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Mining the Soma Cube for Gems: Isomorphic Subgraphs Reveal Equivalence Classes

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Synopsis

Soma cubes are an example of a *dissection puzzle*, where an object is broken down into pieces, which must then be reassembled to form either the original shape or some new design. In this paper, we present some interesting discoveries regarding the Soma Cube. Equivalence classes form aesthetically pleasing shapes in the solution set of the puzzle. These gems are identified by subgraph isomorphisms using SNAP!/Edgy, a simple block-based computer programming language. Our preliminary findings offer several opportunities for researchers from middle school to undergraduate to utilize graphs, group theory, topology, and computer science to discover connections between computation and geometric patterns.

What is K-12 mathematics education about other than learning arithmetic to learn algebra to learn calculus to get a STEM degree and get a STEM job and then not use much calculus?

Mathematics is art.

Mathematics is play.

Mathematics is counting, organizing, estimating, discovering patterns, and discussing those things. We all do this. We are made of mathematics.

Let's enjoy this ride.

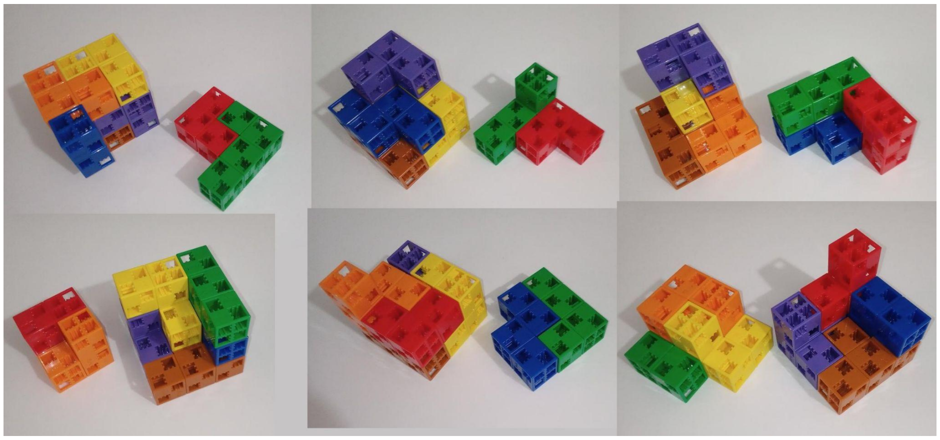


Figure 1: Six Soma Cube Gemstones. Cube puzzle solutions with these properties make up nearly half of the 240 unique solutions.

1. Introduction

The Soma cube is an example of a *dissection puzzle* where an object is broken down into pieces, that must then be reassembled to form either the original shape or some new design (see Figure 1). According to Wikipedia, it was invented in 1933 by Piet Hein during a lecture on quantum physics by Werner Heisenberg. The pieces are the seven different non-convex shapes that can be constructed by joining four or fewer cubes at their faces; see Figure 2 below for the seven shapes.

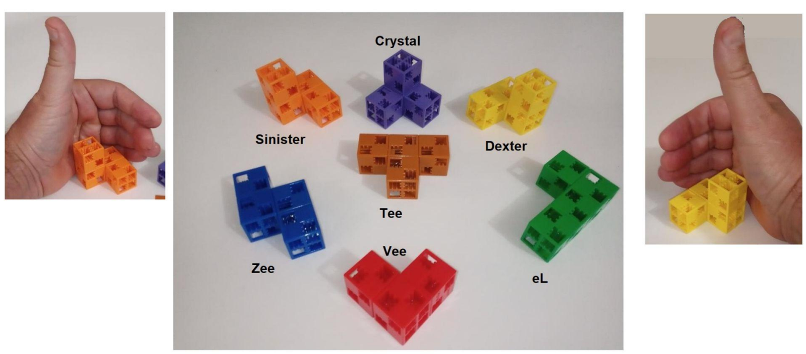
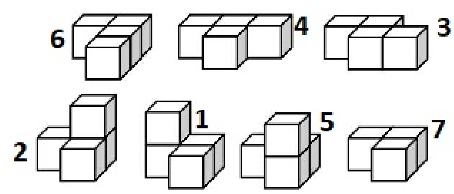


Figure 2: The seven pieces of the Soma cube and our names for them. Out of the seven pieces, two are chiral and mirror each other.

The first enumeration of all solutions was obtained by hand in 1961 by John Horton Conway and M. J. T. Guy "one wet afternoon" [1] when both mathematicians had no more pressing chores at the University of Cambridge. The 240 unique solutions were later verified by computer programs.

Solutions are typically presented similar to "Table 3" from Orth's paper [9], shown in Figure 3. All of the solutions are there but a table doesn't readily show the similarities between solutions. Orth's table [9] was tabulated in such a way to normalize the solution to the Tee piece [Figure 2] to be in just one position. This makes Soma a six-piece puzzle and provides the root for a tree to sort the solutions further.



	611	666	666	337	666	666	666	655	655	115	113	511	155	771	773
top	615	511	551	157	156	615	551	677	633	155	175	517	115	571	511
	665	551	511	155	155	115	511	337	337	773	775	337	773	553	551
	221	226	226	233	233	233	276	615	615	615	613	551	156	611	673
center	327	327	723	227	227	227	223	225	225	223	223	227	223	223	223
	355	351	513	115	115	155	513	233	277	273	255	233	273	253	251
	327	327	723	666	337	337	773	611	611	666	666	666	666	666	666
bottom	347	347	743	246	247	247	243	241	241	243	243	246	243	243	243
	444	444	444	444	444	444	444	444	444	444	444	444	444	444	444

Figure 3: Soma Cube Puzzle Solutions in table format taken from [9]. Verbose but not particularly illuminating. Are there other ways to show the solutions that also show the relationship of the solutions?

After converting the table of results to 240 rows of 27 cells [Figure 4], the results can be sorted and plotted using yEd [15] into a tree diagram [Figure 5] [13]. This tree diagram can be further sorted to reveal more intricate branching [Figure 6].

