Forum of Mathematics, Sigma (2022), **Vol. 10**:e80 1–33
doi:10.1017/fms.2022.70

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RESEARCH ARTICLE

Omega results for cubic field counts via lower-order terms in the one-level density

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Received: 24 February 2021; **Revised:** 7 July 2022; **Accepted:** 12 August 2022

2020 Mathematics Subject Classification: *Primary* – 11R16, 11R42, 11M26; *Secondary* – 11R47, 11M50

Abstract
In this paper, we obtain a precise formula for the one-level density of L-functions attached to non-Galois cubic Dedekind zeta functions. We find a secondary term which is unique to this context, in the sense that no lower-order term of this shape has appeared in previously studied families. The presence of this new term allows us to deduce an omega result for cubic field counting functions, under the assumption of the Generalised Riemann Hypothesis. We also investigate the associated L-functions Ratios Conjecture and find that it does not predict this new lower-order term. Taking into account the secondary term in Roberts's conjecture, we refine the Ratios Conjecture to one which captures this new term. Finally, we show that any improvement in the exponent of the error term of the recent Bhargava–Taniguchi–Thorne cubic field counting estimate would imply that the best possible error term in the refined Ratios Conjecture is $O_\varepsilon(X^{-\frac{1}{3}+\varepsilon})$. This is in opposition with all previously studied families in which the expected error in the Ratios Conjecture prediction for the one-level density is $O_\varepsilon(X^{-\frac{1}{2}+\varepsilon})$.

Contents

1. Introduction

In [KS1, KS2], Katz and Sarnak made a series of fundamental conjectures about statistics of low-lying zeros in families of L-functions. Recently, these conjectures have been refined by Sarnak, Shin and Templier [SaST] for families of parametric L-functions. There is a huge body of work on the confirmation of these conjectures for particular test functions in various families, many of which are harmonic (see, e.g. [ILS, Ru, FI, HR, ST]). There are significantly fewer geometric families that have been studied. In this context, we mention the work of Miller [M1] and Young [Yo] on families of elliptic curve L-functions and that of Yang [Ya], Cho and Kim [CK1, CK2] and Shankar, Södergren and Templier [ShST] on families of Artin L-functions.

In families of Artin L-functions, these results are strongly linked with counts of number fields. More precisely, the set of admissible test functions is determined by the quality of the error terms in such counting functions. In this paper we consider the sets

$$\mathcal{F}^{\pm}(X) := \{K/\mathbb{Q} \text{ non-Galois } : \ [K:\mathbb{Q}] = 3, 0 < \pm D_K < X\},$$

where for each cubic field $K/\mathbb{Q}$ of discriminant D_K, we include only one of its three isomorphic copies. The first power-saving estimate for the cardinality $N^{\pm}(X) := |\mathcal{F}^{\pm}(X)|$ was obtained by Belabas, Bhargava and Pomerance [BBP] and was later refined by Bhargava, Shankar and Tsimerman [BST], Taniguchi and Thorne [TT] and Bhargava, Taniguchi and Thorne [BTT]. The last three of these estimates take the shape

$$N^{\pm}(X) = C_1^{\pm} X + C_2^{\pm} X^{\frac{5}{6}} + O_{\varepsilon}(X^{\theta+\varepsilon}) \tag{1.1}$$

for certain explicit values of $\theta < \frac{5}{6}$, implying, in particular, Roberts's conjecture [Ro]. Here,

$$C_1^{+} := \frac{1}{12\zeta(3)}; \quad C_2^{+} := \frac{4\zeta(\frac{1}{3})}{5\Gamma(\frac{2}{3})^3\zeta(\frac{5}{3})}; \quad C_1^{-} := \frac{1}{4\zeta(3)}; \quad C_2^{-} := \frac{4\sqrt{3}\zeta(\frac{1}{3})}{5\Gamma(\frac{2}{3})^3\zeta(\frac{5}{3})}.$$

The presence of this secondary term is a striking feature of this family, and we are interested in studying its consequences for the distribution of low-lying zeros. More precisely, the estimate (1.1) suggests that one should be able to extract a corresponding lower-order term in various statistics on those zeros.

In addition to (1.1), we will consider precise estimates involving local conditions, which are of the form

$$\begin{aligned} N_p^{\pm}(X,T) :&= \#\{K \in \mathcal{F}^{\pm}(X) : p \text{ has splitting type } T \text{ in } K\} \\ &= A_p^{\pm}(T)X + B_p^{\pm}(T)X^{\frac{5}{6}} + O_{\varepsilon}(p^{\omega}X^{\theta+\varepsilon}), \end{aligned} \tag{1.2}$$

where p is a given prime, T is a splitting type and the constants $A_p^{\pm}(T)$ and $B_p^{\pm}(T)$ are defined in Section 2. Here, θ is the same constant as that in (1.1) and $\omega \geq 0$. Note, in particular, that (1.2) implies (1.1) (take $p = 2$ in (1.2) and sum over all splitting types T).

Perhaps surprisingly, it turns out that the study of low-lying zeros has an application to cubic field counts. More precisely, we were able to obtain the following conditional omega result for $N_p^{\pm}(X,T)$.

Theorem 1.1. *Assume the Generalised Riemann Hypothesis for $\zeta_K(s)$ for each $K \in \mathcal{F}^{\pm}(X)$. If $\theta, \omega \geq 0$ are admissible values in (1.2), then $\theta + \omega \geq \frac{1}{2}$.*

As part of this project, we have produced numerical data which suggest that $\theta = \frac{1}{2}$ and any $\omega > 0$ are admissible values in (1.2) (indicating, in particular, that the bound $\omega + \theta \geq \frac{1}{2}$ in Theorem 1.1 could be the best possible). We have made several graphs to support this conjecture in Appendix A. As a first example of these results, in Figure 1, we display a graph of $X^{-\frac{1}{2}}(N_5^{+}(X,T) - A_5^{+}(T)X - B_5^{+}(T)X^{\frac{5}{6}})$ for the various splitting types T, which suggests that $\theta = \frac{1}{2}$ is admissible and the best possible.

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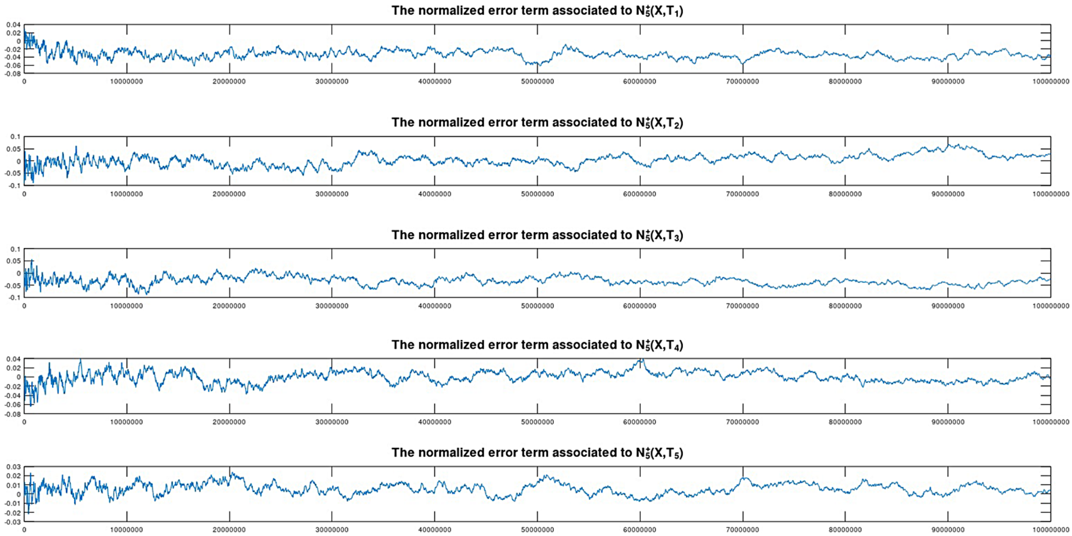

Figure 1. *The normalised error terms $X^{-\frac{1}{2}}(N_5^{+}(X,T) - A_5^{+}(T)X - B_5^{+}(T)X^{\frac{5}{6}})$ for the splitting types $T = T_1, \dots, T_5$ as described in Section 2.*

Let us now describe our unconditional result on low-lying zeros. For a cubic field K, we will focus on the Dedekind zeta function $\zeta_K(s)$, whose one-level density is defined by

$$\mathfrak{D}_\phi(K) := \sum_{\gamma_K} \phi\Big(\frac{\log(X/(2\pi e)^2)}{2\pi}\gamma_K\Big).$$

Here, ϕ is an even, smooth and rapidly decaying real function for which the Fourier transform

$$\widehat{\phi}(\xi) := \int_{\mathbb{R}} \phi(t) e^{-2\pi i \xi t} dt$$

is compactly supported. Note that ϕ can be extended to an entire function through the inverse Fourier transform. Moreover, X is a parameter (approximately equal to $|D_K|$) and $\rho_K = \frac{1}{2} + i\gamma_K$ runs through the nontrivial zeros[1] of $\zeta_K(s)/\zeta(s)$. In order to understand the distribution of the γ_K, we will average $\mathfrak{D}_\phi(K)$ over the family $\mathcal{F}^{\pm}(X)$. Our main technical result is a precise estimation of this average.

Theorem 1.2. *Assume that the cubic field count (1.2) holds for some fixed parameters $\frac{1}{2} \leq \theta < \frac{5}{6}$ and $\omega \geq 0$. Then, for any real even Schwartz function ϕ for which $\sigma := \sup(\operatorname{supp}(\widehat{\phi})) < \frac{1-\theta}{\omega+\frac{1}{2}}$, we have the estimate*

$$\begin{aligned}\frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} \mathfrak{D}_\phi(K) = \widehat{\phi}(0)\Big(1 + \frac{\log(4\pi^2 e)}{L} - \frac{C_2^{\pm}}{5C_1^{\pm}}\frac{X^{-\frac{1}{6}}}{L} + \frac{(C_2^{\pm})^2}{5(C_1^{\pm})^2}\frac{X^{-\frac{1}{3}}}{L}\Big) \\ + \frac{1}{\pi}\int_{-\infty}^{\infty} \phi\Big(\frac{Lr}{2\pi}\Big) \operatorname{Re}\Big(\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}(\tfrac{1}{2}+ir)\Big)dr - \frac{2}{L}\sum_{p,e} \frac{x_p \log p}{p^{\frac{e}{2}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)(\theta_e + \tfrac{1}{p}) \\ - \frac{2C_2^{\pm}X^{-\frac{1}{6}}}{C_1^{\pm}L}\Big(1 - \frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big)\sum_{p,e}\frac{\log p}{p^{\frac{e}{2}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)\beta_e(p) + O_\varepsilon(X^{\theta-1+\sigma(\omega+\frac{1}{2})+\varepsilon}),\end{aligned} \tag{1.3}$$

[1]The Riemann Hypothesis for $\zeta_K(s)$ implies that $\gamma_K \in \mathbb{R}$.

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where $\Gamma_+(s) := \pi^{-s}\Gamma(\frac{s}{2})^2$, $\Gamma_-(s) := \pi^{-s}\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})$, $x_p := (1+\frac{1}{p}+\frac{1}{p^2})^{-1}$, θ_e *and* $\beta_e(p)$ *are defined in (3.4) and (3.6), respectively, and* $L := \log\big(\frac{X}{(2\pi e)^2}\big)$.

Remark 1.3. In the language of the Katz–Sarnak heuristics, the first and third terms on the right-hand side of (1.3) are a manifestation of the symplectic symmetry type of the family $\mathcal{F}^{\pm}(X)$. More precisely, one can turn (1.3) into an expansion in descending powers of L using Lemma 3.4 as well as [MV, Lemma 12.14]. The first result in this direction is due to Yang [Ya], who showed that under the condition $\sigma < \frac{1}{50}$, we have that

$$\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\mathfrak{D}_\phi(K) = \widehat{\phi}(0) - \frac{\phi(0)}{2} + o_{X\to\infty}(1). \tag{1.4}$$

This last condition was relaxed to $\sigma < \frac{4}{41}$ by Cho–Kim [CK1, CK2][2] and Shankar–Södergren–Templier [ShST], independently, and corresponds to the admissible values $\theta = \frac{7}{9}$ and $\omega = \frac{16}{9}$ in (1.1) and (1.2) (see [TT]). In the recent paper [BTT], Bhargava, Taniguchi and Thorne show that $\theta = \frac{2}{3}$ and $\omega = \frac{2}{3}$ are admissible and deduce that (1.4) holds as soon as $\sigma < \frac{2}{7}$. Theorem 1.2 refines these results by obtaining a power saving estimate containing lower-order terms for the left-hand side of (1.4). Note, in particular, that the fourth term on the right-hand side of (1.3) is of order $X^{\frac{\sigma-1}{6}+o(1)}$ (see once more, Lemma 3.4).

The Katz–Sarnak heuristics are strongly linked with statistics of eigenvalues of random matrices and have been successful in predicting the main term in many families. However, this connection does not encompass lower-order terms. The major tool for making predictions in this direction is the L-functions Ratios Conjecture of Conrey, Farmer and Zirnbauer [CFZ]. In particular, these predictions are believed to hold down to an error term of size roughly the inverse of the square root of the size of the family. As an example, consider the unitary family of Dirichlet L-functions modulo q, in which the Ratios Conjecture's prediction is particularly simple. It is shown in [G+] that if η is a real even Schwartz function for which $\widehat{\eta}$ has compact (but arbitrarily large) support, then this conjecture implies the estimate

$$\frac{1}{\phi(q)}\sum_{\chi \bmod q}\sum_{\gamma_\chi}\eta\Big(\frac{\log q}{2\pi}\gamma_\chi\Big) = \widehat{\eta}(0)\Big(1-\frac{\log(8\pi e^\gamma)}{\log q}-\frac{\sum_{p\mid q}\frac{\log p}{p-1}}{\log q}\Big) + \int_0^\infty \frac{\widehat{\eta}(0)-\widehat{\eta}(t)}{q^{\frac{t}{2}}-q^{-\frac{t}{2}}}dt + E(q), \tag{1.5}$$

where $\rho_\chi = \frac{1}{2}+i\gamma_\chi$ is running through the nontrivial zeros of $L(s,\chi)$ and $E(q) \ll_\varepsilon q^{-\frac{1}{2}+\varepsilon}$. In [FM], it was shown that this bound on $E(q)$ is essentially the best possible, in general, but can be improved when the support of $\widehat{\eta}$ is small. This last condition also results in improved error terms in various other families (see, e.g. [M2, M3, FPS1, FPS2, DFS]).

Following the Ratios Conjecture recipe, we can obtain a prediction for the average of $\mathfrak{D}_\phi(K)$ over the family $\mathcal{F}^{\pm}(X)$. The resulting conjecture, however, differs from Theorem 1.2 by a term of order $X^{\frac{\sigma-1}{6}+o(1)}$, which is considerably larger than the expected error term $O_\varepsilon(X^{-\frac{1}{2}+\varepsilon})$. We were able to isolate a specific step in the argument which could be improved in order to include this additional contribution. More precisely, modifying Step 4 in [CFZ, Section 5.1], we recover a refined Ratios Conjecture which predicts a term of order $X^{\frac{\sigma-1}{6}+o(1)}$, in agreement with Theorem 1.2.

Theorem 1.4. *Let* $\frac{1}{2}\le\theta<\frac{5}{6}$ *and* $\omega\ge 0$ *be, such that (1.2) holds. Assume Conjecture 4.3 on the average of shifts of the logarithmic derivative of* $\zeta_K(s)/\zeta(s)$*, as well as the Riemann Hypothesis for* $\zeta_K(s)$*, for*

[2]In [CK1], the condition $\sigma < \frac{4}{25}$ should be corrected to $\sigma < \frac{4}{41}$.

https://doi.org/10.1017/fms.2022.70 Published online by Cambridge University Press

all $K \in \mathcal{F}^{\pm}(X)$. Let ϕ be a real even Schwartz function, such that $\widehat{\phi}$ is compactly supported. Then we have the estimate

$$\frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} \sum_{\gamma_K} \phi\Big(\frac{L\gamma_K}{2\pi}\Big) = \widehat{\phi}(0)\Big(1 + \frac{\log(4\pi^2 e)}{L} - \frac{C_2^{\pm}}{5C_1^{\pm}} \frac{X^{-\frac{1}{6}}}{L} + \frac{(C_2^{\pm})^2}{5(C_1^{\pm})^2} \frac{X^{-\frac{1}{3}}}{L}\Big)$$
$$+ \frac{1}{\pi} \int_{-\infty}^{\infty} \phi\Big(\frac{Lr}{2\pi}\Big) \operatorname{Re}\Big(\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}(\tfrac{1}{2} + ir)\Big) dr - \frac{2}{L} \sum_{p,e} \frac{x_p \log p}{p^{\frac{e}{2}}} \widehat{\phi}\Big(\frac{\log p^e}{L}\Big)(\theta_e + \tfrac{1}{p})$$
$$- \frac{2C_2^{\pm} X^{-\frac{1}{6}}}{C_1^{\pm} L}\Big(1 - \frac{C_2^{\pm}}{C_1^{\pm}} X^{-\frac{1}{6}}\Big) \sum_{p,e} \frac{\log p}{p^{\frac{e}{2}}} \widehat{\phi}\Big(\frac{\log p^e}{L}\Big)\beta_e(p) + J^{\pm}(X) + O_{\varepsilon}(X^{\theta - 1 + \varepsilon}),$$

where $J^{\pm}(X)$ is defined in (5.1). If $\sigma = \sup(\operatorname{supp}(\widehat{\phi})) < 1$, then we have the estimate

$$J^{\pm}(X) = C^{\pm} X^{-\frac{1}{3}} \int_{\mathbb{R}} \Big(\frac{X}{(2\pi e)^2}\Big)^{\frac{\xi}{6}} \widehat{\phi}(\xi) d\xi + O_{\varepsilon}(X^{\frac{\sigma - 1}{2} + \varepsilon}), \tag{1.6}$$

where $C^{\pm}$ is a nonzero absolute constant which is defined in (5.7). Otherwise, we have the identity

$$J^{\pm}(X) = -\frac{1}{\pi i} \int_{(\frac{1}{5})} \phi\Big(\frac{Ls}{2\pi i}\Big)\Big(1 - \frac{C_2^{\pm}}{C_1^{\pm}} X^{-\frac{1}{6}}\Big) X^{-s} \frac{\Gamma_{\pm}(\frac{1}{2} - s)}{\Gamma_{\pm}(\frac{1}{2} + s)} \zeta(1 - 2s) \frac{A_3(-s, s)}{1 - s} ds$$
$$- \frac{1}{\pi i} \int_{(\frac{1}{20})} \phi\Big(\frac{Ls}{2\pi i}\Big) \frac{C_2^{\pm}}{C_1^{\pm}} X^{-s - \frac{1}{6}} \frac{\Gamma_{\pm}(\frac{1}{2} - s)}{\Gamma_{\pm}(\frac{1}{2} + s)} \zeta(1 - 2s) \Bigg\{\Big(1 - \frac{C_2^{\pm}}{C_1^{\pm}} X^{-\frac{1}{6}}\Big) \frac{\zeta(\frac{5}{6} - s)}{\zeta(\frac{5}{6} + s)} \frac{A_4(-s, s)}{1 - \frac{6s}{5}}$$
$$+ \frac{C_2^{\pm}}{C_1^{\pm}} X^{-\frac{1}{6}} \frac{A_3(-s, s)}{1 - s}\Bigg\} ds, \tag{1.7}$$

where $A_3(-s, s)$ and $A_4(-s, s)$ are defined in (5.2) and (4.9), respectively.

