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Expert Cognition During Proof Construction Using the Principle of Mathematical Induction

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by

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ABSTRACT

The purpose of this study was to identify and analyze the observable cognitive processes of experts in mathematics while they work on proof-construction activities using the Principle of Mathematical Induction (PMI). Graduate student participants in the study worked on "nonstandard" mathematical induction problems that did not involve algebraic identities or finite sums. This study identified some of the problem solving-strategies used by the participants during a Cognitive Task Analysis (Feldon, 2007) as well as epistemological obstacles they encountered while working with PMI. After the Cognitive Task Analysis, the graduate students participated in two semi-structured interviews. These interviews explored graduate students' beliefs about proofs and proof techniques and situates their use of PMI within the contexts of these beliefs.

Two primary theoretical frameworks were used to analyze participant cognition and the qualitative data collected. First, the study used Action, Process, Object, Schema (APOS) Theory (Asiala et al., 1996) to to study and analyze the participants' conceptual understanding of the technique of mathematical induction and to test a preliminary genetic decomposition adapted from previous studies on PMI (Dubinsky & Lewin 1996, 1999; Garcia-Martinez & Parraguez, 2017). Second, an Expert Knowledge Framework (Bransford, Brown, & Cocking, 1999; Shepherd & Sande, 2014) was used to classify the participants' responses to the semi-structured interview questions according to several characteristics of expertise. The study identified several results which (1) give insight to the mental constructions used by mathematical experts when solving problem involving PMI; (2) offer some implications for improving the instruction of PMI in introductory proofs classrooms; and (3) provide results that allow for future comparison between expert and novice mathematical learners.

INDEX WORDS: Proofs education, Mathematical induction, APOS, Expertise, Proof techniques, Cognition

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DEDICATION

This is dedicated to the women in mathematics who have gone before me, who walk beside me, and who will follow after me.

"on days when i could not move
it was women
who came to water my feet
it was women
who nourished me
back to life"
sisters- rupi kaur

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1 INTRODUCTION

1.1 Statement of the Problem

The purpose of this study is to identify and analyze the observable cognitive processes of experts in mathematics while they work on proof-construction activities, specifically problems using the Principle of Mathematical Induction (PMI). Graduate student participants in the study work on "nonstandard" mathematical induction problems in which the base case has not been explicitly identified for them. This study seeks to identify the problem solving-strategies used by the participants as well as epistemological obstacles they encounter while working with PMI. In addition, this research explores graduate students' beliefs about proofs and proof techniques and situates their use of PMI within the contexts of these beliefs. Action, Process, Object, Schema (APOS) Theory, including the theory of schema development, is the primary theoretical framework through which participant cognition is evaluated, analyzed, and discussed. In addition, learning theories involving expert learning and knowledge organization also serve as foundational to the data analysis.

1.1.1 *Significance of the Study*

Proof lies at the center of advanced mathematical study and research. It follows that mathematicians should be able to effectively read, write, and evaluate proofs. Krantz (2007) argues "It is the proof concept that makes the subject cohere, that gives it its timelessness, and that enables it to travel well" (p.1). He explores the historical development of mathematical proof and contextualizes its importance to the field of mathematics. Many mathematicians argue that efforts to strengthen, expand, and adapt notions of mathematical proof have positive effects on the mathematical community as a whole (Thurston, 1994). Proof is closely linked to the communication of mathematical ideas and the identification of weaknesses within logical arguments, so a solid grasp of various proof techniques becomes increasingly important for students as they advance in their studies. The 2015 CUPM Curriculum Guide's second Content Recommendation claims that undergraduate math majors should "learn to read, understand, analyze, and produce proofs at increasing depth as they progress through a major." This guide repetitively emphasizes the value of good proof education, especially for students pursuing a graduate degree in mathematics.

The literature is rich with research analyzing student difficulties with proof in general, focusing on issues with the construction, comprehension, or evaluation of mathematical proofs (Baker & Campbell, 2004; Hanna & Barbeau, 2008; Harel & Sowder, 1998, 2007; Inglis & Alcock, 2012; Mejia-Ramos, et al., 2012;Piatek-Jimenez, 2010). Over the past few decades, many math education researchers have tightened their focus to particular proof techniques (Antonini, 2003; Chamberlain & Vidakovic, 2021; Demiray & Bostan, 2017; Harel, 2001). In particular, the use of the Principle of Mathematical Induction as both a concept and a proof technique has been the central focus of many research studies (Ashkenazi & Itzkovitch, 2014; Atwood 2001; Dubinsky, 1986, 1989; Dubinsky & Lewin, 1986; Ernest 1984; Garcia-Martinez & Parraguez, 2017). Avital and Libeskind (1978) argue that a deeper understanding of PMI and the natural numbers is positively related to the "mathematical maturity of the learner" (p. 429). Many curriculum-focused entities for grades K-12 encourage the introduction of PMI at earlier stages of mathematics education as the principle can help students understand iterative and recursive processes as well as introducing them to the natural numbers and their properties (NCTM, 2000). Mathematical Induction is used across numerous disciplines of mathematics, and the common epistemological difficulties associated with the topic make it of particular interest for those who study the pedagogy of mathematics.

Many studies have characterized some students' difficulties with mathematical induction. Lowenthal and Eisenberg (1992) state that students view PMI as "a mechanical procedure triggered by the statement, 'Prove that for all n...'" (p. 238). Other research supports this claim, finding that students perceive mathematical induction as indistinguishable from trial-and-error and as a "technique of drawing a general conclusion from a number of individual cases" (Harel, 2001, p. 11). Most of these studies concerning PMI and its associated epistemological obstacles have focused on subjects in high school, undergraduate subjects, and preservice teachers, but very few have focused on PMI with advanced mathematical students or mathematicians. Further, much of the relevant research focuses on the standard types of mathematical induction problems encountered in introductory courses to proof such as the verification of algebraic equalities or verifications involving sums (Avital & Libeskind, 1976; Movshovitz-Hadar, 1993). In advanced mathematics courses, however, proofs and problems involving PMI do not often fit into one of these two limited categories of mathematical induction. A few studies have analyzed participants or suggested research using PMI in nonstandard mathematical induction problems (geometry, graph theory, etc.), but they are significantly less common (Ashkenazi & Itzkovitch, 2014; Garcia-Martinez & Parraguez, 2017). Relatively little is known about how the concept of PMI develops long-term as learners progress through their mathematical education. More research is needed to understand how advanced mathematical learners think about and use mathematical induction so that this progress can be closely examined in order to develop pedagogical practices that foster the development of a deep understanding of PMI.

1.1.2 *Purpose of the Study*

In a seminal paper discussing the purpose and value of undergraduate math education research, Selden & Selden (1994) say, "Making major changes in curriculum or teaching methods with inadequate knowledge of how students learn is like designing flying machines with little knowledge of aerodynamics" (p. 432) Their work focuses on importance of collegiate mathematics education research, and they emphasize the role of the learner within such work. We cannot, or at the very least should not, advocate for pedagogical and curriculum designs which do not take into account the individual learner. For every mathematical proof, there is a writer and a reader. Effective proof curriculum necessarily requires research in which the learner plays a central role. One common practice to develop effective pedagogical tools is to study learners who have successfully learned a topic and integrate relevant parts of their learning process into the development of curricular materials (Inglis & Alcock, 2012; Styliandes, Sandefur, & Watson, 2016; Weber, 2008). With this practice in mind, this study seeks to add to the existing literature concerning learning as it specifically relates to the Principle of Mathematical Induction by studying graduate students as they work on problems using PMI and by investigating, through semi-structured interviews and Cognitive Task Analysis (See Section 2.1.1), both the solutions and thought processes associated with their work on these problems. By studying students who have been, by certain measures, successful in pursuing mathematical study at a high level, we might gain insight into how those with advanced mathematical skills think about and use mathematical induction. This research provides some results which can be compared to and contrasted with previous novice-participant studies to understand any similarities or differences between how experts and novices think about and work on mathematical induction problems. Furthermore, the findings provide insight into possible pedagogical adjustments to undergraduate curriculum involving PMI.

This study expands on previous work on PMI in three main ways. First, the participants in this study are mathematically advanced with years of proof-writing experience. These participants can be considered experts in their field. For a discussion of experts and novices in the context of knowledge construction and retrieval, see Section 2.1. The choice of expert participants is intended to help isolate mathematical induction as the focus of the research. Several studies (discussed in Section 2.4.2) identify issues with mathematical induction that stem from technical or mathematical issues which have nothing to do with the inductive process itself, but rather gaps in mathematical knowledge (Avital & Libeskind, 1978; Ernest, 1984). Choosing mathematically advanced participants will allow the research to target the proof technique of mathematical induction as the main object of study, with less need to worry that participants will struggle with aspects of content or proof-writing like elementary computations or general proof mechanics. Secondly, this research will explore "nonstandard" examples of PMI. Namely, participants will work on questions which do not primarily involve the verification of algebraic identities or statements involving sums. These are the types of mathematical induction problems most often used in introductory proofs instruction, and research by Styliandes, Sandefur, and Watson (2016) suggests that these types of mathematical induction problems can encourage students to use rote memorization or algorithmic applications of PMI, which are not easily transferable to more difficult problems. By using "nonstandard" mathematical induction problems, this research provides examples of more nuanced applications of mathematical induction and more closely interrogates the participants' understanding of each part of mathematical induction as well as their perceptions of the relationships between these parts. Lastly, in contrast to mathematical induction problems given in introductory proofs courses, participants in this study work on problems in which the base case is not explicitly given to them. This setup provides the opportunity to study the base step of mathematical induction and creates an organic problem-solving situation which more closely mimics the mathematical research process in which the base case may need to be identified.

1.2 Research Questions

This research attempts to illustrate a holistic picture of how graduate students think about PMI and situate it within their overall conceptual understanding of proof and proof technique. The research design focuses on nonstandard problems involving the Principle of Mathematical Induction. The research will be situated in the contexts of problem solving, proof construction, and mathematical discussion. This research is guided and motivated by the following questions:

1. How do experts describe the development of their conceptual understanding of PMI over time?

2. How do experts situate their conceptual understanding of PMI in relation to the notions of proof and proof technique?

3. When viewing a novel problem, how do experts determine whether or not mathematical induction is an appropriate method for proving a statement?

4. What obstacles, if any, do experts face when solving mathematical induction problems in which mathematical induction is not explicitly specified as the technique to use?

5. How do experts explain and define the two primary parts of PMI (the base step and the inductive step) and the perceived relationship, if any, between these two primary parts?

1.3 Epistemological Perspectives

This research is primarily rooted in two fundamental epistemological perspectives which are linked to the theoretical frameworks discussed in Section 1.4. First, many factors motivating this research, as well as the researcher's long-term goals as a collegiate mathematics education researcher are rooted in a postructuralist foundation. Second, several of the primary theoretical perspectives used in this research are inextricably linked to constructivism and constructivist frameworks. Post-structuralism and constructivism both provide unique, though tangent, treatments of the individual and the notion of individual experience. Because these two perspectives strongly inform and influence this research, a brief summary of each of them is included here.

1.3.1 *Post-Structuralism*

Post-Structuralism can be difficult to formally define. This difficulty arises both because poststructuralism, by nature, resists definition, and because it is both rooted in and in opposition to structuralism, from which it gets its name. In his influential text in qualitative research, Crotty (1998) posits that the introduction of the prefix "post" to an epistemological perspective can be interpreted in several distinct ways. In some cases, such as with positivism and post-positivism, the addition of "post" can simply be indicative of a chronological or logical continuation of the initial epistemology with little to no modifications to the core tenants of the perspective. He states that post-positivism "remains in the broad tradition of positivism and retains many of its features" (Crotty, 1998, p. 197). In contrast, Crotty (1998) also highlights post-modernism and modernism, asserting that "post-modernism is a thoroughgoing rejection of modernism and an overturning of the foundation upon which it rests"(p.198). In the case of structrualism and post-structuralism, the latter of these two interpretations of "post" is most appropriate.

Crotty (1998) defines structuralism as "an approach to the study of human culture, centered on the search for constraining patterns or structures which claims that individual phenomena have meaning only by virtue of their relation to other phenomena as elements within a systematic structure" (p. 212). However, these structures were often binary representations of power, and this epistemology privileges established hegemonic structures above the experiences and thoughts of the individual. This is a crucial distinction between structuralism and post-structuralism. While early structuralism viewed social power structures as dichotomous, and an individual as being valuable or relevant only in their role as a piece of the whole, post-structuralism assumes a more spectral approach to power structures and views the individual as important on their own and as being capable of operating independently of the group.

Many of these post-structuralist notions underpin educational research, even if the connection is not always explicitly mentioned. Mathematics education researchers have noted that one difference in qualitative and quantitative research lies in the role and importance of "generalizability." Selden and Selden (1998) note that in math education research "Even very careful observations only suggest, but do not prove, general principles" (p. 432). In this way, Collegiate Mathematics Education (CME) research does not usually seek to situate an individual's cognitive processes or problem solving as being only valuable for how it can generalize to a group of learners or how it can inform pedagogical practice. Instead, it uses more in-depth methods like interviews, longitudinal studies, case studies, etc. to form more comprehensive illustrations of individual learners as valid entities in their own right. Williams (2014) says in the post-structuralist view, "truth becomes a matter of perspective rather than absolute order." This description intimates a link between post-structuralist thought and cognitive models, like APOS Theory, which view learning and cognition as individualized and non-linear.

1.3.2 *Constructivism*

While post-structuralism is primarily concerned with notions of general and individual truths and what knowledge is valued, constructivism is an epistemological perspective which considers the ways in which knowledge is constructed. Crotty (1998) defines constructivism as "the view that all knowledge, and therefore all meaningful reality as such, is contingent upon human practices, being constructed in and out of interaction between human beings and their world." In other words, Constructivism maintains that an individual constructs knowledge in relation to their previous experiences and knowledge. This epistemological viewpoint provides structure for theories, like APOS Theory and Expert/Novice theories, concerned with how new and pre-existing knowledges relate to one another. While constructivism does not argue that there is a single, all-encompassing pattern of learning, it does provide some potential structures within which multiple possible patterns can be observed and cataloged. Some examples of these structures are discussed in the following section.

1.4 Theoretical Frameworks

This research is informed, guided, and influenced by two primary theoretical frameworks, APOS Theory and an Expert Knowledge Framework. APOS Theory, a theoretical framework developed by mathematician Ed Dubinksy, is based on Jean Piaget's theory of reflective abstraction. APOS Theory will be used as the first primary framework in the data analysis of this study. The theory of schema development further expands the APOS framework and will also be used as part of the APOS-based data analysis. In addition to APOS Theory, two existing theoretical frameworks classifying characteristics of expertise will be incorporated into a single categorization framework, which will also be used as part of the data analysis. This framework will be referred to as the 'Expert Knowledge Framework,' since it captures and classifies various characteristics of expert knowledge. An overview of both of these frameworks can be found in the following sections. The existing literature relating to these two frameworks is detailed in Chapter 2, and more information on how these frameworks were used within the data analysis can be found in Chapter 3.

1.4.1 *Piaget's Reflective Abstraction*

Jean Piaget was a Switzerland-born cognitive psychologist whose prodigious research was primarily guided by the question "How does knowledge grow?" (Jean Piaget Society, 2021). While studying the ways in which individuals gain and develop knowledge, Piaget formed his theory of reflective abstraction which can be broken down into two parts- reflection and abstraction. Arnon et. al. (2014) describe reflection as a process of contemplation about content, operations, and concepts, while abstraction is described as reconstruction and reorganization of content and operations. This reconstruction and reorganization results in the operations being modified into content to which new operations can then be applied. There are two characteristics of reflective abstraction which are worth noting in greater detail. First, the restructuring of operations into content is indicative of Piaget's belief that higher level structures can be constructed from low level structures. This notion is crucial for learning theories and pedagogical practices rooted in the idea that new knowledge can be built onto and connected to previous knowledge. Second, Piaget argues that this restructuring is cyclic and repetitive, therefore enabling this process to repeat itself as new knowledge continues to be constructed. These two characteristics of reflective abstraction have been important to those who have used Piaget's work to develop theories used in mathematics education research, like Ed Dubinksy's APOS Theory (Dubinsky, 2000; Dubinsky & McDonald, 2001).

1.4.2 *APOS Theory*

APOS Theory is a theoretical framework based on a refined model of Piaget's reflective abstraction discussed in the previous section. It is a constructivist framework developed by mathematician Ed Dubinsky to study students learning mathematics. This framework is based on the idea that students traverse through four distinct, but related, stages while learning (Arnon et al., 2014). APOS Theory maintains that if a student is able to successfully navigate, sometimes cyclically and non-linearly, through the four stages: Action-Process-Object-Schema, then the student will, to some extent, have a sufficient grasp on a topic. A brief example of each of the four stages follows.

In APOS Theory, an individual is at the action stage when they are able to respond to external stimuli by transforming objects, performing the necessary steps for the transformation. For example, a student at the action stage, when working on mathematical induction problems, might exhibit the ability to use mathematical induction as a formulaic algorithm to verify algebraic identities. The action-stage student may be able to "plug in" the appropriate value for the base case, then show how $P(n)$ leads them algebraically to $P(n+1)$ for the specific algebraic equality given. As the student continues to reflect on the action of verifying a base case or shifting from $P(n)$ to $P(n+1)$ in algebraic uses of mathematical induction and begins to be able to imagine or perform these actions in their head without the use of any external stimuli, we say that the action has been interiorized into a process. A student who is at the process stage of mathematical induction might now be able to imagine verifying a base case for a given statement $P(n)$ in their mind without the need to write their work down. Further, a student at a process stage will be able to think about the inductive implication $P(n) \Rightarrow P(n+1)$ along with the base case P_0 without the visual stimuli of the written algebraic equality grounding and concretizing them. When the individual can think of these Processes of the inductive step and base step as a whole entities to which they could apply actions or other transformations we say that the student exhibits an Object stage of understanding of mathematical induction. Continuing with the mathematical induction example, a student at the object stage might be able to think about and use backwards mathematical induction or double mathematical induction. Here, a reversal (backward mathematical induction) or multiplicative (double mathematical induction) action is applied to the mathematical induction Object. Lastly, when the collection of Actions, Processes, Objects, and other Schemas connected to some initial concept begin to form into a coherent understanding for the individual, this is called a Schema (Dubinsky, 2002). Schemas are uniquely formed based on individual experiences. For instance, once a student has formed an mathematical induction schema, they may be able to think about how this schema connects to other schemas like the Logic Schema, the Function Schema, or the Method-Of-Proof Schema. Schemas may develop alongside or in-tandem with other schemas. For more information on this phenomenon, see Section 1.4.3. Importantly, although APOS Theory is presented in a hierarchical manner, the theory also emphasizes that this cognitive construction is non-linear and often cycles back and forth through the four structures. APOS Theory identifies several mechanisms through which students cycle and progress through the four stages.

The primary mechanisms through which students move through (often non-linearly) the stages in APOS Theory are detailed in the following discussion. The aforementioned transition from action to process is called *interiorization*. Simply, interiorization occurs when a student is able to transition from performing an action to imagining the action without external stimuli. The mechanism through which students can form associations and connections between processes in a particular schema is called *coordination*. Additionally, a process can undergo *reversal* within the student's mind as they continue to reflect on it. Then, a process becomes an object through a mechanism called *encapsulation*. Importantly, students can *de-encapsulate* an object into the process or processes it originated from in order to assimilate new knowledge or to enmesh or combine different objects based on newly acquired information and coordination (Arnon et. al. 2014). An illustration of these stages and processes adapted from the work of Arnon et. al. (2014) is included below.

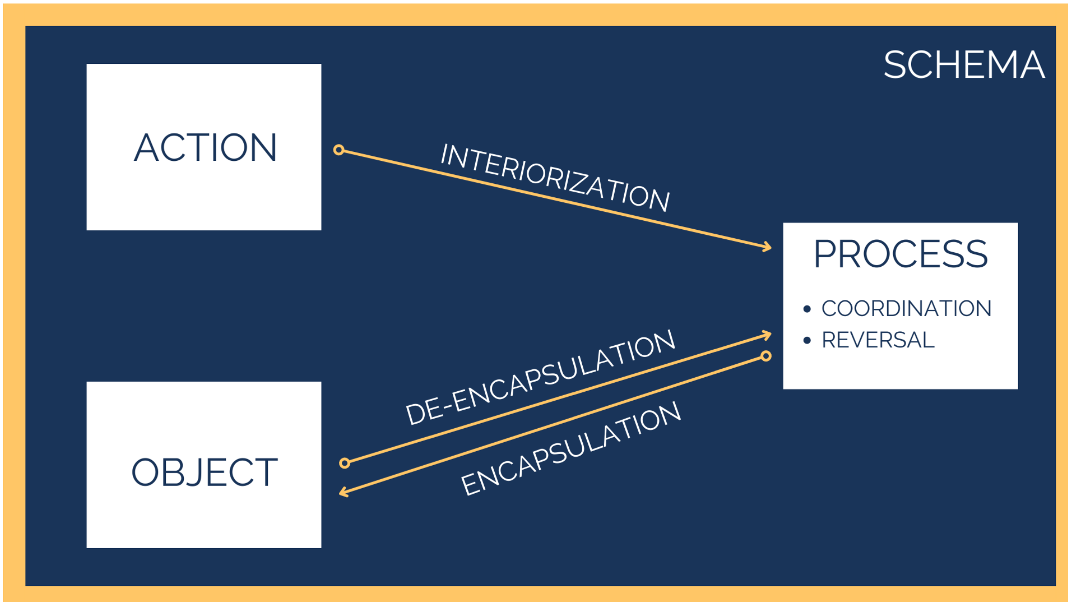


Figure 1.1: APOS Theory illustration adapted from Arnon et. al. (2014)

APOS theory has been used by a host of mathematics education researchers in various branches of mathematics. The most relevant of these works will be covered in Chapter 2. In particular, one important aspect of the APOS research process is the construction of a *genetic decomposition*.

Genetic decompositions operate as rubric-like models for explaining cognitive progression through the APOS cycle. In APOS Theory, they are used as research tools to identify and summarize some perceived necessary conditions students should meet in order to develop sufficient understanding of mathematical concepts (Arnon et al. 2014). It is important to note that genetic decompositions offer a *potential* model of how a concept *may* be constructed in students' minds. The researcher initially creates a preliminary genetic decomposition. This preliminary model is based on the historical development of the concept, known epistemological difficulties, relevant research, the researcher's or researchers' own experiences with the topic, and, occasionally, another author's initial decomposition from a previous study. This initial genetic decomposition is crucial to the APOS research cycle. It informs the creation and implementation of research instruments like surveys, interviews, activities, etc. and further influences the strategies for the analysis of the collected data. The initial genetic decomposition also guides the instructors as they develop curriculum for teaching the concept in question. Often, the research cycle starts with an initial genetic decomposition, followed by the development of teaching material or research instruments and finally, instruction or research using these materials. Afterward, data is collected and analyzed, the genetic decomposition is revised, and the teaching material or research instrument is modified for a repetition of the cycle to start anew.

During data analysis, the preliminary genetic decomposition is evaluated. Researchers check to see if the study's instruments, created based on the initial decomposition, helped students progress through mental constructions suggested by the genetic decomposition. Often, the data indicates that something about student understanding is not completely captured by the preliminary genetic decomposition. Based on the evaluation, the initial genetic decomposition may be revised, refined or enlarged to account for the findings of the study. The final goal of this cyclic evaluation and revision process is a genetic decomposition that closely and accurately describes the cognitive development of the concept in question. The hope is that, while learning and cognition is individual, the genetic decomposition will capture a patterned progression of cognition which is applicable to a large group of individuals. This "final" genetic decomposition can be used to design teaching materials that effectively facilitate student learning in mathematics courses (Arnon et al. 2014). In Sections 2.5.1 and 2.5.2, an initial genetic decomposition of PMI and its revision are presented, respectively.

Because cognitive processes are, by nature, not directly observable, tools like genetic decompositions become necessary to help us create models that link the observable to the unobservable. Research using genetic decompositions accomplishes this by testing for patterns in participants' verbal explanations and physical actions and yielding an appropriate list of cognitive progressions based on these patterns. The fours stages of APOS provide structures and mechanisms into which these genetic decompositions can be grounded. This serves as one way for a researcher to describe and interpret some pieces of learning and mental construction. In particular, throughout the APOS process, a student forms, develops, and refines schemas. One theoretical framework for analyzing and discussing this so-called schema development is discussed in the following section.

1.4.3 *Schema Development and the Triad*

As mentioned in the previous section, knowledge construction is a deeply individualized process. The progression through the stages of APOS theory, detailed in Section 1.4.2, will most likely occur or progress differently for distinct individuals. Schema development is an aspect of APOS Theory that helps describe and explain how students develop and expand their understanding of mathematical concepts. Schemas are complex mental structures, which may consist of a single concept/definition that is applied in various situations or may instead be composed of several interconnected concepts (Arnon et. al, 2014). As an individual's schema develops in their mind, connections are often created between the different components of the schema. Different Actions, Processes, and Objects can be introduced into this existing schema through a process which is often called assimilation. Additionally, multiple schemas can be assimilated into a new schema which uses components from several schemas. For example, the the Implication Schema is involved in the development of the Induction Schema, because the idea of implication is inherently involved in the construction of the inductive step in PMI.

Studying the development of an individual's schema has the potential to aid in pedagogical research in several key ways. First, studying the relationships between concepts within a well-developed schema or connections between different schemas can provide insight for useful ways to structure or organize how we teach these concepts. The study of schema development can also increase our understanding of how individuals use schemas to reason in novel problem-solving situations. Studying how an individual accesses and deploys a schema during problem-solving can illuminate the underlying structure of this schema. In order to provide a framework for naming and categorizing the evolution and progression of an individual's schema, Piaget and García (1989) proposed three stages of this schema development: Intra-, Inter-, and Trans-. These three stages have been adapted from Piaget's work to APOS theory and collectively referred as the triad of schema development, and many authors have conducted studies using this framework (Baker, Clark, et al., 1997; Cooley, & Trigueros, 2000; Cotrill, 1999; McDonald et al., 2000). The triad has been found to be effective as an additional tool of analysis when trying to understand how various schemas interact with one another. In particular, this theoretical orientation is useful in situations where schemas have had the time to become cohesive. As the participants of this study are experts with years of practice reading, writing, and constructing proofs, they theoretically have well-developed schemas which interact with each other in multiple ways. This breakdown of schema development is therefore used in the analysis of the interview data to help classify participants' observable schema organization. Each of the three stages in the triad is briefly outlined below, and applications of this triad in the proposed research will be discussed in Chapter 3.

The initial stage, *Intra-*, is identifiable when Actions, Processes, or Objects within the Schema are viewed as isolated from one another. This stage of development of the Schema can be exemplified by individuals who focus on single, isolated components of a schema. In particular, a student may be able to identify a set of local properties between objects within a specific schema. For example, an individual who can identify similarities in the base cases and inductive implications of a certain class of mathematical induction problems (like routine algebraic verifications for natural numbers) shows evidence of schema development characteristic of the Intra-stage of the mathematical induction schema. An individual at this stage of schema development may also view the various parts of mathematical induction as being isolated. We will refer to this stage as Intra-induction or Intra-PMI for the remainder of the paper.

The next stage, *Inter-*, is characterized when some relationships begin to form between Actions, Processes, or Objects within some Schema. As knowledge continues to develop within the mind of an individual and their schema is refined and expanded, connections between different components of the schema begin to form. These connections indicate that an individual is beginning to operate in the Inter-stage of the triad. For example, when an individual is aware of the connection between the base step and inductive step, they are exhibiting evidence that they are operating in the Inter-stage of the mathematical induction schema. Students at this stage of schema development should also be able to translate the general structure of mathematical induction within a particular problem (i.e. identify what $P(n) \Rightarrow P(n+1)$ means in the context of a given problem. We will refer to this stage of development as Inter-induction or Inter-PMI.

The last stage, *Trans-*, can be characterized by an implicit or explicit coherence and understanding of relationships developed in the Inter- stage. An individual exhibits indicators they are operating in the Trans-stage of development when they can conceptualize the schema as a whole, cohesive unit. Further, the individual is able to determine whether the schema is appropriate for a given scenario. For example, an individual operating in the Trans-induction, or Trans-PMI, stage of development understands the underlying structure of a general argument using PMI, and can determine when the approach is appropriate when given a novel proof-construction problem. An individual at this stage can easily situate the relationships from the Inter-PMI stage within the particular context of the problem.

1.4.4 *Expert Knowledge Classification*

Because this research takes experts as its participants, the data analysis will also take into consideration how expertise may affect and shape the data. This study will categorize participant responses during semi-structured interviews and Cognitive Task Analysis (CTA) using a classification of expert knowledge organization and retrieval. The items in this theoretical framework have been developed by researchers studying expertise. These nine items highlight some of the characteristics classifying aspects of expert knowledge.

1. Pattern Recognition: Experts notice features and meaningful patterns of information that are not noticed by novices.

2. Knowledge Organization: Experts have aquired a great deal of content knowledge that is organized in ways that reflect a deep understanding of their subject matter.

3. Contextual Conditioning: Experts' knowledge cannot be reduced to sets of isolated facts or propositions but, instead, reflects contexts of applicability: that is, the knowledge is "conditionalized" on a set of circumstances.

4. Flexible Retrieval: Experts are able to flexibly retrieve important aspects of their knowledge with little attentional effort.

5. Variable Communication: Experts may or may not be able to teach others effectively, and expertise is not necessarily a good indicator of an individual's ability to communicate their own knowledge.

6. Novel Application: Experts have varying levels of flexibility in their approach to new situations.

7. Mathematical Fluency: Mathematical experts can decode mathematical language and symbols and skim over known mathematical concepts.

8. Comprehension Monitoring: Experts self-check performance, and persist through difficulty when working on tasks.

9. External Exploration: Experts search the relevant text along with outside resources to gain a better understanding of the concept in question.

This classification is adapted from two primary sources which are explored in detail in Section 2.1. A detailed description of this framework's role in the data analysis can be found in Chapter 3.

1.5 Chapter Summary and Outline of the Study

The beginning of this chapter provided a brief explanation of the significance of mathematical proof within the mathematics community at large as well as its importance for students studying mathematics. Secondly, the chapter contained a precursory explanation of some literature regarding the teaching, learning, and comprehension of proofs involving the Principle of Mathematical Induction. Afterward, the chapter situated the current research within the existing body of work, highlighting both the connections to previous research and the ways in which this work seeks to expand the literature. Lastly, this introductory chapter identified both the epistemological perspectives and theoretical frameworks which inform and guide the research, giving a brief description of the primary theories used of interpretation and analysis in the remaining pages.

The remainder of this research report will address the research questions outlined in Section 1.2. Chapter 2 contains a review of literature relevant to mathematical proof, PMI, and the theoretical frameworks outlined in Section 1.4. This chapter is intended to inform the reader of both past and current research adjacent to the main topic of this work as well as relevant background information. Chapter 3 includes a detailed account of methodology, data collection and analysis methods, and study design. Chapter 4 presents the results of the research, and Chapter 5 offers a summary and conclusion of the work along with study limitations and thoughts for future work. Finally, the Appendix includes relevant materials associated with the research project including interview guides and problem solutions.

2 LITERATURE REVIEW

The introductory chapter situated this research proposal within the broader literature around proof and proof using PMI. This chapter will provide more detailed descriptions of the most relevant results in literature on topics associated with the focus of the proposed research. Section 2.1 will outline work done on how experts learn and communicate within their field of expertise, with special attention to studies focusing on mathematical experts. Section 2.2 will provide a description of relevant studies dealing with the broad category of mathematical proof, focusing on work done with proof construction and evaluation, in particular. Section 2.3 will give a comprehensive description of research involving proof using PMI. Lastly, Section 2.4 will cover two works in detail relating APOS Theory and PMI.

2.1 Experts and Novices

As this proposed research focuses on graduate students as its subjects, it is important to examine the literature pertaining to the differences in experts and novices. This study intentionally focuses on graduate students due to the length and breadth of their proof-writing experience as well as their experience with a wide array of mathematical content. This section will discuss general classifications of expert knowledge as well as work done specifically with mathematical experts.

2.1.1 *Expert Knowlege Framework and Cognitive Task Analysis*

What does it mean to be an expert? Numerous studies have explored this topic (Chi, 1978; DeGroot, 1965; Feldon, 2007; Flavell, 1994; Hatano & Inagaki, 2000; Hinsley, et al., 1977; Glaser & Chi, 1988; Robinson & Hayes, 1978; Schneider 1993). Other researchers analyze effective methods for studying expert participants and extracting their knowledge for instructional design purposes (Crandall, Klein, & Hoffman, 2006; Feldon 2007; Feldon & Tofel-Grehl, 2013; McAdams, 2001; McAllister, 1996; Wegner, 2002). Most commonly, studies involving experts find that it is not only what an individual knows that makes them an expert, but also how their knowledge is organized and accessed. Discussion about expert knowledge construction can be situated within the framework proposed in the second chapter of *How People Learn: Brain, Mind, Experience, and School*, an extensive exploration on the science of learning. This framework identifies six key features that are characteristic of expert knowledge, synthesized from several well-known studies on expertise. The six characteristics in the framework are described below.

1. Experts notice features and meaningful patterns of information that are not noticed by novices.

2. Experts have acquired a great deal of content knowledge that is organized in ways that reflect a deep understanding of their subject matter.

3. Experts' knowledge cannot be reduced to sets of isolated facts or propositions but, instead, reflects contexts of applicability: that is, the knowledge is "conditionalized" on a set of circumstances.

4. Experts are able to flexibly retrieve important aspects of their knowledge with little attentional effort.

5. Though experts know their disciplines thoroughly, this does not guarantee that they are able to teach others.

6. Experts have varying levels of flexibility in their approach to new situations (p.19).

These six items were adapted and named to form the initial part of the theoretical framework discussed in Section 1.4.4. These characteristics of expert knowledge are useful to help us identify and classify expert behaviors, which can be useful for the development of pedagogical tools. The work underscores the importance of studying expert knowledge, claiming that "understanding expertise is important because it provides insights to the nature of thinking and problem solving." However, while this framework gives us a way to classify expertise and its characteristics, the methods we use to study expertise and use it to affect pedagogical change must also be carefully considered.

Feldon (2007) navigates the nuances of working with expert participants in a study. In particular, highlights important issues identified by studies on expertise. Research indicates that some individuals, with experts being particularly susceptible, attribute routine or procedural actions to intentional decision making (Wegner, 2002). This attribution can cause them to craft intentionally-reasoned descriptions of behaviors, even if these descriptions are not an accurate explanation for the choices they made. Studies indicate that this phenomena may even cause participants to give such a false explanation regardless of whether or not it is compatible with the reality of the actual events that transpired. This has particularly important implications for experts in an academic setting. Experts, especially academics who operate in an instructional capacity, often explain processes involved in their disciplines to students or colleagues. In these roles, they become accustomed to identifying and describing linkages between successful problem-solving and intentional actions and decisions. This can, and has been shown to, elicit fabrications or exaggerations when self-describing certain choices as calculated or deliberate. Studies show that experts may be even more susceptible to these types of misattributions or hyperboles (McAdams, 2001; McAllister, 1996). After noting these issues, and the research studies which have identified them, Feldon (2007) also identifies strategies to avoid these pitfalls when conducting research with experts.

Feldon (2007) notes that a specific cognitive engineering tool called guided knowledge elicitation shows promise to be an effective instrument for gaining the pedagogical benefits of studying experts while minimizing the impacts of the previously-mentioned issues, and this tool is explored in more detail by Feldon & Tofel-Grehl (2013). Cognitive Task Analysis (CTA), a specific type of guided knowledge elicitation, has been shown to effectively elicit and capture expert knowledge in ways that successfully translate to instructional development. Crandall, Klein, and Hoffman (2006) argue that CTA techniques identify the knowledge and processes experts use while performing complex tasks in their discipline. Feldon & Tofel-

Grehl (2013) perform a meta-analysis on studies using CTA to study expertise. The authors argue this meta-analysis grants "the ability to combine the findings of multiple, independent studies to assess aggregate effects of an independent variable (CTA-based elicitation of instructional content, in this case)" (Feldon & Tofel-Grehl, 2013, p. 294). In this case, the meta-analysis indicates that when CTA is used to elicit expert knowledge, the resulting CTA-based instructional materials are statistically more effective than instructional materials derived from other experimental or study designs (i.e. unguided expert self-report). The authors conclude that the significant statistical effects noted indicate that CTA "offers great value to organizations with human performance needs," giving it the potential to be a useful tool in pedagogical research (Feldon & Tofel-Grehl, 2013, p. 302).

The first part of this section highlights part of a classification framework that will be used during the data analysis of the current study. In addition, this section highlights some common issues arising from studies with expert participants, and identifies CTA as a method which has shown promise for successfully translating research on expertise to instructional materials. The use of this expert framework and CTA in the current study will be detailed in Chapter 3. In particular, this work seeks to study mathematical expert and to use these experts to develop and test new pedagogical tools for mathematics classrooms. Research which specifically involves mathematical experts will therefore also be pertinent to the current work. Several math education research studies have used experts as their subjects. A few of these studies are explored in more detail in the following sections.

2.1.2 *Studies Involving Mathematical Experts*

In addition to the research conducted on expertise in general, there have been several studies conducted that specifically focus on mathematical experts and expertise (Inglis & Alcock 2012; Sella & Cohen-Kadosh, 2018; Shanahan, Shanahan, & Misischia 2011; Shepherd & Sande 2014; Sweller, Mawer, & Ward, 1983; Weber 2008). These studies explore characteristics of expertise that are unique to mathematics and contextualize general results on expertise in mathematically-focused situations. This sectioni will restrict attention to several of these studies that offer results pertinent to the current research. These studies are explored in more detail below.

Shepherd & Sande (2014) studied three undergraduate students, three graduate students and three faculty members to understand how mathematically advanced subjects read for comprehension in mathematical texts and compared their findings to the reading habits of novice readers. The authors conducted two-hour sessions consisting of the participant reading aloud followed by an interview period. The authors identified three main components of reading and proof comprehension in which differences were noted between the undergraduate students and the more advanced participants: Mathematical Fluency, Comprehension Monitoring, and Engagement. First, mathematical fluency consisted of decoding mathematical language and symbols, skimming or not skimming over familiar concepts, and reading verbatim or summarizing. The study found that the more mathematically advanced the reader, the more likely they were to skim and summarize. Second, Comprehension Monitoring consisted of performance checking, time spent on understanding, and willingness to persevere. Experts in the study were more likely to spend more time, perform comprehension checks more frequently, and be more likely to persevere through difficult concepts. Lastly, Engagement consisted of exploring and searching the text and outside resources to gain better understanding of the concept in question. Experts in the study were much more likely to explore and search than their novice counterparts. Although Shepherd & Sande used these three components of analysis to study mathematical reading and proof comprehension, these components can also be applied in the context of proof construction, and they will be used as part of the data analysis framework outlined in Chapter 3. These three items make up the last three parts of the framework discussed in 1.4.4. Together with the six items from the previous section, they form an adapted classification framework which will inform both the study design and the data analysis of the current study. Shepherd and Sande's findings are consistent with other work which has studied the differences in experts and novices in general (Glaser, 2013; Shiffrin & Schneider, 1977; Stehr & Grundmann, 2011). These works most often indicate that identifying differences in novices and experts can be helpful in developing curriculum which effectively teaches novice students to be more successful in problem-solving. Other studies focusing on mathematical experts have identified and analyzed ways in which these experts interact with mathematical proof, specifically.

Inglis & Alcock (2012) conducted a study comparing the proof validation strategies of undergraduate students (novices) and active mathematicians (experts). For a definition of proof validation, refer to Section 2.2.2. The study used eye tracking software to understand proof validation behavior without relying on verbal descriptions of the validation process. While the experts were more consistent than the novices in their ability to accurately validate some of the proofs, the study found some disagreement between experts on the validity of several other proofs presented. However, the key difference discovered by the authors was in the "dwell times" (Inglis & Alcock, 2012, p.371) on different aspects of the proofs. Novices in the study spent significantly more time, proportionally, dwelling on formulas within the proofs. While the actual time spent on formulas was roughly the same for both groups, experts spent proportionally more time dwelling on the non-formulaic portions of the proof (Inglis & Alcock, 2012).

Inglis & Alcock's (2012) study also sought to analyze Weber and Meija-Ramos's (2011) proposed two validation strategies of *zooming in* and *zooming out*. Zooming in is a line-by-line approach which targets the "problematic parts of the proof" (Weber & Meija-Ramos, 2011, p. 340) and zooming out consists of a more holistic approach to proof validation which focuses on the overarching ideas and methods used in the proof rather than individual details. Inglis & Alcock (2012) tried to see if participant eye-movements during their study validated these two proof-validation strategies. The authors tracked the number of saccades, rapid eye movements between two or more fixation points, in relation to line numbers of the proof to determine if the participants seemed to use a zooming-in (linear progression through proof) or a zooming-out (nonlinear progression through proof). The study found that the average number of between-line saccades was significantly larger for experts (78.8 per proof) than for novices (53.3 per proof), suggesting that the mathematicians may employ more zooming out techniques than their undergraduate counterparts (Inglis & Alcock, 2012, p.375).