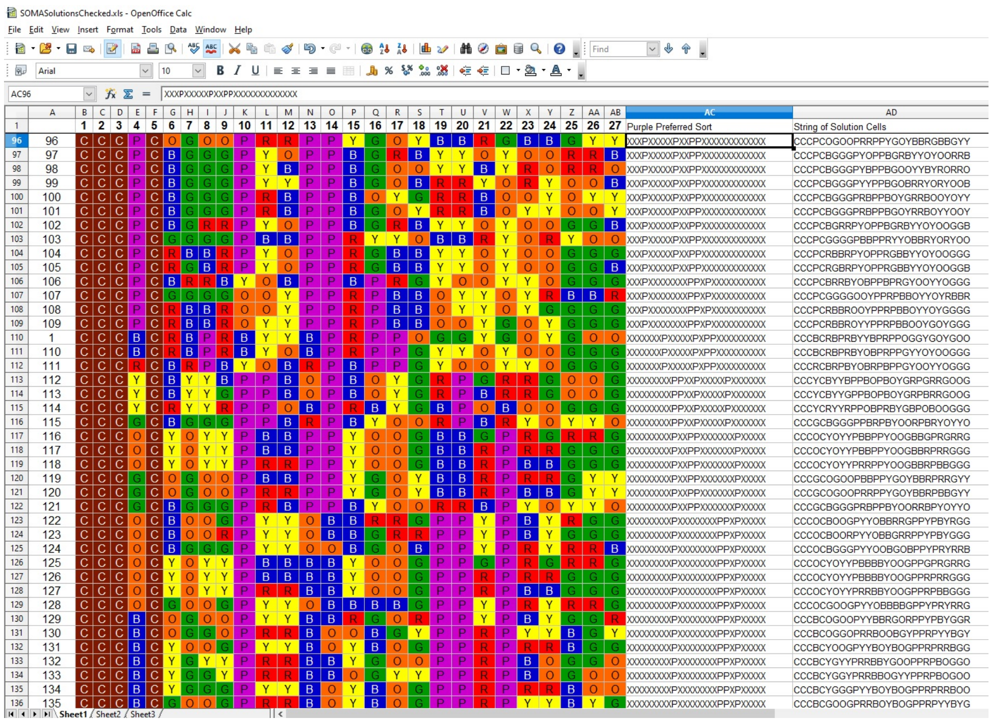


Figure 4: Soma Cube Puzzle solutions arranged as 27 consecutive cells and edited strings that allow for sorting. Add some color to the cells and the similarities start to pop.

At some point though, the branching becomes difficult to continue in a consistent manner via spreadsheet sorting. Other interesting patterns that relate solutions to each other "across branches" begin to emerge. Similar to finding a bunch of peaches growing on a cherry tree. Which is possible with grafting but we didn't do anything like that. A better analogy might be of identical crystal lattices made up of different chemical elements.

And this takes us back to quantum mechanics and the origin story of Soma with Hein listening to Heisenberg talk on his ideas of describing the space-time continuum as a lattice [3].

These patterns are complexes of two or three pieces that can be either transformed through flips and rotations or rearrangements of the pieces into a new complex that still drops back into place in the same cube puzzle solution. They are equivalence classes [Figure 1].

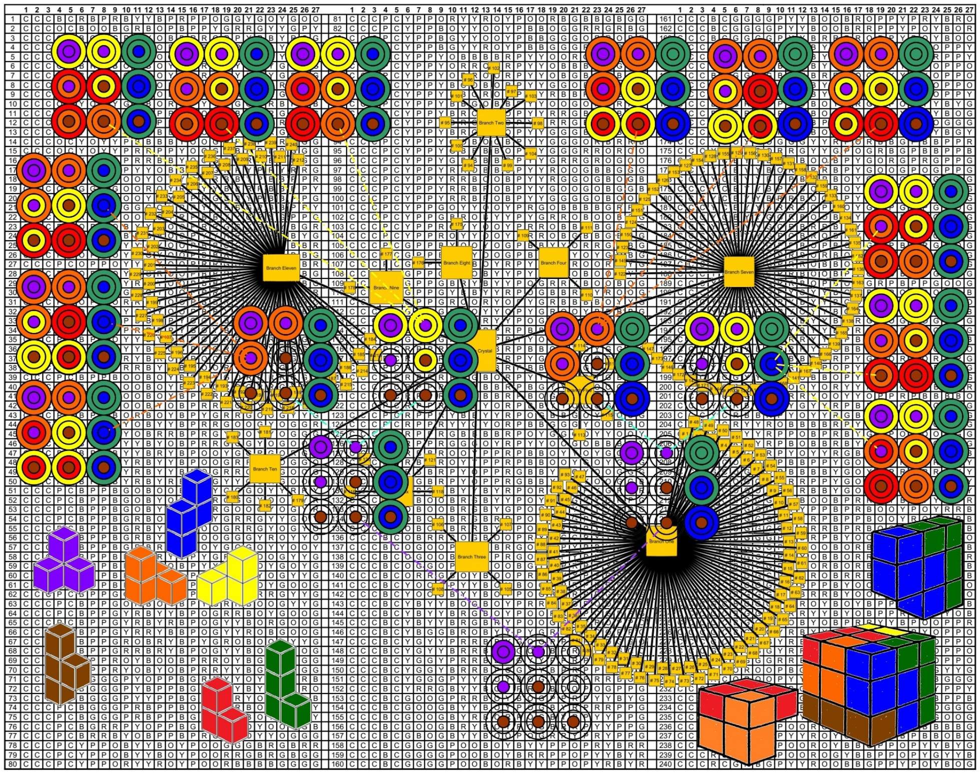


Figure 5: The 240 Unique Soma Cube solutions as a spreadsheet and a tree graph with piece definitions and a concentric circle compact solution notation of twelve closely related solutions. The tree graph was generated with yEd Graph Editor using the sorted spreadsheet in Figure 4 as the input.

Similar work has been done by Hansen on the 6×10 tiling of pentominoes, showing that "[a]lthough there are 2339 pentomino tilings of the 6×10 rectangle, many pairs of solutions are similar. We show that the solutions can be divided into 911 equivalence classes by the similarity transformations-rotate or reflect a subset of the pieces, interchange two congruent subsets, rearrange two pieces" [8].

Conway and Guy [5] developed the SOMAP, a map of the 240 solutions that similarly relates all of the solutions with a very amazing graph (Figure 7). The graph and how to use it is a bit cryptic, and we are still not able to readily use its notation and methods. Some good work clarifying this has been done by Eberhardt [6].

