ 0 & \frac{1}{P} & \cdots & \frac{1}{P} & \frac{1}{P} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad \Pi_2^{\mathrm{U}} = \begin{pmatrix} \frac{1}{P+1}+\kappa & \frac{1}{P+1} & \cdots & \frac{1}{P+1} & \frac{1}{P+1}-\kappa \\ 0 & \frac{1}{P}+\kappa & \cdots & \frac{1}{P} & \frac{1}{P}-\kappa \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2}+\kappa & \frac{1}{2}-\kappa \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

It can be verified that the above transition probability matrices support Assumptions 2.1, 2.2, and 2.3. A modification to Π_2^{U} is to have κ be dependent on the price. In which case for Assumption 2.3 to hold we require $\frac{1}{P+1} \geq \kappa_0 \geq \kappa_1 \geq \ldots \geq \kappa_{P-1}$.

Continuous Uniform

A continuous Uniform version can be constructed as follows. Assume the starting price is 0 and the upper bound is one, $X_i \in [0,1]$. Assume that when there is only one ongoing auction that the price-jump is uniformly distributed between the current price and the upper limit. And that the 'cannibalization' effect is such that with two ongoing auctions, the price-jump is only uniformly distributed between the current price and *half-way* to the upper limit. Formally, if

$$f^1_{X_{i,t_i+1}|X_{i,t_i}}(x|w) = \begin{cases} \frac{1}{1-w} & w \leq x \leq 1 \\ 0 & \text{o/w} \end{cases} \qquad f^2_{X_{i,t_i+1}|X_{i,t_i}}(x|w) = \begin{cases} \frac{2}{1-w} & w \leq x \leq \frac{1+w}{2} \\ 0 & \text{o/w} \end{cases}$$

then it can be verified that Assumptions 2.1, 2.2, and 2.3 are satisfied.

Bernoulli

Suppose that for each period and every price level there are only two possible transitions - remain at same price or jump up one increment. This bidding process is the core of the auction dynamics analyzed by Segev, Beam, and Shantikumar (2001). In this scenario the upper price bound $P \equiv \tau$, and consequently the size of the transition probability matrices are $(\tau+1) \times (\tau+1)$. Let Π_1 and Π_2 be defined as follows,

$$\Pi_1^{\mathrm{Be}} = \begin{pmatrix} 1-\pi_0 & \pi_0 & \cdots & 0 & 0 \\ 0 & 1-\pi_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1-\pi_{\tau-1} & \pi_{\tau-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad \Pi_2^{\mathrm{Be}} = \begin{pmatrix} 1-\rho_0 & \rho_0 & \cdots & 0 & 0 \\ 0 & 1-\rho_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1-\rho_{\tau-1} & \rho_{\tau-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

In other words, $\Pr\{X_{i,t_i+1} = q+1 | X_{i,t_i} = q, Z = 1\} = \pi_q$, and $\Pr\{X_{i,t_i+1} = q+1 | X_{i,t_i} = q, Z = 2\} = \rho_q$. In order for Π_1^{Be} and Π_2^{Be} to satisfy Assumptions 2.1, 2.2, and 2.3, we require,

$$\pi_{\tau-1} \geq \pi_{\tau-2} - \pi_{\tau-1} \geq \ldots \geq \pi_0 - \pi_1 \geq 0 \quad \text{and} \quad \rho_{\tau-1} \geq \rho_{\tau-2} - \rho_{\tau-1} \geq \ldots \geq \rho_0 - \rho_1 \geq 0 \quad (9)$$

$$\pi_0 - \rho_0 \geq \pi_1 - \rho_1 \geq \ldots \geq \pi_{\tau-1} - \rho_{\tau-1} \geq 0 \quad (10)$$

Note that (9) implies that the probability of making a price jump is decreasing in the current price, that is, $\pi_0 \geq \pi_1 \geq \ldots \geq \pi_{\tau-1} \geq 0$ and $\rho_0 \geq \rho_1 \geq \ldots \geq \rho_{\tau-1} \geq 0$. The intuition of this is that as the current price increases, fewer and fewer bidders are actually willing to pay above the current price, or that for each price-jump bidders 'opt out' of the auction. Under inequalities (10), which reflects the diminishing 'cannibalization' effect, Assumption 2.3 holds. A special case of Π_z^{Be}, $z = 1, 2$, are when the transition probabilities are independent of the current price, that is, when $\pi_q = \pi$ and $\rho_q = \rho$ for all $q = 0, 1, \ldots, \tau - 1$. This special case has some interesting consequences which are discussed in Section 3.3.

Exponential

Assume prices are positive and unbounded, $X_i \in \Re_+$, and that the conditional one-period price-increments, C, given $X_i = w$, are exponentially distributed with rate $\lambda^z(w)$, $z = 1, 2$. That is, for $w, x \in \Re_+$,

$$\Pr\{X_{i,t_i+1} \leq x | X_{i,t_i} = w, Z = z\} = G_C^z(x - w|w) = \begin{cases} 1 - \exp(-\lambda^z(w)(x - w)) & w \leq x \\ 0 & o/w \end{cases} \quad (11)$$

The rate $\lambda^z(\cdot)$ is a function both of the current price and the number of ongoing auctions, and that expected price increment is $1/\lambda^z(w)$. In order for the price increments to be decreasing in the current price, we will assume $\lambda^z(w)$ to be increasing in w. In other words, the higher the current price the smaller the expected price increment. In order for $G_C^z(\cdot|\cdot)$, $z = 1, 2$, to satisfy Assumptions 2.1, 2.2, and 2.3, we assume that,

1. $\lambda^z(w)(x - w)$ is decreasing in w, $z = 1, 2$ (equivalently $\frac{\partial \lambda^z(w)}{\partial w}(x - w) - \lambda^z(w) \leq 0$)

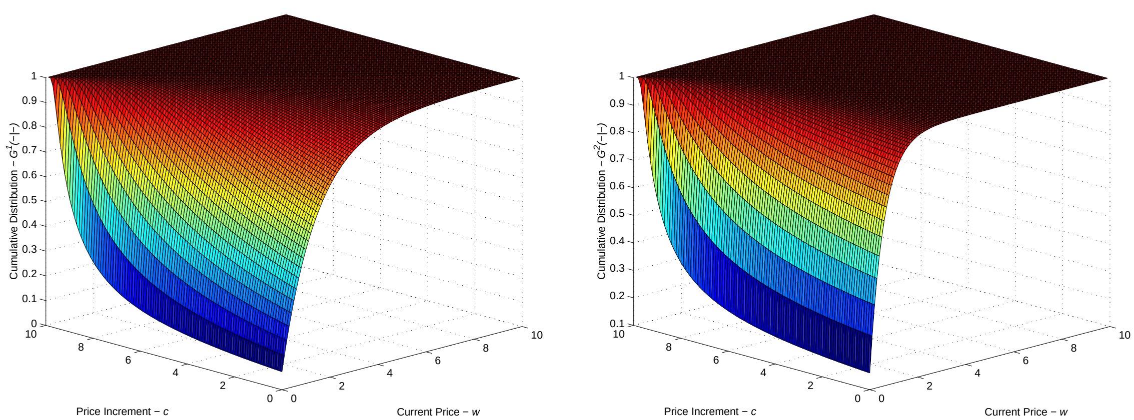
2. $\lambda^1(w) \leq \lambda^2(w)$

3. $\lambda^2(w) - \lambda^1(w)$ is decreasing in w (equivalently $\frac{\partial}{\partial w}\lambda^2(w) \leq \frac{\partial}{\partial w}\lambda^1(w)$)

Stated in this form, Assumptions 2.1, 2.2, and 2.3 can be validated using models fitted to data. For a discussion regarding this, the reader is referred to the PhD thesis Odegaard (2007). Though we assume prices to be unbounded one can constrain the rate-functions, $\lambda^z(\cdot)$, such that $\Pr\{X_{i,t_i+1} > P | X_{i,t_i} = w, Z = z\} \leq \epsilon$, $z = 1, 2$. The following illustrates how this can be done. Let P be the upper bound of bidders' valuation, choose an $\epsilon > 0$ and $K > 0$, then define the rate-functions as follows,

$$\lambda^1(w) = \begin{cases} -\ln(\epsilon) & P \leq w \\ \frac{\ln(\epsilon)}{-(P-w)} & 0 \leq w < P \end{cases} \qquad \lambda^2(w) = \begin{cases} -\ln(\epsilon) & P \leq w \\ \lambda^1(w) + K & 0 \leq w < P \end{cases} \tag{12}$$

Note that the three assumptions listed above are satisfied. A graphical illustration of the resulting conditional distribution functions $G^1_C(c|w)$ and $G^2_C(c|w)$, for $P = 10$, $\epsilon = .001$, and $K = 1$, appears in Figure 5 below. Each graph shows, for a given current price w the conditional distribution function of the price increments c $(= x - w)$. The graph to the left shows when there is only one auction underway, while the graph to the right shows when there are two auctions underway.



$G^1_C(c|w)$ $\qquad$ $G^2_C(c|w)$

Figure 5: Numerical example of Exponentially distributed price increments given rate functions defined according to (12), with $P = 10$, $\epsilon = 0.001$, and $K = 1$. Each graph displays the conditional distribution function for the price increments $c = x - w$ given a current price w.

3. Guaranteed Successful Auctions: The Single Listing Case

The first case we consider is when the auctions are guaranteed to be successful, and hence the seller only has to list an item once. This could occur when the items are such that it is certain a positive bid will arrive ($\pi_{0,0|z} = 0$, $z = 1, 2$), or when the seller decides in advance to immediately salvage items remaining from unsuccessful auctions. As an illustration of the former case is the 4500 laptop and desktop auctions at the eBay store *Dell_Financial_Services*. Out of all the auctions held between December 2005 and August 2006, with a starting price of \$.99, not a single auction was unsuccessful. Due to the additional assumption that auctions are guaranteed to be successful, we can make some modifications to the MDP model in this case.

Decision Epochs As a consequence of the vigilant seller assumption there is no reason to consider a planning horizon beyond two sequential auctions, hence $T = 2\tau$. Furthermore, provided the second item has not been released, non-trivial decisions can only be made in periods $t = 0, 1, 2, \dots, \tau - 1$. At $t = \tau$ the vigilant seller assumption requires that the second item is released immediately.

State Space Since we assume the items will at least sell for p, we will omit the 0 state. Thus for the discrete case $X_i \in \{p, p+1, \dots, P\}$ while for the continuous case $X_i \in [p, P]$.

Rewards In order to facilitate the 'accounting', and since we are not assuming discounting, we assume the seller receives the payment at $t = T$. Therefore, the reward $r_t(s)$ for a given $s \in S$ and period t is as follows,

$$r_t([p_1, t_1; p_2, t_2], z) = \begin{cases} -h1_{\{t_1 < \tau\}} - h1_{\{t_2 < \tau\}} & t = 0, 1, \dots, T-1 \\ p_1 + p_2 & t = T \end{cases}$$

Before we discuss the optimality equations we introduce some notation. When *both items* have been released we define $E[X_{i,\tau}|([X_1, Y_1; X_2, Y_2], Z)]$ to be the conditional expected final price of auction i, $i = 1, 2$,

$$\text{Discrete prices:} \quad E[X_{i,\tau}|([p_1, t_1; p_2, t_2], z)] = \sum_{q=p_i}^{P} q \Pr\{X_{i,\tau} = q|([p_1, t_1; p_2, t_2], z)\} \tag{13}$$

$$\text{Continuous prices:} \quad E[X_{i,\tau}|([w_1, t_1; w_2, t_2], z)] = \int_{w_i}^{P} q f^z_{X_{i,\tau}|X_{i,t_i}}(q|w_i) dq \tag{14}$$

Where $\Pr\{X_{i,\tau} = q|([p_1, t_1; p_2, t_2], z)\}$ and $f^z_{X_{i,\tau}|X_{i,t_i}}(q|w_i)$ are derived using the Chapman-Kolmogorov equations discussed earlier. Note that as a consequence of the assumptions that auctions progress independently and that the current price is increasing, $E[X_{i,\tau}|([X_1, Y_1; X_2, Y_2], Z)]$ is increasing in X_i and independent of X_j, for $i \neq j$. We define $R([X_1, Y_1; X_2, Y_2], Z)$ to represent the total expected profit when both items have been released, for $s = ([p_1, t_1; p_2, t_2], z)$,

$$R([p_1, t_1; p_2, t_2], z) \equiv -h(2\tau - t_1 - t_2) + E[X_{1,\tau}|([p_1, t_1; p_2, t_2], z)] + E[X_{2,\tau}|([p_1, t_1; p_2, t_2], z)]$$

Note that $R(\cdot)$ is not necessarily increasing or decreasing in the elapsed time of the auctions. Though the incurred holding cost will decrease, the expected final price of the items will also decrease. It is this trade-off that is the crux of the problem regarding when to start the second auction. We define $g_2([X_1, Y_1; p, 0])$ to be the gain in expected final price of auction 2 by delaying the release one period, for $t_1 < \tau$,

$$g_2([p_1, t_1; p, 0]) \equiv E[X_{2,\tau}|([p_1, t_1 + 1; p, 0], z)] - E[X_{2,\tau}|([p_1, t_1; p, 0], 2)]$$

where $z = 2$ if $t_1 < \tau - 1$, and 1 otherwise. Due to Assumption 2.2 and that auctions progress independently, $g_2([X_1, Y_1; p, 0]) \geq 0$ and independent of X_1.

For the remainder of this section we will only consider discrete prices. All results will hold for the continuous case as well, with the only change required is to replace the summation with an integral.

3.1 Auction Release Policies

A Markov deterministic policy is a sequence of decision rules which determine what action to take in each decision epoch, possibly contingent on the state of the system but not on the past. Let $\gamma_t(s)$ be the decision rule in period t given a state $s \in S$. As a consequence of the vigilant seller assumption, we only need to consider decision rules for $t = 0, 1, \ldots, \tau$, and hence, a policy γ is defined as follows,

$$\gamma = (\gamma_0(s), \gamma_1(s), \ldots, \gamma_\tau(s)) \qquad \gamma_t(s) \in \{0, 1\}, \forall s \in S, t = 0, 1, \ldots, \tau$$

If all the decision rules, $\gamma_t(s)$, are independent of the price components of state s we refer to the policy γ as an *open loop* policy, while if the decision rules depend on both the price and time components of state s the resulting policy is referred to as a *closed loop* policy. Note that there are only $\tau + 1$ open loop policies of interest. We write $V_{O(j)}$ to denote the total expected profit of releasing the second item j periods after the first, $j = 0, 1, 2, \ldots, \tau$. In Table 1 the four open loop policies and their respective total expected profit for the case when $\tau = 3$ are provided. In the table we see that although we incur an additional unit of h for each additional period we hold the second item, the expected final price for both items increase since there is an additional period when both auctions evolve according to Π_1 instead of Π_2.

γ	Release item 2 j periods after item 1	Total expected profit - $V_{O(j)}$
(1,0,0,0)	$j=0$	$-6h + 2\sum_{l=p}^{P}\sum_{q=p}^{l}\sum_{r=q}^{l} l(\pi_{p,q\|2})(\pi_{q,r\|2})(\pi_{r,l\|2})$
(0,1,0,0)	$j=1$	$-7h + \sum_{l=p}^{P}\sum_{q=p}^{l}\sum_{r=q}^{l} l(\pi_{p,q\|1})(\pi_{q,r\|2})(\pi_{r,l\|2}) + \sum_{l=p}^{P}\sum_{q=p}^{l}\sum_{r=q}^{l} l(\pi_{p,q\|2})(\pi_{q,r\|2})(\pi_{r,l\|1})$
(0,0,1,0)	$j=2$	$-8h + \sum_{l=p}^{P}\sum_{q=p}^{l}\sum_{r=q}^{l} l(\pi_{p,q\|1})(\pi_{q,r\|1})(\pi_{r,l\|2}) + \sum_{l=p}^{P}\sum_{q=p}^{l}\sum_{r=q}^{l} l(\pi_{p,q\|2})(\pi_{q,r\|1})(\pi_{r,l\|1})$
(0,0,0,1)	$j=3$	$-9h + 2\sum_{l=p}^{P}\sum_{q=p}^{l}\sum_{r=q}^{l} l(\pi_{p,q\|1})(\pi_{q,r\|1})(\pi_{r,l\|1})$

Table 1: The four open loop policies and their total expected profit for $\tau = 3$.