Remark 1.5. It is interesting to compare Theorem 1.4 with Theorem 1.2, especially when σ is small. Indeed, for $\sigma < 1$, the difference between those two evaluations of the one-level density is given by

$$C^{\pm} X^{-\frac{1}{3}} \int_{\mathbb{R}} \Big(\frac{X}{(2\pi e)^2}\Big)^{\frac{\xi}{6}} \widehat{\phi}(\xi) d\xi + O_{\varepsilon}\big(X^{\frac{\sigma - 1}{2} + \varepsilon} + X^{\theta - 1 + \sigma(\omega + \frac{1}{2}) + \varepsilon}\big).$$

Selecting test functions ϕ for which $\widehat{\phi} \geq 0$ and σ is positive but arbitrarily small, this shows that no matter how large ω is, any admissible $\theta < \frac{2}{3}$ in (1.1) and (1.2) would imply that this difference is asymptotic to $C^{\pm} X^{-\frac{1}{3}} \int_{\mathbb{R}} (\frac{X}{(2\pi e)^2})^{\frac{\xi}{6}} \widehat{\phi}(\xi) d\xi \gg X^{-\frac{1}{3}}$. In fact, Roberts's numerics [Ro] (see also [B]), as well as our numerical investigations described in Appendix A, indicate that $\theta = \frac{1}{2}$ could be admissible in (1.1) and (1.2). In other words, in this family, the Ratios Conjecture, as well as our refinement (combined with the assumption of (1.1) and (1.2) for some $\theta < \frac{2}{3}$ and $\omega \geq 0$), are not sufficient to obtain a prediction with precision $o(X^{-\frac{1}{3}})$. This is surprising, since Conrey, Farmer and Zirnbauer have conjectured this error term to be of size $O_{\varepsilon}(X^{-\frac{1}{2} + \varepsilon})$, and this has been confirmed in several important families [M2, M3, FPS1, FPS2, DFS] (for a restricted set of test functions).

2. Background

Let $K/\mathbb{Q}$ be a non-Galois cubic field, and let $\widehat{K}$ be the Galois closure of K over $\mathbb{Q}$. Then, the Dedekind zeta function of the field K has the decomposition

$$\zeta_K(s) = \zeta(s) L(s, \rho, \widehat{K}/\mathbb{Q}),$$

where $L(s,\rho,\widehat{K}/\mathbb{Q})$ is the Artin L-function associated to the two-dimensional representation ρ of $\mathrm{Gal}(\widehat{K}/\mathbb{Q}) \simeq S_3$. The strong Artin conjecture is known for such representations; in this particular case, we have an explicit underlying cuspidal representation τ of $GL_2/\mathbb{Q}$, such that $L(s,\rho,\widehat{K}/\mathbb{Q}) = L(s,\tau)$. For the sake of completeness, let us describe τ in more detail. Let $F = \mathbb{Q}[\sqrt{D_K}]$, and let χ be a nontrivial character of $\mathrm{Gal}(\widehat{K}/F) \simeq C_3$, considered as a Hecke character of F. Then $\tau = \mathrm{Ind}_F^{\mathbb{Q}}\chi$ is a dihedral representation of central character $\chi_{D_K} = \left(\frac{D_K}{\cdot}\right)$. When $D_K < 0$, τ corresponds to a weight one newform of level $|D_K|$ and nebentypus χ_{D_K}, and when $D_K > 0$, it corresponds to a weight zero Maass form (see [DFI, Introduction]). In both cases, we will denote the corresponding automorphic form by f_K, and, in particular, we have the equality

$$L(s,\rho,\widehat{K}/\mathbb{Q}) = L(s,f_K).$$

We are interested in the analytic properties of $\zeta_K(s)/\zeta(s) = L(s,f_K)$. We have the functional equation

$$\Lambda(s,f_K) = \Lambda(1-s,f_K). \tag{2.1}$$

Here, $\Lambda(s,f_K) := |D_K|^{\frac{s}{2}}\Gamma_{f_K}(s)L(s,f_K)$ is the completed L-function, with the gamma factor

$$\Gamma_{f_K}(s) = \begin{cases} \Gamma_+(s) & \text{if } D_K > 0 \text{ (that is } K \text{ has signature } (3,0));\\ \Gamma_-(s) & \text{if } D_K < 0 \text{ (that is } K \text{ has signature } (1,1)),\end{cases}$$

where $\Gamma_+(s) := \pi^{-s}\Gamma(\frac{s}{2})^2$ and $\Gamma_-(s) := \pi^{-s}\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})$.

The coefficients of $L(s,f_K)$ have an explicit description in terms of the splitting type of the prime ideal $(p)\mathcal{O}_K$. Writing

$$L(s,f_K) = \sum_{n=1}^{\infty}\frac{\lambda_K(n)}{n^s},$$

we have that

Splitting type	$(p)\mathcal{O}_K$	$\lambda_K(p^e)$
T_1	$\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$	$e+1$
T_2	$\mathfrak{p}_1\mathfrak{p}_2$	$(1+(-1)^e)/2$
T_3	$\mathfrak{p}_1$	τ_e
T_4	$\mathfrak{p}_1^2\mathfrak{p}_2$	1
T_5	$\mathfrak{p}_1^3$	0

where

$$\tau_e := \begin{cases} 1 & \text{if } e \equiv 0 \bmod 3;\\ -1 & \text{if } e \equiv 1 \bmod 3;\\ 0 & \text{if } e \equiv 2 \bmod 3.\end{cases}$$

Furthermore, we find that the coefficients of the reciprocal

$$\frac{1}{L(s,f_K)} = \sum_{n=1}^{\infty}\frac{\mu_K(n)}{n^s} \tag{2.2}$$

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are given by

$$\mu_K(p^k) = \begin{cases} -\lambda_K(p) & \text{if } k = 1; \\ \left(\frac{D_K}{p}\right) & \text{if } k = 2; \\ 0 & \text{if } k > 2. \end{cases}$$

The remaining values of $\lambda_K(n)$ and $\mu_K(n)$ are determined by multiplicativity. Finally, the coefficients of the logarithmic derivative

$$-\frac{L'}{L}(s, f_K) = \sum_{n\geq 1} \frac{\Lambda(n) a_K(n)}{n^s}$$

are given by

Splitting type	(p)	$a_K(p^e)$
T_1	$\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$	2
T_2	$\mathfrak{p}_1\mathfrak{p}_2$	$1+(-1)^e$
T_3	$\mathfrak{p}_1$	η_e
T_4	$\mathfrak{p}_1^2\mathfrak{p}_2$	1
T_5	$\mathfrak{p}_1^3$	0

where

$$\eta_e := \begin{cases} 2 & \text{if } e \equiv 0 \bmod 3; \\ -1 & \text{if } e \equiv \pm 1 \bmod 3. \end{cases}$$

We now describe explicitly the constants $A_p^{\pm}(T)$ and $B_p^{\pm}(T)$ that appear in (1.2). More generally, let $\mathbf{p} = (p_1, \dots, p_J)$ be a vector of primes and let $\mathbf{k} = (k_1, \dots, k_J) \in \{1, 2, 3, 4, 5\}^J$ (when $J = 1$, $\mathbf{p} = (p)$ is a scalar, and we will abbreviate by writing $\mathbf{p} = p$ and similarly for $\mathbf{k}$). We expect that

$$\begin{aligned} N_{\mathbf{p}}^{\pm}(X, T_{\mathbf{k}}) :&= \#\{K \in \mathcal{F}^{\pm}(X) : p_j \text{ has splitting type } T_{k_j} \text{ in } K \ (1 \leq j \leq J)\} \\ &= A_{\mathbf{p}}^{\pm}(T_{\mathbf{k}})X + B_{\mathbf{p}}^{\pm}(T_{\mathbf{k}})X^{\frac{5}{6}} + O_{\varepsilon}((p_1 \cdots p_J)^{\omega} X^{\theta+\varepsilon}), \end{aligned} \tag{2.3}$$

for some $\omega \geq 0$ and with the same θ as in (1.1). Here,

$$A_{\mathbf{p}}^{\pm}(T_{\mathbf{k}}) = C_1^{\pm} \prod_{j=1}^{J} (x_{p_j} c_{k_j}(p_j)), \qquad B_{\mathbf{p}}^{\pm}(T_{\mathbf{k}}) = C_2^{\pm} \prod_{j=1}^{J} (y_{p_j} d_{k_j}(p_j)),$$

$$x_p := \left(1 + \frac{1}{p} + \frac{1}{p^2}\right)^{-1}, \qquad y_p := \frac{1 - p^{-\frac{1}{3}}}{(1 - p^{-\frac{5}{3}})(1 + p^{-1})},$$

and $c_k(p)$ and $d_k(p)$ are defined in the following table:

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k	$c_k(p)$	$d_k(p)$
1	$\frac{1}{6}$	$\frac{(1+p^{-\frac{1}{3}})^3}{6}$
2	$\frac{1}{2}$	$\frac{(1+p^{-\frac{1}{3}})(1+p^{-\frac{2}{3}})}{2}$
3	$\frac{1}{3}$	$\frac{(1+p^{-1})}{3}$
4	$\frac{1}{p}$	$\frac{(1+p^{-\frac{1}{3}})^2}{p}$
5	$\frac{1}{p^2}$	$\frac{(1+p^{-\frac{1}{3}})}{p^2}$

Recently, Bhargava, Taniguchi and Thorne [BTT] have shown that the values $\theta = \omega = \frac{2}{3}$ are admissible in (2.3).

3. New lower-order terms in the one-level density

In this section, we shall estimate the one-level density

$$\frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} \mathfrak{D}_{\phi}(K)$$

assuming the cubic field count (1.2) for some fixed parameters $\frac{1}{2} \leq \theta < \frac{5}{6}$ and $\omega \geq 0$. Throughout the paper, we will use the shorthand

$$L = \log\Big(\frac{X}{(2\pi e)^2}\Big).$$

The starting point of this section is the explicit formula.

Lemma 3.1. *Let ϕ be a real even Schwartz function whose Fourier transform is compactly supported, and let $K \in \mathcal{F}^{\pm}(X)$. We have the formula*

$$\begin{aligned}\mathfrak{D}_{\phi}(K) = \sum_{\gamma_K} \phi\Big(\frac{L\gamma_K}{2\pi}\Big) &= \frac{\widehat{\phi}(0)}{L} \log|D_K| + \frac{1}{\pi} \int_{-\infty}^{\infty} \phi\Big(\frac{Lr}{2\pi}\Big) \operatorname{Re}\Big(\frac{\Gamma'_{\pm}}{\Gamma_{\pm}}(\tfrac{1}{2} + ir)\Big) dr \\ &\quad - \frac{2}{L} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \widehat{\phi}\Big(\frac{\log n}{L}\Big) a_K(n),\end{aligned} \tag{3.1}$$

where $\rho_K = \frac{1}{2} + i\gamma_K$ runs over the nontrivial zeros of $L(s, f_K)$.

Proof. This follows from, for example, [RS, Proposition 2.1], but for the sake of completeness, we reproduce the proof here. By Cauchy's integral formula, we have the identity

$$\begin{aligned}\sum_{\gamma_K} \phi\Big(\frac{L\gamma_K}{2\pi}\Big) &= \frac{1}{2\pi i} \int_{(\frac{3}{2})} \phi\Big(\frac{L}{2\pi i}\Big(s - \frac{1}{2}\Big)\Big) \frac{\Lambda'}{\Lambda}(s, f_K) ds \\ &\quad - \frac{1}{2\pi i} \int_{(-\frac{1}{2})} \phi\Big(\frac{L}{2\pi i}\Big(s - \frac{1}{2}\Big)\Big) \frac{\Lambda'}{\Lambda}(s, f_K) ds.\end{aligned}$$

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These integrals converge since $\phi\left(\frac{L}{2\pi i}\left(s-\frac{1}{2}\right)\right)$ is rapidly decreasing in vertical strips. For the second integral, we apply the change of variables $s \to 1-s$. Then, by the functional equation in the form $\frac{\Lambda'}{\Lambda}(1-s, f_K) = -\frac{\Lambda'}{\Lambda}(s, f_K)$ and since $\phi(-s) = \phi(s)$, we deduce that

$$\sum_{\gamma_K} \phi\left(\frac{L\gamma_K}{2\pi}\right) = \frac{1}{\pi i}\int_{(\frac{3}{2})} \phi\left(\frac{L}{2\pi i}\left(s-\frac{1}{2}\right)\right)\frac{\Lambda'}{\Lambda}(s, f_K)ds.$$

Next, we insert the identity

$$\frac{\Lambda'}{\Lambda}(s, f_K) = \frac{1}{2}\log|D_K| + \frac{\Gamma'_{f_K}}{\Gamma_{f_K}}(s) - \sum_{n\geq 1}\frac{\Lambda(n)a_K(n)}{n^s}$$

and separate into three integrals. By shifting the contour of integration to $\mathrm{Re}(s) = \frac{1}{2}$ in the first two integrals, we obtain the first two terms on the right-hand side of (3.1). The third integral is equal to

$$-2\sum_{n\geq 1}\frac{\Lambda(n)a_K(n)}{\sqrt{n}}\frac{1}{2\pi i}\int_{(\frac{3}{2})}\phi\left(\frac{L}{2\pi i}\left(s-\frac{1}{2}\right)\right)n^{-(s-\frac{1}{2})}ds.$$

By moving the contour to $\mathrm{Re}(s) = \frac{1}{2}$ and applying Fourier inversion, we find the third term on the right-hand side of (3.1) and the claim follows. □

Our goal is to average (3.1) over $K \in \mathcal{F}^{\pm}(X)$. We begin with the first term.

Lemma 3.2. *Assume that (1.1) holds for some* $0 \leq \theta < \frac{5}{6}$. *Then, we have the estimate*

$$\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\log|D_K| = \log X - 1 - \frac{C_2^{\pm}}{5C_1^{\pm}}X^{-\frac{1}{6}} + \frac{(C_2^{\pm})^2}{5(C_1^{\pm})^2}X^{-\frac{1}{3}} + O_{\varepsilon}(X^{\theta-1+\varepsilon} + X^{-\frac{1}{2}}).$$

Proof. Applying partial summation, we find that

$$\sum_{K\in\mathcal{F}^{\pm}(X)}\log|D_K| = \int_1^X (\log t)dN^{\pm}(t) = N^{\pm}(X)\log X - N^{\pm}(X) - \frac{1}{5}C_2^{\pm}X^{\frac{5}{6}} + O_{\varepsilon}(X^{\theta+\varepsilon}).$$

The claimed estimate follows from applying (1.1). □

For the second term of (3.1), we note that it is constant on $\mathcal{F}^{\pm}(X)$. We can now concentrate our efforts on the average of the third (and most crucial) term

$$I^{\pm}(X;\phi) := -\frac{2}{LN^{\pm}(X)}\sum_p\sum_{e=1}^{\infty}\frac{\log p}{p^{e/2}}\widehat{\phi}\left(\frac{e\log p}{L}\right)\sum_{K\in\mathcal{F}^{\pm}(X)}a_K(p^e). \tag{3.2}$$

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It follows from (1.2) that

$$\sum_{K\in\mathcal{F}^{\pm}(X)} a_K(p^e) = 2N_p^{\pm}(X,T_1) + (1+(-1)^e)N_p^{\pm}(X,T_2) + \eta_e N_p^{\pm}(X,T_3) + N_p^{\pm}(X,T_4)$$
$$= C_1^{\pm}X(\theta_e + \tfrac{1}{p})x_p + C_2^{\pm}X^{\frac{5}{6}}(1+p^{-\frac{1}{3}})(\kappa_e(p) + p^{-1} + p^{-\frac{4}{3}})y_p + O_\varepsilon(p^{\omega}X^{\theta+\varepsilon}), \tag{3.3}$$

where

$$\theta_e := \delta_{2|e} + \delta_{3|e} = \begin{cases} 2 & \text{if } e \equiv 0 \bmod 6 \\ 0 & \text{if } e \equiv 1 \bmod 6 \\ 1 & \text{if } e \equiv 2 \bmod 6 \\ 1 & \text{if } e \equiv 3 \bmod 6 \\ 1 & \text{if } e \equiv 4 \bmod 6 \\ 0 & \text{if } e \equiv 5 \bmod 6, \end{cases} \tag{3.4}$$

and

$$\kappa_e(p) := (\delta_{2|e} + \delta_{3|e})(1+p^{-\frac{2}{3}}) + (1-\delta_{3|e})p^{-\frac{1}{3}} = \begin{cases} 2+2p^{-\frac{2}{3}} & \text{if } e \equiv 0 \bmod 6 \\ p^{-\frac{1}{3}} & \text{if } e \equiv 1 \bmod 6 \\ 1+p^{-\frac{1}{3}}+p^{-\frac{2}{3}} & \text{if } e \equiv 2 \bmod 6 \\ 1+p^{-\frac{2}{3}} & \text{if } e \equiv 3 \bmod 6 \\ 1+p^{-\frac{1}{3}}+p^{-\frac{2}{3}} & \text{if } e \equiv 4 \bmod 6 \\ p^{-\frac{1}{3}} & \text{if } e \equiv 5 \bmod 6. \end{cases} \tag{3.5}$$

Here, $\delta_{\mathcal{P}}$ is equal to 1 if $\mathcal{P}$ is true and is equal to 0 otherwise. Note that we have the symmetries $\theta_{-e} = \theta_e$ and $\kappa_{-e}(p) = \kappa_e(p)$. With this notation, we prove the following proposition.

