CODEE Journal

Volume 15 Article 4

9-30-2022

A Generalized Method of Undetermined Coefficients

James S. Cook
Liberty University

William J. Cook
Appalachian State University

The authors would like to express their sincere gratitude to the anonymous referees for their many useful suggestions.

A Generalized Method of Undetermined Coefficients

James S. Cook
Liberty University

William J. Cook
Appalachian State University

Keywords: Undetermined coefficients, Annihilators, Cauchy-Euler
Manuscript received on February 18, 2022; published on September 30, 2022.

Abstract: The method of undetermined coefficients is used to solve constant coefficient nonhomogeneous differential equations whose forcing function is itself the solution of a homogeneous constant coefficient differential equation. In this paper, we show that the classical methods for tackling constant coefficient equations, including the method of undetermined coefficients, generalize to much wider class linear differential equations which, for example, include Cauchy-Euler type equations. This general method includes an explicit construction of the fundamental solution sets of such equations. We also briefly consider where this method can be applied by producing the most general second and third order differential equations that are polynomial in a first order differential operator. In addition, we provide a number of constant coefficient, Cauchy-Euler, and novel examples.

1 Introduction

The method of undetermined coefficients can be applied in an ad hoc fashion to many nonhomogeneous linear differential equations. In some sense, one could apply this technique to any equation as long as they could predict the general form of their solution (see [1, 2, 3]). When constant coefficient equations are under consideration, we have explicit formulas for all solutions of homogeneous equations. Consequently, we can predict the form of any particular solution of a nonhomogeneous constant coefficient linear differential equation if the forcing function is itself a solution of some homogeneous equation of the same type. Keeping this in mind, we first turn our attention to homogeneous equations.

Unfortunately, finding solutions of homogeneous linear differential equations of order at least 2 in terms of elementary functions is not always possible. For example, the Airy equation $y''-ty=0$ looks harmless enough, but its nontrivial solutions are not elementary functions. In fact, its nontrivial solutions cannot be obtained from elementary functions after a finite number of exponentiations, integrations, and algebraic steps (see for example, [4], Example 6.21).

However, we can always solve first order linear differential equations in finite terms. In what follows, we show how to leverage this into a method for solving any homogeneous linear differential equation that is a polynomial in a first order differential operator. This class of equations includes both constant coefficient and Cauchy-Euler equations. Once we know how to solve homogeneous equations of this form, we can also solve related nonhomogeneous equations using the method of undetermined coefficients.

In particular, we study problems of the form $\mathscr{L}[y] = f$ where $\mathscr{L} = p(S)$ is a polynomial in a first order differential operator $S = g\frac{d}{dt}$ and g is a smooth nonzero function defined on some fixed open interval. If f is the solution of $q(S)[f] = 0$ for some polynomial $q(S)$, then we can use the method of undetermined coefficients to construct a particular solution of $\mathscr{L}[y] = f$ and therefore find a general solution for $\mathscr{L}[y] = f$.

While the method of undetermined coefficients and as well as Cauchy-Euler equations are standard fare for introductory differential equations textbooks (see for example, [5]), it does not seem to be widely recognized that everything one does for constant coefficient equations can be adapted to the Cauchy-Euler case. Our hope is that seeing our generalization presented in this paper will help one better understand exactly why the undetermined coefficients technique works and also provides an interesting environment to practice working with differential operators.

After establishing a few linear algebraic tools in Section 2, Section 3 is devoted to developing a technique that allows one to find general solutions of homogeneous equations of the form $p(S)[y] = 0$. In Section 4 we discuss how one can recognize whether their equation is of the type under consideration. Finally, in Section 5, we present a generalized method of undetermined coefficients.

2 Some Linear Algebra

Our forthcoming solution sets involve collections of functions that resemble chains of generalized eigenvectors. Since they fail to be such chains, we need to slightly adjust several standard linear algebra proofs to establish that our collections of functions are linearly independent.

Fix some vector space and let A, B, C, and S be linear operators on that space. Also, recall that the commutator bracket is defined by $[A, B] = AB - BA$.

Lemma 2.1. Let A, B, and C be operators and assume that the commutator $[A, B]$ commutes with B. Then $[A, BC] = [A, B]C + B[A, C]$ and for any positive integer k, $[A, B^k] = k[A, B]B^{k-1}$.

Proof: First, we simply calculate:

$$[A, BC] = ABC - BCA = ABC - BAC + BAC - BCA = [A, B]C + B[A, C].$$

Next, notice that $[A, B^1] = 1[A, B]B^0$. Now assume the second part of the result holds for some positive integer k and, using the first result, find that

$$\begin{aligned} [A, B^{k+1}] &= [A, B]B^k + B[A, B^k] &&= [A, B]B^k + Bk[A, B]B^{k-1} \\ &= [A, B]B^k + k[A, B]B^k &&= (k+1)[A, B]B^k. \end{aligned}$$

The lemma now follows by induction. ♦

Let λ be a scalar. Recall that $v_1, \ldots, v_k$ is a k-chain of generalized eigenvectors of S with eigenvalue λ if $(S-\lambda)v_j = v_{j-1}$ for $j = 2, \ldots, k$ and $(S-\lambda)v_1 = 0$ with $v_1 \neq 0$ (i.e., v_1 is an eigenvector with eigenvalue λ). One can show that such a chain forms a linearly independent set and that chains associated with distinct eigenvalues are independent from each other. We require a slight adjustment for our solution sets that follow.

Definition 2.2. Let λ be a scalar and S a linear operator. We say $v_1, \ldots, v_k$ is a *near-k-chain* associated with eigenvalue λ if $(S-\lambda)v_j = (j-1)\, v_{j-1}$ for $j = 2, \ldots, k$ and $(S-\lambda)v_1 = 0$ with $v_1 \neq 0$ (i.e., v_1 is an eigenvector with eigenvalue λ).

Let $v_1, \ldots, v_k$ be a near-k-chain and, for convenience, define $v_\ell = 0$ for any $\ell \leq 0$. Repeatedly applying $S - \lambda$, we get:

$$(S-\lambda)^m[v_j] = (j-1)(j-2)\cdots(j-m)\, v_{j-m}$$

for all $j = 1, \ldots, k$ and any positive integer m.

Proposition 2.3. Let S be a linear operator with near-k-chain $v_1, \ldots, v_k$ associated with eigenvalue λ. Then $\{v_1, \ldots, v_k\}$ is linearly independent.