These findings indicate that most of these between-line saccades, for both experts and novices, were primarily limited to adjacent lines, suggesting that participants were checking the places where logical or mathematical justification was happening between consecutive lines. Further, experts had significantly more saccades than did the novices in the study overall. This evidence supports the idea that experts practice the zooming in technique more often than novices during proof validation (Inglis & Alcock 2012). The works discussed in this section establish that some known differences exist between the ways in which expert and novice mathematicians study and read mathematical works. This suggests the potential for other differences in mathematical thinking, including in proof-based activities. While the current research is not focused on tracking eye-movements, it is concerned with identifying and understanding which aspects of proof experts might focus on. Therefore, some of the findings of this proof validation study proved useful during the data analysis portion of the research. In another proof validation study, Kieth Weber (2008) also studied the proof validation strategies of expert mathematicians.

Weber's 2008 study involved 8 mathematicians, all faculty with Ph.D's at a regional institution. These participants were given 8 purported proofs for number theory statements. Half of these purported proofs were for basic number theory statements, called the "elementary arguments," and half were more sophisticated arguments, called the "advanced arguments" (Weber 2008, p. 436-437). These mathematicians were asked to determine if each of these 8 proofs constituted valid proofs, then asked follow-up questions associated with proof validation. For a definition and discussion of proof validation, see Section 2.1.2. The study identified "225 instances in which a participant read an assertion whose validity could reasonably be judged" (Weber, 2008, p. 438). From these 255, 77 instances were identified in which the participant determined that the assertion with some explicit analyzable comment after being unsure about it's validity at first. 71 of these 77 instances occurred when participants were analyzing the advanced arguments, and these instances served as the primary basis for Weber's findings discussed below.

Weber (2008) identified two primary types of arguments used in these cases to accept a claim as valid. Property-based arguments were instances when a participant validated an assertion using known properties or concepts pertinent to the proof. Example-based arguments were when the participant accepted the validity of a claim solely by examining the statement in the context of carefully chosen examples. Property-based arguments included the construction of subproofs or the construction of informal justifications. Example-based arguments primarily consisted of identification of systemic patterns or utilization of a specific example to construct a generic proof. In a few cases, participants also based their argument on their failure to find a counterexample or on a single carefully chosen example that convinced them of an assertion's validity. Weber notes that several of these forms of arguments that participants used to convince themselves of an assertion's validity would not be acceptable as a formal proof. Weber (2008) says, "mathematicians would not judge an open theorem to be true simply because they were unable to find a counterexample to this theorem, yet the participants in this study would sometimes accept particular assertions within a proof to be true for this reason" (p. 450). He offers two hypotheses for this phenomena. First, it is possible that the mathematicians understood how their example-based inductive reasoning could be generalized, but did not express this in a way that was observable. Secondly, it is possible that the participants were only requiring a high level of confidence in a statement's validity, rather than absolute certainty. In either case, this study offers insight to some important processes involved in the validation of proofs with expert participants and may be indicative of patterns in the way expert mathematicians think about the validity of assertions and mathematical proofs. In particular, Weber's hypotheses for the expert behavior identified by his study also offer potential explanations for the findings of the current study and will be used to help in interpreting some of the results in Chapter 4. Weber's procedure in this study asks participants to read, comprehend, and validate purported proofs. These three tasks represent only a few of the activities which can be associated with mathematical proofs. The following section explores a detailed description of several kinds of activities associated with proof.

2.2 Mathematical Proof

Because proof plays an integral role in mathematics education and research, there are numerous studies primarily focused on mathematical proof (Hanna, 2002; Jones, 2010; Krantz, 2007; Meija-Ramos & Inglis, 2009; Selden & Selden, 2017; Weber, 2005; Weber & Mejia-Ramos, 2014), proof techniques (Antonini, 2003; Baker, 1996; Chamberlain & Vidakovic, 2021; Demiray & Bostan, 2017; Harel, 2001), and student difficulties associated with proof (Dreyfus, 1999; Ernest, 1982; Moore, 1990; Samkoff & Weber, 2015; Selden & Selden, 2011). It is a common pedagogical practice in most traditional advanced mathematics classrooms for mathematics students to learn the concept of proof by reading and studying proofs presented by their instructors during lecture. However, much of the research regarding proofs in mathematics education indicates that most students do not effectively learn proof in this manner (Dreyfus, 1999; Selden & Selden, 2011; Weber & Mejia-Ramos, 2014). This section will begin by furnishing an operational definition for proof and distinguishing between four different aspects of understanding a mathematical proof as outlined by Selden & Selden (2017). Two of these aspects, construction and validation, are relevant to the current work and will be explored in greater detail. Next, a few studies involving known student difficulties associated with mathematical proof will also be detailed. Lastly, since this study focuses on a particular proof technique, some studies analyzing particular proof techniques will also be summarized.

2.2.1 *Definition of Proof and Four Aspects of Proof*

There have been many studies focused on mathematical proof, but the approaches of these works vary greatly. Likewise, there is not necessarily one universal definition of what constitutes a mathematical proof. This research primarily relies on the definition of proof outlined by Stylianides (2007), although the researcher also acknowledges that this definition is certainly not all-encompassing. Here, proof is defined as "a mathematical argument, a connected sequence of assertions for or against a mathematical claim" (Stylianides, 2007, p. 291) which uses statements, methods of argumentation, and communication representations which are generally accepted by the mathematical community. This definition captures two important aspects of proof. First, it recognizes that mathematical proof requires the use of specific tools and methods. These include, but are not limited to, objects like definitions, logical statements, and axioms. Second, the definition acknowledges that a valid proof should adhere to some set of standards widely accepted by most, or all, of the mathematical community. These two requirements provide a solid foundation for a general description of mathematical proof. In addition to understanding what is considered to constitute a valid proof, another common question involves determining when a student exhibits a solid conceptual grasp on mathematical proof. There are many ways in which researchers choose to evaluate whether a student (1) has a sufficient grasp on this definition of proof and (2) is able to create and evaluate proofs adhering to the two characteristics of proof given by Styliandes (2007). In order to provide a framework to help with this type of evaluation, Selden & Selden (2017) classified aspects of proof into four categories: construction, comprehension, evaluation, and validation. This research focuses primarily on proof construction, but also includes elements of proof validation. These two aspects of proof are defined in the following sections, and a summary of some relevant literature is included.

2.2.2 *Proof Construction*

Selden and Selden (2017) define proof construction as an attempt to "construct correct proofs at the level expected of university mathematics students" (p. 1). Construction is perhaps the most well-studied of the four aspects of proof, with the bibliographic study discussed in Section 2.2 estimating that around 77% of studies involving mathematical proof focus on proof construction (Meija-Ramos & Inglis, 2009). Common difficulties with proof construction have been documented frequently within proofs-focused research (Andrew, 2007; D. Baker & Campbell, 2004; J. D. Baker, 1996; Dubinsky, 1986, 1989; Dubinsky & Lewin, 1986; Harel & Sowder, 1998, 2007; Selden & Selden, 2009; Weber, 2005). Some noted difficulties associated with proof construction are explored in Section 2.2.5. In addition to the classification of common obstacles with proof construction, some research also seeks to classify different kinds of successful proof construction.

Weber (2005) describes three distinct approaches to proof construction that "undergraduates successfully use to construct proofs" (p. 353). Namely, he discusses *procedural* proof production, *syntactic* proof production, and *semantic* proof production. Each of these is described below:

- In *procedural* proof production, a student locates a proof of some statement similar to the statement they are proving, and they use this existing proof as a template for their own proof production. In procedural production, the student uses some external source to construct a procedure or algorithm, a "linear set of steps not directly attached to conceptual knowledge," (Weber 2005 p. 353) that can be used to write a new proof for a similar statement.

- In *syntactic* proof production, a student uses some previously known definitions and assumptions and draws inferences or conclusions regarding these statements by using some set of established theorems and logical rules. Specifically, Weber (2005) refers to this type of proof construction as "logically manipulating mathematical statements without referring to intuitive representations of mathematical concepts" (p. 355).

- In *semantic* proof production, a student uses some informal or intuitive examples of a relevant concept to understand the given statement. The student can then use the informal, intuitive representations of this concept to guide their formal line of inquiry.

Examples of each of these in the contexts of mathematical induction are provided below to give the reader a deeper understanding of the differences between the types of proof productions.

A procedural proof production might happen when students are first exposed to routine algebraic verifications for the natural numbers. Students may become accustomed to the algorithmic versions of the base step "plugging in $n = 1$" and inductive step of "an equation involving n and add something to both sides so as to produce a similar equation with $n+1$" (Woodall, 1981, p. 100). Weber (2005) outlines the benefits and restrictions of this type of proof production. Procedural proof productions can often allow students to become proficient at a particular proof technique or at specific types of proofs, for example, the specific type of mathematical induction arguments described above. However, Weber (2005) also states that procedural productions do *not* necessarily aid in helping convince the student that a proof is true, nor does this type of production lend itself to helping the student understand the underlying logic of a proof.

An example of syntactic proof construction might be when a student knows the definition of PMI and is able to construct a series of logical steps and sequences of deductions without using intuition about the underlying mathematical concepts and structures relevant to the proof. Weber (2005) again outlines the uses and limitations of this kind of proof production. Using this method of production, a student can improve their ability to make correct inferences and deductions based on logical rules and the appropriate application of theorems and definitions. The student is also able to see how the theorem or statement they are proving is connected logically to previous theorems or concepts they have learned. However, syntactic productions are not reliant on intuition or informal understandings of a concept, and therefore, they do not allow students to develop their intuition or their ability to craft a meaningful mathematical explanation of a statement's validity, which is the primary learning opportunity provided by the last of the three types of proof productions.

Consider a statement for all graphs of size n. A student using a semantic proof production method for an inductive proof to prove such a statement may begin by thinking of how the statement works for a specific classification of graphs (e.g. complete graphs or bipartite graphs), and then use this informal exploration to inform their formal proof. Semantic proof productions offer several learning opportunities that the previous two productions do not. Semantic production allows students to develop individual representations of more formal mathematical concepts and ideas (or in Tall's language to further develop their concept image). Further, semantic production allows a student to develop a, albeit intuitive, convincing explanation for why the statement is true, allowing for the proof construction to serve the role of convincing them of the truth of the given statement. While each of these proof productions offer students the opportunity to learn different sets of skills while constructing proofs and each has value in classrooms, Weber (2005) argues that "semantic proof productions provide more important learning opportunities than procedural or syntactic proof productions" (p.358). In particular, he argues that the scaffolding typically present in activities lending themselves to the two former proof productions can often limit students' proof construction competence. Many of these limitations can be seen specifically in the context of PMI, as activities given in transition-to-proof courses involving mathematical induction typically privilege procedural and syntactic proof productions. In this research, expert participants work on mathematical induction proofs which are more semantic in nature. This setup highlights some nuanced differences between how experts and novices may view mathematical induction, and Weber's (2005) classification will be used to interpret some of the results of the study in Chapter 4. In addition to constructing proofs, students and research participants may also be asked to present or explain them in a process Selden & Selden (2017) refer to as proof validation.

2.2.3 *Proof Validation*

Proof validation and evaluation are closely linked aspects of proof and can often be difficult to distinguish (Selden & Selden, 2017). Proof validation is generally said to describe the reading of or reflection on proofs. The Inglis & Alcock (2012) study described in Section 2.1.2 gives one example of research focused on proof validation. Several other studies have also concentrated on proof validation (Selden & Selden, 2003; Weber, 2008). Selden & Selden (2003) provide some examples of activities involved in proof validation. They are listed below.

1. Asking and answering questions and assenting to claims.

2. Constructing subproofs.

3. Remembering or finding and interpreting related theorems and definitions.

4. Feelings of rightness or wrongness.

5. Production of a new text- modification, expansion, or contraction of the original argument (p.5).

While mathematicians are often concerned with evaluating or reflecting on the work of others, self-reflection and self-editing are also important parts of the mathematical process. Most of the studies analyzing proof validation and proof validation strategies have involved participants validating proofs found in textbooks or written by other mathematicians. In contrast, the current research will require participants to validate their own proofs during a post-proof-construction, semi-structured interview. These 5 activities will be used to aid in the interpretations of this study's results in Chapter 4. In addition to the 5 activities involved in proof validation above, a study conducted by Kieth Weber in 2008 also discovered another pattern emerging within a proof validation study he conducted on mathematicians. This study was explored in Section 2.1.2. While Selden & Selden's (2017) work gives us an important and useful framework through which to analyze various aspects of proof, other researchers have also focused more closely on studying particular proof techniques.

The preceding sections have highlighted some relevant research regarding proof in general as well as particular aspects of proving. However, despite the copious research addressing proof and proof comprehension, Selden and Selden (2017) argue that "more is known in the research literature about difficulties that often prevent students from proving a theorem than about interventions that would help students' proving" (p. 1). In particular, the authors identify several obstacles to proof construction noted by the various research studies focused on this aspect of proof. These obstacles are listed below:

1. Difficulties interpreting and using mathematical definitions and theorems.
2. Difficulties interpreting the logical structure of a theorem statement one wishes to prove.
3. Difficulties using existential and universal quantifiers.
4. Difficulties handling symbolic notation.
5. Knowing, but not bringing, appropriate information to mind.
6. Knowing which (previous) theorems are important. (Selden & Selden, 2017, p. 3)

Many of these difficulties are related to the tasks of organizing and accessing previous knowledge. This pattern is noteworthy in the contexts of this proposal due to the discussion of expertise in Section 2.2. It is hypothesized that these issues will not be as common with expert participants, due to the general superiority of experts' knowledge arrangement. Since strategies for improving students' abilities to prove and comprehend proof are less prevalent than those studying issues with proof, it is important to consider frameworks about mathematical proof which prioritize the purpose and goals of proof, rather than associated difficulties. Karen Giaquinto's (2005) work offers one such framework and is explored in the following section.

2.3 Mathematical Activity in Research

While many consider the research of mathematicians to be the only type of mathematical activity, Giaquinto (2005) explores an alternate definition of *mathematical activity* and classifies it into four initial categories, each with an associated goal. The associated category and goal pairings are as follows: discovery and knowledge, explanation and understanding, justification and relative certainty, and application and practical benefits (Giaquinto, 2005, p. 75). She further argues that mathematical proof can be used as part of any one of these four categories. Proof is used for discovering, explaining, justifying, and applying mathematical knowledge. Therefore, instead of being a category of activity, it intersects with each category depending on the situation. Within each of the four categories, Giaquinto (2005) describes three activites: *making, presenting,* and *taking in.* For instance, when participating in the justification activity one can either create justification (constructing a proof), present justification (giving a lecture or a conference talk), or take in justification (listening to justification from a talk, lecture, or colleague). Similar examples of these three activities can be given for the remaining three categories. This structured definition of mathematical activity allows us to analyze how activity might be useful in both the classroom and in math education research.

Meija-Ramos & Inglis (2009) use Giaquinto's mathematical activity framework in the context of proof activity. They specify the general activities of making, presenting, and taking in for proof as "constructing a novel argument, presenting an available argument, and reading a given argument" (p. 88), respectively. But these authors further Giaquinto's (2005) work by arguing that one must also consider the goal of the activity. For example, a mathematician who is reading (taking in) a proof by mathematical induction might do so differently if they are (1) analyzing the proof to apply a similar approach to research she is working on (2) grading a student's proof homework (3) preparing to present the proof in a lecture or (4) trying to understand a new proof for the first time. Based on the notion that goals and contexts can produce different behaviors within the *same* proving activity, the authors adapt the framework of proving activities to include the sub-activities illustrated in Figure 4.

The authors were primarily interested in how each of these designated subactivities appeared in math education research. They performed a bibiographic study on one database (ERIC) to analyze the frequency of each subactivity. They found that 82 of 131 articles dealt primarily with Construction, with 44, 16, and 22 of those relating to Exploration, Estimation, and Justification, respectively. 24 of the 131 articles dealt with Reading, with three focused on Comprehension and 21 focused on Evaluation. None of the articles in this study dealt with the Presenting Activities associated with proof. The authors conclude by saying "researchers in the field have tended to concentrate on understanding a relatively small subset of the activities associated with mathematical argumentation and proof" (Meija-Ramos & Inglis, 2009, p. 93). The current proposed research will explore the construction, exploration, and justification categories discussed by the author through the use of semi-structured interviews in which participants construct, justify, and explicate novel proofs with mathematical induction. For a more detailed description of the interview setup, see Section 3.2.2. The next section will give a better understanding of the research with mathematical proof that *does* currently exist in the literature to help quantify and explicate some of the gaps identified by this bibliographic study.

Figure 2.1: Sub-Activities of Proof Adapted from Meija-Ramos and Inglis (2009)

2.4 Principle of Mathematical Induction

While this study is also concerned with some facets of general proof, it is primarily centered on understanding how experts work on proofs specifically using the Principle of Mathematical Induction. In order to situate the current discussion of mathematical induction within the broader context of inductive reasoning, this section begins by providing an outline of existing literature specifically pertaining to PMI. First, the section gives a historical overview of the development of the technique. Next, common epsitemological obstacles associated with mathematical induction are explored in detail. Lastly, some relevant studies analyzing the use of PMI in Proof Construction Activities are summarized.

2.4.1 *Historical Overview*

While the origins of proof itself have been widely studied (Thurston, 1994; Krantz, 2007; Bramlett & Drake, 2013), the historical development of various proof techniques: direct proof, proof by contrapositive, proof by mathematical induction etc. have been somewhat less researched. Many researchers who study the historical development of mathematical concepts and tools highlight the tendency of mathematical knowledge and technique to develop in non-linear and geographically disparate progressions. Since the conception and historical development of mathematical proof has also adhered to this scattered development pattern, it can be difficult to cleanly chronologize the formation of the rigorous form of mathematical proof which exists today. Since the development of particular proof techniques are inextricably linked to the development of proof in general, it is even more difficult to formulate a neat, straightforward description of the historical evolution of these techniques. Instead, it is often the case that the genesis and history of such concepts is described by fluid patterns in mathematical research and practice rather than rigid timelines of specific historical events. One example of this phenomenon can be found in research regarding the development of mathematical induction. There is no easily identifiable point of origin for the concept of mathematical induction. Instead, we can find many examples of both implicit and explicit uses of this strategy in the work of many early mathematicians, then trace the more rigorous and formal development of the concept as the discipline of mathematics was constructed, developed, and rigorized.

There are several authors who document the genesis and historical development of mathematical induction (Bather, 1994; Bussey, 1917; Burton, 1991; Coughlin and Kerwin, 1985; Ernest, 1982; Movshovitz-Hadar, 1993; Rabinovitch, 1970; Weil, 1983). The earliest examples we have of mathematical induction, from mathematicians including Maurolycus, Pascal, and Fermat, did not use the familiar axiomatic structure of mathematical induction (developed by Peano), but rather looser versions of inductive reasoning. These early examples would later serve as foundational for the axiomization that resulted in the formal definition of mathematical induction we use today. Many scholars, including Bussey (1917), claim that PMI was first used in 1575 by Maurolycus in his book Arithmeticorum Libri Duo, where Maurolycus uses PMI as a method to prove that the sum of the first n odd integers is equal to n^2.

Almost 100 years later, Pascal wrote proofs with PMI using what we now call Pascal's Triangle. One of the first record of Pascal using an inductively-structured argument came as the result of a gambling problem presented to him. This problem, known as "The Problem of Points," is the focus of a 1985 paper published by Coughlin and Kerwin. According to the authors, the problem was presented to Pascal by a prolific gambler named Chevelier de Méré. It can be simplified as follows: Assume there are two players of equal skill gambling in a game in which the winner is the first player to win four games. Now, if the game is interrupted before someone has won, how should the money be split? In summary, Pascal sought to determine how the money might be fairly allocated in an interrupted game based on the known number of wins for each player at the time of interruption. Some work had been done on this problem by other mathematicians, but Pascal wanted to eliminate some of the recursion in the solution, which he did by using mathematical induction and his triangle. He wrote letters to Fermat regarding this problem, and these letters are considered by many to be "fundamental to the development of modern concepts of probability" as well as one of the "earliest examples of the use of mathematical induction" (Coughlin & Kerwin, 1985, p. 376).

Fermat is well-know for his prodigious work as a mathematician, and he has contributed crucial results in various branches of mathematics, but he is perhaps best known for his work in Number Theory. In addition to his correspondence with Pascal, Fermat contributed many other examples of early mathematical induction. In his text *Number Theory: An Approach through History*, André Weil details some of Fermat's work with mathematical induction. In particular, he explores Fermat's frustration with an inductive-type argument used by John Wallis, the mathematician credited with the development of infinetesimal calculus (Weil, 1983, p. 49). In his seminal work, *Arithmetica Infinitorum*, published in 1656, Wallis repeatedly uses an incomplete form of mathematical induction to prove statements. Namely, he claims that a general statement $P(n)$ is true for all natural numbers by proving the statement for some finite list of values. Fermat's (translated) critique of this work follows.

> One might use this method if the proof of some proposition were deeply concealed and if, before looking for it, one wished first to convince oneself more or less of its truth; but one should place only limited confidence in it and apply proper caution. Indeed, one could propose such a statement, and seek to verify it in such a way, that it would be valid in several special cases but nonetheless false and not universally true, so that one has to be most circumspect in using it; no doubt it can still be o f value if applied prudently, but it cannot serve to lay the foundations for some branch of science, as Mr. Wallis seeks to do, since for such a purpose nothing short of a demonstration is admissible. (Weil, 1983, p. 50)

Fermat's own use of mathematical induction, although not yet axiomated, was much more rigorous than Wallis'. Perhaps the most famous example is in a small case of his famous "Last Theorem." It is well-known that around 1637, Fermat wrote a claim in the margins of a copy of Diophantus' *Arithmetica* that there are no positive integers x, y, and z such that $x^n + y^n = z^n$ for $n > 2$. Fermat produced a proof for this statement in the case where $n = 4$ with used a sophisticated and geometric inductive argument (Weil, 1983, p. 88). Mathematical Induction showed up to some extent in the work of other mathematicians around this time, but it began to truly cohere in the 19th century.

The term " Mathematical Induction" first appeared in work by DeMorgan, a mathematician well-known for his work in logic (Burton, 1991, p. 422). Then, in 1888, Dedekind proposed a complete system of axioms for arithmetic, finally formalizing PMI in a more solid manner. Peano was simultaneously working on a similar set of axioms, and we now refer to them as the Peano postulates. Axiom V of these postulates is most closely associated with the formal definition of mathematical induction used today. After the introduction of these postulates, mathematical induction was axiomized and formalized into a technique closely resembling the one we use today. PMI is the fifth of Peano's postulates involving the foundation of natural numbers, and PMI is equivalent to the fact that any (non-empty) subset of $\mathbb{N}$ has a minimum element (Movshovitz-Hadar, 1993). Mathematical Induction became a core proof technique, and is now integral to any study of mathematical proving. As such, there are various difficulties associated with the teaching and learning of this topic. A collection of studies concerned with these difficulties are outlined in the following section.

2.4.2 *Epistemological Obstacles with PMI*

In a 1984 article, Paul Ernest identified "unresolved problems concerning the teaching of mathematical induction which should benefit from a careful analysis." Since then, many researchers have studied the obstacles associated with mathematical induction (Avital & Libeskind, 1978; Baker, 1996; Doyle & Núñez, 2021; Dubinsky, 1986; Harel, 2002; Lane, 2007; Movshovitz-Hadar, 1993; Nardi & Iannone, 2003; Ron & Dreyfus, 2004; Stylianides, Stylianides, & Philippou, 2007). While some of these articles address pedagogical practices pertaining strictly to the natural numbers, many of them seek to address particular epistemological obstacles associated with PMI. Before exploring these obstacles, Ernest (1984) first formulates a list of the skills necessary for a student to be able to write a proof by mathematical induction. It should be noted that he specifically focuses on the skills necessary to use the method of PMI in routine algebraic problems. His three "necessary behavioral skills" are listed below

1. The ability to prove the basis of the mathematical induction. This consists of the ability to verify that fixed numerical properties hold for particular numbers. Under the restrictive conditions considered, this depends on the ability to perform substitution into algebraic expressions in a single variable.

2. The ability to prove the mathematical induction step. This depends on the ability to prove an implication statement by deducing a conclusion from a hypothesis. Under the restrictive conditions considered, this consists of the ability to make deductions from algebraic identities, which in turn depends on the ability to manipulate algebraic expressions and identities.

3. The ability to present a proof by mathematical induction in the correct form. This is manifested in the ability to communicate the knowledge of the correct form of a proof by mathematical induction in some way - be it verbal, written or diagrammatic (Ernest, 1984, p. 176-177).

Ernest's discussion of these necessary skills presents one way of identifying underlying cognitive processes necessary for understanding a complex concept like mathematical induction. Using this discussion, Ernest identifies six key misconceptions associated with mathematical induction. These are listed and described below.

1. Ambiguity in the term "induction": While inductive reasoning is a "heuristic method for arriving at a conjectured generality describing a finite sequence of examples," the Principle of Mathematical Induction is rigorous and deductive in nature. This distinction can cause confusion for students when the difference between these two uses of the term "induction" are not clarified.

2. Misconceptions about the legitimacy of the inductive step: Many students have difficulties with the assumption of the inductive hypothesis $P(n)$, which is used as part of a complex proving process to show $P(n)$ for all $n \in \mathbb{N}$. In other deductive proofs, this would certainly be illegitimate, so students are reasonably confused. However, this confusion is indicative that a student has a limited understanding of the implication and of the underlying structure of PMI.

3. The use of quantifiers: Students struggle with the use of quantifiers, and PMI's reliance on universally quantified variables can cause even greater confusion for many students. The use of such quantifiers is "subtle and abstract" and must be practiced.

4. Components of PMI as being inessential: This most often applies to the base case. Students may often underestimate the importance and logical necessity of the base case of mathematical induction.

5. Linkages to routine versions of PMI: Students are often unable to generalize the method of PMI to examples that differ from the routine algebraic verifications they see associated with PMI in introductory proofs courses.

6. The purpose and use of mathematical induction: Students struggle to understand the usefulness and necessity of PMI. PMI is, in many ways, unlike other principles they may have been previously exposed to. In particular, "mathematical induction is neither self evident nor a generalisation of previous more elementary experience." Therefore, students may struggle with the basis and justification for PMI (Ernest, 1984, p. 181-183).

A few of these difficulties were explored prior to Ernest's (1984) work in greater detail and are discussed below.

Authors Avital and Libeskind (1978) highlight several obstacles students face when learning mathematical induction. The authors categorize these epistemological obstacles into three categories: conceptual, mathematical, and technical (Avital & Libeskind 1978). The authors identify two conceptual difficulties, (1) The Implication $P(k) \Rightarrow P(k+1)$ and (2) The Transition from k to $k+1$ (Avital & Libeskind 1978). They found that advanced students were asking questions like "How can you establish the truth of $P(k+1)$ if you don't even know that $P(k)$ is true?" The authors argue that this difficulty arises from a gap in knowledge regarding the logic of implications. Namely, proving $p \Rightarrow q$ does not inherently show anything about the truth value of p itself. The second conceptual obstacles relates to student's struggles with the cognitive jump from starting with a base case then transitioning from $P(k)$ to $P(k+1)$. As a solution, the authors recommend asking students to first perform some inductive "naive" calculations with actual numbers, then slowly transitioning to more general inductive proofs (Avital & Libeskind 1978).

The second class of difficulties is comprised of what the authors call mathematical difficulties. Again, the authors classify two particular difficulties in this category (1) Underestimating the Importance of the Base Case and (2) Difficulty with Step Size Greater than 1.

For (1), the authors highlight the following (incorrect) statement $P(n)$:

$$\sum_{i=1}^{n} n = \frac{n(n+1)}{2} + 1.$$

For this statement, $P(k) \Rightarrow P(k+1)$ for all $k \geq 1$. However, it is easily seen that this statement does not hold even in the base case when $n = 1$. Many students misunderstand the crucial requirement that $P(1)$ (or $P(m)$) *must* be true in order for the Principle of Mathematical Induction to work. The second mathematical obstacle occurs when students are asked to inductively prove some statement in cases when the "step" size between cases is larger that one. For example, if a statement applies to the even integers, a student would need to show that $P(k) \Rightarrow P(k+2)$ (or equivalently that $P(2k) \Rightarrow P(2(k+1))$. The authors suggest that using appropriate substitution can reduce these cases to the original formulation of mathematical induction, eliminating this particular epistemological difficulty.

Finally, the authors classify two final obstacles as technical problems. These two "technical issues" are (1) Determining what $P(k) \Rightarrow P(k+1)$ is for a given $P(n)$ and (2) Student Ability to Perform the necessary algebra for the $P(k) \Rightarrow P(k+1)$ implication. In particular, the statements $P(k)$ and $P(k+1)$ can sometimes be difficult for students to find. They give the following as an example of this phenomenon:

> A class of 22 high school students with above average in ability were asked to prove by the Principle of Mathematical Induction that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. A vast majority of these students wrote out their proofs of the stage $P(k) \Rightarrow P(k+1)$ in the following way:

$$\frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+2)}{2}$$

$$n^2 + n + 2n + 2 = n^2 + n + 2n + 2$$

$$2 = 2 \text{ (p. 435)}$$

This demonstrates the student's lack of understanding of the underlying logical statement, as they give no indication of assumption or conclusion. Dubinsky refers to this as "difficulty formulating the hypothesis in the induction step" (1989). The second technical difficulty the authors explore is common issues with necessary algebra in the inductive step. Here, they offer several examples, one of which is given below:

> Prove that $(a-b)|(a^n - b^n)$ for all n.
>
> "To prove statements like the last two many textbooks and teachers apply an approach which involves adding and subtracting an appropriate expression" (p. 436). For instance, part of the step $P(k)$ to $P(k+1)$ in a possible proof of the problem above is as follows:

$$\begin{aligned} a^{k+1} - b^{k+1} &= a \cdot a^k - b \cdot b^k \\ &= a \cdot a^k - b \cdot a^k + b \cdot a^k - b \cdot b^k \\ &= (a-b)a^k + b(a^k - b^k) \end{aligned}$$

The authors argue that many algebraic steps can be seen by many students as tricks which "they find difficult to apply in similar problems" (Avital & Libeskind 1978, p. 436). This work introduces the tendency of students to overlook or misunderstand the importance of various parts of the mathematical induction process.

Nitsa Movshovitz-Hadar (1993) gives concrete pedagogical strategies for addressing some of the pedagogical issues discussed in the two previous studies. She categorizes these strategies according to the two steps of mathematical induction which she refers to as the "checking step" (base case) and the "transition step" (inductive step). The author advocates for convincing students of the importance of *both* steps of the mathematical induction process. In particular, she advises instructors to give examples where one holds, but the overall statement is not true for all n. One such task where the base case is not true is shown below:

> Task No. 7: Is checking for $n = 1$ always simple?

1. Is it true for all n that:

$$\sum_{i=2}^{n} \frac{1}{(i-1)(i)} = \frac{3}{2} - \frac{1}{n}$$

2. Check for $n = 6$.

3. Comment on your results.

The author notes that for this problem, students must first recognize that the base case is $n = 2$ (not $n = 1$). Further, if the students fail to check the base case, they will be able to "prove" the statement even though it is not true for $n = 2$. Similarly another of the authors "tasks" demonstrates the importance that the transition step must be true for every k:

Task No. 8: What if the transition from $n = 1$ to $n = 2$ fails?

1. Try to prove by applying the principle of mathematical induction that: For all n, If the maximum of two positive integers is n, then the two integers are equal.

2. Suppose you proved part 1 successfully, show that it implies that all positive integers are equal.

3. Comment on your results.

Here, the author highlights how the transition step between $n = 1$ and $n = 2$ fails and why this causes mathematical induction itself to fail, even though a "convincing" proof might make a student think otherwise. Movshovitz-Hadar argues that these types of activities and tasks can reinforce students understanding of mathematical induction, remind students of the importance of both steps of the inductive process, and address several of the common epistemological obstacles associated with mathematical induction. The next section deals with literature exploring PMI proof construction associated with more semantic proof production.

It is important to note that the primary difficulties identified by the three studies discussed above are tightly linked to the typical mathematical induction problems associated with mathematical induction in transition-to-proof courses. In particular, these types of problems often encourage *procedural* (routine algebraic equality verification or mathematical induction proofs involving finite series) or *semantic* (routine checks using the definition of PMI and relevant theorems) proof construction (See Section 2.3). These routine types of PMI problems have been shown to cause problematic associations for students and can also make it difficult for researchers to accurately isolate epistemological difficulties with PMI.

2.4.3 *General Research with PMI*

In addition to the issues and obstacles involved in the teaching and learning of mathematical mathematical induction discussed in the previous section, several other studies have focused on the use of PMI as a proving technique. Smith (2006) found that some students did not view mathematical induction as explanatory, but "as an algorithm they can apply almost blindly" (pp. 80–81). The work reviewed in the previous two sections suggest that one potential explanation for this belief is the focus on more procedural and syntactic proof production in transition-to-proof courses' treatment of PMI. Many authors have analyzed the explanatory potential of mathematical induction proofs (Hoeltje, Schneider, and Steinberg, 2013; Lange, 2009; Smith, 2006; Styliandes, Sandefur, & Watson, 2016). To further explore this issue, Styliandes, Sandefur, & Watson (2016) analyze mathematical induction in the context of what they refer to as "explanatory proving" or "proving activity that is explanatory for provers" (p. 21). The authors argue that for a proving activity to be explanatory, the prover or provers must receive some level of insight as to *why* the statement is true. This notion of explanatory proving is consistent with Weber's (2005) semantic proof production detailed in Section 2.3. The authors offer an example of explanatory proving in the contexts of PMI and Proof Construction.

Provers could use *recursive reasoning* (that is, reasoning relating to or involv-

> ing the repeated application of a rule or procedure to successive results) in their exploration of a mathematical statement in ways that could help provers see informally the structure of the inductive step in a possible proof by mathematical induction; the provers could subsequently apply mathematical induction to formalize their thinking and verify the truth of the statement. (Styliandes, Sandefur, & Watson, 2016, p. 21).

The authors note that mathematical induction is an appropriate context through which to explore explanatory proving as students often view mathematical induction as verification rather than explanation. While many works focus on the explanatory value of a written proof, the authors argue that experts instead identify explanatory value in how proof "provides new insights into the field of application, new ways of reasoning about particular objects, or new connections between fields of study" (Styliandes, Sandefur, & Watson, 2016, p. 22).

In order to frame their discussion of explanatory proving in the context of mathematical induction, Styliandes, Sandefur, & Watson (2016) identify and describe two perspectives involving the function of proof. The first perspective, called the *subjective perspective*, maintains that the purpose or function of proof is based on how it serves the prover or the reader. The second, called the *absolutist perspective*, considers the function of proof as characteristics of the actual text of the proof (Styliandes, Sandefur, & Watson, 2016). In summary, a subjective perspective focuses on the *proving* and an absolutist perspective focuses on the *proof*. The authors primarily depend on the prior perspective for this proposal, as they seek to explore students' process of proving using mathematical induction. An important characteristic of this research which distinguishes it from previous work on mathematical induction is the way in which statements were worded. Rather than an explicit statement to prove "for all n," the authors described their activity construction below.

> The problems we used in our study were not implicitly recursive: they were posed as "make a conjecture about the conditions under which a statement $P(n)$ is true or false," where it would likely appear to the student as if the statement

> could be true for one n-value but not the next. We consider statements in this latter form as particularly rich as they are less obviously conducive to a proof by mathematical induction – the statement to be proved is not given in the problem. The students in our study therefore had to undertake some justified mathematical exploration to arrive at a statement they needed to prove (Styliandes, Sandefur, & Watson, 2016, p. 23).

This is important to note, as the current study will also follow a similar presentation model for the mathematical induction activities given to participants.

The authors note a pattern in how expert mathematicians often approach a proving exercise. This pattern is summarized below.

1. Attempt to identify a reasonable method or technique to prove the statement. If one can be identified, they may use the technique without necessarily thinking about *why* the statement is true.

2. If no method can be immediately or easily identified, then the expert may try to experiment with some examples to gain insight to possible proving strategies.

3. Use discoveries made in the previous step to inform the formalization of an argument.

The authors wanted to compare the behavior of their undergraduate participants with this expert pattern. They hypothesized that students working on problems that do not offer explicit instruction or scaffolding directing them toward a particular proof technique or method would also likely explore examples to gain insight. They further predicted that these explorations could aid students in constructing an informal inductive step. Such exploration could potentially help students view mathematical induction as explanatory rather than just a tool for verification (Styliandes, Sandefur, & Watson, 2016). The authors had trios of students work on problems related to mathematical induction without specifying mathematical induction as a preferred technique. Their findings supported their hypothesis, and they claim that the use of problems that do not explicitly ask students to use mathematical induction have the "explanatory power for them illustrates the success students can have when given an appropriately phrased problem" (Styliandes, Sandefur, & Watson, 2016, p.33). The work discussed in this section has primarily been focused on novice participants, which gives a well-developed body of literature to which I will compare the findings from my study with expert participants. While the research from this section provides some suggestions on the explanatory potential of PMI, it is also important that we are able to classify how this explanatory potential relates to the multiple associated pieces of cognition in PMI. Therefore, the following section will explore how APOS Theory has decomposed the process of mathematical induction.

2.5 Preliminary Genetic Decomposition of PMI

This section primarily focuses on the work of two pairs of researchers who have developed genetic decompositions of mathematical induction and situated PMI within the APOS Theory framework. The first section presents a genetic decomposition created by Dubinsky & Lewin (1986, 1989). The second section covers a study which uses, then refines, this original genetic decompositions. The two studies outlined below will serve as an integral part of the methodology of the current proposal as discussed in Chapter 3. For a complete explanation of genetic decompositions, please refer to Section 1.4.2.

2.5.1 *Genetic Decomposition of PMI*

Dubinsky and Lewin (1986,1989) conducted research with university students to better understand their difficulties using mathematical induction in proofs. The authors provide the following three steps for teaching a mathematical concept (like mathematical induction):

1. determination of a genetic decomposition of the concept;

2. helping students to perform the required reflexive abstractions; and

3. explanation and practice

They constructed a preliminary genetic decomposition of mathematical induction which is briefly summarized below.

1. Expand the schema of functions to include a function mapping the natural numbers to a propostion-valued output $f : N \to P(N)$.

2. Encapsulate logic into the implication $p \Rightarrow k$. The implication cognitively becomes an object which be the value of the function f.

3. Create the schema of the implication-valued function g where $g(\mathbb{N}) = (P(N) \Rightarrow P(N+1))$.

4. Interiorize the action of logical necessity into a process so that inputs P_0 and $P_A \Rightarrow P_B$ allow one to conclude P_B.

5. Coordinate the function g from Step 3 with Modus Ponens beginning with $P(a)$ for some base case a.

6. Encapsulate this inductive process into an object be connected to the Method of Proof schema so induction can be applied as a proof method.

7. Generalize actions on the induction object within various problem types coordinated with the Method-of-Proof schema until students can apply induction as a proof technique.

Dubinksy (1986, 1989) conducted two studies to evaluate this genetic decomposition. He created a teaching experiment in a Finite Mathematics course in which he used SETL, "a very high level procedural programming language with standard constructs of assignments and procedures..." (Dubinsky, 1986, p. 308). In addition to the typical course material, some activities within the course were specifically designed to test the above genetic decomposition for PMI. SETL was used within the course to help students learn and understand the mathematical syntax associated with mathematical induction. SETL was also used within the teaching experiment to help students apply actions to specific concepts including mathematical induction.

The author evaluated the genetic decomposition by determining student success in two goals: (1) describing and discussing the process of mathematical induction and (2) setting up proper arguments and producing correct proofs using mathematical induction (Dubinsky, 1986). The experiment was assesed with take-home proof activities and short individual interviews focused on mathematical induction. Dubinsky discovered that each of the schemas identified in the preliminary genetic decomposition materialized within these short interviews. The appearance of these predicted schema suggests that this genetic decomposition could be successful in the study of mathematical induction. This genetic decomposition, along with the refined version detailed in the following section, will inform the study design as outlined in Chapter 3.