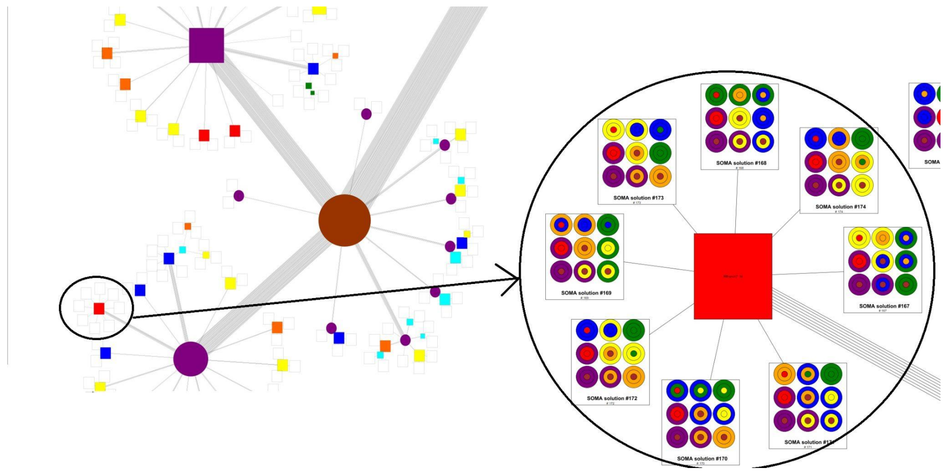


Figure 6: yEd allows nodes of graphs to be associated with scalar vector graphic files. In this case, the leaves of the tree graph are 27 concentric circles arranged to show the three cube layers. Compact, colorful, and verbose.

Our approach is less ambitious: find all of the equivalence classes that can be "cleaved" at the surface of a cube solution like a gemstone. This restriction is both practical and aesthetically pleasing. Practical in that there are going to be fewer to find and categorize also they will be readily identifiable by eye and also pleasing to the eye.

2. Definitions

Before we go mining for gems, we need to know a little bit more about what we are looking for and how it was formed.

We named and colored the Soma pieces as in Figure 2. This scheme gives each piece a descriptive name and color that can be shortened to the unique single capitalized letter for that piece, which makes for a simple representation in graphs, spreadsheets, and tables, as in Figure 4. We used the Latin names Dexter and Sinister for two pieces that are chiral mirrors of each other, in homage to the chiral molecule limonene, the righthanded being the flavor lemon (Dexter and Yellow) and the lefthanded orange (Sinister and Orange).

Constructing the cube into the 240 unique solutions is quite a challenge but as before Berlenkamp, Conway, and Guy provide guidance [1].

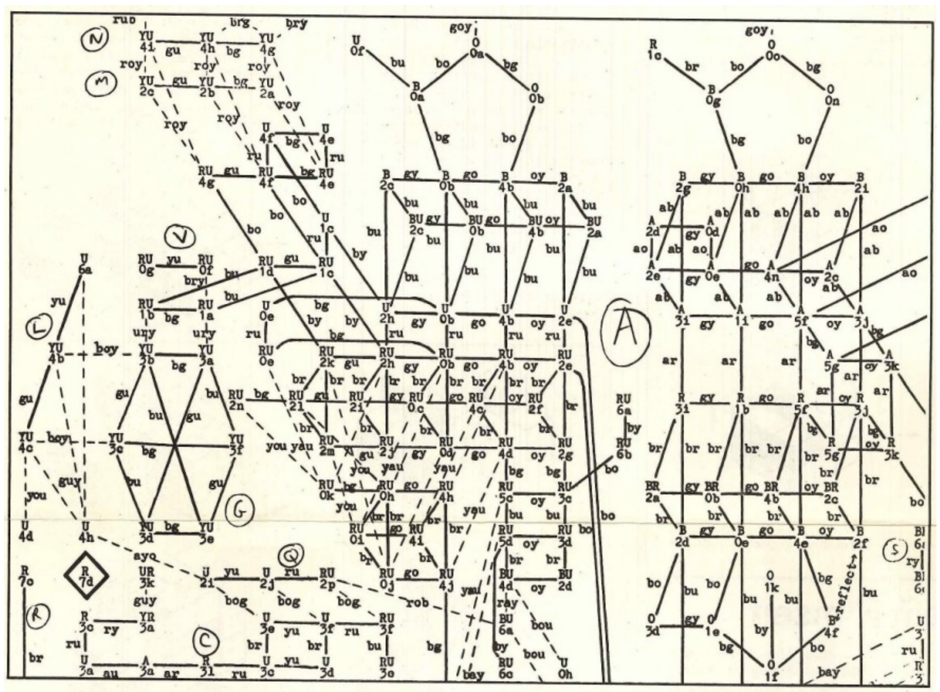


Figure 7: The SOMAP is a graph that shows the solutions to the Soma Cube and their relationships to each other. Conway and Guy developed a very compact notation for solutions which appear at the vertices. The edge labels identify which pieces get “swapped” to form another solution on an edge connected vertex. This partial SOMAP is from the Soma Addict Newsletter published in 1972.

3. Shall We Play a Game?

3.1. Eight Ball Corner Pocket

Figure 8 is a piece by piece illustration of understanding the Soma Cube puzzle mathematically by Berlenkamp, Conway, and Guy [1]. A cube has eight vertices and each of the seven pieces can contribute 0 or 1 in five cases and 0, 1, or 2 in two cases.

What if all seven pieces contribute the maximum? The result is 9 vertices and that is not a cube. All of the pieces must contribute vertices in such a way that one of the pieces must contribute 1 vertex less than it could.