3.2 Optimality Equations

Let $V_t(s)$ denote the expected total future reward (expected total profit) given the system is in state $s \in S$ in period t. Then $V_t(s)$ satisfies the following optimality equations,

$$V_t(s) = \begin{cases} r_t(s) + \max_{a\in A(s)} \sum_{s'\in S} V_{t+1}(s')\pi(s'|s,a) & t = 0,1,\ldots,T-1 \\ r_T(s) & t = T \end{cases} \tag{15}$$

Due to the vigilant seller assumption, the structure of the transition probabilities, and that auctions are guaranteed to be successful, the value function (15) can be summarized and explicitly evaluated according to the three cases listed in the following lemma.

Lemma 3.1. *If we assume a vigilant seller and that auctions are guaranteed to be successful then the value functions of interest are as follows,*

$$\begin{aligned} V_t([p_1,t_1;p_2,t_2],z) &= p_1 + p_2 && t = T \\ V_t([p_1,t_1;p_2,t_2],z) &= R([p_1,t_1;p_2,t_2],z) && t = \tau \\ V_t([p_1,t_1;p_2,t_2],z) &= \max\{-2h + \sum_{q=p_1}^{P} V_{t+1}([q,t_1+1;p,0],1)\pi_{p_1,q|1}\ , R([p_1,t_1;p,0],2)\} && \\ & && t \le \tau - 1 \textit{ and } z = 1 \end{aligned}$$

The above value functions are computed using backward induction. An example for discrete prices and $\tau = 3$ is presented in Table 2 below.

$$
\begin{array}{ll}
V_6([p_1,3;p_2,3],0) & = p_1 + p_2 \\
V_3([p_1,3;p,0],1) & = R([p_1,3;p,0],1) \\
V_2([p_1,2;p,0],1) & = \max\{-2h + \sum_{q=p_1}^{P} V_3([q,3;p,0],1)\pi_{p_1,q|1}, R([p_1,2;p,0],2)\ \} \\
V_1([p_1,1;p,0],1) & = \max\{-2h + \sum_{q=p_1}^{P} V_2([q,2;p,0],1)\pi_{p_1,q|1}, R([p_1,1;p,0],2)\ \} \\
V_0([p,0;p,0],1) & = \max\{-2h + \sum_{q=p}^{P} V_1([q,1;p,0],1)\pi_{p,q|1}, R([p,0;p,0],2)\ \}
\end{array}
$$

Table 2: Optimality equations of interest for $\tau = 3$.

3.3 Structural Results

Given the above MDP and the assumption that auctions are guaranteed to be successful, we derive three monotonicity properties: the value equation is increasing in the current price of the two auctions, the optimal policy is a threshold policy, and the threshold is decreasing in the holding cost. Note that though the proofs are for the case of discrete prices, the results holds for continuous prices as well.

Proposition 3.2. *If Assumptions 2.1 holds and auctions are guaranteed to be successful, then the optimal value function, $V_t^\star([p_1, t_1; p_2, t_2], z)$, is increasing in p_1 and p_2, for $t = 0, 1, \ldots, T$.*

In other words an increase in the current price of either item 1 or item 2 will increase the optimal expected total reward. Though this might seem natural and 'obvious' it is a result of the assumptions made, most notably that at a higher price-level the auction is more likely to advance to the higher prices than at a low price-level.

Theorem 3.3. *If Assumptions 2.1, 2.2, and 2.3 hold and auctions are guaranteed to be successful, then there exist optimal decision rules, $\gamma_t^\star([p_1, t_1; p, 0], 1)$, which are increasing in p_1, for $t = 0, 1, \ldots, \tau - 1$. Consequently, the optimal policy is a threshold policy in p_1.*

Theorem 3.3 states that, for each t there exists a $p_t^\star$ such that if $X_1 \geq p_t^\star$ then it is optimal to release the second item for auction, while if $X_1 < p_t^\star$ it is optimal to hold the second item at least one more period. The reason we are only considering $t = 0, 1, \ldots, \tau - 1$ is because we are only interested in those periods where non-trivial decision can be made. For $t = \tau$ the decision to release is immediate by the vigilant seller assumption. Also note that if $Y_2 > 0$ the second item has already been released and no further decision needs to be made. The main assumption driving the result of Theorem 3.3 is the diminishing cannibalization effect. Our next result summarizes the effect the holding cost has on $p_t^\star$.

Corollary 3.4. *The control limit in Theorem 3.3, $p_t^\star$, is decreasing in the holding cost h.*

Given these three properties it may be natural to ask if the threshold price, $p_t^\star$, is monotone in t (or Y_1). With only the three assumptions stated above the answer is no. And it turns out that monotonicity over time will also depend on the holding cost h. In other words, depending on the holding cost $p_t^\star \leq p_{t+1}^\star$ or $p_t^\star \geq p_{t+1}^\star$ (see the example below). We can, however, derive the following lower bound on h to ensure the optimal policy is to release the items sequentially.

Proposition 3.5. *The optimal policy is to release the second item after the first auction has been completed if and only if*

$$h \leq \min_{t_1 < \tau} \{g_2([p_1, t_1; p, 0])\} \tag{16}$$

Similarly, if h is 'too high' then the optimal closed loop policy is to release them simultaneously. However, the condition for h to be 'too high' is more complicated than the condition for 'too low'. For instance, it is not sufficient that h is such that the best open loop policy is to release them simultaneously, to ensure that this is also the optimal policy (see example below). A 'lower bound' for h to be 'too high ' is $V_{O(1)} - V_{O(0)} \leq 0$. The implication on the holding cost h is better illustrated by the following modified inequality,

$$(V_{O(1)} + (2\tau + 1)h) - (V_{O(0)} + 2\tau h) \leq h \tag{17}$$

In other words, if the minimum gain in revenue by deferring the release one period is greater than h, then the optimal policy can never be to release item 2 immediately. Therefore in order for h to be 'too high' it has to be large enough that the open loop policy of simultaneous release is better than the open loop policy of deferring the release by one period. In other words, that inequality (17) holds. Note though that this is only a necessary condition, it could be optimal to defer the release despite that the above inequality holds. To determine the threshold on h for which the optimal closed loop policy is to release the two items simultaneously one has to solve the dynamic program when (17) holds. The numerical example below illustrates these.

3.4 Examples

In Section 2.5 we provided four common probability distributions and conditions on their parameters that support Assumptions 2.1, 2.2, and 2.3. Consequently, we have the following results.

Corollary 3.6. *If price increments are distributed as discrete or continuous Uniform random variables, as specified in Section 2.5, then Proposition 3.2 and Theorem 3.3 holds.*

Corollary 3.7. *If price increments are distributed as Exponential random variables, as specified by (11) and the rate functions satisfies the three additional assumptions stated in Section 2.5, then Proposition 3.2 and Theorem 3.3 holds.*

Corollary 3.8. *If price increments are distributed as Bernoulli random variables as specified in Section 2.5, and (9) and (10) holds, then Proposition 3.2 and Theorem 3.3 holds.*

In the Bernoulli case with price-independent transition probabilities, $\pi_q = \pi$ and $\rho_q = \rho$ for all $q = 0, 1, \ldots, \tau - 1$, the optimal policies simplify further. First note that due to the special structure of the transition probability matrices the n-period transition matrices are symmetric in the following sense, for $\tau = 3$, $\Pi_{1\cdot 2\cdot 2} = \Pi_{2\cdot 2\cdot 1}$ and $\Pi_{1\cdot 1\cdot 2} = \Pi_{2\cdot 1\cdot 1}$. As a result the expected value for the open loop policy of releasing the second item j periods after the first item has been released is given by,

$$V_{O(j)} = -(2\tau + j)h + 2(p + j\pi + (\tau - j)\rho) \qquad j = 0, 1, 2, \ldots, \tau \tag{18}$$

Furthermore, the total marginal gain by deferring the release one period, $2(\pi - \rho)$, is independent of X_1 and as a result closed loop policies are not required.

Proposition 3.9. *In the case of price-independent Bernoulli increments, the optimal policy is to release both items simultaneously if and only if $h \geq 2(\pi - \rho)$. If $h < 2(\pi - \rho)$ then releasing the two auctions sequentially is the optimal policy.*

The two policies stated in Proposition 3.9 are the only optimal policies - open and closed loop policies included. The interpretation of the condition $h \geq 2(\pi - \rho)$, is that if the holding cost exceeds the expected one-period gain for both auctions by deferring the release it will never be optimal to defer the release of item 2.

Numerical Example

We now provide a numerical example which demonstrates the increase in expected value that can be achieved through a closed loop price dependent policy, compared to a non-adaptive open loop policy. It also shows that depending on the holding cost, the price threshold may be increasing or decreasing over time. Assume prices are discrete, $\tau = 3$, $p = k = \$10$, $P = \$60$, and that the transition probabilities are as follows,

$$
\Pi_1 = \begin{array}{c} \\ 10 \\ 20 \\ 30 \\ 40 \\ 50 \\ 60 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 10 & 20 & 30 & 40 & 50 & 60 \end{array} \\
\begin{pmatrix}
.25 & .25 & .2 & .1 & .1 & .1 \\
0 & .25 & .3 & .25 & .1 & .1 \\
0 & 0 & .25 & .4 & .25 & .1 \\
0 & 0 & 0 & .4 & .45 & .15 \\
0 & 0 & 0 & 0 & .6 & .4 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
\qquad
\Pi_2 = \begin{array}{c} \\ 10 \\ 20 \\ 30 \\ 40 \\ 50 \\ 60 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 10 & 20 & 30 & 40 & 50 & 60 \end{array} \\
\begin{pmatrix}
.3 & .3 & .25 & .1 & .05 & 0 \\
0 & .35 & .35 & .2 & .1 & 0 \\
0 & 0 & .4 & .35 & .2 & .05 \\
0 & 0 & 0 & .5 & .4 & .1 \\
0 & 0 & 0 & 0 & .65 & .35 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

It can be verified that the matrices satisfy Assumptions 2.1, 2.2, and 2.3, so that a threshold policy is optimal. The expected profit for the four open loop policies satisfies,

$$
\begin{aligned}
V_{O(0)} &= -6h + 2 \times 40.76 \\
V_{O(1)} &= -7h + 43.69 + 43.40 \\
V_{O(2)} &= -8h + 46.09 + 45.82 \\
V_{O(3)} &= -9h + 2 \times 47.93
\end{aligned}
$$

While for the closed loop or optimal policy we use backward induction to find the action that maximizes the value equations. In Figure 2 of the Introduction and Table 3 below, the value of each policy with respect to various holding costs appears. The last two columns of Table 3 display the difference between the optimal policy and the best and worst open loop policy for a given h. As illustrated in Figure 2, if the holding cost is 'low' then it is better to release the two auctions sequentially, while if the holding cost is 'high' then it will never be worth holding the second item an additional period so that the optimal policy is to release both immediately. The interesting aspect are the cases in between where we see that the optimal policy performs better than any of the open loop policies. Though the gain at a given h for the optimal policy over the best open loop policy is not that drastic, the gain versus the other open loop policy can be quite large. For instance, if $h = \$2.30$ then the difference between using the optimal policy and open loop policy of simultaneous release is more than \$7 (11% improvement). Furthermore, in a setting where a seller has a large inventory of items to dispose of, even incremental gains on each item may accumulate to large total gains.

Table 4 gives the critical price thresholds $p_t^\star$ for various h. For instance, if $h = 4.00$ and $t = 1$ then for $X_1 < 40$ it is optimal to defer the release of item 2, while for $X_1 \geq 40$ it is optimal to start the second auction; that is $p_1^\star = 40$. Note that when h is 'low' then $p_t^\star > P$

h	$V_{O(0)}$	$V_{O(1)}$	$V_{O(2)}$	$V_{O(3)}$	$V_0^\star$	Max Gain(%)	Min Gain(%)
.10	80.93	86.39	91.10	**94.97**	**94.97**	17.9	0
1.00	75.53	80.09	83.90	**86.87**	**86.87**	15.0	0
2.00	69.53	73.09	75.90	**77.87**	**77.87**	12.0	0
2.30	67.73	70.99	73.50	**75.17**	**75.21**	11.0	.1
4.00	57.53	59.09	**59.90**	59.87	**60.99**	6.0	1.8
5.00	51.53	**52.09**	51.90	50.87	**53.01**	4.2	1.8
5.80	**46.73**	46.49	45.50	43.67	**46.85**	7.3	.3
6.00	**45.53**	45.09	43.90	41.87	**45.53**	8.7	0
9.00	**27.53**	24.09	19.90	14.87	**27.53**	85.1	0
10.00	**21.53**	17.09	11.90	5.87	**21.53**	266.8	0
15.00	**−8.47**	−17.91	−28.10	−39.14	**−8.47**	-	-

Table 3: Expected total profit of the *open loop* policies and the optimal policy as a function of the holding cost evaluated at the start of the planning horizon.

and hence it is always optimal to wait one more period before releasing the second item. Similarly, if h is 'high' then it is never optimal to defer the release. Furthermore, note that depending on the holding cost, the price threshold could be either increasing or decreasing. For instance, if $h = 2.75$ then $p_0^\star = 60$, while $p_1^\star = p_2^\star = 50$. On the other hand, if, for instance, $h = 5.50$ then $p_0^\star = 20$, while $p_1^\star = p_2^\star = 30$. This means that depending on the holding cost, the manager may become more or less 'sensitive' when to release the second item as the first auction evolves. For instance, when h is relatively low, in which case $p_t^\star$ is decreasing in t, then the manager will lower his release threshold for each period and hence be less sensitive to the current price. On the other hand, when h is relatively high, such that $p_t^\star$ is increasing in t, then the manager becomes more sensitive to the current price and requires a higher release threshold.

h	$p_t^\star$		
	$t = 0$	$t = 1$	$t = 2$
1.00	n/a	n/a	n/a
2.50	n/a	60	60
2.75	60	50	50
4.00	40	40	40
5.00	30	30	40
5.50	20	30	30
6.00	10	20	30
8.00	10	10	10

Table 4: Threshold price level as a function of holding cost ('n/a' indicates that $p_t^\star > P$).

With regards to the bounds on h, we have that if $h \leq \min_{t_1<\tau}\{g_2([X_1, t_1; p, 0])\} = 47.93 - 45.81 = 2.12$, then the optimal closed loop policy is to release the second item after the first auction is over; if $h \leq 2.12$ then $p_t^\star > P$ for all $t < \tau$. To find the upper limit of h for which the optimal closed loop policy is to release the two auctions simultaneously we solve for h by backward induction given that $h \geq (43.69 + 43.40) - (2 \times 40.76) = 5.57$, this results in that if $h \geq 5.88$ then the optimal policy is to release the second item immediately; if $h \geq 5.88$ then $p_0^\star = p = 10$.