Proof: Suppose j is the smallest integer such that there exists a linear combination with $c_j \neq 0$ and $\sum_{i=1}^{j} c_i v_i = 0$. Operate by $(S-\lambda)^{j-1}$ to find

$$0 = \sum_{i=1}^{j} c_i (S-\lambda)^{j-1}[v_i] = \sum_{i=1}^{j} c_i (i-1)(i-2)\cdots(i-j+1) v_{i-j+1} = c_j (j-1)(j-2)\cdots 1\, v_1$$

since $v_\ell = 0$ for $\ell \leq 0$. However, $c_j(j-1)(j-2)\cdots 1$ and v_1 are non-zero. We have reached a contradiction. Therefore, no such linear combination exists. ♦

For completeness, we include a proof that near-chains associated with different eigenvalues are linearly independent. However, if we accept that generalized eigenvectors from distinct generalized eigenspaces are linearly independent, the next lemma follows after realizing that elements of near-chains are in fact generalized eigenvectors (i.e., $(S-\lambda)^\ell[v_j] = 0$ when $\ell \geq j$).

Lemma 2.4. Let λ and μ be distinct scalars and suppose $v \neq 0$ satisfies $(S-\lambda)^m[v] = 0$ but $(S-\lambda)^{m-1}[v] \neq 0$ for some positive integer m. If ℓ is a non-negative integer, then $(S-\mu)^\ell[v] \neq 0$.

Proof: Noting that $S-\lambda$ and $S-\mu$ commute, we calculate:

$$\begin{aligned}(S-\lambda)^{m-1}\left[(S-\mu)^\ell[v]\right] &= (S-\mu)^{\ell-1}(S-\lambda)^{m-1}\left[(S-\mu)[v]\right] \\ &= (S-\mu)^{\ell-1}(S-\lambda)^{m-1}\left[(S-\lambda)[v] + (\lambda-\mu)v\right] \\ &= (S-\mu)^{\ell-1}\left[(S-\lambda)^m[v] + (\lambda-\mu)(S-\lambda)^{m-1}[v]\right].\end{aligned}$$

Therefore, $(S-\lambda)^{m-1}\left[(S-\mu)^\ell[v]\right] = (\lambda-\mu)(S-\mu)^{\ell-1}(S-\lambda)^{m-1}[v]$ since $\lambda-\mu$ is a scalar and $(S-\lambda)^m[v] = 0$. Commuting the $S-\lambda$ and $S-\mu$ factors, we get

$$(S-\lambda)^{m-1}\left[(S-\mu)^\ell[v]\right] = (\lambda-\mu)(S-\lambda)^{m-1}\left[(S-\mu)^{\ell-1}[v]\right]$$

Then, repeating the calculation above ℓ-fold times yields

$$(S-\lambda)^{m-1}\left[(S-\mu)^{\ell}[v]\right] = (\lambda-\mu)^{\ell}(S-\lambda)^{m-1}[v]$$

and since $(\lambda-\mu)^{\ell} \neq 0$ and $(S-\lambda)^{m-1}[v] \neq 0$ it follows that $(S-\mu)^{\ell}[v] \neq 0$. ♦

Theorem 2.5. Let $\lambda_1, \lambda_2, \dots, \lambda_s$ be distinct eigenvalues of the linear operator S. Further, suppose for each $i = 1, \dots, s$, we have $v_{i,1}, \dots, v_{i,k_i}$ is a near-k_i-chain associated with λ_i. Then $\{v_{i,j} \mid i = 1, \dots, s \text{ and } j = 1, \dots, k_i\}$ is linearly independent.

Proof: Suppose $\sum_{i,j} c_{i,j} v_{i,j} = 0$ and let $w_i = \sum_{j=1}^{k_i} c_{i,j} v_{i,j}$ so that $w_1 + \cdots + w_s = 0$. Suppose $w_n \neq 0$.

Since $v_{i,1}, \dots, v_{i,k_i}$ is a near-k_i-chain, $(S-\lambda)^{k_i}[w_i] = \sum_{j=1}^{k_i} c_{i,j}(S-\lambda_i)^{k_i}\left[v_{i,j}\right] = 0$. Consider $\mathscr{L} = \prod_{i\neq n}(S-\lambda_i)^{k_i}$. Then $\mathscr{L}[w_i] = 0$ for $i \neq n$ so that $0 = \mathscr{L}[w_1 + \cdots + w_s] = \mathscr{L}[w_n]$.

Now $(S-\lambda_n)^{k_n}[w_n] = 0$. However, $w_n \neq 0$ so there is some m such that $(S-\lambda_n)^m[w_n] = 0$ but $(S-\lambda_n)^{m-1}[w_n] \neq 0$. By Lemma 2.4, for all $i \neq n$, we have $(S-\lambda_i)^{k_i}[w_n] \neq 0$. Keeping in mind that all of our $S-\lambda_j$ factors commute, $(S-\lambda_i)^{k_i}[w_n]$ is annihilated by $(S-\lambda)^m$. Finally, apply Lemma 2.4 repeatedly for each factor $(S-\lambda_i)^{k_i}$ where $i \neq n$ and get $\mathscr{L}[w_n] = \prod_{i\neq n}(S-\lambda_i)^{k_i}[w_n] \neq 0$ which is a contradiction. Therefore, $w_1 = w_2 = \cdots = w_s = 0$.

Finally, since each w_i is a linear combination of a near-chain and near-chains are independent, so all $c_{i,j} = 0$.♦

3 Polynomial Type Differential Equations

Let $g : I \subseteq \mathbb{R} \to \mathbb{C}$ be a smooth nonzero function[1] and $D = d/dt$ and define $S = gD$. Let

$$\mathscr{L} = p(S) = a_n S^n + \cdots + a_1 S + a_0$$

where $a_0, a_1, \dots, a_n$ are constants with $a_n \neq 0$. Recall that $S^0[y] = y$ and for any positive integer n, $S^n[y] = S^{n-1}[S[y]]$. Note that multiplication by constants commutes with $S = gD$, so $a_k S^k = S^k a_k$. Also, if g itself is a constant function, then $gD = Dg$. However, when g is not constant, this will not be the case.

Now $\mathscr{L}[y] = 0$ is an n-th order homogeneous linear differential equation of the form:

$$\mathscr{L}[y] = a_n S^n[y] + \cdots + a_2 S[S[y]] + a_1 S[y] + a_0 y = 0. \tag{3.1}$$

For convenience, we will refer to these as *polynomial differential operators* and *polynomial differential equations*. For example, if $S = D$ with $I = \mathbb{R}$, then $\mathscr{L}[y] = 0$ is a n-th order constant coefficient linear differential equation. Likewise, if $S = tD$ with $I = (0, \infty)$, then $\mathscr{L}[y] = 0$ is the n-th order Cauchy-Euler differential equation:

$$b_n t^n y^{(n)} + b_{n-1} t^{n-1} y^{(n-1)} + \cdots + b_1 t y' + b_0 y = 0$$

See Corollary 4.2 for the relationship between the a_j coefficients in (3.1) and b_j coefficients above.

[1]Ultimately our interest lies in mappings into $\mathbb{R}$, but it helps to work in slightly more generality for now.