2.5.2 *The Base Case and APOS*

While Dubinsky's work with mathematical induction provides a preliminary genetic decomposition for mathematical induction, Garcia-Martinez and Parraguez note that this initial research did not include the base step as a part of the analysis. This study was conducted as an extension of Dubinsky and Lewin's work detailed in the previous section. While Dubinsky and Lewin's article highlight many of the aspects of learning PMI, Garcia-Martinez and Parraguez (2017) note that the research did not include the base step as a central part of the analysis. Seeking to bridge this gap in the research with their article, the authors created a study analyzing mathematical induction with four university student participants in order to assess the formulation of the base step of mathematical induction as a mental process. Seeking to specifically address the base step, Garcia-Martinez and Parraguez (2017) analyzed mathematical induction proofs from four university students to assess the formulation of the base step of mathematical induction as a mental process. The authors refined Dubinsky's genetic decomposition, adding the base step as a process on its own. An illustration of the decomposition is pictured below.

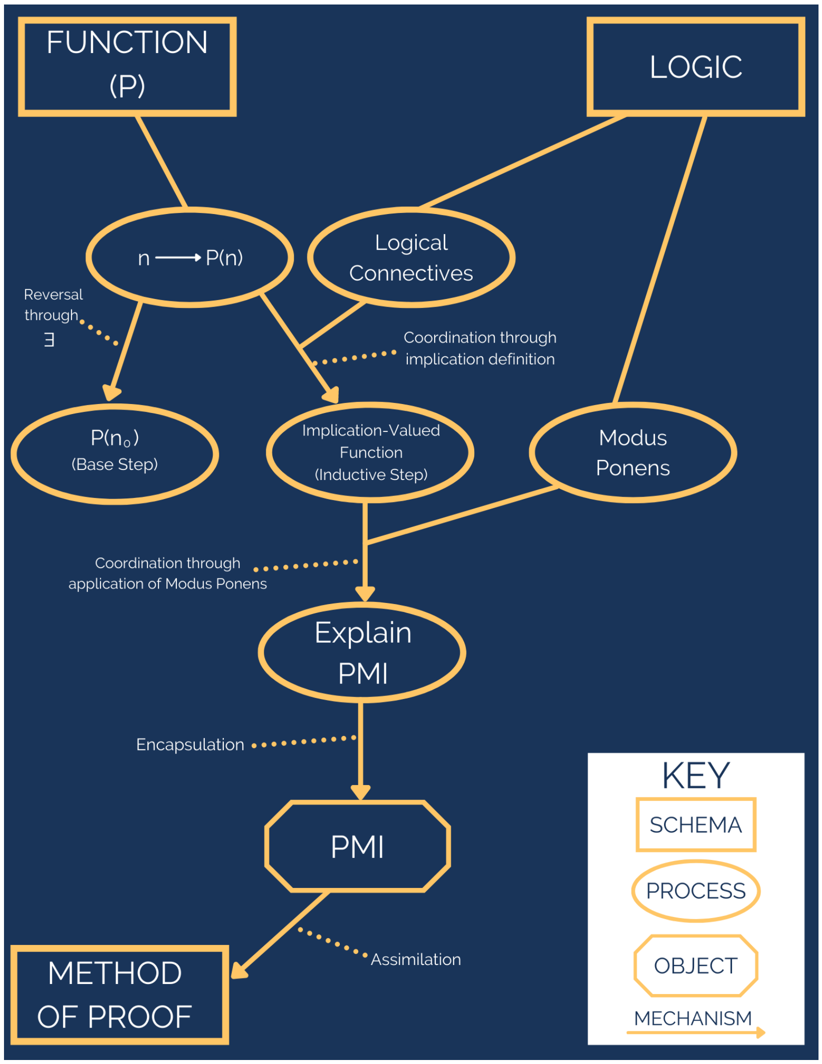


Figure 2.2: Genetic Decomposition Illustration adapted from Garcia-Martinez & Parraguez (2017)

The authors conducted interviews with the four participants to assess their mental constructions of the base step. They summarize their findings in the following three statements:

- Process of logical connectives: The student in Case 2 shows this mental construction, by omitting the use of the truth table for logical connectives to answer question 1.

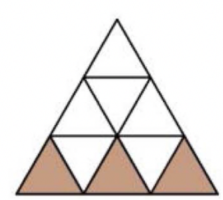
- Process of $n \Rightarrow P(n)$: This construction is seen explicitly. In fact, in answering questions 2 and 3 the student in Case 2 shows that to achieve this construction the student must carry out the action of associating a number with a truth value of the proposition associated with it.

- Process of the Basis Step: Shown only by the student in Case 2, when showing a counterexample and confirming the importance of this step in the construction of explain PMI.

The authors use this adapted genetic decomposition to analyze students through activity and interview. In particular, the addition of the base step as a process allows them to specifically evaluate and describe the cognitive constructions associated with the base step of mathematical induction. The authors then designed interviews based on this genetic decomposition to be used in conjuction with assigned proofs activities involving mathematical induction. The findings of this study highlight the important ways in which the APOS Theory can help pedagogical researchers "determine the constructions that underlie the difficulties students have and the strategies they use when carrying out a mathematics activity with natural numbers or their isomorphic equivalent."

This research is also an important example of an analysis of PMI which does not use routine examples of PMI. The study uses nonstandard, geometric mathematical induction problems. One of the problems from the study is pictured below.

Consider a triangle as in the figure made up of equilateral triangles. In this case, there are 3 triangles along the base in in total there are 3^2 triangles. This property is true for any number of triangles along the base, i.e. if there are n triangles along the base, then there are n^2 triangles in total.



How would you prove this property? And why?
Use your chosen method to prove it.

Figure 2.3: Triangle Induction Problem from Garcia-Martinez & Parraguez (2017)

Although this specific problem is not used in the current study, similar nonstandard mathematical induction problems are used for the CTA. Further, the interview protocol asks participants to identify what method of proof they would use for given problems as well as asking probing questions to understand the reasoning behind these choices, similarly to the study design above. In these ways, the study by Garcia-Martinez and Parraguez (2017) greatly informs the study design discussed in Chapter 3. In addition to these two genetic decomposition studies, this proposed research will also use the ideas of schema development as discussed and defined in Section 1.4.3. A more specific discussion of these ideas in the context of PMI is discussed in the next section.

2.5.3 *PMI and Schema Development*

Schemata are complex and deeply individualized structures. It is almost certainly true that no two people will have identical schemata for any given topic. Further, as with all cognitive structures, a schema can be hard to observe from the outside. However, through the use of cognitive theories, including APOS Theory, researchers can, over several interconnected studies describe properties of a well-developed schema for a given mathematical concept. Consider an excerpt from Ernest's (1984) work which highlights his attempt to identify the concepts related to and held within a student's PMI Schema.

> The first concept to be separated from mathematical induction is that of *implication*. Both the concept of implication as a binary sentential connective and the concept of the proof of an implication statement are entailed. In addition to the proof of an implication, the concept of an *elementary proof* in general is required for an understanding of the method of MI, since it is a particular method of proof itself... Mathematical induction also presupposes the concept of a *defined property of natural numbers*, for mathematical induction ranges over those numbers which have a fixed property. Defined properties of natural numbers arise from algebraic identities, but they also depend on the concept of a *function*, as

> does the concept of an algebraic identity... Out of the concept of a function arises a particular type of function with direct links to mathematical induction, the *inductively defined function*... The notion of an inductively defined function interrelates with another concept which is a direct contributor to mathematical induction, namely the concept of *recurrence*... The concept of recurrence can be built on the notions of *iteration and flow diagram*, which also aid the development of the concept of inductively defined function. Finally, the concepts of flow diagram, iteration and inductively defined function all arise from the *ordering of the natural numbers* which is one of the major contributors to mathematical induction (Ernest, 1984, p. 179).

This conceptual breakdown by Ernest suggests that the PMI schema involves several other mathematical concepts (e.g. implication, function, iteration). His work gives some possible insight to the basic building blocks students may need to initially construct the PMI-Schema, and it is similar to the genetic decomposition of PMI described in the previous chapter. However, while this type of conceptual breakdown informed the development of the current study's instruments, expert participants in the study already demonstrated a solid grasp on these smaller mathematical concepts associated with induction. This study sought to expand on the exploration of the PMI-Schema using the framework of schema development discussed in Section 1.4.3. In particular, this study identifies characteristics of experts' behaviors in the study associated with each stage in the triad of schema development (i.e. Intra, Inter, and Trans), and classifies the participants in the study according to which of the three stages their behavior during the study most closely represents. The findings of this part of the study can be found in Section 4.1.1.4.

2.6 Chapter Summary

This chapter explored the relevant literature pertaining to the elements of this research proposal. The chapter began with research distinguishing experts from novices. In particular, the section explores how experts construct, expand, store, and retrieve knowledge in their field of expertise and highlighted the value of studying experts. Next, the proposal explored works involving the use of mathematical activity within educational research. The section details the value of using mathematical activities as a method of studying student cognition and learning. The third section presents a few of the many studies concerning the construction of proofs in general. This is followed by a section focusing on the research specifically exploring the Principle of Mathematical Induction including its historical development, its two primary components (Base and Inductive Steps), and the epistemological difficulties associated with PMI. Then, the chapter highlighted the two main research studies which have analyzed PMI within the APOS Theory framework. Both of these studies present genetic decompositions of PMI, upon which much of this study is based. Lastly, PMI was situated within works concerning Schema Development. The studies discussed in this chapter, along with the epistemological perspectives and theoretical frameworks discussed in Chapter 1, form the foundation for the methodological choices of this research proposal.

3 METHODOLOGY

In this chapter, I will discuss the methodological choices of this study and situate them within the broader contexts of qualitative research practices and research involving mathematical proof. As previously mentioned, cognitive processes and subjective constructs like "understanding" cannot be entirely observed directly. Instead, research dealing with such subjects must rely on observable actions and theoretical frameworks in tandem when studying such phenomena. As such, this type of exploration usually focuses on *processes* associated with understanding rather than the *products* of understanding. In particular, this research is focused on the *proving* process rather than the *proof* (as discussed in Section 2.4.3). Since this research involves studying a variety of observable sources including dialogue, gestures, and written work, qualitative data collection and analysis methods will be appropriate. Unlike many quantitative methods, qualitative research does not necessarily seek to provide statistical or generalizable results from large samples of people. Instead, qualitative research "emphasizes descriptive data in natural settings and emphasizes understanding the [subject's] point of view" (Bogdan & Biklen, 2007, p. 274). There are three primary elements of qualitative research.

1. Data collection methods (along with the methodology informing these choices)
2. The researcher's epistemology (See Section 1.3)
3. The researcher's theoretical perspective(s) (See Section 1.4)

Items 2 and 3 were discussed in Chapter 1. Methodological choices and the study's methods will be discussed in this chapter, and they will be situated within the contexts of APOS Theory and the Expert theoretical framework discussed in Section 1.4. Section 3.1 will discuss the research setting and participant criteria and selection. Section 3.2 will cover the data collection methods used as well as providing justification for these methodological choices and study design. Section 3.3 will cover the methods of data analysis.

3.1 Research Setting and Participants

This research was primarily conducted in the 2021-2022 academic year. Participants for the study were chosen based on several criteria. All participants were, at the time of interview, graduate students in Ph.D. programs in mathematics. The participants were required to have successfully passed all required qualifying exams for their program of study and to have at least two years of graduate school experience at the time they were interviewed. Participants were taken from two universities in the southeastern United States. An email was sent to all graduate students at both universities informing graduate students about the study. Of the students who responded to the email, six satisfied all selection criteria. Out of these six initial participants who agreed to take part in the study, five of them completed both interviews for the study. Three of the participants came from Institution 1, a large land-grant R1 institution with a student population of approximately 38,000. The remaining two participants were recruited from Institution 2, a large, urban R1 institution with a student body of approximately 32,000.

3.2 Data Collection Methods and Study Design

Data collected for this study included:

- Audio-Video recordings of interviews conducted with each participant. The interviews included proof construction activities as well as semi-structured interviews guided by questions from the protocols in Appendix Items B and C. The interviews are described in detail in Section 3.2.2

- Transcriptions of each recorded interview.

- Copies of any written work created during the interviews.

The audio-video recordings were primarily used to transcribe the interviews, including notations of gestures or other inaudible cues (such as lengthy pauses in dialogue). Audio recordings were used to manually verify that the computer-generated transcriptions from the interviews were accurate. Collected written work consisted of anything written down by the participant during the interviews including pictures, scratch work, and formal proofs or algorithms.

3.2.1 *Cognitive Task Analysis*

Some existing literature about CTA was explored in Section 2.1, and this section also highlighted some of the merits of using CTA in research involving experts. In their textbook on CTA, Shraagen, Chipman, & Shalin (2000) describe CTA as "the extension of traditional task analysis techniques to yield information about the knowledge, thought processes, and goal structures that underlie observable task performance. Some would confine the term exclusively to the methods that focus on the cognitive aspects of tasks, but this seems counterproductive. Overt observable behavior and the covert cognitive functions behind it form an integrated whole" (p. 3). CTA is used to describe a host of qualitative research techniques, but there are some defining characteristics of the method. The basis of CTA is the study of participants as they work on a cognitive task. CTA has most often been used to study expertise. The authors classify the direct observation of subject matter experts (SME's), including audio-visual recordings and careful qualitative coding of these interviews, as one important type of CTA. In particular, the use of think-aloud protocols while SME's work on cognitive tasks has been shown to be an effective method of knowledge elicitation (Schraagen, Chipman, & Shalin, 2000). In order to be most effective, CTA requires both these think-aloud protocols in addition to a well-developed and adaptable set of probing questions rooted in research relevant to the task. This current study has SME's perform the cognitive task of working on two non-standard mathematical induction problems, followed by a semi-structured interview with probing questions developed based on the literature detailed in Chapter 2 of the paper.

3.2.2 *Semi-Structured Interviews*

For each of the five participants who completed all required components in the study, two interviews, Interview 1 and Interview 2, were conducted, with each lasting a little over an hour. Interview 1 consisted of two twenty-minute sessions where the participant worked on a cognitive task associated with PMI, with each followed by a twenty-minute semi-structured interview based on the Interview Protocol in Appendix Item B. During the two problem-solving periods, participants were encouraged to discuss their thought processes out loud and to write down work whenever possible. This allowed the participants' internal cognitive processes to be more easily observed and studied during transcription and analysis. During the post-problem-solving, semi-structured interview portions of Interview 1, the researcher asked participants to discuss their work and solutions from the problem-solving section and asked further probing questions in order to gain a deeper insight to the participants' conceptual understanding. The mathematical induction problems given to participants in Interview 1 are discussed in Section (insert section number here).

During Interview 2, the researcher conducted a semi-structured interview with questions about proof techniques and proof construction in general, as well as several questions focused on the participant's conceptual development of PMI. For the interview protocol and a complete list of the questions and prompts to be used in Interview 2, see Appendix Item C.

The interviews are intentionally arranged in this way so that the participants are not overly predisposed to automatically think of mathematical induction during the problem-solving section in Interview 1. In particular, the work by Styliandes, Sandefur, & Watson (2016) discussed in Section 2.4.3 suggests that it is useful to observe participants when mathematical induction is not explicitly known to be the appropriate method of proof. This allowed the researcher to analyze the participants' reasoning through their choice of proof technique. Each question in the interview protocol was developed according to three primary criteria.

1. The question should be informed by the existing literature on PMI and Proof, particularly the research on epistemological issues with mathematical induction and the

genetic decompositions of PMI.

2. The question should be designed to elicit responses that answer the Research Questions outlined in Section 1.2.

3. Sub-questions for each top-level question should be developed in a manner that anticipates likely student responses so that the elicited knowledge during the interview is maximized.

The questions were also strongly linked to the mathematical induction problems given to the participants as cognitive tasks. These two problems are discussed in the following section.

3.2.3 *Induction Problems*

The participants in this study were given two mathematical induction problems to work on during Interview 1. In order to address the research questions effectively, the problems were chosen and written specifically to satisfy several conditions.

- The problems *can* be solved using mathematical induction, and are worded with phrases typically associated with mathematical induction (e.g. "show for all n").

- The correct base case for the property is not explicitly identified. The participants will be required to determine the base case as part of the proof construction.

- The questions do not involve routine verifications of algebraic equalities or statements involving sums.

- Induction is not explicitly mentioned in the phrasing of the problem.

The work discussed in Section 2.4.3 gives precedent for phrasing mathematical induction questions this way and provides some evidence that this type of phrasing might allow the researcher to observe more parts of the problem-solving process. Additionally, the second requirement encourages the semantic proof production method of construction rather than the procedural and syntactic productions discussed in Section 2.3. Weber (2005) argues that such semantic production provides more learning opportunities than the other two, providing some evidential support to use such semantic-centric proof construction activities in studies seeking to study the problem-solving process. The two problems used for the initial interview are provided below. For example solutions to each problem, see Appendix Item A. The structure of the interview associated with these problems was detailed in the previous section.

1. Show that there exists a minimal $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ with $m \geq n$, a $2^m \times 2^m$ chessboard with one missing tile can be exactly covered (no overhang) with "trominos" that is, three tiles in an L-shape as pictured below (the trominos in the cover may be oriented in any direction).

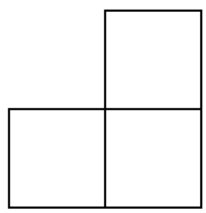


Figure 3.1: Tromino Problem

2. Assume that if you want to send a package, you must pay a certain amount of postage. Show that there exists some minimal $n \in \mathbb{N}$ such that any package with a postage price of m cents for $m \in \mathbb{N}$ and $m \geq n$ can be paid for using only 4 and 5 cent stamps.

3.3 Methodological Choices

3.3.1 *The Role of APOS Theory and Genetic Decomposition*

As this study aims to analyze cognition, it is important to mention that mental processes and learning processes are deeply individualized and primarily internal, meaning that it is impossible to observe them directly. However, theoretical frameworks focused on cognition provide us with strategies for studying cognitive processes via observable actions including dialogue, physical movement, written work, and more. While imperfect, these strategies allow researchers to create, test, and validate cognitive models. APOS Theory, discussed in detail in Section 1.4.2, is one such cognitive framework.

In order to analyze the mental constructions and mechanisms related to PMI, Dubinsky & Lewin's genetic decomposition for mathematical induction and Garcia-Martinez & Parraguez's reformulation of this genetic decomposition were used. These genetic decompositions were detailed in Sections 2.5.1 and 2.5.2, respectively. The decompositions served as models through which the participants might be studied. In particular, Garcia-Martinez & Parraguez's conceptualization of the base step as a process will serve as an important point of reference when analyzing the mathematical induction activities with unspecified base cases. The interview protocols for this research were designed based on these genetic decompositions to help analyze the participants' problem-solving strategies and activity solutions. The genetic decomposition used for the proposed research as an initial genetic decomposition for PMI, is an amalgymation of the two aforementioned decompositions and it is detailed below:

1. Expand the Function Schema to include a function mapping each natural number to a propostion-valued output ($f : N \to P(N)$).

2. Reversal through the existential quantifier to form a process of identifying and testing an appropriate base case.

3. Encapsulate logic into the implication $p \Rightarrow k$. The implication cognitively becomes an object which be the value of the function f.

4. Create the schema of the implication-valued function g where $g(\mathbb{N}) = (P(N) \Rightarrow P(N+1))$

5. Interiorize the action of logical necessity into a process so that inputs P_0 and $P_A \Rightarrow P_B$ allow one to conclude P_B.

6. Coordinate the function g from Step 3 with Modus Ponens beginning with $P(a)$ for some base case a.

7. Coordinate this implication valued function along with the base case process through the use of modus ponens to explain the PMI.

8. Encapsulate this inductive process into an object be connected to the Method of Proof schema so induction can be applied as a proof method.

9. Generalize actions on the induction object within various problem types coordinated with the Method-of-Proof schema until students can apply induction as a proof technique.

10. Generalize actions to the base case object until students can identify an appropriate base case in novel problems where it is not specified.

Although this decomposition has been tested by two research studies (Dubinsky, 1989; Garcia-Martinez & Parraguez, 2017), this proposed research will test the decomposition with *expert* participants to determine if there is a difference between experts' and novices' mathematical induction schema decomposition and if the decomposition needs needs revision to include any actions, processes, or objects found in the analysis of experts' use of mathematical induction.

3.3.2 *Case Study*

Case studies are in-depth examinations of a single setting, a single subject, a single phenomena, etc. (Bogdan & Bikklen, 2007). Case studies are commonly used within qualitative researchers and their use is well-documented. If the researcher conducts these in-depth examination on multiple subjects, the process is sometimes referred to as a multi-case study (Bogdan & Bikklen, 2007). Case studies have been widely used in research involving mathematical proof (Garcia-Martinez & Parraguez, 2017; Maher & Martino, 1996; Schwarz et.

al., 2008; Weber, 2004;) This study will include a multi-case study with an 5 total participants. The multi-case study design was chosen to allow a deep exploration of several experts' problem-solving processes. In addition, many of the studies discussed in Chapter 2 focusing on proof and PMI (Dubinsky, 1986, 1989; Garcia-Martinez & Parraguez, 2017; Inglis & Alcock 2012; Weber 2005) use the multi-case study design, suggesting that it is an appropriate choice for the current proposal. In particular, when evaluating the validity of a genetic decomposition, "Case studies are part of the research cycle of APOS theory to conduct a coherent analysis of the work of participants with the proposed GD" (Garcia-Martinez & Parraguez, 2017). The multi-case study for this proposed research will include interviews, recordings, transcripts, and the participants' written work.

3.4 Data Analysis

A computer-generated transcription was initially created for each interview. These transcriptions had mid-level accuracy. Each interview was re-watched twice during the transcription phase. The first re-watch was used to edit the transcription and correct errors. The second re-watch was used to verify the transcription corrections for accuracy and to note any non-verbal components of the interview, including gestures and lengthy pauses in dialogue. During this second phase of transcription, sections of dialogue were also linked to corresponding sections of the written work for ease of later interpretation.

After transcription was complete, each transcript received several reading and coding cycles. First, the transcript was read and any initial thoughts by the researcher were noted in relation to specific passages, with particular attention paid to recurring themes throughout the interview. The notes created during this iteration of coding provided most of the foundation for the thematic analysis detailed in Chapter 4. Second, the researcher made note of any obstacles or difficulties the participants encountered or described during the interviews. Next, the researcher linked passages of dialogue to any relevant literature from Chapter 2 so that the analysis and interpretation of results could be situated within existing work. Next, the researcher coded passages according to the APOS Theory Framework and Expert Frameworks Discussed in Chapter 1. This included breaking down portions of dialogue during the problem-solving periods according to the four stages of APOS as well as noting passages of dialogue which were indicative of schema activation. After each coding, the resulting coded transcripts were sent to a second researcher to be read and checked. The second researcher included extra codes and clarified pre-existing codings, when necessary. In any cases where there was disagreement, the two researchers discussed the passage and collaboratively decided on a code.

The results of these initial codings were used to construct a coding framework through thematic analysis (discussed in detail in Section 3.4.1) with which a second, more thorough, line-by-line coding cycle could be conducted. For this second round of coding, the transcripts were uploaded to the qualitative research tool, NVivo 12. In total, 538 unique sections of participant dialogue were collected from the ten interviews. These sections ranged in length from one sentence to several paragraphs. Sections which contained no usable data were removed. A section was considered to have "no usable data" if it met one or more of the following criteria.

1. The section included *a* single affirmative negative word or phrase answering a question asked for the interviewer. This included instances like"MmHmmm" or "Yes, that's correct" or "Nope."

2. The section consisted *only* of the participant repeating part of a question asked by the interviewer. For instance, if the interviewer asked, "Is the sky blue?" and the participant responded, "The sky?" this section would have been removed from consideration for the line-by-line analysis.

3. The section consisted *only* of the participant asking a logistical question like "Where should I email my written work?"

After all sections satisfying the conditions above were removed, 460 sections remained. These

460 sections of participant dialogue contained approximately 2,280 unique sentences of dialogue. The line-by-line coding was applied to this group of unique lines. Each line was coded into an appropriate category in the coding framework developed by the initial round of coding. If a line represented more than one category in the coding framework, it was multi-coded into every relevant category. A more in-depth explanation of this coding process is given in the following section.

3.4.1 *Thematic Analysis*

Thematic Analysis is a method of data analysis widely used in qualitative research. Historically, the term thematic analysis has been applied to a broad span of research involving the identification of patterns within qualitative data. However, more recently there has been effort to formalize and structure the approach (Braun & Clarke, 2006, 2012, 2017; Kiger & Varpio, 2020). Broadly speaking, thematic analysis involves searching across a qualitative data set to identify and analyze recurring patterns (Braun & Clarke, 2006). Thematic analysis is an appropriate and powerful method to use when seeking to understand a set of experiences, thoughts, or behaviors across a data set (Braun and Clarke 2012). Integral to any discussion of thematic analysis is an operational definition of the word *theme*. The current study takes a definition for theme derived from the existing literature on thematic analysis. For the purposes of this study, a theme satisfies the following conditions.

1. It is a "patterned response or meaning" derived from the data (Braun & Clarke, 2006, p. 82).

2. It "informs the research question(s)" (Kiger & Varpio, 2020, p.3)

3. It "captures and unifies the nature or basis of the experience into a meaningful whole" (DeSantis & Ugarriza, 2000, p. 362)

It is highly important to note that a theme is not necessarily dependent on quantifiable measures such as its frequency in the data set. Instead, a theme is determined by its ability to capture something important in relation to the overall research question(s) (Braun & Clarke, 2006). Braun and Clarke (2006) offer a six-phase guide to conducting a thematic analysis which is now one of the most widely-used models in thematic analysis (Kiger & Varpio, 2020, p.3, Nowell et al., 2017). The six phases of the guide are shown in the Table 3.1.

Table 3.1: Six Phases of Thematic Analysis by Braun & Clarke, 2006

Phase	Description
Familiarizing yourself with data	Transcribing data (if necessary), reading and re-reading the data, noting down initial ideas.
Generating initial codes	Coding interesting features of the data in a systematic fashion across the entire data set, collating data relevant to each code
Searching for themes	Collating codes into potential themes, gathering all data relevant to each potential theme.
Reviewing themes	Checking if the themes work in relation to the coded extracts (Level 1) and the entire data set (Level 2), generating a thematic 'map' of the analysis.
Defining and naming themes	Ongoing analysis to refine the specifics of each theme, and the overall story the analysis tells, generating clear definitions and names for each theme.
Producing the report	The final opportunity for analysis. Selection of vivid, compelling extract examples, final analysis of selected extracts, relating back of the analysis to the research question and literature, producing a scholarly report of the analysis.

This coding framework can be applied either *inductively*, using pertinent patterns identified in the data to develop the coding framework, or *deductively*, where the coding framework is predetermined and guided by specific theories or theoretical frameworks (Attride-Stirling, 2001; Braun & Clarke, 2006). This six-phase framework was used in two phases of the data analysis. First, the nine-item expert knowledge framework was used as to conduct a deductive thematic analysis on the data. This thematic analysis is discussed in detail in Section 4.2.3. Secondly, an inductive thematic analysis was also conducted on the entire data set to identify any other pertinent themes that address some, or all, of the research questions but were not captured by the APOS Theory or Expert Knowledge frameworks.

This inductive analysis is presented in Section 4.3. Braun and Clarke (2006) note that it is important to acknowledge the type(s) of thematic analysis conducted (inductive or deductive) as well as the epistemological viewpoint through which the analysis was conducted. This study operates from the constructivist view as discussed in Section 1.3, and the thematic analyses were conducted from this perspective.

3.5 Chapter Summary

This chapter presents the methods an methodology of the study. The initial section described the research setting and the selection of participants. The following section detailed the methods of data collection as well as the study design including a description of Cognitive Task Analysis and semi-structured interviews. Next, the methodological choices for the study were described in detail and situated within existing literature. Finally, methods of data analysis were presented including a detailed description of thematic analysis, which was used extensively during two stages of data analysis. The results of this data analysis process, as well as a contextual discussion of these results, is discussed in Chapter 4.

4 RESULTS

This chapter presents an overview of the findings of the study as well as some interpretations of these findings. The primary purpose of this chapter is to link the data and data interpretation to the research questions outlined in Section 1.2. Rather than splitting the chapter by research question, the questions will be connected to the data throughout each section within the contexts of the relevant theoretical frameworks discussed in Section 1.4. First, Section 4.1 provides a description of the data analysis conducted using APOS Theory, as discussed in Section 1.4. This section also includes a discussion of the genetic decomposition discussed in Section 3.3.1 and the triad of Schema Development detailed in 2.5.3. Section 4.2 highlights the results of the deductive thematic analysis (see Section 3.4.1) based on the nine-item expert knowledge framework discussed in Section 1.4. The chapter concludes with Section 4.3, which presents the inductive thematic analysis. Several recurring themes from the data that were not captured by the APOS Theory analyis or the deductive thematic analysis are highlighted and analyzed, and a philosophical discussion of one interesting result arising from the data is also explored.

As discussed in Section 1.3, the notion of broad "generalizability," or the desire to identify results that can be widely generalized across an array of different contexts, is not necessarily the primary goal of qualitative research. It is important to note that, since this study is qualitative in nature, the results of the data analysis may appear different than a standard or traditional quantitative study, and it is appropriate that they should be considered under a different lens. The APA Publications and Communications Board Task Force Report says the following of qualitative research:

> Qualitative data sets typically are drawn from fewer sources (e.g., participants) than quantitative studies, but include rich, detailed, and heavily contextualized descriptions from each source. Following from these characteristics, qualitative research tends to engage data sets in intensive analyses, to value open-ended discovery rather than verification of hypotheses, to emphasize specific histories or settings in which experiences occur rather than expect findings to endure across all contexts, and to recursively combine inquiry with methods that require researchers' reflexivity (i.e., self-examination) about their influence upon research process (Levitt et al., 2018).

With this reminder, the remainder of the chapter will seek to address the research questions using the analysis of the qualitative data collected during the study. The research questions are included again below for convenience and labeled with the designation that will be used in the chapter's discussion (RQ1, RQ2, etc.).

RQ1 How do experts describe the development of their conceptual understanding of PMI over time?

RQ2 How do experts situate their conceptual understanding of PMI in relation to the notions of proof and proof technique?

RQ3 When viewing a novel problem, how do experts determine whether or not mathematical induction is an appropriate method for proving a statement?

RQ4 What obstacles, if any, do experts face when solving mathematical induction problems in which mathematical induction is not explicitly specified as the technique to use?

RQ5 How do experts explain and define the two primary parts of PMI (the base step and the inductive step) and the perceived relationship, if any, between these two primary parts?

These research questions informed all aspects of the study and will likewise guide both the discussion and interpretation of results included in the remainder of the chapter.

4.1 APOS Theory

APOS Theory, including the theory of Schema Development, was discussed in detail in Chapter 2. This section will present the results of the data analysis in the context of APOS Theory. Section 4.1.1 will reiterate the preliminary genetic decomposition discussed in Section and give a worked example of the APOS coding process for a small excerpt. Section 4.1.2 will describe the results of the APOS Coding Process for each participant's work on the cognitive tasks, giving notable and relevant examples from the data. Section 4.1.2 will discuss examples Schema Development identified by the study. Finally, Section 4.1.3 will conclude the section with a summary of results and conclusions based on the findings, including suggested revisions and additions to the preliminary genetic decomposition.

4.1.1 *Analysis Using APOS Theory and Preliminary Genetic Decomposition*

The role of genetic decompositions in work using APOS Theory was discussed in detail in Section 1.4. Recall that a *genetic decomposition* is a contextual description which attempts to capture a sequence of actions, processes, objects, and schemas that students *may* progress through when constructing knowledge for some concept, as well as the mental mechanisms (e.g. interiorization, coordination, encapsulation, etc.) by which those constructions are possibly created. As has already been mentioned, learning is a deeply individualized process, and it is not expected that a genetic decomposition will exactly capture every single individual's knowledge construction. Instead, a quality genetic decomposition will capture a large part of construction which is representative of most learners. As discussed in Chapters 2 and 3, two previous studies have done work with a preliminary genetic decomposition for PMI (Dubinsky & Lewin; Garcia-Martinez & Parraguez). The findings of these two studies were coordinated to identify the genetic decomposition below, which was used as the guiding genetic decomposition for the current study and will be evaluated and modified in in Section 4.1.3.

1. Expand the Function Schema to include a function mapping each natural number to a propostion-valued output ($f : N \to P(N)$).

2. Reversal through the existential quantifier to form a process of identifying and testing an appropriate base case.

3. Encapsulate logic into the implication $p \Rightarrow k$. The implication cognitively becomes an object which is the value of the function f.

4. Create the schema of the implication-valued function g where $g(\mathbb{N}) = (P(N) \Rightarrow P(N+1))$

5. Interiorize the action of logical necessity into a process so that inputs P_0 and $P_A \Rightarrow P_B$ allow one to conclude P_B.

6. Coordinate the function g from Step 3 with Modus Ponens beginning with $P(a)$ for some base case a.

7. Coordinate this implication valued function along with the base case process through the use of modus ponens to explain the PMI.

8. Encapsulate this inductive process into an object be connected to the Method of Proof schema so mathematical induction can be applied as a proof method.

9. Generalize actions on the mathematical induction object within various problem types coordinated with the Method-of-Proof schema until students can apply mathematical induction as a proof technique.

10. Generalize actions to the base case object until students can identify an appropriate base case in novel problems where it is not specified.

This preliminary genetic decompostion informed the study design, including the methodological choices and interview protocols discussed in Chapter 2. In the following sections, the results of data analysis are presented and are interpreted using the APOS Theory Framework as well as this preliminary genetic decomposition. In section 4.1.1.1, the APOS Coding process is explained in greater detail along with a worked example to illustrate the coding process. Then, in 4.1.1.2, notable findings are discussed and interpreted. Finally, in Section 4.1.1.3, conclusions based on the findings of the APOS data analysis are presented and contextualized within the study's guiding research questions.

4.1.1.1 Worked Example of APOS Coding Since the participants in this study are mathematical experts, their work during the CTA was intricate and nuanced. In many cases, one line of spoken dialogue encompassed numerous cognitive processes. Therefore, performing APOS coding on the transcripts was an involved and lengthy process. In order to provide insight to some steps in this process, a worked example is included in this section. In general, the following set of steps served as a guideline for the coding process. Note that these steps take place after the initial transcripts have been computer-generated and manually checked for accuracy several times, and the video-recordings have been reviewed multiple times to ensure consistency between the recordings and transcripts.

- Step 1 Carefully read through the transcript, referencing the video-recordings and written work when dialogue in the transcript is unclear.
- Step 2 The first coding involved identifying large passages where any of the stages of APOS Theory (e.g. action stage, process stage, etc.) appear in the transcript without necessarily elaborating or explicating.
- Step 3 Revisited the codes assigned in Step 2, this time reading the passage associated with a stage and giving a line-by-line description of the passage in the context of the four stages of APOS Theory.
- Step 4 Revisited the codes from Steps 1-3, this time including a discussion of any exhibited APOS Theory mechanisms (e.g. interiorization, coordination, encapsulation, etc.) which link the stages of knowledge construction together in each passage.
- Step 5 Sent coded transcript to a secondary coder for verification and edits or additions.
- Step 6 If any disagreement occurred, the two coders met and discussed the issue(s) in order to arrive at an agreement.

Steps 5 and 6 are to ensure consistency in the data and to reduce bias that can occur if a single researcher is the sole source of data analysis or interpretation. These two steps are not included in the worked example below. Instead, Steps 1-4 are illustrated by this worked example to give the reader insight to what coding an interview using APOS Theory is like. The example that follows analyzes an excerpt from Participant 3's solution to the first cognitive task (tromino problem), following Steps 1-4 of the coding process detailed above.

- STEP 1: First, the excerpt from the transcript was linked to the participant's written work and any non-verbal cues were added to the excerpt. Any additions are coded in **red** to help distinguish them from the original excerpt.
- STEP 2: Additions from the first APOS Coding are coded in **blue**.
- STEP 3: The line-by-line descriptions added in the second coding are coded in **green**.
- STEP 4: Additions made in the third coding, including mechanisms, is coded in **purple**.

Action Stage: [So, I'm taking a copy of the 8 by 8, and I have the empty cell in the top left corner. And I'm going to make a copy of it. And I'm going to rotate it around so I'm just gonna, like, put 4 copies of it.] **[Performs the action of copying an 8×8 grid four times, then creates a 16×16 grid using these four copies (as pictured in the written work below)] This action is linked to the external stimuli of the written work, the participant physically performs these actions.**

[Here, he holds up Part C of the written work to the camera.]

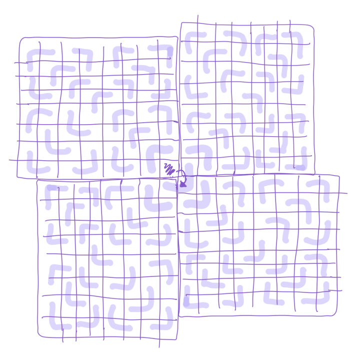


Figure 4.1: Participant 3 Written Work

Process Stage: [Okay. Okay, okay so I took um, my 8 by 8 and that missing tile was in the bottom right corner. I don't know if it's mirrored for you, but it was in one of the bottom sides and then I made a copy of it rotated it. So that the... all of the empty squares, all the empty corners, and either the copies are at the center of the grid.] **[He has reflected on the action of constructing a 16×16 board from four copies of the 8×8 board, and can now visualize this process in his mind.] As he reflects on the action of constructing the board, he *interiorizes* this action and is able to visualize the process in his head without the need for the external stimuli.**

[Here, he is pointing to the four quadrants and making rotational gestures with his hands.]

Process Stage: [And so, if you think of it like that, then okay well, now there's 4 empty cells. Right? But the thing is that reduces it down to a two by 2, which we said we can already tile, which is our base case, right? And we can definitely tile that and leave, like, one empty cell.] **[He has reflected on the action of tiling a 2×2 chessboard, which he performed during the base step, and is now able to visualize the process in his mind.] Instead of having to perform the action of tiling an 8×8 chessboard, he has reflected on the action and is now able to imagine the interiorized action as a process in his mind without the need to perform the action itself.**

[During this section he is pointing to the center 2×2 grid.]

Object Stage: [So what that leaves you with if I tie that one let's just say I'm going to choose. Um, my empty cell to remain in just the top left 8 by 8. Um, from there, we can just go back to the previous case and say, like, okay, well, I know from the previous case that I can get the cell everywhere within that 8 by 8.] **[The participant is able to think of the process of generating tilings for an 8×8 in totality and apply other processes (like rotation and iteration) to this tiled object.] After performing the action of tiling the 8×8 using the external stimuli of his written work and hav-**

ing *interiorized* this action into a process he can imagine performing in his mind, the participant is now able to imagine the tiled 8×8 in totality, and thus deomonstrates that he has *encapsulated* this tiling process into an object. Right. And, um, in order for me to get this thing to a new grid,**[By this, he means shifting the missing tile to a different quadrant.]** **Object Stage:** [I can just, like, rotate by 90 degrees, right? And actually don't even need to rotate the whole grid. You can just rotate the center 2×2 so, once you have the empty cell in the center, then you just rotate that like that two by two in the middle with the L and the empty square. And that'll put it into a new quadrant and then you can do all of your rotations from there.] [**The participant is able to visualize the process of rotation in his mind without needing to perform the action on paper, and as he reflects on these rotations, he is able to think of them in totality and exhibits the ability to compose multiple rotations.**] **Initially, the participant used the external stimuli of his written work to draw rotations of the board. As he reflected on this action of rotation, he was able to interiorize the rotation action into a rotation process that he could visualize inside his mind. It is important to note that there are two different kinds of rotations involved- whole board rotation and sub-board rotation. The participant demonstrates the ability to imagine applying a composition process to the rotations, suggesting that he has encapsulated the rotation processes into objects to which he can apply other processes.**

This excerpt is only part of Participant 3's solution to the tromino problem, but it captures examples and coding of action, process, and object stages of APOS Theory. Within this part of his problem-solving process, he performs actions, reflects on these actions and interiorizes them into processes that he can imagine performing in his mind. After the actions have been interiorized, he continues to reflect on these processes until he can imagine them in totality and has encapsulated these processes into objects to which he can then apply other processes. The worked example above is representative of the complexity and intricacies of APOS coding. This coding process was used for all interviews in the study. The coding steps were used to analyze both the CTA portion of the interviews as well as the semi-structured interviews that followed the CTA. It is clear that, even in the short excerpt above, the participant exhibits several cognitive constructions and employs the use of multiple APOS Theory mechanisms. This was commonly seen among the participants during the CTA. Some relevant findings based on the APOS Theory coding are detailed in the following section.

4.1.1.2 Action, Process, Object Pattern During Cognitive Tasks Recall that Styliandes, Sandefur, and Watson (2016) identified a pattern of behavior exhibited by experts while they worked on tasks using PMI. The three steps are included again below for convenience. In particular, the authors observed that experts would:

1. Attempt to identify a reasonable method or technique to prove the statement. If one can be identified, they may use the technique without necessarily thinking about *why* the statement is true,

2. If no method can be immediately or easily identified, then the expert may try to experiment with some examples to gain insight to possible proving strategies.