Proposition 3.3. *Let ϕ be a real even Schwartz function for which $\widehat{\phi}$ has compact support, and let $\sigma := \sup(\operatorname{supp}(\widehat{\phi}))$. Assume that (1.2) holds for some fixed parameters $0 \le \theta < \frac{5}{6}$ and $\omega \ge 0$. Then we have the estimate*

$$I^{\pm}(X;\phi) = -\frac{2}{L}\sum_{p,e}\frac{x_p \log p}{p^{\frac{e}{2}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)(\theta_e + \tfrac{1}{p})$$
$$+\frac{2}{L}\Big(-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}} + \frac{(C_2^{\pm})^2}{(C_1^{\pm})^2}X^{-\frac{1}{3}}\Big)\sum_{p,e}\frac{\log p}{p^{\frac{e}{2}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)\beta_e(p) + O_\varepsilon(X^{\theta-1+\sigma(\omega+\frac{1}{2})+\varepsilon} + X^{-\frac{1}{2}+\frac{\sigma}{6}}),$$

where

$$\beta_e(p) := y_p(1+p^{-\frac{1}{3}})(\kappa_e(p) + p^{-1} + p^{-\frac{4}{3}}) - x_p(\theta_e + \tfrac{1}{p}). \tag{3.6}$$

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Proof. Applying (3.3), we see that

$$\begin{aligned}
I^{\pm}(X;\phi) &= -\frac{2C_1^{\pm}X}{LN^{\pm}(X)}\sum_{p}\sum_{e=1}^{\infty}\frac{x_p\log p}{p^{\frac{e}{2}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)(\theta_e+\tfrac{1}{p})\\
&\quad -\frac{2C_2^{\pm}X^{\frac{5}{6}}}{LN^{\pm}(X)}\sum_{p}\sum_{e=1}^{\infty}\frac{y_p(1+p^{-\frac{1}{3}})\log p}{p^{\frac{e}{2}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)(\kappa_e(p)+p^{-1}+p^{-\frac{4}{3}})\\
&\quad +O_{\varepsilon}\Big(X^{\theta-1+\varepsilon}\sum_{\substack{p^e\leq X^{\sigma}\\ e\geq 1}}p^{\omega-\frac{e}{2}}\log p\Big)\\
&= -\frac{2}{L}\Big(1-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}+\frac{(C_2^{\pm})^2}{(C_1^{\pm})^2}X^{-\frac{1}{3}}\Big)\sum_{p}\sum_{e=1}^{\infty}\frac{x_p\log p}{p^{\frac{e}{2}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)(\theta_e+\tfrac{1}{p})\\
&\quad -\frac{2}{L}\Big(\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}-\frac{(C_2^{\pm})^2}{(C_1^{\pm})^2}X^{-\frac{1}{3}}\Big)\sum_{p}\sum_{e=1}^{\infty}\frac{y_p(1+p^{-\frac{1}{3}})\log p}{p^{\frac{e}{2}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)(\kappa_e(p)+p^{-1}+p^{-\frac{4}{3}})\\
&\quad +O_{\varepsilon}\big(X^{\theta-1+\sigma(\omega+\frac{1}{2})+\varepsilon}+X^{-\frac{1}{2}+\frac{\sigma}{6}}\big).
\end{aligned}$$

Note, in particular, that the error term $O(X^{-\frac{1}{2}+\frac{\sigma}{6}})$ bounds the size of the contribution of the first omitted term in the expansion of $X^{\frac{5}{6}}/N^{\pm}(X)$ appearing in the second double sum above. Indeed, this follows since $\kappa_1(p)=p^{-\frac{1}{3}}$ and

$$X^{-\frac{1}{2}}\sum_{p\leq X^{\sigma}}\frac{\log p}{p^{\frac{5}{6}}}=O\big(X^{-\frac{1}{2}+\frac{\sigma}{6}}\big).$$

The claimed estimate follows. □

Proof of Theorem 1.2. Combine Lemmas 3.1 and 3.2 with Proposition 3.3. □

We shall estimate $I^{\pm}(X;\phi)$ further and find asymptotic expansions for the double sums in Proposition 3.3.

Lemma 3.4. *Let ϕ be a real even Schwartz function whose Fourier transform is compactly supported, define $\sigma := \sup(\mathrm{supp}(\widehat{\phi}))$, and let ℓ be a positive integer. Define*

$$I_1(X;\phi):=\sum_{p}\sum_{e=1}^{\infty}\frac{x_p\log p}{p^{\frac{e}{2}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)(\theta_e+\tfrac{1}{p}),\qquad I_2(X;\phi):=\sum_{p}\sum_{e=1}^{\infty}\frac{\log p}{p^{\frac{e}{2}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)\beta_e(p).$$

Then, we have the asymptotic expansion

$$I_1(X;\phi)=\frac{\phi(0)}{4}L+\sum_{n=0}^{\ell}\frac{\widehat{\phi}^{(n)}(0)\nu_1(n)}{n!}\frac{1}{L^n}+O_{\ell}\Big(\frac{1}{L^{\ell+1}}\Big),$$

where

$$\begin{aligned}
\nu_1(n) &:= \delta_{n=0}+\sum_{p}\sum_{e\neq 2}\frac{x_pe^n(\log p)^{n+1}}{p^{\frac{e}{2}}}(\theta_e+\tfrac{1}{p})+\sum_{p}\frac{2^n(\log p)^{n+1}}{p}\Big(x_p\Big(1+\frac{1}{p}\Big)-1\Big)\\
&\quad +\int_1^{\infty}\frac{2^n(\log u)^{n-1}(\log u-n)}{u^2}\mathcal{R}(u)du
\end{aligned}$$

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with $\mathcal{R}(u) := \sum_{p\le u}\log p - u$. *Moreover, we have the estimate*

$$I_2(X;\phi) = L\int_0^\infty \widehat{\phi}(u)e^{\frac{Lu}{6}}\,du + O\Big(X^{\frac{\sigma}{6}}e^{-c_0(\sigma)\sqrt{\log X}}\Big),$$

where $c_0(\sigma) > 0$ *is a constant. Under the Riemann Hypothesis, we have the more precise expansion*

$$I_2(X;\phi) = L\int_0^\infty \widehat{\phi}(u)e^{\frac{Lu}{6}}\,du + \sum_{n=0}^{\ell}\frac{\widehat{\phi}^{(n)}(0)\nu_2(n)}{n!}\frac{1}{L^n} + O_\ell\Big(\frac{1}{L^{\ell+1}}\Big),$$

where

$$\nu_2(n) := \delta_{n=0} + \sum_p\sum_{e=2}^{\infty}\frac{e^n(\log p)^{n+1}\beta_e(p)}{p^{\frac{e}{2}}} + \sum_p\frac{(\log p)^{n+1}}{p^{\frac12}}\Big(\beta_1(p) - \frac{1}{p^{\frac13}}\Big)$$
$$+ \int_1^\infty \frac{(\log u)^{n-1}(5\log u - 6n)}{6u^{\frac{11}{6}}}\mathcal{R}(u)du.$$

Proof. We first split the sums as

$$I_1(X;\phi) = \sum_p \frac{\log p}{p}\widehat{\phi}\Big(\frac{2\log p}{L}\Big) + I_1'(X;\phi), \qquad I_2(X;\phi) = \sum_p\frac{\log p}{p^{\frac56}}\widehat{\phi}\Big(\frac{\log p}{L}\Big) + I_2'(X;\phi), \tag{3.7}$$

where

$$I_1'(X;\phi) := \sum_p\sum_{e\neq 2}\frac{x_p\log p}{p^{\frac{e}{2}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)(\theta_e + \tfrac1p) + \sum_p\frac{\log p}{p}\Big(x_p\Big(1+\frac1p\Big) - 1\Big)\widehat{\phi}\Big(\frac{2\log p}{L}\Big),$$
$$I_2'(X;\phi) := \sum_p\sum_{e=2}^{\infty}\frac{\log p}{p^{\frac{e}{2}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)\beta_e(p) + \sum_p\frac{\log p}{p^{\frac12}}\widehat{\phi}\Big(\frac{\log p}{L}\Big)\Big(\beta_1(p) - \frac{1}{p^{\frac13}}\Big). \tag{3.8}$$

We may also rewrite the sums in (3.7) using partial summation as follows:

$$\sum_p\frac{\log p}{p}\widehat{\phi}\Big(\frac{2\log p}{L}\Big) = \int_1^\infty \frac1u\widehat{\phi}\Big(\frac{2\log u}{L}\Big)d(u+\mathcal{R}(u))$$
$$= \frac{\phi(0)}{4}L + \widehat{\phi}(0) - \int_1^\infty\Big(\frac{-1}{u^2}\widehat{\phi}\Big(\frac{2\log u}{L}\Big) + \frac{2}{u^2L}\widehat{\phi}'\Big(\frac{2\log u}{L}\Big)\Big)\mathcal{R}(u)du,$$
$$\sum_p\frac{\log p}{p^{\frac56}}\widehat{\phi}\Big(\frac{\log p}{L}\Big) = L\int_0^\infty\widehat{\phi}(u)e^{Lu/6}du + \widehat{\phi}(0)$$
$$-\int_1^{X^\sigma}\Big(\frac{-5}{6u^2}\widehat{\phi}\Big(\frac{\log u}{L}\Big) + \frac{1}{u^2L}\widehat{\phi}'\Big(\frac{\log u}{L}\Big)\Big)u^{\frac16}\mathcal{R}(u)du. \tag{3.9}$$

Next, for any $\ell \ge 1$ and $|t| \le \sigma$, Taylor's theorem reads

$$\widehat{\phi}(t) = \sum_{n=0}^{\ell}\frac{\widehat{\phi}^{(n)}(0)}{n!}t^n + O_\ell(|t|^{\ell+1}), \tag{3.10}$$

and one has a similar expansion for $\widehat{\phi}'$. The claimed estimates follow from substituting this expression into (3.8) and (3.9) and evaluating the error term using the prime number theorem $\mathcal{R}(u) \ll ue^{-c\sqrt{\log u}}$. □

We end this section by proving Theorem 1.1.

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Proof of Theorem 1.1. Assume that $\theta, \omega \geq 0$ are admissible values in (1.2) and are such that $\theta+\omega < \frac{1}{2}$. Let ϕ be any real even Schwartz function, such that $\widehat{\phi} \geq 0$ and $1 < \sup(\mathrm{supp}(\widehat{\phi})) < (\frac{5}{6}-\theta)/(\frac{1}{3}+\omega)$; this is possible thanks to the restriction $\theta+\omega<\frac{1}{2}$. Combining Lemmas 3.1 and 3.2 with Proposition 3.3, we obtain the estimate[3]

$$\begin{aligned}\frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} \mathfrak{D}_{\phi}(K) &= \widehat{\phi}(0)\Big(1+\frac{\log(4\pi^2 e)}{L} - \frac{C_2^{\pm}}{5C_1^{\pm}}\frac{X^{-\frac{1}{6}}}{L} + \frac{(C_2^{\pm})^2}{5(C_1^{\pm})^2}\frac{X^{-\frac{1}{3}}}{L}\Big)\\ &\quad + \frac{1}{\pi}\int_{-\infty}^{\infty}\phi\Big(\frac{Lr}{2\pi}\Big)\operatorname{Re}\Big(\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}(\tfrac{1}{2}+ir)\Big)dr - \frac{2}{L}\sum_{p,e}\frac{x_p \log p}{p^{\frac{e}{2}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)(\theta_e+\tfrac{1}{p})\\ - \frac{2C_2^{\pm}X^{-\frac{1}{6}}}{C_1^{\pm}L}\Big(1-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big)&\sum_{p,e}\frac{\log p}{p^{\frac{e}{2}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)\beta_e(p) + O_{\varepsilon}(X^{\theta-1+\sigma(\omega+\frac{1}{2})+\varepsilon}+X^{-\frac{1}{2}+\frac{\sigma}{6}}), \end{aligned} \tag{3.11}$$

where $\sigma = \sup(\mathrm{supp}(\widehat{\phi}))$.

To bound the integral involving the gamma function in (3.11), we note that Stirling's formula implies that for s in any fixed vertical strip minus discs centred at the poles of $\Gamma_{\pm}(s)$, we have the estimate

$$\operatorname{Re}\Big(\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}(s)\Big) = \log|s| + O(1).$$

Now, $\phi(x) \ll |x|^{-2}$, and thus,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\phi\Big(\frac{Lr}{2\pi}\Big)\operatorname{Re}\Big(\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}(\tfrac{1}{2}+ir)\Big)dr \ll \int_{-1}^{1}\Big|\phi\Big(\frac{Lr}{2\pi}\Big)\Big|dr + \int_{|r|\geq 1}\frac{\log(1+|r|)}{(Lr)^2}dr \ll \frac{1}{L}.$$

Moreover, Lemma 3.4 implies the estimates

$$-\frac{2}{L}\sum_{p,e}\frac{x_p\log p}{p^{\frac{e}{2}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)(\theta_e+\tfrac{1}{p}) \ll 1$$

and

$$\begin{aligned}&-\frac{2C_2^{\pm}X^{-\frac{1}{6}}}{C_1^{\pm}L}\Big(1-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big)\sum_{p,e}\frac{\log p}{p^{\frac{e}{2}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)\beta_e(p)\\ &\qquad = -\frac{2C_2^{\pm}X^{-\frac{1}{6}}}{C_1^{\pm}}\int_0^{\infty}\widehat{\phi}(u)e^{\frac{Lu}{6}}du + O\big(X^{-\frac{1}{6}}+X^{\frac{\sigma-2}{6}}\big),\end{aligned}$$

since the Riemann Hypothesis for $\zeta_K(s)$ implies the Riemann Hypothesis for $\zeta(s)$. Combining these estimates, we deduce that the right-hand side of (3.11) is

$$\leq -C_{\varepsilon}X^{\frac{\sigma-1}{6}-\varepsilon} + O_{\varepsilon}(1+X^{\frac{\sigma-1}{6}-\delta+\varepsilon}+X^{-\frac{1}{3}+\frac{\sigma}{6}}),$$

where $\varepsilon>0$ is arbitrary, C_{ε} is a positive constant and $\delta := \frac{\sigma-1}{6} - (\theta-1+\sigma(\omega+\frac{1}{2})) > 0$. However, for small enough ε, this contradicts the bound

$$\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\mathfrak{D}_{\phi}(K) = O(\log X),$$

[3]This is similar to the Proof of Theorem 1.2. However, since we have a different condition on θ (that is $\theta+\omega<\frac{1}{2}$), there is an additional error term in the current estimate.

which is a direct consequence of the Riemann Hypothesis for $\zeta_K(s)$ and the Riemann-von Mangoldt formula [IK, Theorem 5.31]. □

4. A refined Ratios Conjecture

The celebrated L-functions Ratios Conjecture [CFZ] predicts precise formulas for estimates of averages of ratios of (products of) L-functions evaluated at points close to the critical line. The conjecture is presented in the form of a recipe with instructions on how to produce predictions of a certain type in any family of L-functions. In order to follow the recipe, it is of fundamental importance to have control of counting functions of the type (1.1) and (2.3) related to the family. The connections between counting functions, low-lying zeros and the Ratios Conjecture are central in the present investigation.

The Ratios Conjecture has a large variety of applications. Applications to problems about low-lying zeros first appeared in the work of Conrey and Snaith [CS], where they study the one-level density of families of quadratic Dirichlet L-functions and quadratic twists of a holomorphic modular form. The investigation in [CS] has inspired a large amount of work on low-lying zeros in different families (see, e.g. [M2, M3, HKS, FM, DHP, FPS1, FPS3, MS, CP, W]).

As part of this project, we went through the steps of the Ratios Conjecture recipe with the goal of estimating the one-level density. We noticed that the resulting estimate does not predict certain terms in Theorem 1.2. To fix this, we modified [CFZ, Step 4], which is the evaluation of the average of the coefficients appearing in the approximation of the expression

$$R(\alpha,\gamma;X) := \frac{1}{N^{\pm}(X)} \sum_{K\in\mathcal{F}^{\pm}(X)} \frac{L\left(\frac{1}{2}+\alpha, f_K\right)}{L\left(\frac{1}{2}+\gamma, f_K\right)}. \tag{4.1}$$

More precisely, instead of only considering the main term, we kept track of the secondary term in Lemma 4.1.

We now describe more precisely the steps in the Ratios Conjecture recipe. The first step involves the approximate functional equation for $L(s, f_K)$, which reads

$$L(s,f_K) = \sum_{n<x} \frac{\lambda_K(n)}{n^s} + |D_K|^{\frac{1}{2}-s} \frac{\Gamma_{\pm}(1-s)}{\Gamma_{\pm}(s)} \sum_{n<y} \frac{\lambda_K(n)}{n^{1-s}} + \text{Error}, \tag{4.2}$$

where x, y are such that $xy \asymp |D_K|(1+|t|)^2$ (this is in analogy with [CS]; see [IK, Theorem 5.3] for a description of the approximate functional equation of a general L-function). The analysis will be carried out assuming that the error term can be neglected and that the sums can be completed.