4. Possibly Unsuccessful Auctions: The Multiple Re-Listing Case

We now consider the case when there is a positive probability an auction is unsuccessful, meaning that there is some chance an auction receives no bids ($\pi_{0,0|z} > 0, z = 1, 2$), and that the seller does not have any alternative salvage channel. Reasons why an auction may not receive any bids include that the seller perhaps posted a too high starting price or that the items simply do not generate enough interest. For instance, a quick search on the completed listing of the eBay stores *Pokerstores* (poker chips), *The_Sharper_Image* (consumer electronics), *uptempoair* (Nike sportswear), *GlobalGolfUSA* (used golf clubs) reveal that a large quantity of their listings do not attract a single bid. In contrast, to the 4500 auctions from *Dell_Financial_Services*'s eBay store we have data on, where not a single auction with a starting price of $.99 was unsuccessful. Bertsimas, Hawkins and Perakis (2003) discuss and provide empirical evidence regarding optimal control of starting price and reserve price. For our purposes, even if the sample of companies listed above are doing something wrong in their administration of auction control parameters, we include this section for mathematical completeness. And it turns out that the managerial consequence is both important and interesting, since unlike the single listing case the optimal policy need not be a control limit policy.

By the positive expected profit assumption, even though the seller at the end of an unsuccessful auction has incured a total cost of τh, it is still optimal to try and auction off the item once again. Therefore the seller has to decide when to re-list items that remain from previous unsuccessful auctions. The main effect of this is that the system state transitions may form loops, as illustrated in Figure 3. And in particular may loop back to the initial starting state. Consequently the time when auction i is successfully completed is not known.

To address this issue, we formulate the problem as an *expected total-reward* MDP (cf. *Chapter 7*, Puterman 1994). However, the problem involves some subtleties such that neither the properties of *positive* or *negative* dynamic programs directly apply. In particular, there need not be unique solutions to the optimality equations, and policy or value iteration need not converge without further assumptions (cf. *Chapter 7*, Puterman 1994, Bertsekas and Tsitsiklis 1991). Our model has the following relevant structural properties,

1. The state space and actions are finite.

2. The expected one-period reward in each state is bounded (above and below).

3. There is a single absorbing state Δ under all policies.

The third property holds due to the vigilant seller assumption, without which there would be an additional absorbing state, $([0,0;0,0],0)$, with total reward of negative infinity, resulting from the policy of never releasing an item for auction. Given these three properties, the problem can be converted to a negative dynamic problem with the following transformation. In each transient state subtract $2P$ from the one-period reward. As a consequence all rewards in the transformed problem are less than or equal to zero (cf. *Proof of Theorem 8.10.1.*, Puterman 1994). Therefore, optimal solutions exists, and value iteration and policy iteration converges (though a modification to policy iteration may be required, cf. *Chapter 10.4.2*, Puterman 1994). Alternatively *Assumption 1* and *2* of Bertsekas and Tsitsiklis (1991) holds (even without the vigilant seller assumption).

4.1 Auction Release Policies

For the multiple re-listing (infinite horizon) case a Markov deterministic policy γ is defined as the following,

$$\gamma = (\gamma_0(s), \gamma_1(s), \gamma_2(s), \ldots) \qquad \gamma_t(s) \in \{0,1\}, \forall s \in S, t \geq 0$$

If $\gamma_t(s) = \gamma_{t'}(s)$, for all $t \neq t'$, and for a given $s \in S$, the policy is referred to as *stationary.* In the multiple re-listing case, we only need to consider stationary policies and therefore use the notation $\gamma(s)$, and interchangeable refer to it as both the decision rule and policy (cf. *Theorem 7.3.6.*, Puterman 1994, *Proposition 2*, Bertsekas and Tsitsiklis 1991). In the multiple re-listing case, *open loop* policies do not apply. Instead we define two types of closed loop policies. We refer to a policy that only depends on Y_1 as *time-based closed loop*, and policies that depends on (X_1, Y_1) as *price-based closed loop.* The reason we need this distinction is because, unlike the single listing case, it is necessary to consider decisions even after the first decision to release item 2 has been made. The reason for this is that auctions may get out of 'sync' with each other. We illustrate with an example, suppose $\tau = 5$ and the seller decides to start the second auction 2 periods after the first. And suppose further that the first auction is unsuccessful, which means that the seller must now decide whether, after the second auction has elapsed three periods, to re-list item 1, and if the decision is not to release what to do when the second auction has elapsed four periods. This reasoning generalizes for any τ, and any policy which specify not to release if $Y_1 < x$ and release if $Y_1 = x$, where $x < \tau/2$, must also specify what to do when $Y_1 \geq \tau - x$. For policies defined such that the first release should occur when $Y_1 = x > \tau/2$, the issue of what decision to make if the first auction is unsuccessful has already been addressed.

4.2 Optimality Equation

Before analyzing the optimality equations for two items we begin by only considering one item. Let the pair of random variables (X, Y) denote the state of an auction, where $X \in \{0, p, p+k, \ldots, P\}$ and $Y \in \{0, 1, \ldots, \tau\}$ or δ, and $E[X_\tau | X_u = q]$ denotes the expected final price given $X = q$ and $Y = u$. Let $v(q, u)$ denote the total expected future reward given the system is in state (q, u). By the positive expected profit and vigilant seller assumptions, the item should be immediately re-listed following an unsuccessful auction. Therefore the value of an item in state (q, u) is,

$$v(q,u) = \begin{cases} -h(\tau - u) + E[X_\tau | X_u = q] + (\pi_{q,0|1})(\pi_{0,0|1})^{(\tau-(u+1))} v(0,0) & u \neq \delta \\ 0 & u = \delta \end{cases}$$

Note that $\pi_{q,0|1} = 0$ for all $q > 0$, and that we are not considering discounting. Consequently the expected value of an item continuously re-listed until the auction is successful is,

$$v(0,0) = \frac{-h\tau + E[X_\tau | X_0 = 0]}{1 - (\pi_{0,0|1})^\tau} \tag{19}$$

We now return to the two item case. In total there are 19 different cases for which the optimality equation needs to be evaluated for. These appear in Table A2 of Appendix 2. Note that there are only non-trivial decisions in those periods for which $Y_1 < \tau$ and $Y_2 = 0$, namely cases 16, 17, and 19. Furthermore, note that under cases 13, 15 and 17 there is a positive probability of looping back to the initial state $([0, 0; 0, 0], 1)$. However, due to the positive expected profit assumption, we have that $\pi_{0,0|z} < 1, z = 1, 2$, and therefore with probability one the system state will eventually reach the recurrent state $\Delta = ([\Delta_1, \Delta_2], 0)$. Recall that $\Delta_i = (X_i, \delta)$ is the state of item i, $i = 1, 2$, when it has been awarded and hence will not incur any additional cost or generate any further revenue. From the discussion above, we note that any solution satisfying the optimality equations in Table A2 is an optimal solution (cf.*Proposition 7.3.4*, Puterman 1994, *Proposition 2*, Bertsekas and Tsitsiklis 1991). Though the optimal policy need in general not be a control limit policy (see the example below), under some of the cases the optimal policy is a control limit.

4.3 Structural Results

Similarly to the single listing case the optimality equations can be simplified and explicitly evaluated for some of the cases.

Lemma 4.1. *If we assume a vigilant seller and that the first auction has received a bid, then the value functions for those states can be evaluated as follows,*

1. *If* $p_1 > 0$, $t_1 = \tau, \delta$ *and* $z = 0, 1$, *or* $p_1 > 0$, $t_1, t_2 < \tau$ *and* $z = 2$, *then*

$$V([p_1, t_1; p_2, t_2], z) = R'([p_1, t_1; p_2, t_2], z)$$

2. *If* $p_1 > 0$, $t_1 < \tau, t_2 = 0, z = 1$ *then*

$$V([p_1, t_1; p_2, t_2], z) = \max\{-2h + \sum_{q=p_1}^{P} V([q, t_1 + 1; 0, 0], 1)\pi_{p_1,q|1} \ , R'([p_1, t_1; 0, 0], 2)\}$$

where $R'(\cdot)$ *represents the value of having both items released and a positive current price in the first auction,* $p_1 > 0$, *and* $z = 0, 1, 2$,

$$\begin{aligned} R'([p_1, t_1; p_2, t_2], z) \equiv & -h(2\tau - t_1 - t_2) + E[X_{1,\tau}|([p_1, t_1; p_2, t_2], z)] \\ & + E[X_{2,\tau}|([p_2, t_1; p_2, t_2], z)] + (\pi_{p_2,0|z})(\pi_{0,0|z})^{\tau - t_1 - 1}(\pi_{0,0|1})^{t_1 - t_2} v(0, 0) \end{aligned}$$

where $E[X_{i,\tau}|([p_1, t_1; p_2, t_2], z)]$ *is defined by (13) (or (14)) and* $v(0, 0)$ *is defined by (19).*

The implication of Lemma 4.1 is that once a bid arrives in the first auction, there are no loops back to the initial state, and hence the problem is reduced to the case of guaranteed successful auctions. Consequently the optimal decision when $X_1 > 0$ and $Y_2 = 0$ follows a control limit policy. This result also holds when both auctions are underway but the second auction has received a bid.

Lemma 4.2. *If we assume a vigilant seller and that both auctions are underway but only the second auction has received a bid, then the value functions can be evaluated as follows, for* $t_1, t_2 < \tau, p_1 = 0, p_2 > 0, z = 2$,

$$V([p_1, t_1; p_2, t_2], z) = R''([0, t_1; p_2, t_2], 2)$$

where $R''(\cdot)$ *represents the value of having both auctions underway and a positive current price in the second auction,* $t_1, t_2 < \tau$, $p_1 = 0, p_2 > 0$, *and* $z = 2$,

$$\begin{aligned} R''([p_1, t_1; p_2, t_2], z) \quad \equiv \quad & -2h(\tau - t_1) + E[X_{1,\tau}|([p_1, t_1; p_2, t_2], z)] \\ & +(1 - (\pi_{p_1,0|z})^{\tau - t_1})(-h(t_1 - t_2) + E[X_{2,\tau}|([p_1, t_1; p_2, t_2], z)]\ ) \\ & +(\pi_{p_1,0|z})^{\tau - t_1}(E[V([X_2, t_2 + \tau - t_1; 0, 0], 1) \mid ([p_1, t_1; p_2, t_2], z)\ ]\ ) \end{aligned}$$

where $E[X_{i,\tau}|([p_1, t_1; p_2, t_2], 2)]$ *is defined by (13) (or (14)) and the conditional expectation of* $V([X_2, t_2 + \tau - t_1; 0, 0], 1)$ *is defined according to Lemma 4.1 since* $X_2 > 0$.

The implication of Lemma 4.2 is that the *possible* decision to re-list item 1, which happens with probability $(\pi_{0,0|2})^{\tau - t_1}$, follows a control limit policy. For all other cases, in order to determine the optimal solution and policy, one has to solve the optimality equations either using value iteration or policy iteration. In Table 5 the resulting 8 cases of the optimality equation are summarized. In the table the value function for those states where a positive bid has arrived have been separated from the states where neither auction has received a bid. The issue with the multiple re-listing case is exactly when no bid has arrived and the potential for looping back to the starting state exist. We will next illustrate with a numerical example.

4.4 Numerical Example

This example shows that the optimal policy in the multiple re-listing case might not be a threshold policy. Let prices be discrete, $\tau = 2$, $p = k = 10$, $P = 30$ and the transition probability matrices be defined as follows,

$$\Pi_1 = \begin{array}{c} \\ 0 \\ 10 \\ 20 \\ 30 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 10 & 20 & 30 \end{array} \\ \begin{pmatrix} .5 & .2 & .2 & .1 \\ 0 & .6 & .3 & .1 \\ 0 & 0 & .6 & .4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \qquad \Pi_2 = \begin{array}{c} \\ 0 \\ 10 \\ 20 \\ 30 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 10 & 20 & 30 \end{array} \\ \begin{pmatrix} .6 & .2 & .2 & 0 \\ 0 & .7 & .3 & 0 \\ 0 & 0 & .65 & .35 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Case	item 1	item 2	z	$V([p_1,t_1;p_2,t_2],z)$
1)	$t_1=\delta$ or τ $t_1<\tau$ & $p_1>0$	 $t_2<\tau$	0,1 2	$=R'([p_1,t_1;p_2,t_2],z)$
2)	$t_1<\tau$ & $p_1>0$	$t_2=0$	1	$=\max\{-2h+\sum_{q=p_1}^{P}V([q,t_1+1;0,0],1)\pi_{p_1,q\|1}$, $R'([p_1,t_1;0,0],2)\}$
3)	$t_1<\tau$ & $p_1=0$	$t_2<\tau$ & $p_2>0$	2	$=R''([0,t_1;p_2,t_2],2)$
4)	$\tau-1$ & $p_1=0$	$\tau-1$ & $p_2=0$	2	$=-2h+\sum_{q=p}^{P}\sum_{r=p}^{P}R'([q,\tau;r,\tau],0)\pi_{0,q\|2}\pi_{0,r\|2}$ $+\sum_{q=p}^{P}R'([q,\tau;0,0],1)\pi_{0,q\|2}\pi_{0,0\|2}$ $+\sum_{r=p}^{P}R'([r,\tau;0,0],1)\pi_{0,0\|2}\pi_{0,r\|2}$ $+\pi_{0,0\|2}\pi_{0,0\|2}V([0,0;0,0],1)$
5)	$\tau-1$ & $p_1=0$	$<\tau-1$ & $p_2=0$	2	$=-2h+\sum_{q=p}^{P}\sum_{r=0}^{P}R'([q,\tau;r,t_2+1],1)\pi_{0,q\|2}\pi_{0,r\|2}$ $+\sum_{r=p}^{P}V([r,t_2+1;0,0],1)\pi_{0,0\|2}\pi_{0,r\|2}$ $+\pi_{0,0\|2}\pi_{0,0\|2}V([0,t_2+1;0,0],1)$
6)	$\tau-1$ & $p_1=0$	0	1	$=-2h+\max\{\sum_{q=p}^{P}V([q,\tau;0,0],1)\pi_{0,q\|1}+\pi_{0,0\|1}V([0,0;0,0],1)$, $\sum_{q=p}^{P}\sum_{r=0}^{P}R'([q,\tau;r,1],1)\pi_{p,q\|2}\pi_{0,r\|2}$ $+\sum_{r=p}^{P}V([r,1;0,0],1)\pi_{0,0\|2}\pi_{0,r\|2}$ $+\pi_{0,0\|2}\pi_{0,0\|2}V([0,1;0,0],1)\ \}$
7)	$<\tau-1$ & $p_1=0$	$<\tau-1$ & $p_2=0$	2	$=-2h+\sum_{q=p}^{P}\sum_{r=0}^{P}R'([q,t_1+1;r,t_2+1],2)\pi_{0,q\|2}\pi_{0,r\|2}$ $+\sum_{r=p}^{P}R''([0,t_1+1;r,t_2+1],2)\pi_{0,0\|2}\pi_{0,r\|2}$ $+V([0,t_1+1;0,t_2+1],2)\pi_{0,0\|2}\pi_{0,0\|2}$
8)	$<\tau-1$ & $p_1=0$	0	1	$=-2h+\max\{\sum_{q=p}^{P}V([q,t_1+1;0,0],1)\pi_{p_1,q\|1}$ $+V([0,t_1+1;0,0],1)\pi_{0,0\|1}$, $\sum_{q=p}^{P}\sum_{r=0}^{P}R'([q,t_1+1;r,1],2)\pi_{0,q\|2}\pi_{0,r\|2}$ $+\sum_{r=p}^{P}R''([0,t_1+1;r,1],1)\pi_{0,0\|2}\pi_{0,r\|2}$ $+\pi_{0,0\|2}\pi_{0,0\|2}V([0,t_1+1;0,1],2)\ \}$

Table 5: Resulting optimality equation of interest for multiple re-listing case.