3. Use discoveries made in the previous step to inform the formalization of an argument.

This pattern of behavior was consistently modeled by participants in the current study, and as a result, this model is used in several sections throughout this chapter to help interpret the data. The three steps above can clearly be seen in the worked example in Section 4.1.1.1. In fact, these three steps can be translated to the following mental constructions in APOS Theory.

1. If no method or proof can be easily or immediately identified, an expert may begin by performing the action of solving small examples.

2. The individual may reflect on these actions until they can be interiorized into processes that they can imagine in their mind.

3. The individual may search for relationships or patterns between successive examples. This could also be equivalent to coordinating the process from Step 2 with the proccess of succession to identify a relationship between cases n and $n + 1$.

4. Once the individual has identified this pattern or relationship, they may be able to reflect on it until they are able to coherently understand the process in totality and encapsulate it into an inductive step object, which can serve as a formalized argument.

This pattern offers a general description of behaviors that captures the majority of the problem-solving behavior exhibited by experts in the study. The remainder of this section will present examples of this pattern as it appeared in the data. Since Schemata are explored in detail in the following section on Schema Development, this section will primarily focus on the Action, Process, and Object stages of APOS Theory as they pertain to the pattern above along with relevant examples from the data which contextualize these steps in the context of the study's two mathematical induction problems.

Action Stage

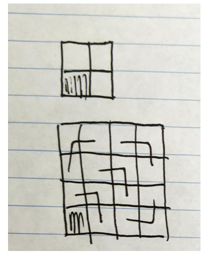
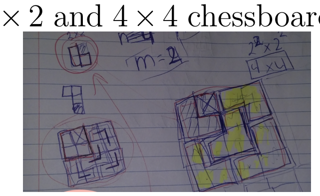
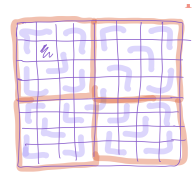
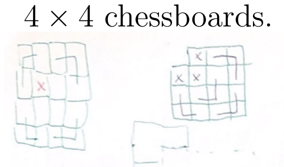
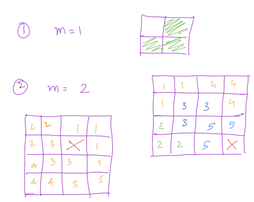
As indicated by Styliandes, Sandefur and Watson (2016), experts demonstrated a tendency to use examples when they could not immediately identify a proving strategy. This study corroborates this finding, and examples of this behavior observed during the CTA are detailed below.

1. *Problem 1- Tiling Action*: While working on Problem 1 (tromino problem), every participant performed the action of tiling a chessboard or multiple chessboards. This action was grounded in the physical act of tiling the board in their written work. Table 4.1 shows an example of each participant's tilings. The tiling actions pictured took place at the beginning of the problem solving section for Problem 1 for all participants

except Participant 5, who began by proving an adjacent counting problem on the number of tiles and trominos. After constructing this proof, she returned to the tiling problem and performed the tiling action for the 2×2 and 4×4 chessboards. The largest board that was physically drawn and tiled was an 8×8 board, drawn by Participant 3 and pictured in the table below. The other four participants did not manually tile boards larger than 4×4.

There is one notable pattern of behavior related to the tiling action. Since the missing tile could be located anywhere, a few of the participants tiled a couple of 4×4 chessboards (Participants 2, 3, and 5). However, some of the participants only tiled a single 4×4 board with the corner tile missing (Participants 1 and 4), and seemed convinced that the statement held for $n = 2$, despite not verifying this for other missing tiles. No participant performed the action of tiling for more than two different 4×4 boards. The participants seemed to accept one or two examples of a tiling to be sufficient to accept the claim as true. This suggests participants may have been able to interiorize a single tiling action into a process of tiling with a missing tile. That is, they were able to imagine what would happen if the missing tile was located elsewhere. For instance, after performing the action of tiling the two 4×4 boards pictured in Table 4.1, Participant 5 said, "I think I'm like... after these 2 examples, I think if I choose a missing tile, then that thing makes like a two cross two square. And then for the remaining, I think that there should be a way to figure out the tiling. I guess what I'm trying to say is like, uh, if I choose a blank tile anywhere, it should work."

Table 4.1: Images of Tiling Action

Participant	Image of Tiling Action
Participant 1	2×2 and 4×4 chessboards. 
Participant 2	2×2 and 4×4 chessboards. 
Participant 3	8×8 chessboard. 
Participant 4	4×4 chessboards. 
Participant 5	2×2 and 4×4 chessboards. 

This phenomenon illustrates findings by Weber (2008) that were discussed in Section

2.1.2. Weber (2008) noted that the mathematical experts in his study would use what he called example-based arguments, or arguments when the participant accepted the validity of a claim solely by examining the statement in the context of carefully chosen examples. He notes that, while proof by example is certainly not a valid proof technique, experts seemed to be convinced by the exploration of a single example. This behavior was exhibited by the participants who accepted the claim that a 4×4 board missing any tile could be covered with trominos based solely on their ability to tile the board in the case of one or two specific missing tiles. Weber (2008) offered two hypotheses for this behavior. He believes that either (1) it is possible that the mathematicians understood how their example-based inductive reasoning could be generalized, but did not express this in a way that was directly observable or (2) it is possible that the participants were only requiring a high level of confidence in a statement's validity, rather than absolute certainty. The second hypothesis is explored further in Section 4.3 in the section about formal and informal types of proving. Only Participant 3 offered a rigorous explanation of how a tiling with one missing tile could be used to generate tilings for a board missing a tile in a different location. This will be explored in the following item.

2. *Problem 1- Rotation Action*: As seen in the worked example in Section 4.1.1.1, Participant 3 used rotational symmetry as part of his argument. While one other participant (Participant 4) also spoke about rotational symmetry, Participant 3 was the only one to link rotation to an external stimuli by drawing several examples of rotation in his written work. He exhibited three different rotation actions: Full-Board rotation by $90°$, Quadrant Rotation by $90°$, and 2×2 Sub-board Rotation. These are pictured below in Participant 3's work.

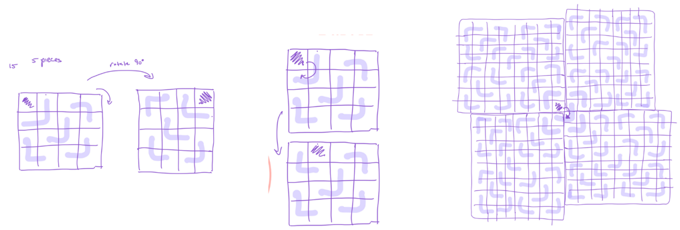


Figure 4.2: Participant 3 Rotation Actions

After initially reading the problem, Participant 3 was immediately interested in the fact that the missing tile could be anywhere on the board. These rotational actions served as the foundation for what became a rigorous explanation to account for the location of the missing tile. Initially, the participant relied on the images in Figure 8 to help him see where a missing tile would shift to for a given rotation. That is to say, he relied on this external stimuli in order to understand the relationship between the tilings for different missing tiles. For example, when working on the 4×4 board pictured in the center of Figure 8, he said, "And then rotate so that the empty square is like, um, inside. So it's over... let me draw it. So, it's over there. Well, now there's a new two by two square that I can rotate, um. I'll try to, like, highlight it so then that now I can rotate that two by two little grid." This illustrates his initial reliance on the visual stimuli of his drawings to help him visualize the results of these rotations. Later, he was able to begin interiorizing this action so that he could imagine it in his mind. This will be discussed further in the section on the Process Stage. Examples of action-stage behavior for the second problem are explored in the following item.

3. *Problem 2- Linear Combination Action*: For the stamp problem, all five participants began by explicitly writing out small package prices and attempting to cover them

with the given stamp values. Consider Participant 5's work below along with the corresponding excerpt from the transcript.

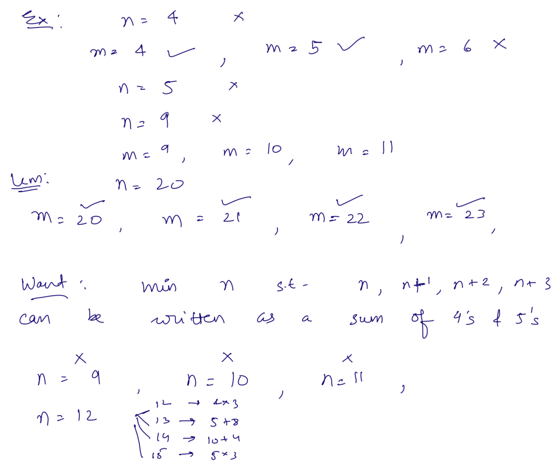


Figure 4.3: Participant 5 Enumeration Actions

P5: So n=4 works. Then, um, so. . . . And of course 5 works. But $n = 6$ doesn't work, because we can't make up 6 out of 4 and 5. Um, so, $n = 5$ doesn't work as the minimal. Okay, so now I'm thinking maybe I should start with the $n = 9$, like, as an example. So 9 works, 10 works, but $n = 11$ doesn't work. . . Maybe I should try with like the lcm. So if $m = 20$... So, 20 works. 21 works. Does 22 work? Yeah, 22 because I can do $12 + 10$. 23. . . works too. And then 24 will work because 20 worked. So, I guess if I check just like 4 consecutive numbers. Then, like, each higher would work. So I guess the question is, um. What is the minimal n, such that n, $n + 1$, $n + 2$, and $n + 3$ can be written as a sum of fours and fives? Okay, so I have a candidate which is $n = 20$. I guess at this point, I would just like, uh, do, like a trial and error. Because I checked up until, like, $n = 9$. And then I would just, like, check after that. Okay, because I only have to

> check numbers between 9 and 20 and, like, because I think it's going to be a fast process. I know $n = 9$ doesn't work, $n = 10$ doesn't work and $n = 11$ doesn't work. And all 3 of these don't work... Don't work because of um, 11. Because 11... yeah... yeah 11 doesn't work. Does $n = 12$ work? 12 works, 13 works. Does 14 work? 14 works. Does 15 work? Yeah 15 works too. Okay, then, um. Yeah, so my answer is $n = 12$ is the minimal n.

Here, Participant 5 exhibited an action stage of of construction, since she used examples to figure out two parts of the solution. First, she used the examples to perform the action of identifying the necessary number of consecutive numbers needed for her argument. Second, she used example action to identify the minimal number satisfying the condition. Similar written-work and dialogue pairs were observed during Problem 2 for the other participants, although not all participants identified the need for four consecutive integers. While only three of the participants (Participants 3, 4, and 5) were able to construct a complete proof for Problem 2, all five participants successfully identified $n = 12$ as the minimal n using the Linear-Combination Action. However, the same type of example-based argument also appeared in this problem. Participant 1, who was unable to construct an argument showing that any package price greater than $n = 12$ could be exactly paid still believed that $n = 12$ was correct based on a limited number of examples. In particular, he performed the action of covering the integers 12-40 using 4 and 5 cent stamps, then made the claim that $n = 12$ was the minimal n. Again, this exhibits the same type of behavior identified by Weber (2008), since this argument is not logically sufficient to prove that 12 is the minimal n. This case seems to better support Weber's second hypothesis explaining this behavior. In particular, Participant 2 was unable to construct any argument for why an arbitrary package could be covered. It is therefore unlikely that he understood how his small examples could be generalized and much more likely that the 28 consecutive true cases were enough to give him a high degree of confidence.This indicates that Participant 1 was not able to interiorize these actions into a process, keeping him from identifying an argument for the remainder of the problem.

Process Stage

Most of the items associated with the Process Stage of APOS Theory are the processes resulting from the interiorization of the actions mentioned above, with one exception. The participants who recognized the relationship between successive chessboards (namely, that increasing n by 1 yields a chessboard that is four times the previous board size) did not necessarily perform any observable action or use any visual stimuli related to this relationship, but were able to imagine the process in their minds and describe this process out loud. The themes linked to the Process Stage of development are listed below. The processes described below illustrate the second behavior in the pattern of behavior discussed at the beginning of this section: "The individual may reflect on these actions until they can be interiorized into processes that they can imagine in their mind."

1. *Problem 1- Tiling Process*: Once the participants reflected on the tiling action and began to be able to imagine the process of tiling in their minds, the action was thought to be interiorized into a tiling process. Consider the excerpt from Participant 1's interview below.

   > P1: And then I drew the 8×8 grid and then saw that if I could do large "L's," **[Here, he means three of the four quadrants of a chessboard.]** then that would limit me down to a smaller square. Um. And I could work from there, so I guess I never never even drew the 8×8. I just understood what it should look like from a 4×4.

   This excerpt is indicative that the participant is operating at the process stage of tiling, since he is able to visualize the 8×8 tiling in his mind. Participants 2 and 3 also demonstrated the ability to imagine the tiling process in their minds without continuing to rely on the external stimuli of their drawings. In contrast, Participants 4 and 5 did not exhibit signs of progressing past the tiling action. While they were able to generate the tiled boards pictured in Table 4.1, they were unable to progress further in the problem and gave no observable sign that they could imagine this tiling process without the use of their drawings. It is important to note that Participants 4 and 5 did not successfully construct a proof for this problem, indicating that the ability to interiorize the actions associated with examples may be important for constructing proofs by PMI.

2. *Problem 1- Rotation Process*: Consider again a small subsection of the excerpt used for the worked example in Section 4.1.1.1.

> P3: I can just, like, rotate by 90 degrees, right? And actually don't even need to rotate the whole grid. You can just rotate the center 2×2. so, once you have the empty cell in the center, then you just rotate that like that 2×2 in the middle with the L and the empty square. And that'll put it into a new quadrant and then you can do all of your rotations from there.

Again, Participant 3's description here indicates that he is able to imagine these rotations in his mind, which indicates that the rotation action has been interiorized into a rotation process. Participant 3 was the only participant who used rotation as part of his mathematical induction argument for Problem 1. He did not draw any boards or refer to any of his previously drawn boards or rotations during the excerpt above. This suggests that the rotational process for multiple kinds of rotations (he refers to both whole-board rotation and 2×2 sub-board rotation in the excerpt above) has been interiorized into a rotation process so that he is able to imagine these various kinds of rotations in his mind. Further, he exhibits the ability to *coordinate* the processes of multiple kinds of rotations (e.g. he talks about rotating the center 2×2 and *then* performing a whole-board rotation). Later, he exhibits the ability to encapsulate these rotation processes into rotation objects, and this is discussed in the section on Object Stage.

3. *Problem 1- Quadrupling Process*: The relationship between successive boards was a crucial part of the tromino problem. Namely, for any $n \geq 2$, a $2^n \times 2^n$ chessboard consists of four copies of a $2^{n-1} \times 2^{n-1}$ boards. Four of the five participants exhibited behaviors that indicated they were able to imagine this quadrupling process in their minds without having to perform an associated action or to rely on external stimuli. Consider the excerpt from Participant 2's interview below.

> P2: I started to think about the size of the chessboard because it's not increasing arbitrarily. It's increasing by doubling the rows and the columns. Every time you have doubled the amount of area... Is it double the area? no it is four times the area...

When referencing this growth, Participant 2 was not referencing any written work or relying on any external stimuli that could be observed in the video recording. Instead, he is able to imagine the growth of the chessboard in his mind, indicating that he is operating at a process-stage of construction. Participants 1, 3, and 5 also exhibited behavior that indicated they could imagine this quadrupling behavior in their minds, as demonstrated by their verbal discussions of the process of growth similar to the excerpt above. However, Participant 5 was unable to identify this relationship and exhibited behavior that indicated he *could not* imagine this process in his mind. Consider the drawing from his written work below.

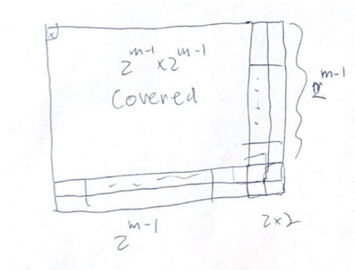


Figure 4.4: Participant 4 Issues with Chessboard Growth

This image was supposed to represent a $2^m \times 2^m$ chessboard. However, the picture drawn actually has dimension $(2^{m-1}+2) \times (2^{m-1}+2)$. This suggests that, not only was the participant unable to imagine the growth process in his mind, but he was also unable to perform the action of representing the growth in his written work. The ability to visualize this process of chessboard growth was strongly linked to success in constructing a cohesive mathematical induction proof, and this is explored further in Section 4.2.1. In general, it is important to note that the inability to identify the relationship in a single problem is not necessarily indicative of the individuals ability to identify similar patterns in general, and it is possible that the Participant would have been able to identify the pattern of chessboard growth with certain prompting.

4. *Problem 2- Linear Combination Process*: There were 3 participants who successfully constructed a proof for the stamp problem (Participants 3, 4, and 5). In order to construct this proof, these three participants interiorized the linear combination action discussed above in the Action Stage section. In particular, they were able to imagine the process of covering an arbitrary package price with stamps. Consider the written work and corresponding transcript excerpt from Participant 3's interview below.

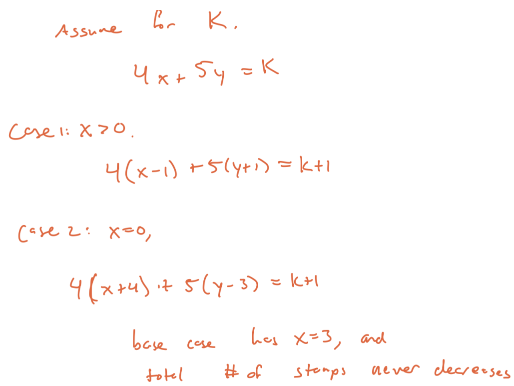


Figure 4.5: Participant 3 Stamp Problem Linear Combination

> P3: Okay, like $4x + 5y = m$ and I'm adding one to both sides. Well, I can make one by taking... adding a 5 and subtracting a 4. Because 5 minus 4 of course is equal to one, and then as long as the number of fours, I have is bigger than zero. then I can actually do this like, add one to the y. Because we want to make sure that we're actually using, like, positive number of postage stamps. Um, so the issue then arises if x is equal to 0, in the case where x is equal to 0, then you have to make one another way. And that is by, um let's see. Okay this is for x greater than 0 and then this is for x equals 0. Okay. Wait a minute no, no, I'm not crazy. Okay, so this should work, um, when... when x is actually equal to 0, um, then you take away 3 of the um, fives and then add... is that right? Or it should be four fours. Ah four times four is 16. That's 4. Okay. There we go.

The participant exhibits the ability to imagine the process of removing and adding stamps to cover a package price (contextualizing this addition and removal algebraically using linear combinations). It is important to note that the written work above was written *after* the dialogue above. The participant was reasoning aloud, but the calculations were taking place primarily inside his head, suggesting that he was operating at the process stage.

Object Stage

The Object Stage was primarily observed toward the end of problem-solving sessions and primarily involved participants being able to reflect on the processes discussed in the previous section and to imagine them in totality until they were able to encapsulate them into objects. This stage is primarily associated with steps 3 and 4 in the pattern of behaviors associated with PMI detailed at the beginning of the section.

1. *Problem 1- Tiling Object*: After continuing to reflect on the process of tiling a board, the participants were able to imagine this process in totality and encapsulate the process into a tiling object. Namely, participants exhibited the ability to think of a tiled board as its own entity without imagining the process of tiling. Further they were able to apply the quadrupling process discussed in the previous section to these tiled board objects in order to generalize the tiling to larger boards and thus, develop an inductive argument. Consider the image from Participant 1's written work below.

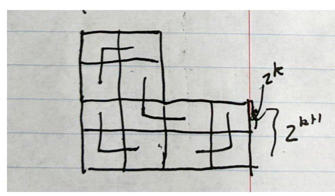


Figure 4.6: Participant 1 Tiling Object

The "tiles" in this image are not single unit tiles. Instead, each tile in the picture represents a tiled 2^k board. This indicates that Participant 1 has encapsulated the tiling process into a tiled board object, to which he can apply other processes, like the quadrupling growth process discussed in the previous section. Participants 2 and 3 also exhibited this same Object Stage of tiling. Participants 5, though she showed that she had interiorized the action of tiling a 4×4, seemed unable to encapsulate this process in order to be able to think about a tiled board as an object to which she could apply other processes. Participant 5, as previously discussed, was unable to interiorize the action of tiling.

2. *Problem 1- Rotation and Mapping Object*: After continuing to reflect on the process of tiling a board, Participant 3 was able to imagine this process in totality and encapsulate the process into a rotation object. Namely, he exhibited the ability to think of rotations as functions. This was demonstrated by his ability to apply the process of composition to these rotations. These composition processes were coordinated to create a Mapping object which takes a tiling as an input and outputs a tiled board with a different missing tile. Consider the excerpt below which illustrates this mapping object.

   > P3: I guess this 8×8 is actually split up into four 4×4s, and they are all very similar. Um, if we're ignoring... there's one piece that you kind of have to ignore for this, but the pattern in all of these is pretty much the same. Just rotated versions of it. So, I think if you can show that you can move it anywhere in one of these little sub grids, right? If I can move it anywhere in one of these sub grids, then I can just rotate the whole thing. Right, and then the empty Square just goes to the rest of it.

   Here, we can see him imagining the various rotational processes in his mind, then coordinating them to form a chain of rotations, which is then encapsulated into a mapping that he can visualize in totality as taking one tiled board and mapping it to a tiled board with a different missing tile. As previously stated, Participant 3 was the only participant in the study to use these rotations as part of his argument.

Participant 3's proof to Problem 1 most successfully captured the pattern of behavior discussed by Styliandes, Sandefur, and Watson (2016), though all the participants participated in some or all of these steps, as indicated in the examples given throughout this section. The results presented in this discussion identify some aspects of the mental constructions involved in proofs by PMI which may not be completely captured by the preliminary genetic decomposition used for the study. This will be explored in Section 4.1.3. While this section offered some descriptions of results collected during the CTA portion of the interview, the following section explores the Base Step and Inductive Step of PMI in more generality.

4.1.1.3 The Base Case and Inductive Step Research Question 5 of this study asks, "How do experts explain and define the two primary parts of PMI (the base step and the inductive step) and the perceived relationship, if any, between these two primary parts?" While the CTA-specific examples given in the previous section provide some insight to this question, this section will further explore the base case and inductive step in broader contexts using the APOS Theory framework. In particular, this section explores results associated with the two steps of mathematical induction as well as the participants' perceived relationship, if any, between the two parts. Results discussed in this section cover results discussed outside of the CTA, during the semi-structured question and answer portion of the interviews.

Base Step

When asked about how the base case fits in to the overall technique of mathematical induction, Participant 2 said, "It depends on how trivial it is if it's like a really basic base case, I almost don't think about it at all. I'm like, It's clearly true and then I move on."This sentiment was echoed by other participants. Consider Participant 1's answer to the same question below.

> P1: In principle I know that you need the base case, because you can write things where you're trying to prove something inductively and a base case doesn't work. Sorry. And it, the proof fails because the base case doesn't work and you're proving something. So, like, in practice the base case is really for me, almost a triviality because what I'm... If I'm trying to prove something, I will have already checked that it works in the most trivial case I can check, which would be the base case... Actually. Um, most of the time the base case for me, usually has almost no content. It's, it's checking that the thing works at the very sort of most trivial level, and often at the... well, I'm calling it the trivial level, but often that the, the level of the base case, it's hard to understand what what is

> actually going on, whereas all the content content and inductive proof for me is contained in the inductive step, which really tells you how to go from one situation to another.

These statements represent a limited and rudimentary description of the base case, consisting only of checking a small case and requiring little to no attentional effort or thought. However, even this simple description of the base case is indicative that the participants are operating at least at a process stage of construction of the base case, since they are able to imagine the process of checking a base case in their minds. While Participant 1 said, "at the level of the base case, it's hard to understand what's going on." Participant 4 offered a somewhat differing opinion and indicated that, "sometimes the base case shows you how to do the general construction." This statement is supported by the examples from the CTA, where participants used the base case to inform the rest of the inductive proof. The relationship between the base case and the rest of PMI is explored later in the section.

Participant 5 offered a more nuanced description of the base case. She was asked how she might identify a base case if it isn't identified for her. Consider her answer in the excerpt below.

> P5: I think like, depending... Okay. I think look at, like, the simplest things in that area and I... that's generally the base case, like n=0 and n=1 if it's like natural numbers. Or like a point if it's geometric. And if it's like groups, then the trivial group.
>
> I: Okay. And then if those very, very low-level ones don't work, then what?
>
> P5: Okay. I will keep trying like, 2 and 3, if it's not working, then probably the problem is wrong. Try like a line if it's geometric or a circle. So increment up in level of difficulty. Yeah. See if something happens there.

In this discussion of the base step, the participant is not relying on any external stimuli (like a specific problem or a statement $P(0)$ or $P(1)$) to discuss the base case. Instead, she is able to imagine the *process* of checking a base case in numerous scenarios(e.g. trivial group in algebra, a point in a geometric proof, etc.). In particular, this description is indicative that she has, at minimum, interiorized the base case action into a process, since she demonstrates the ability to imagine the process of proving a base case in her mind. It is crucial to the development of conceptual understanding of PMI that an individual is able to at least operate at the process stage of the base case, since successful use of PMI requires an individual to coordinate the base case process with the inductive step process. Despite giving a variety of different explanations of the base case, all participants exhibited at least a Process Stage of understanding of the base case, since all participants were exhibited the ability to imagine the process of proving a base case without actually performing the associated action. This is demonstrated in the two excerpts above for Participants 1 and 5, but was also exhibited by the remaining three participants in similar base case discussions. Participants' conceptual understanding of the base case was also highlighted when they were asked how they would explain mathematical induction to someone with no mathematical background. Three of the participants (Participants 1, 2, and 4) responded by giving analogies of PMI. These are explored in greater detail in Section 4.2.5.1. However, they are briefly mentioned here since these analogies did demonstrate an object stage of construction for the base case. In particular, in order to analogize a concept, an individual must be able to apply the process of comparison to the concept, requiring at least an object-stage of construction. Thus, these participants' ability to construct solid analogies for the base case provides evidence that they have at least an object-stage of understanding of this step of PMI. For example, Participant 2 used the analogy of walking up a staircase to represent the technique of mathematical induction. His description of the base case using this analogy follows.

> It is like the foundation that your build needs. Stairs... again, coming back with that analogy, they don't start at the second step or at the third one, they start at the first one, which is right at the floor level... We need to be sure is that we have that first step from where we begin. That's where we need... that's

> why we need that particular case where this actually happens. So then, because if I'm on one step and if I can go to the next step, then I can go one next step and one next step... so that is the idea.

Here, the participant compares the base case to the first stair on a staircase. In terms of APOS Theory, his ability to compare the base case to the first step in a staircase would require him to be able to imagine the base case as a total object to which he can compare other known objects (e.g. a staircase). Therefore, the ability to create a successful analogy relies (in part) to a Participants ability to conceptualize the concept they are analogizing as an entire object. While most the participants primarily exhibited a strong level of understanding of understanding of the base case, in general, most of the data indicates that other participants agree with Participant 1's statement, "all the content of an inductive prood for me is contained in the inductive step, which really tells you how to go from one situation to another." A discussion of the inductive step follows.

Inductive Step

Many of the participants were able to describe the general process of the inductive step without relying on external stimuli, suggesting that they were operated on at least an process-stage of construction. In some cases, participants specifically visualized the process of the inductive step in the context of standard algebraic PMI problems. Consider the excerpt from Participant 2 below. He was asked about the strategies he uses when trying to prove the inductive step. He responded as follows.

> P2: Well, I talk about like, two things I can do. Uh, one is the common thing to do that is like, okay, take $P(n+1)$, let's see how we can get $P(n)$ over there plus whatever up here. Apply the inductive hypothesis to that $P(n)$ and then try to do some other things and play with that other thing that he was like over there and see how we can get there? So that's one thing. The other one that is like, actually, um. Because usually you have, like, equality over there. So I'm

> trying to see what's happened on the left side of the equals. See, what you can get going to on the right, and then try to see how to connect those two when you are, like, trying to play around with those. Those are two things that I would do.

Taken alone, this excerpt could be indicative of an underdeveloped mathematical induction schema, since the process of the inductive step is specifically linked here to a limited subset of problems using PMI. The use of standard mathematical induction problems is explored in depth in Section 4.3. However, regardless of the specific context, Participant 2 still demonstrated the ability to imagine the process of proving the inductive step in his mind without the use of external stimuli. The discussion of the participants' PMI analogies above also applies to the inductive step of PMI. Namely, the ability to analogize the technique requires at least an object stage of understanding of the inductive step. As mentioned in the Base Case discussion, participants in general expressed that the inductive step is the more difficult of the two. They also discussed some difficulties associated with the inductive step that they faced when learning mathematical induction for the first time. Participants were asked what part of the inductive step they struggled with when learning or what parts of this step students may struggle with.

Avital and Libeskind's (1978) work identified several obstacles associated with PMI. Once conceptual difficulty they identified related to the inductive step. They found that advanced students were asking questions like "How can you establish the truth of$P(k+1)$ if you don't even know that $P(k)$ is true?" The authors argue that this difficulty arises from a gap in knowledge regarding the logic of implications. Namely, proving $p \Rightarrow q$ does not inherently show anything about the truth value of p itself. In APOS Theory language, this indicates that participants experiencing this difficulty may be operating at the pre-Implication stage of development, and may not have interiorized the actions of antecedent and consequence of an implication. Both Participants 4 and 5 identified the same issue when speaking about their experience learning PMI for the first time. Participant 5's response to the question is below.

> P5: I think this used to happen to me too. It's like, if it says $P(k)$ implies $P(k+1)$... What I would try to do is like, figure out how I even get $P(k)$, like how to get to the step that we are assuming- that $P(k)$ is true. And understanding that we are not trying to show it. I think that's like a leap of faith. And that makes a lot of students uncomfortable. Like, they don't... And, like... even I didn't understand that, like, no, no, you don't have to check this or like show that it's true. You just like work with the fact that it's true, and then try to show the next thing. You don't have to show that both $P(k)$ and $P(k+1)$ are true. But it's like, there's the thought of like you, you've already checked the base case and, like, if you assume the $P(k)$ case too, why can't you assume like, the problem that you're given is true?

This discussion brings to light three important points.

1. The participant indicates that she no longer struggles with this issue, suggesting that her understanding of the inductive step has developed and matured over time.

2. The described difficulty indicates an underdeveloped logic schema, since the misunderstanding is based on a misconception about the role of the implication in the inductive step. This will be discussed further in the following section.

3. This excerpt corroborates Ernest's (1984) work identifying common epistemological obstacles with PMI, as discussed in Section 2.4.2. Namely, he says that "mathematical induction is neither self evident nor a generalization of previous, more elementary experience." Therefore, students may struggle with the basis and justification for PMI (Ernest, 1984, p. 181-183). This is validated by the struggles discussed above, as the participant is describing a fundamental misconception concerning the way PMI is justified.

In addition to analyzing Participants' views on each step individually, the study also sought to understand participants' perceived relationship between the two.

Relationship between Base Case and Inductive Step

The participants demonstrated varying levels of understanding relating to the relationship between the base case and inductive step. For instance, when asked about the perceived relationship between the base case and inductive step, Participant 1 replied, "I guess I, I almost think of them very not very related at all." It is important to reiterate that this study can only present *observable* behaviors in an attempt to link them to the *unobservable* cognitive processes happening inside a participants mind. With this being said, it is highly possible that the excerpts discussed here do not fully capture the entirety of the participants' understanding of this relationship. This is important since, even when participants expressed underdeveloped understanding of the relationship between the base case and inductive step, this expression was not aligned with their behavior during the CTA. In particular, Participant 1 *did* use the relationship between the base case and successive cases to make his argument.

Other participants responded to this question by simply restating the two parts of mathematical induction without explicating the relationship. Consider Participant 2's response to the question.

> P2: So. We might have our first case, and we know that that is true for that. But then we don't have the tools to say, hey, the next one is true as well. And why is that? Because the first one was true, we don't actually have the proof. So the induction is actually telling you that, hey, you know, for this case, then for the next one is also true.

Here, while the participant demonstrates the ability to imagine both the process of the base case and the process of the inductive step in his mind, there is no explicit evidence that these processes are coordinated, and there is no verbal description that illuminates how they work together to form the basis for an argument using PMI. In contrast, Participant 3 offered a response to the same question that specifically exhibits this type of coordination. Consider the excerpt below.

> P3: I think in a good proof by induction they're related, um. I feel like in this case um, like, the base case is like heavily related to the inductive step, right? Because like, how did we show it was true for the 2×2? Oh, well. I can just put an L in let's say the empty spare the empty spaces in the corner. I can put an L in, and then I can just rotate it around to get the empty square everywhere else, right? If I want to do it for 4×4, I took those 2×2's and just tile them around. That original idea of putting an L and tiling it somehow and then rotating it around was actually how my, like, inductive argument worked, right? I started with some base case that I knew, I could tile and rotate around so I can get the empty cell and or empty cell on every single thing um or empty space in every single cell. Uh, and then I use that fact to build a larger one by making up a bunch of grids that are copies of the original one that I knew, um, in that case, they're very related, um, sometimes in proof by induction arguments that I think are kind of bad, for the base case, you don't have to do anything for so it doesn't feel... it always feels fishy to me.

There are two observations of note in relation to this excerpt. First, this excerpt indicates that the base case process and the inductive step process have been coordinated in the participant's mind since he is able to discuss the linkages between them. Second, the participant mentioned that when he *cannot* see a relationship between cases, a proof using mathematical induction feels "fishy" to him. He indicated that this usually happens when he cannot see the relationship between the base case and the inductive step, suggesting that this relationship may be central to his understanding of PMI. This section focused primarily on the Action, Process, and Object stages of construction. The following section will address the Schema stage.

4.1.1.4 Schema Development of PMI The three stages of schema development, *intra, inter, and trans* were identified and discussed in Section 2.5.3. First, this section gives a brief description of the three stages of schema development in the context of the Mathematical Induction Schema (also referred to as the PMI Schema). Second, the section briefly classifies each participants' overall behavior in the study into one of the three levels of schema development, providing relevant details for each. Recall, the initial stage, *Intra-*, is identifiable when Actions, Processes, or Objects within the Schema are viewed as isolated from one another. The next stage, *Inter-*, is characterized when some relationships begin to form between Actions, Processes, or Objects within some Schema. The last stage, *Trans-*, can be characterized by an implicit or explicit coherence and understanding of relationships developed in the Inter- stage. Consider the table below, which identifies the triad of schema development in the context of PMI, and describes *some* of the characteristics that may be representative of the three stages. The first column identifies the stage, the second gives descriptions of characteristics indicative of the stage of development, and the third column lists the mental structures necessary for an individual to satisfy the descriptions. The table was developed based on the results outlined in the previous section.

Table 4.2: PMI Triad of Schema Development

Triad Level	Description	Necessary Mental Structures
Intra-PMI	• An inductive proof is analyzed in terms of its technical properties (e.g. algebraic manipulations) or in an algorithmic way (e.g. plug in $n = 1$, then plug in n and rearrange to find $n + 1$). • Explanations of PMI are linked to specific classes of mathematical induction problems (e.g. algebraic equalities or sum properties). • The components of the PMI schema (e.g. base step, inductive step, implication) are *isolated* structures.	At least an action conception of **base step, inductive step, implication**.
Inter-PMI	• Connections are identified between some isolated components from Intra-PMI (e.g. Base Step and Inductive Step form the antecedent of an implication which results in a claim for all $n \in \mathbb{N}$). • The necessity of the isolated components from the Intra-PMI stage is recognized in the broader context of the proving technique. • Relationships are formed between the algorithmic process associated with more routine problems from Intra-PMI stage and more general mathematical induction problems (e.g. geometrical, abstract).	At least an object conception of the **base step** and **inductive step** and at least a process conception of the **implication**.
Trans-PMI	• Constructs a complete understanding of the PMI schema and perceives more global applications of the principle to a wide array of problems. • Can coherently explain each step in the inductive process without referring to a specific problem type (e.g. the purpose of the base case without mentioning "plugging in") and can give examples of each step in various contexts (e.g. what the base case may look like in different fields). • Can identify how each separate part of the mathematical induction coheres to form an effective proof technique and how mathematical induction relates to the natural numbers. • Can compare and contrast the method of PMI with other proving techniques and classify scenarios when PMI is an appropriate technique.	At least an object understanding of the **Natural Numbers, Implication, Base Step,** and **Inductive Step**.

The participants in the study exhibited behaviors consistent with different levels of schema development. A brief description of each participant in the contexts of the findings in Table 4.2 is included in the discussion that follows.

- *Participant 1*: Some statements made by Participant 1 were indicative of an Intra-PMI level of schema development, including the excerpt below.

  > P1: I guess I, I almost think of them very not very related at all. Actually. Um, most of the time the base case for me, usually has almost no content. It's, it's checking that the thing works at the very sort of most trivial level, and often at the... well, I'm calling it the trivial level, but often that the, the level of the base case, it's hard to understand what what is actually going on, whereas all the content content and inductive proof for me is contained in the inductive step, which really tells you how to go from one situation to another.

  Here, he described the components of PMI as being isolated from one another and also exhibited a shallow conceptualization of the base case as "having no content." However, in other instances, he exhibited characteristics of an Inter-PMI level of schema development, saying, "In principle I know that you need the base case, because you can write things where you're trying to prove something inductively and a base case doesn't work, and the proof fails because the base case doesn't work." This exhibits that he understands the necessity of the base case within the broader context of PMI. He also exhibited the ability to apply the technique of induction in nonstandard PMI problems (as exhibited by his ability to solve Problem 1 during the CTA). In general, Participant 1 did not exhibit many of the characteristics of the Trans-level of schema development. His explanations of PMI during the semi-structured interview were primarily limited to references to specific problems in the CTA (i.e. linked to specific problem types, rather tht general explanations of the theorems). His inability to identify PMI as an appropriate technique for Problem 2 is also indicative that he may still struggle applying PMI in a wide array of scenarios. Taken together, these findings are indicative that Participant 1 was primarily operating in the Inter-PMI level of schema development during the study.

- *Participant 2*: Participant 2 primarily exhibited behaviors consistent with the Itra-PMI Level of Schema Development. Consider his description of the inductive step below.

> P2: Well, I talk about like, two things I can do. Uh, one is the common thing to do that is like, okay, take $P(n+1)$, let's see how we can get $P(n)$ over there plus whatever up here...the other one that is like, actually, um. Because usually you have, like, equality over there. So I'm trying to see what's happened on the left side of the equals. See, what you can get going to on the right, and then try to see how to connect those two when you are, like, trying to play around with those.

  Here, his description of the inductive step is strongly linked to standard mathematical induction problems involving algebraic manipulations, as illustrated by his reference to "what's on the left side of the equals." This indicates that strong relationships between routine problems and more general applications of PMI may not yet have been formed. This was also seen elsewhere in his description of the base case when he said, "...usually the initial step, you can see it from the statement, like, from $n = 1$ or something or they give you like the endpoint." Again, this exhibits a strong association with the components of PMI and standard mathematical induction problems involving sums. While he was able to capture the necessity of both steps of induction in his staircase analogy discussed in a previous section, the descriptions of each step above still indicate an understanding that is strongly linked to a specific type of problem and are indicative that the participant is primarily operating at the Intra-level during most of the interview.

- *Participant 3*: Out of all the participants, Participant 3 best exemplified a student at the Trans-PMI level of schema development. He successfully identified PMI as an appropriate method of proof in *both* CTA Problems and was able to successfully apply the principle in both cases to successfully construct a complete proof. He frequently emphasized the relationship between the base case and the inductive step of PMI during his explanations saying that "in a good proof by induction, they are related to each other." Further, he emphasized the importance of being able to apply PMI outside of the standard types of problems associated with mathematical induction. Consider the excerpt below:

> I don't know, I mean, to me, like for these mathematical induction proofs to make sense I need to feel in my like, core that, like the reason it's true for the next thing is because of the previous case and it was very clear why it was true. For that problem it was only clear why it was true for the next case, when I literally built it up from the previous case. Whereas like, with these standard mathematical induction ones, it's like, oh, like you start with this $k + 1$ case and then if you, you know, boil things out and move things around it, like, magically pops out, but it doesn't make me feel to my core that, like the k... or that the fact that it's true for the kth iteration means that it has to be true for the $k + 1$st iteration. So I feel like those problems aren't very useful.

Here, he indicates that the algorithmic approaches to standard PMI problems do not necessarily represent the full essence of PMI, and instead he prefers more global applications of the principle that are more illuminating to the technique. This preference could be seen in his approach to the problems during the CTA. Instead of approaching them using an algorithmic application of PMI, he began exploring the problem via the action of working examples (e.g. tiling action, rotation action). He was quickly able to reflect on and interiorize these actions into process which he used to inform the construction of a solid proof by mathematical induction. His success in the CTA and his ability to coherently explain steps in the PMI (as exhibited by his communication strategies detailed in Section 4.2.5) both exemplify a student at the Trans-PMI level of schema development.