Following [CFZ], we replace the numerator of (4.1) with the approximate functional equation (4.2) and the denominator of (4.1) with (2.2). We will need to estimate the first sum in (4.2) evaluated at $s = \frac{1}{2}+\alpha$, where $|\mathrm{Re}(\alpha)|$ is sufficiently small. This gives the contribution

$$R_1(\alpha,\gamma;X) := \frac{1}{N^{\pm}(X)} \sum_{K\in\mathcal{F}^{\pm}(X)} \sum_{h,m} \frac{\lambda_K(m)\mu_K(h)}{m^{\frac{1}{2}+\alpha} h^{\frac{1}{2}+\gamma}} \tag{4.3}$$

to (4.1). This infinite sum converges absolutely in the region $\mathrm{Re}(\alpha) > \frac{1}{2}$ and $\mathrm{Re}(\gamma) > \frac{1}{2}$, however, later in this section, we will provide an analytic continuation to a wider domain. We will also need to evaluate the contribution of the second sum in (4.2), which is given by

$$R_2(\alpha,\gamma;X) := \frac{1}{N^{\pm}(X)} \sum_{K\in\mathcal{F}^{\pm}(X)} |D_K|^{-\alpha} \frac{\Gamma_{\pm}\left(\frac{1}{2}-\alpha\right)}{\Gamma_{\pm}\left(\frac{1}{2}+\alpha\right)} \sum_{h,m} \frac{\lambda_K(m)\mu_K(h)}{m^{\frac{1}{2}-\alpha} h^{\frac{1}{2}+\gamma}} \tag{4.4}$$

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(Once more, the series converges absolutely for $\mathrm{Re}(\alpha) < -\frac{1}{2}$ and $\mathrm{Re}(\gamma) > \frac{1}{2}$, but we will later provide an analytic continuation to a wider domain).

A first step in the understanding of the $R_j(\alpha,\gamma;X)$ will be achieved using the following precise evaluation of the expected value of $\lambda_K(m)\mu_K(h)$.

Lemma 4.1. *Let $m,h \in \mathbb{N}$, and let $\frac{1}{2} \le \theta < \frac{5}{6}$ and $\omega \ge 0$ be, such that (2.3) holds. Assume that h is cubefree. We have the estimate*

$$\frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} \lambda_K(m)\mu_K(h)$$
$$= \prod_{p^e \| m, p^s \| h} f(e,s,p)x_p + \Bigg(\prod_{p^e \| m, p^s \| h} g(e,s,p)y_p - \prod_{p^e \| m, p^s \| h} f(e,s,p)x_p \Bigg) \frac{C_2^{\pm}}{C_1^{\pm}} X^{-\frac{1}{6}} \Big(1 - \frac{C_2^{\pm}}{C_1^{\pm}} X^{-\frac{1}{6}}\Big)$$
$$+ O_{\varepsilon}\Bigg(\prod_{p | hm, p^e \| m} ((2e+5)p^{\omega}) X^{\theta-1+\varepsilon} \Bigg),$$

where

$$f(e,0,p) := \frac{e+1}{6} + \frac{1+(-1)^e}{4} + \frac{\tau_e}{3} + \frac{1}{p};$$
$$f(e,1,p) := -\frac{e+1}{3} + \frac{\tau_e}{3} - \frac{1}{p};$$
$$f(e,2,p) := \frac{e+1}{6} - \frac{1+(-1)^e}{4} + \frac{\tau_e}{3};$$
$$g(e,0,p) := \frac{(e+1)(1+p^{-\frac{1}{3}})^3}{6} + \frac{(1+(-1)^e)(1+p^{-\frac{1}{3}})(1+p^{-\frac{2}{3}})}{4}$$
$$+ \frac{\tau_e(1+p^{-1})}{3} + \frac{(1+p^{-\frac{1}{3}})^2}{p};$$
$$g(e,1,p) := -\frac{(e+1)(1+p^{-\frac{1}{3}})^3}{3} + \frac{\tau_e(1+p^{-1})}{3} - \frac{(1+p^{-\frac{1}{3}})^2}{p};$$
$$g(e,2,p) := \frac{(e+1)(1+p^{-\frac{1}{3}})^3}{6} - \frac{(1+(-1)^e)(1+p^{-\frac{1}{3}})(1+p^{-\frac{2}{3}})}{4} + \frac{\tau_e(1+p^{-1})}{3}.$$

Proof. We may write $m = \prod_{j=1}^J p_j^{e_j}$ and $h = \prod_{j=1}^J p_j^{s_j}$, where $p_1,\dots,p_J$ are distinct primes and for each j, e_j and s_j are nonnegative integers but not both zero. Then we see that

$$\sum_{K \in \mathcal{F}^{\pm}(X)} \lambda_K(m)\mu_K(h) = \sum_{K \in \mathcal{F}^{\pm}(X)} \prod_{j=1}^J \Big(\lambda_K(p_j^{e_j})\mu_K(p_j^{s_j})\Big) = \sum_{\mathbf{k}} \sum_{\substack{K \in \mathcal{F}^{\pm}(X) \\ \mathbf{p}:\ type\ T_{\mathbf{k}}}} \prod_{j=1}^J \Big(\lambda_K(p_j^{e_j})\mu_K(p_j^{s_j})\Big),$$

where $\mathbf{k} = (k_1,\dots,k_J)$ runs over $\{1,2,3,4,5\}^J$ and $\mathbf{p} = (p_1,\dots,p_J)$. When each p_j has splitting type T_{k_j} in K, the values $\lambda_K(p_j^{e_j})$ and $\mu_K(p_j^{s_j})$ depend on p_j, k_j, e_j and s_j. Define

$$\eta_{1,p_j}(k_j,e_j) := \lambda_K(p_j^{e_j}), \qquad \eta_{2,p_j}(k_j,s_j) := \mu_K(p_j^{s_j})$$

for each $j \le J$ with p_j of splitting type T_{k_j} in K, as well as

$$\eta_{1,\mathbf{p}}(\mathbf{k},\mathbf{e}) := \prod_{j=1}^J \eta_{1,p_j}(k_j,e_j), \qquad \eta_{2,\mathbf{p}}(\mathbf{k},\mathbf{s}) := \prod_{j=1}^J \eta_{2,p_j}(k_j,s_j). \tag{4.5}$$

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We see that

$$\begin{aligned}\sum_{K\in\mathcal{F}^\pm(X)}\lambda_K(m)\mu_K(h) &= \sum_{\mathbf{k}}\eta_{1,\mathbf{p}}(\mathbf{k},\mathbf{e})\eta_{2,\mathbf{p}}(\mathbf{k},\mathbf{s})\sum_{\substack{K\in\mathcal{F}^\pm(X)\\ \mathbf{p}:\, type\ T_{\mathbf{k}}}}1\\ &= \sum_{\mathbf{k}}\eta_{1,\mathbf{p}}(\mathbf{k},\mathbf{e})\eta_{2,\mathbf{p}}(\mathbf{k},\mathbf{s})N^\pm_{\mathbf{p}}(X,T_{\mathbf{k}}),\end{aligned}$$

which by (2.3) is equal to

$$\begin{aligned}&\sum_{\mathbf{k}}\eta_{1,\mathbf{p}}(\mathbf{k},\mathbf{e})\eta_{2,\mathbf{p}}(\mathbf{k},\mathbf{s})\left(C_1^\pm\prod_{j=1}^J(x_{p_j}c_{k_j}(p_j))X + C_2^\pm\prod_{j=1}^J(y_{p_j}d_{k_j}(p_j))X^{\frac56} + O_\varepsilon\left(\prod_{j=1}^J p_j^\omega X^{\theta+\varepsilon}\right)\right)\\ &= C_1^\pm X\left(\sum_{\mathbf{k}}\eta_{1,\mathbf{p}}(\mathbf{k},\mathbf{e})\eta_{2,\mathbf{p}}(\mathbf{k},\mathbf{s})\prod_{j=1}^J(x_{p_j}c_{k_j}(p_j))\right) + C_2^\pm X^{\frac56}\left(\sum_{\mathbf{k}}\eta_{1,\mathbf{p}}(\mathbf{k},\mathbf{e})\eta_{2,\mathbf{p}}(\mathbf{k},\mathbf{s})\prod_{j=1}^J(y_{p_j}d_{k_j}(p_j))\right)\\ &\quad + O_\varepsilon\left(\sum_{\mathbf{k}}|\eta_{1,\mathbf{p}}(\mathbf{k},\mathbf{e})\eta_{2,\mathbf{p}}(\mathbf{k},\mathbf{s})|\prod_{j=1}^J p_j^\omega X^{\theta+\varepsilon}\right).\end{aligned}$$

We can change the last three **k**-sums into products by (4.5). Doing so, we obtain that the above is equal to

$$\begin{aligned}&C_1^\pm X\prod_{j=1}^J\left(x_{p_j}\widetilde{f}(e_j,s_j,p_j)\right) + C_2^\pm X^{\frac56}\prod_{j=1}^J\left(y_{p_j}\widetilde{g}(e_j,s_j,p_j)\right) + O_\varepsilon\left(\prod_{j=1}^J\left(p_j^\omega(2e_j+5)\right)X^{\theta+\varepsilon}\right)\\ &\quad = C_1^\pm X\prod_{p^e\|m,\,p^s\|h}\widetilde{f}(e,s,p)x_p + C_2^\pm X^{\frac56}\prod_{p^e\|m,\,p^s\|h}\widetilde{g}(e,s,p)y_p + O_\varepsilon\left(\prod_{j=1}^J\left(p_j^\omega(2e_j+5)\right)X^{\theta+\varepsilon}\right),\end{aligned}$$

where

$$\widetilde{f}(e,s,p) := \sum_{k=1}^5\eta_{1,p}(k,e)\eta_{2,p}(k,s)c_k(p), \qquad \widetilde{g}(e,s,p) := \sum_{k=1}^5\eta_{1,p}(k,e)\eta_{2,p}(k,s)d_k(p).$$

A straightforward calculation shows that $\widetilde{f}(e,s,p) = f(e,s,p)$ and $\widetilde{g}(e,s,p) = g(e,s,p)$ (see the explicit description of the coefficients in Section 2; note that $\eta_{2,p}(k,0) = 1$) and the lemma follows. □

We now proceed with the estimation of $R_1(\alpha,\gamma;X)$. Taking into account the two main terms in Lemma 4.1, we expect that

$$R_1(\alpha,\gamma;X) = R_1^M(\alpha,\gamma) + \frac{C_2^\pm}{C_1^\pm}X^{-\frac16}\left(1 - \frac{C_2^\pm}{C_1^\pm}X^{-\frac16}\right)(R_1^S(\alpha,\gamma) - R_1^M(\alpha,\gamma)) + \text{Error}, \tag{4.6}$$

where

$$\begin{aligned}R_1^M(\alpha,\gamma) &:= \prod_p\left(1 + \sum_{e\ge1}\frac{x_pf(e,0,p)}{p^{e(\frac12+\alpha)}} + \sum_{e\ge0}\frac{x_pf(e,1,p)}{p^{e(\frac12+\alpha)+(\frac12+\gamma)}} + \sum_{e\ge0}\frac{x_pf(e,2,p)}{p^{e(\frac12+\alpha)+2(\frac12+\gamma)}}\right),\\ R_1^S(\alpha,\gamma) &:= \prod_p\left(1 + \sum_{e\ge1}\frac{y_pg(e,0,p)}{p^{e(\frac12+\alpha)}} + \sum_{e\ge0}\frac{y_pg(e,1,p)}{p^{e(\frac12+\alpha)+(\frac12+\gamma)}} + \sum_{e\ge0}\frac{y_pg(e,2,p)}{p^{e(\frac12+\alpha)+2(\frac12+\gamma)}}\right)\end{aligned}\tag{4.7}$$

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for $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > \frac{1}{2}$. Since

$$R_1^M(\alpha,\gamma) = \prod_p \Bigg(1 + \frac{1}{p^{1+2\alpha}} - \frac{1}{p^{1+\alpha+\gamma}} + O\Big(\frac{1}{p^{\frac{3}{2}+\mathrm{Re}(\alpha)}} + \frac{1}{p^{\frac{3}{2}+3\,\mathrm{Re}(\alpha)}} + \frac{1}{p^{\frac{3}{2}+\mathrm{Re}(\gamma)}} + \frac{1}{p^{\frac{3}{2}+\mathrm{Re}(2\alpha+\gamma)}} + \frac{1}{p^{\frac{5}{2}+\mathrm{Re}(3\alpha+2\gamma)}}\Big)\Bigg),$$

we see that

$$A_3(\alpha,\gamma) := \frac{\zeta(1+\alpha+\gamma)}{\zeta(1+2\alpha)} R_1^M(\alpha,\gamma) \tag{4.8}$$

is analytically continued to the region[4] $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > -\frac{1}{6}$. Similarly, from the estimates

$$\sum_{e\geq 1} \frac{y_p g(e,0,p)}{p^{e(\frac{1}{2}+\alpha)}} = \frac{1}{p^{\frac{5}{6}+\alpha}} + \frac{1}{p^{1+2\alpha}} + O\Big(\frac{1}{p^{\mathrm{Re}(\alpha)+\frac{3}{2}}} + \frac{1}{p^{2\,\mathrm{Re}(\alpha)+\frac{4}{3}}} + \frac{1}{p^{3\,\mathrm{Re}(\alpha)+\frac{3}{2}}}\Big),$$
$$\sum_{e\geq 0} \frac{y_p g(e,1,p)}{p^{e(\frac{1}{2}+\alpha)+(\frac{1}{2}+\gamma)}} = -\frac{1}{p^{\frac{5}{6}+\gamma}} - \frac{1}{p^{1+\alpha+\gamma}} + O\Big(\frac{1}{p^{\frac{4}{3}+\mathrm{Re}(\alpha+\gamma)}}\Big),$$
$$\sum_{e\geq 0} \frac{y_p g(e,2,p)}{p^{e(\frac{1}{2}+\alpha)+2(\frac{1}{2}+\gamma)}} = O\Big(\frac{1}{p^{\frac{3}{2}+\mathrm{Re}(\alpha+2\gamma)}}\Big),$$

we deduce that

$$A_4(\alpha,\gamma) := \frac{\zeta(\frac{5}{6}+\gamma)\zeta(1+\alpha+\gamma)}{\zeta(\frac{5}{6}+\alpha)\zeta(1+2\alpha)} R_1^S(\alpha,\gamma) \tag{4.9}$$

is analytic in the region $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > -\frac{1}{6}$. Note that by their defining product formulas, we have the bounds

$$A_3(\alpha,\gamma) = O_\varepsilon(1), \quad A_4(\alpha,\gamma) = O_\varepsilon(1) \tag{4.10}$$

for $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) \geq -\frac{1}{6} + \varepsilon > -\frac{1}{6}$. Using this notation, (4.6) takes the form

$$R_1(\alpha,\gamma;X) = \frac{\zeta(1+2\alpha)}{\zeta(1+\alpha+\gamma)}\Big(A_3(\alpha,\gamma) + \frac{C_2^{\pm}}{C_1^{\pm}} X^{-\frac{1}{6}}\Big(1 - \frac{C_2^{\pm}}{C_1^{\pm}} X^{-\frac{1}{6}}\Big)\Big(\frac{\zeta(\frac{5}{6}+\alpha)}{\zeta(\frac{5}{6}+\gamma)} A_4(\alpha,\gamma) - A_3(\alpha,\gamma)\Big)\Big) + \mathrm{Error}.$$

The above computation is sufficient in order to obtain a conjectural evaluation of the average (4.3). However, our goal is to evaluate the one-level density through the average of $\frac{L'}{L}(\frac{1}{2}+r, f_K)$; therefore, it is necessary to also compute the partial derivative $\frac{\partial}{\partial\alpha} R_1(\alpha,\gamma;X)|_{\alpha=\gamma=r}$. To do so, we need to make sure that the error term stays small after a differentiation. This is achieved by applying Cauchy's integral formula for the derivative

$$f'(a) = \frac{1}{2\pi i}\int_{|z-a|=\kappa} \frac{f(z)}{(z-a)^2} dz$$

[4]To see this, write $\frac{\zeta(1+\alpha+\gamma)}{\zeta(1+2\alpha)}$ as an Euler product and expand out the triple product in (4.8). The resulting expression will converge in the stated region.

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(valid for all small enough $\kappa > 0$) and bounding the integrand using the approximation for $R_1(\alpha, \gamma; X)$ above. As for the main terms, one can differentiate them term by term and obtain the expected approximation

$$\begin{aligned}\frac{\partial}{\partial\alpha} R_1(\alpha, \gamma; X)\Big|_{\alpha=\gamma=r} &= A_{3,\alpha}(r, r) + \frac{\zeta'}{\zeta}(1+2r)A_3(r, r) + \frac{C_2^{\pm}}{C_1^{\pm}} X^{-\frac{1}{6}}\Big(1 - \frac{C_2^{\pm}}{C_1^{\pm}} X^{-\frac{1}{6}}\Big) \\ &\times \Big(A_{4,\alpha}(r, r) + \frac{\zeta'}{\zeta}(\tfrac{5}{6}+r)A_4(r, r) - A_{3,\alpha}(r, r) + \frac{\zeta'}{\zeta}(1+2r)(A_4(r, r) - A_3(r, r))\Big) \\ &+ \text{Error}, \end{aligned} \tag{4.11}$$

where $A_{3,\alpha}(r, r) = \frac{\partial}{\partial\alpha} A_3(\alpha, \gamma)\Big|_{\alpha=\gamma=r}$ and $A_{4,\alpha}(r, r) = \frac{\partial}{\partial\alpha} A_4(\alpha, \gamma)\Big|_{\alpha=\gamma=r}$.