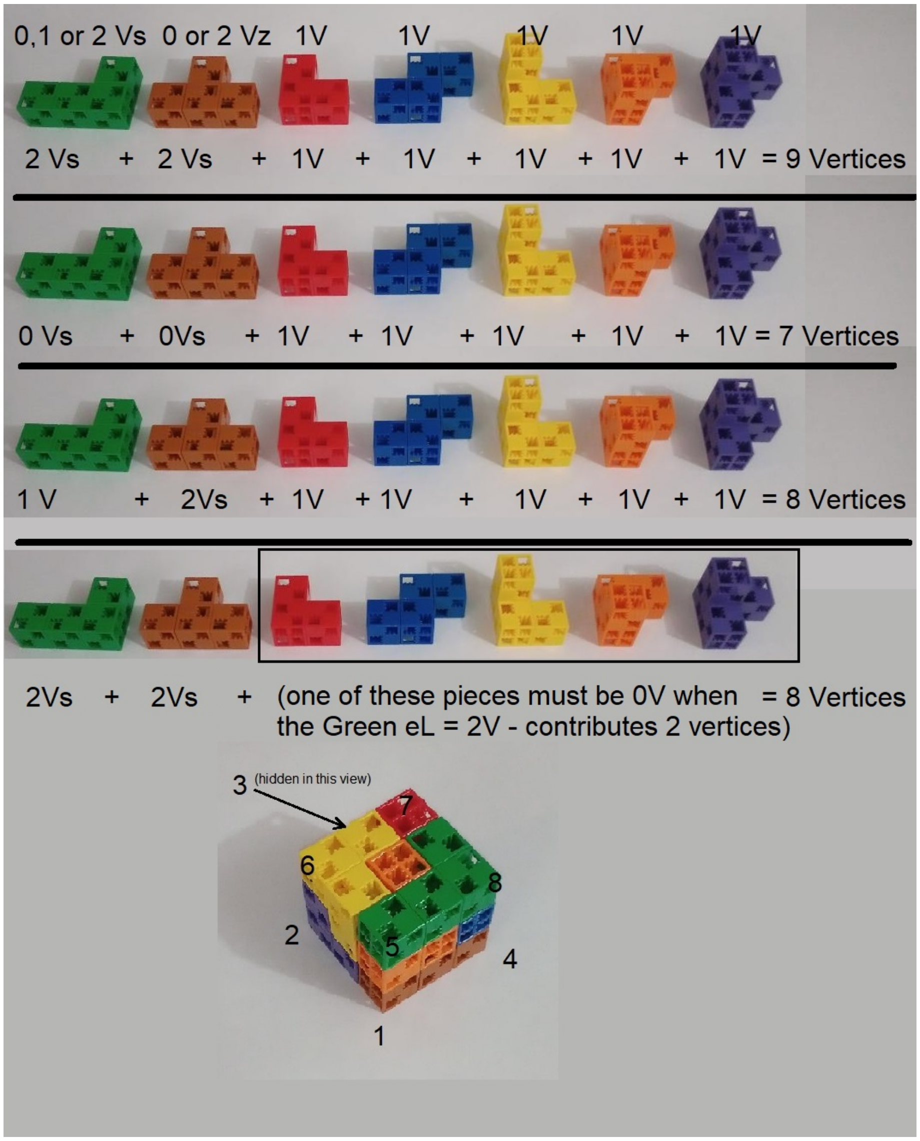


Figure 8: A cube has eight vertices. The vertex contribution of each piece can vary but the sum of the contributions of the seven pieces must be eight for a cube solution. This leads to some piece placement constraints that reduce Soma to a six-piece puzzle.

Let's try taking the exceptional pieces out of the vertex equation, the Tee and eL contributing 0 vertices. That adds up to 7 vertices-also not a cube.

The Tee, as it turns out, must always contribute 2 vertices, which means it must always be on an edge of the cube. This is good for several reasons, the puzzle is now a six-piece puzzle and by fixing the Tee to one and only one of the twelve possible edges of the cube we eliminate the need to check solutions as we find them to see if it is a symmetric rotation of another one.

3.2. How About Checkers

Constraining the Tee has greatly reduced the scope of our search. However, there is more to do per Conway and company [5]. Let's dig a little deeper into the properties of the cube before digging for gems.

Here we also introduce another way of thinking about and representing the Soma cube — as a graph; see Figure 9. Each circle is a vertex representing one of the 27 unit cubes of a $3 \times 3 \times 3$ cube. The lines connecting the vertices are the edges of the graph. For the moment they are here to make the picture a bit more clear. They also serve another purpose: the edges provide a way to relate the pieces to each other, independent of their position in the overall cube. We will explore this later using the block programming language SNAP! [X] and an add on application called Edgy[X].

Let's "checker" the cube with "emeralds and flames"; an alternating pattern emerges which isolates the vertices (in the larger graph theoretic sense) into two groups

8 Vertices and 6 Faces make 14 "flames"

12 Edges and 1 Center make 13 "emeralds"

14 VF + 13 EC = 27 VFEC

which are the 27 unit cubes of a $3 \times 3 \times 3$ *cube*

This means each piece must contribute the same number of "emeralds and flames" in every cube solution. Look at Figure 10 for an example.

The green eL in A is contributing 2 Vertices making its VF = 2 and 2 Edges making its EC = 2 so VFEC = 4.

In B the green eL is contributing 1 Vertex and 1 Face VF = 2 and 2 Edges making its EC = 2 so VFEC = 4.

The red Vee is the piece changing its Vertex contribution in this case.

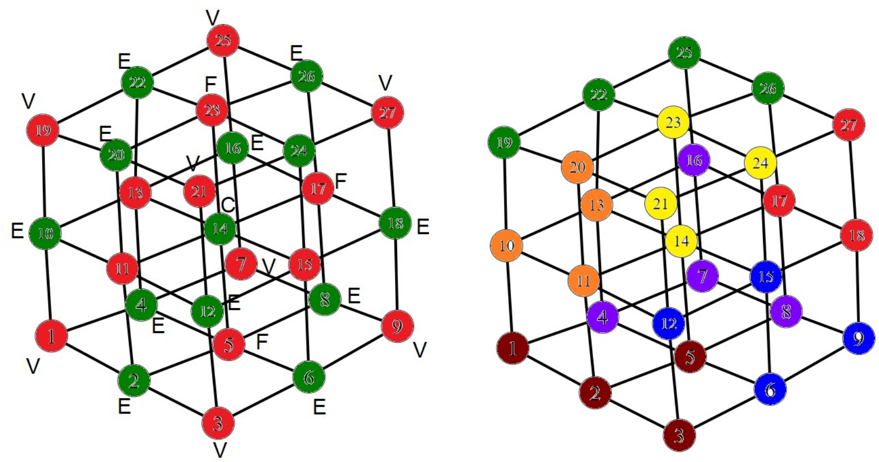


Figure 9: Ball and stick representation of the Soma cube allows a 2D view of all 27 sub cubes of the $3 \times 3 \times 3$ puzzle. Left: Emerald edges and centers join with red flame vertices and faces to show another way of looking at the cube and how each piece contributes to a solution. Right: A sample solution that allows for seeing all sub cubes. This perspective also informs the SNAP!/Edgy computer programming that will help us find gems in the solution set.