In Table 6 below the optimal policy, derived using policy iteration with $V(\Delta)=0$, for various holding costs is shown. Note in particular that there are instances when it may be optimal to release the second item when $X_1=0$ yet defer if $X_1>0$. For example, if $h=2.75$ and the first auction elapsed one period, $Y_1=1$, then we see that it is optimal to release the item 2 if $X_1=0$ (or $X_1\geq \$20$), but optimal to defer the release if $X_1=\$10$. Note that this scenario can occur since at the start of the first auction it is optimal to defer the release of the item 2 and $\pi_{0,0|1}>0$, but that this can not occur if, for instance, $h=3$, since then the optimal decision at the start of the first auction is to release item 2, and hence there is no decision to be made when $Y_1 = 1$. If, however, both auctions are unsuccessful, which happens with probability $(.6)^{2*2} = .1296$ then the problem is back to its original state at which it was optimal to release both items.

	$Y_1 = 0$		$Y_1 = 1$		
	$X_1 = 0$	$X_1 = 0$	$X_1 = 10$	$X_1 = 20$	$X_1 = 30$
$3.6 \leq h$	Release	Release	Release	Release	Release
$2.9 \leq h \leq 3.5$	Release	Release	Defer	Release	Release
$2.7 \leq h \leq 2.8$	Defer	Release	Defer	Release	Release
$1.9 \leq h \leq 2.6$	Defer	Defer	Defer	Release	Release
$1.3 \leq h \leq 1.8$	Defer	Defer	Defer	Defer	Release
$h \leq 1.2$	Defer	Defer	Defer	Defer	Defer

Table 6: Optimal decision as a function of various holding costs for numerical example.

5. Conclusions and Future Research

In this paper we have analyzed the problem of strategically releasing items for auction in order to maximize profit. Our objective has been to provide a framework for modeling the dynamics of competing auctions and derive structural properties on the optimal auction release policy. The two main underlying assumptions that formed the basis for our analysis were: 1) each period an item remains unsold it incurs a holding cost, and 2) competing auctions 'cannibalize' on each other and therefore decrease the expected final price of each auction. Two scenarios were analyzed - guaranteed successful auctions and possibly unsuccessful auctions. For the first case the problem reduces to a finite horizon MDP, while the second case results in an infinite horizon negative dynamic program. Given behavioral assumptions on the bidding dynamics of ongoing auctions, we were able to show that in the first case the optimal release policy is a control limit policy in the current price of the ongoing auction. Furthermore, we showed that the control limit is decreasing in the holding cost. However, for the case when there is a positive probability that an auction may be unsuccessful, the optimal policy does not have to be a control limit policy. The problem that arises is that the optimal decision when the ongoing auction has not received any bids may or may not be consistent with a control limit policy.

The main managerial insight and contribution of this paper is that there is a significant value of understanding the special dynamics of online auctions. And that by using a price adaptive or closed loop policy a seller can improve his expected total profit. Because online auctions are rather inexpensive to conduct and administer, they are becoming more and more popular as alternative salvage channels. In industries where the value of 'old' items depreciates quickly, such as consumer electronics or fashion goods, being able to optimally sell excess inventory quickly can be of great importance. In addition, even though the gain on each individual item may be small, the overall impact can be quite substantial as the size of the inventory grows.

There are many open research questions to pursue. The perhaps most obvious and important regards the general N item case. This paper has been restricted to the two items case. Though the model formulation and optimality equations remains the same for the general N item case, it is not immediate how to define and derive optimal monotone policies. Our current research efforts concern this. Another direction for future research is to model the problem in continuous time. In this paper, we assumed the seller makes the decision at the beginning of a period. A perhaps more realistic scenario is where the seller tracks his ongoing auctions and each time a bid arrives decides whether to release another item. A third direction for future research is to combine the ideas presented here with the results from previous research. Namely to consider a seller that not only has to decide on the optimal timing, or release, of each auction, but also has to determine the optimal auction control parameters. Our hope is that the framework presented in this paper provides a basis for analyzing such a problem.

Appendix 1: Proofs

Proof of Lemma 3.1 Due to the vigilant seller assumption and that we are considering the case when auctions are guaranteed to be successful we can explicitly write out the value function (15) according to Table A1 below. Note that there are only non-trivial decisions to be made for $t < \tau$ and $z = 1$. Consequently once the second auction has started, we can evaluate the expected total future reward (= total remaining cost - expected final price for both items). The implication of this is summarized in following two lemmas which will facilitate the 'book keeping' and establish Lemma 3.1.

Period	Condition	$V_t([p_1, t_1; p_2, t_2], z)$
$t = T$	$z = 0$	$= p_1 + p_2$
$\tau \leq t < T$	$z = 0$	$= V_{t+1}([p_1, t_1; p_2, t_2], z)$
	$z = 1$	$= -h + \sum_{q=p_2}^{P} V_{t+1}([p_1, t_1; q, t_2 + 1], z')\pi_{p_2,q\|z}$
$t < \tau$	$z = 2$	$= -2h + \sum_{q=p_1}^{P} \sum_{r=p_2}^{P} V_{t+1}([q, t_1 + 1; r, t_2 + 1], z')\pi_{p_1,q\|2}\pi_{p_2,r\|2}$
	$z = 1$	$= -2h + \max\{ \sum_{q=p_1}^{P} V_{t+1}([q, t_1 + 1; p_2, t_2], z)\pi_{p_1,q\|1} ,$ $\sum_{q=p_1}^{P} \sum_{r=p}^{P} V_{t+1}([q, t_1 + 1; r, t_2 + 1], z')\pi_{p_1,q\|2}\pi_{p_2,r\|2} \}$

Table A1: Optimality equations for the single listing case.

Lemma A 1. *If we assume a vigilant seller and each auction is guaranteed to be successful, then once item 2 has been released we can explicitly evaluate the value function, for 1) $\tau \leq t \leq T, z = 0, 1,$ or 2) $t < \tau, z = 2$,*

$$V_t([p_1, t_1; p_2, t_2], z) = R([p_1, t_1; p_2, t_2], z) \tag{20}$$

Comment: Recall that $R([p_1, t_1; p_2, t_2], z) \equiv -h(2\tau - t_1 - t_2) + E[X_{1,\tau}|([p_1, t_1; p_2, t_2], z)] + E[X_{2,\tau}|([p_1, t_1; p_2, t_2], z)]$, and note that there is a slight abuse of notation for the cases when $t_i = \delta$. In these cases we implicitly assume that $\tau - t_i = 0$ and $E[X_{i,\tau}|([p_1, t_1; p_2, t_2], z)] = p_i$, $i = 1, 2$.

Proof of Lemma A1 - There are three cases to consider.

1) For $\tau \leq t \leq T$ and $z = 0$, proof by backward induction on t. Let $t = T$ then $t_1 = \delta$ and $t_2 = \tau$ or δ, and therefore $V_t([p_1, t_1; p_2, t_2], z) = -h0 + p_1 + p_2 = -h(2\tau - \delta - \delta) + E[X_{1,\tau}|([p_1, \delta; p_2, \delta], 0)] + E[X_{2,\tau}|([p_1, \delta; p_2, \delta], 0)]$, and the result holds. Assume the result holds for $t = l+1, l+2, \ldots, T$. Let $\tau \leq t = l$, then $t_1 = \tau$ or δ and $t_2 = \tau$ or δ, and therefore $V_t([p_1, t_1; p_2, t_2], z) = V_{t+1}([p_1, \delta; p_2, \delta], 0) = -h(2\tau - \delta - \delta) + E[X_{1,\tau}|([p_1, \delta; p_2, \delta], 0)] + E[X_{2,\tau}|([p_1, \delta; p_2, \delta], 0)]$, where the second equality holds due to the induction hypothesis. Therefore the result holds for all $\tau \leq t \leq T$ and $z = 0$.

2) For $\tau \leq t \leq T$ and $z = 1$, proof by backward induction on t. Let $t = T - 1$ and $z = 1$, then $t_1 = \delta$ and $t_2 = \tau - 1$, therefore $V_t([p_1, t_1; p_2, t_2], z) =$

$$\begin{aligned}
&= -h + \sum_{q=p_2}^{P} V_T([p_1, \delta; q, \tau], 0)\pi_{p_2,q|1} = -h + \sum_{q=p_2}^{P} p_1 \pi_{p_2,q|1} + \sum_{q=p_2}^{P} q\pi_{p_2,q|1} \\
&= -h + p_1 + \sum_{q=p_2}^{P} q\pi_{p_2,q|1} \\
&= -h(2\tau - \delta - (\tau - 1)) + E[X_{1,\tau}|([p_1, \delta; p_2, \tau - 1], 1)] + E[X_{2,\tau}|([p_1, \delta; p_2, \tau - 1], 1)]
\end{aligned}$$

And the result holds (note that if $t = T$ then due to the vigilant seller assumption all auctions are completed and $z \neq 1$). Assume the result holds for $t = l+1, l+2, \ldots, T$. Let $\tau \leq t = l$ and $z = 1$, then $t_1 = \tau$ or δ, therefore $V_t([p_1, t_1; p_2, t_2], z) =$

$$\begin{aligned}
&= -h + \sum_{q=p_2}^{P} V_{t+1}([p_1, \delta; q, t_2 + 1], z')\pi_{p_2,q|1} \\
&= -h - h(2\tau - \delta - (t_2 + 1)) + \sum_{q=p_2}^{P} E[X_{1,\tau}|([p_1, \delta; q, t_2 + 1], z')]\pi_{p_2,q|1} \\
&\qquad + \sum_{q=p_2}^{P} E[X_{2,\tau}|([p_1, \delta; q, t_2 + 1], z')]\pi_{p_2,q|1} \\
&= -h(2\tau - \delta - t_2+) + E[X_{1,\tau}|([p_1, \delta; p_2, t_2], 1)] + E[X_{2,\tau}|([p_1, \delta; p_2, t_2], 1)]
\end{aligned}$$

Where the second equality holds due to the induction hypothesis when $z' = 1$, or case 1) above when $z' = 0$. Therefore the result holds for all $\tau \leq t \leq T$ and $z = 1$.

3) For $t < \tau$ and $z = 2$, proof by backward induction on t. Let $t = \tau - 1$ and $z = 2$, then

$t_1 = \tau - 1$, therefore $V_t([p_1, t_1; p_2, t_2], z) =$

$$\begin{aligned}
&= -2h + \sum_{q=p_1}^{P}\sum_{r=p_2}^{P} V_{t+1}([q, \tau; r, t_2+1], z')\pi_{p_1,q|2}\pi_{p_2,r|2} \\
&= -2h - h(2\tau - \tau - (t_2+1)) + \sum_{q=p_1}^{P}\sum_{r=p_2}^{P} E[X_{1,\tau}|([q, \tau; r, t_2+1], z')]\pi_{p_1,q|2}\pi_{p_2,r|2} \\
&\quad + \sum_{q=p_1}^{P}\sum_{r=p_2}^{P} E[X_{2,\tau}|([q, \tau; r, t_2+1], z')]\pi_{p_1,q|2}\pi_{p_2,r|2} \\
&= -h(2\tau - (\tau-1) - t_2) + \sum_{q=p_1}^{P} E[X_{1,\tau}|([q, \tau; p_2, t_2+1], z')]\pi_{p_1,q|2} \\
&\quad + \sum_{r=p_2}^{P} E[X_{2,\tau}|([p_1, \tau; r, t_2+1], z')]\pi_{p_2,r|2} \\
&= -h(2\tau - (\tau-1) - t_2) + E[X_{1,\tau}|([p_1, \tau-1; p_2, t_2], 2)] + E[X_{2,\tau}|([p_1, \tau-1; p_2, t_2], 2)]
\end{aligned}$$

Where the second equality follows from case 1) above when $z' = 0$, or case 2) above when $z' = 1$, while the third equality follows from that we assume each auction progress independently of the price of the other auction. Therefore the result holds for $t = \tau - 1$ and $z = 2$. Assume the result holds for $t = \tau - (l-1), \tau - (l-2), \ldots, \tau - 1$. Let $t = \tau - l$ and $z = 2$, then $t_1 = \tau - l$ and $V_t([p_1, t_1; p_2, t_2], z) =$

$$\begin{aligned}
&= -2h + \sum_{q=p_1}^{P}\sum_{r=p_2}^{P} V_{t+1}([q, \tau-(l-1); r, t_2+1], z')\pi_{p_1,q|2}\pi_{p_2,r|2} \\
&= -2h - h(2\tau - (\tau-(l-1)) - (t_2+1)) \\
&\quad + \sum_{q=p_1}^{P}\sum_{r=p_2}^{P} E[X_{1,\tau}|([q, \tau-(l-1); r, t_2+1], z')]\pi_{p_1,q|2}\pi_{p_2,r|2} \\
&\quad + \sum_{q=p_1}^{P}\sum_{r=p_2}^{P} E[X_{2,\tau}|([q, \tau-(l-1); r, t_2+1], z')]\pi_{p_1,q|2}\pi_{p_2,r|2} \\
&= -h(2\tau - (\tau-l) - t_2) + \sum_{q=p_1}^{P} E[X_{1,\tau}|([q, \tau-(l-1); p_2, t_2+1], z')]\pi_{p_1,q|2} \\
&\quad + \sum_{r=p_2}^{P} E[X_{2,\tau}|([p_1, \tau-(l-1); r, t_2+1], z')]\pi_{p_2,r|2} \\
&= -h(2\tau - (\tau-l) - t_2) + E[X_{1,\tau}|([p_1, \tau-l; p_2, t_2], 2)] + E[X_{2,\tau}|([p_1, \tau-l; p_2, t_2], 2)]
\end{aligned}$$

Where the second equality follows from the induction hypothesis when $z' = 2$, case 1) above when $z' = 0$, or case 2) above when $z' = 1$, and the third equality holds due to the assumption that each auction progress independently of the price in the other auction. Therefore the result holds for all $t < \tau$ and $z = 2$. $\square$

Lemma A 2. *If we assume a vigilant seller and each auction is guaranteed to be successful then for $t < \tau$ and $z = 1$,*

$$V_t([p_1, t_1; p_2, t_2], z) = \max\{-2h + \sum_{q=p_1}^{P} V_{t+1}([q, t_1 + 1; p, 0], 1)\pi_{p_1,q|1} \;, R([p_1, t_1; p, 0], 2)\} \quad (21)$$

Proof of Lemma A2 - Proof by backward induction on t. Let $t = \tau - 1$ and $z = 1$, then $t_1 = \tau - 1$ and
$-2h + \sum_{q=p_1}^{P} \sum_{r=p}^{P} V_{t+1}([q, \tau; r, 1], 1)\pi_{p_1,q|2}\pi_{p_2,r|2} =$

$$\begin{aligned}
&= -2h + \sum_{q=p_1}^{P} \sum_{r=p}^{P} (-h(2\tau - \tau - 1) + E[X_{1,\tau}|([q, \tau; r, 1], 1)] \\
&\quad + E[X_{2,\tau}|([q, \tau; r, 1], 1)])\pi_{p_1,q|2}\pi_{p_2,r|2} \\
&= -h(2\tau - (\tau - 1) - 0) + \sum_{q=p_1}^{P} E[X_{1,\tau}|([q, \tau; p, 1], 1)]\pi_{p_1,q|2} \\
&\quad + \sum_{r=p}^{P} E[X_{2,\tau}|([p_1, \tau; r, 1], 1)]\pi_{p_2,r|2} \\
&= -h(2\tau - (\tau - 1) - 0) + E[X_{1,\tau}|([p_1, \tau - 1; p, 0], 2)] + E[X_{2,\tau}|([p_1, \tau - 1; p, 0], 2)] \\
&= R([p_1, \tau - 1; p, 0], 2)
\end{aligned}$$

where the first equality holds due to Lemma A1 with $z' = 1$, and the second equality holds due to that each auction progress independently of the price in the other auction. Assume the result holds for $t = \tau - (l - 1), \tau - (l - 2), \ldots, \tau - 1$. Let $t = \tau - l$ and $z = 1$, then $t_1 = \tau - l$ and

$-2h + \sum_{q=p_1}^{P} \sum_{r=p}^{P} V_{t+1}([q, \tau-(l-1); r, 1], 2)\pi_{p_1,q|2}\pi_{p_2,r|2} =$

$$
\begin{aligned}
&= -2h + \sum_{q=p_1}^{P} \sum_{r=p}^{P} (-h(2\tau - (\tau-(l-1)) - 1) + E[X_{1,\tau}|([q, \tau-(l-1); r, 1], 2)] \\
&\qquad\qquad + E[X_{2,\tau}|([q, \tau-(l-1); r, 1], 2)])\pi_{p_1,q|2}\pi_{p_2,r|2} \\
&= -h(2\tau - (\tau-l) - 0) + \sum_{q=p_1}^{P} E[X_{1,\tau}|([q, \tau-(l-1); p, 1], 2)]\pi_{p_1,q|2} \\
&\quad + \sum_{r=p}^{P} E[X_{2,\tau}|([p_1, \tau-(l-1); r, 1], 2)]\pi_{p_2,r|2} \\
&= -h(2\tau - (\tau-l) - 0) + E[X_{1,\tau}|([p_1, \tau-l; p, 0], 2)] + E[X_{2,\tau}|([p_1, \tau-l; p, 0], 2)] \\
&= R([p_1, \tau-l; p, 0], 2)
\end{aligned}
$$

where the first equality holds due to Lemma A1 with $z' = 2$, and the second equality holds due to that each auction progress independently of the price in the other auction. □

Due to Lemma A1 and A2, and that we mainly are interested in states $s \in S$ such that $A(s) = \{0, 1\}$, we have the value functions listed in Lemma 3.1. □

For the proofs of Proposition 3.2 and Theorem 3.3 we require the following corollaries.