Now, from the definition of $f(e, j, p)$ and $g(e, j, p)$ (see Lemma 4.1) as well as (3.4) and (3.5), we have

$$\begin{aligned} f(1, 0, p) + f(0, 1, p) &= g(1, 0, p) + g(0, 1, p) = 0, \\ f(e, 0, p) + f(e-1, 1, p) + f(e-2, 2, p) &= g(e, 0, p) + g(e-1, 1, p) + g(e-2, 2, p) = 0, \\ f(e, 0, p) - f(e-2, 2, p) &= \theta_e + p^{-1}, \\ g(e, 0, p) - g(e-2, 2, p) &= (1 + p^{-\frac{1}{3}})(\kappa_e(p) + p^{-1} + p^{-\frac{4}{3}}). \end{aligned}$$

By the above identities and the definition (4.7), we deduce that

$$R_1^M(r, r) = A_3(r, r) = R_1^S(r, r) = A_4(r, r) = 1.$$

It follows that for $\mathrm{Re}(r) > \frac{1}{2}$,

$$\begin{aligned} R_{1,\alpha}^M(r, r) &= \frac{R_{1,\alpha}^M(r, r)}{R_1^M(r, r)} = \frac{\partial}{\partial\alpha} \log R_1^M(\alpha, \gamma)\Big|_{\alpha=\gamma=r} \\ &= \sum_p \Big(-\frac{x_p \log p}{p^{\frac{1}{2}+r}} f(1, 0, p) - \sum_{e \geq 2} \frac{x_p \log p}{p^{e(\frac{1}{2}+r)}} (f(e, 0, p) - f(e-2, 2, p)) \Big) \\ &+ \sum_p \Big(-\sum_{e \geq 2} \frac{x_p \log p}{p^{e(\frac{1}{2}+r)}} (e-1)\big(f(e, 0, p) + f(e-1, 1, p) + f(e-2, 2, p)\big) \Big) \\ &= -\sum_p \sum_{e \geq 1} \frac{x_p \log p}{p^{e(\frac{1}{2}+r)}} \Big(\theta_e + \frac{1}{p}\Big) \end{aligned}$$

and

$$\begin{aligned} R_{1,\alpha}^S(r, r) &= \sum_p \Big(-\frac{y_p \log p}{p^{\frac{1}{2}+r}} g(1, 0, p) - \sum_{e \geq 2} \frac{y_p \log p}{p^{e(\frac{1}{2}+r)}} (g(e, 0, p) - g(e-2, 2, p)) \Big) \\ &+ \sum_p \Big(-\sum_{e \geq 2} \frac{y_p \log p}{p^{e(\frac{1}{2}+r)}} (e-1)\big(g(e, 0, p) + g(e-1, 1, p) + g(e-2, 2, p)\big) \Big) \\ &= -\sum_p \sum_{e \geq 1} \frac{y_p \log p}{p^{e(\frac{1}{2}+r)}} (1 + p^{-\frac{1}{3}})\big(\kappa_e(p) + p^{-1} + p^{-\frac{4}{3}}\big) \\ &= -\sum_p \sum_{e \geq 1} \frac{\log p}{p^{e(\frac{1}{2}+r)}} \Big(\beta_e(p) + x_p\Big(\theta_e + \frac{1}{p}\Big)\Big), \end{aligned}$$

by (3.6). Thus, we have

$$A_{3,\alpha}(r,r) = R^M_{1,\alpha}(r,r) - \frac{\zeta'}{\zeta}(1+2r) = -\sum_{p,e\geq 1}\Big(\theta_e + \frac{1}{p}\Big)\frac{x_p \log p}{p^{e(\frac{1}{2}+r)}} - \frac{\zeta'}{\zeta}(1+2r)$$

and

$$A_{4,\alpha}(r,r) - A_{3,\alpha}(r,r) = -\sum_{p,e\geq 1}\frac{(\beta_e(p) - p^{-\frac{e}{3}})\log p}{p^{e(\frac{1}{2}+r)}}, \tag{4.12}$$

which are now valid in the extended region $\mathrm{Re}(r) > 0$. Coming back to (4.11), we deduce that

$$\begin{aligned}\frac{\partial}{\partial\alpha}R_1(\alpha,\gamma;X)\Big|_{\alpha=\gamma=r} &= A_{3,\alpha}(r,r) + \frac{\zeta'}{\zeta}(1+2r)\\ &\quad + \frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big(1 - \frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big)\Big(A_{4,\alpha}(r,r) - A_{3,\alpha}(r,r) + \frac{\zeta'}{\zeta}(\tfrac{5}{6}+r)\Big) + \text{Error}\\ &= -\sum_{p,e\geq 1}\Big(\theta_e + \frac{1}{p}\Big)\frac{x_p\log p}{p^{e(\frac{1}{2}+r)}} - \frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big(1 - \frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big)\sum_{p,e\geq 1}\frac{(\beta_e(p) - p^{-\frac{e}{3}})\log p}{p^{e(\frac{1}{2}+r)}}\\ &\quad + \frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big(1 - \frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big)\frac{\zeta'}{\zeta}(\tfrac{5}{6}+r) + \text{Error},\end{aligned}$$

where the second equality is valid in the region $\mathrm{Re}(r) > 0$.

We now move to $R_2(\alpha,\gamma;X)$. We recall that

$$R_2(\alpha,\gamma;X) = \frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}|D_K|^{-\alpha}\frac{\Gamma_{\pm}(\frac{1}{2}-\alpha)}{\Gamma_{\pm}(\frac{1}{2}+\alpha)}\sum_{h,m}\frac{\lambda_K(m)\mu_K(h)}{m^{\frac{1}{2}-\alpha}h^{\frac{1}{2}+\gamma}}, \tag{4.13}$$

and the Ratios Conjecture recipe tells us that we should replace $\lambda_K(m)\mu_K(h)$ with its average. However, a calculation involving Lemma 4.1 suggests that the terms $|D_K|^{-\alpha}$ and $\lambda_K(m)\mu_K(h)$ have nonnegligible covariance. To take this into account, we substitute this step with the use of the following corollary of Lemma 4.1.

Corollary 4.2. *Let $m,h\in\mathbb{N}$, and let $\frac{1}{2}\leq\theta<\frac{5}{6}$ and $\omega\geq 0$ be, such that (2.3) holds. For $\alpha\in\mathbb{C}$ with $0<\mathrm{Re}(\alpha)<\frac{1}{2}$, we have the estimate*

$$\begin{aligned}\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}|D_K|^{-\alpha}\lambda_K(m)\mu_K(h) &= \frac{X^{-\alpha}}{1-\alpha}\prod_{p^e\|m,\,p^s\|h} f(e,s,p)x_p\\ &+ X^{-\frac{1}{6}-\alpha}\Bigg(\frac{1}{1-\frac{6\alpha}{5}}\prod_{p^e\|m,\,p^s\|h} g(e,s,p)y_p - \frac{1}{1-\alpha}\prod_{p^e\|m,\,p^s\|h} f(e,s,p)x_p\Bigg)\frac{C_2^{\pm}}{C_1^{\pm}}\Big(1-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big)\\ &+ O_{\varepsilon}\Bigg((1+|\alpha|)\prod_{p|hm,\,p^e\|m}\big((2e+5)p^{\omega}\big)X^{\theta-1-\mathrm{Re}(\alpha)+\varepsilon}\Bigg).\end{aligned}$$

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Proof. This follows from applying Lemma 4.1 and (1.1) to the identity

$$\sum_{K\in\mathcal{F}^\pm(X)} |D_K|^{-\alpha}\lambda_K(m)\mu_K(h) = \int_1^X u^{-\alpha} d\Big(\sum_{K\in\mathcal{F}^\pm(u)} \lambda_K(m)\mu_K(h)\Big)$$
$$= X^{-\alpha}\sum_{K\in\mathcal{F}^\pm(X)} \lambda_K(m)\mu_K(h) + \alpha\int_1^X u^{-\alpha-1}\Big(\sum_{K\in\mathcal{F}^\pm(u)} \lambda_K(m)\mu_K(h)\Big)du.$$

□

Applying this lemma, we deduce the following heuristic approximation of $R_2(\alpha,\gamma;X)$:

$$\frac{\Gamma_\pm(\frac12-\alpha)}{\Gamma_\pm(\frac12+\alpha)}\sum_{h,m}\frac{1}{m^{\frac12-\alpha}h^{\frac12+\gamma}}\Bigg\{\frac{X^{-\alpha}}{1-\alpha}\prod_{p^e\|m,p^s\|h} f(e,s,p)x_p$$
$$+X^{-\frac16-\alpha}\frac{C_2^\pm}{C_1^\pm}\Big(1-\frac{C_2^\pm}{C_1^\pm}X^{-\frac16}\Big)\Big(\frac{1}{1-\frac{6\alpha}{5}}\prod_{p^e\|m,p^s\|h} g(e,s,p)y_p-\frac{1}{1-\alpha}\prod_{p^e\|m,p^s\|h} f(e,s,p)x_p\Big)\Bigg\}$$
$$=\frac{\Gamma_\pm(\frac12-\alpha)}{\Gamma_\pm(\frac12+\alpha)}\Bigg\{X^{-\alpha}\frac{R_1^M(-\alpha,\gamma)}{1-\alpha}+X^{-\frac16-\alpha}\frac{C_2^\pm}{C_1^\pm}\Big(1-\frac{C_2^\pm}{C_1^\pm}X^{-\frac16}\Big)\Big(\frac{R_1^S(-\alpha,\gamma)}{1-\frac{6\alpha}{5}}-\frac{R_1^M(-\alpha,\gamma)}{1-\alpha}\Big)\Bigg\}$$
$$=\frac{\Gamma_\pm(\frac12-\alpha)}{\Gamma_\pm(\frac12+\alpha)}\frac{\zeta(1-2\alpha)}{\zeta(1-\alpha+\gamma)}\Bigg\{X^{-\alpha}\frac{A_3(-\alpha,\gamma)}{1-\alpha}$$
$$+X^{-\frac16-\alpha}\frac{C_2^\pm}{C_1^\pm}\Big(1-\frac{C_2^\pm}{C_1^\pm}X^{-\frac16}\Big)\Big(\frac{A_4(-\alpha,\gamma)}{1-\frac{6\alpha}{5}}\frac{\zeta(\frac56-\alpha)}{\zeta(\frac56+\gamma)}-\frac{A_3(-\alpha,\gamma)}{1-\alpha}\Big)\Bigg\}.$$

If $\mathrm{Re}(r)$ is positive and small enough, then we expect that

$$\frac{\partial}{\partial\alpha}R_2(\alpha,\gamma;X)\Big|_{\alpha=\gamma=r}=-\frac{\Gamma_\pm(\frac12-r)}{\Gamma_\pm(\frac12+r)}\zeta(1-2r)\Bigg\{X^{-r}\frac{A_3(-r,r)}{1-r}$$
$$+X^{-\frac16-r}\frac{C_2^\pm}{C_1^\pm}\Big(1-\frac{C_2^\pm}{C_1^\pm}X^{-\frac16}\Big)\Big(\frac{\zeta(\frac56-r)}{\zeta(\frac56+r)}\frac{A_4(-r,r)}{1-\frac{6r}{5}}-\frac{A_3(-r,r)}{1-r}\Big)\Bigg\}+\text{Error}.$$

We arrive at the following conjecture.

Conjecture 4.3. *Let $\frac12\le\theta<\frac56$ and $\omega\ge0$ be, such that (2.3) holds. There exists $0<\delta<\frac16$, such that for any fixed $\varepsilon>0$ and for $r\in\mathbb{C}$ with $\frac1L\ll\mathrm{Re}(r)<\delta$ and $|r|\le X^{\frac{\varepsilon}{2}}$,*

$$\frac{1}{N^\pm(X)}\sum_{K\in\mathcal{F}^\pm(X)}\frac{L'(\frac12+r,f_K)}{L(\frac12+r,f_K)}$$
$$=-\sum_{p,e\ge1}\Big(\theta_e+\frac1p\Big)\frac{x_p\log p}{p^{e(\frac12+r)}}-\frac{C_2^\pm}{C_1^\pm}X^{-\frac16}\Big(1-\frac{C_2^\pm}{C_1^\pm}X^{-\frac16}\Big)\sum_{p,e\ge1}\frac{(\beta_e(p)-p^{-\frac{e}{3}})\log p}{p^{e(\frac12+r)}}$$
$$+\frac{C_2^\pm}{C_1^\pm}X^{-\frac16}\Big(1-\frac{C_2^\pm}{C_1^\pm}X^{-\frac16}\Big)\frac{\zeta'}{\zeta}(\tfrac56+r)-X^{-r}\frac{\Gamma_\pm(\frac12-r)}{\Gamma_\pm(\frac12+r)}\zeta(1-2r)\frac{A_3(-r,r)}{1-r}$$
$$-\frac{C_2^\pm}{C_1^\pm}X^{-r-\frac16}\Big(1-\frac{C_2^\pm}{C_1^\pm}X^{-\frac16}\Big)\frac{\Gamma_\pm(\frac12-r)}{\Gamma_\pm(\frac12+r)}\zeta(1-2r)\Big(\frac{\zeta(\frac56-r)}{\zeta(\frac56+r)}\frac{A_4(-r,r)}{1-\frac{6r}{5}}-\frac{A_3(-r,r)}{1-r}\Big)+O_\varepsilon(X^{\theta-1+\varepsilon}).\tag{4.14}$$

Note that the two sums on the right-hand side are absolutely convergent.

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Traditionally, when applying the Ratios Conjecture recipe, one has to restrict the real part of the variable r to small enough positive values. For example, in the family of quadratic Dirichlet L-functions [CS, FPS3], one requires that $\frac{1}{\log X} \ll \mathrm{Re}(r) < \frac{1}{4}$. This ensures that one is far enough from a pole for the expression in the right-hand side. In the current situation, we will see that the term involving $X^{-r-\frac{1}{6}}$ has a pole at $s = \frac{1}{6}$.

Proposition 4.4. *Assume Conjecture 4.3 and the Riemann Hypothesis for $\zeta_K(s)$ for all $K \in \mathcal{F}^{\pm}(X)$, and let ϕ be a real even Schwartz function, such that $\widehat{\phi}$ is compactly supported. For any constant $0 < c < \frac{1}{6}$, we have that*

$$\begin{aligned}\frac{1}{N^{\pm}(X)} \sum_{K\in\mathcal{F}^{\pm}(X)} \sum_{\gamma_K} \phi\Big(\frac{L\gamma_K}{2\pi}\Big) &= \widehat{\phi}(0)\Big(1 + \frac{\log(4\pi^2 e)}{L} - \frac{C_2^{\pm}}{5C_1^{\pm}}\frac{X^{-\frac{1}{6}}}{L} + \frac{(C_2^{\pm})^2}{5(C_1^{\pm})^2}\frac{X^{-\frac{1}{3}}}{L}\Big)\\
&+ \frac{1}{\pi}\int_{-\infty}^{\infty} \phi\Big(\frac{Lr}{2\pi}\Big)\mathrm{Re}\Big(\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}(\tfrac{1}{2}+ir)\Big)dr - \frac{2}{L}\sum_{p,e}\frac{x_p\log p}{p^{\frac{e}{2}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)(\theta_e + \tfrac{1}{p})\\
&- \frac{2C_2^{\pm}X^{-\frac{1}{6}}}{C_1^{\pm}L}\Big(1 - \frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big)\sum_{p,e}\frac{\log p}{p^{\frac{e}{2}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)(\beta_e(p) - p^{-\frac{e}{3}})\\
&- \frac{1}{\pi i}\int_{(c)}\phi\Big(\frac{Ls}{2\pi i}\Big)\Big\{-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big(1 - \frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big)\frac{\zeta'}{\zeta}(\tfrac{5}{6}+s) + X^{-s}\frac{\Gamma_{\pm}(\frac{1}{2}-s)}{\Gamma_{\pm}(\frac{1}{2}+s)}\zeta(1-2s)\frac{A_3(-s,s)}{1-s}\\
&+ \frac{C_2^{\pm}}{C_1^{\pm}}X^{-s-\frac{1}{6}}\Big(1 - \frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big)\frac{\Gamma_{\pm}(\frac{1}{2}-s)}{\Gamma_{\pm}(\frac{1}{2}+s)}\zeta(1-2s)\Big(\frac{\zeta(\frac{5}{6}-s)}{\zeta(\frac{5}{6}+s)}\frac{A_4(-s,s)}{1-\frac{6s}{5}} - \frac{A_3(-s,s)}{1-s}\Big)\Big\}ds\\
&+ O_{\varepsilon}(X^{\theta-1+\varepsilon}).\end{aligned} \tag{4.15}$$

Proof. By the residue theorem, we have the identity

$$\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\mathfrak{D}_{\phi}(K) = \frac{1}{2\pi i}\Big(\int_{(\frac{1}{L})} - \int_{(-\frac{1}{L})}\Big)\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\frac{L'(s+\frac{1}{2},f_K)}{L(s+\frac{1}{2},f_K)}\phi\Big(\frac{Ls}{2\pi i}\Big)ds. \tag{4.16}$$

Under Conjecture 4.3 and well-known arguments (see, e.g. [FPS3, Section 3.2]), the part of this sum involving the first integral is equal to

$$\begin{aligned}-\frac{1}{2\pi i}\int_{(\frac{1}{L})}\phi\Big(\frac{Ls}{2\pi i}\Big)\Big\{&\sum_{p,e\geq 1}\Big(\theta_e + \frac{1}{p}\Big)\frac{x_p\log p}{p^{e(\frac{1}{2}+s)}} + \frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big(1 - \frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big)\sum_{p,e\geq 1}\frac{(\beta_e(p) - p^{-\frac{e}{3}})\log p}{p^{e(\frac{1}{2}+s)}}\\
&- \frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big(1 - \frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big)\frac{\zeta'}{\zeta}(\tfrac{5}{6}+s) + X^{-s}\frac{\Gamma_{\pm}(\frac{1}{2}-s)}{\Gamma_{\pm}(\frac{1}{2}+s)}\zeta(1-2s)\frac{A_3(-s,s)}{1-s}\\
&+ \frac{C_2^{\pm}}{C_1^{\pm}}X^{-s-\frac{1}{6}}\Big(1 - \frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big)\frac{\Gamma_{\pm}(\frac{1}{2}-s)}{\Gamma_{\pm}(\frac{1}{2}+s)}\zeta(1-2s)\Big(\frac{\zeta(\frac{5}{6}-s)}{\zeta(\frac{5}{6}+s)}\frac{A_4(-s,s)}{1-\frac{6s}{5}} - \frac{A_3(-s,s)}{1-s}\Big)\Big\}ds\\
&+ O_{\varepsilon}(X^{\theta-1+\varepsilon}),\end{aligned}$$

where we used the bounds (4.10) and

$$\phi\Big(\frac{Ls}{2\pi i}\Big) = \frac{(-1)^{\ell}}{L^{\ell}s^{\ell}}\int_{\mathbb{R}} e^{L\,\mathrm{Re}(s)x}e^{iL\,\mathrm{Im}(s)x}\widehat{\phi}^{(\ell)}(x)dx \ll_{\ell} \frac{e^{L|\mathrm{Re}(s)|\sup(\mathrm{supp}(\widehat{\phi}))}}{L^{\ell}|s|^{\ell}} \tag{4.17}$$

for every integer $\ell > 0$, which is decaying on the line $\mathrm{Re}(s) = \frac{1}{L}$. We may also shift the contour of integration to the line $\mathrm{Re}(s) = c$ with $0 < c < \frac{1}{6}$.