Since $S = gD$ is linear, S commutes with scalars and so we can treat S like a commuting indeterminate. Thus over the algebraically closed field of complex numbers, $\mathscr{L} = p(S)$ factors into linear factors: $\mathscr{L} = p(S) = a_n(S-\lambda_1)^{k_1} \cdots (S-\lambda_\ell)^{k_\ell}$ for some positive integers k_j and distinct complex roots λ_j ($j = 1, \dots, \ell$). Notice that since the factors commute, if $(S - \lambda_j)^{k_j}[f(t)] = 0$, then $\mathscr{L}[f(t)] = (S-\lambda_1)^{k_1} \cdots (S-\lambda_\ell)^{k_\ell}[f(t)] = 0$ as well. In other words, solving $(S - \lambda_j)^{k_j}[y] = 0$ for each j yields solutions for $\mathscr{L}[y] = 0$.

If we can construct n linearly independent solutions $y_1, y_2, \dots, y_n$ of $\mathscr{L}[y] = 0$, then the general solution, sometimes called the complementary function of the associated homogeneous linear differential equation, is given by the superposition of this fundamental solution set: $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$.

Our goal in this section is to construct the fundamental solution set for a polynomial differential equation: $p(S)[y] = 0$. We accomplish this task in two stages. In Subsection 3.1, we show how to find the part of the solution which stems from factors in $p(S)$ of the form $(S - \lambda)^k$ where $\lambda \in \mathbb{C}$. Theorem 3.4 then gives a (possibly complex) fundamental solution set for (3.1). In Subsection 3.2, Theorem 3.9 provides a set of real solutions derived from a conjugate pair of roots. Thus we provide an effective way of finding a real general solution of any real equation of the form (3.1). In fact, these theorems allow one, just as in the case of constant coefficient equations, to read off a set of solutions merely from the factorization of the polynomial $p(S)$.

3.1 General Factors

Let us construct the solution set for a single factor with multiplicity k: $(S - \lambda)^k[y] = 0$ where $\lambda \in \mathbb{C}$. When $k = 1$, since $S = g\frac{d}{dt}$, we have $(S - \lambda)[y] = 0$. This is equivalent to the separable equation $g\frac{dy}{dt} = \lambda y$. Since g is a nonzero function on our domain I, we have $\frac{dy}{y} = \frac{\lambda\, dt}{g}$ and integration yields the general solution $y = C \exp\left(\lambda \int \frac{dt}{g}\right)$ where $\int dt/g$ denotes any fixed antiderivative of $1/g$ and C is an arbitrary constant. For convenience, define $h = \int \frac{dt}{g}$ and let

$$y_1 = \exp(\lambda h)\,. \tag{3.2}$$

Therefore, $S[y_1] = \lambda y_1$ and $y_1 \neq 0$ thus y_1 is an eigenvector for S with eigenvalue λ.

When considering the repeated root case (i.e., $k \geq 2$), it is helpful to study the interplay between the partial derivative operator $\partial_\lambda = \frac{\partial}{\partial\lambda}$ and $(S - \lambda)^k$. On the space of smooth functions, Clairaut's theorem tells us that ∂_λ and $D = \frac{d}{dt}$ commute. Thus, since g only depends on t, the operators ∂_λ and $S = gD$ commute as well. However, the commutator of ∂_λ and $S - \lambda$ is nontrivial:

$$\begin{aligned} [\partial_\lambda, S-\lambda]y &= [\partial_\lambda, S]y - [\partial_\lambda, \lambda]y &&= 0 - [\partial_\lambda, \lambda]y \\ &= -\partial_\lambda[\lambda y] + \lambda\partial_\lambda y &&= -y - \lambda\partial_\lambda y + \lambda\partial_\lambda y \\ &= -y. \end{aligned}$$

Consequently, we have the operator equation $[\partial_\lambda, S - \lambda] = -1$.

Since our commutator is a scalar (and thus commutes with every operator), we can apply Lemma 2.1 with $A = \partial_\lambda$ and $B = S - \lambda$; we thus get the commutation relation

$$[\partial_\lambda, (S-\lambda)^k] = -k\,(S-\lambda)^{k-1} \tag{3.3}$$

for any positive integer k. Likewise, since $[S-\lambda, \partial_\lambda] = 1$, we also have the commutation relation

$$[(S-\lambda), \partial_\lambda^k] = k\,\partial_\lambda^{k-1} \tag{3.4}$$

for any positive integer k. Equation (3.3) gives us

$$(S-\lambda)^k[\partial_\lambda y] = \partial_\lambda (S-\lambda)^k[y] + k\,(S-\lambda)^{k-1}[y].$$

From this we obtain our first main result:

Theorem 3.1. If y solves $(S-\lambda)^{k-1}[y] = 0$, then $\partial_\lambda y$ satisfies $(S-\lambda)^k[\partial_\lambda y] = 0$.

This means we can construct a solution for $(S-\lambda)^k[y] = 0$ by repeated partial differentiation (with respect to λ) of a solution for $(S-\lambda)[y] = 0$.

Corollary 3.2. Suppose y_1 is a solution of $(S-\lambda)[y] = 0$ and let $y_j = \partial_\lambda^{j-1}[y_1]$. Then $y_1, \ldots, y_k$ are solutions of $(S-\lambda)^k[y] = 0$.

Recall the solution y_1 of $(S-\lambda)[y] = 0$ defined by (3.2). Note that

$$\partial_\lambda^{j-1}[y_1] = \partial_\lambda^{j-1}\left[\exp(\lambda h)\right] = h^{j-1}\exp(\lambda h)$$

because h only depends on t. Applying the above corollary, $(S-\lambda)^k[y] = 0$ has a solution set:

$$y_1 = e^{\lambda h},\ y_2 = he^{\lambda h},\ y_3 = h^2 e^{\lambda h},\ \ldots\ ,\ y_k = h^{k-1}e^{\lambda h}. \tag{3.5}$$

We now note that if y_1 is a nonzero solution of $(S-\lambda)[y] = 0$, then Equation (3.4) implies

$$(S-\lambda)[y_{j+1}] = (S-\lambda)[\partial_\lambda^j y_1] = \partial_\lambda^j (S-\lambda)[y_1] + j\,\partial_\lambda^{j-1} y_1 = j\,y_j \tag{3.6}$$

for any positive integer j. Therefore, $y_1, \ldots, y_k$ form a near-k-chain. Recalling that near-chains yield linearly independent sets and since the differential equation $(S-\lambda)^k[y] = 0$ is of order k, we immediately have the following:

Corollary 3.3. The functions $y_1, \ldots, y_k$ as defined in Equation (3.5) form a fundamental solution set for $(S-\lambda)^k[y] = 0$.