Corollary A 3. *If Assumption 2.1 holds then, in period t with $Z_t = z$ ongoing auctions, the n-period conditional expected final price, $E[X_{i,\tau}|X_{i,\tau-n} = p_i, Z_t = z]$, is increasing in p_i, that is for $t_i = \tau - n$ and $p_i < P$,*

$$
E[X_{i,\tau}|X_{i,t_i} = p_i, Z_t = z] \leq E[X_{i,\tau}|X_{i,t_i} = p_i + 1, Z_t = z] \tag{22}
$$

Proof of Corollary A3 - By induction on the number of remaining periods n. For $n = 1$, $E[X_{i,\tau}|X_{i,\tau-n} = p_i, Z_t = z] = \sum_{q=p_i}^{P} q\pi_{p_i,q|z} \leq \sum_{q=p_i+1}^{P} q\pi_{p_i+1,q|z} = E[X_{i,\tau}|X_{i,\tau-n} = p_i + 1, Z_t = z]$, which holds due to (3). Assume (22) holds for $n = 1, 2, \ldots, l-1$. For $n = l$, $t_i + 1 = \tau - (l-1)$, and therefore from the induction assumption and (3) we have $E[X_{i,\tau}|X_{i,\tau-n} = p_i, Z_t = z] =$ $\sum_{q=p_i}^{P} E[X_{i,\tau}|X_{i,\tau-(l-1)} = q, Z_{t+1}]\pi_{p_i,q|z} \leq \sum_{q=p_i+1}^{P} E[X_{i,\tau}|X_{i,\tau-(l-1)} = q, Z_{t+1}]\pi_{p_i+1,q|z} =$ $E[X_{i,\tau}|X_{i,\tau-n} = p_i + 1, Z_t = z]$. □

Corollary A 4. *If Assumption 2.1 holds then* $R([p_1, t_1; p_2, t_2], z)$ *is increasing in* p_1 *and* p_2*, for all* t_1*,* t_2*, and* $z = 0, 1, 2$*.*

Proof of Corollary A4 - Each auction progress independently of the price in the other auction, the result is therefore immediate by Corollary A3. $\square$

Proof of Proposition 3.2 - By Lemma A1 and A2 there are only two cases to consider.
Case 1) If $t < \tau$ and $z = 2$, or $\tau \leq t \leq T$, then by Lemma A1, $V_t^\star([p_1, t_1; p_2, t_2], z) = R([p_1, t_1; p_2, t_2], z)$, and the result follows from Corollary A4.
Case 2) For $t < \tau$ and $z = 1$, by Lemma A2,

$$V_t^\star([p_1, t_1; p_2, t_2], z) = \max\{-2h + \sum_{q=p_1}^{P} \pi_{p_1,q|1} V_{t+1}^\star([q, t_1 + 1; p, 0], 2)\ , R([p_1, t_1; p, 0], 2)\}$$

Proof by backward induction on t. Note that for $t < \tau$ and $z = 1$, $p_2 = p$. Let $t = \tau - 1$ and hence $t_1 + 1 = \tau$, by Case 1) above and (3), $\sum_{q=p_1}^{P} \pi_{p_1,q|1} V_{t+1}^\star([q, \tau; p, 0], 1)$ is increasing in p_1, and by Corollary 4, $R([p_1, t_1; p, 0], 2)$ is increasing in p_1. Since $V_t^\star([p_1, t_1; p_2, t_2], z)$ is the maximum of two increasing functions it is also increasing in p_1 and the result holds. Assume Proposition 3.2 holds for $t = \tau - (l - 1), \ldots, \tau - 2, \tau - 1$. Let $t = \tau - l$ and hence $t_1 + 1 = \tau - (l-1)$, and again by Case 1) above and (3), $\sum_{q=p_1}^{P} \pi_{p_1,q|1} V_{t+1}^\star([q, t_1+1; p, 0], 2)$ is increasing in p_1, and by Corollary 4, $R([p_1, t_1; p, 0], 2)$ is increasing in p_1. Since $V_t^\star([p_1, t_1; p_2, t_2], z)$ is the maximum of two increasing functions it is also increasing in p_1 and the result holds. $\square$

Corollary A 5. *If Assumption 2.1, 2.2, and 2.3 holds, then*
$V_t^\star([p_1, t_1; p, 0], 1) - R([p_1, t_1; p, 0], 2)$ *is decreasing in* p_1*, for all* $t < \tau$*.*

Proof of Corollary A5 - Note that by Corollary A4 and Proposition 3.2, $R([p_1, t_1; p, 0], 2)$ and $V_t^\star([p_1, t_1; p, 0], 1)$ are increasing in p_1. Also note the following relationship,

$$R([p_1, t_1; p_2, t_2], 2) = -h + \sum_{q=p_1}^{P} R([q, t_1 + 1; p_2, t_2], z)\pi_{p_1,q|2} - g_2([p_1, t_1; p_2, t_2]) \tag{23}$$

Proof by backward induction on t.
Induction Step - Let $t = \tau - 1$, then $t_1 = \tau - 1$ and by Lemma A1, A2, and (23), $V_t^\star([p_1, t_1; p, 0], 1) - R([p_1, t_1; p, 0], 2) =$

$$= \max\{-h + \sum_{q=p_1}^{P} R([q, \tau; p, 0], 1)(\pi_{p_1,q|1} - \pi_{p_1,q|2})\ + g_2([p_1, \tau - 1; p, 0])\ , 0\}$$

Since $g_2([p_1, \tau - 1; p, 0])$ is independent of p_1, and by Corollary A4 and Assumption 2.3, $\sum_{q=p_1}^{P} R([q, \tau; p, 0], 1)(\pi_{p_1,q|1} - \pi_{p_1,q|2})$ is decreasing in p_1. And the result holds.
Induction Hypothesis - Assume Corollary A5 holds for $t = \tau - (l-1), \tau - (l-2), \ldots, \tau - 1$.
Induction Argument - Let $t = \tau - l$. Then by Lemma A1, A2, and (23),

$$V_t^\star([p_1, t_1; p, 0], 1) - R([p_1, t_1; p, 0], 2) =$$

$$= \max\{-h + \sum_{q=p_1}^{P} V_{t+1}^\star([q, t_1+1; p, 0], 1)\pi_{p_1,q|1} - R([q, t_1+1; p, 0], 2)\pi_{p_1,q|2} \; + g_2([p_1, t_1; p, 0]) \; , 0\}$$

Where $g_2([p_1, t; p, 0])$ is independent of p_1. And by Corollary A4, Proposition 3.2, and the induction hypothesis $V_{t+1}^\star([q, t+1; p, 0], 1) - R([q, t+1; p, 0], 1)$ is decreasing in q, and therefore by Assumption 2.3 the result holds. □

Proof of Theorem 3.3 Let $t < \tau$, then $t_1 = t$ and by Lemma A2 and (23),

$$V_t^\star([p_1, t_1; p, 0], 1) =$$

$$\max\{-h + \sum_{q=p_1}^{P} V_{t+1}^\star([q, t+1; p, 0], 1)\pi_{p_1,q|1} - R([q, t+1; p, 0], 2)\pi_{p_1,q|2} \; + g_2([p_1, t_1; p, 0]) \; , 0\}$$

By Corollary A5, $V_t^\star([p_1, t_1; p, 0], 1) - R([p_1, t_1; p, 0], 2)$ is decreasing in p_1 for all $t < \tau$ (note that at the border case for $t = \tau - 1$, $V_{t+1}^\star([q, t+1; p, 0], 1) = R([q, t+1; p, 0], 1)$), and since $g_2([p_1, t; p, 0])$ is independent of p_1. Therefore by Assumption 2.3,
$\sum_{q=p_1}^{P} V_t^\star([q, t; p, 0], 1)\pi_{p_1,q|1} - R([q, \tau; p, 0], 1)\pi_{p_1,q|2}$ is decreasing in p_1. And the result follows. □

Proof of Corollary 3.4 For a given decision epoch t we know that for $X_1 \geq p_t^\star$ any additional holding cost by deferring the release is not compensated by the gain in expected final price for the two items. Therefore, if h increases and since the expected final prices remains the same, then any additional holding cost will still not be compensated (in fact it is even less compensated), and the result follows. □

Proof of Proposition 3.5 ($\Leftarrow$) If (16) holds then the additional gain in expected final price for item 2, by deferring the release one period, alone compensates for the additional holding cost. Since that gain holds for all periods that the first auction is still ongoing and independently of X_1, it is always optimal to release the second auction after the first is finished.

(⇒) If (16) does not hold then there exist a period for which the gain in expected final price for item 2 does not compensate the additional holding cost. Consequently, in order for it to be optimal to defer the release in that period, there must be some gain in expected final price of item 1. However, this gain is dependent on the current price and hence the optimal policy is not independent of the current price. Therefore, the optimal policy might not be to release the items sequentially. $\square$

Proof of Proposition 3.9 From equation (18) we can compare various open loop policies and determine when each one dominates another. Let OPj and OP$(j+m)$ be the open loop policies of releasing the second auction j and $(j+m)$ periods respectively after the first auction. We then have, $V_{O(j)} \geq V_{O(j+m)} \Leftrightarrow -(2\tau + j)h + 2(p + j\pi + (\tau - j)\rho) \geq -(2\tau + j + m)h + 2(p + (j+m)\pi + (\tau - j - m)\rho) \Leftrightarrow h \geq 2(\pi - \rho)$.
Since this condition is independent of j and $j+m$ the result is that simultaneous release is optimal iff $h \geq 2(\pi - \rho)$. By symmetry (non-overlapping) sequential release is optimal iff $h < 2(\pi - \rho)$ and there are no other optimal Open Loop policies. $\square$

Proof of Corollary 3.6, 3.7, and 3.8 - For each of the cases we have that Assumptions 2.1, 2.2, and 2.3 holds, it therefore follows that Proposition 3.2 and Theorem 3.3 holds. $\square$

Proof of Lemma 4.1
Comment: Recall that $R'([p_1,t_1;p_2,t_2],z) = -h(2\tau - t_1 - t_2) + E[X_{1,\tau}|([p_1,t_1;p_2,t_2],z)] + E[X_{2,\tau}|([p_1,t_1;p_2,t_2],z)] + (\pi_{p_2,0|z})(\pi_{0,0|z})^{\tau-t_1-1}(\pi_{0,0|1})^{t_1-t_2}v(0,0)$, and note that there is a slight abuse of notation for the cases when $t_i = \delta$. In these cases we implicitly assume that $\tau - t_i = 0$ and $E[X_{i,\tau}|([p_1,t_1;p_2,t_2],z)] = 0$, $i = 1,2$.

1) $p_1 > 0, t_1 = \tau, \delta$ and $z = 0,1$, or $p_1 > 0, t_1, t_2 < \tau$, and $z = 2$.

1a) For $p_1 > 0, t_1 = \delta$ and $z = 0,1$, $R'([p_1,\delta;p_2,t_2],z) = -h(\tau - t_2) + E[X_{2,\tau}|([p_1,\delta;p_2,t_2],z)]$ $+(\pi_{p_2,0|z})(\pi_{0,0|z})^{\tau-t_1-1}(\pi_{0,0|1})^{t_1-t_2}v(0,0)$.
Let $p_1 > 0, t_1 = \delta$, and $z = 0$. If $t_2 = \delta$ then $V([p_1,t_1;p_2,t_1],z) = 0 = R'([p_1,\delta;p_2,\delta],0)$. If $t_2 = \tau$ then $z = 0$ and $V([p_1,t_1;p_2,t_1],z) = p_2 + V(\Delta) = -h(\tau - \tau) + E[X_{2,\tau}|([p_1,\delta;p_2,\tau],0)] = R'([p_1,\delta;p_2,\tau],0)$.
Let $p_1 > 0, t_1 = \delta$, and $z = 1$, that is $t_2 < \tau$. Proof by backward induction on t_2. Let

$t_2 = \tau - 1$ and $p_2 > 0$ then $V([p_1, t_1; p_2, t_1], z) =$

$$
\begin{aligned}
&= -h + \sum_{q=p_2}^{P} V([p_1, \delta; q, \tau], 0)\pi_{p_2,q|1} = -h + \sum_{q=p_2}^{P} R'([p_1, \delta; q, \tau], 0)\pi_{p_2,q|1} \\
&= -h + \sum_{q=p_2}^{P} (-h(\tau - \tau) + E[X_{2,\tau}|([p_1, \delta; q, \tau], 0)])\pi_{p_2,q|1} \\
&= -h(\tau - (\tau - 1)) + E[X_{2,\tau}|([p_1, \delta; p_2, \tau - 1], 1)]) = R'([p_1, \delta; p_2, \tau - 1], 1)
\end{aligned}
$$

Where the second equality holds due to the case above with $t_1 = \delta, z = 0$.

Let $t_2 = \tau - 1$ and $p_2 = 0$ then $V([p_1, t_1; p_2, t_1], z) =$

$$
\begin{aligned}
&= -h + \sum_{q=p}^{P} V([p_1, \delta; q, \tau], 0)\pi_{0,q|1} + V([p_1, \delta; 0, 0], 1)\pi_{0,0|1} \\
&= -h + \sum_{q=p_2}^{P} R'([p_1, \delta; q, \tau], 0)\pi_{p_2,q|1} + v(0, 0)\pi_{0,0|1} \\
&= -h + \sum_{q=p_2}^{P} (-h(\tau - \tau) + E[X_{2,\tau}|([p_1, \delta; q, \tau], 0)])\pi_{p_2,q|1} + v(0, 0)\pi_{0,0|1} \\
&= -h(\tau - (\tau - 1)) + E[X_{2,\tau}|([p_1, \delta; p_2, \tau - 1], 1)]) + v(0, 0)\pi_{0,0|1} \\
&= R'([p_1, \delta; 0, \tau - 1], 1)
\end{aligned}
$$

Where the second equality holds due to the case above with $t_1 = \delta$ and $z = 0$, and that $V([p_1, \delta; 0, 0], 1) = v(0, 0)$, since the first item has been awarded and by the vigilant seller assumption the second item will be continuously re-listed until the auction is successful.