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For the second integral in (4.16) (over the line $\mathrm{Re}(s) = -\frac{1}{L}$), we treat it as follows. By the functional equation (2.1), we have

$$
\begin{aligned}
&-\frac{1}{2\pi i}\int_{(-\frac{1}{L})}\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\frac{L'(s+\frac{1}{2},f_K)}{L(s+\frac{1}{2},f_K)}\phi\Big(\frac{Ls}{2\pi i}\Big)ds\\
&=\frac{1}{2\pi i}\int_{(\frac{1}{L})}\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\frac{L'(s+\frac{1}{2},f_K)}{L(s+\frac{1}{2},f_K)}\phi\Big(\frac{Ls}{2\pi i}\Big)ds\\
&\quad+\frac{1}{2\pi i}\int_{(-\frac{1}{L})}\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\Big(\log|D_K|+\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}(\tfrac{1}{2}+s)+\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}(\tfrac{1}{2}-s)\Big)\phi\Big(\frac{Ls}{2\pi i}\Big)ds.
\end{aligned}
$$

The first integral on the right-hand side is identically equal to the integral that was just evaluated in the first part of this proof. As for the second, by shifting the contour to the line $\mathrm{Re}(s) = 0$, we find that it equals

$$
\begin{aligned}
&\Big(\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\log|D_K|\Big)\frac{1}{2\pi i}\int_{(0)}\phi\Big(\frac{Ls}{2\pi i}\Big)ds+\frac{1}{2\pi i}\int_{(0)}\Big(\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}(\tfrac{1}{2}+s)+\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}(\tfrac{1}{2}-s)\Big)\phi\Big(\frac{Ls}{2\pi i}\Big)ds\\
&=\Big(\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\log|D_K|\Big)\frac{\widehat{\phi}(0)}{L}+\frac{1}{\pi}\int_{-\infty}^{\infty}\phi\Big(\frac{Lr}{2\pi}\Big)\mathrm{Re}\Big(\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}(\tfrac{1}{2}+ir)\Big)dr.
\end{aligned}
$$

By applying Lemma 3.2 to the first term, we find the leading terms on the right-hand side of (4.15).

Finally, by absolute convergence, we have the identity

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{(c)}\phi\Big(\frac{Ls}{2\pi i}\Big)\sum_{p,e\geq 1}\Big(\theta_e+\frac{1}{p}\Big)\frac{x_p\log p}{p^{e(\frac{1}{2}+s)}}ds&=\sum_{p,e\geq 1}\Big(\theta_e+\frac{1}{p}\Big)\frac{x_p\log p}{p^{\frac{e}{2}}}\frac{1}{2\pi i}\int_{(c)}\phi\Big(\frac{Ls}{2\pi i}\Big)p^{-es}ds\\
&=\frac{1}{L}\sum_{p,e\geq 1}\Big(\theta_e+\frac{1}{p}\Big)\frac{x_p\log p}{p^{\frac{e}{2}}}\widehat{\phi}\Big(\frac{e\log p}{L}\Big),
\end{aligned}
$$

since the contour of the inner integral can be shifted to the line $\mathrm{Re}(s) = 0$. The same argument works for the term involving $\beta_e(p) - p^{-\frac{e}{3}}$. Hence, the proposition follows. □

5. Analytic continuation of $A_3(-s, s)$ and $A_4(-s, s)$

The goal of this section is to prove Theorem 1.4. To do so, we will need to estimate some of the terms in (4.15), namely,

$$
\begin{aligned}
J^{\pm}(X) :=&\ \frac{2C_2^{\pm}X^{-\frac{1}{6}}}{C_1^{\pm}L}\Big(1-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big)\sum_{p,e}\frac{\log p}{p^{\frac{5e}{6}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)\\
&-\frac{1}{\pi i}\int_{(c)}\phi\Big(\frac{Ls}{2\pi i}\Big)\Big\{-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big(1-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big)\frac{\zeta'}{\zeta}(\tfrac{5}{6}+s)+X^{-s}\frac{\Gamma_{\pm}(\frac{1}{2}-s)}{\Gamma_{\pm}(\frac{1}{2}+s)}\zeta(1-2s)\frac{A_3(-s,s)}{1-s}\\
&+\frac{C_2^{\pm}}{C_1^{\pm}}X^{-s-\frac{1}{6}}\Big(1-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac{1}{6}}\Big)\frac{\Gamma_{\pm}(\frac{1}{2}-s)}{\Gamma_{\pm}(\frac{1}{2}+s)}\zeta(1-2s)\Big(\frac{\zeta(\frac{5}{6}-s)}{\zeta(\frac{5}{6}+s)}\frac{A_4(-s,s)}{1-\frac{6s}{5}}-\frac{A_3(-s,s)}{1-s}\Big)\Big\}ds, \qquad (5.1)
\end{aligned}
$$

for $0 < c < \frac{1}{6}$. The idea is to provide an analytic continuation to the Dirichlet series $A_3(-s, s)$ and $A_4(-s, s)$ in the strip $0 < \mathrm{Re}(s) < \frac{1}{2}$ and to shift the contour of integration to the right.

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Lemma 5.1. *The product formula*

$$A_3(-s,s) = \zeta(3)\zeta(\tfrac{3}{2}-3s)\prod_p\left(1-\frac{1}{p^{\frac{3}{2}+s}}+\frac{1}{p^{\frac{5}{2}-s}}-\frac{1}{p^{\frac{5}{2}-3s}}-\frac{1}{p^{3-4s}}+\frac{1}{p^{\frac{9}{2}-5s}}\right) \tag{5.2}$$

provides an analytic continuation of $A_3(-s,s)$ to $|\operatorname{Re}(s)|<\frac{1}{2}$ except for a simple pole at $s=\frac{1}{6}$ with residue

$$-\frac{\zeta(3)}{3\zeta(\frac{5}{3})\zeta(2)}.$$

Proof. From (4.7) and (4.8), we see that in the region $|\operatorname{Re}(s)|<\frac{1}{6}$,

$$\begin{aligned}
A_3(-s,s) &= \prod_p\left(1-\frac{1}{p^3}\right)^{-1}\left(1-\frac{1}{p^{1-2s}}\right)\\
&\quad\times\left(1+\frac{1}{p}+\frac{1}{p^2}+\sum_{e\geq 1}\frac{f(e,0,p)}{p^{e(\frac{1}{2}-s)}}+\sum_{e\geq 0}\frac{f(e,1,p)}{p^{e(\frac{1}{2}-s)+\frac{1}{2}+s}}+\sum_{e\geq 0}\frac{f(e,2,p)}{p^{e(\frac{1}{2}-s)+1+2s}}\right)\\
&= \zeta(3)\prod_p\left(1-\frac{1}{p^{1-2s}}\right)\left(\frac{1}{p^2}+\sum_{e\geq 0}\frac{1}{p^{e(\frac{1}{2}-s)}}\left(f(e,0,p)+\frac{f(e,1,p)}{p^{\frac{1}{2}+s}}+\frac{f(e,2,p)}{p^{1+2s}}\right)\right).
\end{aligned} \tag{5.3}$$

The sum over $e\geq 0$ on the right-hand side is equal to

$$\begin{aligned}
&\frac{1}{6}\left(1-\frac{1}{p^{\frac{1}{2}+s}}\right)^2\sum_{e\geq 0}(e+1)\frac{1}{p^{e(\frac{1}{2}-s)}}+\frac{1}{2}\left(1-\frac{1}{p^{1+2s}}\right)\sum_{e\geq 0}\frac{1+(-1)^e}{2}\frac{1}{p^{e(\frac{1}{2}-s)}}\\
&\quad+\frac{1}{3}\left(1+\frac{1}{p^{\frac{1}{2}+s}}+\frac{1}{p^{1+2s}}\right)\sum_{e\geq 0}\tau_e\frac{1}{p^{e(\frac{1}{2}-s)}}+\frac{1}{p}\left(1-\frac{1}{p^{\frac{1}{2}+s}}\right)\sum_{e\geq 0}\frac{1}{p^{e(\frac{1}{2}-s)}}\\
&=\frac{1}{6}\cdot\frac{\left(1-\frac{1}{p^{\frac{1}{2}+s}}\right)^2}{\left(1-\frac{1}{p^{\frac{1}{2}-s}}\right)^2}+\frac{1}{2}\cdot\frac{1-\frac{1}{p^{1+2s}}}{1-\frac{1}{p^{1-2s}}}+\frac{1}{3}\cdot\frac{1+\frac{1}{p^{\frac{1}{2}+s}}+\frac{1}{p^{1+2s}}}{1+\frac{1}{p^{\frac{1}{2}-s}}+\frac{1}{p^{1-2s}}}+\frac{1}{p}\cdot\frac{1-\frac{1}{p^{\frac{1}{2}+s}}}{1-\frac{1}{p^{\frac{1}{2}-s}}}.
\end{aligned} \tag{5.4}$$

Here, we have used geometric sum identities, for example,

$$\sum_{k=0}^{\infty}\tau_k x^k = \sum_{k=0}^{\infty}x^{3k}-\sum_{k=0}^{\infty}x^{3k+1} = \frac{1-x}{1-x^3} = \frac{1}{1+x+x^2} \qquad (|x|<1).$$

Inserting the expression (5.4) in (5.3) and simplifying, we obtain the identity

$$A_3(-s,s) = \zeta(3)\zeta(\tfrac{3}{2}-3s)\prod_p\left(1-\frac{1}{p^{\frac{3}{2}+s}}+\frac{1}{p^{\frac{5}{2}-s}}-\frac{1}{p^{\frac{5}{2}-3s}}-\frac{1}{p^{3-4s}}+\frac{1}{p^{\frac{9}{2}-5s}}\right)$$

in the region $|\operatorname{Re}(s)|<1/6$. Now, this clearly extends to $|\operatorname{Re}(s)|<1/2$ except for a simple pole at $s=1/6$ with residue equal to

$$-\frac{\zeta(3)}{3}\prod_p\left(1-p^{-\frac{5}{3}}-p^{-2}+p^{-\frac{11}{3}}\right) = -\frac{\zeta(3)}{3}\frac{1}{\zeta(\frac{5}{3})\zeta(2)},$$

as desired. □

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Lemma 5.2. *Assuming the Riemann Hypothesis, the function $A_4(-s,s)$ admits an analytic continuation to the region $|\operatorname{Re}(s)| < \frac{1}{2}$, except for a double pole at $s = \frac{1}{6}$. Furthermore, for any $0 < \varepsilon < \frac{1}{4}$ and in the region $|\operatorname{Re}(s)| < \frac{1}{2} - \varepsilon$, we have the bound*

$$A_4(-s,s) \ll_\varepsilon (|\operatorname{Im}(s)| + 1)^{\frac{2}{3}}.$$

Proof. By (4.7) and (4.9), for $|\operatorname{Re}(s)| < \frac{1}{6}$, we have that

$$A_4(-s,s) = \prod_p \frac{\left(1 - \frac{1}{p^{1-2s}}\right)\left(1 - \frac{1}{p^{\frac{5}{6}-s}}\right)\left(1 - \frac{1}{p^{\frac{1}{3}}}\right)}{\left(1 - \frac{1}{p^2}\right)\left(1 - \frac{1}{p^{\frac{5}{6}+s}}\right)\left(1 - \frac{1}{p^{\frac{5}{3}}}\right)} \left(\frac{1}{p^2}\left(1 + \frac{1}{p^{\frac{1}{3}}}\right) + \sum_{e \geq 0} \frac{g(e,0,p) + \frac{g(e,1,p)}{p^{\frac{1}{2}+s}} + \frac{g(e,2,p)}{p^{1+2s}}}{p^{e(\frac{1}{2}-s)}}\right),$$

since $y_p^{-1} - g(0,0,p) = \frac{1}{p^2}\left(1 + \frac{1}{p^{\frac{1}{3}}}\right)$. Recalling the definition of $g(e,j,p)$ (see Lemma 4.1), a straightforward evaluation of the infinite sum over $e \geq 0$ yields the expression

$$\begin{aligned} A_4(-s,s) = \zeta(2)\zeta(\tfrac{5}{3}) \prod_p \frac{\left(1 - \frac{1}{p^{1-2s}}\right)\left(1 - \frac{1}{p^{\frac{5}{6}-s}}\right)\left(1 - \frac{1}{p^{\frac{1}{3}}}\right)}{\left(1 - \frac{1}{p^{\frac{5}{6}+s}}\right)} &\left(\frac{\left(1 + \frac{1}{p^{\frac{1}{3}}}\right)^3\left(1 - \frac{1}{p^{\frac{1}{2}+s}}\right)^2}{6\left(1 - \frac{1}{p^{\frac{1}{2}-s}}\right)^2}\right. \\ + \frac{\left(1 + \frac{1}{p^{\frac{1}{3}}}\right)\left(1 + \frac{1}{p^{\frac{2}{3}}}\right)\left(1 - \frac{1}{p^{1+2s}}\right)}{2\left(1 - \frac{1}{p^{1-2s}}\right)} + \frac{\left(1 + \frac{1}{p}\right)\left(1 + \frac{1}{p^{\frac{1}{2}+s}} + \frac{1}{p^{1+2s}}\right)}{3\left(1 + \frac{1}{p^{\frac{1}{2}-s}} + \frac{1}{p^{1-2s}}\right)} &+ \frac{\left(1 + \frac{1}{p^{\frac{1}{3}}}\right)^2\left(1 - \frac{1}{p^{\frac{1}{2}+s}}\right)}{p\left(1 - \frac{1}{p^{\frac{1}{2}-s}}\right)} + \left.\frac{1 + \frac{1}{p^{\frac{1}{3}}}}{p^2}\right). \end{aligned}$$

Isolating the 'divergent terms' leads us to the identity

$$A_4(-s,s) = \zeta(2)\zeta(\tfrac{5}{3}) \prod_p \left(D_{4,p,1}(s) + A_{4,p,1}(s)\right),$$

where

$$\begin{aligned} D_{4,p,1}(s) := \frac{1 - \frac{1}{p^{\frac{5}{6}-s}}}{1 - \frac{1}{p^{\frac{5}{6}+s}}} &\left(\frac{\left(1 + \frac{2}{p^{\frac{1}{3}}} - \frac{2}{p}\right)\left(1 - \frac{1}{p^{\frac{1}{2}+s}}\right)^2\left(1 + \frac{1}{p^{\frac{1}{2}-s}}\right)}{6\left(1 - \frac{1}{p^{\frac{1}{2}-s}}\right)}\right. \\ &+ \left.\frac{1 - \frac{1}{p^{1+2s}}}{2} + \frac{\left(1 - \frac{1}{p^{\frac{1}{3}}} + \frac{1}{p}\right)\left(1 + \frac{1}{p^{\frac{1}{2}+s}} + \frac{1}{p^{1+2s}}\right)\left(1 - \frac{1}{p^{1-2s}}\right)}{3\left(1 + \frac{1}{p^{\frac{1}{2}-s}} + \frac{1}{p^{1-2s}}\right)} + \frac{1}{p}\right) \end{aligned}$$

and

$$\begin{aligned} A_{4,p,1}(s) := \frac{1 - \frac{1}{p^{\frac{5}{6}-s}}}{1 - \frac{1}{p^{\frac{5}{6}+s}}} &\left(- \frac{\left(1 - \frac{1}{p^{\frac{1}{2}+s}}\right)^2\left(1 + \frac{1}{p^{\frac{1}{2}-s}}\right)}{6p^{\frac{4}{3}}\left(1 - \frac{1}{p^{\frac{1}{2}-s}}\right)} - \frac{1 - \frac{1}{p^{1+2s}}}{2p^{\frac{4}{3}}} - \frac{\left(1 + \frac{1}{p^{\frac{1}{2}+s}} + \frac{1}{p^{1+2s}}\right)\left(1 - \frac{1}{p^{1-2s}}\right)}{3p^{\frac{4}{3}}\left(1 + \frac{1}{p^{\frac{1}{2}-s}} + \frac{1}{p^{1-2s}}\right)}\right. \\ &+ \left.\frac{\left(1 + \frac{1}{p^{\frac{1}{3}}} - \frac{1}{p^{\frac{2}{3}}} - \frac{1}{p}\right)\left(1 - \frac{1}{p^{\frac{1}{2}+s}}\right)\left(1 + \frac{1}{p^{\frac{1}{2}-s}}\right) - 1}{p} + \frac{1}{p^2}\left(1 - \frac{1}{p^{\frac{2}{3}}}\right)\left(1 - \frac{1}{p^{1-2s}}\right)\right). \end{aligned}$$