Moreover, combining this corollary with Theorem 2.5, we have the following:

Theorem 3.4. Suppose $\mathscr{L} = p(S) = (S-\lambda_1)^{k_1}\cdots(S-\lambda_s)^{k_s}$ where $S = g\frac{d}{dt}$ (g is a smooth nonzero function defined on some open interval I). Then the polynomial differential equation, $\mathscr{L}[y] = 0$, has a fundamental solution set formed by the following $k_1+k_2+\cdots+k_s$ functions:

$$e^{\lambda_1 h}, he^{\lambda_1 h}, \ldots, h^{k_1-1}e^{\lambda_1 h}, e^{\lambda_2 h}, he^{\lambda_2 h}, \ldots, h^{k_2-1}e^{\lambda_2 h}, \ldots, e^{\lambda_s h}, he^{\lambda_s h}, \ldots, h^{k_s-1}e^{\lambda_s h}$$

where $h = \int \frac{dt}{g}$ (i.e., h is some fixed antiderivative of $1/g$).

We will refer to $p(S)$ as our *characteristic polynomial.* At this point, if we consider examples whose characteristic polynomials have real roots, we can find complete (real) general solutions. We illustrate our result with a few examples.

Example 3.5. Constant Coefficients: Suppose $g(t) = 1$ so that $S = d/dt$. Then $h = \int \frac{dt}{1} = t$. The solution associated with $(S - \lambda)[y] = 0$ is then $y_1 = \exp(\lambda t)$. Thus repeated root solutions obtained by differentiating with respect to λ are $te^{\lambda t}$, $t^2 e^{\lambda t}$, etc.

In particular, the equation $y^{(4)} - 11y''' + 45y'' - 81y' + 54y = 0$ factors: $(S-2)(S-3)^3[y] = 0$. Its solution is given by

$$y = c_1 e^{2t} + c_2 e^{3t} + c_3 t e^{3t} + c_4 t^2 e^{3t}.$$

Example 3.6. Cauchy-Euler: Suppose $g(t) = t$ (for $t > 0$) so that $S = t\frac{d}{dt}$. Then $h = \int \frac{dt}{t} = \ln t$. Here $y_1 = \exp(\lambda \ln(t)) = t^{\lambda}$ solves $(S - \lambda)[y] = 0$. Differentiating with respect to λ now yields $\ln(t) \cdot t^{\lambda}$, $(\ln(t))^2 \cdot t^{\lambda}$, etc.

In particular, $t^4 y^{(4)} - 5t^3 y''' + 19t^2 y'' - 46ty' + 54y = 0$ factors: $(S-2)(S-3)^3[y] = 0$ (we will come back to the issue of factoring such equations in Section 4). Its solution is

$$y = c_1 t^2 + c_2 t^3 + c_3 \ln(t) t^3 + c_4 (\ln(t))^2 t^3.$$

The preceding pair of examples are commonly covered in an introductory differential equations course. In contrast, the examples below are much more novel.

Example 3.7. Suppose $g(t) = 2\sqrt{t}$ (for $t > 0$) so that $S = 2\sqrt{t}\frac{d}{dt}$. Then $h = \int \frac{dt}{2\sqrt{t}} = \sqrt{t}$. We have $y_1 = e^{\lambda\sqrt{t}}$ solves $(S - \lambda)[y] = 0$ and repeated root solutions are of the form $\sqrt{t}e^{\lambda\sqrt{t}}$, $te^{\lambda\sqrt{t}}$, etc.

In particular, the equation $(S-2)(S-3)^3[y] = 0$ expanded out becomes $16t^2 y^{(4)} + (-88t\sqrt{t} + 48t)y''' + (180t - 132\sqrt{t} + 12)y'' + (-162\sqrt{t} + 90)y' + 54y = 0$. Its solution is given by

$$y = c_1 e^{2\sqrt{t}} + c_2 e^{3\sqrt{t}} + c_3 \sqrt{t} e^{3\sqrt{t}} + c_4 t e^{3\sqrt{t}}.$$

Example 3.8. Suppose $g(t) = \cos t$ (for $-\pi/2 < t < \pi/2$) so that $S = \cos(t)\frac{d}{dt}$. Then $h = \int \frac{dt}{\cos(t)} = \int \sec(t)dt = \ln(\sec(t) + \tan(t))$. Thus we have that $e^{\lambda h} = e^{\lambda \ln(\sec(t)+\tan(t))} = (\sec(t) + \tan(t))^{\lambda}$ solves $(S - \lambda)[y] = 0$.

In particular, since $\cos^2(t)y'' - \cos(t)(\sin(t)+1)y' + \frac{1}{4}y = 0$ factors as $(S-1/2)^2[y] = 0$, its solution is

$$y = c_1(\sec(t) + \tan(t))^{1/2} + c_2(\sec(t) + \tan(t))^{1/2}\ln(\sec(t) + \tan(t)).$$

3.2 Making Complex Solutions Real

We now assume that g is real valued on I and that the polynomial $p(S)$ has real coefficients. Thus the operator $\mathscr{L} = p(S)$ maps real valued functions to real valued functions and so

$$\mathscr{L}[x + iy] = \mathscr{L}[x] + i\mathscr{L}[y]$$

for real valued functions x and y. Since both $\mathscr{L}[x]$ and $\mathscr{L}[y]$ are then real valued on our domain, $\mathscr{L}[x + iy] = 0$ implies both $\mathscr{L}[x] = 0$ and $\mathscr{L}[y] = 0$. In other words, the real and imaginary parts of a complex solution are real solutions.

Now since $p(S)$ has real coefficients, its complex roots come in conjugate pairs with matching multiplicities. Notice that $h = \int \frac{dt}{g}$ is a real function. Consider a complex root

$\lambda = \alpha + i\beta$ with $\beta \neq 0$ so that $\overline{\lambda} = \overline{\alpha + i\beta} = \alpha - i\beta$ is also a root (distinct from λ). Then $\exp(\lambda h) = e^{\alpha h}\cos(\beta h) + ie^{\alpha h}\sin(\beta h)$ and $\exp(\overline{\lambda} h) = e^{\alpha h}\cos(\beta h) - ie^{\alpha h}\sin(\beta h)$.

Recall that our solutions coming from repeated roots are of the form $h^\ell \exp(\lambda h)$ for some non-negative integer ℓ. By superposition, given solutions $y_1 = h^\ell e^{\lambda h}$ and $y_2 = h^\ell e^{\overline{\lambda} h}$, we also have solutions $u_1 = \frac{1}{2}y_1 + \frac{1}{2}y_2 = h^\ell e^{\alpha h}\cos(\alpha h)$ and $u_2 = \frac{1}{2i}y_1 - \frac{1}{2i}y_2 = h^\ell e^{\alpha h}\sin(\alpha h)$. Notice that since we can recover y_1 and y_2 by taking linear combinations of u_1 and u_2, replacing our complex exponential solutions with cosine/sine pairs of solutions will still yield a complex solution set. Therefore, for a conjugate pair of roots $\alpha \pm i\beta$ of multiplicity k, we get $2k$ linearly independent solutions.