- *Participant 4*: Participant 4 demonstrated some behaviors associated with both the Intra-PMI and Inter-PMI level of development during the study. For instance, consider the excerpt below.

  > P4: I still find induction harder to find the holes in them. Just for me, I struggle to be able to... um, to as closely be able to say like, oh, this is like a problem in this inductive argument.

  This difficulty may be indicative that Participant 4 struggles with some part of the underlying structure of a proof by PMI. This struggle could be with understanding an isolated part (e.g. base case or inductive step) or the in understanding the connections between isolated parts (e.g. relationship between the base case and inductive step). In either case, his understanding of PMI is underdeveloped enough that he sometimes struggles to be able to identify what he refers to as “holes” in the inductive argument. However, in other cases, he demonstrated the ability to make connections between these isolated parts. When speaking about the base case, he mentioned that “sometimes the base case shows you how to do the general construction.” Which indicates that he, to some extent recognizes the necessity of the isolated component of the base case. Additionally, he described the relationship between a standard problem using PMI and a nonstandard problem using PMI. Consider the excerpt below.

  > P4: Yeah, so the only other area where I’ve ever done. . . where I really do induction is in discrete math, and I would say the difference is. . . especially like, in graph theory, those kinds of number problems. A lot of the inductive steps are algebraic in nature. You do a lot of them simply by algebraic operations and in graph theory, you do very little of that. So your inductive

> steps are more like manipulating abstract objects. And that's very different. And in some ways, I like the abstract object one more than the algebraic manipulation ones.

This is indicative that he has formed relationships between routine PMI problems and more general PMI problems, which is indicative of the Inter-PMI level of schema development. In addition to these observations taken from the semi-structured interviews, the CTA also gave some insight to Participant 4's PMI-schema. While he was able to solve Problem 2 (stamp problem), he struggled with recognizing the pattern in Problem 1 (i.e. was unable to interiorize the tiling process), which suggests that he may still struggle to apply PMI in more general scenarios. When all behaviors during the study are considered holistically, the data indicates that he operates somewhere between the Intra-PMI and Inter-PMI levels of schema development.

- *Participant 5*: Participant 5 primarily demonstrated behaviors consistent with the Inter-PMI level of schema development. For instance, recall her discussion of the base case presented in the previous section.

> P5: I think like, depending... Okay. I think look at, like, the simplest things in that area and I... that's generally the base case, like $n = 0$ and $n = 1$ if it's like natural numbers. Or like a point if it's geometric. And if it's like groups, then the trivial group.

This exemplifies the ability to identify relationships between algorithmic processes associated with standard problems involving PMI and more general mathematical induction problems. Namely, she is able to translate the base case in a standard PMI problem ($n = 0$ or $n = 1$) to the base case involved in less routine problems (e.g. trivial group or point). Further, she demonstrated that she recognized the importance of the base case within the overall technique of mathematical induction. When asked about the importance of the base case, she said, "I mean, the base case is pretty important because if it fails for the base case, then the whole thing fails. So it's usually easy, but it's also important." Further, during the CTA, Participant 5 translated Problem 1 into a related number theoretic problem which she solved using PMI, then translated that proof back to the original context to help her work on the tromino problem. This will be discussed in greater detail in Section 4.2. However, this does demonstrate relationships between number theoretic applications of PMI with more nonstandard applications of the technique, which is another characteristic of the Inter-PMI level of schema development. Taken altogether, the results of the study indicated that Participant 5 is likely operating at the Inter-PMI level.

There are a few important notes related to the participant classifications above. First, the same participant can and did exhibit behaviors characteristic of more than one level of schema development. As discussed with the progression through the four stages of APOS Theory, schema development is not necessarily linear. Individuals may oscillate between different levels as they develop their schemata. Second, as has been mentioned several times, it is impossible to directly observe cognitive processes and structures since they are internal. The classifications above are based on the observable characteristics demonstrated by the experts during the study. The following sections detail the findings identified during the data analysis as they relate to the guiding theoretical framework of APOS Theory.

4.1.1.5 Conclusions The results of the data analysis presented in this section provide several findings relevant to the current study. First, the observations from the CTA detailed in Section 4.1.1.2 corroborate the work done by Styliandes, Sandefur, and Watson (2016). In particular, this study's participants exhibited behavior that suggests the following model may effectively capture most of the proving strategies of experts working on mathematical induction problems.

1. If no method or proof can be easily or immediately identified, an expert may begin by performing the action of solving small examples.

2. The individual may reflect on these actions until they can be interiorized into processes that they can imagine in their mind.

3. The individual may search for relationships or patterns between successive examples. This could also be equivalent to coordinating the process from Step 2 with the proccess of succession to identify a relationship between cases n and $n + 1$.

4. Once the individual has identified this pattern or relationship, they may be able to reflect on it until they are able to coherently understand the process in totality and encapsulate it into an inductive step object, which can serve as a formalized argument.

This model will be further explored together with the preliminary genetic decomposition in Section 4.1.3.

In addition to this model, the section highlights six other relevant findings pertinent to RQ5. These are listed below.

1. Experts demonstrate behaviors indicative of varying levels of development within the APOS Theory framework for both the base step and inductive step of PMI.

2. Most of the data indicates that the experts in the study have at least a process-stage of understanding of both of the primary parts of PMI.

3. While some participants view the base case as being easy, all of the participants demonstrated recognition of the necessity of the base case as part of the technique of mathematical induction.

4. Experts in the study demonstrate growth in their conceptual understanding of PMI when compared to their initial conceptual understanding of the technique.

5. Some participants are more successful at describing the relationship between the two steps of mathematical induction, but all of the participant behavior during CTA indicates that all participants have, to some extent, coordinated the base case process and the inductive step process.

6. While experts in the study demonstrate behaviors indicative of varying levels of schema development, they all demonstrate a cohesive PMI-schema which includes a solid grasp on both parts of PMI, the base case and inductive step, as well as the ability to apply the technique in *some* novel cases.

These six items primarily give insight to RQ1 and RQ5, since they deal with participants' understanding of the two parts of mathematical induction and how it has developed over time. Overall, while the findings do indicate differences among participants, all participants exhibit the ability to describe the purpose of each step within the context of mathematical induction. Some participants were able to reflect on difficulties or obstacles they experienced when learning mathematical induction for the first time, but were able to identify the misconceptions, and their behavior indicated that they no longer experience these same difficulties, suggesting that they have reached higher levels of conceptual development over time. In particular, the analysis of each participant's demonstrated level of schema development provides insight to their understanding of PMI. Those who exhibited higher levels of schema development were better able to (1) discuss the relationship between some isolated parts of PMI (like the base case and inductive step); (2) apply PMI in a brad range of contexts; and (3) situate PMI outside of routine contexts of application (like standard PMI problems.

4.1.2 ***Revised Genetic Decomposition***

Although the findings of this study support all of the constructions and associated mechanisms described by the preliminary genetic decomposition, they also indicate that some steps in the preliminary genetic decomposition need to be refined and additional steps should be included. In particular, the preliminary genetic decomposition does not seem to effectively capture some of the behavior exhibited by expert participants in both previous studies (Styliandes, Sandefur, & Watson, 2016; Weber, 2008) and the current study. Namely, there is behavior exhibited during the transition from proving the base case to proving the inductive step which seems overlooked by the preliminary genetic decomposition. This section presents the following revised genetic decomposition. The proposed additions and modifications are indicated in bold.

1. Reversal through the existential quantifier to form a process of identifying and testing an appropriate base case $P(a)$.

2. **Interiorize the action of a logical statement $P(N)$ for a given statement P and an arbitrary $N \in \mathbb{N}$.**

3. **Coordinate the process of $P(N)$ from Step 2 with the process of identifying and testing an appropriate base case from Step 1 to form a process of testing a statement $P(N)$.**

4. **Encapsulate the coordinated processes from Step 2 into the statement object $P(N)$ for any $N \in \mathbb{N}$.**

5. Expand the Function Schema to include a function mapping each natural number to a propostion-valued output $(f : N \to P(N))$.

6. Encapsulate logic into the implication $p \Rightarrow k$. The implication cognitively becomes an object which is the value of the function f.

7. **Encapsulate $P(N)$ and $P(N+1)$ into the logical implication $p \Rightarrow k$ to form the implication $P(N) \Rightarrow P(N+1)$**

8. Create the schema of the implication-valued function g where $g(\mathbb{N}) = (P(N) \Rightarrow P(N+1))$

9. Interiorize the action of logical necessity into a process so that inputs P_0 and $P_A \Rightarrow P_B$ allow one to conclude P_B.

10. Coordinate the function g from Step 7 with Modus Ponens beginning with $P(a)$ from Step 1 for an appropriate case a.

11. Coordinate this implication valued function along with the base case process through the use of modus ponens to explain the PMI.

12. Encapsulate this inductive process into an object be connected to the Method of Proof schema so mathematical induction can be applied as a proof method.

13. Generalize actions on the mathematical induction object within various problem types coordinated with the Method-of-Proof schema until students can apply mathematical induction as a proof technique.

14. Generalize actions to the base case object until students can identify an appropriate base case in novel problems where it is not specified.

The four items added to the genetic decomposition (Items 2, 3, 4, and 7) above are supported by several sources.

- These items capture the phemomena identified during the cognitive tasks of the current study. Participants routinely exhibited with behavior corresponding to these cognitive constructions.

- These items are also related to expert behaviors identified by Styliandes, Sandefur, and Watson (2016). Namely, they capture the actions of using small examples of the given statement $P(N)$, coordinated with the other mechanisms involved with mathematical induction, and using these small examples to generalize to a formal argument using PMI.

- These added items are also supported by Ernest (1984), who says, in order to construct a proof for the inductive step of PMI, students should "be able to prove an implication statement by deducing a conclusion from a hypothesis" he argues that, in general, this consists of the ability to make deductions from small examples of the given statement (p.177).

- These added steps in the genetic decomposition also broadly encompass Weber's (2008) finding that the use of example-based argumentation may be integral to how experts convince themselves of a statements validity and may be used to inform their formal use of proof techniques.

This revised preliminary genetic decomposition should be tested in future studies in order to test its validity and to evaluate the need for any more potential modifications. This is discussed further in Chapter 5. The next section discusses the results of data analysis in the context of the Expert Knowledge Framework.

4.2 Expert Cognition and Knowledge Organization

The Expert Knowledge Framework, discussed in Chapters 2 and 3, provides a method for discussing several ways in which expert knowledge is organized and retrieved. Each of the items in the framework has been identified and validated by the various studies on expert knowledge discussed in Section 2.1, and this chapter will further contextualize each item within the specific context of the current study. This framework offers one important lens through which to analyze the research questions outlined in Section 1.2. RQ1 deals with the development of knowledge and understanding over time, and this framework will allow for comparison between the experts demonstrated knowledge organization as graduate students and their recollections of knowledge organization, or lack thereof, when they were initially learning about proof and PMI. RQ2 and RQ5 deal with the relationships between different concepts and ideas associated with proof, proof technique, and PMI. These relationships are intrinsically linked to how experts' knowledge about proof PMI is organized and accessed during proof-related activities, making this framework of particular interest. RQ3 concentrates on the determination of how appropriate PMI is for a given problem. This determination is related to several of the items in this framework including, but not limited to, Pattern Recognition and Contextual Conditioning. Lastly, RQ4 deals with potential obstacles experts may face when solving problems related to PMI. This Expert Knowledge Framework importantly includes items addressing gaps in expert knowledge (Variable Communication, Novel Application, and Comprehension Monitoring). These items will be useful in the discussion of RQ4 and the epistemological difficulties identified in the data. A description of how the framework was used as part of the data analysis follows.

As discussed in Section 3.4.1, this nine-item framework was used to conduct a deductive thematic analysis according to the six-phase framework by Braun and Clarke (2006). Two of the nine items in the framework, Flexible Retrieval and External Exploration had significant overlap in the contexts of the study, and they were collapsed into a single item under the umbrella term Flexible Retrieval. The remaining seven items in the framework were treated separately, though the interaction between items is discussed at the end of the section. It is crucial to emphasize that this first thematic analysis was deductive in nature. When working through the six-phase process for this thematic analysis, the nine-item framework informed each phase. A worked example of these six phases in the contexts of this deductive analysis is detailed below. In the excerpt used for the table below, Participant 1 was asked how he chooses a proof technique for a given proof construction problem.

Phase	Worked Example
Familiarizing Yourself with the Data	Below is an excerpt from Participant 1's transcript that we will use to illustrate the steps of the deductive thematic analysis: "Yeah, that is, I think, hard to analyze. Somehow it's just whatever feels right about the problem. I guess like, for example, in this case, it felt like there was... so I guess I have some, like like, proof strategy toolbox things. Yeah. So one is induction, right? So, here, when I could relate something to a smaller version of itself, then I'm immediately thinking induction that says that type of induction flavor argument. Other times, I'm particularly. . . like the playing around to try and prove something idea lends itself to thinking about contradiction. Because what a proof by contradiction can do is often give you an example that you're trying to find some problem with. Um, so if I, if I start there, then probably. The main idea is contradiction though, eventually I could use the contrapositive. Often sometimes what happens with contradiction is you find why something doesn't work and then you can see how you could prove the thing directly. If you switch your perspective to the contrapositive."
Generating initial codes	Codes Identified in the Excerpt **Toolbox of Proof Techniques**: 'So I guess I have some, like like, proof strategy toolbox things. Yeah. So one is induction, right?' **Relationship between Small and Large Cases**: 'So, here, when I could relate something to a smaller version of itself... **PMI Linked to Case Relationship**: then I'm immediately thinking induction that says that type of induction flavor argument.' **Contradiction Linked to Examples**: 'like the playing around to try and prove something idea lends itself to thinking about contradiction. Because what a proof by contradiction can do is often give you an example that you're trying to find some problem with.' **Links between Proof Techniques**: 'The main idea is contradiction though, eventually I could use the contrapositive. Often sometimes what happens with contradiction is you find why something doesn't work and then you can see how you could prove the thing directly. Or you could switch your perspective to the contrapositive.'
Searching for themes	In this step, I noticed that codes 1 and 5 each deal with the way the participant's knowledge about proof techniques is organized. Code 2 deals with identifying a relationship between 2 cases, while codes 3 and 4 both deal with the contexts in which the participant uses a particular technique. I developed three initial themes: **Noticing Relationships in Examples**: This theme included the sub-case 'relationship between small and large cases' **Proof Technique Organization**: This theme included sub-themes of 'distinguishing techniques' (toolbox) and 'linking techniques' (code 4) **Contexts that Trigger Proof Techniques**: From this excerpt, two sub-themes were identified. 'Contradiction' and 'Induction'

Table 4.3: Worked Example of Six Phases of Thematic Analysis by Braun & Clarke, 2006

Phase	Worked Example
Reviewing themes	In this step, additional codes from different portions of the transcripts were incorporated into the theme and its sub-themes. Some, but not all, examples of this are given below Several other sub-themes were added to the theme **Noticing Relationships between Examples** including 'identifying generalizable patterns in examples' and 'relationships between consecutive examples'. The initial sub-themes of 'distinguishing techniques' and 'linking techniques' sufficiently captured the remaining codes for this theme, so no additional sub-themes were created for the theme **Proof Technique Organization**. 'Direct Proof' and 'Proof by Contrapositive' were added to the theme **Contexts that Trigger Proof Techniques** as other participants spoke about these techniques and the contexts in which they use them. Each proof technique sub-theme also includes excerpts when participants discussed when they *would not* use a certain proof technique. Note: Many other themes were identified from the other passages in the transcripts during this thematic analysis. These will be detailed in the following discussion and are not included in this worked example.
Defining and naming themes	Since this thematic analysis was deductive and based on the expert knowledge framework, this phase consisted of determining whether themes identified in Phases 1-3 could be appropriately matched with any of the 9 items in the framework. If an identified theme did not have significant overlap or was not fundamentally compatible with any of the nine items, they were not included in this part of data analysis and were considered instead as part of the inductive thematic analysis discussed in Section 4.3. From this excerpt, the theme **Noticing Relationships in Examples** was linked to Item 1: Pattern Recognition since it deals with identifying patterns in a problem or proof. **Proof Technique Organization** was linked to Item 2: Knowledge Organization since it deals with arranging knowledge in ways that reflect deep understanding. The theme **Contexts that Trigger Proof Techniques** was linked to Item 3: Contextual Conditioning, since it deals with contexts of applicability.
Producing the report	The report and interpretations for this worked example and the remainder of this deductive analysis can be found in the sections that follow. The report for this section of data analysis is organized according to the nine items from the expert knowledge framework.

Although a theme's value should not be determined solely by its prevalence or frequency in the data, frequency can be used as *one* measure that a theme is appropriate for the data (Braun & Clarke, 2006). Each of the nine items in the classification frequently appeared in the data set, which, together with pre-existing literature, suggests that the framework likely captures at least a subset of characteristics of expertise. As part of the data analysis, the researcher noted the level of commonality for each of the eight items (recall that Flexible Retrieval and External Exploration were combined). First, the number of interviews each item appeared in was noted (out of a total of $n = 10$ interviews). Next, out of the $n = 460$ usable sections of dialogue (see Section 3.4), each was analyzed to see if any of the expert classification items were related to that section. It is important to note that the nine items are not necessarily independent of one another, so one section of dialogue may have been linked to more than one item. The interaction between items in the classification is explored further in Section 4.2.3.9. The information is summarized in the table below.

Table 4.4: Summary of Expert Classification Item Prevalence in Dialogue Sections

Item Name	Number of Interviews	Number of Associated Sections
Item 1: Pattern Recognition	8	30
Item 2: Knowledge Organization	10	29
Item 3: Contextual Conditioning	9	20
Item 4: Flexible Retrieval	6	13
Item 5: Variable Communication	9	16
Item 6: Novel Application	5	11
Item 7: Mathematical Fluency	7	23
Item 8: Comprehension Monitoring	8	33

The section is organized by the items in the classification framework, giving any relevant examples of each item found during data analysis, when appropriate, and contextualizing the item according to the guiding research questions. The definition of each item is reiterated in each section. For a more in-depth discussion of the framework and its corresponding source material, see Section 2.1.

4.2.1 *Pattern Recognition*

Pattern Recognition refers to the idea that experts notice features and meaningful patterns of information that are not noticed by novices. There were two primary sub-themes identified for this item. These are listed below.

1. Noticing Relationships between Examples

2. Visually Identifying a Pattern

'Noticing Relationships between Examples' was a sub-theme used to describe moments when participants gave verbal descriptions of either a concrete or abstract relationship between two or more examples. Codes in this sub-theme often, though not always, dealt with either abstract or formulaic relationships between examples. In contrast, the 'Visually Identifying a Pattern' sub-theme encompassed instances when participants drew pictures of a pattern, visualized some pattern in their minds, or described some visual pattern verbally or by using gestures.

In previous research with expert participants, it has been noted that mathematics experts are able to effectively recognize informational patterns including specific classes of mathematical solutions (Hinsley et al., 1977; Robinson & Hayes, 1978). The notion of pattern recognition occurred frequently in the data. However, it was most closely associated with two concepts in particular. First, it was closely associated with the inductive step of PMI and was commonly seen when participants were working on the mathematical induction problems during the problem-solving section of the interviews. Second, it was highly prevalent in sections of the interviews when the participants were discussing how to identify whether or not mathematical induction is a useful technique for the problem, suggesting some interaction between Pattern Recognition and Contextual Conditioning (item 3). This interaction will be discussed in Section 4.3.10. Some of the most pertinent and noteworthy examples of each sub-theme of pattern recognition are included in this section.

4.2.1.1 Relationships between Examples The act of identifying relationships between two, usually consecutive, cases was named as an important part of proving the inductive step by several of the participants. When discussing this type of identification, the language of "patterns" and "generalizing patterns" was frequently used, suggesting that the participants associate the act of pattern-finding with the inductive step of PMI. Participant 4 was asked to imagine that he was working on constructing a proof for which he had already identified mathematical induction as an appropriate technique. He was then asked to describe how he would approach the proof construction. The excerpt below is a portion of his response.

> P4: ...induction works sometimes well when how you manipulate a small object is the same way I manipulate a big object and so when I'm thinking about induction, I often think about the small example and then ask myself, okay, how did I show this result for a reasonably small not the smallest... but like a reasonably small example and can that same kind of pattern or manipulation work for the more abstract one?... Like, if I do the same algorithm here, it does it work again? If so then, maybe it will work in abstract. And so that kind of toying with small examples is really how I like to think about doing induction.

The strategy described here by Participant 4 is echoed in various instances by most of the other participants in relation to the inductive step of PMI. It is clear that, in order to employ this strategy, the participant must feel confident in his ability to identify or recognize a pattern in the chosen small examples. There are two primary notes to consider regarding this example. First, the pattern recognition described in this excerpt is not grounded in any concrete mathematical induction example, but it is instead described as an abstract strategy used in general in problems involving mathematical induction. This suggests that the notion of pattern recognition is linked, in the participants' minds, to the technique of mathematical induction, rather than to the context in which it is being applied. Second, this strategy of identifying a pattern through small examples and generalizing or abstracting served as the bulk of the participant's description of how he writes the proof of the inductive step. This suggests that his ability to successfully write the inductive step of a proof using PMI may heavily depend on his ability to recognize the pattern for a given proof.

The hypothesis that effective pattern recognition may be linked to success in the inductive step is supported by the data, since participants who were unable to identify a solution to the tromino problem (Participants 4 and 5) were the only two participants who did not recognize the pattern of growth between consecutive chessboards. On the other hand, participants who were able to successfully identify the pattern in small examples were able to construct a sound argument for the inductive step without needing to perform cumbersome tasks like tiling an 8×8 chessboard. For instance, Participant 1 said this during his work on the tromino problem: "I never even drew the 8×8. I just understood what it should look like from a 4×4." However, when Participant 1 worked on the stamp problem immediately afterward, he was unable to recognize the relationship between package prices, and did not successfully come up with a solution to that problem independently. However, given a slight prompt in the post-problem-solving period, he was able to almost immediately craft a valid argument for the inductive proof, which he had been unable to do in the allocated 20-minutes of problem-solving. In the post-problem-solving section of the interview, the participant was asked why he chose to use PMI for the first problem (trominos) but not the second. Part of his response is included below.

> P1: I guess the main, um, argument against induction is knowing that I can do, like, suppose I knew how to, um write down 102 in terms of fours and fives, and I knew the number of fours I needed and the number of fives I needed... I... Huh. Hold up.. maybe I want an inductive proof. Okay. Well, I was about to say, I don't see a way to, uh to get to 103 using my knowledge of the number of fours and the number of fives I used, but couldn't I use one fewer 5... Or sorry, one more 5 and one less 4 to increase by 1? I guess I could have. So, maybe... Maybe induction could work so now this is giving me an entirely different proof

idea.

Here, with minimal prompting, the participant was able to rapidly recognize the pattern in the relationship between package prices. As soon as the pattern was identified, the participant expressed confidence that he could construct an inductive proof. This supports the existence of a relationship between pattern recognition and the inductive step of PMI. This phenomenon gives further evidence of a link between successful pattern recognition and the ability to prove the inductive step. On the other hand, it also suggests that difficulties with pattern recognition might inhibit success with the inductive step. While this section focused on data associated with identifying general or abstract relationships and patterns, the following section highlights some of the more visually-based pattern recognition found in the data.

4.2.1.2 Visually Identifying a Pattern Since the participants' written work was also collected for use during data analysis, it was included when the data was coded for the deductive thematic analysis. From the written work, several instances of visually-based pattern identification were noted. One interesting example of this phenomenon was identified during Participant 3's work on the tromino problem during the problem solving section. In order to deal with the fact that the missing tile could be located anywhere on the $2^m \times 2^m$ chessboard, he identified a rotational pattern that would allow him to generate a tiling for any missing tile from a tiling with a missing corner tile (his argument was explored in detail in Section 4.1). Consider his explanation from the transcript below (irrelevant portions of the dialogue are redacted for brevity and are indicated by "...").

> P3: ...And then rotate so that the empty square is like, um, inside. So it's over... let me draw it. So, it's over there. Well, now there's a new two by two square that I can rotate... Um, I can move the empty square pretty much all all the way around... And it becomes like this pattern, right? So you can just kind of rotate it around, shifting it until it gets where you needed to be... and then the place

where the tile is missing, I can rotate these square pieces to move that missing tile anywhere that I need. Yeah, because then, so then that's just giving you a tiling no matter where the missing square is, like, just generating it from the tiling you have for the original one, right?

Participant 3 drew two figures (Figures 7 and 8) to illustrate the patterns of rotation he is discussing here. His identification of this rotational pattern was crucial to his inductive argument for an arbitrary tiling. He first used PMI to show a tiling existed for all $2^m \times 2^m$ boards with a *corner* tile missing. Then he used a second argument using mathematical induction to show that he could generate a tiling for any missing tile using the missing corner tiling together with this rotational pattern.

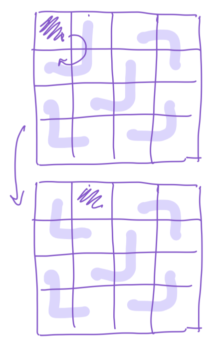


Figure 4.7: Participant 3 Visual Pattern Identification Part 1

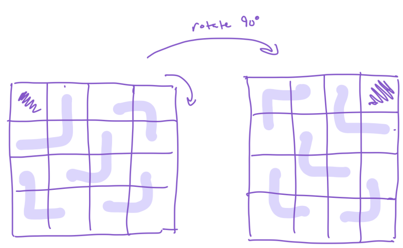


Figure 4.8: Participant 3 Visual Pattern Identification Part 2

One important note in this example of visual pattern recognition is that the participant was able to coordinate two different instances of pattern recognition into one argument. Namely, the participant first identified that the missing tile could be moved anywhere on the board using rotational patterns. Then, the participant linked this pattern to the pattern of the chessboard's growth (this pattern is discussed further in the following section). This exhibits not only the ability to recognize patterns, but also the ability to coordinate patterns in nuanced ways within mathematical arguments. In addition to Participant 3's use of this rotational pattern in the tromino problem, Participant 1 also exhibited the use of visual pattern identification in his solution to the stamp problem.

When trying to identify the appropriate base case for the stamp problem, Participant 1 decided to create a table. The horizontal axis represents the number of 4 cent stamps used, and the vertical axis represents the number of 5 cent stamps used. The numbers in the chart represent the resulting paid package price. The illustration is shown in Figure 9 below.

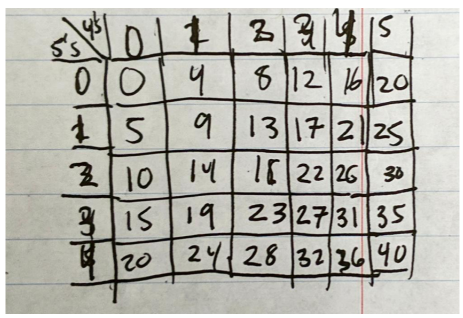


Figure 4.9: Participant 1 Visual Pattern Identification

The participant used the table to identify the package prices which could be created, and used the diagonal pattern to (correctly) conclude that 12 was the appropriate minimal value for the question. While the participant was able to successfully recognize one pattern in his table, he was unable to recognize the modular pattern across the rows of the table which may have helped him successfully make an argument for creating any package price. As previously mentioned, the participant was unable to do so in the second problem. Again, this reinforces the connection between successful pattern recognition and the proof of the inductive step. In addition to pattern recognition linked to images and tables, some notable examples of pattern recognition found in the data were closely related to functions.

4.2.1.3 Conclusions The data supports two primary findings associated with Pattern Recognition as it relates to the current study.

1. Mathematical experts exhibit success at recognizing and using patterns in problem-solving and proof construction.

2. Pattern recognition is linked to the inductive step of PMI.

These two findings offer some insight to RQ3 and RQ4. If mathematical induction is not specified as the appropriate technique for a proof construction problem, the participant must determine it on their own. The data indicates that this determination may rely, at least in part, on a proof-writer's ability to recognize and generalize a pattern or relationship between two cases (regardless of the mathematical context). It therefore follows that difficulty with pattern recognition may also contribute to difficulty with using PMI in novel situations. This relationship has some potential implications for teaching which will be discussed further in Chapter 5. The next section deals with Item 2 in the expert knowledge framework, Knowledge Organization.

4.2.2 *Knowledge Organization*

Experts have acquired a great deal of content knowledge. Knowledge Organization refers to the idea that experts often organize or store their knowledge in ways that reflect a deep understanding of their subject matter. Existing research studying Knowledge Organization has often used the language of "chunking" which refers to a phenomenon in which experts group related pieces of information into cohesive units referred to as chunks, which allow them to more effectively identify relevant pieces of information in a problem-solving context (Chi et al., 1981). It is important to note that novices may also use chunking strategies. However, the important distinction between experts and novices lies in the way in which knowledge is organized. Novices tend to make associations based on surface-level relationships, while experts are more likely to associate knowledge based on big-picture, holistic linkages (Chi et al., 1981). In the case of this study, both surface-level and holistic chunking styles were identified. The two primary sub-themes noted in relation to Knowledge Organization were 'Organization of Proof Techniques' and 'Knowledge Organization by Field or Discipline'.

The first sub-theme, 'Organization of Proof Techniques' consists of instances when the participants discussed or exhibited particular ways of organizing their knowledge relating to proof techniques. This sub-theme included both relationships between the proof techniques and distinguishing factors which make them distinct from one another in participants' minds. The second sub-theme, 'Knowledge Organization by Field or Discipline' pertains to participants' tendency to chunk knowledge and create knowledge associations based on various fields of mathematics. This included relationships between proof techniques, problem-solving strategies, and various disciplines of mathematics (i.e. analysis, graph theory, etc.). Some notable examples from the data along with the corresponding analyses are detailed in the sections below.

4.2.2.1 Organization of Proof Techniques Two of the codes developed in the worked example in Figure 2 deal with knowledge organization, and in particular, focus on the ways in which Participant 1 organizes his knowledge of proof techniques. As discussed in the worked example, there were two primary parts of this proof technique organization noted during data analysis. First, some comments described ways of 'distinguishing techniques' from each other. Second, other comments dealt with 'linking techniques' to each other by discussing the relationships between them. Recall one excerpt from the worked example when Participant 1 said, "I guess I have some, like proof strategy toolbox things." The act of comparing proof techniques to tools was prevalent among many of the participants. When discussing how his understanding of PMI developed over time, Participant 4 said he struggled to use the technique at first, but now he is able to "recognize its usefulness and it's just another tool in my toolbox."

This toolbox analogy lends itself to describing both the distinctions between each technique and the relationships between them. In a toolbox, each tool is naturally appropriate for different scenarios. However, the toolbox itself is a mechanism which holds all the tools in the same place, indicating that the tools may share some common purposes or often be used in similar situations. Analogously, while proof techniques may be used in different scenarios, it is also certainly true that they are interconnected and interdependent in many ways. Participant 1 further used the tool/toolbox analogy when describing the development of his understanding of proof technique. He said the techniques he uses are exactly the same, but his understanding of them has grown and evolved. He described this evolution in the following way.

> P1: I already knew it a wrench was, but now I have like, 10 different sizes of wrenches or something. I think of technique... It's just more an extension of how to formally explain things correctly that you know are true.

Here, he referred to a single, arbitrary proof technique (wrench), but said that he felt as though he has learned new ways of applying that same technique, or tool, which he describes as "different sizes of wrenches." This suggests that the participant does not view the technique as having fundamentally changed as he has developed as a mathematician, but instead he feels he has learned how to better use or apply the technique in a greater range of scenarios.

This finding is noteworthy since it indicates that an individual's perception of the mechanics underlying a technique may be fairly static, even as they develop new ways of applying the technique. This can also be seen in Participant 5's discussion of mathematical induction. When asked if she feels like the technique of PMI changes based on the field she is working in, she said the following.

> P5: I think because it's like a technique, like how people use it might differ, but I think it's like the same... Also it's like a very, it's not like a very flexible technique. Right? You need some hypothesis. Um, to be true before you apply

> it, like, we need like, some a set, which is well-ordered. So I think it's like a bit rigid in that sense. Like, if you if your problem has that setting, then I think then, using induction would be the same in, like, different fields.

This excerpt gives some important insight to Participant 5's knowledge organization. In particular, much of the existing literature with PMI and novices suggests that students may strongly associate PMI with particular phrases like "for all $n \in \mathbb{N}$" (Avital & Libeskind, 1978; Ernest, 1984; Movshovitz-Hadar, 1993). This association is certainly related to Item 3: Contextual Conditioning, and will be discussed in the next section. However, the linkage is also likely related to how novices are chunking information associated with mathematical induction and proving techniques. Participant 5's discussion above shows a different, more nuanced way of chunking this same information. Rather than linking mathematical induction to surface-level characteristics (like phrases involving the natural numbers), she instead focuses on broader characteristics like a "well-ordered" set. This type of chunking allows her to conceptualize the technique of mathematical induction as being the same regardless of the context, and it allows her to be able to recognize when mathematical induction may be appropriate in a broader range of contexts. Participant 5's discussion above does not necessarily mean that experts have no associations between mathematical fields and PMI, but instead suggests that the fundamental characteristics of the technique are not context-dependent. However, some of the data indicated that participants might associate certain proof techniques more strongly with certain disciplines based on experience. This is discussed in the section that follows.

4.2.2.2 Knowledge Organization By Field or Discipline The belief that a proof technique's fundamental characteristics are context-independent was discussed in the previous section. While this consensus was shared by most of the participants, it was also the case that some of the experts' knowledge is chunked or organized according to discipline. Consider the excerpt from Participant 2 below.

> P2: That's a good question. Because I remember when I got here [he's referring to graduate school], I was thinking, like, okay, I know a lot of about proofs and techniques, but then I took that class in geometry and I started to see the proofs that came from geometry and I was like, okay, this is something completely different... So, yeah, I was thinking that I. . . that I had a good development of proof techniques, but there is so much difference between the fields. At the end, what you know is with respect to your field. . . kind of. It's a little attached to your field. I mean, of course, we can generalize and move techniques from 1 field to another 1, but still there is like a little relation between the field and the technique of proving that we are using.

While this quote reinforces the finding that experts may not view proof techniques as context-dependent (he acknowledged that "of course we can generalize and move techniques from one field to another"), there may still be some relationship between the technique and the field in which it is being applied. An important detail to note is that, despite a perceived field-based difference in application or appearance of the technique, he is still able to recognize that the technique itself is the same. This suggests that the knowledge organization he is discussing here may be more significantly related to problem characteristics or nuances of the proof than to the proof technique itself.

This type of problem or situation-based organization was seen elsewhere in the data. Consider the excerpt from Participant 4. Here, he was asked what his first step is when working on a proof construction problem.

> P4: Well, I categorize it. So, this problem is very clearly discrete math. And so I think to myself, okay, it's a discrete problem. So what are the typical methods of proof that discrete mathematicians use and is it likely that one of those methods is going to work here?... Since I started learning mathematics, I've always categorized things, but your ability to know proof strategies is something that just kind of develops the more techniques that you see and the more proofs

that you see and have yourself discovered.

Within this excerpt, we are able to more closely analyze this field-based organization. It is clear here that the participant referred to using an organizational system to classify problems. First, he categorizes it based on what field he thinks the problem is coming from. Next, it appears that for each field or discipline, he has some number of techniques he thinks are commonly used in that field, likely based on experience and exposure. He begins exploring the problem by first trying these associated strategies. This finding is consistent with the pattern identified by Styliandes, Sandefur, and Watson (2016), which was discussed in Section 2.4.3. In particular, the first step in the expert proving exercise pattern noted that experts often begin by attempting " to identify a reasonable method or technique to prove the statement. If one can be identified, they may use the technique without necessarily thinking about *why* the statement is true" (Styliandes, Sandefur, and Watson, 2016, p.23). This approach is exemplified in the excerpt above, since the participant's main focus seemed to be identifying a technique that would work for the given problem, and there was little to no discussion of why the given statement might be true. This may suggest that experts use their knowledge organization in ways that help them more rapidly identify appropriate techniques for given classes of problems, validating other work done with mathematical experts (Hinsley et al.,1977; Robinson & Hayes, 1978; Styliandes, Sandefur, and Watson, 2016). The primary findings of this and the previous section are concisely summarized in the following section.

4.2.2.3 Conclusions The data supports three primary findings associated with knowledge organization as it relates to the current study.

1. Mathematical experts recognize the similarities and differences between various proof techniques and generally group their knowledge of techniques together, adding to this knowledge as they gain mathematical maturity.

2. Mathematical Experts recognize that the fundamental components and characteristics

of a given proof technique are context-independent.

3. Some mathematical experts may organize their knowledge according to mathematical discipline and may associate certain proof techniques more closely with particular disciplines.

Item 1 offers some insight to RQ1. The experts in the study generally expressed that their definition and fundamental understanding of proof techniques, including PMI, has remained generally stable, but the number of contexts they can apply the technique in and the number of ways they know how to implement the techniques has grown in tandem with their mathematical maturity. Item 2 offers insight to both RQ1 and RQ2. First, experts described their recognition that proof techniques are context-independent as developing over time ("I knew what a wrench was already, but now I have 10 different sizes of wrenches"), which is indicative of conceptual growth after gaining experience. Second, the fact that experts view techniques as context-independent gives some enlightenment to the relationships between specific techniques, including mathematical induction, and the act of proving. Finally, Item 3 gives insight to RQ3. Evidence suggests that experts may organize their knowledge based on mathematical disciplines, and they may associate certain techniques with specific fields. The evidence also indicates that this organizational strategy may be used in order to help the expert more rapidly identify an appropriate proof technique for a given problem. This last finding is also closely linked to the idea that expert knowledge may be linked to context. Item 3 of the framework focuses on contextual conditioning, and it is explored in the following section.

4.2.3 *Contextual Conditioning*

Contextual Conditioning refers to the idea that experts' knowledge often cannot be reduced to sets of isolated facts or propositions but, instead, reflects contexts of applicability. In other words, expert knowledge is frequently "conditionalized" on some set of circumstances. According to Bransford, Brown, and Cocking (1999), the concept of conditionalized knowledge offers important implications for pedagogical practice, but “many forms of curricula and instruction do not help students conditionalize their knowledge” (p. 31). As exemplified in the worked example in Table 4.3, Contextual Conditioning was primarily seen during discussions of when participants choose to use particular proof techniques. Although all proof techniques (i.e. direct, contrapositive, contradiction, etc.) were referenced in various codes throughout the transcripts, this section will restrict attention specifically to codes associated with this item which primarily dealt with PMI, since that is the primary focus of the current study. With this restriction in mind, two primary sub-themes were identified in relation to Contextual Conditioning. These sub-themes are listed below.

1. PMI Associated with Specific Mathematical Fields

2. PMI Associated with Problem Characteristics

Sub-theme 1, ‘PMI Associated with Specific Mathematical Fields’ encompasses instances when participants contextually conditionalized mathematical induction by linking it to specific fields or disciplines (i.e. graph theory, discrete math, etc.). This also includes any sections of the transcripts where participants mentioned fields that they *do not* associate with mathematical induction. Sub-theme 2, ‘PMI Associated with Problem Characteristics’ focuses on moments when participants associated the method of mathematical induction with certain characteristics of a problem or problem statement. These sub-themes are each explored in more detail below, and they will be discussed using the language of “condition-action” pairs, which is a phrase used to describe the link created by an individual between some condition and a corresponding action that occurs when the condition is met (Bransford, Brown, & Cocking, 1999, p. 31).

4.2.3.1 PMI Associated with Specific Mathematical Fields As mentioned in the discussion of knowledge organization in the preceding section, experts may organize parts of their knowledge according to mathematical field. There were strong links in the data between knowledge organization and contextual conditioning, and those are discussed carefully in

Section 4.3.10. However, this section will strictly discuss the contextualized conditioning elements of the data. Specifically, the discussion will begin with noteworthy examples of proof techniques being contextually situated in relation to specific mathematical disciplines. Participants were asked to explain how they might choose an appropriate proof technique for an arbitrary proving exercise. Participant 2's response to the question is below.

> P2: Now um, if I see that the problem looks like a real analysis problem, most likely I will go for constructive. Because in my experience, in that field, most of the problems are actually constructive. So, I will go with that. I know that in complex analysis, a lot of the proofs are by contradiction, so if I'm studying something like that, and I need to do a proof, most likely, I will try contradiction. So, it's more like um... an experience thing.

Here, the participant indicated that, based on his experience, certain fields of mathematics automatically trigger him to attempt to use specific proving techniques. This study does not seek to make a judgment on whether these associations are "good." That is to say, evaluating whether or not most real analysis proofs "are actually constructive" or whether or not "in complex analysis, a lot of the proofs are by contradiction" are true statements is outside the scope of this study. Instead, we will focus on the act of using experience to create a set of conditions that, when they are met, prompt an individual to perform a certain action. In this case, the condition is a problem in a specific discipline and the action is using a particular proving technique. As previously mentioned, particular focus will be placed on PMI. The following quote from Participant 4 highlights the same condition (a problem in a specific field of math) which is associated with the action of using mathematical induction as a proving technique.