Berlekamp, Conway, and Guy [1] summarize the allowable positions in the $3 \times 3 \times 3$ cube for each of the piece's characteristic VFEC combinations with a table of drawings. Orth [9] further develops a complete table of VFEC contributions for each piece and their possible combinations [Figure 11]. This is then used to simplify the search for solutions in a Fortran program.

Inspired by this partitioning, we decided to extend this approach further as shown in Figure 13. The motivation for this is yet another issue of finding all unique solutions. Because the Orange Sinister and Yellow Dexter pieces are mirrors of each other, if you find one solution there is another that can be found by mirroring the other pieces and swapping Sinister and Dexter (as shown in Figure 12). A mirror solution is not a unique solution. Our idea as part of an effort to find all 240 solutions "by hand and eye" was to begin from all of the possible positions that the Purple Crystal can be placed in as the second piece placed after the Caramel Tee while keeping it to the left side of the $3 \times 3 \times 3$ cube, thus eliminating the mirror solution issue. There are 12 such positions that are possible.

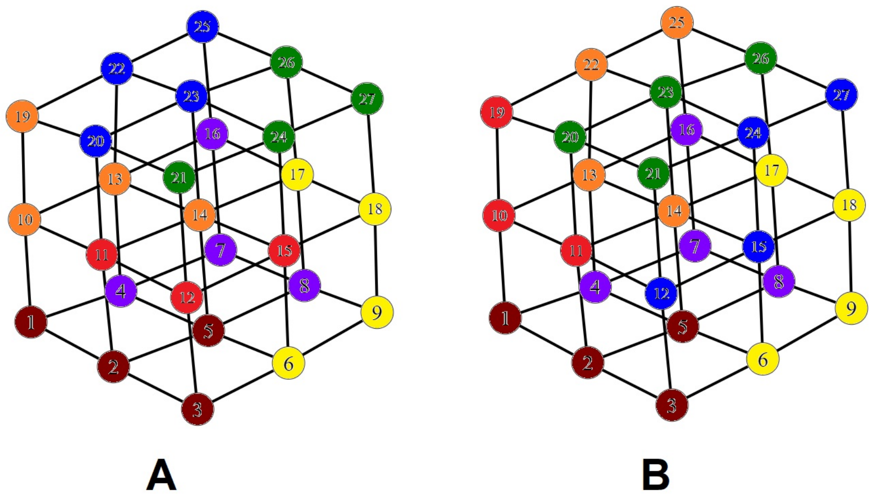


Figure 10: VFEC (Vertex, Face, Edge, Center) contributions of each piece varies by piece position in a cube puzzle solution. Here the green eL contributes two vertices in the left cube and one vertex on the right. Can you see which piece goes from contributing 0 vertices to contributing 1 to make up for green eL's reduced vertex contribution?

Orth [9] computed his solutions to be "left justified" with respect to the Purple Crystal (Piece 2 or tripod for Orth) piece. We converted his solutions to 240 rows of 27 spreadsheet cells. Then we converted each of the solution rows [Figure 4] into strings and replaced all of the piece color letters with "X" except the "P" for the Crystal. These are now unique strings that can be used to sort the other columns. After spreadsheet sorting the solutions are ordered into eleven but not twelve groups which was somewhat expected. Prior to sorting we tried each of the twelve starting combinations shown in Figure 13 and found one (lower right hand corner of Figure 13) did not "bear fruit" and yield a cube solution. (Figure 15 shows a binder of business card sized cube graphs that were used in an attempt to find and categorize all of the soma cube solutions by hand. 170 were found.)

Why doesn't that Tee and Crystal starting position admit a cube solution? The Crystal's position meets the VFEC criteria of VF=1 and EC=3. It doesn't look impossible; in fact, at first glance, it looks like a better prospect than the one to its left. However, that one only yields three cube solutions.

Piece	1	2	3	4	5	6	7	8	9	10	11
1	FEEV	FEEV	FEEV	FEEV	CFEV	FEEV	CFEV	FEEV	CFFE	FFEE	CFEV
2	CFEE	EEEV	EEEV	EEEV	EEEV	EEEV	EEEV	EEEV	EEEV	EEEV	EEEV
3	FEEV	CFFE	FEEV	FEEV	FEEV	FEEV	FEEV	FEEV	FEEV	FEEV	FEEV
4	FEVV	FEVV	FEVV	FEVV	FEVV	FEVV	FEVV	FEVV	FEVV	FEVV	FEVV
5	FEEV	FEEV	FEEV	CFEV	FEEV	CFEV	FEEV	CFFE	FEEV	CFEV	FFEE
6	EEVV	EEVV	EEVV	EEVV	EEVV	EEVV	EEVV	EEVV	EEVV	EEVV	EEVV
7	FEV	FEV	CFF	FFE	FFE	FEV	FEV	FEV	FEV	FEV	FEV
all	74	66	38	14	14	21	21	51	51	65	65
left	37	33	19	6	8	9	12	26	25	22	43
right	37	33	19	8	6	12	9	25	26	43	22

Figure 11: This table summarizes the VFEC (Vertex, Face, Edge, and Center) placement combinations for all of the Soma Cube pieces. Orth uses these properties to further constrain the search space of his Fortran program to more rapidly find all of the Soma Cube puzzle solutions.

The starting combination in the upper left corner admits over one hundred solutions, it does look more inviting for dropping in pieces after all. Perhaps topology may lend some insights here, a figure of merit summarizing the unfilled space with a sort of Hausdorff Hein Dimension.