These solutions form a solution set for the $2k$-order differential equation

$$(S - \overline{\lambda})^k (S - \lambda)^k [y] = ((S - \alpha)^2 + \beta^2)^k [y] = 0.$$

In summary, the arguments of this section demonstrate:

Theorem 3.9. If $S = g\frac{d}{dt}$ where g is a smooth nonzero real valued function and $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$, then $((S - \alpha)^2 + \beta^2)^k [y] = 0$ has the fundamental solution set:

$$e^{\alpha h}\cos(\beta h), e^{\alpha h}\sin(\beta h), he^{\alpha h}\cos(\beta h), he^{\alpha h}\sin(\beta h), \ldots, h^{k-1}e^{\alpha h}\cos(\beta h), h^{k-1}e^{\alpha h}\sin(\beta h)$$

where $h = \int \frac{dt}{g}$ (i.e., h is some fixed antiderivative of $1/g$).

Example 3.10. Constant Coefficients: The equation $y^{(4)} - 12y''' + 86y'' - 300y' + 625y = 0$ factors as $((S - 3)^2 + 16)^2 [y] = 0$ where $g = 1$ so $S = d/dt$. This has a general solution

$$y = c_1 e^{3t}\cos(4t) + c_2 e^{3t}\sin(4t) + c_3 te^{3t}\cos(4t) + c_4 te^{3t}\sin(4t).$$

Example 3.11. Cauchy-Euler: The equation $t^4 y^{(4)} + 4t^3 y''' + \frac{41}{2}t^2 y'' + \frac{1369}{16}y = 0$ factors as $((S - 1/2)^2 + 9)^2 [y] = 0$ where $g = t$ so $S = t\,d/dt$. This has a general solution

$$y = c_1\sqrt{t}\cos(3\ln(t)) + c_2\sqrt{t}\sin(3\ln(t)) + c_3(\ln(t))\sqrt{t}\cos(3\ln(t)) + c_4(\ln(t))\sqrt{t}\sin(3\ln(t)).$$

Once again, the preceding examples are commonplace, but the following are probably unfamiliar.

Example 3.12. Given $S = \frac{1}{2t}\frac{d}{dt}$, then $g = 1/2t$. Thus $h = \int 2t\,dt = t^2$. We solve $(S^2 + 25)^3 [y] = 0$ (i.e., $\alpha = 0$ and $\beta = 5$ with multiplicity 3) and find

$$y = c_1\cos(5t^2) + c_2\sin(5t^2) + c_3 t^2\cos(5t^2) + c_4 t^2\sin(5t^2) + c_5 t^4\cos(5t^2) + c_6 t^4\sin(5t^2).$$

We conclude with an example which treats both real and complex roots simultaneously.

Example 3.13. Set $S = \frac{1}{2t}\frac{d}{dt}$ as in Example 3.12. We solve $(S^2 + 1)(S - 1)^2(S + 2)[y] = 0$. Observe $\lambda = \pm i, 1, -2$ with $\lambda = 1$ repeated. Thus a general solution is given by

$$y = c_1\cos(t^2) + c_2\sin(t^2) + c_3 e^{t^2} + c_4 t^2 e^{t^2} + c_5 e^{-2t^2}.$$

4 Recognizing Polynomial Type Differential Equations

Recognizing which linear differential equations can be expressed as a polynomial in a linear differential operator is beyond the scope of this paper. However, in the case that $S = g\frac{d}{dt} = gD$ where g is linear (i.e., $g'' = 0$), it is easy both to recognize such equations and to recover our characteristic polynomial $p(S)$ from the expanded linear operator $\mathscr{L}$. This allows us in the constant coefficient (i.e., $g = 1$) and Cauchy-Euler (i.e., $g = t$) cases to move between our differential equation and its characteristic polynomial with ease.

Theorem 4.1. Let $g(t)$ be linear (i.e., $g(t) = bt + c$ for some constants b and c). Also, for constants $a_0, \ldots, a_n$, let

$$\mathscr{L} = a_n g^n D^n + \cdots + a_2 g^2 D^2 + a_1 gD + a_0.$$

Then there exists a polynomial p such that $\mathscr{L} = p(S)$ where $S = g\frac{d}{dt} = gD$. In particular,

$$p(S) = a_n S(S - g')(S - 2g') \cdots (S - (n-1)g') + \cdots + a_2 S(S - g') + a_1 S + a_0.$$

Proof: A quick calculation shows that

$$S[gy] = gD[gy] = gg'y + ggD[y] = g(gD + g')[y] = g(S + g')[y]$$

where we applied the product rule to get the second equality. This operator identity, $g(S + g') = Sg$, after subtracting kgg', becomes $g(S - (k-1)g') = (S - kg')g$.

To establish our result we show $g^k D^k$ can be replaced by $S(S - g') \cdots (S - (k-1)g')$. Since $g^1 D^1 = S$, our identity holds for $k = 1$. Assume $g^k D^k = S(S - g') \cdots (S - (k-1)g')$ for some positive integer k. Thus

$$\begin{aligned} g^{k+1} D^{k+1}[y] &= gg^k D^k[D[y]] \\ &= g(S(S - g') \cdots (S - (k-1)g')[D[y]] \\ &= (S - g')(S - 2g') \cdots (S - kg')[gD[y]] \\ &= (S - g')(S - 2g') \cdots (S - kg')[S[y]] \\ &= S(S - g') \cdots (S - kg')[y] \end{aligned}$$

where the second equality holds since, as we commute g past each $S - \ell g'$ factor, that factor becomes $S - (\ell + 1)g'$. The last equality holds since S commutes with all $S - \ell g'$ factors because g' is constant (this is where we finally used our assumption $g'' = 0$). Therefore, the desired identity holds for all positive integers k and the result follows. ♦

We have two familiar special cases. First, constant coefficient equations are those where $g = 1$. Here $g' = 0$, so $g^k D^k = D^k = S(S - 0) \cdots (S - (k-1)0) = S^k$. Thus the characteristic polynomial $p(S)$ is (after replacing D^k with S^k) just the left-hand side of our equation. The other familiar case is that of Cauchy-Euler equations where $g = t$. Here $g' = 1$, so we have the following:

Corollary 4.2. Let $a_0, \ldots, a_n$ be constants and $\mathscr{L} = a_n t^n D^n + \cdots + a_2 t^2 D^2 + a_1 tD + a_0$. If we let $S = t\frac{d}{dt} = tD$, then

$$\mathscr{L} = p(S) = a_n S(S-1)(S-2) \cdots (S - (n-1)) + \cdots + a_2 S(S-1) + a_1 S + a_0.$$

Example 4.3. Consider the equation $t^2y'' - 5ty' + 9y = 0$. This is $\mathscr{L}[y] = 0$ where $\mathscr{L} = t^2D^2 - 5tD + 9 = S(S-1) - 5S + 9 = S^2 - 6S + 9 = (S-3)^2$. Therefore, this equation has the following general solution: $y = c_1t^3 + c_2\ln(t)\cdot t^3$.