> P4: I was probably biased towards induction, because my first thought was that this is a discrete problem and I... use induction a lot in discrete proofs. So I... I categorized where it fit and then just using some of the intuition from what I've

> seen. Really... you know, you're like, okay, there's certain tactics that... tools or methods of arguing that are often successful and so that's at least a good first place to start.

Here, the broad field of discrete math seems to be linked strongly to the technique of mathematical induction. Therefore, without necessarily even mentioning the problem statement, the participant already seemed primed to try PMI in the proving activity. This excerpt provides further support for the claim that experts likely use mathematical fields as one context through which their knowledge of proof techniques can be conditioned. In addition, this data further corroborates the claim by Styliandes, Sandefur, and Watson (2016) that experts may begin a proving exercise by attempting to identify a proving technique without necessarily thinking about why the statement is true. While several of the participants briefly mentioned mathematical fields as being associated with PMI, an even stronger contextual conditioning pattern was seen based on problem characteristics. This is explored in the following section.

4.2.3.2 PMI Associated with Problem Characteristics The transcripts contained numerous examples the technique of mathematical induction being contextually triggered by specific problem characteristics. Some of these examples were surface level characteristics, like the phrase "all natural numbers," that even novices have been shown to associate with PMI (Avital & Libeskind, 1978; Ernest, 1984; Movshovitz-Hadar, 1993). For instance, Participant 4 said, "Whenever I see natural numbers, the first thing I always think of is: is induction going to work on this?" Likewise, Participant 2 said, "So it will depend on the problem. If I see like, for example, the natural numbers and something that is going step by step, okay, it is induction, no questions asked." Further corroborating this well-documented association, Participant 3 stated, "I mean, like, in this one, in particular, the, the hint is that you're trying to show that there's like, some numbers of stuff for everything, bigger than it. Like this thing is true well, then that probably, like, suggests that there's gonna be some induction going on." In all of these examples, the condition is a proving problem mentioning natural numbers and the resulting action is the participant identifying PMI as an appropriate proving technique. As mentioned, this type of contextual conditioning has been noted by several studies, so these examples simply corroborate a fairly well-known result. However, a more interesting example of contextual conditioning associated with problem characteristics also appeared in the data.

In addition to the natural, rudimentary association between PMI and the natural numbers, other problem characteristics were also shown to trigger the use of PMI. The most commonly expressed characteristic associated with mathematical induction an identifiable relationship between a small case or example and a larger case or example. This type of relationship was explored in Section 4.3.1, but this section will focus on the ways in which this relationship is used as a contextual trigger that signals an individual to use mathematical induction. Consider the excerpt from Participant 1.

> “So one is induction, right? So, here, when I could relate something to a smaller version of itself, then I'm immediately thinking induction that says that type of induction flavor argument.”

A relationship between smaller and larger cases was the most common contextual trigger for PMI described by participants. Participant 4 said, “whenever I think of induction now, what I often think about is... induction works sometimes well when how you manipulate a small object is the same way I manipulate a big object.” Another finding that is equally as important is that participants also used the *lack of* this relationship in a problem as a signal that mathematical induction may not be an appropriate choice. This can also be seen in an excerpt from Participant 1.

> P1: And it is quite often the case that I don't see a way to relate sort of, the smaller cases, and then induction's off the table and go and do other stuff.

This notable trend represents a “lack of condition-inaction” pair. Namely, if the participant does not identify a relationship between cases, then the action of using PMI as a proving technique is not triggered and, more importantly, may be disregarded immediately. While contextual conditioning has many benefits, this "lack of condition-inaction" phenomenon gives one example of a pitfall of conditionalized knowledge. Consider the example below.

Three out of the five participants were able to effectively identify the relationship between successive chessboards in the tromino problem, and in each case, they were able to almost immediately identify PMI as an appropriate proof strategy. A fourth participant (Participant 5) first considered an adjacent number theoretic argument (counting tiles) and successfully used PMI to prove that statement. She was also able to recognize the link between successive chessboards, but ran out of time before finishing the proof. In the last case, Participant 4 was unable to successfully link the condition-action pair. Namely, he attempted to use mathematical induction, but was unable to identify the correct relationship between successive chessboards. See an image from his written work below.

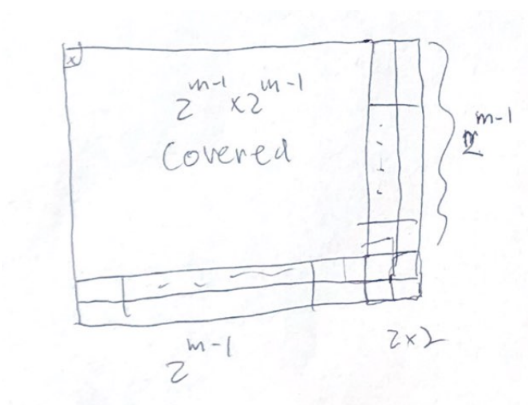


Figure 4.10: Participant 4 Issues with Condition-Action

This image was supposed to represent a $2^m \times 2^m$ chessboard. However, the picture drawn actually has dimension $(2^{m-1}+2) \times (2^{m-1}+2)$. Because of this mistake, the participant was unable to effectively find a useful relationship between cases, meaning that the condition in the aforementioned condition-action pair was not satisfied. Immediately, he decided that mathematical induction would not work and was ultimately unable to answer the question. This result was also seen in the second problem, albeit with a different subset of participants getting correct responses. However, the primary takeaway is that participants who immediately identified the relationship between cases, rapidly identified PMI as an effective proof technique. Those who did not meet the condition overwhelmingly failed to perform the action of proving. It is particularly important to note that in several cases, a single participant would successfully use the condition-action pair in one problem and fail to identify the condition in the other. This supports the claim that if the condition is not recognized, it may result in difficulties with adaptability. This relationship between expertise, conditionalized knowledge, and adaptability is well-documented in the literature.

An important aspect of conditionalized knowledge is its automative nature. The condition-action pairs occur, in many ways, without conscious thought. This has been studied in the literature, and there are several problems related to this phenomenon. The term automaticity refers to the mostly effortless execution of cognitive procedures that have been developed through the repeated association of condition-action pairs (Schneider & Shiffrin, 1977). Because this automated behavior often occurs without conscious thought, it has sometimes been found to inhibit experts when confronted with environments not well-suited to their pre-developed condition-action pairs (Ericsson, 2004). Hatano and Inagaki (1986, 2004) describe two disctinct kinds of expertise, routine expertise and adaptive expertise. Adaptive experts are able to successfully perform even in changing conditions, while routine experts are successful in established, predictable situations. The current study validates work done on the relationship between automaticity and adaptibility, and suggests that the participants in the study may be primarily demonstrating routine expertise in relation to their PMI condition-action pairs. The following section summarizes the findings relating to conditionalized knowledge.

4.2.3.3 Conclusions The data supports three primary findings associated with contextual conditioning as it relates to the current study.

1. Some experts may conditionalize their use of specific proving techniques based on the field of mathematics the problem is in, and certain proving techniques may be perceived

to be more closely related to specific disciplines as a result of this conditionalization.

2. Most mathematical experts still strongly link language involving the natural numbers with the technique of mathematical induction.

3. Experts likely conditionalize their use of PMI on the identification of a relationship between small and large cases in a given problem, and the lack of this identification can result in the expert actively dismissing PMI as an appropriate technique.

All three of these items offer insight to RQ3. The data suggests that these mathematical experts have created condition-action pairs that link PMI to particular problem characteristics, and that these pairs are likely utilized in the determination of whether or not PMI is an appropriate method of proving in a novel scenario. In addition, these items also provide some enlightenment regarding RQ1. Namely, previous studies have identified links between the natural numbers and mathematical induction, but these findings suggest that conditionalized knowledge associated with mathematical induction may become more nuanced over time. Clearly, the original associations with $\mathbb{N}$ still exist, but there is evidence of more mature associations with the expert participants, as seen in findings 1 and 3 above. The relationship between adaptability and conditionalized knowledge discussed in the previous section also provides insight to RQ4, since limited adaptability of their condition-action pairs likely inhibits their ability to identify PMI as an appropriate technique in some scenarios. In addition to contextualizing their copious amounts of knowledge, experts also need to be able to quickly and effectively retrieve this knowledge. This is explored in the following section.

4.2.4 *Flexible Retrieval*

Experts are able to flexibly retrieve important aspects of their knowledge with little attentional effort. This is referred to as Flexible Retrieval. As mentioned at the beginning of Section 4.3, there was significant overlap in this item and Item 9: External Exploration. Experts often search known relevant texts along with other outside resources in order to gain a better understanding of the concept in question. External Exploration is the term ascribed to this practice, and it was strongly linked to Flexible Retrieval during the deductive thematic analysis. In particular, Flexible Retrieval was primarily linked to two sub-themes in the data. First, the sub-theme 'Recalling Relevant Facts' was used to describe moments when participants recalled specific facts or theorems which they perceived to be associated with a given problem. Second, 'Linking Novel Problems to Known Proofs' referred to a common occurrence in the data when participants discussed how they use known proofs and proving strategies when approaching novel problems. While related, the two sub-themes are distinct. The first sub-theme strictly encompasses moments when general facts or pieces of knowledge were retrieved, and the second sub-theme focuses on when knowledge associated with the act of proving or proving strategies was accessed and used. Both sub-themes had strong ties to the idea of external exploration, and almost all codes assigned to the external exploration item were also linked to flexible retrieval. Because of this significant overlap, these two items were collapsed into the single item named Flexible Retrieval in order to avoid redundancy. Both sub-themes are explored in more detail in the following sections.

4.2.4.1 Recalling Relevant Facts Data associated with the sub-theme of 'Recalling Relevant Facts' most often appeared in the problem-solving sections of the interviews, when participants were working on novel problems and thinking about potential solutions. Consider the excerpt below from Participant 1.

> P1: Yep. All right. Okay, um. So, my first thought is, this is a little similar to the last question in that. I kind of recognize it, meaning I've thought about problems like this before so that that helps with the framework. So I'm trying to figure out. Okay, I've, in fact, even remembered the answer which. My my cheap, um, by the answer, I mean, the smallest number for which you can't express it as a linear combination... With two numbers, I think should be those two numbers minus the sum of the two numbers or something. So I guess, like,

I've remembered a claim that the smallest number you can't write should be 11.

In this piece of dialogue, Participant 1 was beginning to work on the stamp problem. Although the participant had not seen this specific problem before, he recognized some elements of the problem and was quickly able to recall some relevant information with no apparent difficulty. These are the first few lines spoken by the participant after being given the problem. This demonstrates the ease with which he is able to identify relevant information. It is important to note that, while it is expected that experts usually retrieve information that is generally relevant to a given task, not all retrieved information will necessarily be useful. For example, when working on the stamp problem, Participant 4 quickly recalled a relevant fact which did not necessarily help her find a solution. After noting that the problem dealt with linear equations of the form $4x + 5y = m$, she immediately recalled a number theory fact stating, "so like the gcd of 4 and 5 is 1, so, like a linear solution… like a solution to a linear equation 4x+5y=1 exists." However, she quickly judged that this fact was not useful and proceeded to try other strategies. Likewise, other participants often accessed and retrieved relevant knowledge quickly, regardless of it's usefulness to the problem. Participant 1 fluidly recalled the Chinese Remainder Theorem and some of its implications when working with modular arithmetic in the stamp problem, even though it was not necessary to prove the statement. These examples of retrieval, whether the retrieved information was "useful" or"not useful," give insight to its importance in conversations involving the study of expertise.

In the context of Flexible Retrieval, is not necessarily the use-value of a retrieved piece of knowledge that is most important. Instead, there are two primary characteristics of retrieval that make it valuable to experts- speed and effort. Speed and cognitive effort tend to be strongly correlated. Schneider and Schiffrin (1977) characterized three primary levels of retrieval: effortful, relatively effortless, and automatic. Effortful retrieval is primarily seen in novices. It requires significant attentional effort and nontrivial amounts of time. Experts most often fluctuate between relatively effortless retrieval and automatic retrieval.

It therefore takes significantly less time to retrieve the information, and because it also requires little to no conscious thought, experts are able to retrieve relevant knowledge while still actively working on the task at hand. It is important to note that the speed of retrieval does not necessarily mean that experts will always perform the overall task faster (Bransford, Brown, and Cocking, 1999). Experts often attempt to gain a deep understanding of a problem, which may take more time than simply identifying a solution. The data of this study supports this finding, since participants often retrieved and explored relevant knowledge that was not necessarily linked to the solution and used a nontrivial amount of their problem-solving time to explore this retrieved knowledge. The participants still demonstrated retrieval that was fast and visibly effortless, even if the solution to the problem itself did not necessarily develop quickly. In work discussed in Section 2.2, Selden & Selden (2017) identified some obstacles associated with proof construction. Item 5 on this list involves "knowing, but not bringing, appropriate information to mind," and Item 6 deals with "knowing which (previous) theorems are important" (p. 3). The experts in this study demonstrated that their flexible retrieval can help preclude them from these types of obstacles. In addition to relevant facts and isolated pieces of knowledge, experts in the study also demonstrated the ability to flexibly retrieve larger pieces of information like entire proofs or overarching proof strategies. This is explored further in the following section

4.2.4.2 Linking Novel Problems to Known Proofs Participants frequently referred to the retrieval of previously known proofs or proving strategies. The participants may have either previously read the proof or constructed the proof, or both. The data suggests that the participants were able to quickly retrieve both entire proofs and overarching proof strategies from their memory. When asked what his first step is when working on a proving task, Participant 2 immediately replied, "It will depend on two things. Do I know the problem? If I know the problem, then I will already answer with the proof that I know." This is indicative that, in some contexts, he is able to recall entire proofs for statements with little to no need to "re-work" the proof. Participant 3 discussed a similar ability in the excerpt below. When asked how quickly he identifies an appropriate proving technique for a given problem he responded as follows.

> P3: Typically really early on. Um, I mean, at least for like, elements like more elementary proofs, I guess usually pretty early on, but that's just because, like, I usually know how it's going to go instantly. Like, I, at least now when I look back at, like, you know some of the old problems that I used to do an undergrad that got assigned to me for homework. Like, I don't really have to think that hard about them anymore. And so it was like, oh, this is an inductive argument. I know what they're looking for, because I'm just familiar with this.

There are two phrases of note in this excerpt. First, he said, "I usually know how it's going to go instantly" indicating that the retrieval of his knowledge of a given proof is instantaneous. Second, he indicated that he doesn't "really have to think that hard about them anymore" indicating that the retrieval of this knowledge does not require much of his attention. These phrases exemplify the two primary characteristics of Flexible Retrieval, speed and effort, detailed in the previous section. Although the retrieval of proofs could arguably be linked to memorization, it is important to note that experts are exposed to innumerable proofs over their years of study.

While flexible retrieval is certainly linked to memorization in some ways, this level of retrieval is also more nuanced and extensive than the rote memorization of algorithms we often see with novices in mathematics. This nuance can be more easily seen in examples where participants retrieve proof strategies, rather than entire proofs themselves. Participant 5 was asked about her strategies when dealing with a novel proving task. Her response can be seen in the excerpt below.

> P5: Thinking of a similar example. Like the problems that I dealt with in my research is like… You already know how to get the estimate on this sort of set.

> And like, the set that I'm looking at, it's sort of similar, but not really exactly the same. So go through that proof and then, um, see if this proof is like, maybe 2 different cases of that one, something like that.

Here, she did not refer to the recollection of a proof in its entirety, but instead referred to a known proving strategy in her field. Participant 2 gave a similar example.

> So, sometimes it's actually doing proof, but a proof I've seen from someone that already did it, only to get the feeling that I understand that. And then I would start to do the proof of my own conjecture. Uh, based on things that I read on that paper, it can be as easy as trying to generalize one thing, I don't know, like, what I did with rectangular cases from the argument about square matrices.

Again, the participant referred to the recollection of a known proof in terms of the *strategy* involved in that proof. Both of these excerpts exemplify some of the distinctions between rote memorization and flexible retrieval. The retrieved knowledge in these cases is more than just a line-by-line memory, but instead reflects an understanding of the structural aspects of an argument as well an ability to adapt the recalled argument to a current task. This finding is linked to existing work discussed in Section 2.3. In particular, studies with mathematicians show that when reading (taking in) a proof, a mathematician may file that proof away for use in future, relevant work (Giaquinto, 2005; Meija-Ramos & Inglis, 2009). The data in this study that was linked to Flexible Retrieval corroborates this claim. The findings associated with this item are detailed and summarized in the following section.

4.2.4.3 Conclusions The data supports three primary findings associated with Flexible Retrieval as it relates to the current study.

1. Mathematical experts exhibit the ability to flexibly retrieve relevant facts and theorems when working on proving problems.

2. Mathematical Experts exhibit the ability to flexibly retrieve and recall both entire proofs and overarching proof strategies.

3. The flexible retrieval exhibited by mathematics experts in this study has characteristics which distinguish it from rote memorization.

These findings, in totality, offer insight to RQ2 and RQ3. First, knowledge of specific proving techniques, including PMI, may be stored and retrieved quickly and with little attentional effort by experts. Experts in the study demonstrated fluency with identifying relevant proof techniques for given types of problems and retrieving associated knowledge quickly. It is likely that this type of fluent retrieval is related to how proof and specific proof techniques are linked and organized in experts' minds. Second, the determination of PMI as an appropriate proof technique for a given problem may be related to fluency and flexible retrieval. In order to appropriately identify PMI as an appropriate technique, for example, an expert will likely need to retrieve known examples related to a current task where PMI has been an appropriate technique. Additionally, Item 3 offers some enlightenment to RQ1. Namely, studies have identified that novices often exhibit rote memorization and restrictive algorithmic approaches to proving problems, particularly problems involving PMI (Avital & Libeskind, 1978; Ernest, 1984). In contrast, the flexible retrieval documented in the preceding sections is significantly more nuanced in nature. This offers evidence that experts' fluency surrounding both proof and proof techniques deepens over time. While experts may be able to easily and fluently retrieve known information, it is not necessarily the case that they can effectively communicate this knowledge. This phenomenon is explored in the following section.

4.2.5 *Variable Communication*

The term Variable Communication is used to describe a phenomenon associated with expertise. In particular, experts may or may not be able to teach others effectively, and expertise is not necessarily a good indicator of an individual's ability to communicate their own knowledge. This item is an important part of the framework, since it highlights the difference in expertise in a field and expertise in the pedagogy or communication within the same field. There were two primary ways that variable communication appeared in the data. First, participants were asked how they might describe PMI to a student who had no mathematical background. Almost all the participants chose to answer this question with a mathematical induction analogy, and a few of these are described in more detail in Section 4.3.5.1. Second, there were several instances when participants discussed how they might teach mathematical induction to students learning the technique for the first time. Some illustrative examples of this are discussed further in Section 4.3.5.2.

4.2.5.1 Induction Analogies As previously mentioned, during the interview, participants were asked how they would describe the technique of mathematical induction and why it works to a student with no mathematical background. Three participants chose to use analogies in their answers. Each of these three distinct analogies explicitly describes the purpose and role of the two primary parts of mathematical induction, the base step and the inductive step. Implicitly, two other aspects of PMI are also addressed by the analogies. First, the analogies address the necessity of each part of mathematical induction. Second, each analogy includes an understood ability to continue in perpetuity. That is, the notion of "infinity" is understood to be involved in the process of using mathematical induction. Each of the analogies, along with relevant excerpts from the transcripts, is included in the table below.

Table 4.5: Participant PMI Analogies

Participant	Summary of Analogy	Relevant Excerpt(s)
Participant 1	"Traveling along a Path" Base Step: Ability to make it to some step on the path Inductive Step: Ability to make it from one step to the next	"Induction is like, uh, traveling along a path and claiming that you could go to infinity if you wanted to. So, the idea is that if you can make it to the kth step on the path, and from every kth step, you have a way to get to the next step. Well, in principle you could go as long as you wanted."
Participant 2	"Stairway to Heaven" Base Step: Some beginning step in the stairway Inductive Step: Ability to move from one step to the following step	"An example with the stairs is that in induction, we are showing that if I am on one step, I'm able to construct the next step and go onto that step. So then, because I'm in the next step, I can construct one next step again and one next step and one next step... so that is the idea. But the only thing that we… we need to be sure is that we have that first step from where we begin. And that's where we need… that's why we need that particular cases where this actually happens."
Participant 4	"Dominoes" Base Step: The first domino falls Inductive Step: Every domino knocks over the following domino.	"If the first domino fell over, and if you assume that every previous domino is going to knock over the next one, Then as long as the first one falls over, you know that the rest of them are going to fall over. And so, as long as you know that one of them implies the next one, if the first thing is true, then that implies the second one, implies the third one, implies the fourth one, and so on."

Some research with expert participants indicates that expertise can cause difficulties with teaching, as experts can have a difficult time determining what parts of the content may be difficult for students (Bransford, Brown, & Cocking, 1999). The analogies in the table above indicate that some of the experts in this study are able, on some level, to effectively identify important characteristics of PMI and to cohere them into a form that would be understandable to someone with little to no mathematical background. They communicate complex mathematical ideas, like the concept of infinite iteration, in ways that could be understood by a layperson, like stairs or dominos. In contrast, Participant 5 struggled with this question. Her response is included below.

> P5: Induction definitely, uh, relies on the well ordered property of natural numbers. And then you want to check that it's true in the base case like the prob, like, even in the simplest case, it's... the problem is true. So you do that. And then, uh. What you do is you pick like, a random number, and you assume it's true for that and then you try to show that it's true for, like, the immediate next 1. And if that works out good, and the thing is... because the number that you chose was random, like, you can apply that for anything. Like, you can take the base case, and then the next 1, and then. This 1, and the next 1, and you can keep doing it forever, but you don't have to because the number you chose was random. So, in a way are, like, proving it for, like, all the natural numbers.

Although her response is generally correct, it lacks coherence, and it would certainly not be comprehensible to someone with no mathematical background. Terms like "natural numbers" and "base case" would likely be confusing for someone who had not been exposed to that language, and they are introduced without a recognizable example to ground them. Taken together with the analogies described above, this example provides insight to the varying levels of success experts may have when attempting to communicate a mathematical concept to someone else. It is important to note, that the ability to communicate concepts can vary depending on the concept. For instance, Participant 5 may have been able to explain a different concept more successfully than Participant 2. The ability to communicate one concept is not necessarily indicative of the ability to communicate mathematical ideas as a whole. Importantly, these responses were associated with learners who had no mathematical background. Participants also spoke about how they would communicate the technique of mathematical induction to learners in an introductory proofs course who have a general understanding of mathematical terms and logic. These findings are discussed in the following section.

4.2.5.2 Communicating PMI to Math Students Graduate students represent a specific subset of the set of mathematical experts. They are both students and teachers. They have years of mathematical experience, but most of them are still able to clearly remember their experiences learning mathematical concepts for the first time. As a result, graduate students offer a unique perspective when asked how to teach mathematical concepts to novice mathematics students. Participants were asked how they would teach PMI to students in an introductory proofs course. Consider two different responses which are representative of two different types of answers given by the participant. The excerpt below is Participant 2's response.

> P2: I will introduce this like hey, what we are doing on the abstract level is that, hey, if I can show that if I have one step, then I can get another one and another one. So, basically, I'm saying, hey, starting from this level, I'm proving that I can actually reach the next one now. Um, then after I show that, I will explain that I will say, okay. Now that we have in mind these concepts that takes once I have this level, I can go to the next one, we have to say, hey, what is the first level? What... where do we start? Like do we start with one, do we start with two? Or where? Because there should be one case where our thing is true.

The following excerpt is Participant 3's response to the same question.

> P3: I mean, my hot take is that you really shouldn't actually tell students the real definition of principle of mathematical induction because it's too general. Like, to me, I feel like, would actually make most sense is show a bunch of proofs in different fields that are mathematical induction proofs but like, aren't you know.. we wouldn't say it explicitly and then say, like, oh, these are all kind of doing the same thing and then, like, introduce the principle mathematical

> induction. I think part of the problem is that most students I feel like, have a hard time grasping abstract definitions, or abstract like theorems or whatever... those general statements of things um. And so the hardest part for them, I imagine, would just be that they don't actually understand what the statement means fundamentally. And so then it makes it kind of impossible to apply it on a like deeper than surface level.

Participant 2's response essentially consists of restating the definition of PMI with little elaboration. It does not seem to reflect any intentional thought about delivery or offer any reason or justification for the approach. In contrast, Participant 3's response is a deeper, more nuanced exploration of a communication strategy. Instead of leading with a definition, the participant believes in showing various examples from multiple fields that use PMI and introducing the definition afterward. He said that this approach is based on his belief that, "part of the problem is that most students I feel like, have a hard time grasping abstract definitions, or abstract like theorems." He also mentioned that he believes the approach of exposing students to a variety of mathematical induction examples will help reduce the strong association between mathematical induction and standard PMI problems. This is indicative that his communication strategy reflects some level of understanding of the epistemological obstacles associated with mathematical induction.

These two different responses illustrate an important point related to expertise. It is crucial to note that expertise in a field is not equivalent to expertise in effectively teaching content in that field. Research indicates that expertise in a content area together with generic teaching strategies is not sufficient, but that there are content-specific pedagogical strategies that expert teachers should know (Shulman, 1986, 1987). Many mathematics education researchers emphasize the importance of this difference (Selden & Selden, 2003, 2017). The two excerpts discussed above can provide insight into these concepts. While Participant 2's response may be indicative that he understands PMI and is able to use it and describe the process of using it, it solely reflects content knowledge, rather than knowledge of effective pedagogy. However, Participant 3 demonstrated content-specific decisions about his communication strategies. He noted student difficulties with abstract definitions and also referred to the value of exposing students to multiple examples of proofs using PMI. While the two participants have similar backgrounds and levels of experience in their content area, their responses indicate different levels in teaching expertise. This exemplifies the concept of variable communication in relation to expertise. The findings of this and the preceding section are detailed and discussed in the following section.

4.2.5.3 Conclusions The data supports two primary findings associated with Variable Communication as it relates to the current study.

1. Some experts are able to simplify and explain the two primary parts of mathematical induction and to describe the overall principle in the context of these two parts.

2. Experts, even experts with similar backgrounds, exhibit varying levels of success when communicating concepts associated with PMI to students.

Both these findings provide insight to RQ5. The three mathematical induction analogies detailed in Section 4.3.5.1 each detail both the base case and the inductive step of PMI. The analogies illustrate that the participants understand the necessity of both parts of mathematical induction, indicating that they do not experience the same difficulties identified by studies on PMI with novice participants. In particular, the work done by Ernest (1984) suggests that novice students may underestimate the importance of one or both parts of mathematical induction. In addition to providing answers to the research question, the data presented in this section also exemplifies some potential limitations of expertise. Namely, expertise in mathematics does not necessarily guarantee the ability to effectively communicate mathematical concepts. This fact supports the need for mathematical experts who have also developed expertise in teaching mathematics or studying mathematical pedagogy. In addition to having varying levels of success communicating mathematical concepts, the participants in the study also demonstrated differing abilities in applying known concepts to novel situations. This is explored further in the following section.

4.2.6 *Novel Application*

Similarly to Variable Communication, discussed in the preceding section, Novel Application can refer to a potential limitation of expertise. Namely, experts may have varying levels of flexibility in their approach to new situations, and expertise is not necessarily a guarantee that an individual will always be able to apply their knowledge in a new situation or environment. Since the current study deals primarily with mathematical induction, the discussion of novel application will be restricted to instances in the data dealing with PMI. There were several examples during problem-solving sections of the interview where participants exhibited different degrees of success in applying PMI to novel proving exercises. Some notable examples are discussed in detail in the following section.

4.2.6.1 Applying PMI in Novel Problems As detailed in Chapter 3, one significant part of the study was a CTA on the participants as they worked on two questions that *could be* solved using PMI. In all cases, the participants had never seen either problem in the form given. The table below shows which of the two given problems each participant solved during the problem solving section. A participant was said to have solved the problem if they crafted a valid and complete proof that completely answered the question given. There are two important notes about the table below. First, this reflects only the proofs created during the initial problem-solving period. Some participants were able to finish or revise their arguments during the post-solving interview, but did not solve the problem initially during the allocated problem-solving time. This table only reflects a successful outcome for those who were able to construct a complete proof without help or prompting. Second, the participants were not required to write the formal proof down in order to have been considered as creating a valid proof, though many of them did. The primary point of this

CTA was not for the participants to successfully complete a proof for the given problems, and they were told before their allocated problem-solving period that they should not feel pressure to do so. The table is included merely to help frame the discussion of Novel Application. Note that a ✓ represents a successful solution to the problem, while a ✗ means that the participant was unsuccessful in proving the problem.

Table 4.6: Participant Success on Induction Problems

Participant	Problem 1	Problem 2
Participant 1	✓	✗
Participant 2	✓	✗
Participant 3	✓	✓
Participant 4	✗	✓
Participant 5	✗	✓

Note that one participant (Participant 3) successfully answered both questions, while the remaining four participants either answered the first or second problem correctly, but not both. For example, Participant 2 identified an algorithmic approach to proving the tromino problem and indicated that if writing a formal proof down, he would reformulate the algorithm into an argument using PMI. However, he struggled to come up with a valid proof for the second problem. When asked why he was able to quickly identify an inductive argument for the first problem, but not the second, he replied, “Because the first one had this natural way of going back to a previous step. With this one, I don’t see anything like that.” In contrast, Participant 3, who answered both problems successfully, almost immediately identified mathematical induction as a strategy for both.

Aside from just the ability to identify a pattern leading them to an inductive argument, which was discussed in Section 4.3.1, participants also described other strategies for approaching unfamiliar problems. Consider the following excerpt from Participant 5.

> P5: When I first learned how to do questions, like, in the beginning, I would... once I’d written down, like, the basic things, I would just stare at the problem because I wouldn’t even know where to begin. But now, I’ve learned to like, at

> least figure out what the easiest thing is to do. I mean, sometimes it's also like doing examples, but like, in this case, I could have started with the 2 cross 2 grid, for instance... I at least like, write down the problem. Or like use real numbers and try to figure that piece. Usually I try to do that, but sometimes I still forget.

Here, she spoke about how her approach to novel problems has evolved over time. She described the feeling of being frozen when confronted with an unfamiliar problem. In order to prevent that feeling, she developed a strategy of identifying a single, simple task within the problem and starting the problem by working on that task. In the case of the tromino problem, she began by considering the more simple problem of counting the tiles on the chessboard and calculating the number of trominos she would need to cover a chessboard with a missing tile. The strategy of identifying part of a problem to work on, even if an entire solution is not immediately apparent, was used by many of the participants, regardless of whether or not they explicitly mentioned it. For instance, even though he was unable to completely solve Problem 2, Participant 1 began by enumerating small package prices and was able to at least identify his candidate for the minimal n. This suggests that experts may chunk novel problems into known or manageable pieces in order to solve the problem in its entirety. In research discussed in Section 2.4, Styliandies, Sandefur, and Watson (2016) suggest that when experts cannot immediately identify a technique to prove a given statement, they may begin to experiment with examples to gain insight into an appropriate proving technique. The data in the current study corroborates this. Many participants who were successful in quickly identifying a solution to one or both of the problems did so after first identifying a proving technique. Those who were unable to do this immediately began working on smaller examples in order to better understand the mechanics of the problem, as exhibited in the excerpt above. Based on the data, is is hypothesized that this progression of proof construction develops as experts gain experience.

As discussed, most of the experts followed the same pattern of proof construction identified by Styliandies, Sandefur, and Watson (2016). However, the data indicates that this model may be unique to experts and may not be the same model that novices use when confronted with a novel proof construction activity using PMI. As indicated in Participant 5's description above, novel problems were more difficult for her when she initially learned how to construct proofs. Other participants also shared anecdotes of their experience when first learning how to write proofs. Consider the excerpt from Participant 2 below.

> It was in my first year of undergraduate, actually. And it was like really easy proofs, now that I'm thinking about it, but I didn't have that kind of like the logical thinking yet. So, my first attempt to prove something was following what the teacher was telling me. It was following the path of someone else, it was not me trying to understand what is going on or trying to play with the problem like now. It was actually me... Me remembering... okay the professor started like this, so I need to start like it like this. And then what are the next step? And I was like, I know the professor did a problem similar in class. I was like, okay, then he did this. Okay. What do I need to do here? Then here what is, like... close enough? Or what is similar that I can like, use here and I was doing that. Of course, when the problem was really different, I was screwed. It was like, okay, I cannot do this. But it was always trying to follow something that I knew.

Participant 2's recollection is consistent with studies that suggest students may view proving, especially proofs by mathematical induction, in terms of algorithms they apply with little to no conceptual understanding of the proof technique they are using (Avital & Libeskind, 1978; Weber, 2005). This description of the proving process is both incompatible with the proving behavior exhibited by participants in this study as well as previous research with mathematical experts that suggests experts have a more nuanced approach to proof production (Weber 2005, 2008). This implies that experts' proof construction approaches develop and evolve over time. In particular, participants' descriptions of their proof construction strategies when they were novices are compatible with Weber's (2005) definition of procedural proof production, where a student uses previously seen theorems as a template for a "linear set of steps not directly attached to conceptual knowledge" (p.353). In contrast, the proving behavior of the participants during the CTA more closely mirrored Weber's (2005) description of semantic proof production, where a student uses some informal or intuitive examples of a relevant concept to understand the given statement. The significant results linked to novel application are discussed in the following section.

4.2.6.2 Conclusions The data supports three primary findings associated with Novel Application as it relates to the current study.

1. Experts, even experts with similar backgrounds and levels of experience, may exhibit varying levels of success on cognitive tasks involving proof construction for novel problem statements.

2. Over time, experts have developed strategies for approaching novel proof construction problems, including those involving the use of PMI.

3. The proof production behaviors of mathematical experts differs from the proof production behaviors of novices.

Item 1 provides insight to RQ4. Even if experts exhibit a solid conceptual understanding of PMI, they may not always be able to correctly identify it as an appropriate proving technique. Expertise does not guarantee the ability to effectively apply knowledge in all scenarios, as exhibited by participants who were able to successfully use PMI in one cognitive task, but not the other. Items 2 and 3 also have implications for RQ1. The data offers many anecdotes associated with participants' experience with proof production when they were novices. These anecdotes corroborate previous work done with novices that suggest surface-level, algorithmic approaches to proof construction (Avital & Libeskind 1978). In contrast, experts in the current study discussed and exhibited strategies for solving novel proof construction problems and demonstrated more nuanced, advanced approaches to constructing proofs during cognitive tasks. This type of mathematical development over time is further explored in the following section, which focuses on Mathematical Fluency.

4.2.7 *Mathematical Fluency*

Mathematical Fluency may refer to a broad range of characteristics of mathematical experts. For the purposes of this study, Mathematical Fluency most often refers to an expert's ability to decode mathematical language and symbols, skim over known mathematical concepts, and refine or condense mathematical arguments. As would be expected, experts in the study demonstrated well-developed mathematical fluency in numerous ways, including a deep understanding of logical reasoning and a solid grasp on mathematical language and notation. This section will restrict focus to two primary sub-themes associated with mathematical fluency that are linked to proof construction, proof techniques, and PMI. First, there were multiple instances where participants exhibited mathematical fluency during proof validation. These instances include several of the proof validation activities discussed by Selden and Selden (2003). Second, participants also exhibited mathematical fluency through their demonstrated ability to translate problems. This sub-theme includes examples when participants translated a problem from one discipline to another as well as examples when the participant altered the problem to gain more insight to the structure and underlying argument. Both of these sub-themes are explored in the sections that follow.

4.2.7.1 Proof Validation Proof Validation was explored in detail in Section 2.2.3. Seldon and Seldon (2003) identified five primary activities associated with proof validation. They are included again in the table below along with examples of each activity from the data.

Table 4.7: Examples of Proof Validation

1. Asking and answering questions and assenting to claims.	P3:"Yeah, because then, so then that's just giving you a tiling no matter where the missing square is, like, just generating it from the tiling you have for the original one, right? Um, and now that I'm looking at this more. I need to be careful probably. But it looks like. Oh, okay. I think I know how to make the other… my claim is that n is two. Um. How I can make tilings for all the higher, um order grids? "
2. Constructing subproofs.	P1: "what I've actually done is sort of 2 inductions. I proved a Lemma for building the L's, starting at a small L, and then I did a lemma for starting at a big step, and then going downwards when I was dealing with the with the grids."
3. Remembering or finding and interpreting related theorems and definitions.	P1: "I mean, the smallest number for which you can't express it as a linear combination... With two numbers, I think should be those two numbers minus the sum of the two numbers or something. So I guess, like, I've remembered a claim that the smallest number you can't write should be 11."
4. Feelings of rightness or wrongness.	P2: "So basically they are saying that there is a minimum value n such that for any m bigger than that n, that number will be the divisible by 4 and 5, if I'm understanding is correct. But I don't think that is true because just take any minimal n if it exists. A quantity that is divisible by 9 or whatever it would make the quantity something such that… that m will be divisible by 9. and then we… Do we have that? 4 plus 5 is… Oh, no... it's not divisible. Okay. I assume this is true actually."
5. Production of a new text- modification, expansion, or contraction of the original argu- ment (Selden & Selden, 2003, p.5).	P1: "Yeah, so I think the way I proved it, it's sort of well, now now, I feel like I'm seeing there's a better way to do it with a single induction, but what I've actually done is sort of 2 inductions... But now I think I could see, I could have built up with a missing tile grid from a small group to the grid.

While most proof validation studies ask participants to validate the proofs of others, the data from the current study indicates that participants are capable of self-validation and that they participate in the same kinds of validation activities associated with validating the work of others. More importantly, the experts in this study performed these validation checks without prompting, suggesting that self-validation may be an automatic part of their proving process. While the table above offers one example of each validation activity, these occurrences were prevalent throughout the data. A summary of each activity's appearance in data analysis is included below.

1. *Asking and answering questions and assenting to claims*: This item is related to the self-checking behavior discussed in the section on Comprehension Monitoring. In summary, participants frequently make a claim, ask themselves about the validity of the claim, provide some sort of justification for the claim (this may be external or internal), and assent or dissent to the claim. This happened frequently throughout the problem-solving section.

2. *Constructing subproofs*: It was common for participants to construct small subproofs to provide justification for claims or to break a larger argument down into smaller, more manageable pieces. For example, Participant 1 constructed a Lemma to help him understand part of the tromino question. It is pictured below.

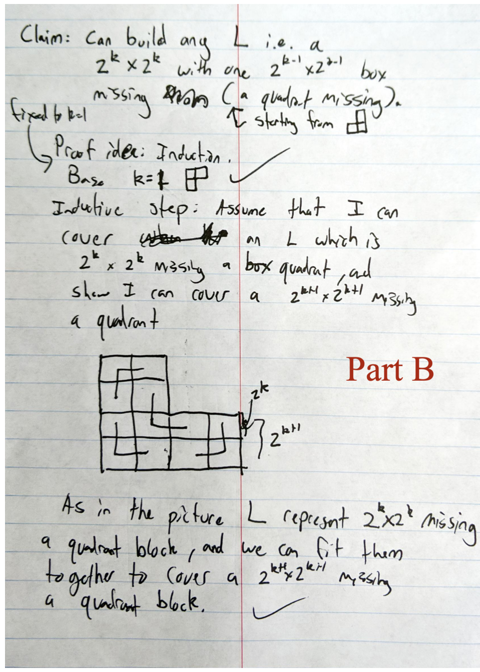


Figure 4.11: Participant 1 Lemma

3. *Remembering or finding and interpreting related theorems*: In general, this occured in two places in the data. First, as exemplified in Table 4.7, participants often recalled a theorem that was useful to them during the cognitive task and interpreted it in the context of the problem. Second, participants spoke about proofs of theorems and interpreted those proofs in the context of a different proof. This phenomenon was explored further in the section on Flexible Retrieval.

4. *Feelings of rightness or wrongness*: Participants often read a claim or made an assertion and immediately exhibited or verbalized feelings of rightness or wrongness, then followed with justification for the feeling. This activity seemed closely linked to intuition in the data. In places when participants discussed intuition, they also indicated feelings of rightness and wrongness.

5. *Production of a new text- modification, expansion, or contraction of the original argument*: This process occurred almost continuously throughout problem-solving and during the post-solving interview. Participants consistently checked their argument

> and made arguments more concise. This ranged in extremity from condensing notation in a proof (Participant 3) to merging two completely separate inductive proofs into a single, cohesive proof by PMI (Participant 1).