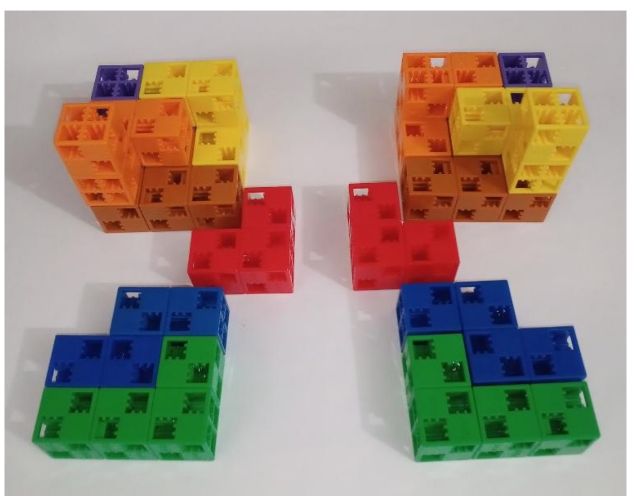


Figure 12: Two Soma Cube puzzle solutions that demonstrate the mirroring properties of the Yellow Dexter and Orange Sinister pieces. The other five pieces can flip and rotate to mirror themselves, Dexter and Sinister cannot. For such a small system it is interesting that a sort of “symmetry breaking” can happen and be so fundamental to the combinatorial analysis.

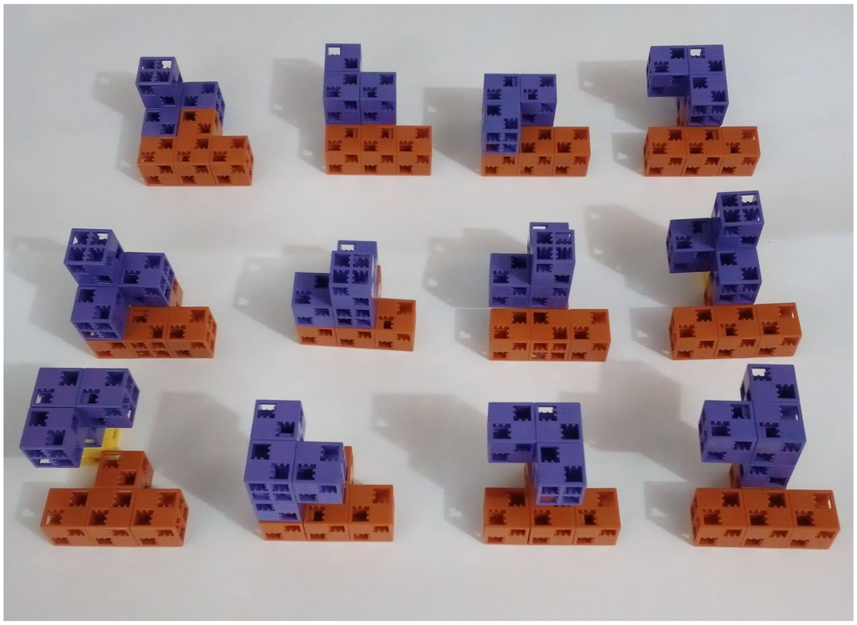


Figure 13: The twelve possible combinations of the Caramel Tee and "left justified" Purple Crystal. Now we have twelve puzzles of five pieces. This makes writing a SNAP!/Edgy program to find the solutions a bit easier and informs our tree graph structure. Caramel Tee as the "trunk" and Purple Crystal as the "main branches".

3.3. Precious Gems Don't Just Grow On Trees You Know

We have discovered that both finding all of the unique solutions by hand and sorting an existing table of them is complicated by the many similar solutions. An interesting case of this is shown in Figure 14. Starting with Tee and Crystal positions in the upper right of Figure 13, we add Dexter forming a "chaise lounge" shape. This can be completed into the cube in at least seven (just checked by hand at this typing) ways. While finding solutions by hand makes it difficult to find unique ones without lots of backtracking to see if a solution was repeated. During Spreadsheet sorting this becomes an issue because aesthetically speaking, shouldn't this be a branching point? Two of the seven also contain one of the equivalence classes shown in Figure 1: the "Blue Green Corner_1". Doesn't that make those solutions a separate branch as well in some aesthetic sense?

If we adopt strict criteria sorting on “the next piece placed/sorted is always the same piece”, the similar final solutions get spread all over the tree which dilutes the sorting to being only a little better than a table of results. How can we present the results so all of the interesting properties of the solution set are presented concisely and compactly? How can we find all of the solutions with little or no backtracking? We don’t know . . . yet. We do know, however, that there are many of these gems and I have a way to make maps to find them.

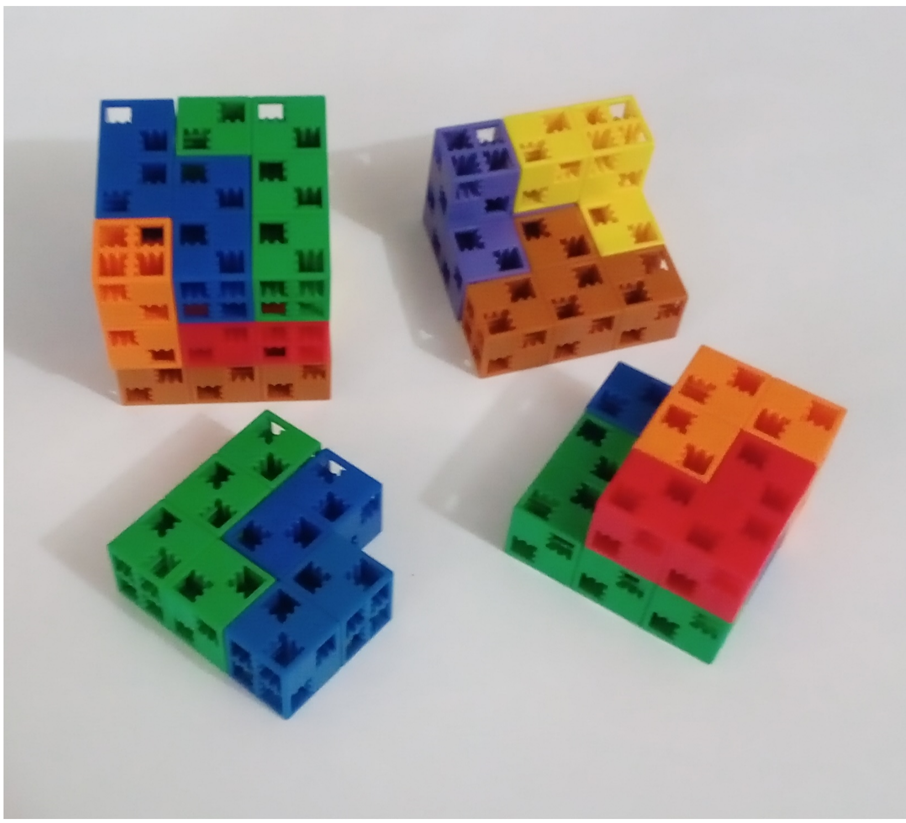


Figure 14: A problem—how do we define gems? Particularly when a cleavable two-piece gem (left) is part of another four-piece cleavable gem (right). We decided three pieces is the maximum for a gem to limit this sort of overlap.