In general, if $g = bt + c$ with $b \neq 0$ (i.e., not the constant coefficient case), then $h = \int \frac{dt}{bt+c} = \frac{1}{b}\ln|bt+c|$ and $e^{\lambda h} = e^{\frac{\lambda}{b}\ln|bt+c|} = |bt+c|^{\frac{\lambda}{b}}$. In particular, $(S-\lambda)^n[y] = 0$ has solutions

$$|bt+c|^{\frac{\lambda}{b}},\ \left(\frac{1}{b}\ln|bt+c|\right)|bt+c|^{\frac{\lambda}{b}},\ \ldots,\ \left(\frac{1}{b}\ln|bt+c|\right)^{n-1}|bt+c|^{\frac{\lambda}{b}}.$$

Example 4.4. We can solve $(4t+1)^2y'' + 3y = 0$ by setting $g = 4t+1$ so our equation is $(g^2D^2+3)[y] = 0$. As a polynomial in S, this equation is $[S(S-4)+3][y] = 0$ since $g' = 4$. We factor our characteristic polynomial: $S(S-4)+3 = S^2 - 4S + 3 = (S-1)(S-3)$. Therefore, we have the following general solution: $y = c_1|4t+1|^{1/4} + c_2|4t+1|^{3/4}$.

4.1 Equations Corresponding to Quadratic and Cubic Polynomials

When g is non-linear, the relationship between an expanded operator and its characteristic polynomial is much more complicated. Let us consider the general case for a second order equation. Let $S = g\frac{d}{dt} = gD$ and consider

$$(S-a)(S-b)[y] = (S^2 - (a+b)S + ab)[y] = 0.$$

Notice that $S^2[y] = gD[gD[y]] = gD[gy'] = gg'y' + g^2y''$. Thus our equation is

$$g^2y'' + (k+g')gy + \ell = 0$$

where $k = -(a+b)$ and $\ell = ab$. Thus solving such an equation amounts to factoring the characteristic polynomial $p(S) = S^2 + kS + \ell$ and applying results from previous sections.

In summary, let $S = g\frac{d}{dt} = gD$. The equation

$$ag^2\frac{d^2y}{dt^2} + g\left(ag' + b\right)\frac{dy}{dt} + cy = 0 \tag{4.1}$$

corresponds to the characteristic polynomial $p(S) = aS^2 + bS + c$. Likewise, a similar calculation shows that the equation

$$jg^3\frac{d^3y}{dt^3} + g^2\left(3jg' + k\right)\frac{d^2y}{dt^2} + g\left(j[(g')^2 + gg''] + kg' + \ell\right)\frac{dy}{dt} + my = 0 \tag{4.2}$$

corresponds to the characteristic polynomial $p(S) = jS^3 + kS^2 + \ell S + m$.

Clearly the relations between the coefficients exhibited in Equations (4.1) and (4.2) will not be realized by most second or third order differential equations. We finish this section by exhibiting an example where our factoring technique does apply.

Example 4.5. Consider $\cot^2(t)y'' - \cot(t)(\cot^2(t)+3)y' + 2y = 0$. If this factors, we need to have $S = g\frac{d}{dt}$ where $g = \cot(t)$. Comparing with Equation (4.1) we must have $a = 1$ and $c = 2$. Then, considering the coefficient of y' in our equation, $\cot(t)(1\cdot(-\csc^2(t)) + b) =$

$-\cot(t)(\cot^2(t)+3)$. In other words, we need $-\csc^2(t)+b=-\cot^2(t)-3$. Taking into account that $\cot^2(t)+1=\csc^2(t)$, we find that $b=-2$. Thus our characteristic polynomial is $p(S)=S^2-2S+2$ whose roots are $\lambda=1\pm i$.

We have $S=\cot(t)\frac{d}{dt}$. To find the solution we calculate $h=\int\frac{dt}{\cot(t)}=-\ln|\cos(t)|$ thus $e^{\lambda h}=\exp(-\lambda\ln|\cos(t)|)=|\cos(t)|^{-\lambda}$. Since $\lambda=1\pm i$ and referring to Theorem 3.9, we have

$$y=c_1|\cos(t)|^{-1}\cos\left(|\cos(t)|^{-1}\right)+c_2|\cos(t)|^{-1}\sin\left(|\cos(t)|^{-1}\right). \tag{4.3}$$

5 Undetermined Coefficients

We now turn to non-homogeneous equations. In this section, we prove Theorem 5.1 which establishes a generalized method of undetermined coefficients that extends the technique traditionally presented in introductory differential equations texts.

Given a linear differential equation $\mathscr{L}[y]=f$, we first solve $\mathscr{L}[y]=0$ finding a general solution (sometimes called the homogeneous or complementary solution), denoted y_h, and then find a particular solution of $\mathscr{L}[y]=f$, denoted y_p. Thus by the principle of superposition, we have $y=y_h+y_p$ is a general solution of $\mathscr{L}[y]=f$.

Suppose that we have a polynomial differential equation $\mathscr{L}[y]=f$ where $\mathscr{L}=p(S)$ is a polynomial in the first order differential operator S. At this point, we know how to solve the complementary equation $\mathscr{L}[y]=0$. In general, we could find a particular solution of $\mathscr{L}[y]=f$ by using variation of parameters, but if f is itself a solution of a homogeneous equation arising from a polynomial in S, we could use the more tractable method of undetermined coefficients.

In particular, suppose that $\mathscr{K}[f]=0$ where $\mathscr{K}=q(S)$ is a polynomial in S. We call $\mathscr{K}$ an *annihilator* of f. Since both $\mathscr{L}$ and $\mathscr{K}$ are polynomials in S, they commute:

$$\mathscr{L}\mathscr{K}=p(S)q(S)=q(S)p(S)=\mathscr{K}\mathscr{L}.$$

Suppose that y solves $\mathscr{L}[y]=f$, then $\mathscr{K}[\mathscr{L}[y]]=\mathscr{K}[f]=0$. So y is also a solution of $\mathscr{K}\mathscr{L}[y]=0$. In other words, each solution of the nonhomogeneous equation $\mathscr{L}[y]=f$ is also a solution of the homogeneous differential equation $\mathscr{K}\mathscr{L}[y]=0$. It follows that any particular solution of $\mathscr{L}[y]=f$ must appear in a general solution of $\mathscr{K}\mathscr{L}[y]=0$.

Suppose that $\mathscr{L}=p(S)$ is a polynomial of degree n and $\mathscr{K}=q(S)$ has degree m. Select a solution set for $\mathscr{L}[y]=0$, denoted $y_1,\ldots,y_n$. Then complete this to a solution set for $\mathscr{K}\mathscr{L}[y]=0$, denoted $y_1,\ldots,y_n,z_1,\ldots,z_m$. Suppose y_o is a solution of $\mathscr{L}[y]=f$, then y_o is also a solution of $\mathscr{K}\mathscr{L}[y]=0$ and consequently may be written as a linear combination of the solution set: $y_o=y_h+y_p$ where $y_h=\sum\limits_{i=1}^{n}c_iy_i$ and $y_p=\sum\limits_{j=1}^{m}b_jz_j$ for some constants $c_1,\ldots,c_n,b_1,\ldots,b_m$. But we notice that y_h is a solution of $\mathscr{L}[y]=0$. Therefore,

$$f=\mathscr{L}[y_o]=\mathscr{L}\left[y_h\right]+\mathscr{L}\left[y_p\right]=0+\mathscr{L}\left[y_p\right].$$

Therefore, y_p is in fact a particular solution of $\mathscr{L}[y]=f$. This means that we do not require the y_h part which solves $\mathscr{L}[y]=0$ when seeking our particular solution of $\mathscr{L}[y]=f$.