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The term $A_{4,p,1}(s)$ is 'small' for $|\operatorname{Re}(s)| < \frac{1}{2}$, hence, we will concentrate our attention on $D_{4,p,1}(s)$. We see that

$$D_{4,p,1}(s) = \frac{1 - \frac{1}{p^{\frac{5}{6}-s}}}{1 - \frac{1}{p^{\frac{5}{6}+s}}} D_{4,p,2}(s) + \frac{1}{p} + A_{4,p,2}(s),$$

where

$$D_{4,p,2}(s) := \frac{\left(1 + \frac{2}{p^{\frac{1}{3}}}\right)\left(1 - \frac{1}{p^{\frac{1}{2}+s}}\right)^2\left(1 + \frac{1}{p^{\frac{1}{2}-s}}\right)}{6\left(1 - \frac{1}{p^{\frac{1}{2}-s}}\right)} + \frac{1 - \frac{1}{p^{1+2s}}}{2}$$
$$+ \frac{\left(1 - \frac{1}{p^{\frac{1}{3}}}\right)\left(1 + \frac{1}{p^{\frac{1}{2}+s}} + \frac{1}{p^{1+2s}}\right)\left(1 - \frac{1}{p^{1-2s}}\right)}{3\left(1 + \frac{1}{p^{\frac{1}{2}-s}} + \frac{1}{p^{1-2s}}\right)}$$

and

$$A_{4,p,2}(s) := \frac{\left(1 - \frac{1}{p^{\frac{5}{6}-s}}\right)}{p\left(1 - \frac{1}{p^{\frac{5}{6}+s}}\right)}\left(-\frac{\left(1 - \frac{1}{p^{\frac{1}{2}+s}}\right)^2\left(1 + \frac{1}{p^{\frac{1}{2}-s}}\right)}{3\left(1 - \frac{1}{p^{\frac{1}{2}-s}}\right)} + \frac{\left(1 + \frac{1}{p^{\frac{1}{2}+s}} + \frac{1}{p^{1+2s}}\right)\left(1 - \frac{1}{p^{1-2s}}\right)}{3\left(1 + \frac{1}{p^{\frac{1}{2}-s}} + \frac{1}{p^{1-2s}}\right)} + 1\right) - \frac{1}{p},$$

which is also 'small'. Taking common denominators and expanding out shows that

$$D_{4,p,2}(s) = \frac{1}{1 - \frac{1}{p^{\frac{3}{2}-3s}}}\left(1 - \frac{1}{p} - \frac{1}{p^{\frac{5}{6}+s}} + \frac{1}{p^{\frac{5}{6}-s}} + \frac{1}{p^{\frac{4}{3}-2s}} + A_{4,p,3}(s)\right),$$

where

$$A_{4,p,3}(s) := -\frac{1}{p^{\frac{3}{2}-s}} + \frac{1}{p^{\frac{5}{2}-s}} - \frac{1}{p^{\frac{4}{3}}} + \frac{1}{p^{\frac{11}{6}+s}} - \frac{1}{p^{\frac{11}{6}-s}} + \frac{1}{p^{\frac{7}{3}}} - \frac{1}{p^{\frac{7}{3}-2s}}$$

is 'small'. More precisely, for $|\operatorname{Re}(s)| \leq \frac{1}{2} - \varepsilon < \frac{1}{2}$ and $j = 1, 2, 3$, we have the bound $A_{4,p,j}(s) = O_\varepsilon\left(\frac{1}{p^{1+\varepsilon}}\right)$. Therefore,

$$A_4(-s, s) = \zeta(2)\zeta(\tfrac{5}{3})\zeta(\tfrac{3}{2} - 3s)\widetilde{A}_4(s) \prod_p \left(\frac{\left(1 - \frac{1}{p^{\frac{5}{6}-s}}\right)}{\left(1 - \frac{1}{p^{\frac{5}{6}+s}}\right)}\left(1 - \frac{1}{p^{\frac{5}{6}+s}} + \frac{1}{p^{\frac{5}{6}-s}} + \frac{1}{p^{\frac{4}{3}-2s}}\right)\right), \tag{5.5}$$

where

$$\widetilde{A}_4(s) := \prod_p \left(1 + \frac{\frac{1}{p}\left(1 - \frac{1}{p^{\frac{3}{2}-3s}} - \frac{1-p^{-\frac{5}{6}+s}}{1-p^{-\frac{5}{6}-s}}\right) + \frac{1-p^{-\frac{5}{6}+s}}{1-p^{-\frac{5}{6}-s}} A_{4,p,3}(s) + \left(1 - \frac{1}{p^{\frac{3}{2}-3s}}\right)(A_{4,p,2}(s) + A_{4,p,1}(s))}{\frac{1-p^{-\frac{5}{6}+s}}{1-p^{-\frac{5}{6}-s}}\left(1 - \frac{1}{p^{\frac{5}{6}+s}} + \frac{1}{p^{\frac{5}{6}-s}} + \frac{1}{p^{\frac{4}{3}-2s}}\right)}\right)$$

is absolutely convergent for $|\operatorname{Re}(s)| < \frac{1}{2}$. Hence, the final step is to find a meromorphic continuation for the infinite product on the right-hand side of (5.5), which we will denote by $D_3(s)$. However, it is

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straightforward to show that

$$A_{4,4}(s) := D_3(s)\frac{\zeta(\frac{8}{3}-4s)\zeta(\frac{5}{3}-2s)\zeta(\frac{13}{6}-3s)}{\zeta(\frac{4}{3}-2s)} \tag{5.6}$$

converges absolutely for $|\operatorname{Re}(s)| < \frac{1}{2}$. This finishes the proof of the first claim in the lemma.

Finally, the growth estimate

$$A_4(-s,s) \ll_\varepsilon (|\operatorname{Im}(s)|+1)^\varepsilon |\zeta(\tfrac{3}{2}-3s)\zeta(\tfrac{4}{3}-2s)| \ll_\varepsilon (|\operatorname{Im}(s)|+1)^{\frac{2}{3}}$$

follows from (5.5), (5.6), as well as [MV, Theorems 13.18 and 13.23] and the functional equation for $\zeta(s)$. □

Now that we have a meromorphic continuation of $A_4(-s,s)$, we will calculate the leading Laurent coefficient at $s=\frac{1}{6}$.

Lemma 5.3. *We have the formula*

$$\lim_{s\to\frac{1}{6}}(s-\tfrac{1}{6})^2 A_4(-s,s) = \frac{1}{6}\frac{\zeta(2)\zeta(\frac{5}{3})}{\zeta(\frac{4}{3})}\prod_p \Big(1-\frac{1}{p^{\frac{2}{3}}}\Big)^2\Big(1-\frac{1}{p}\Big)\Big(1+\frac{2}{p^{\frac{2}{3}}}+\frac{1}{p}+\frac{1}{p^{\frac{4}{3}}}\Big).$$

Proof. By Lemma 5.2, $A_4(-s,s)$ has a double pole at $s=\frac{1}{6}$. Moreover, by (5.5) and (5.6), we find that $\frac{A_4(-s,s)}{\zeta(\frac{3}{2}-3s)\zeta(\frac{4}{3}-2s)}$ has a convergent Euler product in the region $|\operatorname{Re}(s)| < \frac{1}{3}$ (this allows us to interchange the order of the limit and the product in the calculation below), so that

$$\begin{aligned}
\lim_{s\to\frac{1}{6}}(s-\tfrac{1}{6})^2 A_4(-s,s) &= \frac{1}{6}\lim_{s\to\frac{1}{6}}\frac{A_4(-s,s)}{\zeta(\frac{3}{2}-3s)\zeta(\frac{4}{3}-2s)}\\
&=\frac{\zeta(2)\zeta(\frac{5}{3})}{6}\prod_p\Big(1-\frac{1}{p}\Big)\Big(1-\frac{1}{p^{\frac{2}{3}}}\Big)^2\Big(1-\frac{1}{p^{\frac{1}{3}}}\Big)\Bigg\{\frac{\Big(1+\frac{1}{p^{\frac{1}{3}}}\Big)^3\Big(1-\frac{1}{p^{\frac{2}{3}}}\Big)^2}{6\Big(1-\frac{1}{p^{\frac{1}{3}}}\Big)^2}\\
&\quad+\frac{\Big(1+\frac{1}{p^{\frac{1}{3}}}\Big)\Big(1+\frac{1}{p^{\frac{2}{3}}}\Big)\Big(1-\frac{1}{p^{\frac{4}{3}}}\Big)}{2\Big(1-\frac{1}{p^{\frac{2}{3}}}\Big)}+\frac{\Big(1+\frac{1}{p}\Big)\Big(1+\frac{1}{p^{\frac{2}{3}}}+\frac{1}{p^{\frac{4}{3}}}\Big)}{3\Big(1+\frac{1}{p^{\frac{1}{3}}}+\frac{1}{p^{\frac{2}{3}}}\Big)}+\frac{\Big(1+\frac{1}{p^{\frac{1}{3}}}\Big)^2\Big(1-\frac{1}{p^{\frac{2}{3}}}\Big)}{p\Big(1-\frac{1}{p^{\frac{1}{3}}}\Big)}+\frac{1+\frac{1}{p^{\frac{1}{3}}}}{p^2}\Bigg\}.
\end{aligned}$$

The claim follows. □

We are now ready to estimate $J^\pm(X)$ when the support of $\widehat{\phi}$ is small.

Lemma 5.4. *Let ϕ be a real even Schwartz function, such that $\sigma = \sup(\operatorname{supp}(\widehat{\phi})) < 1$. Let $J^\pm(X)$ be defined by (5.1). Then we have the estimate*

$$J^\pm(X) = C^\pm\phi\Big(\frac{L}{12\pi i}\Big)X^{-\frac{1}{3}} + O_\varepsilon\Big(X^{\frac{\sigma-1}{2}+\varepsilon}\Big),$$

where

$$C^\pm := \frac{5}{12}\frac{C_2^\pm}{C_1^\pm}\frac{\Gamma_\pm(\frac{1}{3})}{\Gamma_\pm(\frac{2}{3})}\frac{\zeta(\frac{2}{3})^2\zeta(\frac{5}{3})\zeta(2)}{\zeta(\frac{4}{3})}\prod_p\Big(1-\frac{1}{p^{\frac{2}{3}}}\Big)^2\Big(1-\frac{1}{p}\Big)\Big(1+\frac{2}{p^{\frac{2}{3}}}+\frac{1}{p}+\frac{1}{p^{\frac{4}{3}}}\Big). \tag{5.7}$$

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Proof. We rewrite the integral in $J^{\pm}(X)$ as

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(c)}(-2)\phi\Big(\frac{Ls}{2\pi i}\Big)\bigg\{\Big(1-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac16}\Big)\Big(-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac16}\frac{\zeta'}{\zeta}(\tfrac56+s)+X^{-s}\frac{\Gamma_{\pm}(\frac12-s)}{\Gamma_{\pm}(\frac12+s)}\zeta(1-2s)\frac{A_3(-s,s)}{1-s}\Big)\\&\quad+\frac{C_2^{\pm}}{C_1^{\pm}}X^{-s-\frac16}\Big(1-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac16}\Big)\frac{\Gamma_{\pm}(\frac12-s)}{\Gamma_{\pm}(\frac12+s)}\zeta(1-2s)\frac{\zeta(\frac56-s)}{\zeta(\frac56+s)}\frac{A_4(-s,s)}{1-\frac{6s}{5}}\\&\quad+\Big(\frac{C_2^{\pm}}{C_1^{\pm}}\Big)^2X^{-s-\frac13}\frac{\Gamma_{\pm}(\frac12-s)}{\Gamma_{\pm}(\frac12+s)}\zeta(1-2s)\frac{A_3(-s,s)}{1-s}\bigg\}ds\end{aligned}\tag{5.8}$$

for $0<c<\frac16$. The integrand has a simple pole at $s=\frac16$ with residue

$$\begin{aligned}&-2\phi\Big(\frac{L}{12\pi i}\Big)\Big(1-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac16}\Big)X^{-\frac16}\Big(\frac{C_2^{\pm}}{C_1^{\pm}}-\frac25\frac{\Gamma_{\pm}(\frac13)}{\Gamma_{\pm}(\frac23)}\frac{\zeta(\frac23)\zeta(3)}{\zeta(\frac53)\zeta(2)}\Big)\\&-2\phi\Big(\frac{L}{12\pi i}\Big)\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac13}\frac{\Gamma_{\pm}(\frac13)}{\Gamma_{\pm}(\frac23)}\frac{5\zeta(\frac23)^2}{4}\lim_{s\to\frac16}(s-\tfrac16)^2A_4(-s,s)+O\Big(\phi\Big(\frac{L}{12\pi i}\Big)X^{-\frac12}\Big)\\&\quad=-C^{\pm}\phi\Big(\frac{L}{12\pi i}\Big)X^{-\frac13}+O(X^{\frac{\sigma}{6}-\frac12})\end{aligned}\tag{5.9}$$

by Lemma 5.3, as well as the fact that the first line vanishes. Due to Lemmas 5.1 and 5.2, we can shift the contour of integration to the line $\mathrm{Re}(s)=\frac12-\frac{\varepsilon}{2}$, at the cost of -1 times the residue (5.9).

We now estimate the shifted integral. The term involving $\frac{\zeta'}{\zeta}(\frac56+s)$ can be evaluated by interchanging sum and integral; we obtain the identity

$$\frac{1}{\pi i}\int_{(\frac12-\frac{\varepsilon}{2})}\phi\Big(\frac{Ls}{2\pi i}\Big)\frac{\zeta'}{\zeta}(\tfrac56+s)ds=-\frac{2}{L}\sum_{p,e}\frac{\log p}{p^{\frac{5e}{6}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big).\tag{5.10}$$

The last step is to bound the remaining terms, which is carried out by combining (4.17) with Lemmas 5.1 and 5.2. □

Finally, we complete the Proof of Theorem 1.4.

Proof of Theorem 1.4. Given Proposition 4.4 and Lemma 5.4, the only thing remaining to prove is (1.7). Applying (5.8) with $c=\frac{1}{20}$ and splitting the integral into two parts, we obtain the identity

$$\begin{aligned}J^{\pm}(X)&=\frac{2C_2^{\pm}X^{-\frac16}}{C_1^{\pm}L}\Big(1-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac16}\Big)\sum_{p,e}\frac{\log p}{p^{\frac{5e}{6}}}\widehat{\phi}\Big(\frac{\log p^e}{L}\Big)\\&-\frac{1}{\pi i}\int_{(\frac{1}{20})}\phi\Big(\frac{Ls}{2\pi i}\Big)\bigg\{\Big(1-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac16}\Big)\Big(-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac16}\frac{\zeta'}{\zeta}(\tfrac56+s)+X^{-s}\frac{\Gamma_{\pm}(\frac12-s)}{\Gamma_{\pm}(\frac12+s)}\zeta(1-2s)\frac{A_3(-s,s)}{1-s}\Big)\bigg\}ds\\&-\frac{1}{\pi i}\int_{(\frac{1}{20})}\phi\Big(\frac{Ls}{2\pi i}\Big)\bigg\{\frac{C_2^{\pm}}{C_1^{\pm}}X^{-s-\frac16}\Big(1-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-\frac16}\Big)\frac{\Gamma_{\pm}(\frac12-s)}{\Gamma_{\pm}(\frac12+s)}\zeta(1-2s)\frac{\zeta(\frac56-s)}{\zeta(\frac56+s)}\frac{A_4(-s,s)}{1-\frac{6s}{5}}\\&+\Big(\frac{C_2^{\pm}}{C_1^{\pm}}\Big)^2X^{-s-\frac13}\frac{\Gamma_{\pm}(\frac12-s)}{\Gamma_{\pm}(\frac12+s)}\zeta(1-2s)\frac{A_3(-s,s)}{1-s}\bigg\}ds.\end{aligned}$$

By shifting the first integral to the line $\mathrm{Re}(s)=\frac15$ and applying (5.10), we derive (1.7). Note that the residue at $s=\frac16$ is the first line of (5.9), which is equal to zero.

□

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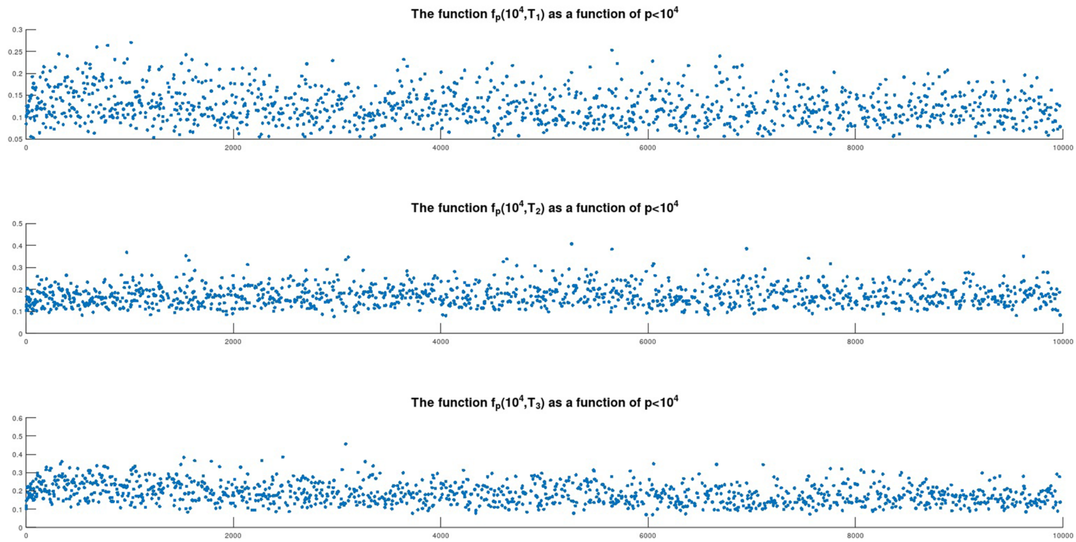

Figure 2. *A plot of* $(p, f_p(10^4, T_j))$ *for* $p < 10^4$ *and* $j = 1, 2, 3$.