Assume the result holds for $t_2 = \tau - (l - 1), \tau - (l - 2), \ldots, \tau - 1$. Let $t_2 = \tau - l$ then $V([p_1, t_1; p_2, t_1], z) =$

$$
\begin{aligned}
&-h + \sum_{q=p_2}^{P} V([p_1, \delta; q, \tau - (l - 1)], 1)\pi_{p_2,q|1} = -h + \sum_{q=p_2}^{P} R'([p_1, \delta; q, \tau - (l - 1)], 1)\pi_{p_2,q|1} \\
&= -h + \sum_{q=p_2}^{P} (-h(\tau - (\tau - (l - 1))) + E[X_{2,\tau}|([p_1, \delta; q, \tau - (l - 1)], 1)])\pi_{p_2,q|1} \\
&= -h(\tau - (\tau - l)) + E[X_{2,\tau}|([p_1, \delta; p_2, \tau - l], 1)]) = R'([p_1, \delta; p_2, \tau - l], 1)
\end{aligned}
$$

Where the second equality holds due to induction hypothesis.

Therefore Lemma 4.1 holds for the case 1a) $p_1 > 0, t_1 = \delta$ and $z = 0, 1$.

1b) For $p_1 > 0, t_1 = \tau$ and $z = 0, 1$, $R'([p_1, \tau; p_2, t_2], z) =$
$-h(2\tau - \tau - t_2) + E[X_{1,\tau}|([p_1, \tau; p_2, t_2], z)] + E[X_{2,\tau}|([p_1, \tau; p_2, t_2], z)]$
$+(\pi_{p_2,0|z})(\pi_{0,0|z})^{\tau-t_1-1}(\pi_{0,0|1})^{t_1-t_2}v(0, 0)$.
Let $p_1 > 0, t_1 = \tau$, and $z = 0$, that is $t_2 = \tau$. Therefore $V([p_1, t_1; p_2, t_1], z) = p_1+p_2+V(\Delta) = R'([p_1, \tau; p_2, \tau], 0)$.
Let $p_1 > 0, t_1 = \tau$, and $z = 1$, that is $t_2 < \tau$. Proof by backward induction on t_2. Let $t_2 = \tau - 1$ and $p_2 > 0$ then $V([p_1, t_1; p_2, t_1], z) =$

$$
\begin{aligned}
&= -h + p_1 + \sum_{q=p_2}^{P} V([p_1, \delta; q, \tau], 0)\pi_{p_2,q|1} = -h + p_1 + \sum_{q=p_2}^{P} R'([p_1, \delta; q, \tau], 0)\pi_{p_2,q|1} \\
&= -h + E[X_{1,\tau}|([p_1, \tau; q, \tau - 1], 1) + \sum_{q=p_2}^{P}(-h(\tau - \tau) + E[X_{2,\tau}|([p_1, \delta; q, \tau], 0)])\pi_{p_2,q|1} \\
&= -h(2\tau - \tau - (\tau - 1)) + E[X_{1,\tau}|([p_1, \tau; p_2, \tau - 1], 1)]) + E[X_{2,\tau}|([p_1, \tau; p_2, \tau - 1], 1)]) \\
&= R'([p_1, \tau; p_2, \tau - 1], 1)
\end{aligned}
$$

Where the second equality holds due to case 1a) above.
Let $t_2 = \tau - 1$ and $p_2 = 0$ then $V([p_1, t_1; p_2, t_1], z) =$

$$
\begin{aligned}
&= -h + p_1 + \sum_{q=p}^{P} V([p_1, \delta; q, \tau], 0)\pi_{0,q|1} + V([p_1, \delta; 0, 0], 1)\pi_{0,0|1} \\
&= -h + p_1 + \sum_{q=p_2}^{P} R'([p_1, \delta; q, \tau], 0)\pi_{p_2,q|1} + v(0, 0)\pi_{0,0|1} \\
&= -h + E[X_{1,\tau}|([p_1, \tau; q, \tau - 1], 1)]) + \sum_{q=p_2}^{P}(-h(\tau - \tau) + E[X_{2,\tau}|([p_1, \delta; q, \tau], 0)])\pi_{p_2,q|1} \\
&\quad +v(0, 0)\pi_{0,0|1} \\
&= -h(2\tau - \tau - (\tau - 1)) + E[X_{1,\tau}|([p_1, \tau; p_2, \tau - 1], 1)]) + E[X_{2,\tau}|([p_1, \tau; p_2, \tau - 1], 1)]) \\
&\quad +v(0, 0)\pi_{0,0|1} \\
&= R'([p_1, \tau; 0, \tau - 1], 1)
\end{aligned}
$$

Where the second equality holds due to case 1a) above and that $V([p_1, \delta; 0, 0], 1) = v(0, 0)$, which holds since the first item has been awarded and by the vigilant seller assumption the second item will be continuously re-listed until the auction is successful.
Assume the result holds for $t_2 = \tau - (l - 1), \tau - (l - 2), \ldots, \tau - 1$. Let $t_2 = \tau - l$ then

$V([p_1, t_1; p_2, t_1], z) =$

$$
\begin{aligned}
&= -h + p_1 + \sum_{q=p_2}^{P} V([p_1, \delta; q, \tau - (l-1)], 1)\pi_{p_2,q|1} \\
&= -h + p_1 + \sum_{q=p_2}^{P} R([p_1, \delta; q, \tau - (l-1)], 1)\pi_{p_2,q|1} \\
&= -h + E[X_{1,\tau}|([p_1, \tau; q, \tau - (l-1)], 1)]) \\
&\qquad + \sum_{q=p_2}^{P} (-h(\tau - (\tau - (l-1))) + E[X_{2,\tau}|([p_1, \delta; q, \tau - (l-1)], 1)])\pi_{p_2,q|1} \\
&= -h(2\tau - \tau - (\tau - l)) + E[X_{1,\tau}|([p_1, \tau; p_2, \tau - l], 1)]) + E[X_{2,\tau}|([p_1, \tau; p_2, \tau - l], 1)]) \\
&= R'([p_1, \tau; p_2, \tau - l], 1)
\end{aligned}
$$

Where the second equality holds due to case 1a) above.
Therefore Lemma 4.1 holds for the case 1b) $p_1 > 0, t_1 = \tau$ and $z = 0, 1$.

Note that for $p_1 > 0, t_1, t_2 < \tau$ and $z = 2$, $R'([p_1, t_1; p_2, t_2], z) = -h(2\tau - t_1 - t_2) + E[X_{1,\tau}|([p_1, t_1; p_2, t_2], 2)] + E[X_{2,\tau}|([p_1, t_2; p_2, t_2], 2)] + (\pi_{p_2,0|z})(\pi_{0,0|z})^{\tau - t_1 - 1}(\pi_{0,0|1})^{t_1 - t_2} v(0, 0)$.

1c) Let $t_1 = \tau - 1$. Proof by backward induction on t_2. Let $t_2 = \tau - 1$ and $p_2 > 0$, then $V([p_1, t_1; p_2, t_1], z) =$

$$
\begin{aligned}
&= -2h + \sum_{q=p_1}^{P} \sum_{r=p_2}^{P} V([q, \tau; r, \tau], 0)\pi_{p_1,q|2}\pi_{p_2,r|2} \\
&= -2h + \sum_{q=p_1}^{P} \sum_{r=p_2}^{P} R'([q, \tau; r, \tau], 0)\pi_{p_1,q|2}\pi_{p_2,r|2} \\
&= -2h + \sum_{q=p_1}^{P} E[X_{1,\tau}|([q, \tau; p_2, \tau], 2)]\pi_{p_1,q|2} + \sum_{q=p_2}^{P} E[X_{2,\tau}|([p_1, \tau; q, \tau], 2)]\pi_{p_1,q|2} \\
&= -h(2\tau - (\tau - 1) - (\tau - 1)) + E[X_{1,\tau}|([p_1, \tau - 1; p_2, \tau - 1], 2)] \\
&\qquad + E[X_{2,\tau}|([p_1, \tau - 1; q, \tau - 1], 2)] \\
&= R'([p_1, \tau - 1; p_2, \tau - 1], 2)
\end{aligned}
$$

Where the second equality holds due to case 1b) above, and the third equality holds due to that each auction progress independently of the price in the other auction.

Let $t_2 = \tau - 1$ and $p_2 = 0$, then $V([p_1, t_1; p_2, t_1], z) =$

$$
\begin{aligned}
&= -2h + \sum_{q=p_1}^{P}\sum_{r=p}^{P} V([q,\tau;r,\tau],0)\pi_{p_1,q|2}\pi_{p_2,r|2} + \sum_{q=p_1}^{P} V([q,\tau;0,0],1)\pi_{p_1,q|2}\pi_{0,0|2} \\
&= -2h + \sum_{q=p_1}^{P} \pi_{p_1,q|2}(\sum_{r=p}^{P} R'([q,\tau;r,\tau],0)\pi_{0,r|2} + R'([q,\tau;0,0],1)\pi_{0,0|2}) \\
&= -2h + \sum_{q=p_1}^{P} \pi_{p_1,q|2}(\sum_{r=p}^{P}(E[X_{1,\tau}|([q,\tau;r,\tau],0)] + E[X_{2,\tau}|([q,\tau;r,\tau],0)])\pi_{0,r|2} \\
&\qquad\qquad +(-h(2\tau-\tau-0) + E[X_{1,\tau}|([q,\tau;0,0],1)] + E[X_{2,\tau}|([q,\tau;0,0],1)] \\
&\qquad +(\pi_{0,0|1})^{\tau} v(0,0))\pi_{0,0|2}) \\
&= -h(2\tau-(\tau-1)-(\tau-1)) + \sum_{q=p_1}^{P} E[X_{1,\tau}|([q,\tau;p_2,\tau],0)]\pi_{p_1,q|2} \\
&\qquad +\sum_{r=p}^{P} E[X_{2,\tau}|([p_1,\tau;r,\tau],0)])\pi_{0,r|2} + ((1-(\pi_{0,0|1})^{\tau})v(0,0) + (\pi_{0,0|1})^{\tau} v(0,0))\pi_{0,0|2} \\
&= -h(2\tau-(\tau-1)-(\tau-1)) + E[X_{1,\tau}|([p_1,\tau-1;p_2,\tau-1],2)] \\
&\qquad +E[X_{2,\tau}|([p_1,\tau-1;0,\tau-1],2)] + v(0,0)\pi_{0,0|2} \\
&= R'([p_1,\tau-1;0,\tau-1],2)
\end{aligned}
$$

Where the second equality holds due to case 1b) above, and the fourth equality holds due to (19). Assume the result holds for $t_2 = \tau-(l-1), \tau-(l-2), \ldots, \tau-1$. Let $t_2 = \tau - l$ then $V([p_1, t_1; p_2, t_1], z) =$

$$
\begin{aligned}
&= -2h + \sum_{q=p_1}^{P}\sum_{r=p_2}^{P} V([q,\tau;r,\tau-(l-1)],1)\pi_{p_1,q|2}\pi_{p_2,r|2} \\
&= -2h + \sum_{q=p_1}^{P}\sum_{r=p_2}^{P} R'([q,\tau;r,\tau-(l-1)],1)\pi_{p_1,q|1}\pi_{p_2,r|2} \\
&= -2h + \sum_{q=p_1}^{P}\sum_{r=p_2}^{P}(-h(2\tau-\tau-(\tau-(l-1)))) + E[X_{1,\tau}|([q,\tau;r,\tau-(l-1)],1)] \\
&\qquad +E[X_{2,\tau}|([q,\tau;r,\tau-(l-1)],1)] + (\pi_{r,0|1})(\pi_{0,0|1})^{\tau-(\tau-(l-1))-1} v(0,0))\pi_{p_1,q|2}\pi_{p_2,r|2}
\end{aligned}
$$

$$= -h(2\tau - (\tau - 1) - (\tau - l)) + \sum_{q=p_1}^{P} E[X_{1,\tau}|([q,\tau;p_2,\tau-(l-1)],1)]\pi_{p_1,q|2}$$

$$+ \sum_{r=p_2}^{P} E[X_{2,\tau}|([p_1,\tau;r,\tau-(l-1)],1)]\pi_{p_2,r|2} + (\pi_{p_2,0|1})(\pi_{0,0|1})^{\tau-(\tau-(l-1))}v(0,0)$$

$$= -h(2\tau - (\tau-1) - (\tau-1)) + E[X_{1,\tau}|([p_1,\tau-1;p_2,\tau-l],2)]$$

$$+ E[X_{2,\tau}|([p_1,\tau-1;q,\tau-l],2)] + (\pi_{p_2,0|1})(\pi_{0,0|1})^{\tau-(\tau-(l-1))}v(0,0)$$

$$= R'([p_1,\tau-1;p_2,\tau-l],2)$$

Where the second equality holds due to the case above with $t_1 = \tau$, and the third equality holds due to that each auction progress independently of the price in the other auction and that $\pi_{p_2,r|z} = 0$ for $r < p_2$.
Therefore Lemma 4.1 holds for the case 1c) $p_1 > 0$, $t_1 = \tau - 1$, $t_2 < \tau$, and $z = 2$.

1d) Let $t_1 < \tau - 1$. Proof by backward induction on t_1. Let $p_1 > 0$, $t_1 = \tau - 2$, and $z = 2$, then $V([p_1, t_1; p_2, t_2], 2) =$

$$= -2h + \sum_{q=p_1}^{P}\sum_{r=p_2}^{P} V([q,\tau-1;r,t_2+1],2)\pi_{p_1,q|2}\pi_{p_2,r|2}$$

$$= -2h + \sum_{q=p_1}^{P}\sum_{r=p_2}^{P} R'([q,\tau-1;r,t_2+1],2)\pi_{p_1,q|2}\pi_{p_2,r|2}$$

$$= -2h + \sum_{q=p_1}^{P}\sum_{r=p_2}^{P}(-h(2\tau-(\tau-1)-(t_2+1)) + E[X_{1,\tau}|([q,\tau-1;r,t_2+1],2)]$$

$$+E[X_{2,\tau}|([q,\tau-1;r,t_2+1],2)]$$

$$+(\pi_{r,0|2})(\pi_{0,0|1})^{\tau-1-t_2+1}v(0,0)\ )\pi_{p_1,q|2}\pi_{p_2,r|2}$$

$$= -h(2\tau-(\tau-2)-t_2) + \sum_{q=p_1}^{P} E[X_{1,\tau}|([q,\tau-1;p_2,t_2+1],2)]\pi_{p_1,q|2}$$

$$+ \sum_{r=p_2}^{P} E[X_{2,\tau}|([p_1,\tau-1;r,t_2+1],2)]\pi_{p_2,r|2} + (\pi_{p_2,0|2})(\pi_{0,0|2})(\pi_{0,0|1})^{\tau-1-t_2+1}v(0,0)$$

$$= -h(2\tau-(\tau-2)-t_2) + E[X_{1,\tau}|([p_1,\tau-2;p_2,t_2],2)]$$

$$+E[X_{2,\tau}|([p_1,\tau-2;p_2,t_2],2)] + (\pi_{p_2,0|2})(\pi_{0,0|2})(\pi_{0,0|1})^{\tau-1-t_2+1}v(0,0)$$

$$= R'([p_1,\tau-2;p_2,t_2],2)$$

Where the second equality holds from case 1c) above, and the third equality holds due to that each auction progress independently of the price in the other auction and that $\pi_{p_2,r|z} = 0$ for $r < p_2$. Assume the result holds for $t_1 = \tau - (l-1), \tau - (l-2), \ldots, \tau - 2$. Let $t_1 = \tau - l$, $p_1 > 0$, and $z = 2$, then $V([p_1, t_1; p_2, t_2], 2) =$

$$
\begin{aligned}
&= -2h + \sum_{q=p_1}^{P} \sum_{r=p_2}^{P} V([q, \tau - (l-1); r, t_2 + 1], 2)\pi_{p_1,q|2}\pi_{p_2,r|2} \\
&= -2h + \sum_{q=p_1}^{P} \sum_{r=p_2}^{P} R'([q, \tau - (l-1); r, t_2 + 1], 2)\pi_{p_1,q|2}\pi_{p_2,r|2} \\
&= -2h + \sum_{q=p_1}^{P} \sum_{r=p_2}^{P} (-h(2\tau - (\tau - (l-1)) - (t_2 + 1)) \\
&\qquad +E[X_{1,\tau}|([q, \tau - (l-1); r, t_2 + 1], 2)] \\
&\qquad +E[X_{2,\tau}|([q, \tau - (l-1); r, t_2 + 1], 2)] \\
&\qquad +(\pi_{r,0|2})(\pi_{0,0|1})^{\tau-(l-1)-t_2+1} v(0,0)\ )\pi_{p_1,q|2}\pi_{p_2,r|2}
\end{aligned}
$$

$$
\begin{aligned}
&= -h(2\tau - (\tau - l) - t_2) + \sum_{q=p_1}^{P} E[X_{1,\tau}|([q, \tau - (l-1); p_2, t_2 + 1], 2)]\pi_{p_1,q|2} \\
&\qquad + \sum_{r=p_2}^{P} E[X_{2,\tau}|([p_1, \tau - (l-1); r, t_2 + 1], 2)]\pi_{p_2,r|2} \\
&\qquad +(\pi_{p_2,0|2})(\pi_{0,0|2})(\pi_{0,0|1})^{\tau-(l-1)-t_2+1} v(0,0) \\
&= -h(2\tau - (\tau - l) - t_2) + E[X_{1,\tau}|([p_1, \tau - l; p_2, t_2], 2)] \\
&\qquad +E[X_{2,\tau}|([p_1, \tau - l; p_2, t_2], 2)] + (\pi_{p_2,0|2})(\pi_{0,0|2})(\pi_{0,0|1})^{\tau-l-t_2+1} v(0,0) \\
&= R'([p_1, \tau - l; p_2, t_2], 2)
\end{aligned}
$$

where the second equality holds from the induction hypothesis, and the third equality holds due to that each auction progress independently of the price in the other auction and that $\pi_{p_2,r|z} = 0$ for $r < p_2$.