The findings discussed in this section corroborate work already done on proof validation and mathematical fluency. First, the experts in the study participated in all of the proof validation activities identified in work by Selden and Selden (2003). Second, experts exhibited the same characteristics of mathematical fluency named by Shepherd and Sande (2014). Experts were able to skim and summarize over complicated mathematical concepts and notation while working on their cognitive tasks. However, this study also indicates that, in addition to validating the work of others, experts also participate in proof validation activities when working on and reviewing their own proofs. Additionally, while Shepherd and Sande's (2014) work primarily looked at mathematical fluency in terms of *reading proof*, this study identified similar characteristics of mathematical fluency during proof construction. In addition to the findings mentioned in this section, participants also displayed mathematical fluency through translating and altering given proving exercises. This is discussed in the following section.

4.2.7.2 Problem Translation and Alteration In addition to the proof validation activities discussed in the previous section, participants also participated in or discussed two other activities linked to mathematical fluency. First, participants exhibited the ability to translate mathematical problems and known proofs from one field or situation to another. Second, participants exhibited the ability to alter problems in order to better understand them. Each of these two activities is discussed in this section. Consider the excerpt from Participant 5 below.

> P5: Okay. Okay, I guess so if it's covering, then this number of tiles, I guess, um has to be divisible by 3, is what I'm thinking. So, I guess the new question is... we want a minimal n such that for all m greater than equal to n, $2^{2m} - 1$

> is divisible by 3. Like, I think that's like the first necessary condition.

Here, Participant 5 began her solution for the tromino problem by translating the question to number theory (her primary field of study). She noted that the reformulation is not logically equivalent, but that it is certainly a necessary condition for the assigned cognitive task. She indicated that she uses this strategy often. When asked how she approaches a novel question, she answered in the following way.

> Turning the question into something familiar... Thinking of a similar example. Like the problems that I dealt with in my research is like. . . You already know how to get the estimate on this sort of set. And like, the set that I'm looking at, it's sort of similar, but not really. So go through that proof and then, um, see if this proof is like, maybe 2 different cases of that one, something like that.

While at first glance this strategy may seem simple, the act of translating either a problem or proof into a new context is a difficult and nuanced skill. It requires the individual to first have a solid conceptual grasp on the initial question or proof and the knowledge required from both the initiating field and terminal field in order to correctly complete the translation. For instance, in order to translate the tromino problem to the tile counting problem, Participant 5 had to have an understanding of the initial problem statement, translate that into a counting argument, and compare and evaluate the logical relationship between the two formulations to determine that the reformulation was necessary, but not sufficient. The participant then proceeded to construct a proof for her counting argument, demonstrating mathematical fluency in her use of number theoretic argumentation and notation. Following this, she again translated back to the original context in order to proceed with the more geometric approach to the tiling. This type of translation occurred often in the data. The stamp problem was translated into the language of modular arithmetic and linear combinations, and participants frequently discussed taking argumentation from a proof and translating it into a different context. In addition to translating problems and proofs across mathematical disciplines, participants also demonstrated the ability to alter different aspects of a problem in order to gain better understanding of the underlying structure of the given exercise.

When asked how he approaches proving exercises, Participant 2 responded as follows.

> I might start to change their assumptions of problem. Like, okay, like, for example, here they were saying that there was one tile missing. In that part, I will maybe say, like, hey, what, if there is no tile missing? What if it is the complete chessboard? How will that tile affect my approach to the previous cases that already played with?...So if I'm in a point in where my first approach didn't work, I will try to change the assumptions to see what is going on.

This action of altering assumptions, similarly to the previously discussed translation process, appears to be simple on the surface, but is actually reflective of deep mathematical fluency. The interrogation of assumptions in a problem indicates a solid conceptual understanding of the underlying logic and reflects a deep level of mathematical knowledge. Participant 2 indicated that understanding the necessity of assumptions in a given problem is something that he did not have as a novice mathematician. He described the following memory.

> I have noticed that when you are, like, actually writing the proof. Uh, you can actually see if all the assumptions are important or not. I remember one time that I was doing a proof, and I was like, hey, I'm not using this. Apparently, I was not using that, and then I went to a professor, this was when I was an undergraduate, and I was like, professor, why is that? We are assuming this, and we are not actually using it. And then the professor said, like, hey, you are using it, but you just don't know, because it's not like visible. But, if you take out this assumption, this will happen to this and this to this, and eventually you end up with a contradiction to what we were like, trying to prove.

This excerpt indicates that the participant has developed these skills over time. In particular, the notion of understanding assumptions is important when considering PMI, since many of the epistemological obstacles identified by previous studies deal with misunderstandings of the fundamental assumptions and underlying logic of mathematical induction (Avital & Libeskind, 1978; Ernest, 1984; Movshovitz-Hadar, 1993). The examples of mathematical fluency identified and discussed in this section provide some insight to the study's research questions. These insights are discussed in the following section.

4.2.7.3 Conclusions The data supports three primary findings associated with Mathematical Fluency as it relates to the current study.

1. Mathematical experts demonstrate numerous behaviors associated with mathematical fluency during proof construction and proof validation of problems involving PMI.

2. Experts demonstrate mathematical fluency through the actions of translation and alteration of proving exercises.

3. Evidence suggests that the mathematical fluency demonstrated by experts is developed over a long period of time.

Items 1 and 2 offer information relevant to RQ3. During problem-solving, the five participants employed many of the mathematical strategies discussed in this section, including problem translation, proof validation activities, and assumption alteration. When approaching novel problems using PMI, the participants often questioned how small alterations to the problem would change the question. Some of them translated the problem into another mathematical induction problem that was more comprehensible to them, and throughout their work, they asked questions, and verified assertions, demonstrating many characteristics of mathematical fluency identified by previous work. Finally, Item 3 gives further enlightenment to RQ1. Namely, the data indicates that the actions associated with mathematical fluency detailed in this section have developed over time. The next section explores the final item in the Expert Knowledge Framework and discusses instances when participants monitored their own comprehension during cognitive tasks.

4.2.8 *Comprehension Monitoring*

Experts were found to self-check performance, and persist through difficulty when working on tasks much more often than novice counterparts (Shepherd & Sande, 2014). This tendency is referred to as Comprehension Monitoring. The participants in this study performed the action of comprehension monitoring consistently throughout the problem-solving section. Two sub-themes, based on the two actions associated with comprehension monitoring, were created to classify data linked to this item. First, examples of participants self-checking their performance were classified into one sub-theme, named 'Self-Checking and Performance Monitoring'. Second, instances where participants continued to work on the cognitive task even when they struggled were included in the second sub-theme, called 'Persisting through Difficulty'. Both sub-themes associated with Comprehension Monitoring occurred frequently throughout the data, but a sub-collection of notable examples are detailed and analyzed in the following sections.

4.2.8.1 Self-Checking and Performance Monitoring The notion of self-monitoring is strongly linked to some of the proof validation activities discussed in the previous section. One of the proof validation activites identified by Selden & Selden's (2003), and discussed in the previous section, involves asking and answering questions and assenting to claims. This was a large part of the self-checking and performance monitoring processes, and it happened frequently throughout the interviews. These performance checks were unprompted, and appeared natural and almost automatic for all participants. Participants asked themselves questions and paused between claims to either think or justify the claim out loud. The research by Shepherd and Sande (2014) suggests that these types of reflections are more common among experts than novices. Five sub-actions were identified frequently in sections of the interview linked to self-checking and performance-monitoring. Note that some of these (Checking for Understanding, Asking and Answering Questions, and Justifying an Assertion) were also identified by previous studies (Selden& Selden, 2003; Shepherd & Sande, 2014) while the other two were developed by the current study (Identifying Mistakes or Irrelevant Steps and Self-Correcting Errors). These sub-actions are are listed below.

1. **Checking for Understanding**
2. **Identifying Mistakes or Irrelevant Steps.**
3. **Self-Correcting Errors**
4. **Justifying an Assertion**
5. **Asking and Answering Questions**

These same performance monitoring actions were identified throughout the data. Consider some excerpts below with these sub-actions identified within the text, color-coded using the key above. Consider the excerpt below from Participant 1.

> P1: Yeah, so I'm, I'm happy with how I started working. How I started was remembering what I think the answer should be, which I guess I guess I can't be sad about but, uh. As far as actually trying to work on the problem writing on the table did seem to be a good idea to get me an idea for how the numbers fit together. And it gave me some confidence that the right answer here for n was 12 the smallest number for which any number of bigger than equal to it can be written as a combination of fours and fives. **And I made a table to verify. Um, so that that table looks like this. It just sort of like addition table for the number of fours I was using and the number of fives I was using.** Um, then I wanted to do some... **I had a sense that I needed to understand what was happening in in the additive structure of fours and fives, which maybe you want to consider things mod 4 and mod 5**, **and I think what I've done is spent about 15 minutes dithering on something that wasn't super useful**, **though, maybe messing around with it to help me think about what I really need to think about**. **And**

> **then about the last 3 minutes or whatever, doing something kind of useful.**

In this excerpt, he was asked to reflect on his performance on the second cognitive task (stamp problem), for which he did not identify a complete solution. Even though he did not fully construct a proof for the given cognitive task, he still performed several of the sub-actions associated with self-checking behavior. An excerpt from Participant 5 is also included below, again color-coded to reflect the sub-actions listed above.

> P5: Okay. **I guess I'm writing it down, and I'm a little confused, uh, because, uh, this question seems similar to the first question and the first questions uh, like the language used was that for any... for all m and n with $m \geq n$. So I'm trying to figure out if the language over here is the same. Like, is the quantifier "for all" as well here? So my first thing I thought was like, is the quantifier for all? Yeah, and it is. Okay, so basically we want a solution to like a linear equation, right?** Like $n = 4x + 5y$ where x and y are integers. **So, I'm like trying to rewrite the question now.**

These five sub-activities effectively describe and account for the self-checking and performance monitoring behavior found in the data. As discussed, self-checking behavior was prevalent in the data, corroborating Shepherd and Sande's (2014) claim that experts commonly demonstrate these behaviors during proving activities. Importantly, these behaviors indicate a willingness to acknowledge mistakes, to adjust an argument, and to interrogate and justify their own claims. Additionally, these self-checking behaviors corroborate the findings by Inglis and Alcock (2012) discussed in Section 2.1.2. Namely, the data of the current study indicates that experts are willing to participate in "zooming in" strategies when validating their own proofs during proof construction (p.340). Zooming in behaviors consist of considering problematic parts of a given proof by performing line-by-line checks.

This behavioral description is consistent with some of the sub-activities identified above. In addition to these self-checking and performance-monitoring behaviors, the participants in the study also demonstrated other Comprehension Monitoring behaviors. A discussion of the participants' willingness to persist through difficulty is discussed in the following section.

4.2.8.2 Persisting through Difficulty Mathematics often involves long, intricate and detailed problems. It is rarely the case that difficult problems are solved immediately or with little thought. Therefore, it makes sense that persistence would be a valuable characteristic for a mathematical expert. As previously mentioned, Shepherd and Sande's (2014) study concluded that mathematical experts were far more likely than novices to persist through difficulties when working on a proving exercise. This study's findings are consistent with their claim. A few relevant examples from the data are included below. After working on the first cognitive task (trominoes), Participant 3 was asked what the hardest part of the problem was. He responded in the excerpt below.

> P3: Well finding the tiling for the 4×4. Because I was ready to give up, um, the, the, the tricky part for me was thinking exactly how to set it up. So what I can make copies and rotate around cause, you know, the original drawing I had um, for the 8×8 with something like... This where the missing cell is in the top, it should be in the top left corner not wherever it is right now but, um and at least in that drawing um, or with that set up the, the other, um, 4×4 is weren't exact copies they were, like, slightly different cause they all each had a different, um, piece of that L, shape in the center. Um, so, for me, a tricky part was trying to figure out how to, like, actually piece the 4×4 grids together um, in a true, like, oh, this is a copy being rotated around as opposed to, like, well, this is like, kind of a copy being rotated around, then you have this extra piece somewhere.

During the problem-solving period for the first cognitive task, Participant 3 initially struggled to tile the 4×4 board. He spent a large portion of the time dwelling on that part of the proof, which he reflected on in the excerpt above. However, rather than giving up on the problem he persisted. Importantly, his reflection above is indicative that he not only continued working on the problem, but also used the difficulty to give him a deeper insight to the problem after he successfully identified the tiling. Later in his interview, he referred to the importance of this perseverance and exploration. When asked what skills he felt novice students were lacking in relation to proving, he responded as follows.

> P3: Oh, they just unwilling to just try something. Um. I guess, like, okay, I've actually thought about this. So, um, I think that students often conflate something that is correct with something that is useful, like, when I talk to my students and they're trying to solve for x or whatever, um, and they'll, like, do something and they'll ask me is that correct? Well, so let's say they're trying to solve some like, quadratic thing and they square root both sides or whatever. Like that is correct, it just isn't necessarily useful, but they, they want me to say, no, it's not correct. And so they are just like always looking for the correct thing to do. Like... the thing that moves them in the correct direction and not just like something that will move them in some direction and see what happens. Um. I think actually, that's the biggest thing.

Here, he identified the unwillingness to persist through difficulty as a stumbling block that negatively impacts novice students, and claimed that the willingness to "just try something" can be of benefit when working on difficult proving problems. Altogether, the participants in the study exhibited consistent willingness to persist through struggle. Participant 1 spent the majority of his allocated problem solving time for Problem 2 struggling with the details of modular arithmetic, and Participant 4 did the same on the first problem. However, when faced with difficulties, they exhibited the willingness to explore multiple avenues of thought, as mentioned in the excerpt by Participant 3 above. The findings of this and the previous section are summarized in the following section.

4.2.8.3 Conclusions The data supports two primary findings associated with comprehension monitoring as it relates to the current study.

1. Experts demonstrate the ability and willingness to perform consistent self-checking and performance-monitoring behaviors when working on proving problems.

2. Experts demonstrate the ability and desire to persist through difficulties when working on cognitive tasks

While these two items are not necessarily directly related to any of the guiding research questions, they still provide valuable insight to the ways in which experts work on proving problems, including those involving PMI. In particular, the Comprehension Monitoring Behaviors discussed in the preceding sections illustrate experts' self-monitoring behavior and provide concrete examples of the ways in which experts check their own performance. These behaviors have implications for teaching that will be discussed in Chapter 5. In addition, these results corroborate other studies involving mathematical experts, including the work by Shepherd and Sande (2014). The following section discusses the interactions between various items in this framework to further elucidate the some of the findings discussed in Section 4.3.

4.2.9 *Notable Item Interactions*

While the items in the Expert Knowledge Framework represent unique aspects of expert knowledge, it is certainly true that they are not independent of one another. Depending on the research context, it is expected that various subsets of these eight items will interact with one another in different ways. This was certainly the case for the current study. While not every instance of item interaction will be discussed, some commonly noted interactions are detailed in this section. Specific attention was given to the interactions which relate to the study's research questions. It is important to note that some of these interactions are nuanced, and a more careful analysis of these interactions may be pertinent to future study using the framework. This is discussed further in Section 5.4.

4.2.9.1 Pattern Recognition, Knowledge Organization and Contextual Conditioning in PMI There was significant interaction between the items of Pattern Recognition, Knowledge Organization, and Contextual Conditioning in the contexts of this study. In the case of proof techniques, including PMI, experts in the study demonstrated several methods of organization. These included discipline-based organization strategies as well as organization based on problem characteristics. In particular, the recognition of patterns between small and large cases and examples was the primary problem characteristic associated with PMI. The experts demonstrated the formation of condition-action pairs based on this Knowledge Organization system. In particular, the action of using PMI as a proving technique was associated with the two conditions of associated discipline and pattern-based problem characteristics. In summary, Pattern Recognition was found to inform the Knowledge Organization of knowledge associated with PMI and to as one of the conditions in a condition-action pair.

4.2.9.2 Mathematical Fluency and Comprehension Monitoring Since both items involved activities associated with proof validation, there was nontrivial overlap in the items of Mathematical Fluency and Comprehension Monitoring. The most notable interaction between these two items occurred in relation to two of the proof validation activities identified by Selden and Selden (2003): ‘Asking and Answering Questions and Assenting to Claims” and ‘Production of a New Text: Modification, Expansion, or Contraction of the Original Argument.’ First, In order to perform the actions of asking, answering, and assenting, participants needed mathematical fluency. In particular, it was often the case that these three actions required rigorous or nuanced mathematical justification. Additionally, these actions are closely tied to the self-checking and performance-monitoring behaviors associated with Comprehension Monitoring. Second, in order to produce a new text from an original argument, a participant needs to have a solid handle on the associated logical processes, theorems, and definitions. These are all linked to demonstrated Mathematical fluency. Additionally, the modification of an existing argument requires the ability to meaningfully reflect on the argument as well as the ability to persist through difficult details. Both of these abilities are linked to Comprehension Monitoring. In summary, these two items are inextricably linked to one another in the contexts of this study and provide a second example of how different characteristics of expertise interact with one another.

4.2.10 *Section Summary*

Section 4.2 provided a summary of the deductive thematic analysis conducted based on the Expert Knowledge theoretical framework. Each of the eight items appeared frequently in the data, and provided a guide for exploring mathematical expertise in the context of the current study on PMI. The section detailed findings for each of the eight items in the framework, and situated these findings within the existing literature discussed in Chapter 2. These findings were linked to the study's five guiding research questions, and the section concluded with an analysis of notable item interactions. Section 4.3 will explore the findings of the second thematic analysis conducted for this study.

4.3 Inductive Thematic Analysis

In addition to the analysis using the two primary theoretical frameworks, APOS Theory and Expert Knowledge Classification, several recurring themes were identified in the data each of these themes helps address the study's research questions in various ways. It is important to note, that while inductive and deductive thematic analyses share many characteristics, the primary difference is related to how themes are developed. In a deductive thematic analysis, the themes are pre-determined by an existing framework, and the data is classified by these pre-existing themes. In contrast, there are no pre-developed themes in an inductive analysis, rather, overarching themes are developed *from* the data. These overarching themes were identified and analyzed through the six-phase thematic analysis process detailed in Section

3.4.1. Section 4.3.1 deals with how participants think about formality and rigor in proofs as well as the role of convincing in proof. Section 4.3.2 discusses the role of examples in participants' work on novel problems. Next, Section 4.3.3 details participants' views on the standard mathematical induction problems that are often presented in introductory proofs courses. Finally, Section 4.3.4 summarizes the thematic analysis and further contextualizes these themes in relation to the study's research questions.

4.3.1 *Informal and Formal Proof: Convincing Oneself and Convincing Others*

The first recurring theme identified during the thematic inductive analysis involves two types of proofs participants mentioned during the interview. First, participants used the term *informal proof* to describe the act of creating a proof or sequence of justifications intended to *convince oneself* that something is true. These informal proofs can be written down, be spoken aloud, or be mentally constructed. Second, participants used the term *formal proof* to describe the more rigorous construction of a proof, and this type of proof was strongly linked with the act of *convincing others* of the validity of either the statement or proof. Recall that this study takes Stylianides (2007) definition that a proof is "a mathematical argument, a connected sequence of assertions for or against a mathematical claim" (p. 291). It is important to note that *both* the informal and formal types of proofs described by the participants may satisfy this definition. Consider the excerpt from Participant 1's interview below. He was asked what skills are necessary in order for someone to be able to construct a good proof. His response follows.

> P1: Yeah, so I think somehow there, at least 2 distinct phases of good proof construction at least for me, not that I've also… I've haven't like super formalized it in my head or anything, but first, the initial stage is really convincing yourself that something's true um, which can be a lot looser than writing a formal proof... and then and then there's actually writing it down, which is somehow…

> it's supposed to be like the... the full justification to the reader that something's true. So really dealing with every possible issue a reader might have. Yeah, so, um, and that second stage is really where you're trying to communicate a proof, which is what most people would think of as, as the proof.

The concepts of convincing oneself and convincing others were also referenced by other participants. Participant 3, in particular, linked these ideas to PMI. Consider the excerpt below.

> P3: I think... I mean, this is true for induction and it's just true in general. I think too many people are trying to go from here's the problem statement to I'm going to write on the formal proof immediately. And that's it. And, like, I just think that is unproductive, I mean, for me, the way I understand things that there's a proof for myself, and there's a proof for everyone else, and I'm going to understand the proof that convinces me first, before I ever try to convince someone else or something is true. And, like, when I see students struggling with mathematical induction argument it's, because they, like, show their base case and then they like, you know, assume it's true for the k thing, but they don't actually have a feel of what's going on in the problem or why it should be true.

There are commonalities in the two excerpts above. First, the informal proof seems to be a precursor to the formal proof in the participants' minds. That is to say, both participants refer to convincing *themselves* first. Participant 3 indicated that he thinks this ordering is crucially important, and he mentioned that students may struggle when they attempt to construct a formal proof before they have convinced themselves of a statement's validity. The data indicates that the action of convincing oneself that a statement is true or that a proof is valid may encompass many different approaches. During CTA, it was noted that none of the five participants immediately began constructing a formal, rigorous proof in either cognitive task (tromino problem or stamp problem). Instead, the participants began by working small examples. This behavior corroborates the work done by Styliandes, Sandefur, and Watson (2016), and the act of working with examples is explored further in the Section 4.3.2. It was during this self-justification that participants participated in many of the proof validation activities identified by previous studies, including asking and answering questions, assenting to claims, and constructing subproofs (Harel & Sowder, 1998; Selden & Selden, 2003).

Not all of the participants wrote down formal proofs. In the case of participants who chose to write the formal proof down, they did so at the end of the problem-solving section *after* they had successfully reasoned through all aspects of the assigned problem. While other proof validation activities took place primarily during the informal phase of proof construction, it was often during the process of writing down a formal version of their constructed proof that participants modified, refined, or condensed the structure of their argument. Selden & Selden (2003) identify these behaviors as activities associated with proof validation. The participants' demonstrated tendency to perform these modifications during the formal proof stage may be related to the perceived role of the reader in proof construction. As previously mentioned, when a participant is constructing an informal proof, the justification and argumentation may take place internally, externally, or both. As such, the thread of the argument needs only be coherent to the individual constructing the proof during this informal stage of proof construction. However, when a formal proof needs to be created, the participant must translate this informal line of reasoning into a form that can be easily understood and evaluated by others. When referring to the formal proof in the excerpt above, Participant 2 said that a formal proof should deal with "every possible issue a reader might have." Altogether, the findings of this section seem to suggest that, while both informal and formal proofs may satisfy the study's definition of proof, informal proof may be primarily concerned with *self-justification only*, and formal proof is likely concerned with both *rigorous justification* and *clear communication*. Further, the data suggests that participants feel that both informal and formal proofs are integral parts of the overall proof construction process. The results presented and interpreted in this section are summarized below.

4.3.1.1 Conclusions The data supports three primary findings associated with Formal and Informal Proof as they relate to the current study.

1. Mathematical experts view both informal proofs (self-justification) and formal proofs (justification for others) as important components of proof construction.

2. The data indicates that experts likely view informal proof as a precursor to formal proof.

3. Proof validation activities identified by existing studies likely take place during both the informal and formal phases of proof construction.

These three results give insight to both RQ3 and RQ4. The data indicates that the determination of an appropriate proving technique likely occurs during the informal phase of proof construction. The excerpt from Participant 3 in the previous section highlights the obstacles that may occur if an individual immediately tries to use a technique to write a formal proof without first going through some self-justification that provides them with some underlying structure to inform the proof. While previous studies indicate that novices are unlikely to participate in self-justification behaviors, the current study found that experts use these behaviors frequently during cognitive tasks. This suggests that informal proving may be a skill that develops along with mathematical maturity and the development of expertise. Styliandes, Sandefur, and Watson (2016) found that novice students primiraly demonstrate behaviors associated with absolutist perspective of proof, which focuses on the product of the proof. In contrast, this study found that experts may operate primarily from the subjective perspective, which focuses on the proving process. In particular, during the informal phase of the proving process discussed in this section, participants relied heavily on the use of examples. This phenomenon is explored further in the following section.

4.3.2 *"Playing Around" and "Getting your Hands Dirty": The Role of Examples in Proof Construction and Problem-Solving*

Recall the study by Styliandes, Sandefur, and Watson (2016) discussed in Section 2.4 in which the researchers studied the proving behavior of experts working on proving exercises with PMI. The authors note a pattern in how expert mathematicians often approach a proving exercise. This pattern is summarized below.

1. Attempt to identify a reasonable method or technique to prove the statement. If one can be identified, they may use the technique without necessarily thinking about why the statement is true.

2. If no method can be immediately or easily identified, then the expert may try to experiment with some examples to gain insight to possible proving strategies.

3. Use discoveries made in the previous step to inform the formalization of an argument.

The proving behaviors observed during the CTA in this study strongly adhere to this pattern. This section primarily focuses on Items 2 and 3 in this set of steps, in which experts used examples to provide insight to a problem. This behavior was common to all participants during problem-solving, *and* the act of using examples was discussed frequently by participants in other parts of the interview. When discussing the use of examples in proof construction, participants frequently used the language of "playing around" or "getting your hands dirty." When asked what he does when he gets stuck on a problem, Participant 2 responded as follows.

> If I don't know the problem, the thing that I will do is start to, like, play with the problem. And by play with the problem, I mean do examples and small cases for motiviation and see how it works or see what I can see from that problem. It's like... I like this part because it's having fun with the problem. It's just not being worried about proving, just see what is going on with the problem. And

> then after I play a little bit comes... well, the part that I don't like too much, because I want to actually find a pattern. I want to find what is going on.

This excerpt illustrates two important aspects of the role of examples in proof construction and problem solving. First, the participant describes the act of "playing around" and exploring examples as a strategy to familiarize himself with the problem and give him insight to the problem, as discussed in Step 2 of the pattern described above. Second, he mentions that during this time he is not "worried about proving." This reflects the subjective perspective on proof identified by Styliandes, Sandefur, and Watson (2016), in which an individual is focused on the process of proving, rather than the proof itself. The participant says that he enjoys working with examples because "it's having fun with the problem." In terms of APOS theory, this could be indicative that he enjoys working on a problem at the action stage, but he finds it more difficult to reflect on those actions and interiorize them into processes, since this requires a higher level of cognitive activity.

The three step process identified above also requires a high level of cognitive activity. These example-based behaviors were demonstrated by all five participants during the CTA. This approach seems so integral to experts' proving strategies that it can cause difficulty if the pattern is disrupted. When asked what the hardest part of proving a statement using mathematical induction, Participant 3 responded as follows.

> Um, actually the, the hardest thing is if I can't... if... there are some of these problems where I can't come up with an example, right? If there's really no way to do a worked example um, that illuminates like a broader picture, like you're kind of forced to work in full generality, and I really uncomfortable doing that. Um, because I like getting my hands dirty on with examples and, like. That happens more often than not that, like, you know, I could in theory, simplify it, but if I could prove the simplified version of this, it would be a direct translation. There's no, like, content difference between, um, the proofs... like the proof of the specific example and the general proof.

In this excerpt, Participant 3 indicated that if he cannot come up with an example, his proof construction strategy is disrupted, and this disruption causes him to have difficulties when solving the problem. In particular, this description exemplifies Step 3 in the three-step pattern above. The participant stated that there is often "no content difference between the proof of the specific example and the general proof." This corroborates the findings by Styliandes, Sandefur, and Watson (2016), who found that experts in their study used examples to inform the general argument. In terms of APOS Theory, Participant 3's response above is similar to the previous quote from Participant 2. Namely, Participant 3 also enjoys starting with an action stage by working examples and while performing them, he reflects on these actions, which he can then interiorize into processes that eventually gives him insight to the "general proof." Other participants describe a similar process. Consider the excerpt from Participant below, who also mentioned the phrase "playing around" when referencing the act of exploring examples. When asked about the role this strategy plays in his problem-solving processes, he responded as follows.

> P1: For me, it's it's basically the entirety of my problem-solving process I need to have some sort of hands on feel for what's going on. If there's if it's like an algebraic question about an arbitrary group, what I'm going to do to start with is play with a toy group first and check whatever property works there and see if they can understand what pieces are fitting together to make it work there. Um, that sort of thing, and then once you, once you understand the like, small hands on things, you can think about how it could work in more general cases.

Again, this description mirrors Steps 2 and 3 of the model above almost exactly, providing further evidence that the model effectively captures the problem-solving behaviors of experts. In terms of APOS theory, his response highlights the action of working on a small example and reflecting on that action ("play with a toy group first and check whatever property works there and see if they can understand what pieces are fitting together to make it work") and continuing to reflect on the action until it becomes interiorized into a process ("you can think about how it could work in more general cases").

In addition to discussing this behavioral pattern, the experts also exhibited this behavior during the CTA. All five of the participants began both questions by exploring examples, albeit with varying levels of success. For Problem 1 (tromino problem), participants experimented with tiling small boards. Participants who successfully constructed a valid and complete proof were able to generalize the identified pattern from their examples to a general case, while those who failed to identify the pattern were unable to generate the proof, further supporting the claim that a disruption in the three-step process may inhibit proof construction. Similarly, for the second problem, participants began by testing small package prices to see if they could be exactly covered using the available stamp problem. Again, those who were able to generalize the example-based patterns were the same participants who successfully completed the second proof. These behaviors during CTA, together with the discussion above give solid evidence that suggests the three-step model captures most of the problem-solving strategies employed by experts during tasks involving mathematical induction. The findings of this section are summarized below.

4.3.2.1 Conclusions The data supports two primary findings associated with the Role of Examples in the contexts of the current study.

1. When working on problems involving PMI, experts use small examples to identify a generalizable pattern in order to construct a proof.

2. Experts may struggle to construct proofs using PMI in scenarios where relevant examples cannot be identified or when no pattern can be identified from small examples.

Both of these items relate to RQ2 and RQ4. First, it is likely that the example-based proof construction strategies discussed by participants in relation to problems involving PMI are also used in broader proof construction contexts, so this may illustrate one way proof by PMI is linked to other techniques and the action of proving. Second, the data provided strong evidence that difficulties associated with identifying relevant examples or generalizing example-based patterns contributed to difficulties constructing proofs using mathematical induction. In terms of APOS Theory, this is indicative that the participants may experience difficulties with interiorization (i.e. transitioning from the action stage to the process stage). This suggests that the worked examples of small cases may be integral to proof construction involving PMI. This discussion is tangentially related to the pattern recognition discussion in section 4.2 as well as some of the revisions to the preliminary genetic decompositions discussed in Section 4.1. The following section continues the discussion of examples by exploring the use of standard or routine examples of PMI used in introductory proofs courses.

4.3.3 *Standard Induction Examples*

This research uses nonstandard examples involving mathematical induction (see Section 3.2.3). Recall that this study considers algebraic verifications and statements involving finite sums to be "standard" mathematical induction problems. A common theme identified by the inductive thematic analysis involved the use of standard mathematical induction problems in introductory proofs courses. During data analysis, three trends emerged as sub-themes in relation to standard PMI examples. First, individual participants offered some differing opinions of the use-value of these standard problems, and this trend formed the 'Use-Value of Standard PMI Problems' sub-theme. Second, although perceptions of usefulness differed, participants agreed that these standard problems were not representative of the types of mathematical induction problems they encounter now as graduate students. This phenomenon was encoded as the 'Relevance of Standard PMI Problems.' Third, the data indicates a consensus that intro to proofs courses should include at least some examples of non-standard proofs using PMI, and this common occurrence was coded as the 'Need for Nonstandard Examples of PMI'. Some illuminating discussions and interpretations of these three sub-themes are included in this section.

4.3.3.1 Use-Value of Standard Problems As mentioned, there were differing opinions regarding the value of standard mathematical induction problems in introductory proofs courses. Participant 3 indicated that he did not feel these problems effectively convey the power of mathematical induction. When asked why he felt these types of problems were not very useful for understanding mathematical induction, he responded as follows.

> P3: I mean, I think they're good once or twice. So you're seeing a specific application of the principle of mathematical induction to like, a number whatever... number theory problem. But like, mathematical induction as a principle goes beyond just like, oh, let me look at this algebraic expression and expand. You know, you have some polynomial thing that you expand collect some terms on the side. We'll look at the previous case. Like, it's much deeper than that. Like using mathematical induction for, um. The problem with the L shaped tiles that we did... from last week. Um, like, I think that... that's like, so far removed from the algebraic number theory, like the $x^3 - x$ is divisible by 6 problem. Like. I... it wouldn't shock me actually if you went to an intro to proofs student and told them, like, assuming they had mostly been learning mathematical induction through those, like, number of theoretical problems, if you told them that you could prove that problem from the first interview by mathematical induction, they would not understand why or how. Because it just becomes too intrinsically tied to these like silly problems about divisibility and whatever else.

This response validates work discussed in Chapter 2 which suggests that the sole use of standard examples of PMI can create harmful associations between rote algebraic manipulations and mathematical induction which may not easily generalize to broader applications of the principle (Avital & Libeskind, 1978; Ernest, 1984; Smith, 2006). In particular, the fifth epistemological obstacle Ernest (1984) identified in his work with PMI was students inability "to generalize the method of PMI to examples that differ from the routine algebraic verification they see associated with PMI in introductory proofs courses" (p. 182). The excerpt from Participant 3's Interview above not only substantiates this claim, but also adds to it. In particular, Participant 3 not only feels that the student will be unable to generalize the method, but also feels that they would not even be able to recognize the method in a non-standard context. In fact, several of the participants in the study experienced this difficulty with generalizing the method to nonstandard contexts, as Participants 4 and 5 were unable to successfully use PMI in Problem 1 (tromino) and and Participants 1 and 2 were unable to successfully use PMI in Problem 2 (stamp problem). This has implications not only for proof-construction activities in the classroom, but also for activities involving proof-reading and proof-comprehension. While Participant 3 expressed concerns with these negative associations and questioned the overall usefulness of these standard problems, Participant 2 offered some differing opinions on the role and value of these problems.

When asked about the types of problems he primarily saw associated with mathematical induction in his intro to proofs courses, Participant 2 gave the following response.

> P2: It was sums. So it was like the sum of the first n numbers is... what was it? Something like $\frac{n(n+1)}{2}$? So it was like those kinds of examples. I think they're good examples, because it's easy to see where to apply the induction hypothesis, you just need to like try to cut the sum into the previous number, and then try to make this from algebra. And doing algebra at this point where you learn induction is something that I think everyone already know. So yeah. I remember sums and I'm pretty sure there are more exercises that were not sums, but every time I think of induction, I think of sums.

This quote illustrates two primary points. First, although he does not view it negatively like Participant 3, Participant 2 exhibits the same strong association between PMI and these standard problems when he says "every time I think of induction, I think of sums." Second, the primary benefit and value he identifies in these standard problems is their accessibility to younger students with less mathematical training. This is important to note, since the problems are not perceived to have value because they necessarily teach the technique of mathematical induction effectively, but rather because they offer an accessible way of using mathematical induction. In addition to the usefulness of these standard problems, their relevance to more advanced work using mathematical induction was also discussed.

4.3.3.2 Relevance of Standard PMI Problems When participants were asked to describe how the standard examples of mathematical induction compare to the types of mathematical induction they have seen in their graduate studies or their research, most of the participants responded by saying that standard problems were trivial by comparison. Participant 1 said, "Those types of problems are absolutely not what I do now." Similarly, when asked if he thought standard mathematical induction problems were representative of the types of mathematical induction problems he sees in his work now, he replied, " No, no, they're toy problems. They're silly. Yeah. No, not even close." This sentiment was echoed by Participants 2 and 3. However, Participant 5 had a slightly different perspective. When asked the same question, she responded as follows.

> P5: I would say they were just different.... Like, how, um. So I'm thinking like in precalc, maybe we first teach our students how to solve quadratic equations. But, like, the really difficult problems are, like, the word problems where they have to do the whole set up of the quadratic equation, and the quadratic equation has a meaning to it. And then you.... and, like, sometimes the solution makes sense sometimes like, both solutions don't make sense. So I think that I would say, like, proof by induction grew as well. In grad school, like, the problem comes with some context.

Participant 5 expressed the view that these standard problems are important, relevant, and related to the more advanced examples of PMI she has seen. In particular, she felt that standard problems serve the purpose of introducing students to the technique of PMI in the same way that we might introduce the most basic level of a concept in an undergraduate mathematics course. This perspective should be closely considered in light of existing research involving PMI. Ernest's theoretical research on PMI (1984) claims that students may struggle to understand the usefulness and necessity of PMI. PMI is, in many ways, unlike other the other principles and proving techniques they may have been previously exposed to. In particular, he claims that "mathematical induction is neither self evident nor a generalisation of previous more elementary experience" (Ernest, 1984, p. 181-183). The act of using standard PMI problems as introductory examples, as suggested by Participant 5 above, may therefore have merit.

Standard examples of mathematical induction represent straightforward applications of the principle with few complications from the contexts. These characteristics make them well-suited to help students during as they adjust to the novel technique. However, several authors have also identified strong associations between PMI and these standard problems as harmful (Avital & Libeskind, 1978; Ernest, 1984; Movshovitz-Hadar, 1993). In her analogy above, Participant 5 discusses the process of gradually increasing the level of conceptual difficulty as a teaching strategy, giving context to a mathematical concept like the quadratic formula. This same approach could be used when teaching PMI to students for the first time. Namely, initially using standard examples to ease them into using the technique could serve to alleviate the difficulties associated with the foreign nature of PMI. However, in order to prevent such strong, rigid associations with these types of problems, nonstandard examples of mathematical induction should also be incorporated into instruction and curriculum. This type of incorporation, along with associated difficulties, is discussed in the following section.

4.3.3.3 Need for Nonstandard Examples of PMI As discussed in the previous section, the sole use of standard mathematical induction problem can lead to epistemological difficulties with PMI. Smith (2006) found that some students did not view mathematical induction as explanatory, but "as an algorithm they can apply almost blindly" (pp. 80–81). Ernest (1984) also claims that students are often unable to generalize the method of PMI to examples that differ from the routine algebraic verifications they see associated with PMI in introductory proofs courses. Not only do these issues cause students difficulty when approaching novel scenarios requiring mathematical induction, they can also inhibit them from developing a robust understanding of the underlying technique itself. This phenomenon can be seen in the excerpt below.

> P2: So, yeah, that would be nice to actually get a little taste of how you use mathematical induction in other things that are not so attached to algebra. Because you do not want the students to um link those things in their minds or else, they maybe will not be able to do mathematical induction in other scenarios. I remember that happened to me. When I saw an mathematical induction problem that was not with algebra, it was hard for me to understand because it was not in the so-called "right format." **(air quotes)**

Here, Participant 2 indicated that he developed such a strong association between PMI and standard mathematical induction that he was completely unable to apply or think about the principle in other scenarios. Participant 3 also discussed the harm with only using standard mathematical induction problems.

> P3: If it's one of those number, like the divisibility problems or number theoretical ones, I immediately just write down what the $k+1$ case, and then just start trying to, like, simplify like, there's actually no thoughts going on in my head because I just assume it's I'm algebra trick, and if that fails, then I'll go and think about it more but, like, that's usually my first bet for those type of problems.

According to Woodall (1981), students may become accustomed to the algorithmic versions of the base step "plugging in $n = 1$" and inductive step of "an equation involving n and add something to both sides so as to produce a similar equation with $n + 1$" (p. 100). This is illustrated above in Participant 3's excerpt. The lack of thought that occurs when working on the standard problems indicates an algorithmic approach to PMI that, while it may not be harmful for an expert who otherwise has a solid grasp on the technique of mathematical induction, can have harmful consequences for a novice. Styliandes, Sandefur, and Watson (2016) believe that PMI can be introduced to students in a way that highlights what they call its explanatory proving power. Namely, they advocate for using nonstandard mathematical induction problems that (1) do not explicitly tell students to use PMI and (2) Are worded in nonstandard ways (i.e. do not uses phrases like "show for all $n \in \mathbb{N}$."

One common argument against the use of nonstandard problems in introductory proofs courses is difficulty finding problems which do require significant prerequisite content knowledge. This difficulty is captured in the excerpt from Participant 1 below.

> I was trying to think of of. I don't know, quote, unquote, low level, examples of mathematical induction, and I was coming up blank, except that picture problem with you.

Here, he referred to the tromino problem. It is true that many nonstandard examples of PMI would require content knowledge beyond what is known by most students in an introductory proofs course. However, there are still plenty of viable options. The two problems in this study require only minimal content knowledge, as well as the nonstandard problems used in the work on PMI done by Garcia-Martinez and Parraguez (2017). Later in his interview, Participant 1 recalled another example of one such nonstandard problem.

> P1: You know, actually, I think another early problem that's like, actually kind of quite like the tiling problem. Well, okay. It's also a tiling problem, but maybe it works quite differently. Um, is there's, there's a question about, um. Where you have n boxes in a line, better phrased a $1 \times n$ grid, um and you're trying to cover that with tiles of either size 2 or size 1 I don't know if it's quite familiar. If this is a problem that's familiar to you. I mean, the question is how many how many ways can you tile and $n \times 1$ object with tiles of size 2×1 and 1×1. So, for example, if you've got length 3 tile, there are 3 ways to do it. Yeah, you can either have all 1 length tiles, or you can have a 1 then a 2 or then a 2 then a

> 1. All. Okay, and the way to go about it, um, actually figuring it out in general is an inductive argument, you get the Fibonacci numbers and the idea is that, like... But the idea is, like, you can, um cut... you can think about how to build up, um, the nth case from the previous case, and the case before that.