A. Numerical investigations

In this section, we present several graphs[5] associated to the error term

$$E_p^+(X,T) := N_p^+(X,T) - A_p^+(T)X - B_p^+(T)X^{\frac{5}{6}}.$$

We recall that we expect a bound of the form $E_p^+(X,T) \ll_\varepsilon p^\omega X^{\theta+\varepsilon}$ (see (1.2)). Moreover, from the graphs shown in Figure 1, it seems likely that $\theta = \frac{1}{2}$ is admissible and the best possible. Now, to test the uniformity in p, we consider the function

$$f_p(X,T) := \max_{1 \le x \le X} x^{-\frac{1}{2}} |E_p^+(x,T)|;$$

we then expect a bound of the form $f_p(X,T) \ll_\varepsilon p^\omega X^{\theta - \frac{1}{2} + \varepsilon}$ with θ possibly equal to $\frac{1}{2}$. To predict the smallest admissible value of ω, in Figure 2, we plot $f_p(10^4, T_j)$ for $j = 1, 2, 3$, as a function of $p < 10^4$. From this data, it seems likely that any $\omega > 0$ is admissible. Now, one might wonder whether this is still valid in the range $p > X$. To investigate this, in Figure 3, we plot the function $f_p(10^4, T_3)$ for every 10^4-th prime up to 10^8, revealing similar behaviour. Finally, we have also produced similar data associated to the quantity $N_p^-(X, T_j)$ with $j = 1, 2, 3$, and the result was comparable to Figure 2.

However, it seems like the splitting type T_4 behaves differently (see Figure 4 for a plot of $p \cdot f_p(10^4, T_4)$ for every $p < 10^5$). One can see that this graph is eventually essentially constant. This is readily explained by the fact that in the range $p > X$, we have $N_p^\pm(X, T_4) = 0$. Indeed, if p has splitting type T_4 in a cubic field K of discriminant at most X, then p must divide D_K, which implies that $p \le X$. As a consequence, $p f_p(X, T_4) \asymp X^{\frac{1}{2}}$, which is constant as a function of p. As for the more interesting range $p \le X$, it seems like $f_p(X, T_4) \ll_\varepsilon p^{-\frac{1}{2}+\varepsilon} X^\varepsilon$ (i.e. for $T = T_4$, the values $\theta = \frac{1}{2}$ and any $\omega > -\frac{1}{2}$ are admissible in (1.2)). In Figure 5, we test this hypothesis with larger values of X by plotting $p^{\frac{1}{2}} \cdot f_p(10^5, T_4)$ for all $p < 10^4$. This seems to confirm that for $T = T_4$, the values $\theta = \frac{1}{2}$ and any $\omega > -\frac{1}{2}$ are admissible in (1.2). In

[5] The computations associated to these graphs were done using development version 2.14 of pari/gp (see https://pari.math.u-bordeaux.fr/Events/PARI2022/talks/sources.pdf), and the full code can be found here: https://github.com/DanielFiorilli/CubicFieldCounts.

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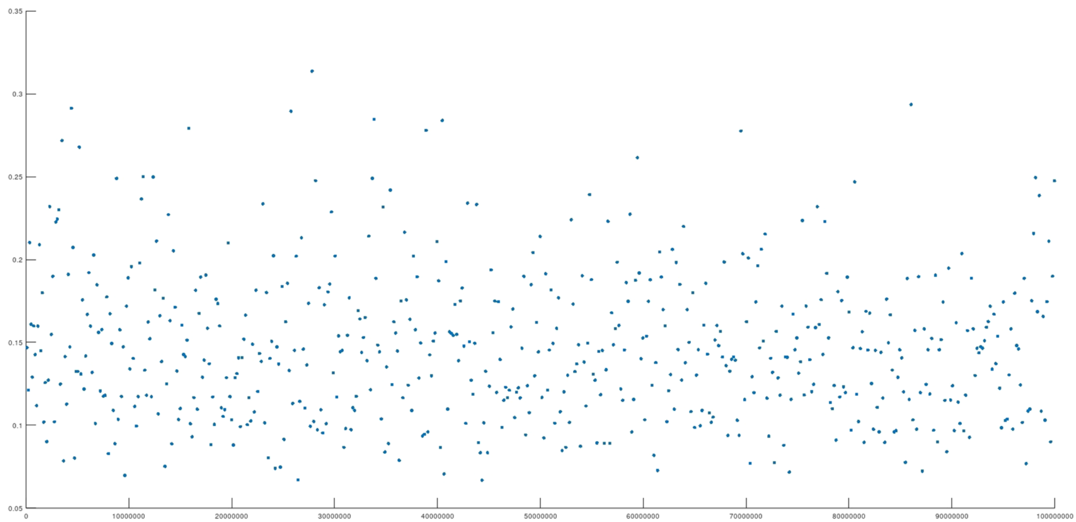

Figure 3. *A plot of some of the values of* $(p, f_p(10^4, T_3))$ *for* $p < 10^8$.

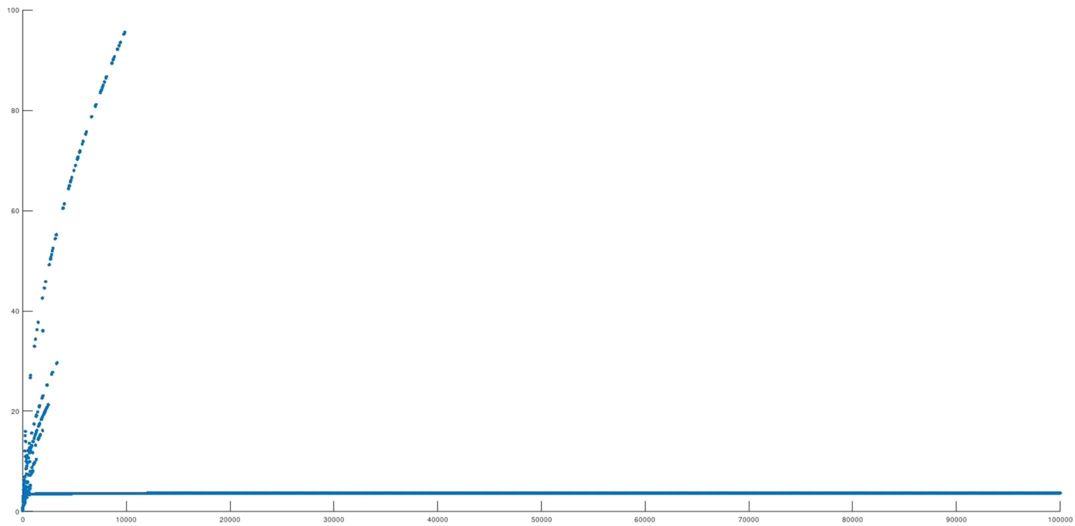

Figure 4. *A plot of* $(p, p f_p(10^4, T_4))$ *for* $p < 10^5$.

other words, it seems like we have $E_p^+(X, T_4) \ll_\varepsilon p^{-\frac{1}{2}+\varepsilon} X^{\frac{1}{2}+\varepsilon}$, and the sum of the two exponents here is 2ε, which is significantly smaller than the sum of exponents in Theorem 1.1, which is $\omega + \theta \geq \frac{1}{2}$. Note that this is not contradictory, since in that theorem we are assuming such a bound uniformly for all splitting types and, from the discussion above, we expect that $E_p^+(X, T_1) \ll_\varepsilon p^\varepsilon X^{\frac{1}{2}+\varepsilon}$ is essentially the best possible. Finally, we have also produced data for the quantity $N_p^-(X, T_4)$. The result was somewhat similar but far from identical. We would require more data to make a guess as strong as the one we made for $E_p^+(X, T_4)$.

For the splitting type T_5, it seems like the error term is even smaller (probably owing to the fact that these fields are very rare). Indeed, this is what the graph of $p^2 \cdot f_p(10^6, T_5)$ for all $p < 10^3$ in Figure 6 indicates. Again, there are two regimes. Firstly, by [B, p. 1216], $p > 2$ has splitting type T_5 in the cubic field K if and only if $p^2 \mid D_K$, hence, $N_p^\pm(X, T_5) = 0$ for $p > X^{\frac{1}{2}}$ (that is $p^2 \cdot f_p(X, T_5) \asymp X^{\frac{1}{2}}$). As

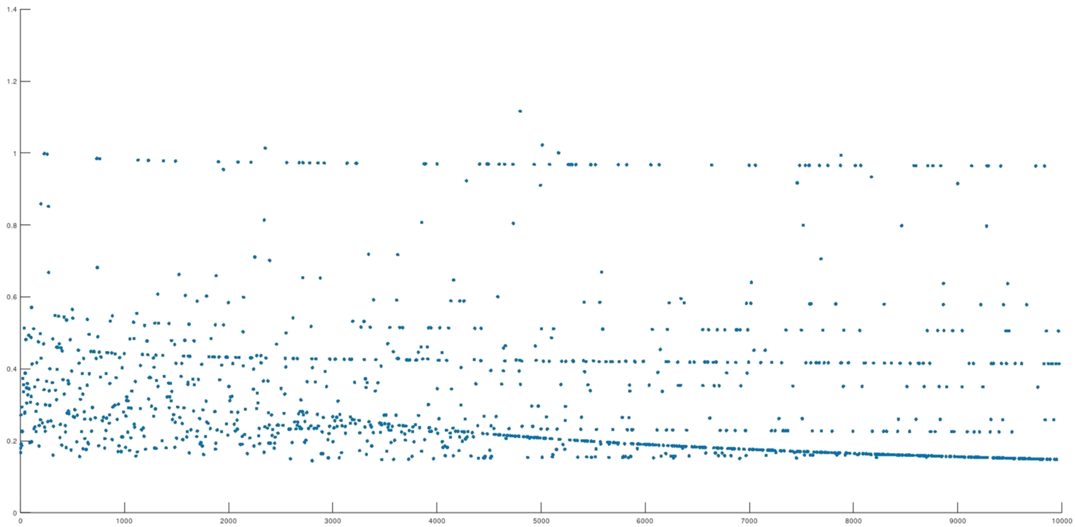

Figure 5. *A plot of* $(p, p^{\frac{1}{2}} f_p(10^5, T_4))$ *for* $p < 10^4$.

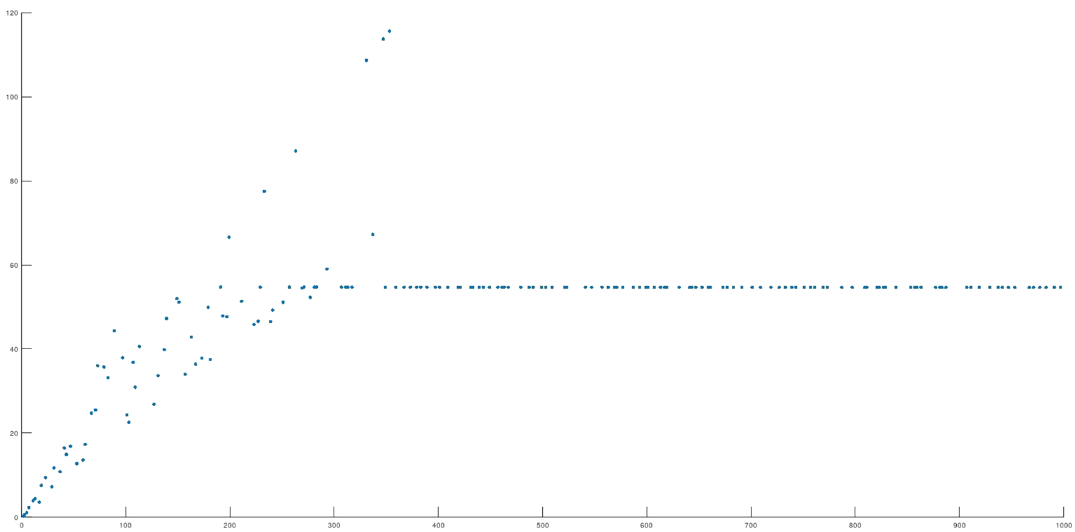

Figure 6. *A plot of* $(p, p^2 f_p(10^6, T_5))$ *for* $p < 10^3$.

for $p \le X^{\frac{1}{2}}$, Figure 6 indicates that $f_p(X, T_5) \ll_\varepsilon p^{-1+\varepsilon} X^\varepsilon$ (e.g. for $T = T_5$, the values $\theta = \frac{1}{2}$ and any $\omega > -1$ are admissible in (1.2)). Once more, it is interesting to compare this with Theorem 1.1, since it seems like $E_p^+(X, T_5) \ll_\varepsilon p^{-1+\varepsilon} X^{\frac{1}{2}+\varepsilon}$, and the sum of the two exponents is now $-\frac{1}{2} + 2\epsilon$. We have also produced analogous data associated to the quantity $N_p^-(X, T_5)$. The result was somewhat similar.

Finally, we end this section with a graph (see Figure 7) of

$$E^+(X) := X^{-\frac{1}{2}} \left(N_{\text{all}}^+(X) - C_1^+ X - C_2^+ X^{\frac{5}{6}} \right)$$

for $X < 10^{11}$ (which is the limit of Belabas' program[6] used for this computation). Here, $N_{\text{all}}^+(X)$ counts all cubic fields of discriminant up to X, including Galois fields (by Cohn's work [C], $N_{\text{all}}^+(X) - N^+(X) \sim c X^{\frac{1}{2}}$,

[6]The program, based on the algorithm in [B], can be found here: https://www.math.u-bordeaux.fr/~kbelabas/research/cubic.html.

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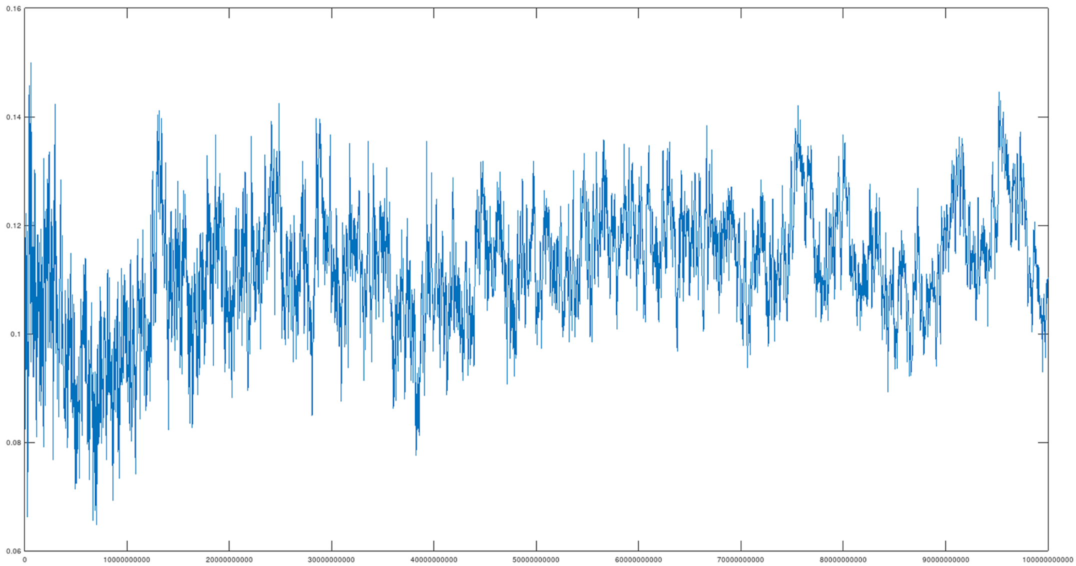

Figure 7. *A plot of $E^+(X)$ for $X < 10^{11}$.*

with $c = 0.1585\ldots$). This strongly supports the conjecture that $E^+(X) \ll_\varepsilon X^{\frac{1}{2}+\varepsilon}$ and that the exponent $\frac{1}{2}$ is the best possible. It is also interesting that the graph is always positive, which is not without reminding us of Chebyshev's bias (see, for instance, the graphs in the survey paper [GM]) in the distribution of primes.

Given this numerical evidence, one may summarise this section by stating that in all cases, it seems like we have square-root cancellation. More precisely, the data indicate that the bound

$$N_p^+(X,T) - A_p^+(T)X - B_p^+(T)X^{\frac{5}{6}} \ll_\varepsilon (pX)^\varepsilon \big(A_p^+(T)X\big)^{\frac{1}{2}} \tag{A.1}$$

could hold, at least for almost all p and X. This is reminiscent of Montgomery's conjecture [Mo] for primes in arithmetic progressions, which states that

$$\sum_{\substack{n\le x\\ n\equiv a \bmod q}} \Lambda(n) - \frac{x}{\phi(q)} \ll_\varepsilon x^\varepsilon\Big(\frac{x}{\phi(q)}\Big)^{\frac{1}{2}} \qquad (q\le x, \qquad (a,q)=1).$$

Precise bounds such as (A.1) seem to be far from reach with the current methods, however, we hope to return to such questions in future work.

Acknowledgements. We would like to thank Frank Thorne for inspiring conversations and for sharing the preprint [BTT] with us. We also thank Keunyoung Jeong for providing us with preliminary computational results. The computations in this paper were carried out on a personal computer using pari/gp as well as Belabas's CUBIC program. We thank Bill Allombert for his help regarding the latest development version of pari/gp. Peter J. Cho is supported by an National Research Foundation (NRF) grant funded by the Korea government (Ministry of Science and ICT (MSIT)) (No. 2019R1F1A1062599) and the Basic Science Research Program (2020R1A4A1016649). Yoonbok Lee is supported by an NRF grant funded by the Korea government (MSIT) (No.2019R1F1A1050795). Daniel Fiorilli was supported at the University of Ottawa by an Natural Sciences and Engineering Research Council of Canada (NSERC) discovery grant. Anders Södergren was supported by a grant from the Swedish Research Council (grant 2016-03759).

Conflict of Interest. The authors have no conflicts of interest to declare.

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