Therefore Lemma 4.1 holds for the case 1d) $p_1 > 0$, $t_1 < \tau - 1$, $t_2 < \tau$, and $z = 2$.

And consequently Lemma 4.1 holds for 1) $p_1 > 0, t_1 = \tau, \delta$ and $z = 0, 1$, or $p_1 > 0, t_1, t_2 < \tau$ and $z = 2$.

It remains to show that Lemma 4.1 also holds for 2) $p_1 > 0, t_1 < \tau, t_2 = 0, z = 1$. Proof by backward induction on t_1. Let $t_1 = \tau - 1$ and $z = 1$, then

$-2h + \sum_{q=p_1}^{P} \sum_{r=0}^{P} V([q,\tau;r,1],1)\pi_{p_1,q|2}\pi_{0,r|2} =$

$$\begin{aligned}
&= -2h + \sum_{q=p_1}^{P}\sum_{r=0}^{P} R'([q,\tau;r,1],1)\pi_{p_1,q|2}\pi_{0,r|2} = \\
&= -2h + \sum_{q=p_1}^{P}\sum_{r=0}^{P}\{-h(2\tau-\tau-1) + E[X_{1,\tau}|([q,\tau;r,1],1)] \\
&\qquad\qquad + E[X_{2,\tau}|([q,\tau;r,1],1)] + (\pi_{r,0|1})^{\tau-1}v(0,0)\}\pi_{p_1,q|2}\pi_{0,r|2} \\
&= -h(2\tau-(\tau-1)-0) + \sum_{q=p_1}^{P} E[X_{1,\tau}|([q,\tau;0,1],1)]\pi_{p_1,q|2} \\
&\qquad + \sum_{r=0}^{P}(E[X_{2,\tau}|([p_1,\tau;r,1],1)] + (\pi_{r,0|1})^{\tau-1}v(0,0))\pi_{0,r|2} \\
&= -h(2\tau-(\tau-1)-0) + E[X_{1,\tau}|([p_1,\tau-1;0,0],2)] \\
&\qquad + E[X_{2,\tau}|([p_1,\tau-1;0,0],2)] + (\pi_{0,0|2})(\pi_{0,0|1})^{\tau-1}v(0,0) \\
&= R([p_1,\tau-1;0,0],2)
\end{aligned}$$

where the first equality holds due to case 1) above, and the second equality holds due to that each auction progress independently of the price in the other auction. Assume the result holds for $t_1 = \tau-(l-1), \tau-(l-2), \ldots, \tau-1$. Let $t_1 = \tau - l$ and $z = 1$, then $-2h + \sum_{q=p_1}^{P}\sum_{r=0}^{P} V_{t+1}([q,\tau-(l-1);r,1],2)\pi_{p_1,q|2}\pi_{0,r|2} =$

$$\begin{aligned}
&= -2h + \sum_{q=p_1}^{P}\sum_{r=0}^{P} R'([q,\tau-(l-1);r,1],2)\pi_{p_1,q|2}\pi_{0,r|2} \\
&= -2h + \sum_{q=p_1}^{P}\sum_{r=0}^{P}\{-h(2\tau-(\tau-(l-1))-1) + E[X_{1,\tau}|([q,\tau-(l-1);r,1],2)] \\
&\qquad\qquad + E[X_{2,\tau}|([q,\tau-(l-1);r,1],2)] \\
&\qquad\qquad + \pi_2(0|[q,\tau-(l-1);r,1],2)v(0,0)\ \}\pi_{p_1,q|2}\pi_{0,r|2} \\
&= -h(2\tau-(\tau-l)-0) + \sum_{q=p_1}^{P} E[X_{1,\tau}|([q,\tau-(l-1);0,1],2)]\pi_{p_1,q|2} \\
&\qquad + \sum_{r=0}^{P} E[X_{2,\tau}|([p_1,\tau-(l-1);r,1],2)]\pi_{0,r|2} + (\pi_{0,0|2})^{l-1}(\pi_{0,0|1})^{\tau-(l-1)}v(0,0) \\
&= -h(2\tau-(\tau-l)-0) + E[X_{1,\tau}|([p_1,\tau-l;0,0],2)] + E[X_{2,\tau}|([p_1,\tau-l;0,0],2)] \\
&\qquad + (\pi_{0,0|2})^{l-1}(\pi_{0,0|1})^{\tau-(l-1)}v(0,0) \\
&= R([p_1,\tau-l;0,0],2)
\end{aligned}$$

where the first equality holds due to case 1) above, and the second equality holds due to that each auction progress independently of the price in the other auction. Therefore Lemma 4.1 holds for the case 2) $p_1 > 0, t_1 < \tau, t_2 = 0$ and $z = 1$. $\square$

Proof of Lemma 4.2

Comment: Recall that

$$\begin{aligned} R''([p_1,t_1;p_2,t_2],z) &= -h(\tau-t_1) + E[X_{1,\tau}|([p_1,t_1;p_2,t_2],z)] \\ &\quad +(1-(\pi_{p_1,0|2})^{\tau-t_1})(-h(t_1-t_2) + E[X_{2,\tau}|([p_1,t_1;p_2,t_2],z)]) \\ &\quad +(\pi_{p_1,0|2})^{\tau-t_1} E[V([X_2,t_2+\tau-t_1;0,0],1)|([p_1,t_1;p_2,t_2],2)] \end{aligned}$$

Let $t_1 = \tau - 1$, $p_1 = 0$, $p_2 > 0$ and $z = 2$. Proof by backward induction on t_2. Let $t_2 = \tau - 1$, then $V([p_1,t_1;p_2,t_2],z) =$

$$\begin{aligned} &= -2h + \sum_{r=p_2}^{P}(\sum_{q=p}^{P} V([q,\tau;r,\tau],0)\pi_{0,q|2} + \pi_{0,0|2}V([r,\tau;0,0],1))\pi_{p_2,r|2} \\ &= -2h + \sum_{r=p_2}^{P}\sum_{q=p}^{P} R'([q,\tau;r,\tau],0)\pi_{0,q|2}\pi_{p_2,r|2} + \pi_{0,0|2}\sum_{r=p_2}^{P} V([r,\tau;0,0],1)\pi_{p_2,r|2} \\ &= -2h + \sum_{r=p_2}^{P}\sum_{q=p}^{P}(E[X_{1,\tau}|([q,\tau;r,\tau],0)]) + E[X_{2,\tau}|([q,\tau;r,\tau],0)]))\pi_{0,q|2}\pi_{p_2,r|2} \\ &\quad +\pi_{0,0|2}E[V([X_2,\tau;0,0],1)|([0,\tau-1;p_2,\tau-1],2)] \\ &= -2h + \sum_{r=p_2}^{P}\sum_{q=p}^{P} q\pi_{0,q|2}\pi_{p_2,r|2} + \sum_{r=p_2}^{P}\sum_{q=p}^{P} r\pi_{0,q|2}\pi_{p_2,r|2} \\ &\quad +\pi_{0,0|2}E[V([X_2,\tau;0,0],1)|([0,\tau-1;p_2,\tau-1],2)] \end{aligned}$$

$$\begin{aligned} &= -2h + \sum_{q=p}^{P} q\pi_{0,q|2} + \sum_{q=p}^{P}\pi_{0,q|2}\sum_{r=p_2}^{P} r\pi_{p_2,r|2} \\ &\quad +\pi_{0,0|2}E[V([X_2,\tau;0,0],1)|([0,\tau-1;p_2,\tau-1],2)] \\ &= -2h + E[X_{1,\tau}|([0,\tau-1;p_2,\tau-1],2)] + (1-\pi_{0,0|2})E[X_{1,\tau}|([p_1,\tau-1;p_2,\tau-1],2)] \\ &\quad +\pi_{0,0|2}E[V([X_2,\tau;0,0],1)|([0,\tau-1;p_2,\tau-1],2)] \\ &= R''([0,\tau-1;p_2,\tau-1],2) \end{aligned}$$

Where the second equality holds due to Lemma 4.1 above, the fifth equality holds because $\sum_{r=p_2}^{P}\pi_{p_2,r|2} = 1$, and the sixth equality holds because $\sum_{q=p}^{P}\pi_{0,q|2} = 1 - \pi_{0,0|2}$. Assume the result holds for $t_2 = \tau-(l-1), \tau-(l-2), \ldots, \tau-1$. Let $t_2 = \tau - l$, then $V([p_1,t_1;p_2,t_2],z) =$

$$
\begin{aligned}
&= -2h + \sum_{r=p_2}^{P}(\sum_{q=p}^{P} V([q,\tau;r,t_2+1],1)\pi_{0,q|2} + \pi_{0,0|2}V([r,t_2+1;0,0],1))\pi_{p_2,r|2} \\
&= -2h + \sum_{r=p_2}^{P}\sum_{q=p}^{P} R'([q,\tau;r,t_2+1],1)\pi_{0,q|2}\pi_{p_2,r|2} + \pi_{0,0|2}\sum_{r=p_2}^{P} V([r,t_2+1;0,0],1)\pi_{p_2,r|2}
\end{aligned}
$$

$$
\begin{aligned}
&= -2h + \sum_{r=p_2}^{P}\sum_{q=p}^{P}\{-h(2\tau-\tau-(t_2+1)+E[X_{1,\tau}|([q,\tau;r,t_2+1],1)]) \\
&\qquad +E[X_{2,\tau}|([q,\tau;r,t_2+1],0)])\}\pi_{0,q|2}\pi_{p_2,r|2} \\
&\quad +\pi_{0,0|2}E[V([X_2,t_2+1;0,0],1)|([0,\tau-1;p_2,t_2],2)] \\
&= -2h + E[X_{1,\tau}|([0,\tau-1;p_2,t_2],2)]) \\
&\quad +\sum_{q=p}^{P}\{-h(2\tau-\tau-(t_2+1)+E[X_{2,\tau}|([0,\tau-1;p_2,t_2],2)])\}\pi_{0,q|2} \\
&\quad +\pi_{0,0|2}E[V([X_2,t_2+1;0,0],1)|([0,\tau-1;p_2,t_2],2)] \\
&= -2h(\tau-(\tau-1)) + E[X_{1,\tau}|([0,\tau-1;p_2,t_2],2)]) \\
&\quad +(1-\pi_{0,0|2})(-h(\tau-1-t_2)+E[X_{2,\tau}|([0,\tau-1;p_2,t_2],2)]) \\
&\quad +\pi_{0,0|2}E[V([X_2,t_2+1;0,0],1)|([0,\tau-1;p_2,t_2],2)] \\
&= R''([0,\tau-1;p_2,t_2],2)
\end{aligned}
$$

Where the second equality holds due to Lemma 4.1 above, the fourth equality holds because $\sum_{r=p_2}^{P}\pi_{p_2,r|2}=1$, and the fifth equality holds because $\sum_{q=p}^{P}\pi_{0,q|2}=1-\pi_{0,0|2}$. Therefore Lemma 4.2 holds for $t_1=\tau-1, p_1=0, p_2>0$, and $z=2$.

For $t_1<\tau-1, p_1=0, p_2>0, z=2$, proof by backward induction on t_1. Let $t_1=\tau-2$, then $V([p_1,t_1;p_2,t_2],z)=$

$$
\begin{aligned}
&= -2h + \sum_{q=0}^{P}\sum_{r=p_2}^{P} V([q,\tau-1;r,t_2+1],2)\pi_{0,q|2}\pi_{p_2,r|2} \\
&= -2h + \sum_{q=0}^{P}\sum_{r=p_2}^{P} R''([q,\tau-1;r,t_2+1],2)\pi_{0,q|2}\pi_{p_2,r|2} \\
&= -2h + \sum_{q=0}^{P}\sum_{r=p_2}^{P}\{-2h(\tau-(\tau-1)) + E[X_{1,\tau}|([q,\tau-1;r,t_2+1],2)] \\
&\qquad +(1-(\pi_{q,0|2})^{\tau-(\tau-1)})(-h(\tau-1-(t_2+1))+E[X_{2,\tau}|([q,\tau-1;r,t_2+1],2)]) \\
&\qquad +(\pi_{q,0|2})^{\tau-(\tau-1)}(E[V([X_2,t_2+1+\tau-(\tau-1);0,0],1)|([q,\tau-1;r,t_2+1],2)]) \\
&\qquad\ \}\pi_{0,q|2}\pi_{p_2,r|2}
\end{aligned}
$$