We just mentioned the equivalence class “Blue Green Corner_1”. This used the notation for a gem/equivalence class that we have not yet formally introduced. So now a definition is in order.

We have decided to use the new par

1. Colors of the pieces in the complex of pieces
2. Corner or Face to indicate where it was cleaved from the cube
3. Order 1 or 2_X to indicate -
   a. 1 - the piece can be picked up then flipped, twisted, or rotated to drop back in and complete the cube.
   b. 2_X - the pieces must be rearranged to drop back in and complete the cube. X being the number of rearrangements.

For example, the equivalence class in the lower left of Figure 1 is "Orange Red Corner2_3" because it is cleaved from a corner, must be rearranged to drop back into the cube and there are three ways to do that.

3.4. Graph Theory in a SNAP!

A graph representation of Soma Cube solutions was introduced previously [Figure 10]. This presentation provides some nice features such as being able to see what piece is where for all 27 unit cubes. This would be blocked from view if looking at an actual Soma Cube made from blocks. If presented as a table such as the one in Figure 3, that requires interpretation and hopping visually and mentally through three cube layers. Graph representation is a good thing.

There's more — each solution has another graph that describes the relationship of puzzle pieces to each other [Figure 16]. Seven vertices, one for each piece with edges that connect two vertices if any of the unit cubes of those pieces touch at unit cube faces. The edge weights are equal to the number of faces that are touching. The gems can then be found as subgraphs with the same sum of edge weights for the vertices (pieces) of interest.

Not all subgraphs meeting gem-like criteria turn out to be gems, but the search space is greatly reduced. This is accomplished appropriately enough by a block programming language called SNAP! [10] and an add-on application for graph theory properties called Edgy [2] that provides a hands-on interface to manipulate graphs via algorithms.

In Figure 16, the SNAP!/Edgy code has detected (possibly) what we were calling at the time a "Red Green Corner Leaf."

How does it do that?

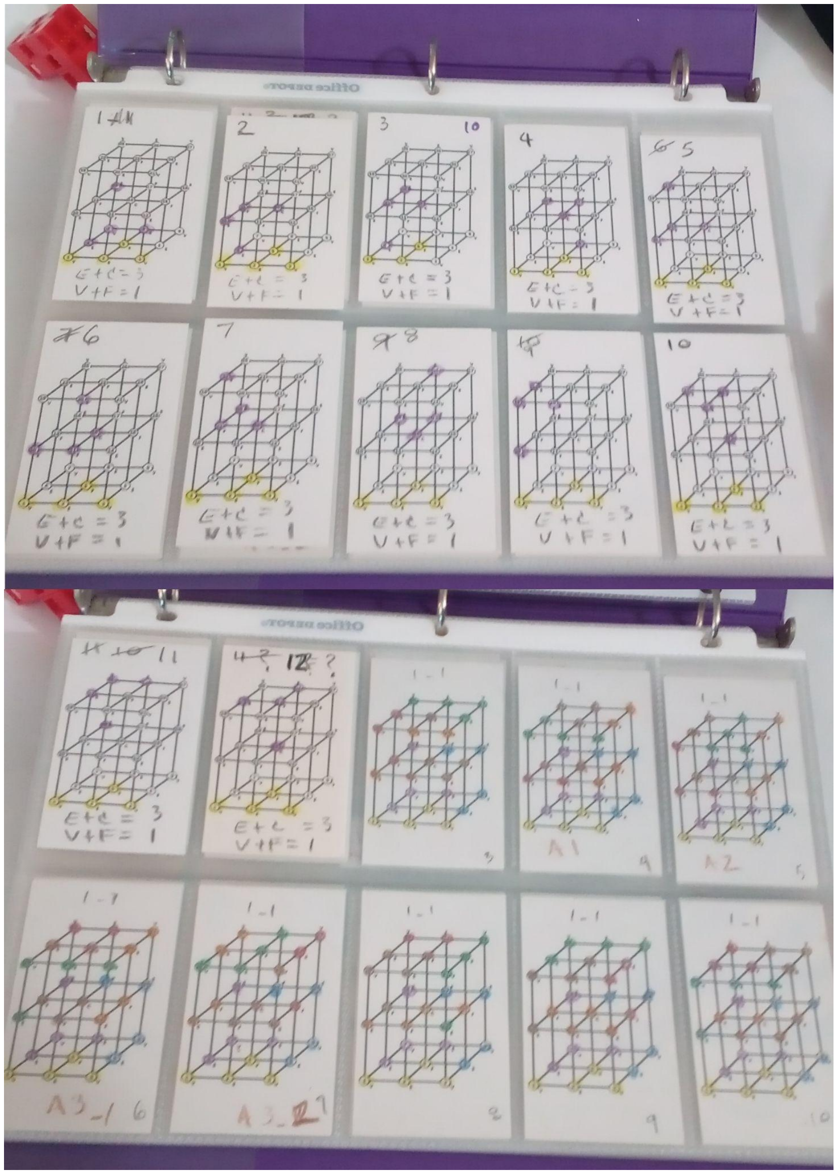


Figure 15: Trying to manually find all of the solutions to the trunk and branching scheme from Figure 12 led us to the discovery of the beautiful gems but also to many questions about how to arrange the sub-branches. Branches of very different structures were sprouting the same leaf. How can we show both the progression of pieces that yield cube solutions and the same leaves sprouting from those differing branches?