We now arrive at the method of undetermined coefficients. First, solve $\mathscr{K}\mathscr{L}[y] = 0$. Then let $y_h = \sum_{i=1}^{n} c_i y_i$ and $y_p = \sum_{j=1}^{m} b_j z_j$ be the *homogeneous* and *particular* parts of our solution. By the argument above, we know that some choice of constants $b_1, \ldots, b_m$ will yield a particular solution y_p such that $\mathscr{L}[y_p] = f$. Notice that any two particular solutions of $\mathscr{L}[y] = f$ differ by a homogeneous solution of the form $y_h = \sum_{i=1}^{n} c_i y_i$ where $c_1, \ldots, c_n$ are arbitrary constants. Therefore, $y = y_h + y_p$ is a general solution of $\mathscr{L}[y] = f$.

At this point we know that there is a particular solution of the form $y_p = \sum_{j=1}^{m} b_j z_j$. We now show that any two particular solutions constructed in this fashion must match. Suppose $y_q = \sum_{j=1}^{m} a_j z_j$ is another such solution. Then $\mathscr{L}[y_p - y_q] = \mathscr{L}[y_p] - \mathscr{L}[y_q] = f - f = 0$ so that $y_p - y_q$ solves $\mathscr{L}[y] = 0$. This means that $y_p - y_q$ must be a linear combination of $y_1, \ldots, y_n$. However, $y_p - y_q = \sum_{j=1}^{m} (b_j - a_j) z^j$ is a linear combination of $z_1, \ldots, z_m$ which are independent from $y_1, \ldots, y_n$. Thus $y_p - y_q = 0$ and so $y_p = y_q$. Therefore, there are *unique* constants $b_1, \ldots, b_m$ such that $y_p = \sum_{j=1}^{m} b_j z_j$ is a solution of $\mathscr{L}[y] = f$.

Finally, notice that since f is a linear combination of the *linearly independent* functions $y_1, \ldots, y_n, z_1, \ldots, z_m$, if we plug $y = \sum_{j=1}^{m} b_j z_j$ into $\mathscr{L}$, we will be able to solve for our *unique* constants $b_1, \ldots, b_m$ by equating coefficients of the functions $y_1, \ldots, y_n, z_1, \ldots, z_m$ appearing in the required equation: $\mathscr{L}[y] = f$.

Theorem 5.1. Let $S = g\frac{d}{dt}$ where g is any nonzero smooth function and suppose $\mathscr{L} = p(S)$ is a polynomial of degree n in S. If f is a function for which there exists a polynomial $\mathscr{K} = q(S)$ of degree m with $\mathscr{K}[f] = 0$, then $\mathscr{L}[y] = f$ is solved as follows:

1. Find a solution set $y_1, \ldots, y_n$ for $\mathscr{L}[y] = 0$,
2. Extend the solution set $y_1, \ldots, y_n$ by $z_1, \ldots, z_m$ to form a solution set for $\mathscr{K}\mathscr{L}[y] = 0$,
3. Set $y_p = \sum_{j=1}^{m} b_j z_j$, compute $\mathscr{L}[y_p] = f$, and equate coefficients to determine our previously undetermined coefficients $b_1, \ldots, b_m$.

Then $y = c_1 y_1 + \cdots + c_n y_n + y_p$ where $c_1, \ldots, c_n$ are arbitrary constants is a general solution.

Example 5.2. Let us solve $y'' + y = t + 2\sin(3t) + \cos(t)$. We notice that our equation is $\mathscr{L}[y] = f$ where $\mathscr{L} = D^2 + 1$ and $f = t + 2\sin(3t) + \cos(t)$. The function f can be annihilated by $\mathscr{K} = D^2(D^2 + 9)(D^2 + 1)$ since D^2 annihilates t, $D^2 + 9$ annihilates $2\sin(3t)$, and $D^2 + 1$ annihilates $\cos(t)$.

The characteristic polynomial for $\mathscr{K}\mathscr{L}$ is $D^2(D^2 + 9)(D^2 + 1)^2$. We thus get solutions $y_1 = \cos(t)$, $y_2 = \sin(t)$, as well as $z_1 = 1$, $z_2 = t$, $z_3 = \cos(3t)$, $z_4 = \sin(3t)$, $z_5 = t\cos(t)$, and $z_6 = t\sin(t)$ (the last two have a factor t since $D^2 + 1$ is repeated). Our desired particular solution has the form $y_p = b_1 z_1 + \cdots + b_6 z_6$. After a short calculation we find $\mathscr{L}[y_p] = y_p'' + y_p = f$ requires that

$$b_1 + b_2 t - 8b_3 \cos(3t) - 8b_4 \sin(3t) - 2b_5 \sin(t) + 2b_6 \cos(t) = t + 2\sin(3t) + \cos(t).$$

Equating coefficients yields $b_1 = 0, b_2 = 1, -8b_3 = 0, -8b_4 = 2, -2b_5 = 0, 2b_6 = 1$. Therefore, the general solution of $y'' + y = t + 2\sin(3t) + \cos(t)$ is

$$y = c_1 \cos(t) + c_2 \sin(t) + t - \frac{1}{4}\sin(3t) + \frac{1}{2} t \sin(t).$$

Now we move away from familiar examples to unfamiliar examples.