$$
\begin{aligned}
&= -2h(\tau-(\tau-2))) + E[X_{1,\tau}|([0,\tau-2;p_2,t_2],2)] \\
&\quad + \sum_{q=0}^{P}\sum_{r=p_2}^{P}\{(1-(\pi_{q,0|2})^{\tau-(\tau-1)})(-h(\tau-1-(t_2+1)) \\
&\qquad\qquad +E[X_{2,\tau}|([q,\tau-1;r,t_2+1],2)])\ \}\pi_{0,q|2}\pi_{p_2,r|2} \\
&\ +\pi_{0,0|2}(\pi_{0,0|2})^{\tau-(\tau-1)}\sum_{r=p_2}^{P} E[V([X_2,t_2+1+\tau-(\tau-1);0,0],1)|([0,\tau-1;r,t_2+1],2)]\pi_{p_2,r|2} \\
&= -2h(\tau-(\tau-2)) + E[X_{1,\tau}|([0,\tau-2;p_2,t_2],2)] \\
&\quad + \sum_{q=p}^{P}\sum_{r=p_2}^{P}\{-h(\tau-1-(t_2+1)) \\
&\qquad\qquad +E[X_{2,\tau}|([q,\tau-1;r,t_2+1],2)])\}\pi_{0,q|2}\pi_{p_2,r|2} \\
&\quad + \sum_{r=p_2}^{P}\{(1-(\pi_{0,0|2})^{\tau-(\tau-1)})(-h(\tau-1-(t_2+1)) \\
&\qquad\qquad +E[X_{2,\tau}|([0,\tau-1;r,t_2+1],2)])\}\pi_{0,0|2}\pi_{p_2,r|2} \\
&\quad +(\pi_{q,0|2})^{\tau-(\tau-2)}E[V([X_2,t_2+\tau-(\tau-2);0,0],1)|([0,\tau-2;p_2,t_2],2)] \\
&= -2h(\tau-(\tau-2)) + E[X_{1,\tau}|([0,\tau-2;p_2,t_2],2)] \\
&\quad +(1-\pi_{0,0|2})(-h(\tau-1-(t_2+1)) + E[X_{2,\tau}|([0,\tau-2;p_2,t_2],2)]) \\
&\quad +\pi_{0,0|2}(1-(\pi_{0,0|2})^{\tau-(\tau-1)})(-h(\tau-1-(t_2+1)) + E[X_{2,\tau}|([0,\tau-2;p_2,t_2],2)]) \\
&\quad +(\pi_{q,0|2})^{\tau-(\tau-2)}E[V([X_2,t_2+\tau-(\tau-2);0,0],1)|([0,\tau-2;p_2,t_2],2)] \\
&= -2h(\tau-(\tau-2)) + E[X_{1,\tau}|([0,\tau-2;p_2,t_2],2)] \\
&\quad +(1-\pi_{0,0|2})(-h(\tau-2-t_2) + E[X_{2,\tau}|([0,\tau-2;p_2,t_2],2)]) \\
&\quad +\pi_{0,0|2}(1-\pi_{0,0|2})(-h(\tau-2-t_2) + E[X_{2,\tau}|([0,\tau-2;p_2,t_2],2)]) \\
&\quad +(\pi_{q,0|2})^{\tau-(\tau-2)}E[V([X_2,t_2+\tau-(\tau-2);0,0],1)|([0,\tau-2;p_2,t_2],2)] \\
&= -2h(\tau-(\tau-2)) + E[X_{1,\tau}|([0,\tau-2;p_2,t_2],2)] \\
&\quad +(1+\pi_{0,0|2})(1-\pi_{0,0|2})(-h(\tau-2-t_2) + E[X_{2,\tau}|([0,\tau-2;p_2,t_2],2)]) \\
&\quad +(\pi_{q,0|2})^{\tau-(\tau-2)}(E[V([X_2,t_2+\tau-(\tau-2);0,0],1)|([0,\tau-2;p_2,t_2],2)]) \\
&= -2h(\tau-(\tau-2)) + E[X_{1,\tau}|([0,\tau-2;p_2,t_2],2)] \\
&\quad (1-\pi_{0,0|2})^{\tau-(\tau-2)})(-h(\tau-2-t_2) + E[X_{2,\tau}|([0,\tau-2;p_2,t_2],2)]) \\
&\quad +(\pi_{q,0|2})^{\tau-(\tau-2)}E[V([X_2,t_2+\tau-(\tau-2);0,0],1)|([0,\tau-2;p_2,t_2],2)] \\
&= R''([0,\tau-2;p_2,t_2],2)
\end{aligned}
$$

Where the second equality holds due to the case above with $t_1 = \tau - 1$, the fourth equality holds because the expected final price of the first auction is independent of the price in the second auction and that $\pi_{q,0|2} = 0$ for $q > 0$, the fifth equality holds because $\pi_{q,0|2} = 0$ for $q > 0$ and the second auction progress independently of the price in the first auction, and the sixth equality holds due to that $\sum_{q=p}^{P} \pi_{0,q|2} = 1 - \pi_{0,0|2}$ and that the second auction progress independently of the price in the first auction.

Assume the result holds for $t_1 = \tau - (l-1), \tau - (l-2), \ldots, \tau - 2$. Let $t_1 = \tau - l$, then $V([p_1, t_1; p_2, t_2], z) =$

$$
\begin{aligned}
&= -2h + \sum_{q=0}^{P} \sum_{r=p_2}^{P} V([q, \tau - (l-1); r, t_2 + 1], 2) \pi_{0,q|2} \pi_{p_2,r|2} \\
&= -2h + \sum_{q=0}^{P} \sum_{r=p_2}^{P} R''([q, \tau - (l-1); r, t_2 + 1], 2) \pi_{0,q|2} \pi_{p_2,r|2} \\
&= -2h + \sum_{q=0}^{P} \sum_{r=p_2}^{P} \{-2h(\tau - (\tau - (l-1))) + E[X_{1,\tau} | ([q, \tau - (l-1); r, t_2 + 1], 2)] \\
&\quad +(1 - (\pi_{q,0|2})^{\tau - (\tau - (l-1))})(-h(\tau - (l-1) - (t_2 + 1)) + E[X_{2,\tau} | ([q, \tau - (l-1); r, t_2 + 1], 2)]) \\
&\quad +(\pi_{q,0|2})^{\tau - (\tau - (l-1))} (E[V([X_2, t_2 + 1 + \tau - (\tau - (l-1)); 0, 0], 1) | ([q, \tau - (l-1); r, t_2 + 1], 2)]) \\
&\qquad\qquad \} \pi_{0,q|2} \pi_{p_2,r|2} \\
&= -2h(\tau - (\tau - l)) + E[X_{1,\tau} | ([0, \tau - l; p_2, t_2], 2)] \\
&\qquad + \sum_{q=0}^{P} \sum_{r=p_2}^{P} \{(1 - (\pi_{q,0|2})^{\tau - (\tau - (l-1))})(-h(\tau - (l-1) - (t_2 + 1)) \\
&\qquad\qquad + E[X_{2,\tau} | ([q, \tau - (l-1); r, t_2 + 1], 2)])\} \pi_{0,q|2} \pi_{p_2,r|2} \\
&\qquad + \pi_{0,0|2} (\pi_{0,0|2})^{\tau - (\tau - (l-1))} (\\
&\sum_{r=p_2}^{P} \{E[V([X_2, t_2 + 1 + \tau - (\tau - (l-1)); 0, 0], 1) | ([0, \tau - (l-1); r, t_2 + 1], 2)]\} \pi_{p_2,r|2}) \\
&= -2h(\tau - (\tau - (l-1))) + E[X_{1,\tau} | ([0, \tau - (l-1); p_2, t_2], 2)] \\
&\qquad + \sum_{q=p}^{P} \sum_{r=p_2}^{P} (-h(\tau - (l-1) - (t_2 + 1)) + E[X_{2,\tau} | ([q, \tau - (l-1); r, t_2 + 1], 2)]) \pi_{0,q|2} \pi_{p_2,r|2} \\
&\qquad + \sum_{r=p_2}^{P} \{(1 - (\pi_{0,0|2})^{\tau - (\tau - (l-1))})(-h(\tau - (l-1) - (t_2 + 1)) \\
&\qquad\qquad + E[X_{2,\tau} | ([0, \tau - (l-1); r, t_2 + 1], 2)])\} \pi_{0,0|2} \pi_{p_2,r|2} \\
&\qquad + (\pi_{q,0|2})^{\tau - (\tau - l)} E[V([X_2, t_2 + \tau - (\tau - l); 0, 0], 1) | ([0, \tau - l; p_2, t_2], 2)]
\end{aligned}
$$

$$
\begin{aligned}
&= -2h(\tau - (\tau - l)) + E[X_{1,\tau}|([0, \tau - l; p_2, t_2], 2)] \\
&\quad +(1 - \pi_{0,0|2})(-h(\tau - (l - 1) - (t_2 + 1)) + E[X_{2,\tau}|([0, \tau - l; p_2, t_2], 2)]) \\
&\quad +\pi_{0,0|2}(1 - (\pi_{0,0|2})^{\tau-(\tau-(l-1))})(-h(\tau - (l - 1) - (t_2 + 1)) + E[X_{2,\tau}|([0, \tau - l; p_2, t_2], 2)]) \\
&\quad +(\pi_{q,0|2})^{\tau-(\tau-l)} E[V([X_2, t_2 + \tau - (\tau - l); 0, 0], 1)|([0, \tau - l; p_2, t_2], 2)] \\
&= -2h(\tau - (\tau - l))) + E[X_{1,\tau}|([0, \tau - l; p_2, t_2], 2)] \\
&\quad +(1 - \pi_{0,0|2})(-h(\tau - l - t_2) + E[X_{2,\tau}|([0, \tau - l; p_2, t_2], 2)]) \\
&\quad +\pi_{0,0|2}(1 - \pi_{0,0|2})(-h(\tau - l - t_2) + E[X_{2,\tau}|([0, \tau - l; p_2, t_2], 2)]) \\
&\quad +(\pi_{q,0|2})^{\tau-(\tau-l)} E[V([X_2, t_2 + \tau - (\tau - l); 0, 0], 1)|([0, \tau - l; p_2, t_2], 2)] \\
&= -2h(\tau - (\tau - l)) + E[X_{1,\tau}|([0, \tau - l; p_2, t_2], 2)] \\
&\quad +(1 + \pi_{0,0|2})(1 - \pi_{0,0|2})(-h(\tau - l - t_2) + E[X_{2,\tau}|([0, \tau - l; p_2, t_2], 2)]) \\
&\quad +(\pi_{q,0|2})^{\tau-(\tau-l)} E[V([X_2, t_2 + \tau - (\tau - l); 0, 0], 1)|([0, \tau - l; p_2, t_2], 2)] \\
&= -2h(\tau - (\tau - l)) + E[X_{1,\tau}|([0, \tau - l; p_2, t_2], 2)] \\
&\quad (1 - \pi_{0,0|2})^{\tau-(\tau-l)})(-h(\tau - l - t_2) + E[X_{2,\tau}|([0, \tau - l; p_2, t_2], 2)]) \\
&\quad +(\pi_{q,0|2})^{\tau-(\tau-l)} E[V([X_2, t_2 + \tau - (\tau - l); 0, 0], 1)|([0, \tau - l; p_2, t_2], 2)] \\
&= R''([0, \tau - l; p_2, t_2], 2)
\end{aligned}
$$

Where the second equality holds due to the induction hypothesis, and the other equalities due to the same reasoning as above. Therefore Lemma 4.2 holds for $t_1 < \tau - 1, p_1 = 0, p_2 > 0$, and $z = 2$. □

Appendix 2: Optimality Equations - Infinite Horizon

Case	Item 1 t_1	Item 2 t_2	z	$V([p_1,t_1;p_2,t_2],z)$
1)	δ	δ	0	$= 0$
2)	δ	τ	0	$= p_2 + V(\Delta)$
	δ	$\tau - 1$	1	
3)		$p_2 > 0$		$= -h + \sum_{r=p_2}^{P} V([p_1,\delta;r,\tau],0)\pi_{p_2,r\vert 1}$
4)		$p_2 = 0$		$= -h + \sum_{r=p}^{P} V([p_1,\delta;r,\tau],0)\pi_{0,r\vert 1} + V([p_1,\delta;0,0],1)\pi_{0,0\vert 1}$
5)	δ	$< \tau - 1$	1	$= -h + \sum_{r=p_2}^{P} V([p_1,\delta;r,t_2+1],1)\pi_{p_2,r\vert 1}$
6)	τ	τ	0	$= p_1 + p_2 + V(\Delta)$
	τ	$\tau - 1$	1	
7)		$p_2 > 0$		$= p_1 - h + \sum_{r=p_2}^{P} V([p_1,\delta;r,\tau],0)\pi_{p_2,r\vert 1}$
8)		$p_2 = 0$		$= p_1 - h + \sum_{r=p}^{P} V([p_1,\delta;r,\tau],0)\pi_{0,r\vert 1} + V([p_1,\delta;0,0],1)\pi_{0,0\vert 1}$
9)	τ	$< \tau - 1$	1	$= p_1 - h + \sum_{r=p_2}^{P} V([p_1,\delta;r,t_2+1],1)\pi_{p_2,r\vert 1}$
	$\tau - 1$	$\tau - 1$	2	
10)	$p_1 > 0$	$p_2 > 0$		$= -2h + \sum_{q=p_1}^{P} \sum_{r=p_2}^{P} V([q,\tau;r,\tau],0)\pi_{p_1,q\vert 2}\pi_{p_2,r\vert 2}$
11)	$p_1 > 0$	$p_2 = 0$		$= -2h + \sum_{q=p_1}^{P} (\sum_{r=p}^{P} V([q,\tau;r,\tau],0)\pi_{0,r\vert 2} + \pi_{0,0\vert 2}V([q,\tau;0,0],1))\pi_{p_1,q\vert 2}$
12)	$p_1 = 0$	$p_2 > 0$		$= -2h + \sum_{r=p_2}^{P} (\sum_{q=p}^{P} V([q,\tau;r,\tau],0)\pi_{0,q\vert 2} + \pi_{0,0\vert 2}V([r,\tau;0,0],1))\pi_{p_2,r\vert 2}$
13)	$p_1 = 0$	$p_2 = 0$		$= -2h + \sum_{q=p}^{P} \sum_{r=p}^{P} V([q,\tau;r,\tau],0)\pi_{0,q\vert 2}\pi_{0,r\vert 2} + \pi_{0,0\vert 2}\pi_{0,0\vert 2}V([0,0;0,0],1)$
				$+ \sum_{q=p}^{P} V([q,\tau;0,0],1)\pi_{0,q\vert 2}\pi_{0,0\vert 2} + \sum_{r=p}^{P} V([r,\tau;0,0],1)\pi_{0,0\vert 2}\pi_{0,r\vert 2}$
	$\tau - 1$	$< \tau - 1$	2	
14)	$p_1 > 0$			$= -2h + \sum_{q=p_1}^{P} \sum_{r=p_2}^{P} V([q,\tau;r,t_2+1],1)\pi_{p_1,q\vert 2}\pi_{p_2,r\vert 2}$
15)	$p_1 = 0$			$= -2h + \sum_{r=p_2}^{P} (\sum_{q=p}^{P} V([q,\tau;r,t_2+1],1)\pi_{0,q\vert 2} + \pi_{0,0\vert 2}V([r,t_2+1;0,0],1))\pi_{p_2,r\vert 2}$
	$\tau - 1$	0	1	
16)	$p_1 > 0$			$= -2h + \max\{\sum_{q=p_1}^{P} V([q,\tau;0,0],1)\pi_{p_1,q\vert 1}\ , \sum_{q=p_1}^{P} \sum_{r=0}^{P} V([q,\tau;r,1],1)\pi_{p_1,q\vert 2}\pi_{0,r\vert 2}\}$
17)	$p_1 = 0$			$= -2h + \max\{\sum_{q=p}^{P} V([q,\tau;0,0],1)\pi_{0,q\vert 1} + \pi_{0,0\vert 1}V([0,0;0,0],1)\ ,$
				$\sum_{r=0}^{P} (\sum_{q=p}^{P} V([q,\tau;r,1],1)\pi_{p,q\vert 2} + \pi_{0,0\vert 2}V([r,1;0,0],1))\pi_{0,r\vert 2}\}$
18)	$< \tau - 1$	$< \tau - 1$	2	$= -2h + \sum_{q=p_1}^{P} \sum_{r=p_2}^{P} V([q,t_1+1;r,t_2+1],2)\pi_{p_1,q\vert 2}\pi_{p_2,r\vert 2}$
19)	$< \tau - 1$	0	1	$= -2h + \max\{\sum_{q=p_1}^{P} V([q,t_1+1;0,0],1)\pi_{p_1,q\vert 1}\ ,$
				$\sum_{q=p_1}^{P} \sum_{r=0}^{P} V([q,t_1+1;r,1],2)\pi_{p_1,q\vert 2}\pi_{0,r\vert 2}\}$

Table A2: Optimality equations for multiple re-listing case (infinte planning horizon)

Acknowledgments

The authors wish to thank Mahesh Nagarajan for his many helpful comments and suggestions, and Andrew Gray for his code to automate the extraction and analysis of auction data from eBay. This research was supported through NSERC Discovery Grant 5527-201.

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