Here is an outline of the code

1. Get a solution from an array of all the solutions.
2. Plot the solution on the 3x3x3 cube graph.
3. Compute a Relationship Graph of the puzzle pieces [7].
4. Search the Relationship Graph for edge weight sums that correspond to previously recognized gems.
5. The code will stop when it finds one of the specified edge weight sums prompting you to build the solution with Soma Cube blocks and see if you have uncovered a gem.
6. If yes, take a screenshot and save it to a folder. Click the "yellow resume arrow" in the upper right corner of the screen to find more "gem possibles".
7. If no, click the "yellow resume arrow" in the upper right to find more "gem possibles".

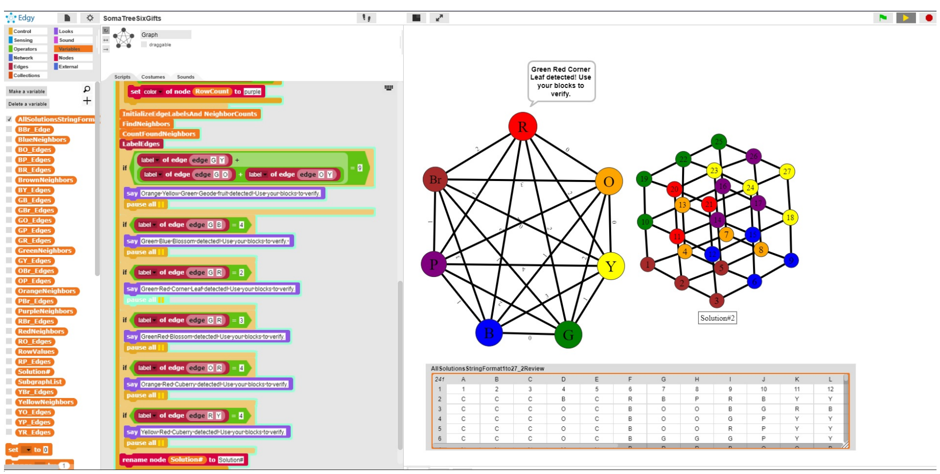


Figure 16: SNAP!/Edgy block programming is great for visualizing program flow, graphical outputs, and data. The table in the bottom right was loaded into the user interface simply by right clicking on a variable block and loading the associated CSV file which contains the 240 unique Soma Cube puzzle solutions. The table can be resized and scrolled.

The results so far show that of the 240 unique Soma Cube puzzle solutions 105 of them have equivalence class properties that meet the added criteria of being "cleavable" from a face or corner of a cube [Figure 17].

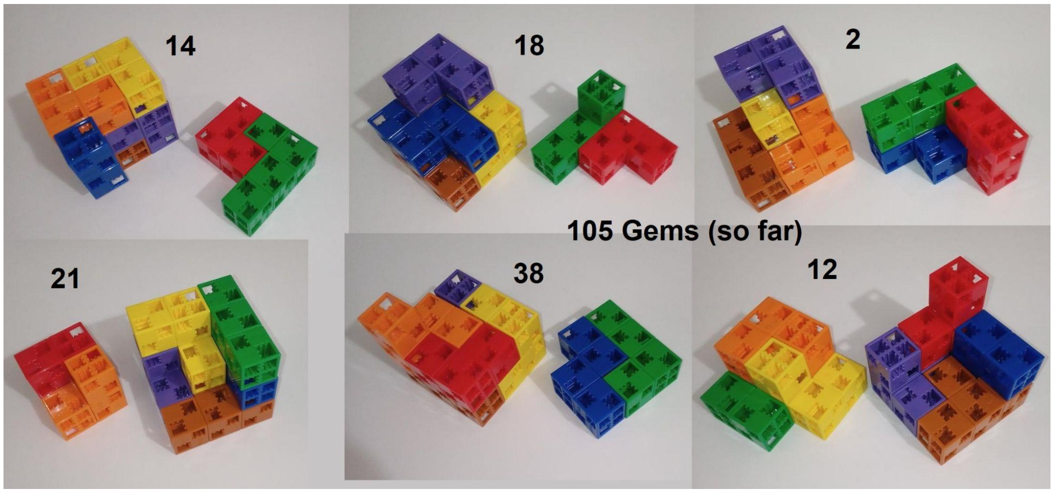


Figure 17: Nearly half of the 240 Soma Cube puzzle solutions have vertex or face cleavable gems. Perhaps there are more gems to discover or other ways to define similarities between solutions.

We encountered these gems during our attempts to find the cube solutions by hand and eye. There could be more. We are not counting the "chaise lounge" complex mentioned in the "Precious Gems" section because we don't find it an appealing gem-like shape. Other explorers may have a different perspective. We hope so. We relish being shown new things particularly if it means we have been limited in our own vision.

4. Topics to Explore

We have, we hope, sparked some interest in other math explorers (to borrow a term from Francis Su [11]) to take these concepts and methods and extend them further. We use the term math explorer here because much of this is doable by students from middle school on up who have an interest in shapes, patterns, and puzzles. For those looking for a reason to explore coding programs, SNAP! was designed for them.

Here is a list of topics we think may be appealing:

- Pentominoes. They continue to be a popular puzzle and have many manifestations of different size rectangles. Hansen [8] has only analyzed the 6×10 problem; there are many more [14].

- Tetrominoes (aka Tetris). Are there gems to be found here? This could make for an interesting twist on the many Tetris games and activities.
- Converting SNAP!/Edgy programs to Python. Our source code is available via GitHub [12]. This code could very likely be improved to run faster and incorporate more sophisticated algorithms. Edgy has some good documentation on this topic [4].

A personal note from the first author (EV)

On Christmas Eve 1969, I received a Soma Cube. My mom likely noticed my keen interest in it when it was advertised on television. As my parents watched me open it and spill the pieces onto the floor my dad asked me "How long do you think it will take you to figure this out?" Fifty years later, I can confidently say "I think I am making some progress".

Financial Disclosure

The blocks used in this paper are manufactured by Artec (https://www.artec-kk.co.jp/en/blocks/). Edward Vogel previously used their blocks in a commercial product "BLOKL - Building Blocks Puzzles and Games".

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