Example 5.3. Let $t > 0$. The Cauchy-Euler equation $t^2 y'' - 3ty' + 13y = \sqrt{t}$ may be rewritten as $\mathscr{L}[y] = f$ where $\mathscr{L} = (S-2)^2 + 9$, $S = t\frac{d}{dt}$, and $f = \sqrt{t}$. The function f is annihilated by $\mathscr{K} = S - 1/2$. Thus our solution set for $\mathscr{K}\mathscr{L}[y] = 0$ is $y_1 = t^2\cos(3t)$, $y_2 = t^2\sin(3t)$, and $z_1 = \sqrt{t}$. Thus we set $y_p = b_1\sqrt{t}$. Then $\mathscr{L}[y_p] = t^2 y_p'' - 3ty_p' + 13y_p = \sqrt{t}$ yields $\frac{45}{4} b_1 \sqrt{t} = \sqrt{t}$ thus $b_1 = \frac{4}{45}$. Therefore, $t^2 y'' - 3ty' + 13y = \sqrt{t}$ has general solution:

$$y = c_1 t^2 \cos(3t) + c_2 t^2 \sin(3t) + \frac{4}{45}\sqrt{t}.$$

Example 5.4. Let $t > 0$. The Cauchy-Euler equation $t^3 y''' - t^2 y'' + 2ty' - 2y = 9/t + 2t^2$ can be recast as $\mathscr{L}[y] = f$ where $\mathscr{L} = S(S-1)(S-2) - S(S-1) + 2S - 2$, $S = t\frac{d}{dt}$, and $f = 9t^{-1} + 2t^2$. Notice that $\mathscr{L} = S^3 - 4S^2 + 5S - 2 = (S-2)(S-1)^2$. Also, since $9t^{-1}$ is annihilated by $S + 1$ and $2t^2$ is annihilated by $S - 2$, the operator $\mathscr{K} = (S+1)(S-2)$ annihilates f. Thus we get the following solution set for $\mathscr{K}\mathscr{L}[y] = 0$: $y_1 = t$, $y_2 = t\ln(t)$, and $y_3 = t^2$ as well as $z_1 = t^{-1}$, $z_2 = t^2\ln(t)$. Therefore, we have a particular solution of the form $y_p = b_1 t^{-1} + b_2 t^2 \ln(t)$. After some calculation we find $t^3 y_p''' - t^2 y_p'' + 2ty_p' - 2y_p = 9t^{-1} + 2t^2$ requires that

$$-(12b_1)t^{-1} + b_2 t^2 = 9t^{-1} + 2t^2.$$

Equating coefficients yields equations $-12b_1 = 9$ and $b_2 = 2$. Thus $b_1 = -3/4$ and $b_2 = 2$. Therefore,

$$y = c_1 t + c_2 t\ln(t) + c_3 t^2 - \frac{3}{4t} + 2t^2\ln(t)$$

is the general solution of $t^3 y''' - t^2 y'' + 2ty' - 2y = \frac{9}{t} + 2t^2$.

We conclude with an example outside the realm of constant coefficient and Cauchy-Euler type equations.

Example 5.5. Consider the equation $\frac{1}{4t^2} y'' - \frac{1}{4t^3} y' + y = 8\cos(t^2) + 20e^{3t^2}$. Referring to Equation (4.1), if we let $g = \frac{1}{2t}$ so $S = g\frac{d}{dt} = gD$, then setting $a = 1$ and $c = 1$ we need $\frac{1}{2t}\left(1 \cdot \frac{-1}{2t^2} + b\right) = -\frac{1}{4t^3}$ so $b = 0$. Thus our equation is $\mathscr{L}[y] = f$ where $f = 8\cos(t^2) + 20e^{3t^2}$ and $\mathscr{L} = \frac{1}{4t^2}D^2 - \frac{1}{4t^3}D + 1 = S^2 + 1$.

Recall that when $S = \frac{1}{2t}\frac{d}{dt}$, our multiplier is $h = \int \frac{dt}{1/2t} = \int 2t\,dt = t^2$, so, for example, $y = c_1 e^{\lambda t^2} + c_2 t^2 e^{\lambda t^2}$ is a general solution of $(S-\lambda)^2[y] = 0$. Now when $\lambda = \pm i$, we get $y = c_1\cos(t^2) + c_2\sin(t^2) + c_3 t^2\cos(t^2) + c_4 t^2 \sin(t^2)$ is a general solution for $(S^2+1)^2[y] = 0$.

Next, notice that $8\cos(t^2)$ is annihilated by S^2+1 and $20e^{3t^2}$ is annihilated by $S-3$, so $\mathcal{K} = (S^2+1)(S-3)$ annihilates f. This means that we can apply the method of undetermined coefficients. To that end, a solution set for $\mathcal{K}\mathcal{L}[y] = 0$ is $y_1 = \cos(t^2)$ and $y_2 = \sin(t^2)$ as well as $z_1 = t^2\cos(t^2)$, $z_2 = t^2\sin(t^2)$, and $z_3 = e^{3t^2}$. Therefore, we can find a particular solution of the form $y_p = b_1t^2\cos(t^2) + b_2t^2\sin(t^2) + b_3e^{3t^2}$. A short calculation reveals that

$$(S^2+1)[y_p] = (2b_2 - b_1t^2 + b_1t^2)\cos(t^2) + (-2b_1 - b_2t^2 + b_2t^2)\sin(t^2) + 10b_3e^{3t^2}.$$

Thus $\mathcal{L}[y_p] = 8\cos(t^2) + 20e^{3t^2}$ requires $2b_2\cos(t^2) - 2b_1\sin(t^2) + 10b_3e^{3t^2} = 8\cos(t^2) + 20e^{3t^2}$. Equating coefficients forces $2b_2 = 8$, $-2b_1 = 0$, and $10b_3 = 20$. Thus $b_1 = 0$, $b_2 = 4$, and $b_3 = 2$. Putting this together, we find that

$$y = c_1\cos(t^2) + c_2\sin(t^2) + 4t^2\sin(t^2) + 2e^{3t^2} \tag{5.1}$$

is a general solution of our differential equation.

In summary, the generalized method of undetermined coefficients laid out in Theorem 5.1 applies to any a differential equation of the form $p(S)[y] = f$ where f is a solution of $q(S)[y] = 0$ (i.e., f is a solution of a homogeneous equation of the same type). Surprisingly, we can find a general solution of our complementary equation $p(S)[y] = 0$ from the factorization of the polynomial $p(S)$ alone (see Theorems 3.4 and 3.9). Then solving $q(S)p(S)[y] = 0$ by considering the factorization of $q(S)p(S)$, we can find a template for a particular solution of $p(S)[y] = f$. We can then find the previously undetermined coefficients in our template solution by solving a system of linear equations. Finally, putting together our complementary and particular solutions, we get a general solution of the equation $p(S)[y] = f$.

References

[1] Doreen De Leon. Euler-Cauchy using undetermined coefficients. College Math Journal, **41** (2010), 236–237.

[2] Doreen De Leon. Using undetermined coefficients to solve certain classes of variable-coefficient equations. *American Mathematical Monthly* **122** (2015), no. 3, 246–255.

[3] Adnan H. Sabuwala and Doreen De Leon. Particular solution to the Euler-Cauchy equation with polynomial non-homogeneities. Discrete and Continuous Dynamical Systems 2011. *Dynamical systems, differential equations and applications.* 8th AIMS Conference. Suppl. Vol. II, 1271–1278. ISBN: 978-1-60133-008-6.

[4] Andy R. Magid. *Lectures on differential Galois theory.* University Lecture Series, 7. American Mathematical Society, Providence, RI, 1994. ISBN: 0-8218-7004-1.

[5] R. Kent Nagle, Edward B. Saff, and Arthur David Snider. *Fundamentals of Differential Equations.* 9th edition. Pearson, New York, NY, 2017. ISBN: 978-0321977069.