Again, this demonstrates that there *are* nonstandard examples of mathematical induction that could be appropriate to use in an introductory proofs course. The addition of these nonstandard problems has the potential to help students develop both their understanding of the underlying structure of an argument by PMI as well as the ability to apply the technique in novel, nonstandard problems. The implementation of these nonstandard problems into instruction and curriculum is explored in more detail in Chapter 5. The following section provides a summary of the findings associated with standard PMI problems.

4.3.3.4 Conclusions The data supports three primary findings associated with Standard Mathematical Induction Problems as they relate to the current study.

1. While experts have varying opinions on the use-value and relevance of standard examples of PMI, there is a consensus that nonstandard examples of PMI should also be incorporated into introductory proofs curriculum and instruction.

2. Some experts do not believe standard mathematical induction problems effectively illuminate the underlying technique of PMI and can cause students to have difficulty generalizing the technique.

3. Experts have experienced difficulties resulting from strong associations between PMI and standard mathematical induction problems.

Together, the findings in this section provide insight to RQ1 and RQ4. First, the data corroborates existing literature which says novices strongly associate PMI with standard mathematical induction problems and are unable to generalize. However, the expert participants discussed how exposure to nonstandard PMI problems, coupled with years of study, have allowed them to develop their ability to apply PMI in novel scenarios. This illustrates one way in which expert knowledge associated with PMI may develop over time. Second, the section's findings suggest that some experts believe that the incorporation of nonstandard PMI problems into introductory proofs curriculum may help students develop the ability to (1) identify situations where PMI is appropriate and (2) effectively apply PMI in novel situations. These findings are consistent with the work done by Styliandes, Sandefur, and Watson (2016). The following section concludes the chapter with a summary of the results and findings detailed in Chapter 4.

4.4 Chapter Summary

This chapter presented the results of the study, summarizing the findings of the data analysis. Section 4.1 presented findings associated with the APOS Theory framework, including a suggested revised version of the preliminary genetic decomposition. Section 4.2 provided a revised eight-item version of the Expert Knowledge Framework. The section discussed the results of the deductive thematic analysis conducted based on this revised framework, and gave relevant examples from the data for each item. Finally, section 4.3 explored the findings of the inductive thematic analysis and presented three recurring themes identified in the data, giving relevant examples associated with each theme. Holistically, this chapter presented the data, linked the findings to both the guiding research questions introduced in Chapter 1 and the existing literature discussed in Chapter 2. The chapter also offered interpretations of the data based on the study's guiding theoretical frameworks and epistemological perspectives. The following chapter will conclude the paper.

5 SUMMARY AND CONCLUSIONS

This chapter offers a summary of the results discussed in detail in Chapter 4 as well as closing remarks and implications of the study. Section 5.1 offers a summary of the study's findings as they relate to the research questions presented in Section 1.2 and the primary theoretical framework, APOS Theory. Since the primary focus of this study involves learning and cognition in mathematics, Section 5.2 discusses potential implications of the research for the teaching of PMI. Section 5.3 identifies potential limitations of the current research, and Section 5.4 discusses potential future work that might expand on the work of this study. Section 5.4 concludes the paper with closing remarks and final points for consideration.

5.1 Summary of Results

The following sections discuss how the results of the study address the research questions outlined in Chapter 1. Note that some of the findings of the study relate to more than one of the research questions, so they may be listed and discussed more than once in this section.

5.1.1 *Research Question 1*

RQ1 How do experts describe the development of their conceptual understanding of PMI over time?

This research question was primarily addressed by the study in two ways. First, the instruments and study design were created in a way to elicit responses associated with participants memories of learning PMI for the first time. This allowed for direct comparison of those memories with the participants current demonstrated conceptual understanding of the technique of mathematical induction. Second, this study used expert participant, which allows the data collected in this study to be compared with existing literature studying PMI with novice participants. During data analysis, several of the findings offer information related to RQ1. These findings are listed below, along with a brief description of how each finding relates to RQ1.

1. *Mathematical experts recognize the similarities and differences between various proof techniques and generally group their knowledge of techniques together, adding to this knowledge as they gain mathematical maturity.* Participants in the study demonstrated

a nuanced understanding of proof techniques that includes the ways in which they are related to one another in the broader context of proof. This contrasts with participants descriptions of their initial exposure to proof techniques, when they viewed them as isolated from one another and had a limited understanding when a proof technique was appropriate for a given context.

2. *Mathematical Experts recognize that the fundamental components and characteristics of a given proof technique are context-independent.* While experts demonstrate the ability to apply mathematical induction in a broad array of contexts, they still recognize that the fundamental nature of the technique is the same despite the context in which it is applied. In contrast, novices have difficulties extracting the technique from the context of standard problems and may not recognize the similarities between a standard and nonstandard use of PMI (Ernest, 1984).

3. *Most mathematical experts still strongly link language involving the natural numbers with the technique of mathematical induction.* Various studies support the claim that students strongly associate PMI with statements involving the natural numbers (Avital & Libeskind, 1978; Ernest, 1984; Movshovitz-Hadar, 1993). This study corroborates this well-known link and notes that experts still maintain this strong association after years of study.

4. *Experts likely conditionalize their use of PMI on the identification of a relationship between small and large cases in a given problem, and the lack of this identification can result in the expert actively dismissing PMI as an appropriate technique.* While it is well-known that experts and novices alike associate PMI with the natural numbers, this study found that experts have developed other associations, including the association between PMI and patterned relationships between successive cases. This is indicative that as individuals gain mathematical maturity, they may develop more nuanced associations between problem characteristics and PMI.

5. *The flexible retrieval exhibited by mathematics experts in this study has characteristics which distinguish it from rote memorization.* While much of the research on PMI conducted with novices suggest that they use algorithmic approaches along with rote memorization to construct PMI proofs (Avital & Libeskind, 1978; Ernest, 1984; Movshovitz-Hadar, 1993), this study found that experts demonstrated retrieval that was more advanced and reflected a deep knowledge of the technique of mathematical induction.

6. *Over time, experts have developed strategies for approaching novel proof construction problems, including those involving the use of PMI.* While novices have been shown to struggle to generalize the technique of mathematical induction to nonstandard PMI problems (Avital & Libeskind 1978, Ernest 1984), experts demonstrated the ability to apply the technique of PMI broadly in various mathematical fields and to discuss the technique's use in multiple contexts.

7. *The proof production behaviors of mathematical experts differs from the proof production behaviors of novices.* Previous studies note that novice students often participate in what Weber (2005) calls procedural proof production, in which the student mimics previously seen arguments with no real understanding of the statement they are proving. In contrast, the participants in this study were more likely to use what Weber (2005) refers to as semantic proof production in which a student uses some informal or intuitive examples of a relevant concept to understand the given statement. Based on the memories of the participant this proof production behavior developed over time as they gained mathematical maturity.

8. *Evidence suggests that the mathematical fluency demonstrated by experts is developed over a long period of time.* By nature, the ability to gain a solid grasp on mathematical language and concepts develops over a long period of time. The mathematical fluency displayed by the participants in this study was developed over several years of study.

5.1.2 *Research Question 2*

RQ2 How do experts situate their conceptual understanding of PMI in relation to the notions of proof and proof technique?

The Principle of Mathematical Induction is used as a method of proof. Therefore, the concept of PMI is inextricably linked to the general notion of proof and to other proving techniques. This study sought to elucidate these linkages and to situate the technique of mathematical induction within the broader literature involving proof. During data analysis, several of the findings offer information associated with RQ2. These findings are listed below, along with a brief description of how each finding relates to RQ2.

1. *Mathematical Experts recognize that the fundamental components and characteristics of a given proof technique are context-independent.* While experts demonstrate the ability to apply mathematical induction in a broad array of contexts, they still recognize that the fundamental nature of mathematical induction remains unchanged, regardless of context. They are also able to compare the fundamental characteristics of the technique of mathematical induction with other proof techniques. In particular, they exhibit the ability to understand how PMI is similar to and distinct from other proof techniques and to use these comparisons to classify when PMI may be appropriate for a given problem.

2. *Mathematical experts exhibit the ability to flexibly retrieve relevant facts, theorems, entire proofs, and overarching proof strategies of known-proofs when working on proving problems.* The participants in the study demonstrated the ability to rapidly recall relevant details related to a given novel problem with little to no attentional effort. This is indicative that experts may organize their knowledge of proof and specific proof techniques, like PMI in ways that allow them to easily access relevant information quickly when working on novel problems.

3. *The flexible retrieval exhibited by mathematics experts in this study has characteristics which distinguish it from rote memorization.* The flexible retrieval demonstrated by experts in the study reflects a deep understanding of how the concepts of proof technique and PMI are related, rather than just a set of memorized facts.

5.1.3 *Research Question 3*

RQ3 When viewing a novel problem, how do experts determine whether or not mathematical induction is an appropriate method for proving a statement?

This study sought to better understand how experts determine whether or not mathematical induction is an appropriate technique for a given problem statement. This research question was most closely associated with the items of Pattern Recognition and Contextual Condition in the Expert Knowledge Framework. During data analysis, several of the findings offer information associated with RQ3. These findings are listed below, along with a brief description of how each finding relates to RQ3.

1. *Pattern recognition is linked to the inductive step of PMI.* Experts most often identified PMI as an appropriate technique when they identified a pattern between small and large cases of the given statement. This is likely an important way that experts determine when to use PMI in novel problems.

2. *Some mathematical experts may organize their knowledge according to mathematical discipline and may conditionalize their use of certain proof techniques according to which discipline a given problem is associated with.* Some experts strongly associate PMI with particular mathematical disciplines (e.g. graph theory, combinatorics) and indicate that the mathematical field of a given problem may serve as part of their determination of whether or not PMI is an appropriate technique.

3. *Most mathematical experts still strongly link language involving the natural numbers with the technique of mathematical induction.* While this finding was also associated

with RQ1, it also provides insight to RQ3. Namely, this finding indicates that experts may still use phrases involving the natural numbers as an indicator that PMI may be an appropriate proof technique for a given problem.

4. *Experts likely conditionalize their use of PMI on the identification of a relationship between small and large cases in a given problem, and the lack of this identification can result in the expert actively dismissing PMI as an appropriate technique.* While this item was also linked to RQ1, since it highlights a difference between how experts think about PMI when compared to novices, it also gives insight to RQ3. As mentioned in Item 1, the recognition of an inductive pattern may be integral to an expert's determination that PMI is an appropriate technique for a given problem. This finding also indicates that the experts in the study use the **lack of** such a pattern as an indicator that PMI may not be appropriate. In terms of APOS Theory, this may mean that an experts ability to appropriately identify PMI as a proving technique for a given problem depends, in part, on their ability to interiorize actions associated with small examples into processes they can use to generalize patterns between small cases.

5. *Mathematical Experts exhibit the ability to flexibly retrieve and recall both entire proofs and overarching proof strategies.* Because experts demonstrate the ability to easily recall known proofs and proof strategies, the decision to use PMI in a given scenario may be based on a related proof or problem that is known to the expert.

5.1.4 *Research Question 4*

RQ4 What obstacles, if any, do experts face when solving mathematical induction problems in which mathematical induction is not explicitly specified as the technique to use?

The literature identifies several epistemological associated with PMI (Avital& Libeskind, 1978; Ernest, 1984; Movshovitz-Hadar, 1993). This study sought to determine if some, all, or none of these difficulties were also common to expert participants. During data analysis, several of the findings offer information associated with RQ4. These findings are listed below, along with a brief description of how each finding relates to RQ4.

1. *Experts demonstrate behaviors indicative of varying levels of development within the APOS Theory framework for both the base step and inductive step of PMI.* As discussed in Section 4.1, not all participants in the study demonstrated a high-level of conceptual development for both parts of PMI. While the observed behavior may not be wholly indicative of the cognitive constructions in an individual's mind, it was certainly the case that students exhibiting higher levels of development in the context of the APOS Theory framework were, in general, more successful on the cognitive tasks. This suggests that students with lower levels of understanding of one or both parts of mathematical induction or of the relationship between the two parts may struggle to some extent applying the technique in novel scenarios.

2. *Mathematical experts exhibit success at recognizing and using patterns in problem- solving and proof construction, and Pattern Recognition is linked to success proving the inductive step of PMI.* As previously mentioned, successfully identifying a pattern linking small cases to larger cases was a primary indicator for participants that PMI may be an appropriate proving technique. In the mathematical induction problems used for this study, PMI was not specified as the proving technique to use. Participants who struggled to identify a pattern relating small cases to large cases during the cognitive tasks were significantly less likely to solve the problems. This is indicative that lack of pattern recognition skills may be one obstacle for experts working on novel problems where PMI is not specified for them. In terms of APOS Theory, these difficulties with pattern recognition are likely indicative of difficulties with the mechanism of interiorization. That is to say, this may reflect participant difficulties with reflecting on the actions they perform when working on small examples, which then prohibits them from interiorizing these actions into processes. As a result, they may be unable to generalize patterns identified in small examples into broader arguments for the given problem.

3. *Experts, even experts with similar backgrounds and levels of experience, may exhibit varying levels of success on cognitive tasks involving proof construction for novel problem statements.* As discussed in Section 4.2, expertise is not a guarantee that an individual will be able to effectively apply their knowledge in all scenarios. This was demonstrated in the study. Only one of the participants (Participant 3) was able to successfully solve both mathematical induction problems. The remaining four participants were only able to solve one of the two. This is indicative that, even when they demonstrate a solid conceptual understanding of a concept, there is no guarantee they will be able to apply it in all scenarios.

5.1.5 *Research Question 5*

RQ5 How do experts explain and define the two primary parts of PMI (the base step and the inductive step) and the perceived relationship, if any, between these two primary parts?

Since the primary focus of the study was understanding how experts think about and use mathematical induction, it was natural to try to understand how these experts conceptualize each component of PMI. The insight to this research question primarily came from the analysis with the APOS Theory Framework, since APOS Theory is useful in deconstructing complex mental structures and concepts. During data analysis, several of the findings offer information associated with RQ5. These findings are listed below, along with a brief description of how each finding relates to RQ5.

1. *Some experts are able to simplify and explain the two primary parts of mathematical induction and to describe the overall principle in the context of these two parts.* Many of the participants demonstrated the ability to explain mathematical induction well, even to individuals with no mathematical background. These participants used analogies (explored in Detail in Section 4.2) that grounded both steps of mathematical induction, the base case and the inductive step, in simple terms. As discussed in Section 4.1, the

ability to form successful analogies is indicative that an individual is operating at an object stage of understanding, since the ability to compare two separate ideas (like a chain of dominos and PMI) requires the individual to have encapsulated the process into an object they can think about in totality.

2. *Experts, even experts with similar backgrounds, exhibit varying levels of success when communicating concepts associated with PMI to students.* Not all participants were able to effectively communicate the two primary parts of mathematical induction in ways that would be comprehensible or illuminating for students. Some participants merely re-worded the formal definition without actually explaining the technique. In contrast, some participants were able to explain the principle in ways that reflected deep knowledge of both mathematics and pedagogy. In particular, participants who gave the most informed communication strategies indicated that they would use several different examples of PMI in various contexts in order to motivate students' understanding of the concept before giving them a rigorous definition. This approach allows students to first perform the actions associated with PMI (via working on examples using PMI) and then to reflect on those actions before ever trying to understand a more abstract definition of PMI.

3. *Experts demonstrate behaviors indicative of varying levels of development within the APOS Theory framework for both the base step and inductive step of PMI, but the data indicates that the experts in the study have at least a process-stage of understanding of both of the primary parts of PMI.* The participants all exhibited the ability to imagine the process of proving a statement using mathematical induction in their minds, indicating that they have at least a process stage conception of both parts of the technique. Many of the participants exhibited behavior suggesting they had progressed past the process stage of mathematical induction. In general, the participants were all able to provide both a description of each step *and* to explicate each step's purpose in

the broader technique.

4. *While some participants view the base case as being easy, all of the participants demonstrated recognition of the necessity of the base case as part of the technique of mathematical induction.* Existing research indicates that novices tend to underestimate the importance and neccesity of the base case of mathematical induction (Avital & Libeskind, 1978; Ernest, 1984; Movshovitz-Hadar, 1993). While the participants in this study indicated various levels of depth in their conceptual understanding of the base case (some viewed it as trivial, while others viewed it as informing the overall proof), all participants demonstrated a full recognition of the necessity of the base case within PMI. In terms of APOS Theory, this is indicative that the participants have successfully coordinated the base case process and the inductive step process to form the process of PMI, so that they recognize both the role that each process plays as well as the necessity of each within the broader technique of mathematical induction.

5. *Some participants are more successful at describing the relationship between the two steps of mathematical induction, but all of the participant behavior during CTA indicates that all participants have, to some extent, coordinated the base case process and the inductive step process.* Some participants were unable to communicate the relationship between the base step and inductive step, while others gave nuanced descriptions of this relationship and how it operates as part of their proving strategies. However, regardless of their ability to verbally communicate their understanding of the relationship, all the participants demonstrated that they had coordinated the process of the base case and the process of the inductive step in their minds, as demonstrated in their work on the cognitive tasks when they were able to describe or construct an argument using the technique of mathematical induction. The varying degrees of success in describing the relationship may be correlated with the different levels of schema development demonstrated by the participant during the CTA and interviews. Those

who demonstrated a more advanced level of PMI-schema development were able to more effectively describe the relationships between the base case and the inductive step of mathematical induction.

5.1.6 *Revised Genetic Decomposition*

Recall that genetic decompositions operate as rubric-like models for explaining cognitive constructions associated with a particular concept. In APOS Theory, they are used to identify mental constructions students should be able to make in order to develop sufficient understanding of mathematical concepts (Arnon et al. 2014). It is important to note that genetic decompositions offer a *potential* model of how a concept *may* be constructed in students' minds. The researcher initially uses a preliminary genetic decomposition, which may be a novel decomposition or may have been tested in previous research studies. The preliminary genetic decomposition used for this study informed the creation and implementation of the research instruments including the CTA activities and the interviews. During data analysis, the preliminary genetic decomposition for this study was evaluated. The data indicated that a few constructions demonstrated by participants in the study were not completely captured by the preliminary genetic decomposition. The revised version of the genetic decomposition is below (with revisions in bold).

1. Reversal through the existential quantifier to form a process of identifying and testing an appropriate base case $P(a)$.

2. **Interiorizing the action of a logical statement $P(N)$ for a given statement P and an arbitrary $N \in \mathbb{N}$.**

3. **Coordinate the process of $P(N)$ from Step 2 with the process of identifying and testing an appropriate base case from Step 1 to form a process of testing a statement $P(N)$.**

4. **Encapsulate the coordinated processes from Step 2 into the statement object $P(N)$ for any $N \in \mathbb{N}$.**

5. Expand the Function Schema to include a function mapping each natural number to a propostion-valued output ($f : N \to P(N)$).

6. Encapsulate logic into the implication $p \Rightarrow k$. The implication cognitively becomes an object which is the value of the function f.

7. **Encapsulate $P(N)$ and $P(N+1)$ into the logical implication $p \Rightarrow k$ to form the implication $P(N) \Rightarrow P(N+1)$**

8. Create the schema of the implication-valued function g where $g(\mathbb{N}) = (P(N) \Rightarrow P(N+1))$

9. Interiorize the action of logical necessity into a process so that inputs P_0 and $P_A \Rightarrow P_B$ allow one to conclude P_B.

10. Coordinate the function g from Step 7 with Modus Ponens beginning with $P(a)$ from Step 1 for an appropriate case a.

11. Coordinate this implication valued function along with the base case process through the use of modus ponens to explain the PMI.

12. Encapsulate this inductive process into an object be connected to the Method of Proof schema so induction can be applied as a proof method.

13. Generalize actions on the induction object within various problem types coordinated with the Method-of-Proof schema until students can apply induction as a proof technique.

14. Generalize actions to the base case object until students can identify an appropriate base case in novel problems where it is not specified.

The testing and revision of genetic decompositions is a crucial part of research using APOS Theory. Recall that the APOS Theory research cycle starts with some preliminary genetic decomposition (like the one described in Chapter 3), followed by the development of teaching material or research instruments and finally, instruction or research using these materials. Afterward, data is collected and analyzed, the genetic decomposition is revised, and the teaching material or research instrument is modified for a repetition of the cycle to start anew. The revised genetic decompostion presented by this paper should be tested by future studies to evaluate the validity of the added steps and to determine if any further modifications are required to fully capture the constructions involved in the technique of PMI.

5.2 Implications for Instruction

At the heart of mathematics education research is the desire to improve pedagogical practices in mathematics classrooms. This research identifies several potential implications for instruction at the undergraduate level. Each of these implications are discussed in detail in this section, along with some concrete suggestions for pedagogical and curricular adjustments that may help address the issues identified in this research.

5.2.1 *The Use of Nonstandard PMI Problems*

Potentially the most prevailing connection to instruction identified within the data relates to the use of standard mathematical induction problems discussed in detail in Section 4.3.1. This study validates existing literature discussed in Chapter 2 that indicates the isolated use of standard mathematical induction examples, including algebraic verifications and equalities involving finite sums, can create and reinforce harmful associations. These associations can create issues with generalizing the technique of mathematical induction to broader contexts as a student progresses through a major (Ernest, 1984). One common argument against rectifying this issue is that other inductive proofs require high-level knowledge that students in an introductory proofs course may not have. However, this study offers two problems using mathematical induction that do not require highly-specific knowledge and demonstrate the use of PMI in a nonstandard problem. Other such examples have been used in research, as discussed in Section 2.5.2. One such example was discussed by Participant 1 (see Section 4.4.1), and it is likely that numerous other examples could be easily constructed and implemented into activities for the classroom. The pre-existing literature, in addition to the current study's results, suggest that the use of these nonstandard problems *in tandem with* the standard problems typically seen in an intro to proofs course may lessen the negative associations with PMI and allow students to more easily generalize the technique of mathematical induction to other contexts. The use of nonstandard mathematical induction problems also has the potential to alleviate another issue identified by the study, which is discussed in the following section.

5.2.2 ***Teaching Expertise***

Expertise can often be perceived as elusive or unattainable. Each of the eight items in the revised Expert Knowledge Framework are concrete characteristics, and the abstract concept of expertise can become more tangible and grounded when linking it to the framework. These tangible, more manageable characteristics are often measurable skills, allowing educators to use them to inform instructional and curricular design. Some suggestions for incorporating each of the eight items in the framework into into instruction of PMI, based on the findings of the study, are included below.

1. Pattern Recognition: When introducing PMI, proof construction and proof reading activities can be scaffolded in ways that emphasize patterns between large and small cases. This has the potential to help students to develop their ability to recognize these kinds of patterns.

2. Contextual Conditioning: The pattern recognition discussed in Item 1 should be linked to the technique of mathematical induction. This link can be reinforced through lecture and activities, by providing several nonstandard examples of PMI and emphasizing

the relationship between problem characteristics and the technique of mathematical induction.

3. Knowledge Organization: When homework is assigned section-by-section, students are not forced to practice identifying the appropriate strategy on their own. As students are exposed to various proof techniques, exercises which ask them to compare and contrast the techniques and to identify appropriate techniques for given problems may help them develop a well-organized system associated with proof techniques and strategies.

4. Flexible Retrieval: Active learning activities, when carefully and intentionally constructed, can help students develop their retrieval skills. When topics are introduced in class, they should be followed first with scaffolded activities to help students develop links from problem contexts to learned knowledge. Then, the scaffolding should be removed so that the student can practice accessing the appropriate knowledge independently. Continually providing new situations requiring the same piece of knowledge can help further develop this skill.

5. Variable Communication: The ability to communicate both concepts and mathematical arguments is a crucial skill for mathematicians. This skill can be developed by incorporating presentational elements into the classroom. This can be in the form of group discussion, where individuals must present and defend their arguments in small groups or on a wider scale where participants present a proof to the entire class.

6. Novel Application: The ability to apply the technique of mathematical induction in a novel scenario is best developed by giving students where they (1) must determine when PMI is appropriate on their own (2) are asked to apply PMI in a variety of nonstandard contexts.

7. Mathematical Fluency: This item should develop naturally as long as students are given intentionally developed instruction, like the examples in the previous items. The data

indicates that Mathematical Fluency develops over time with exposure and experience.

8. Comprehension Monitoring: Assigning proof validation activities will likely help students develop their ability to monitor comprehension. Modeling this behavior during lecture and providing opportunities for them to question themselves, each other, and you during the learning process will allow students to improve their Comprehension Monitoring Skills.

These are only a few, general suggestions of how the findings in this study may be used to inform teaching practices, and these strategies should be implemented and tested during future studies. In general, however, expertise should not be seen as unattainable for the average student. Expertise is not innate or inborn. Instead, most experts have carefully and intentionally cultivated their expertise over years of study, and the characteristics that define expertise can certainly be included into instructional design. Further work should be done to explore other implications for teaching PMI since, by nature, all research studies are limited in scope. The limitations of the current study are explored in the following section.

5.3 Limitations of the Study

Every research study has limitations. Acknowledging and describing these limitations is a crucial component of the research process. This section will explore the limitations of the current study. First, although generalizability is not the primary purpose of this research, it is important to note that findings from a case-study or multi-case study with a smaller number of participants will not always provide results which are easily applicable to a broader set of individuals. But, collectively, this study along with previous qualitative research studies, provide us with valuable insights about students understanding of PMI. Therefore, the results from this study offer a deeper and richer account of several expert participants' views and experiences. Secondly, for data consistency, the participants in this study were selected from the same regional area of the United States with several other selection criteria (detailed in Section 3.1), and there is potential bias in the sample as a result. Although these criteria were used intentionally, more research should be conducted with a variety of participants in order to further triangulate the data with a broader pool of participants. This is explored further in the following section.

As with all research, whether quantitative or qualitative, the researcher's own bias is an integral part of the research process and should always be acknowledged. I conducted the interviews, verified the transcriptions, coded the results, and interpreted the data presented in this study. In order to account for this bias and to ensure that it did not unduly or negatively impact the findings and results of the study, several methods of triangulation were used. A second researcher checked, validated, and critiqued all initial codings as well as the interpretations of the data. The written work of the participants is included as part of the data analysis to corroborate and enhance the interpretation and dialogue data. The interpretations are informed heavily by existing peer-reviewed research and theoretical frameworks to ensure rigorous analysis in the work. These actions help to lessen any negative impacts the primary researcher's bias may have contributed to the research and data analysis. Lastly, this study intentionally used questions which did not require any highly specific mathematical knowledge associated with any branches of mathematics. While this was an important part of the current work, it is likely that exploring mathematical induction in even more specific cases with experts in various fields of mathematics may offer even more insight to PMI and the cognitive processes associated with it. Potential future studies with such specificity are discussed in the following section.

5.4 Future Research

This study offers three natural avenues for future study. First, the revised genetic decomposition outlined in Section 4.2 can be used to develop teaching materials for PMI, and the outcomes of these materials could be tested to check if the genetic decomposition may need further refinement. Second, the expert knowledge framework, used in tandem with CTA or comparable methods, has the potential to serve as an effective method for analyzing mathematical expertise in various contexts. Future work could further investigate each item in the framework to identify any redundancies or gaps in the framework. Additionally, the more work could be done on item interaction for the framework similar to the discussion in Section 4.3.10 to further our understanding about how various characteristics of expert knowledge inform and influence each other. Finally, good mathematics education research should have concrete applications to the teaching and learning of mathematics. The implications for teaching discussed in Section 5.2 offer various strategies for adjusting teaching practices associated with mathematical induction. Each of these suggestions offers the potential for implementation and evaluation research studies. I will conclude the paper with a quote from Henkin (1961) which motivated this work.

> "Of what real good is this principle anyhow?" you may ask. Of course one answer is that it can be used to establish many general statements about positive integers, but perhaps you are not really interested in general statements about positive integers. You have heard that mathematics can be used to build bridges or guide rockets, and you may wonder if mathematical induction can be applied to problems in such domains. As a matter of fact there are very few direct applications of mathematical induction to what we might call "engineering problems"; most of these arise in connection with computations in the elementary theory of probability. But in spite of this, mathematical induction is really of great importance to engineering, for it enters into the proofs of a great many of the fundamental theorems in the branch of mathematics we call analysis - and these theorems are used over and over by engineers. And yet, to me, the true significance of mathematical induction does not lie in its importance for practical applications. Rather I see it as a creation of man's intellect which symbolizes his ability to transcend the confines of his environment. After all, wherever we go, wherever we look in our universe, we see only finite sets: The eggs in a market, the people in a room, the leaves in a forest, the stars in a galaxy - all of these are

finite. But somehow man has been able to send his imagination soaring beyond anything he has ever seen, to create the concept of an infinite set. And mathematical induction is his most basic tool of discovery in this abstract and distant realm. To me, this conception gives to mathematical study a sense of excitement, and I hope that some of you will carry your study of mathematics to the point where you too can experience the unique excitement which mathematics affords to its devoted student. (p.10)

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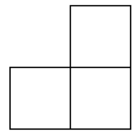
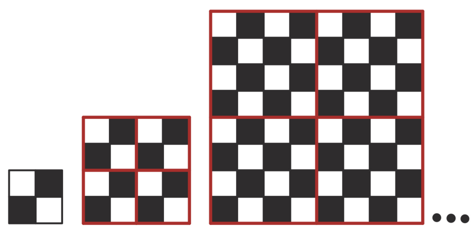
A Induction Problem Solutions

1. Show that there exists a minimal n such that for all m with $m \geq n$, a $2^m \times 2^m$ chessboard with one missing tile can be exactly covered with "trominos" that is, three tiles in an L-shape as pictured below (the trominos in the cover may be oriented in any direction):



Proof: First, note that for $n = 1$, a 2×2 chessboard with one tile removed is a tromino, so the property holds trivially when $n = 1$. Now, assume the property holds for some $n \geq 1$. Consider a chessboard C of size $2^{n+1} \times 2^{n+1}$. Note that we can think of this chessboard as four copies of a $2^n \times 2^n$ chessboard, with one in each quadrant as illustrated below.



If we remove one tile from C, what remains is 3 complete copies of $2^n \times 2^n$ chessboards and one copy with a single tile missing. Now, the inductive hypothesis ensures that the $2^n \times 2^n$ board with a missing tile can be covered by trominos. For the remaining three complete boards, we can place a tromino covering the three squares where these boards meet at the center of C. This will cover exactly one tile in each of the three boards, leaving boards which can be covered by trominos by the inductive hypothesis. ■

2. Assume that if you want to send a package, you must pay a certain amount of postage. Show that there exists some minimal $n \in \mathbb{N}$ such that any package with a postage price of m cents for $m \geq n$ can be paid for exactly using only 4 and 5 cent stamps.

Proof: First, note that 11 cannot be written as a linear combination of 4 and 5. Thus, $n \geq 12$. Next, we can see that the property holds for 12, 13, 14, and 15. Namely, we have that $12 = 4(3)$, $13 = 4(2) + 5(1)$, $14 = 4(1) + 5(2)$, and $15 = 5(3)$. Now, let $m \geq 15$, we have that m is congruent to one of these four base cases modulo 4. Therefore, $m = y + 4k$ for $y \in \{12, 13, 14, 15\}$ and $k \in \mathbb{Z}$. Since we know y can be written as a linear combination of 4 and 5, we can see that m can also be written in this way, as desired. ■

OR

Proof: For the basis step, we will prove not only $P(12)$, but also $P(13)$, $P(14)$, and $P(15)$. Namely, we have that $12 = 4(3)$, $13 = 4(2) + 5(1)$, $14 = 4(1) + 5(2)$, and $15 = 5(3)$.

For the inductive step, assume for all j with $12 \leq j \leq n$, the statement holds. Now consider a package whose postage costs $n + 1$ cents. Consider a package which costs $n+1-4 = n-3$ cents. Then as long as $12 \leq n-3$, we can cover this cost with 4 and 5 cent stamps by the inductive hypothesis. Therefore, adding one 4 cent stamp will cover the cost of the $n + 1$ cent package. Therefore, as long as $n \geq 15$, the property holds. Together with the four base cases, this completes the induction. ■

B Interview 1 Questions and Protocol

B.1 Interview Guide

Initial Prompt:

This initial interview will include a problem solving section. I will provide you with two problems, one at a time. For each problem, you will have around 20 minutes to read the question and think about, talk about, and write out a potential solution to each of them. If you are able to come up with a complete proof by the end of the allotted time, that is great. However, it is not necessary, and you don't have to feel pressured to do so. The primary point of these exercises is to get a feel for your thought process as you work on the problem. You may use any method you wish to solve the problems. It is helpful if you talk out loud as you work on a solution.

After the problem solving period, I will ask you some questions about your thought process, work, and ideas. You are not required to do so, but if you choose to write things down, it would be helpful for me to have a copy of your written work after the interview. It is helpful if you have two differently colored writing utensils, so that your original work can be distinguished from any edits you make afterward. We will repeat this process for each of the two problems. Do you have any questions before we get started?

Question 1 (Trominos)

Prompt: I would like you to read the prompt, and take about 20 minutes to work on the problem. You can take notes, think out loud, think to yourself, or any combination of these three things. Then, I will ask you some questions.
"Stuck" Prompts (Optional): Use if participant is struggling at a particular step.

1. Base Case:
   (a) If you are having difficulties identifying the initial m, what are some strategies you could use to find a candidate?
   (b) Are there any natural numbers you know *won't* work? Why?
2. Inductive Step:
   (a) So what is the inductive hypothesis you are using?
   (b) Can you see any way that your inductive hypothesis links to the "n+1" statement?
   (c) It seems like you may be having difficulties linking the nth step to the $n+1$ step. Could you reduce it to the case where we move from $n = 1$ to $n = 2$. Does this generalize somehow?
   (d) I see you have shown this works in the case of a particular tile being removed. Does your exact argument still work no matter which tile we remove from the chessboard?

Post-Solution Questions:

1. This is a problem is asking for a proof. What is the first step in your process when you work on a problem asking for a proof?
   Further prompts (as needed):
   (a) Why do you think it's important to start with this step?
   (b) How does this step affect or inform how you approach the rest of a problem?
   (c) Have you always started your proof construction process this way?
   (d) What did this initial step look like for you in the context of this particular problem?
2. How do you identify what proof strategy you use for a given problem?
   Further prompts (as needed):
   (a) Can you give me some examples of words or phrases that you associate with particular proof strategies?
   (b) For this question, how did you decide on a proof strategy?
3. You indicated that you used ________________________ as your strategy for this question. If you were explaining this proof strategy to someone in an introductory proofs course, how would you explain the process of using this strategy?
   Further prompts (as needed):

(a) Using your explanation of that proof strategy, can you walk me through each part of your solution and how it fits into your description of the process?

(b) And in the context of this strategy, what was the purpose of __________________________ in your solution?

(c) Thinking about this proof strategy, are there any parts of your proof which are extraneous or unnecessary? Are there any crucial components which are missing?

4. (Induction Questions)

   (a) How would you describe the way the base case fits in with the overall induction proof for this question?

   (b) In general, how related do you think the proof of the base case and the proof of the inductive step are?

   (c) For this question, can you talk about which part of the induction you found most difficult? Why was that difficult?

5. (Optional) Since you did not finish, can you walk me through what your plan was for the rest of the problem?

Question 2 (Postage)

Prompt: Now, we are going to follow the same process for one more problem. Again, I would like you to read the prompt, and take about 20 minutes to work on the problem. You can take notes, think out loud, think to yourself, or any combination of these three things. Then, I'll ask you some questions.

"Stuck" Prompts (Optional): Use if participant is struggling at a particular step.

1. Base Case:

   (a) If you are having difficulties identifying the initial m, what are some strategies you could use to find a candidate?

   (b) Are there any natural numbers you know *will not* work? Why?

2. Inductive Step:

   (a) So what is the inductive hypothesis you are using?

   (b) Can you see any way that your inductive hypothesis links to the "$n+1$" statement?

   (c) It seems like you may be having difficulties linking the nth step to the $n+1$ step. Could you reduce it to the case where we move from $n = 1$ to $n = 2$. Does this generalize somehow?

   (d) I see you have shown this works in the case of a particular tile being removed. Does your exact argument still work no matter which tile we remove from the chessboard?

<u>Post-Solution Questions:</u>

1. How did you identify what proof strategy you wanted to use for this problem?

2. You indicated that you used ________________________ as your strategy for this question. (If not same strategy as before): If you were explaining this proof strategy to someone in an introductory proofs course, how would you explain the process of using this strategy?
   Further prompts (as needed):
   (a) Using your explanation of that proof strategy, can you walk me through each part of your solution and how it fits into your description of the process?
   (b) And in the context of this strategy, what was the purpose of ________________________ in your solution?
   (c) Thinking about this proof strategy, are there any parts of your proof which are extraneous or unnecessary? Are there any crucial components which are missing?

3. Are there any notable similarities or differences between this problem and the previous problem? Further prompts (as needed):
   (a) When you use the same proof technique for different problems, what are some of the ways the structure is similar and/or different?

4. (Induction Questions)
   (a) How would you describe the way the base case fits in with the overall induction proof for this question?
   (b) For this question, can you talk about which part of the induction you found most difficult? Why was that difficult.

5. (Optional) Since you did not finish, can you walk me through what your plan was for the rest of the problem?

C Interview 2 Questions and Protocol

<u>Initial Prompt:</u>
The initial interview consisted of a problem-solving section followed by questions and discussion about your solutions. During this interview, I will be asking you several questions. We will start by talking about your mathematical background, current position, and general questions about your teaching and research. Next, I will ask several questions about about proof writing and construction as well as the Principle of Mathematical Induction. There are no right or wrong answers. The purpose of this interview is to better understand your mathematical background, how you approach certain types of problems, and how you think about certain concepts. Before we get started, do you have any questions for me?
<u>Interview Questions (Graduate Student Version):</u>

1. Before we get started, can you tell me what your current position is and what your main areas of current research are?

2. How long have you been in your program?

3. To what extent and in what capacity would you say mathematical proof shows up in your day-to-day life?
   Further Prompts (if necessary):

   (a) How often do you write proofs? Read them?

   (b) How comfortable do you feel constructing proofs for problems you haven't seen before?

4. This interview will primarily deal with proof writing, construction, and comprehension. You were chosen for this study in part due to the length and scope of your mathematical and proof-writing experience. What are the primary skills you think are necessary for good proof-construction?
   Further Prompts (if necessary):

   (a) How and when did you start developing these skills?

   (b) Do you feel that you are still developing these skills? If so, how and when do you work on these abilities?

5. There are a lot of different proof techniques we use as mathematicians. Over the years you have studied mathematics, how has your understanding of these different techniques changed or grown?
   Further Prompts (if necessary):

   (a) Are there any proof techniques which were difficult for you to learn at first?

   (b) Are there any proof techniques that you still sometimes struggle with in the context of your current work?

   (c) What was your experience of proof-writing/proof-comprehension like during the transition from undergraduate to graduate coursework?

   Now, I would like to ask you some questions about the technique of induction, in particular.

6. If you were trying to explain how and why induction works as a proving technique to someone with no mathematical background at all, how would you do that?

7. If you had to give me a formal definition of the Principle of Mathematical Induction, what would it be? Feel free to say it out loud, or write it down, or both.

8. When you first learned induction, what kinds of examples do you most often remember being exposed to? What kinds of examples do you see now in your own studies or research?
   Further Prompts (if necessary):

   (a) Do you think the kinds of induction problems you saw in your intro to proofs course were representative of the kinds of induction problems you see now?

9. Can you think of an example of a difficult or challenging induction problem you've seen, solved, or read recently? Tell me about how that problem compares to the more simple or "classic" induction problems we see in intro courses. Further Prompts (if necessary):

   (a) What is it about the problem you thought of that makes it trickier or more difficult to understand?

   (b) What was the most difficult part of the proof (e.g. base case, inductive step, technical components, etc.)?

10. If you are tackling an induction problem you have not seen before, walk me through an outline of the process you would go through.

11. Could you give me a list of concepts/facts/definitions/skills/etc. that you feel you need/need to know in order to be able to successfully write an induction proof from start to finish?

12. The base step of PMI usually involves proving (and sometimes identifying) a statement for some particular value. What concepts/facts/definitions/skills/etc. do you think are involved with understanding and proving this base step?

13. The inductive step is usually written in the form $P(n) \Rightarrow P(n+1)$. What concepts/facts/definitions/skills/etc. do you think are involved with understanding and proving this inductive step?

14. How do you feel that other proof techniques relate to induction?
    Further Prompts (if necessary):

    (a) Do you need to understand other techniques to be able to perform induction?

    (b) Do you use other techniques within induction?

15. Are there any interesting anecdotes/thoughts you have from your experience either learning or using induction that you feel offer insight to the process or potential issues?